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I. *On the Foundations of the Theory of Probabilities.* *By* R. L. Ellis, Esq. M.A., *Fellow of Trinity College.*

[Read *Feb.* 14, 1842.]

The Theory of Probabilities is at once a metaphysical and a mathematical science. The mathematical part of it has been fully developed, while, generally speaking, its metaphysical tendencies have not received much attention.

This is the more remarkable, as they are in direct opposition to the views of the nature of knowledge, generally adopted at present.

(2.) The theory received its present form during the ascendancy of the school of Condillac. It rejects all reference to *à priori* truths as such, and attempts to establish them as mathematical deductions from the simple notion of probability. Are we prepared to admit, that our confidence in the regularity of nature is merely a corollary from Bernouilli's theorem? That until this theorem was published, mankind could give no account of convictions they had always held, and on which they had always acted? If we are not, what refutation have we to give? For these views are entitled to refutation, from the general reception they have met with, from the authority of the great writers by whom they were propounded, and even from the imposing form of the mathematical demonstration in which they are invested.

I shall be satisfied if the present essay does no more than call attention to the inconsistency of the theory of probabilities with any other than a *sensational* philosophy.

(3.) As the first principles of the mathematical theory are familiar to every one, I shall merely recapitulate them.

If on a given trial, there is no reason to expect one event rather than another, they are said to be equally possible.

The probability of an event is the number of equally possible ways in which it may take place, divided by the total number of such ways which may occur on the given trial.

If a_1, b_1, ... m_1, denote equally possible cases which may occur on one trial, $a_2 b_2 \ldots . k_2$ those which may occur on a second trial, $a_3 b_3 \ldots . p_3$ those belonging to a third, &c.: then a_1, $b_2 a_3 \ldots .$ $a_1 a_2 b_3 \ldots .$&c. &c. are all equally possible complex results.

Hence it follows that on the repetition of the same trial k times, the probability that an event whose simple probability is m will occur p times is

$$\frac{1 . 2 \ldots . k}{1 . 2 \ldots p \, 1 . 2 \ldots (k-p)} m^p (1-m)^{k-p}:$$

this follows merely by the doctrine of combinations. These are all the propositions to which I shall have occasion to refer.

(4.) If the probability of a given event be correctly determined, the event will on a long run of trials, tend to recur with frequency proportional to this probability.

This is generally proved mathematically. It seems to me to be true *à priori.*

When on a single trial we expect one event rather than another, we necessarily believe that on a series of similar trials the former event will occur more frequently than the latter. The connection between these two things seems to me to be an ultimate fact, or rather, for I would not be understood to deny the possibility of farther analysis—to be a fact, the evidence of which must rest upon an appeal to consciousness. Let any one endeavour to frame a case in which he may expect one event on a single trial, and yet believe that on a series of trials another will occur more frequently; or a case in which he may be able to divest himself of the belief that the expected event will occur more frequently than any other.

For myself, after giving a painful degree of attention to the point, I have been unable to sever the judgment that one event is more likely to happen than another, or that it is to be expected in preference to it, from the belief that on the long run it will occur more frequently.

(5.) It follows as a limiting case, that when we expect two events equally, we believe they will recur equally on the long run. In this belief we may of course be mistaken: if we are, we are wrong in expecting the two events equally, and in thinking them equally possible. Conversely, if the events are truly equally possible, they really will tend to recur equally on a series of trials. But this proves the proposition placed at the head of the section: for if any event can occur in a out of b equally possible ways, its probability is $\frac{a}{b}$: and if all these b cases tend to recur equally on the long run, the event must tend to occur a times out of b; or in the ratio of its probability. Which was to be proved.

(6.) Let us now examine the mathematical demonstration of this proposition. In entering upon it, we are supposed to have no reason whatever to believe that equally possible events tend to occur with equal frequency.

It is well known that what is called Bernouilli's theorem, relates to the comparative magnitudes of the several terms of the binomial expansion.

The general term of $\{m + (1 - m)\}^k$, is $\frac{[k]}{[p][k-p]} m^p (1 - m)^{k-p}$, which is the probability that an event whose simple probability is m will recur p times on k trials; and hence the connexion between the binomial expansion and the theory of probabilities.

(7.) A particular example will suffice to illustrate what seems to me to be the essential defect of the mathematical proof of the proposition in question.

A coin is to be thrown 100 times: there are 2^{100} definite sequences of heads and reverses, all equally possible if the coin is fair. One only of these gives an unbroken series of 100 heads. A very large number give 50 heads and 50 reverses; and Bernouilli's theorem shows that an absolute majority of the 2^{100} possible sequences give the difference between the number of heads and reverses less than 5.

If we took 1000 throws, the absolute majority of the 2^{1000} possible sequences give the difference less than 7, which is proportionally smaller than 5. And so on.

Now all this is not only true, but important.

But it is not what we want. We want a reason for believing that on a series of trials, an event tends to occur with frequency proportional to its probability; or in other words, that generally speaking, a group of 100 or 1000 will afford an approximate estimate of this probability.

But, although a series of 100 heads can occur in one way only, and one of 50 heads and 50 reverses in a great many, there is not the shadow of a reason for saying that therefore, the former series is a rare and remarkable event, and the latter, comparatively at least, an ordinary one.

Non constat, but the single case producing 100 heads may occur so much oftener than any case which produces 50 only, that a series of 100 heads may be a very common occurrence, and one of 50 heads and 50 reverses may be a curious anomaly.

Increase the number of trials to 1000, or to 10,000. Precisely the same objection applies: namely, that in Bernouilli's theorem, it is merely proved that one event is more probable than another, *i. e.* by the definition can occur in more equally possible ways, and that there is no ground whatever for saying, it will therefore occur oftener, or that it is a more natural occurrence. On the contrary, the event shown to be improbable may occur 10,000 times for once that the probable one is met with.

To deny this, is to admit that if an event can take place in more equally possible ways, it will take place more frequently. But if this is admitted, Bernouilli's theorem is unnecessary. It leaves the matter just where it was before, and introduces no new element into the question.

(8.) Thus, both by an appeal to consciousness, and by the impossibility of dispensing with such an admission, we are led to recognize the principle, that when an event is expected rather than another, we believe it will occur more frequently on the long run. And thus we perceive that we are in the habit of forming judgments as to the comparative frequency of recurrence of different possible results of similar trials. These judgments are founded, not on the fortuitous and varying circumstances of each trial, but on those which are permanent—on what is called the nature of the case. They involve the fundamental axiom, that on the long run, the action of fortuitous causes disappears. Associated with this axiom is the idea of an average among discordant results, &c.

I conceive this axiom to be an *à priori* truth, supplied by the mind itself, which is ever endeavouring to introduce order and regularity among the objects of its perceptions.

(9.) With a view to conciseness, I omit several interesting points which here present themselves—namely, the connection between the axiom just stated, and the inductive principle; the real utility of Bernouilli's theorem; and what seems to me to be the true definition of probability, founded on a reference to the ratios developed on the long run.

I proceed to illustrate what has been said by a few passages from Laplace's "*Essai Philosophique sur les Probabilités.*"

(10.) It seems obvious that no mathematical deduction from premises which do not relate to laws of nature, can establish such laws. Yet it is beyond doubt that Laplace thought Bernouilli's theorem afforded a demonstration of a general law of nature, extending even to the moral world.

At p. XLII. of the Essay, prefixed as an Introduction to the third edition of the Théorie des Probabilités, after giving some account of the theorem of James Bernouilli, Laplace proceeds: "On peut tirer du théorème précédent cette conséquence qui doit être regardée comme une loi générale, savoir que les rapports des effets de la nature, sont à fort peu près constants, quand ces effets sont considérés en grand nombre.... Je n'excepte pas de la loi précédente, les effets dus aux causes morales."

It appears not to have occurred to Laplace, that this theorem is founded on the mental phenomenon of expectation. But it is clear that expectation never could exist, if we did not believe in the general similarity of the past to the future, *i.e.* in the regularity of nature, which is here deduced from it.

A little further on,..."Il suit encore de ce théorème que dans une série d'évènemens indéfiniment prolongée, l'action des causes régulières et constantes doit l'emporter à la longue, sur celle, des causes irrégulières....Ainsi des chances favorables et nombreuses etant constamment attachées à l'observation des principes éternels de raison de justice et d'humanité, qui fondent et qui maintiennent les societés; il y a un grand avantage à se conformer à ces principes, et de graves inconvéniens à s'en écarter. Que l'on consulte les histoires, et sa propre expérience on y verra tous les faits venir à l'appui de ce résultat du calcul." Without disputing the truth of the conclusion, we may doubt whether it is to be considered as a "résultat du calcul."

The same expression occurs immediately afterwards in another passage, in which the writer seems to allude to the history of his own times, and to the ambition of the great chieftain whom he at one time served.

Indeed it would seem as if to Laplace all the lessons of history were merely confirmations of the "résultats du calcul." We are tempted to say with Cicero—"hic ab artificio suo non recessit."

(11.) The results of the theory of probabilities express the number of ways in which a given event can occur, or the proportional number of times it will occur on the long run: they are not to be taken as the measure of any mental state; nor are we entitled to assume that the theory is applicable wherever a presumption exists in favour of a proposition whose truth is uncertain.

Nevertheless it has been applied to a great variety of inductive results; with what success and in what manner, I shall now attempt to enquire.

(12.) Our confidence in any inductive result varies with a variety of circumstances; *one* of these is the number of particular cases from which it is deduced. Now the measure of this confidence which the theory professes to give, depends on this number exclusively. Yet no one can deny, that the force of the induction may vary, while this number remains unchanged. This consideration appears almost to amount to a *reductio ad absurdum*.

(13.) If, on m occasions, a certain event has been observed, there is a presumption that it will recur on the next occasion. This presumption the theory of probabilities estimates at $\frac{m+1}{m+2}$. But here two questions arise; What shall constitute a "next occasion?" What degree of similarity in the new event to those which have preceded it, entitles it to be considered a recurrence of the same event?

Let me take an example given by a late writer:—

Ten vessels sail up a river. All have flags. The presumption that the next vessel will have a flag is $\frac{11}{12}$. Let us suppose the ten vessels to be Indiamen. Is the passing up of any vessel whatever, from a wherry to a man of war, to be considered as constituting a "next occasion?" or will an Indiaman only satisfy the conditions of the question?

It is clear that in the latter case, the presumption that the next *Indiaman* would have a flag is much stronger, than that, as in the former case, the next *vessel* of any kind would have one. Yet the theory gives $\frac{11}{12}$ as the presumption in both cases. If right in one, it cannot be right in the other. Again, let all the flags be red. Is it $\frac{11}{12}$ that the next vessel will have a red flag, or a flag at all? If the same value be given to the presumption in both cases, a flag of any other colour must be an impossibility.

It is to be noticed, that I only refer to the visible differences among different kinds of vessels, and not to any knowledge we may have about them from previous acquaintance.

(14.) I turn to a more celebrated application of the theory.

All the movements of the planetary system, known as yet, are from west to east. This undoubtedly affords a strong presumption in favour of some common cause producing motion in that direction. But this presumption depends not merely upon the number of observed movements, but also on the natural affinity which in a greater or less degree appears to exist among them.

This is so natural a reflection, that Lacroix, in calculating the mathematical value of the presumption, omits the rotatory movements, and, I believe, those of the secondary planets, in order, as he expressly says, to include none but similar movements. But in the admission thus by implication made, that regard must be had to the similarity of the movements, too much is conceded for the interests of the theory. For are the retained movements absolutely similar? The planets move in orbits of unequal eccentricity and in different planes: they are themselves bodies of very various sizes; some have many satellites and others none. If these points of difference were diminished or removed, the presumption in favour of a common cause determining the direction of their movements would be strengthened; its calculated value would not increase, and *vice versa*.

Again, up to the close of 1811, it appears (Laplace) that 100 comets had been observed, 53 having a direct and 47 a retrograde movement. If these comets were gradually to lose the peculiarities which distinguish them from planets—we should have 64 planets with direct movement, 47 with retrograde. The presumption we are considering would, in such a case, be very much weakened. At present, we unhesitatingly exclude the comets on account of their striking peculiarities: in the case supposed we should with equal confidence include them in the induction. But at what precise point of their transition-state are we abruptly, from giving them no weight at all in the induction, to give them as much as the old planets?

(15.) It is difficult to acquiesce in a theory which leads to so many conclusions seemingly in opposition to the common sense of mankind.

One of the most singular of them may, perhaps, serve as a key to explain their nature. When any event, whose cause is unknown, occurs, the probability that its *à priori* probability was greater than $\frac{1}{2}$ is $\frac{3}{4}$. Such at least is the received result. But in reality, the *à priori* probability of a given event has no absolute determinate value independent of the point of view in which it is considered. Every judgment of probability involves an analysis of the event contemplated. We toss a die, and an ace is thrown. Here is a complex event. We resolve it into, (1) the tossing of the die; (2) the coming up of the ace. The first constitutes the 'trial,' on which different possible results might have occurred; the second is the particular result which actually did occur. They are in fact related as *genus* and *differentia*. Beside both, there are many circumstances of the event; as how the die was tossed, by whom, at what time, rejected as irrelevant.

This applies in every case of probability. Take the case of a vessel sailing up a river. The vessel has a flag. What was the *à priori* probability of this? Before any answer can by possibility be given to the enquiry, we must know (1) what circumstances the person who makes it rejects as irrelevant. Such as, *e. g.* the colour of which the vessel is painted, whether it is sailing on a wind, &c. &c.; (2) what circumstances constitute in his mind the 'trial;' the experiment which is to lead to the result of flag or no flag; must the vessel have three masts? must it be square rigged? (3) What idea he forms to himself of a flag. Is a pendant a flag? Must the flag have a particular form and colour? Is it matter of indifference whether it is at the peak or the main? Unless all such points were clearly understood, the most perfect acquaintance with the nature of the case would not enable us to say what was the *à priori* probability of the event: for this depends, not only on the event, but also on the mind which contemplates it.

The assertion therefore that $\frac{3}{4}$ is the probability that any observed event had on an *à priori* probability greater than $\frac{1}{2}$, or that three out of four observed events had such an *à priori* probability, seems totally to want precision. *A priori* probability to what mind? In relation to what way of looking at them?

(16.) Let us see if this will throw any light on the question. Let h be a large number. And suppose we took h trials and that the probability of a certain event from each (considered in a determinate manner) was $\frac{1}{m}$; let us take a second set of h trials for which the same quantity is $\frac{2}{m}$: and so on to $\frac{m-1}{m}$ and 1.

When the trials have taken place, we shall have approximately,

$$h\left(\frac{1}{m}+\frac{2}{m}+\ldots\ldots+\frac{m-1}{m}+1\right)$$

of the sought events. Of these

$$h\left\{\left(\frac{1}{2}+\frac{1}{m}\right)+\left(\frac{1}{2}+\frac{2}{m}\right)+\ldots\ldots+1\right\},$$

had *à priori* a probability greater than $\frac{1}{2}$. Summing these series and dividing the second by the first, we get $\frac{3m+2}{4m+4}$, for the ratio which the latter class of events bears to the total number. The limit of this, when m is infinite, or when we take an infinite number of sets of trials is $\frac{3}{4}$, which is the received result.

(17.) Thus, it appears this result is based upon some thing equivalent to the following assumption:—There are an infinity of events whose simple probability *à priori* is x, and another infinite number for which it is x'. These two infinities bear to one another the definite ratio of equality, (x and x' may represent *any* quantity from 0 to 1.) Now in reality, as we have seen, these numbers are not only infinite, but *in rerum naturâ* indeterminate, and therefore the assumption that they bear to one another a definite ratio is illusory.

And this assumption runs through all the applications of the theory to events whose causes are unknown.

This position could be *directly* proved only by an analysis of the various ways in which this part of the subject has been considered, which would require a good deal of detail. Those who take an interest in the question, may without much difficulty satisfy themselves, whether the view I have taken (which at least avoids the manifest contradictions of the received results) is correct.

(18.) I will add only one remark. If in (16) instead of taking one event from each of the trials there specified, we had taken p in succession, and kept account only of those sequences of p events each, which contained none but events of the kind sought; we should have had of such sequences

$$h\left(\frac{1}{m^p}+\frac{2^p}{m^p}+\ldots\ldots 1\right),$$

of which

$$h\left\{\left(\frac{1}{2}+\frac{1}{m}\right)^p+\ldots\ldots 1\right\}$$

would have belonged to trials where the simple *à priori* probability was $>\frac{1}{2}$: the ratio of these two expressions is ultimately

$$\frac{\int_{\frac{1}{2}}^{1} x^p dx}{\int_0^1 x^p dx}=1-\left(\frac{1}{2}\right)^{p+1}.$$

This is the expression applied to determine the probability of a common cause among similar phenomena, as in the case already mentioned of the planets.

But this application is founded on *a petitio principii:* we *assume* that all the phenomena are allied: that they are the results of repetitions of the same trial, that they have the same simple probability; all that, setting other objections aside, we really determine, is the probability, that this simple probability common to all these allied phenomena is $>\frac{1}{2}$.

But how does this determine the force of the presumption that the phenomena *are* allied, or to use Condorcet's illustration, that they all come out of the same infinite lottery?

(19.) The object of this little essay being to call attention to the subject rather than fully to discuss it, I have omitted several questions which entered into my original design.

The principle on which the whole depends, is the necessity of recognizing the tendency of a series of trials towards regularity, as the basis of the theory of probabilities.

I have also attempted to show that the estimates furnished by what is called the theory *à posteriori* of the force of inductive results are illusory.

If these two positions were satisfactorily established, the theory would cease to be, what I cannot avoid thinking it now is, in opposition to a philosophy of science which recognizes ideal elements of knowledge, and which makes the process of induction depend on them.

II. *On the Reflexion and Refraction of Light at the Surface of an Uncrystallized Body.* *By the* REV. M. O'BRIEN, *late Fellow of Caius College.*

[Read *Nov.* 28, 1842.]

THE object of the present paper is to determine completely the Laws of Reflexion and Refraction of Light, without introducing any empirical considerations, or omitting to take into account the normal vibrations which are generated in cases of oblique incidence. Though several eminent mathematicans have written upon this subject, I believe that most of what is here contained is new. I must state, however, that I have not been able to procure a Memoir by M. Cauchy, which he constantly refers to in his *Exercices d'Analyse et de Physique Mathématique* (for 1840), and in which he has given a general method of arriving at the equations of condition relative to the limits of bodies. I can therefore only guess at the physical principles upon which he obtains his equations of condition, which equations, in the form he has given them in the *Exercices* for 1840, are particular cases of those obtained in the present paper. As M. Cauchy states that he has made use of some new principles in obtaining his equations of condition (see the *Nouveaux Exercices, Prague* 1835, p. 203), I am justified in assuming that the method employed in the present paper is different from his; for I have deduced my equations of connection, not from any new physical principal, but from an old and obvious one, which has been either directly used, or tacitly assumed by all the writers upon the reflexion and refraction of Sound and Light, that I am acquainted with. This principle is very clearly stated by Poisson, in the *Mémoires de l'Institut*, Tom. x. p. 320.

The following is a brief outline of the course pursued in the present paper.

In Section I. I have proved some very simple theorems by means of which I have afterwards deduced the laws of reflexion and refraction, without assuming the integrals of the equations of motion, or supposing the waves to be plane.

In Section II. I have deduced the equations of connection of the vibratory motion of two media, separated by a plane, from the principle above alluded to. These equations of connection are apparently the same as those given by Mr Green in the *Cambridge Transactions*, Vol. VII. p. 11.; but they differ from them very materially with respect to the constants involved in them, and on that account they, and the results deduced from them, are perfectly free from difficulties* which seem to me to be fatal to the correctness of Mr Green's equations, and which he appears to have felt himself. I shall not however enter into this subject now, as I shall be obliged to do so on a future occasion.

I have shewn that these equations of connection are considerably simplified when we suppose the ether to have the same constitution as ordinary gases, and neglect the variation of temperature.

In Section III. I have applied these equations of connection to determine completely the laws of reflexion and refraction of polarized light, both as regards direction, colour, and intensity, taking fully into account the production of normal as well as transversal waves in the

* One difficulty I have mentioned a little farther on. Another difficulty is this, that there are just the same constants (A) and (B) in Mr Green's Equations of Connection, as those in his Equations of Motion: which arises, first, from an error in the form of the function ϕ_2 (*Cambridge Transactions*, Vol. VII. p. 7), and secondly, because ϕ_2 is not symmetrical round the axes of y and z at the plane of separation, as Mr Green assumes it to be.

case of oblique incidence, where the vibrations take place parallel to the plane of incidence. The laws, which the directions of the normal rays obey, are curious, and have not been noticed before, so far as I am aware; nor indeed can I perceive that these rays have been taken into account in a satisfactory manner by writers upon this subject.

In this section I have shewn that if we take the equations of connection in their simplest form, Fresnal's formulæ will result from them on two suppositions; first, that normal waves are propagated very slowly compared with transversal waves; and secondly, that normal waves are propagated with the same, or nearly the same, velocity in vacuum and in transparent media. The former hypothesis seems to me to be very improbable, for it is very difficult to conceive a *stable* medium in which normal waves are propagated more slowly than transversal. I may observe here, that M. Cauchy's equations and results are obtained by assuming the truth of this hypothesis, (see his *Exercices* for 1840, p. 135), and appear, on this accoun., liable to objection.

In Section IV. I have shewn that Fresnal's formulæ may be applied, without making any vague use of the symbol $\sqrt{-1}$, to the case of Total Internal Reflexion, and that he was fully justified in the very remarkable interpretation he put upon his formulæ in this case.

In Section V. I have shewn that normal waves will never produce any sensible effect on the eye by producing transversal vibrations, provided the velocity of propagation of no..nal waves be either very great, or very small, compared with that of transversal waves.

In Section VI. I have attempted to prove, from well established experimental laws, that polarized light consists of vibrations at right angles to the plane of polarization.

In Section VII. I have briefly shewn how we must proceed when the equations of connection are not taken in their simplest form, in which they are used in Section III.

Lastly, in Section VIII. I have obtained expressions which apply to substances of high refractive power, such as the diamond, and from which I have deduced results in exact accordance with the experiments of Mr Airy. These expressions are different from those of Mr Green, which certainly cannot be correct, since they give (see *Cambridge Transactions*, Vol. VII. p. 23,) $\frac{\beta^2}{a^2}$ = more than $\frac{1}{10}$, for plate-glass; and $\frac{\beta^2}{a^2}$ = more than $\frac{1}{2}$, for diamond: which results are utterly at variance with experiment. The fact is, Mr Green's original mistake respecting the constants (A) and (B), mentioned above, obliges him to suppose that the index of refraction is the same for normal and for transversal waves, and this makes his results true only for substances of very low refractive power; for instance, they are quite at fault in the case of common plate-glass, both as regards the intensity and the rotation of the plane of a polarized ray. If ν is put $= \mu$ in my result, it agrees with Mr Green's, which confirms the correctness of what I have just stated.

SECTION I.

Preliminary Observations.

BEFORE we proceed to the direct investigation of the laws of Reflexion and Refraction, we shall make a few observations, which will be found useful hereafter.

(1.) Let a, β, γ be the small displacements at any point (xyz) of a wave propagated with a normal velocity (v); p, q, s the direction of the cosines of v, and V the actual velocity of the vibrating particle, *i.e.* the resultant of the velocities $\frac{da}{dt}$, $\frac{d\beta}{dt}$, $\frac{d\gamma}{dt}$.

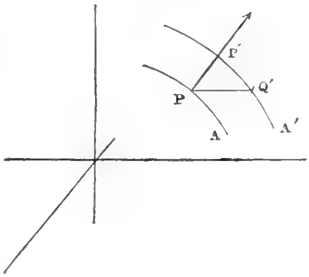
Then we have the following relations, viz.

$$(A)\ldots\ldots\left\{\begin{array}{l} \dfrac{d\alpha}{dx} = -\dfrac{p}{v}\dfrac{d\alpha}{dt}, \qquad \dfrac{d\alpha}{dy} = -\dfrac{q}{v}\dfrac{d\alpha}{dt}, \qquad \dfrac{d\alpha}{dz} = -\dfrac{s}{v}\dfrac{d\alpha}{dt}, \\ \text{and similar equations connecting the partial differential coefficients of } \beta, \gamma, \text{ and } V, \\ \text{or any function of these quantities.} \end{array}\right.$$

It is easy to prove these equations, without assuming the integrals of the equations of motion, in the following manner:

Let P be the point (xyz), AP the wave-surface which contains P at any time t, $A'P'$ the position of this wave-surface at the time $t + dt$, PP' the normal to the wave at P, PQ' a line parallel to the axis of x meeting the wave $A'P'$ in Q'. Then assuming dx to represent PQ', we have

$$PP' = PQ' \cos P'PQ' = p\,dx.$$



Also, since the space PP' is described in the time dt with the velocity v, $PP' = v\,dt$; hence we have

$$v\,dt = p\,dx \ldots\ldots (1).$$

Now at the time $(t + dt)$, any point of the wave $A'P'$ is in the same phase of vibration as any point of the wave AP at the time (t); therefore α, β, γ, V, or any function of these quantities will not be altered by putting $x + dx$, and $t + dt$, for x and t. We have therefore

$$\frac{d\alpha}{dx}dx + \frac{d\alpha}{dt}dt = 0, \text{ which by (1) becomes } \frac{d\alpha}{dx} = -\frac{p}{v}\frac{d\alpha}{dt}.$$

In the same way we may shew that

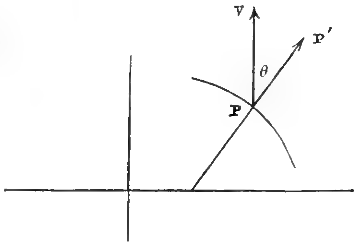
$$\frac{d\alpha}{dy} = -\frac{q}{v}\frac{d\alpha}{dt}, \quad \text{and} \quad \frac{d\alpha}{dz} = -\frac{s}{v}\frac{d\alpha}{dt},$$

and thus the truth of the equations (A) is proved.

(2.) Suppose the wave-surface to be a cylindrical surface perpendicular to the plane of xz, the vibrations to take place parallel to that plane, and therefore $\beta = 0$, $q = 0$, and α and γ independent of y: then we have the following relations between V, v, and the partial differential coefficients of α and γ, viz.

$$(B)\ldots\ldots\left\{\begin{array}{l} \dfrac{d\alpha}{dt} = Vp, \quad \dfrac{d\gamma}{dt} = Vs \\ \dfrac{d\alpha}{dx} + \dfrac{d\gamma}{dz} = -\dfrac{V}{v} \\ \dfrac{d\alpha}{dz} - \dfrac{d\gamma}{dx} = 0 \end{array}\right\}\ldots\ldots\text{for normal vibrations.}$$

$$(C)\ldots\ldots\left\{\begin{array}{l} \dfrac{d\alpha}{dt} = -Vs, \quad \dfrac{d\gamma}{dt} = Vp \\ \dfrac{d\alpha}{dx} + \dfrac{d\gamma}{dz} = 0 \\ \dfrac{d\alpha}{dz} - \dfrac{d\gamma}{dx} = \dfrac{V}{v} \end{array}\right\}\ldots\ldots\text{for transversal vibrations.}$$



To prove these formulæ, let PP' be the normal, and θ the angle which the direction of V makes with PP', then V is equivalent to $V\cos\theta$ along PP', and $V\sin\theta$ perpendicular to PP'; therefore, since s and p are the sine and cosine of the angle which PP' makes with the axis of x, we have

$$\frac{d\alpha}{dt} = V\cos\theta \, . \, p - V\sin\theta \, . \, s \ldots\ldots (1),$$

$$\frac{d\gamma}{dt} = V\cos\theta \, . \, s + V\sin\theta \, . \, p \ldots\ldots (2),$$

$(1)\,\dfrac{p}{v} + (2)\,\dfrac{s}{v}$, and $(1)\,\dfrac{s}{v} - (2)\,\dfrac{p}{v}$, give (since $p^2 + q^2 = 1$),

$$\frac{p}{v}\frac{d\alpha}{dt} + \frac{s}{v}\frac{d\gamma}{dt} = \frac{V}{v}\cos\theta, \qquad \frac{s}{v}\frac{d\alpha}{dt} - \frac{p}{v}\frac{d\gamma}{dt} = -\frac{V}{v}\sin\theta,$$

which, by the equations (A) last article, reduce to

$$\frac{d\alpha}{dx} + \frac{d\gamma}{dx} = -\frac{V}{v}\cos\theta, \qquad \frac{d\alpha}{dz} - \frac{d\gamma}{dx} = \frac{V}{v}\sin\theta.$$

In these two equations, and in (1) and (2), put $\theta = 0$, and we immediately obtain the formulæ (B); again, put $\theta = \dfrac{\pi}{2}$, and we obtain the formulæ (C)*.

(3.) If u_1, u_2, u_3, &c.....u_n be any functions of x and t, such that the equations

$$u_1 + u_2 + u_3 \ldots\ldots\ldots + u_n = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{du_1}{dx} = a_1\frac{du_1}{dt}, \quad \frac{du_2}{dx} = a_2\frac{du_2}{dt} \ldots\ldots\ldots \frac{du_n}{dx} = a_n\frac{du_n}{dt} \ldots\ldots\ldots (2)$$

are true for all values of x and t; $a_1, a_2, a_3, \ldots$ &c. being any †constants; then must

$$a_1 = a_2 = a_3 \ldots\ldots\ldots = a_n.$$

For $\dfrac{d(1)}{dx} - a_n\dfrac{d(1)}{dt}$ gives by (2)

$$(a_1 - a_n)\frac{du_1}{dt} + (a_2 - a_n)\frac{du_2}{dt} \ldots\ldots + (a_{n-1} - a_n)\frac{du_{n-1}}{dt} = 0 \ldots (3).$$

Again, $\dfrac{d(3)}{dx} - a_{n-1}\dfrac{d(3)}{dt}$ gives by $\dfrac{d(2)}{dt}$,

$$(a_1 - a_n)(a_1 - a_{n-1})\frac{d^2u_1}{dt^2} + (a_2 - a_n)(a_2 - a_{n-1})\frac{d^2u_2}{dt^2} \ldots\ldots + (a_{n-2} - a_n)(a_{n-2} - a_{n-1})\frac{d^2u_{n-2}}{dt^2} = 0.$$

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* The formulæ (B) and (C) are particular cases of the following, viz.

$$\frac{d\alpha}{dx} + \frac{d\alpha}{dy} + \frac{d\alpha}{dz} = -\frac{V}{v}\cos\theta$$

$$\left(\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right)^2 + \left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right)^2 + \left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right)^2 = \frac{V^2}{v^2}\sin^2\theta,$$

which may be easily proved.

† The result of this Article is also true when $a_1, a_2, a_3 \ldots$ &c. are variables, provided they vary very slowly compared with u_1, u_2, u_3, &c.; in which case $\dfrac{da_1}{dt}\,\dfrac{da_1}{dx}$ &c. will be extremely small compared with $\dfrac{du_1}{dt}\,\dfrac{du_1}{dx}$, &c.

And by repeating this process, we find finally that

$$(a_1 - a_n)(a_1 - a_{n-1})(a_1 - a_{n-2})(\ldots\ldots)(a_1 - a_2) = 0,$$

which shews that some one of the constants a_2, a_3, &c. must be equal to a_1. Suppose that $a_1 = a_2$, and put $u_1 + u_2 = u^1$; then we have u^1 instead of the first two terms of (1), and $\frac{du^1}{dx} = a_1 \frac{du^1}{dt}$ instead of the first two equations of (2), and therefore, just as before, we may shew that

$$(a_1 - a_n)(a_1 - a_{n-1})(\ldots\ldots)(a_1 - a_3) = 0.$$

Therefore a_1 must be equal to some one of the quantities a_3, a_4, a_5, &c.; let it be a_3, then proceeding as before, we may shew that $a_1 = a_4$, and again, that $a_1 = a_5$, and so on. We have therefore

$$a_1 = a_2 = a_3 \ldots\ldots = a_n.$$

(4.) The equations (A), (B), (C) in the preceding articles, may be very readily obtained from the integrals of the equations of motion in the case of plane-waves of polarized light. For when the wave-surface is a plane and the light polarized, we have

$$\alpha = au, \quad \beta = bu, \quad \gamma = cu,$$

where $u = f(vt - px - qy - sz)$, and a, b, c, any constants.

By differentiating these expressions with respect to x, y, z, and t, observing that p, q, s are now constants, we have immediately the equations (A).

To obtain the equations (B) and (C), we must put $q = 0$, $b = 0$, and then we have

$$V^2 = \left(\frac{d\alpha}{dt}\right)^2 + \left(\frac{d\gamma}{dt}\right)^2 = (a^2 + c^2)\left(\frac{du}{dt}\right)^2.$$

$$\text{Now } a^2 + c^2 = (ap + cs)^2 + (as - cp)^2,$$

$$\text{also } ap\frac{du}{dt} = va\frac{du}{dx} = v\frac{d\alpha}{dx}, \quad cs\frac{du}{dt} = v\frac{d\gamma}{dz}, \quad as\frac{du}{dt} = v\frac{d\alpha}{dz}, \quad cp\frac{du}{dt} = v\frac{d\gamma}{dx}.$$

$$\text{hence } \frac{V^2}{v^2} = \left(\frac{d\alpha}{dx} + \frac{d\gamma}{dz}\right)^2 + \left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right)^2.$$

Now for transverse vibrations, we have $ap + cs = 0$, or $\frac{d\alpha}{dx} + \frac{d\gamma}{dz} = 0$, and for normal, $as - cp = 0$, or $\frac{d\alpha}{dz} - \frac{d\gamma}{dx} = 0$: hence the truth of the equations (B) and (C) is manifest.

(5.) If any of the quantities p, q or s, be imaginary, (a case we shall have to consider hereafter) the first method of proving the formulæ (A), (B), (C), fails, but the second method does not. In such a case we call the vibrations transversal when the condition $ap + cs = 0$ holds; and normal when the condition $as - cp = 0$ holds; and it follows easily from the equations of motion, (see *Cambridge Transactions*, Vol. VII. p. 416) that transversal and normal waves, thus defined, are in general propagated with different velocities: *i. e.* the constant v is different for these two species of vibration.

(6.) It is important to observe that, in articles (1), (2), the wave is supposed to be propagated in the direction PP', *i. e.* from P towards P'. If therefore p, q, s be positive quantities, the motion of the disturbance along PP' tends to *increase* x, y, and z; if p be negative it tends to *diminish* x, if q negative y, and if s negative z.

SECTION II.

The general Equations of Connection of the Vibratory Motion of two Elastic Media, separated by a Plane Surface.

(7.) THE two media are supposed to consist of discrete particles symmetrically arranged, and acting upon each other with forces varying according to any law which ensures stable equilibrium. By the Surface of Separation, we simply mean an imaginary plane described between the two media, the particles of one medium lying on one side of it, and those of the other on the other side. In the immediate vicinity of this plane, the media are supposed to exercise a mutual repulsion, so that no mixture takes place. We shall take the plane of separation to be the plane of xy.

(8.) We shall obtain the general Equations of Connection of the vibratory motion of the two media, by means of the following self-evident *Principle.*

When a very small vibratory motion is communicated to a stable system of particles, such as the two media just described, we may assume that the vibratory motion will *always* remain very small, and, at most, of the same order of magnitude as the original motion.

This principle is either tacitly assumed, or employed as self evident, by all the writers who have treated of the problem of the transmission of waves from one medium into another. Poisson states it very clearly in the *Mémoires de l'Institut*, Tom. x. p. 320, and makes use of it precisely as we shall do in the present paper. It is evidently assumed in the Article Sound, *Ency. Metrop.* p. 776; for by saying that the two media must have a common elasticity at their junction, and that that elasticity is expressed by $E(1+\beta s)$, and $E'(1+\beta' s')$, the writer supposes that there is the same slow variation of elasticity at the surface of junction as elsewhere, and therefore the same slow variation of pressure, and consequently the same small vibratory motion.

(9.) To apply this principle to the case we are at present concerned with, let xyz ($z=0$) be the co-ordinates of the equilibrium position of any particle (P) of the lower medium in the immediate vicinity of the plane of separation, α, β, γ its displacements at the time t, and let $x+\delta x$, $y+\delta y$, $z+\delta z$, $\alpha+\delta\alpha$, $\beta+\delta\beta$, $\gamma+\delta\gamma$ be the co-ordinates and displacements of any other neighbouring particle (Q) of the lower medium; also let $x+\Delta x$, $y+\Delta y$, $z+\Delta z$, $\alpha+\Delta\alpha$, $\beta+\Delta\beta$, $\gamma+\Delta\gamma$ be those of any particle (P') of the upper medium.

Put $r^2=\delta x^2+\delta y^2+\delta z^2$, and $r'^2=\Delta x^2+\Delta y^2+\Delta z^2$,

and let $mf(r)$, $m'\phi(r')$ be, respectively, the forces exercised by Q and P' on P. Then, if X be the whole force, parallel to the axis of x, brought into action upon P by the vibration, we have (see *Cambridge Transactions*, Vol. VII. p. 403)

$$X=\Sigma m\left\{f(r)\,\delta\alpha+\frac{1}{r}f'(r)\,\delta x\,(\delta x\delta\alpha+\delta y\delta\beta+\delta z\delta\gamma)\right\}$$
$$+\Sigma' m'\left\{\phi(r')\,\Delta\alpha+\frac{1}{r'}\phi'(r')\,\Delta x\,(\Delta x\Delta\alpha+\Delta y\Delta\beta+\Delta z\Delta\gamma)\right\},$$

Σ referring to the lower medium and Σ' to the upper.

In this expression we shall substitute for $\delta\alpha$ the series

$$\frac{d\alpha}{dx}\delta x+\frac{d\alpha}{dy}\delta y+\frac{d\alpha}{dz}\delta z+\&c.\ldots\ldots$$

Also, let α', β', γ' be the values which the displacements $\alpha+\Delta\alpha$, $\beta+\Delta\beta$, $\gamma+\Delta\gamma$, assume when x, y, z ($=0$) are substituted in them for $x+\Delta x$, $y+\Delta y$, $z+\Delta z$, then we have

$$\alpha+\Delta\alpha=\alpha'+\frac{d\alpha'}{dx}\Delta x+\frac{d\alpha'}{dy}\Delta y+\frac{d\alpha'}{dz}\Delta z+\&c.\ldots\ldots$$

Now the differences of the corresponding displacements of two contiguous particles at a distance from the plane of separation must be indefinitely small, (supposing of course, as is always done, that the interval between two contiguous particles is extremely small, compared with the length of a wave); therefore, by the principle stated in Article 8, the same must be true of the displacements in the immediate vicinity of the plane of separation, which cannot be the case unless we have $\alpha' = \alpha$. Hence

$$\Delta\alpha = \frac{d\alpha'}{dx}\Delta x + \frac{d\alpha'}{dy}\Delta y + \frac{d\alpha'}{dz}\Delta z + \&c. \ldots\ldots$$

Substituting these expressions for $\delta\alpha$, and $\Delta\alpha$, and similar expressions for $\delta\beta$, $\delta\gamma$, $\Delta\beta$, $\Delta\gamma$, and observing that all sums involving odd powers of δx, δy, Δx or Δy, must vanish, in consequence of the symmetrical arrangement of the system about the axis of z, but that sums involving odd powers of δz or Δz do not vanish, since the particles are not arranged symmetrically with respect to the plane of xz, we have

$$X = -(C + D)\frac{d\alpha}{dz} - D\frac{d\gamma}{dx} + (C' + D')\frac{d\alpha'}{dz} + D'\frac{d\gamma'}{dx} + \text{higher differential coefficients},$$

$$\text{where } -C = \Sigma m f(r)\,\delta z, \qquad -D = \Sigma m\,\frac{1}{r}f'(r)\,\delta x^2\,\delta z,$$

$$C' = \Sigma' m'\,\phi(r')\,\Delta z, \qquad D' = \Sigma' m'\,\frac{1}{r'}\phi'(r')\,\Delta x^2\,\Delta z.$$

(We assume the two first constants in a negative form, because δz is negative, whereas Δz is positive).

It is evident that $-C + C' = 0$, is one of the conditions of previous equilibrium, therefore we have $C' = C$ in the expression for X.

Now since the length of the wave is extremely large compared with the sphere of action of the molecular forces, the part of X involving first differential coefficients, has its several terms extremely large compared with those of the part involving second and higher differential coefficients (see *Cambridge Transactions*, Vol. VII. p. 408): therefore, unless the former terms mutually destroy each other, X will be extremely large compared with the corresponding force which acts upon a particle at a distance from the plane of separation (for this force involves only second and higher differential coefficients, see *Cambridge Transactions*, Vol. VII. p. 408): and if this be the case, the vibratory motion of the particles at the plane of separation will be extremely large compared with that at a distance from it, contrary to the principle stated in Article (8). Hence the terms of X involving first differential coefficients must destroy each other, and we therefore have

$$(C + D)\frac{d\alpha}{dz} + D\frac{d\gamma}{dx} = (C + D')\frac{d\alpha'}{dz} + D'\frac{d\gamma'}{dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In exactly the same way we may shew that $\beta' = \beta$, and

$$(C + D)\frac{d\beta}{dz} + D\frac{d\gamma}{dy} = (C + D')\frac{d\beta'}{dz} + D'\frac{d\gamma'}{dy} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Lastly, the force parallel to the axis of z is

$$\Sigma m\left\{f(r)\,\delta\gamma + \frac{1}{r}f'(r)\,\delta z\,(\delta x\,\delta\alpha + \delta y\,\delta\beta + \delta z\,\delta\gamma)\right\}$$

$$+ \Sigma' m'\left\{\phi(r')\,\Delta\gamma + \frac{1}{r'}\phi'(r')\,\Delta z\,(\Delta x\,\Delta\alpha + \Delta y\,\Delta\beta + \Delta z\,\Delta\gamma)\right\},$$

which, treated as above, gives $\gamma' = \gamma$, and

$$(C + E)\frac{d\gamma}{dz} + D\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy}\right) = (C + E')\frac{d\gamma'}{dz} + D'\left(\frac{d\alpha'}{dx} + \frac{d\beta'}{dy}\right) \ldots\ldots (3).$$

Observing that $\Sigma m \frac{1}{r} f'(r)\, \delta y^2\, \delta z = -D$, $\quad \Sigma' m' \frac{1}{r'} \phi'(r')\, \Delta y^2\, \Delta z = D'$,

and putting $-E$ and E' to represent $\Sigma m \frac{1}{r} f'(r)\, \delta z^3$ and $\Sigma' m' \frac{1}{r'} \phi'(r')\, \Delta z^3$ respectively.

(10.) Hence it appears that, if α, β, γ be the displacements at any point (xyz) of the lower medium, and α', β', γ' those at any point $(x'y'z')$ of the upper, and if we put $x' = x$, $y' = y$, $z' = z = 0$, then the equations (1), (2), (3), and the equations, $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, will hold for all values of x and y. Now this being the case, we may differentiate these equations with respect to x or y; therefore $\frac{d\gamma}{dx} = \frac{d\gamma'}{dx}$, and therefore (1) may be put in the form

$$(C + D)\left(\frac{d\alpha}{dz} + \frac{d\gamma}{dx}\right) = (C + D')\left(\frac{d\alpha'}{dz} + \frac{d\gamma'}{dx}\right),$$

and a similar alteration may be made in (2) and (3).

Hence, if we put $C + D = M$, $C + D' = M'$, $C + E = N$, $C + E' = N'$, we have the following equations:

$$\alpha = \alpha', \quad \beta = \beta', \quad \gamma = \gamma' \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (D),$$

$$\left.\begin{aligned} M\left(\frac{d\alpha}{dz} + \frac{d\gamma}{dx}\right) &= M'\left(\frac{d\alpha'}{dz} + \frac{d\gamma'}{dx}\right) \\ M\left(\frac{d\beta}{dz} + \frac{d\gamma}{dy}\right) &= M'\left(\frac{d\beta'}{dz} + \frac{d\gamma'}{dy}\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (E),$$

$$N\frac{d\gamma}{dz} + M\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy}\right) = N'\frac{d\gamma'}{dz} + M'\left(\frac{d\alpha'}{dx} + \frac{d\beta'}{dy}\right) \ldots\ldots (F).$$

These are the general equations of connection of the vibratory motion of the two media; they hold at all points of the plane of separation, *i. e.* they are true for all values of x and y, z being put equal to zero.

(11.) We shall now compare with the last of these equations the equation of connection furnished by the common law of elasticity, in the case of two ordinary elastic fluids separated by plane surface.

Let p be the pressure at any point of the lower medium when at rest, considered as a common elastic fluid; then the pressure when it is in a state of vibration, will (by the law of elasticity) be (See Airy's *Tracts*, note, p. 278, 2nd Ed.)

$$p\left\{\frac{\delta x\, \delta y\, \delta z}{(\delta x + \delta\alpha)(\delta y + \delta\beta)(\delta z + \delta\gamma)}\right\}^n, \quad \text{or } p\left\{1 - n\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right)\right\},$$

n being a constant nearly equal to unity, depending upon the alteration of temperature during the vibration.

Similarly, the pressure in the upper medium will be

$$p'\left\{1 - n'\left(\frac{d\alpha'}{dx} + \frac{d\beta'}{dy} + \frac{d\gamma'}{dz}\right)\right\}.$$

Now these two pressures ought to be equal at the plane of separation; also, by the conditions of previous equilibrium, $p = p'$.

Hence, when $z = 0$, we have

$$n\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right) = n'\left(\frac{d\alpha'}{dx} + \frac{d\beta'}{dy} + \frac{d\gamma'}{dz}\right).$$

Comparing this with the equation (F), we see that

$$N = M, \quad N' = M', \quad \text{and } \frac{M}{M'} = \frac{n}{n'}.$$

(12.) In our ignorance of the constitution of the liminiferous ether, it is natural to assume that it is of the same nature as ordinary elastic fluids, and that it accordingly obeys the common law of elasticity; we shall, in the first instance, make this assumption, and therefore put $M = N$, $M' = N'$, and $M' = \epsilon M$, where $\epsilon = \frac{n'}{n}$, a quantity not differing much from unity; and then the equations (E) and (F) become

$$\left.\begin{aligned} \frac{d\alpha}{dz} + \frac{d\gamma}{dx} &= \epsilon\left(\frac{d\alpha'}{dz} + \frac{d\gamma'}{dx}\right) \\ \frac{d\beta}{dz} + \frac{d\gamma}{dy} &= \epsilon\left(\frac{d\beta'}{dz} + \frac{d\gamma'}{dy}\right) \end{aligned}\right\} \dots\dots\dots\dots (G),$$

$$\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right) = \epsilon\left(\frac{d\alpha'}{dx} + \frac{d\beta'}{dy} + \frac{d\gamma'}{dz}\right) \dots\dots (H).$$

Further, if we neglect the variation of temperature, and therefore put $\epsilon = 1$, these equations, in virtue of the equations (D) differentiated with respect to x and y, assume the simple forms

$$\frac{d\alpha}{dz} = \frac{d\alpha'}{dz}, \quad \frac{d\beta}{dz} = \frac{d\beta'}{dz}, \quad \frac{d\gamma}{dz} = \frac{d\gamma'}{dz} \dots\dots\dots\dots (I).$$

(13.) The equations of connection just obtained, along with the equations of motion given in the *Cambridge Transactions*, Vol. VII. p. 409, are sufficient to solve all problems respecting the propagation of waves from one medium into the other. We shall assume that these equations of motion hold up to the very plane of separation: which of course is not accurately true, since there will most probably be a variation of density in the media in the immediate vicinity of that plane. If we describe two planes parallel to the plane of separation, one above it and the other below it, including between them the slice of the two media in which this variation of density is sensible, it is easy to see that, in consequence of the smallness of the sphere of action of the molecular forces compared with the length of a wave, the thickness of this slice will be extremely small compared with the length of a wave. Indeed, if one medium exercised a sensible action only upon those particles of the other which are immediately contiguous to the plane of separation, the thickness of this slice would be actually zero. We shall therefore consider this slice to be of insensible thickness, and regard it as a physical plane. This being assumed, we may, without sensible error, suppose that the equations of motion hold up to the very plane of separation. All therefore that is proved of the propagation of waves in a symmetrical medium in the *Cambridge Transactions*, Vol. VII. p. 416, &c., we shall assume to be true up to the very plane of separation.

We shall in the following Section, use the equations of connection in their simplest form, viz. (D) and (I); and afterwards, in Section VII., shew how we must proceed when they are taken in their most general form, viz. (D), (E), (F).

SECTION III.

Application of the Equations of Connection just obtained to the case of ordinary Reflexion and Refraction.

(14.) WE shall first consider the case of cylindrical or plane-waves perpendicular to the plane of xz (which will therefore be the plane of incidence), the vibrations taking place at *right angles* to that plane.

In this case $\alpha = 0$, $\gamma = 0$, $\alpha' = 0$, $\gamma' = 0$, and β is independent of y: therefore the six equations of connection, (D) and (I), Section II, reduce to two, viz.

$$\beta = \beta', \qquad \frac{d\beta}{dz} = \frac{d\beta'}{dz}.$$

We shall suppose that the whole motion consists of three sets of waves (for we shall shew presently that it cannot in general consist of only two), one set in the upper medium, and two in the lower. Let $\beta + \beta_{,}$ be the whole displacement at any point of the lower medium, the part β arising from one of the sets of waves, and the part $\beta_{,}$ from the other; then we must write $\beta + \beta_{,}$ instead of β in the two equations of connection, which therefore become

$$\beta + \beta_{,} = \beta' \ldots\ldots (1), \qquad \frac{d}{dz}(\beta + \beta_{,}) = \frac{d\beta_{,}}{dz} \ldots\ldots (2).$$

Now, using the notation in Article (2), we have

$$\frac{d\beta}{dt} = V, \qquad \frac{d\beta_{,}}{dt} = V_{,}, \qquad \frac{d\beta_{,}}{dt} = V'.$$

Hence, and by the equations (A), Article (1), $\frac{d(1)}{dt}$ and (2) immediately give us

$$V + V_{,} = V' \ldots\ldots (3), \qquad \frac{s}{v} V + \frac{s_{,}}{v_{,}} V_{,} = \frac{s'}{v'} V' \ldots\ldots (4).$$

Again, since by the equations (A), Article (1), we have

$$\frac{dV}{dx} = -\frac{p}{v}\frac{dV}{dt}, \qquad \frac{dV_{,}}{dx} = -\frac{p_{,}}{v_{,}}\frac{dV_{,}}{dt}, \qquad \frac{dV'}{dx} = -\frac{p'}{v'}\frac{dV'}{dt},$$

and by (3), $V + V_{,} - V' = 0$;

and since $\frac{p}{v}$, $\frac{p_{,}}{v_{,}}$, $\frac{p'}{v'}$, are either constants (in the case of plane-waves), or vary very slowly compared with V, $V_{,}$, $V'_{,}$, on account of the extreme smallness of the length of a wave of light; we have by Article (3), (see Note),

$$\frac{p}{v} = \frac{p_{,}}{v_{,}} = \frac{p'}{v'}.$$

Now $v_{,} = v$, therefore

$$p_{,} = p \ldots\ldots (5), \qquad p = \mu p' \ldots\ldots (6) \left\{\text{where } \mu = \frac{v}{v'}\right\}.$$

Hence, observing that q, $q_{,}$, q', are each zero, we have

$$s_{,} = \pm s, \qquad s' = \pm\sqrt{1 - \frac{p^2}{\mu^2}}.$$

If we take $s_{,} = s$, $\frac{(4)}{(3)}$ gives $\frac{s}{v} = \frac{s'}{v'}$, which is in general inconsistent with (6); we must therefore take $s_{,} = -s$. We may suppose s' to have either sign. Hence by (3) and (4), we have

$$V + V_{,} = V', \qquad s(V - V_{,}) = \mu s' V', \quad \text{which give}$$

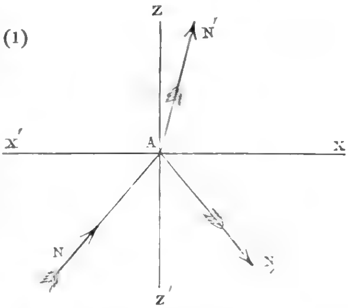
$$V_{,} = \frac{s - \mu s'}{s + \mu s'} V \ldots\ldots (7), \qquad V' = \frac{2s}{s + \mu s'} V \ldots\ldots (8).$$

(15.) We shall now interpret these results.

Supposing p and s positive, the normal propagation of the wave V tends to increase x and z (see Article 6); and since $p_{,} = p$, $s_{,} = -s$, that of the wave $V_{,}$ tends to increase x and diminish z; and since $p' = \mu p$, $s' = \pm\sqrt{1 - \frac{p^2}{\mu^2}}$, that of the wave V' tends to increase x, and to increase or diminish z. Hence we have two cases according as we take the upper or lower sign of s'.

Fig. (1) represents the first case; $X'AX$ and $Z'AZ$ are the co-ordinate axes; NA, $N_{,}A$, $N'A$ are the normals to the waves V, $V_{,}$, V' respectively, the arrows representing the direction of normal propagation, N and N' tending to increase x and z, and $N_{,}$ to increase x and diminish z. Since $p_{,} = p$, and $p' = \mu p$, we have $\angle NAZ' = \angle N_{,}AZ'$, and $\sin N'AZ = \mu \sin NAZ'$. This is the ordinary case of reflexion and refraction. If a, $a_{,}$, a' be the maximum values of V, $V_{,}$, and V' respectively, the intensities of the three rays N $N_{,}$ and N' will be proportional to a^2, $a_{,}^2$, a'^2. Now by (7) and (8), we have

(1)

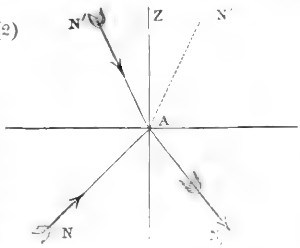


$$a_{,} = \frac{s - \mu s'}{s + \mu s'} a, \qquad a' = \frac{2s}{s + \mu s'} a.$$

These are Fresnel's formulæ for the *intensities* of the reflected and refracted rays of a ray polarized in the plane of incidence.

Fig. (2) represents the second case, in which s' is negative; and therefore N' tends to diminish z. This case may occur in the following manner. An incident ray along NA will produce a reflected ray along $AN_{,}$, and a refracted ray along AN'', $\angle N''AZ$ being equal to $\angle N'AZ$; and another incident ray along $N'A$ will produce a reflected ray along AN'', and a refracted ray along $AN_{,}$. Now let the intensities of the two rays along AN'' be equal, and let one of these rays be half a wave behind the other; then they will interfere and destroy each other, and we shall have remaining only a ray along NA, one along $N'A$, and one along $AN_{,}$ (namely, the sum of the two along $AN_{,}$). This is exactly the second case.

(2)



(16.) It is evident, that, in the ordinary case where the rays $N_{,}$ and N' are the effects produced by the ray N, the normal propagation of N' will be *from* and not *towards* the plane of separation: therefore s' must have its positive value, and consequently the second of the above cases cannot occur.

(17.) If we suppose either $V_{,}$ or V' equal to zero, the equations (7) and (8) give us either $s - \mu s' = 0$, or $s = 0$, neither of which equations can be generally true. Hence the incident ray must, in general, be accompanied by a refracted and a reflected ray, or the equations of connection cannot be satisfied.

(18.) It appears from (7) and (8) that $V_{,}$ and V' have the same periodic time as V; it follows, therefore, that the *colour* of the reflected and refracted ray is the same as that of the incident.

(19.) We shall now in the second place consider the case of cylindrical or plane-waves perpendicular to the plane of xz, the vibrations taking place *parallel* to that plane.

In this case $\beta = 0$, and α and γ are independent of y: therefore the six equations of connection, (D) and (E) Section II, reduce to four, viz.

$$\alpha = \alpha', \quad \gamma = \gamma',$$
$$\frac{d\alpha}{dz} = \frac{d\alpha'}{dz}, \quad \frac{d\gamma}{dz} = \frac{d\gamma'}{dz}.$$

If we attempt to satisfy these equations by three sets of waves, as in the preceding case, we shall immediately arrive at the conclusion, $\mu^2 = 1$; which shews that these equations cannot be satisfied in this manner. The reason of this is obvious; for, in the case of vibrations perpendicular to the plane of incidence, it is clear that no normal waves will be produced by the refraction and reflexion: but in the present case, supposing, as of course we do, that the incident vibrations are transversal, we have every reason to suppose that normal vibrations will be generated by the reflexion and refraction. Therefore, since normal waves are in general propagated with a different velocity from that of transversal, we shall have to take into account a set of normal waves in the lower medium, and one in the upper also, not coinciding with the transversal waves.

Let $\alpha + \alpha_1 + \alpha_2$, and $\gamma + \gamma_1 + \gamma_2$, be the whole displacements at any point of the lower medium, and $\alpha' + \alpha''$, $\gamma' + \gamma''$, at any point of the upper; the parts α_2, γ_2, α'', γ'', arising from the normal waves brought into existence by the reflexion and refraction. Then, the four equations of connection become,

$$\alpha + \alpha_1 + \alpha_2 = \alpha' + \alpha'' \;\ldots\ldots\ldots\ldots\ldots\ldots (1), \qquad \gamma + \gamma_1 + \gamma_2 = \gamma' + \gamma'' \;\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\frac{d}{dz}(\alpha + \alpha_1 + \alpha_2) = \frac{d}{dz}(\alpha' + \alpha'') \;\ldots\ldots (3), \qquad \frac{d}{dz}(\gamma + \gamma_1 + \gamma_2) = \frac{d}{dz}(\gamma' + \gamma'') \;\ldots\ldots (4).$$

From (1), or (2), by the equations (A), and by Article (3), we have, as in the preceding case, (Article 14),

$$\frac{p}{v} = \frac{p_1}{v_1} = \frac{p_2}{v_2} = \frac{p'}{v'} = \frac{p''}{v''} \;\ldots\ldots (5).$$

Also $(3) - \frac{d(2)}{dx}$, and $(4) + \frac{d(1)}{dx}$, give us, by the equations (B) and (C), Article (2),

$$\frac{V}{v} + \frac{V_1}{v_1} = \frac{V'}{v'} \;\ldots\ldots (6), \qquad \frac{V_2}{v_2} = \frac{V''}{v''} \;\ldots\ldots (7).$$

Also $\frac{d(1)}{dt}$ and $\frac{d(2)}{dt}$, give us, by the equations (B) and (C),

$$Vs + V_1 s_1 - V_2 p_2 = V's' - V''p'' \;\ldots\ldots (8),$$
$$Vp + V_1 p_1 + V_2 s_2 = V'p' + V''s'' \;\ldots\ldots (9).$$

These equations, namely (5), (6), (7), (8), (9), completely solve the problem as in the preceding case.

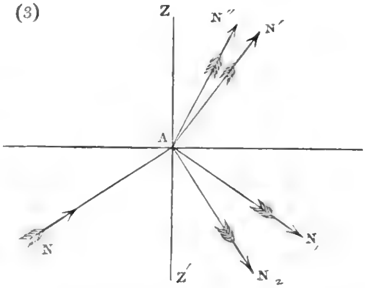
From (5) we get $p_1 = p$, and therefore $s_1 = \pm s$. As in the previous case we must take the lower sign; for otherwise V and V_1 would enter into each of the equations, (6), (7), (8), (9), in the form $V + V_1$, and therefore we might eliminate altogether the quantities V, V_1, V', V_2, V'', from these equations, and so obtain an equation which would not be generally true, since it would contain no disposable quantity, p_2, p', p'', s_2, s', s'', being all given in terms of p by (5). We have therefore $s_2 = -s$.

For the reasons given in Article (16), we must take the positive values of s' and s'', and the negative value s_2; *i. e.* we must put

$$s' = +\sqrt{1 - \frac{p^2}{\mu^2}}, \quad s'' = +\sqrt{1 - \left(\frac{v''}{v}p\right)^2} \ldots\ldots (10), \qquad s_2 = -\sqrt{1 - \left(\frac{v_2}{v}p\right)^2} \ldots\ldots (11).$$

Hence, if we take the arrows N, N_1, N_2, N', N'', to mark the directions of the rays, as before, Fig. (3) will represent the circumstances of the case, and we have

(3)



$$\angle\, N_1AZ' = \angle\, NAZ, \quad \sin N'AZ = \mu \sin NAZ',$$

$$\sin N_2AZ' = \frac{v_2}{v} \sin NAZ', \quad \sin N''AZ = \frac{v''}{v} \sin NAZ'.$$

Thus, both the reflected and the refracted normal ray obey the law of refraction, putting $\frac{v_2}{v}$ and $\frac{v''}{v}$ instead of μ. The transversal rays are circumstanced just as in the preceding case.

(20.) We now proceed to compare the intensities of the rays N, N_1, and N', and we shall do this, first, on the hypothesis that normal waves are propagated very slowly compared with transversal.

On this hypothesis we may suppose that $\frac{v_2}{v}$ and $\frac{v''}{v}$ are zero, and then, by (5), we have, $p_2 = 0$, $p'' = 0$. Hence, writing a, a_1, a', for V, V_1, V', as before, we have by (6) and (8),

$$a + a_1 = \mu a', \quad s(a - a_1) = s'a',$$

$$\text{and therefore} \quad a_1 = \frac{\mu s - s'}{\mu s + s'}\, a, \quad a' = \frac{2s}{\mu s + s'}\, a.$$

These are identical with Fresnel's formulæ for light polarized at right angles to the plane of incidence.

To determine a_2 and a'', we have, by (7) and (9), (observing that $s_2 = -1$, $s'' = 1$ by (10) and (11))

$$\frac{a_2}{v_2} = \frac{a''}{v''},$$

$$p(a + a_1) - a_2 = p'a' + a'', \qquad \text{or} \quad a_2 + a'' = (\mu p - p')\, a' \text{ by (6)};$$

$$\text{therefore} \quad a_2 = \frac{v_2}{v_2 + v''}\, \frac{\mu^2 - 1}{\mu}\, \frac{2ps}{\mu s + s'} \cdot a, \qquad a'' = \frac{v''}{v_2}\, a_2.$$

(21.) We shall now make a different hypothesis, and suppose that v_2 is equal or very nearly equal to v''.

On this hypothesis, we have by (5) $p_2 = p''$, and by (10) and (11) $s_2 = -s''$; therefore by (6) and (8) we obtain

$$a + a_, = \mu a', \qquad s(a - a_,) = s'a',$$

which give us Fresnel's formulæ just as before.

Also by (7) and (9) we have

$$a_2 = a'', \qquad p(a + a_1) - s''a_2 = p'a' + s''a'';$$

$$\text{therefore} \quad a_2 = a'' = \frac{\mu^2 - 1}{\mu s''}\, \frac{2ps}{\mu s + s'} \cdot a.$$

(22.) Lastly, we shall try whether any other hypothesis leads to Fresnel's formulæ in this case. If Fresnel's formulæ hold, it is easy to see that

$$s(a - a_{,}) = s'a', \quad \text{and therefore } s(V - V_{,}) = s'V';$$

therefore by (8) we have $V_2 p_2 = V'' p''$, and therefore by (7) and (5)

$$v_2^2 = v''^2,$$

from which it is evident that no other hypothesis except those in Articles (20) and (21) will lead to Fresnel's formulæ in this case.

The second hypothesis here employed seems to me to be the only one we can adopt; for it is extremely difficult to conceive how normal vibrations could be propagated more slowly than transversal in a stable medium.

If we do not suppose that v_2 is very nearly equal to v'', we may proceed to find $V_{,}$ and V' in terms of V in the following manner.

Substitute for V' and V_2 their values got from (6) and (7) in the equations (8) and (9); then, putting $\frac{v_2}{v''} = \nu$, we have

$$(\mu s - s')V - (\mu s + s')V_{,} = V''(\nu p_2 - p''), \qquad (\mu p - p')(V + V_{,}) = V''(s'' - \nu s_2).$$

Hence, if for brevity we put $\frac{\nu p_2 - p''}{s'' - \nu s_2}(\mu p - p') = \eta$, we have

$$(\mu s - s')V - (\mu s + s')V_{,} = \eta(V + V_{,}),$$

$$\text{and therefore } V_{,} = \frac{\mu s - s' - \eta}{\mu s + s' + \eta} V,$$

and from this expression we may easily find V', since $V' = \frac{V + V_{,}}{\mu}$.

Since $p = \mu p'$, and $p_2 = \nu p''$, we have

$$\eta = (\nu^2 - 1)(\mu^2 - 1) \cdot \frac{p'p''}{s'' - \nu s_2}.$$

SECTION IV.

Explanation of the case of Total Internal Reflexion.

(23.) SINCE $p = \mu p'$, p' will be > 1 when p is $> \mu$ (which it may be when μ is < 1), and then s' will be impossible; and Fresnel's formulæ become imaginary; which indicates that the equations of connection cannot be satisfied by the three rays in Article (15), or the five rays in Article (19). We shall now consider how the equations of connection may be satisfied under such circumstances, and first in the case of vibrations perpendicular to the plane of incidence.

Let us suppose that the general value of V is

$$V = \frac{a}{2} e^{k(vt - px - sz)\sqrt{-1}} \;\ldots\ldots\; (1).$$

It is allowable to give V this value, though it is imaginary, since it is an integral of the equations of motion, and is capable of satisfying, analytically, the equations of connection, and the equations (A), (B), (C), Section 1, (see Article 5). Moreover, by superposing two such imaginary values of V, viz. $a e^{k(vt-px-sz)\sqrt{-1}}$ and $a e^{-k(vt-px-sz)\sqrt{-1}}$, we obtain a real value, viz. $a \cos k(vt - px - sz)$, which will of course satisfy the same *linear* equations as the two expressions of which it is the sum, *i.e.* the equations of motion and of connection.

Assuming then this value of V, we have at the surface of separation,

$$V = \frac{a}{2} e^{k(vt-px)\sqrt{-1}},$$

and therefore $V_{,} = \frac{s - \mu s'}{s + \mu s'} \frac{a}{2} e^{k(vt-px)\sqrt{-1}}$, $\quad V' = \frac{2s}{s + \mu s'} \frac{a}{2} e^{k(vt-px)\sqrt{-1}}$.

Hence if $\frac{1}{2} a_{,} e^{k(vt-px+sz)\sqrt{-1}}$, and $\frac{1}{2} a' e^{k'(v't-p'x-s'z)\sqrt{-1}}$, be the general values of $V_{,}$ and V', we have

$$a_{,} = \frac{s - \mu s'}{s + \mu s} a, \qquad a' = \frac{2s}{s + \mu s} a,$$

and of course $k'v' = kv$, $\quad p' = \frac{v'}{v} p = \frac{p}{\mu}$, $\quad$ and $s' = \pm \sqrt{1 - \frac{p^2}{\mu^2}}$.

Now let us assume

$\sqrt{1 - \frac{p^2}{\mu^2}} = \sigma . \sqrt{-1}$ (supposing $p > \mu$), and $\frac{\mu s'}{s} = \pm \frac{\mu\sigma}{s}\sqrt{-1} = \pm \sqrt{-1} \tan \omega$,

then $a_{,} = e^{\mp 2\omega\sqrt{-1}} a$, $\qquad a' = 2 \cos \omega \, e^{\mp \omega \sqrt{-1}} a$,

and the general values of $V_{,}$ and V' become

$$V_{,} = \tfrac{1}{2} a e^{\{k(vt-px+sz)\mp 2\omega\}\sqrt{-1}} \ldots\ldots (2), \qquad V' = a \cos \omega e^{\{k'(v't-p'x)\mp\omega\}\sqrt{-1} \pm k'\sigma z} \ldots\ldots (3).$$

Now let us superpose the system (1) (2) (3), taking the lower of the double signs, with another system formed from (1) (2) (3), by putting $-k$ for k and therefore $-k'$ for k' and taking the upper of the double signs. The result of this superposition will be the following real system, viz.

$$V = a \cos k (vt - px - sz),$$

$$V_{,} = a \cos \{k (vt - px + sz) + 2\omega\}, \qquad V' = 2a \cos \omega e^{-k'\sigma z} \cos \{k' (v't - p'x) + \omega\}.$$

These values of V, $V_{,}$, and V', since they are real, and satisfy the equations of motion and of connection, represent a possible case of motion. The expression for $V_{,}$ shews that there is a reflected ray, of the same intensity as the incident ray, but having its *phase* altered by the quantity 2ω. The expression for V' gives $4a^2 \cos^2 \omega e^{-2k'\sigma z}$ for the intensity of the refracted ray, which quantity rapidly diminishes as z increases, since $k' = \frac{2\pi}{\lambda'}$, and λ' is extremely small. This indicates a complete extinction of the refracted ray. If we had taken the upper signs instead of the lower, and the lower instead of the upper, in the above process of superposition (as we might have done), we should have obtained $e^{2k'\sigma z}$, instead of $e^{-2k'\sigma z}$, in the expression for the intensity of V'. Now this represents an intensity which increases rapidly with z, and therefore a vibratory motion which becomes extremely large compared with that which gave rise to it, contrary to the principle stated in Article (8). We must therefore take the signs as we have done above.

The alteration of the phase of the reflected ray is given by the equation

$$\tan \omega = \frac{\mu\sigma}{s} = \frac{\sqrt{p^2 - \mu^2}}{s}.$$

This is exactly the first case of total internal reflexion considered in Airy's *Tracts*, p. 361. (second edition)*.

(24.) We shall, in the second place, apply the same method to the case of vibrations parallel to the plane of incidence. To do this we have only to put $\tan \omega = \frac{\sigma}{\mu s} = \frac{\sqrt{p^2 - \mu^2}}{\mu^2 s}$, and

* The μ in Airy's *Tracts* is the same as the $\frac{1}{\mu}$ here.

we arrive at exactly the same expressions as before, for $V_{,}$ and V', with this difference, that the coefficient of V' is $\frac{2a\cos\omega}{\mu}$ instead of $2a\cos\omega$.

This is exactly the second case considered in Airy's *Tracts*, p. 361.

SECTION V.

Why Normal Waves never produce any sensible Effect on the Eye directly or indirectly.

(25.) WE must suppose of course that normal waves cannot produce vision directly, (*i. e.*) that when such waves are incident on the retina they do not affect the optic nerve in such a manner as to give rise to the sensation of light. But we have proved that when a transversal ray undergoes oblique refraction it brings into existence normal rays, and it would be easy to shew that, in the same manner, the oblique refraction of a normal ray will produce transversal rays. Therefore, though normal waves cannot affect the retina *directly*, they may do so *indirectly*, by giving rise to transversal waves. Now it is a matter of fact that they do not produce this indirect effect, and it therefore becomes necessary to explain theoretically why they do not.

(26.) If we take the hypothesis in Article (20), it is easy to do this. For suppose the normal ray, generated by the oblique refraction of a transversal ray at the first surface of a prism or lens, to fall on the second surface at an angle of incidence $\sin^{-1}p$, and let the transversal ray produced by this oblique refraction emerge at an angle $\sin^{-1}p'$, then, as in Article (19), we may shew that $\frac{p}{v''} = \frac{p'}{v}$, and therefore $p' = \frac{v}{v''}p$. Now by our hypothesis $\frac{v}{v''}$ is very large, therefore, unless p is very small (in which case the transversal ray will not be produced at all), p' will be > 1, and s' impossible; *i.e.* the transversal ray will be extinguished. (See Article 23).

Thus the normal waves generated by the first refraction, will not produce transversal waves at the second refraction.

Again, if we take the hypothesis in Article (21), and assume moreover that v'' and v_2 are large compared with v and v', it is easy to see, by similar reasoning, that the normal rays will be extinguished immediately after their production by the first refraction.

It is evident, therefore, that on either hypothesis (adding to the latter, that v_2 and v'' are large compared with v and v'), normal waves will produce no sensible effect on the eye, even indirectly.

SECTION VI.

Whether Polarized Light consists of Vibrations at Right Angles to, *or* Parallel to *the Plane of Polarization.*

(27.) THERE can be no doubt of the truth of Sir D. Brewster's *law of tangents*, and the laws of the *rotation* of the plane of polarization given by M. Arago, and Sir D. Brewster. From these laws we shall attempt to prove, that polarized light consists of vibrations perpendicular to the plane of polarization, in the following manner.

If possible, let polarized light consist of vibrations parallel to the plane of polarization; then taking the equations of connection in their most general form, viz. (D), (E), (F), Section II., and proceeding as in Articles (14) and (15), we find, for light polarized perpendicularly to the plane of incidence, the following formulæ:

$$V_1 = \frac{s - \epsilon\mu s'}{s + \epsilon\mu s'} V, \qquad V' = \frac{2s}{s + \epsilon\mu s'} V,$$

$$\text{where } \epsilon = \frac{C + D'}{C + D}.$$

Now, by the law of tangents, V_1 ought to become zero when the tangent of the angle of incidence is μ, *i. e.* when $s' = \mu s$. Therefore we have

$$s - \epsilon\mu^2 s = 0, \quad \text{and therefore } \epsilon = \frac{1}{\mu^2},$$

and this gives us

$$V_1 = \frac{\mu s - s'}{\mu s + s'} V, \qquad V' = \frac{2\mu s}{\mu s + s'} V.$$

Now if U, U_1, U', U_2, U'', be the velocities, when the light is polarized *in* the plane of incidence, we have, by the laws of the rotation of the plane of polarization,

$$\frac{U_1}{V_1} = \frac{s - \mu s'}{s + \mu s'} \cdot \frac{\mu s + s'}{\mu s - s'} \frac{U}{V}, \qquad \frac{U'}{V'} = \frac{\mu s + s'}{s + \mu s'} \frac{U}{V}.$$

Hence, by the above expressions for V_1 and V', we have

$$U_1 = \frac{s - \mu s'}{s + \mu s'} U, \qquad U' = \frac{2\mu s}{s + \mu s'} U,$$

and from these equations we find

$$\mu (U + U_1) = U' \ldots\ldots (1), \qquad s (U - U_1) = s' U' \ldots\ldots (2).$$

Now from the equations of connection (D), we have, as in Article 19, equations (8) and (9),

$$s (U - U_1) - U_2 p_2 = s' U' - U'' p'',$$

$$p (U + U_1) + U_2 s_2 = p' U' + U'' s'',$$

which, by (1) and (2), and since $p = \mu p'$, become

$$U_2 p_2 = U'' p'', \qquad U_2 s_2 = U'' s''.$$

Now since $\frac{p''}{v''} = \frac{p_2}{v_2}$, and v'', v_2 are essentially positive, p'' and p_2 have the same sign; therefore U_2 and U'', and therefore s_2 and s'' have the same sign. Now by Article (19), equations (10) and (11), s_2 and s'' have opposite signs, which is absurd. The only way to get over this, is to suppose that U_2 and U'' are zero, but then it will be impossible to satisfy the equations of connection for all values of p. (See Article 19).

Hence it follows that, if we adopt the hypothesis which supposes polarized light to consist of vibrations parallel to the plane of polarization, and take into account the experimental laws above stated, we arrive at an absurd result.

We may therefore conclude, that Fresnel's hypothesis is true.

SECTION VII.

How we must proceed when the Equations of Connection are taken in their most general Form.

(28.) THE equations (5), (8), and (9), Article (19), are still true, being deduced from the equations (D), Section II., without making any hypothesis respecting the constants. But instead of (6) and (7), we shall arrive at two equations, somewhat more complicated in form, as follows.

The equations (E) and (F), Section II., may, in virtue of the equations $\frac{d\alpha}{dx}=\frac{d\alpha'}{dx}$, $\frac{d\gamma}{dx}=\frac{d\gamma'}{dx}$, be put in the forms

$$(C+D)\left(\frac{d\alpha}{dz}-\frac{d\gamma}{dx}\right)=(C+D')\left(\frac{d\alpha'}{dz}-\frac{d\gamma'}{dx}\right)+2(D'-D)\frac{d\gamma'}{dx},$$

$$(C+E)\left(\frac{d\gamma}{dz}+\frac{d\alpha}{dx}\right)=(C+E')\left(\frac{d\gamma'}{dz}+\frac{d\alpha'}{dx}\right)+(D'-E'-D+E)\frac{d\alpha'}{dx}.$$

Now, in these equations, as in Article (19), we shall put $\alpha+\alpha_1+\alpha_2$, $\gamma+\gamma_1+\gamma_2$, $\alpha'+\alpha''$, $\gamma'+\gamma''$, for α, γ, α', γ', respectively, and then, as in Article (19), and by the equations (A), (B), (C), Section I., we obtain immediately the following equations,

$$(C+D)\frac{V+V_{\prime}}{v}=(C+D')\frac{V'}{v'}-2(D'-D)\left(s'p'\frac{V'}{v'}+s''^2\frac{V''}{v''}\right)$$

$$(C+E)\frac{V_2}{v_2}=(C+E')\frac{V''}{v''}+(D'-E'-D+E)\left(s'p'\frac{V'}{v'}-p''^2\frac{V''}{v''}\right).$$

From these equations, and the equations (8) and (9) Article (19), we may find $V_{\prime}$, V', V_2, and V'' in terms of V. We shall not calculate these values, as they are rather complicated, and not necessary to the object of the present paper. The last of the equations just obtained considerably simplifies when we suppose the ether to obey the common law of elasticity, in which case we have $D'-E'-D+E=0$. (See Article 12.)

It is easy to see that Fresnel's formulæ cannot be deduced from these equations, unless $D=D'$, and $E=E'$, and therefore it will be useless to employ the equations of connection in their most general form, as it is highly probable that Fresnel's formulæ are experimentally true for a great number of substances.

SECTION VIII.

Intensity and Phase of the Reflected Ray, in the case of highly Refractive Substances.

(29.) THERE are some substances, such as the diamond and other bodies of high refractive power, for which Fresnel's formulæ do not appear to be accurately true. It is easy to account for this in the following manner.

When we do not assume that $v_2=v''$, we have, by Article (22),

$$V_{\prime}=\frac{\mu s-s'-\eta}{\mu s+s'+\eta}V, \text{ where } \eta=(\nu^2-1)(\mu^2-1)\frac{p'p''}{s''-\nu s_2}.$$

Now, by Article (26), we must suppose that v_2 and v'' are very large compared with v and v', and consequently that s_2 and s'', and therefore η, are impossible quantities. Let us accordingly put

$$\frac{\eta}{\mu s - s'} = \tan\psi\sqrt{-1} \;\ldots\ldots(1), \qquad \frac{\eta}{\mu s + s'} = \tan\omega\sqrt{-1} \;\ldots\ldots(2),$$

and we then have

$$V_{,} = \frac{\mu s - s'}{\mu s + s'}\,\frac{\cos\omega}{\cos\psi}\, e^{-(\psi+\omega)\sqrt{-1}}\, V.$$

Now this value of $V_{,}$ indicates, as in Section IV, that the intensity of the reflected ray is

$$\left\{\frac{\mu s - s'}{\mu s + s'}\,\frac{\cos\omega}{\cos\psi}\right\}^2 a^2, \text{ or } \frac{(\mu s - s')^2 - \eta^2}{(\mu s + s')^2 - \eta^2}\, a^2, \text{ (by (1) and (2)).}$$

And that its phase differs from the phase of the incident ray by the quantity $(\psi + \omega)$.

To calculate η, we observe that $s'' = \sqrt{1 - \left(\frac{p v''}{v}\right)^2} = p\,\frac{v''}{v}\sqrt{-1}$ very nearly, since $\frac{v''}{v}$ is very large: and similarly, $s_2 = -p\,\frac{v_2}{v}\sqrt{-1}$. We here give s'' and s_2 opposite signs, because the expressions for V_2 and V'' will contain the factors $e^{k s_2 z\sqrt{-1}}$ and $e^{k' s'' z\sqrt{-1}}$*. Now one of these (namely, V'') ought to diminish rapidly as z increases, and the other (V_2) ought to do so as z decreases; but this cannot be if s'' and s_2 have the same sign, therefore we must take these quantities with different signs.

$$\text{Hence}\quad \eta = (\nu^2 - 1)(\mu^2 - 1)\,\frac{\dfrac{p}{\mu}\,\dfrac{p v''}{v}}{\left(\dfrac{p v''}{v} + \nu\,\dfrac{p v_2}{v}\right)\sqrt{-1}},$$

and therefore, since $\nu = \dfrac{v_2}{v''}$, we have

$$-\eta^2 = \left\{\frac{\nu^2 - 1}{\nu^2 + 1}\,\frac{\mu^2 - 1}{\mu}\, p\right\}^2.$$

If we suppose the light to be incident at the polarizing angle, the expression for the intensity of the reflected ray becomes (since at that angle $\mu s = s' = p$)

$$a_{,}^2 = \frac{-\eta^2}{4p^2 - \eta^2}\, a^2 = \frac{\left(\dfrac{\nu^2 - 1}{\nu^2 + 1}\,\dfrac{\mu^2 - 1}{2\mu}\right)^2}{1 + \left(\dfrac{\nu^2 - 1}{\nu^2 + 1}\,\dfrac{\mu^2 - 1}{2\mu}\right)^2}\, a^2.$$

For common plate-glass we may put $\mu = \frac{3}{2}$, and therefore $\left(\frac{\mu^2 - 1}{2\mu}\right)^2 = \frac{1}{6}$ nearly; and for diamond we may put $\mu = \frac{12}{5}$, and therefore $\left(\frac{\mu^2 - 1}{2\mu}\right)^2 = 1$ very nearly. Hence, supposing that ν is the same for both, the intensity of the reflected ray at the polarizing angle is about six times greater for diamond than for plate-glass. But we have every reason to suppose that ν (the index of refraction for normal waves) and μ (that for transversal) will increase together.

* See the process of superposition in Section IV.

Let us suppose, at a venture, that the value of ν is $\frac{11}{10}$ for glass, and $\frac{4}{3}$ for diamond: then the value of $\left(\frac{\nu^2 - 1}{\nu^2 + 1}\right)^2$ is about $\frac{1}{100}$ for glass, and $\frac{1}{15}$ for diamond; therefore we have

$$a_{,}^2 = \frac{a^2}{600} \text{ for glass,}$$

$$a_{,}^2 = \frac{a^2}{15} \text{ for diamond.}$$

These expressions, if correct, would indicate that the reflected ray was scarcely visible for glass; and faint, though decidedly visible, for diamond: which, I believe, is the case. From this example it is clear that if we suppose the normal index of refraction to be less than about $\frac{11}{10}$ when the transversal index is less than $\frac{3}{2}$, the reflected ray at the polarizing angle will be scarcely visible for plate-glass and substances of lower refractive power: and if we suppose the normal index not less than about $\frac{4}{3}$ when the transversal is greater than 2, the reflected ray will be decidedly visible.

Supposing this to be true, $-\eta^2$ will be very small for substances of moderate refractive power, and therefore Fresnel's formulæ will hold for such substances, at least the deviation from Fresnel's formulæ will be insensible.

Hence, for substances of moderate refractive power ω will be always small; but ψ, and therefore the phase $(\omega + \psi)$ will increase rapidly by very nearly 180^0 while the angle of incidence is passing through the polarizing angle; this is evident from (1).

For substances of high refractive power, $-\eta^2$ will not be very small; therefore there will be a sensible deviation from Fresnel's formulæ. Moreover ω will not be very small, and ψ, and therefore the phase $(\omega + \psi)$ will increase by a quantity somewhat less than 180^0, while the angle of incidence is passing through the polarizing angle.

These results are in strict accordance with the experiments of Mr Airy; see the *Cambridge Transactions*, Vol. IV, p. 422.

III. *On the Possibility of accounting for the Absorption of Light, by supposing it due to the Motion of the Particles of Matter. By the* Rev. M. O'Brien, *late Fellow of Caius College.*

[Read *Feb.* 14, 1843.]

When we take into account the motion of the particles of matter (see *Cambridge Transactions*, Vol. vii. p. 421*), we arrive at the following equation for determining the velocity of propagation (v^2), viz.

$$k^2 = \frac{m_{,}C}{v^2 - mB} + \frac{mC}{v^2 - m_{,}B_{,}},$$

the disturbance being proportional to $\cos k\,(vt - u)$.

If we put $kv = n$, this equation becomes

$$n^2\,(v^2 - mB)\,(v^2 - m_{,}B_{,}) = C\,\{m_{,}\,(v^2 - m_{,}B_{,}) + m\,(v^2 - mB)\}\,v^2 \text{.........}(1).$$

which is a quadratic equation for determining v^2 when n is given, *i. e.* when the colour is given, since $\frac{2\pi}{n}$ is the time of vibration.

This equation affords a complete explanation of the dispersion of light, and it may also be applied to account, apparently in a satisfactory manner, for the absorption, as follows.

Suppose that the roots of the equation are impossible, then we shall obtain four values of v, which we may put in the form, $\frac{1}{v} = \pm\,\epsilon \pm \eta\sqrt{-1}$.

Now $\alpha = ae^{n\left(t-\frac{u}{v}\right)\sqrt{-1}}$ is an integral of the equations of motion; hence we have four integrals included in the formula

$$\alpha = ae^{n(t-(\pm\epsilon\pm\eta\sqrt{-1})u)\sqrt{-1}}, \text{ or } a\,e^{\pm n\eta u}\,.\,e^{n(t\pm\epsilon u)\sqrt{-1}}.$$

From these imaginary integrals we obtain the real integrals

$$\alpha = ae^{\pm n\eta u}\,.\cos n\,(t \pm \epsilon u).$$

Now we must not suppose a continually increasing intensity of vibration; and therefore the upper sign of the exponential coefficient must be rejected, as is usually done in similar cases: we have therefore

$$\alpha = ae^{-n\eta u}\cos n\,(t \pm \epsilon u).$$

This expression indicates a continually decreasing intensity of vibration different for different colours (since η is evidently a function of n), and thus the supposition that the roots of (1) are impossible, leads to an explanation of the absorption of light.

It is easy to follow out this explanation into detail, and to shew that it agrees with experiment so far as it goes; but the object of the present paper is to prove, very briefly, that there is a serious objection against the supposition that the equation (1) has impossible roots, and therefore against the explanation of absorption depending on the motion of the particles of matter. To do this, we

* Since the paper here referred to was printed, I have been informed that Professor Lloyd had previously read a paper on the same subject, in which he gave an explanation of the Dispersion and Absorption of Light; but I am not aware that his paper has been printed, for I have not been able to procure it.

must investigate the equations of connection of the vibratory motion of two media separated by a plane (as in a previous paper in the present part of the *Cambridge Transactions*), supposing that one of the media is composed of material as well as ethereal particles.

We shall make the same suppositions, and use the same notation, as in the paper just referred to; assuming the upper medium to contain material particles, and $\alpha_{,}$, $\beta_{,}$, $\gamma_{,}$ to belong to any one of them, α, β, γ, and α', β', γ' belonging (as before) to the etherial particles.

Then the force parallel to the axis of x on any particle of ether at the plane of separation will be

$$(C+D)\frac{d\alpha}{dz}+D\frac{d\gamma}{dx}-(C'+D')\frac{d\alpha'}{dz}-D'\frac{d\gamma'}{dx}+\text{terms of superior order.}$$

The terms of superior order here alluded to consist, in the first place, of higher differential coefficients of α, β, γ, α', β', γ', and secondly of terms arising from the action of the material particles, the largest of which we have assumed, in obtaining the equation (1), to be of the same order of magnitude as the second differential coefficients of α, β, γ. Hence we have at the plane of separation

$$(C+D)\frac{d\alpha}{dz}+D\frac{d\gamma}{dx}=(C'+D')\frac{d\alpha'}{dz}+D'\frac{d\gamma'}{dx}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

In the same way we obtain

$$(C+D)\frac{d\beta}{dz}+D\frac{d\gamma}{dy}=(C'+D')\frac{d\beta'}{dz}+D'\frac{d\gamma'}{dx}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

$$(C+E)\frac{d\gamma}{dz}+D\left(\frac{d\alpha}{dx}+\frac{d\beta}{dz}\right)=(C'+E')\frac{d\gamma'}{dz}+D'\left(\frac{d\alpha'}{dx}+\frac{d\beta'}{dy}\right)\quad\ldots\ldots (4).$$

We also find, just as before,

$$\alpha'=\alpha,\qquad \beta'=\beta,\qquad \gamma'=\gamma\ldots\ldots\ldots\ldots\ldots (5).$$

In addition to these six equations of connection, we obtain three others in the following manner.

At the plane of separation the force acting on any particle of matter is

$$\Sigma m\left\{\psi(r_{,})\,\delta\alpha_{,}+\frac{1}{r_{,}}\psi'(r_{,})\,\delta x_{,}(\delta x_{,}\delta\alpha_{,}+\delta y_{,}\delta\beta_{,}+\delta z_{,}\delta\gamma_{,})\right\}$$

+ a part arising from the action of the ethereal particles.

This may be reduced, as the force on an ethereal particle, to the form

$$(C_{,}+D_{,})\frac{d\alpha_{,}}{dz}+D_{,}\frac{d\gamma_{,}}{dx}+\text{terms of superior order;}$$

observing that we include the part arising from the action of the ethereal particles among the terms of superior order for the same reason as before. We have, therefore, at the surface of separation,

$$\left.\begin{aligned}&(C_{,}+D_{,})\frac{d\alpha_{,}}{dz}+D_{,}\frac{d\gamma_{,}}{dx}=0\\ \text{and similarly }&(C_{,}+D_{,})\frac{d\beta_{,}}{dz}+D_{,}\frac{d\gamma_{,}}{dy}=0\\ &(C_{,}+E_{,})\frac{d\gamma_{,}}{dz}+D_{,}\left(\frac{d\alpha_{,}}{dx}+\frac{d\beta_{,}}{dy}\right)=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

These nine equations, namely, (2), (3), (4), (5), and (6), are the complete equations of connection which apply to the case of reflexion and refraction, the motion of the material particles being taken into account. It is evidently not allowable to simplify these equations by putting $E = E'$ and $D = D'$, as we did in the previous paper. Moreover, instead of having $C = C'$, we have $C = C' + C_,$.

Our present purpose requires us to apply these equations only to the case of rays incident directly on a refracting surface; we shall therefore suppose that the quantities $a, \gamma, a_,, \gamma_,, a', \gamma'$, are each zero, and that $\beta, \beta_,, \beta'$, are functions of z only: then the nine equations of connection reduce to three, viz.

$$\beta = \beta' \ldots\ldots (7), \qquad \frac{d\beta}{dz} = h\frac{d\beta'}{dz} \ldots\ldots (8), \qquad \frac{d\beta_,}{dz} = 0 \ldots\ldots (9),$$

$$\text{where } h = \frac{C' + D'}{C + D}.$$

We shall assume v to be the velocity of propagation in the lower medium, and v', v'' the two velocities in the upper, namely, the two roots of the equation (1). We shall suppose that the waves in the upper medium are an incident and a reflected, and in the lower, two refracted waves, one propagated with the velocity v', and the other with the velocity v'', for it will be impossible to satisfy the three equations (7), (8), (9), with only one refracted wave. Hence, using the same notation as in the previous paper, we have from (7), (8), and (9),

$$V + V_, = V' + V'' \ldots\ldots\ldots (10),$$

$$(V - V_,)\frac{1}{v} = \frac{V'}{v'} + \frac{V''}{v''} \ldots\ldots\ldots (11).$$

$$\frac{V_,'}{v'} + \frac{V_,''}{v''} = 0 \ldots\ldots\ldots\ldots\ldots (12).$$

When V' V'' belong to the two refracted waves, and $V_,'$ $V_,''$ to the corresponding waves of the particles of matter, observing that the two latter waves are propagated respectively with the same velocities as the two former (See Vol. VII. p. 421).

Hence, if we assume in general that

$$V = a e^{n\left(t - \frac{z}{v}\right)\sqrt{-1}}, \qquad V_, = a_, e^{n\left(t + \frac{z}{v}\right)\sqrt{-1}}, \qquad V' = a' e^{n\left(t - \frac{z}{v'}\right)\sqrt{-1}} \text{ \&c., \&c.}$$

We have from (10) (11) and (12), putting $z = 0$,

$$a + a_, = a' + a'' \ldots\ldots\ldots (13).$$

$$a - a_, = \frac{v}{v'}a' + \frac{v}{v''}a'' \ldots\ldots\ldots (14).$$

$$a_,'' = -\frac{v''}{v'}a_,' \ldots\ldots\ldots (15).$$

Also by the two equations in the middle of page 423, Vol. VII.* we have (a' and $a_,'$ or a'' and $a_,''$ here, correspond to a and $a_,$ there)

$$a_,' = -\frac{m}{m_,}\frac{v'^2 - mB}{v'^2 - m_,B_,}a', \qquad a_,'' = -\frac{m}{m_,}\frac{v''^2 - mB}{v''^2 - m_,B_,}a''.$$

Hence by (15) we have

$$a'' = -\frac{v'^2 - mB}{v'^2 - m_,B_,} \cdot \frac{v''^2 - m_,B_,}{v''^2 - mB}\frac{v''}{v'}a' \ldots\ldots\ldots (16).$$

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* In the second of these $a - a_,$ is written by mistake instead of $a_, - a$.

Also (13) + (14) gives

$$2a = \left(1 + \frac{v}{v'}\right) a' + \left(1 + \frac{v}{v''}\right) a'' \quad \text{.........(17).}$$

And from (16) and (17) we have

$$2\,a = \left\{1 + \frac{v}{v'} - \left(1 + \frac{v}{v''}\right)\frac{v''}{v'} \cdot \frac{v'^2 - mB}{v'^2 - m_{\prime}B_{\prime}}\,\frac{v''^2 - m_{\prime}B_{\prime}}{v''^2 - mB}\right\} a' \quad \text{.........(18).}$$

Now if we suppose the roots of (1) to be impossible, we have

$$\frac{1}{v'} = \epsilon + \eta\sqrt{-1}, \qquad \frac{1}{v''} = \epsilon - \eta\sqrt{-1},$$

Making these substitutions in (18), we evidently find an expression for a' of the form

$$a' = f(\cos\omega + \sqrt{-1}\sin\omega)\,a, \qquad \text{or, } f e^{\omega\sqrt{-1}}\,.\,a;$$

where f and ω are real quantities, the former of which does not change sign when $-\eta$ is put for η, but the latter does. Now a' becomes a'' when v'' is put in place of v', and v' in place of v''; *i.e.* when $-\eta$ is put for η. Hence we have

$$a'' = f e^{-\omega\sqrt{-1}}\,.\,a.$$

Hence the general expression for $V' + V''$ is

$$f\,a\,\left\{e^{\omega\sqrt{-1}}.\,e^{n\left(t-\frac{z}{v'}\right)\sqrt{-1}} + e^{-\omega\sqrt{-1}}.\,e^{n\left(t-\frac{z}{v''}\right)\sqrt{-1}}\right\};$$

$$\text{or, } f\,a\,\left\{e^{n\eta z}.\,e^{\{n(t-\epsilon z)+\omega\}\sqrt{-1}} + e^{-n\eta z}.\,e^{\{n(t-\epsilon z)-\omega\}\sqrt{-1}}\right\} \quad \text{.........(19).}$$

(19) therefore is the symbolical disturbance in the upper medium arising from the symbolical disturbance $a\,e^{n\left(t-\frac{z}{v}\right)\sqrt{-1}}$ in the lower. By changing the signs of n and η, and therefore of ω, we find that the symbolical disturbance $a\,e^{-n\left(t-\frac{z}{v}\right)\sqrt{-1}}$ in the lower medium gives rise, in the upper, to

$$f\,a\,\left\{e^{n\eta z}.\,e^{-\{n(t-\epsilon z)+\omega\}\sqrt{-1}} + e^{-n\eta z}.\,e^{-\{n(t-\epsilon z)-\omega\}\sqrt{-1}}\right\}.$$

Hence, superposing these two sets of disturbances, we find that the real disturbance

$$a\cos n\left(t - \frac{z}{v}\right)$$

in the lower medium gives rise to the real disturbance

$$f\,a\left[e^{n\eta z}.\cos\{n(t-\epsilon z)+\omega\} + e^{-n\eta z}.\cos\{n(t-\epsilon z)-\omega\}\right]$$

in the upper.

Now this latter expression indicates a continually increasing intensity, and therefore if the roots of (1) were impossible, light after refraction would continually increase in intensity in passing through the refracting substance; a result which is quite at variance with experiment. Hence we may conclude that the roots of (1) cannot be impossible, and that the explanation of absorption given above is not true. In fact, that explanation falls to the ground if we be not at liberty to reject the integral $a\,e^{n\eta z}\cos n(t-\epsilon z)$ and retain $a\,e^{-n\eta z}\cos n(t-\epsilon z)$, which we cannot do without violating the equations of connection, as is evident from the process just gone through.

It appears, therefore, that though the action of the material upon the ethereal particles affords a complete and satisfactory explanation of dispersion, we must look to some other source for an explanation of absorption.

IV. *On a new Fundamental Equation in Hydrodynamics. By the* REV. JAMES CHALLIS, MA., *Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge.*

[Read *March* 6, 1843.]

THE object of this communication is to shew, that in addition to the two fundamental equations of Hydrodynamics already recognised, a third is necessary to complete the analytical principles of the science.

For the purpose of reference I shall call the two known equations, *the dynamical equation*, and, *the equation of continuity of the fluid*. The same notation will be made use of as in my last paper: p is the pressure and ρ the density of a particle whose co-ordinates at the time t are x, y, z, and the components of whose velocity V are u, v, w, in the directions of the axes of co-ordinates. X, Y, Z are the impressed forces in the same directions. A *differential coefficient* is put in brackets to indicate that the differentiation refers both to the co-ordinates and the time: a *differential* in brackets means that the co-ordinates alone are differentiated. All differential coefficients not in brackets are partial.

1. It will be assumed that in any case of fluid motion an unlimited number of surfaces may be drawn at each instant, cutting at right angles the directions of motion. In other words, it is assumed that the directions of motion at every instant fulfil the condition of *geometrical continuity*. In my last paper it was shewn that if

$$d\psi = \frac{u}{N}dx + \frac{v}{N}dy + \frac{w}{N}dz, \ldots\ldots\ldots\ldots (1).$$

the factor $\frac{1}{N}$ being such that the right-hand side of the above equality is an exact differential, the general differential equation of all these surfaces at all times is $d\psi = 0$. It is not necessary that the surfaces should be continuous: that is, it is not necessary that the equation of a given surface should be the same function of the co-ordinates through its whole extent. But that the condition of the geometrical continuity of the directions of the motion may be maintained, each surface must be made up of parts, either finite or indefinitely small, which are surfaces of continuous curvature. Hence the quantity N has a real value for every part of the fluid in motion; at least, motions for which this is not the case, if there are such, do not come under our consideration.

2. Let the integral of the equation $d\psi = 0$ be $\psi(x, y, z, t) = 0$, the arbitrary function of the time being included in the function ψ. The surfaces of which this is the general equation I shall continue to call *surfaces of displacement*. Since the equation $\psi(x, y, z, t) = 0$ embraces all the surfaces of displacement at all times, it will include the surfaces of displacement of a given element of the fluid at two successive instants of its motion, *if the path of the element in the interval be continuous.* It is not necessary that the path of an element through its whole extent should be determined by the same equations, but it is necessary for the continuity of the motion that it should be made up of parts, either finite or indefinitely small, which are geometrically continuous, and that the directions of motion at two successive instants should not make a finite angle with each other. The condition of the continuity of the motion of each element is therefore expressed analytically by the equation $\delta\psi(x, y, z, t) = 0$, the symbol δ having reference,

as in the Calculus of Variations, to the function ψ, while the co-ordinates and the time vary with the varying position of a given element. Hence,

$$\frac{d\psi}{dt}\delta t + \frac{d\psi}{dx}\delta x + \frac{d\psi}{dy}\delta y + \frac{d\psi}{dz}\delta z = 0.$$

But $\delta x = u\delta t$, $\delta y = v\delta t$, and $\delta z = w\delta t$. Consequently,

$$\frac{d\psi}{dt} + \frac{d\psi}{dx}u + \frac{d\psi}{dy}v + \frac{d\psi}{dz}w = 0. \dots\dots\dots\dots\dots (2).$$

The main object of the arguments in this paper will be, to shew that the equation just obtained is a necessary and fundamental equation of Hydrodynamics. I propose to call it, with reference to the principle on which it was investigated, the *equation of continuity of the motion*, to distinguish it from the equation of continuity of the *fluid*.

It may here be remarked, that in the place of the actual surface of displacement we might have reasoned in the same manner on a surface having with it a contact of the second order at the point xyz; for instance, the surface whose equation is,

$$\frac{(x-\alpha)^2}{m^2} + \frac{(y-\beta)^2}{n^2} + \frac{(z-\gamma)^2}{p^2} - 1 = 0,$$

the six parameters $\alpha, \beta, \gamma, m, n, p$, being functions in general, both of the co-ordinates and the time. Writing $F = 0$ for this equation, it is clear, that when the co-ordinates and parameters vary with the change of position of an element, we shall have $\delta F = 0$, provided there be no *abrupt* change of the parameters, and consequently no abrupt change of the curvature of the surface of displacement and of the directions of the lines of motion. This equation, therefore, to which the equation $\delta\psi(x, y, z, t) = 0$ is equivalent, expresses the condition of continuity of the motion.

3. Before entering on the consideration of equation (2), it will be shewn by an example that the two recognised fundamental equations are insufficient for the general determination of fluid motion. One instance of contradictory results legitimately deduced from those equations will suffice for this purpose. The example I have chosen is as simple as possible.

Let the fluid be incompressible, and the motion be parallel to the plane of xy. The equation of the continuity of the fluid for this case is $\frac{du}{dx} + \frac{dv}{dy} = 0$. If $u = mx$ and $v = -my$, that equation is satisfied. These values make $udx + vdy$ an exact differential. Hence the dynamical equation gives, $p = C - \frac{m}{2}(x^2 + y^2)$, the arbitrary quantity being either constant or a function of the time. By putting $p = 0$, we obtain $x^2 + y^2 = \frac{2C}{m}$ for the equation of the free surface of the fluid, which is therefore at *all* times cylindrical, and hence the velocity is every instant the same at all points of the surface. But the differential equation of a line of motion is $\frac{dy}{dx} = \frac{v}{u} = -\frac{y}{x}$. The lines of motion are therefore rectangular hyperbolas having the axes of co-ordinates for asymptotes, and the directions of motion are consequently different at different points of the cylindrical boundary. Hence it is impossible that the boundary can be constantly cylindrical. This contradiction proves that the equations on which the reasoning was founded are either erroneous or insufficient. We have no reason to suspect any error in the principles from which they were derived, and must therefore conclude that they are insufficient. It will appear afterwards that this instance does not satisfy the conditions of continuity.

4. Since $u = N\frac{d\psi}{dx}$, $v = N\frac{d\psi}{dy}$, and $w = N\frac{d\psi}{dz}$, we readily obtain from equation (2),

$$\frac{d\psi}{dt} + N\left(\frac{d\psi^2}{dx^2} + \frac{d\psi^2}{dy^2} + \frac{d\psi^2}{dz^2}\right) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

which equation determines N.

A remark is important here. It appears from the equality (1), that $udx + vdy + wdz = Nd\psi$, and it might hence be supposed that when the left-hand side of this equality is integrable, we are at liberty to assume $N = 1$, and to consider ψ identical with the quantity which is usually called ϕ in Hydrodynamics. But it is clear from the reasoning by which equation (2) was obtained, that N is a quantity of the same kind as the velocity, and that ψ is supposed to be freed from any factors which do not verify the equation $\psi = 0$, whilst $d\phi$ is merely a substitution for $udx + vdy + wdz$, and its integral ϕ is subject to no such operation. It is not, therefore, allowable in any case to suppose the two quantities to be the same, on which account I have here employed the letter ψ in the place of the ϕ of my former paper. When $udx + vdy + wdz$ is integrable, in general $N = f(t) \,.\, F(\phi)$.

5. For the purposes of the reasoning on which we shall presently enter, it is required to shew, first, that when $udx + vdy + wdz$ is an exact differential $(d\phi)$, the integral of the dynamical equation may be taken from any one point of the fluid to any other, and that the arbitrary quantity to be added is either a constant or a function of the time only. This will appear as follows.

The general dynamical equation is equivalent to the three equations,

$$\frac{dP}{dx} - X + \left(\frac{du}{dt}\right) = 0, \ (4). \qquad \frac{dP}{dy} - Y + \left(\frac{dv}{dt}\right) = 0, \ (5). \qquad \frac{dP}{dz} - Z + \left(\frac{dw}{dt}\right) = 0, \ (6).$$

in which P is substituted for $\frac{p}{\rho}$, or for k^2 Nap. log ρ, according as the fluid is incompressible or compressible. Assuming $Xdx + Ydy + Zdz$ to be an exact differential, putting $(d\lambda)$ for $\left(\frac{dP}{dx} - X\right)dx + \left(\frac{dP}{dy} - Y\right)dy + \left(\frac{dP}{dz} - Z\right)dz$, and adding the above equations after multiplying them respectively by dx, dy, dz, it is known that we obtain for the case in question,

$$(d\lambda) + \left(d \,.\, \frac{d\phi}{dt}\right) + \tfrac{1}{2}\left(d \,.\, \left\{\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right\}\right) = 0, \ldots\ldots\ldots\ldots (7).$$

which, if V^2 be substituted for $\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dx^2}$, is equivalent to

$$\left(\frac{d\lambda}{dx} + \frac{d^2\phi}{dxdt} + V\frac{dV}{dx}\right)dx + \left(\frac{d\lambda}{dy} + \frac{d^2\phi}{dydt} + V\frac{dV}{dy}\right)dy + \left(\frac{d\lambda}{dz} + \frac{d^2\phi}{dzdt} + V\frac{dV}{dz}\right)dz = 0.$$

But the quantities in brackets must be respectively identical with the quantities on the left-hand sides of the equations (4), (5), (6). Hence by reason of those equations,

$$\frac{d\lambda}{dx} + \frac{d^2\phi}{dxdt} + V\frac{dV}{dx} = 0, \qquad \frac{d\lambda}{dy} + \frac{d^2\phi}{dydt} + V\frac{dV}{dy} = 0, \qquad \frac{d\lambda}{dz} + \frac{d^2\phi}{dzdt} + V\frac{dV}{dz} = 0.$$

Hence, dividing the foregoing equation by dx, it will be seen that $\frac{dy}{dx}$ and $\frac{dz}{dx}$ may be of any arbitrary values. The integral of that equation may consequently be taken from any one point to any other of the fluid, and the arbitrary quantity to be added is independent of co-ordinates.

6. It is next required to introduce into the equation of continuity of the fluid, by means of equation (3), the condition of continuity of the motion. For this purpose the process must be gone through which is given in my former paper (*Camb. Phil. Trans.* Vol. VII. Part III. pp. 385, 386). The result there arrived at is,

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = \frac{u}{V} \cdot \frac{dV}{dx} + \frac{v}{V} \cdot \frac{dV}{dy} + \frac{w}{V} \cdot \frac{dV}{dz} + V\left(\frac{1}{r} + \frac{1}{r'}\right),$$

where r, r' are the principal radii of curvature of the surface of displacement at the point xyz. If ds be the increment of the line of motion, we have $\frac{u}{V} = \frac{dx}{ds}$, $\frac{v}{V} = \frac{dy}{ds}$, $\frac{w}{V} = \frac{dz}{ds}$. Hence if dV be the increment of velocity along the line of motion corresponding to the increment ds, the required equation becomes for incompressible fluids,

$$\frac{dV}{ds} + V\left(\frac{1}{r} + \frac{1}{r'}\right) = 0 \ldots\ldots (8).$$

When the fluid is compressible we have the equation,

$$\frac{d\rho}{dt} + \frac{d.\rho u}{dx} + \frac{d.\rho v}{dy} + \frac{d.\rho w}{dz} = 0;$$

$$\text{or, } \frac{d\rho}{dt} + \frac{d\rho}{dx}u + \frac{d\rho}{dy}v + \frac{d\rho}{dz}w + \rho\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0.$$

Now $u = V\frac{dx}{ds}$, $v = V\frac{dy}{ds}$, $w = V\frac{dz}{ds}$; and, as before,

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = \frac{dV}{ds} + V\left(\frac{1}{r} + \frac{1}{r'}\right).$$

By substituting these values in the equation above, it will readily be found that

$$\frac{d\rho}{dt} + \frac{d.V\rho}{ds} + V\rho\left(\frac{1}{r} + \frac{1}{r'}\right) = 0, \ldots\ldots (9).$$

in which $d.V\rho$ is the increment of $V\rho$ along the line of motion corresponding to the increment ds of the line of motion. I have obtained equation (9) in my former paper (pp. 387 and 388) by elementary considerations, and equation (8) might clearly be obtained in a similar manner. That method, being independent, may be adduced in confirmation of the reasoning here employed, and of the general equation (2), by means of which the reasoning has been conducted. It also has the advantage of shewing distinctly that the increment $d.V\rho$ in (9) must be limited to the direction of the line of motion, unless $V\rho$ has the same value at all points of a given surface of displacement; and that dV in (8) must be similarly limited, unless the velocity be the same at all points of a given surface of displacement.

The equations (8) and (9) may be called equations of *absolute continuity*. When they are satisfied consistently with the respective dynamical equations, there can be no breach of continuity and the motion is possible. Examples will hereafter be adduced to illustrate the use of these equations.

7. I propose now to determine by means of equations (8) and (9) in what cases of possible motion $udx + vdy + wdz$ is an exact differential. This important question has not yet received a satisfactory answer.*

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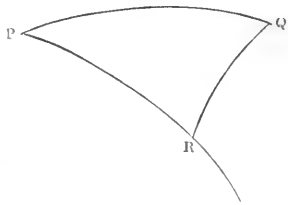
* Lagrange in the *Mécanique Analytique* argues that $udx + vdy + wdz$ is an exact differential when the motion is so small that powers of the velocity above the first may be neglected; and again, when the motion begins from rest. These theorems occur in the Edition of Poisson's *Traité de Mécanique* of 1811, but are omitted in that of 1833. Lagrange's reasoning

First, let the fluid be incompressible. It has been shewn in Art. 5 that when $u\,dx + v\,dy + w\,dz$ is an exact differential, the dynamical equation may be integrated from any one to any other point of the fluid. But the result obtained by integrating that equation in this manner, does not give a possible motion unless the equation (8) be similarly integrable. Let this be the case. Then the first condition that must be satisfied is, that each surface of displacement be a surface of equal velocity. For on no other supposition can the differential coefficient $\frac{dV}{ds}$ remain the same, in passing from a given point to another indefinitely near in an unlimited number of directions. In the annexed figure P and Q are any two points of the fluid; QR is an orthogonal trajectory to the surfaces of displacement situated at a given instant between P and Q; PR is a line drawn on the surface of displacement which passes through P, and intersecting QR in R. Now by hypothesis the integral of equation (8) may be taken between arbitrary limits. Therefore the integral from P to Q along PQ is the same as the integral along PR and RQ. But the integral along PR is nothing, because PR is on a surface of equal velocity. Therefore the integral from P to Q is the same as the integral from R to Q. Supposing therefore the surface of displacement through P and the velocity in this surface to be given at a given instant, the velocity at any point Q is a function of the line QR. Let $QR = s$. Then $V ds$ is a differential of a function of s and the time. Since, therefore, $d\phi = V ds$, ϕ is also a function of s and the time. But the equation $\phi = 0$ is the equation of a surface of displacement. Hence for a given surface of displacement s is constant. This proves that *the surfaces of displacement are parallel to each other, the orthogonal trajectories are straight lines, and the motion is rectilinear.*



Again, let $d\sigma$ be the increment of any line drawn arbitrarily on any surface of displacement. Then since the direction of the variation of co-ordinates in the equation (7) may be any whatever, we shall have,

$$\frac{d\lambda}{d\sigma} + \frac{d^2\phi}{d\sigma\,dt} + V\frac{dV}{d\sigma} = 0.$$

But since $\frac{d\lambda}{d\sigma}$ is the effective accelerative force perpendicular to the direction of motion, and since, as we have seen, the motion is rectilinear, it follows that $\frac{d\lambda}{d\sigma} = 0$. Also $\frac{dV}{d\sigma} = 0$. Consequently $\frac{d^2\phi}{d\sigma\,dt} = 0$. This proves that the equations $u\,dx + v\,dy + w\,dz = 0$, and $\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz = 0$ are true at the same time. The latter equation is the former differentiated with respect to t, on the supposition that dx, dy, dz do not vary with the time. It follows

with respect to the first is liable to this objection:—he concludes that $\frac{du}{dy} = \frac{dv}{dx}$, $\frac{du}{dz} = \frac{dw}{dx}$, $\frac{dv}{dz} = \frac{dw}{dy}$, from *approximate* equations, whence it follows that those equalities are approximate; whilst the inference that $u\,dx + v\,dy + w\,dz$ is a complete differential, requires that they should be *exact*. No reason is assigned by Lagrange for the other Theorem. The following argument shews it to be without foundation. If each of the quantities u, v, w vanishes for a certain value h of t, they must each contain $t-h$ as a factor. We may therefore assume that $u\,dx + v\,dy + w\,dz = (t-h)^{\alpha}(U\,dx + V\,dy + W\,dz)$, one at least of the quantities U, V, W not vanishing when $t = h$. Since $t-h$ is unaffected by the sign of differentiation, if the left-hand side of the equality be an exact differential, $U\,dx + V\,dy + W\,dz$ must be an exact differential also. But the latter quantity is not necessarily an exact differential when $t = h$; therefore neither is the other.

that the direction of motion through a given point remains the same in successive instants. This is rectilinear motion, and it thus appears that the rectilinearity of the motion is in accordance with the dynamical equation.*

When the motion is perpendicular to a plane, r and r' are each infinite, and equation (8) becomes $\frac{dV}{ds} = 0$. This is true whether the motion be in parallel straight lines or in concentric circles about a fixed axis. But equation (8) does not enable us to determine whether in the latter of these two kinds of motion, $udx + vdy + wdz$ can be an exact differential. This question will be considered further on.

Reserving then the case just mentioned, the following will be the conclusion to which the foregoing reasoning conducts:—*The only motions of an incompressible fluid which are possible when* $udx + vdy + wdz$ *is an exact differential of a function of three independent variables, are rectilinear motions.*

8. Now let the fluid be compressible. For the same reason as that adduced in the case of incompressible fluids, equation (9) cannot be integrated between limits entirely arbitrary unless $V\rho$ is constant along a given surface of displacement. And again, as before, if s be drawn at a given instant the orthogonal trajectory to surfaces of displacement from any point to a given surface of displacement, then $V\rho$ at that point is a function of s. Hence, since $\rho\,(udx + vdy + wdz) = V\rho ds$, it follows that the left-hand side of this equality is integrable. But by hypothesis $udx + vdy + wdz$ is an exact differential $d\phi$. Hence, since $\rho d\phi = V\rho ds$, ρ is a function of ϕ, and ρ and ϕ are each functions of s. But $V\rho$ is a function of s. Therefore V is also a function of s. It is thus shewn that the surfaces of displacement are surfaces both of equal velocity and equal density. By reasoning precisely as in the case of incompressible fluids a like conclusion is arrived at; viz. that *the only motions of a compressible fluid which are possible when* $udx + vdy + wdz$ *is an exact differential of a function of three independent variables, are rectilinear motions.*

The above result and the analogous one respecting incompressible fluids, are evidently dependent on the fact that when $udx + vdy + wdz$ is an exact differential $d\phi$, both ϕ and V are functions of the variable s, which is a line drawn at a given instant in the direction of the motion of the particles through which it passes, commencing at an arbitrary origin and terminating at the point xyz. And again, this fact is a direct consequence from the general equation (3), as may be thus concisely shewn. That equation, on multiplying by N, becomes $N\frac{d\psi}{dt} + V^2 = 0$: or, since $V = N.\frac{d\psi}{ds}$, it becomes $\frac{d\psi}{dt} + N\frac{d\psi^2}{ds^2} = 0$. Now when N is a function of t only, and consequently $udx + vdy + wdz$ is integrable of itself, the last equation by integration gives ψ a function of s and t. Therefore also $N\frac{d\psi}{ds}$, or V, is a function of s and t. And since $d\phi = Vds$, ϕ is also a function of s and t.

9. It remains to consider what are the forms of the surfaces of displacement which satisfy the condition of rectilinear motion.

* If all the parts of the fluid have a common motion in a common direction, the surfaces of displacement will partake of this motion, and the motion of the particles in space will not be rectilinear. Such common motions are not the proper subject of consideration in Hydrodynamics. When they exist, it must be under *given* circumstances, and their amount may be calculated in the same manner as for a solid body. These motions may therefore always be considered to be eliminated by impressing equal motions on all the parts of the fluid in a contrary direction.

Since V and $\frac{dV}{ds}$ are the same at all points of a given surface of displacement when the motion is rectilinear, it follows from equation (8) that in incompressible fluids $\frac{1}{r} + \frac{1}{r'}$ is the same at all points of the same surface of displacement. This is true also when the fluid is compressible. For since ρ is given for a given surface of displacement, and since the equation $\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz = 0$, obtained in Art. 7, proves that the surface of displacement through a given point does not vary its position, it follows that $\frac{d\rho}{dt}$ is the same at all points of the same surface of displacement. It has been already shewn that this is the case with $\frac{d.V\rho}{ds}$ and $V\rho$. Hence equation (9) shews that $\frac{1}{r} + \frac{1}{r'}$ is the same at all points of a given surface of displacement. Again, if δr be an indefinitely small constant quantity, $\frac{1}{r+\delta r} + \frac{1}{r'+\delta r}$ is constant for the next contiguous surface of displacement. Hence if $\frac{1}{r} + \frac{1}{r'} = c$ and $\frac{1}{r+\delta r} + \frac{1}{r'+\delta r} = c + \delta c$, we have $\frac{1}{r^2} + \frac{1}{r'^2} = -\frac{\delta c}{\delta r} =$ a constant. It follows that r and r' must each be constant for a given surface, and consequently that not only is the curvature the same, but the principal radii of curvature the same at all points of the surface. The only surfaces that possess this property are the surface of a sphere and that of the common cylinder. Hence the only motions, whether of incompressible or compressible fluids, that are possible when $udx + vdy + wdz$ is an exact differential, *are in straight lines drawn from a fixed centre or perpendicular to a fixed axis.*

10. By reviewing the reasoning which has conducted to the above conclusion it will be seen that after proving the dynamical equation to be integrable from any one point of the fluid to any other whenever $udx + vdy + wdz$ is an exact differential, the equations (8) and (9) were assumed to be integrable in like manner. It is necessary therefore to inquire under what circumstances the result obtained in the preceding Article is consistent with that assumption.

Let $udx + vdy + wdz$ be an exact differential, ϕ be a function of r, and $r^2 = x^2 + y^2 + z^2$. Then the equation of continuity of a compressible fluid becomes,

$$\left(k^2 - \frac{d\phi^2}{dr^2}\right)\frac{d^2\phi}{dr^2} - \frac{d^2\phi}{dt^2} - 2\frac{d\phi}{dr}\cdot\frac{d^2\phi}{drdt} + \frac{d\phi}{dr}\left(\frac{2k^2}{r} + \frac{Xx}{r} + \frac{Yy}{r} + \frac{Zz}{r}\right) = 0,$$

which does not agree in giving ϕ a function of r unless the impressed force either be nothing or a function of r. No such limitation is necessary with reference to incompressible fluids, because the equation of continuity applicable to them becomes,

$$\frac{d^2\phi}{dr^2} + 2\frac{d\phi}{dr} = 0,$$

which gives ϕ a function of r, whatever be the impressed force. It is, however, necessary that $Xdx + Ydy + Zdz$ be integrable.

11. The investigation I have now gone through, shews that there are several defects in the reasoning of my last paper, which I will endeavour to point out as distinctly as possible. The first occurs in Art. 6 (p. 377), where it is asserted that "Vdr is not an exact differential, unless the variation of V from one point of space to another at a given instant, depends only

on the change of position in the direction normal to the surface of displacement." This is not true as a general proposition, but it is true with reference to fluid motion, solely in consequence of the condition expressed by the general equation (2), as appears by the reasoning in Arts. 7 and 8 of this paper. Hence the Proposition proved in the 'Note' added to the former Paper fails in giving support to the above cited assertion, because it takes no account of that equation. In fact, the proof neglects the curvature of the lines of motion, and therefore only amounts to shewing that in rectilinear motion a surface of displacement is a surface of equal velocity when $udx + vdy + wdz$ is an exact differential, or the converse.—In the same Article (p. 378) it is said incorrectly, that "$\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz = 0$, because for a surface of displacement $udx + vdy + wdz = 0$." This is true only when the position of the surface of displacement through a given point is invariable, which should first have been shewn to be the case. The correct reasoning is given in Art. 7 of the present paper.—At the beginning of Art. 7 of the former paper (p. 379), it is supposed that in rectilinear motion the lines of motion may pass through "fixed focal lines." The more complete investigation of the present Essay shews that they must be limited to passing through a fixed centre, or a fixed axis.—The assertion (in p. 382) that "$\frac{d\phi}{dt}$ and V are constant for a given surface of displacement at a given time, when $udx + vdy + wdz$ is an exact differential," is true, but, on account of the defects already mentioned, does not follow from any previous reasoning.—It is not generally true as asserted in p. 389, that "the variation of V at a given point is the same as if r and r' were constant," and consequently the equation derived from that supposition is of no value. I am not aware of any other points that require adverting to.

I proceed now to make some uses of equation (2) which will shew the importance and necessity of it.

12. First, let it be required to determine on what hypotheses the general dynamical equation is integrable. To do this it is necessary to introduce into the dynamical equation the condition expressed by the equation (2), or by its equivalent equation (3). I have already gone through the process for this purpose in Arts. 10 and 12 of my former paper. The result there obtained, expressed in the notation of this paper, is

$$\lambda + \frac{d.\int N d\psi}{dt} + \frac{V^2}{2} = F(t).$$

It is supposed in this equation that $Xdx + Ydy + Zdz$ is an exact differential. This condition being fulfilled by the impressed forces, the equation is integrable either if the second term vanishes, or if $Nd\psi$ be integrable. Since $\frac{d.\int N d\psi}{dt} = \int \frac{dV}{dt} ds$, in the first case, $\frac{dV}{dt} = 0$ and the motion is steady; in the other, $udx + vdy + wdz$ is an exact differential. These are the only cases in which the general dynamical equation is integrable.

13. Next let it be required to find the factor $\frac{1}{N}$ in proposed instances of motion, and to determine whether the motions are possible.

To make the equation (2), viz.

$$\frac{d\psi}{dt} + \frac{d\psi}{dx}u + \frac{d\psi}{dy}v + \frac{d\psi}{dz}w = 0,$$

convenient for this purpose, it will be transformed into another equivalent equation in the manner following. The equation $\psi = 0$, being by hypothesis the equation of a curve surface, may be supposed to contain besides the variables x, y, z explicitly, certain parameters a, b, c, &c. which are functions of the co-ordinates and the time, and vary with the varying position of a given element of the fluid, but which are constant in passing from point to point of a surface of displacement through either a finite or an indefinitely small space. Let, therefore, the equation $\psi = 0$ be equivalent to $f(x, y, z, a, b, c, \&c.) = 0$. Then

$$\frac{d\psi}{dt} = \frac{df}{da} \cdot \frac{da}{dt} + \frac{df}{db} \cdot \frac{db}{dt} + \frac{df}{dc} \cdot \frac{dc}{dt} + \&c.$$

$$u\frac{d\psi}{dx} = u\frac{df}{dx} + \frac{df}{da} \cdot \frac{da}{dx} \cdot \frac{dx}{dt} + \frac{df}{db} \cdot \frac{db}{dx} \cdot \frac{dx}{dt} + \frac{df}{dc} \cdot \frac{dc}{dx} \cdot \frac{dx}{dt} + \&c.$$

and so for $v\frac{d\psi}{dy}$ and $u\frac{d\psi}{dz}$. Hence, substituting in equation (2), we have

$$\frac{df}{da} \cdot \left(\frac{da}{dt}\right) + \frac{df}{db} \cdot \left(\frac{db}{dt}\right) + \frac{df}{dc} \cdot \left(\frac{dc}{dt}\right) + \&c. + u\frac{df}{dx} + v\frac{df}{dy} + w\frac{df}{dz} = 0.$$

Now differentiating the equation $f(x, y, z, a, b, c, \&c.) = 0$, we obtain, since a, b, c, &c. are constant for a given surface of displacement,

$$\frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz = 0 = \frac{u}{N}dx + \frac{v}{N}dy + \frac{w}{N}dz.$$

Hence, $u = N\frac{df}{dx}$, $v = N\frac{df}{dy}$, $w = N\frac{df}{dz}$: and consequently,

$$\frac{df}{da} \cdot \left(\frac{da}{dt}\right) + \frac{df}{db} \cdot \left(\frac{db}{dt}\right) + \frac{df}{dc} \cdot \left(\frac{dc}{dt}\right) + \&c. + N\left(\frac{df^2}{dx^2} + \frac{df^2}{dy^2} + \frac{df^2}{dz^2}\right) = 0. \quad \ldots\ldots\ldots(10).$$

We shall presently illustrate the use of this equation in finding N.

It is plain that the equation $f(x, y, z, a, b, c, \&c.) = 0$, may be that of a surface having a contact of the second order with the surface of displacement at any point xyz, the parameters in the equation of such a surface being a, b, c, &c. For instance, let the equation of the surface of contact be

$$\frac{(x-\alpha)^2}{m^2} + \frac{(y-\beta)^2}{n^2} + \frac{(z-\gamma)^2}{p^2} - 1 = 0,$$

then we have for determining N the general equation,

$$\frac{x-\alpha}{m^2} \cdot \left(\frac{d\alpha}{dt}\right) + \frac{y-\beta}{n^2} \cdot \left(\frac{d\beta}{dt}\right) + \frac{z-\gamma}{p^2} \cdot \left(\frac{d\gamma}{dt}\right) + \frac{(x-\alpha)^2}{m^3} \cdot \left(\frac{dm}{dt}\right) + \frac{(y-\beta)^2}{n^3} \cdot \left(\frac{dn}{dt}\right) + \frac{(z-\gamma)^2}{p^3} \cdot \left(\frac{dp}{dt}\right) =$$

$$2N\left(\frac{(x-\alpha)^2}{m^4} + \frac{(y-\beta)^2}{n^4} + \frac{(z-\gamma)^2}{p^4}\right).$$

I proceed now to adduce some examples of finding N, and of applications of the equations (8), (9), and (10), to determine whether proposed instances of motion are possible.

Ex. 1. Let the case of motion be that considered in Art. 3. This instance gives $u = mx$, $v = -my$, and satisfies the equation $\frac{du}{dx} + \frac{dv}{dy} = 0$. Also $udx + vdy = m(xdx - ydy)$, and $\frac{v}{u} = \frac{dy}{dx} = -\frac{y}{x}$. Hence the general equation of the surfaces of displacement may be assumed to be $x^2 - y^2 - a^2 = 0$, and the general equation of the lines of motion, $xy = c^2$.

Let $f = x^2 - y^2 - a^2$. Then $\frac{df}{da} \cdot \left(\frac{da}{dt}\right) = -2a\left(\frac{da}{dt}\right)$; $\frac{df}{dx} = 2x$, $\frac{df}{dy} = -2y$.

$$\text{Hence} \quad -a\left(\frac{da}{dt}\right) + 2N(x^2+y^2) = 0, \quad \text{and } N = \frac{a\left(\frac{da}{dt}\right)}{2(x^2+y^2)}.$$

But by the equation $x^2 - y^2 - a^2 = 0$, (since x, y, and a vary simultaneously with the position of the element,) we have $x\left(\frac{dx}{dt}\right) - y\left(\frac{dy}{dt}\right) - a\left(\frac{da}{dt}\right) = 0$, and $\left(\frac{dx}{dt}\right) = u = mx$, $\left(\frac{dy}{dt}\right) = v = -my$. Hence $a\left(\frac{da}{dt}\right) = m(x^2+y^2)$, and consequently $N = \frac{m}{2}$. This value makes $\frac{u}{N}dx + \frac{v}{N}dy$ an exact differential, and the equation (3) is therefore verified. We have now to see in what manner equation (8) is verified. This equation for the instance before us becomes $\frac{dV}{V} + \frac{ds}{r} = 0$, r being the radius of curvature of the curve of displacement at the point xy, and ds the increment of the line of motion at the same point. Hence $r = -\frac{(x^2+y^2)^{\frac{3}{2}}}{x^2-y^2}$, and $ds = \frac{(x^2+y^2)^{\frac{1}{2}}dx}{x}$. Therefore $\frac{dV}{V} - \frac{x^2-y^2}{x^2+y^2}\cdot\frac{dx}{x} = 0$. This equation cannot be integrated unless y is eliminated by means of the equation $xy = c^2$; that is, it can be integrated only along a line of motion. The dynamical equation must therefore be integrated in the same manner, and the arbitrary quantity to be added is a function of co-ordinates as well as the time. The fluid must be conceived to be included between two hyperbolic surfaces indefinitely near each other. This explains the contradiction met with in Art. 3.

Ex. 2. Let the equation of the surfaces of displacement be $\theta - \tan^{-1}\frac{y}{x} = 0$. Putting therefore f for $\theta - \tan^{-1}\frac{y}{x}$, we have

$$\frac{df}{d\theta}\cdot\left(\frac{d\theta}{dt}\right) = \left(\frac{d\theta}{dt}\right), \quad \frac{df}{dx} = \frac{y}{x^2+y^2}, \quad \text{and } \frac{df}{dy} = -\frac{x}{x^2+y^2}.$$

$$\text{Hence} \quad \left(\frac{d\theta}{dt}\right) + N\cdot\frac{x^2+y^2}{(x^2+y^2)^2} = 0, \quad \text{and } N = -\left(\frac{d\theta}{dt}\right)(x^2+y^2).$$

Now since $y = x\tan\theta$, the motion is evidently parallel to the plane of xy in concentric circles about a fixed axis. Hence at any distance r from the axis $V = r\left(\frac{d\theta}{dt}\right)$. Consequently $N = -\frac{V}{r}(x^2+y^2)$ $= -Vr$. Therefore if $V = \frac{f(t)}{r}$, we have N a function of t, and $udx + vdy$ an exact differential, although the motion is curvilinear. This is the case of motion alluded to at the end of Art. 7.

Ex. 3. Let it be required to determine whether in an incompressible fluid the surfaces of displacement can be concentric spherical surfaces, the centre of which is always on the axis of x, and at the same time the motion be such that a given particle in successive instants is at the same distance from the common centre.

Here if α = the co-ordinate of the centre, and a = the radius of any surface of displacement, we have $f = (x-\alpha)^2 + y^2 + z^2 - a^2$.

$$\text{Hence} \quad \frac{df}{d\alpha}\cdot\left(\frac{d\alpha}{dt}\right) = -2(x-\alpha)\left(\frac{d\alpha}{dt}\right) = -2V_{,}(x-\alpha) \text{ suppose.}$$

$$\frac{df}{da}\cdot\left(\frac{da}{dt}\right) = 0, \text{ because by hypothesis } \left(\frac{da}{dt}\right) = 0.$$

$$\frac{df}{dx} = 2(x-a), \quad \frac{df}{dy} = 2y, \quad \frac{df}{dz} = 2z.$$

$$\therefore\ -2V_{,}(x-a) + 4N\{(x-a)^2 + y^2 + z^2\} = 0, \text{ and } N = \frac{V_{,}(x-a)}{2a^2}.$$

$$\text{Hence } u = N\frac{df}{dx} = \frac{V_{,}(x-a)^2}{a^2}; \quad v = N\frac{df}{dy} = \frac{V_{,}(x-a)y}{a^2};$$

$$w = N\frac{df}{dz} = \frac{V_{,}(x-a)z}{a^2}; \quad \text{and } V = \frac{V_{,}(x-a)}{a}.$$

We have now to find $V_{,}$ by means of equation (8). Since the above value of N shews that $udx + vdy + wdz$ is not for this instance an exact differential, V must be differentiated along a line of motion. Hence putting $\cos\theta$ for $\frac{x-a}{a}$, we have $V = V_{,}\cos\theta$, and $dV = dV_{,}\cos\theta$; so that $\frac{dV}{V} = \frac{dV_{,}}{V_{,}}$. Also $ds = da$ and $\frac{1}{r} + \frac{1}{r'} = \frac{2}{a}$. Equation (8) therefore becomes $\frac{dV_{,}}{V_{,}} + \frac{2da}{a} = 0$. Hence $V_{,} = \frac{f(t)}{a^2}$, and $V = \frac{f(t)}{a^2}\cos\theta$. Thus the motion is completely determined. It is plain that this motion would be produced by a smooth solid sphere moving in an arbitrary manner in the fluid, with its centre always in a given straight line.

Ex. 4. Poisson's determination of the simultaneous motions of a sphere and the surrounding fluid (*Memoirs of the Paris Academy*, Tom. XI, and *Connaissance des Tems*, An. 1834) differs from the foregoing. Let us therefore inquire, assuming the motion to be such as Poisson has found, whether the conditions of continuity are satisfied.

For the sake of simplicity I shall consider the fluid to be incompressible. Poisson assumes that $udx + vdy + wdz = d\phi$, and finds values of the velocities which, if $R^2 = (x-a)^2 + y^2 + z^2$, may be thus expressed:

$$u = -\frac{Tc^3}{2R^3}\left(1 - \frac{3(x-a)^2}{R^2}\right), \quad v = \frac{3Tc^3}{2R^5}y(x-a), \quad w = \frac{3Tc^3}{2R^5}z(x-a);$$

T being an arbitrary function of the time, and c the radius of the sphere. These values make $udx + vdy + wdz$ an exact differential, and satisfy the equation $\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0$. By integrating $udx + vdy + wdz = 0$, the equation of the surfaces of displacement will be found to be $R^3 - h^2(x-a) = 0$ for the positive values of $x-a$, and $R^3 + h^2(x-a) = 0$ for the negative values. This change of equation implies a breach of continuity*. If we put $R\cos\theta$ for $x-a$, we obtain $R^2 - h^2\cos\theta = 0$ for the polar equation of the curve which by its revolution about the axis of x generates a surface of displacement. The lines of motion lie in planes passing through the axis of x. The general polar equation of these lines will be found to be $R - e\sin^2\theta = 0$. From the latter of these equations the value of ds is to be found, and from the other the values of r and r', for the purpose of ascertaining whether the equation (8), viz. $\frac{dV}{V} + ds\left(\frac{1}{r} + \frac{1}{r'}\right) = 0$, is satisfied by these values so as to allow of its being integrated between arbitrary limits, the dynamical equation having been already integrated in this manner.

* According to this solution the fluid in contact with the sphere and in a plane passing through its centre perpendicular to the axis of x, moves *backwards* with half the velocity with which the sphere moves *forwards*, a result, to say the least, very improbable.

It will be found that $ds = \frac{dR}{2\cos\theta}(1 + 3\cos^2\theta)^{\frac{3}{2}}$, $\frac{1}{r} = \frac{6\cos\theta\,(1+\cos^2\theta)}{R(1+3\cos^2\theta)^{\frac{3}{2}}}$, and $\frac{1}{r'} = \frac{3\cos\theta}{R(1+3\cos^2\theta)^{\frac{1}{2}}}$

$$\text{Hence, } \frac{dV}{V} + ds\left(\frac{1}{r} + \frac{1}{r'}\right) = \frac{dV}{V} + \frac{3dR}{2R} \cdot \frac{3+\cos^2\theta}{1+3\cos^2\theta} = 0.$$

This equation cannot be integrated independently of the equation $R - e\sin^2\theta = 0$; that is, it can be integrated only along a line of motion. Hence the conditions of continuity are not satisfied.

Ex. 5. Let it be supposed that the motion is in straight lines drawn from the vertex of a cone, and let the fluid move in parallel slices so that the motion parallel to the axis of the cone is the same at all points of any section perpendicular to this axis: it is required to determine whether this motion is possible.

The equation of the surfaces of displacement is $x^2 + y^2 + z^2 - R^2 = 0$, and the equation of the lines of motion $R - x\sec\theta = 0$. Hence $ds = dR = dx\sec\theta$, and $R = r = r'$. Equation (8) consequently becomes $\frac{dV}{V} + \frac{2dx}{x} = 0$. Now putting f for $x^2 + y^2 + z^2 - R^2$, we have $\frac{df}{dR}\left(\frac{dR}{dt}\right) = -2RV$, $\frac{df}{dx} = 2x$, $\frac{df}{dy} = 2y$, $\frac{df}{dz} = 2z$. Hence $-2RV + 4N(x^2+y^2+z^2) = 0$, and $N = \frac{V}{2R}$. Therefore $u = N\frac{df}{dx} = \frac{Vx}{R} = \phi(x)$ by hypothesis. Consequently $v = \frac{y\phi(x)}{x}$, $w = \frac{z\phi(x)}{x}$, and $V = \frac{R\phi(x)}{x} = \phi(x)\sec\theta$. The above values of u, v, w, do not make $udx + vdy + wdz$ an exact differential. Hence the dynamical equation must be integrated along a line of motion, and the equation (8) with the same limitation. Consequently $\frac{dV}{dx} = \phi'(x)\sec\theta$, and the above equation becomes $\frac{\phi'(x)}{\phi(x)} + \frac{2}{x} = 0$, which by integration gives $\phi(x) = \frac{f(t)}{x^2}$. Hence $V = \frac{f(t)}{x^2}\sec\theta$, and the motion is completely determined. This solution agrees with the one I deduced from particular considerations in the Cambridge Philosophical Transactions (Vol. V. Part II. p. 186).

The preceding example is instructive as shewing that the motion may be rectilinear when $udx + vdy + wdz$ is not an exact differential. Another inference may also be drawn from it. Let the motion be steady, and let W be the velocity at a point of the axis distant by h from the vertex. Then $W = \frac{f(t)}{h^2}$; whence $f(t) = h^2 W$, and $V = \frac{Wh^2}{x^2}\sec\theta$. If gravity ($g$) be supposed to act parallel to the axis of x, the dynamical equation gives for the pressure (p),

$$p = C - gx - \frac{W^2 h^4}{2x^4}\sec^2\theta,$$

and if $p = 0$ where $x = H$, $C = gH + \frac{W^2 h^4}{2H^4}\sec^2\theta$. If now H be assumed to be so large that the second term of the expression for C may be neglected, we shall have, $C = gH$, and

$$p = g(H - x) - \frac{W^2 h^4}{2x^4}\sec^2\theta.$$

It might hence be argued that $udx + vdy + wdz$ is an exact differential for this case, since C is independent of $\sec\theta$. But the objection to this inference is, that if it were true, the above value of p might be differentiated supposing $\sec\theta$ variable, which would manifestly be incorrect, for the result would be at variance with the differential from which this value was derived. The fact is, the neglected quantity has no effect on the numerical computation of p, but as it contains $\sec\theta$, we cannot regard C as independent of co-ordinates.

Ex. 6. Let it be required to determine whether in a compressible fluid the surfaces of displacement can be spherical surfaces, the centres of which are always on the axis of x, and at the same time the motion be such that the radius of the surface of displacement of a given particle remains the same in successive instants.

Let $(x-\alpha)^2 + y^2 + z^2 - R^2 = 0$. Then by reasoning as in Ex. 3, $V = \left(\frac{d\alpha}{dt}\right)\cos\theta = V_{,}\cos\theta$.

By hypothesis $\left(\frac{d\alpha}{dt}\right)$ is the same for all points of the same surface of displacement. Hence $V_{,}$ is the velocity at any point of the axis of x. The equation (9) may be put under the form,

$$\frac{d\rho}{\rho dt} + \frac{dV}{ds} + \frac{V}{\rho} \cdot \frac{d\rho}{ds} + \frac{2V}{R} = 0,$$

subject to the limitation of integrating along the line of motion s. The dynamical equation, subject to the same limitation, is

$$k^2 \text{ Nap. log } \rho + \int \frac{dV}{dt} ds + \frac{V^2}{2} = f(t).$$

Hence, carrying the approximation only to the first power of the velocity, we have

$$\frac{k^2 d\rho}{\rho ds} = -\frac{dV}{dt}, \text{ and } \frac{k^2 d\rho}{\rho dt} = -\int \frac{d^2V}{dt^2} ds + f'(t). \text{ Therefore}$$

$$-\frac{1}{k^2}\int \frac{d^2V}{dt^2} ds + \frac{dV}{ds} + \frac{2V}{R} + f'(t) = 0;$$

and differentiating with respect to s,

$$\frac{d^2V}{dt^2} - k^2 \cdot \frac{d^2V}{ds^2} - 2k^2\left(\frac{dV}{Rds} - \frac{V}{R^2} \cdot \frac{dR}{ds}\right) = 0.$$

The motion is symmetrical about the axis of x, which is plainly a line of motion. Hence the above equation is true when $V_{,}$ is put for V and x for s. It thus becomes,

$$\frac{d^2V_{,}}{dt^2} - k^2 \cdot \frac{d^2V_{,}}{dx^2} - 2k^2\left(\frac{dV_{,}}{Rdx} - \frac{V_{,}}{R^2} \cdot \frac{dR}{dx}\right) = 0.$$

Now in this equation $R = x - \alpha$, and $\frac{dR}{dx} = 1 - \frac{d\alpha}{dx}$, for α is a function both of x and t. The differential coefficient $\frac{d\alpha}{dx}$ will in all cases be very small, if the velocity of the particles be small compared to the velocity of propagation of the motion. Hence $\frac{V_{,}dR}{dx} = V_{,}\left(1 - \frac{d\alpha}{dx}\right) = V_{,}$ nearly, regarding $\frac{d\alpha}{dx}$ a quantity of the same order as $V_{,}$. Also as $V_{,}$ may be considered a function of R and t, $\frac{dV_{,}}{dx} = \frac{dV_{,}}{dR} \cdot \frac{dR}{dx} = \frac{dV_{,}}{dR}$ nearly. And $\frac{d^2V_{,}}{dx^2} = \frac{d^2V_{,}}{dR^2} \cdot \frac{dR}{dx} = \frac{d^2V_{,}}{dR^2}$ nearly. By substitution in the foregoing equation, we have

$$\frac{d^2V_{,}}{dt^2} - k^2 \cdot \frac{d^2V_{,}}{dR^2} - 2k^2\left(\frac{dV_{,}}{RdR} - \frac{V_{,}}{R^2}\right) = 0.$$

This equation gives $V_{,}$ by integration, whence V is known from the equation $V = V_{,}\cos\theta$. Thus the motion is completely determined consistently with equation (9), and this is the proof of the possibility of the assumed kind of motion, so far, at least, as regards small motions. The above solution is that which I have employed for finding the resistance of the air to the vibrations of a ball-pendulum.

CAMBRIDGE OBSERVATORY.
March 2, 1843.

V. *Observations on the Nature of the Biliary Secretion;—the object being to shew, that the Bile is essentially composed of an Electro-negative body in chemical combination with one or more inorganic bases.* By GEORGE KEMP, M.B. *St. Peter's College.*

[Read *March* 6, 1843.]

THE following observations on the nature of the Bile, form a portion of some researches into the elementary composition of that secretion, commenced in the laboratory at Giessen. Professor Liebig suggested the following mode of conducting the inquiry.

A portion of ox-bile as received from the gall-bladder was to be evaporated to dryness, and then submitted to ultimate analysis, without any farther manipulation.

This plan was abandoned for the following reasons.

The gall-bladder of every animal yet examined contains, in addition to the bile, another body, always varying in quantity, and possessing physical properties differing so essentially from the biliary secretion, that I determined in the first place to separate and examine this body, to which the name of Mucus of the gall-bladder has been given. The analysis proved that this body contains 15·4 per cent of nitrogen, while the bile itself contains only 3·5 per cent of that element, so that the results obtained in the manner originally proposed would have been constantly varying, and always erroneous. The fats and fatty acids also, contained in the bile, would have led us still farther astray; eventually, therefore, I determined on removing the mucus and fatty acids before attempting the analysis of the fluid. Previously, however, to entering on the manipulation employed, it will be proper to give a sketch of the principal opinions which have been hitherto entertained on the nature of the bile.

The first proximate analysis of this fluid, of any importance, seems to have been made by Thénard in the year 1806, with the results contained in the note *. According to his opinion the bile is principally composed of biliary resin and picromel; the biliary resin he supposed to be held in solution by the picromel. Berzelius in 1807 instituted an analysis of which the table † below gives a summary view. He considered the biliary resin and picromel as one body, altered by the manipulation of Thénard, who made use of nitric acid in his analysis. To this body, composed of biliary resin and picromel, Berzelius applies the name *biliary matter.* An analysis instituted by Dr. Prout about the same time confirms the analysis of Berzelius in every essential point. At a subsequent period Gmelin undertook the investigation of this secretion; his results induced him to imagine that the opinion of Thénard with reference to the existence of biliary resin and picromel, was correct, although the substance described as picromel by Gmelin differs very essentially from that body as described by Thénard. He,

* Water	876·6
Biliary resin	30·0
Picromel	75·4
Yellow colouring matter	5·0
Soda	5·0
Phosphate of Soda	2·5
Chloride of Sodium	4·0
Sulphate of Soda	1·0
Sulphate of Lime	1·5
A trace of Oxide of Iron.	
	1000·0

† Water	90·44
Biliary matter with fat	8·00
Mucus of the gall-bladder	0·30
Osmazome, Chloride of Sodium, and Lactate of Soda	0·74
Soda	0·41
Phosphate of Soda, Phosphate of Lime, and traces of a substance insoluble in alcohol	0·11
	100·00

moreover, found other substances denominated cholic acid and taurin, which have since been proved to be products of manipulation. In the year 1826, Demarçay employed himself in the laboratory of Liebig in preparing, and submitting to analysis, a substance obtained from the bile when treated with diluted sulphuric acid, and to which he subsequently gave the name choleic acid. This body was in all probability the picromel of Thénard, and the matter which remained after removing the choleic acid, denominated choloïdic acid by Demarçay, bears a striking resemblance to the biliary resin of Thénard; as, however, no elementary analysis was made by that chemist, the matter must remain in doubt. The choleic acid of Demarçay is an important body, as Professor Liebig has acceded to the opinion that it is the essential organic ingredient of the bile; a conclusion, however, which subsequent researches tend to overthrow; indeed, the opinion of Demarçay was grounded on the following circumstance. After he had prepared his choleic acid, and combined it with soda, the compound possessed a considerable number of the physical characters of the Bile, and in estimating the quantity of soda which combined with a given quantity of his choleic acid, he found the quantity of the base almost precisely the same as that contained in the same quantity of the dried bile. One unfortunate oversight, however, occasioned this erroneous inference of the identity of the two bodies. The choleate was converted into the sulphate of soda, in order to estimate the quantity of the base. On applying the same method to the bile, the chloride of sodium contained in that fluid became converted into sulphate of soda, and thus the quantity of soda combined with the organic body was supposed to be considerably greater than it really was; for on looking over the analysis of Thénard it will be seen that the quantity of chloride of sodium stands in the proportion of 4 : 5 to the soda; a quantity much too large to be overlooked, as it would occasion an error in the second whole number of the atomic weight. Indeed, the circumstance of the similarity found in the dried bile and in the choleate of soda in this one experiment, was evidently purely accidental, as the chloride of sodium is always present in bile, and that in constantly varying proportions. In fact, the choleic acid of Demarçay seems to be a product of decomposition of the bile effected by means of sulphuric acid; and the errors in the late work of Professor Liebig on the subject have arisen from not taking into account the other product of manipulation, the body which Demarçay has denominated choloïdic acid. The labours of Demarçay were however exceedingly valuable, as they directed the attention from proximate to ultimate analysis, and were the means of inducing the illustrious Berzelius to make one more effort towards effecting the solution of this difficult problem*. A paper, which has recently appeared I believe in an English form, was in the year 1841 laid before the Royal Academy of Stockholm, purporting to be an analysis of the bile of the ox, and the characteristic properties of its component parts. This elaborate research was conducted in the same manner as his former analysis, the object being to eliminate what he considered the proximate principles of the bile; and the results confirmed all his former investigations on this subject. He concludes by stating the theory, that the bile in its healthy and perfectly fresh state is essentially composed of bilin, a body agreeing in every physical character with the biliary matter of his former analysis, and that this body is continually undergoing a change into two acids, fellic acid and cholic acid; that at the same time these two acids form *binary* compounds with bilin, to which compounds he has given the names Bilifellic Acid and Bilicholic Acid. To this theory we shall have occasion again to advert.

These last researches of Berzelius seemed to discourage any farther attempt to elicit facts by means of proximate analysis; and at the request of Professor Liebig, I commenced a series of ultimate analyses, with the limitations alluded to above. It appeared desirable also to extend

* Ueber die Analyse der Ochsengalle, und die characterisirenden Eigenshaften ihrer Bestandtheile; von J. Berzelius. (Aus den Kongl. Vet. Acad. Handl. 1841. S. 1—64, übersetzt von Dr. Wiggers.

the inquiry, which had hitherto been confined almost exclusively to the bile of the ox; I therefore proposed to examine the bile of that animal as the type of the graminivorous, the human bile as the type of the omnivorous, and the bile of some decidedly carnivorous animal, the lion or tiger for instance, as the type of that class of animals. It was further proposed to institute an inquiry into the differences which exist in the bile of different species of fish. Thus I hoped that some general character at least would be found to illustrate the nature of the secretion in its relation to the researches of physiology and pathology. The results of the first investigation which were made at Giessen have since been published in the Journal of Erdman and Marchand at Liepsig, and in the London Medical Gazette; it will, therefore, merely be necessary to give a general outline of the manner in which the investigation was pursued; the subsequent portions of the research have, by the kind permission of Professor Cumming, been carried on in the laboratory of this University. At the onset of the inquiry it seemed most important to take a large average, and the bile obtained from twelve oxen killed at the same time at Frankfort was evaporated in a water-bath to dryness; the mass was reduced to powder and treated with alcohol sp. gr. ·840, in order to remove the mucus; the clear fluid obtained by filtration was again evaporated to dryness, powdered, and treated with ether, in order to remove the fats and fatty acids in combination with soda, and this treatment continued until the ether on evaporation gave no residue. The substance was now dried at a temperature of 110^0 of the centigrade thermometer, reduced to a powder and submitted to analysis. The solution of this substance was perfectly neutral; on burning it however in a platinum crucible, an alkaline ash was left, which consisted of carbonate of soda, and chloride of sodium. The carbonic acid which was found combined with soda was of course the result of the combination of the carbon of the organic portion of the bile during combustion with the oxygen of the atmospheric air. In the bile therefore soda itself was present in combination with organic matter, and as in the bile the alkaline property of the soda is suspended, we have positive proof that the soda in the ox-bile is combined with an electro-negative body; for in no other way can we account for the perfectly neutral character of the bile. Those who are acquainted with the description of the bile in physiological works, will remember that it has been described as an alkaline fluid; and Schulz has made the statement that one ounce requires half a dram of acetic acid for its saturation. His account is, however, much too vague to place any dependence upon, for what is usually called *acetic acid* is merely a solution of acetic acid, and the strength of the solution has not been recorded by this author. It is certain that a portion of a strong acid may be added to the bile without any acid reagency taking place before the quantity of soda combined with the electro-negative body (naturally contained in the secretion) has become saturated. And we must not be surprised that the electro-negative body set at liberty produces no change on litmus paper, as bodies of very high atomic weight seldom produce any visible reaction on test-paper. We may instance the new alkaloid Berberin, which has an atomic weight of more than 4000 ($O = 100$);—the combining weight of the body which we are about to examine is between 5000 and 6000. But to come to actual experiment on the subject.

I have tested the fresh bile of more than forty oxen, the human bile, the bile from the tiger, * the fox, the cat, * several kinds of monkey, the dog, * the wolf, * an Indian bull [a], and the secretion as found in the codfish; in all these cases, with two exceptions, the bile was perfectly neutral. One of these exceptions was the bile from a child which had been burnt to death, and which was not examined until three days after its removal from the body, in which state, it is needless to remark, that decomposition had already commenced, and even in this case the alkaline reaction was barely perceptible. The other was that of the bile obtained from an Indian bull, in which the secretion was not only decomposed, but absolutely putrid. To return however to the ox-bile.

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[a] The bile of the animals marked (*) was obtained through the kindness of Dr. Clark, Professor of Anatomy in this University.

Having now decided that the bile of the twelve oxen under examination contained an alkaline base, the physical properties of which had been suspended by combination with a body in an opposite electrical condition, the next point was to determine the quantity of soda contained in a given quantity, and thus estimate the combining weight of the organic body with which the base was combined. On this being ascertained, an analysis was made to determine the quantity of organic elements. For the sake of brevity the analyses will be given together, after the description of the manipulations. A further portion of bile obtained from twelve more oxen was now submitted to examination, and the results, both with respect to combining weight and proportion of organic elements, were as nearly identical with the former portion as our present modes of analysis, and the nature of the research, warrant us to expect. It may here be remarked, that animal bodies in general present great obstacles to minute analysis, from the difficulty with which they are burnt, and from the readiness with which they attract moisture. The body contained in the bile is so hygroscopic, that even in the act of mixing and introducing it into the tube a sufficient quantity of moisture is absorbed to render the estimation of hydrogen always too high. Having now determined analytically that the bile of the ox contained an organic electro-negative body in combination with soda, it seemed desirable to attempt a synthetical proof. Here serious objections presented themselves. The bile is more or less influenced by every chemical reagent yet tried, or, to use the words of Berzelius, "it has so great a tendency to undergo changes in its composition, that the action of different reagents upon it converts it into different compounds, which vary according to the processes employed to extract them; exactly as oils and fats are converted into sugar and fatty acids by the action of the oxides of lead and zinc." It appeared probable, however, on consideration, that by extreme dilution of reagents, and carefully avoiding a greater excess than necessary, we might succeed, if not in isolating the body for analysis, yet in separating it from the soda with which it was originally combined; uniting it again with a fresh portion of soda, and thus in forming the bile artificially. If the composition of the body thus formed should by subsequent research furnish us with results identical with those obtained from the bile in a natural state, I conceived that no candid person would reject the evidence either as unsatisfactory or unsound. A portion of the dried extract of the bile freed from mucus and fatty acids was dissolved in alcohol of as great a strength as could easily be obtained, and then treated drop by drop with diluted sulphuric acid. The sulphate of soda thus formed being insoluble in alcohol, could of course be separated by filtration, the organic elements previously combined with the soda remaining in solution. The sulphuric acid was added in the slightest possible excess, in order to ensure the complete separation of the soda, and the clear solution obtained by filtration was now treated with an excess of carbonate of soda deprived of its water of crystallization. The excess of sulphuric acid was precipitated in the form of sulphate of soda, while a portion of the carbonate readily combined with the electro-negative body remaining in solution. The solution obtained by filtration was now evaporated to dryness, and submitted to analysis. As no change in the physical characters of the body had been made by this process, I was not surprised to find that the combining weight and ratio of organic elements were found by analysis to be identical with the bile in its natural state. But the question may be asked, Why not (having separated the soda by means of sulphuric acid) have evaporated the solution, and then analyzed the body thus isolated? My reason for not doing so was, that it was necessary to add sulphuric acid in slight excess, and this in proportion as the solution became concentrated by evaporation would have rendered the result unsatisfactory, as we know that sulphuric acid of moderate strength decomposes the bile, and converts it into the choleic and choloïdic acids of Demarçay. Thus it would have been as much a matter of probable evidence whether the isolated body was the matter contained in the bile, as whether the body separated as above and recombined with a base, was the electro-negative substance, the composition of which we wished to determine. It now only remains to give a summary of the general results obtained from the analysis of the ox-bile, before passing on to the consideration of the bile of other animals.

1. A portion of the substance was burnt in a platinum crucible, and an ash remained consisting of

Carbonate of Soda = 11·16 per cent.
Chloride of Sodium = 0·54 per cent.

2. Another portion treated in the same manner gave

Carbonate of Soda = 11·13 per cent.
Chloride of Sodium = 0·37 per cent.

The organic portion gave on combustion with chromate of lead:—

	1	2
Carbon	= 64·60	64·85
Hydrogen	= 9·62	9·40
Nitrogen	= 3·40	3·40
Oxygen	= 22·38	22·35
	100·00	100·00

The human bile, from the smallness of its quantity, presents us with still greater difficulties than the ox-bile, the portion obtained from an adult under the most favourable circumstances being barely sufficient for the necessary number of analyses. The first portion of human bile which I examined was removed about eight hours after death from a man who died suddenly under an attack of delirium tremens. Having separated the mucus and fat as above described, it was submitted to analysis with the following results. On burning a portion in a platinum crucible it was found to contain 6·6 per cent of soda and 1·87 per cent of chloride of sodium. The organic elements were in the following proportions:

Carbon	= 68·80
Hydrogen	= 10·40
Nitrogen	= 3·44
Oxygen	= 17·36
	100·00

The general conclusion from the above analysis is, that human bile, as well as the ox-bile, is an electro-negative body in combination with soda. Two other cases of bile obtained from children who died in consequence of severe burns confirmed this conclusion. The next examination was into the nature of the bile of fishes. I have not yet been able to obtain a sufficient quantity of this secretion for anything more than a cursory examination, the results however, so far as they went, were exceedingly satisfactory. The bile of four large codfish gave 2·61 per cent of chloride of sodium, 1·8 per cent of lime, 4·3 per cent of soda with a trace of magnesia. I had merely substance enough to estimate the quantity of carbon and hydrogen, which were 68·60 and 10·8 per cent respectively. From this analysis we see that this species of bile is also an electro-negative body, but combined with three bases, lime, soda, and magnesia. I have recently obtained the bile from a tiger, which was treated in the usual manner to remove the mucus and fat. A portion burnt in a platinum crucible gave an alkaline ash, the nature of which I have yet to determine. The solution of the bile itself was perfectly neutral; we therefore conclude that its nature is similar to all the others which we have examined, and that in the carnivorous, as well as the graminivorous and omnivorous animals, the bile is essentially composed of an electro-negative body in combination with one or more inorganic bases. That the body is not the choleic acid of Demarçay, is evident from the difference of the elementary composition which exists between them. The bile of the ox contains nearly 65 per cent of carbon, human bile upwards of 68 per cent, while the acid of Demarçay contains between 63 and 64; and the difference in the quantity of hydrogen is so great that we cannot construct any formula under which bodies differing so widely from each other can be included. The choleic acid also when combined with soda is precipitated by acetic acid; the body contained in the bile is not precipitated by that reagent. We know also that ox-bile, treated in the manner directed by Demarçay for the preparation of choleic acid, is resolved into two bodies, the choleic acid and the chloïdic, the latter forming a very large proportion of the results, probably as much as one half; and it is remarkable, that adding the quantity of elements found in the two, and taking the mean, we have almost exactly the quantity as given by the analysis of human bile. The highest authority on all subjects connected with chemical research is undoubtedly Berzelius, and he has lately given it as his opinion that the bile is essentially composed of bilin, bilifellic acid, and bilicholic acid. Considering, however, that these bodies were eliminated by means of reagents which he himself has acknowledged as more likely to yield *products* than *educts*, we are perhaps justified in supposing that these bodies were the results of manipulation; it is at any rate highly improbable that in the very large number of elementary analyses made, we should in each case have accidentally procured bile in which precisely the same point of transformation of bilin into the other products should have been arrived at. One experiment was however made which proved that the body described by Berzelius as bilin does not always exist at all in the bile. I obtained the biliary secretion from an ox immediately as slaughtered, and while it was quite warm: the mucus and fatty acids were removed with as great dispatch as practicable, the dried bile was then dissolved in alcohol freed from water as thoroughly as possible, and through the solution a stream of carbonic acid gas was transmitted for the space of three hours, without the slightest precipitate or even opacity occurring. Now one of the principal characters of bilin, according to Berzelius, is, that if combined with a base its tendency to combine is so slight that the combination is destroyed by carbonic acid. In the above experiment, therefore, if bilin had been present, the carbonic acid would have combined with the soda, forming the carbonate of soda, which is insoluble in alcohol, while the bilin would have remained in solution.

Such are the principal facts which I beg to lay before you. It remains yet to be determined whether the electro-negative body in the bile is the same in all animals. A certain analogy seems to exist between the bile of the ox and that of man; but it would be premature to place on record any reasonings which, however probable at the present stage of the investigation, more accumulated evidence may not confirm. The subject is in progress, and bids fair to give decided and satisfactory results.

G. KEMP.

St. Peter's College.

VI. *On the Motion of Glaciers.* *By* WILLIAM HOPKINS, M.A. *and* F.R.S., *Fellow of the Cambridge Philosophical Society, of the Geological Society, and of the Royal Astronomical Society.*

[Read *May* 1, 1843.]

SECTION I.

On the Present State of Theories of Glacial Motion.

DE SAUSSURE appears to have been the first to examine with accuracy, and to describe in detail, the various phenomena which the Alpine glaciers present to us. The phenomena connected with the motion of glaciers, constituting the class with which alone we are concerned in the present communication, engaged their share of his attention, though his observations did not aim at that degree of exactness with which observers of the present day are conducting their researches. Nor did he fail to speculate on the causes of glacial movements. He considered glaciers to *slide* along the surfaces over which they move, the motion being due to the inclination of those surfaces to the horizon, and the action of gravity on the moving mass; and though he was not the first who adopted this theory of glacial movement, it is now usually associated with his name, from his having been the first to acquire any exact knowledge of such movements, or to form, perhaps, any very definite conception of the mechanical causes to which they might be referrible. From his time to a recent period the subject seems to have excited little comparative interest; but within the last few years glacial phenomena have been investigated with great care, and attention has been again directed to them, not only as forming an interesting branch of physical enquiry, but also as pregnant with geological inferences of the first importance. We are especially indebted to M. Agassiz for his active researches among the Alpine glaciers. The influence of his name has awakened an interest in them which might otherwise have long slumbered; and whether some of the opinions he has promulgated respecting the motion of glaciers be ultimately established or refuted, geology must continue equally indebted to him for the manner in which he has directed our attention to the importance of the subject in its geological bearings.

One of the consequences of these renewed researches has been to cast great doubt on the adequacy of De Saussure's theory to account for the motion of glaciers. The inclination of the surface over which some of the Alpine glaciers move is found to be so small as to render it apparently inconceivable that such glaciers should not only descend, but overcome powerful obstacles to their descent, if there be no other moving force than that of gravity. The mean inclination of the surface of the Aar glacier is stated not to exceed 3° (and that of its bed must be still less), an inclination much smaller than that at which a very smooth hard body will descend down an equally smooth and hard plane*. Nor is the difficulty diminished by the consideration of the great

* The following results are given by Professor Whewell in his *Mechanics of Engineering*, on the authority of Morin. If θ be the angle of the plane down which sliding will just take place, and μ the coefficient of friction, we have for

	Values of μ	Values of θ
Hard Limestone on Hard Limestone............	,38	20° 50
Brass on Brass..	,20	11° 20′
Brass on Iron	,16	9° 5′
Cast Iron on Cast Iron	,15	8° 30′
Cast Iron on Cast Iron, greased..................	,10	5° 45′
Brass on Iron, greased	,08	4° 35′

weight of the moving mass, or of the extent of its surface in contact with that over which it moves; for, according to the observed laws of sliding bodies, the motion is independent of both these circumstances. This difficulty has been hitherto regarded, and with reason, as a most serious if not an insuperable one to the sliding theory. Another has also been frequently urged, for which, however, there is no real foundation. It has been contended that if a glacier moved by sliding over its bed from the mere action of gravity, it ought to move with an *accelerated* motion, whereas the motion is observed to be *unaccelerated*. If the force retarding the motion were solely that of ordinary *friction* of the surface over which it moves, the objection would be valid, because the retarding force of friction is *independent of the velocity acquired*; but in the case of a glacier moving down an irregular valley and over an irregular surface, all the retarding forces do not act on the mass in the same manner as friction in the ordinary cases of sliding bodies. Besides the friction, there will be retarding forces acting at an indefinite number of projecting points along the sides or bottom of the glacial valley. Such forces will depend on the velocity of the glacier, and therefore the whole accelerating force on the mass will be some function of the velocity, and the motion will not necessarily be an accelerated motion*. The difficulty now spoken of, therefore, seems to have arisen from an imperfect conception of the problem; but the one first mentioned is sufficient to shew that the solution afforded by De Saussure's theory is far from being satisfactory.

The rejection of the *sliding theory* has led to the adoption, by different persons, of two other theories, which have been denominated respectively the *dilatation* and *expansion theories*. They both rest on the same principle—the expansion of water in the act of freezing. The former has had recently for its principal advocate M. Agassiz. It is found that a portion of the water arising from the dissolution of the superficial ice of the glacier by the direct rays of the sun and the warmth of the summer atmosphere, infiltrates into the minute pores and cavities of the ice, where, it is contended, it is frozen by the cold of the glacier, and, in freezing, expands and produces a *dilatation* and consequent onward motion of the whole mass. According to the expansion theory, the motion is due to the freezing and consequent expansion of water collected, not in minute pores and crevices, but in cavities or fissures of considerable dimensions. A repetition of these processes is supposed to keep up the continuous motion of the glacier.

These theories appear to me to involve insuperable difficulties, both physical and mechanical. Supposing the capillary cavities in the one case, and the large ones in the other, to become full of water, and that water to be frozen, the cavities will be completely filled with solid ice. How is another set of cavities to be formed for a repetition of the process? Such an effect cannot be ascribed to an internal dissolution of the ice as a consequence of external temperature, for though the internal temperature of the glacier might be depressed far below the freezing point in winter, it cannot possibly be raised above that point, or even up to it, except at the extreme surface, during the summer. That water does percolate through the pores of glacial ice with extraordinary freedom, M. Agassiz has proved by making the percolation evident to the eye, but he has not proved that it freezes there. The temperature of the upper portion of a glacier, where the percolation has been observed, is, in fact, very little below that of freezing, and does not appear to be sufficiently low to convert water into ice while moving with the freedom with which it descends through the glacier. Wherever congelation does take place the capillary pores must necessarily, I conceive, be filled up, and where it does not, the percolating water must proceed till it meets with the larger fissures, through which it will descend freely to the bottom of the glacier. The existence of the larger internal cavities of the expansion theory is purely hypothetical;

* The descent of water along a river-course, or of ice floating down its current, is not necessarily with an *accelerated* motion, and for a reason exactly similar to that assigned in the text.

and a repetition of the process to which the motion is referred is perhaps still more inexplicable than in the dilatation theory.

If, however, we chose to allow the alternations of congelation and dissolution required by these theories, it might still be shewn (as I have done elsewhere*) that the effectiveness of the causes of glacial motion assigned by them must probably be very much less than that of gravity whenever the inclination of the bed of the glacier is not much less than that of any known glacier. I think it unnecessary, however, to repeat such investigations in this communication, or to insist on other difficulties involved in these theories, because there is an obvious and conclusive test to which they will doubtless be soon subjected. It is manifest, that, according to either theory, the velocity with which any proposed point of a glacier will move must be approximately proportional to its distance from the upper and fixed extremity. If, therefore, it should be found, on the contrary, that the motion near the two extremities of a glacier is nearly the same, the refutation of both these theories will be complete. M. Agassiz has been engaged in the most careful determination of all the circumstances connected with the motion of the glacier of the Aar, and Professor Forbes has in like manner been occupied with the Mer de Glace of Mont Blanc. The results in the latter case are already partially known through Professor Forbes's letters to Professor Jameson†, and appear to be totally inconsistent with both the theories of which we are now speaking. The full details of the surveys of these two glaciers will form most important additions to our knowledge of glacial phenomena. In the mean time sufficient has been said to indicate the great, and, as I believe, insuperable difficulties both of the expansion and dilatation theories.

A conviction of the inadequacy of any of the three theories above mentioned to account for the motion of glaciers, has led Professor Forbes to suggest another theory. In common with that of De Saussure, it attributes the motion of a glacier to the action of gravity; but whereas, according to the sliding theory, gravity is enabled to act effectively in communicating motion to the glacial mass in consequence of *the facility with which the lower surface of the glacier moves over the bed on which it rests*, the theory now alluded to attributes the efficiency of gravity to *the facility with which contiguous particles of the ice itself may move with reference to each other.* Such at least is my conception of the theory, and it is only in this sense that I can understand it as a *mechanical* theory: for if it be merely meant to assert that certain phenomena of glacial motion are similar to those which would present themselves if the glacial mass were really a *viscous fluid*, the assertion is only equivalent to a particular geometrical representation of the phenomena in question. In this sense the theory asserts nothing respecting mechanical causes, and therefore cannot be classed with the theories already mentioned.

Regarding this view of glacial motion, however, (in the absence of its more complete development) according to my conception of it as a mechanical theory, it may be asked, what reason have we to suppose that the adhesion of contiguous particles of glacial ice is much less than that of a particle of ice in the lower surface of the mass to the contiguous particle of the bed of the glacier? The general mass of glacial ice is extremely hard and compact, and has unquestionably a great cohesive power, so that when we consider the probable effects of terrestrial heat and subglacial currents in destroying the adhesion between the glacier and its bed, it would appear the more probable that this adhesion should be much less instead of being much greater than that between contiguous portions of the ice itself. I do not insist on the absolute conclusiveness of this reasoning, but on its sufficiency to shew the necessity of proving, by independent experimental evidence, that glacial ice does possess this property of *semi-fluidity* or *viscosity*, if we would attribute to that property the effectiveness of gravity in setting a glacier in motion.

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* The investigations alluded to were printed and privately circulated among most persons interested in glacial researches. The object was to compare the degrees of efficiency of the causes of glacial motion assigned respectively by the three theories mentioned in the text.

† Edinburgh Quarterly Journal of Science.

It may perhaps be answered, that the best way of making such experiments is by observing the glaciers themselves, or in other words, that it is better to make our theory depend on observation than on direct experiment; and, undoubtedly, it is thus that we do arrive at the highest order of evidence which the greatest problems of physical science admit of. We set out with some determinate hypothesis, of which we calculate the consequences. These calculated results are then compared with the results of observation, and the degree of accordance between them will constitute the evidence in favour of our original hypothesis. The conclusiveness, however, of this inductive process of reasoning must depend on the rigorousness with which we can calculate our results, and the accuracy with which the phenomena to be accounted for can be observed. If our methods possess, in both these particulars, the requisite degree of exactness, we shall be certain of demonstrating the truth or detecting the fallacy of our original hypotheses, and of thus eliminating, as it were, all but the true one. In the case before us, however, the required exactness is not attainable, for it will appear, in the course of this paper that the particular phenomena to which Professor Forbes would seem to appeal in evidence of the truth of his theory, are equally consistent with that which I shall offer. Consequently, the necessity of direct experimental proof of the *viscosity* of glacial ice assumed in this theory cannot be superseded, in the present state of our knowledge of the motion of viscous fluids and of glacial movements, by an appeal to phenomena which those movements themselves present to us.

This review of the existing state of glacial theories is sufficient to shew how imperfect a solution of the problem of glacial motion has yet been offered. All the above theories repose more or less on hypotheses unsupported by the direct evidence of experiment or observation. The theory of De Saussure is apparently in opposition to the ascertained facts respecting the motion of sliding bodies; in the theories of dilatation and expansion, the alternations of thawing and freezing is an unsupported assumption, and the mechanical adequacy of the causes assigned by these theories (supposing them to be real causes) a pure hypothesis; and in the last-mentioned of the above theories, the viscosity of the glacial mass necessary to give effectiveness to the moving force of gravity, seems to be opposed to the evidence of our senses. It would be difficult perhaps to conceive the solution of any mechanical problem in a much more unsatisfactory state than the one before us; for, of the different solutions which have been proposed, each involves some difficulty, which, if not removed, must ensure its ultimate rejection.

In considering these difficulties it occurred to me, that the motion down an inclined plane of a mass of ice *having its lower surface in a state of disintegration*, might take place according to laws different from those observed in the sliding motion of rigid bodies, and, without forming any very definite conception of the manner in which the motion might be modified under this new condition, I determined to try the experiment. The results were such as to remove entirely, I conceive, what appeared to be an insuperable objection to the sliding theory, by shewing that ice, under the condition above stated, is capable of descending with a slow unaccelerated motion, by the action of gravity alone, down planes of much smaller inclination than those over which known glaciers are observed to move. In the next section I shall describe the experiments which leave no doubt, in my estimation, as to the real cause of glacial motion.

SECTION II.

On the Cause of Glacial Motion.

1. *Experiments.*—A slab of sandstone was so arranged that the inclination of its surface to the horizon could be slowly and continuously varied by the elevation of one edge. The surface was in the state in which it had been sent from the quarry, and in which such stones are sometimes laid down as paving stones, retaining the marks of the pick with which the

quarry-man has shaped them, without any subsequent process for rendering the surface smooth. The slab thus presented a grooved surface (the grooves running in very nearly parallel directions), having some resemblance to those over which existing glaciers move, but having little of the smoothness of *roches poliés*. The best measure, however, of the degree of its roughness is this —when placed at an inclination of about 20°, a piece of polished marble would just rest upon it.

The slab was so placed that the direction of the grooves coincided with that of greatest inclination. A frame of about 9 inches square and 6 inches in depth, without top or bottom, was then placed on the slab and filled with lumps of ice from a neighbouring ice-house, in such a manner that the ice, and not the frame (which merely served to keep the ice together as one mass) was in contact with the slab. In the experiments in which the following results were obtained, weights were placed on the ice such that the pressure on the slab was at the rate of about 150lbs on the square foot.

Inclination of the Slab.	Spaces in decimals of an inch though which the loaded ice descended in successive intervals of 10 minutes.									Mean Space in inches for 1 hour.
3°	,08	,05	,07	,03	,04	,05	,07	,06	,04	,31
6°	,09	,10	,09	,07	,08					,52
9°	,14	,12	,17	,14	,19	,20				,96
12°	,38	,34	,36	,27						2,0
15°	,43	,41								2,5
20°	The mass descended with an *accelerated motion*.									

When the inclination was 9° about two-thirds of the weight was removed; the velocity was diminished by nearly one half.

When the inclination of the slab did not exceed one degree, there was a small but very appreciable motion.

On the surface a slab of the same kind of stone *smooth* but not *polished*, there was appreciable motion at an angle of 40 minutes. Nor am I prepared to say that either in this, or the preceding case, the angle was the least at which sensible motion would take place.

When the surface used was that of *polished marble*, there was sensible motion with the smallest possible inclination. The motion, in fact, afforded almost as sensitive a test of deviation from horizontality as the spirit level itself.

In all these experiments the ice melted continually but very slowly at its lower surface in immediate contact with the slab. During the night the temperature descended below that of freezing, and the motion entirely ceased.

The angle at which the accelerated motion just begins to take place is that at which the ice would just rest upon the inclined plane, if the temperature of the slab and of the air were at or below the freezing temperature, so that no disintegration of the ice should take place. This angle appears to be nearly the same in the case of ice, on the grooved slab I made use of, as for that in which polished marble was the sliding body, and is that whose tangent determines the coefficient of friction between the slab in question and *solid* ice. When the slab was of polished marble this angle was very small.

2. In the experiment above detailed we have these results:—

(1). For all angles less than that just mentioned the motion was *not an accelerated motion.* This result was verified in every experiment I made.

(2). For inclinations not exceeding 9 or 10 degrees, the velocity, *cæteris paribus*, was approximately *proportional to the inclination.* This, I doubt not, would hold in all cases in which the inclinations should be sufficiently small compared with the angle of accelerated motion. It is manifestly equivalent to the assertion, that the velocity is proportional to the moving force.

(3). The velocity of the mass was increased by an increase of weight.

3. It is not very difficult to give a general explanation of the mechanism of this motion. Conceive a very thin slice of the sliding body in contact with the inclined plane on which the motion takes place to become instantaneously fluid: an indefinitely small motion would necessarily take place, by which the lower surface of the portion of the mass retaining its solidity would be brought in contact with the plane. If the plane were horizontal, it is manifest that this indefinitely small motion would be vertical; but it appears sufficiently evident, that if the plane be inclined the motion will be compounded of a vertical motion by the action of gravity, with a motion parallel to the plane arising from what may be termed a momentary *floating* of the solid body on the small portion which has been supposed to become fluid or disintegrated, and depending partly on the inclination of the plane. The instant the solid portion of the body comes in contact with the plane, the motion will be arrested. At that instant, suppose another thin slice of the body to become fluid; the same motion will be repeated, and so on. A discontinuous motion would be thus produced: but if the successive slices which become disintegrated be indefinitely thin, *i. e.* if the liquefaction or disintegration be continuous, the resulting motion will be continuous, and it will, moreover, be uniform if the disintegration be so.

The fact that motion takes place down planes of such small inclination compared with that necessary to make the ice slide independently of its disintegration at the lower surface, may simply be stated as due to this circumstance—that, whereas the particles of ice in contact with the plane are capable, so long as they remain a part of the *solid* mass, of exerting a considerable force to prevent sliding, they are incapable of exerting any sensible force when they become detached from the mass by the liquefaction or disintegration of its lower surface.

When the sliding mass is small (as in the experiments above described) the exact uniformity of the motion will be destroyed by local irregularities in different parts of the inclined plane down which it takes place, or temporary irregularities in the disintegration; but where the whole inclined surface on which the motion takes place is always the same (as in the case of a glacier), and the mass is sufficiently large, all local or temporary irregularities will, in a great measure, counteract each other, and will therefore not materially disturb the uniformity of the motion, which will be preserved so long as the intensity of the causes of disintegration remains unaltered.

4. *Temperature of the Lower Surface of a Glacier.*—The essential condition under which gravity becomes effective in putting the loaded ice in motion in the experiments above described, is that the lower surface of the ice shall be in a state of disintegration, or that its temperature shall be that of zero of the centigrade thermometer. In order, therefore, that our results may be applicable to any proposed glacier, we must shew that the temperature of its lower surface must be zero. For this purpose, let us conceive the earth to be covered with a superficial crust of ice, and, for the greater simplicity of explanation, let us suppose the conducting power for heat within the icy shell, and in passing into it from the earthy nucleus, to be the same as in the interior of the nucleus. The temperature of the ice, to a certain depth beneath the external surface, would be subject to sensible annual variations of temperature, which would become insensible at a certain depth (x_1), where the temperature (u_1) would be constant. The mathematical determination of x_1 and u_1 will be given in the concluding section. The temperature u_1 would necessarily be less than zero (centigrade) and at greater depths than x_1, the increase of

temperature would be proportional to the increase of depth, the rate of increase (with our present supposition respecting the conductive power of the ice) being exactly the same as in the actual case of the earth, provided the ice should always remain solid, *i. e.* if the temperature, thus increasing with the depth, should not rise to zero at the lower surface of the icy crust. Now, though more accurate observations on the internal temperature of glaciers are wanting, it is probable from those of M. Agassiz, that the internal temperature of glaciers in those regions in which their motions have been observed, and at depths below the influence of external variations, is not less than -1^0 (cent.). The least depth in the actual case of the earth at which the temperature is sensibly constant may be stated generally at about 60 feet, below which the rate of increase of temperature in descending may be taken at about 1^0 (cent.) for every 100 feet. Hence, supposing the same to hold for ice, the internal temperature of our icy shell, where exposed to the same external temperature as an actual glacier, would be below zero at every point, provided its depth were less than 160 feet. If the thickness of the shell were greater than that quantity, the temperature of its lower part would be higher than zero if ice were capable of receiving such higher temperature; but since that is impossible, the heat which would be employed in raising the temperature of the lower portion of the shell above zero if it could retain its solidity, would be actually employed in converting into water its lower surface, which would thus be retained at the constant temperature of zero, and in a state of perpetual disintegration.

If instead of supposing the icy shell to cover the whole surface of the earth, we suppose it to be of comparatively small extent, the same conclusions will hold, provided its linear superficial dimensions be sufficiently great with reference to the depth, which in the above case has been estimated at 160 feet. Such is the case in all considerable glaciers. Hence, assuming the truth of our data, if a glacier in those regions in which it is accessible to observation, exceed 150 or 160 feet in thickness, its lower surface must be in a constant state of disintegration, as a consequence of the internal heat of the earth. This result is liable to error, depending on our imperfect knowledge of the internal temperature of glaciers, and the conductivity of glacial ice; but in those parts at least, where the thickness of a glacier is considerably greater than 150 feet*, it leaves no reasonable doubt, I conceive, of the truth of our conclusion respecting the state of slow perpetual disintegration of the lower surface.

5. *Agency of Subglacial Currents.*—The internal heat of the earth, however, is not the only cause producing this constant disintegration. Another and probably very effective agency exists in the subglacial currents, which, during the summer, are principally produced by the rapid melting of the ice at the upper surface of the glacier, whence they descend through open fissures, and afterwards force their way between the glacier and the bed on which it rests. I cannot appeal to any direct experiments to determine the effect of water at the temperature of zero in dissolving ice at the same temperature, when running in contact with its surface, but its efficiency in this respect is sufficiently proved by its action on the upper surface of a glacier when the direct rays of the sun and the temperature of the atmosphere are sufficient to dissolve the superficial ice, and thus to create innumerable rivulets running upon the surface till they meet with a fissure into which the water is precipitated, and finds it way to the bed of the glacier. These little superficial streams shew their effect in disintegrating the ice by the manner in which they cut out for themselves their own channels, thus assisting greatly in the degradation of the surface. Its effect on the lower surface of the glacier is probably greater than on the upper, on account of the hydrostatic pressure under which it must there act. The descending water must reach the bed of the glacier at almost every point of it, and cannot

* That such is the case throughout extensive portions of large glaciers, there seems to be no doubt. M. Agassiz informed me that he had discovered a nearly vertical hole in the ice, not far from his *cabane* on the glacier of the Aar, of which the depth could not be much less than 780 feet.

afterwards collect and proceed in uninterrupted channels, because if such channels were once formed they must necessarily be immediately destroyed, or at least impeded at numerous points by the motion of the glacier. The existence of such impediments to the motion of the water, and the consequent formation of subglacial reservoirs, is proved by the continued flow of the streams which issue from the lower extremities of glaciers during the night, though the supply from the upper surface is entirely stopped immediately after sunset, when the melting ceases, and does not recommence till a considerable time after sunrise the next morning. During the intervening ten or twelve hours the whole of the water beneath the glacier at sunset would necessarily discharge itself if its course were unimpeded, even from the longest and least inclined of the Alpine glaciers, before sunrise the next morning; whereas the volume of water issuing from the glacier of the Aar is very little less in the morning than in the evening. This equable supply can only arise from the discharge during the night from reservoirs formed during the day. Hence it will follow that these subglacial currents, commencing from almost every point of the glacier, will be forced under every part of it by hydrostatic pressure, by which, as above asserted, its disintegrating action on the lower surface of the ice will doubtless be increased.

SECTION III.

Phenomena depending upon the Motion of Glaciers.

6. *Relative Velocities of the Central and Lateral Portions of a Glacier*.—The central part of a glacier moves considerably faster than its sides, but, according to Professor Forbes*, the change of velocity takes place not far from the lateral boundaries, the whole central portion moving with nearly the same velocity. In the month of August last summer, the central part of the Aar glacier, near the *cabane* of M. Agassiz, was moving at about the rate of a foot a day, while near the sides it was less by one third or one half. On the Mer de Glace the motion appears to be generally greater, in the ratio of about 3 : 2, but varying in different parts of the glacier†. The difference between the central and lateral motions seems to be less than in the former case.

On the Mer de Glace the velocity near the lower extremity appears to be somewhat greater than near the upper one. On other glaciers no adequate observations on this point have yet been made.

7. *Crevasses or Fissures*—The fissures which traverse a glacier are among its most distinct and striking phenomena. When the glacial valley *contracts in descending*, the following facts appear to be established.

The fissures are *transverse* and *curved*, having their convexity turned towards the upper extremity of the glacier.

Systems of fissures, preserving a certain identity of character with respect to number and form, remain fixed in position, not with reference to the moving mass, but with reference to the fixed objects around. It is not however to be understood that each fissure of a system remains absolutely stationary, but that each system remains so in the same sense in which what may be termed a system of breakers on the sea-shore may be said to be stationary, although every successive wave is in constant motion. In like manner every fissure must move through a certain space with the glacier, and then disappear by closing, or be so modified as to lose its identity‡;

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* Letters to Professor Jameson—Edinburgh Journal of Science.

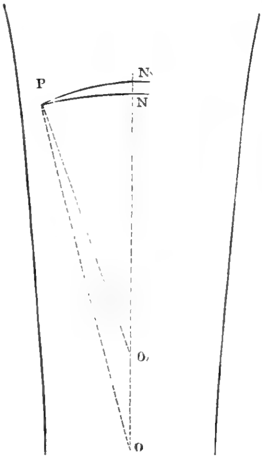
† Ibid.

‡ I consider a fissure to remain identically the same so long as it continues to intervene between the same identical portions of ice.

and when, during this motion, it has passed forward a certain distance, a succeeding one originates at the same point, moves forward in the same manner, and ultimately disappears at the same point as those which have preceded it.

If the sides of the containing valley be *divergent*, the longitudinal fissures predominate, and diverge from the axis of the glacier in a manner accordant with the divergency of the sides.

8. The continued convexity of a crevasse turned towards the upper extremity for a great length of time would manifestly be inconsistent with the fact already stated, that the central portion of a glacier moves considerably faster than its sides; for such relative motion must have the effect of continually lessening and ultimately destroying the convexity. Let us examine how long a time it might require to produce this effect.



Let $PN_{\prime}$ be a transverse fissure when first formed in a glacier, of which $N_{\prime}O$ is the axis. We may, for an approximation, suppose $PN_{\prime}$ to be the arc of a circle whose center is $O_{\prime}$. Since $N_{\prime}$ will move faster than P, the position and form of the fissure will change, but, as the change will depend only on the *relative* motions of different points of the $PN_{\prime}$, we may here suppose P to remain at rest, and the other points of the fissure to move only with their relative motion. It will be sufficiently near for our purpose if we suppose this motion such that the fissure shall always retain the form of the arc of a circle. Suppose it to come into the position PN after a time t, and let O then be its center of curvature. We may first examine what change of curvature will take place in the fissure in the time t, the curvature being measured by the angle PON.

Let $PON = \theta$, $PO_{\prime}N_{\prime} = \theta_{\prime}$, and $PO = r$, $PO_{\prime} = r_{\prime}$; and let v be the relative velocity of N. Then

$$N_{\prime}N = vt,$$

$$\text{and} \quad r \text{ vers. } \theta = r_{\prime} \text{ vers. } \theta_{\prime} - vt \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Also, if $b = \sin$ of the arc $PN_{\prime}$,

$$r \sin\theta = b,$$

$$\text{and} \quad r_{\prime} \sin\theta_{\prime} = b.$$

Hence (1) becomes

$$\frac{\text{vers. }\theta}{\sin\theta} = \frac{\text{vers. }\theta_{\prime}}{\sin\theta_{\prime}} - \frac{v}{b}t,$$

$$\text{or} \quad \tan\frac{\theta}{2} = \tan\frac{\theta_{\prime}}{2} - \frac{v}{b}t,$$

which gives the curvature at the time t.

If $\theta_{\prime}$ and, therefore, θ be not too large, we shall have approximately

$$\theta = \theta_{\prime} - \frac{2v}{b}t;$$

or if θ and $\theta_{\prime}$ be expressed in degrees,

$$\theta^{0} = \theta_{\prime}^{0} - \left(\frac{180}{\pi}\cdot\frac{2v}{b}t\right)^{0}.$$

To take a numerical example, let us suppose PN to be 2000 feet, where the relative velocity (v) is ,3 feet*, the unit of time being one day; we shall then have

$$\theta^0 = \theta_,^0 - \left(\frac{t}{58}\right)^0$$

nearly. Consequently it would in this case require nearly two months to diminish the angle θ by one degree. If $\theta_, = 8^0$ or 10^0, the change of curvature during a whole summer will scarcely be sensible to the eye.

When the whole curvature is destroyed we must have $\theta = 0$, or

$$\theta_, - \frac{180}{\pi} \cdot \frac{2v}{b} t = 0;$$

$$\therefore t = \frac{\pi}{180} \cdot \frac{b}{20} \cdot \theta_,$$

$$= 58\,\theta_,$$

nearly with the above values of v and b.

If $\theta_, = 10^0$, $t = 580$ days, supposing the relative motion to be ,3 feet each day.

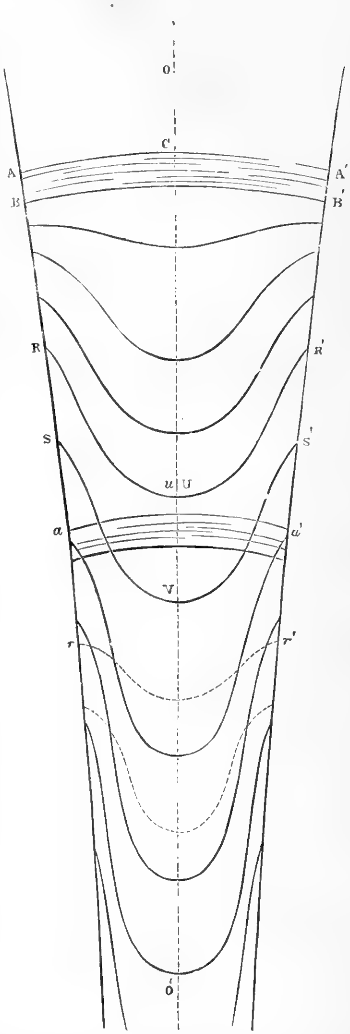
If the central part of the glacier move through a foot each day, the curvature in the above case would be destroyed after the highest point (N) of the fissure should have moved through 580 feet.

9. Professor Forbes has shewn by his observations on the Mer de Glace, that there is little variation of velocity except at points near the sides of the glacier. Consequently, if we take a fissure of which the extremities do not approach too near the sides, the relative velocities of different points of it may be much less than supposed in the above example, and a fissure might remain for several years without losing its convexity. The period during which these crevasses preserve their identity as open fissures, has not yet been made sufficiently a matter of observation. Whatever this period, however, may be, it is manifest, that since it is too short for the convexity above described to be destroyed, the crevasses must generally close after moving with the general mass of ice through a space extremely small compared with the length of the glacier. After being closed they will form *surfaces of discontinuity* within the glacier, i.e. surfaces along which there is a discontinuity in the cohesive power of the ice. There will, in fact, be no cohesion along such a surface when the crevasse first closes, though it may be afterwards partially restored. As the existence of these surfaces may exercise an important influence on the relative motions of different parts of the glacier, it may be well to examine more particularly the positions they will assume in consequence of the observed relative motions of the center and sides of a glacier.

10. *Surfaces of Discontinuity.*—Let AA' in the annexed diagram represent a fissure immediately after its first formation. For the greater distinctness of explanation it is supposed to extend from one side of the glacier to the other. Suppose it to move to BB' before it closes. During that time other fissures will have been successively formed at AA', and will in like manner have moved forward; so that between AA' where the fissures are formed and BB' where they finally close, there will be a system of open fissures as represented in the figure. Below BB' the fissures will no longer be open, but will form *surfaces of discontinuity*, as above described. The

* If the central velocity be represented by unity, the velocity of the sides will probably be very frequently between ,6 and ,8, and therefore the relative velocity of the center between ,2 and ,4.

successive lines in the diagram represent successive positions of any one of these surfaces, or simultaneous positions of successive surfaces originating in the same system of fissures. If we suppose RUR' to have been an open fissure at AA' when SVS' was so at BB', there will be a number of surfaces of discontinuity between RUR' and SVS' corresponding to the number of open fissures between AA' and BB'; and the same would hold between each consecutive pair of lines which at a previous epoch coincided simultaneously with AA' and BB'.



If another system of fissures be formed at aa', they will give rise to a corresponding system of surfaces of discontinuity, of which the dotted line rr' may be taken as the general type. These surfaces will intersect those of the former system at angles more acute as they become more remote from aa'*.

Hence then it follows, as a simple geometrical consequence of the existence of transverse fissures and of the more rapid movement of the central portion of the glacier, that the whole mass must be traversed by numerous surfaces of discontinuity; all those originating near the higher extremity of the glacier becoming very nearly longitudinal as they descend, and others being less so, according as their origin is more remote from that extremity. The whole mass will thus be divided by these intersecting surfaces into innumerable portions. Cohesion, as before intimated, may be partially restored along the surfaces of discontinuity, but the difference of velocity in the central and lateral portions will have a constant tendency to give slightly different motions to contiguous portions, and thus to prevent the restoration of cohesion. The whole glacier will thus become a *dislocated* mass; and that it actually is so is indicated by the facility with which it breaks up into vertical masses whenever irregularity of motion is superinduced by irregularities in the bottom or sides of the glacial valley. I consider a glacier, therefore, as an aggregate of numerous parts, cohering so imperfectly as to allow a much greater facility of motion among themselves, than if the mass were perfectly continuous. The glacier will thus derive a much greater facility of adapting itself to the configuration of the valley through which it descends, than if its power of adaptation depended merely on the *plasticity* and *compressibility* of glacial ice—properties which it must doubtless possess, though possibly in so small a degree that they may only become sensible under the action of the enormous pressure to which I shall hereafter shew the glacier must be subjected whenever its motion is considerably impeded.

* This exposition respecting the surfaces of discontinuity is similar to that given by Professor Forbes with reference to the alternate layers of ice of different structure, which constitute the *ribboned structure*.

SECTION IV.

Explanation of Phenomena depending on the Motion of Glaciers.

11. *Relative Velocities of different parts of a Glacier.*—According to our theory, the velocity of any portion of a glacier will depend (1) on the inclination of its bed, (2) the disintegration of its lower surface by the internal heat of the earth, (3) on subglacial currents, (4) the depth of the mass, and (5) local and lateral obstacles. The first and second causes will generally have nearly the same effect both in the central and lateral portions; but the third cause will manifestly produce in general the greatest acceleration in the central parts, and the fourth cause will produce a similar effect, if the glacier be deeper in the center than at its sides [Art. 2 (3)], while the greatest retardation will be produced on the lateral portions by the last of the above-mentoned causes. These causes sufficiently account for the greater velocity of the center of the glacier.

Again, the second of the above causes will probably act with approximate uniformity throughout the whole length of the glacier, but the third cause will act with the greatest energy at the lower extremity, because the subglacial currents will be increased by innumerable tributaries as they descend. This cause, therefore, will tend to make the velocity greater, as we approach the lower end of the glacier, while the greater depth of the mass at the upper extremity will tend to give the greater velocity to that part of the glacier [Art. 2 (3)]. In winter the effect of the currents must be very inconsiderable, and we should consequently expect that there would be a tendency in the portions of the glacier in the higher regions to move faster than those in the lower, in which case there must be a longitudinal compression, and consequent closing up of transverse fissures in a greater or less degree. During the summer, on the contrary, the subglacial currents will be most efficient, and we should expect that they would give the greater velocity to the lower extremity of the glacier, in which case the mass would be brought into a state of longitudinal tension, by which new transverse fissures would be formed, or old ones reopened.

12. *Internal Tensions and Compressions arising from the unequable Motion of the Glacier.*—The mathematical determination of the internal state of tension or pressure of a solid, but extensible and compressible body, acted on by external forces, presents difficulties which are at present insuperable, except in the most simple cases; nor can demonstrable conclusions of a less determinate character be arrived at except by an exact knowledge and careful application of mechanical principles. The cases I shall consider are the simplest of the kind, and admit of simple and conclusive reasoning. Let us first suppose a glacier to be a continuous mass, and to descend down a gradually contracting valley, so that the mass may be everywhere subjected to *lateral compression;* and let us also suppose that points near the upper extremity of the glacier tend to move with a smaller velocity than those more remote from it, and the central with a greater velocity than the lateral portions, from the causes above explained. Our first object is to determine the direction of greatest tension at any proposed point.

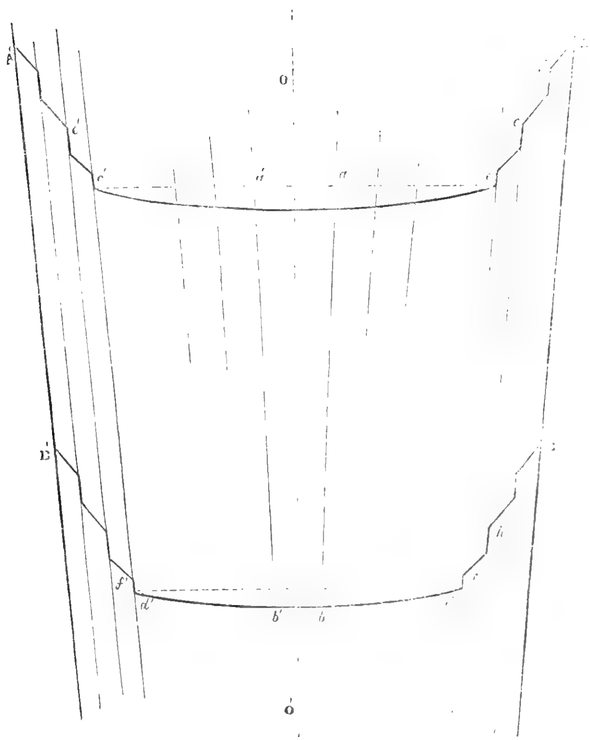
Conceive the mass divided into two portions by an imaginary surface, which, for the greater distinctness, may be supposed vertical or nearly so at every point. The mechanical action between any two contiguous particles, situated on opposite sides of this geometrical surface, may be resolved into two forces, the one normal and the other tangential to it. The normal force may be either a pressure or tension; in the latter case there must be *cohesion* between the particles. The tangential force may arise from cohesion, or may be of the nature of friction, and independent of the existence of cohesive power. Now let us conceive the *normal cohesion* at every point of our imaginary surface to be destroyed. Then, since the part of the mass near the lower extremity tends to move faster than the other part, these two portions will necessarily separate, if the surface intersect the lines of motion of the particles through which it passes, and the internal state of pressure and tension will be altered. But, again, let us suppose this surface to coincide with the line of motion of every particle situated in it, and while the normal cohesion is destroyed, conceive the tangential force between contiguous particles to be still maintained by friction. Since, by hypothesis, the mass is in a state of *transverse compression*, it is manifest that the destruction of the cohesion along the internal surface in the position now supposed will cause no separation of the two portions into which the mass is thus divided, or any modification of the previous motion, or of the internal pressures and tensions due to it. The same will be true if another such surface existed as near as we please to the former. But in this case, it is manifest that the direction of greatest tension at any point between these two surfaces must be in the direction of these surfaces, *i.e.* in the direction of motion; for since by hypothesis no cohesion exists between the portion of the mass included by those surfaces and the contiguous portions, it is impossible that any tensions should be impressed upon it in directions transverse to its bounding surfaces of no cohesion. Consequently, the same must hold when the cohesion of the mass is unbroken, since it has been shewn that the destruction of the cohesion would not affect the state of internal pressure or tension.

13. From the former part of the preceding paragraph, it follows that if any surface be described within the mass, perpendicular at every point to the direction of motion, there will be a maximum tendency to destroy the cohesion along such surfaces, so far as that tendency depends on the relative motions of the portions of the mass near the upper and lower extremities respectively.

14. Again, if our imaginary surface be longitudinal, and coincide with the direction of motion of the particles through which it passes, it is manifest that the greater motion of the central parts will cause an action of the particles on one side of the surface, on those on the opposite side of it, and in directions tangential to it. This force will depend on the tendency of the one set of particles to move faster than the other, and will evidently be greatest in the direction in which that tendency is greatest, *i.e.* in the direction of the motion. If it be sufficiently great the cohesion will be destroyed. There will be no tendency to produce open fissures in the case we are considering, on account of the lateral compression to which the mass is assumed to be subjected, but there will be a tendency to produce longitudinal surfaces of discontinuity. To investigate the effects of the internal forces thus called into action, let the following diagram represent a portion of a glacier bounded by the transverse sections AA' and BB', originally plane, but brought into the positions there represented by the relative motions of the center and sides of the mass. Let ab, cd, &c. be any longitudinal surfaces along which the tangential forces are called into action, and therefore in the direction of motion; and for the greater simplicity suppose every thing approximately symmetrical with respect to the axis OO'. Also let $w_1\, w_2 \ldots\ldots w_n \ldots\ldots w_s$ be the weights of these longitudinal portions into which the mass is thus divided; $V_1\, V_2 \ldots\ldots V_n \ldots\ldots V_s$ the velocities with which they would respectively move, independently of the action of adjoining portions on each other; they may be supposed to diminish from the center to the side; $v_1\, v_2 \ldots\ldots v_n \ldots\ldots v_s$ their actual velocities; $f_1\, f_2 \ldots\ldots f_n \ldots\ldots f_s$ the tangential longitudinal forces of contiguous portions on each other.

Now if W denote the weight of a mass of ice moving down an inclined plane, of which the inclination $= \alpha$, in the manner described in my experiments, the moving force of gravity along the plane will be $W \sin \alpha$. Let a retarding force (f) be applied to the mass, and let the velocity of descent then $= v$. Then, if V be the velocity when f does not act, we shall have, by the second observed law of motion in such cases (Art. 2),

$$\frac{v}{V} = \frac{W \sin \alpha - f}{W \sin \alpha},$$



and therefore,

$$f = \left(1 - \frac{v}{V}\right) W \sin \alpha.$$

The central portion $abb'a'$ in the preceding diagram will be retarded by the force $2f_1$, and therefore, since its weight $= 2w_1$, we shall have

$$f_1 = \left(1 - \frac{v_1}{V_1}\right) w_1 \sin \alpha,$$

α being the inclination of the bed of the glacier. The second portion, of which the weight $= w_2$, will be retarded by $f_2 - f_1$, and, therefore,

$$f_2 - f_1 = \left(1 - \frac{v_2}{V_2}\right) w_2 \sin \alpha.$$

We shall therefore have the following series of equations :

$$f_1 - 0 = \left(1 - \frac{v_1}{V_1}\right) w_1 \sin \alpha,$$

$$f_2 - f_1 = \left(1 - \frac{v_2}{V_1}\right) w_2 \sin \alpha,$$

$$\ldots\ldots = \ldots\ldots\ldots\ldots$$

$$f_n - f_{n-1} = \left(1 - \frac{v_n}{V_n}\right) w_n \sin \alpha,$$

$$f_{n+1} - f_n = \left(1 - \frac{v_{n+1}}{V_{n+1}}\right) w_{n+1} \sin \alpha,$$

$$\ldots\ldots = \ldots\ldots\ldots\ldots$$

$$f_s - f_{s-1} = \left(1 - \frac{v_s}{V_s}\right) w_s \sin \alpha,$$

$$\ldots\ldots = \ldots\ldots\ldots\ldots$$

Suppose ϕ to be the greatest value of the tangential forces (f) which one portion of the mass, on account of its nature and structure, is capable of exerting on a contiguous portion, without destroying their cohesion, and the one sliding past the other. The above equations shew that $f_1\ f_2$, &c. are in ascending order of magnitude. Let $f_n = \phi$; then will $f_1\ f_2 \ldots\ldots f_{n-1}$ be less than ϕ, which will also be the limiting value of $f_{n+1}\ f_{n+2} \ldots\ldots f_s \ldots\ldots$, and will be their actual values if we suppose ϕ to be the same for every two contiguous portions. In this case, we shall have

$$f_{n+1} - f_n = 0,$$

$$\ldots\ldots\ldots\ldots$$

$$f_s - f_{s-1} = 0,$$

$$\ldots\ldots\ldots\ldots\ ;$$

and therefore

$$v_{n+1} = V_{n+1}$$

$$\ldots = \ldots\ldots$$

$$v_s = V_s$$

$$\ldots\ldots\ldots\ldots$$

Hence if cd, $c'd'$ represent the boundaries of the n^{th} portions on each side of the axis, the portions between these lines and the boundaries, AB, $A'B'$ respectively of the glacier will move with the same velocities as if they were not affected on the one hand by the lateral action of the central portion, and on the other by that of the side of the containing valley, assuming this latter action also $= \phi$. If each longitudinal portion of the mass were perfectly rigid, the central portion $cdd'c'$ would remain unbroken (since $f_1 f_2 \ldots f_{n-1}$ are less than ϕ), and would move with a common velocity, sliding past the adjoining portions, as these portions would again slide past those contiguous to them. The central part would thus be brought into the position represented in the diagram by the dotted lines at its upper and lower boundaries; but if the mass have some degree of plasticity (as is doubtless the case with ice), it will be brought into the position defined by the dark transverse lines; for any such portion as cf will be acted on by a tangential longitudinal force ϕ on one side in the direction cd, and on the other by an equal force in the direction fe; and these forces, while they counteract each other with respect to the progressive motion, will *twist* the mass from the form of a rectangular into that of an oblique-angled parallelopiped. The forces $f_1\ f_2$, &c. will in like manner twist the component portions of the central mass $cdd'c'$ in a degree proportional to their intensities, and therefore in the least degree those parts nearest the axis; so that the central parts of cc' and dd' will have little curvature. A small additional motion, however, will thus be given to the middle of the central portion, but with the degree of plasticity here supposed, it may be considered as much less than that due to the sliding of one portion past another.

If ϕ be considerably smaller near the sides than at points more remote from them, the width cc' will be large, and there will be little variation in the velocity of the glacier except at points near its edges, as stated by Professor Forbes to be generally the case.

15. If we add together the two sides respectively of the first n of the above equations, we have

$$f_n = \left(1 - \frac{v_1}{V_1}\right) w_1 \sin\alpha + \ldots\ldots + \left(1 - \frac{v_n}{V_n}\right) w_n \sin\alpha.$$

But $v_1 = v_2 = \ldots\ldots = v_n$ nearly; and if we suppose $V_1\, V_2 \ldots\ldots V_n$ not to differ much from each other (as will probably be the case in most glaciers), we may substitute for each a mean value V. Then we have

$$\phi = f_n = \left(1 - \frac{v}{V}\right) (w_1 + w_2 + \ldots\ldots w_n) \sin\alpha,$$

where v is the common velocity of the central portion; or, if

$$w_1 + w_2 + \ldots\ldots + w_n = W_n,$$

$$\phi = \left(1 - \frac{v}{V}\right) W_n \sin\alpha.$$

If the whole mass be very wide, like that of a glacier, and α be equal to the ordinary inclination of a glacier (from 3^0 to 10^0 or 12^0), and if the retardation $V - v$ be considerable, ϕ may become a force of enormous magnitude. In order that the motion of the mass may be entirely destroyed, *cd* must coincide with *AB*, and we shall have

$$\phi = W \sin\alpha,$$

where W = weight of the whole mass.

This explains the prodigious power which large glaciers are capable of exerting to overcome local obstacles to their motion, arising from irregularities along the sides or bottoms of the valley down which they move.

16. If the tangential action along *gh*, instead of being equal to ϕ, be equal to ϕ' less than ϕ, the portion *eh* will be accelerated by the difference of the lateral actions, ϕ and ϕ', and similarly for any other portion; but it will be observed that the portion *cf*, the nearest to the center of those against which sliding takes place, will be neither accelerated nor retarded by these lateral actions. Hence, if, in any proposed glacier, the velocity is nearly the same for the central portion (dd'), but diminishes with considerable rapidity on approaching the sides, we shall have two points (d, d') which may be determined approximately, in any transverse section, at which the velocity will be the same as that of a glacier whose thickness should be the same as the depth at these points, and in which the conditions at its lower surface should be the same as for the longitudinal portions through d and d', but whose motion should be unimpeded by any lateral obstacles. This conclusion is not unimportant as shewing that the slowness of glăcial motion does not result from lateral or local impediments, but is a necessary consequence of the action of the bed of the glacier on the lower surface of the mass, as in the experiments above detailed. It is this unimpeded or *mean motion* which ought in strictness to be compared with the motion in these experiments.

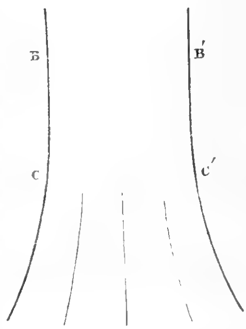
17. In the preceding investigations the mass has been supposed to be continuous, but it is easily seen that similar reasoning will apply if the mass be more or less dislocated. In such case its cohesion will oppose comparatively little resistance to the formation of transverse fissures; and the greatest tangential force (ϕ) which can be exerted will be much less than when the cohesion is continuous. The sliding of one longitudinal portion past another, and the more rapid motion of the central portions, will thus, as already remarked, be much facilitated.

18. *Formation of Crevasses.*—It has been shewn (Art. 13), that if the mass of a glacier were continuous, there would be the greatest tendency to form fissures in directions perpendicular to those of motion, when the lower extremity of the glacier moves faster than the upper one. Hence, if the tension becomes sufficient to overcome the cohesion, fissures would be formed in

these directions, and would therefore be curved with their convexity towards the higher extremity of the glacier, the glacial valley being convergent. The degree of curvature would depend on the convergency of the lines of motion. If the mass be more or less dislocated, there will still be a prevailing tendency to cause fissures to open in the same direction, though their formation will necessarily be modified by the pre-existing dislocation. There will be the greatest tendency to form these transverse fissures, or crevasses, where the change of velocity is most rapid, or where lateral or other obstacles produce the greatest irregularity of motion. This accounts for the permanent existence of systems of crevasses in particular localities, as already noticed (Art. 7). Particular local causes may produce tensions which are not longitudinal, and, therefore, crevasses which deviate from the general law of formation; but the general transversal directions of these fissures proves beyond doubt the predominance of a general longitudinal tension during the period of their formation.

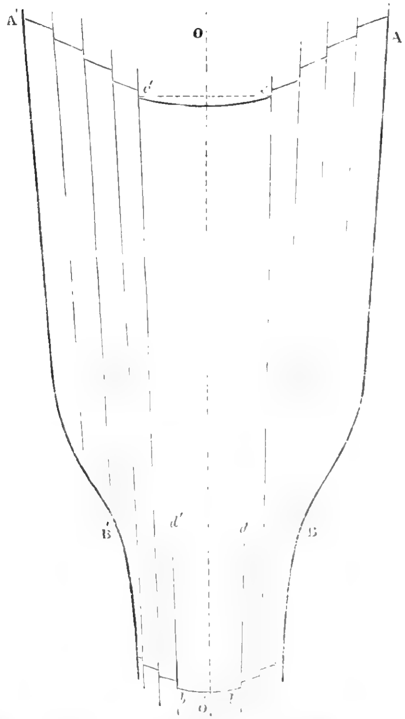
This period, according to our theory, would be the summer, as already shewn (Art. 11). In the winter, it has been also shewn, the motion must probably tend to produce in general an internal longitudinal pressure, and therefore to close previously existing fissures. And here it should be remarked, that it is not essential in order to produce these latter effects, that the motion of the glacier near its upper extremity should be absolutely greater than that near its lower end, but that the former of these motions should bear a greater proportion to the latter during winter, than during summer.

19. In our previous reasoning the glacial valley has been supposed to be convergent in descending. Let us now suppose its width to increase, and its sides to become divergent below CC'. It is a very general law in such valleys, that where the valley expands its descent becomes less rapid. Assuming such to be the case, the part of the glacier below CC' will tend to move more slowly than the part above that line. Consequently, the former of these portions will be in a state of *longitudinal compression*, which will prevent the general formation of transverse fissures. Also the pressure along CC', which will be greatest in the centre, will push forward the mass below, so as to make it tend to move along diverging lines of motion. Hence if the mass remain continuous it will be in a state of *transversal tension*, or if the continuity be broken, a system of longitudinal diverging crevasses will be formed. Such systems have been recognized both by M. Agassiz and Professor Forbes.



20. *Passage of a Glacier through a narrow Strait.*—Let us suppose the glacial valley to contract suddenly at BB' in the following diagram, and consider the motion of the glacier after it arrives at that section. Conceive the mass divided into different portions by longitudinal planes of discontinuity, as in the figure. The central portion $cdd'c'$ represents, as before, that in which no sliding of one part of it past another takes place, the planes where this relative motion begins being cd and $c'd'$. The more the central motion is impeded the greater will be the force f_{μ} (Art. 14), and the narrower will be the breadth dd'. The motion of the lateral portions will be much impeded in such a case as that represented, and near to B and B' may be entirely arrested, but there will be no action which can destroy the motion of the part $cdd'c'$. The central portion, bounded by the planes of discontinuity through B and B', will in fact move very much in the same manner as if those planes were the immoveable boundaries of the glacial valley.

Unless BB' be too narrow, therefore, the motion of the glacier will be only retarded and not destroyed; but even this retardation may be counteracted by other causes. The effectiveness of the subglacial currents will be increased by the contraction of the valley, and very generally the inclination of a valley increases as its width diminishes. These causes may compensate for the



retarding effects of the lateral action on the flanks of the glacier. The same explanation will apply whether we suppose the cohesion along the planes of separation to be entirely destroyed or not. It is only necessary that the tangential action between the central and contiguous portions should not be sufficient to prevent the former from sliding past the latter.

21. *Position of the Surface of a Glacier.*— Let P, Q be two points on the surface of a glacier situated on the same line of motion. C a point fixed in space in the vertical line through Q. Draw Pp parallel to the bed (AB) of the glacier. If the thickness BP of the glacier at P remained constant while P moved to the vertical line QC, P would come to p, and the thickness of the glacier along AC would be increased by Qp. Draw PM horizontal, and let $MPQ = \alpha$, the inclination of the surface of the glacier, and $MPp = \beta =$ that of the bed of the glacier. Then if $PM = a$,

$$Qp = MQ - Mp$$
$$= a(\tan\alpha - \tan\beta).$$

If we suppose an upward expansion of the mass to take place in consequence of the freezing and consequent expansion of infiltrated water, according to the theory of dilatation, this expansion will also increase the thickness of the glacier above A. Let ϵ denote this increase for a unit of thickness, while P moves through the horizontal space PM; then will ϵh be the whole increase, h being the thickness AQ of the glacier. On the contrary, the thickness will be diminished by the melting of the superficial ice during summer, occasioned by external influences, and of the ice in contact with the bed of the glacier, as the effect of internal heat and subglacial currents. Let Δ and δ denote the depressions of the surface below the point C, due to these causes respectively, in the time (t) of moving through PM. Then if D denote the whole depression of the surface in the vertical through C in the same time, we shall have

$$D = \Delta + \delta - \epsilon h - a(\tan \alpha - \tan \beta).$$

Of the quantities involved in this equation D, Δ, a and α may be easily observed. For this purpose conceive two vertical poles fixed firmly in the ice at P and Q in the same line of motion, their upper extremities coinciding as nearly as possible with the mean level of the glacier at the time. The inclination to the horizon of the line joining them would give the value of α; and the height to which the poles should project above the surface of the glacier after the time (t) would give the value of Δ for that time. To determine the corresponding value of D, we might observe the vertical distance of the surface of the glacier from the fixed point C when the poles should be first fixed, and after the time t of moving through PM, repeat the observation. The difference between the observed vertical distances below C would give the required value of D.

The only attempts at the independent determination of ϵ have been made, I believe, by observing the distances at different times of fixed points on the surface of the ice. Such determinations I consider entirely valueless, on account of the impossibility of separating the effects of dilatation from those of pressures and tensions depending on other and independent causes. If, however, instead of horizontal we should make vertical admeasurements, the value of ϵ for a given depth of ice might, I conceive, be determined with great accuracy. If two short horizontal poles were firmly fixed in a vertical line in the vertical wall of a crevasse, and an inextensible line or chain were fixed to the lower one, any variation of the known distance between the two poles might be ascertained with great accuracy by observations made at the upper one, and thence the value of ϵ might be accurately determined*.

Supposing the quantities D, Δ, a, α and ϵ to be determined, our equation will still contain three unknown quantities, β, δ, and h, which cannot be determined by direct observation. I think it probable, however, that ϵ might be found to be inappreciable, or, at least, extremely small, so that the term ϵh might either be neglected or expressed approximately by means of an assumed value of h. We might also eliminate δ from the above equation by making one of the observations for the determination of D as late as possible in the autumn, and the other as early as possible in the following spring, since the corresponding value of δ would doubtless be very small on account of the absence of subglacial currents during the winter. The value of D in this case would probably indicate an elevation of the surface. Let this value therefore be denoted by $-D_{,}$. We should thus have

$$D_{,} = \epsilon h + a(\tan \alpha - \tan \beta) - \Delta,$$

$$\text{or,} \quad \tan \beta = \tan \alpha - \frac{D_{,} + \Delta - \epsilon h}{a} = \tan \alpha - \frac{D_{,} + \Delta}{a}$$

nearly, the value of ϵh for the winter being small enough to be neglected. If $\tan \beta$ were thus determined, the value of δ corresponding to any observed values of D and Δ, would be given by our previous equation.

Also if β were known we should have immediately the difference of thickness at P and Q; for this difference $= Qp = a(\tan \alpha - \tan \beta)$.

* It appears singular that those who insist so much on glacial dilatation should never have subjected their views to this simple test.

The determination of β would afford an obvious means of approximating to the thickness of the glacier at any proposed point. For suppose β determined for all those different portions of the glacier where a difference of inclination of the upper surface might indicate a corresponding difference in that of the lower one. Let the length of the successive portions, beginning at the lower extremity, be $a_1\ a_2 \ldots\ldots a_n$; and let $\alpha_1\ \alpha_2 \ldots\ldots \alpha_n$, $\beta_1\ \beta_2 \ldots\ldots \beta_n$ be the corresponding values of α and β. Then if h_1 be the vertical thickness at the lower extremity, and h the required thickness at a distance $= a_1 + a_2 + \ldots\ldots a_n$ from that extremity, we shall have

$$h = h_1 + a_1 (\tan \alpha_1 - \tan \beta_1) + \ldots\ldots + a_1 (\tan \alpha_n - \tan \beta_n).$$

The chief practical difficulty in the application of this formula would be in the determina- of $\beta_1\ \beta_2$, &c. with sufficient accuracy. It appears not improbable, however, that the limits of error in determining β by the formula above given for $\tan \beta$, would be such as to render the determination a sure approximation to the real value; and, at all events, if it were found impracticable to determine all the quantities $\beta_1\ \beta_2 \ldots\ldots \beta_n$, and therefore the complete thickness of the glacier, such of them as should correspond to the more accessible and least irregular parts of the glacier, might probably be determined with considerable accuracy, and thus the rate of increase of thickness in these parts would be known.

SECTION V.

Internal Temperature of a Glacier.

22. In a previous section I have given the general reasoning by which we conclude that the temperature at the lower surface of a glacier of considerable thickness cannot be higher than zero of the centigrade thermometer. Since this conclusion, however, is of the first importance in the theory which has now been offered of glacial motion, I shall give the mathematical investigation of the problem. The case taken for direct investigation will be that of a large sphere, like the earth, of which the temperature increases as we descend, coated with an external shell of ice, the temperature of the shell being at every point below zero (cent.), that the ice may in every part remain perfectly solid. We shall thus be able to deduce the limiting thickness of the icy crust compatible with this condition of perfect solidity. If the thickness exceed this limit, then must its lower surface be in a state of constant disintegration, as already explained (Art. 4).

We have no exact knowledge of the conductive power of ice, but there is no reason to doubt its being very small. I shall suppose it (for the greater simplicity of investigation) to be the same as that of the earthy matter supposed to constitute the nucleus of the sphere; and for the same reason I shall also suppose the conductive power from the nucleus to the icy envelope to be the same as in the interior of the nucleus, or in that of the icy crust. I shall also assume the *external* temperature to be represented by $V + C \cos \left(2\pi t + \frac{\pi}{2}\right)$. So long as this is less than zero, the problem will present no peculiarity arising from the circumstance of the exterior crust being composed of ice; but however much the external temperature may exceed zero, the *superficial* temperature of the crust cannot, from the nature of ice, rise higher than zero. Hence while the external temperature is below zero, we shall have the ordinary case of a solid body placed in a medium of which the temperature varies according to a given law; but when the external temperature rises above zero, the condition at the surface will be that the *superficial* temperature of the mass shall be constantly at zero. Instead of this last condition, however, we may suppose that, during the time it would hold, the *external* temperature shall be zero; for it is manifest that the two condtions will in the case we are contemplating be very approximately the same. Hence, then, the case for investigation will be that of a sphere of large dimensions cooling in a medium of which the temperature is $V + C\cos\left(2\pi t + \frac{\pi}{2}\right)$ when this quantity is negative, and zero for those values of t which render the expression positive. If $V = 0$ the first of these conditions will be satisfied from $t = 0$ to $t = \frac{1}{2}$, from $t = 1$ to $t = \frac{3}{2}$, &c.; and the second from $t = \frac{1}{2}$ to $t = 1$, from $t = \frac{3}{2}$ to $t = 2$, &c. If V do not $= 0$, the former of these periods will be shortened and the latter lengthened, or the converse, according as V is positive or negative; if, however, V be small compared with C, the periods will be approximately as above stated, and such, therefore, we shall consider them. They will be semi-annual, if we take one year as the unit of time.

The theorems given by Poisson, in his *Théorie de la Chaleur*, Articles 194, 195 and 196, will enable us to obtain the required solution.

23. If the external temperature be represented by the general formula

$$B + A\cos(mt + \epsilon) + A_1\cos(m_1 t + \epsilon_1) + A_2(\cos m_2 t + \epsilon_2) + \&c. \ldots\ldots\ldots (1),$$

and u' denote that part of the internal temperature which depends on the external, we shall have, at the depth x beneath the surface,

$$\begin{aligned} u' = B &+ \frac{b}{D} A \epsilon^{-\frac{x}{a}\sqrt{\frac{m}{2}}} \cos\left(mt + \epsilon - \frac{x}{a}\sqrt{\frac{m}{2}} - \delta\right) \\ &+ \frac{b}{D_1} A_1 \epsilon^{-\frac{x}{a}\sqrt{\frac{m_1}{2}}} \cos\left(m_1 t + \epsilon_1 - \frac{x}{a}\sqrt{\frac{m_1}{2}} - \delta_1\right) \\ &+ \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2). \end{aligned}$$

$$\text{Where } D\cos\delta = b + \frac{1}{a}\sqrt{\frac{m}{2}}, \quad D\sin\delta = \frac{1}{a}\sqrt{\frac{m}{2}};$$

and therefore

$$D^2 = b^2 + \frac{b\sqrt{2m}}{a} + \frac{m}{a^2};$$

with similar formulæ connecting $D_1 m_1 \delta_1$, $D_2 m_2 \delta_2$, &c. Also $a^2 = \frac{k}{c}$, where k represents the conductivity and c the specific heat of the matter constituting the globe; and b is a quantity depending on the conductivity and radiating power of the surface.

Now generally if $\phi(t)$ denote any function whatever, continuous or discontinuous, whose values recur whenever t is increased by θ, so that $\phi(t + \theta) = \phi(t)$, we have the general formulæ

$$\begin{aligned} \phi(t) &= \int_0^\theta \phi(t')\,dt' \\ &+ \frac{2}{\theta}\int_0^\theta \phi(t')\cos\frac{2\pi t'}{\theta}\,dt' . \cos\frac{2\pi t}{\theta} + \frac{2}{\theta}\int_0^\theta \phi(t')\sin\frac{2\pi t'}{\theta}\,dt' . \sin\frac{2\pi t}{\theta} \\ &+ \frac{2}{\theta}\int_0^\theta \phi(t')\cos\frac{4\pi t'}{\theta}\,dt' . \cos\frac{4\pi t}{\theta} + \frac{2}{\theta}\int_0^\theta \phi(t')\sin\frac{4\pi t'}{\theta}\,dt' . \sin\frac{4\pi t}{\theta} \\ &+ \&c. \end{aligned}$$

which will coincide with (1) when the following equations are satisfied,

$$m = \frac{2\pi}{\theta}, \quad m_1 = \frac{4\pi}{\theta}, \quad m_2 = \frac{6\pi}{\theta}, \quad \&c.$$

$$B = \frac{1}{\theta}\int_0^\theta \phi(t')\,dt'$$

$$A \cos \epsilon = \frac{2}{\theta}\int_0^\theta \phi(t') \cos \frac{2\pi t'}{\theta} dt', \quad A \sin \epsilon = \frac{2}{\theta}\int_0^\theta \phi(t') \sin \frac{2\pi t'}{\theta} dt',$$

$$A_1 \cos \epsilon_1 = \frac{2}{\theta}\int_0^\theta \phi(t') \cos \frac{4\pi t'}{\theta} dt', \quad A_1 \sin \epsilon_1 = \frac{2}{\theta}\int_0^\theta \phi(t') \sin \frac{4\pi t'}{\theta} dt',$$

&c. = &c. &c. = &c.

$$A_n \cos \epsilon_n = \frac{2}{\theta}\int_0^\theta \phi(t') \cos \frac{2(n+1)\pi t'}{\theta} dt', \quad A_n \sin \epsilon_n = \frac{2}{\theta}\int_0^\theta \phi(t') \sin \frac{2(n+1)\pi t'}{\theta} dt'.$$

&c. = &c. &c. = &c.

In the application of these formulæ to the case before us we have

$$B = \int_0^{\frac{1}{2}} \phi(t')\, dt' + \int_{\frac{1}{2}}^1 \phi(t')\, dt'$$

$$= \int_0^{\frac{1}{2}} \left\{ V + C \cos\left(2\pi t' + \frac{\pi}{2}\right)\right\} dt'$$

$$= \frac{V}{2} - \frac{C}{\pi}.$$

$$A \cos \epsilon = 2 \int_0^{\frac{1}{2}} \left\{ V + C \cos\left(2\pi t' + \frac{\pi}{2}\right)\right\} \cos . 2\pi t'\, dt'$$

$$= 2 \int_0^{\frac{1}{2}} \left\{ V \cos 2\pi t' + \frac{C}{2} \cos\left(4\pi t' + \frac{\pi}{2}\right)\right\} dt'$$

$$= 0$$

$$A \sin \epsilon = 2 \int_0^{\frac{1}{2}} \left\{ V + C \cos\left(2\pi t' + \frac{\pi}{2}\right)\right\} \sin 2\pi t'\, dt'$$

$$= 2 \int_0^{\frac{1}{2}} \left[V \sin 2\pi t' + \frac{C}{2} \left\{ \sin\left(4\pi t' + \frac{\pi}{2}\right) - \sin\frac{\pi}{2}\right\}\right] dt$$

$$= \frac{2V}{\pi} - \frac{C}{2}.$$

Hence, $\epsilon = \frac{\pi}{2}, \quad A = \frac{2V}{\pi} - \frac{C}{2}.$

Also taking the general term,

$$A_n \cos \epsilon_n = 2 \int_0^{\frac{1}{2}} \left\{ V + C \cos\left(2\pi t' + \frac{\pi}{2}\right)\right\} \cos 2(n+1)\pi t'\, dt$$

$$= 2 \int_0^{\frac{1}{2}} \left\{ V \cos 2(n+1)\pi t' + \frac{C}{2}\left[\cos\left(2(n+2)\pi t' + \frac{\pi}{2}\right) + \cos\left(2n\pi t' - \frac{\pi}{2}\right)\right]\right\} dt'$$

$= 0$ when n is *even*;

$= \frac{C}{\pi} \cdot \frac{1}{n(n+2)}$ when n is *odd*.

$$A_n \sin \epsilon_n = 2 \int_0^{\frac{1}{2}} \left\{ V + C \cos\left(2\pi t' + \frac{\pi}{2}\right)\right\} \sin 2(n+1)\pi t'\, dt'$$

$$= 2 \int_0^{\frac{1}{2}} \left\{ V \sin 2(n+1)\pi t' + \frac{C}{2}\left[\sin\left(2(n+2)\pi t' + \frac{\pi}{2}\right) + \sin\left(2n\pi t' - \frac{\pi}{2}\right)\right]\right\} dt'$$

$$= -\frac{V}{\pi}\cdot\frac{2}{n+1} \text{ when } n \text{ is } even;$$

$$= 0 \text{ when } n \text{ is odd.}$$

Consequently when n is *even*

$$\epsilon_n = \frac{\pi}{2}, \qquad A_n = -\frac{V}{\pi}\cdot\frac{2}{n+1};$$

and when n is *odd*,

$$\epsilon_n = 0, \qquad A_n = \frac{C}{\pi}\cdot\frac{1}{n(n+2)}.$$

Hence, substituting in (2) we have

$$u' = \frac{V}{2} - \frac{C}{\pi} + \frac{b}{D}\left(\frac{2V}{\pi} - \frac{C}{2}\right)\epsilon^{-\frac{x}{a}\sqrt{\pi}}\cos\left(2\pi t + \frac{\pi}{2} - \frac{x}{a}\sqrt{\pi} - \delta\right)$$

$$+ \frac{b}{D_1}\,\frac{C}{1.3.\pi}.\epsilon^{-\frac{x}{a}\sqrt{2\pi}}\cos\left(4\pi t - \frac{x}{a}\sqrt{2\pi} - \delta_1\right)$$

$$- \frac{b}{D_2}\cdot\frac{2V}{3\pi}\epsilon^{-\frac{x}{a}\sqrt{3\pi}}\cos\left(6\pi t + \frac{\pi}{2} - \frac{x}{a}\sqrt{3\pi} - \delta_2\right)$$

$$+ \&c.$$

24. If v denote the temperature which would exist at any point within the sphere at the depth x beneath its surface, if the external temperature were always equal zero, we have (x being small compared with the radius of the sphere)

$$v = v_0 + \gamma x,$$

where v_0 is the superficial temperature of the sphere, and γ the rate at which the temperature depending on the original heat of the sphere, increases with the depth.

Let n denote the temperature of the sphere at the depth x, as depending both on the original heat, and the variable external temperature; then

$$u = v + u',$$

or

$$u = \frac{V}{2} - \frac{C}{\pi} + v_0 + \gamma x + \frac{b}{D}\left(\frac{2V}{\pi} - \frac{C}{2}\right)\epsilon^{-\frac{x}{a}\sqrt{\pi}}\cos\left(2\pi t + \frac{\pi}{2} - \frac{x}{a}\sqrt{\pi} - \delta\right)$$

$$+ \frac{b}{D_1}\cdot\frac{C}{1.3.\pi}\epsilon^{-\frac{x}{a}\sqrt{2\pi}}\cos\left(4\pi t - \frac{x}{a}\sqrt{2\pi} - \delta_1\right)$$

$$+ \&c. \qquad (3)$$

the complete expression for the temperature required.

25. I am not aware of any experiments for the determination of a and b for ice. Poisson has given their values for the case of the earth, deduced from observations made at Paris on the annual variations of temperature at different depths. They are

$$\left.\begin{aligned} a &= 5{,}11655 \\ b &= 1{,}05719 \end{aligned}\right\} \text{ in metres.}$$

He also gives

$$\left.\begin{aligned} v_0 &= 0^0{,}0265 \\ \gamma &= 0^0{,}0281 \end{aligned}\right\} \text{ (centigrade).}$$

Substituting the above values in the expressions for D, D_1, &c. we obtain

$$\frac{b}{D} = ,7 \quad \text{nearly}$$

$$\frac{b}{D_1} = ,63 \;\ldots\ldots$$

$$\&c. = \&c.$$

A year is taken as the unit of time.

26. In the preceding investigation the sphere has been supposed to have a complete shell of ice. The result will also be sensibly the same if, instead of the whole surface of the sphere being covered with ice, a small portion only of it be so covered, provided the thickness of the ice be small compared with its superficial dimensions. This is the actual case of a glacier, to which therefore equation (3) will be approximately applicable. Let us proceed then to the interpretation of that equation.

We observe that when x = a few multiples of a, the value of the periodical terms becomes insensible, on account of the exponential involved in them. Let x_1 be the least value of x for which we may neglect these terms. Then, if u_1 be the temperature at that depth,

$$u_1 = \frac{V}{2} - \frac{C}{\pi} + v_0 + \gamma x_1$$

$$= \frac{V}{2} - \frac{C}{\pi} + \gamma x_1 \;\ldots\ldots\ldots\ldots\ldots \;(4),$$

neglecting the small quantity v_0. Consequently the temperature at a certain depth is independent of annual variations, and lower by $\frac{C}{\pi} - \frac{V}{2}$ than it would be if the exterior shell were composed of rock instead of ice; for, in that case the value of B (Art. 23) would be the mean external temperature V, instead of $\frac{V}{2} - \frac{C}{\pi}$.

If x_2 be the depth for which the temperature = 0, we shall have

$$0 = \frac{V}{2} - \frac{C}{\pi} + \gamma x_2,$$

$$\therefore \; x_2 = \frac{1}{\gamma}\left(\frac{C}{\pi} - \frac{V}{2}\right), \;\ldots\ldots\ldots\ldots\ldots \;(5)$$

which, if we give to γ the value above stated (Art. 25), will be the numerical value of x_2 in metres.

If x_2 be less than the thickness of the glacier, the formula (3), and therefore (5), will be no longer applicable; for (3) would give the temperature of the ice at depths greater than x_2, higher than zero, which from the nature of ice is impossible. In such cases the lower surface of the ice, at whatever depth it might be, would be necessarily at zero, because the heat which, if the superficial crust were not ice, would elevate its temperature, will be employed in melting the ice at its lower surface, which will thus be kept at the zero temperature.

With the value of γ above given, equation (5) gives the value of x_2 supposing the ratio of the conductive power of ice to its specific heat to be the same as for the rocky crust of the earth. If this be not the case, the equation (5) will still give the depth at which the temperature = zero, by assigning the proper value to γ as depending on the ratio just mentioned for ice.

As a numerical example, suppose $V = 0$, and $C = 15^0$ (cent.) We shall have at the depth x

$$u_1 = -5^0 \text{ nearly};$$

and

$$x_2 = \frac{5}{,028} \text{ feet},$$

$$= 178 \text{ feet nearly}.$$

27. The temperature -5^0 (cent.) appears, however, to be much lower than that observed at different depths by M. Agassiz, and which did not exceed half a degree. The difference may, I conceive, be easily accounted for. In our investigation the surface of the glacier has been supposed to be exposed to the winter temperature, whereas, as soon probably as the mean temperature of the twenty-four hours descends to zero, the surface is protected from the external cold by a coating of snow, which increases as the temperature diminishes, and thus it is probable that the temperature of the surface of the *ice** may descend but little below zero during the whole winter. If we suppose its lowest temperature to be about $-1^0,5$ (cent.) we shall have $u_1 = -0^0,5$, and $x_2 = 54$ feet nearly. If the conductive power of ice be less than that of common rock, the value of x_2 will be proportionally less.

Taking this last value of x_2, it follows that if the thickness of a glacier should exceed 50 or 60 feet †, the temperature of its lower surface would necessarily be zero, as already explained. Now the thickness of glaciers is doubtless much greater in general than 50 or 60 feet‡, and therefore we conclude, *that generally the temperature of the lower surface of a glacier cannot be less than zero, and must, consequently, be in a state of constant disintegration, unless the conductive power of glacial ice be much greater than that of the ordinary matter forming the crust of the globe.*

28. From the conclusion of the last article it appears, that if we would investigate accurately the internal temperature of a glacier of considerable thickness, we must take, besides the condition given by the superficial temperature, the additional one that the temperature at the lower surface shall always = zero. In this case, however, the resulting expression for the temperature would become so complicated, that it would be useless, I think, to give it, especially with the uncertainty which exists respecting the superficial temperature of the *ice* during winter. The conclusion above enunciated, which is not invalidated by this uncertainty, is all that is requisite for the theory of glacial motion which has now been offered.

W. HOPKINS.

* It appears to be established, that the snow which falls on all but the higher regions of a glacier is again dissolved in the spring or early summer, and does not contribute to any permanent increase of the glacier.

† As a deduction from the general reasoning of Art. 4, this thickness was estimated roughly at about 150 feet, that there might be no doubt of its being an extreme value. The thickness of 50 or 60 feet as deduced above, is probably much nearer the truth.

‡ See Note, Art. 4.

CAMBRIDGE,
May 1, 1843.

VII. *On the Theory of Determinants.* *By* A. CAYLEY, Esq. *Fellow of Trinity College.*

[Read *Feb.* 20, 1843.]

THE following Memoir is composed of two separate investigations, each of them having a general reference to the Theory of Determinants, but otherwise perfectly unconnected. The name of "Determinants" or "Resultants" has been given, as is well known, to the functions which equated to zero express the result of the elimination of any number of variables from as many linear equations, without constant terms. But the same functions occur in the resolution of a system of linear equations, in the general problem of elimination between algebraic equations, and particular cases of them in algebraic geometry, in the theory of numbers, and, in short, in almost every part of mathematics. They have accordingly been a subject of very considerable attention with analysts. Occurring, apparently for the first time, in Crenner's *Introduction à l'Analyse des Lignes Conches*, 1750. They are afterwards met with in a Memoir *On Elimination*, by Bezont, *Mémoires de l'Académie*, 1764. In two Memoirs by Laplace and Verndermonde in the same collection, 1772. In Bezont's *Theory of Equations*, and in Memoirs by Binet, *Journal Polytechnique*, Vol. IX.; by Cauchy, *ditto*, Vol. X.; by Jacobi, *Crelles Journal*, Vol. XXII.; Lebesgue *Liouville*, Vol. VI. &c. The Memoirs of Cauchy and Jacobi contain the greatest part of their known properties, and may be considered as constituting the general theory of the subject. In the first part of the present paper, I consider the properties of certain derivational functions of a quantity U, linear in two separate sets of variables (by the term "Derivational Function," I would propose to denote those functions, the nature of which depends upon the form of the quantity to which they refer, with respect to the variables entering into it, *e.g.* the differential coefficient of any quantity, is a derivational function. The theory of derivational functions is apparently one that would admit of interesting developements.) The particular functions of this class which are here considered, are closely connected with the theory of the reciprocal polars of surfaces of the second order, which latter is indeed a particular case of the theory of these functions.

In the second part, I consider the notation and properties of certain functions resolvable into a series of determinants, but the nature of which can hardly be explained independently of the notation.

In the first section I have denoted a determinant, by simply writing down in the form of a square the different quantities of which it is made up. This is not concise, but it is clearer than any abridged notation. The ordinary properties of determinants, I have throughout taken for granted; these may easily be learnt by referring to the Memoirs of Cauchy and Jacobi, quoted above. It may however be convenient to write down the following fundamental property, demonstrated by these authors, and by Binet.

$$\left|\begin{matrix} \alpha, & \beta .. \\ \alpha', & \beta' \\ \vdots & \end{matrix}\right| \; \left|\begin{matrix} \rho, & \sigma .. \\ \rho', & \sigma' \\ \vdots & \end{matrix}\right| = \left|\begin{matrix} \rho\alpha + \sigma\beta .., & \rho\alpha' + \sigma\beta' .., .. \\ \rho'\alpha + \sigma'\beta .., & \rho'\alpha' + \sigma'\beta' .. \\ \vdots & \end{matrix}\right| \quad ......(\odot).$$

An equation, particular cases of which are of very frequent occurrence, *e.g.* in the investigations on the forms of numbers in Gauss' *Disquisitiones Arithmetica*, in Lagrange's *Determination of the Elements of a Comit's Orbit, &c.* I have applied it in the *Cambridge Mathematical Journal* to Carnot's problem, of finding the relation between the distances of five points in space, and to another geometrical problem. With respect to the notation of the second section, this is so fully explained there, as to render it unnecessary to say any thing further about it at present.

§ I. On the properties of certain determinants, considered as Derivational Functions.

Consider the function

$$\begin{aligned} U = \; & x\,(\alpha\xi + \beta\eta + \dots) + \dots\dots(1). \\ & x'\,(\alpha'\xi + \beta'\eta + \dots) + \\ & \vdots \end{aligned}$$

(n lines, and n terms in each line).

And suppose

$$KU = \begin{vmatrix} \alpha, & \beta & .. \\ \alpha', & \beta' & .. \\ \vdots & \vdots & \end{vmatrix} \quad \dots\dots(2).$$

(The single letter κ being employed instead of KU, in cases where the quantity (KU), rather than the functional symbol K, is being considered).

$$FU = -\begin{vmatrix} & \text{A}x + \text{A}'x' + .., & \text{B}x + \text{B}'x' + .., & .. \\ \text{R}\xi + \text{S}\eta + .., & \alpha & , \;\beta & , .. \\ \text{R}'\xi + \text{S}'\eta + .., & \alpha' & , \;\beta' & , \\ \vdots & & & \end{vmatrix} \quad \dots\dots(3).$$

$$\digamma U = -\begin{vmatrix} & \text{R}x + \text{R}'x' + .., & \text{S}x + \text{S}'x' + .., & .. \\ \text{A}\xi + \text{B}\eta + .., & \alpha & , \;\beta & , .. \\ \text{A}'\xi + \text{B}'\eta + .., & \alpha' & , \;\beta' & , .. \\ \vdots & & & \end{vmatrix} \quad \dots\dots(4).$$

The symbols K, F, $\digamma$ possess properties which it is the object of this section to investigate.

Let $A, B, .., A', B', .., ..$ be given by the equations:

$$A = \begin{vmatrix} \beta', & \gamma' .. \\ \beta'', & \gamma'' \\ \vdots & \end{vmatrix}, \quad B = \pm \begin{vmatrix} \gamma', & \delta' .. \\ \gamma'', & \delta'' \\ \vdots & \end{vmatrix} \quad .. \quad \dots\dots(5).$$

$$A' = \pm \begin{vmatrix} \beta'', & \gamma'' .. \\ \beta''', & \gamma''' .. \\ \vdots & \end{vmatrix}, \quad B' = \begin{vmatrix} \gamma'', & \delta'' .. \\ \gamma''', & \delta''' \end{vmatrix}$$

(The upper or lower signs according as (n) is odd or even).

These quantities satisfy the double series of equations,

$$\begin{aligned} & A\alpha + B\beta + .. = \kappa, \dots\dots(6). \\ & A\alpha' + B\beta' + .. = 0, \\ & \vdots \\ & A'\alpha + B'\beta + .. = 0, \\ & A'\alpha' + B'\beta' + .. = \kappa, \\ & \vdots \\ & \&c. \end{aligned}$$

$$\begin{aligned}
&A\alpha + A'\alpha' + \ldots = \kappa, \ \ldots\ldots (7).\\
&A\beta + A'\beta' + \ldots = 0,\\
&\ \vdots\\
&B\alpha + B'\alpha' + \ldots = 0,\\
&B\beta + B'\beta' + \ldots = \kappa,\\
&\ \vdots\\
&\&c.
\end{aligned}$$

The second side of each equation being (0), except for the r^{th} equation of the r^{th} set of equations in the systems.

Let $\lambda, \mu, \ldots$ represent the r^{th}, $\overline{r+1}^{\text{th}}, \ldots$ of the series $\alpha, \beta, \ldots$, $L, M, \ldots$ the corresponding terms of the series $A, B \ldots$, r being any number less than (n), and consider the determinant

$$\left| \begin{matrix} A, & \ldots & L \\ \vdots & & \\ A^{(r-1)} & \ldots & L^{(r-1)} \end{matrix} \right| \ \ldots\ldots (8),$$

which may be expressed as a determinant of the n^{th} order, in the form

$$\left| \begin{matrix} A & \ldots & L, & 0 & 0 \ldots \\ \vdots & & & & \\ A^{(r-1)} & \ldots & L^{(r-1)} & 0 & 0 \\ 0 & & 0 & 1 & 0 \\ 0 & & 0 & 0 & 1 \\ \vdots & & & & \end{matrix} \right| \ \ldots\ldots (9).$$

Multiplying this by the two sides of the equation

$$\kappa = \left| \begin{matrix} \alpha, & \beta \ldots \\ \alpha', & \beta' \\ \vdots & \end{matrix} \right| \ \ldots\ldots (10),$$

and reducing the result by the equation ($\odot$), and the equations (6), the second side becomes

$$\left| \begin{matrix} \kappa & 0 \ldots & & & \\ 0 & \kappa & & & \\ \vdots & & \ddots & & \\ & & \kappa & 0 & 0 \ \ldots \\ & & 0 & \mu^{(r)} & \nu^{(r)} \\ & & 0 & \mu^{(r+1)}, & \nu^{(r+1)} \\ & & \vdots & & \end{matrix} \right| \ \ldots\ldots (11),$$

which is equivalent to

$$\kappa^r \left| \begin{matrix} \mu^{(r)}, & \nu^{(r)} & \ldots \\ \mu^{(r+1)}, & \nu^{(r+1)} & \\ \vdots & & \end{matrix} \right| \ \ldots\ldots (12).$$

Or we have the equation

$$\left| \begin{matrix} A & \ldots\ldots & L \\ \vdots & & \vdots \\ A^{(r-1)} & \ldots & L^{(r-1)} \end{matrix} \right| = \kappa^{r-1} \left| \begin{matrix} \mu^{(r)}, & \nu^{(r)} & \ldots \\ \mu^{(r+1)}, & \nu^{(r+1)} & \\ \vdots & & \end{matrix} \right| \ \ldots\ldots (13),$$

which in the particular case of $r = n$, becomes

$$\left| \begin{matrix} A, & B \ldots \\ A', & B' \\ \vdots & \end{matrix} \right| = \kappa^{r-1} \ \ldots\ldots (14),$$

which latter equation is given by M. Cauchy in the Memoirs already quoted; the proof in the "*Exercises*," being nearly the same with the above one of the more general equation (13). The equation (13) itself has been demonstrated by Jacobi, somewhat less directly. Consider now the function FU, given by the equation (3). This may be expanded in the form

$$\begin{aligned} FU = &(\mathrm{r}\xi + \mathrm{s}\eta + ..)\,[A\,.\,(\mathrm{a}x + \mathrm{a}'x' + ..) + B\,(\mathrm{b}x + \mathrm{b}'x' + ..) + ..] + \\ &(\mathrm{r}'\xi + \mathrm{s}'\eta + ..)\,[A'\,.\,(\mathrm{a}x + \mathrm{a}'x' + ..) + B'\,(\mathrm{b}x + \mathrm{b}'x' + ..) + ..] + \\ &\vdots \end{aligned} \qquad ......(15),$$

which may be written

$$\begin{aligned} FU = &x\,.\,(\mathrm{A}\xi + \mathrm{B}\eta + ..) + \\ &x'\,.\,(\mathrm{A}'\xi + \mathrm{B}'\eta + ..) + \\ &\vdots \end{aligned} \qquad ......(16).$$

By putting

$$\begin{aligned} \mathrm{A} &= \mathrm{a}\,.\,(\mathrm{r}A + \mathrm{r}'A' + ..) + \mathrm{b}\,.\,(\mathrm{r}B + \mathrm{r}'B' + ..) \qquad ......(17). \\ \mathrm{B} &= \mathrm{a}\,.\,(\mathrm{s}A + \mathrm{s}'A' + ..) + \mathrm{b}\,.\,(\mathrm{s}B + \mathrm{s}'B' + ..) \\ &\vdots \\ \mathrm{A}' &= \mathrm{a}'\,.\,(\mathrm{r}A + \mathrm{r}'A' + ..) + \mathrm{b}'\,.\,(\mathrm{r}B + \mathrm{r}'B' + ..) \\ \mathrm{B}' &= \mathrm{a}'\,.\,(\mathrm{s}A + \mathrm{s}'A' + ..) + \mathrm{b}'\,.\,(\mathrm{s}B + \mathrm{s}'B' + ..) \\ &\vdots \end{aligned}$$

Hence,

$$KFU = \left|\begin{matrix} \mathrm{A}, & \mathrm{B}\,.. \\ \mathrm{A}', & \mathrm{B}' \\ \vdots & \end{matrix}\right| \qquad ......(18).$$

$$= \left|\begin{matrix} \mathrm{a}, & \mathrm{b}\,.. \\ \mathrm{a}', & \mathrm{b}' \\ \vdots & \end{matrix}\right| \; \left|\begin{matrix} \mathrm{r}, & \mathrm{s}\,.. \\ \mathrm{r}', & \mathrm{s}' \\ \vdots & \end{matrix}\right| \; \left|\begin{matrix} A, & B\,.. \\ A', & B' \\ \vdots & \end{matrix}\right| \qquad ......(19).$$

Or observing the equation (14), and writing

$$\left|\begin{matrix} \mathrm{a}, & \mathrm{b}\,.. \\ \mathrm{a}', & \mathrm{b}' \\ \vdots & \end{matrix}\right| = J \qquad ......(20).$$

$$\left|\begin{matrix} \mathrm{r}, & \mathrm{s}\,.. \\ \mathrm{r}', & \mathrm{s}' \\ \vdots & \end{matrix}\right| = \Gamma \qquad ......(21).$$

This becomes

$$KFU = J\,.\,\Gamma\,.\,(KU)^{n-1} \qquad ......(22).$$

Whence likewise

$$K\Finv U = J\,.\,\Gamma\,.\,(KU)^{n-1} \qquad ......(23).$$

Consider next the equation

$$\Finv FU = -\left|\begin{matrix} & \mathrm{r}x + \mathrm{r}'x' + .., & \mathrm{s}x + \mathrm{s}'x' + .. \\ \mathrm{a}\xi + \mathrm{b}\eta + .., & \mathrm{A}, & \mathrm{B} \\ \mathrm{a}'\xi + \mathrm{b}'\eta + .., & \mathrm{A}', & \mathrm{B}' \\ \vdots & & \end{matrix}\right| \qquad ......(24).$$

$$= -\left|\begin{matrix} 1 & .. & \\ . & \mathrm{a} & \mathrm{b}\,.. \\ . & \mathrm{a}' & \mathrm{b}' \end{matrix}\right| \; \left|\begin{matrix} 1 & . & ... \\ . & \mathrm{r} & \mathrm{s} \\ . & \mathrm{r}', & \mathrm{s}' \\ \vdots & & \end{matrix}\right| \; \left|\begin{matrix} & \xi & \eta\,.. \\ x & A, & B \\ x' & A', & B' \\ \vdots & & \end{matrix}\right| \qquad ......(25).$$

$$= -J\Gamma \left| \begin{array}{lll} & \xi, & \eta \;.\,. \\ x & A & B \\ x' & A' & B' \\ \vdots & & \end{array} \right| \quad \ldots\ldots (26).$$

If the two sides of this equation are multiplied by the two sides of the equation (2), written under the form

$$\kappa = \left| \begin{array}{llll} 1 & . & . & .\,. \\ . & \alpha & \beta & \\ . & \alpha' & \beta' & \\ \vdots & & & \end{array} \right| \quad \ldots\ldots (27).$$

The second side is reduced to

$$-J\Gamma \left| \begin{array}{llll} . & \alpha\xi + \beta\eta \;.\,.\,, & \alpha'\xi + \beta'\eta \;.\,. & \\ x & \kappa & . & \\ x' & . & \kappa & \\ \vdots & & & \end{array} \right| \quad \ldots\ldots (28).$$

$$= -J\Gamma \,.\, (\kappa)^{n-1},\ U. \quad \ldots\ldots (29).$$

And hence

$$\Finv FU = J\Gamma \,.\, (KU)^{n-2}.\ U. \quad \ldots\ldots (30).$$

And similarly

$$F\Finv U = J\Gamma \,.\, (KU)^{n-2}.\ U. \quad \ldots\ldots (31).$$

Also combining these with the equations (22), (23),

$$\frac{\Finv FU}{KFU} = \frac{F\Finv U}{K\Finv U} = \frac{U}{KU} \quad \ldots\ldots (32).$$

It may be remarked here that if U, V are functions connected by the equation

$$FU = c\,FV, \quad \text{or } \Finv U = c\,\Finv V. \quad \ldots\ldots (33).$$

Then in general

$$U = c^{\frac{1}{n-1}}.\ V. \quad \ldots\ldots (34).$$

To prove this, observing that the first of the equations (33) may be written

$$FU = F \,.\, (c^{\frac{1}{n-1}}.\ V), \quad \ldots\ldots (35),$$

we have

$$\Finv \,.\, FU = \Finv \,.\, F \,.\, (c^{\frac{1}{n-1}}.\ V), \quad \ldots\ldots (36),$$

or

$$J\Gamma \,.\, (KU)^{n-2}.\ U = J\Gamma\, [K\,(c^{\frac{1}{n-1}}.\ V)]^{n-2}.\ c^{\frac{1}{n-1}}.\ V. \quad \ldots\ldots (37).$$

Or, if neither J, Γ nor (KU) vanish, this equation is of the form

$$U = kV, \quad \ldots\ldots (38).$$

whence substituting in (33),

$$k^{n-1} = c, \quad \ldots\ldots (39).$$

which demonstrates the equation (34); and this equation might be proved in like manner from the second of the equations (33). If however, $J = 0$, or $\Gamma = 0$, the above proof fails, and if $KU = 0$, the proof also fails, unless at the same line $n = 2$. In all these cases probably, certainly in the case of $KU = 0$, $n \neq 2$, the equation (34) is not a necessary consequence of (33): In fact, FU, or $\Finv U$ may be given, and yet U remain indeterminate.

Let $U_{,}$, $\alpha_{,}$, $\beta_{,}$, ... $A_{,}$, $B_{,}$, &c... be analogous to U, α, β..., $A_{,}$, B, &c... and consider the equation

$$K.(KU_{,}FU + g.KU.FU_{,}) \quad \ldots\ldots\ldots\ldots (40).$$
$$= \begin{vmatrix} \kappa_{,}A + g\kappa A_{,}, & \kappa_{,}B + g\kappa B_{,} .. \\ \kappa_{,}A' + g\kappa A'_{,}, & \kappa_{,}B + g\kappa B'_{,} \\ \vdots & \end{vmatrix}$$

Multiply the two sides by the two sides of the equation (2), the second side becomes, after reduction,

$$\begin{vmatrix} \kappa_{,}\kappa + g\kappa.(A_{,}\alpha + B_{,}\beta + ..), & g\kappa.(A'_{,}\alpha + B'_{,}\beta + ..) .. \\ g\kappa.(A_{,}\alpha' + B_{,}\beta' + ..), & \kappa_{,}\kappa + g\kappa.(A'_{,}\alpha' + B'_{,}\beta' + ..) \\ \vdots & \end{vmatrix} \quad \ldots\ldots (41).$$

Multiplying by the two sides of the analogous equation

$$\kappa_{,} = \begin{vmatrix} \alpha_{,}, & \alpha'_{,} .. \\ \beta_{,}, & \beta_{,} .. \\ \vdots & \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots (42).$$

and reducing, the second side becomes

$$\begin{vmatrix} \kappa\kappa_{,}.(\alpha_{,} + g\alpha), & \kappa\kappa_{,}.(\beta_{,} + g\beta) .. \\ \kappa\kappa_{,}.(\alpha'_{,} + g\alpha'), & \kappa\kappa_{,}.(\beta'_{,} + g\beta') \\ \vdots & \end{vmatrix} \quad \ldots\ldots (43).$$

$$= \kappa^n.\kappa_{,}^n.K(U_{,} + gU), \quad \ldots\ldots (44).$$

whence

$$K.(KU_{,}.FU + gKUFU_{,}) = (KU)^{n-1}.(KU_{,})^{n-1}.K(U_{,} + gU). \quad \ldots\ldots (45).$$

and similarly

$$K.(KU_{,}\daleth U + gKU\daleth U_{,}) = (KU)^{n-1}.(KU_{,})^{n-1}.K.(U_{,} + gU). \quad \ldots\ldots (46).$$

In a similar manner is the following equation to be demonstrated,

$$\daleth.(KU_{,}FU + gKUFU_{,}) = F(KU_{,}\daleth U + gKU\daleth U_{,}) = \quad \ldots\ldots (47).$$
$$- Jf.(KU)^{n-2}(KU_{,})^{n-2} \times \begin{vmatrix} & \alpha_{,}x + \alpha'_{,}x' .., & \beta_{,}x + \beta'_{,}x' .. \\ \alpha\xi + \beta\eta .. & \alpha_{,} + g\alpha, & \beta_{,} + g\beta \\ \alpha'\xi + \beta'\eta .. & \alpha'_{,} + g\alpha', & \beta'_{,} + g\beta' \\ \vdots & & \end{vmatrix}$$

Suppose

$$\overline{U} = \Sigma(\rho\xi + \sigma\eta + \ldots)(\alpha x + \alpha' x' + \ldots) \quad \ldots\ldots (48).$$

This expression being the abbreviation of

$$\overline{U} = (\rho\xi + \sigma\eta + ..)(\alpha x + \alpha' x' + ..) + \quad \ldots\ldots (49).$$
$$(\rho_{,}\xi + \sigma_{,}\eta + ..)(\alpha_{,}x + \alpha'_{,}x' + ..) +$$
$$+$$
$$\vdots$$

[$(n-1)$ lines, or a smaller number].

$$K\overline{U} = \begin{vmatrix} \Sigma\alpha\rho, & \Sigma\alpha\sigma .. \\ \Sigma\alpha'\rho, & \Sigma\alpha'\sigma .. \\ \vdots & \end{vmatrix} = 0, \quad \ldots\ldots (50),$$

which follows from the equation ($\odot$).

Conversely, whenever $K\overline{U} = 0$, $\overline{U}$ is of the above form.

Also

$$F\bar{U} = - \left|\begin{array}{ccc} & Ax + A'x' + \ldots, & Bx + B'x' + \ldots \\ R\xi + S\eta & \Sigma a\rho \quad , & \Sigma a\sigma \\ R'\xi + S'\eta & \Sigma a'\rho \quad , & \Sigma a'\sigma \end{array}\right| \quad \ldots\ldots (51),$$

which may be transformed into

$$F\bar{U} = \left|\begin{array}{cc} Ax + A'x' \ldots, & Bx + B'x' \ldots \\ \rho \quad , & \sigma \\ \rho' \quad , & \sigma' \\ \vdots & \end{array}\right| \; \left|\begin{array}{cc} R\xi + S\eta \ldots, & R'\xi + S'\eta \ldots, \\ a \quad , & a' \\ b \quad , & b' \\ \vdots & \end{array}\right| \quad \ldots\ldots (52),$$

(for shortness, I omit the demonstration of this equation).

And similarly,

$$\digamma\bar{U} = \left|\begin{array}{cc} Rx + R'x', \ldots, & Sx + S'x' \ldots. \\ \rho \quad , & \sigma \\ \rho' \quad , & \sigma' \\ \vdots & \end{array}\right| \; \left|\begin{array}{cc} A\xi + B\eta \ldots, & A'\xi + B'\eta \ldots, \\ a \quad , & a' \\ b \quad , & b' \\ & \end{array}\right| \quad \ldots\ldots (53),$$

where it is obvious that if the sum Σ contain fewer than $(n-1)$ terms, $FU = 0$, $\digamma U = 0$.

The equations (52), (53) express the theorem, that whenever $K\bar{U} = 0$, the functions $F\bar{U}$, $\digamma\bar{U}$ are each of them the product of two determinants.

If next

$$U_1 = U + \bar{U}.$$

Taking $g = -1$, in the formulæ

$$K.\left(K(U+\bar{U})\,FU - KUF(U+\bar{U})\right) = K.\left(K(U+\bar{U})\,\digamma U - KU\digamma(U+\bar{U})\right) \ldots\ldots (56).$$

$$= (KU)^{n-1}.\left(K(U+\bar{U})\right)^{n-1}.K\bar{U}.$$

Or observing the equation (50),

$$K.\left(K(U+\bar{U})\,FU - KU.F(U+\bar{U})\right) = K.\left(K(U+\bar{U})\,\digamma U - KU\digamma(U+\bar{U})\right) = 0. \ldots\ldots 57.$$

Hence $F.\left(K(U+\bar{U})\,\digamma U - KU\digamma(U+\bar{U})\right) = \digamma.\left(K(U+\bar{U})\,FU - KUF(U+\bar{U})\right)$ are each of them the product of two determinants. But this result admits of a further reduction. We have

$$F.\left(K(U+\bar{U})\,\digamma U - KU\digamma(U+\bar{U})\right) = \digamma.\left(K(U+\bar{U})\,FU - KU.F(U+\bar{U})\right) \ldots\ldots (58)$$

$$= -J\Gamma(KU)^{n-2}.\left(K(U+\bar{U})\right)^{n-2} \left|\begin{array}{ccc} & a_{,}x + a_{,}'x' \ldots, & \beta_{,}x + \beta_{,}'x' \ldots \\ a\xi + \beta\eta \ldots & a_{,} - a, & \beta_{,} - \beta \\ a'\xi + \beta'\eta & a_{,}' - a', & \beta_{,}' - \beta' \\ \vdots & & \end{array}\right|$$

Substituting $a_{,} = a + \Sigma\rho a$, &c. .. also observing that if the second line be multiplied by x, the third by x', .. and the sum subtracted from the first line, the value of the determinant is not altered, and that the effect of this is simply to change $a_{,}$, $a_{,}'$.. into a, a'.. in the first line, and introduce into the corner place a quantity $-U$, which in the expansion of the determinant is multiplied by zero. This may be written in the form

$$-J\Gamma(KU)^{n-2}\left(K(U+\bar{U})\right)^{n-2} \left|\begin{array}{ccc} & ax + a'x' + \ldots, & \beta x + \beta' x' \ldots \\ a\xi + \beta\eta & \Sigma\rho a \quad , & \Sigma\sigma a \\ a'\xi + \beta'\eta & \Sigma\rho a' \quad , & \Sigma\sigma a' \\ \vdots & & \end{array}\right| \quad \ldots\ldots (59),$$

which may be reduced to

$$J\Gamma . (KU)^{n-2} . (K(U+\bar{U}))^{n-2} \times \quad \ldots\ldots (60),$$

$$\left|\begin{array}{ll} ax + a'x' + \ldots, & \beta x + \beta' x' + \ldots \\ \rho \quad , & \sigma \\ \vdots & \end{array}\right| \left|\begin{array}{ll} a\xi + \beta\eta \ldots, & a'\xi + \beta'\eta \ldots \ldots \\ a \quad , & a' \\ \vdots & \end{array}\right|$$

If each of these determinants were multiplied by the quantity $(KU)^{n-1}$, expressed under the two forms

$$\left|\begin{array}{l} A,\ B \ldots \\ A',\ B' \ldots \\ \vdots \end{array}\right| , \quad \left|\begin{array}{l} A,\ A' \ldots \\ B,\ B' \\ \vdots \end{array}\right| \quad \ldots\ldots (61).$$

They would become respectively

$$KU \left|\begin{array}{ll} x \quad , & x' \quad \ldots \\ A\rho + B\sigma \quad , & A'\rho + B'\sigma' \\ \vdots & \end{array}\right| , \quad KU . \left|\begin{array}{ll} \xi \quad , & \eta \\ Aa + A'a' \ldots, & Ba + B'a' \ldots \\ \vdots & \end{array}\right| \quad \ldots\ldots (62).$$

So that finally

$$F . (K(U+\bar{U}) \Finv U - KU . \Finv . (U+\bar{U})) = \Finv . (K(U+\bar{U}) FU - KU . F(U+\bar{U})) =$$

$$J\Gamma . \left(\frac{K(U+\bar{U})}{KU}\right)^{n-2} \times \left|\begin{array}{lll|lll} x,\ x' \ldots & & ,\ \ldots & \xi \quad , & \eta & ,\ \ldots \\ A\rho + B\sigma \ldots, & A'\rho + B'\sigma \ldots, & \ldots & Aa + A'a' \ldots, & Ba + B'a' \ldots, & \\ \vdots & & & \vdots & & \end{array}\right| \quad \ldots (63).$$

The second side of which may be written under the forms

$$\left(\frac{K(U+\bar{U})}{KU}\right)^{n-2} \left|\begin{array}{lll} \mathrm{A}x + \mathrm{A}'x' \ldots & , \quad \mathrm{B}x + \mathrm{B}'x' \ldots & ,\ \ldots \\ \mathrm{A} . (A\rho + B\sigma \ldots) + \mathrm{A}' . (A'\rho + B'\sigma \ldots) \ldots, & \mathrm{B} . (A\rho + B\sigma \ldots) + \mathrm{B}' . (A'\rho + B'\sigma \ldots) & \ldots \\ \vdots & & \end{array}\right.$$

$$\left.\begin{array}{lll} R\xi + S\eta \ldots & , \quad R'\xi + S'\eta \ldots & ,\ \ldots \\ R . (Aa + A'a' \ldots) + S . (Ba + B'a' \ldots) \ldots, & R' . (Aa + A'a' \ldots) + S' . (Ba + B'a' \ldots) & \end{array}\right| \quad \ldots (64).$$

And

$$\left(\frac{K(U+\bar{U})}{KU}\right)^{n-2} \left|\begin{array}{ll} Rx + R'x' \ldots & , \quad Sx + S'x' \quad , \\ R . (A\rho + B\sigma \ldots) + R' . (A'\rho + R'\sigma \ldots) \ldots, & S . (A\rho + B\sigma \ldots) + S' . (A'\rho + B'\sigma \ldots) \ldots, \\ \vdots & \end{array}\right.$$

$$\left|\begin{array}{lll} A\xi + B\eta \ldots & , \quad A'\xi + B'\eta \ldots & ,\ \ldots \\ A . (Aa + A'a' \ldots) + B . (Ba + B'a' \ldots) \ldots, & A' . (Aa + A'a' \ldots) + B' . (Ba + B'a' \ldots) & \\ \vdots & & \end{array}\right| \quad \ldots (65).$$

And again, by the equations (52) (53), in the new forms

$$\left(\frac{K(U+\bar{U})}{KU}\right)^{n-2} . F . \Sigma \{[(A\rho + B\sigma \ldots)(\mathrm{A}\xi + \mathrm{B}\eta \ldots) + (A'\rho + B'\sigma \ldots)(\mathrm{A}'\xi + \mathrm{B}'\eta + \ldots) \ldots] \times$$
$$[(Aa + A'a' \ldots)(\mathrm{R}x + \mathrm{R}'x' \ldots) + (Ba + B'a' \ldots)(\mathrm{S}x + \mathrm{S}'x' \ldots) \ldots]\} \quad \ldots (66),$$

$$\left(\frac{K(U+\bar{U})}{(KU)}\right)^{n-2} \Finv \Sigma . \{[(A\rho + B\sigma \ldots)(\mathrm{R}\xi + \mathrm{S}\eta \ldots) + (A'\rho' + B'\sigma \ldots)(\mathrm{R}'\xi + \mathrm{S}'\eta + \ldots) \ldots] \times$$
$$[(Aa + A'a' \ldots)(\mathrm{A}x + \mathrm{A}'x' \ldots) + (Ba + B'a' \ldots)(\mathrm{B}x + \mathrm{B}'x' \ldots) \ldots]\}. \quad \ldots (67).$$

Comparing these latter forms with the two equivalent quantities forming the first side of (53), and observing (33), (34). It would appear at first sight that

$$K(U+\bar{U})\,.\,\mathit{\Psi} U - KU\,.\,\mathit{\Psi}(U+\bar{U}) =$$

$$\left(\frac{K(U+\bar{U})}{KU}\right)^{\frac{n-2}{n-1}}\{\Sigma[A\rho + B\sigma\,..)(\text{A}\xi + \text{B}\eta) + (A'\rho + B'\sigma\,..)(\text{A}'\xi + \text{B}'\eta\,..)\,..]\times$$
$$[(Aa + A'a'\,..)(\text{R}x + \text{R}'x'\,..) + (Ba + B'a'\,..)(\text{S}x + \text{S}'x'\,..)\,..]\}$$

$$K(U+\bar{U})FU - KU\,.\,F(U+\bar{U}) =$$

$$\left(\frac{K(U+\bar{U})}{KU}\right)^{\frac{n-2}{n-1}}.\,\Sigma\{[(A\rho + B\sigma\,..)(\text{R}\xi + \text{S}\eta\,..) + (A'\rho + B'\sigma\,..)(\text{R}'\xi + \text{S}'\eta\,..) + ..]$$
$$\times[(Aa + A'a'\,..)(\text{A}x + \text{A}'x'\,..) + (Ba + B'a'\,..)(\text{B}x + \text{B}'x'\,..)\,..]\},$$

which however are not true, except for $n=2$, on account of the equation (57). In the case of $n=2$, these equations become

$$K(U+\bar{U})\,.\,\mathit{\Psi} U - KU\,.\,\mathit{\Psi}(U+\bar{U}) =$$
$$[(A\rho + B\sigma\,..)(\text{A}\xi + \text{B}\eta\,..) + (A'\rho + B'\sigma\,..)(\text{A}'\xi + \text{B}'\eta\,..) + ..]\times$$
$$[(Aa + A'a'\,..)(\text{R}x + \text{R}'x'\,..) + (Ba + B'a'\,..)(\text{S}x + \text{S}'x'\,..) + ..] \quad ......(68),$$

$$K(U+\bar{U})FU - KU\,.\,F(U+\bar{U}) =$$
$$[(A\rho + B\sigma\,..)(\text{R}\xi + \text{S}\eta\,..) + (A'\rho + B'\sigma\,..)(\text{R}'\xi + \text{S}'\eta\,..) + ..]\times$$
$$[(Aa + A'a'\,..)(\text{A}x + \text{A}'x'\,..) + (Ba + B'a'\,..)(\text{B}x + \text{B}'x' + ..) + ..] \quad ......(69),$$

and it is remarkable that these equations ((68), (69)) are true whatever be the value of (n), provided Σ contains a single term only. The demonstration of this theorem is somewhat tedious, but it may perhaps be as well to give it at full length. It is obvious that the equation (69) alone need be proved, (68) following immediately when this is done.

I premise by noticing the following general property of determinants. The function

$$\left|\begin{array}{ll} a+\Sigma\rho a, & \beta+\Sigma\sigma a, \;.. \\ a'+\Sigma\rho a', & \beta'+\Sigma\sigma' a, \\ \vdots & \end{array}\right| \quad ......(70),$$

(where $\Sigma\rho a = \rho_1 a_1 + \rho_2 a_2 \ldots + \rho_s a_s$), contains no term whose dimension in the quantities $a, a'\ldots$, or in the other quantities $\rho, \sigma\ldots$, is higher than s. (Of course if the order of the determinant be less than s or equal to it, this number becomes the limit of the dimension of any term in $a, a'\ldots$ or $\rho, \sigma\ldots$, and the theorem is useless). This is easily proved by means of a well known theorem,

$$\left|\begin{array}{ll} \Sigma\rho a, & \Sigma\sigma a\;.. \\ \Sigma\rho' a, & \Sigma\sigma' a \\ \vdots & \end{array}\right| = 0 \quad ......(71),$$

whenever (s) is less than the number expressing the order of the determinant. Hence in the formula (70), if Σ contain a single term only, the first side of the equation is linear in $\rho, \sigma\ldots$, and also in $a, a'\ldots$, *i.e.* it consists of a term independent of all these quantities, and a second term linear in the products $\rho a, \rho a'\ldots \sigma a, \sigma a'\ldots$. This is therefore the form of $K(U+\bar{U})\ldots$

Consider the several equations

$$\kappa = KU = Aa + B\beta + .. \quad ......(72).$$
$$= A'a' + B'\beta' + +$$
$$\vdots$$

It is easy to deduce

$$\kappa_{,} = K.(U+\bar{U}) = KU + A\rho a + B\sigma a + \quad ......(73).$$
$$+ A'\rho a' + B'\sigma a' +$$
$$\vdots$$

To find the values of A, B, &c... Corresponding to $U + \overline{U}$, we must write

$$\begin{aligned} A &= \text{M}'\beta + \text{N}'\gamma' + \qquad \ldots\ldots (74), \\ &= \text{M}''\beta + \text{N}''\gamma'' + \\ &= \ldots \end{aligned}$$

where

$$\text{M}' = \begin{vmatrix} \gamma'', & \delta'' & .. \\ \gamma''' & \delta''' & \\ \vdots & & \end{vmatrix}, \quad \text{N}' = \pm \begin{vmatrix} \delta'', & \epsilon'' & .. \\ \delta''' & \epsilon''' & \\ \vdots & & \end{vmatrix} \quad .. \qquad \ldots\ldots (75),$$

$$\text{M}'' = \pm \begin{vmatrix} \gamma''', & \delta''' & .. \\ \gamma'''' & \delta'''' & \\ \vdots & & \end{vmatrix}, \quad \text{N}'' = \begin{vmatrix} \delta''', & \epsilon''' & .. \\ \delta'''' & \epsilon'''' & \\ \vdots & & \end{vmatrix}$$

The order of each of these determinants being $\overline{n-2}$, and the upper or lower signs being used according as $\overline{n-1}$ is odd or even, *i. e.* as n is even or odd. Hence

$$\begin{aligned} A_{\prime} = A + \text{M}'.\sigma\alpha' + \text{N}'.\tau\alpha' + \qquad \ldots\ldots (76). \\ \text{M}''.\sigma\alpha'' + \text{N}''.\tau\alpha'' + .. \end{aligned}$$

And therefore

$$\begin{aligned} \kappa_{\prime}A - \kappa A_{\prime} = A^2\rho\alpha + AB.\sigma\alpha + AC.\tau\alpha + & \qquad \ldots\ldots (77). \\ AA'\rho\alpha' + (AB' - \kappa\text{M}')\sigma\alpha' + (AC' - \kappa\text{N}')\tau\alpha' + .. & \\ AA''\rho\alpha'' + (AB'' - \kappa\text{M}'')\sigma\alpha'' + (AC'' - \kappa\text{N}'')\tau\alpha'' + & \\ \vdots & \end{aligned}$$

The additional quantities C, τ having been introduced for greater clearness. Now the equations

$$\begin{aligned} AB' - \kappa M' = A'B, \quad AC' - \kappa N' = A'C, .. \qquad \ldots\ldots (78), \\ AB'' - \kappa M'' = A''B, \quad AC'' - \kappa N'' = A''C, \\ \vdots \end{aligned}$$

written under the form

$$\begin{aligned} AB' - A'B = \kappa M', \quad AC' - A'C = \kappa N', .. \qquad \ldots\ldots (79), \\ AB'' - A''B = \kappa M'', \quad AC'' - A''C = \kappa N'' \\ \vdots \end{aligned}$$

are particular cases of the equation (13), and are therefore identically true. Hence, substituting in (77),

$$\begin{aligned} \kappa_{\prime}A - \kappa A_{\prime} = A^2\rho\alpha + AB\sigma\alpha + AC\tau\alpha \ldots + & \qquad \ldots\ldots (80), \\ AA'\rho\alpha' + A'B\sigma\alpha' + A'C\tau\alpha' \ldots + & \\ A''A\rho\alpha'' + A''B\sigma\alpha'' + A''C\tau\alpha'' \ldots + & \\ \vdots & \\ = (\rho A + \sigma B + \ldots)(A\alpha + A'\alpha' + \ldots). & \end{aligned}$$

Forming in a similar manner, the combinations $\kappa_{\prime}B - \kappa B_{\prime} \ldots \kappa_{\prime}A' - \kappa A'_{\prime}$, $\kappa_{\prime}B - \kappa B'_{\prime}, \ldots$, multiplying by the products of the different quantities $Ax + A'x' \ldots$, $Bx + B'x' \ldots, \ldots R\xi + S\eta + \ldots$, $R'\xi + S'\eta, \ldots$ and adding so as to form the function $K(U + \overline{U}).FU - KU.F(U + \overline{U})$, we obtain the required formula, viz. that the value of this quantity is

$$\begin{aligned} [(\rho A + \sigma B + ..)(\text{R}\xi + \text{S}\eta + ..) + (A'\rho + B'\sigma ..)(\text{R}'\xi + \text{S}'\eta ..) + ..] \times \qquad \ldots\ldots (81), \\ [(A\alpha + A'\alpha' ..)(\text{A}x + \text{A}'x' ..) + (B\alpha + B'\alpha' ..)(\text{B}x + \text{B}'x' + ..) + -]. \end{aligned}$$

with this theorem, I conclude the present section,—noticing only, as a problem worthy of investigation, the discovery of the forms of the second sides of the equations (68), (69), in the case of Σ containing more than a single term.

§ II. On the notation and properties of certain functions resolvable into a series of determinants.

Let the letters

$$r_1,\ r_2 \ldots r_k. \quad \ldots\ldots (1),$$

represent a permutation of the numbers

$$1,\ 2 \ldots k. \quad \ldots\ldots (2).$$

Then if in the series (1), if one of the letters succeeds mediately or immediately a letter representing a higher number than its own, then for each line this happens there is said to be a "derangement" or "inversion" in the series (1). It is to be remarked that if any letter succeed (s) letters representing higher numbers, this is reckoned for the same number (s) of inversions.

Suppose next the symbol

$$\pm_r \quad \ldots\ldots (3),$$

denotes the sign + or −, according as the number of inversions in the series (1) is even or odd.

This being premised, consider the symbol

$$\left\{\begin{matrix} A\rho_1,\ \sigma_1 \ldots (n) \\ \vdots \\ \rho_k,\ \sigma_k \ldots \end{matrix}\right\} \quad \ldots\ldots (4),$$

denoting the sum of all the different terms of the form

$$\pm_r \pm_s \ldots A\rho_{r_1},\ \sigma_{s_1}, \ldots A\rho_{r_k},\ \sigma_{s_k}, \ldots \quad \ldots\ldots (5).$$

The letters

$$r_1,\ r_2 \ldots r_k;\quad s_1,\ s_2 \ldots s_k:\ \&c. \ldots\ldots (6),$$

denoting any permutations whatever, the same or different, of the series of numbers (2). The number of terms represented by the symbol (5) is evidently

$$(1 . 2 \ldots k)^n. \quad \ldots\ldots (7).$$

In some cases it will be necessary to leave a certain number of the vertical rows $\rho, \sigma \ldots$ unpermuted. This will be represented by writing the mark (†) immediately above the rows in question. So that for instance

$$\left\{\begin{matrix} A\rho_1\sigma_1 \ldots \overset{\dagger}{\theta_1}\overset{\dagger}{\phi_1}\ .\ (n) \\ \vdots \\ \rho_k\sigma_k \ldots \theta_k\phi_k \end{matrix}\right\} \quad \ldots\ldots (8).$$

The number of rows with the (†) being (x), denotes the sum of the

$$(1 . 2 \ldots k)^{n-x} \quad \ldots\ldots (9),$$

terms, of the form

$$\pm_r \pm_s \ldots A\rho_{r_1},\ \sigma_{s_1} \ldots \theta_1,\ \phi_1 \ldots \ldots A\rho_{r_k},\ \sigma_{s_k} \ldots \theta_k,\ \phi_k \ldots \quad \ldots\ldots (10).$$

This is obvious, that if all the rows have the mark (†) the notation (8) denotes a single product only, and if the mank (†) be placed over all the rows but (1), the notation (8) belongs to a determinant. It is obvious also that we may write

$$\left\{\begin{matrix} A\rho_1 . \sigma_1 \ldots \theta\phi_1 \ldots (n) \\ \vdots \\ \rho_k\sigma_k \ldots \theta_k\phi_k \end{matrix}\right\} = \Sigma \pm_u \pm_v \ldots \left\{\begin{matrix} A\rho_1\sigma_1 \ldots \overset{\dagger}{\theta_{u_1}}\overset{\dagger}{\phi_{v_1}} \ldots (n) \\ \vdots \quad\quad \vdots \\ \rho_k\sigma_k \ldots \theta_{u_k}\phi_{v_k} \end{matrix}\right\} \quad \ldots\ldots (11).$$

where Σ refers to the different permutations,

$$u_1,\ u_2 \ldots u_k;\quad v_1,\ v_2 \ldots v_k;\ \&c. \ldots\ldots\ldots\ldots\ldots (12).$$

which can be formed out of the numbers (2). The equation (11) would still be true, if the mark (†) were placed over any number of the columns $\rho, \sigma \ldots$.

Suppose in this equation a single column only is left without the mark (†) on the second side of the equation; the first side is then expressed as the sum of a number

$$(1\,.\,2\ldots k)^{n-1}, \text{ or generally } (1\,.\,2\ldots k)^{n-x-1}, \quad \ldots\ldots (13),$$

of determinants, according as we consider the symbol (4) or the more general one (8). And this may be done in (n) or $\overline{n-x}$ different ways respectively.

It may be remarked, that the symbol (8) is the same in form as if a single column only had the mark (†) over it; the number (n) being at the same time reduced from (n) to $(n-x+1)$. For, the marked columns of symbols may be replaced by a single marked column of new symbols. Hence, without loss of generality, the theorems which follow may be stated with reference to a single marked column only.

Suppose the letters

$$\rho_1, \rho_2 \ldots \rho_k; \quad \sigma_1, \sigma_2 \ldots \sigma_k; \ \&c. \quad \ldots\ldots (14)$$

denote certain permutations of

$$\alpha_1, \alpha_2 \ldots \alpha_k; \quad \beta_1, \beta_2 \ldots \beta_k; \ \&c. \quad \ldots\ldots (15),$$

in such a manner that

$$\rho_1 = \alpha_{g_1}, \ \rho_2 = \alpha_{g_2} \ldots \rho_k = \alpha_{g_k}; \qquad \sigma_1 = \beta_{h_1}, \ \sigma_2 = \beta_{h_2}, \ldots \sigma_k = \beta_{h_k}; \ \ldots \quad \ldots\ldots (16).$$

Then the two following theorems may be proved:

$$\left\{\begin{matrix} \overset{\dagger}{A}\rho_1\,\sigma_1\ldots(n) \\ \vdots \quad \vdots \\ \rho_k\,\sigma_k \end{matrix}\right\} = \pm_g \pm_h \ldots \left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\,\beta_1\ldots(n) \\ \vdots \\ \alpha_k\,\beta_k \end{matrix}\right\} \quad \ldots\ldots (17).$$

If (n) be even, but in the contrary case

$$\left\{\begin{matrix} \overset{\dagger}{A}\rho_1\,\sigma_1\ldots(n) \\ \vdots \\ \rho_k\,\sigma_k \end{matrix}\right\} = + \pm_g \ldots \left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\,\beta_1\ldots(n) \\ \vdots \\ \alpha_k\,\beta_k \end{matrix}\right\} \quad \ldots\ldots (18).$$

By means of these, and the equation (11), a fundamental property of the symbol (3) may be demonstrated. We have

$$\left\{\begin{matrix} A\alpha_1\,\beta_1\ldots(n) \\ \vdots \quad \vdots \\ \alpha_k\,\beta_k \end{matrix}\right\} = \Sigma \pm_g . \left\{\begin{matrix} \overset{\dagger}{A}\rho_1\,\beta_1\ldots(n) \\ \vdots \\ \rho_k\,\beta_k \end{matrix}\right\} \quad \ldots\ldots (19),$$

which when (n) is even, reduced itself by (17) to

$$\left\{\begin{matrix} A\alpha_1\,\beta_1\ldots(n) \\ \vdots \quad \vdots \\ \alpha_k\,\beta_k \end{matrix}\right\} = \left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\,\beta_1\ldots(n) \\ \vdots \\ \alpha_k\,\beta_k \end{matrix}\right\} \Sigma\,(\pm_g \pm_g . 1) \quad \ldots\ldots (20)$$

$$= 1\,.\,2\ldots k \left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\,\beta_1\ldots(n) \\ \vdots \\ \alpha_k\,\beta_k \end{matrix}\right\}.$$

But when (n) is odd, from the equation (18),

$$\left\{\begin{matrix} A\alpha_1,\ \beta_1\ldots(n) \\ \vdots \\ \alpha_k\beta_k \end{matrix}\right\} = \left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\beta_1\ldots(n) \\ \vdots \\ \alpha_k,\ \beta_k \end{matrix}\right\} \Sigma\,(\pm_g 1) = 0. \quad \ldots\ldots (21).$$

Since the number of negative and positive values of $\pm_g$ are equal.

From the equation (20), it follows that when (n) is even, the values of a symbol of the form

$$\left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\beta_1, & (n) \\ \vdots & \\ \alpha_k\beta_k & \end{matrix}\right\} \quad \ldots\ldots (22),$$

is the same, over whichever of the columns α, β.. the mark (†) is placed. To denote this indifference, the preceding quantity is better represented by

$$\left\{\begin{matrix} \overset{\dagger}{A}\alpha_1, \ \beta_1 \ldots (n) \\ \vdots \\ \alpha_k\beta_k \end{matrix}\right\} \quad \ldots\ldots (23),$$

this last form being never employed when (n) is odd, in which case the same property does not hold. Hence also an ordinary determinant is represented by

$$\left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\beta_1 \\ \vdots \\ \alpha_k, \ \beta_k \end{matrix}\right\}, \quad \left\{\begin{matrix} \overset{\dagger}{A}\,1\,1 \\ \vdots \ \vdots \\ k\,k \end{matrix}\right\} \quad \ldots\ldots (24),$$

the latter form being obviously equally general with the former one.

It is obvious from the equations (17), (18), that the expression (22) vanishes, in the case of (n) even whenever any two of the symbols (α) are equivalent, or any two of the symbols (β), &c.; but if (n) be odd, this property holds for the symbols (β), &c., but not for the marked ones (α). In fact, the interchange of the two equal symbols, in each case, changes the sign of the expression (22), but they evidently leave it unaltered, *i. e.* the quantity in question must be zero.

Consider now the symbol

$$\left\{\begin{matrix} \overset{\dagger}{A}\,1\,1 \ldots (2p) \\ \vdots \ \vdots \\ k\,k \end{matrix}\right\} \quad \ldots\ldots (25),$$

which, for shortness, may be denoted by

$$\{\overset{\dagger}{A} \,.\, k \,.\, 2p\} \quad \ldots\ldots (26).$$

I proceed to prove a theorem, which may be expressed as follows:

$$\{\overset{\dagger}{A} \,.\, k \,.\, 2p\} \,.\, \{\overset{\dagger}{B} \,.\, k \,.\, 2q\} = \{\overline{AB}|\, k \,.\, 2p + 2q \,.\, 2\} \quad \ldots\ldots (27),$$

where

$$\overline{AB}|_{r, s \ldots x, y \ldots} = S \,.\, A_{r, s \ldots l}\, S_{x, y \ldots l} \quad \ldots\ldots (28).$$

The number of the symbols $r, s \ldots$ being obviously $2p - 1$, and that of $x, y \ldots 2q - 1$. The summatory sign S refers to l, and denotes the sum of several terms corresponding to values of l from $l = 1$ to $l = k$. Also the theorem would be equally true if l had been placed in any position whatever in the series $r, s \ldots l$; and again, in any position whatever in the series $x, y \ldots l$, instead of at the end of each of these. With a very slight modification this may be made to suit the case of an odd number instead of one of the numbers $2p$, $2q$; (in fact, it is only to place the mark (†) in $\{\overline{AB}|\ldots\}$ over the column corresponding to the marked column in $\{A\ldots\}$, $\{A\ldots\}$ being the one for which the number is odd), but it is inapplicable where the two numbers are odd. Consider the second side of (27). This may be expanded in the form

$$\Sigma + \pm_s \ldots \pm_x \pm_y \ldots \overline{AB}|_{1 . s_1 \ldots x_1 y_1 \ldots} \,.\, \overline{AB}|_{2 . s_2 \ldots x_2 y_2 \ldots} \ldots \overline{AB}|_{k . s_k \ldots x_k y_k \ldots} \quad \ldots\ldots (29),$$

where Σ refers to the different quantities $s, \ldots, x, y, \ldots$ as in (11).

Substituting from (28), this becomes

$$\Sigma \,.\, S_{l_1} \,..\, S_{l_k} \,..\, (+ \pm_s \,..\, \pm_x \pm_y A_{l\,.\,s_1\,..\,l_1} \,..\, A_{k\,.\,s_k\,..\,l_k}\, B_{x_1\,.\,y_1\,..\,l_1} \,..\, B_{x_k\,.\,y_k\,..\,l_k}). \quad \text{......} (30).$$

Effecting the summation with respect to x, y.. this becomes

$$\Sigma \,.\, S_{l_1} \,..\, S_{l_k} \,..\, + \pm_s \,..\, A_{l\,.\,s_1\,..\,l_1} \,..\, A_{k\,.\,s_k\,..\,l_k} \left\{\begin{matrix} \overset{\dagger}{B} 1\,.\,1\,..\,l_1 \\ \vdots \\ k k\,..\,l_k \end{matrix}\right\} \quad \text{......} (31),$$

Σ now referring to s,... only. The quantity under the sign Σ vanishes if any two of the quantities l are equal, and in the contrary case, we have

$$\left\{\begin{matrix} \overset{\dagger}{B} 1 1\,..\,l_1 \\ \vdots \\ k k\,..\,l_k \end{matrix}\right\} = \pm_l \{\overset{\dagger}{B}\,.\,k\,.\,2q\}, \quad \text{......} (32),$$

which reduces the above to

$$\{\overset{\dagger}{B}\,.\,k\,.\,2q\}\,.\,\Sigma + \pm_s \,..\, \pm_l A_{l\,.\,s_1\,..\,l_1} \,..\, A_{k\,.\,s_k\,..\,l_k}, \quad \text{......} (33),$$

Σ referring to the quantities s ..., and also to the quantities l. And this is evidently equivalent to

$$\{\overset{\dagger}{A}\,.\,k\,.\,2p\}\,\{\overset{\dagger}{B}\,.\,k\,.\,2q\}, \quad \text{......} (34),$$

the theorem to be proved. It is obvious that when $p = 1$, $q = 1$, the equation (27), coincides with the theorem (⊙), quoted in the introduction to this paper.

VIII. *On Small Finite Oscillations. By the Rev.* H. HOLDITCH, *Fellow of Caius College, and of the Cambridge Philosophical Society.*

[Read *May* 15, 1843.]

THE system of bodies here considered is supposed to be such, that their position, and the forces acting upon them in that position, depend upon a single variable; and the object is to find general expressions, which may be applied to any particular case, without performing any integration, for the length of the isochronous pendulum and the time of oscillation, rocking or sliding, when the body or system of bodies is slightly disturbed from its position of equilibrium, the approximation including the square of the variation of the independent variable.

By the principle of *vis viva*,

$$mv^2 + m_1v_1^2 + \dots = 2m\int Pdp + 2m_1\int P_1dp_1 + \dots$$

Let u be the independent variable, and a its value when the system is in equilibrium, and $a + z$ its value at the end of the time t; also let β be the value of z at the beginning of the motion, when the system is disturbed and left to the action of the forces upon it.

$$\text{Let } \frac{Pdp}{du} = f(u) = f(a+z),$$

$$\frac{P_1dp_1}{du} = \phi(u) = \phi(a+z),$$

$$\text{and } U = \frac{mPdp + m_1P_1dp_1 + \dots}{du} \quad\dots\dots\dots\dots (1),$$

$$\therefore\ U_0 = mf(a) + m_1\phi(a) + \dots$$

$$U_1 = mf_1(a) + m_1\phi_1(a) + \dots$$

$$U_2 = mf_2(a) + m_1\phi_2(a) + \dots$$

$$\text{and } mPdp + m_1P_1dp_1 + \dots = \left(U_0 + U_1z + U_2\frac{z^2}{1.2} + \dots\right)du \dots\dots\dots (2);$$

but, when the system is in equilibrium, $u = a$ and $U_0 = 0$, or $mf(a) + m_1\phi(a) + \dots = 0$, which determines its position when at rest, and as $du = dz$, the integration of (2) will give

$$m\int Pdp + m_1\int P_1dp_1 + \dots = U_1 . \frac{z^2-\beta^2}{2} + U_2 . \frac{z^3-\beta^3}{2.3} + U_3 . \frac{z^4-\beta^4}{2.3.4} + \dots$$

$$\text{Again, let } \frac{ds}{du} = \psi(u) = \psi(a+z),$$

$$\frac{ds_1}{du} = \xi(u) = \xi(a+z),$$

$$\text{and } V = \frac{mds^2 + m_1ds_1^2 + \dots}{du^2} \quad\dots\dots\dots\dots (3);$$

$$\therefore\ m ds^2 + m_1 ds_1^2 + \ldots = \left(V_0 + V_1 z + V_2 \cdot \frac{z^2}{2} + \ldots\right) dz^2,$$

$$\text{and}\ \left(V_0 + V_1 . z + V_2 . \frac{z^2}{2}\right)\frac{dz^2}{dt^2} = U_1 (z^2 - \beta^2) + U_2 . \frac{z^3 - \beta^3}{3} + U_3 . \frac{z^4 - \beta^4}{3 . 4} + \ldots\ldots\ldots \ (4).$$

For a first approximation, $V_0 . \frac{dz^2}{dt^2} = U_1 . (z^2 - \beta^2)$, from which it appears that $\frac{dz}{dt}$ vanishes when $z = \pm \beta$, and since $s = \int \psi(u) du = R(u) = R(a \pm \beta) = R_0(a) \pm R_1(a) . \beta$, $\frac{ds}{dt}$ vanishes, when $s = R_0(a) \pm R_1(a) . \beta$; which shews that each body of the system vibrates to an equal distance on each side of its position of equilibrium.

The time of oscillation $= \pi \sqrt{\frac{-V_0}{U_1}} = \pi . \sqrt{-\frac{m\psi(a)^2 + m_1 \xi(a)^2 + \ldots}{m f_1(a) + m_1 \phi_1(a) + \ldots}}$.

And L the length of the pendulum $= -g . \frac{m\psi(a)^2 + m_1 \xi(a)^2 + \ldots}{m f_1(a) + m_1 \phi_1(a) + \ldots}$.

In the case of gravity, $Pdp = -gdy$, and $f(u) = -\frac{gdy}{du}$,

$$\text{and} \therefore\ L = \frac{m ds^2 + m_1 ds_1^2 + \ldots}{m d^2y + m_1 d^2y_1 + \ldots} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (5),$$

the position of equilibrium being determined from $m dy + m_1 dy_1 + \ldots = 0$.

If the body be rigid, and X, Y the co-ordinates of its centre of gravity, and Mk^2 the moment of inertia about the centre, and θ the angle of rotation be made the independent variable; then

$$m\frac{ds^2}{d\theta^2} + m_1 \frac{ds_1^2}{d\theta^2} + \ldots = Mk^2 + M . \frac{dX^2 + dY^2}{d\theta^2}$$

$$= Mk^2 + M\frac{dX^2}{d\theta^2},$$

$$\text{and} \therefore\ L = \frac{k^2 + \frac{dX^2}{d\theta^2}}{\frac{d^2Y}{d\theta^2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (6).$$

When a rigid body oscillates about a point of suspension, the expression $\frac{m ds^2 + m_1 ds_1^2 + \ldots}{m d^2y + m_1 d^2y_1 + \ldots}$ becomes $L = \frac{m\delta^2 + m_1 \delta_1^2 + \ldots}{(m + m_1 + \ldots) Y}$, the point of suspension being made the origin. .... (7).

The equation (4) for the purpose of integration may be more conveniently put under the form

$$\frac{dz^2}{dt^2} . (p + qz + rz^2) = a . (\beta^2 - z^2) + b . (\beta^3 - z^3) + c . (\beta^4 - z^4)$$

$$= (\beta - z) . \{(a + bz + cz^2 + c\beta^2) . (z + \beta) + b\beta^2\}.$$

Assume $(a + bz + cz^2 + c\beta^2) . (z + \beta) + b\beta^2 = (a + bz + cz^2 + c\beta^2) . (z + \beta + \delta\beta^2 + e\beta^2 z)$,

$$\therefore\ (a + bz + cz^2 + c\beta^2) . (\delta + ez) = b,$$

or, $a\delta + b\delta z + aez = b$, omitting the squares of β and z;

$$\therefore\ a\delta = b, \text{ and } b\delta + ae = 0,$$

$$\text{or, } \delta = \frac{b}{a}, \text{ and } e = -\frac{b^2}{a^2};$$

$$\text{and therefore } (a + bz + cz^2 + c\beta^2).(z + \beta + \frac{b}{a}\beta^2 - \frac{b^2}{a^2}\beta^2 z),$$

$$\text{or } (a + bz + cz^2 + c\beta^2).\left(1 + \frac{b}{a}\beta\right).\left\{\beta + z.\left(1 - \frac{b}{a}.\beta\right)\right\}$$

differs only from the factor $(a + bz + cz^2 + c\beta^2).(z + \beta) + b\beta^2$ by quantities of the fourth degree of β and z: or

$$\frac{dz^2}{dt^2}.(p + qz + rz^2) = (\beta - z).\left(1 + \frac{b}{a}\beta\right).\left\{\beta + z.\left(1 - \frac{b}{a}.\beta\right)\right\}.(a + bz + cz^2 + c\beta^2)$$

is true to the fourth order of those quantities; and the limits of the oscillation of the system are.

$$z = \beta, \text{ and } z = \frac{-\beta}{1 - \frac{b}{a}.\beta}, \text{ or } -\gamma.$$

Again, as $\beta = \gamma.\left(1 - \frac{b}{a}.\beta\right)$

$$\beta + z.\left(1 - \frac{b}{a}.\beta\right) = (\gamma + z).\left(1 - \frac{b}{a}.\beta\right), \text{ and}$$

$$\therefore \frac{dz^2}{dt^2}.(p + qz + rz^2) = (\beta - z).(\gamma + z).\left(1 - \frac{b^2}{a^2}.\beta^2\right).(a + bz + cz^2 + c\beta^2).$$

and if $a + c\beta^2 = \alpha$

$$-dt = \frac{dz}{\sqrt{(\beta - z).(\gamma + z)}}.\frac{1}{\sqrt{1 - \frac{b^2\beta^2}{a^2}}}.\sqrt{\frac{p + qz + rz^2}{\alpha + bz + cz^2}}.$$

The position of equilibrium must be a stable one, and therefore $\int mPdp + \int m_1P_1dp_1 + \ldots$ a maximum, or U_1 is negative, and $\therefore \alpha = a + c\beta^2 = -U_1 + c\beta^2$ is positive, and $\therefore$ also $\sqrt{\frac{p}{\alpha}}$ is positive.

Expanding therefore the last term of the above expression,

$$-dt = \frac{dz}{\sqrt{(\beta - z).(\gamma + z)}}.\frac{\sqrt{\frac{p}{\alpha}}}{\sqrt{1 - \frac{b^2\beta^2}{a^2}}}.\left\{1 + z.\left(\frac{q}{2p} - \frac{b}{2\alpha}\right) + z^2.\left(\frac{r}{2p} - \frac{q^2}{8p^2} - \frac{c}{2\alpha} - \frac{qb}{4p\alpha} + \frac{3b^2}{8\alpha^2}\right)\right\}.. \quad (8)$$

to be integrated between $z = \beta$ and $z = -\gamma$, excluding the powers of β above the second.

For this purpose, taking $\frac{z^n.dz}{\sqrt{(\beta - z).(\gamma + z)}}$, let $\sqrt{(\beta - z).(\gamma + z)} = (\beta - z).x$,

$$\therefore z = \frac{\beta x^2 - \gamma}{1 + x^2}, \text{ and } \frac{dz}{\sqrt{(\beta - z).(\gamma + z)}} = \frac{2dx}{1 + x^2},$$

$$\text{and } \frac{z^n dz}{\sqrt{(\beta - z).(\gamma + z)}} = \frac{(\beta x^2 - \gamma)^n.2dx}{(1 + x^2)^{n+1}} = \frac{2dx}{1 + x^2}.\left(\beta - \frac{\beta + \gamma}{1 + x^2}\right)^n$$

$$= \frac{2dx}{1+x^2} \cdot \left\{\beta^n - n \cdot \beta^{n-1} \cdot \frac{\beta+\gamma}{1+x^2} + n \cdot \frac{n-1}{2} \cdot \beta^{n-2} \cdot \frac{(\beta+\gamma)^2}{(1+x^2)^2}\right\}$$

the other terms vanishing as n is not greater than 2.

The limits of z being β and $-\gamma$, those of x are ∞ and 0, and

$$\int \frac{dx}{(1+x^2)^n} \text{ between these limits } = -\frac{(2n-3)\cdot(2n-5)\ldots 3 \cdot 1}{(2n-2)\cdot(2n-4)\ldots 4 \cdot 2} \cdot \frac{\pi}{2};$$

$$\therefore \int \frac{z^n dz}{\sqrt{(\beta-z)\cdot(\gamma+z)}} = -\pi \cdot \left\{\beta^n - n\beta^{n-1} \cdot \frac{\beta+\gamma}{2} + n \cdot \frac{n-1}{2} \cdot \beta^{n-2} \cdot (\beta+\gamma)^2 \cdot \frac{3}{4 \cdot 2}\right\}$$

$$= -\pi \cdot \{\beta^n - n\beta^{n-1} \cdot (\beta + \frac{b}{2a} \cdot \beta^2) + n \cdot \frac{n-1}{4} \cdot 3\beta^n\}$$

$$= -\pi\beta^n \cdot \left\{\left(\frac{3n}{4} - 1\right) \cdot (n-1) - \frac{nb\beta}{2a}\right\}.$$

$$\text{Hence } \int \frac{dz}{\sqrt{(\beta-z)\cdot(\gamma+z)}} = -\pi,$$

$$\int \frac{z\,dz}{\sqrt{(\beta-z)\cdot(\gamma+z)}} = \pi \cdot \frac{b\beta^2}{2a},$$

$$\int \frac{z^2 dz}{\sqrt{(\beta-z)\cdot(\gamma+z)}} = -\pi \cdot \frac{\beta^2}{2}:$$

$$\therefore t = \frac{\pi\sqrt{\frac{p}{a}}}{\sqrt{1-\frac{b^2\beta^2}{a^2}}} \cdot \left\{1 + \frac{\beta^2}{2} \cdot \left(\frac{r}{2p} - \frac{q^2}{8p^2} - \frac{c}{2a} - \frac{qb}{4pa} + \frac{3b^2}{8a^2} - \frac{qb}{2pa} + \frac{b^2}{2a^2}\right)\right\},$$

$$\text{and as } \frac{\sqrt{\frac{p}{a}}}{\sqrt{1-\frac{b^2\beta^2}{a^2}}} = \sqrt{\frac{p}{a}} \cdot \left(1 - \frac{c\beta^2}{2a}\right) \cdot \left(1 + \frac{b^2\beta^2}{2a^2}\right) = \sqrt{\frac{p}{a}} \cdot \left\{1 + \left(\frac{b^2}{2a^2} - \frac{c}{2a}\right) \cdot \beta^2\right\}$$

$$t = \pi\sqrt{\frac{p}{a}} \cdot \left\{1 + \frac{\beta^2}{4} \cdot \left(\frac{r}{p} - \frac{3c}{a} - \frac{q^2}{4p^2} - \frac{3qb}{2pa} + \frac{15b^2}{4a^2}\right)\right\},$$

or, restoring the quantities U and V, and their differential coefficients,

$$t = \pi\sqrt{\frac{V_0}{-U_1}} \cdot (1 + C\beta^2) \ldots\ldots (9),$$

$$\text{where } C = \frac{1}{8} \cdot \left(\frac{V_2}{V_0} - \frac{U_3}{2U_1} - \frac{1}{2} \cdot \frac{V_1^2}{V_0^2} - \frac{V_1 U_2}{V_0 U_1} + \frac{5}{6} \cdot \frac{U_2^2}{U_1^2}\right);$$

$$\text{and } L = \frac{gV_0}{-U_1} \ldots\ldots (10),$$

where L and t are expressed by quantities and their differentials.

The times of descent to the position of equilibrium and of ascent from it, will be found by the integration of equation (8) between the limits, $z = (\beta, 0)$, and $z = (0, -\gamma)$; but as the first powers of the arc will appear, it will be sufficient to integrate

$$-dt = \frac{dz}{\sqrt{(\beta - z)\cdot(\gamma + z)}} \cdot \sqrt{\frac{p}{a}} \cdot \left\{1 + \frac{z}{2} \cdot \left(\frac{q}{p} - \frac{b}{a}\right)\right\};$$

$$\therefore\ t = \sqrt{\frac{p}{a}} \cdot \left\{\left(\frac{q}{p} - \frac{b}{a}\right) \cdot \frac{\beta x}{1 + x^2} - 2\tan^{-1} x + C\right\}.$$

$$\text{Time of descent} = \sqrt{\frac{p}{a}} \cdot \left\{\left(\frac{q}{p} - \frac{b}{a}\right) \cdot \frac{\beta}{2} - 2\tan^{-1}\sqrt{\frac{\gamma}{\beta}} + \pi\right\},$$

$$\text{.........} \quad \text{ascent} = \sqrt{\frac{p}{a}} \cdot \left\{2\tan^{-1}\sqrt{\frac{\gamma}{\beta}} - \left(\frac{q}{p} - \frac{b}{a}\right) \cdot \frac{\beta}{2}\right\};$$

$$\text{but } \tan^{-1}\sqrt{\frac{\gamma}{\beta}} = \tan^{-1}\sqrt{1 + \frac{b}{a} \cdot \beta} = \frac{\pi}{4} + \frac{b\beta}{4a};$$

$$\therefore\ \text{time of descent} = \sqrt{\frac{p}{a}} \cdot \left\{\frac{\pi}{2} + \left(\frac{q}{2p} - \frac{b}{a}\right) \cdot \beta\right\} \quad \text{..............} \quad (11),$$

$$\text{.........} \quad \text{ascent} = \sqrt{\frac{p}{a}} \cdot \left\{\frac{\pi}{2} - \left(\frac{q}{2p} - \frac{b}{a}\right) \cdot \beta\right\} \quad \text{..............} \quad (12).$$

Excess of time of ascent over the time of descent, or

$$E = \sqrt{\frac{p}{a}} \cdot \left(\frac{2b}{a} - \frac{q}{p}\right) \cdot \beta = \sqrt{\frac{V_0}{-U_1}} \cdot \left(\frac{2U_2}{3U_1} - \frac{V_1}{V_0}\right) \cdot \beta \text{.........} \quad (13),$$

which is remarkable as not involving π.

The excess of the arc or angle of ascent $= \gamma - \beta = \frac{b}{a} \cdot \beta^2 = \frac{U_2}{3U_1} \cdot \beta^2$. ...... (14).

These results are on the supposition that the displacement of the system was by an increase of the independent variable; in the opposite case, the odd differential coefficients of V and the even ones of U must have their signs changed.

Example. Two bodies m and m_1, moving in a circle and connected by a rod subtending an angle $4a$ at the centre are acted upon by a repulsive force in the circumference, varying as the n^{th} power of the distance.

Let 2θ be the angle at the centre between the radii passing through S the centre of force and m, $\therefore\ 2\theta + 2\theta_1 = 4a$,

$$P = k \cdot (2a\sin\theta)^n, \quad \text{and} \quad p = 2a\sin\theta;$$

$$\therefore\ U = \frac{Pdp + P_1 dp_1}{d\theta}, \quad \text{or} \quad \frac{U}{k \cdot (2a)^{n+1}} = m\sin^n\theta\cos\theta - m_1\sin^n\theta_1\cos\theta_1.$$

$$V = \frac{m\,ds^2 + m_1\,ds_1^2}{d\theta^2} = 4a^2(m + m_1).$$

If the bodies are equal, $V = 8ma^2$,

$$U_1 = 8ma^2k \cdot (2a\sin a)^{n-1} \cdot (n\cos^2 a - \sin^2 a),$$

$$\text{and } L = \frac{g}{k \cdot (2a\sin a)^{n-1} \cdot (\sin^2 a - n\cos^2 a)},$$

$$\text{and } t = \pi\sqrt{\frac{L}{g}} \cdot \left\{1 - \frac{\Delta a^2}{256} \cdot \left[(n+1)^2 - \frac{(n-1)\cdot(n-2)}{\sin^2 a} + \frac{2n-2}{n\cos^2 a - \sin^2 a}\right]\right\}.$$

where Δa is the whole angle of oscillation.

When the body is rigid, the general expressions may be put under more convenient forms: for if the differential coefficients be taken with respect to the angle of rotation round the centre of gravity, X and Y being its co-ordinates, and Mk^2 the moment of inertia round the centre,

$$V = Mk^2 + M(X_1^2 + Y_1^2),$$
$$V_1 = 2M(X_1X_2 + Y_1Y_2),$$
$$V_2 = 2M(X_2^2 + X_1X_3 + Y_2^2 + Y_1Y_3).$$

$$\text{And } U = -MY_1,$$
$$U_1 = -MY_2,$$
$$U_2 = -MY_3,$$
$$U_3 = -MY_4;$$

$$\therefore\ V_0 = M(k^2 + X_1^2),\quad V_1 = 2MX_1X_2,\quad \text{and } V_2 = 2M(X_2^2 + X_1X_3 + Y_2^2);$$

$$\therefore\ L = \frac{k^2 + X_1^2}{Y_2} \quad \ldots\ldots (15),$$

$$\left.\begin{aligned} C = \frac{X_2^2 + X_1X_3 + Y_2^2}{4(k^2 + X_1^2)} - \frac{Y_4}{16Y_2} - \frac{1}{4}\cdot\frac{X_1^2X_2^2}{(k^2 + X_1^2)^2} \\ - \frac{X_1X_2Y_3}{4Y_2(k^2 + X_1^2)} + \frac{5}{48}\cdot\frac{Y_3^2}{Y_2^2}. \end{aligned}\right\} \ldots\ldots (16).$$

In the case of a particle, $L = \dfrac{X_1^2}{Y_2}$,

$$\text{and } C = \frac{X_3}{4X_1} + \frac{Y_2^2}{4X_1^2} - \frac{Y_4}{16Y_2} - \frac{X_2Y_3}{4X_1Y_2} + \frac{5}{48}\cdot\frac{Y_3^2}{Y_2^2} \ldots\ldots (17).$$

Example. A rod oscillates upon two planes, inclined at the angles α and α_1 to the horizon: the centre of gravity being at the distances a and a_1 from the extremities of the rod.

$$\text{Here } X = A\sin\theta + B\cos\theta,$$
$$Y = M\sin\theta + N\cos\theta,$$

where θ is the inclination of the rod to the horizon, and

$$A = \frac{(a + a_1)\,.\cos\alpha\cos\alpha_1}{\sin(\alpha + \alpha_1)},\quad B = \frac{a\sin\alpha\cos\alpha_1 - a_1\cos\alpha\sin\alpha_1}{\sin(\alpha + \alpha_1)},$$

$$M = \frac{a\cos\alpha\sin\alpha_1 - a_1\sin\alpha\cos\alpha_1}{\sin(\alpha + \alpha_1)},\quad \text{and } N = \frac{(a + a)_1\,.\sin\alpha\sin\alpha_1}{\sin(\alpha + \alpha_1)};$$

$$\therefore\ Y_1 = M\cos\theta - N\sin\theta = 0,$$

$$\sin\theta = -\frac{M}{\sqrt{M^2 + N^2}},\quad \text{and } \cos\theta = -\frac{N}{\sqrt{M^2 + N^2}}.$$

$$\text{Let } M^2 + N^2 = \frac{(a + a_1)\,.(a\sin^2\alpha_1 + a_1\sin^2\alpha)}{\sin^2(\alpha + \alpha_1)} - aa_1 = Q,$$

$$\text{and } AM + BN = \frac{(a + a_1)\,.(a\sin 2\alpha_1 - a_1\sin 2\alpha)}{2\sin^2(\alpha + \alpha_1)} = P,$$

$$\left.\begin{aligned} \text{then } X_1 = -X_3 = \frac{aa_1}{\sqrt{Q}} \\ Y_2 = -Y_4 = \sqrt{Q} \\ X_2 = \frac{P}{\sqrt{Q}} \end{aligned}\right\},\quad \text{and } L = \frac{a^2a_1^2 + Qk^2}{Q^{\frac{3}{2}}};$$

$$\text{and } T = \pi\sqrt{\frac{L}{g}}.\left\{1 + \beta^2.\left[\frac{Q^2 - a^2 a_1^2}{4(k^2 Q^2 + a^2 a_1^2)} + \frac{1}{16} + \frac{k^2 P^2 Q}{4(k^2 Q^2 + a^2 a_1^2)^2}\right]\right\}.$$

If the planes include a right angle, and the centre of gravity be in the middle of the rod,

$$L = a + \frac{k^2}{a},$$

$$T = \pi\sqrt{\frac{L}{g}}.\left(1 + \frac{\beta^2}{16}\right),$$

both of which are independent of the inclination of the planes to the horizon.

If a particle move in a curve, by a constant force (g), making a given angle ϕ with the axis of x; then,

$$U = g\frac{dp}{du} = g.\frac{\cos\phi dx + \sin\phi dy}{du} = g.\sin(\phi - \theta).\frac{ds}{du},$$

$$V = \frac{ds^2}{du^2},$$

where θ is the angle made by the normal with the axis of x; and making this the variable,

$V = R^2$, where R is the radius of curvature;

$V_1 = 2RR_1$,

$V_2 = 2R_1^2 + 2RR_2$,

$U = g.\sin(\phi - \theta).R$,

$U_1 = -g\cos(\phi - \theta).R + g.\sin(\phi - \theta).R_1$,

$U_2 = -g.\sin(\phi - \theta).R - 2g\cos(\phi - \theta).R_1 + g.\sin(\phi - \theta).R_2$,

$U_3 = g\cos(\phi - \theta).R - 3g\sin(\phi - \theta).R_1 - 3g\cos(\phi - \theta).R_2 + g.\sin(\phi - \theta).R_3$

In the position of equilibrium $U = 0$, or $\phi - \theta = 0$;

$$\therefore U_1 = -gR,$$
$$U_2 = -2gR_1,$$
$$U_3 = -gR - 3gR_2.$$

Hence $L = \dfrac{gV}{-U_1} = R$, and

$$T = \pi\sqrt{\frac{R}{g}}.\left\{1 + \Delta\theta^2.\left(\frac{1}{16} + \frac{R_2}{16R} - \frac{R_1^2}{12R^2}\right)\right\}\text{.........} \quad (18).$$

$$\text{Excess of time of ascent} = -\frac{2R_1.\Delta\theta}{3\sqrt{gR}}.$$

$$\text{Excess of angle of ascent} = \frac{2R_1.\Delta\theta^2}{3R}.$$

If the arc be made the independent variable,

$$T = \pi\sqrt{\frac{L}{g}}.\left\{1 + \Delta s^2.\left(\frac{1}{16R^2} + \frac{R_2}{16R} - \frac{R_1^2}{48R^2}\right)\right\}\text{.........} \quad (19).$$

$$\text{Excess of time} = -\frac{2R_1.\Delta s}{3\sqrt{gR}}.$$

$$\text{Excess of arc} = \frac{-R_1.\Delta s^2}{3R}.$$

This last result compared with the former, shews that an increase of the angle of vibration is attended with a diminution of the arc, and *vice versa*.

Example. In an ellipse, $R = \dfrac{b^2}{a(1 - e^2 \sin\phi)^{\frac{3}{2}}}$,

$$R_1 = \frac{3b^2e^2}{a} . \sin\theta . \cos\theta . (1 - e^2\sin^2\theta)^{-\frac{5}{2}},$$

$$R_2 = \frac{3b^2e^2}{a} . (1 - e^2\sin^2\theta)^{-\frac{7}{2}} . (\cos^2\theta + e^2\sin^4\theta - \sin^2\theta + 4e^2\sin^2\theta . \cos^2\theta),$$

and by substituting these values in (18),

$$T = \pi\sqrt{\frac{R}{g}} . \left\{1 + \frac{\Delta\theta^2}{16} . \left[4 - 3 . \left(\frac{R^2}{ab}\right)^{\frac{2}{3}}\right]\right\}.$$

If the ellipse become a circle, $R^2 = ab$,

$$\text{and } T = \pi\sqrt{\frac{R}{g}} . \left(1 + \frac{\Delta\theta^2}{16}\right).$$

If the axis of a cycloid be inclined at θ to the vertical,

$$R = 2a\cos\theta,$$
$$R_1 = -2a\sin\theta,$$
$$R_2 = -2a\cos\theta.$$
$$L = 2a\cos\theta,$$
$$T = \pi\sqrt{\frac{2a\cos.\theta}{g}} . \left(1 - \frac{\Delta\theta^2 . \tan^2\theta}{12}\right).$$

$$\text{Increase of angle of ascent} = -\frac{2}{3} . \tan\theta . \Delta\theta^2.$$

The time of oscillation in a cycloid therefore decreases, as the arc increases, when the axis is not vertical.

If a central force $kf(\delta)$, varying according to any law act on a particle in a given curve, the co-ordinates of the centre of force being α, β; then, taking the arc for the independent variable,

$$U = kf(\delta) . \frac{d\delta}{ds} = k . \frac{f(\delta)}{2\delta} . \frac{d . \delta^2}{ds}.$$

$$\text{Let } \delta^2 = z, \text{ and } \frac{f(\delta)}{2\delta} = \phi(z);$$

$\therefore$ $U = k . \phi(z) . z_1 = 0$, at the position of equilibrium,

$$U_1 = k . \phi(z) . z_2,$$
$$U_2 = k . \phi(z) . z_3,$$
$$U_3 = 3k\phi_1(z) . z_2^2 + k\phi(z) . z_4;$$

but $z = (x - \alpha)^2 + (y - \beta^2)$;

$\therefore$ $z_1 = 2 . (x - \alpha) . \dfrac{dx}{ds} + 2 . (y - \beta) . \dfrac{dy}{ds}$, and if θ be the angle made by the normal with the axis of x, $\dfrac{dx}{ds} = \sin\theta$ and $\dfrac{dy}{ds} = \cos\theta$;

$$\therefore z_1 = 2 . (x - \alpha . \sin\theta + 2 . (y - \beta) . \cos\theta,$$

$$z_2 = 2 + 2 . \{(x-a) . \cos\theta - (y-\beta) . \sin\theta)\} . \frac{d\theta}{ds}$$

$$= 2 + 2 . \{(x-a) . \cos\theta - (y-\beta) . \sin\theta\} . R^{-1},$$

$$z_3 = -2 . \{(x-a) . \sin\theta + (y-\beta) . \cos\theta\} . R^{-2} - 2 . \{(x-a) . \cos\theta - (y-\beta) . \sin\theta) . R^{-2} . R_1,$$

$$z_4 = -2R^{-2} - 2 . \{(x-a) . \cos\theta - (y-\beta) . \sin\theta\} . (R^{-3} + R^{-2} . R_2 - 2R^{-3}R_1^2)$$

omitting the terms which vanish; but since $U = 0$, $(y-\beta) . \cos\theta + (x-a) . \sin\theta = 0$,

and therefore $(x-a) . \cos\theta - (y-\beta) . \sin\theta = -\delta$.

Hence $$\frac{z_2}{2} = 1 - \frac{\delta}{R},$$

$$\frac{z_3}{2} = \delta . \frac{R_1}{R^2},$$

$$\frac{z_4}{2} = \frac{\delta - R}{R^3} + \frac{\delta . (RR_2 - 2R_1^2)}{R^3}.$$

Also $V = \dfrac{ds^2}{ds^2} = 1$, and $V_1 = V_2 = 0$;

$$\therefore L = \frac{g}{-U_1} = \frac{R\delta . g}{kf(\delta) . (\delta - R)} \text{ .... } (20).$$

$$T = \pi\sqrt{\frac{L}{g}} . \left\{1 + \frac{\Delta s^2}{16} . \left[\frac{5z_3^2}{3z_2^2} - \frac{z_4}{z_2} - 3z_2 . d_z . \log\phi(z)\right]\right\},$$

or $$T = \pi\sqrt{\frac{L}{g}} . \left\{1 + \frac{\Delta s^2}{16} . \left[\frac{1}{R^2} + \frac{\delta R_2}{R\delta - R^2} + \frac{R_1^2\delta . (6R - \delta)}{3(R\delta - R^2)^2} + \frac{3 . (\delta - R)}{R\delta} . d_\delta . \log\frac{f(\delta)}{\delta}\right]\right\} \text{.... } (21)$$

If the force vary as the distance, $d_\delta . \log\dfrac{f(\delta)}{\delta} = 0$, and the force does not affect the correction of the expression for the time.

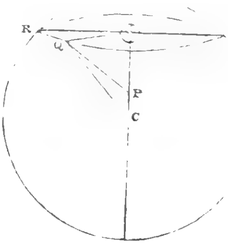
The excess of the time of ascent $= -\dfrac{2R_1}{3} . \dfrac{\delta^{\frac{3}{2}} . \Delta s}{(\delta - R)^{\frac{3}{2}} . \sqrt{kRf(\delta)}}$;

.... arc ...... $= \dfrac{-\delta R_1}{R\delta - R^2} . \dfrac{\Delta s^2}{3}$;

.... angle ...... $= \dfrac{2\delta - 3R}{R\delta - R^2} . d_\theta . R . \dfrac{\Delta\theta}{3}$.

If the force be constant and act in parallel lines, δ is infinite, and $f(\delta)$ constant, and formula (21) becomes the same as (19).

(1). To find the time of oscillation of a particle in the centre of a hollow sphere, the force varying as the n^{th} power of the distance, and the density being $= \mu . r^m$.



Let $QOR = \phi$ and $QCP = \theta$; then the volume of a particle at $Q = r^2 dr \sin\theta . d\theta . d\phi$, and its force on the particle at P, where $CP = x$ and $QP = p$, is

$$\mu . r^{m+2} dr . \sin\theta . d\theta . d\phi . p^n;$$

$$\therefore U = \frac{Pdp}{dx} + \ldots = \mu . r^{m+2} dr . \sin\theta . d\theta . d\phi . p^n \frac{dp}{dx} + \ldots$$

$= \mu . r^{m+2} dr . \sin\theta . d\theta . d\phi . p^{n-1} . (x - r . \cos\theta) + \ldots$ since $p^2 = r^2 + x^2 - 2rx\cos\theta.$

Let $k = \mu r^{m+2} dr . \sin\theta d\theta . d\phi$;

$\therefore\ U = kp^{n-1} . (x - \cos\theta) + \ldots$

$U_1 = k . (n-1) . p^{n-3} . (x - r . \cos\theta)^2 + kp^{n-1} + \ldots$

$= \mu . r^{m+n+1} dr \sin\theta . d\theta . d\phi \{(n-1) . \cos^2\theta + 1\}$, when $x = 0$;

which, being integrated from $\theta = 0$ to $\theta = \pi$; from $\phi = 0$ to $\phi = 2\pi$, and from $r = r_1$ to $r = r_2$, we have finally

$$U_1 = \frac{4\mu\pi . (n+2)}{3 . (m+n+2)} . (r_2^{m+n+2} - r_1^{m+n+2}),$$

and $$T = \frac{\pi}{\sqrt{U_1}} = \sqrt{\frac{3\pi . (m+n+2)}{4\mu . (n+2) . (r_2^{m+n+2} - r_1^{m+n+2})}};$$

or if M be the mass of the hollow sphere,

$$T = \pi . \sqrt{\frac{3 . (m+n+2) . (r_2^{m+3} - r_1^{m+3})}{(n+2) . (m+3) . M . (r_2^{m+n+2} - r_1^{m+n+2})}}.$$

If the force varies as the distance, $T = \dfrac{\pi}{\sqrt{M}}$.

If the force varies inversely as the square, T is infinite.

If $m + n + 2 = 0$, $\dfrac{r_2^{m+n+2} - r_1^{m+n+2}}{m+n+2} = \log\left(\dfrac{r_2}{r_1}\right)$.

And if $m + 3 = 0$, $$T = \pi\sqrt{\frac{3 . (n-1) . \log\dfrac{r_2}{r_1}}{(n+2) . M . (r_2^{n-1} - r_1^{n-1})}}.$$

(2). To find the correction for the time of oscillation, we have

$$U_3 = k . (n-1) . (n-3) . (n-5) . p^{n-7} . (x - r . \cos\theta)^4 + 6k . (n-1) . (n-3) . p^{n-5} . (x - r\cos\theta)^2 + 3k . (n-1) . p^{n-3},$$

or, making $x = 0$, there results for the attracting particle

$$U_3 = \mu . (n-1) . r^{m+n-1} dr . \sin\theta . d\theta . d\phi [(n-3) . (n-5) . \cos^4\theta + 6 . (n-3) . \cos^2\theta + 3],$$

which, integrated between $\theta = (0, \pi)$, is $2\mu . \dfrac{n . (n-1) . (n+2)}{5} . r^{m+n-1} dr d\phi$,

and again between $\phi = (0, 2\pi)$ and $r = (r_1, r_2)$ is for the hollow sphere

$$U_3 = \frac{4\mu . \pi . n . (n-1) . (n+2)}{5 . (m+n)} . (r_2^{m+n} - r_1^{m+n}),$$

$$\therefore\ T = \pi . \sqrt{\frac{3 . (m+n+2) . (r_2^{m+3} - r_1^{m+3})}{(n+2) . (m+3) . M . (r_2^{m+n+2} - r_1^{m+n+2})}} . \left\{1 - \frac{3\beta^2}{80} . \frac{n . (n-1) . (m+n+2)}{m+n} . \frac{r_2^{m+n} - r_1^{m+n}}{r_2^{m+n+2} - r_1^{m+n+2}}\right\}.$$

If the sphere be solid and density uniform, and the force inversely as the square of the distance, $T = \sqrt{\dfrac{3\pi}{4\mu}}$.

If r equal forces $2k\delta . \phi(\delta^2)$, be placed in the angles of a regular polygon, the time of oscillation of a particle at the centre will be found to be

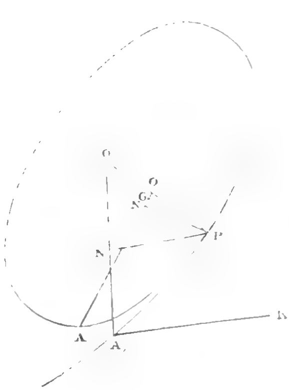
$$= \frac{\pi}{\sqrt{-2kr . \{\phi(a^2) + \phi_1(a^2) . a^2\}}} . \left\{1 - \frac{3\beta^2}{16} . \frac{2\phi_1(a^2) + 4a^2\phi_2(a^2) + a^4\phi_3(a^2)}{\phi(a^2) + a^2\phi_1(a^2)}\right\},$$

and as kr is the whole quantity of attracting matter, the time is the same while kr is, and therefore if the matter be in the form of a ring, the above is still the time of oscillation.

If the force $= k . \delta^n$, then $\phi(\delta^2) = \frac{\delta^{n-1}}{2}$, and

$$T = \frac{\pi\sqrt{2}}{\sqrt{-rk . (n+1) . a^{n-1}}} . \left\{1 - \frac{3\beta^2 . (n-1)^2}{64a^2}\right\}.$$

On Rocking Bodies.



In the position of equilibrium, the centre G of the rocking body, will be in the same vertical line as the point of support; that is, when A is at A_1, AG will be vertical.

Let $AG = a$, $AN = y$, $NP = x$,

$A_1N_1 = y_1$, $N_1P = x_1$,

and PO being a common normal,

let $AOP = \theta$, and $A_1O_1P = \theta_1$;

then ϕ the angle rocked through $= \theta - \theta_1$;

and if X, Y be the co-ordinates of G, measured from A_1,

$$\left.\begin{aligned} X &= x_1 - x . \cos\phi + (a-y) . \sin\phi \\ Y &= y_1 + x . \sin\phi + (a-y) . \cos\phi \end{aligned}\right\} \ldots\ldots (22),$$

also, $\sin\theta = \frac{dy}{ds}$, $\cos\theta = \frac{dx}{ds}$, and $ds = ds_1$;

$$\therefore \frac{dX}{d\phi} = \frac{dx_1}{d\phi} - \frac{dx}{d\phi} . \cos\phi + x . \sin\phi - \frac{dy}{d\phi} . \sin\phi + (a-y) . \cos\phi$$

$$= \cos\theta_1 . \frac{ds}{d\phi} - \cos\theta . \cos\phi \frac{ds}{d\phi} + x . \sin\phi - \sin\theta \sin\phi . \frac{ds}{d\phi} + (a-y) . \cos\phi$$

$$= x . \sin\phi + (a-y) . \cos\phi,$$

$$\frac{dY}{d\phi} = \frac{dy_1}{d\phi} + \frac{dx}{d\phi} . \sin\phi + x . \cos\phi - \frac{dy}{d\phi} . \cos\phi - (a-y) . \sin\phi$$

$$= \sin\theta_1 . \frac{ds}{d\phi} + \cos\theta . \sin\phi . \frac{ds}{d\phi} + x . \cos\phi - \sin\theta . \cos\phi . \frac{ds}{d\phi} - (a-y) . \sin\phi$$

$$= x\cos\phi - (a-y) . \sin\phi;$$

$$\left.\begin{aligned} \text{or } \frac{dX}{d\phi} &= Y - y_1 \\ \frac{dY}{d\phi} &= x_1 - X \end{aligned}\right\} \ldots\ldots (23).$$

To find X, Y and their differential coefficients with respect to ϕ; we have, from (22)

$$X = 0,$$

$$Y = a.$$

From (23), $X_1 = Y = a,$

$$Y_1 = 0,$$

$$X_2 = Y_1 - \frac{dy_1}{d\phi} = -\frac{dy_1}{d\phi},$$

$$Y_2 = \frac{dx_1}{d\phi} - X_1 = \frac{dx_1}{d\phi} - a,$$

$$X_3 = Y_2 - \frac{d^2y_1}{d\phi^2} = \frac{dx_1}{d\phi} - a - \frac{d^2y_1}{d\phi^2},$$

$$Y_3 = \frac{d^2x_1}{d\phi^2} - X_2 = \frac{d^2x_1}{d\phi^2} + \frac{dy_1}{d\phi},$$

$$Y_4 = \frac{d^3x_1}{d\phi^3} - X_3 = \frac{d^3x_1}{d\phi^3} - \frac{dx_1}{d\phi} + a + \frac{d^2y_1}{d\phi^2},$$

and it will be necessary to express $\frac{dx_1}{d\phi}$, $\frac{dy_1}{d\phi}$, &c. in terms derivable from the separate curves; let R and r be their radii of curvature at P;

$$\therefore\ R = \frac{ds_1}{d\theta_1}, \text{ and } r = \frac{ds}{d\theta},$$

$$\text{let } R_1 = \frac{d^2s_1}{d\theta_1^2},\quad r_1 = \frac{d^2s}{d\theta^2}, \text{ \&c.}$$

then $\frac{dy_1}{d\phi} = \sin\theta_1 \cdot \frac{ds_1}{d\phi}$,

$$\frac{d^2y_1}{d\phi^2} = \cos\theta_1 \cdot \frac{d\theta_1}{d\phi}\cdot\frac{ds_1}{d\phi} + \sin\theta_1 \cdot \frac{d^2s_1}{d\phi^2},$$

$$\frac{dx_1}{d\phi} = \cos\theta_1 \cdot \frac{ds_1}{d\phi},$$

$$\frac{d^2x_1}{d\phi^2} = -\sin\theta_1 \cdot \frac{d\theta_1}{d\phi}\cdot\frac{ds_1}{d\phi} + \cos\theta_1 \cdot \frac{d^2s_1}{d\phi^2},$$

$$\frac{d^3x_1}{d\phi^3} = -\cos\theta_1 \cdot \frac{d\theta_1^2}{d\phi^2}\cdot\frac{ds_1}{d\phi} - \sin\theta_1 \cdot \frac{d^2\theta_1}{d\phi^2}\cdot\frac{ds_1}{d\phi} - 2\sin\theta_1 \cdot \frac{d\theta_1}{d\phi}\cdot\frac{d^2s_1}{d\phi^2} + \cos\theta_1 \cdot \frac{d^3s_1}{d\phi^3}.$$

Now $d\phi = d\theta - d\theta_1 = d\theta - d\theta \cdot \frac{r}{R}$;

$$\therefore\ \frac{d\theta}{d\phi} = \frac{R}{R-r}$$

$$\frac{d\theta_1}{d\phi} = \frac{r}{R-r}.$$

Then $\frac{ds}{d\phi} = \frac{ds}{d\theta}\cdot\frac{d\theta}{d\phi} = \frac{Rr}{R-r}$

$$\frac{d^2s}{d\phi^2} = \frac{R^2 \cdot \frac{dr}{d\phi} - r^2 \cdot \frac{dR}{d\phi}}{(R-r)^2} = \frac{R^2 \cdot \frac{dr}{d\theta}\cdot\frac{d\theta}{d\phi} - r^2 \cdot \frac{dR}{d\theta_1}\cdot\frac{d\theta_1}{d\phi}}{(R-r)^2}$$

$$= \frac{R^3 r_1 - r^3 R_1}{(R-r)^3}$$

$$\frac{d^3 s}{d\phi^3} = \frac{R^3 r_2 \cdot \dfrac{d\theta}{d\phi} - r^3 R_2 \cdot \dfrac{d\theta_1}{d\phi}}{(R-r)^3} + 3 R_1 r_1 \cdot \frac{R^2 \cdot \dfrac{d\theta_1}{d\phi} - r^2 \cdot \dfrac{d\theta}{d\phi}}{(R-r)^3}$$

$$- 3 \cdot \frac{(R^3 \cdot r_1 - r^3 R_1) \cdot \left(R_1 \cdot \dfrac{d\theta_1}{d\phi} - r_1 \cdot \dfrac{d\theta}{d\phi}\right)}{(R-r)^4}$$

$$= \frac{R^4 r_2 - r^4 R_2}{(R-r)^4} + 3 \cdot \frac{(R^2 r_1 - r^2 R_1)^2}{(R-r)^5}.$$

If then, a be the angle, the common normal makes with the vertical in the position of equilibrium; since $s = s_1$, there results,

$$\frac{dy_1}{d\phi} = \sin a \cdot \frac{Rr}{R-r}$$

$$\frac{d^2 y_1}{d\phi^2} = \cos a \cdot \frac{Rr^2}{(R-r)^2} + \sin a \cdot \frac{R^3 r_1 - r^3 R_1}{(R-r)^3}$$

$$\frac{dx_1}{d\phi} = \cos a \cdot \frac{Rr}{R-r}$$

$$\frac{d^2 x_1}{d\phi^2} = -\sin a \cdot \frac{Rr^2}{(R-r)^2} + \cos a \cdot \frac{R^3 r_1 - r^3 R_1}{(R-r)^3}$$

$$\frac{d^3 x_1}{d\phi^3} = -\cos a \cdot \frac{Rr^3}{(R-r)^3} - \sin a \cdot Rr \cdot \frac{R^2 r_1 - r^2 R_1}{(R-r)^4} - 2 \sin a \cdot r \cdot \frac{R^3 r_1 - r^3 R_1}{(R-r)^4}$$

$$+ \cos a \cdot \frac{R^4 r_2 - r^4 R_2}{(R-r)^4} + 3 \cos a \cdot \frac{(R^2 r_1 - r^2 R_1)^2}{(R-r)^5}$$

which values being substituted above, we have finally

$$X_1 = a, \quad X_2 = -\sin a \cdot \frac{Rr}{R-r},$$

$$Y_1 = 0, \quad Y_2 = \cos a \frac{Rr}{R-r} - a,$$

$$X_3 = \cos a \cdot Rr \cdot \frac{R-2r}{(R-r)^2} - a - \sin a \cdot \frac{R^3 r_1 - r^3 R_1}{(R-r)^3},$$

$$Y_3 = \sin a \cdot Rr \cdot \frac{R-2r}{(R-r)^2} + \cos a \cdot \frac{R^3 r_1 - r^3 R_1}{(K-r)^3},$$

$$Y_4 = a - \cos a \cdot \frac{Rr}{R-r} + Rr^2 \cos a \cdot \frac{R-2r}{(R-r)^3} + \sin a \cdot \frac{R^3 r_1}{(R-r)^3},$$

$$+ 3r \cdot \sin a \cdot \frac{r^3 R_1 - R^3 r_1}{(R-r)^4} + \cos a \cdot \frac{R^4 r_2 - r^4 R_2}{(R-r)^4} + 3 \cos a \cdot \frac{(R^2 r_1 - r^2 R_1)^2}{(R-r)^5}.$$

$$\text{Length of pendulum} = \frac{k^2 + a^2}{\cos a \cdot \dfrac{Rr}{R-r} - a} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (24),$$

and $T = \pi \sqrt{\frac{L}{g}} \cdot (1 + C \cdot \Delta\phi^2)$, where C is to be determined by the substitution of X, Y, &c. in (16).

If the pendulum be suspended by a point, r and its differential coefficients vanish, and

$$X_1 = -X_3 = -Y_2 = Y_4 = a, \text{ and } \therefore\ C = \frac{1}{16},$$

$$L = a + \frac{k^2}{a}, \text{ measuring } a \text{ downwards, and}$$

$$T = \pi \sqrt{\frac{L}{g}} \cdot \left(1 + \frac{\Delta\phi^2}{16}\right).$$

$$\text{If } \alpha = 0, \quad C = \frac{1}{16} - \frac{R r^2 \cdot (R - 2r)}{16(R-r)^2 \cdot \{Rr - (R-r) \cdot a\}} - \frac{R^4 r_2 - r^4 R_2}{16(R-r)^3 \cdot \{Rr - (R-r) \cdot a\}} \ldots (25),$$

$$- \frac{3(R^2 r_1 - r^2 R_1)^2}{16(R-r)^4 \cdot \{Rr - (R-r) \cdot a\}} + \frac{R^2 r \cdot (r - a)}{4(R-r)^2 \cdot (k^2 + a^2)} + \frac{5}{48} \cdot \frac{(R^3 r_1 - r^3 R_1)^2}{(R-r)^4 \cdot \{Rr - (R-r) \cdot a\}^2}.$$

If R and r be constant, or the curves be circles,

$$C = \frac{R^2 r \cdot (r - a\cos\alpha)}{4(R-r)^2 \cdot (k^2 + a^2)} + \frac{1}{16} + \frac{\cos\alpha \cdot (2r^3 R - r^2 R^2)}{16(R-r)^2 \cdot \{Rr\cos\alpha - (R-r) \cdot a\}} - \frac{a^2 \sin^2\alpha\, R^2 r^2}{4(R-r)^2 \cdot (k^2 + a^2)^2}$$

$$+ \frac{a \sin^2\alpha \cdot R^2 r^2 \cdot (R - 2r)}{4(R-r)^3 \cdot \{Rr\cos\alpha - (R-r) \cdot a\} \cdot (k^2 + a^2)} + \frac{5}{48} \cdot \frac{\sin^2\alpha \cdot (R^2 r - 2r^2 R)^2}{(R-r)^4 \cdot \{\cos\alpha \cdot Rr - (R-r) \cdot a\}^2} \ldots (26).$$

If R and r be constant, and also $\alpha = 0$;

$$C = \frac{1}{16} + \frac{R^2 r \cdot (r - a)}{4(R-r)^2 \cdot (k^2 + a^2)} + \frac{R r^2 \cdot (2r - R)}{16(R-r)^2 \cdot \{Rr - (R-r) \cdot a\}} \ldots\ldots\ldots\ldots (27).$$

Ex. One sphere within another,

$$L = \frac{7}{5} \cdot (R - r)$$

$$T = \pi \sqrt{\frac{L}{g}} \cdot \left(1 + \beta^2 \cdot \frac{r^2}{16(R-r)^2}\right).$$

Ex. If an ellipse whose semiaxis a is horizontal rock within another ellipse whose semiaxis a_1 is also horizontal,

$$L = \frac{(k^2 + b^2) \cdot (a_1^2 b - a^2 b_1)}{a^2 a_1^2 - a_1^2 b^2 + a^2 b b_1}.$$

$$\text{And } r = \frac{a^2 b^2}{(b^2 + a^2 e^2 \sin^2\theta)^{\frac{3}{2}}}; \ \therefore\ r_0 = \frac{a^2}{b}, \quad r_1 = 0, \quad r_2 = -\frac{3a^4 e^2}{b^3};$$

$$\therefore\ C = \frac{a^4 a_1^4 e^2}{4(a_1^2 b - a^2 b_1)^2 \cdot (k^2 + b^2)} + \frac{1}{16} + \frac{a^4 a_1^2 b b_1 \cdot (a_1^2 - a^2)}{16(a_1^2 b - a^2 b_1)^2 \cdot (b^2 a_1^2 - a^2 b b_1 - a^2 a_1^2)};$$

$$+ \frac{3a^4 a_1^4}{16} \cdot \frac{a^4 e_1^2 b_1 - a_1^4 e^2 b}{(a_1^2 b - a^2 b_1)^3 \cdot (a_1^2 b^2 - a^2 b b_1 - a^2 a_1^2)}.$$

If the bowl becomes a plane,

$$L = \frac{(k^2 + b^2) \cdot b}{a^2 - b^2}.$$

$$T = \pi\sqrt{\frac{L}{g}}\cdot\left\{1 + \Delta\phi^2\cdot\left(\frac{a^4 e^2}{4b^2\,.\,(k^2+b^2)} + \frac{1}{16} + \frac{3}{16}\cdot\frac{a^2}{b^2}\right)\right\}.$$

If a body be suspended by an axis whose radius is r on a circular support, whose radius is R: and a_1 be the distance of the centre of gravity below the point of support,

$$L = \frac{k^2 + a^2}{\cos a\,.\,\dfrac{Rr}{R-r} + a};$$

and, if the pendulum be suspended on another axis, the radius of which is r_1, and be isochronous with respect to these axes;

$$L = \frac{a_1^2 - a^2}{\cos a\,.\,\dfrac{Rr_1}{R-r_1} - \cos a\,.\,\dfrac{Rr}{R-r} + a_1 - a},$$

and if the axes are equal, $L = \dfrac{a_1^2 - a^2}{a_1 - a} = a_1 + a$,

$$\text{and } k^2 = (a + a_1)\,.\,\frac{Rr\cos a}{R-r} + aa_1,$$

and therefore if Kater's pendulum be supported on a concave or convex surface, the length is independent of the curvature of the surface.

If $A = \dfrac{Rr}{R-r}$, and it rests on the first axis,

$$T = \pi\sqrt{\frac{L}{g}}\cdot\left\{1 + \theta^2\cdot\left(\frac{1}{16} + \frac{A^2\,.\,(r+a)}{4rL\,.\,(A+a)} + \frac{(A^2 - 2Ar)\,.\,(A-r)}{16r^2\,.\,(A+a)}\right)\right\},$$

which is not independent of a, unless R is infinite, and therefore $A = r$, in which case,

$$T = \pi\sqrt{\frac{L}{g}}\cdot\left\{1 + \theta^2\left(\frac{1}{16} + \frac{r}{4L}\right)\right\}.$$

On Sliding Bodies.

When a body oscillates by sliding contact on a horizontal plane, X and its differential coefficients vanish, and by (15),

$$L = \frac{k^2}{Y_2},$$

$$T = \pi\sqrt{\frac{L}{g}}\cdot\left\{1 + \Delta\theta^2\cdot\left(\frac{Y_2^2}{4k^2} - \frac{Y_4}{16Y_2} + \frac{5}{48}\cdot\frac{Y_3^2}{Y_2^2}\right)\right\}.$$

The equation (2) becomes $Y = x\sin\theta + (a-y)\,.\cos\theta$, since $\theta_1 = 0$, and $y_1 = 0$, and $\therefore\ \phi = \theta$:

$$\therefore\ Y_1 = x\,.\cos\theta - (a-y)\,.\sin\theta + \sin\theta\,.\frac{dx}{d\theta} - \cos\theta\,.\frac{dy}{d\theta}$$

$$= x\,.\cos\theta - (a-y)\,.\sin\theta,$$

$$Y_2 = -x\sin\theta - (a-y)\,.\cos\theta + \cos\theta\,.\frac{dx}{d\theta} + \sin\theta\,.\frac{dy}{d\theta},$$

$$= -Y + \frac{ds}{d\theta} = -Y + r;$$

$$\therefore\ Y_3 = -Y_1 + r_1,$$

$$Y_4 = -Y_2 + r_2;$$

and taking the limits, $Y = a$,

$$Y_1 = 0,$$

$$Y_2 = r - a,$$

$$Y_3 = r_1,$$

$$Y_4 = a - r + r_2,$$

$$\text{and } T = \frac{\pi}{\theta} \cdot \sqrt{\frac{r-a}{g}} \cdot \left\{1 + \Delta\theta^2 \cdot \left(\frac{(r-a)^2}{4k^2} + \frac{1}{16} - \frac{r_2}{16 \cdot (r-a)} + \frac{5}{48} \cdot \frac{r_1^2}{(r-a)^2}\right)\right\}.$$

Example. An ellipse with its axis major horizontal.

$$\text{Here } r = \frac{a^2 b^2}{(b^2 + a^2 e^2 \sin^2 . \theta)^{\frac{3}{2}}} = \frac{a^2}{b} - \frac{3a^4 e^2 \theta^2}{2b^3} + \ldots\ldots$$

$$= r_0 + r_1 . \theta + r_2 . \frac{\theta^2}{2} + \ldots\ldots;$$

$$\therefore\ L = \frac{k^2 b}{a^2 e^2},$$

$$\text{and } T = \frac{\pi k}{ae} \cdot \sqrt{\frac{b}{g}} \cdot \left\{1 + \Delta\theta^2 \cdot \left(\frac{a^4 e^4}{4k^2 b^2} + \frac{3a^2 + b^2}{16b^2}\right)\right\}.$$

The same principles may also be applied, with great facility, to the oscillations of floating bodies.

H. HOLDITCH.

IX. *On some Cases of Fluid Motion.* *By* G. G. Stokes, B.A., *Fellow of Pembroke College.*

[Read *May* 29, 1843.]

The equations of Hydrostatics are founded on the principles that the mutual action of two adjacent elements of a fluid is normal to the surface which separates them, and that the pressure is equal in all directions. The latter of these is a necessary consequence of the former, as has been shewn by Mr. Airy*. An exactly similar proof may be employed in Hydrodynamics, by which it may be shewn that, if the mutual action of two adjacent elements of a fluid in motion is normal to their common surface, the pressure must be equal in all directions, in order that the accelerating force which acts on the centre of gravity of an element may not become infinite, when we suppose the dimensions of the element indefinitely diminished. In Hydrostatics, the accurate agreement of the results of our calculations with experiments, (those phenomena which depend on capillary attraction being excepted), fully justifies our fundamental assumption. The same assumption is made in Hydrodynamics, and from it are deduced the fundamental equations of fluid motion. But the verification of our fundamental law in the case of a fluid at rest, does not at all prove it to be true in the case of a fluid in motion, except in the very limited case of a fluid moving as if it were solid. Thus, oil is sufficiently fluid to obey the laws of fluid equilibrium, (at least to a great extent), yet no one would suppose that oil in motion ought to be considered a perfect fluid. It would appear from the following consideration, that the fluidity of water and other such fluids is not quite perfect. When a mass of water contained in a vessel of the form of a solid of revolution is stirred round, and then left to itself, it presently comes to rest. This, no doubt, is owing to the friction against the sides of the vessel. But if the fluidity of water were perfect, it does not appear how the retardation due to this friction could be transmitted through the mass. It would appear that in that case a thin film of fluid close to the sides of the vessel would remain at rest, the remaining part of the fluid being unaffected by it. And in this respect, that part of Poisson's solution of the problem of an oscillating sphere, which relates to friction, appears to me in some degree unsatisfactory. A term enters into the equation of motion of the sphere depending on the friction of the fluid on the sphere, while no such term enters into the equations of motion of the fluid, to express the equal and opposite friction of the sphere on the fluid. In fact, as long as we regard the fluidity of the fluid as perfect, no such term can enter. The only way by which to estimate the extent to which the imperfect fluidity of fluids may modify the laws of their motion, without making any hypothesis as to the molecular constitution of fluids, appears to be, to calculate according to the hypothesis of perfect fluidity some cases of fluid motion, which are of such a nature as to be capable of being accurately compared with experiment. The cases of that nature which have hitherto been calculated, are by no means numerous. My object in the present paper which I have the honour to lay before the Society, has been partly to calculate some such cases which may be useful in determining how far we are justified in regarding fluids as perfectly fluid, and partly to give examples of the methods by which the solution of problems depending on partial differential equations may be effected.

In the first seven articles, I have mentioned and explained some general principles, which are afterwards applied. Some of these are not new, but it was convenient to state them for the sake of reference. Others are I believe new, at least in their developement. In the remaining articles, I have given different problems, of which I have succeeded in obtaining the solutions. As the pro-

* See also Professor Miller's *Hydrostatics*, page 2.

blem to be solved is usually stated at the head of each article, I shall here only mention some of the results. As a particular case of the problem given in Art. 8, I find that, when a cylinder oscillates in an infinitely extended fluid, the effect of the inertia of the fluid is to increase the mass of the cylinder by that of the fluid displaced. In part of Art. 9, I find that when a ball pendulum oscillates in a concentric spherical envelope, the effect of the inertia of the fluid is to increase the mass of the ball by $\frac{b^3 + 2a^3}{2(b^3 - a^3)}$ times that of the fluid displaced, a being the radius of the ball, and b that of the envelope. Poisson, in his solution of the problem of the sphere, arrives at the strange result that the envelope does not at all retard the oscillating sphere. I have pointed out the erroneous step by which he was led to this conclusion, which I am clearly called upon to do, in venturing to differ from so high an authority. Of the different cases of fluid motion which I have given, that which appears to be capable of the most accurate and varied comparison with experiment, is the motion of fluid in a rectangular box which is closed on all sides, given in Art. 13. The experiment consists in comparing the calculated and observed times of oscillation. I find that when the motion is small, the effect of the fluid on the motion of the box is the same as that of a solid having the same mass, centre of gravity, and principal axes, but having different moments of inertia, these moments being given by infinite series, which converge with great rapidity. I have also in Art. 11, given some cases of progressive motion, deduced on the supposition that the same particles of fluid remain in contact with the solid, which do not at all agree with experiment.

In almost all the cases given in this paper, the problem of finding the permanent state of temperature in the several solids considered, supposing the surfaces of those solids kept up to constant temperatures varying from point to point, may be solved by a similar analysis. I find that some of these cases have been already solved by M. Duhamel in a paper inserted in the 22nd *Cahier* of the *Journal de l'Ecole Polytechnique*. The cases alluded to are those of the temperature in a solid sphere, and in a rectangular parallelopiped. Since, however, the application of the formulæ in the two cases of fluid motion and of the permanent state of temperature is different, as well as the formulæ themselves to a certain extent, I thought it might be worth while to give them.

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1. The investigations in this paper apply directly to *incompressible* fluids, as the fluids spoken of will be supposed to be, unless the contrary is stated. The motions of elastic fluids may in most cases be divided into two classes, one consisting of those condensations on which sound depends, the other, of those motions which the fluid takes in consequence of the motion of solid bodies in it. Those motions of the fluid, which take place in consequence of very rapid motions of solids, (such as those of bullets), form a connecting link between these two classes. The motions of the second class are, it is true, accompanied by condensations, and propagated with the velocity of sound, but if the motions of the solids are not great we may, without sensible error, suppose the motions of the fluid propagated instantaneously to distances where they cease to be sensible, and may neglect the condensation. The investigations in this paper will apply without sensible error to this kind of motion of elastic fluids.

In all cases also the motion will be supposed to begin from rest, which allows us to suppose that $udx + vdy + wdz$ is an exact differential $d\phi$, where u, v and w are the components, parallel to the axes of x, y, and z, of the whole velocity of any particle. In applying our investigations however to fluids such as they exist in nature, this principle must not be strained too far. When a body is made to revolve continually in a fluid, the parts of the fluid near the body will soon acquire a rotatory motion, in consequence, in all probability, of the mutual friction of the parts of the fluid; so that after a time $udx + vdy + wdz$ could no longer be taken an exact differential. It is true that in motion in two dimensions there is one sort of rotatory motion for which that quantity is an exact differential; but if a close vessel, filled with fluid at first at rest, be made to revolve uniformly round a fixed axis, the fluid will soon do so too, and therefore that quantity will cease to be an exact differential. For the same reason, in the progressive motion of a solid in a fluid, the effect of friction continually accumulating, the motion might at last be sensibly different from what it would be if there were no friction, and that, even if the friction were very small. In the case of small oscillatory motions however it would appear that the effect of friction in the forward oscillation, supposing that friction small, would be counteracted by its effect in the backward oscillation, at least if the two were symmetrical. In this case then we might expect our results to agree very nearly with experiment, so far at least as the *time* of oscillation is concerned.

The forces which act on the fluid are supposed in the following investigations to be such that $Xdx + Ydy + Zdz$ is the exact differential of a function of x, y and z, where X, Y, Z, are the components, parallel to the axes, of the accelerating force acting on the particle whose co-ordinates are x, y, z. The only effect of such forces, in the case of a homogeneous, incompressible fluid, being to add the quantity $\rho \int (Xdx + Ydy + Zdz)$ to the pressure, the forces, as well as the pressure due to them, will for the future be omitted for the sake of simplicity.

2. It is a recognised principle, and one of great importance in these investigations, that when a problem is determinate any solution which satisfies all the requisite conditions, no matter how obtained, is the solution of the problem. In the case of fluid motion, when the initial circumstances and the conditions with respect to the boundaries of the fluid are given, the problem is determinate. If it were required to find what sort of steady motion could take place between given surfaces, the problem would not be determinate, since different kinds of steady motion might result from different initial circumstances.

It may be well here to enumerate the conditions which must be satisfied in the case of a homogeneous incompressible fluid without a free surface, the case which is considered in this paper. We have first the equations,

$$\frac{1}{\rho}\frac{dp}{dx} = -\varpi_1, \quad \frac{1}{\rho}\frac{dp}{dy} = -\varpi_2, \quad \frac{1}{\rho}\frac{dp}{dz} = -\varpi_3, \ldots\ldots\ldots\ldots\ldots\ldots (A);$$

putting ϖ_1 for $\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}$, and ϖ_2, ϖ_3, for the corresponding quantities for y and z, and omitting the forces.

We have also the equation of continuity,

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B);$$

(A) and (B) hold at all times for all points of the fluid mass.

If σ be the velocity of the point (x, y, z) of the surface of a solid in contact with the fluid resolved along the normal, and ν the velocity, resolved along the same normal, of the fluid particle, which at the time t is in contact with the above point of the solid, we must have

$$\nu = \sigma \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a)^*,$$

at all times and for all points of the fluid which are in contact with a solid.

If the fluid extend to infinity, and the motion at first be zero at an infinite distance, we must have

$$u = 0, \quad v = 0, \quad w = 0, \text{ at an infinite distance. } \ldots\ldots\ldots\ldots (b).$$

An analagous condition is, that the motion shall not become infinitely great about a particular point, as the origin.

* For greater clearness, those equations which must hold for all values of the variables within limits depending on the problem are denoted by capitals, while those which hold only for certain values of the variables, or of some of them, are denoted by small letters. The latter class serve to determine the forms of the arbitrary functions contained in the integrals of the former.

Lastly, if u_0, v_0, w_0, be the initial velocities, subject of course to satisfy equations (*B*) and (*a*), we must have

$$u = u_0,\quad v = v_0,\quad w = w_0,\quad \text{when } t = 0. \quad\ldots\ldots\ldots (c).$$

In the most general case the equations which u, v and w are to satisfy at every point of the mass and at every time are (*B*) and the three equations

$$\frac{d\varpi_1}{dy} = \frac{d\varpi_2}{dx},\quad \frac{d\varpi_2}{dz} = \frac{d\varpi_3}{dy},\quad \frac{d\varpi_3}{dx} = \frac{d\varpi_1}{dz} \quad\ldots\ldots\ldots (C).$$

These equations being satisfied, the quantity $\varpi_1 dx + \varpi_2 dy + \varpi_3 dz$ will be an exact differential, whence p may be determined by integrating the value of dp given by equations (*A*). Thus the condition that these latter equations shall be satisfied is equivalent to the condition that the equations (*C*) shall be satisfied.

In nearly all the cases considered in this paper, and in all those of which the complete solution is given, the motion is such that $udx + vdy + wdz$ is an exact differential $d\phi$. This being the case, the equations (*C*) are, as it is well known, always satisfied, the value of p being given by the equation

$$\frac{p}{\rho} = \psi(t) - \frac{d\phi}{dt} - \frac{1}{2}\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right\} \quad\ldots\ldots\ldots (D),$$

$\psi(t)$ being an arbitrary function of t, which may if we please be included in ϕ. In this case, therefore, the *single* condition which has to be satisfied at all times, and at every point of the mass is (*B*), which becomes in this case

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0. \quad\ldots\ldots\ldots (E).$$

In the case of impulsive motion, if u_0, v_0, w_0, be the velocities just before impact, u, v, w, the velocities just after, and q the impulsive pressure, the equations (*A*) are replaced by the equations

$$\frac{1}{\rho}\frac{dq}{dx} = -u + u_0,\quad \frac{1}{\rho}\frac{dq}{dy} = -v + v_0,\quad \frac{1}{\rho}\frac{dq}{dz} = -w + w_0 \quad\ldots\ldots\ldots (F);$$

and in order that these equations may be satisfied it is necessary and sufficient that $(u - u_0)dx + (v - v_0)dy + (w - w_0)dz$ be an exact differential $d\phi$, which gives

$$q = C - \rho\phi.$$

The only equation which must be satisfied at every point of the mass is (*B*), which is equivalent to (*E*), since by hypothesis u_0, v_0, and w_0 satisfy (*B*). The conditions (*a*) and (*b*) remain the same as before.

One observation however is necessary here. The values of u, v and w are always supposed to alter continuously from one point in the interior of a fluid mass to another. At the extreme boundaries of the fluid they may however alter abruptly. Suppose now values of u, v and w to have been assigned, which do not alter abruptly, which satisfy equations (*B*) and (*C*) as well as the conditions (*a*), (*b*) and (*c*), or, to take a particular case, values which do not alter abruptly, which satisfy the equation (*B*) and the same conditions, and which render $udx + vdy + wdz$ an exact differential. Then the values of $\frac{dp}{dx}$, $\frac{dp}{dy}$ and $\frac{dp}{dz}$ will alter continuously from one point to another, but it does not follow that the value of p itself cannot alter abruptly. Similarly in impulsive motion the value of q may alter abruptly, although those of $\frac{dq}{dx}$, $\frac{dq}{dy}$ and $\frac{dq}{dz}$ alter continuously. Such abrupt alterations are, however, inadmissible; whence it follows as an additional condition to be satisfied,

that the value of p or q, obtained by integrating equations (*A*) or (*F*), shall not alter abruptly from one point of the fluid to another. .... (*d*).

An example will make this clearer. Suppose a mass of fluid to be at rest in a finite cylinder, whose axis coincides with that of z, the cylinder being entirely filled, and closed at both ends. Suppose the cylinder to be moved by impact with an initial velocity C in the direction of x; then shall

$$u = C, \quad v = 0, \quad w = 0.$$

For these values render $u\,dx + v\,dy + w\,dz$ an exact differential $d\phi$, where ϕ satisfies (E); they also satisfy (a); and, lastly, the value of q obtained by integrating equations (F), namely, $C' - C\rho x$, does not alter abruptly. But if we had supposed that ϕ were equal to $Cx + C'\theta$, where $\theta = \tan^{-1}\dfrac{y}{x}$, the equation (E) and the condition (a) would still be satisfied, but the value of q would be $C'' - \rho(Cx + C'\theta)$, in which the term $\rho C'\theta$ alters abruptly from $2\pi\rho C'$ to 0, as θ passes through the value 2π. The condition (d) then alone shews that the former and not the latter is the true solution of the problem.

The fact that the analytical conditions of a problem in fluid motion, as far as those conditions depend on the velocities, may be satisfied by values of those velocities, which notwithstanding correspond to a pressure which alters abruptly, may be thus explained. Conceive two masses of the same fluid contained in two similar and equal close vessels A and B. For more simplicity, suppose these vessels and the fluid in them to be at first at rest. Conceive the fluid in B to be divided by an infinitely thin lamina which is capable of assuming any form, and, at the same time, of sustaining pressure. Suppose the vessels A and B to be moved in exactly the same manner, the lamina in B being also moved in any arbitrary manner. It is clear that, except for one particular motion of the lamina, the motion of the fluid in B will be different from that of the fluid in A. The velocities u, v, w, will in general be different on opposite sides of the lamina in B. For particular motions of the lamina however the velocities u, v, w, may be the same on opposite sides of it, while the pressures are different. The motion which takes place in B in this case might, only for the condition (d), be supposed to take place in A.

It is true that equations (A) or (F), could not strictly speaking be said to hold good at those surfaces where such a discontinuity should exsist. Still, to avoid the liability to error, it is well to state the condition (d) distinctly.

When the motion begins from rest, not only must $u\,dx + v\,dy + w\,dz$ be an exact differential $d\phi$, and u, v, w, not alter abruptly, but also ϕ must not alter abruptly, provided the particles in contact with the several surfaces remain in contact with those surfaces; for if this condition be not fulfilled, the surface for which it is not fulfilled will as it were cut the fluid into two. For it follows from the equation (D) that $\dfrac{d\phi}{dt}$ must not alter abruptly, since otherwise p would alter abruptly from one point of the fluid to another; and $\dfrac{d\phi}{dt}$ neither altering abruptly nor becoming infinite, it follows that ϕ will not alter abruptly. Should an impact occur at any period of the motion, it follows from equations (F) that that cannot cause the value of ϕ to alter abruptly, since such an abrupt alteration would give a corresponding abrupt alteration in the value of q.

3. A result which follows at once from the principle laid down in the beginning of the last article is this, that when the motion of a fluid in a close vessel which is at rest, and is completely filled, is of such a kind that $u\,dx + v\,dy + w\,dz$ is an exact differential, it will be steady. For let u, v, w, be the initial velocities, and let us see if the velocity at the same point can remain u, v, w. First, $u\,dx + v\,dy + w\,dz$ being an exact differential, equations (A) will be satisfied by a suitable value of p, which value is given by equation (D). Also equation (B) is satisfied since it is so at first. The condition (a) becomes $v = 0$, which is also satisfied since it is satisfied at first. Also the value of p given by equation (D) will not alter abruptly, for $\dfrac{d\phi}{dt} = 0$, or a function of t, and the velocities $\dfrac{d\phi}{dx}$ &c.,

are supposed not to alter abruptly. Hence, all the requisite conditions are satisfied; and hence, (Art. 2) the hypothesis of steady motion is correct.

4. In the case of an incompressible fluid, either of infinite extent, or confined, or interrupted in any manner by any solid bodies, if the motion begin from rest, and if there be none of the cutting motion mentioned in Art. 2, the motion at the time t will be the same as if it were produced instantaneously by the impulsive motion of the several surfaces which bound the fluid, including among these surfaces those of any solids which may be immersed in it. For let u, v, w, be the velocities at the time t. Then by a known theorem $udx + vdy + wdz$ will be an exact differential $d\phi$, and ϕ will not alter abruptly (Art. 2). ϕ must also satisfy the equation (E), and the conditions (a) and (b). Now if u', v', w', be the velocities on the supposition of an impact, these quantities must be determined by precisely the same conditions as u, v and w. But the problem of finding u', v' and w', being evidently determinate, it follows that the identical problem of finding u, v and w is also determinate, and therefore the two problems have the same solution; so that

$$u = u', \quad v = v', \quad w = w'.$$

This principle has been mentioned by M. Cauchy, in a memoir entitled *Mémoire sur la Théorie des Ondes*, in the first volume of the memoirs presented to the French Institute, page 14. It will be employed in this paper to simplify the requisite calculations *by enabling us to dispense with all consideration of the previous motion*, in finding the motion of the fluid at any time in terms of that of the bounding surfaces. One simple deduction from it is that, when all the bounding surfaces come to rest, each element of the fluid will come to rest. Another is, that if the velocities of the bounding surfaces are altered in any ratio the value of ϕ will be altered in the same ratio.

5. *Superposition of different motions.*

In calculating the inital motion of a fluid, corresponding to given initial motions of the bounding surfaces, we may resolve the latter into any number of systems of motions, which when compounded give to each point of each bounding surface a velocity, which when resolved along the normal is equal to the given velocity resolved along the same normal, provided that, if the fluid be enclosed on all sides, each system be such as not to alter its volume. For let u', v', w', ν', σ', be the values of u, v, &c., corresponding to the first system of motions; u'', v'', &c., the values of those quantities corresponding to the second system, and so on; so that

$$u = u' + u'' + \ldots, \quad v = v' + v'' + \ldots, \quad w = w' + w'' + \ldots, \quad \nu = \nu' + \nu'' + \ldots, \quad \sigma = \sigma' + \sigma'' + \ldots .$$

Then since we have by hypothesis $u'dx + v'dy + w'dz$ an exact differential $d\phi'$, $u''dx + v''dy + w''dz$ an exact differential $d\phi''$, and so on, it follows that $udx + vdy + wdz$ is an exact differential. Again by hypothesis $\nu' = \sigma'$, $\nu'' = \sigma''$, &c., whence $\nu = \sigma$. Also, if the fluid extend to an infinite distance, u, v, and w must there vanish, since that is the case with each of the systems u', v', w', &c. Lastly, the quantities ϕ', ϕ'', &c., not altering abruptly, it follows that ϕ, which is equal to $\phi' + \phi'' + \ldots$, will not alter abruptly. Hence the compounded motion will satisfy all the requisite conditions, and therefore, (Art. 2) it is the actual motion.

It will be observed that the pressure p will not be obtained by adding together the pressures due to each of the above systems of velocities. To find p we must substitute the complete value of ϕ in equation (D). If, however, the motion be very small, so that the square of the velocity is neglected, it will be sufficient to add together the several pressures just mentioned.

In general the most convenient systems into which to decompose the motion of the bounding surfaces are those formed by considering the motion of each surface, or of a certain portion of each surface, separately. Such a portion may be either finite or infinitesimal. In fact, in some of the cases of motion that will be presently given, where ϕ is expressed by a double integral with a function under the integral sign expressing the motion of the bounding surfaces, it will be found that each element of the integral gives a value of ϕ such that, except about the corresponding element of the bounding surface, the motion of all particles in contact with those surfaces is tangential.

A result which follows at once from this principle, and which appears to admit of comparison with experiment, is the following. Conceive an ellipsoid, or any body which is symmetrical with respect to three planes at right angles to each other, to be made to oscillate in a fluid in the direction of each of its three axes in succession, the oscillations being very small. Then, in each case, as may be shown by the same sort of reasoning as that employed in Art. 8, in the case of a cylinder, the effect of the inertia of the fluid will be to increase the mass of the solid by a mass having a certain unknown ratio to that of the fluid displaced. Let the axes of co-ordinates be parallel to the axes of the solid; let x, y, z, be the co-ordinates of the centre of the solid, and let M, M', M'', be the imaginary masses which we must suppose added to that of the solid when it oscillates in the direction of the axes of x, y, z, respectively. Let it now be made to oscillate in the direction of a line making angles α, β, γ, with the axes, and let s be measured along this line. Then the motions of the fluid due to the motions of the solid in the direction of the three axes will be superimposed. The motion being supposed to be small, the resultant of the pressures of the fluid on the solid will be three forces, equal to $M\cos\alpha\,\dfrac{d^2s}{dt^2}$, $M'\cos\beta\,\dfrac{d^2s}{dt^2}$, $M''\cos\gamma\,\dfrac{d^2s}{dt^2}$, respectively, in the directions of the three axes. The resultant of these in the direction of the motion will be $M_{\prime}\,\dfrac{d^2s}{dt^2}$ where

$$M_{\prime} = M\cos^2\alpha + M'\cos^2\beta + M''\cos^2\gamma.$$

Each of the quantities M, M', M'' and $M_{\prime}$, may be determined by observation, and we may find whether the above relation holds between them. Other relations of the same nature may be deduced from the principle explained in this article.

6. *Reflection.*

Conceive two solids, A and B, immersed in a fluid of infinite extent, the whole being at rest. Suppose A to be moved in any manner by impulsive forces, while B is held at rest. Suppose the solids A and B of such forms that, if either were removed, and the several points of the surface of the other moved instantaneously in any given manner, the motion of the fluid could be determined: then the actual motion can be approximated to in the following manner. Conceive the place of B to be occupied by fluid, and A to receive its given motion; then by hypothesis the initial motion of the fluid can be determined. Let the velocity with which the fluid in contact with that which is supposed to occupy B's place penetrates into the latter be found, and then suppose that the several points of the surface of B are moved with normal velocities equal and opposite to those just found, A's place being supposed to be occupied by fluid. The motion of the fluid corresponding to the velocities of the several points of the surface of B can then be found, and A must now be treated as B has been, and so on. The system of velocities of the particles of the fluid corresponding to the first system of velocities of the particles of the surface of B, form what may be called *the motion of A reflected from B;* the motion of the fluid arising from the second system of velocities of the particles of the surface of A may be called *the motion of A reflected from B and again from A*, and so on. It must be remembered that all these motions take place simultaneously. It is evident that these reflected motions will rapidly decrease, at least if the distance between A and B is considerable compared with their diameters, or rather with the diameter of either. In this case the calculation of one or two reflections will give the motion of the fluid due to that of A with great accuracy. It is evident that the principle of reflection will extend to any number of solid bodies immersed in a fluid; or again, the body B may be supposed to be hollow, and to contain the fluid and A, or else A to contain B. In some cases the series arising from the successive reflections can be summed, in which case the motion will be determined exactly. The principle explained in this article has been employed in other subjects, and appears likely to be of great use in this. It is the same for instance as that of *successive influences* in Electricity.

7. If a mass of fluid be at rest or in motion in a close vessel which it entirely fills, the vessel being either at rest or moving in any manner, any additional motion of translation communicated to the vessel will not affect the relative motion of the fluid. For it is evident that on the supposition that the relative motion is not affected the equation (B) and the condition (a) will still be satisfied. Also, if ϖ_1, ϖ_2, ϖ_3, be the components of the effective force of any particle in the first case, and U, V, W, be the components of the velocity of translation, then

$$\varpi_1 + \frac{dU}{dt}, \quad \varpi_2 + \frac{dV}{dt}, \quad \varpi_3 + \frac{dW}{dt},$$

will be the components of the effective force of the same particle in the second case. Now since by hypothesis $\varpi_1 dx + \varpi_2 dy + \varpi_3 dz$ is an exact differential, as follows from equations (C), and U, V, W, are functions of t only, it follows at once that

$$\left(\varpi_1 + \frac{dU}{dt}\right) dx + \left(\varpi_2 + \frac{dV}{dt}\right) dy + \left(\varpi_3 + \frac{dW}{dt}\right) dz$$

is an exact differential, where x, y, z, are the co-ordinates of any particle referred to the old axes, which are themselves moving in space with velocities U, V, W. But if x_1, y_1, z_1, be the co-ordinates of the same particle referred to parallel axes fixed in space, we have

$$x_1 = x + \int U dt, \quad y_1 = y + \int V dt, \quad z_1 = z + \int W dt,$$

whence, supposing the time constant, $dx = dx_1$, $dy = dy_1$, $dz = dz_1$, and therefore

$$\left(\varpi_1 + \frac{dU}{dt}\right) dx_1 + \left(\varpi_2 + \frac{dV}{dt}\right) dy_1 + \left(\varpi_3 + \frac{dW}{dt}\right) dz_1$$

is an exact differential. Hence, equations (A) can be satisfied by a suitable value of p. Denoting by p the pressure about the particle whose co-ordinates are x, y, z, in the first case, the pressure about the same particle in the second case will be

$$p + \psi(t) - \rho\left\{\frac{dU}{dt}x + \frac{dV}{dt}y + \frac{dW}{dt}z\right\},$$

none of the terms of which will alter abruptly, since by hypothesis p does not.

Since then the present hypothesis satisfies all the requisite conditions, it follows from Art. 2 that that hypothesis is correct. If F be the additional effective force of any particle of the vessel in consequence of the motion of translation, and we take new axes of x', y', z', of which the first is in the direction of F, the additional term introduced into the value of the pressure will be $-\rho F x'$, omitting the arbitrary function of the time. The resultant of the additional pressures on the sides of the vessel will be equal to F multiplied by the mass of the fluid, and will pass through the centre of gravity of the fluid, and act in the direction of $-x'$.

8. *Motion between two cylindrical surfaces having a common axis.*

Let us conceive a mass of fluid at rest, bounded by two cylindrical surfaces having a common axis, these surfaces being either infinite or bounded by two planes perpendicular to their axis. Let us suppose the several generating lines of these cylindrical surfaces to be moved parallel to themselves in any given manner consistent with the condition that the volume of the fluid be not altered: it is required to determine the initial motion at any point of the mass.

Since the motion will take place in two dimensions, let the fluid be referred to polar co-ordinates r, θ, in a plane perpendicular to the axis, r being measured from the axis. Let a be the radius of the inner surface, b that of the outer, $f(\theta)$ the normal velocity of any point of the inner surface, $F(\theta)$ the corresponding quantity for the outer.

Since for any particular radius vector between a and b the value of ϕ is a periodic function of θ which does not become infinite, (for the motion at each point of each bounding surface is supposed to be finite), and which does not alter abruptly, it may be expanded in a converging series of sines and cosines of θ and its multiples. Let then

$$\phi = P_0 + \Sigma_1^\infty (P_n \cos n\theta + Q_n \sin n\theta) \;\ldots\ldots\ldots\ldots (1).$$

Substituting the above value in the equation

$$r\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \frac{d^2\phi}{d\theta^2} = 0 \;\ldots\ldots\ldots\ldots (2),$$

which ϕ is to satisfy, and equating to zero the coefficients of corresponding sines and cosines, which is allowable, since a given function can be expanded in only one series of the form (1), we find that P_0 must satisfy the equation

$$r\frac{d}{dr}\left(r\frac{dP_0}{dr}\right) = 0,$$

of which the general integral is

$$P_0 = A\log r + B,$$

the base being ϵ, and P_n and Q_n must both satisfy the same equation, viz.

$$r\frac{d}{dr}\left(r\frac{dP_n}{dr}\right) - n^2 P_n = 0,$$

of which the general integral is

$$P_n = Cr^{-n} + C'r^n.$$

We have then, omitting the arbitrary constant in ϕ, as will be done for the future, since we have occasion to use only the differential coefficients of ϕ,

$$\phi = A_0 \log r + \Sigma_1^\infty \{(A_n r^{-n} + A'_n r^n)\cos n\theta + (B_n r^{-n} + B'_n r^n)\sin n\theta\} \;\ldots\ldots\ldots (3)$$

with the conditions

$$\frac{d\phi}{dr} = f(\theta) \text{ when } r = a \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$\frac{d\phi}{dr} = F(\theta) \text{ when } r = b \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Let $f(\theta) = C_0 + \Sigma_1^\infty (C_n \cos n\theta + D_n \sin n\theta)$,
$F(\theta) = C'_0 + \Sigma_1^\infty (C'_n \cos n\theta + D'_n \sin n\theta)$;

so that $C_0 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(\theta')\,d\theta'$, $\quad C_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\theta')\cos n\theta'\,d\theta'$, $\quad D_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\theta)\sin n\theta'\,d\theta'$.

with similar expressions for C'_0, &c. Then the condition (4) gives

$$\frac{A_0}{a} + \Sigma_1^\infty n\{(-A_n a^{-(n+1)} + A'_n a^{n-1})\cos n\theta + (-B_n a^{-(n+1)} + B'_n a^{n-1})\sin n\theta\}$$
$$= C_0 + \Sigma_1^\infty (C_n \cos n\theta + D_n \sin n\theta);$$

whence,

$$A_0 = aC_0,$$

$$A_n a^{-(n+1)} - A'_n a^{n-1} = -\frac{1}{n}C_n,$$

$$B_n a^{-(n+1)} - B'_n a^{n-1} = -\frac{1}{n}D_n.$$

Similarly, from the condition (5), we get

$$A_0 = bC'_0,$$

$$A_n b^{-(n+1)} - A'_n b^{n-1} = -\frac{1}{n} C'_n,$$

$$B_n b^{-(n+1)} - B'_n b^{n-1} = -\frac{1}{n} D'_n.$$

It will be observed that $aC_0 = bC'_0$, by the condition that the volume of fluid remain unchanged, which gives

$$a\int_0^{2\pi} f(\theta')\,d\theta' = b\int_0^{2\pi} F(\theta')\,d\theta'.$$

From the above equations we easily get

$$A_n = \frac{a^{2n}b^{2n}}{n(b^{2n}-a^{2n})}\{b^{-n+1}C'_n - a^{-n+1}C_n\},$$

and, changing the sign of n,

$$A'_n = \frac{1}{n(b^{2n}-a^{2n})}\{b^{n+1}C'_n - a^{n+1}C_n\},$$

with similar expressions for B_n and B'_n, involving D in place of C.

We have then

$$\phi = aC_0\log r + \Sigma_1^\infty \frac{1}{n}(b^{2n}-a^{2n})^{-1}\{[(b^{-n+1}C'_n - a^{-n+1}C_n)\cos n\theta$$
$$+ (b^{-n+1}D'_n - a^{-n+1}D_n)\sin n\theta]\,a^{2n}b^{2n}r^{-n}$$
$$+ [(b^{n+1}C'_n - a^{n+1}C_n)\cos n\theta + (b^{n+1}D'_n - a^{n+1}D_n)\sin n\theta]\,r^n\} \quad\ldots\ldots\ldots\ldots (6),$$

which completely determines the motion.

It will be necessary however, (Art. 2), to shew that this value of ϕ does not alter abruptly for points within the fluid, as may be easily done. For the quantities C_n, D_n cannot be greater than $\frac{1}{\pi}\int_0^{2\pi} \pm f(\theta)\,d\theta$, where each element of the integral is taken positively; and since by hypothesis $f(\theta)$ is finite for all values of θ from 0 to 2π, it follows that neither C_n nor D_n can be numerically greater than a constant quantity which is independent of n. The same will be true of C'_n and D'_n. Remembering then that $r > a$ and $< b$, it can be easily shewn that the series which occur in (6) have their terms numerically less than those of eight geometric series respectively whose ratios are less than unity; and since moreover the terms of the former set of series do not alter abruptly, it follows that ϕ cannot alter abruptly. The same may be proved in a similar manner of the differential coefficients of ϕ. The other infinite series expressing the value of ϕ which occur in this paper may be treated in the same way: and in Art. 10, where ϕ is expressed by a definite integral, the value of ϕ and its differential coefficients will alter continuously, since that is the case with each element of the integral. It will be unnecessary therefore to refer again to the condition (d).

If the fluid be infinitely extended, we must suppose C'_n and D'_n to vanish in (6), since the velocity vanishes at an infinite distance; we must then make b infinite, which reduces the above equation to

$$\phi = aC_0\log r - \Sigma_1^\infty \frac{a^{n+1}}{nr^n}\{C_n\cos n\theta + D_n\sin n\theta\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

This value of ϕ may be put under the form of a definite integral: for, replacing C_0, C_n and D_n by their values, it becomes

$$\frac{a}{2\pi}\log r\int_0^{2\pi} f(\theta')d\theta' - \frac{a}{\pi}\Sigma_1^\infty \frac{1}{n}\left(\frac{a}{r}\right)^n \int_0^{2\pi} f(\theta')\cos n(\theta-\theta')d\theta',$$

which becomes on summing the series

$$\frac{a}{2\pi}\log r\int_0^{2\pi} f(\theta')d\theta' + \frac{a}{\pi}\int_0^{2\pi}\log\left\{1 - 2\frac{a}{r}\cos(\theta-\theta') + \frac{a^2}{r^2}\right\}^{\frac{1}{2}} f(\theta')d\theta';$$

$$\text{whence } \frac{d\phi}{dr} = \frac{a}{\pi r}\int_0^{2\pi}\left\{\frac{1}{2} + \frac{ar\cos(\theta-\theta') - a^2}{r^2 - 2ar\cos(\theta-\theta') + a^2}\right\} f(\theta')d\theta'.$$

If we suppose r to become equal to a the quantity under the integral sign vanishes, except for values of θ', which are indefinitely near to θ. The value of the integral itself becomes $f(\theta)$*. Hence it appears, that to the disturbance of each element of the surface, there corresponds a normal velocity of the particles in contact with the surface, which is zero, except just about the disturbed element. The whole disturbance of the fluid will be the aggregate of the disturbances due to those of the several elements of the surface. The case of the initial motion of fluid within a cylinder, and the analogous cases of motion within and without a sphere, which will be given in the next article, may be treated in the same manner.

The velocity in the direction of r given by equation (7), $\left(=\frac{d\phi}{dr}\right)$,

$$= \frac{aC_0}{r} + \Sigma_1^\infty \left(\frac{a}{r}\right)^{n+1}\{C_n\cos n\theta + D_n\sin n\theta\},$$

and that perpendicular to r, and reckoned positive in the same direction as θ, $\left(=\frac{d\phi}{rd\theta}\right)$,

$$= \Sigma_1^\infty \left(\frac{a}{r}\right)^{n+1}\{C_n\sin n\theta - D_n\cos n\theta\}.$$

Conceive a mass of fluid comprised between two infinite parallel planes, and suppose that a certain portion of this fluid contains solid bodies bounded by cylindrical surfaces perpendicular to these planes. The whole being at first at rest, suppose that the surfaces of these solids are moved in any manner, the motion being in two dimensions. Conceive a circular cylindrical surface described perpendicular to the parallel planes, and with a radius so large that all the solids are comprised within it. Then, (Art. 4), we may suppose the motion of the fluid at any time to have been produced directly by impact. On this supposition the initial motion of the part of the fluid without the above cylindrical surface will be determined in terms of the normal motion of the fluid forming that surface, as has just been done. If C_0 be different from zero, then, at a great distance in the fluid, the velocity will be ultimately $\frac{aC_0}{r}$, and directed to or from the axis of the cylinder, and alike in all directions. Since the rate of increase of volume of a length l of the cylinder is equal to $la\int_0^{2\pi} f(\theta')d\theta' = 2\pi laC_0$, it appears that the velocity at a great distance is proportional to the expansion or contraction of a unit of length of the solids. If however there should be no expansion or contraction, or if the expansion of some of the solids should make up for the contraction of the rest, then in general the most important part of the motion at a great distance will consist of a velocity $\frac{C'\cos\theta_1}{r^2}$ directed to or from the centre, and another $\frac{C'\sin\theta_1}{r^2}$ perpendicular to the radius vector, the value of C' and the direction from which θ_1 is measured varying from one instant to another. The resultant of these velocities will vary inversely as the square of the distance.

* Poisson, *Théorie de la Chaleur*, Chap. vii.

Resuming the value of ϕ given by equation (6), let us suppose that the interior cylindrical surface is rigid, and moved with a velocity C in the direction from which θ is measured, the outer surface being at rest: then $f(\theta) = C\cos\theta$, $F(\theta) = 0$; whence $C_1 = C$, and the other coefficients are each zero. We have then

$$\phi = -\frac{Ca^2}{b^2 - a^2}\left(\frac{b^2}{r} + r\right)\cos\theta \text{ .... (8).}$$

Suppose now that the inner cylinder has a small oscillatory motion about an axis parallel to the axes of the cylinders, the cylinders having their axes coincident in the position of equilibrium. Let ψ be the angle which a plane drawn through the axis of rotation, and that of the solid cylinder at any time makes with a vertical plane drawn through the former. The motion of translation of the axis of the cylinder will differ from a rectilinear motion by quantities depending on ψ^2: the motion of rotation about its axis will be of the order ψ, but will have no effect on the fluid. Therefore in considering the motion of the fluid we may, if we neglect squares of ψ, consider the motion of the cylinder rectilinear. The expression given for ϕ by equation (8) will be accurately true only for the instant when the axes of the cylinders coincide; but since the whole resultant pressure on the solid cylinder in consequence of the motion is of the order ψ, we may, if we neglect higher powers of ψ than the first, employ the approximate value of ϕ given by equation (8). Neglecting the square of the velocity, we have

$$p = -\rho\frac{d\phi}{dt}.$$

In finding the complete value of $\dfrac{d\phi}{dt}$ it would be necessary to express ϕ by co-ordinates referred to axes fixed in space, which after differentiation we might suppose to coincide with others fixed in the body. But the additional terms so introduced depending on the square of the velocity, which by hypothesis is neglected, we may differentiate the value of ϕ given by equation (8) as if the axes were fixed in space. We have then, to the first order of approximation,

$$\frac{d\phi}{dt} = -\frac{a^2\dfrac{dC}{dt}}{b^2 - a^2}\left\{\frac{b^2}{r} + r\right\}\cos\theta.$$

If l be the length of the cylinder, the pressure on the element $l\,a\,d\theta$, resolved parallel to x and reckoned positive when it acts in the direction of x,

$$= -\frac{\rho l a^3\dfrac{dC}{dt}}{b^2 - a^2}\left\{\frac{b^2}{a} + a\right\}\cos^2\theta\, d\theta;$$

and integrating from $\theta = 0$ to $\theta = 2\pi$, we have the whole resultant pressure parallel to x

$$= -\frac{b^2 + a^2}{b^2 - a^2}\pi\rho l a^2\frac{dC}{dt}.$$

Since $\dfrac{dC}{dt}$ is the effective force of the axis, parallel to x, and that parallel to y is of the order ψ^2, we see that the effect of the inertia of the fluid is to increase the mass of the cylinder by $\dfrac{b^2 + a^2}{b^2 - a^2}\mu$, where μ is the mass of the fluid displaced. This imaginary additional mass must be supposed to be collected at the axis of the cylinder.

If the cylinder oscillate in an infinitely extended fluid $b = \infty$, and the additional mass becomes equal to that of the fluid displaced. This appears to be a result capable of being compared with experiment, though not with very great accuracy. Two cylinders of the same material, and of the same radius, but whose lengths differ by several radii, might be made to oscillate in succession in a fluid, at a depth sufficiently great to allow us to neglect the motion of the surface of the fluid. The time of oscillation of each might then be calculated as if the cylinder oscillated in vacuum, acted on by a moving force equal to its weight *minus* that of the fluid displaced, acting downwards through its centre of gravity, and having its mass increased by an unknown mass collected in the axis. Equating the time of oscillation so calculated to that given by observation, we should determine the unknown mass. The difference of these masses would be very nearly equal to the mass which must be added to that of a cylinder whose length is equal to the difference of the lengths of the first two, when the motion is in two dimensions. This evidently comes to supposing that, at a distance from the middle of the longer cylinder not greater than half the difference of the lengths of the two, the motion may be taken as in two dimensions. The ends of the cylinders may be of any form, provided that they are all of the same. They may be suspended by fine equal wires, in which case we should have a compound pendulum, or attached to a rigid body oscillating above the fluid by means of thin flat bars of metal, whose plane is in the plane of motion. Another way of getting rid of the motion in three dimensions about the ends would be, to make those ends plane, and to fix two rigid planes parallel to the plane of motion, which should be almost in contact with the ends of the cylinder.

9. *Motion between two concentric spherical surfaces.—Motion of a ball pendulum enclosed in a spherical case.*

Let a mass of fluid be at rest, comprised between two concentric spherical surfaces. Let the several points of these surfaces be moved in any manner consistent with the condition that the volume of the fluid be not changed: it is required to determine the initial motion at any point of the mass.

Let a, b, be the radii of the inner and outer spherical surfaces respectively; then employing the co-ordinates r, θ, ω, where r is the distance from the centre, θ the angle which r makes with a fixed line passing through the centre, ω the angle which a plane passing through these two lines makes with a fixed plane through the latter, the value of ϕ corresponding to any radius vector comprised between a and b can be expanded in a converging series of Laplace's coefficients. Let then

$$\phi = V_0 + V_1 \ldots\ldots + V_n + \ldots\ldots,$$

V_n being a Laplace's coefficient of the n^{th} order.

Substituting in the equation,

$$r\frac{d^2 r\phi}{dr^2} + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\phi}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2\phi}{d\omega^2} = 0,$$

which ϕ is to satisfy, employing the equation

$$n(n+1)V_n + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV_n}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2V_n}{d\omega^2} = 0 \ \ldots\ldots\ldots\ (9),$$

and then equating to zero the Laplace's coefficients of the several orders, we find

$$r\frac{d^2 rV_n}{dr^2} - n(n+1)V_n = 0.$$

The general integral of this equation is

$$V_n = Cr^n + \frac{C'}{r^{n+1}},$$

where C and C' are functions of θ and ω. Substituting in the equation (9), and equating coefficients of the two powers of r which enter into it separately to zero, we find that both C and C' satisfy it, and therefore are both Laplace's coefficients of the n^{th} order. We have then

$$\phi = \Sigma_0^\infty (Y_n r^n + Z_n r^{-(n+1)}) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (10),$$

where Y_n and Z_n are each Laplace's coefficients of the n^{th} order, and do not contain r. Let $f(\theta, \omega)$ be the normal velocity of the point of the inner surface corresponding to θ and ω, $F(\theta, \omega)$ the corresponding quantity for the outer; then the conditions which ϕ is to satisfy are that

$$\frac{d\phi}{dr} = f(\theta, \omega) \text{ when } r = a,$$

$$\frac{d\phi}{dr} = F(\theta, \omega) \text{ when } r = b.$$

Let $f(\theta, \omega)$, expanded in a series of Laplace's coefficients, be

$$P_0 + P_1 \ldots + P_n + \ldots$$

which expansion may be performed by the usual formula, if not by inspection: then the first condition gives

$$\Sigma_0^\infty (nY_n a^{n-1} - (n+1) Z_n a^{-(n+2)}) = \Sigma_0^\infty P_n;$$

and equating Laplace's coefficients of the same order, we get

$$nY_n a^{n-1} - (n+1) Z_n a^{-(n+2)} = P_n \ldots\ldots\ldots\ldots (11).$$

Let $F(\theta, \omega)$, expanded in a series of Laplace's coefficients, be

$$P'_0 + P'_1 \ldots P'_n + \ldots;$$

then from the second condition, we get

$$nY_n b^{n-1} - (n+1) Z_n b^{-(n+2)} = P'_n \ldots\ldots\ldots\ldots (12).$$

From (11) and (12) we easily get

$$Y_n = \frac{P_n' b^{n+2} - P_n a^{n+2}}{n(b^{2n+1} - a^{2n+1})},$$

$$Z_n = \frac{a^{2n+1} b^{2n+1} \{P'_n b^{-(n-1)} - P_n a^{-(n-1)}\}}{(n+1)(b^{2n+1} - a^{2n+1})},$$

provided n be greater than 0. If $n = 0$, we have

$$-a^{-2} Z_0 = P_0, \quad -b^{-2} Z_0 = P'_0.$$

But the condition that the volume of the fluid be not altered, gives

$$a^2 \int_0^\pi \int_0^{2\pi} f(\theta, \omega) \sin\theta d\theta d\omega = b^2 \int_0^\pi \int_0^{2\pi} F(\theta, \omega) \sin\theta d\theta d\omega,$$

$$\text{or } 4\pi a^2 P_0 = 4\pi b^2 P'_0,$$

which reduces the two equations just given to one.

We have then, omitting the constant Y_0,

$$\phi = -\frac{P_0 a^2}{r} + \Sigma_1^\infty \{b^{2n+1} - a^{2n+1}\}^{-1} \left\{\frac{1}{n}(P'_n b^{n+2} - P_n a^{n+2}) r^n \right.$$
$$\left. + \frac{a^{2n+1} b^{2n+1}}{n+1} (P'_n b^{-(n-1)} - P_n a^{-(n-1)}) r^{-(n+1)} \right\} \ldots\ldots\ldots (13),$$

which determines the motion.

When the fluid is infinitely extended, we have $P'_n = 0$ since the velocity vanishes at an infinite distance, and $b = \infty$, whence

$$\phi = -\frac{P_0 a^2}{r} - \Sigma_1^\infty \frac{a^{n+2} P_n}{(n+1) r^{n+1}}.$$

It may be proved, precisely as was done, (Art. 8), for motion in two dimensions, that if any portion of an infinitely extended fluid be disturbed by the motion of solid bodies, or otherwise, if all the fluid beyond a certain distance from the part disturbed were at first at rest, the velocity at a great distance will ultimately be directed to or from the disturbed part, and will be the same in all directions, and will vary as $\frac{1}{r^2}$. The coefficient of $\frac{1}{r^2}$ will be proportional to the rate of gain or loss of volume of the part disturbed. If however this rate should be zero, then the most important part of the velocity at a great distance will in general be that depending on the term $-\frac{a^3 P_1}{2r^2}$ in ϕ. Since the general form of P_1 is

$$A \cos\theta + B \sin\theta \cos\omega + C \sin\theta \sin\omega,$$

we easily find, by making use of rectangular co-ordinates, changing the direction of the axes, and then again adopting polar co-ordinates, that the above term in ϕ takes the form $\frac{D\cos\theta_1}{r^2}$, θ_1 being measured from same line passing through the origin. The motion will therefore be the same as that round a ball pendulum in an incompressible fluid, the centre of the ball being in the origin; a case of motion which will be considered immediately. In order to represent the motion at different times, we must suppose the velocity and direction of motion of the ball to change with the time.

The value of ϕ given by equation (13) is applicable to the determination of the motion of a ball pendulum enclosed in a spherical case which is concentric with the ball in its position of equilibrium. If C be the velocity of the centre of the ball at the instant when the centres of the ball and case coincide, and if θ be measured from the direction in which it is moving, we shall have

$$f(\theta) = C\cos\theta, \quad F(\theta) = 0;$$

$$\therefore \; P_0 = 0, \quad P_1 = C\cos\theta, \quad P_2 = 0, \; \&c., \quad P'_1 = 0, \; \&c.,$$

and the value of ϕ for this instant is accurately

$$-\frac{Ca^3}{b^3 - a^3}\left(r + \frac{b^3}{2r^2}\right)\cos\theta,$$

which, when $b = \infty$, becomes

$$-\frac{Ca^3\cos\theta}{2r^2},$$

which is the known expression for the value of ϕ for a sphere oscillating in an infinitely extended, incompressible fluid.

It may be shewn, by precisely the same reasoning as was employed in the case of the cylinder, that in calculating the small oscillations of the sphere the value of $\frac{d\phi}{dt}$ to be employed is

$$-\frac{a^3\frac{dC}{dt}}{b^3 - a^3}\left(a + \frac{b^3}{2a^2}\right)\cos\theta;$$

and from the equation $p = -\rho\frac{d\phi}{dt}$, we easily find that the whole resultant pressure on the sphere in the direction of its centre, and tending to retard it is

$$\frac{4}{3}\frac{\pi\rho a^5}{b^3 - a^3}\left(a + \frac{b^3}{2a^2}\right)\frac{dC}{dt},$$

and that perpendicular to this direction is zero. Since $\frac{dC}{dt}$ is the effective force of the centre in the direction of the motion, and that perpendicular to this direction is of the second order, the effect of the inertia of the fluid will be to increase the mass of the sphere by a mass

$$= \frac{4}{3}\frac{\pi\rho a^5}{b^3 - a^3}\left(a + \frac{b^3}{2a^2}\right) = \frac{b^3 + 2a^3}{b^3 - a^3}\frac{\mu}{2},$$

μ being the mass of the fluid displaced; so that the effect of the case is, to increase the mass which we must suppose added to that of the ball in the ratio of $b^3 + 2a^3$ to $b^3 - a^3$.

Poisson, in his solution of the problem of the oscillating sphere given in the *Mémoires de l'Institut*, *Tome* XI. arrives at a different conclusion, viz. that the case does not at all affect the motion of the sphere. When the elimination which he proposes at p. 563 is made, the last term of equation (f) p. 550 becomes $\dfrac{\delta\gamma}{2a^2c\lambda(l - \delta\gamma)}\left(\dfrac{d^3\zeta}{dt^3} + \dfrac{d^3\zeta'}{dt^3}\right)$, where a is the velocity of propagation of sound, and δ the ratio of the density of air to that of the ball, ζ and ζ' being functions derived from others which enter into the value of ϕ by putting $r = c$, where c is the radius of the ball. He then argues that this term may be neglected as insensible, since it involves δ in the numerator and a^2 in the denominator, tacitly assuming that $\dfrac{d^3\zeta}{dt^3} + \dfrac{d^3\zeta'}{dt^3}$ is not large since ϕ is not large. Now for the disturbances of the air which have the same period as those of the pendulum $\dfrac{d\phi}{dt}$ is not large compared with ϕ, as it is for those on which sound depends. Let then Poisson's solution of equation (a), p. 547 of the volume already mentioned, be put under the form

$$\phi = \frac{1}{r^2}\left\{f\left(t - \frac{r}{a}\right) + F\left(t + \frac{r}{a}\right)\right\} + \frac{1}{ar}\left\{f'\left(t - \frac{r}{a}\right) - F'\left(t + \frac{r}{a}\right)\right\},$$

f' and F' denoting the derived functions, and all the Laplace's coefficients except those of the first order being omitted, the value of ϕ just given being supposed to be a Laplace's coefficient of that order. Then if we expand the above functions in series ascending according to powers of $\dfrac{r}{a}$, we find

$$\phi = \frac{1}{r^2}\{f(t) + F(t)\} - \frac{1}{2a^2}\{f''(t) + F''(t)\} + \frac{r}{3a^3}\{f'''(t) - F'''(t)\} + \dots;$$

and in order that when $a = \infty$ this equation may coincide with (10), when all the Laplace's coefficients except those of the first order are omitted in that equation, it will be seen that it is necessary to suppose $f'''(t) - F'''(t)$, and therefore $f(t) - F(t)$, to be of the order a^3, while $f(t) + F(t)$ is not large. Putting then

$$f(t) = \chi(t) + a^3\varpi(t),$$
$$F(t) = \chi(t) - a^3\varpi(t),$$

we shall have

$$\zeta + \zeta' = \chi\left(t - \frac{c}{a}\right) + \chi\left(t + \frac{c}{a}\right) + a^3\left\{\varpi\left(t - \frac{c}{a}\right) - \varpi\left(t + \frac{c}{a}\right)\right\};$$

so that $\dfrac{d^3(\zeta + \zeta')}{dt^3}$ will contain a term of the order a^2, and the term which Poisson proposes to leave out will be of the same order of magnitude as those retained.

In making the experiment of determining the resistance of the air to an oscillating sphere, it would appear to be desirable to enclose the sphere in a concentric spherical case, which would at the same time exclude currents of air, and facilitate in some measure the experiment by increasing the small quantity which is the subject of observation. The radius of the case however ought not to be nearly as small as that of the ball, for if it were, in the first place a small error in the position of the centre of the ball when at rest might not be insensible, and in the second place the oscillations would have to be inconveniently small, in order that the value of ϕ which has been given might be sufficiently approximate. The effect of a small slit in the upper part of the case, sufficient to allow the wire by which the ball is supported to oscillate, would evidently be insensible, for the condensation being insensible in a vertical plane passing through the axis of rotation, since the alteration of pressure in that plane is insensible, the air would not have a tendency alternately to rush in and out at the slit.

10. *Effect of a distant rigid plane on the motion of a ball pendulum.*

Although this problem may be more easily solved by an artifice, it may be well to give the direct solution of it by the method mentioned in Article 6. In order to calculate the motion reflected from the plane, it will be necessary to solve the following problem:

To find the initial motion at any point of a mass of fluid infinitely extended, except where it is bounded by an infinite solid but not rigid plane, the initial motion of each point of the solid plane being given.

It is evident that motion directed to or from a centre situated in the plane, the velocity being the same in all directions, and varying inversely as the square of the distance from that centre, would satisfy the condition that $u\,dx + v\,dy + w\,dz$ is an exact differential, and would give to the particles in contact with the plane a velocity directed *along* the plane, except just about the centre. Let us see if the required motion can be made up of an infinite number of such motions directed to or from an infinite number of such centres.

Let x, y, z, be the co-ordinates of any particle of fluid, the plane xy coinciding with the solid plane, and the axis of z being directed into the fluid. Let x', y', be the co-ordinates of any point in the solid plane: then the part of ϕ corresponding to the motion of the element $dx'\,dy'$ of the plane will be

$$\frac{\psi(x', y')\,dx'\,dy'}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}},$$

and therefore the complete value of ϕ will be given by the equation

$$\phi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\psi(x', y')\,dx'\,dy'}{\sqrt{\{(x-x')^2 + (y-y')^2 + z^2\}}} \quad \text{.................. (14).}$$

The velocity parallel to z at any point $= \dfrac{d\phi}{dz}$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\psi(x', y')\,z\,dx'\,dy'}{\{(x-x')^2 + (y-y')^2 + z^2\}^{\frac{3}{2}}}.$$

Now when z vanishes the quantity under the integral signs vanishes, except for values of x' and y' indefinitely near to x and y respectively, the function $\psi(x', y')$ being supposed to vanish when x' or y' is infinite. Let then $x' = x + \xi$, $y' = y + \eta$, then, $\xi_{,}$ and $\eta_{,}$ being as small as we please, the value of the above expression when $z = 0$ becomes

$$-\text{ the limit of } \int_{-\xi_{,}}^{\xi_{,}}\int_{-\eta_{,}}^{\eta_{,}} \frac{z\psi(x+\xi, y+\eta)\,d\xi\,d\eta}{(\xi^2 + \eta^2 + z^2)^{\frac{3}{2}}} \text{ when } z = 0.$$

Now if $\psi(x', y')$ does not alter abruptly between the limits $x - \xi_{,}$ and $x + \xi_{,}$ of x', and $y - \eta_{,}$ and $y + \eta_{,}$ of y', the above expression may be replaced by

$$-\psi(x, y) \times \text{the limit of } \int_{-\xi_{,}}^{\xi_{,}}\int_{-\eta_{,}}^{\eta_{,}} \frac{z\,d\xi\,d\eta}{(\xi^2 + \eta^2 + z^2)^{\frac{3}{2}}},$$

which is $= -2\pi\,\psi(x, y)$.

If now $f(x', y')$ be the given normal velocity of any point (x', y') of the solid plane, the expression for ϕ given by equation (14) may be made to give the required normal velocity of the fluid particles in contact with the solid plane by assuming

$$\psi(x', y') = -\frac{1}{2\pi} f(x', y'),$$

whence

$$\phi = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x', y')\,dx'\,dy'}{\{(x-x')^2+(y-y')^2+z^2\}^{\frac{1}{2}}}.$$

This expression will be true for any point at a finite distance from the plane xy even when $f(x', y')$ does alter abruptly; for we may first suppose it to alter continuously, but rapidly, and may then suppose the rapidity of alteration indefinitely increased: this will not cause the value of ϕ just given to become illusory for points situated without the plane xy.

If it be convenient to use polar co-ordinates in the plane xy, putting $x = q\cos\omega$, $y = q\sin\omega$, $x' = q'\cos\omega'$, $y' = q'\sin\omega'$, and replacing $f(x', y')$ by $f(q', \omega')$, the equation just given becomes

$$\phi = -\frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi}\frac{f(q', \omega')\,q'\,dq'\,d\omega'}{\{q'^2+q^2-2qq'\cos(\omega-\omega')+z^2\}^{\frac{1}{2}}}.$$

To apply this to the case of a sphere oscillating in a fluid perpendicularly to a fixed rigid plane, let a be the radius of the sphere, and let its centre be moving towards the plane with a velocity C at the time t. Then, (Art. 4), we may calculate the motion as if it were produced directly by impact. Let h be the distance of the centre of the sphere from the fixed plane at the time t, and let the line h be taken for the axis of z, and let r, θ, be the polar co-ordinates of any point of the fluid, r being the distance from the centre of the sphere, and θ the angle between the lines r and h. Then if the fluid were infinitely extended around the sphere we should have

$$\phi = -\frac{Ca^3\cos\theta}{2r^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15).$$

The velocity of any particle, resolved in a direction towards the plane, $= \dfrac{d\phi}{dr}\cos\theta - \dfrac{d\phi}{rd\theta}\sin\theta$

$$= \frac{Ca^3}{r^3}\{\cos^2\theta - \tfrac{1}{2}\sin^2\theta\}.$$

For a particle in the plane xy we have

$$r\cos\theta = h, \quad r\sin\theta = q',$$

and the above velocity becomes

$$\frac{Ca^3(2h^2-q'^2)}{2(h^2+q'^2)^{\frac{5}{2}}}.$$

We must now, according to the method explained in (Art. 6), suppose the several points of the plane xy moved with the above velocity parallel to z. We have then

$$f(q', \omega') = \frac{Ca^3(2h^2-q'^2)}{2(h^2+q'^2)^{\frac{5}{2}}};$$

whence, for the motion of the sphere reflected from the plane,

$$\phi = -\frac{Ca^3}{4\pi}\int_0^{\infty}\int_0^{2\pi}\frac{(2h^2-q'^2)\,q'\,dq'\,d\omega'}{(h^2+q'^2)^{\frac{5}{2}}\{q^2+q'^2-2qq'\cos(\omega-\omega')+z^2\}^{\frac{1}{2}}} \quad\ldots\ldots\ldots (16).$$

We must next find the velocity, corresponding to this value of ϕ, with which the fluid penetrates the surface of the sphere. We have in general

$$z = h - r\cos\theta, \quad q = r\sin\theta,$$

whence $\{q^2 + q'^2 - 2qq' \cos(\omega - \omega') + z^2\}^{-\frac{1}{2}} = \{h^2 + r^2 + q'^2 - 2hr\cos\theta - 2q'r\sin\theta\cos(\omega - \omega')\}^{-\frac{1}{2}}$.
Now supposing the ratio of a to h to be very small, and retaining the most important term, the value of $\dfrac{d\phi}{dr}$ when $r = a$ will be equal to the coefficient of r when ϕ is expanded in a series ascending according to powers of r,

$$= -\frac{Ca^3}{4\pi}\int_0^\infty \int_0^{2\pi} \frac{(2h^2 - q'^2)\{h\cos\theta + q'\sin\theta\cos(\omega - \omega')\}q'dq'd\omega'}{(h^2 + q'^2)^4}$$

$$= -\tfrac{1}{2}Ca^3h\cos\theta\int_0^\infty \frac{(2h^2 - q'^2)q'dq'}{(h^2 + q'^2)^4} = -\frac{Ca^3\cos\theta}{8h^3} \quad\ldots\ldots\ldots\ldots (17).$$

In order now to determine the motion reflected from the plane and again from the sphere, we must suppose the several points of the sphere to be moved with a normal velocity $\dfrac{Ca^3\cos\theta}{8h^3}$, or, which is the same, we must suppose the whole sphere to be moved towards the plane with a velocity $\dfrac{Ca^3}{8h^3}$. Hence the value of ϕ corresponding to this motion will be given by the equation

$$\phi = -\frac{Ca^6\cos\theta}{16h^3r^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18).$$

For points at a great distance from the centre of the sphere, the motion which is twice reflected will be very small compared with that which is but once reflected. For points close to the sphere however, with which alone we are concerned, those motions will be of the same order of magnitude, and if we take account of the one we must take account of the other.

Putting $q = r\sin\theta$, $z = h - r\cos\theta$ in (16), expanding, and retaining the two most important terms, we have

$$\phi = C\left(K - \frac{a^3r\cos\theta}{8h^3}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots (19),$$

K being a constant, the value of which is not required, and the second term being evidently found by multiplying the quantity at the second side of (17) by r. Adding together the parts of ϕ given by equations (15), (18) and (19), putting $r = a$, replacing C by $\dfrac{dC}{dt}$, and taking for h the value which it has in equilibrium, just as in the case of the oscillating cylinder in Article 8, we have for the small motion of the sphere

$$\frac{d\phi}{dt} = K\frac{dC}{dt} - \frac{a}{2}\left(1 + \frac{3a^3}{8h^3}\right)\frac{dC}{dt}\cos\theta.$$

The resultant of the part of the pressure due to the first term is zero: that due to the second term is greater than if the plane were removed in the ratio of $1 + \dfrac{3a^3}{8h^3}$ to 1. Consequently, if we neglect quantities of the order $\dfrac{a^4}{h^4}$, the effect of the inertia of the fluid is, to add a mass equal to $\left(1 + \dfrac{3}{8}\dfrac{a^3}{h^3}\right)\dfrac{\mu}{2}$ to that of the sphere, without increasing the moment of inertia of the latter about its diameter. The effect therefore of a large spherical case is eight times as great as that of a tangent plane to the case, perpendicular to the direction of the motion of the ball.

The effect of a distant rigid plane parallel to the direction of motion of an oscillating sphere might be calculated in the same manner, but as the method is sufficiently explained by the first case, it will be well to employ the artifice before alluded to, an artifice which is frequently employed in this subject. It consists in supposing an exactly symmetrical motion to take place on the opposite side of the rigid plane, by which means we may evidently conceive the plane removed.

Let the sphere be oscillating in the direction of the axis of x, the oscillations in this case, as in the last, being so small that they may be taken as rectilinear in calculating the motion of the fluid; and instead of a rigid plane conceive an equal sphere to exist at an equal distance on the opposite side of the plane xy, moving in the same direction and with the same velocity as the actual sphere. Let r, θ, ω, be the polar co-ordinates of any particle measured from the centre of the sphere, θ being the angle between r and a line drawn through the centre parallel to the axis of x, and ω the angle which the plane passing through these lines makes with the plane xz. Let r', θ', ω', be the corresponding quantities symmetrically measured from the centre of the imaginary sphere.

If the fluid were infinite we should have for the motion corresponding to that of the given sphere

$$\phi = -\frac{Ca^3 \cos\theta}{2r^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots (20).$$

The motion reflected from the plane is evidently the same as that corresponding to the motion of the imaginary sphere in an infinite mass of fluid, for which we have

$$\phi = -\frac{Ca^3 \cos\theta'}{2r'^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots (21).$$

Now $r' \cos\theta' = r\cos\theta$, $r' \sin\theta' \sin\omega' = r\sin\theta\sin\omega$, $r'\sin\theta'\cos\omega' + r\sin\theta\cos\omega = 2h$;

$$\text{whence } r'^2 = r^2 + 4h^2 - 4hr\sin\theta\cos\omega,$$

and equation (21) is reduced to

$$\phi = -\frac{Ca^3 r\cos\theta}{2\{r^2 + 4h^2 - 4hr\sin\theta\cos\omega\}^{\frac{3}{2}}}.$$

Retaining only the terms of the order $\frac{a^3 r}{h^3}$ or $\frac{r^4}{h^3}$, so as to get the value of $\frac{d\phi}{dr}$ to the order $\frac{a^3}{h^3}$, the above equation is reduced to

$$\phi = -\frac{Ca^3 r\cos\theta}{16h^3} \quad\ldots\ldots\ldots\ldots\ldots\ldots (22),$$

and the value of $\frac{d\phi}{dr}$ when $r = a$ is, to the required degree of approximation,

$$-\frac{Ca^3\cos\theta}{16h^3}.$$

For the value of ϕ corresponding to the motion of the imaginary sphere reflected from the real sphere, we shall therefore have

$$\phi = -\frac{Ca^6\cos\theta}{32h^3r^2}. \quad\ldots\ldots\ldots\ldots\ldots\ldots (23).$$

Adding together the values of ϕ given by (20), (22) and (23), putting $r = a$, and replacing C by $\frac{dC}{dt}$, we have, to the requisite degree of approximation,

$$\frac{d\phi}{dt} = -\frac{a}{2}\left(1 + \frac{3}{16}\frac{a^3}{h^3}\right)\frac{dC}{dt}.\cos\theta.$$

Hence in this case the motion of the sphere will be the same as if an additional mass equal to $\left(1 + \frac{3}{16}\frac{a^3}{h^3}\right)\frac{\mu}{2}$ were collected at its centre. The effect therefore of a distant rigid plane which is parallel to the direction of the motion of a ball pendulum will be half that of a plane at the same distance, and perpendicular to that direction. It would seem from Poisson's words at page 562 of the eleventh volume of the *Mémoires de l'Institut*, that he supposed the effect in the former case to depend on a higher order of small quantities than that in the latter.

If the ball oscillate in a direction inclined to the plane, the motion may be easily deduced from that in the two cases just given, by means of the principle of superposition.

11. The values of ϕ which have been given for the motion of translation of a sphere and cylinder, do not require us to suppose that either the velocity, or the distance to which the centre of the sphere or axis of the cylinder has been moved is small, provided the same particles remain in contact with the surface. The same indeed is true of the values corresponding to a motion of translation combined with a motion of contraction or expansion which is the same in all directions, but varies in any manner with the time. The value of ϕ corresponding to a motion of translation of the cylinder is $-\dfrac{Ca^2\cos\theta}{r}$, C being the velocity of the axis, and θ being measured from a line drawn in the direction of its motion. The whole resultant of the part of the pressure due to the square of the velocity is zero, since the velocity at the point whose co-ordinates are r, θ, is the same as that at the point whose co-ordinates are r and $\pi-\theta$. To find the resultant of the part depending on $\dfrac{d\phi}{dt}$, it will be necessary to express ϕ by means of co-ordinates referred to axes fixed in space. Let Ox, Oy, be rectangular axes passing through the centre of any section of the cylinder, ϖ the angle which the direction of motion of the axis makes with Ox, θ' the inclination of any radius vector to Ox; then

$$\phi = -\frac{Ca^2}{r^2}(r\cos\theta'\cos\varpi + r\sin\theta'\sin\varpi)$$

$$= -\frac{a^2(C'x + C''y)}{x^2+y^2},$$

putting C' and C'' for the resolved parts of the velocity C along the axes of x and y respectively. Taking now axes Ax', Ay', parallel to the former and fixed in space, putting α and β for the co-ordinates of O, differentiating ϕ with respect to t, and replacing $\dfrac{d\alpha}{dt}$ by C', and $\dfrac{d\beta}{dt}$ by C'', and then supposing α and β to vanish, we have

$$\frac{d\phi}{dt} = \frac{a^2C^2}{x^2+y^2} - \frac{2a^2(C'x+C''y)^2}{(x^2+y^2)^2} - \frac{a^2\left(x\dfrac{dC'}{dt} + y\dfrac{dC''}{dt}\right)}{x^2+y^2}.$$

The resultant of the part of the pressure due to the first two terms is zero, since the pressure at the point (x, y) depending on these terms is the same as that at the point $(-x, -y)$. It will be easily found that the resultant of the whole pressure parallel to x, and acting in the negative direction, on a length l of the cylinder, is equal to $\pi\rho l a^2\dfrac{dC'}{dt}$, and that parallel to y equal to $\pi\rho l a^2\dfrac{dC''}{dt}$. The resultant of these two will be $\pi\rho l a^2 F$, where F is the effective force of a point in the axis of the cylinder, and will act in a direction opposite to that of F. Hence the only effect of the motion of the fluid will be, to increase the mass of the cylinder by that of the fluid displaced. In a similar manner it may be proved that, when a solid sphere moves in any manner in an infinite fluid, the only effect of the motion of the fluid is to increase the mass of the sphere by half that of the fluid displaced.

A similar result may be proved to be true for any solid symmetrical with respect to two planes at right angles to each other, and moving in the direction of the line of their intersection in an infinitely extended fluid, the solid and fluid having been at first at rest. Let the planes of symmetry be taken for the planes of xy and xz, the origin being fixed in the body: then it is evident that the resultant of the pressure on the solid due to the motion will be in the direction of the axis of x, and that there will be no resultant couple. Let C be the velocity of the solid at any time; then the value of ϕ at that time will be of the form $C\psi(x, y, z)$, where C alone contains t, (Art. 4), and the velocity of the particle whose co-ordinates are x, y, z, being proportional to C, the *vis viva* of the solid and fluid together will be proportional to C^2. Now if no forces act on the fluid and solid, except the pressure of the fluid, this *vis viva* must be constant*; therefore C must be constant; therefore the resultant of the fluid pressure on the solid must be zero. If now C be a function of t we shall have

$$p = -\rho\psi(x, y, z)\frac{dC}{dt} + p',$$

p' being the pressure when C is constant. Since therefore the resultant of the fluid pressure varies for the same solid and fluid as $\frac{dC}{dt}$ the effective force, and for different fluids varies as ρ, the effect of the inertia of the fluid will be, to increase the mass of the solid by n times that of the fluid displaced, n depending only on the particular solid considered.

Let us consider two such solids, similar to each other, and having the co-ordinates planes similarly situated, and moving with the same velocities. Let the linear dimensions of the second be greater than those of the first in the ratio of m to 1. Let u, v, w, be the velocities, parallel to the axes, of the particle (x, y, z) in the fluid about the first; then shall the corresponding velocities at the point (mx, my, mz) in the fluid about the second be also u, v, w. For

$$udmx + vdmy + wdmz = m(udx + vdy + wdz) \ldots\ldots\ldots\ldots\ldots (24),$$

and is therefore an exact differential, since $udx + vdy + wdz$ is one: also the normal at the point (x, y, z) in the first surface will be inclined to the axes at the same angles as the normal at the point (mx, my, mz) of the second surface is inclined to its axes, and therefore the normal velocities of the two surfaces at these points are the same; and the velocities of the fluid at these two points parallel to the axes being also the same, it follows that the normal velocity of each point of the second surface is equal to that of the fluid in contact with it. Lastly, the motion about the first solid being supposed to vanish at an infinite distance from it, that about the second will vanish also. Hence the supposition made with respect to the motion of the fluid about the second surface is correct. Now putting ϕ for $\int(udx + vdy + wdz)$ for the fluid in the first case, the corresponding integral for the fluid in the second case will be $m\phi$, if the constant be properly chosen, as follows from equation (24). Consequently the value of that part of the expression for the pressure, on which the resistance depends, will be m times as great for any point in the

* If an incompressible fluid which is homogeneous or heterogeneous, and contains in it any number of rigid bodies, be in motion, the rigid bodies being also in motion, if the rigid bodies are perfectly smooth, and no contacts are formed or broken among them, and if no forces act except the pressure of the fluid, the principle of *vis viva* gives

$$\frac{d\Sigma mv^2}{dt} = 2\iint p_{,}\nu dS \ldots\ldots\ldots\ldots\ldots\ldots (a),$$

where v is the whole velocity of the mass m, and the sign Σ extends over the whole fluid and the rigid bodies spoken of, and where dS is an element of the surface which bounds the whole, $p_{,}$ the pressure about the element dS, and ν the normal velocity of the particles in that element, reckoned positive when tending into the fluid, and where the sign $\iint$ extends to all points of the bounding surface. To apply equation (a) to the case of motion at present considered, let us first confine ourselves to a spherical portion of the fluid, whose radius is r, and whose centre is near the solid, so that dS refers to the surface of this portion. Let us now suppose r to become infinite: then the second side of (a) will vanish, provided $p_{,}$ remain finite, and ν decrease in a higher ratio than $\frac{1}{r^2}$. Both of these will be true, (Art. 9.); for ν will vary ultimately as $\frac{1}{r^3}$, since there is no alteration of volume. Hence if the sign Σ extend to infinity, we shall have Σmv^2 constant.

second case as it is for the corresponding point in the first. Also, each element of the surface of the second solid will be m^2 times as great as the corresponding element of the surface of the first. Hence the whole resistance on the second solid will be m^3 times as great as that on the first, and therefore the quantity n depends only on the *form*, and not on the *size* of the solid.

When forces act on the fluid, it will only be necessary to add the corresponding pressure. Hence when a sphere descends from rest in a fluid by the action of gravity, the motion will be the same as if a moving force equal to that of the sphere *minus* that of the fluid displaced acted on a mass equal to that of the sphere *plus* half that of the fluid displaced. For a cylinder which is so long that we may suppose the length infinite, descending horizontally, every thing will be the same, except that the mass to be moved will be equal to that of the cylinder *plus* the whole of the fluid displaced. In these cases, as well as in that of any solid which is symmetrical with respect to two vertical planes at right angles to each other, the motion will be uniformly accelerated, and similar solids of the same material will descend with equal velocities. These results are utterly opposed even to the commonest observation, which shews that large solids descend much more rapidly than small ones of the same shape and material, and that the velocity of a body falling in a fluid, (such as water), does not sensibly increase after a little time. It becomes then of importance in the theory of resistances to inquire what may be the cause of this discrepancy between theory and observation. The following are the only ways of accounting for it which suggest themselves to me.

First. It has been supposed that the same particles remain in contact with the solid throughout the motion. It must be remembered that we suppose the ultimate molecules of fluids, (if such exist), to be so close that their distance is quite insensible, a supposition of the truth of which there can be hardly any doubt. Consequently we reason on a fluid as if it were infinitely divisible. Now if the motion which takes place in the cases of the sphere and cylinder be examined, supposing for simplicity their motions to be rectilinear, it will be found that a particle in contact with the surface of either moves along that surface with a velocity which at last becomes infinitely small, and that it does not reach the end of the sphere or cylinder from which the whole is moving until after an infinite time, while any particle not in contact with the surface is at last left behind. It seems difficult to conceive of what other kind the motion can be, without supposing a line, (or rather surface) of particles to make an abrupt turn. If it should be said that the particles may come off in tangents, it must be remembered that this sort of motion is included in the condition which has been assumed with respect to the surface.

Secondly. The discrepancy alluded to might be supposed to arise from the friction of the fluid against the surface of the solid. But, for the reason mentioned in the beginning of this paper, this explanation does not appear to me satisfactory.

Thirdly. It appears to me very probable that the *spreading out* motion of the fluid, which is supposed to take place behind the middle of the sphere or cylinder, though dynamically possible, nay, the *only* motion dynamically possible when the conditions which have been supposed are accurately satisfied, is unstable; so that the slightest cause produces a disturbance in the fluid, which accumulates as the solid moves on, till the motion is quite changed. Common observation seems to shew that, when a solid moves rapidly through a fluid at some distance below the surface, it leaves behind it a succession of eddies in the fluid. When the solid has attained its terminal velocity, the product of the resistance, or rather the mean resistance, and any space through which the solid moves, will be equal to half the *vis viva* of the corresponding portion of its *tail of eddies*, so that the resistance will be measured by the *vis viva* in the length of two units of that tail. So far therefore as the resistance which a ship experiences depends on the disturbance of the water, which is independent of its elevation or depression, that ship which leaves the least wake ought, according to this view, to be *cæteris paribus* the best sailer. The resistance on a ship differs from that on a solid in motion immersed in a fluid in the circumstance that part of the resistance is employed in producing a wave.

Fourthly, the discrepancy alluded to may be due to the mutual friction, or imperfect fluidity of the fluid.

12. *Motion about an elliptic cylinder of small eccentricity.*

The value of ϕ, which has been deduced, (Art. 8), for the motion of the fluid about a circular cylinder, is found on the supposition that for each value of r there exists, or may be supposed to exist, a real and finite value of ϕ. This will be true, in any case of motion in two dimensions where $udx + vdy$ is an exact differential, *for those values of* r *for which the fluid is not interrupted*, but will be true for values of r for which it is interrupted by solids only when it is possible to replace those solids at any instant by masses of fluid, without affecting the motion of the fluid exterior to them, those masses moving in such a manner that the motion of the whole fluid might have been produced instantaneously by impact. In some cases such a substitution could be made, while in others it probably could not. In any case however we may try whether the expansion given by equation (3) will enable us to get a result, and if it will, we need be in no fear that it is wrong, (Art. 2). The same remarks will apply to the question of the possibility of the expansion of ϕ in the series of Laplace's coefficients given in equation (10), for values of r for which the fluid is interrupted. They will also apply to such a question as that of finding the permanent temperature of the earth due to the solar heat, the earth being supposed to be a homogeneous oblate spheroid, and the points of the surface being supposed to be kept up to constant temperatures, given by observation, depending on the latitude.

In cases of fluid motion such as those mentioned, the motion may be determined by conceiving the whole mass of fluid divided into two or more portions, taking the most general value of ϕ for each portion, this value being in general expressed in a different manner for the different portions, then limiting the general value of ϕ for each portion so as to satisfy the conditions with respect to the surfaces of solids belonging to that portion, and lastly introducing the condition that the velocity and direction of motion of each pair of contiguous particles in any two of the portions are the same. The question first proposed will afford an example of this method of solution.

Let an elliptic cylinder be moving with a velocity C, in the direction of the major axis of a section of it made by a plane perpendicular to its axis. The motion being supposed to be in two dimensions, it will be sufficient to consider only this section. Let

$$r = c(1 + \epsilon \cos 2\theta)$$

be the approximate equation to the ellipse so formed, the centre being the pole, and powers of ϵ above the first being neglected. Let a circle be described about the same centre, and having a radius γ equal to $(1 + k)c$, k being $\not< \epsilon$, and being a small quantity of the order ϵ. Let the portions of fluid within and without the radius γ be considered separately, and putting

$$r = c + z,$$

let the value of ϕ corresponding to the former portion be

$$P + Qz + Rz^2,$$

P, Q and R being functions of θ, and the term in z^2 being retained, in order to get the value of $\dfrac{d\phi}{dr}$ true to the order ϵ, while the terms in z^3, &c. are omitted. Substituting this value of ϕ in equation (2), and equating to zero coefficients of different powers of z, we have

$$R = -\frac{Q}{2c} - \frac{1}{2c^2}\frac{d^2P}{d\theta^2},$$

which is the only condition to be satisfied, since the other equations would only determine the coefficients of z^3, &c. in terms of the preceding ones. We have then

$$\phi = P + Qz - \frac{1}{2c}\left(Q + \frac{1}{c}\frac{d^2P}{d\theta^2}\right)z^2 \ldots\ldots\ldots\ldots (25).$$

Now if ξ be the angle between the normal at any point of the ellipse, and the major axis, we have

$$\xi = \theta + 2\epsilon \sin 2\theta,$$

and the velocity of the ellipse resolved along the normal

$$= C\cos\xi = C(1-\epsilon)\cos\theta + C\epsilon\cos 3\theta \ldots\ldots\ldots (26).$$

The velocity of the fluid at the same point resolved along the normal is

$$\frac{d\phi}{dr} + 2\epsilon \sin 2\theta \frac{d\phi}{rd\theta},$$

$$\text{or} \quad \frac{d\phi}{dz} + \frac{2\epsilon}{c}\sin 2\theta \frac{d\phi}{d\theta} \ldots\ldots\ldots\ldots (27).$$

Let P and Q be expanded in series of cosines of θ and its multiples, so that

$$P = \Sigma_0^\infty P_n \cos n\theta, \quad Q = \Sigma_0^\infty Q_n \cos n\theta,$$

there being no sines in the expansions of P and Q, since the motion is symmetrical with respect to the major axis; then

$$\phi = \Sigma_0^\infty \left\{P_n + Q_n z - \frac{1}{2c}\left(Q_n - \frac{n^2}{c}P_n\right)z^2\right\}\cos n\theta \ldots\ldots\ldots\ldots (28);$$

$$\frac{d\phi}{dz} = \Sigma_0^\infty \left\{Q_n - \frac{1}{c}\left(Q_n - \frac{n^2}{c}P_n\right)z\right\}\cos n\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (29);$$

$$\frac{1}{c+z}\frac{d\phi}{d\theta} = -\Sigma_0^\infty n\left\{\frac{P_n}{c} + \left(\frac{Q_n}{c} - \frac{P_n}{c^2}\right)z\right\}\sin n\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (30).$$

For a point in the ellipse, $z = c\epsilon\cos 2\theta$, whence from (27), (29) and (30), we find that the normal velocity of the fluid

$$= \Sigma_0^\infty \left\{Q_n\cos n\theta + \frac{\epsilon}{2}\left[n(n-2)\frac{P_n}{c} - Q_n\right]\cos(n-2)\theta + \frac{\epsilon}{2}\left[n(n+2)\frac{P_n}{c} - Q_n\right]\cos(n+2)\theta\right\},$$

which is the same thing as

$$\Sigma_0^\infty \left\{\frac{\epsilon}{2}\left[n(n-2)\frac{P_{n-2}}{c} - Q_{n-2}\right] + Q_n + \frac{\epsilon}{2}\left[n(n+2)\frac{P_{n+2}}{c} - Q_{n+2}\right]\right\}\cos n\theta \ldots\ldots (31),$$

if we suppose P and Q to be zero when affected with a negative suffix. This expression will have to be equated to the value of $C\cos\xi$ given by equation (26).

For the part of the fluid without the radius γ we have

$$\phi = A_0 \log r + \Sigma_1^\infty \frac{A_n}{r_n}\cos n\theta\,^*,$$

since there will be no sines in the expression for ϕ, because the motion is symmetrical with respect to the major axis, and no positive powers of r, because the velocity vanishes at an infinite distance.

From the above value of ϕ we have, for the points at a distance γ from the centre,

$$\frac{d\phi}{dr} = \frac{A_0}{\gamma} - \Sigma_1^\infty \frac{nA_n}{\gamma^{n+1}}\cos n\theta,$$

* The first term of this expression is accurately equal to zero, since there is no expansion or contraction of the solid, (Art. 8). I have however retained it, in order to render the solution of the problem in the present article independent of the proposition referred to.

$$\frac{d\phi}{rd\theta} = -\Sigma_1^\infty \frac{nA_n}{\gamma^{n+1}} \sin n\theta.$$

Equating the above expressions to the velocities along and perpendicular to the radius vector given by equations (29) and (30), when z is put $= kc$, and then equating coefficients of corresponding sines and cosines we have

$$(1-k)\,Q_n + kn^2 \frac{P_n}{c} = -\frac{nA_n}{\gamma^{n+1}} \text{ .... } (32),$$

$$(1-k)\frac{P_n}{c} + kQ_n = \frac{A_n}{\gamma^{n+1}} \text{ .... } (33),$$

when $n > 0$, and equating constant terms we have

$$(1-k)\,Q_0 = \frac{A_0}{\gamma},$$

from which equation with (32) and (33) we have, putting $\gamma = (1+k)c$,

$$\frac{P_n}{c} = \frac{A_n}{c^{n+1}}, \quad Q_n = -\frac{nA_n}{c^{n+1}}, \text{ when } n > 0, \text{ and } Q_0 = \frac{A_0}{c}.$$

Substituting these values in the expression (31), it becomes

$$\Sigma_0^\infty \left\{ \frac{\epsilon}{2}(n+1)(n-2)\frac{A_{n-2}}{c^{n-1}} - \frac{nA_n}{c^{n+1}} + \frac{\epsilon}{2}(n+1)(n+2)\frac{A_{n+2}}{c^{n+3}} \right\} \cos n\theta + \frac{A_0}{c} - \frac{\epsilon A_0}{2c}\cos 2\theta.$$

In the case of a circular cylinder the quantities A_0, A_2, A_3, &c. are each zero. In the present case therefore they are small quantities depending on ϵ. Hence, neglecting quantities of the order ϵ^2 in the above expression, it becomes

$$\frac{A_0}{c} + \frac{2\epsilon A_1}{c^2}\cos 3\theta - \Sigma_1^\infty \frac{nA_n}{c^{n+1}} \cos n\theta,$$

which must be equal to $C\{(1-\epsilon)\cos\theta + \epsilon\cos 3\theta\}$. Equating coefficients of corresponding cosines, we have

$$A_1 = -C(1-\epsilon)c^2,$$
$$A_3 = -C\epsilon c^4,$$

and the other quantities A_0, A_2, &c. are of an order higher than ϵ. Hence, for the part of the fluid which lies without the radius γ, we have

$$\phi = -C\left\{(1-\epsilon)\frac{c^2}{r}\cos\theta + \frac{\epsilon c^4}{r^3}\cos 3\theta\right\} \text{ .... } (34),$$

and for the part which lies between that radius and the ellipse we have from (28)

$$\phi = -Cc\{(1-\epsilon)\cos\theta + \epsilon\cos 3\theta\} + C\{(1-\epsilon)\cos\theta + 3\epsilon\cos 3\theta\}z$$
$$-\frac{C}{c}\cos\theta\, z^2 \text{ .... } (35).$$

The value of ϕ given by equation (35) may be deduced from that given by equation (34) by putting $r = c + z$, and expanding as far as to z^2. In the case of the elliptic cylinder then it appears that the same value of ϕ serves for the part of the fluid without, and the part within the radius γ. If the cylinder be moving with a velocity C' in the direction of the minor axis of a section, the value of ϕ will be found from that given by equation (34) by changing the sign of ϵ, putting C' for C, and supposing θ to be measured from the minor axis.

If the cylinder revolve round its axis with an angular velocity ω, the normal velocity of the surface at any point will be $2\omega\epsilon c \sin 2\theta$. Since ϵ^2 is neglected, we may suppose this normal velocity to take place on the surface of a circular cylinder whose radius is c; whence, (Art. 8), the corresponding value of ϕ will be

$$-\frac{\omega\epsilon c^4}{r^2}\sin 2\theta.$$

If we suppose all these motions to take place together, we have only, (Art. 5), to add together the values of ϕ corresponding to each. If we suppose the motion very small, so as to neglect the square of the velocity, we need only retain the terms depending on $\frac{d\omega}{dt}$, $\frac{dC}{dt}$ and $\frac{dC'}{dt}$, in the value of $\frac{d\phi}{dt}$, and we may calculate the pressure due to each separately. The resultant of the pressure due to the term $\frac{d\omega}{dt}$ will evidently be zero, on account of the symmetry of the corresponding motion, while the resultant couple will be of the order ϵ^2, since the pressure on any point of the surface, and the perpendicular from the centre on the normal at that point, are each of the order ϵ. The pressure due to the term $\frac{dC}{dt}$ will evidently have a resultant in the direction of the major axis of a section of the cylinder; and it will be easily proved that the resultant pressure on a length l of the cylinder is $\pi\rho c^2 l(1-2\epsilon)\frac{dC}{dt}$. That due to the term $\frac{dC'}{dt}$ will be $\pi\rho c^2 l(1+2\epsilon)\frac{dC'}{dt}$, acting along the minor axis. If the cylinder be constrained to oscillate so that its axis oscillates in a direction making an angle a with the major axis, and if C'' be its velocity, which is supposed to be very small, the resultant pressures along the major and minor axes will be $\mu(1-2\epsilon)\cos a\frac{dC''}{dt}$ and $\mu(1+2\epsilon)\sin a\frac{dC''}{dt}$ respectively, where μ is the mass of the fluid displaced. Resolving these pressures in the direction of the motion, the resolved part will be $\mu(1-2\epsilon\cos 2a)\frac{dC''}{dt}$, or $\mu(1-\frac{e^2}{2}\cos 2a)\frac{dC''}{dt}$, e being the eccentricity; so that the effect of the inertia of the fluid will be, to increase the mass of the solid by a mass equal to $\mu(1-\frac{e^2}{2}\cos 2a)$, which must be supposed to be collected at the axis.

A similar method of calculation would apply to any given solid differing little either from a circular cylinder or from a sphere. In the latter case it would be necessary to use expansions in series of Laplace's coefficients, instead of expansions in series of sines and cosines.

13. *Motion of fluid in a closed box whose interior is of the form of a rectangular parallelopiped.*

The motion being supposed to begin from rest, the motion at any time may be supposed to have been produced by impact (Art. 4). The motion of the box at any instant may be resolved into a motion of translation and three motions of rotation about three axes parallel to the edges, and passing through the centre of gravity of the fluid, and the part of ϕ due to each of these motions may be calculated separately. Considering any one of the motions of rotation, we shall see that the normal velocity of each face in consequence of it will ultimately be the same as if that face revolved round an axis passing through its centre, and that the latter motion would not alter the volume of the fluid. Consequently, in calculating the part of

ϕ due to any one of the angular velocities, we may calculate separately the part due to the motion of each face.

Let the origin be in a corner of the box, the axes coinciding with its edges, Let a, b, c, be these edges, U, V, W, the velocities, parallel to the axes, of the centre of gravity of the interior of the box, ω', ω'', ω''', the angular velocities of the box about axes through this point parallel to those of x, y, z. Let us first consider the part of ϕ due to the motion of the face xz in consequence of the angular velocity ω'''.

The value of ϕ corresponding to this motion must satisfy the equation

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0 \dots\dots\dots \quad (36),$$

with the conditions

$$\frac{d\phi}{dx} = 0, \text{ when } x = 0 \text{ or } a \dots\dots\dots \quad (37),$$

$$\frac{d\phi}{dy} = 0, \text{ when } y = b \dots\dots\dots \quad (38),$$

$$\frac{d\phi}{dy} = \omega''' \left(x - \frac{a}{2}\right), \text{ when } y = 0 \dots\dots\dots \quad (39),$$

within limits corresponding to those of the box.

Now, for a given value of y, the value of ϕ between $x = 0$ and $x = a$ can be expanded in a convergent series of cosines of $\frac{\pi x}{a}$ and its multiples; and, since (37) is satisfied, the series by which $\frac{d\phi}{dx}$ will be expressed will also hold good for the limiting values of x, and will be convergent. The general value of ϕ then will be of the form $\Sigma_0^\infty Y_n \cos \frac{n\pi x}{a}$. Substituting in (36), and equating coefficients of corresponding cosines, which may be done, since any function of x can be expanded in but one such series of cosines between the limits 0 and a, we find that the general value of Y_n is $C\epsilon^{\frac{n\pi y}{a}} + C'\epsilon^{-\frac{n\pi y}{a}}$, or, changing the constants,

$$Y_n = A_n \left(\epsilon^{\frac{n\pi(b-y)}{a}} + \epsilon^{-\frac{n\pi(b-y)}{a}}\right) + B_n \left(\epsilon^{\frac{n\pi y}{a}} + \epsilon^{-\frac{n\pi y}{a}}\right),$$

when $n > 0$, and for $n = 0$,

$$Y_0 = A_0 y + B_0.$$

From the condition (38) we have

$$A_0 + \frac{\pi}{a} \Sigma_1^\infty n B_n \left(\epsilon^{\frac{n\pi b}{a}} - \epsilon^{-\frac{n\pi b}{a}}\right) \cos \frac{n\pi x}{a} = 0:$$

whence $A_0 = 0$, $B_n = 0$, and, omitting B_0,

$$\phi = \Sigma_1^\infty A_n \left(\epsilon^{\frac{n\pi(b-y)}{a}} + \epsilon^{-\frac{n\pi(b-y)}{a}}\right) \cos \frac{n\pi x}{a}.$$

From the condition (39), we have

$$-\frac{\pi}{a} \Sigma_1^\infty n A_n \left(\epsilon^{\frac{n\pi b}{a}} - \epsilon^{-\frac{n\pi b}{a}}\right) \cos \frac{n\pi x}{a} = \omega''' \left(x - \frac{a}{2}\right).$$

Determining the coefficients in the usual manner, we have

$$A_n = \frac{2a^2\omega'''}{n^3\pi^3} \{1 - (-1)^n\} \left(\epsilon^{\frac{n\pi b}{a}} - \epsilon^{-\frac{n\pi b}{a}}\right)^{-1};$$

whence

$$\phi = \frac{4a^2\omega'''}{\pi^3}\Sigma_0 \frac{1}{n^3}\frac{\epsilon^{\frac{n\pi(b-y)}{a}} + \epsilon^{-\frac{n\pi(b-y)}{a}}}{\epsilon^{\frac{n\pi b}{a}} - \epsilon^{-\frac{n\pi b}{a}}}\cos\frac{n\pi x}{a},$$

putting Σ_0, for shortness, to denote the sum corresponding to *odd* integral values of n from 1 to ∞.

It is evident that the value of ϕ corresponding to the motion of the opposite face in consequence of the angular velocity ω''' will be found from that just given by putting $b - y$ for y, and changing the sign of ω'''; whence the value corresponding to the motion of these two faces in consequence of ω''' will be

$$\frac{4\omega''' a^2}{\pi^3}\Sigma_0 \frac{1}{n^3}\frac{(\epsilon^{\frac{n\pi b}{a}} - 1)\,\epsilon^{-\frac{n\pi y}{a}} + (\epsilon^{-\frac{n\pi b}{a}} - 1)\,\epsilon^{\frac{n\pi y}{a}}}{\epsilon^{\frac{n\pi b}{a}} - \epsilon^{-\frac{n\pi b}{a}}}\cos\frac{n\pi x}{a}.$$

Let this expression be denoted by $\omega'''\psi(x, a, y, b)$. It is evident that the part of ϕ due to the motion of the two faces parallel to the plane yz will be got by interchanging x and y, a and b, and changing the sign of ω''' in the last expression, and will therefore be $-\omega'''\psi(y, b, x, a)$. The parts of ϕ corresponding to the angular velocities ω', ω'', will be got by interchanging the requisite quantities. Also the part of ϕ due to the velocities U, V, W, will be $Ux + Vy + Wz$, (Art. 7), and therefore we have for the complete value of ϕ

$$Ux + Vy + Wz + \omega'''\{\psi(x, a, y, b) - \psi(y, b, x, a)\} + \omega'\{\psi(y, b, z, c) - \psi(z, c, y, b)\} + \omega''\{\psi(z, c, x, a) - \psi(x, a, z, c)\}.$$

According to Art. 7 we may consider separately the motion of translation of the box and fluid, and the motion of rotation about the centre of gravity of the latter; and the whole pressure will be compounded of the pressures due to each. The pressures at the several points of the box due to the motion of translation will have a single resultant, which will be the same as if the mass of the fluid were collected at its centre of gravity. Those due to the motion of rotation will have a single resultant couple, to calculate which we have

$$\phi = \omega'''\{\psi(x, a, y, b) - \psi(y, b, x, a)\} + \&c.$$

Since for the motion of rotation there is no resultant force, we may find the resultant couple of the pressures round *any* origin, that for instance which has been chosen. If now we suppose the motion very small, so as to neglect the square of the velocity, we may find $\frac{d\phi}{dt}$ as if the axes were fixed in space. We have then for the motion of rotation

$$p = -\rho\frac{d\omega'''}{dt}\{\psi(x, a, y, b) - \psi(y, b, x, a)\} - \&c.$$

Hence we may calculate separately the couples due to each of the quantities $\frac{d\omega'''}{dt}$, $\frac{d\omega'}{dt}$ and $\frac{d\omega''}{dt}$. It is evident from the symmetry of the motion that that due to $\frac{d\omega'''}{dt}$ will act round the axis of z, and that the pressures on the two faces perpendicular to that axis will have resultants which are equal and opposite. Also, since $\psi(a, a, y, b) = -\psi(0, a, y, b)$ and $\psi(x, a, b, b) = -\psi(x, a, 0, b)$, it will be seen that the couples due to the pressures on the faces perpendicular to the axes of x and y will be twice as great respectively as those due to the pressures on the planes yz and xz. The pressure on the element $dydz$ of the plane yz will be $p_{,,}dydz$, and the moment of this pressure round the axis of z, reckoned positive when it tends to turn the box from x to y, will be

$$-\rho\frac{d\omega'''}{dt}y\{\psi(0, a, y, b)-\psi(y, b, 0, a)\}dydz.$$

Substituting the values of the functions, integrating from $y=0$ to $y=b$, and from $z=0$ to $z=c$, replacing $\Sigma_0\frac{1}{n^4}$ by its value $\frac{\pi^4}{96}$, and reducing the other terms, it will be found that the couple due to the pressure on the plane yz is

$$\frac{\rho a^3bc}{24}\frac{d\omega'''}{dt}-\frac{8\rho a^4c}{\pi^5}\frac{d\omega'''}{dt}\Sigma_0\frac{1}{n^5}\frac{1-\epsilon^{-\frac{n\pi b}{a}}}{1+\epsilon^{-\frac{n\pi b}{a}}}-\frac{8\rho b^4c}{\pi^5}\frac{d\omega'''}{dt}\Sigma_0\frac{1}{n^5}\frac{1-\epsilon^{-\frac{n\pi a}{b}}}{1+\epsilon^{-\frac{n\pi a}{b}}}.$$

We shall get the couple due to the pressure on the plane xz by interchanging a and b, changing the sign of ω''', and measuring the couple in the opposite direction, or, which is the same, by merely interchanging a and b. Adding together these two couples and doubling their sum we shall find that the couple due to $\frac{d\omega'''}{dt}$ is $-C\frac{d\omega'''}{dt}$, where

$$C=\frac{32\rho c}{\pi^5}\Sigma_0\frac{1}{n^5}\left\{a^4\frac{1-\epsilon^{-\frac{n\pi b}{a}}}{1+\epsilon^{-\frac{n\pi b}{a}}}+b^4\frac{1-\epsilon^{-\frac{n\pi a}{b}}}{1+\epsilon^{-\frac{n\pi a}{b}}}\right\}-\frac{\rho abc}{12}(a^2+b^2)\ \ldots\ldots\ldots\ldots\ (40).$$

Similarly, the couple due to $\frac{d\omega'}{dt}$ will be $-A\frac{d\omega'}{dt}$, tending to turn the box from y to z, and that due to $\frac{d\omega''}{dt}$ will be $-B\frac{d\omega''}{dt}$, tending to turn the box from z to x, where A and B are derived from C by interchanging the requisite quantities. Hence, considering the motions both of translation and rotation of the box, we see that the small motions of the box will take place as if the fluid were replaced by a solid having the same mass, centre of gravity, and principal axes, and having A, B and C for its principal moments. This will be true whether forces act on the fluid or not, provided that if there are any they are of the kind mentioned in Art. 1.

Putting $A_{,}$, $B_{,}$, $C_{,}$, for the principal moments of inertia of the solidified fluid we have

$$C_{,}=\frac{\rho abc}{12}(a^2+b^2).$$

Taking the ratio of C to $C_{,}$, replacing each term such as

$$\frac{1-\epsilon^{-\frac{n\pi b}{a}}}{1+\epsilon^{-\frac{n\pi b}{a}}}\text{ by }1-\frac{2\epsilon^{-\frac{n\pi b}{a}}}{1+\epsilon^{-\frac{n\pi b}{a}}},\text{ putting for }\frac{384}{\pi^5}\Sigma_0\frac{1}{n^5}$$

its approximate value 1·260497, and for $\frac{384}{\pi^5}$ its approximate value 1·254821, and employing subsidiary angles, we have

$$\frac{C}{C_{,}}=1{\cdot}260497\frac{a^4+b^4}{ab(a^2+b^2)}-1{\cdot}254821\left\{\frac{a^3}{b(a^2+b^2)}\Sigma_0\frac{1}{n^5}\text{ versin }2\theta_n\right.$$

$$\left.+\frac{b^3}{a(a^2+b^2)}\Sigma_0\frac{1}{n^5}\text{ versin }2\theta'_n\right\}-1,$$

$$\text{where }\tan\theta_n=\epsilon^{-\frac{n\pi b}{2a}},\quad\tan\theta'_n=\epsilon^{-\frac{n\pi a}{2b}},\text{ so that}$$

$$L\tan\theta_n=10-k\frac{nb}{a},\quad L\tan\theta'_n=10-k\frac{na}{b},\text{ where }k=\cdot6821882.$$

The numerical calculation of this ratio is very easy, on account of the great rapidity with which the series contained in it converge, both on account of the coefficients, and on account of the rapid diminution of the angles θ_n and θ'_n. The values of $\frac{A}{A_{\prime}}$ and $\frac{B}{B_{\prime}}$ will be derived from that of $\frac{C}{C_{\prime}}$ by putting c for a in the first case, and c for b in the second. The calculation of the small motions of the box will thus be reduced to a question of ordinary rigid dynamics.

These results appear capable of being accurately compared with experiment. For this purpose it will only be necessary to attach a box, capable of containing fluid, to a rigid body oscillating as a pendulum. The box may itself form the rigid body. The centre of gravity of the interior of the box should be in a vertical plane passing through the axis of suspension, which will be known by observing whether the position of equilibrium of the whole is affected by filling the box with fluid. The mass, moment of inertia, and depth of the centre of gravity of the solid, including the box, must first be found. The last of these may be found by loading the upper part of the oscillating body till the equilibrium just becomes unstable: the moment of inertia will then be found by means of the time of oscillation when the weight is removed; or else both may be determined by the times of oscillation when the solid is loaded with another of known mass and form and placed in a known position, and again when it is not loaded. The same must then be done when the box is filled with fluid. We shall thus determine the moment of inertia and depth of the centre of gravity of the fluid; and, subtracting the moment of inertia due to the motion of translation of the fluid, we shall thus get that due to the motion of rotation of the box, and thus determine in succession by observation the quantities A, B and C, or any one of them. These quantities might also be determined by making the box oscillate by torsion, and observing the time of oscillation. It must be remembered that the moment of inertia due to the motion of translation of the centre of gravity of the fluid, being capable of being derived from the general dynamical principle, that the motion of the centre of gravity of any system is the same as if the whole mass were collected there, and the external moving forces applied there, is of no use whatever in determining the question of the equality of the pressure in all directions, or that of the amount of friction. It would seem to be most convenient to have the centre of gravity of the fluid *in* the axis of suspension. In this case if M, M', be the masses of the solid and fluid, μ, μ', their moments of inertia, t, t', the times of oscillation, in seconds, when the box is empty and when it is full respectively, h the depth of the centre of gravity of the solid, l the length of the second's pendulum, we have

$$\mu = lt^2 Mh,$$

$$\mu + \mu' = lt'^2 Mh;$$

$$\text{whence } \mu' = l(t'^2 - t^2) Mh.$$

If the centre of gravity of the fluid be at a depth h' below the axis of suspension, we shall have $\mu' = l(t'^2 - t^2) Mh + lt'^2 M'h'$; in this case $\mu' - M'h'^2$ will be the moment of inertia due to the motion of rotation of the box.

When one of the quantities a, b, becomes infinitely great compared with the other, the ratio $\frac{C}{C_{\prime}}$ becomes 1, as will be seen from equation (40). This result might have been expected. When $a = b$ the value of $\frac{C}{C_{\prime}}$ is ·156537.

The experiment of the box appears capable of great variety as well as accuracy. We may take boxes in which the edges have various ratios to each other, and may make the same box oscillate in various positions.

15. *Initial motion in a rectangular box, the several points of the surface of which are moved with given velocities, consistent with the condition that the volume of the fluid is not altered.*

Employing the same notation as in the last case, let $F(x, y)$ be the given normal velocity at any point of the face in the plane xy. Let $\int_0^a \int_0^b F(x, y)\, dx\, dy = Wab$, and let

$$F(x, y) = f(x, y) + W:$$

then, since the normal motion of the above face due to the function $f(x, y)$ does not alter the volume of the fluid, we may consider separately the part of ϕ due to this quantity. For this part we have

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0 \quad \dots\dots\dots\dots\dots\dots (41),$$

with the conditions

$$\frac{d\phi}{dx} = 0, \text{ when } x = 0 \text{ or } a \quad \dots\dots\dots\dots\dots (42),$$

$$\frac{d\phi}{dy} = 0, \text{ when } y = 0 \text{ or } b \quad \dots\dots\dots\dots\dots (43),$$

$$\frac{d\phi}{dz} = 0, \text{ when } z = c \quad \dots\dots\dots\dots\dots\dots\dots (44),$$

$$\frac{d\phi}{dz} = f(x, y), \text{ when } z = 0 \quad \dots\dots\dots\dots\dots (45),$$

within limits corresponding to those of the box.

For a given value of z the value of ϕ from $x = 0$ to $x = a$ and from $y = 0$ to $y = b$ may be expanded in a series of the form

$$\Sigma_0^\infty \Sigma_0'^\infty P_{m,n} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b},$$

the sign Σ referring to m, and Σ' to n: and since the values of ϕ, $\frac{d\phi}{dx}$ and $\frac{d\phi}{dy}$ do not alter abruptly, and equations (42) and (43) are satisfied, it follows that the series by which ϕ, $\frac{d\phi}{dx}$ and $\frac{d\phi}{dy}$ are expressed are convergent, and hold good for the limiting values of x and y. Substituting the value of ϕ just given in (41), equating to zero coefficients of corresponding cosines, and introducing the condition (44), we have, omitting the constant, or supposing $A_{0,0} = 0$,

$$\phi = \Sigma_0^\infty \Sigma_0'^\infty A_{m,n} \{\epsilon^{\frac{p\pi(c-z)}{c}} + \epsilon^{-\frac{p\pi(c-z)}{c}}\} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b},$$

$$\text{where } \frac{p^2}{c^2} = \frac{m^2}{a^2} + \frac{n^2}{b^2}.$$

Determining the coefficients such as $A_{m,n}$ from the condition (45) in the usual manner we have, m and n being > 0,

$$A_{m,n} = -\frac{4c}{\pi pab}(\epsilon^{p\pi} - \epsilon^{-p\pi})^{-1} \int_0^a \int_0^b f(x, y) \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}\, dx\, dy,$$

$$A_{0,n} = -\frac{2}{\pi na}(\epsilon^{\frac{n\pi c}{b}} - \epsilon^{-\frac{n\pi c}{b}})^{-1} \int_0^a \int_0^b f(x, y) \cos \frac{n\pi y}{b}\, dx\, dy^*,$$

with a similar expression for $A_{m,0}$, whence the value of ϕ corresponding to $f(x, y)$ is known. In a similar manner we may find the values corresponding to the similar functions belonging to each of the other faces. If W' be the quantity corresponding to W for the face opposite to the plane xy,

* The function $f(x, y)$ in these integrals may be replaced by $F(x, y)$, since $\int_0^a \int_0^b W \cos \frac{n\pi y}{b} \cos \frac{n\pi x}{a}\, dx\, dy = 0$, unless $m = n = 0$.

and U, U', correspond to W, W', for the faces perpendicular to the axis of x, and if V, V', be the corresponding quantities for y, there remains only to be found the part of ϕ due to these six quantities. Since U, U', are the velocities parallel to the axis of x of the faces perpendicular to that axis, and so for V, V', &c., the motion corresponding to these six quantities may be resolved into three motions of translation parallel to the three axes, the velocities being U, V and W, and that motion which is due to the motions of the faces opposite to the planes yz, xz, xy, moving with velocities $U' - U$, $V' - V$, $W' - W$, parallel to the axes of x, y, z, respectively. The condition that the volume of the fluid remains the same requires that

$$\frac{1}{a}(U' - U) + \frac{1}{b}(V' - V) + \frac{1}{c}(W' - W) = 0.$$

It will be found that the velocities

$$u = \frac{x}{a}(U' - U), \quad v = \frac{y}{b}(V' - V), \quad w = \frac{z}{c}(W' - W),$$

satisfy all the requisite conditions. Hence the part of ϕ due to the six quantities U, U', V, V', W, W', is

$$Ux + Vy + Wz + (U' - U)\frac{x^2}{2a} + (V' - V)\frac{y^2}{2b} + (W' - W)\frac{z^2}{2c}.$$

This quantity, added to the six others which have already been given, gives the value of ϕ which contains the complete solution of the problem.

The case of motion which has just been given seems at first sight to be an imaginary one, capable of no practical application. It may however be applied to the determination of the small motion of a ball pendulum oscillating in a case in the form of a rectangular parallelopiped, the dimensions of the case being great compared with the radius of the ball. For this purpose it will be necessary to calculate the motion of the ball reflected from the case, by means of the formulæ just given, and then the motion again reflected from the sphere, exactly as has been done in the case of a rigid plane Art. 10. In the present instance however the result contains definite integrals, the numerical calculation of which would be very troublesome.

G. G. STOKES.

PEMBROKE COLLEGE,
May, 1843.

X. *Notice on the Occurrence of Land and Freshwater Shells with Bones of some extinct Animals in the Gravel near Cambridge. By* P. B. BRODIE, F.G.S., *of Emmanuel College.*

[Read, *April* 30, 1838.]

THE discovery of recent shells associated with bones of some extinct mammalia, and other animals, is a subject of considerable interest, especially as the same fact has also been noticed in several other distant localities. The shells in question were found in a gravel pit at Barnwell, adjoining the river, in a bed of fine sandy gravel, about fourteen feet from the surface, the whole bed consisting of alternating layers of fine white sand and pebbly gravel, resting upon a thin bed of brown clay; altogether amounting to a thickness of about twenty feet. The stratum in which most of the shells occur is composed of a thin bed of shelly gravel, abounding in many perfect specimens, and comminuted fragments of the same fossils. To this succeeds an equally thin bed of fine white loam, containing shells far more perfect but less numerous. This gravel, though of course derivative, appears to differ from the coarser beds of the same formation; for while the latter chiefly consist of rolled fragments of older rocks, the former, on the other hand, contains but a small proportion of such materials, and appears to be more immediately derived from a finer sediment formed by local inundations. Indeed, many of the terrestrial and aquatic shells are of so fragile and delicate a texture, that they must have been inevitably injured had they been swept away by any *violent* aqueous action. In most of the specimens, the mouths of the Univalves, and the hinges of the Bivalves, are in excellent preservation, whilst the associated bones exemplify the same fact. The shells are also very abundant, and generally of small size; *all* the *genera*, and *most* of the species being identical with those now living, though one or two species do not appear to be so. Among the terrestrial specimens the following genera and species may be enumerated.

Helix hortensis.	Bulimus clavulus.	Pupa umbilicata.
...... carthusiana.	Clausilia.	...... sex-dentata.

The aquatic shells afford examples of the following genera:

Cyclas, a new species.	Valvata obtusa.	Lymnæa auricularis.
	...... spirorbis.	.... glutionosa.
Succinea amphibia.		.... species undetermined.
.... oblonga.	Planorbis marginatus.	
.... species undetermined.	.... and some others.	
Paludina, species undetermined.	Testacellus.	
.... Operculæ of		

The above undetermined species may not, perhaps, have any living representative. The Rev. Leonard Jenyns has decided the Cyclas to be a new species. Seed-vessels of Chara or Gyrogonite, and wood partly charred accompany them.

The bones discovered in the shelly gravel consist of the following specimens. A large tibia and a small molar tooth of an elephant. Tibia of the gigantic ox. Lower portion of the horn of a stag. Tibia of a deer, with teeth and vertebræ of the same animal. From the brown clay forming the basis of the gravel, and overlying the chalk marl, was obtained the pelvis of a small elephant; but no shells occur in this bed. Some of the other localities, in which I have also observed the same facts, are in the neighbourhood of Maidstone in Kent, and Salisbury in

Wiltshire. In the former place a bed of brown clay fills up fissures in the lower green sand, containing bones of mammalia and other animals. The shells accompanying them belong chiefly to the genus Pupa. In the latter locality a thick bed of brown clay affords the bones and teeth of elephants, with remains of the horse and deer, a jaw of a fox, and some others. The only shells hitherto found associated with them belong to the genus Helix. Recent shells also occur with bones of numerous quadrupeds in clay and gravel near Ilford, Essex, where several of the shells appear to be identical with those above mentioned. (See Loudon's Magazine, Vol. IX. p. 263, and Lyell, Vol. III. p. 140.) Recent marine shells have also been discovered by Sir P. Egerton in a bed of gravel in Cheshire, which are described in the second Volume of the Geological Proceedings. From the occurrence then of the same facts in these distant localities, it may be asked, whether any conclusions might be drawn with regard to the *probable* contemporaneous origin of these respective deposits; and what argument might be founded on the excellent preservation of many recent land and freshwater shells associated with bones of some extinct animals, in strata, evidently of diluvial origin.

Since writing the above, I have observed that there are *two distinct* beds containing shells. The uppermost, is the fine, white sandy stratum, containing Helix and Paludina in great abundance, with other shells. While the lower one, is a hard white marl (resembling chalk), charged with numerous Pupa, small Planorbes, some Clausilia, and a very few Seed-vessels of Chara. Large and small fragments of wood abound. This distinction of the two shell beds is necessary to be observed, because they do not contain shells common to both. No Pupa, Chara or wood occur in the upper sandy layer; indeed the general characters of each are very different; one being a fine sandy shelly bed; the other, a hard white marl, and in this latter formation the bones were found. These two beds however lie within a few inches of each other, so that the distinction is chiefly necessary, with reference to the different Testacea and Mollusca which they each contain.

P. B. BRODIE.

EMMANUEL COLELGE,
April 28, 1838.

The following NOTES to the above communication are added by Professor SEDGWICK.

IN a paper by J. Okes, Esq., published in the first Volume of the Cambridge Transactions (p. 175), there is a description of some fossil remains of a beaver dug up from the bed of the Old West Water about three miles south of Chatteris: and in a subsequent communication he described numerous fossil bones found in beds of gravel which extend from Barnwell Abbey to Jesus Common. All the specimens were subsequently deposited in the Woodwardian Museum: and, with those derived from the Barnwell gravel, were some species of land and fresh water shells (Helix hortensis, &c.) well preserved and in a few instances retaining traces of their original colours. Mr Okes considered these shells to belong to the period when the bones and gravel were deposited. But the conclusion admitted of some doubt, as the pits from which the bones were derived gave no clear sections; and it *was just possible* that the shells might have fallen down among the bones (during the progress of the excavations), from the superficial part of the gravel.

Similar phenomena fell under my own notice, a year or two afterwards, while workmen were employed in excavating the foundations of the new houses at the west end of Barnwell. But there was still a difficulty; because the sections did not shew the exact position of the shells, so as to prove that they were strictly contemporaneous with the deposit of the bones. The observations of Mr Brodie have settled this question, and there can now be no doubt that the shells above mentioned were as old as the period of the gravel.

In my last paper, the bases of length and direction were on the unit-line and its perpendicular, and the lengths of these bases were units.

If the logometer be $p + q\sqrt{-1}$, its primitive, (r, ρ), is found from

$$\log r = \frac{\nu p - \mu q}{m\nu - n\mu}, \qquad \rho = \frac{mq - np}{m\nu - n\mu}.$$

We are now to express X^Y: it is convenient to express the radical letter X by its length and direction (x, ξ), and the exponent by its projections, $v + w\sqrt{-1}$. If X^Y, which by definition is $\lambda^{-1}(Y\lambda X)$, be called Z or (z, ζ), we have

$$\log z = (v - bw)\log x - cw\xi, \quad \zeta = (v + bw)\,\xi + aw\log x,$$

$$\text{where } a = \frac{m^2 + n^2}{m\nu - n\mu}, \qquad b = \frac{m\mu + n\nu}{m\nu - n\mu}, \qquad c = \frac{\mu^2 + \nu^2}{m\nu - n\mu}.$$

If we prefer to express the bases of length and direction by *their* lengths and directions, as

$$m + n\sqrt{-1} = (g, \gamma), \qquad \mu + \nu\sqrt{-1} = (k, \kappa),$$

we have

$$a = \frac{g}{k}\,\frac{1}{\sin(\kappa - \gamma)}, \qquad b = \frac{\cos(\kappa - \gamma)}{\sin(\kappa - \gamma)}, \qquad c = \frac{k}{g}\,\frac{1}{\sin(\kappa - \gamma)},$$

which are connected by $ac - b^2 = 1$.

Some mode of expressing X^Y should be contrived, such as $X^Y_{mn\mu\nu}$, which may show its dependence on the arbitrary constants in the bases; this will allow us to reserve X^Y for its common signification, as an abridged form of X^Y_{1001}. But, before proceeding further, I may notice that the logometer of my last paper is not as general as it might be, even on the supposition that X^Y is to have no extended meaning. For if $\kappa - \gamma$ be a right angle, and if $k = g$, then $a = 1$, $b = 0$, $c = 1$, $\log z = v\log x - w\xi$, $\zeta = v\xi + w\log x$, which two last equations simply express that X^Y has the ordinary meaning. That is to say, every result in the last paper remains if, instead of the bases of length and direction being units, they be any equal lines, and if instead of being on the unit-line and its perpendicular, they be on any lines which are at right angles to one another, provided only that the base of direction be a right angle in advance of that of length.

Returning to the most general definition, we have

$$X^Y_{mn\mu\nu} \text{ or } (x, \xi)^{v + w\sqrt{-1}}_{mn\mu\nu} = \epsilon^{(v - bw)\log x - cw\xi + [(v + bw)\xi + aw\log x]\sqrt{-1}}$$

$$= \epsilon^{[v - (b - a\sqrt{-1})w]\log x + [v + (b + c\sqrt{-1})w]\xi\sqrt{-1}} = x^{v - (b - a\sqrt{-1})w}\;\epsilon^{[v + (b + c\sqrt{-1})w]\xi\sqrt{-1}}.$$

Of the three fundamental equations $A^B A^C = A^{B+C}$, $A^B C^B = (AC)^B$ and $(A^B)^C = A^{BC}$, it is instantly seen that the two first are satisfied by this new signification of the exponent; and that they are satisfied independently of the relation between a, b, and c, or $ac - b^2 = 1$. The third is a little more intricate: the formation of $(X^{v + w\sqrt{-1}})^{v' + w'\sqrt{-1}}$ requires us to write $(v - bw)\log x - cw\xi$ for $\log x$ and $(v + bw)\,\xi + aw\log x$ for ξ, v' for v and w' for w in the first or second of the preceding expressions for X^Y. This being done, it is found that in consequence of $ac - b^2 = 1$, the result is precisely the same as if $vv' - ww'$ had been written for v and $vw' + v'w$ for w, without any substitutes being employed for x and ξ. But these last changes turn

$$v + w\sqrt{-1} \text{ into } (v + w\sqrt{-1})(v' + w'\sqrt{-1}).$$

The theory of quantities once called real admits of no extension; for if ξ and w vanish, $x^v_{mn\mu\nu} = \epsilon^{v\log x}$, or x^v. But the following deductions,

$$\epsilon^{\theta\sqrt{-1}}_{mn\mu\nu} = \epsilon^{-b\theta + a\theta\sqrt{-1}}, \quad \epsilon^{\theta\sqrt{-1}} = \epsilon^{\frac{b}{a}\theta + \frac{1}{a}\theta\sqrt{-1}}_{mn\mu\nu},$$

show that the signification of ordinary exponentials involving $\sqrt{-1}$ is completely changed: thus $\epsilon_{mn\mu\nu}^{\theta\sqrt{-1}}$ signifies a line of the length $\epsilon^{-b\theta}$ inclined at the angle $a\theta$ to the unit-line.

Without going further into details we may see that, as before remarked, it is not necessary to retain this extended notion of A^B, since the consequences of the extension can be expressed by the particular case in general use; which cannot be said of AB as compared with $A+B$, or of A^B as compared with AB. This rejection is a generalization of the rejection of all logarithmic bases in favor of ϵ, and the extended definition of A^B is itself a substitution of logarithmic bases in their most general form. For whereas, in the common system, ϵ and $\epsilon^{\sqrt{-1}}$ are the logarithmic bases* employed for ordinary and periodic magnitude, we have, in the system above described, employed

$$\epsilon^{(m+n\sqrt{-1})^{-1}}, \quad \text{and} \quad \epsilon^{(\nu-\mu\sqrt{-1})^{-1}}.$$

Great care will be necessary, in verifying the conclusions, not to confound the meanings of $A^B_{mn\mu\nu}$ and A^B, or the operations performed upon them. Thus the function whose $mn\mu\nu$-logometer is 1, may be represented by

$$\epsilon_{mn\mu\nu}^{\frac{1}{m+n\sqrt{-1}}}, \quad \text{or by} \quad \epsilon^{\frac{\nu}{m\nu-n\mu}-\frac{n}{m\nu-n\mu}\sqrt{-1}},$$

and $\epsilon_{mn\mu\nu}^{\frac{\lambda X}{m+n\sqrt{-1}}} = X$. Without such care, the inquirer will infallibly be led to equations of condition between m, n, μ, and ν, which he will find are satisfied by $m=1$, $n=0$, $\mu=0$, $\nu=1$: that is, he will imagine he has proved the system of my last paper to be necessary.

From the expression of X in terms of its logometer, we derive the following, ϵ meaning $(\epsilon, 0)$;

$$X = (x, \xi) = \epsilon_{mn\mu\nu}^{\log x + \frac{\mu+\nu\sqrt{-1}}{m+n\sqrt{-1}}\xi} = \epsilon_{mn\mu\nu}^{\log x + \left(\frac{b}{a}+\frac{1}{a}\sqrt{-1}\right)\xi}.$$

On this it is to be observed, that the notion formed from the ordinary modes of expression, namely, that in $\epsilon^{p+q\sqrt{-1}}$ there is a peculiar reference to length in p, and to direction in q, is not altogether correct. The imaginary part (it may perhaps be allowed to retain the nominal distinction of real and imaginary) determines the direction, but the length depends upon both parts. The interpretation of $\epsilon_{mn\mu\nu}^{p+q\sqrt{-1}}$ is, that it represents a line of the length ϵ^{p-bq} inclined to the unit-line at an angle aq; or (ϵ^{p-bq}, aq). One case, and one only is indefinite, when $(\mu+\nu\sqrt{-1}) \div (m+n\sqrt{-1})$ is real, that is, when $m=0$, $\mu=0$, or when $n=0$, $\nu=0$, or when $m:n::\mu:\nu$, which last includes the others. In this case the line takes the form $(0, \infty)$ or (∞, ∞) the indefinite character of the result arising from the coincidence of the bases of length and direction; it resembles the attempt in common algebra to form a system of logarithms to the base unity. But when $(\mu+\nu\sqrt{-1}) \div (m+n\sqrt{-1}) = -\sqrt{-1}$, which gives $b=0$, $a=-1$, we find (x, ξ) represented by $x\epsilon_{mn\mu\nu}^{-\xi\sqrt{-1}}$. Here the bases of length and direction are at right angles to one another, but that of length is in advance of that of direction. This case requires that $\mu=n$, $\nu=-m$, and the logometer is $(m+n\sqrt{-1})(\log x - \xi\sqrt{-1})$.

There would be little use in entering into more detail than is necessary to illustrate the general meaning of the symbol A^B. But it must be considered necessary, in all future explanations of the elements of algebra, to point out the complete meaning of this symbol, not only to avoid defective reasoning, but to prevent the student from attaching an undue weight to the connexion of $\sqrt{-1}$ with the representation of direction. It is a strong corroboration of what seems to have been pointed out by the course of the complete science up to the present time, namely, that we must not expect any new *imaginary* or *impossible* quantities. I must own that

* As far as I know the bases actually employed are four, ϵ and $\epsilon^{\sqrt{-1}}$ in analysis as above described, 10 in the facilitation of computations, and $\sqrt[12]{2}$ in the numerical consideration of the musical scale.

I rather expected to find something of the sort in the present inquiry: remembering that the first great difficulty arose from the inverse process to addition, the next from *an* inverse process to multiplication, I should not have been surprised to have found a third in the most general direct and inverse consideration of A^B. But though we are not to look for any new inexplicables from $A+B$, AB, or A^B, it should be remembered that there is a scale of ascent in the fundamental mode of deriving them from one another which does stop anywhere. Addition being obtained, and the general notion of operation, the solution of $\phi(x+1)=\phi x+c$ gives $\phi x = cx$, and introduces multiplication. Next $\phi(x+1)=c\phi x$ gives $\phi x = c^x$, and introduces involution. But $\phi(x+1)=c^{\phi x}$, the solution of which gives the next step, gives for ϕx a function which has not been considered; though its particular cases

$$\phi 1 = a, \quad \phi 2 = c^a, \quad \phi 3 = c^{c^a}, \quad \phi 4 = c^{(c^{c^a})}, \text{ \&c.}$$

are known. If ϕx could be completely inverted, new inexplicables might, and perhaps would arise, either from this or some succeeding case.

A. DE MORGAN.

UNIVERSITY COLLEGE, LONDON,
October 7, 1843.

XII. *On the Measure of the Force of Testimony in Cases of Legal Evidence.* By JOHN TOZER, Esq. M.A., *Barrister-at-Law, Fellow of Gonville and Caius College.*

[Read *Nov.* 27, 1843.]

ON the question of the possibility or advantage of measuring numerically the force of testimony, the opinions which pervade the legal literature of the English language differ almost invariably from the conclusions of science. This paper contains an attempt to trace the effect of those conclusions in their application to a practical example, and to shew that they afford the best means of analysing the processes which are necessarily adopted in such examples. The mere purpose of rendering demonstrated truths more accessible, might seem to assign to the observations which follow a place in professional rather than in scientific literature: it must however be remembered, that practical men are concerned with practical rules, and with principles no further than may be sufficient to render those rules intelligible. The occasional devotion of time to higher pursuits can scarely be regarded by them as other than treasonable to their personal interests; the assertion of the supremacy of science over art they must for the most part leave to the cultivators of science.

The proposition that a moral certainty is a mathematical probability whose numerical measure lies between unity and some definite numerical fraction, puts in issue either directly or indirectly every question that can be raised on the subject treated of in this paper, though the subject itself is of a much more limited extent than the proposition. The vague way in which the processes by which this proposition, and those which must stand or fall with it, can alone be established or disproved, have been described by even the ablest of our legal authorities, removes every feeling of diffidence in approaching the subject. Professor Starkie, in speaking of the mode of estimating the weight of the united testimony of numbers, says, "If definite degrees of probability could be attached to the testimony of each witness, the resulting probability in favour of their united testimony would be obtained not by the mere addition of the numbers expressing the several probabilities, *but by a process of multiplication.*" 1 Starkie, 3rd ed. 554. And in a work there cited occurs this passage: "*On one side of the equation are mentally collected all the facts and circumstances which have an affirmative value; and on the other, all those which either lead to an opposite inference, or tend to diminish the weight or to shew the non-relevancy of all or any of the circumstances which have been put into the opposite scale. The value of each separate portion of the evidence is separately estimated, and, as in algebraic addition, the opposite quantities, positive and negative, are united, and the balance of probabilities is what remains as the ground of human belief and judgment.*" Wills *on Circumstantial Evidence*, 14.

Symbolical language has given expression to no processes of greater refinement and beauty than those employed in the investigations of the theory of probabilities. No elaborate ones are required in this particular application of its principles; but the expression, "a process of multiplication," conveys to the mind no adequate idea of the simplest of them. Subjects which have been deemed worthy of their attention by Laplace and Poisson cannot be thus dismissed.

"The notions of those who have supposed that mere moral probabilities or relations could ever be represented by numbers or space, and thus be subjected to arithmetical analysis, cannot but be regarded as visionary and chimerical." Starkie 571.

"Whenever the probability is of a definite and limited nature, (whether in the proportion of one hundred to one, or of one thousand to one, or any other ratio, is immaterial), it cannot be safely made the ground of conviction; for to act upon it in any case would be to decide, that for the sake of convicting many criminals, the life of one innocent man might be sacrificed." 574.

"The distinction between evidence of a conclusive tendency which is sufficient for the purpose, and that which is inconclusive, appears to be this: the latter is limited and concluded by some degree or other of finite probability, beyond which it cannot go; the former, though not demonstrative, is attended with a degree of probability of an indefinite and unlimited nature" Ibid.

The above short passages are cited as containing a clear enunciation of the propositions dissented from, and not as affording a complete exposition of the author's views, for which the work itself is referred to.

A passage from Lord Brougham's *Natural Theology* is also cited to by Mr. Wills, as including the noble author among the advocates of the truth of the last of these propositions; it does not however appear to do so. If the propositions are true, the conclusions here arrived at must be erroneous.

The expression of the value of a probability numerically is a necessary consequence of any attempt to express that value accurately: if a certain event has been observed to accompany a certain set of appearances more frequently than the appearances have been observed to occur without the occurrence of the event, we may say that a repetition of the appearances creates a probability of the repetition of the event—we may even say that that probability is great or small; but if we wish to say how great or how small, we are immediately forced on the enquiry, how many times have the appearances to our knowledge occurred, and, out of these, how many times has the event accompanied them. That the fraction which expresses the ratio of these numbers measures the probability of the occurrence of the event accompanying the appearances, is a consequence of the definition of the term "probability;" and if the term "moral probability" have any other definition, that definition remains yet to be enunciated.

If the appearances are of ordinary occurrence, or capable of being resolved into others which are so, the fact that the particular combination may never before have been presented to the senses of the person deciding, is not material; the conceiving that if they were repeated a certain number of times the event would accompany them a certain other number of times, is a process essential to the conception of measuring the probability at all. If, again, the appearances afford some probability of the event, but are so unusual that the judgment hesitates to assign the definite numbers it assigns in the previous cases, the process is only varied to this extent: instead of assigning a numerical measure to the probability itself, we assign numerical values to the limits within which it lies. The measure here then is indefinite, but it is so because, to the imperfect experience of the observer, the probability is so; the indefiniteness has not been introduced in the process of measurement: the least value also that the judgment assigns to the measure of the probability may be large enough to measure a moral certainty, or the greatest so small that the probability must in ordinary occurrences be disregarded, without expanding or narrowing the limits through which indefiniteness may range. If the probability be conclusive, its conclusiveness depends on the magnitude of the least possible value of its measure; if it raise but a "light presumption," it would do so if the measure of the highest limit were that of the probability itself. Suppose, for example, a medical witness to assert, that certain appearances had led him to the conclusion that a person had died from taking hydrocyanic acid. To determine then whether the allegation possesses the degree of probability which would warrant our treating the fact alleged as true, we estimate the ability generally of the witness to judge, his opportunities of judging in the particular case, and his sincerity. The phenomenon then that we witness is that of a man possessing the ability and the inclination to speak correctly, which the values we assign to these would confer on this particular witness making this particular allegation; if then in our opinion this phenomenon would in 997 cases out of 1000 be produced by the fact asserted, and in three cases out of 1000 by some other cause, and if we have assured ourselves that to suspend our judgments from a fear of erring no more than three times out of 1000 would be to defeat the purposes for which laws were instituted ; $\frac{997}{1000}$ measures a probability which we consider large enough to warrant decision ; and the testimony of this witness does therefore warrant decision; and it would do no more if in our judgments the witness would be impelled to give his evidence by any other cause than the presence of the acid no more than three times in a million ; the actual value of the probability in such a case, perhaps cannot, and certainly need not, be assigned, but the value of its inferior limit is definite, and its measure is a numerical fraction. The mind of the person deciding may have done no more than perceive that the probability equalled or exceeded in magnitude those on which he habitually decided in affairs of equal importance; but if called on to assign measures to the probabilities he has employed, he must say that his decision would not be withheld from a fear of erring three times in 1000, and that the chance of erring in the case before him was within that limit : the employment of numbers is a consequence of the effort to be definite. If, again, we wish to compare the effect of evidence on different minds, though each may say in a particular instance that enough has or has not been adduced to produce conviction, the answer to the question, how much has been adduced ? or, how much will produce conviction ? is, and is necessarily, a numerical fraction. The conclusiveness or inconclusiveness of evidence is then altogether independent of the definiteness or indefiniteness of the probability it raises ; the only condition necessary to conclusiveness is, that that probability should be measured by a numerical fraction which exceeds some given definite magnitude. As regards criminal cases, the nature of the evidence does not admit that demonstration can be obtained ; we cannot therefore ensure that out of some definite number of persons punished one innocent person will not be punished as guilty ; the only effect of making the standard of conviction indefinite, is to make the number of cases indefinite in which the wrongful decision has occurred; but it leaves us in doubt as to whether the injustice is increased or diminished.

It is humiliating to intellectual pride to admit that our best exertions will not protect us from inflicting wrong on others, but nothing can be gained by shrinking from measuring the extent of our ability to do so. "Selon Condorcet, la chance d'être condamné injustement pourrait être équivalente à celle d'un danger que nous jugeons assez petite pour ne pas même chercher à nous y soustraire dans les habitudes de la vie ; car, dit il, la société a bien le droit, pour sa sûreté, d'exposer un de ses membres à un danger dont la chance lui est, pour ainsi dire, indifférente : mais cette consideration est beaucoup trop subtile dans une question aussi grave. Laplace donne une définition, bien plus propre à éclairer la question, de la chance d'erreur qu'on est forcé d'admettre dans les jugements en matière criminelle. Selon lui cette probabilité doit être telle qu'il y ait plus de danger pour la sureté publique, à l'acquittement d'un coupable, que de crainte de la condamnation d'un innocent." Poisson *sur la Probabilité des Jugements. 5.*

Condorcet assumes that a man has no more fear of dying at 25 than at 20, and that he therefore neglects a probability measured by $\frac{1}{1900}$, and infers that we may neglect this in our decisions. Condorcet, *Probabilité des Décisions.*

If in the term "danger to the public" we include the danger arising from a callousness or indifference to the infliction of wrong, or from a diminution of respect for the laws, the definition of Laplace seems unexceptionable.

In the example taken below, the formula first obtained applies to all facts the truth of which may be established or disproved by experiment; it assumes that the witnesses giving their testimony have no wish to deceive. The peculiarity in facts of this nature is, that the repeating and varying of the experiments tends successively to eliminate the several causes by which the appearances could have been produced, and to leave the fact attested as the only known cause by which they can be accounted for. If the tribunal be competent to judge of the skill and success with which the experiments have been conducted, their detail is submitted to its consideration ; if it be not, the conclusions are partly arrived at by the witnesses themselves, and taken on trust by the tribunal.

The phenomenon then actually witnessed is that of a witness alleging that, from appearances which his experiments have produced, he infers the existence of a certain fact, and the object is to determine the probability of that fact being true. First then we consider, for each separate experiment, on the hypothesis that the fact is true, what is the probability that it would have produced appearances sufficient to convince the mind of the witness, and induce him to give the testimony he has given. We then take each of the known possible causes of such appearances, and similarly calculate the probabilities that each one of those severally would, if it existed, have produced them in such a way as to have impelled the giving of the testimony. And, lastly, the probability of some unknown cause having so acted. The probability of the hypothesis that the alleged fact is the true cause, is then determined by the known processes of the science. If the operations have been conducted in symbolical language, no step has thus far been taken without the sanction of rigid demonstration; the effect has been to resolve the probability whose value is sought into the elementary probabilities of which it is composed. To the next step therefore, which is that of assigning numerical values to the symbols in which the result is expressed, has been given all the facility of which it is capable. In the particular case of persons accused of crime, the minimum value of the probabilities which favour the accusation alone are required, the precise numerical value of their measures never need therefore be assigned. The values which in our judgments those which favour the hypothesis cannot fall short of, and which those that favour any other hypothesis cannot exceed, are all that are necessary to be decided; the result is a number which is not greater than the numerical value of the measure of the probability whose value is sought: and as far as this particular fact is concerned, conviction or acquittal must follow, as this measure does or does not exceed the standard which justifies decision. The actual measure of the value of the probability is left indefinite in magnitude; its least possible value alone is defined, but the assigning of accurate values to the elementary probabilities, and thus defining the actual measure, will not in the slightest degree affect the result.

The next formula applies to allegations of facts the truth of which cannot be tested by experiment; the consideration of the credibility of the witness is also introduced; the modification by which it is made to differ slightly from that given by Poisson, does not affect the principle by which it is obtained. The hypothesis that the fact alleged is true will account for its being alleged, first, when the witness is neither deceived, nor intending to deceive; and secondly, when both the one and the other, provided that among the various allegations which he may make for the purpose of deceiving, he should chance to make that which is in fact true. The various ways in which he may be deceived without intending to deceive, endeavour to deceive without being himself deceived, and being himself deceived also endeavour to deceive without alleging the fact which did occur; all suggest hypotheses which will or may, with some degree of probability, account for the testimony being given, though the fact which it alleges is not true. The probability that the hypothesis which assumes the fact alleged to be true is the correct one, is then as before given by the scientific process, and this whether its truth be alleged by one or more witnesses, or alleged by some and denied by others. The antecedent probability, of the fact alleged having occurred, is also taken account of in the formula.

The same process applies to ascertaining the probability that a fact is true which is alleged, but which is not material to the issue, of which an example also occurs. We then have a witness alleging a fact the probability of whose truth we have measured; and also other facts, the probability of whose truth we wish to measure; and the former modifies the values of the probabilities, that the witness deceives, or is deceived, which are involved in the equations which express the latter.

When the measures of the probabilities of those facts which must be *proved* to sustain the accusation have been ascertained, their product will measure the probability of a series of facts being true, from which the truth of the accusation is an inference; the probability of the accusation being true will therefore be this product, or this product multiplied by the fraction which expresses the probability of the inference being true, on the assumption that all the facts of the series are so, as the inference is or is not a necessary one: and the numerical values of the component fractions, or of their limits being assigned a numerical measure of the probability of the accusation being true, or of its inferior limit, will be obtained; and the evidence will or will not warrant conviction as this number does or does not exceed the certain prescribed value; and whether the precise value be or be not definitely assigned, that is, whether the probability be definite or indefinite, will be immaterial, so long as this condition is fulfilled.

In civil cases, the questions to be decided having been elicited by the parties in their pleadings, the value of the evidence by which they are to be determined is estimated in the same way; but it will frequently be necessary to assign the numerical values with greater accuracy. The following paragraph applies to such cases, and seems also to involve an admission of all that is contended for in favour of scientific investigation. "In some instances, nevertheless, where from paucity of circumstances the usual means of judging of the credit due to conflicting witnesses fail, it is possible that the abstract principles adverted to may operate by way of approximation, especially in those cases where the decision is to depend on a mere preponderance of evidence." Starkie 554. A paucity of circumstances or incompleteness of data is what distinguishes the evidence in favour of events which are merely probable, from that which supports those which are certain, and it is the business of the science to determine the probability of the truth of the event from the data which are offered to support or disprove it, however limited in extent these data may be. When the numerical measure of this probability is precisely $\frac{1}{2}$, the data are insufficient for decision, and in no other case; in criminal cases, this *punctum indifferens* is claimed by the legal presumption in favour of innocence.

If, therefore, in a case where the mere preponderance is to decide, we obtain a result by assigning to the probabilities which favour the claim of A the least values of which in our judgments they are capable, and to those which favour the claim of B the greatest values of which in our judgments they are capable; and another result, from the greatest which favour A's claim, and the least which favour B's; then if each of these exceed $\frac{1}{2}$, the decision is in favour of A, and if each be less than $\frac{1}{2}$, in favour of B; but if one be greater and the other less than $\frac{1}{2}$, more accurate values must be assigned to the numerical limits, till both the limiting values of the probability be made to exceed or fall short of $\frac{1}{2}$, or till on assigning what in our judgments are correct values, a result precisely equal to $\frac{1}{2}$ is obtained; in which latter case no decision can be arrived at. The only peculiarity then in a case whose decision must depend on a mere preponderance of evidence is this, that a more accurate estimate of the probabilities it involves must be made.

A consideration of the investigations by which these remarks are illustrated, will shew that the mode of estimating the force of evidence employed in a court, is a process which algebraic investigation analyses, and of which it explains the theory; and an approximation, (in most cases, scientifically speaking, a rude one,) to a result which is obtained with accuracy by assigning numerical values to the algebraic symbols. The complication which exhibits itself in the algebraic process is in the nature of the subject, and is not in any degree introduced by the operation employed. The difficulties are difficulties which belong to the act of in any way eliciting truth from a complicated series of circumstances; the practical process, to a certain extent, evades, and necessarily evades, these; the algebraic encounters them, and resolves them into their elements. The employment of symbolical language facilitates the processes of deductive reasoning, but does not change them; the assigning of numerical measures to the probabilities involved defines with accuracy their magnitudes, but in nowise modifies them.

Again, the analytical process does not exclude considerations other than those which result from the bare probabilities. Presumptions of law may be adopted in its formulæ, and these may be dictated by reasons of policy, or other motives, as well as by the necessity for substituting approximations in practice. They are inferences to which legislative enactment or judicial decision has attached the legal consequences which properly belong to facts, and analysis therefore assigns to them the measure of certainty. At the commencement of a criminal proceeding the law presumes that the accused is innocent; and the analyst therefore assigns unity as a measure of the probability of his innocence, though it may merely represent the confidence which the state reposes in the integrity of its individual members. The claim to the property in waste lands beside a road is advanced by the owner of the adjoining freehold, with a probability in favour of its justice measured by unity, which must be reduced below one half by any claimant who would deprive him of the benefit of the presumption. The consequences attached by the statutes of limitation to the expiration of the periods which they assign, cause 1 or 0 to be employed as the measures of probabilities imperceptibly near in value to those to which by the non-expiration of the periods 0 or 1 would be immediately before assigned. Some legal presumptions have, however, the effect of modifying the probability, that the inference which they establish is a just one; it is perhaps immaterial whether that raised by the production of a subsequent receipt in favour of the payment of previous rent be absolute or capable of being rebutted. If it were absolute, and generally known to be such, the knowledge that a conclusive presumption existed would diminish the probability of such a receipt being given when any previous rent was unpaid. In the presumptions raised in criminal cases against the innocence of a prisoner, the probability that the inference is just can never be less than that which justifies conviction.

The presumption of guilt in the case of stolen property of which the possession by an accused party is unaccounted for is defined by decisions on actual cases, and becomes more accurately so as the number of these decisions is increased: the test of consistency among these appears to be simply this, that for each case the probability that the guilt of the accused is the cause of the unaccounted-for possession of the property should have the same numerical measure.

Proceeding to the investigation of the reasoning processes by the algebraic solution of an example. In a case of alleged poisoning by arsenic, to determine from the testimony of the witnesses the probability of the presence of the poison.

Let there be n witnesses, who respectively allege, with a greater or less degree of confidence, that they discovered $\dot{A}s$; and m others that they were unable to do so; and suppose there is no doubt about the veracity of any of them.

An ordinary jury is not competent, from a detail of the processes of experiment, to decide on the success with which they have been conducted. The phenomenon, therefore, which they witness is the delivery of the testimony by a number of witnesses, whose respective abilities to judge it is a part of their duty to estimate.

In the case where $\dot{A}s$ is present:

Let $p_1 \ldots p_n$ be the probabilities that the first n witnesses would elicit from its presence such appearances as to induce them to allege its presence.

$q_1 \ldots q_m$ that the m latter ones would do so.

Then $1 - p$, $1 - q$ would be the probabilities that they would not succeed in doing so.

Where $\dot{A}s$ is not present:

Let $r_1, r_2 \ldots r_n$ be the probabilities that some other substance has caused the appearances in the case of the first n witnesses.

$s_1, s_2 \ldots s_m$ in the case of the latter m witnesses.

Then $1 - r$, $1 - s$ are the probabilities that appearances causing the testimony to be given would not be exhibited by any other substance than $\dot{A}s$.

Then if P be the probability that $\dot{A}s$ was present,

$$P = \frac{p_1 \ldots p_n . (1 - q_1) \ldots\ldots (1 - q_m)}{p_1 \ldots p_n . (1 - q_1) \ldots (1 - q_m) + r_1 \ldots r_n (1 - s_1) \ldots (1 - s_m)}.$$

If the jury were capable of judging of the evidence as furnished by the immediate result of the experiments, selecting among the various causes of appearances which might be mistaken for those produced by $\dot{A}s$, would be performed by its members instead of by the witnesses, or as well as by them. In the particular case, antimony, the persalts of tin, and probably some other substances, exhibit with some tests what to inferior skill would be such appearances.

Suppose there are n such causes of fallacious results, and let m several experiments be made,

$p_1, q_1, r_1,$ be the respective probabilities that *As*, if it were present, and each of the other substances severally, if it were present, would produce the appearances witnessed by the application of any one test.

t_1 that some substance, other than those known and enumerated, would do so if it were present.

$p_2, q_2 \ldots t_2 \ldots\ldots\ldots p_n\, q_n \ldots t_n$ the corresponding probabilities for the other tests.

P the probability that *As* is present.

$$\text{Then } P = \frac{p_1 \cdot p_2 \ldots\ldots p_n}{p_1 \ldots p_n + q_1 \ldots q_n + \&c. \ldots\ldots + t_1 \cdot t_2 \ldots t_n}.$$

If this substance exist in moderate quantity, and even an ordinary degree of skill be employed, the experiments may be varied so as to produce appearances which could not have been produced by some one or more of the causes other than the presence of *As*, and therefore a factor 0 will be successively introduced into the terms $q_1 \ldots q_n$, $r_1 \ldots r_n$, &c. of the denominator, and the expression is reduced to

$$P = \frac{1}{1 + \frac{t_1}{p_1} \cdots \frac{t_n}{p_n}}.$$

It may be observed generally, that it is the presence of this term $\frac{t_1}{p_1} \cdots \frac{t_n}{p_n}$ in the denominator representing possible hypotheses, yet unthought of, that distinguishes the proof of a physical fact from a mathematical demonstration.

The successive elimination of the known causes of error is precisely that which common sense employs in arriving at a moral certainty; when this cannot be effected, the previous expression remains, and the probability of the fact alleged being true is arrived at by assigning numerical values to its elementary probabilities.

The evidence which is the subject of the following formulæ is that, or nearly that which was given in a case which occurred of a woman who was accused of having caused the death of her husband, by administering *As*; it is merely used as an illustration, and therefore no particular pains is taken to state the evidence very accurately. The death and its cause were not disputed, the probabilities therefore of the presence of *As*, and of its having caused the death, are taken in the investigation as measured by 1.

The first witness, whose evidence is here considered, alleged that she had seen the accused on the morning of a particular day making some pills.

Consider, therefore, first, the probability of a fact being true which depends for its evidence on the testimony of a single witness. In such a case the allegation may have been made, either because the event alleged took place, and the witness saw and believed it to do so; or because the witness believed it to have taken place, though it did not in fact do so; or because the witness was actuated by a wish to deceive, and made the allegation without believing in its truth. Call the event alleged E_1, and as a convenient mode of expressing the probabilities involved in the investigation, let there be $n - 1$ other events $E_2 \ldots\ldots E_n$, which include, first, all those by the belief in the occurrence of which the disposition, on the part of the witness, to make the particular allegation could be influenced, whether they might in fact have occurred or not: and secondly, all those which the witness might be induced to allege on the particular subject, without believing them whether they could or could not have occurred.

Take then,

$$\frac{h_1}{\Sigma h}, \quad \frac{h_2}{\Sigma h}, \, \dots\dots\dots \, \frac{h_n}{\Sigma h},$$

the respective probabilities of the happening of the events $E_1 \dots E_n$ as derived from our knowledge of the nature of the events themselves;

$$\frac{b_1}{\Sigma b}, \quad \frac{b_2}{\Sigma b}, \, \dots\dots\dots \, \frac{b_n}{\Sigma b},$$

the probabilities that they will be respectively believed, the witness being deceived;

$$\frac{a_1}{\Sigma a}, \quad \frac{a_2}{\Sigma a}, \, \dots\dots\dots \, \frac{a_n}{\Sigma a},$$

the probabilities that they will respectively be alleged, the witness not believing the one alleged to have occurred;

u the probability that the witness is not deceived;

v that the testimony is not given with a knowledge that it is false;

p_1 that the occurrence of E_1 would cause the allegation to be made;

p_i that the occurrence of E_i any one of the events $E_2 \dots E_n$ would do so;

π_1 that the fact alleged is true.

Now the occurrence of E_1 will cause the allegation to be made, if the witness be neither deceived nor intending to deceive, of which the probability is uv; and also, if both the one and the other, provided the event chosen for the purpose of deceiving be that which in fact occurred.

The probability of the hypothesis is $(1 - u)(1 - v)$, and the probability of the particular mode of being deceived being the believing in the occurrence of some one of the events E_r, other than E_1 is $\frac{b_r}{\Sigma b - b_1}$, since E_1 cannot be believed, but if E_r be believed, the probability that E_1 will be alleged, is $\frac{a_1}{\Sigma a - a_r}$, since E_r will not be alleged; the probability therefore that E_r will be believed, and E_1 alleged is $\frac{a_1}{\Sigma b - b_1} \cdot \frac{b_r}{\Sigma a - a_r}$.

And the whole probability that E_1 will be alleged on the hypothesis is

$$\frac{a_1}{\Sigma b - b_1} \left\{ \Sigma \frac{b}{\Sigma a - a} - \frac{b_1}{\Sigma a - a_1} \right\}.$$

Hence,

$$p_1 = uv + (1 - u)(1 - v) \frac{a_1}{\Sigma b - b_1} \left\{ \Sigma \frac{b}{\Sigma a - a} - \frac{b_1}{\Sigma a - a_1} \right\}.$$

Again, the occurrence of E_i will cause the allegation to be made, first when the witness is deceived, and does not intend to deceive, but believes E_1 to have occurred.

The probability of the hypothesis is $(1 - u)v$.

that E_1 will be believed, and therefore alleged $\frac{b_1}{\Sigma b - b_i}$.

Secondly, when not deceived, but intending to deceive.

The probability of the hypothesis is $u(1 - v)$;

that E_1 will be alleged $\frac{a_1}{\Sigma a - a_i}$.

Thirdly, when both deceiving and deceived provided among the modes of deceiving the allegation of the occurrence of E_1 be not chosen.

The probability of the hypothesis is $(1-u)(1-v)$;

that the belief in the occurrence of E_r is the mode of being deceived, $\dfrac{b_r}{\Sigma b - b_i}$;

that E_1 will be alleged, E_r being believed, $\dfrac{a_1}{\Sigma a - a_r}$;

that E_r will be believed and E_1 alleged, $\dfrac{a_1}{\Sigma b - b_i} \cdot \dfrac{b_r}{\Sigma a - a_r}$;

the whole probability, on the hypotheses, that E_1 will be alleged, is

$$\frac{a_1}{\Sigma b - b_i}\left\{\Sigma \frac{b}{\Sigma a - a} - \frac{b_1}{\Sigma a - a_1} - \frac{b_i}{\Sigma a - a_i}\right\}:$$

E_1 and E_i being excluded in the several values of E_r, because one is alleged and the other happened.

Hence,

$$p_i = (1-u)v\frac{b_1}{\Sigma b - b_i} + u(1-v)\frac{a_1}{\Sigma a - a_i} + (1-u)(1-v)\frac{a_1}{\Sigma b - b_i}\left\{\Sigma \frac{b}{\Sigma a - a} - \frac{a_1}{\Sigma a - a_1} - \frac{a_i}{\Sigma a - a_i}\right\}.$$

And
$$\pi_1 = \frac{p_1 \dfrac{h_1}{\Sigma h}}{\dfrac{p_1 h_1}{\Sigma h} + \Sigma \dfrac{p_i h_i}{\Sigma h}} = \frac{1}{1 + \dfrac{1}{p_1 h_1} \Sigma p_i h_i}.$$

The value of $\Sigma p_i h_i$ being

$$\Sigma p_i h_i = (1-u)vb_1\left\{\Sigma \frac{h}{\Sigma b - b} - \frac{h_1}{\Sigma b - b_1}\right\} + u(1-v)a_1\left\{\Sigma \frac{h}{\Sigma a - a} - \frac{h_1}{\Sigma a - a_1}\right\}$$

$$+(1-u)(1-v)a_1\left[\left\{\Sigma \frac{b}{\Sigma a - a} - \frac{b_1}{\Sigma a - a_1}\right\}\left\{\Sigma \frac{h}{\Sigma b - b} - \frac{h_1}{\Sigma b - b_1}\right\} - \left\{\Sigma \frac{ah}{(\Sigma b - b)(\Sigma a - a)} - \frac{a_1 h_1}{(\Sigma b - b_1)(\Sigma a - a_1)}\right\}\right].$$

It is here supposed that the occurrence of an event, which is not believed to have occurred, will not affect the disposition to believe in the occurrence of any one event which did not occur in preference to any other; and that the disposition to allege the occurrence of any event which is not believed to have occurred in preference to any other, is independent of the event which is believed; if this assumption be not made, the values of a will be different for different values of r, and the values of b different for the different values of i, but the process will be the same.

The expression for $\Sigma p_i h_i$ is adapted to the case of all the circumstances by which the belief or veracity of the witness can be influenced being known; when the data are less complete the expression becomes much simplified, the result of course becoming less accurate, as from the insufficiency of the data it must do.

Taking

$$\frac{1}{h_1}\Sigma p_i h_i = (1-u)v\left\{\Sigma \frac{h}{h_1}\cdot\frac{b_1}{\Sigma b - b} - \frac{b_1}{\Sigma b - b_1}\right\} + u(1-v)\left\{\Sigma \frac{h}{h_1}\frac{a_1}{\Sigma a - a} - \frac{a_1}{\Sigma a - a_1}\right\}$$

$$+ (1-u)(1-v)\left[\left\{\Sigma \frac{b}{\Sigma a - a} - \frac{b_1}{\Sigma a - a_1}\right\}\left\{\Sigma \frac{h}{h_1}\frac{a_1}{\Sigma b - b} - \frac{a_1}{\Sigma b - b_1}\right\}\right.$$

$$\left. - \left\{\Sigma \frac{h}{h_1}\frac{a_1 . b}{(\Sigma b - b)(\Sigma a - a)} - \frac{a_1 b_1}{(\Sigma a - a_1)(\Sigma b - b_1)}\right\}\right].$$

If the data be so incomplete, that among the events by the occurrence of which, either the belief of the witness, or his disposition to allege one fact in preference to another, is influenced, no reason be afforded for thinking that one rather than another has occurred, $h = h_1$; and therefore $\frac{h}{h_1}$ disappears from the formula.

If among the events, the belief in which may have prompted the allegation, no reason be shewn why one rather than another should be believed, $b = b_1$; and the multiplier of $(1-u)v$ becomes $\frac{\Sigma b - b}{\Sigma b - b} = 1$.

Similarly, if no reason be shewn why the witness in attempting to deceive should make any particular allegation rather than another, $\Sigma \frac{a_1}{\Sigma a - a} - \frac{a_1}{\Sigma a - a_1} = 1$.

And lastly, if in the case of a witness both deceived and intending to deceive, if there be no reason why the probability that he would allege any one fact should differ from the probability that he would believe any other, $a = b$; and the multiplier of $(1-u)(1-v)$ becomes $\left(1 - \frac{a}{\Sigma a - a}\right)$.

With these hypotheses we therefore have

$$\frac{1}{h_1}\Sigma h_i p_i = (1-u)v + (1-v)u + (1-u)(1-v)\left(1 - \frac{a}{\Sigma a - a}\right).$$

And with the same assumptions,

$$p_1 = uv + (1-u)(1-v)\frac{a}{\Sigma a - a}.$$

And

$$\frac{1}{p_1 h_1}\Sigma h_i p_i = \frac{\left(\frac{1}{u} - 1\right) + \left(\frac{1}{v} - 1\right) + \left(\frac{1}{u} - 1\right)\left(\frac{1}{v} - 1\right)\left(1 - \frac{a}{\Sigma a - a}\right)}{1 + \left(\frac{1}{u} - 1\right)\left(\frac{1}{v} - 1\right)\frac{a}{\Sigma a - a}} \quad \ldots\ldots\ldots\ldots (A).$$

The next material allegation was made by another witness to the effect, that she saw the accused exchange some pills which she had procured for others: the evidence of this fact, as of the former, is contained in the testimony of a single witness; but the antecedent probability of its occurrence is different as we do or do not believe that previously alleged. If then π_2 be the probability that this allegation is true,

$$\pi_2 = \frac{1}{1 + \frac{1}{p_1\{h_1\pi_1 + h_1'(1-\pi_1)\}}\Sigma p_i\{h_i\pi_1 + h_i'(1-\pi_1)\}},$$

the previous notation being preserved and adapted as regards the value of its symbols to this particular allegation,

$$\frac{h_1}{\Sigma(h+h')}, \ldots\ldots\ldots \frac{h_n}{\Sigma(h+h')}$$

being the probabilities of the occurrence of $E_1 \ldots E_n$, on the supposition that the previous fact is true, and

$$\frac{h_1'}{\Sigma(h+h')}, \ldots\ldots\ldots \frac{h_n'}{\Sigma(h+h')},$$

on the supposition that it is not true.

Among the means of assigning numerical values to the probabilities of the accuracy and sincerity of a witness, the comparison of the allegations of different witnesses as to immaterial facts is one of the most important. This witness also alleged, that she saw the accused procure the pills for which she substituted others from a surgeon, the surgeon however alleged, that if she had done so, an entry should appear in a book that he produced, which entry did not appear.

Let p_1 p_i be determined as in the value of π_1, and let

e be the probability that the surgeon omitted the entry from design ;

f that he did so from neglect ;

Q that the fact alleged is true.

Then the truth of the allegation is consistent with the non-appearance of the entry ;

firstly : if he designed the omission, of which the probability is e ;

secondly : if he did not design the omission, but neglected to make the entry of this, the probability is $(1 - e)f$.

Again, whatever may have caused the allegation, any hypothesis which excludes its truth may be taken also to exclude the possibility of an entry being made ; and therefore 1 will measure the probability of its not appearing, on every hypothesis but that of the fact having occurred.

$$\text{Hence } Q = \frac{1}{1 + \frac{1}{h_1 p_1 \{e + f(1 - e)\}} \Sigma h_i p_i}.$$

Referring to the expression for π_1 the probability of a fact being true, which has no other support than the testimony of one witness, we see that the values of u and v which satisfy the equation

$$Q = \frac{1}{1 + \frac{1}{p_1 h_1} \Sigma p_i h_i},$$

are those which they would possess if the probability Q were raised by the testimony of this witness alone, and this equation therefore affords the means of correcting the values of those quantities.

The next independent fact was, that the accused bought $\dot{A}s$; it depended for its evidence on the testimony of a single witness ; if therefore π_3 measure the probability that she did so. π_3 will be determined by the formula for π_1 ; the numerical values being adapted to the particular allegation.

The witness, who spake to the making of pills, also alleged that she saw some given by the accused to the deceased ; and that she herself took one of them which produced effects similar to those produced by $\dot{A}s$: as far as this testimony is concerned, the probability that poisonous pills were administered is compounded of the probabilities that any were administered, and that those given, if any were so, were poisonous. In this example it is assumed, that the probability that poisonous pills were given by the accused to the deceased is, as far as the testimony of this particular witness is concerned, the same as the probability that her previous allegation is true. or π_1.

Let then P_1 be the probability that the accused knowingly possessed poison,

P_2 that she administered it,

P_g the probability of guilt.

Then, as far as these facts are concerned,

$$P_g = P_1 . P_2.$$

Now each of the facts, whose probabilities are measured by π_1, π_2, π_3 and π, afford some probability that each of the facts whose probabilities are measured by P_1, P_2 is true, and the falsehood of any number of them less than the whole does not render either P_1 or P_2 0. The complete expression for each of these quantities will therefore be,

$$\pi_1^2 \pi_2 \pi_3 + k_1 \pi_1^2 \pi_2 (1-\pi_3) + k_2 \pi_1^2 (1-\pi_2) \pi_3 + k_3 (1-\pi_1) \pi_1 \pi_2 \pi_3$$
$$+ k_4 (1-\pi_1)^2 \pi_2 \pi_3 + k_5 \pi_1 (1-\pi_1)(1-\pi_2) \pi_3 + k_6 \pi_1 (1-\pi_1) \pi_2 (1-\pi_3) + k_7 \pi_1^2 (1-\pi_2)(1-\pi_3)$$
$$+ k_8 (1-\pi_1)^2 \pi_2 (1-\pi_3) + k_9 (1-\pi_1)^2 (1-\pi_2) \pi_3 + k_{10} \pi_1 (1-\pi_1)(1-\pi_2)(1-\pi_3).$$

Both the facts being certain when all the circumstances concur, the factor of the first term is 1; k_1, k_2, k_3 are the probabilities that the inferences are true when three only out of the four elementary facts concur, and the remaining one is false; k_4, k_5, k_6, k_7, when two are true and two false; and k_8, k_9, k_{10} when one only is true; k_1, &c. are not or not necessarily the same in the value of P_1 as in that of P_2.

The evidence in this case is so far complete, and would or would not warrant the conviction of the accused, as P_g did or did not exceed or equal the standard of conviction; there was however in the particular case a subsequent chain of facts spoken to.

First, a witness alleged that he sold the accused *As* after the administering first spoken to: if ρ_1 be the probability that this allegation is true, ρ_1 will be determined by the formula for π_1.

Let Q_1 be the probability of the possession of *As* by the accused, with knowledge, after this allegation.

Then $Q_1 = 1 - (1 - P_1)(1 - \rho_1)$.

With regard to the administering subsequent to the second purchase, three witnesses severally alleged, that they saw the accused administer a white powder, whose appearance, from their description, corresponded with that of *As*.

For the probability that this fact is true, let q_1, r_1, s_1 be the respective values of p in the formula for π_1 for each of these witnesses respectively, and let ρ_2 be the probability sought. Then, preserving the remainder of the notation,

$$\rho_2 = \frac{1}{1 + \frac{1}{q_1 r_1 s_1} \Sigma \frac{h}{h_1} q_i r_i s_i}.$$

But the fact of possession, with knowledge, of which the probability is Q_1, concurring with the admitted cause of death, affords, independently of the last fact, some probability of the second administering.

Hence, if Q_2 be this latter probability,

$$Q_2 = \{l_1 \rho_2 + l_2 (1 - \rho_2)\} . Q_1 + l_3 (1 - Q_1) . \rho_2;$$

l_1, l_2, and l_3 being the respective probabilities that *As* was administered, when it was possessed; and something like it was administered, when it was simply possessed, and simply when something like it was administered; the cause of the death being in all the cases assumed.

After this second series of facts, we get

$$P_g = 1 - (1 - P_1 P_2)(1 - Q_1 Q_2) = P_1 P_2 + Q_1 Q_2 (1 - P_1 P_2).$$

The numerical values below are assigned for the purpose of completing the illustration, and not with a view to obtain the actual numerical result in the particular case, the assigning of those values is no part of the scientific process, but is determined by a consideration of the situation and character of the witnesses, and of the manner in which they give their testimony.

The operation is also completed for the purpose of shewing, by the attempt to assign numerical values, that the practical approximation to a correct result must necessarily be a rude one. Though the elementary probabilities are expressed by low numbers, the resulting numbers rapidly become very large; and to assign at once the value of the resulting probability, without the assistance of the processes of calculation, would be necessarily to assign them very inaccurately; and the process of at once determining the consequences of that value must be affected with at least as great a chance of error. We may, perhaps, in criminal cases, make as small as we please the chance of an innocent person being convicted; but it must be either by increasing the chance of a guilty person escaping, or by rendering the practical process more perfect; if we were to conceive the elementary probabilities to be elicited by the skill of science, and presented to the jury as separate issues, they would then have to decide on simple facts instead of on complex series of facts, and the remainder of the process would be logical deduction, and therefore would exclude the possibility of error. A result so obtained would possess all the accuracy of which the subject is capable: by as much as the practical process differs from this, by so much is it, as far as mere accuracy is concerned, inferior: the difference is the price of practicability.

In the value of π_1

$$u \nless \frac{10}{11} \ngtr 1, \quad \therefore \ v \nless \frac{100}{101} \ngtr 1, \quad \therefore \ \frac{a}{\Sigma a - a} \ngtr \frac{1}{500} \nless 0.$$

$$\text{Then } \frac{1}{p_1 h_1} \Sigma p_i h_i \ngtr \frac{111}{1000},$$

$$\text{and } \pi_1 \nless \frac{1000}{1111}.$$

For determining the value of π_2.

$$\text{First, } Q = \frac{1}{1 + \dfrac{1}{h_1 p_1 \{e + f(1-e)\}}} \Sigma h_i p_i, \quad \frac{1}{u} - 1 \ngtr \frac{1}{200} \nless 0; \quad \left(\frac{1}{v} - 1\right) \ngtr \frac{1}{100} \nless 0; \quad \frac{a}{\Sigma a - a} \ngtr \frac{1}{100} \nless 0.$$

The values of u and v being assigned independently of this particular allegation.

$$\text{Then } \frac{1}{h_1 p_1} \Sigma h_i p_i \ngtr \frac{301}{20000},$$

$$\text{and } e \nless \frac{3}{4} \ngtr 1, \quad f \nless \frac{1}{2} \ngtr 1,$$

$$\text{and } \frac{1}{h_1 p_1 \{e + f(1-e)\}} \Sigma h_i p_i \ngtr \frac{43}{2500}, \quad Q \nless \frac{2500}{2543}.$$

Since Q differs so little from 1, the values of u and v are not materially diminished by the evidence as to this collateral fact.

$$\text{Assuming, therefore, } \frac{1}{u} - 1 = \frac{1}{200}, \text{ and } \frac{1}{v} - 1 = x, \quad \frac{a}{\Sigma a - a} = \frac{1}{100},$$

$$\frac{43}{2500} = \frac{100 + x(20000 + 99)}{20000 + x},$$

$$\text{whence } x \ngtr \frac{1}{82}.$$

Employing then this value of $\frac{1}{v} - 1$ in the expression for π_2, which is

$$\frac{1}{1 + \dfrac{1}{\left\{\pi_1 + \dfrac{h_1'}{p_1}(1 - \pi_1)\right\} p_1 h_1}} \Sigma h_i p_i,$$

$$\text{and putting } \frac{a}{\Sigma a - a} = \frac{1}{50}, \text{ and } \frac{h_1'}{h_1} \nless \frac{1}{100},$$

$$\frac{1}{p_1 h_1 \left\{\pi_1 + \frac{h_1'}{h_1}(1 - \pi_1)\right\}} \Sigma h_i p_i \ngtr \frac{314413}{16418204},$$

$$\text{and } \pi_2 \nless \frac{1641}{1673}.$$

For π_3 $\quad u \nless \frac{3}{4} \ngtr 1, \; v = 1,$

$$\frac{1}{h_1 p_1} \Sigma h_i p_i \ngtr \frac{1}{3},$$

$$\text{and } \pi_3 \nless \frac{3}{4}.$$

For the values of P_1, P_2, then $\pi_1 \nless \frac{1000}{1111}$, $\pi_2 \nless \frac{1641}{1673}$, $\pi_3 \nless \frac{3}{4}$.

And for P_1 alone, substituting the values

$$k_1 \nless \frac{99}{100}, \quad k_2 = 1, \quad k_3 = 2, \quad k_4 = 1, \quad k_5 = 2, \quad k_6 = \frac{18}{10}, \quad k_7 = \frac{8}{10}, \quad k_8 = \frac{49}{50}, \quad k_9 = 1, \quad k_{10} = \frac{11}{100},$$

and observing that each of the coefficients k_3, k_5, k_6, and k_{10} multiplies the sum of two terms, we get $P_1 \nless \frac{121}{123}$.

For P_2,

$$k_1 = 1, \quad k_2 = 1, \quad k_3 = \frac{199}{100}, \quad k_4 = \frac{98}{100}, \quad k_5 = \frac{195}{100}, \quad k_6 = \frac{19}{10}, \quad k_7 = 1, \quad k_8 = \frac{1}{100}, \quad k_9 = \frac{1}{100}, \quad k_{10} = \frac{101}{100}.$$

$$\text{Whence } P_2 \nless \frac{122}{123}, \quad \text{and } P_1 P_2 \nless \frac{14762}{15129}.$$

Again, for ρ_1 $\quad \frac{1}{u} - 1 \nless \frac{1}{40}; \quad \frac{1}{v} - 1 = 0;$

$$\therefore \frac{1}{p_1 h_1} \Sigma p_i h_i \ngtr \frac{1}{40}, \text{ and } \rho_1 \nless \frac{40}{41},$$

$$\text{and } Q_1 \nless \frac{5041}{5043}.$$

Also in the value of ρ_2 for each of the quantities q, r, s,

$$\frac{1}{u} - 1 \ngtr \frac{1}{15}, \quad \frac{1}{v} - 1 \ngtr \frac{1}{30}, \text{ and } \frac{a}{\Sigma a - a} \ngtr \frac{1}{10};$$

$$\therefore \frac{q}{q_1} \ngtr \frac{459}{4501}.$$

Whence results

$$\frac{1}{q_1 r_1 s_1} \Sigma \frac{h}{h_1} q_i r_i s_i \ngtr \frac{96702579}{91282466080},$$

$$\text{and } \rho_2 \nless \frac{911}{912}.$$

And for $Q_2 = \{l_1\rho_2 + l_2(1-\rho_2)\}\, Q_1 + l_3\rho_2(1-Q_1)$,

$$l_1 \nless \frac{99}{100},\quad l_2 \nless \frac{4}{10},\quad l_3 \nless \frac{9}{10},\quad \text{and } Q_1 \nless \frac{5041}{5043},$$

$$\text{which give } Q_2 \nless \frac{45500}{45992},\quad \text{and } Q_1 Q_2 \nless \frac{2293}{2319};$$

$$\therefore\ P_g = P_1 P_2 + Q_1 Q_2 (1 - P_1 P_2) \nless \frac{3508}{3509}.$$

In the following cases, Professor Starkie has assumed that a probability to be conclusive must be indefinite; they are inserted here for the purpose of shewing that conclusiveness is independent of this property.

Ex. Two pieces of cloth, on being compared, correspond with each other at the junction; to determine the probability that they were originally one piece. St. 570. *n.*

Let the edges be divided into n corresponding portions, and let $p_1, p_2 \ldots p_n$ be the probabilities that any cause, other than the pieces having been originally one, would have produced the correspondence of the several portions; then, it being certain that if they were originally one piece, the edges would correspond. The probability that this is the true cause of the corresponding, is $\frac{1}{1 + p_1 \cdot p_2 \ldots p_n}$, and the conclusiveness of the evidence depends on the smallness of the fraction $p_1, p_2 \ldots p_n$, and not on the question of its value being or not being accurately determinable.

For example, if the breadth be 18 inches, and this be divided into as many equal portions, and if the values of $p_1, p_2 \ldots p_n$ can be accurately assigned, and are each $= \frac{1}{1000}$, then the probability that the pieces were originally one, is $\frac{1}{1 + \frac{1}{10^{54}}}$, which is a definite measure. But if, as is practically true, it would be difficult or impossible to assign these measures with accuracy, and we can only with certainty define their limits, let $p_1 \ldots p_n$ be each $\ngtr \frac{1}{100}$, and $\nless \frac{1}{500}$, the probability will then be $\nless \frac{1}{1 + \frac{1}{10^{36}}} \ngtr \frac{1}{1 + \frac{1}{5^{18} \,.\, 10^{36}}}$, and the measure will be indefinite. In either case the evidence is conclusive; but the probability whose measure is definite, is many thousand times as great as the other.

Ex. *A* is robbed of 1 penny, 2 sixpences, 3 shillings, 4 half-crowns, 5 crowns, 6 half-sovereigns and 7 sovereigns; *B* is found in the same fair or market in possession of the same combination of coins. No part of the coin can be identified, and no other circumstances operate against *B*.

"Although the circumstances raise a high probability of identity, it is still one of a *definite* and inconclusive nature." St. *ib.*

The hypothesis that *B* is innocent of the theft is opposed by the extraordinary coincidence of the coins in number and value: the hypothesis that he is guilty, by the fact, scarcely less extraordinary that there should be guilt which did not afford any other circumstances of suspicion. It is submitted, that the want of conclusiveness is a consequence of the probability that guilt, if it existed, would have left some other evidence of its existence, being as great, or nearly as great, as the probability that the concurring of the coins in number and value was due to their identity. It would further appear, that extreme *indefiniteness* is the distinguishing character of this problem. All the data, by which the probabilities that either A or B would possess this particular combination of coins could be determined, are wanting. The algebraic solution of the problem must therefore involve a summation through every possible hypothesis for each datum, and no judgment could venture to assign limits to the resulting probability which did not leave a very wide interval to indefiniteness. It seems impossible to conceive any addition to the data which would render the evidence of guilt conclusive which would not also diminish this interval, although therefore the conclusiveness would not be a consequence of the greater degree of definiteness, the progress towards the former would necessarily be accompanied by a corresponding progress towards the latter.

JOHN TOZER.

TEMPLE, *March*, 1844.

XIII. *On the Motion of Glaciers.* *By* WILLIAM HOPKINS, M.A., *Fellow of the Cambridge Philosophical Society, of the Geological Society, and of the Royal Astronomical Society.* [Second Memoir.]

[Read *Dec.* 11, 1843.]

1. IN my former Memoir on the Theory of Glacial Motion, I have given a full development of the *sliding theory* as supported by my own experiments. According to the views there advocated, a glacier is a dislocated mass, all the planes of dislocation, or of discontinuity in the cohesive power being vertical or nearly so, and thus facilitating the more rapid motion of the center of the glacier with reference to its flanks, but not that of its superficial with reference to its inferior portion. It was shewn that the lower surface must be in a constant state of disintegration, and it was thence inferred, that the adhesion between the glacier and its bed must be almost indefinitely less than that between contiguous particles of the solid ice, and that, consequently, the velocities of the superficial and inferior portions of the mass must be equal, or differ from each other by quantities small compared with that of either portion. In my present communication, I propose to investigate the nature of the motion on certain other hypotheses respecting the constitution of the glacial mass, that we may compare such motion, or the effects of it, with observed phenomena, and thus be enabled to judge of the admissibility of our hypotheses. I shall not include amongst these hypotheses those which belong to the *dilatation* or *expansion theories*, because, after the facts observed by Professor Forbes respecting the relative velocities at different distances from the origin of a glacier, and the continuance of glacial motion during the winter*, it appears to me impossible not to recognise the total fallacy of those theories. I shall only therefore consider hypotheses appertaining to views of the subject which, in common with those developed in my former memoir, agree in assigning gravity as the immediate cause of glacial motion, but differ as to the circumstances which render it effective in producing that motion down planes of such small inclination. The hypotheses I shall take are as follows.

(1.) A glacier may be conceived to be divided into *strata*, of which the surfaces are approximately parallel to the upper or lower surface of the mass. In such case, each stratum might slide over that immediately subjacent to it, while the lowest stratum should slide in a similar manner over the bed of the glacier, or remain firmly attached to it. In this motion each stratum must be supposed to preserve its form and continuity as a solid mass, while between two contiguous strata there is *discontinuity*, in the sense in which I shall here use the term, *i. e.* particles originally in contact along the common surface of two contiguous strata do not remain in contact during the motion.

(2.) While the upper part of the mass retains its solidity the inferior portions may be conceived to become disintegrated, so that while the component particles retain their solidity they shall lose their cohesion; the disintegrated portion thus assuming a character similar to that of a mass of sand. In this case, we may conceive the motion of the disintegrated portion to take place by a sliding of the elementary component particles past each other, each particle or element of the mass retaining its original form, like the hard grains of sand during the motion of a mass of that substance.

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* *Travels through the Alps of Savoy, &c.*, by Professor Forbes, p. 361.—This work is full of admirable and well-digested details, founded on the most careful observations and admeasurements, and cannot be too strongly recommended for the perusal of all persons who wish to obtain a knowledge of glacial phenomena, or who feel interested in the many objects of beauty and sublimity which the Alpine regions present to the traveller.

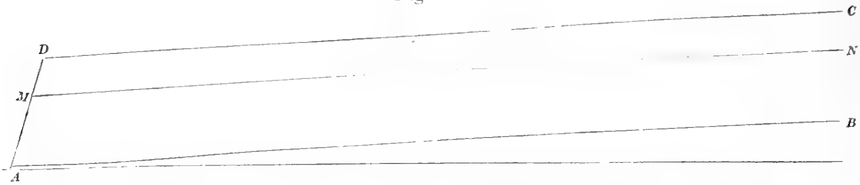
(3.) The glacial mass may be conceived to have the property of great *plasticity*, and to move by a change of form in the elementry particles composing it, the continuity of the mass, in the sense above defined, being strictly preserved. It is in this sense that the continuity of a fluid is assumed to be preserved in those cases of fluid motion which have been subjected to mathematical analysis.

(4.) The mass may be supposed to be *viscous*, and the motion to take place partly by a change of form in the elementary portions of the mass, and partly by the destruction of the continuity supposed to be strictly preserved in the preceding hypothesis.

My immediate purpose in this communication is to investigate certain properties of the motion which would exist in glaciers constituted according to these several hypotheses, and to examine somewhat more in detail than in my former memoir, the state of internal pressure and tension superinduced by the unequal velocities of the central and lateral portions of a glacier. The explanation of this inequality of motion given in my former memoir, will apply with little alteration if we should adopt any of the preceding hypotheses; it will not therefore be necessary to recur to that part of our problem. We shall have to examine more especially the relative motions of the superficial and inferior portions of the mass, to ascertain how far they may be consistent with observed phenomena, and thus to test the truth of our hypotheses.

2. Let us first suppose the glacier stratified as in (1).

Fig. 1.



Let $ABCD$ represent the vertical section of a mass reposing on the inclined plane AB, making an angle a with the horizon; and let MN represent the surface of one of the strata of which we here suppose the mass to consist. We have first to consider under what condition the upper part $CDMN$ would slide over that on which it is superincumbent. Assuming the absence of all cohesion between contiguous strata, the only force opposing the sliding motion will be the friction along the plane MN. Now so long as the original texture of the lower surface of the sliding body and that of the surface on which the motion takes place, remain unaltered by the weight of the sliding mass or other cause, it is well known that the inclination at which the sliding will begin is independent of the weight of the sliding body, and that, if the inclination be a, we must have

$$\tan a = \mu,$$

where μ is the constant ratio which friction bears to the normal pressure. If $\tan a$ were greater than μ, the whole mass would begin to move; and (supposing the friction the same throughout) in such a manner that the relative motion of each stratum to the one immediately subjacent to it would be the same for all the strata. Consequently, if we could ascertain from observation that no such relative motion existed in the upper strata, we should be certain that none existed among the inferior strata, unless at depths at which the assumed condition that the texture of the sliding surfaces shall remain unaltered, may be no longer satisfied.

Now judging from the observations I have made on the descent of ice down inclined planes, I much doubt whether it be possible that two surfaces of ice at a temperature below that of freezing could, under any circumstances, be so smooth as to admit of the sliding motion above contemplated at an inclination so small as that of some observed glaciers; and therefore I believe that no such motion would take place in such a glacier, for instance as that of the Lower Aar, even if it were perfectly stratified, and there were no adhesion whatever between the strata. Much more then is such motion impossible in the actual case of a glacier in which there is little or no indication of stratification, and none whatever of the want of powerful cohesion between two contiguous portions separated by any nearly horizontal plane, such as MN. If, then, any motion of the upper portion take place by its sliding over a lower portion, it must be at a depth at which the hard and crystalline structure of the ice is destroyed*. This brings us to the second of the cases above specified, as possible modes in which glacial motion may take place.

3. There are three causes, I conceive, which may tend to destroy the crystalline structure of the mass—temperature, moisture, and pressure. With respect to the first we may observe, that during the summer the interior temperature, except at points very near the lower surface, must necessarily be lower than that immediately beneath the upper surface, where however, there is no such disintegration of the ice as we are now contemplating. Consequently there can be no such disintegration, as the result of temperature, in the interior of the glacier. Similarly with respect to moisture, if no sensible disintegration result from it near the upper surface where it is most completely disseminated by immediate infiltration, it is not to be supposed that any such effect will be produced in the interior of the mass, except at points so near its lower surface as to be within the influence of the sub-glacial reservoirs and currents.

It would seem then that the only cause to which we can refer any disintegration of the mass, except at points very near the lower surface, must be the pressure of the superincumbent portion. And this must be allowed to be a possible cause of such an effect, for it cannot be doubted that if ice formed under a small pressure were exposed to a very great pressure, its crystalline structure would be effectively destroyed. Still it does not follow that we can assert it to be probable that such is actually the case in existing glaciers; for the hard crystalline structure of glacial ice is doubtless acquired gradually, and probably, in its ultimate degree, under a pressure which bears a considerable ratio to the greatest pressure to which it afterwards becomes subjected; and on this account I should deem it the more probable hypothesis that no part of a glacier becomes disintegrated merely by the pressure which it sustains. Without dwelling, however, on the assertion of probabilities, we may, to a certain extent, appeal to observation. M. Agassiz has descended a vertical fissure to the depth of nearly 200 feet, but we hear of no appearance of a change of structure in the ice, such as here supposed, and which, had it existed, could hardly have escaped his observation. But more conclusive evidence is found in the bore which M. Agassiz sunk to the depth of nearly 200 feet. At the bottom of it the ice was found to be excessively hard, and so little had its structure yielded to the pressure which it sustained, that its specific gravity could scarcely have exceeded that of the superficial ice, as proved by the facility with which the broken fragments rose from the bottom of the bore to the surface when the bore was filled with water. At the depth of the bore, then, we may assert the absence of even the smallest tendency to disintegration, and therefore we are justified in concluding by induction, that no very sensible effect of that kind existed at considerably greater depths, as for instance, at the depth of 300 feet or upwards.

4. Nor does it appear to me possible that glacial ice, retaining its crystalline structure, should possess a degree of *plasticity* sufficient to admit of a motion of the kind above specified in (3). It may be conceived to be possible that the elementary particles of a fluid mass should change their form indefinitely, and that a continuous motion might result from such change; but solid bodies are susceptible of a relative motion of their parts, from this change of form, by the action

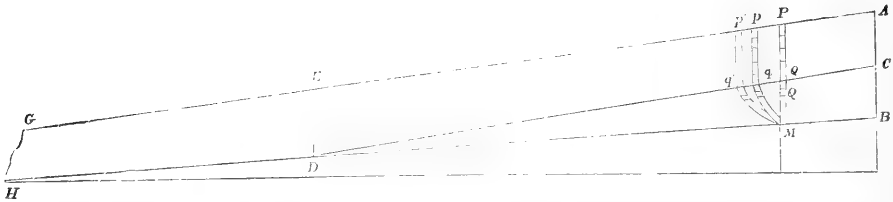
* It was stated by M. Agassiz, in his *Etudes sur les Glaciers*, on the authority of M. Hugi, that the upper portion of glaciers may be observed in deep transverse fissures to project in strata over the lower portion, so as to indicate the relative motion above described. It is now well known that there is a remarkable absence of such appearances.

of external forces, only to a very limited extent, especially when in large masses. Thus if a force be applied to lengthen a given solid mass, a small extension will be the immediate effect; but however long that force may be continued, or however slowly it may be increased, we know of no hard solid substance capable of more than very small extension, so long as it retains that structure on which its hardness and solidity depend. Metals, for instance, with a hard crystalline structure, are susceptible of very small extension, until that structure is destroyed by a sufficiently elevated temperature, when their *ductility* may become indefinitely great, till it becomes fluidity. In the same manner it would seem impossible to believe that glacial ice, a substance of very hard and highly crystalline structure, can have more than an extremely small degree of extensibility; nor when it approaches that temperature which dissolves it, does it appear to acquire the property of ductility above mentioned in metals, but to pass almost immediately from a hard crystalline texture with powerful cohesion, to a state of dissolution in which the cohesion is entirely destroyed. Reasoning thus from the known properties of ice, and from the analogies furnished by other substances, it would seem extremely improbable that a glacier should be susceptible of a continuous motion due to a change of form in its component particles, independently of all sliding of one particle past a contiguous one, and of the sliding of the whole mass over the bed on which it reposes.

Though the two causes of motion considered in this and the preceding article are, when strictly analysed, essentially distinct, the motions resulting from them, so far as such motions can be subjected to observation in glaciers, would be nearly the same. Disintegration of the mass would seem to be essential for the effectiveness of either cause. No evidence whatever of such disintegration has been obtained from observations made at accessible depths in glaciers; but supposing it to exist at greater depths, it would seem to me far the more probable that it should reduce the mass to a state more analogous to that of an aggregation of sand, than to that of an extremely plastic or semifluid substance. But whether we adopt either of these hypotheses, or that of (4) (Art. 1), which may be regarded as a combination of the other two, it is easy to shew, as I shall proceed to do, that the whole mass must necessarily, during its motion, be in a state of *longitudinal compression;* a conclusion which I conceive to be inconsistent with observed appearances.

5. Let the annexed diagram represent a longitudinal section of a glacier, *BDH* being that

Fig. 2.



of the bed on which it reposes. Let MQP be a line of particles vertical at any proposed instant. In the motion we are now contemplating each particle will have a velocity infinitesimally greater than that of the particle immediately below it, the lowest particle at M having no motion if there be no sliding, as I am now supposing, along the bed BH. Thus the physical line MQP will, at successive times, form the continuous physical lines Mqp, $Mq'p'$. These lines, to a certain depth, will sensibly retain their vertical position; for it has been shewn that to a depth of about 300 feet *at least* the texture of the ice is such as to admit of scarcely a sensible change of form, or, consequently, of a sensible difference of velocity in different particles to that depth. In fact, the almost invariable and continued verticality of all transverse fissures to depths not unfrequently of from

100 to 200 feet establishes this fact beyond doubt. Hence if we draw CD parallel to AEG at the depth of about 300 feet, the portion of the mass above that line will have no motion except that which arises from the motion of the subjacent portions CDB. But the cause of motion we are now examining is greatest at BC where the glacier is thickest, and diminishes as we approach to D, where it vanishes. Consequently, the tendency to move will be greatest at the upper extremity of the glacier, and therefore the whole mass must necessarily be throughout the greater part of its extent in a *state of longitudinal compression*. In fact, a large portion DH of the glacier towards its lower extremity could have no sensible motion from the cause under consideration, (since its depth is less than PQ), except that produced by the pushing force exerted upon it by the other portion.

Now this state of longitudinal compression appears to be quite inconsistent with observed facts, at least during the summer-months, when the motion is probably always greatest. During that season, the velocity on the Mer de Glace of Mont Blanc appears to be considerably greatest near the lower extremity, and all observed glaciers, as already stated, are traversed by numerous transverse fissures—facts which indicate unequivocally a state of longitudinal extension, and not of compression. M. Elie de Beaumont has remarked the obvious appearances of extension which glaciers present, and Professo· Forbes has borne testimony to the truth of the remark. In winter, it is probable that there may be a tendency to more rapid motion near the upper extremity of the glacier, as explained in my former memoir (Art. 11), and a consequent tendency to produce compression; but if the principal part of the motion were due to the particular constitution of the mass above supposed, the tendency to compression would be most obvious during summer, when the motion is greatest; a conclusion totally at variance with the results of observation.

Hence, then, it appears that any theory resting on any of the four hypotheses respecting the constitution of a glacier above stated (Art. 1), is not only raised on a foundation unsupported by direct experiment, but leads to results opposed to those of direct observation. The theory which assigns the *viscosity* of the mass as the principal cause of glacial motion necessarily involves these difficulties, so far as it pretends to any distinctive character which may separate it from other theories, which, in common with it, assign gravity as the primary cause of the motion to be accounted for. The absence of longitudinal compression in a glacier is equally opposed also to the theories of *dilatation* and *expansion*.

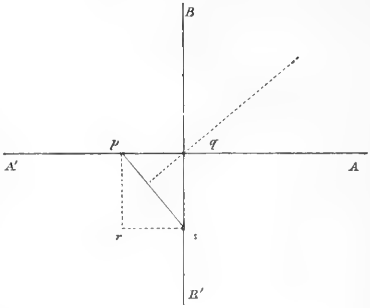
Formation of Transverse Fissures. Since the publication of my former memoir, I have discovered that the explanation there given of the origin of transverse fissures, and of the fact of the convexity of the curves which they form being towards the upper extremity of the glacier, is imperfect. I shall now offer what appears to me to render the explanation complete.

In this investigation we shall only be concerned with the difference of the velocities of the central and lateral portions, for, at least to the depth to which observed fissures extend, there is certainly no difference of velocity for particles in the same vertical line. We may therefore consider the glacier independently of its thickness, or as a *lamina* of ice. The explanation will thus, in some degree, be simplified.

6. When a plain solid lamina having a small degree of compressibility and extensibility, is brought into a position of constraint by forces acting in the plane of the lamina, the particles on one side of a geometrical line will exert certain forces on the contiguous particles on the opposite side of the line. If the lamina were formed of fluid particles the resultant action at each point of this *line of separation* would be normal to it; but when the lamina is solid this will not be generally the case, and therefore the force at any point of the line may be resolved into two forces, one being normal and the other tangential to the line of separation; all forces being supposed to act in the plane of the lamina. Suppose the line of separation to be a straight line $A'A$ parallel to the axis of x, and let pq be a portion of it so small that the action on each point of pq may be considered equal. Let $Y_1 . pq$ denote the normal force exerted by the

particles immediately above pq in the annexed figure, on those immediately below it, estimated in the direction qB; and let $f.pq$ represent the tangential action on pq. Again, let the line of separation coincide with $B'B$, parallel to the axis of y, and perpendicular to $A'A$; and let $X_1.qs$ denote the normal force exerted by the particles immediately on the right of qs on the contiguous particles immediately on the left of it, and $f'.qs$ the tangential action. Join p and s, and let a perpendicular to ps make an angle θ with $A'A$ or the axis of x. Then if $X.ps$ and $Y.ps$ be the resolved parts of the forces which the particles on one side of ps exert on those on the opposite side, estimated in the direction qA and qB respectively, we shall have

Fig. 3.



$$X = X_1 \cos\theta + f \sin\theta,$$
$$Y = Y_1 \sin\theta + f' \cos\theta.$$

To prove these formulæ we have only to observe that the forces acting on the sides pq and qs of the triangular element pqs must be in equilibrium with the forces $-X$ and $-Y$ acting externally on the side ps, neglecting small quantities of the third order. Hence we have

$$-X.ps + X_1.qs + f.pq = 0,$$
$$-Y.ps + Y_1.pq + f'.qs = 0,$$

which, since $\frac{pq}{ps} = \sin\theta$, and $\frac{qs}{ps} = \cos\theta$, prove the above formulæ*.

We have also the relation

$$f' = f.$$

To prove this equation, complete the rectangular element $pqsr$. A tangential force will act on the element along the side rs in a direction opposite to that of the tangential force (f) acting along pq, the intensity of which will not differ from f by any finite quantity; and similarly, a force (f') will act on the side pr in the direction opposite to that on qs. The moments of these forces with respect to the middle point of the rectangular element, will be

$$\tfrac{1}{2}f.pq.qs, \quad \text{and } \tfrac{1}{2}f'.pq.qs.$$

The direction of the resultant of the normal forces on qs will pass at a distance from the middle point of the element small compared with qs; that distance will therefore not exceed a quantity of the second order; and consequently the moment of the force $X_1.qs$ about the middle point of the element will not exceed a quantity of the third order, and may be neglected in comparison with the moments of the tangential forces f and f', which are of the second order. Hence, the equilibrium of the element requires that we should have

$$\tfrac{1}{2}f'.pq.qs = \tfrac{1}{2}f.pq.qs,$$

or

$$f' = f.$$

With this condition we have

$$X = X_1 \cos\theta + f \sin\theta,$$
$$Y = Y_1 \sin\theta + f \cos\theta.$$

If a line be drawn through q parallel and equal to ps, the distance between the two lines will be a small quantity of the first order, and therefore the action on the line through q may

* See Poisson's memoir "Sur le Mouvement des Corps élastiques," in the *Mémoires de l'Institut*, Vol. III. p. 383.

be considered to have for its resolved parts the forces X and Y, from which they cannot differ by quantities exceeding infinitesimals of the first order.

7. Let the length of ps, or of an equal and parallel line through q, $= \lambda$; the resolved parts of the forces upon it will be λX and λY. Let λR be the force on λ estimated in a direction making an angle ϕ with the axis of x, then shall we have

$$\lambda R = \lambda X . \cos\phi + \lambda Y . \sin\phi,$$

or

$$R = X\cos\phi + Y\sin\phi;$$

R is therefore a function of the two independent variables θ and ϕ; and I shall now proceed to find the values of θ and ϕ which render R a maximum or a minimum. Differentiating with respect to ϕ, we have

$$0 = X\sin\phi - Y\cos\phi,$$

which shews that for any assigned value of θ, or position of the line of separation, the maximum value of R will be that of the resultant of X and Y, and the corresponding value of ϕ, that of the angle which the direction of that resultant makes with the axis of x. Differentiating with respect to θ, we have

$$0 = \frac{dX}{d\theta}\cos\phi + \frac{dY}{d\theta}\sin\phi.$$

Substituting for X and Y in these two equations, we obtain

$$(X_1\cos\theta + f\sin\theta)\sin\phi - (Y_1\sin\theta + f\cos\theta)\cos\phi = 0,$$

$$(X_1\sin\theta - f\cos\theta)\cos\phi - (Y_1\cos\theta - f\sin\theta)\sin\phi = 0.$$

Eliminating ϕ, we have

$$(X_1\cos\theta + f\sin\theta)(X_1\sin\theta - f\cos\theta) - (Y_1\sin\theta + f\cos\theta)(Y_1\cos\theta - f\sin\theta) = 0,$$

$$\therefore\ (X_1 f + Y_1 f)(\sin^2\theta - \cos^2\theta) + (X_1^2 - Y_1^2)\sin\theta\cos\theta = 0;$$

$$\therefore\ \tan 2\theta = \frac{2f}{X_1 - Y_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again, from the two preceding equations containing θ and ϕ, we have

$$(X_1 + f\tan\theta)\tan\phi - (Y_1\tan\theta + f) = 0,$$

$$(X_1\tan\theta - f) - (Y_1 - f\tan\theta)\tan\phi = 0,$$

or

$$X_1\tan\phi - Y_1\tan\theta + f\tan\theta\tan\phi - f = 0,$$

$$X_1\tan\theta - Y_1\tan\phi + f\tan\theta\tan\phi - f = 0.$$

θ and ϕ enter exactly in the same manner in these two equations, and must therefore be equal. Hence

$$\tan 2\phi = \frac{2f}{X_1 - Y_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Equation (1) shews that there are two positions of the line of separation through any proposed point, at right angles to each other, for one of which the resultant action between the particles on opposite sides of the line at the proposed point is a maximum, and for the other a minimum; and since ϕ determines the direction of the resultant action, equation (2) proves that direction to coincide with the normal to the line of separation, whenever that line is in a position for which the resultant action is a maximum or minimum. These conclusions may also be arrived at by somewhat different though equivalent reasoning, as follows.

8. First, to find the value of θ which gives R a maximum or minimum, we have

$$R^2 = X^2 + Y^2,$$

and therefore

$$0 = X\frac{dX}{d\theta} + Y\frac{dy}{d\theta},$$

which by substitution and reduction gives

$$(X_1 f + Y_1 f)(\sin^2\theta - \cos^2\theta) + (X_1^2 - Y_1^2)\sin\theta\cos\theta = 0,$$

or

$$\tan 2\theta = \frac{2f}{X_1 - Y_1}.$$

And, secondly, taking ϕ as the angle which the resultant of X and Y makes with the axis of x, we have

$$\tan\phi = \frac{Y}{X} = \frac{Y_1\sin\theta + f\cos\theta}{X_1\cos\theta + f\sin\theta};$$

and if we put $\phi = \theta$, we shall determine that position of the line of separation for which the direction of the resultant action at any proposed point of it coincides with the normal. We thus obtain

$$\sin\theta\{X_1\cos\theta + f\sin\theta\} = \cos\theta\{Y_1\sin\theta + f\cos\theta\},$$

or

$$(X_1 - Y_1)\sin\theta\cos\theta = f(\cos^2\theta - \sin^2\theta);$$

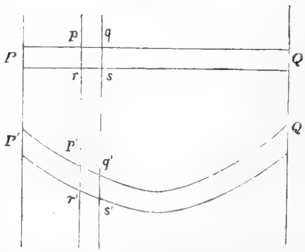
$$\therefore \tan 2\theta = \frac{2f}{X_1 - Y_1}.$$

This equation shews that that position of the line of separation for which $\phi = \theta$, is that which corresponds to the maximum or minimum action between the contiguous particles on opposite sides of the line, as before proved.

9. The maximum action here spoken of is the maximum *tension* at the proposed point, and since it is perpendicular to the corresponding line of separation, *there will manifestly be the greatest tendency to form a fissure along that line, and a fissure will be formed along it if the maximum tension be greater than the cohesive power at the proposed point.*

10. To apply the investigation to the case of a glacier, let PQ (fig. 4) be a portion of the mass contained between two parallel vertical planes perpendicular to the axis of the glacier and indefinitely near to each other. By the more rapid motion of the central part, the element PQ will be brought into the position $P'Q'$; and if $pqrs$ be an infinitesimal rectangular portion of PQ, it will be brought into the position $p'q'r's'$. Let the longitudinal axis of the glacier be that of x. The tangential force f will arise from the greater velocity of the central portion of the mass. It will be of the same intensity, as above proved, for each side of the element, and will manifestly act on the sides respectively in the directions $q's'$ and $r'p'$, $q'p'$ and $r's'$. It is this force which distorts the element from its rectangular form. The longitudinal force X_1 will generally be a tension arising from the greater velocity near the lower extremity. The transverse force Y_1 in actual glaciers, in which the sides have so generally some degree of convergency, may be

Fig. 4.



frequently large, and in the preceding formulæ must be made negative, because it will generally be a *pressure*, and not a tension. Equation (1) thus becomes

$$\tan 2\theta = \frac{2f}{X_1 + Y_1}.$$

If $X_1 = 0$, and $Y_1 = 0$, $\theta = 45^0$ or 135^0. In the case before us it is easily seen that the former of these values corresponds to the maximum and the latter to the minimum value of R; and, therefore, the direction of greatest tension at p' will bisect the angle $r'p'q'$, that angle being supposed, as in the previous reasoning, to differ but little from a right angle. Consequently the greatest tendency to form a fissure will be along a line bisecting the exterior angle $rp'q'$. If the value of $X_1 + Y_1$ be finite, that of θ will be less than 45^0, and the direction of the fissure will so deviate from the above-mentioned position as to approximate more nearly to perpendicularity to the sides and axis of the glacier.

If the angle $q'p'r'$ should deviate from a right angle by a finite quantity before the fissure should be formed, it would not be difficult, to shew that the line of greatest tension might be still considered to bisect that angle. This would cause a still further deviation in the direction of the fissure towards perpendicularity to the sides.

Since the relative motion of particles situated in a transverse line varies most rapidly in the lateral portions, the value of f will be greatest near the sides, and vanish along the axis of the glacier; while the value of $X_1 + Y_1$ will be approximately the same at the sides and center. Consequently, the value of θ will diminish as the distance from the sides increases, and the fissures will be curved; the curvature being most rapid near the sides of the glacier, and the convexity being turned towards the upper extremity of the glacier. The force f will probably be much more effective than X_1 in producing the fissures near the sides of the glacier, while X_1 will possibly be the more effective in the central portion. The incompleteness of my former explanation consisted in ascribing the phenomena to the latter force only, to which alone the reasoning there applied is applicable. The above investigation appears to me to offer the complete solution of the problem.

11. *Riband or Laminated Structure.*—I have made no attempt to account for this curious structure in glacial ice; but I would observe that it appears to me impossible that it should be due, as some persons, I think, have supposed, to internal tensions or pressures, producing, as their direct and immediate effect, an almost infinite number of parallel fissures, into which water percolates, and forms, when frozen, the bands of blue ice. It is conceivable, as an abstract hypothesis, that a mass should be accurately homogeneous, and that the external and internal forces should be such as to have exactly the same tendency to produce a fissure at one point of the mass as at another; but practically, this state is no more possible than that a body should rest in a position of unstable equilibrium—that a cone should rest permanently on its vertex, or a needle on its point. Allowing the nearest practical approximation to this state of the mass, fissures would necessarily begin to be formed, first at particular points, after which the uniformity of condition throughout would be instantly destroyed, and irregular fissures at intervals, large, compared with those between consecutive bands of blue ice in the riband structure, would be the necessary consequence. I repeat, that the formation of a system of parallel fissures, of sensible or insensible width, at distances not exceeding a few inches, in the mass of a glacier, is no more possible than that the mass should permanently maintain a position of unstable equilibrium.

The internal pressures and tensions here spoken of are the consequences of external forces acting on the mass, such as gravity and the resistances of the rocks with which the glacial mass may be in contact. There is, however, another class of internal forces, the molecular forces, the existence and nature of which may be considered independent of the external conditions to which the mass is subjected, though their action and effects may very probably be modified by those conditions. I have investigated the effects of the first kind of forces, and have explained how transverse and longitudinal fissures may result from their action; it is to the molecular forces that I am disposed to attribute the veined or riband structure, their action being modified in some unknown manner by the general conditions under which the glacier exists. In expressing this opinion I am offering no theory of the curious structure in question, but only meeting the theoretical difficulty which it presents to us by a confession of profound ignorance of the nature and action of those forces, to which the peculiarity of crystalline structure is generally due. The mechanical solution of the problem I conceive to be utterly hopeless, till we shall have arrived at some solution of the general problem which crystallization presents to us*.

12. In conclusion, I will state the principal objections which have been urged against the *sliding theory*, and indicate the answers which the preceding investigations afford. In doing this, I shall refer principally to the work of Professor Forbes, already mentioned, as that in which those objections are most systematically stated.

(1.) The *enormous friction* when a glacier moves over a bed of rock, is spoken of by all opponents of this theory as an insurmountable objection to it. My experiments shew that the friction, or rather the force analogous to friction, is extremely small.

(2.) Professor Forbes remarks (p. 362), "As I understand the Gravitation theory, it supposes the mass of the glacier to be a *rigid* one, sliding over its trough or bed in the manner of solid bodies."—I am not aware that any advocate of this theory has fallen into the absurdity of considering a glacier as a *rigid*, when he has spoken of it as a *solid* mass. I have considered it as a *dislocated* mass, glacial ice itself having some degree of plasticity.

(3.) When a glacier passes out of a wider into a more contracted channel, Professor Forbes says that "the idea of *sliding*, in the common legitimate sense of the word, is wholly out of the question."—The term "sliding" is certainly not restricted to the motion of a *rigid* body; it is applicable to any *solid* body, in the sense in which a glacier is considered to be such, and on this hypothesis I have distinctly explained, in my former memoir, how it may pass from a wider into a narrower channel. In objections of this nature the distinction between solidity and rigidity would seem to be forgotten.

(4.) "The inclination of the bed is seldom such as to render the overcoming of such obstacles as the elbows and prominences, contractions and irregularities of the beds of glaciers, even conceivable, being, on an average of the entire Mer de Glace, only 9°, a slope practicable to loaded carts; but the greater part of the surface inclines less than 5°," (p. 363.) This difficulty has arisen in an imperfect conception of the enormous pressure which, according to our theory, must be thrown on abrupt local obstacles†.

(5.) Another objection is founded on the fact that changes in the rapidity of glacial movements are found to be simultaneous with changes of external temperature. "In order to reconcile this to

* At the last meeting of the British Association at Cork, Mr. Phillips mentioned a curious fact, which seems calculated to throw some light on one of the modes in which external conditions may modify the action of molecular forces, assuming the lamellar structure of rocks to be due to such forces. It appeared that certain Trilobites were frequently found in some of the older rocks in South Wales, so deformed as to their general proportions as to present, to a casual observer, the appearance of different species. On comparing, however, a number of cases, it became evident that the specimens had been *compressed* in a direction *perpendicular to the planes of structure*, from which it was justly inferred, that the general mass in which these remains were imbedded had probably been subjected to a great pressure in the direction above mentioned. It would seem to be a legitimate inference from this fact, that the position of the planes of structure had probably been mainly determined by the direction of greatest pressure. Perhaps some of the facts mentioned by Professor Forbes might admit of a similar interpretation.

I may here mention a curious effect of crystallization in the structure of hailstones, which may possibly have some bearing on the question before us. I had an opportunity of witnessing it at Cambridge, on the 9th of August, 1843, during one of the most desolating hail-storms ever known in this country. Many of the hailstones were of the form of rather flat double convex lenses, nearly as large as the palm of the hand, and consisted of white opaque ice in the center, surrounded by a ring of dark transparent ice, with an exterior ring of ice like that in the center. In some cases there were two or three dark rings, the central part and the exterior ring being always opaque. These successive rings (with the exception of their circular form) exactly resembled the alternate opaque and transparent bands in glacial ice, where the riband structure is best developed.

† Art. 15. of my former memoir.

the sliding theory, it should be shewn that the disengagement of the glacier from its bed depends on the kind of weather which affects its surface and temperature." The action of the subglacial currents does fully account, I conceive, for the phenomenon in question.

(6.) It has been contended that, according to the sliding theory, the glacier ought to descend with an *accelerated* motion. This objection never had any real foundation, but only arose, as I have shewn in my former memoir, from an erroneous conception of the nature of the retarding forces which must act on the glacier during its sliding motion, whatever might be the cause of such motion. My experiments, however, afford the most complete answer to the objection.

(7.) It is said that the flow of heat from the earth is not sufficient to produce the effect which this theory ascribes to it:—I reply, that all which the theory requires is, that the lower surface of the glacier should be constantly kept at the temperature at which the disintegration of ice commences. The tangential action of the bed on the bottom of the glacier will in such case be so modified as to render it impossible for that action to prevent all motion.

(8.) Another objection has been founded on the existence of glaciers of the *secondary order*, which are observed to rest on surfaces of great inclination. Professor Forbes remarks, "M. de Charpentier has very justly quoted several examples as proving, that if glaciers really slide over the soil, as De Saussure supposed, these could not for a moment sustain their position at an angle of 30° or more," (p. 79). M. de Charpentier, I presume, would contend that if gravity were the primary cause of glacial motion, such a glacier would descend with the rapidity of an avalanche. But it appeared from my experiments, that a mass of ice might be placed on a surface as smooth as that of a paving slab at an angle of nearly 20°, without descending *with an accelerated motion*, even when the lower surface of the ice was lubricated by its being in a state of dissolution. Now these secondary glaciers are generally at a great elevation, and of no great thickness, so that it is highly probable that a considerable portion of their lower surfaces may be frozen to the rocks on which they rest. This circumstance, together with the probable inequalities of the surface of those rocks, leaves no difficulty in accounting for the want of accelerated and precipitous movements in such glaciers as those above spoken of, nor even in those of still greater inclination. They will descend down their highly-inclined beds with an unaccelerated motion, and will then be precipitated, as avalanches, down the precipices which usually form their lower boundaries.

In another part of his work, Professor Forbes appears to give an opposite phase to the objection derived from secondary glaciers, and to make it rest on the assumed fact of these secondary glaciers being frozen to the rocks throughout the whole of their lower surfaces. That these glaciers are partly frozen to their beds, I have above stated to be probable; that they are entirely so, no proof has been or can be offered. We possess no knowledge of them which does not justify the application of the *sliding theory* to them, as well as to other glaciers.

W. HOPKINS.

CAMBRIDGE,
Dec. 11, 1843.

XIV. *On the Fundamental Antithesis of Philosophy.* By W. Whewell, D.D., *Master of Trinity College, and Professor of Moral Philosophy.*

[Read *Feb.* 5, 1844.]

I have upon former occasions laid before the Society dissertations on certain questions which may be termed metaphysical:—on the nature of the truth of the laws of motion:—on the question whether all matter is heavy:—and on the question whether cause and effect are successive or simultaneous. As these dissertations have not failed to excite some interest, I hope that I shall have the indulgence of the Society in making a few remarks on another question of the same kind. In doing this, as my object is to throw some light if possible on a matter of considerable obscurity and difficulty, I shall not attempt to avoid the occasional repetition of a sentence or two which I may have, in substance, delivered elsewhere.

1. All persons who have attended in any degree to the views generally current of the nature of reasoning are familiar with the distinction of necessary truths and truths of experience; and few such persons, or at least few students of mathematics, require to have this distinction explained or enforced. All geometricians are satisfied that the geometrical truths with which they are conversant are necessarily true: they not only are true, but they must be true. The meaning of the terms being understood, and the proof being gone through, the truth of the proposition must be assented to. That parallelograms upon the same base and between the same parallels are equal;—that angles in the same segment are equal;—these are propositions which we learn to be true by demonstrations deduced from definitions and axioms; and which, when we have thus learnt them, we see could not be otherwise. On the other hand, there are other truths which we learn from experience; as for instance, that the stars revolve round the pole in one day; and that the moon goes through her phases from full to full again in thirty days. These truths we see to be true; but we know them only by experience. Men never could have discovered them without looking at the stars and the moon; and having so learnt them, still no one will pretend to say that they are necessarily true. For aught we can see, things might have been otherwise; and if we had been placed in another part of the solar system, then, according to the opinions of astronomers, experience would have presented them otherwise.

2. I take the astronomical truths of experience to contrast with the geometrical necessary truths, as being both of a familiar definite sort; we may easily find other examples of both kinds of truth. The truths which regard numbers are necessary truths. It is a necessary truth, that 27 and 38 are equal to 65; that half the sum of two numbers added to half their difference is equal to the greater number. On the other hand, that sugar will dissolve in water; that plants cannot live without light; and in short, the whole body of our knowledge in chemistry, physiology, and the other inductive sciences, consists of truths of experience. If there be any science which offer to us truths of an ambiguous kind, with regard to which we may for a moment doubt whether they are necessary or experiential, we will defer the consideration of them till we have marked the distinction of the two kinds more clearly.

3. One mode in which we may express the difference of necessary truths and truths of experience, is, that necessary truths are those of which we cannot distinctly conceive the contrary. We can very readily conceive the contrary of experiential truths. We can conceive the stars moving about the pole or across the sky in any kind of curves with any velocities; we can conceive the moon always appearing during the whole month as a luminous disk, as she might do if her

light were inherent and not borrowed. But we cannot conceive one of the parallelograms on the same base and between the same parallels larger than the other; for we find that, if we attempt to do this, when we separate the parallelograms into parts, we have to conceive one triangle larger than another, both having all their parts equal; which we cannot conceive at all, if we conceive the triangles distinctly. We make this impossibility more clear by conceiving the triangles to be placed so that two sides of the one coincide with two sides of the other; and it is then seen, that in order to conceive the triangles unequal, we must conceive the two bases which have the same extremities both ways, to be different lines, though both straight lines. This it is impossible to conceive: we assent to the impossibility as an axiom, when it is expressed by saying, that two straight lines cannot inclose a space; and thus we cannot distinctly conceive the contrary of the proposition just mentioned respecting parallelograms.

4. But it is necessary, in applying this distinction, to bear in mind the terms of it;—that we cannot *distinctly* conceive the contrary of a necessary truth. For in a certain loose, indistinct way, persons conceive the contrary of necessary geometrical truths, when they erroneously conceive false propositions to be true. Thus, Hobbes erroneously held that he had discovered a means of geometrically doubling the cube, as it is called, that is, finding two mean proportionals between two given lines; a problem which cannot be solved by plane geometry. Hobbes not only proposed a construction for this purpose, but obstinately maintained that it was right, when it had been proved to be wrong. But then, the discussion showed how indistinct the geometrical conceptions of Hobbes were; for when his critics had proved that one of the lines in his diagram would not meet the other in the point which his reasoning supposed, but in another point near to it; he maintained, in reply, that one of these points was large enough to include the other, so that they might be considered as the same point. Such a mode of conceiving the opposite of a geometrical truth, forms no exception to the assertion, that this opposite cannot be distinctly conceived.

5. In like manner, the indistinct conceptions of children and of rude savages do not invalidate the distinction of necessary and experiential truths. Children and savages make mistakes even with regard to numbers; and might easily happen to assert that 27 and 38 are equal to 63 or 64. But such mistakes cannot make such arithmetical truths cease to be necessary truths. When any person conceives these numbers and their addition distinctly, by resolving them into parts, or in any other way, he sees that their sum is necessarily 65. If, on the ground of the possibility of children and savages conceiving something different, it be held that this is not a necessary truth, it must be held on the same ground, that it is not a necessary truth that 7 and 4 are equal to 11; for children and savages might be found so unfamiliar with numbers as not to reject the assertion that 7 and 4 are 10, or even that 4 and 3 are 6, or 8. But I suppose that no persons would on such grounds hold that these arithmetical truths are truths known only by experience.

6. Necessary truths are established, as has already been said, by demonstration, proceeding from definitions and axioms, according to exact and rigorous inferences of reason. Truths of experience are collected from what we see, also according to inferences of reason, but proceeding in a less exact and rigorous mode of proof. The former depend upon the relations of the ideas which we have in our minds: the latter depend upon the appearances or phenomena, which present themselves to our senses. Necessary truths are formed from our thoughts, the elements of the world within us; experiential truths are collected from things, the elements of the world without us. The truths of experience, as they appear to us in the external world, we call Facts; and when we are able to find among our ideas a train which will conform themselves to the apparent facts, we call this a Theory.

7. This distinction and opposition, thus expressed in various forms; as Necessary and Experiential Truth, Ideas and Senses, Thoughts and Things, Theory and Fact, may be termed the *Fundamental Antithesis of Philosophy*; for almost all the discussions of philosophers have been employed in asserting or denying, explaining or obscuring this antithesis. It may be ex-

pressed in many other ways; but is not difficult, under all these different forms, to recognize the same opposition: and the same remarks apply to it under its various forms, with corresponding modifications. Thus, as we have already seen, the antithesis agrees with that of Reasoning and Observation: again, it is identical with the opposition of Reflection and Sensation: again, sensation deals with Objects; facts involve Objects, and generally all things without us are Objects:—Objects of sensation, of observation. On the other hand, we ourselves who thus observe objects, and in whom sensation is, may be called the Subjects of sensation and observation. And this distinction of Subject and Object is one of the most general ways of expressing the fundamental antithesis, although not yet perhaps quite familiar in English. I shall not scruple however to speak of the Subjective and Objective element of this antithesis, where the expressions are convenient.

8. All these forms of antithesis, and the familiar references to them which men make in all discussions, shew the fundamental and necessary character of the antithesis. We can have no knowledge without the union, no philosophy without the separation, of the two elements. We can have no knowledge, except we have both impressions on our senses from the world without, and thoughts from our minds within:—except we attend to things, and to our ideas;—except we are passive to receive impressions, and active to compare, combine, and mould them. But on the other hand, philosophy seeks to distinguish the impressions of our senses from the thoughts of our minds;—to point out the difference of ideas and things;—to separate the active from the passive faculties of our being. The two elements, sensations and ideas, are both requisite to the existence of our knowledge, as both matter and form are requisite to the existence of a body. But philosophy considers the matter and the form separately. The properties of the form are the subject of geometry, the properties of the matter are the subject of chemistry or mechanics.

9. But though philosophy considers these elements of knowledge separately, they cannot really be separated, any more than can matter and form. We cannot exhibit matter without form, or form without matter; and just as little can we exhibit sensations without ideas, or ideas without sensations;—the passive or the active faculties of the mind detached from each other.

In every act of my knowledge, there must be concerned the things whereof I know, and thoughts of me who know: I must both passively receive or have received impressions, and I must actively combine them and reason on them. No apprehension of things is purely ideal: no experience of external things is purely sensational. If they be conceived as *things*, the mind must have been awoke to the conviction of things by sensation: if they be *conceived* as things, the expressions of the senses must have been bound together by conceptions. If we *think* of any *thing*, we must recognize the existence both of thoughts and of things. *The fundamental antithesis of philosophy is an antithesis of inseparable elements.*

10. Not only cannot these elements be separately exhibited, but they cannot be separately conceived and described. The description of them must always imply their relation; and the names by which they are denoted will consequently always bear a relative significance. And thus *the terms which denote the fundamental antithesis of philosophy cannot be applied absolutely and exclusively in any case.* We may illustrate this by a consideration of some of the common modes of expressing the antithesis of which we speak. The terms Theory and Fact are often emphatically used as opposed to each other: and they are rightly so used. But yet it is impossible to say absolutely in any case, This is a Fact and not a Theory; this is a Theory and not a Fact, meaning by Theory, true Theory. Is it a fact or a theory that the stars appear to revolve round the pole? Is it a fact or a theory that the earth is a globe revolving round its axis? Is it a fact or a theory that the earth revolves round the sun? Is it a fact or a theory that the sun attracts the earth? Is it a fact or a theory that a loadstone attracts a needle? In all these cases, some persons would answer one way and some persons another. A person who has never

watched the stars, and has only seen them from time to time, considers their circular motion round the pole as a theory, just as he considers the motion of the sun in the ecliptic as a theory, or the apparent motion of the inferior planets round the sun in the zodiac. A person who has compared the measures of different parts of the earth, and who knows that these measures cannot be conceived distinctly without supposing the earth a globe, considers its globular form a fact, just as much as the square form of his chamber. A person to whom the grounds of believing the earth to revolve round its axis and round the sun, are as familiar as the grounds for believing the movements of the mail-coaches in this country, conceives the former events to be facts, just as steadily as the latter. And a person who, believing the fact of the earth's annual motion, refers it distinctly to its mechanical course, conceives the sun's attraction as a fact, just as he conceives as a fact the action of the wind which turns the sails of a mill. We see then, that in these cases we cannot apply absolutely and exclusively either of the terms, Fact or Theory. Theory and Fact are the elements which correspond to our Ideas and our Senses. The Facts are Facts so far as the Ideas have been combined with the sensations and absorbed in them: the Theories are Theories so far as the Ideas are kept distinct from the sensations, and so far as it is considered as still a question whether they can be made to agree with them. A true Theory is a fact, a Fact is a familiar theory.

In like manner, if we take the terms Reasoning and Observation; at first sight they appear to be very distinct. Our observation of the world without us, our reasonings in our own minds, appear to be clearly separated and opposed. But yet we shall find that we cannot apply these terms absolutely and exclusively. I see a book lying a few feet from me: is this a matter of observation? At first, perhaps, we might be inclined to say that it clearly is so. But yet, all of us, who have paid any attention to the process of vision, and to the mode in which we are enabled to judge of the distance of objects, and to judge them to be distant objects at all, know that this judgment involves inferences drawn from various sensations;—from the impressions on our two eyes;—from our muscular sensations; and the like. These inferences are of the nature of reasoning, as much as when we judge of the distance of an object on the other side of a river by looking at it from different points, and stepping the distance between them. Or again: we observe the setting sun illuminate a gilded weathercock; but this is as much a matter of reasoning as when we observe the phases of the moon, and infer that she is illuminated by the sun. All observation involves inferences, and inference is reasoning.

11. Even the simplest terms by which the antithesis is expressed cannot be applied: ideas and sensations, thoughts and things, subject and object, cannot in any case be applied absolutely and exclusively. Our sensations require ideas to bind them together, namely, ideas of space, time, number, and the like. If not so bound together, sensations do not give us any apprehension of things or objects. All things, all objects, must exist in space and in time—must be one or many. Now space, time, number, are not sensations or things. They are something different from, and opposed to sensations and things. We have termed them ideas. It may be said they are *relations* of things, or of sensations. But granting this form of expression, still a *relation* is not a thing or a sensation; and therefore we must still have another and opposite element, along with our sensations. And yet, though we have thus these two elements in every act of perception, we cannot designate any portion of the act as absolutely and exclusively belonging to one of the elements. Perception involves sensation, along with ideas of time, space, and the like; or, if any one prefers the expression, involves sensations along with the apprehension of relations. Perception is sensation, along with such ideas as make sensation into an apprehension of things or objects.

12. And as perception of objects implies ideas, as observation implies reasoning; so, on the other hand, ideas cannot exist where sensation has not been: reasoning cannot go on when there has not been previous observation. This is evident from the necessary order of development of the human faculties. Sensation necessarily exists from the first moments of our existence, and is

constantly at work. Observation begins before we can suppose the existence of any reasoning which is not involved in observation. Hence, at whatever period we consider our ideas, we must consider them as having been already engaged in connecting our sensations, and as modified by this employment. By being so employed, our ideas are unfolded and defined, and such development and definition cannot be separated from the ideas themselves. We cannot conceive space without boundaries or forms; now forms involve sensations. We cannot conceive time without events which mark the course of time; but events involve sensations. We cannot conceive number without conceiving things which are numbered; and things imply sensations. And the forms, things, events, which are thus implied in our ideas, having been the objects of sensation constantly in every part of our life, have modified, unfolded and fixed our ideas, to an extent which we cannot estimate, but which we must suppose to be essential to the processes which at present go on in our minds. We cannot say that objects create ideas; for to perceive objects we must already have ideas. But we may say, that objects and the constant perception of objects have so far modified our ideas, that we cannot, even in thought, separate our ideas from the perception of objects.

We cannot say of any ideas, as of the idea of space, or time, or number, that they are absolutely and exclusively ideas. We cannot conceive what space, or time, or number would be in our minds, if we had never perceived any thing or things in space or time. We cannot conceive ourselves in such a condition as never to have perceived any thing or things in space or time. But, on the other hand, just as little can we conceive ourselves becoming acquainted with space and time or numbers as objects of sensation. We cannot reason without having the operations of our minds affected by previous sensations; but we cannot conceive reasoning to be merely a series of sensations. In order to be used in reasoning, sensation must become observation; and, as we have seen, observation already involves reasoning. In order to be connected by our ideas, sensations must be things or objects, and things or objects already include ideas. And thus, as we have said, none of the terms by which the fundamental antithesis is expressed can be absolutely and exclusively applied.

13. I now proceed to make one or two remarks suggested by the views which have thus been presented. And first I remark, that since, as we have just seen, none of the terms which express the fundamental antithesis can be applied absolutely and exclusively, the absolute application of the antithesis in any particular case can never be a conclusive or immoveable principle. This remark is the more necessary to be borne in mind, as the terms of this antithesis are often used in a vehement and peremptory manner. Thus we are often told that such a thing is a *Fact* and not a Theory, with all the emphasis which, in speaking or writing, tone or italics or capitals can give. We see from what has been said, that when this is urged, before we can estimate the truth, or the value of the assertion, we must ask to whom is it a fact? what habits of thought, what previous information, what ideas does it imply, to conceive the fact as a fact? Does not the apprehension of the fact imply assumptions which may with equal justice be called theory, and which are perhaps false theory? in which case, the fact is no fact. Did not the ancients assert it as a fact, that the earth stood still, and the stars moved? and can any fact have stronger apparent evidence to justify persons in asserting it emphatically than this had? These remarks are by no means urged in order to shew that no fact can be certainly known to be true; but only to shew that no fact can be certainly shown to be a fact merely by calling it a fact, however emphatically. There is by no means any ground of general skepticism with regard to truth involved in the doctrine of the necessary combination of two elements in all our knowledge. On the contrary, ideas are requisite to the essence, and things to the reality of our knowledge in every case. The proportions of geometry and arithmetic are examples of knowledge respecting our ideas of space and number, with regard to which there is no room for doubt. The doctrines of astronomy are examples of truths not less certain respecting the external world.

14. I remark further, that since in every act of knowledge, observation or perception, both the elements of the fundamental antithesis are involved, and involved in a manner inseparable even

in our conceptions, it must always be possible to derive one of these elements from the other, if we are satisfied to accept, as proof of such derivation, that one always co-exists with and implies the other. Thus an opponent may say, that our ideas of space, time, and number, are derived from our sensations or perceptions, because we never were in a condition in which we had the ideas of space and time, and had not sensations or perceptions. But then, we may reply to this, that we no sooner perceive objects than we perceive them as existing in space and time, and therefore the ideas of space and time are not derived from the perceptions. In the same manner, an opponent may say, that all knowledge which is involved in our reasonings is the result of experience; for instance, our knowledge of geometry. For every geometrical principle is presented to us by experience as true; beginning with the simplest, from which all others are derived by processes of exact reasoning. But to this we reply, that experience cannot be the origin of such knowledge; for though experience shows that such principles are true, it cannot show that they *must be* true, which we also know. We never have seen, as a matter of observation, two straight lines inclosing a space; but we venture to say further, without the smallest hesitation, that we never shall see it; and if any one were to tell us that, according to his experience, such a form was often seen, we should only suppose that he did not know what he was talking of. No number of acts of experience can add to the certainty of our knowledge in this respect; which shows that our knowledge is not made up of acts of experience. We cannot test such knowledge by experience; for if we were to try to do so, we must first know that the lines with which we make the trial *are* straight; and we have no test of straightness better than this, that two such lines cannot inclose a space. Since then, experience can neither destroy, add to, nor test our axiomatic knowledge, such knowledge cannot be derived from experience. Since no one act of experience can affect our knowledge, no numbers of acts of experience can make it.

15. To this a reply has been offered, that it is a characteristic property of geometric forms that the ideas of them exactly resemble the sensations; so that these ideas are as fit subjects of experimentation as the realities themselves; and that by such experimentation we learn the truth of the axioms of geometry. I might very reasonably ask those who use this language to explain how a particular class of ideas can be said to resemble sensations; how, if they do, we can know it to be so; how we can prove this resemblance to belong to geometrical ideas and sensations; and how it comes to be an especial characteristic of those. But I will put the argument in another way. Experiment can only show what is, not what must be. If experimentation on ideas shows what must be, it is different from what is commonly called experience.

I may add, that not only the mere use of our senses cannot show that the axioms of geometry *must be* true, but that, without the light of our ideas, it cannot even show that they *are* true. If we had a segment of a circle a mile long and an inch wide, we should have two lines inclosing a space; but we could not, by seeing or touching any part of either of them, discover that it was a bent line.

16. That mathematical truths are not derived from experience is perhaps still more evident, if greater evidence be possible, in the case of numbers. We assert that 7 and 8 are 15. We find it so, if we try with counters, or in any other way. But we do not, on that account, say that the knowledge is derived from experience. We refer to our conceptions of seven, of eight, and of addition, and as soon as we possess these conceptions distinctly, we see that the sum must be fifteen. We cannot be said to make a trial, for we should not believe the apparent result of the trial if it were different. If any one were to say that the multiplication table is a table of the results of experience, we should know that he could not be able to go along with us in our researches into the foundations of human knowledge; nor, indeed, to pursue with success any speculations on the subject.

17. Attempts have also been made to explain the origin of axiomatic truths by referring them to the association of ideas. But this is one of the cases in which the word *association* has been applied so widely and loosely, that no sense can be attached to it. Those who have written with any degree of distinctness on the subject, have truly taught, that the habitual association of the

ideas leads us to believe a connexion of the things: but they have never told us that this association gave us the power of forming the ideas. Association may determine belief, but it cannot determine the possibility of our conceptions. The African king did not believe that water could become solid, because he had never seen it in that state. But that accident did not make it impossible to conceive it so, any more than it is impossible for us to conceive frozen quicksilver, or melted diamond, or liquefied air; which we may never have seen, but have no difficulty in conceiving. If there were a tropical philosopher really incapable of conceiving water solidified, he must have been brought into that mental condition by abstruse speculations on the necessary relations of solidity and fluidity, not by the association of ideas.

18. To return to the results of the nature of the Fundamental Antithesis. As by assuming universal and indissoluble connexion of ideas with perceptions, of knowledge with experience, as an evidence of derivation, we may assert the former to be derived from the latter, so might we, on the same ground, assert the latter to be derived from the former. We see all forms in space; and we might hence assert all forms to be mere modifications of our idea of space. We see all events happen in time; and we might hence assert all events to be merely limitations and boundary-marks of our idea of time. We conceive all collections of things as two or three, or some other number: it might hence be asserted that we have an original idea of number, which is reflected in external things. In this case, as in the other, we are met at once by the impossibility of this being a complete account of our knowledge. Our ideas of space, of time, of number, however distinctly reflected to us with limitations and modifications, must be reflected, limited and modified by something different from themselves. We must have visible or tangible forms to limit space, perceived events to mark time, distinguishable objects to exemplify number. But still, in forms, and events, and objects, we have a knowledge which they themselves cannot give us. For we know, without attending to them, that whatever they are, they will conform and must conform to the truths of geometry and arithmetic. There is an ideal portion in all our knowledge of the external world; and if we were resolved to reduce all our knowledge to one of its two antithetical elements, we might say that all our knowledge consists in the relation of our ideas. Wherever there is necessary truth, there must be something more than sensation can supply: and the necessary truths of geometry and arithmetic show us that our knowledge of objects in space and time depends upon necessary relations of ideas, whatever other element it may involve.

19. This remark may be carried much further than the domain of geometry and arithmetic. Our knowledge of matter may at first sight appear to be altogether derived from the senses. Yet we cannot derive from the senses our knowledge of a truth which we accept as universally certain;—namely, that we cannot by any process add to or diminish the quantity of matter in the world. This truth neither is nor can be derived from experience; for the experiments which we make to verify it pre-suppose its truth. When the philosopher was asked what was the weight of smoke, he bade the inquirer subtract the weight of the ashes from the weight of the fuel. Every one who thinks clearly of the changes which take place in matter, assents to the justice of this reply: and this, not because any one had found by trial that such was the weight of the smoke produced in combustion, but because the weight lost was assumed to have gone into some other form of matter, not to have been destroyed. When men began to use the balance in chemical analysis, they did not prove by trial, but took for granted, as self-evident, that the weight of the whole must be found in the aggregate weight of the elements. Thus it is involved in the idea of matter that its amount continues unchanged in all changes which takes place in its consistence. This is a necessary truth: and thus our knowledge of matter, as collected from chemical experiments, is also a modification of our idea of matter as the material of the world incapable of addition or diminution.

20. A similar remark may be made with regard to the mechanical properties of matter. Our knowledge of these is reduced, in our reasonings, to principles which we call the laws of motion.

These laws of motion, as I have endeavoured to shew in a paper already printed by the Society, depend upon the idea of Cause, and involve necessary truths, which are necessarily implied in the idea of cause;—namely, that every change of motion must have a cause—that the effect is measured by the cause;—that re-action is equal and opposite to action. These principles are not derived from experience. No one, I suppose, would derive from experience the principle, that every event must have a cause. Every attempt to see the traces of cause in the world assumes this principle. I do not say that these principles are anterior to experience; for I have already, I hope, shewn, that neither of the two elements of our knowledge is, or can be, anterior to the other. But the two elements are co-ordinate in the development of the human mind; and the ideal element may be said to be the origin of our knowledge with the more propriety of the two, inasmuch as our knowledge is the relation of ideas. The other element of knowledge, in which sensation is concerned, and which embodies, limits, and defines the necessary truths which express the relations of our ideas, may be properly termed experience; and I have, in the Memoir just quoted, endeavoured to shew how the principles concerning mechanical causation, which I have just stated, are, by observation and experiment, limited and defined, so that they become the laws of motion. And thus we see that such knowledge is derived from ideas, in a sense quite as general and rigorous, to say the least, as that in which it is derived from experience.

21. I will take another example of this; although it is one less familiar, and the consideration of it perhaps a little more difficult and obscure. The objects which we find in the world, for instance, minerals and plants, are of different kinds; and according to their kinds, they are called by various names, by means of which we know what we mean when we speak of them. The discrimination of these kind of objects, according to their different forms and other properties, is the business of chemistry and botany. And this business of discrimination, and of consequent classification, has been carried on from the first periods of the development of the human mind, by an industrious and comprehensive series of observations and experiments; the only way in which any portion of the task could have been effected. But as the foundation of all this labour, and as a necessary assumption during every part of its progress, there has been in men's minds the principle, that objects are so distinguishable by resemblances and differences, that they may be named, and known by their names. This principle is involved in the idea of a Name; and without it no progress could have been made. The principle may be briefly stated thus:—Intelligible Names of kinds are possible. If we suppose this not to be so, language can no longer exist, nor could the business of human life go on. If instead of having certain definite kinds of minerals, gold, iron, copper and the like, of which the external forms and characters are constantly connected with the same properties and qualities, there were no connexion between the appearance and the properties of the object;—if what seemed externally iron might turn out to resemble lead in its hardness; and what seemed to be gold during many trials, might at the next trial be found to be like copper: not only all the uses of these minerals would fail, but they would not be distinguishable kinds of things, and the names would be unmeaning. And if this entire uncertainty as to kind and properties prevailed for all objects, the world would no longer be a world to which language was applicable. To man, thus unable to distinguish objects into kinds, and call them by names, all knowledge would be impossible, and all definite apprehension of external objects would fade away into an inconceivable confusion. In the very apprehension of objects as intelligibly sorted, there is involved a principle which springs within us, contemporaneous, in its efficacy, with our first intelligent perception of the kinds of things of which the world consists. We assume, as a necessary basis of our knowledge, that things are of definite kinds; and the aim of chemistry, botany, and other sciences is, to find marks of these kinds; and along with these, to learn their definitely-distinguished properties. Even here, therefore, where so large a portion of our knowledge comes from experience and observation, we cannot proceed without a necessary truth derived from our ideas, as our fundamental principle of knowledge.

22. What the marks are, which distinguish the constant differences of kinds of things (definite marks, selected from among many unessential appearances), and what their definite properties are, when they are so distinguished, are parts of our knowledge to be learnt from observation, by various processes; for instance, among others, by chemical analysis. We find the differences of bodies, as shown by such analysis, to be of this nature:—that there are various elementary bodies, which, combining in different definite proportions, form kinds of bodies definitely different. But, in arriving at this conclusion, we introduce a new idea, that of Elementary Composition, which is not extracted from the phenomena, but supplied by the mind, and introduced in order to make the phenomena intelligible. That this notion of elementary composition is not supplied by the chemical phenomena of cumbustion, mixture, &c. as merely an observed fact, we see from this; that men had in ancient times performed many experiments in which elementary composition was concerned, and had not seen the fact. It never was truly seen till modern times; and when seen, it gave a new aspect to the whole body of known facts. This idea of elementary composition, then, is supplied by the mind, in order to make the facts of chemical analysis and synthesis intelligible *as* analysis and synthesis. And this idea being so supplied, there enters into our knowledge along with it a corresponding necessary principle;—That the elementary composition of a body determines its kind and proportions. This is, I say, a principle assumed, as a consequence of the idea of composition, not a result of experience; for when bodies have been divided into their kinds, we take for granted that the analysis of a single specimen may serve to determine the analysis of all bodies of the same kind: and without this assumption, chemical knowledge with regard to the kinds of bodies would not be possible. It has been said that we take only one experiment to determine the composition of any particular kind of body, because we have a thousand experiments to determine that bodies of the same kind have the same composition. But this is not so. Our belief in the principle that bodies of the same kind have the same composition is not established by experiments, but is assumed as a necessary consequence of the ideas of Kind and of Composition. If, in our experiments, we found that bodies supposed to be of the same kind had not the same composition, we should not at all doubt of the principle just stated, but conclude at once that the bodies were *not* of the same kind;—that the marks by which the kinds are distinguished had been wrongly stated. This is what has very frequently happened in the course of the investigations of chemists and mineralogists. And thus we have it, not as an experiential fact, but as a necessary principle of chemical philosophy, that the Elementary Composition of a body determines its Kind and Properties.

23. How bodies differ in their elementary composition, experiment must teach us, as we have already said that experiment has taught us. But as we have also said, whatever be the nature of this difference, kinds must be definite, in order that language may be possible: and hence, whatever be the terms in which we are taught by experiment to express the elementary composition of bodies, the result must be conformable to this principle, That the differences of elementary composition are definite. The law to which we are led by experiment is, that the elements of bodies continue in definite proportions according to weight. Experiments add other laws; as for instance, that of multiple proportions in different kinds of bodies composed of the same elements; but of these we do not here speak.

24. We are thus led to see that in our knowledge of mechanics, chemistry, and the like, there are involved certain necessary principles, derived from our ideas, and not from experience. But to this it may be objected, that the parts of our knowledge in which these principles are involved has, in historical fact, all been acquired by experience. The laws of motion, the doctrine of definite proportions, and the like, have all become known by experiment and observation; and so far from being seen as necessary truths, have been discovered by long-continued labours and trials, and through innumerable vicissitudes of confusion, error, and imperfect truth. This is perfectly true: but does not at all disprove what has been said. Perception of external objects

and experience, experiment and observation are needed, not only, as we have said, to supply the objective element of all knowledge—to embody, limit, define, and modify our ideas; but this intercourse with objects is also requisite to unfold and fix our ideas themselves. As we have already said, ideas and facts can never be separated. Our ideas cannot be exercised and developed in any other form than in their combination with facts, and therefore the trials, corrections, controversies, by which the matter of our knowledge is collected, is also the only way in which the form of it can be rightly fashioned. Experience is requisite to the clearness and distinctness of our ideas, not because they are derived from experience, but because they can only be exercised upon experience. And this consideration sufficiently explains how it is that experiment and observation have been the means, and the only means, by which men have been led to a knowledge of the laws of nature. In reality, however, the necessary principles which flow from our ideas, and which are the basis of such knowledge, have not only been inevitably assumed in the course of such investigations, but have been often expressly promulgated in words by clear-minded philosophers, long before their true interpretation was assigned by experiment. This has happened with regard to such principles as those above mentioned; That every event must have a cause; That reaction is equal and opposite to action; That the quantity of matter in the world cannot be increased or diminished: and there would be no difficulty in finding similar enunciations of the other principles above mentioned;—That the kinds of things have definite differences, and that these differences depend upon their elementary composition. In general, however, it may be allowed, that the necessary principles which are involved in those laws of nature of which we have a knowledge become then only clearly known, when the laws of nature are discovered which thus involve the necessary ideal element.

25. But since this is allowed, it may be further asked, how we are to distinguish between the necessary principle which is derived from our ideas, and the law of nature which is learnt by experience. And to this we reply, that the necessary principle may be known by the condition which we have already mentioned as belonging to such principles:—that it is impossible distinctly to conceive the contrary. We cannot conceive an event without a cause, except we abandon all distinct idea of cause; we cannot distinctly conceive two straight lines inclosing space; and if we seem to conceive this, it is only because we conceive indistinctly. We cannot conceive 5 and 3 making 7 or 9; if a person were to say that he could conceive this, we should know that he was a person of immature or rude or bewildered ideas, whose conceptions had no distinctness. And thus we may take it as the mark of a necessary truth, that we cannot conceive the contrary distinctly.

26. If it be asked what is the test of distinct conception (since it is upon the distinctness of conception that the matter depends), we may consider what answer we should give to this question if it were asked with regard to the truths of geometry. If we doubted whether any one had these distinct conceptions which enable him to see the necessary nature of geometrical truth, we should inquire if he could understand the axioms as axioms, and could follow, as demonstrative, the reasonings which are founded upon them. If this were so, we should be ready to pronounce that he had distinct ideas of space, in the sense now supposed. And the same answer may be given in any other case. That reasoner has distinct conceptions of mechanical causes who can see the axioms of mechanics as axioms, and can follow the demonstrations derived from them as demonstrations. If it be said that the science, as presented to him, may be erroneously constructed; that the axioms may not be axioms, and therefore the demonstrations may be futile, we still reply, that the same might be said with regard to geometry: and yet that the possibility of this does not lead us to doubt either of the truth or of the necessary nature of the propositions contained in Euclid's Elements. We may add further, that although, no doubt, the authors of elementary books may be persons of confused minds, who present as axioms what are not axiomatic truths; yet that in general, what is presented as an axiom by a thoughtful man, though it may include some false interpretation or application of our ideas, will also generally include some principle which really is necessarily true, and which would still be involved in the axiom, if it were cor-

rected so as to be true instead of false. And thus we still say, that if in any department of science a man can conceive distinctly at all, there are principles the contrary of which he cannot distinctly conceive, and which are therefore necessary truths.

27. But on this it may be asked, whether truth can thus depend upon the particular state of mind of the person who contemplates it; and whether that can be a necessary truth which is not so to all men. And to this we again reply, by referring to geometry and arithmetic. It is plain that truths may be necessary truths which are not so to all men, when we include men of confused and perplexed intellects; for to such men it is not a necessary truth that two straight lines cannot inclose a space, or that 14 and 17 are 31. It need not be wondered at, therefore, if to such men it does not appear a necessary truth that reaction is equal and opposite to action, or that the quantity of matter in the world cannot be increased or diminished. And this view of knowledge and truth does not make it depend upon the state of mind of the student, any more than geometrical knowledge and geometrical truth, by the confession of all, depend upon that state. We know that a man cannot have any knowledge of geometry without so much of attention to the matter of the science, and so much of care in the management of his own thoughts, as is requisite to keep his ideas distinct and clear. But we do not, on that account, think of maintaining that geometrical truth depends merely upon the state of the student's mind. We conceive that he knows it because it is true, not that it is true because he knows it. We are not surprized that attention and care and repeated thought should be requisite to the clear apprehension of truth. For such care and such repetition are requisite to the distinctness and clearness of our ideas: and yet the relations of these ideas, and their consequences, are not produced by the efforts of attention or repetition which we exert. They are in themselves something which we may discover, but cannot make or change. The idea of space, for instance, which is the basis of geometry, cannot give rise to any doubtful propositions. What is inconsistent with the idea of space cannot be truly obtained from our ideas by any efforts of thought or curiosity; if we blunder into any conclusion inconsistent with the idea of space, our knowledge, so far as this goes, is no knowledge: any more than our observation of the external world would be knowledge, if, from haste or inattention, or imperfection of sense, we were to mistake the object which we see before us.

28. But further: not only has truth this reality, which makes it independent of our mistakes, that it must be what is really consistent with our ideas; but also, a further reality, to which the term is more obviously applicable, arising from the principle already explained, that ideas and perceptions are inseparable. For since, when we contemplate our ideas, they have been frequently embodied and exemplified in objects, and thus have been fixed and modified; and since this compound aspect is that under which we constantly have them before us, and free from which they cannot be exhibited; our attempts to make our ideas clear and distinct will constantly lead us to contemplate them as they are manifested in those external forms in which they are involved. Thus in studying geometrical truth, we shall be led to contemplate it as exhibited in visible and tangible figures;—not as if these could be sources of truth, but as enabling us more readily to compare the aspects which our ideas, applied to the world of objects, may assume. And thus we have an additional indication of the reality of geometrical truth, in the necessary possibility of its being capable of being exhibited in a visible or tangible form. And yet even this test by no means supersedes the necessity of distinct ideas, in order to a knowledge of geometrical truth. For in the case of the duplication of the cube by Hobbes, mentioned above, the diagram which he drew made two points appear to coincide, which did not really, and by the nature of our idea of space, coincide; and thus confirmed him in his error.

Thus the inseparable nature of the Fundamental Antithesis of Ideas and Things gives reality to our knowledge, and makes objective reality a corrective of our subjective imperfections in the pursuit of knowledge. But this objective exhibition of knowledge can by no means supersede a complete development of the subjective condition, namely, distinctness of ideas. And that there is a subjective condition, by no means makes knowledge altogether subjective, *and thus deprives it of reality; because, as we have said, the subjective and the objective elements are inseparably bound together in the fundamental antithesis.*

29. It would be easy to apply these remarks to other cases, for instance, to the case of the principle we have just mentioned, that the differences of elementary composition of different kinds of bodies must be definite. We have stated that this principle is necessarily true;—that the contrary proposition cannot be distinctly conceived. But by whom? Evidently, according to the preceding reasoning, by a person who distinctly conceives Kinds, as marked by intelligible names, and Composition, as determining the kinds of bodies. Persons new to chemical and classificatory science may not possess these ideas distinctly; or rather, cannot possess them distinctly; and therefore cannot apprehend the impossibility of conceiving the opposite of the above principle; just as the schoolboy cannot apprehend the impossibility of the numbers in his multiplication table being other than they are. But this inaptitude to conceive, in either case, does not alter the necessary character of the truth: although, in one case, the truth is obvious to all except schoolboys and the like, and the other is probably not clear to any except those who have attentively studied the philosophy of elementary compositions. At the same time, this difference of apprehension of the truth in different persons does not make the truth doubtful or dependent upon personal qualifications; for in proportion as persons attain to distinct ideas, they will see the truth; and cannot, with such ideas, see anything as truth which is not truth. When the relations of elements in a compound become as familiar to a person as the relations of factors in a multiplication table, he will then see what are the necessary axioms of chemistry, as he now sees the necessary axioms of arithmetic.

30. There is also one other remark which I will here make. In the progress of science, both the elements of our knowledge are constantly expanded and augmented. By the exercise of observation and experiment, we have a perpetual accumulation of facts, the materials of knowledge, the objective element. By thought and discussion, we have a perpetual development of man's ideas going on: theories are framed, the materials of knowledge are shaped into form: the subjective element is evolved; and by the necessary coincidence of the objective and subjective elements, the matter and the form, the theory and the facts, each of these processes furthers and corrects the other: each element moulds and unfolds the other. Now it follows, from this constant development of the ideal portion of our knowledge, that we shall constantly be brought in view of new Necessary Principles, the expression of the conditions belonging to the Ideas which enter into our expanding knowledge. These principles, at first dimly seen and hesitatingly asserted, at last become clearly and plainly self-evident. Such is the case with the principles which are the basis of the laws of motion. Such may soon be the case with the principles which are the basis of the philosophy of chemistry. Such may hereafter be the case with the principles which are to be the basis of the philosophy of the connected and related polarities of chemistry, electricity, galvanism, magnetism. That knowledge is possible in these cases, we know; that our knowledge may be reduced to principles gradually more simple, we also know; that we have reached the last stage of simplicity of our principles, few cultivators of the subject will be disposed to maintain; and that the additional steps which lead toward very simple and general principles will also lead to principles which recommend themselves by a kind of axiomatic character, those who judge from the analogy of the past history of science will hardly doubt. That the principles thus axiomatic in their form, do also express some relation of our ideas, of which experiment and observation have given the true and real interpretation, is the doctrine which I have here attempted to establish and illustrate in the most clear and undoubted of the existing sciences; and the evidence of this doctrine in those cases seems to be unexceptionable, and to leave no room to doubt that such is the universal type of the progress of science. Such a doctrine, as we have now seen, is closely connected with the views here presented of the nature of the Fundamental Antithesis of Philosophy, which I have endeavoured to illustrate.

W. WHEWELL.

XV. *On Divergent Series, and various Points of Analysis connected with them.* By AUGUSTUS DE MORGAN, V.P.R.A.S., F.C.P.S., *of Trinity College; Professor of Mathematics in University College, London.*

[Read, *March* 4, 1844.]

I BELIEVE it will be generally admitted that the heading of this paper describes the only subject yet remaining, of an elementary character, on which a serious schism exists among mathematicians as to absolute correctness or incorrectness of results. When such a question arises upon a method of pure mathematics, there can be little doubt that it must be one which is likely to lead to error if not cautiously used; and it is probable that the contending parties have not made any close agreement upon the use of terms. A review of the leading points of the controversy may be useful, accompanied by an examination of the maxims which have been adopted, but I think not very plainly stated, in the rejection of the series called divergent. The manner in which the rejection just alluded to has been made will require that, instead of dividing series into convergent and divergent, we should make a more general division, say into convergent and non-convergent. Non-convergent series may be divided into those of infinite and finite divergence: the former of which, as in the cases of $a + a + a + \ldots$ and $1 - 2 + 3 - 4 + \ldots$ can be made, by summation of terms, to differ from a given quantity to any extent; the latter, as in the cases $1 - 1 + 1 - \ldots$ and $\cos\theta + \cos 2\theta + \ldots$ cannot be made to differ from a finite quantity by more than an amount which can be ascertained. It is obvious that only the converging series can, properly speaking, be the objects of arithmetical calculation, in which they occur early, of which $\frac{1}{3} = \cdot 33333 \ldots$ is a sufficient instance. All others, whether of finite or infinite divergence, are equally out of the pale of arithmetic to those who do not acknowledge different degrees of impossibility. I do not here argue with those who reject everything which is not within the province of arithmetic, but only with those others who abandon the use of *infinitely* diverging series, and yet appear to employ *finitely* diverging series with confidence. Such appears to be the practice of those analysts who object to diverging series, both at home and abroad. They seem perfectly reconciled to $1 - 1 + 1 - 1 + \ldots = \frac{1}{2}$, but cannot admit $1 + 2 + 4 + 8 + \ldots = -1$.

Many of an earlier school took an opposite position; they freely used infinitely diverging series, but, with Euler, considered finitely diverging forms as indeterminate. To use a common phrase, they spoke as they found: they could actually obtain by rules of algebra, finite expressions from which they could evolve infinitely diverging series: but they were not able to find, or to satisfy themselves they had found, similar equivalents for most cases, particularly the trigonometrical ones, of the remaining species. They made an unguarded use of the word 'indeterminate:' sometimes it meant *undeterminable*, in the same manner as $\frac{0}{0}$ when looked at as the solution of an identical equation; sometimes only *undetermined*, either with reference to the state of science at the time the word was used, or to the state of a particular question at some one particular stage of the solution (as in the method called that of *indeterminate* coefficients). The moderns seem to me to have made a similar confusion in regard to their rejection of divergent series: meaning sometimes that they cannot be safely used under existing ideas as to their meaning and origin, sometimes that the mere idea of any one applying them at all, under any circumstances, is an absurdity.

We must admit that many series are such as we cannot at present safely use, except as means of discovery, the results of which are to be subsequently verified: and the most determined rejector of all divergent series doubtless makes this use of them in his closet. But to say that what we cannot use no others ever can, to refuse that faith in the future prospects of algebra which has already realised so brilliant a harvest, and to train the future promoter of analysis in a notion which will necessarily prevent him from turning his steps to quarters from whence his predecessors have never returned empty-handed, seems to me a departure from all rules of prudence. The motto which I should adopt against a course which seems to me calculated to stop the progress of discovery would be contained in a word and a symbol—remember $\sqrt{-1}$.

I do not pretend to have that confidence in series which, to judge from elementary writers on algebra, is common among mathematicians: not even in convergent series. A few great forms, which have had substantive and finite expressions assigned to represent the remnants after any given term may, no doubt, be perfectly trustworthy. But as for the rest, I cannot bring myself to that positive assurance with respect to any general class of series which the writers to whom I shall presently allude appear to have with respect to such divergency as they do admit. The main object of this paper is to show that they have underrated the character of most of what they reject, and overrated that of all they receive.

I shall now proceed to the different points of discussion in order.

SECTION I.

All Divergent Series, whether their divergence be finite or infinite, stand upon the same basis, and ought to be accepted or rejected together, as far as any grounds of confidence are concerned which are not directly derived from experience.

I SHALL first examine the general arguments on which Poisson supports the contradictory of the preceding assertion. This great analyst was at the head of the school in which definite integration had been made in a great measure to take the place of expansion into *algebraical* series. A definite integral is a particular kind of series, and has its converging and diverging cases, the latter being either of infinite or of finite divergence. Thus $\int_0^x \epsilon^{-x} dx$ is convergent, $\int_0^x \epsilon^x dx$ is infinitely divergent, and $\int_0^\infty \cos x dx$ is finitely divergent. Perhaps in the natural bias derived from a continual contemplation of integration under the form of summation, not of inverse differentiation, may be seen the reason for the opinion of divergent series adopted by the definite integrators. Let it only be granted that integration is as fully defined and as generally understood, as any of the fundamental operations of arithmetic, and the question on diverging series seems to be settled at once, and by a much easier argument than any of those usually proposed against them. To take an instance;—$\int_0^\infty 2^x dx$ cannot be other than $\int_0^1 2^x dx + \int_1^2 2^x dx + \int_2^3 2^x dx + \ldots$: but the first is (on the above assumption) infinite, and the second is $(\log 2)^{-1}(1+2+4+\ldots)$ which is therefore infinite. Consequently $1+2+4\ldots$ cannot, as usually held in algebra, represent -1. It must certainly be charged upon those who have hitherto used divergent series, that they have never reflected upon and explained, perhaps have never perceived, the singular apparent inconsistency which they were every day committing; namely, treating those very forms as representatives of infinity when they were consequences of integration, which they accepted as finite, when they were results of algebraical development. Referring further discussion of this point to a subsequent section, I now make two citations from memoirs by Poisson in the *Journal de l'Ecole Polytechnique*, Cahier 19, pp. 408, 409, 501.

Page 501 "On enseigne dans les élémens, qu'une série divergente ne peut servir à calculer la valeur approchée de la fonction dont elle résulte par le développement: mais *quelquefois* on a paru croire qu'une telle série peut être employée dans les calculs analytiques à la place de la fonction; et *quoique cette erreur soit loin d'être générale parmi les géomètres*, il n'est cependant pas inutile de la signaler, car les résultats auxquels on parvient par l'intermédiaire des séries divergentes, sont *toujours incertains et le plus souvent inexacts*."

Pages 408, 409. "On peut voir dans les Mémoires de Pétersbourg (*Novi Commentarii*, tom. XVII et XVIII) la discussion qui s'est élevée autrefois entre Euler et D. Bernouilli au sujet des séries de sinus ou de cosinus prolongées à l'infini. Les détails dans lesquels nous venons d'entrer, ne semblent devoir laisser aucune obscurité sur ce point d'analyse: nous admettrons avec Euler que les sommes de ces séries considérées en elle-mêmes, n'ont pas des valeurs déterminées; mais nous ajouterons que chacune d'elles a une valeur unique et qu'on peut employer dans l'analyse, lorsqu'on les regarde comme les limites des séries convergentes, c'est-à-dire, quand on suppose implicitement leurs termes successifs multipliés par les puissances d'une fraction infiniment peu différente de l'unité."

I hardly know which of the passages in my Italics ought to excite most surprise. Divergent series, at the time Poisson wrote, had been nearly universally adopted for more than a century, and it was only here and there that a difficulty occurred in using them. As to the second passage, we may clear Poisson of absolute mistatement by remembering that he had both head and hands full of a subject which had tasked his great powers to their utmost, namely, the substitution of definite integrals for series in questions of mathematical physics. As far as integration is concerned, I admit, and even think I shall presently show, that he was fully justified in what he said: in the meantime I attend to his argument in favour of finitely diverging series.

Let us take the series $1-1+1-1+\ldots$, a remarkable specific case of both algebraical and trigonometrical series. I collect from what I have quoted, and from numerous other parts of his writings, that Poisson is content to equate $\frac{1}{2}$ to $1-1+\ldots$, considering the latter as a mere form indicative of $1-g+g^2-\ldots$, where g is a fraction infinitely near to unity, but less. He will consent to use the limiting form of convergency, to walk on the line which separates convergency from divergency, but not to cross that line, even by an infinitely small quantity.

In using the language of infinitely small quantities, I do not intend to direct any part of my argument against the ideas connected with the phraseology, because both Poisson's statements and my comment on them might easily be translated into the language of the theory of limits. Let us then take $1-1+1-\ldots$ as indicating $1-g+g^2-\ldots$ where $1-g$ is infinitely small and positive. How can $1-g+g^2-\ldots$ be called convergent? Because the terms diminish without limit, and g^n, *if* n *be infinitely great*, becomes infinitely small. The departure from finite divergence, and commencement of real convergence, is infinitely distant. Now all that is wanted to make $1+2+4+\ldots$ equal to -1 is the presence of the infinitely great negative remainder, which might be considered as not destroyed, but only removed, when the second side of $(1-2)^{-1}=1+2+2^2+\ldots+2^n+2^{n+1}(1-2)^{-1}$ is made an infinite series by $n=\infty$. If suppositions which only take effect at an infinite distance from the beginning of the series are allowed to be made with regard to series of finite divergence, why may not the same be conceded in the case of infinite divergence? Both $1-1+\ldots$ and $1+2+\ldots$ are equally irreducible to their finite equivalents by the arithmetical computer; both are equally creatures of algebra: if a reason can be shown for the distinction between them, those who adhere to infinitely divergent series have a right to ask for it; but if, as I suspect, that reason be *experience*, I am prepared to contend that, when integration is not employed, there has not been produced one single instance in which divergency, properly treated, has led to error.

That experience is the guide may be safely inferred in all cases of rejection, when those who reject do it to different extents. Poisson would admit $1^2-2^2+3^2-4^2+\ldots=0$, since there is no question that, g being less than unity, the mere arithmetical computer might establish, to any number of decimal places, the identity of $1^2-2^2g+3^2g^2-\ldots$ and $(1-g)(1+g)^{-3}$. But on this equation, $1^2 - 2^2 + \ldots = 0$, Abel, another rejector, remarks (Works, II. 266), "Peut-on rien imaginer de plus horrible?"

Poisson's mode of allowing $\frac{1}{2} = 1 - 1 + \ldots$ is clearly equivalent to an adoption of the maxim that *whatever is true up to the limit is true at the limit.* When relations of pure magnitude are in question, there is no doubt of the truth of this principle. But the words *up to* must not be understood inclusively, since then the principle would merely assert that what is true at the limit and elsewhere, is true at the limit. With this caution, it is impossible to prove that a relation of magnitude is true at the limit, if at the limit we have no longer calculable magnitude. We may not say that what is calculable up to the limit is calculable at the limit, nor that what is complicated up to the limit is complicated at the limit, &c.: but only that relations which are *quantitatively* true up to the limits are so at the limits, *if the limits be quantities.* Assume $1 - 1 + \ldots$ to be quantity, determinate quantity, and that quantity may possibly be shown to be $\frac{1}{2}$ and no other: but it may not be assumed that $1 - 1 + \ldots$ *is* a quantity, because $1 - g + g^2 - \ldots$ is a quantity, up to its limit; or at least if such assumption may be made, no reason has been given for confining it to any one class of limiting forms.

Again, it is clear enough from the manner in which Fourier, Poisson, Cauchy, &c. use the limiting form $1 - 1 + \ldots$, that they intend it to signify $\frac{1}{2}$ in an absolute manner. The whole fabric of periodic series and integrals, which all have had so much share in erecting, would fall instantly if it were shown to be possible that $1 - 1 + \ldots$ might be one quantity as a limiting form of $A_0 - A_1 + \ldots$ and another as a limiting form of $B_0 - B_1 + \ldots$. Fourier's celebrated expression of a function by means of a definite integral, that of Poisson by means of a series of periodic integrals, &c., are all stated as absolute truths, and used as such, though they are proved only as limiting forms of one particular class of convergent series. A person who is much versed in the writings of the above-mentioned analysts must feel to his finger's ends that one well-established instance in which $1 - 1 + \ldots$ means other than $\frac{1}{2}$ would throw doubt upon all they have written. Now we have Poisson's assurance that these series, though indeterminate, have each a *unique* value, which can be employed in analysis when the series are considered as the limits of convergent series. Here the word 'indeterminate' is loosely used, in the sense of not determinable by actual summation: a unique value, which can be employed (and therefore of course first found) is not indeterminate in any correct sense. But who is to assure us of this uniqueness of value? How could Poisson undertake to make the assertion? By an induction—an extensive one I grant—but still an induction. From $(1 + x)^{-1} = 1 - x + \ldots$ to*

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{\epsilon^{-v^2} dv}{1 + \epsilon^{-2v\sqrt{\log x}}} = 1 - x + x^4 - x^9 + x^{16} - \ldots$$

it is always observed that where the series-side of an attainable developement gives $1 - 1 + \ldots$ the finite side gives $\frac{1}{2}$. But this induction may be overturned: and if the stability of form which really has hitherto characterized series of finite divergency should be found not to belong equally to those of infinite divergency, it should teach us rather to suspect the former than to content ourselves with merely empirical rejection. There are two ways of considering a series: absolutely, as a given algebraical expression, and relatively, as the development of a given function, from which it actually was produced. I do not defend the former mode of considering either convergent or non-convergent series; and I fully believe that analysts have been led into error, as to both classes, by incautiously reasoning on series of which the involopments were unknown. I do not dispute that the arithmetical value of a specific case of a series may, when that particular case is convergent, be calculated: but, speaking of general series, it seems to me that it is dangerous to reason upon them

* This instance is very good for the purpose, since one side or the other must have all the difficulties of divergency: either the integral or the series is divergent.

until as general an invelopment is found; after which, I incline to think that all conclusions upon the series should be upon them considered as the developments of those particular functions which produce them. My reasons are as follows.

1. Discontinuity of form is not perceptible in the series itself, though it may very possibly exist; to reason upon a series as a continuous function, without knowing from its invelopment that it is so, is pure assumption. This remark applies particularly to series which are always convergent, and most of all to series which are convergent and also begin to diminish from the first term. If we spoke of mathematical results in the same sort of language as of physical phenomena, we should say that there is inaptitude in developments to be the permanent *arithmetical* representatives of finite continuous functions, and that series which must of necessity be always convergent, shew this inaptitude by discontinuity, while the others escape from arithmetic altogether by divergency.

2. When divergent series are employed independently of their invelopments, it is impossible to distinguish the cases in which they really represent infinity from those in which they are developed forms of finite quantity. No one can actually calculate with the symbol ∞, even when its sign is determinate: for even if $\infty + \infty'$ and $\infty \times \infty'$ would not puzzle him, it is certain that $\infty - \infty'$ and $\infty \div \infty'$ would require reference to the producing functions. As soon as ∞ is attained, we must stop for examination: this cannot be done if, when attained, it is seen under the divergent form which equally belongs to finite quantities, that is, is not seen at all.

3. It cannot be questioned that series which are infinitely divergent, at least, may appear as very different things in different cases. For instance, an algebraist would be inclined almost to assert that $1 + 2 + 4 + \ldots$ must be -1; for he would say, if it be the object of algebra at all, it must satisfy the equation $z = 1 + 2z$. But now let us consider the series $1 + 2a^{-n} + 2^2a^{-n^2} + 2^3a^{-n^3} + \ldots$ which is certainly convergent, if a and n be both greater than unity, and as certainly increases without limit, as $a - 1$ diminishes without limit. When $n = 1$, the limiting form $1 + 2 + 4 + \ldots$ is clearly the representation, not of -1, but of ∞. The series $\epsilon^{-b} + x\epsilon^{-bn} + x^2\epsilon^{-bn^2} + \ldots$ satisfies the equation

$$x\frac{d^2U}{dx\,db} = \frac{n}{b}\frac{dU}{dn} \quad \text{or } U = \int_\alpha^\beta \phi\theta . x^\theta b^{\psi\theta} n^{\theta\psi\theta}\, d\theta,$$

where $\phi\theta$ and $\psi\theta$ are arbitrary, and α and β are any constants independent of a, b, and n. In taking this form for U, I follow the example of Poisson, Cauchy, &c., who are always content with such a form, provided only that it contain the requisite* number of arbitrary functions. To make the form of U an *algebraical* equivalent of the series, we must determine $\phi\theta$ and $\psi\theta$ from

$$\frac{1}{1-x} = \int_\alpha^\beta \phi\theta x^\theta 0^{\psi\theta} n^{\theta\psi\theta} d\theta, \qquad \frac{\epsilon^{-b}}{1-x} = \int_\alpha^\beta \phi\theta x^\theta b^{\psi\theta} d\theta;$$

a useless attempt, even when $x < 1$, unless discontinuous forms of $\psi\theta$ be introduced. Here is a clear case in which $1 + 2 + 4 + \ldots$ represents ∞: are we then really to abandon the assertion that it satisfies the equation $1 + 2z = z$? If so, the opponents of divergent series have gained their point, for those developments are not even to be trusted as to their symbolical properties. But I rather argue that it is not so, in the following manner. Every equation, it is very well known, has as many roots as units of dimension, only on the supposition that its problem is absolutely of that dimension, and not a degenerate case of a higher dimension. Plenty of simple problems may be proposed which illustrate this known result of common algebraical reasoning. Now the equation which stands related to the series in question in the same manner as $1 + 2z = z$ to $1 + 2 + 4 + \ldots$ is $\phi z = x^z \epsilon^{-bn^z} + \phi(z + 1)$. If this last could be generally solved, then $\phi 0$ would be the series

* They assume that $\Sigma\phi\theta\epsilon^{x\theta}$, or $\phi_1\theta_1\epsilon^{x\theta_1} + \phi_2\theta_2\epsilon^{x\theta_2} + \ldots$ can always be represented by $\int\phi\theta\,\epsilon^{x\theta}d\theta$, which I believe to be true if $\phi\theta$ may be discontinuous. But it has not been proved: should it happen to be false, 'toujours incertains' may be applied to many of their results, and 'le plus souvent inexacts,' may follow.

required: if, after solution, b were made $= 0$, we should see that $1 + 2z = z$, the result for $x = 2$, would be only a degenerate form of a more complicated form.

This remark will illustrate my opinion that a series is to be considered strictly in relation to the function from which it is developed. If $x^z + x^{z+1} + \dots$ be absolutely under consideration, the equation $\phi z = x^z(1 - x)^{-1}$ may be strictly obtained, and thence $(1 - x)^{-1}$ for $1 + x + \dots$ But there is no saying what further degeneracy of form may be seen in passing from $\phi z = x^z \epsilon^{-bz^2} + \phi(z + 1)$ to $1 + 2z = z$, which is not seen in passing from $\phi z = x^z + \phi(z + 1)$ to the same.

My conclusion is, that a divergent series may have for its proper value either that which is usually so considered, or infinity, according to the nature of the function from which it is expanded. And since every equation has as many roots as it has algebraical dimensions, so many of them being infinite as there are vanishing coefficients which precede the first finite coefficient, there can be no right to say that the symbolical character of divergent series is forfeited, until either the symbol ∞ takes the place of the ordinary value in a case in which there is no degeneracy, or until some *finite value*, different from the ordinary one, is shown, in some one particular case, to be the proper representative of the series. Let $1 + 2 + 4 + \dots$ be shown to be any thing but a root of either $1 + 2z = z$, or of another equation which has degenerated into $1 + 2z = z$; that is, let it come out any thing but -1 or ∞, and as a result of any process which does not involve *integration performed on a divergent series*—and I shall then be obliged to confess that divergent series must be abandoned, or rather, that the generalizations frequently made on the subject must be much curtailed. But nevertheless, there is nothing to lead us to doubt that divergent series of all classes, whether of finite or infinite divergence, must be treated alike. If any one say that such a difficulty as the preceding *cannot* occur in series of finite divergence, he must prove it.

It might perhaps be supposed that, in every doubt which has been raised in the preceding remarks, the finitely diverging series have been much less hardly borne upon than the others—to an extent which may make it seem to be almost admitted by myself that the foreign analysts, if not justified in their dogmatical rejection of infinitely diverging series, have nevertheless chosen, and judiciously chosen, to confine themselves to the safer of two paths. But it is to be remembered that I have been obliged, as yet, to mention only their practical division, which really consists in the separation of all finitely diverging series from the rest. Had I had to make my own division of series, I should have admitted that there was one of two paths which was much safer than the other: but I should have asserted that the labors of the writers in question did not extend over the whole of that path. From the sort of appeal to induction which unfortunately must, in the present state of our knowledge, help us to a part of our results on series, backed by considerably more of demonstration than has been applied to the remaining cases, it seems to me pretty clear that the proper line of demarcation does not separate series of finite and infinite divergence, but series having all their signs alike from those of terms alternately positive and negative, or consisting of parcels of terms alternately positive and negative. This will be the subject of a subsequent section.

SECTION II.

The Operation of Integration as at present understood, is one of Arithmetic, as distinguished from Algebra, and must not be applied unreservedly to Divergent Series.

ACCORDING to elementary notions, we differentiate when we find the value of $\{\phi(x+h) - \phi x\} h^{-1}$ in a calculable form when $h = 0$. Integration is usually defined as the inverse question, which must be, required ϕx when the calculable form of $\{\phi(x + h) - \phi x\} h^{-1}$ is given for $h = 0$. This demands the solution of a functional equation, and it is easy to say, Let this equation be considered as solved, and let the process of solution have a name. But the state of our knowledge makes it of no use whatever to express a conventional solution, since our power of translating our convention into ordinary language is confined to a small number of cases, all rendered backwards from the direct process. Common integration is only the *memory of differentiation:* and the process of parts, and the few other artifices by which it is effected, are changes, not from the unknown to the known, but from the forms in which memory will not serve us to those in which it will. We may assume that any function has an integral, and we may write down $\int \cos x^x\, dx$ or $\int \epsilon^{-x^2} dx$; we may also have recourse to series, and by assuming an unlimited use of divergency, we may procure abundance of nominal answers to any question. But we cannot be so much as sure of the fact that every continuous function *has* an integral, except by recourse to the summatory definition, namely,

$$\int_a^x \phi v\, dv = \left\{\phi a + \phi\left(a + \frac{x-a}{n}\right) + \phi\left(x + 2\frac{x-a}{n}\right) + \ldots + \phi\left(x + n\frac{x-a}{n}\right)\right\}\frac{x-a}{n}$$

in which n is made infinite. This definition, as is well known, never fails, nor can fail, to give one value for every value of a and x, applied to one *branch* of the function, except only when ϕv becomes infinite at or between $v = a$ and $v = x$. In this last case, we have not even the means of universally defining $\int \phi v\, dv$: all the difficulties of divergent series meet us again.

In confining ourselves to this *arithmetical* definition of an integral, when one of the limits is infinite, we must, as to a large number of cases, act precisely as if we separated a class of divergent series from the rest, and insisted upon their retaining for their values the idea which the attempt at arithmetical summation gives, infinity. The early problems by which the nature and use of integration is suggested, being problems on concrete (mostly on space) magnitude, cannot afford the means of generalizing our definition. No doubt the area of the curve $y = \epsilon^x$, represented by $\int_0^\infty \epsilon^x dx$, is greater than any surface which can be assigned: no doubt also that the series of inscribed rectangles $1 + \epsilon + \epsilon^2 + \ldots$ is the same. When we shall have obtained the definition of an integral by which we can state such a value for $\int_0^\infty \epsilon^x dx$ as is the true correlative to $(1-\epsilon)^{-1}$ considered as the value of $1 + \epsilon + \ldots$ then, and not till then, shall we be entitled to claim integration as an instrument of algebra in the widest sense. Some of the objections raised against divergent series, indeed most of those which are very plausible, are grounded upon the supposition that integration may be as unreservedly applied to divergent as to convergent series, if the former are to be used at all. That this cannot be done may be satisfactorily shown by instances, as follows:

$$\text{Let } \phi v = \frac{1 - x\cos av}{1 - 2x\cos av + x^2} = 1 + x\cos av + x^2\cos 2av + \ldots\ldots$$

which never becomes infinite for any value of v, except only when $x = \pm 1$; and the series is convergent when x lies between -1 and $+1$. Multiply both sides by $\epsilon^{-v^2}dv$, and integrate from $v = 0$ to $v = \infty$, in which case there cannot be any doubt about the purely arithmetical (or convergent) character of every integration. This gives us, t being $\epsilon^{-\frac{a^2}{4}}$

$$\frac{2}{\sqrt{\pi}}\int_0^\infty \epsilon^{-v^2}\phi v\, dv = 1 + xt + x^2t^4 + x^3t^9 + x^4t^{16} + \ldots\ldots$$

This resulting series is convergent for all values of x: for, since t is less than 1, xt^n must become less than unity after a certain value of n, and thenceforward $S(xt^n)^n$ must be more convergent than any series of simple powers. If x lie between -1 and $+1$, the whole of this process is purely arithmetical, and the identity of the two sides of the last equation might be approximately verified by actual computation: if not, the original series, though divergent, is changed into a convergent one by the process. Change x into x^{-1}, and let ϕv then become $\phi_1 v$; we find $\phi v + \phi_1 v = 1$, and

$$\frac{2}{\sqrt{\pi}}\int_0^\infty \epsilon^{-t^2}\phi v\,dv + \frac{2}{\sqrt{\pi}}\int_0^\infty \epsilon^{-t^2}\phi_1 v\,dv = \frac{2}{\sqrt{\pi}}\int_0^\infty \epsilon^{-t^2}dv = 1.$$

Accordingly, if all that precedes be correct, we have

$$2 + \left(x+\frac{1}{x}\right)t + \left(x^2+\frac{1}{x^2}\right)t^4 + \left(x^3+\frac{1}{x^3}\right)t^9 + \dots = 1,$$

which is certainly false, unless a convergent series can represent less than half the sum of its terms. This last series is always convergent, except only when $a = 0$, or $t = 1$, in which case the last equation is found to be algebraically true. If for x we write $-x$, the equation is found to be true when t is equal to the least of x and x^{-1}, but is certainly not universally true.

Apply the same process to

$$\frac{(1-x)\cos av}{1-2x\cos 2av + x^2} = \cos av + x\cos 3av + x^2\cos 5av + \dots\dots\dots\dots$$

and the result is

$$0 = \left(1+\frac{1}{x}\right)t + \left(x+\frac{1}{x^2}\right)t^9 + \left(x^2+\frac{1}{x^3}\right)t^{25} + \dots\dots$$

on which precisely the same remarks might be made. I might multiply instances of this kind to any extent; but the following consideration will render them needless, as showing that what we have seen is precisely what we ought to have expected.

Integration, though only capable of an arithmetical definition, is the most decided changer of form which we ever use. A change of value in a constant may introduce a totally different form into an integral; and in particular, the assumption of infinite value for a constant has this effect almost without exception. And in regard to definite integrals, there is hardly any end to the *known* instances in which complete and apparently arbitrary changes of form (such as cannot pass one into another through $\frac{0}{0}$ or the like) arise from alteration of the specific value of a constant. If then V be expanded into the series $P_0 + P_1 + P_2 + \dots$ and if the sum of n terms,

$$P_0 + P_1 + \dots + P_{n-1} \text{ be called } Q_n\text{; we obviously have}$$

$$\int_0^a V\,dv = \int_0^a P_0\,dv + \int_0^a P_1\,dv + \dots + \int_0^a (V - Q_n)\,dv$$

where n is made infinite after integration. When the series $P_0 + P_1 + \dots$ is convergent, then, even granting that $\int(V - Q_\infty)\,dv$ may have circumstances peculiar to $n = \infty$, it is of no consequence, since considerations of form are rendered useless by evanescence of value: the elements of $\int(V-Q_n)\,dv$ must, by the hypothesis of convergency, diminish without limit as compared with the corresponding elements of $\int P_0\,dv$, $\int P_1\,dv$, &c. Even if integration converted the convergent series into a divergent one, this would still be the case. But if $P_0 + P_1 + \dots$ be *divergent*, we have no longer any right to draw any conclusion about $\int(V - Q_n)\,dv$ from observing what takes place with $\int P\,dv$, $\int P_1\,dv$, &c. Applying this to our first example above, we have

$$\frac{1 - x\cos av}{1 - 2x\cos av + x^2} = 1 + x\cos av + \dots + x^n\cos nav + x^{n+1}\frac{\cos(n+1)av - x\cos nav}{1-2x\cos av + x^2}$$

change x into x^{-1}, and add; which gives

$$1 = 2 + \left(x+\frac{1}{x}\right)\cos av + \dots + \left(x^n+\frac{1}{x^n}\right)\cos nav + \frac{\left(x^{n+1}+\frac{1}{x^{n-1}}\right)\cos(n+1)av - \left(x^{n+2}+\frac{1}{x^n}\right)\cos nav}{1-2x\cos av + x^2}$$

This equation is identically true, the only restriction being that n must be a positive integer (0 included). Consequently, we have, as specimens of legitimate inference from integrating a divergent series,

$$\frac{2}{\sqrt{\pi}}\int_0^{\infty}\frac{\left(x^{n+2}+\frac{1}{x^n}\right)\cos nav-\left(x^{n+1}+\frac{1}{x^{n-1}}\right)\cos(n+1)av}{1-2x\cos av+x^2}\epsilon^{-v^2}dv$$

$$=1+\left(x+\frac{1}{x}\right)\epsilon^{-\frac{a^2}{4}}+\left(x^2+\frac{1}{x^2}\right)\epsilon^{-\frac{4a^2}{4}}+\ldots+\left(x^n+\frac{1}{x^n}\right)\epsilon^{-\frac{n^2a^2}{4}}$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{(x^{n+1}+x^{-(n+1)})\cos nav-(x^n+x^{-n})\cos(n+1)av}{(x+x^{-1})-2\cos av}\epsilon^{-v^2}dv$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{\frac{1}{2}(x^{n+1}+x^{-(n+1)})-\frac{1}{2}(x^n+x^{-n})\cos av}{\frac{1}{2}(x+x^{-1})-\cos av}\cos nav\,\epsilon^{-v^2}dv\,.$$

The series continued *ad infinitum* is expressed by the value of the integral just found, in which n is made infinite, being the very remainder which is called nothing in the original and fallacious process. Many interesting forms might be derived from the preceding and similar cases, but having no reference to the subject of this paper.

When the terms of a divergent series separately vanish, the series having remained divergent up to the time of evanescence, it is customary, in elementary works, to assume that the series itself vanishes: or $0+0+0+\ldots$ is taken to represent 0. Very frequently, no doubt, the invelopment shows that this is correct; and I think I shall be able to show that if the function be perfectly continuous on both sides of the epoch of evanescent form, a reason can be given why it must be so. But so far as the series itself is concerned, we have no right to come to such a conclusion, unless we can shew that as the evanescent form is approached, the invelopment diminishes without limit. The following instance will show the necessity of this caution.

The integral $\int_0^{\infty}\frac{\epsilon^{-a^2t}\cos bt\,dt}{1+t^2}$ is convergent for all values of a, however small, and certainly is not comminuent with a, but approaches the limit $\frac{1}{2}\pi\epsilon^{-b}$, the well-known value of $\int_0^{\infty}\cos bt\,(1+t^2)^{-1}dt$. Expand the first side into

$$\int_0^{\infty}\epsilon^{-a^2t}\cos bt\left(1-t^2+t^4-\ldots\ldots+(-1)^n t^{2n}+(-1)^{n+1}\frac{t^{2n+2}}{1+t^2}\right)$$

which, from

$$\int_0^{\infty}\epsilon^{-a^2t}\cos bt\,.\,t^{2n}dt=\frac{\Gamma(2n+1)\cos\{(2n+1)\tan^{-1}(ba^{-2})\}}{(b^2+a^4)^{\frac{1}{2}(2n+1)}},$$

gives, making $\tan^{-1}(ba^{-2})=\theta$

$$(b^2+a^4)^{-\frac{1}{2}}\cos\theta-1.2\,(b^2+a^4)^{-\frac{3}{2}}\cos 3\theta+1.2.3.4\,(b^2+a^4)^{-\frac{5}{2}}\cos 5\theta-\ldots\ldots$$

$$+(-1)^n\,1.2.3\ldots 2n\,(b^2+a^4)^{-\frac{2n+1}{2}}\cos(2n+1)\theta+(-1)^{n+1}\int_0\frac{\epsilon^{-a^2t}\cos bt\,.\,t^{2n+2}dt}{1+t^2}.$$

If we neglect the last term, or suppose n infinite, we have expanded the given integral into a divergent series of which all the terms are comminuent with a: for $a=0$ gives $\theta=\frac{1}{2}\pi$. When we have the remainder, we may, by retaining its proper value, allow the preceding form $0+0+0+\ldots$ to stand for 0: but otherwise the appearance of that form must be a warning, when it arises from the value of a divergent series, that there may be some finite equivalent which is not to be neglected.

It is worth noting that immediately before the terms of the preceding series vanish, they are all of one sign, or $\cos\theta$, $\cos 3\theta$, &c. are of alternate signs. This is one out of the constantly recurring cases in which it happens that the difficulties of series are mostly incident to the divergent case in which all the signs are the same: the illustration of which is the subject of the next section but one.

SECTION III.

It generally happens that the real analytical equivalent of the different values of an indeterminate expression, is the mean of those different values.

This principle must rest at present upon induction. When Leibnitz pronounced $\frac{1}{2}$ to be the value of $1 - 1 + \ldots$ because there was no apparent reason why either 1 or 0 should be preferred, he was not only right in his conclusion, but had a glimpse (though not in solid reasons) of a principle which admits of such frequent confirmation that it may be suspected to be general.

In the first place, if we take any algebraical series, such as $a + bx - cx^2 + ax^3 + bx^4 - cx^5 + \ldots$ in which $c = a + b$, so that when $x = 1$, the successive results of summation are a, $a + b$, 0, a, $a + b$, 0, &c. we find by common processes that the analytical equivalent is the mean of a, $a + b$, 0, or $\frac{1}{3}(2a + b)$. The same thing happens if we take other forms which produce the same limiting form, as $a + b\cos\theta - c\cos 2\theta + \ldots$

Secondly, if we take a series $A_0 + A_1\cos\theta + A_2\cos 2\theta + \ldots$ or Fourier's integral $\int_0^\infty \int_{-\infty}^{+\infty} \cos w(x - v)\, \phi v\, dw\, dv$, in such manner that it may represent the ordinate of a discontinuous curve, the branches of which do not join at the common ordinate, it is found that for the abscissa of the common ordinate the series and the integral represent in both cases, not either or both of the ordinates, but the mean between them.

Thirdly, the indeterminate symbols $\sin\infty$ and $\cos\infty$ are found in numberless cases to represent, each of them, 0, the mean value of both $\sin x$ and $\cos x$. The mean value of any function ϕx, between a and b, is $\int_a^b \phi x\, dx$ divided by $b - a$.

Fourthly, if x lie between $-l$ and $+l$, Poisson has shewn that

$$\phi x = \frac{1}{2l}\int_{-l}^{+l} \phi v\, dv + \frac{1}{l}\Sigma\left\{\int_{-l}^{+l} \cos\frac{m\pi(x - v)}{l} \cdot \phi v\, dv\right\} (\text{from } m = 1 \text{ to } m = \infty),$$

the second side of which is not changed in value, by changing the sign of l. And this second side is the same whether we make $x = -l$, or $x = +l$; consequently it is wholly undecided whether it is then to represent $\phi(-l)$ or $\phi(l)$. Poisson has shown that in either case it represents the mean of $\phi(l)$ and $\phi(-l)$.

Fifthly, if we extend the term *mean value*, and, in cases in which the function becomes infinite, define it as $\int_a^b \phi x\, dx \div (b - a)$, the same principle applies, in a very peculiar manner, to the remaining trigonometrical functions, if the part of the integral at which ϕx becomes infinite, be examined in the manner which occurs so frequently in the writings of M. Cauchy. Let us take for instance, $\tan x$. In $\int_0^{2\pi} \tan x\, dx$, the finite parts destroy one another: and to obtain the expression for it we must examine the integral from $\frac{1}{2}\pi - \mu$ to $\frac{1}{2}\pi + \mu$, and from $\frac{3}{2}\pi - \mu$ to $\frac{3}{2}\pi + \mu$, μ being infinitely small. Now the indefinite integral is $-\log\cos x$, so that we have to examine

$$\log\frac{\cos(\frac{1}{2}\pi - \mu)}{\cos(\frac{1}{2}\pi + \mu)} \text{ and } \log\frac{\cos(\frac{3}{2}\pi - \mu)}{\cos(\frac{3}{2}\pi + \mu)}$$

each of which is $\log(-1)$ or $\pi\sqrt{-1}$, when $\mu = 0$. Hence $\int_0^{2\pi} \tan x\, dx$ is $2\pi\sqrt{-1}$, which divided by 2π, gives $\sqrt{-1}$, the proper representation of $\tan\infty$, if this principle be true. Now if we examine the equation $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$ and make x infinite, presuming that ∞ and $\infty + y$ are the same angles, we find $\tan\infty = \pm\sqrt{-1}$. In the same manner $\cot\infty$ is $\pm\sqrt{-1}$. It cannot be argued that since the values of $\tan x$, from $x = 0$ to $x = \pi$, have signs contrary to those from $x = \pi$ to $x = 2\pi$, therefore if $\pi\sqrt{-1}$ be taken for the first, $-\pi\sqrt{-1}$ should be taken for the second: the reason being that the signs in the second semicircle are really repetitions of those in the first, and only contrary in an inverted order. And it must be remembered that, A being the mean value of X, ϕA is not therefore that of ϕX: thus $\sin^2 x$ has the mean value $\frac{1}{2}$, not 0^2. Also that, when a quantity is, at one or more epochs, infinite, its mean value is not necessarily positive because all its values are positive. Thus $\tan^2 x$ has -1 for its mean value. The mean value of $\sec x$ or $\operatorname{cosec} x$ is 0. This remarkable coincidence of two modes so remote from each other of determining the analytical meaning of $\tan \infty$ and $\cot \infty$, depends at last upon $\epsilon^{\pm \infty \sqrt{-1}} = 0$, an equation which more writers have virtually used than have openly dared to state it. The apparent disturbance of the law of continuity when $x = \infty$, as in $\cos^2 \infty + \sin^2 \infty = 0$, &c. is perhaps what has prevented the formal recognition of these relations: nevertheless they will, it may confidently be asserted, not only obtain universal reception, but finally a rational and consistent explanation.

The following is a glimpse, perhaps, of the explanation, as applied to series. In every convergent series, the limit of the sum of all its terms *is* the mean value obtained from all the summations: the mean of n partial summations A_1, $(A_1 + A_2)$, ... $(A_1 + A_2 + \ldots + A_n)$

$$\text{is } A_1 + \frac{n-1}{n} A_2 + \frac{n-2}{n} A_3 + \ldots + \frac{1}{n} A_n,$$

which, as n is increased without limit, has $A_1 + A_2 + \ldots$ *ad inf.* for its limit. Hence, by Poisson's principle, by which I mean the assumption of the right to apply the maxim, "that which is quantitatively true up to the limit, is true in the same sense at the limit, when the limit presents an *incalculable form*"—we may assert most positively, that $1 - 1 + 1 - \ldots$ must be $\frac{1}{2}$ whenever it is the limiting form of convergency: not on the metaphysical doctrine (probably suggested by the known result) of Leibnitz, namely, that we can see no reason to prefer 0 to 1, or 1 to 0, and must therefore take a mean; but because n partial summations give the mean $\frac{1}{n} \times \frac{n}{2}$ or $\frac{1}{n} \times \frac{n+1}{.2}$ according as n is even or odd, and the limit of both is $\frac{1}{2}$. At the same time it is easily proved that whenever the partial summation gives recurrences in which 0 occurs at stated equal intervals, the limit of the means must be the mean of one period.

As in other cases, the diverging series whose terms are all of one sign is not elucidated by this process, which nevertheless, provided we adhere to our principle, brings out the true algebraical result for series which have terms alternately positive and negative. The mean of 1, $1 - a$, $1 - a + a^2$, &c. (n summations), is

$$\frac{1}{1+a} + \frac{a}{n(1+a)^2} + (-1)^{n+1} \frac{a^{n+1}}{n(1+a)^2}:$$

if, when n is infinite, we take $(-a)^{n+1}$ as 0, the mean of the values between which we cannot then choose, we have $(1+a)^{-1}$ as the limit.

SECTION IV.

Series of alternately positive and negative signs stand upon a much safer basis than those in which all the terms have the same signs, and that whether their divergence be finite or infinite.

AT the very outset, namely, in the mode of finding whether the series is convergent or divergent, there is every possible difference between the two species above-named, which we may term *progressing* and *alternating*. The progressing series $\phi(1) + \phi(2) + \ldots$ is convergent when the first of the set

$$P_0 = -x \frac{\phi' x}{\phi x}, \quad P_1 = \log x (P_0 - 1), \quad P_2 = \log \log x (P_1 - 1) \ldots \quad P_n = (\log)^n x (P_{n-1} - 1) \ldots$$

which is not equal to unity is greater than unity; and divergent when the first which is not equal to unity is less than unity. But $\phi(1) - \phi(2) + \ldots$ is necessarily convergent, provided only that $\phi(\infty) = 0$ *continuously*, or that the terms ultimately diminish without limit.

A progressing series must be either convergent or infinitely divergent; an alternating series may be convergent, or either finitely or infinitely divergent: but the infinite divergence of the latter is of a different character from that of the former. I very much doubt whether it is quite correct to apply the same phrases to both kinds of series.

It is easy to apply Poisson's principle to alternating series, even when they are of infinitely diverging form. We can always contrive to find positive quantities B_0, B_1, &c. in such a manner that $A_0B_0 - A_1B_1 + A_2B_2 - \ldots$ is convergent, up to a certain value of a variable contained in B_0 &c., which makes them become severally $= 1$. Thus $1 - a + a^2 -$ is a limiting form of $1 - ax^{-n} + a^2x^{-n^2} - a^3x^{-n^3} + \ldots$ which, n being > 1 is certainly convergent down to $x = 1$, exclusive; and this whatever the value of a may be. Whether this limiting form is always $(1 + a)^{-1}$ may be a question; but, as I think is sufficiently shown in various parts of this paper, the question may also be asked about the finitely diverging series which have been so confidently allowed.

When an alternating series is convergent, and a certain number of its terms are taken as an approximation, the first term neglected is a superior limit of the error of approximation. This very useful property was *observed* to belong to large classes of alternating series, when finitely or even infinitely divergent: I do not remember that any one has *denied* that it is universally true, while many have implicitly asserted it. When the series is convergent for a certain number of terms, particularly if the terms become very small before they begin to increase again, it obviously makes the divergent alternating series practically as useful as the converging series, perhaps even more so, for it is very frequent that the greater the ultimate divergence, the greater also is the primitive tendency towards convergence.

In any series $P_0 - P_1 + P_2 - \ldots$ this theorem is obviously true as long as the remnant $P_n - P_{n+1} + \ldots$ has the same sign as P_n, or the positive sign. Thus, if $P_n - P_{n+1} + \ldots = Q_n$, we have for the series $P_0 - P_1 + Q_2$ and $P_0 - P_1 + P_2 - Q_3$: if Q_2 and Q_3 be positive, the series is greater than $P_0 - P_1$ and less than $P_0 - P_1 + P_2$; which is a case of the theorem. It is also clear that if either Q_2 or Q_3 be negative this case is not true.

That the theorem is not universally true will appear in the following instances:

$$\tfrac{1}{2} = \cos^2 a - \cos^2 2a + \cos^2 3a - \ldots$$

$$\frac{1 - 3t}{1 - t^2} = 1 - 3t + t^2 - 3t^3 + t^4 - 3t^5 + \ldots$$

It is not true that $\frac{1}{2}$ always lies between $\cos^2 a$ and $\cos^2 a - \cos^2 2a$, or that $(1 - 3t)(1 - t^2)^{-1}$ always lies between 1 and $1 - 3t$, whenever t is positive. The following investigations, though they will fully explain why it is that the theorem is so often true, are insufficient to distinguish accurately between those in which it is and is not true.

When ϕx can be expanded into $A - Bx + Cx^2 - \ldots$ (A, B, &c. being positive), we take the known form

$$\phi 0 + \phi' 0 \,.\, x + \phi'' 0 \frac{x^2}{2} + \ldots\ldots + \phi^{(n)} 0 \frac{x^n}{2.3\ldots n} + \phi^{(n+1)}(\theta x) \frac{x^{n+1}}{2.3\ldots n}$$

in which $\theta < 1$. If then $\phi' 0$, $\phi''' 0$, &c. be negative, and $\phi 0$, $\phi'' 0$, &c. positive, and if ϕx, $\phi' x$, &c. each preserve, up to $x = a$, the sign it starts with when $x = 0$, there is no question that the theorem is true from $x = 0$ to $x = a$. Thus common differentiation with respect to x will prove the theorem for the case of

$$\int_0^\infty \frac{\epsilon^{-v} dv}{1 + xv} = 1 - x + 2x^2 - 2.3x^3 + \ldots$$

For any particular series $A_0 - A_1 + \ldots$ it is enough that $A_0 - A_1 x + \ldots$ should be a continuous function of x whose differential coefficients preserve their initial signs from $x = 0$ to $x = 1$. But though some of them should change sign, the theorem obviously remains unaffected as to summation stopping at parts of the series in which no change takes place. It is then no wonder that the theorem should be so frequently true.

Whatever value a function may have when $x = 0$, it is obvious that if the commencing series of signs, namely, those of $\phi\, 0$, $\phi'\, 0$, &c. be $+ - + - + -$ &c. ad infinitum, the function itself, and all its differential coefficients, are at the first instant in a state of *numerical diminution*. The reason is that those which begin negative are *algebraically increasing*, while those which begin positive are *algebraically diminishing:* this follows from the well-known (but much too scantily used) theorem that a function is in a state of algebraical increase or decrease according as its differential coefficient is for the moment positive or negative. Adopting for convenience the mechanical idea of the differential coefficient representing the *velocity* of the function, and supposing x to be the time elapsed, say in seconds, let $\phi x = A_0 - A_1 x + A_2 x^2 - \ldots$ be a function of x, A_0, A_1, &c. all being positive. And first let A_0, A_1, A_2, &c. present an unbroken series of diminutions, or $A_0 - A_1$, $A_1 - A_2$, &c. an unbroken series of positive signs. Then ϕx begins $= A_0$, with retardation at the rate of $-A_1$ per second. But A_1 is less than A_0; therefore this rate of retardation cannot change the sign of ϕx in one second, unless it receive an increase. But this there is no symptom of at the commencement, since $\phi''0$ is positive, and the retardation begins by being checked. Hence, if a function start with a differential coefficient of a sign different from its own, and numerically less, it cannot change sign within the next unit of increase of the variable, without the second differential coefficient first changing sign. Nor can it even change sign before x becomes $\dfrac{A_0}{A_1}$ without a change of sign in $\phi'' x$ previously occurring. For if the velocity had continued uniform, it would then have been $A_0 - \dfrac{A_0}{A_1} A_1$ or 0, and would not have changed sign till after $x = \dfrac{A_0}{A_1}$ at least; but since the velocity of retardation begins by being diminished ($\phi''0$ being positive), it must make this up before $x = \dfrac{A_0}{A_1}$ if a change of sign be to take place; that is, *increase of retardation* must come on, or $\phi'' x$ must become negative. All this will be very plainly pictured in the curve $y = \phi x$.

Again, if $\phi_1 x = A_1 - A_2 x + \ldots$ and if $A_1 > A_2$ similar reasoning shows that $\phi_1 x$ cannot change sign before $x = 1$, unless $\phi_1'' x$ first change sign. If neither $\phi'' x$ nor $\phi_1'' x$ change sign from $x = 0$ to $x = 1$, then it is easily collected that $A_0 - A_1 + \ldots$ lies between A_0 and $A_0 - A_1$. And if we suppose A_0, A_1, &c. to diminish until we come to A_n, then if $\phi_n x = A_n - A_{n+1} x + \ldots$ we see that if neither $\phi''_{n-2} x$ nor $\phi''_{n-1} x$ vanish before $x = 1$, we are sure that A_{n-2} and $A_{n-2} - A_{n-1}$ contain $A_{n-2} - A_{n-1} + \ldots$ between them; from which it may readily be proved that the theorem is true up to the last but one of the converging terms, under the preceding pair of conditions.

The useful part of this theorem in calculation, is undoubtedly its *usual* truth for all the apparently converging terms of the series. And we see from the above that if these converging terms last up to A_n, then m not being $> n$, the theorem is true up to A_{m-1}, inclusive, if neither $\phi''_{m-2} x$ nor $\phi''_{m-1} x$ vanish before $x = 1$. But the theorem is not *universally* true even for converging terms. Let $\phi x = 3 - 2x + x^2 - 20 x^3 + 20 x^4 - 20 x^5 + \ldots$ which has three terms converging, and is of finite divergence; so that Poisson would admit $-8 = 3 - 2 + 1 - 20 + 20 - \ldots$ as the limiting form of the above when $x = 1$. But -8 does not lie between 3 and $3 - 2$. This series is the development of $(3 + x - x^2 - 19 x^3)(1 + x)^{-1}$ and its second differential coefficient will be found to change sign before $x = 1$.

We will now look at the theorem in another point of view. Every alternating series may be reduced to a case of $\phi x - \phi(x+1) + \phi(x+2) - \ldots$ in which ϕv is a positive function from $v = x$ to $v = \infty$. If this be the proper developement of ψx, then $\psi x + \psi(x+1) = \phi x$; consequently $\psi v + \psi(v+1)$ must be always positive from $v = x$ to $v = \infty$. Hence ψv cannot change from + to − when $v = a$, without changing again from − to + before $v = a + 1$. Now the theorem can only be disturbed by ψv becoming negative: for $\psi x = \phi x - \psi(x+1)$, or, $\psi(x+1)$ being positive, $\psi x > \phi x$; again $\psi x = \phi x - \phi(x+1) + \psi(x+2)$, or, $\psi(x+2)$ being positive, $\psi x < \phi x - \phi(x+1)$, and so on.

Hence 1. No function ψx can be expanded as above unless it be one in which its changes of sign go in pairs, the − + change following the + − change before the variable has received an additional unit: 2. except at those epochs at which $\psi(x+n)$ happens to be negative, the theorem must be true. As long as ϕx, $\phi(x+1)$, &c. continue diminishing, the theorem must either be true, or there must be a minimum value of ψx within a unit-change of the variable, reckoning from the last change of sign. When ψx changes from + to −, $\psi' x$ is negative, and when from − to +, $\psi' x$ is positive: there must then be a minimum value of ψx between the two changes. Now as long as ϕx diminishes, or $\phi' x$ is negative, $\psi' x + \psi'(x+1)$ is also negative. After the minimum is past, then, $\psi' x$ cannot continue positive until x has increased by a whole unit, or there must be a maximum value within a unit-change of the variable, reckoning from the minimum. If then the terms continue diminishing as far as $\phi(x+n)$, it may be collected from the above that the theorem is true for the several summations up to $\phi(x+n-1)$, except for those in the neighbourhood of the last terms of which are found two roots of ψx for values of x not differing by a unit, followed by a maximum value of ψx, for a value of x not a unit in advance of that which gives the intermediate minimum of the roots. And if ψx can ever become infinite, ϕx being finite, then ... $\psi(x+2)$, $\psi(x+1)$, ψx, $\psi(x-1)$, ... are all infinite, with alternate signs. From this it will readily be seen that in the greater number of cases the theorem must be strictly true.

Again, it is now known that every function ϕx can be expressed in the form $\Sigma A\epsilon^{ax}$, provided that integration be included under the sign Σ, and also the finite summation of terms in which A is infinitely great, and a infinitely small, and which give a finite sum by difference of sign. Whether many cases of this reduction do not involve much greater difficulty than those of divergent series, may be a question. However this may be, it is clear that in whatsoever manner ϕx may be represented by $\Sigma A\epsilon^{ax}$, in the same manner $\phi x - \phi(x+1) + \ldots$ may be represented by $\Sigma \dfrac{A\epsilon^{ax}}{1+\epsilon^{a}}$. In all cases, then, in which the several terms of $\Sigma A\epsilon^{ax}$ are severally positive, and, if infinite in number, can be arithmetically summed, it follows that ψx or $\Sigma \dfrac{A\epsilon^{ax}}{1+\epsilon^{a}}$ is also positive. Thus for all cases in which ϕx can be expressed by $\int_a^\beta \epsilon^{vx}\chi v\, dv$, χv being always positive between the limits, it follows that the theorem is true.

We find then that this theorem must be true in the great majority of cases: as far as observation goes it is not known to have failed in any one of the instances in which its use is of importance. It is enough, without any thing else, to draw a great distinction between the progressing and alternating series. But this is not all: it is also matter of observation that there is great difficulty in finding alternating series which become infinite for one or more values of their variable, without having recourse to those in which the law of the coefficients is discontinuous. It is most easy, both to make the above theorem fail, and to procure a case in which infinity of value can be obtained, by means of the development of common algebraic functions, presenting discontinuous coefficients; but it is not easy with coefficients following a continuous law.

It cannot of course be proved that $A_0 - A_1 + A_2 - \ldots$ is necessarily a finite quantity, since cases of exception may be procured: but some illustrations may be given of the tendency of this form to represent only finite quantity. Probably nothing but the collection of such *tendencies* will ever lead to a rigorous criterion for ascertaining in what cases it can represent infinite quantity.

In a great many cases, a large majority of those usually considered, the complete alternating series $A_0 - A_1 x + A_2 x^2 - \ldots$ diminishes without limit, as x increases without limit: and the faster

$A_0\, A_1$ &c. increase, the more rapidly does this diminution take place. We shall see, in the next section, that this comminuence of $A_0 - A_1 x + \ldots$ and x^{-1} is to be looked for as the rule, its failure being the exception.

Let the series be transformed into

$$\frac{1}{1+ax}\left\{A_0 - (A_1 - aA_0)\frac{x}{(1+ax)} + (A_2 - 2aA_1 + a^2A_0)\frac{x^2}{(1+ax)^2}\right.$$
$$\left. - (A_3 - 3aA_2 + 3a^2A_1 - a^3A_0)\frac{x^3}{(1+ax)^3} + \ldots\right\}$$

which is easily done. Let a be taken so small that the series just obtained shall still be alternating, which can generally be done, though not always, and then, on account of the factor $(1+ax)^{-1}$, it is clear that the original series and x^{-1} are comminuent except only when the second series and x increase without limit together: that is, instead of supposing, as *à priori* we should do, that the alternating form with terms increasing without limit has an equal facility of approaching any given limit, we are rather to look upon it that its facility of approaching any other limit except 0, as x increases without limit, is only equal to that of its approaching ∞, or increasing without limit. I am not, of course, disposed to attach much weight to reasoning which rather resembles that of the theory of probabilities than of pure mathematics: but I do say that it must be better to take such considerations at their proper value, as suggestions for the conversion of results of observation into demonstrated theorems, than to allow isolated facts which evidently point at *something*, to remain in their state of separation.

This inaptitude to represent infinity, and this tendency to comminuence with x^{-1} are both circumstances which render the operation of integration much safer as applied to alternating than to progressing series. But the principal distinction between the two kinds of series seems to me to depend upon our present knowledge of the meaning of integration, as explained in a previous section, being imperfect. The progressing series cannot be expressed differentially without the operation of what we may call progressing integration; the alternating series can. This is exemplified in the two following remarkable theorems, given by Poisson:

$$\phi 0 + \phi 1 + \phi 2 + \ldots = \tfrac{1}{2}\phi 0 + \int_0^\infty \phi z\, dz + 2\Sigma_{m=1}^{m=\infty}\left\{\int_0^\infty \cos 2m\pi z\, \phi z\, dz\right\},$$
$$\phi 0 - \phi 1 + \phi 2 - \ldots = \tfrac{1}{2}\phi 0 + 2\Sigma_{m=1}^{m=\infty}\left\{\int_0^\infty (\cos m\pi z - \cos 2m\pi z)\,\phi z\, dz\right\}.$$

We may now examine the sort of proof which we can obtain of the usual values of divergent series, with the view of comparing finite and infinite divergence. Let $V = P_0 - P_1 + P_2 - \ldots$ and let $P_n = 1$ when $x = 1$, independently of n. Also, before $x = 1$, let the series be convergent; afterwards divergent. Let $P_n = P_{n-1} - p_{n-1}$, whence $p_{n-1} = 0$ when $x = 1$. And

$$V = P_0 - (P_0 - p_0) + (P_1 - p_1) - \ldots \text{ or } V = \tfrac{1}{2}P_0 + (p_0 - p_1 + p_2 - \ldots).$$

Again, let $W = Q_0 + Q_1 + Q_2 + \ldots$ and let $Q_n = 2^n$ when $x = 1$. Let $Q_n = 2Q_{n-1} - q_{n-1}$; whence $q_n = 0$ when $x = 1$. And $W = Q_0 + (2Q_0 - q_0) + (2Q_1 - q_1) + \ldots$ or $W = -Q_0 - (q_0 + q_1 + q_2 + \ldots)$. When $x = 1$, we have

$$V = \tfrac{1}{2}P_0 + (0 - 0 + 0 - \ldots);\quad W = -1 - (0 + 0 + 0 + \ldots)$$

and on the proper equivalents of the two evanescent forms it depends whether $1 - 1 + 1 - \ldots = \frac{1}{2}$ and $1 + 2 + 4 + \ldots = -1$ are true or not. Now instances enough may be produced in which $0 + 0 + 0 + \ldots$ is not an equivalent of 0: though, by instances merely, it would be found exceedingly difficult to overturn $0 - 0 + 0 \ldots = 0$, as long as the common operations of algebra only are used. But here again, when the forms of the integral calculus are employed, instances may be produced in which, though the form $0 - 0 + \ldots$ may still be called 0, it is only by means of a discontinuity which, occurring as it does at the limiting form of an alternating series of finite divergency, has a tendency to destroy the exclusive confidence which many modern analysts have placed in them.

The very foundation of this confidence is, as we have seen from the expressions of Poisson, a full belief in the maxim that whatever is numerically true up to the limit is true at the limit. To this principle, reasonable and convincing as it is, let us join the remembrance of a fact so well ascertained, were it merely as a matter of observation, that alternating series are more safe and more easily calculated than progressing series, and also the simplest of all theorems on convergency, namely, that an alternating series is rendered convergent by mere diminution, if unlimited, of its terms. With these premises let us consider the integral $\int_0^\infty \frac{\sin ax}{x} dx$. I believe that this one integral might be made to throw a case of exception in the way of those who have claimed privileges for the finitely diverging series over other non-arithmetical forms, in every particular as to which their superiority has been asserted.

Poisson, agreeing in this point with all other analysts, asserts that $\int_0^\infty \frac{\sin ax}{x} dx$ is $\frac{1}{2}\pi$, 0, or $-\frac{1}{2}\pi$, according as a is positive, nothing, or negative: any computer using the method of quadratures would confirm this result in all its parts. But this integral is clearly the same as

$$\int_0^{\frac{\pi}{a}} \frac{\sin ax}{x} dx + \int_{\frac{\pi}{a}}^{\frac{2\pi}{a}} \frac{\sin ax}{x} dx + \int_{\frac{2\pi}{a}}^{\frac{3\pi}{a}} \frac{\sin ax}{x} dx + \dots$$

which is an alternating series, since the second, fourth, &c. integrals are composed entirely of negative elements. Moreover the terms diminish without limit, since the numerators of the elements are recurrent, but the denominators constantly increasing, and without limit. However small a may be, if it be positive, $\frac{1}{2}\pi$ is the real value of the series, obtainable by the computer: and yet if a be absolutely $= 0$, each of the terms is also absolutely $= 0$. But if $1 - 1 + 1 - \dots$ is to be taken as having the *unique* value $\frac{1}{2}$, *which may be employed in analysis* (the Italics are Poisson's expressions) because $1 - g + g^2 - \dots$ is certainly $(1 + g)^{-1}$, however little g may fall short of unity, then surely $0 - 0 + 0 - 0 + \dots$ may here represent either $-\frac{\pi}{2}$ or $+\frac{\pi}{2}$, since, however small a may be, when negative it gives the first, and when positive the second: notwithstanding which, it is certain that $0 - 0 + 0 - \dots$ is in this case $= 0$.

Here then we have $0 - 0 + 0 - \dots$, a limiting form, and that which is true up to the limit is not true at the limit. But why is this principle abandoned, being, as it is, the very point on the assumed clearness of which the line is drawn between the accepted and the rejected cases of non-convergency? Is it that an infinite series of zeros *must* represent zero? I think I have shown sufficient cause against that assumption. Is it by the principle of mean value discussed in the last section? No one that I know of, except Leibnitz on grounds purely metaphysical, has ever used this principle, and no one has hitherto stated it in general terms: and moreover the modern analysts appear to require strictly arithmetical foundations, and would acknowledge no identity of principle between their methods and one which produces $\tan \infty = \sqrt{-1}$; they seem also to suppose that they are quite free of the use of principles established by induction. Either then the principle that whatever is numerically true up to the limit is to be held true at the limit must be abandoned, or exceptions of discontinuity, in questions involving integration, must be admitted to be possible in a manner which renders the cases to which Poisson and others have confined themselves subject to as great difficulties as those which they have abandoned.

In a preceding part of this paper I spoke of it as a strong presumption that $A_0 + A_1 x + A_2 x^2 + \dots$ should represent A_0 when $x = 0$, or that the form $0 + 0 + 0 + \dots$ which follows A_0 should $= 0$. If A_0, A_1, &c. be all positive, and if the series be always divergent, however small x may be, it is obvious that where the preceding represents a function of complete continuity, we may look for its value at $x = 0$, from the limit of $A_0 - A_1 x + A_2 x^2 - \dots$ as well as from that of $A_0 + A_1 x + \dots$ Accordingly, when there is continuity, all the presumptions of superior safety which the alternating series presents may be applied to this intermediate case.

SECTION V.

On Double Infinite Series, in which the Terms are infinitely continued in both Directions.

ONE look at the series

$$\dots + \phi(x-3) + \phi(x-2) + \phi(x-1) + \phi x + \phi(x+1) + \phi(x+2) + \phi(x+3) + \dots$$

will show that, whenever it can represent a definite function of x, which preserves its properties for different values of x, it must be a solution of the equation $\psi(x+1) = \psi x$. Various modes of proof, applicable however only to functions and processes of complete continuity, show that, in all cases to which those proofs apply, the representation of the above is simply 0, or rather $\dfrac{0}{\chi x}$, either 0, or, in particular cases, $\dfrac{0}{0}$. And certainly, in all cases, it can be reduced to the limiting form $0 + 0 + 0 + \dots$, so that, if not always $= 0$, the warning given in another part of this paper is confirmed. Throughout this section, let $S\phi x$ stand for a double series of the above form.

For ϕx write $\phi x . a^x$ and divide by a^x which gives $\dots + \phi(x-1)a^{-1} + \phi x + \phi(x+1)a + \dots$ Now

$$\phi x + \phi(x+1).a + \dots = \frac{1}{1-a}\phi x + \frac{a}{(1-a)^2}\phi' x + \frac{a+a^2}{(1-a)^3}\frac{\phi'' x}{2} + \frac{a+4a^2+a^3}{(1-a)^4}\frac{\phi''' x}{2.3} + \dots$$

in which it need only be noted here that the numerators of the functions of a all read backwards and forwards the same in their coefficients. Now by the same rule

$$\phi(-x) + \phi(-x-1).a + \dots = \frac{1}{1-a}\phi(-x) - \frac{a}{(1-a)^2}\phi'(-x) + \dots$$

change x into $-x$ in the last, and a into a^{-1}, add the result to the preceding equation, and substract ϕx, which gives $S(\phi x . a^x) = a^x(0 + 0 + 0 + \dots)$. Again, taking the calculus of operations, let $E\phi x = \phi(x+1)$, then, of all perfectly continuous answers, $E^{-1}\phi x$ must mean $\phi(x-1)$. The whole operation performed upon ϕx in $S\phi x$ is $\dots + E^{-1} + E^0 + E^1 + \dots$ or $\dfrac{E^{-1}}{1-E^{-1}} + \dfrac{1}{1-E}$, or 0. But it must not be forgotten that, in cases in which discontinuity is possible, it does not follow that $E^{-n}\phi x$ always signifies $\phi(x-n)$. For if we were to assume, for instance

$$E^{-3}\phi x = \phi(x-3) + \left(\frac{\pi}{2} + \int_0^\infty \frac{\sin(a-x)v}{v}dv\right)\psi x$$

we should be justified by the result $E^3 E^{-3}\phi x = \phi x$, whenever $a - (x+3)$ is negative, though when $a - x$ is positive, the preceding would not be the same as $\phi(x-3)$.

This is an important point, not only in reference to the calculus of operations, but to every case in which inverse operations are employed. There is, I am well aware, among mathematicians, something like a disinclination to provide beforehand for discontinuity, which first showed itself in the struggle against admitting discontinuous functions into the solution of partial differential equations. But it should be remembered that, in our time, trigonometrical series of the most continuous form have been shown to represent functions of the most capricious discontinuity. A mathematician has lately amused himself with preserving the first part of the air of 'God save the King' for posterity by means of a case of Fourier's integral; and any one who has studied the properties of the series $A\cos x + B\cos 2x + \ldots$ knows that a sturdy computer, who is not afraid of the method of quadratures, might hand down the means of recovering the profile of his own face from its equation: and that in a form which no analyst could tell at sight from the equation of a circle, a parabola, or some other continuous curve. Nor is such discontinuity a mere possibility: it is constantly occurring in the higher branches of mathematics, and its detection and treatment forms the most distinctive feature of the most recent school. Surely then it is time to pay attention at the outset of every plan of investigation to the possibility of the occurrence of discontinuity in inverse operations.

I do not see how absolute error is to be avoided without such a precaution. Defining $E\phi x$ as $\phi(x+1)$, nothing is clearer than the right to use the symbol E, and those derived from it, algebraically: all the fundamental symbolic definitions are satisfied by it. If we are to assume, as of necessity, that $E^{-n}\phi x$ can be nothing but $\phi(x-n)$, the symbol $S\phi x$ must represent 0, as shown: and experience points out that it actually does so in every case in which there is no discontinuity. But in certain cases, as I shall show, $S\phi x$ does not represent 0, but another solution of $\psi(x+1) = \psi x$: there is then some flaw in the demonstration, which I take to be the assumption without reserve of $E^{-n}\phi x = \phi(x-n)$.

I might give other ways of expressing $S\phi x$, all ending in the same result, that, unless some special mode of introducing and allowing for discontinuity be adopted, it represents 0. But this paper is already too long, and I therefore pass on to some cases in which it does *not* represent 0.

Let us consider the series,

$$\ldots + \frac{1}{1+(b-2c)^2} + \frac{1}{1+(b-c)^2} + \frac{1}{1+b^2} + \frac{1}{1+(b+c)^2} + \frac{1}{1+(b+2c)^2} + \ldots$$

which is both ways convergent. We have the two following results,

$$\int_0^\infty \epsilon^{-(b\pm kc)v}\sin v\,dv = \frac{1}{1+(b\pm kc)^2}, \qquad \int_0^\infty \epsilon^{-(kc-b)v}\sin v\,dv = \frac{1}{1+(b-kc)^2},$$

that one being taken in which ϵ^v is raised to a negative power. Let b lie between mc and $(m+1)c$: then we have

$$\frac{1}{1+(b-mc)^2} + \frac{1}{1+(b-\overline{m-1}\,c)^2} + \ldots = \int_0^\infty \left(\epsilon^{-(b-mc)v} + \epsilon^{-(b-\overline{m-1}c)v} + \ldots\right)\sin v\,dv$$

$$\ldots + \frac{1}{1+(b-\overline{m+2}c)^2} + \frac{1}{1+(b-\overline{m+1}\,c)^2} = \int_0^\infty \left(\epsilon^{-(\overline{m+1}c-b)v} + \epsilon^{-(\overline{m+2}c-b)v} + \ldots\right)\sin v\,dv,$$

in which integration is performed on convergent series only. Hence,

$$S\frac{1}{1+(b+pc)^2} = \int_0^\infty \frac{\epsilon^{-(b-mc)v} + \epsilon^{-(\overline{m+1}c-b)v}}{1-\epsilon^{-cv}}\sin v\,dv = \int_0^\infty \frac{\epsilon^{(m'c-b)v} + \epsilon^{-(m'c-b)v}}{\epsilon^{\frac{1}{2}cv} - \epsilon^{-\frac{1}{2}cv}}\sin v\,dv,$$

where $m' = m + \frac{1}{2}$. Now $m'c - b$ is numerically less than $\frac{1}{2}c$; and Legendre has shown (see my 'Differential Calculus,' page 669) that if g be not greater than h,

$$\int_0^\infty \frac{\epsilon^{gv} + \epsilon^{-gv}}{\epsilon^{hv} - \epsilon^{-hv}}\sin v\,dv = \frac{\pi}{2h}\,\frac{\epsilon^{\frac{\pi}{h}} - \epsilon^{-\frac{\pi}{h}}}{\epsilon^{\frac{\pi}{h}} + \epsilon^{-\frac{\pi}{h}} + 2\cos\frac{\pi g}{h}},$$

$$\text{whence } \frac{\pi}{c}\,\frac{\epsilon^{\frac{2\pi}{c}} - \epsilon^{-\frac{2\pi}{c}}}{\epsilon^{\frac{2\pi}{c}} + \epsilon^{-\frac{2\pi}{c}} + 2\cos\left(2m+1-\frac{2b}{c}\right)\pi} \quad\text{or}\quad \frac{\pi}{c}\,\frac{\epsilon^{\frac{2\pi}{c}} - \epsilon^{-\frac{2\pi}{c}}}{\epsilon^{\frac{2\pi}{c}} + \epsilon^{-\frac{2\pi}{c}} - 2\cos\frac{2\pi b}{c}}$$

is the value of the series, which is, as it should be, a solution of $\psi(b+c) = \psi b$.

Before showing some consequences of this and similar results which will be interesting as extensions of known theorems, I proceed to verify my assertion that this series, being double, and not $=0$, will show signs of discontinuity. Let us consider the series

$$y = \frac{\epsilon^{bx}}{1+b^2} + \frac{\epsilon^{(b+c)x}}{1+(b+c)^2} + \frac{\epsilon^{(b+2c)x}}{1+(b+2c)^2} + \ldots,$$

which is convergent when cx is 0 or negative. This series is a solution of

$$y + \frac{d^2 y}{dx^2} = \frac{\epsilon^{bx}}{1-\epsilon^{cx}},$$

$$\text{whence } y = \sin x \int_{-\infty}^{x} \frac{\cos x\, \epsilon^{bx}\, dx}{1-\epsilon^{cx}} - \cos x \int_{-\infty}^{x} \frac{\sin x\, \epsilon^{bx}\, dx}{1-\epsilon^{cx}}.$$

There is nothing in this result, as long as the final value of x is negative, to hinder the computer from finding the value of y by the method of quadratures, and comparing it with the result of the convergent series. And even when $x=0$, the part of the first integral which comes from between $x=-a$ and $x=0$, a being infinitely small; is rendered evanescent by the factor $\sin x$, as special examination will show. If then we make $x=0$, and if we venture to change the sign of c, and put the two results together, we have, remembering that the term $(1+b^2)^{-1}$ occurs twice,

$$S\frac{1}{1+(b+pc)^2} + \frac{1}{1+b^2} = -\int_{-\infty}^{0} \left(\frac{1}{1-\epsilon^{cx}} + \frac{1}{1-\epsilon^{-cx}}\right) \sin x\, \epsilon^{bx}\, dx$$

$$= -\int_{-\infty}^{0} \sin x\, \epsilon^{bx}\, dx = \frac{1}{1+b^2}; \text{ or } S\frac{1}{1+(b+xc)^2} = 0;$$

a false result, but agreeing with the theorem already discussed, and which I think may now be described as follows. The double series $S\phi x$ is, if its two sides be perfectly continuous, $=0$: and any method which proceeds by neglecting discontinuity will end in $S\phi x = 0$, true or false.

But perhaps it may not be evident at once why I say we have neglected discontinuity in the preceding process: if so, the following explanation will be necessary.

A continuous equation is one in which the two sides are *algebraical* equivalents, that is, in which the right to use the sign of equality is independent of the value of any letter or letters. If this right be destroyed by the passage of any one letter over a given limit, there is obviously discontinuity. Now if $\psi x = \phi x + \phi(x+1) + \ldots$ be a continuous equation, or if $\psi x = \phi x + \psi(x+1)$ be universally true, we may convert it into

$$\psi x = -\phi(x-1) + \psi(x-1), \text{ or } \psi x = -\phi(x-1) - \phi(x-2) - \ldots:$$

if this be granted, then $S\phi x = 0$. Conversely, if $S\phi x$ be not $=0$, then $\psi x = \phi x + \ldots$ being true, $\psi x = -\phi(x-1) - \ldots$ is false. Also, if the assumption of the permanence of any equation make $S\phi x = 0$, then, whenever this last is not true, it follows that such assumption of permanence is erroneous. In the preceding result, we have assumed the permanence of the equation

$$-\int_{-\infty}^{0} \frac{\sin x\, \epsilon^{bx}\, dx}{1-\epsilon^{cx}} = \frac{1}{1+b^2} + \frac{1}{1+(b+c)^2} + \ldots$$

for all values of c. The error of our result is manifest: this permanence then has no existence. And the warning is that when c is made negative, *we integrate over a diverging series:* in fact, our process assumes the ordinary development of $(1-\epsilon^{cx})^{-1}$ when c and x are both negative, or $\epsilon^{cx} > 1$, and integrates that development.

There have been two discontinuities occurring in the preceding; the first dependent upon the introduction of m, the second that just considered. The first may be treated as merely incidental to one particular process; we were not bound to Legendre's integral; and this discontinuity disappears in the result. But the second is essential to the problem; the series satisfies a certain differential equation, the complete solution of that equation is ascertained, and therefore the series *must* be represented by its equivalent solution of the equation. No other equivalent could have been anything but the one we found, or the same in a different form. As matters stand, then, we cannot have a continuous relation between the series and its invelopment: and this, I will venture to prognosticate, will continue until the definition of integration is extended.

Let us now try $\ldots - \frac{1}{1+(b-c)^2} + \frac{1}{1+b^2} - \frac{1}{1+(b+c)^2} + \ldots$ which call $S'\frac{1}{1+(b+pc)^2}$.

Proceeding just as before, and, b lying between mc and $(m+1)c$, we shall find, m' being $m+\frac{1}{2}$, as before,

$$S'\frac{1}{1+(b+pc)^2} = (-1)^m \int_0^\infty \frac{\epsilon^{(m'c-b)v} - \epsilon^{-(m'c-b)v}}{\epsilon^{\frac{1}{2}cv} + \epsilon^{-\frac{1}{2}cv}} \sin v\, dv.$$

But Legendre has also shown the following, g being not greater than h:

$$\int_0^\infty \frac{\epsilon^{gv} - \epsilon^{-gv}}{\epsilon^{hv} + \epsilon^{-hv}} \sin v\, dv = \frac{\pi}{h} \frac{\left(\epsilon^{\frac{\pi}{2h}} - \epsilon^{-\frac{\pi}{2h}}\right)\sin\frac{\pi g}{2h}}{\epsilon^{\frac{\pi}{h}} + \epsilon^{-\frac{\pi}{h}} + 2\cos\frac{\pi g}{h}},$$

whence the series in question is

$$\frac{2\pi(-1)^m}{c} \frac{\left(\epsilon^{\frac{\pi}{c}} - \epsilon^{-\frac{\pi}{c}}\right)\sin\left(m+\frac{1}{2}-\frac{b}{c}\right)\pi}{\epsilon^{\frac{2\pi a}{c}} + \epsilon^{-\frac{2\pi}{c}} + 2\cos\left(2m+1-\frac{2b}{c}\right)\pi}, \quad \text{or} \quad \frac{2\pi}{c} \frac{\left(\epsilon^{\frac{\pi}{c}} - \epsilon^{-\frac{\pi}{c}}\right)\cos\frac{\pi b}{c}}{\epsilon^{\frac{2\pi}{c}} + \epsilon^{-\frac{2\pi}{c}} - 2\cos\frac{2\pi b}{c}},$$

which is, as it ought to be, a solution of $\psi(b+c) = -\psi b$. Now consider the series.

$$y = \frac{\epsilon^{bx}}{1+b^2} - \frac{\epsilon^{(b+c)x}}{1+(b+c)^2} + \frac{\epsilon^{(b+2c)x}}{1+(b+2c)^2} - \ldots$$

which is a solution of $y + \frac{d^2y}{dx^2} = \frac{\epsilon^{bx}}{1+\epsilon^{cx}}$,

$$\text{whence } y = \sin x \int_{-\infty}^{x} \frac{\cos x\, \epsilon^{bx}\, dx}{1+\epsilon^{cx}} - \cos x \int_{-\infty}^{x} \frac{\sin x\, \epsilon^{bx}\, dx}{1+\epsilon^{cx}}$$

on which may be repeated all the remarks on the preceding case. But in this case, when $c = 0$, the value of y gives, as it should do, $\frac{1}{2}\epsilon^{bx}(1+b^2)^{-1}$.

The danger of integrating over a diverging series is thus shown to be incident to alternating as well as progressing series. It cannot be denied that Poisson has separated the only case in which integration can be used with some freedom and safety on non-arithmetical series: namely, the finitely diverging series which lies between the convergent and divergent cases. Whether the freedom is entire and the safety absolute is more than can be determined at present: unacquainted as we are with all the varieties of the discontinuity which appears in limiting cases of integration, as now understood. On this point, I must refer to the preceding part of this paper.

With regard to the alternating double series $+ A_{-2}x^{-2} - A_{-1}x^{-1} + A_0 - A_1x + A_2x^2 - \ldots$ we now learn that, whenever complete continuity exists, $A_0 - A_1x + \ldots$, x being infinite, must have the same value as $A_{-1}x^{-1} - A_{-2}x^{-2} + \ldots$ when x^{-1} is nothing; that is, must vanish, generally speaking. This observed tendency of $A_0 - A_1x + \ldots$ has been already noticed.

I now take some results of the two series here discussed which are interesting in the way of verification and extension, though not illustrative of the points on which I am specially writing. If for b and c we write $b : a$ and $c : a$, we have

$$S\frac{1}{a^2+(b+pc)^2}=\frac{\pi}{ac}\frac{\epsilon^{\frac{2\pi a}{c}}-\epsilon^{-\frac{2\pi a}{c}}}{\epsilon^{\frac{2\pi a}{c}}+\epsilon^{-\frac{2\pi a}{c}}-2\cos\frac{2\pi b}{c}};\quad S'\frac{1}{a^2+(b+pc)^2}=\frac{2\pi}{ac}\frac{(\epsilon^{\frac{\pi a}{c}}-\epsilon^{-\frac{\pi a}{c}})\cos\frac{\pi b}{c}}{\epsilon^{\frac{2\pi a}{c}}+\epsilon^{-\frac{2\pi a}{c}}-2\cos\frac{2\pi b}{c}}.$$

Make $a = 0$, $c = 1$, and we have

$$\left(\frac{\pi}{\sin\pi b}\right)^2=\ldots+\frac{1}{(b-2)^2}+\frac{1}{(b-1)^2}+\frac{1}{b^2}+\frac{1}{(b+1)^2}+\frac{1}{(b+2)^2}+\ldots$$

$$\left(\frac{\pi}{\sin\pi b}\right)^2\cos\pi b=\ldots+\frac{1}{(b-2)^2}-\frac{1}{(b-1)^2}+\frac{1}{b^2}-\frac{1}{(b+1)^2}+\frac{1}{(b+2)^2}-\ldots$$

$$\frac{(-1)^n}{\Gamma n}\frac{d^{n-2}}{db^{n-2}}\left\{\left(\frac{\pi}{\sin\pi b}\right)^2\right\}=\ldots+\frac{1}{(b-2)^n}+\frac{1}{(b-1)^n}+\frac{1}{b^n}+\frac{1}{(b+1)^n}+\frac{1}{(b+2)^n}+\ldots$$

$$\frac{(-1)^n}{\Gamma n}\frac{d^{n-1}}{db^{n-1}}\left\{\left(\frac{\pi}{\sin\pi b}\right)\right\}=\ldots+\frac{1}{(b-2)^n}-\frac{1}{(b-1)^n}+\frac{1}{b^n}-\frac{1}{(b+1)^n}+\frac{1}{(b+2)^n}-\ldots$$

$$\frac{\pi}{\sin\pi b}=\ldots+\frac{1}{b-2}-\frac{1}{b-1}+\frac{1}{b}-\frac{1}{b+1}+\frac{1}{b+2}-\ldots$$

If we had commenced with $\{(b+pc)^2-1\}^{-1}$, and had used the formula $\int_0^\infty \epsilon^{mv\sqrt{-1}}\sin v\,dv = (1-m^2)^{-1}$, which Poisson would have admitted as a limiting form of $\int_0^\infty \epsilon^{(-k+m\sqrt{-1})v}\sin v\,dv$, we should have seen in the final result a right to substitute $a\sqrt{-1}$ for a in the preceding formulæ; giving

$$S\frac{1}{(b+pc)^2-a^2}=\frac{\pi}{ca}\frac{\sin\frac{2\pi a}{c}}{\cos\frac{2\pi a}{c}-\cos\frac{2\pi b}{c}};\quad S'\frac{1}{(b+pc)^2-a^2}=\frac{2\pi}{ac}\frac{\sin\frac{\pi a}{c}\cos\frac{\pi b}{c}}{\cos\frac{2\pi a}{c}-\cos\frac{2\pi b}{c}}.$$

Various formulæ might be obtained by differentiating these with respect to a, b, or c; and various others by integration, one set of which is remarkable. Multiply the two first equations severally by $a\,da$, and it will be seen that the second sides become integrable: integrate from $a = 0$, and make the antilogarithmic change, which gives the following continued products, of which the well-known formulæ for the sine and cosine are particular cases.

$$\ldots\left(1+\frac{a^2}{(b-c)^2}\right)\left(1+\frac{a^2}{b^2}\right)\left(1+\frac{a^2}{(b+c)^2}\right)\left(1+\frac{a^2}{(b+2c)^2}\right)\ldots=\frac{\frac{1}{2}(\epsilon^{\frac{2\pi a}{c}}+\epsilon^{-\frac{2\pi a}{c}})-\cos\frac{2\pi b}{c}}{1-\cos\frac{2\pi b}{c}}$$

$$\frac{\ldots\left(1+\frac{a^2}{(b-2c)^2}\right)\left(1+\frac{a^2}{b^2}\right)\left(1+\frac{a^2}{(b+2c)^2}\right)\ldots}{\ldots\left(1+\frac{a^2}{(b-c)^2}\right)\left(1+\frac{a^2}{(b+c)^2}\right)\ldots}=\frac{\frac{1}{2}(\epsilon^{\frac{\pi a}{c}}+\epsilon^{-\frac{\pi a}{c}})-\cos\frac{\pi b}{c}}{\frac{1}{2}(\epsilon^{\frac{\pi a}{c}}+\epsilon^{-\frac{\pi a}{c}})+\cos\frac{\pi b}{c}}\cdot\frac{1+\cos\frac{\pi b}{c}}{1-\cos\frac{\pi b}{c}}$$

the second of which is readily deducible from the first. It is needless to write down the forms arising from the substitution of $a\sqrt{-1}$ for a.

From what precedes, we are warned to expect some discontinuity arising in the treatment of any series $\phi(x) \pm \phi(x+1) + \dots$ if $\phi(x+n) = \phi(x-n)$, unless that series be an analytical equivalent of 0. And even in this latter case, it is to be remembered that $\frac{0}{\chi x}$ is the real form, and that when $\chi x = 0$, there may arise cases of exception in which the series represents a finite quantity, and even infinity. This particular point has been so beautifully illustrated by Poisson, in his treatment of the series $\frac{1}{2} + \cos\theta + \cos 2\theta + \dots$ that nothing is left for any one else to say, at present.

In mentioning once more the name of this distinguished analyst, I may state that the point in which I have freely ventured to question his judgment is not as to the wisdom of the course he took, in rejecting divergency from the integral calculus as he found it, but as to the grounds on which he asserted a final and fundamental difference between what he adopted and what he rejected.

A. DE MORGAN.

UNIVERSITY COLLEGE, LONDON,
January 15, 1843.

ADDITIONS.

Page 192, line 8. IT is not asserted that $\cos^2\infty + \sin^2\infty = 0$, for the mean value of each of the terms is $\frac{1}{2}$, and $\cos^2\infty + \sin^2\infty = 1$. Many errors may be made by forgetting that $\phi \sin x\ (x = \infty)$ or $\phi(0)$ is not the mean value of $\phi \sin x$, but $\int_0^{2\pi} \phi(\sin x)\, dx \div 2\pi$.

Page 201, last four lines. If it should seem for a moment that this reasoning would apply equally to $A_0 + A_1 x + \dots$ and $-A_{-1}x^{-1} - A_{-2}x^{-2} - \dots$, remember that the theorem in Section IV (to which the exceptions are only occasional) shows that $A_{-1}x^{-1} - A_{-2}x^{-2} + \dots$ lies between $A_{-1}x^{-1}$ and $A_{-1}x^{-1} - A_{-2}x^{-2}$ when x is great: but that we have no such argument from which to infer the comminuence of $-A_{-1}x^{-1} - A_{-2}x^{-2} - \dots$ and x^{-1}. Still however, the equality of this last to $A_0 + A_1 x + \dots$, when there is no discontinuity, would enable us to predict the very large number of cases in which $A_0 + A_1 x + \dots$ is infinite and negative when x is infinite and positive.

XVI. *On the Method of Least Squares.* By R. L. Ellis, Esq., M.A., *Fellow of the Cambridge Philosophical Society.*

[Read *March* 4, 1844.]

The importance attached to the method of least squares is evident from the attention it has received from some of the most distinguished mathematicians of the present century, and from the variety of ways in which it has been discussed.

Something, however, remains to be done—namely, to bring the different modes in which the subject has been presented into juxta-position, so that the relations which they bear to one another may be clearly apprehended. For there is an essential difference between the way in which the rule of least squares has been demonstrated by Gauss, and that which was pursued by Laplace. The former of these mathematicians has in fact given two different demonstrations of the method, founded on quite distinct principles. The first of these demonstrations is contained in the *Theoria Motûs*, and is that which is followed by Encke in a paper of which a translation appeared in the *Scientific Memoirs.* At a later period Gauss returned to the subject, and subsequently to the publication of Laplace's investigation gave his second demonstration in the *Theoria Combinationis Observationum.*

The subject has been also discussed by Poisson in the *Connaissance des Tems* for 1827, and by several other French writers. Poisson's analysis is founded on the same principle as Laplace's: it is more general, and perhaps simpler. It is not, however, my intention to dwell upon mere differences in the mathematical part of the enquiry.

The consequence of the variety of principles which have been made use of by different writers has naturally been to produce some perplexity as to the true foundation of the method. As the results of all the investigations coincided, it was natural to suppose that the principles on which they were founded were essentially the same. Thus Mr. Ivory conceived that if Laplace arrived at the same result as Gauss, it was because in the process of approximation he had introduced an assumption which reduced his hypothesis to that on which Gauss proceeded. In this I think Mr. Ivory was certainly mistaken; it is at any rate not difficult to show that he had misunderstood some part at least of Laplace's reasoning: but that so good a mathematician could have come to the conclusion to which he was led, shows at once both the difficulty of the analytical part of the inquiry, and also the obscurity of the principles on which it rests. Again, a recent writer on the Theory of Probabilities has adopted Poisson's investigation, which, as I have said, is the development of Laplace's, and which proves in the most general manner the superiority of the rule of least squares, whatever be the law of probability of error, provided equal positive and negative errors are equally probable. But in a subsequent chapter we find that he coincides in Mr. Ivory's conclusion, that the method of least squares is not established by the theory of probabilities, unless we assume one particular law of probability of error.

These two results are irreconcilable; either Poisson or Mr. Ivory must be wrong. The latter indeed expressed his dissent from all that had been done by the French mathematicians on the subject, and in a series of papers in the *Philosophical Magazine* gave several demonstrations of the method of least squares, which he conceived ought not to be derived from the theory of probabilities. In this conclusion I cannot coincide; nor do I think Mr. Ivory's reasoning at all satisfactory.

From this imperfect sketch of the history of the subject, we perceive that the methods which have been pursued may be thus classified.

(1). Gauss's method in the *Theoria Motûs*, which is followed and developed by Encke and other German writers.

(2). That of Laplace and Poisson.

(3). Gauss's second method.

(4). Those of Mr. Ivory.

I proceed to consider these separately, and in detail.

For the analysis of Laplace and Poisson, I have substituted another, founded on what is generally known as Fourier's theorem, having been first given by him in the *Théorie de la Chaleur*. It will be seen that the mathematical difficulty is greatly diminished by the change.

GAUSS'S FIRST METHOD.

This method is founded on the assumption that in a series of direct observations, of the same quantity or magnitude, the arithmetical mean gives the most *probable* result. This seems so natural a postulate that no one would at first refuse to assent to it. For it has been the universal practice of mankind to take the arithmetical mean of any series of equally good direct observations, and to employ the result as the approximately true value of the magnitude observed.

The principle of the arithmetical mean seems therefore to be true *à priori*. Undoubtedly the conviction that the effect of fortuitous causes will disappear on a long series of trials, is an immediate consequence of our confidence in the permanence of nature. And this conviction leads to the rule of the arithmetical mean, as giving a result which as the number of observations increases *sine limite*, tends to coincide with the true value of the magnitude observed. For let a be this value, x the observed value, e the error, then we have

$$x_1 - a = e_1$$
$$x_2 - a = e_2$$
$$\&c. \quad = \&c.$$

And as on the long run the action of fortuitous causes disappears, and there is no permanent cause tending to make the sum of the positive differ from that of the negative errors, $\Sigma e = 0$, and therefore

$$\Sigma (x_1 - a) = 0;$$

$$\text{or,} \qquad a = \frac{1}{n} \Sigma x_1;$$

which expresses the rule of the arithmetical mean, and which is thus seen to be absolutely true ultimately when n increases *sine limite*.

In this sense therefore the rule in question is deducible from *à priori* considerations. But it is to be remarked, that it is not the only rule to which these considerations might lead us. For not only is $\Sigma e = 0$ ultimately, but $\Sigma fe = 0$, where fe is any function such that $fe = -f(-e)$; and therefore we should have

$$\Sigma f(x - a) = 0,$$

as an equation which ultimately would give the true value of x when the number of observations increases *sine limite*, and which therefore for a finite number of observations may be looked on in precisely the same way as the equation which expresses the rule of the arithmetical mean. There is no discrepancy between these two results. At the limit they coincide: short of the limit both are approximations to the truth. Indeed, we might form some idea how far the action of fortuitous causes had disappeared from a given series of observations by assigning different forms to f, and comparing the different values thus found for a.

No satisfactory reason can be assigned why, setting aside mere convenience, the rule of the arithmetical mean should be singled out from the other rules which are included in the general equation $\Sigma f(x-a)=0$.

Let us enquire, therefore, whether there is any sufficient reason for saying that the rule of the arithmetical mean gives the *most probable* value of the unknown magnitude. In the first place, it is only one rule out of many among which it has no prerogative but that of being in practice more convenient than any other: in the second place, if this were not so, it would not follow that in the accurate sense of the words it gave the *most probable* result. This objection I shall defer for a moment, and proceed to consider the manner in which Gauss makes use of the postulate on which his method is founded.

From the first principles of what is called the theory of probabilities *à posteriori*, it appears that the most probable value which can be assigned to the magnitude which our observations are intended to determine, is that which shall make the *à priori* probability of the observed phenomena a maximum. That is to say, if a be the true value sought, x_1 being the value observed at the first observation, x_2 the corresponding quantity for the second, and so on, the errors at the first, second, &c. observation must be x_1-a, x_2-a, &c., respectively; and if $\phi\epsilon \,.\, d\epsilon$ be the probability of an error ϵ in any observation of the series, the quantity which is to be made a maximum for a is proportional to

$$\phi(x_1-a)\,\phi(x_2-a)\ldots\,\phi(x_n-a).$$

Equating to zero the differential of this with respect to a, we find

$$\frac{\phi'(x_1-a)}{\phi(x_1-a)}+\text{\&c.}+\frac{\phi'(x_n-a)}{\phi(x_n-a)}=0,$$

as the equation for determining a in x. Let $\dfrac{\phi'}{\phi}=\psi$, then it becomes

$$\Sigma_1^n\,\psi(x-a)=0.$$

Now we have assumed that the most probable value of a is given by the equation

$$\Sigma_1^n(x-a)=0:$$

and it is impossible to make these equations generally coincident, without assuming that

$$\psi\epsilon=m\epsilon,\ m \text{ being any constant};$$

$$\text{hence } \frac{\phi'\epsilon}{\phi\epsilon}=m\epsilon,$$

$$\text{and } \phi\epsilon=Ce^{\frac{1}{2}m\epsilon^2}.$$

Now as the error ϵ is necessarily included in the limits $-\infty+\infty$, we must have

$$\int_{-\infty}^{+\infty}\phi\epsilon\,d\epsilon=\frac{C\sqrt{2}\sqrt{\pi}}{\sqrt{m}}=1,$$

$$C=\sqrt{\frac{m}{2\pi}}:$$

or if we adopt the usual notation, and replace m by $2h^2$,

$$C=\frac{h}{\sqrt{\pi}},\text{ and } \phi\epsilon=\frac{h}{\sqrt{\pi}}e^{-h^2\epsilon^2}.$$

Consequently, we are thus led to adopt one particular law of probability of error as alone congruent with the rule of the arithmetical mean.

But, in fact, we are perfectly sure that in different classes of observations the law of probability of error must vary, and we have no direct proof that in any class it coincides with the form assigned to it. Therefore one of two things must be true, either the rule of the arithmetical mean rests on a mere illusory prejudice, or, if it has a valid foundation, the reasoning now stated must be incorrect. Either alternative is opposed to Gauss's investigation. For the reasons already given, we are, I think, led to adopt the latter, and then the question arises, wherein does the incorrectness of the reasoning reside? It resides in the ambiguity of the words *most probable*. For let us consider what they imply in the theory of probabilities *à posteriori*.

Suppose there were m different magnitudes $a_1\, a_2 \dots a_m$, and that each of these were observed n times in succession. Let this process be repeated p times, p being a large number which increases *sine limite*. Thus we shall have pm sets of observations each containing n observations.

Of these a certain number K will coincide with the set of observations supposed to be actually under discussion; and we shall have the equation.

$$k_1 + k_2 + \dots k_m = K:$$

where k is that portion of K which is derived from observations of a_k.

Then, ultimately, the *most probable* value which the given series of observations leads us to assign to a, is (supposing a is susceptible only of the values $a_1\, a_2 \dots a_m$) equal to a_r, r being such that the corresponding quantity k_r is the maximum value of k.

To make the case now stated entirely coincident with the one which we are in the habit of considering, we have only to suppose (making m infinite) that the series of magnitudes $a_1 \dots a_m$ includes all possible magnitudes from $-\infty$ to $+\infty$.

Now from this statement, it is clear there is no reason for supposing that because the arithmetical mean would give the true result if the number of observations were increased *sine limite*, it must give the *most probable* result the number of observations being finite.

The two notions are heterogeneous: the conditions implied by the one may be fulfilled without introducing those required by the other: and we have already seen that by losing sight of this distinction, we are led to the inadmissible conclusion, that a principle recognised as true *à priori* necessarily implies a result, viz. the universal existence of a special law of error, not only not true *à priori*, but not true at all.

Having stated what seem to me to be the objections in point of logical accuracy to this mode of considering the subject, I will briefly point out the manner in which, from the law of error already obtained, the method of least squares is to be deduced.

Let

$$\left.\begin{aligned} \epsilon_1 &= a_1 x + b_1 y + \&c. - V_1 \\ \epsilon_2 &= a_2 x + b_2 y + \&c. - V_2 \\ \&c. &= \&c. \\ \epsilon_n &= a_n x + b_n y + \&c. - V_n \end{aligned}\right\} \dots\dots\dots\dots (a)$$

be the system of equations of condition, which are to be combined together so as to give the values of x, y, &c. The error committed at the first observation is ϵ_1, at the second ϵ_2, and so on; each observation corresponding to an equation of condition.

The probability of the concurrence of all these errors is, (according to the law of error already arrived at) proportional to

$$e^{-h^2 [(a_1 x + b_1 y + \&c. - V_1)^2 + (a_2 x + b_2 y + - V_2)^2 + \&c.]},$$

and it is to be made a maximum by the most probable values of x, y, &c. These values will therefore make

$$(a_1 x + b_1 y + \&c. - V_1)^2 + (a_2 x + b_2 y + \dots - V_2)^2 + \dots,$$

a minimum: thas is to say, they will make the sum of the squares of errors a minimum.

Hence *the method of least squares*. The conditions of the minimum give the linear equations:

$$\left.\begin{array}{r} x\Sigma a^2 + y\Sigma ab + \&c. = \Sigma a V \\ x\Sigma ab + y\Sigma b^2 + \&c. = \Sigma b V \\ \&c. \qquad = \&c. \end{array}\right\} \dots\dots\dots\dots(\beta),$$

in which system there are always the same number of equations as there are unknown quantities to be determined.

The next investigation of the principle of the method of least squares which I shall attempt to analyze is that of Laplace.

LAPLACE'S DEMONSTRATION.

If, in order to determine x from the equations of condition stated in the last paragraph, we multiply the first by μ_1, the second by μ_2, &c., and add: ($\mu_1\,\mu_2$, &c. fulfilling the conditions

$$\Sigma\mu a = 1, \;\; \Sigma\mu b = 0, \;\; \&c. = 0)$$

we find
$$x = \Sigma\mu V - \Sigma\mu\epsilon;$$

and if we assume that $\Sigma\mu\epsilon$ is equal to zero, then the resulting value of x is $\Sigma\mu V$: the error of this determination being the quantity $\Sigma\mu\epsilon$, which we have assumed to be equal to zero, without knowing whether it really is so or not.

Now supposing there are n equations of condition, and p quantities to be determined, and that n is greater than p, then we see that there are n factors $\mu_1, \mu_2 \dots \mu_n$, and p conditions for them to fulfil. They may therefore be subjected to $n-p$ additional conditions.

This being premised, let us consider the probability that the quantity $\Sigma\mu\epsilon$ will not be less than α, or greater than β, α and β being any quantities whatever. The law of probability of error at each observation being given, the question is evidently analogous to the common problem of finding the chance, that with a given set of dice the number of points thrown shall not be less than one given number or greater than another.

We may therefore suppose that the probability in question has been determined: call it P. Suppose also that we have taken $\alpha = -l$ and $\beta = l$, l being any positive quantity.

Then P is a function of l, and of $\mu_1 \dots \mu_n$.

Let us now so determine $\mu_1 \dots \mu_n$, (subject to the conditions already specified,) that P may be a maximum. When this is done, it follows that there is a greater probability that the error in our determination of x, viz. $\Sigma\mu\epsilon$, lies within the limits $\pm l$, than if we had made use of any other set of factors whatever.

On this principle Laplace determines what he calls the most advantageous system of factors.

It does not follow that the value thus obtained for x is the most *probable* value that could be assigned for it. But if we consider a large number of sets of observations, (the quantities a, b, &c. being the same for all) then the error which we commit by using Laplace's factors will in a greater proportion of cases lie between $\pm l$, than if we had used any other system of factors.

The investigation has reference merely to the different ways in which by the method of factors a given set of linear equations may be solved.

We now enter on the analysis requisite to determine P.

Let the probability that $\Sigma\mu\epsilon$ will be precisely equal to u, be $p\,du$. Then manifestly

$$P = \int_{-l}^{+l} p\,du;$$

and we have therefore only to determine p.

Let $\epsilon_1\,\epsilon_2\ldots\epsilon_n$ be the errors which occur at the first second &c. observation; $\phi_1\,\epsilon_1\,d\,\epsilon_1$, $\phi_2\,\epsilon_2\,d\,\epsilon_2\ldots$ $\phi_n\epsilon_n d\epsilon_n$ be the probabilities of their occurrence: the form of the function ϕ determining the law of probability of error, which, for greater generality, we suppose different at each observation. The probability of the concurrence of these errors is of course

$$\phi_1\,\epsilon_1\,\phi_2\epsilon_2\ldots\phi_n\epsilon_n d\epsilon_1\ldots d\epsilon_n \ldots\ldots (1),$$

and the first principles of the theory of probabilities show that the value of $p\,du$ will be obtained by integrating (1), $\epsilon_1\ldots\epsilon_n$ being subjected to the condition $\Sigma\mu\epsilon = u$.

Thus

$$p\,du = \int\phi_1\,\epsilon_1\,\phi_2\,\epsilon_2\ldots\phi_n\,\epsilon_n\,d\epsilon_1\ldots d\epsilon_n \ldots\ldots (2)$$

with the relation

$$\mu_1\,\epsilon_1 + \mu_2\,\epsilon_2\ldots + \mu_n\,\epsilon_n = u.$$

Consequently

$$p\,du = d\epsilon_n\int\phi_1\epsilon_1\ldots\phi_{n-1}\epsilon_{n-1}\phi_n\frac{u-\mu_1\epsilon_1\ldots-\mu_{n-1}\epsilon_{n-1}}{\mu_n}\,d\epsilon_1\ldots d\epsilon_{n-1}\ldots\ldots (3).$$

Now by Fourier's theorem

$$\phi_n\,\frac{u-\mu_1\epsilon_1-\ldots-\mu_{n-1}\epsilon_{n-1}}{\mu_n} = \frac{1}{\pi}\int_0^\infty da\int_{-\infty}^{+\infty}\phi\epsilon_n\cos\left(a\frac{u-\mu_1\epsilon_1-\ldots-\mu_{n-1}\epsilon_{n-1}-\mu_n\epsilon_n}{\mu_n}.\right)d\epsilon_n,$$

which, replacing $\frac{a}{\mu_n}$ by a, becomes

$$\frac{\mu_n}{\pi}\int_0^\infty da\int_{-\infty}^{+\infty}\phi\epsilon_n\cos a\,(u-\Sigma\mu\epsilon)\,d\epsilon_n,$$

Therefore

$$p\,du = \frac{\mu_n d\epsilon_n}{\pi}\int_0^\infty da\int_{-\infty}^{+\infty}d\epsilon_1\ldots\int_{-\infty}^{+\infty}d\epsilon_n\,\phi_1\epsilon_1\ldots\phi_n\epsilon_n\cos a\,(u-\Sigma\mu\epsilon)\ldots\ldots (4).$$

Now if u and ϵ_n are to vary together

$du = \mu_n d\epsilon_n$, and therefore

$$p = \frac{1}{\pi}\int_0^\infty da\int_{-\infty}^{+\infty}d\epsilon_1\ldots\int_{-\infty}^{+\infty}d\epsilon_n\phi_1\epsilon_1\ldots\phi_n\epsilon_n\cos a\,(u-\Sigma\mu\epsilon)\ldots\ldots (5).$$

And finally,

$$P = \frac{1}{\pi}\int_{-l}^{+l}du\int_0^\infty da\int_{-\infty}^{+\infty}d\epsilon_1\ldots\int_{-\infty}^{+\infty}d\epsilon_n\phi_1\epsilon_1\ldots\phi_n\epsilon_n\cos a\,(u-\Sigma\mu\epsilon)\ldots\ldots (6).$$

Now let us suppose that equal positive and negative errors are equally probable. In this case $\phi\epsilon = \phi\,(-\epsilon)$, and consequently,

$$\int_{-\infty}^{+\infty}\phi\epsilon\sin a\mu\epsilon d\epsilon = 0.$$

Hence (6) will become

$$P = \frac{2}{\pi}\int_0^l du\int_0^\infty\cos au\,da\int_{-\infty}^{+\infty}\phi_1\epsilon_1\cos a\mu_1\epsilon_1 d\epsilon_1\ldots\int_{-\infty}^{+\infty}\phi_n\epsilon_n\cos a\mu_n\epsilon_n d\epsilon_n\ldots\ldots (7).$$

The next step is to find an approximate value of this expression.

When $a = 0$ $\int_{-\infty}^{+\infty}\phi\epsilon\cos a\mu\epsilon d\epsilon = \int_{-\infty}^{+\infty}\phi\epsilon d\epsilon = 1$,

as the error ϵ must have some value lying between $\pm\infty$.

It is clear this is the greatest value the integral in question can have, and therefore as n increases *sine limite*, the continued product

$$\int_{-\infty}^{+\infty} \phi_1 \epsilon_1 \cos \mu_1 a \epsilon_1 d\epsilon_1 \ldots \int_{-\infty}^{+\infty} \phi_n \epsilon_n \cos \mu_n a \epsilon_n d\epsilon_n$$

decreases *sine limite*, (being the product of n factors each less than unity) except for values of a differing infinitesimally from zero.

$$\text{Let } k^2 = \int_0^\infty \phi \epsilon \,.\, \epsilon^2 d\epsilon, \quad \kappa^4 = \int_0^\infty \phi \epsilon \,.\, \epsilon^4 d\epsilon,$$

and develope each of the cosines in the above-written continued product. It is thus seen to be equal to

$$1 - a^2 \Sigma \mu^2 k^2 + a^4 \left(\frac{1}{12} \Sigma \mu^4 \kappa^4 + \Sigma \mu_1^2 \mu_2^2 k_1^2 k_2^2\right) - \&c.$$

Again, n being very large and ultimately infinite, it is evident that $\Sigma \mu^4 \kappa^4$ is of the same order of magnitude as n, while $\Sigma \mu_1^2 \mu_2^2 k_1^2 k_2^2$ is of the order of n^2, the former term of the coefficient of a^4 may therefore be neglected in comparison with the latter, which again may be replaced by $\frac{1}{2} (\Sigma \mu^2 k^2)^2$, from which it differs by a quantity of the order of n. Similar remarks apply with respect to the higher powers of a.

Thus the continued product may be replaced by

$$1 - a^2 \Sigma \mu^2 k^2 + \frac{1}{2} a^4 (\Sigma \mu^2 k^2)^2 - \frac{1}{2\,.\,3} a^6 (\Sigma \mu^2 k^2)^3 + \&c.$$

or by $e^{-a^2 \Sigma \mu^2 k^2}$; a function which is coincident with it when a is infinitesimal. When a is finite both are, as we have seen, infinitesimal.

Consequently,

$$P = \frac{2}{\pi} \int_0^l du \int_0^\infty \cos au\, da \,.\, e^{-a^2 \Sigma \mu^2 k^2} \ldots\ldots\ldots\ldots\ldots (8),$$

or

$$P = \frac{1}{(\pi \Sigma \mu^2 k^2)^{\frac{1}{2}}} \int_0^l e^{-\frac{u^2}{4 \Sigma \mu^2 k^2}} du = \frac{2}{\sqrt{\pi}} \int_0^{\frac{l}{2\sqrt{\Sigma \mu^2 k^2}}} e^{-v^2} dv \ldots\ldots (9),$$

where we have supposed

$$u = 2 (\Sigma \mu^2 k^2)^{\frac{1}{2}} v.$$

It is evident, that whatever l may be, this expression for P is a maximum when

$$\Sigma \mu^2 k^2 \text{ is a minimum.}$$

Hence we get the following remarkable conclusion: When the number of observations increases *sine limite* the most advantageous system of factors are those which make

$$\Sigma \mu^2 k^2 \text{ a minimum.}$$

It remains to determine μ from the condition of the minimum taken in connexion with those already stated, viz. $\Sigma \mu a = 1$, $\Sigma \mu b = 0$, &c. $= 0$. We have

$$\left.\begin{aligned} \Sigma k^2 \mu d\mu &= 0 \\ \Sigma a d\mu &= 0 \\ \Sigma b d\mu &= 0 \\ \&c. &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (A).$$

Let $\lambda_1, \lambda_2 \ldots \lambda_p$ be indeterminate factors, then we may put

$$\left.\begin{aligned} k_1^2 \mu_1 &= a_1 \lambda_1 + b_1 \lambda_2 + \&c. \\ k_2^2 \mu_2 &= a_2 \lambda_1 + b_2 \lambda_2 + \&c. \\ \&c. &= \&c. \end{aligned}\right\} \ldots\ldots\ldots\ldots (B).$$

From the n equations (B) we deduce a new system of p equations. To obtain the first of these, we multiply equations (B) by $\frac{a_1}{k_1^2}$, $\frac{a_2}{k_2^2}$, &c. respectively, and add the results. For the second, we employ instead of the factors $\frac{a_1}{k_1^2}$, &c., the factors $\frac{b_1}{k_1^2}$, $\frac{b_2}{k_2^2}$, &c. and then proceed as before. And similarly for the others.

In consequence of the relations

$$\Sigma \mu a = 1, \quad \Sigma \mu b = 0, \quad \&c. = 0,$$

the new system of equations will be

$$\left.\begin{array}{l} 1 = \lambda_1 \Sigma \frac{a^2}{k^2} + \lambda_2 \Sigma \frac{ab}{k^2} + \&c. \\ 0 = \lambda_1 \Sigma \frac{ab}{k^2} + \lambda_2 \Sigma \frac{b^2}{k^2} + \&c. \\ 0 = \&c. \end{array}\right\} \ldots\ldots\ldots\ldots (C).$$

These p equations determine $\lambda_1, \lambda_2 \ldots \lambda_p$, and thus in virtue of (B) the values of $\mu_1, \mu_2 \ldots \mu_n$ become known. Finally as

$$x = \mu_1 V_1 + \mu_2 V_2 + \ldots + \mu_n V_n,$$

x will be completely determined.

Now let us recur to the original equations of condition stated in the last paragraph.

$$\left.\begin{array}{l} \epsilon_1 = a_1 x + b_1 y + \&c. - V_1 \\ \epsilon_2 = a_2 x + b_2 y + \&c. - V_2 \\ \&c. = \&c. \\ \epsilon_n = a_n x + b_n y + \&c. - V_n \end{array}\right\} \ldots\ldots\ldots\ldots (a).$$

From this system we deduce a new one, containing p equations. The first of these is got by multiplying equations (a) by $\frac{a_1}{k_1^2}$, $\frac{a_2}{k_2^2}$, &c., and adding the results: the second by using the factors $\frac{b_1}{k_1^2}$, $\frac{b_2}{k_2^2}$, &c.: and so on as before. The resulting system will be, neglecting all errors,

$$\left.\begin{array}{l} x \Sigma \frac{a^2}{k^2} + y \Sigma \frac{ab}{k^2} + \&c. = \Sigma \frac{a}{k^2} V \\ x \Sigma \frac{ab}{k^2} + y \Sigma \frac{b^2}{k^2} + \&c. = \Sigma \frac{b}{k^2} V \\ \&c. \qquad\qquad = \&c. \end{array}\right\} \ldots\ldots\ldots\ldots (\beta').$$

The system (β') contains as many equations as there are unknown quantities x, y, &c. I proceed to show that if x be determined from this system, its value will be the same as if it had been obtained from the most advantageous system of factors, namely, that which is determined by means of (B) and (C). In order to prove this, we multiply equations (β') by λ_1, λ_2, &c., and add the results. Then, in virtue of (C)

$$x = \lambda_1 \Sigma \frac{a}{k^2} V + \lambda_2 \Sigma \frac{b}{k^2} V + \&c.$$

Or,
$$x = (\lambda_1 a_1 + \lambda_2 b_1 + \&c.) \frac{V_1}{k_1^2} + (\lambda_1 a_2 + \lambda_2 b_2 + \&c.) \frac{V_2}{k_2^2} + \&c.$$

that is to say, as is seen on referring to (B),
$$x = \mu_1 V_1 + \mu_2 V_2 + \ldots + \mu_n V_n,$$

as before; which proves that the system (β') gives the same value for x as the most advantageous system of factors. Moreover, as (β') is symmetrical in x and a, y and b, &c. it is clear that it will also give the most advantageous values for y and the other unknown quantities.

When the law of probability of error is the same at every observation $k_1 = k_2 =$ &c. and (β') reduces itself to (β) given at p. 208 as the result of the method of least squares. In the general case, it expresses the modification which the method of least squares must undergo, when all the observations are not of the same kind, namely, that instead of making the function $\Sigma\,(ax + by + \&c. - V)^2$ a minimum with respect to xy, &c., we must substitute for it the function $\Sigma \frac{1}{k^2} (ax + by + \&c. - V)^2$, and then proceed as before.

Such, in effect, is Laplace's demonstration, except that he supposed the law of error the same at each observation. The form in which I have presented it is wholly unlike his. The introduction of Fourier's theorem enables us to avoid the theory of combinations, and also the use of imaginary symbols. It must be admitted that there are few mathematical investigations less inviting than the fourth chapter of the *Théorie des Probabilités*, which is that in which the method of least squares is proved.

It may be worth while to recur to the general formula:
$$P = \frac{1}{\pi} \int_{-l}^{+l} du \int_0^{\infty} da \int_{-\infty}^{+\infty} d\epsilon_1 \ldots \int_{-\infty}^{+\infty} d\epsilon_n \, \phi_1 \epsilon_1 \ldots \phi_n \epsilon_n \cos a\,(u - \Sigma\mu\epsilon).$$

It is certain that $\Sigma\mu\epsilon$ lies between the limits $\pm\,\infty$. Therefore when $l = \infty$, P should be equal to unity. I proceed to show that this is the case.
$$P_\infty = \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{-m^2u^2} du \int_0^{\infty} da \int_{-\infty}^{+\infty} d\epsilon_1 \ldots \int_{-\infty}^{+\infty} d\epsilon_n \, \phi_1 \epsilon_1 \ldots \phi_n \epsilon_n \cos a\,(u - \Sigma\mu\epsilon)$$

when $m = 0$.

Effecting the integration for u,
$$P_\infty = \frac{1}{m\sqrt{\pi}} \int_0^{\infty} e^{-\frac{a^2}{4m^2}} da \int_{-\infty}^{+\infty} d\epsilon_1 \ldots \int_{-\infty}^{+\infty} d\epsilon_n \, \phi_1 \epsilon_1 \ldots \phi_n \epsilon_n \cos a\,\Sigma\mu\epsilon \ldots (10)$$

when $m = 0$,

since
$$\int_{-\infty}^{+\infty} e^{-m^2u^2} \cos au\, du = \frac{\sqrt{\pi}}{m} e^{-\frac{a^2}{4m^2}}$$
$$\text{and} \int_{-\infty}^{+\infty} e^{-m^2u^2} \sin au\, du = 0.$$

Integrating for a, we see that when $m = 0$
$$P_\infty = \int_{-\infty}^{+\infty} d\epsilon_1 \ldots \int_{-\infty}^{+\infty} d\epsilon_n \, \phi_1 \epsilon_1 \ldots \phi_n \epsilon_n \, e^{-(m\Sigma\mu\epsilon)^2} \ldots (11).$$

Or,
$$P_\infty = \int_{-\infty}^{+\infty} \phi_1 \epsilon_1 \, d\epsilon_1 \ldots \int_{-\infty}^{+\infty} \phi_n \epsilon_n \, d\epsilon_n \ldots\ldots\ldots\ldots (12).$$

and as each of these integrals is separately equal to unity,
$$P_\infty = 1, \text{ which was to be proved.}$$

I proceed to show that in a particular case in which the value of P can be accurately determined, Laplace's approximation is correct. It has sometimes been thought that the introduction of the negative exponential involves a *petitio principii*, and is equivalent to assuming a particular law of error. It is therefore desirable, and I am not aware that it has hitherto been done, to verify his result in an individual case.

Let the law of error be the same in all the observations, and such that $\phi\epsilon = \frac{1}{2}\epsilon^{\mp\epsilon}$, the upper sign to be taken when ϵ is positive.

Let $\mu_1 = \mu_2 = \&c. = 1$, then

$$p = \frac{1}{\pi}\int_0^\infty da \int_{-\infty}^{+\infty} (\tfrac{1}{2}\epsilon^{\mp\epsilon_1}) d\epsilon_1 \ldots\ldots \int_{-\infty}^{+\infty} (\tfrac{1}{2} e^{\mp\epsilon_n})\, d\epsilon_n \cos a\,(u - \Sigma\epsilon),$$

or,

$$p = \frac{1}{\pi}\int_0^\infty \frac{\cos ua}{(1+a^2)^n}\,da, \text{ since } \int_0^\infty e^{-\epsilon}\cos a\epsilon\, d\epsilon = \frac{1}{1+a^2}.$$

The value of p is thus given by a known definite integral, which has been discussed by M. Catalan in the fifth volume of *Liouville's Journal*.

It may be developed in a series of powers of u. Up to $u^{2(n-1)}$ no odd power of u can appear in this development, for $\int_0^\infty \frac{a^{2p}}{(1+a^2)^n}\,da$ is finite while p is less than n, and therefore the integral may be developed by Maclaurin's theorem. For higher powers the method ceases to be applicable, and we must complete the development by other means. But as we suppose n to increase *s. l.* the integral tends to become developable in a series of even powers only of u. Thus

$$\int_0^\infty \frac{\cos ua}{(1+a^2)^n}\,da = \int_0^\infty \frac{da}{(1+a^2)^n} - \tfrac{1}{2}u^2\int_0^\infty \frac{a^2}{(1+a^2)^n}\,da + \&c.$$

Let
$$\int_0^\infty \frac{da}{(1+a^2)^n} = f(n).$$

Then
$$\int_0^\infty \frac{a^2\,da}{(1+a^2)^n} = -\Delta f(n-1);$$

and generally

$$\int_0^\infty \frac{a^{2p}\,da}{(1+a^2)^n} = \pm\,\Delta^p f(n-p).$$

Now

$$f(n) = \frac{\pi}{2}\cdot\frac{1\,.\,3\ldots 2n-3}{2\,.\,4\ldots 2n-2};$$

$$-\Delta f(n-1) = \frac{\pi}{2}\cdot\frac{1\,.\,3\ldots 2n-5}{2\,.\,4\ldots 2n-2}\,.\,1,$$

$$\Delta^2 f(n-2) = \frac{\pi}{2}\cdot\frac{1\,.\,3\ldots 2n-7}{2\,.\,4\ldots 2n-2}\,.\,1\,.\,3;$$

and generally,

$$\pm\,\Delta^p f(n-p) = \frac{\pi}{2}\cdot\frac{1\,.\,3\ldots\overline{2n-3-2p}}{2\,.\,4\ldots 2n-2}\,.\,1\,.\,3\ldots\overline{2p-1}$$

$$= f(n)\,.\,\frac{1\,.\,3\ldots 2p-1}{2n-1-2p\ldots\overline{2n-3}}.$$

Thus

$$\int_0^\infty \frac{\cos ua\,da}{(1+a^2)^n} = fn\left\{1 - \frac{1}{2}\cdot\frac{1}{2n-3}u^2 + \frac{1}{2.3.4}\quad\frac{1.3}{\overline{2n-5}.\overline{2n-3}}u^4 - \&c.\right\}$$

The coefficient of u^{2p} is

$$\pm fn\,.\,\frac{1.3\ldots\overline{2p-1}}{2.3\ldots 2p}\cdot\frac{1}{\overline{2n-1-2p}\ldots\overline{2n-3}}$$

or,

$$\pm fn\frac{1}{1.2\ldots p}\cdot\frac{1}{2^p}\cdot\frac{1}{\overline{2n-1-2p}\ldots\overline{2n-3}}.$$

Let n become infinite, this becomes

$$f(n)\frac{1}{1.2\ldots p}\,\frac{1}{(4n)^p};$$

and we have only to determine what $f(n)$ then becomes.

Now by Wallis's theorem

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} = \frac{1.3\ldots(2n-1)^{\frac{1}{2}}}{2.4\ldots 2n-2}\text{ . ult.}$$

$$\text{Therefore } fn = \frac{1}{2}\left(\frac{2\pi}{2n-1}\right)^{\frac{1}{2}}\text{ when } n \text{ is infinite,}$$

$$\text{or, } fn = \frac{1}{2}\left(\frac{\pi}{n}\right)^{\frac{1}{2}}.$$

Consequently,

$$\int_0^\infty \frac{\cos ua\,da}{(1+a^2)^n} = \frac{1}{2}\left(\frac{\pi}{n}\right)^{\frac{1}{2}}\left\{1 - \frac{u^2}{4n} + \frac{1}{2}\cdot\frac{u^4}{(4n)^2} - \&c.\right\}$$

$$= \frac{1}{2}\left(\frac{\pi}{n}\right)^{\frac{1}{2}} e^{-\frac{u^2}{4n}}\text{ when } n \text{ is infinite.}$$

Therefore,

$$p = \frac{1}{2\sqrt{n\pi}}e^{-\frac{u^2}{4n}},$$

$$\text{and } P = \frac{1}{\sqrt{n\pi}}\int_0^l e^{-\frac{u^2}{4n}}du.$$

Now the value given for P at p. 210 is

$$P = \frac{1}{\sqrt{\pi\Sigma\mu^2k^2}}\int_0^l e^{-\frac{u^2}{4\Sigma\mu^2k^2}}du.$$

In the present case $\mu = 1$.

$$k^2 = \tfrac{1}{2}\int_0^\infty e^{-\epsilon}\epsilon^2 d\epsilon = 1:\text{ and consequently } \Sigma\mu^2k^2 = n.$$

Thus

$$P = \frac{1}{\sqrt{n\pi}}\int_0^l {}^{-\frac{u^2}{4n}}du,\text{ as before.}$$

Thus Laplace's approximation coincides with the result obtained by an independent method. This example serves to shew distinctly the nature of the approximation in question.

The function p having been developed in a series of powers of u, we take the *principal term* in the coefficient of each power of u; that is the term divided by the lowest power of n. We neglect for instance every such term as $\frac{1}{n^{p+\delta}}u^{2p}$, because we have a term in u^{2p} divided by n^p. Thus we retain $\frac{u^{2p}}{n^p}$ and neglect $\frac{u^{2(p-\delta)}}{n^p}$, although, unless u be large, the former term is of the same or a lower order of magnitude than the latter. That Laplace's method does in a very general manner give an approximation of this kind cannot, I think, be questioned, especially after the verification we have just gone through. But some doubt may perhaps remain, whether such an approximation to the *form* of the function P, if such an expression may be used, is also an approximation to its numerical value, when we consider that in obtaining it we have neglected terms demonstrably larger than those retained.

For two recognized exceptions to the generality of Laplace's investigation, viz. where $\phi\epsilon = \frac{1}{\pi}\frac{1}{1+\epsilon^2}$, and the case in which $\mu_1, \mu_2 \ldots$, decrease *in infinitum sine limite*, I shall only refer to p. 10 of Poisson's paper in the *Connaissance des Tems* for 1827. Neither affects the general argument. We now come to Gauss's second method, which is given in the *Theoria Combinationis Observationum*.

GAUSS'S SECOND DEMONSTRATION.

The connexion between the method of Laplace, and that which Gauss followed in the *Theoria Combinationis Observationum*, will be readily understood from the following remarks.

After determining $\mu_1 \ldots \mu_n$ by the condition that P should be a minimum, Laplace remarked that the same result would have been obtained (viz., that $\Sigma\mu^2k^2$ must be a minimum), if the assumed condition had been that the mean error of the result, *i.e.* the mean arithmetical value of $\Sigma\mu\epsilon$ should be a minimum. (I should rather say that he makes a remark equivalent to this, and differing from it only in consequence of a difference of notation, &c.) It is in fact easy to see that the mean value in question is equal to

$$\frac{\int_0^\infty updu}{\int_0^\infty pdu}, \text{ or to } 2\int_0^\infty updu;$$

and as

$$p = \frac{1}{2(\pi\Sigma\mu^2k^2)^{\frac{1}{2}}}e^{-\frac{u^2}{4\Sigma\mu^2k^2}}$$

$$2\int_0^\infty updu = \frac{2(\Sigma\mu^2k^2)^{\frac{1}{2}}}{\sqrt{\pi}}$$

which is of course a minimum when $\Sigma\mu^2k^2$ is so.

Gauss, adopting this way of considering the subject, pointed out that it involved the postulate that the importance of the error $\Sigma\mu\epsilon$, *i.e.* the detriment of which it is the cause, is proportional to its arithmetical magnitude. Now, as he observes, the importance of the error may be just as well supposed to vary as the square of its magnitude: in fact, it does not, strictly speaking, admit of arithmetical evaluation at all. We must *assume* that it is represented by some direct function of its magnitude, such that both vanish together. One assumption is not more arbitrary than another. Let us suppose, therefore, that the importance of the error is represented by $(\Sigma\mu\epsilon)^2$. That is, that $(\Sigma\mu\epsilon)^2$ is the function whose mean value is to be made a minimum. I now proceed to find it.

$$(\Sigma\mu\epsilon)^2 = \Sigma\mu^2\epsilon^2 + 2\Sigma\mu_1\mu_2\epsilon_1\epsilon_2 \ldots\ldots\ldots (13).$$

The mean value of ϵ^2 is $\int_{-\infty}^{+\infty} \epsilon^2 \phi \epsilon \, d\epsilon = 2\,k^2$.

Hence, that of $\Sigma \mu^2 \epsilon^2$ is $2\,\Sigma \mu^2 k^2$.

The mean value of $\Sigma \mu_1 \mu_2 \epsilon_1 \epsilon_2$ is zero, positive and negative errors of the same magnitude occurring with equal frequency on the long run.

Consequently,

$$\text{mean of } (\Sigma \mu \epsilon)^2 = 2\,\Sigma \mu^2 k^2 \;\ldots\ldots\ldots\; (14);$$

and, therefore, as before, $\Sigma \mu^2 k^2$ is to be made a minimum. The rest of the investigation is of course the same as that of Laplace.

Nothing can be simpler or more satisfactory than this demonstration. It is free from all analytical difficulty, and applicable whatever be the number of observations, whereas that of Laplace requires this number to be very large.

Recurring to equation (11), differentiating it for m^2, and then making $m = 0$, we find

$$\int_{-\infty}^{+\infty} p u^2 \, du = \int_{-\infty}^{+\infty} \phi \epsilon_1 \, d\epsilon_1 \ldots \int_{-\infty}^{+\infty} \phi \epsilon_n \, d\epsilon_n \, (\Sigma \mu \epsilon)^2 = 2\,\Sigma \mu^2 k^2;$$

and as the first member of this equation is evidently the mean value of u^2 or of $(\Sigma \mu \epsilon)^2$, this is a new verification of our analysis.

As an illustration of Gauss's principle, let the fourth power of the error be taken as the measure of its importance;

$$(\Sigma \mu \epsilon)^4 = \Sigma \mu^4 \epsilon^4 + 6\,\Sigma \mu_1^{\,2} \mu_2^{\,2} \epsilon_1^{\,2} \epsilon_2^{\,2} + \text{terms involving odd powers of } \epsilon.$$

Therefore,

$$\text{mean of } (\Sigma \mu \epsilon)^4 = 2\,\Sigma \mu^4 \kappa^4 + 24\,\Sigma \mu_1^{\,2} \mu^2_{\,2} k_1^{\,2} k_2^{\,2} \;\ldots\ldots\ldots\; (15)$$

and $\mu_1 \ldots \mu_n$ must be so determined that this may be a minimum.

I have already said that the results given by what Laplace called the most advantageous system of factors are not strictly speaking the most probable of all possible results.

As the distinction involved in this remark seems to me to be essential to a right apprehension of the subject, I will endeavour to illustrate it more fully.

Recurring to the equations of condition, as given in p. 208, we see that the values Laplace assigns to the factors μ_1 μ_2 &c., are independent of V_1 V_2 &c. They depend merely on the coefficients a b &c., which are quantities known *à priori*, *i. e.* before observation has assigned certain more or less accurate values to the magnitudes V_1 V_2 &c. All we then can say is, that if we employ Laplace's system of factors, and also any other, in a large number of cases (the coefficients a b &c., being the same in all) we shall be right within certain limits in a larger proportion of cases when the former system of factors is made use of than when we employ the latter. And this conclusion is wholly irrespective of the values of V_1 V_2 &c., and consequently of those which we are led in each particular case to assign to x y &c. The comparison is one of *methods*, and not at all one of *results*. But when V_1 V_2 &c. are known, another way of considering any particular case presents itself. We can then compare the probability of different *results*. For, let us consider a large number of sets of equations of condition (in each of which not only are a b &c. equal, as in the former case, but also V_1 V_2 &c.) The true values of the elements x y &c. may be different in each. But in affirming that ξ η &c., are the most *probable* values of x y &c., we affirm that the true values of x y &c. are more frequently equal to ξ η &c. than to any other quantities whatever. Here we have no concern with the method by which the values ξ η &c. were obtained. The comparison is merely one of *results*.

As for one particular law of error (that considered in p. 206), the results of the method of least squares are the most *probable* possible; and as the function by which this law of error is expressed occurs in Laplace's demonstration of that method, it has been thought that his approximation involved an undue assumption, and that in fact his proof was invalid unless that particular law of error was supposed to obtain.

It is easily seen that the method of least squares can give the most probable results only for that law of error (if we except another which involves a discontinuous function). Mr. Ivory attempted to shew that Laplace's conclusions might be applied to prove that the results of the method were, in effect, the most probable possible, and thence drew the inference which I have already mentioned. After some consideration, I have decided on not entering on an analysis of his reasoning, which it would be difficult to make intelligible, without adding too much to the length of this communication. It is set forth with a good deal of confidence; Laplace's conclusions are pronounced invalid on the authority of an indirect argument, and without any examination of the process by which he was led to them. I may just mention that in the whole of Mr. Ivory's reasoning, the probability that $\Sigma\mu\epsilon$ is precisely equal to any assigned magnitude, is, to all appearance at least, considered a finite quantity, though it is perfectly certain that it must be infinitesimal.

It would seem as if he had taken Laplace's expression of the probability in question, viz.

$$\frac{c^{-\frac{kl^2}{4k''a^2 Sm^{(1)2}}}}{2a\sqrt{\pi}\sqrt{\frac{k''}{k}Sm^{(1)2}}},$$

without being aware that in Laplace's notation l and a are infinite, and that consequently the expression is infinitesimal. (Vide *Tilloch's Magazine*, LXV. p. 81.)

MR. IVORY'S DEMONSTRATIONS.

They are three in number. Two appeared in the sixty-fifth, and a third in the sixty-seventh volumes of *Tilloch's Magazine*.

The aim of all three is the same, namely, to demonstrate the rule of least squares without recourse to the theory of probabilities, which appeared to him to be foreign to the question. The grounds of this opinion he has not clearly developed: perhaps the best refutation of it will be found in the unsatisfactory character of the demonstrations which he proposed to substitute for the methods of Laplace and Poisson. In common with many others, Mr. Ivory appears to have looked with some distrust on the results obtained by means of this theory: a not unnatural consequence of the extravagant pretensions sometimes advanced on its behalf.

The first of his demonstrations rests upon what I cannot help considering a vague analogy. In the equation of condition

$$e = ax - V,$$

he remarks that the influence of the error e on the value of x increases as a decreases, and *versâ vice:* that consequently the case is precisely similar to that of a lever which is to produce a given effect, as of course the length of the arm must vary inversely as the weight which it supports.

Consequently, he argues, the condition to be fulfilled, in order that the equations of condition may be combined in the most advantageous manner, is the same as what would be the condition of equilibrium, were $a\ a'\ a''$ &c. weights on a lever, acting at arms $e\ e'\ e''$ &c. This condition is of course

$$\Sigma ae = 0, \text{ whence } \Sigma(ax - V)\,a = 0,$$

the result given by the method of least squares.

But, granting that the influence of an error e, ought to be greater when a is less, and *versâ vice*, how are we entitled to assume that the case is *precisely similar* to that of equilibrium on a lever? Apart from this assumption, there seems to be no reason for inferring that because this influence increases as a decreases, it must therefore vary inversely as a. By what function of a the influence of e ought to be represented, is the very essence of the question; to determine, by introducing the extraneous idea of equilibrium on a lever, that $\frac{1}{a}$ is the function required, seems to be little else than a *petitio principii*, concealed by a metaphor*.

The second demonstration may be thus briefly stated.

The values of different sets of observations might be compared if we knew the average error in each set, or if we knew the average value of the squares of the errors in each. In either case that would be the best set of observations in which the quantity taken as the *measure of precision* was the smallest.

Similarly, by assigning different values to the unknown quantities x, y, &c. involved in a system of equations of condition, we can make it *appear* that the mean of the squares of the errors has a greater or less value. Therefore as of sets of observations, that is the best in which this quantity is least; so of different sets of results deduced from one set of observations, the same is also true; and therefore the sum of the squares of the apparent errors is to be made a minimum.

There seems to be involved in this reasoning a confusion of two distinct ideas; the precision of a set of observations is undoubtedly measured by the average of the errors *actually committed*, and if we knew this average, we should be able to compare the values of different sets of observations. But it is not measured by the average of the *calculated* errors, namely, those which are determined from the equations of condition when particular values have been assigned to x, y, &c.

The problem to be solved may be stated thus. Given that the single observations of which the set is composed are liable to a certain average of error, to combine them so that the resulting values of the unknown quantities may be liable to the smallest average of error.

This problem Laplace and Gauss have both solved. Their solutions differ, because they estimated the average error in different manners.

But how are we justified in assuming that to be the best mode of combining the observations which merely gives the *appearance* of precision *not* to the final results, but only to the individual observations, and which, with reference to them, gives no estimation of the probability that this appearance of accuracy is not altogether illusory?

The third of Mr. Ivory's demonstrations is not, I think, more satisfactory than the other two.

The kind of observations to which the method of least squares is applicable, are such, Mr. Ivory observes, that there exists no bias tending regularly to produce error in one direction, and that the error in one case is supposed to have no influence whatever on the error in any other case.

From this principle he attempts to show that the method of least squares is the only one which is consistent with the independence of the errors.

When, however, we speak of the errors as being independent of one another, only this can be meant, that the circumstances under which one observation takes place do not affect the others. In *rerum naturâ* the errors are independent of one another. Nevertheless, with reference to our knowledge they are not so, that is to say, if we know one error we know all, at least in the case in which the equations of condition involve only one unknown quantity, which is that considered by Mr. Ivory. For the knowledge of one error would imply the knowledge of the true value of the unknown quantity, and thence that of all the other errors.

* I have omitted to notice some remarks which Mr. Ivory appends to this demonstration, as they do not appear to affect the view taken in the text.

Mr. Ivory states the following equations of condition:

$$e = a\,x - m$$
$$e' = a'\,x - m'$$
$$\&c. = \&c.$$

He thence deduces the following value of x:

$$x = \frac{\Sigma a e}{\Sigma a^2} + \frac{\Sigma a m}{\Sigma a^2}, \text{ and those of } e\ e' \text{ are}$$

$$e = -m + \frac{a\,\Sigma a m}{\Sigma a^2} + \frac{a\,\Sigma a e}{\Sigma a^2}$$

$$e' = -m' + \frac{a'\,\Sigma a m}{\Sigma a^2} + \frac{a'\,\Sigma a e}{\Sigma a^2}\ .\ \&c. = \&c.$$

He remarks that these errors are not independent of one another, as all depend on the single quantity $\Sigma a e$, which may be eliminated between any two of the last-written equations: but that there is one case in which they are independent of one another, namely, when we assume $\Sigma a e = 0$, which of course leads to the method of least squares, and that in this case, as we shall have

$$e = -m + \frac{a\,\Sigma a m}{\Sigma a^2}\ \&c. = \&c.$$

each error is determined by "the quantities of its own experiment." But this reasoning is perfectly inconclusive. In the case supposed, $e\,e'$ &c. are as much connected together as in any other, as may be shown by eliminating $\Sigma a m$ between the equations

$$e = -m + \frac{a\,\Sigma a m}{\Sigma a^2},\quad e' = -m' + \frac{a'\,\Sigma a m}{\Sigma a^2}\ \&c. = \&c.;$$

and besides, apart from any mathematical reasoning, it is clear that as if we know one error we know all, so also if we assign any value to one, we have in effect assigned values to all, whether we use the method of least squares or any other.

Moreover, e is not determined by the quantities of its own experiment alone, since $\Sigma a m$ involves the results of all the experiments; there is no difference between this and the general case, except that $\Sigma a e$ has ceased to appear in the equations. But suppose we multiplied the equations of condition by any function of a, we might deduce the following values of x and e:

$$x = \frac{\Sigma \phi a\,.\,e}{\Sigma a\,.\,\phi a} + \frac{\Sigma \phi a\,.\,m}{\Sigma a\,.\,\phi a}$$

$$e = -m + \frac{a\,\Sigma \phi a\,.\,m}{\Sigma a\,.\,\phi a} + \frac{a\,\Sigma \phi a\,.\,e}{\Sigma a\,.\,\phi a}$$

$$e' = -m' + \frac{a'\,\Sigma \phi a\,.\,m}{\Sigma a\,.\,\phi a} + \frac{a'\,\Sigma \phi a\,.\,e}{\Sigma a\,.\,\phi a}\ .$$

Mr. Ivory's reasoning would apply word for word as before, and would show that the best mode of combining the equations of condition was to employ the factors ϕa, $\phi a'$, &c. whatever be the form of ϕ. As it thus would serve to establish, at least apparently, an infinity of contradictory results, the inference is that in no case has it any validity.

I have now completed, though in an imperfect manner, the design indicated at the outset of this paper, namely, to give an account of the different modes in which the subject has been treated, and to simplify the analytical investigations. If I have succeeded in doing this, the present communication may tend to make a very curious subject more accessible than it has hitherto been.

XVII. *On the Transport of Erratic Blocks.* *By* William Hopkins, M.A. *and* F.R.S., *Fellow of the Cambridge Philosophical Society, of the Royal Astronomical Society, and of the Geological Society.*

[Read *April* 29, 1844.]

The instability of opinion which usually, and perhaps necessarily, characterizes the earlier researches into any new and extended branch of philosophical enquiry, is strongly exemplified in the different views which have been entertained respecting the causes to which the transport of erratic blocks is to be referred. In the first stages of the enquiry rapid currents of water were generally recognized as the most probable agents in these phenomena. No attempt, however, was made to calculate the power of this agency, and the theory was associated with hypotheses far too extravagant to bear the test of careful investigation. The natural consequence was the very general abandonment of the theory on the suggestion of another possible cause of the phenomena in question. It was represented that floating ice might have acted as vehicles of transport, and many facts were collected, from the reports of those who had visited the colder latitudes, confirmative of this opinion. Again, this latter theory has been lately endangered by the recognition, on the part of some geologists, of a third theory, which attributes the transport of blocks to the sole action of glaciers; a view of the subject which has arisen out of the curious and interesting observations recently made on the movements of existing glaciers, and the phenomena indicating their far greater extension at some preceding geological epoch.

The entire rejection of any one of these theories would imply a forgetfulness of the fact, that geology is, in a peculiar sense, a mixed science, not merely as involving investigations which properly belong to widely different branches of physical and natural science, but also as treating in some instances of phenomena, (as in the cases of erratic blocks of different kinds, or in different localities,) which, while they possess a great community of character, may be referrible to totally dissimilar causes. Both glaciers and floating ice are manifestly adequate with respect to their motive powers, to produce the phenomena in question. In the following communication I shall investigate the transporting powers of currents of water, and shall shew that, under certain conditions, such currents would be generated of sufficient velocity for the transport of boulders, and consequently that this cause is also adequate to produce the removal of at least a large portion of the boulders which have travelled from their original sites; and that, therefore, the theory is not to be rejected on account of any apparent inefficiency in the cause of transport assigned by it, or the extravagances which have been formerly associated with it. We shall thus, I conceive, be constrained to recognize the general adequacy of each of the three causes of transport above mentioned; and in the further examination of the problem it will only remain for the geologist to ascertain, as far as possible, the share which each cause has had in producing the actual phenomena of transport by a careful comparison of observed facts with the probable results of each mode of transport. Each group of erratic blocks, or each mass of transported materials, may present in this respect a separate problem; in the present communication I shall only offer on this branch of the subject a few general observations, without entering into any discussion of particular examples, beyond what may be necessary for the elucidation of general views.

SECTION I.

Transporting Currents.

1. These currents may be divided into *River Currents*, *Tidal Currents*, *Ocean Currents*, and *Elevation Currents*. By the latter, I mean those currents which would be produced by the more or less sudden elevations of determinate portions of that part of the surface of the earth which is covered with water. They are the only currents among those above mentioned of which it will here be necessary for me to speak. Currents of this kind are always accompanied by a corresponding temporary elevation of the surface of the water, constituting a *wave*. We are indebted to Mr. Russell for all the experimental knowledge we possess of the nature and properties of this wave, of the laws of its motion, and of the current which attends it. He has denominated it the *great wave of translation*. The details of his experiments will be found in the *Proceedings of the British Association*. It will only be necessary for me here to state his general results.

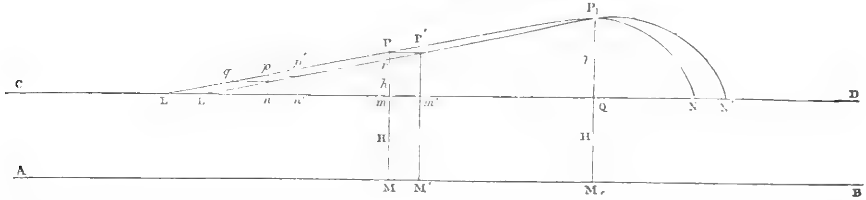
2. Suppose a long canal to be filled with water, and, for the greater simplicity, let it be supposed to be of uniform width and depth. There are various ways in which a *wave of translation* might be produced in this canal. One of the simplest, and most appropriate for our immediate object, would be the *sudden elevation* of a determinate portion of the bottom of the canal, which portion, for distinctness, may be conceived to be about its middle point, and of small extent as compared with the length of the canal. Two waves will thus be sent off in opposite directions. Each wave will move with uniform velocity, preserving very approximately the same form. Its length will depend, in great measure, on that of the portion of the bottom elevated to produce the wave. Each particle of water begins to move when the front of the wave reaches the vertical transverse section in which the particle is situated, and continues in motion till the wave has passed over it, when it is again left at rest. Its motion therefore is not oscillatory, but one of translation in the direction of the wave's motion. Mr. Russell has established experimentally the following law of this motion:

(1) *Every particle in the same vertical transverse section of the canal has the same motion.*

He has also established the following law respecting the propagation of the wave:

(2) *The velocity with which the wave is propagated is equal to that due to half the height of the crest or highest point of the wave above the bottom of the canal**.

3. From these laws we easily deduce the expression for the *velocity of each particle*, i.e. for the *velocity of the current* which accompanies the wave. Let LPN represent the position of a longitudinal section of the wave, at the time t, and $L'P'N'$ at the time $t + \delta t$, AB being the bottom



of the channel, and CFD the level of the general surface of the water. Let P_1 be the crest of the wave, $QP_1 = h_1$; P any other point on the surface of the wave at time t, P' the corresponding

* It should be stated that the experiments and observations by which these laws were established, were made on canals not many feet in depth. There appears, however, to be no reasonable doubt that the same laws hold, at least approximately, for much greater depth, as I have assumed them to do in the application of these investigations to the transport of erratic blocks.

point at time $t+\delta t$, and $mP = m'P' = h$, and $M_1 Q = Mm = H$, the depth of the canal, supposed uniform. Also let V be the velocity of propagation of the wave; then will $LL' = nn' = mm' = NN' = V\delta t$; and let v be the velocity of the current at P, and therefore also (by the first law) at every point of the vertical transverse section through P. Also let b be the breadth of the canal; the area of the transverse section through P will $= (H+h)\,b$.

Now it is manifest that a volume equal to that whose vertical longitudinal section is $LPrL'$ (or, in the limit $LPP'L'$) and breadth b, must have passed through the transverse section MP in the time δt.

Let this volume $= U$; then if $np = y$,

$$\begin{aligned} \delta U &= b \,.\, \text{area}\, qp' \\ &= b \,.\, pp' \,.\, \delta y \\ &= bV\delta t \,.\, \delta y; \\ \therefore\ U &= bVh\delta t, \end{aligned}$$

integrating from $y = 0$ to $y = mP = h$.

But by the first law we must have

$$U = vb\,(H+h)\,\delta t;$$

and therefore equating these values of U, we have

$$v = V\frac{h}{H+h} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1).$$

Also by the second law

$$\begin{aligned} V &= \sqrt{g\,(H+h_1)}\,; \\ \therefore\ v &= \sqrt{g\,(H+h_1)}\,\frac{h}{H+h} \\ &= \frac{\sqrt{g\,(H+h_1)}}{H}\,.\,h\left(1+\frac{h}{H}\right)^{-1} \quad \ldots\ldots\ldots \ (2). \end{aligned}$$

If v_1 be the velocity of the current in the transverse section through the crest of the wave,

$$\begin{aligned} v_1 &= V.\frac{h_1}{H+h_1}, \quad [\text{by } (1)] \\ &= \sqrt{gh_1}\,\sqrt{\frac{h_1}{H+h_1}}\,. \end{aligned}$$

4. Let us now suppose the wave to diverge from a center; then assuming the breadth of the wave to remain constant, and therefore the velocity of propagation (V) to be the same for every part of the wave, we shall have

$$\begin{aligned} \delta U &= 2\pi\rho \,.\, pp' \,.\, \delta y \\ &= 2\pi V\delta t\rho\delta y, \end{aligned}$$

where $\rho = Cn$, C representing the point from which the wave is diverging. U cannot be found generally without knowing the relation between ρ and y, *i.e.* without knowing the form of the wave; but if we suppose the space CL (r) through which the wave has diverged to be much greater than the breadth (l) of the wave, we shall have approximately $\rho = r$, and therefore

$$\delta U = 2\pi V\delta t \,.\, r\delta y,$$

and integrating from $y = 0$ to $y = mP = h$,

$$U = 2\pi Vrh\delta t.$$

Again, since U is now the volume which passes through the cylindrical surface whose radius $= Cm(\rho')$ and height $= MP(H+h)$, in time δt, we must have by the first law

$$U = 2\pi\rho'(H+h)\,v\delta t$$
$$= 2\pi r(H+h)\,v\delta t \text{ nearly.}$$

Equating these values of U we obtain

$$v = V.\frac{h}{H+h},$$

$$v_1 = V.\frac{h_1}{H+h_1}.$$

These approximate expressions for v and v_1 are of the same form as the accurate expressions obtained in the preceding case, but h and h_1 are not here independent of the distance through which the wave has travelled; they are functions of r. To determine them let us assume the *vis viva* of each wave to remain constant during its motion. The element (δm) of the mass in motion at the time t, will be the portion of the fluid included between the two cylindrical surfaces whose radii are ρ' and $\rho'+\delta\rho'$ and height $H+h$ (MP). Therefore

$$\delta m = 2\pi\rho'(H+h)\,\delta\rho'$$
$$= 2\pi r H\delta\rho' \text{ nearly,}$$

if r be much greater than l, and H than h. Also

$$v = V.\frac{h}{H+h}$$
$$= V\frac{h}{H} \text{ nearly.}$$

Hence

$$\Sigma v^2\delta m = 2\pi\frac{V^2}{H}r\int_r^{r+l}h^2\delta\rho'.$$

Now let

$$h = \phi(\rho'-r)$$

be the equation to the curve LPN when $CL = a$, a particular value of r; then assuming the form of the curve so to change that each ordinate shall be diminished in the same ratio, we shall have generally when $CL = r$,

$$h = \psi\left(\frac{r}{a}\right).\phi(\rho'-r):$$

and

$$\Sigma v^2\delta m = 2\pi\frac{V^2}{H}r\left\{\psi\left(\frac{r}{a}\right)\right\}^2\int_r^{r+l}\{\phi(\rho'-r)\}^2 d\rho';$$

or putting $\rho'-r=\rho$, and $\chi' = (\phi)^2$,

$$\Sigma v^2\delta m = 2\pi\frac{V^2}{H}r\left\{\psi\left(\frac{r}{a}\right)\right\}^2\int_0^l\chi'(\rho)\,d\rho,$$

which will be independent of r, if

$$r\left\{\psi\left(\frac{r}{a}\right)\right\}^2 = c = \text{a constant,}$$

or,

$$\psi\left(\frac{r}{a}\right) = \sqrt{\frac{c}{r}}.$$

Hence

$$h = \sqrt{\frac{c}{r}} \cdot \phi(\rho);$$

or, for any assigned value of ρ

$$h \propto \frac{1}{\sqrt{r}}.$$

Consequently,

$$v \text{ and } v_1 \propto \frac{1}{\sqrt{r}}.$$

5. A diverging wave, such as above described, would manifestly be produced in the midst of the ocean by the elevation of a portion of its bottom. The height and breadth of the wave will depend on the area of the elevated portion, the height through which it is raised, and the time occupied in the process of elevation. Suppose this area to be circular, and its radius $= R$; and first suppose the elevation to be *instantaneous*, and the height $= h_1$. The resulting wave will have a steep front, like that of the tidal wave called a *bore*, the height of its crest being at first equal to that of the elevated surface of the water above the level of the general surface $= h_1$ in the case before us; and the breadth of the wave will be the space through which its front shall have diverged from the boundary of the original disturbance, when that boundary shall have been reached by the inner circular boundary of the wave.

6. Let us next suppose the elevation to take place *gradually*, its amount being still $= h_1$. The surface of the water above the elevated area will be raised to a height less than h_1, and consequently the height of the crest of the wave will be less than h_1, and the velocity of the current produced by it will be proportionally less than in the former case. If R be small, a small increase in the time occupied by the elevatory movement may make a great difference in h_1, and consequently in the velocity and transporting power of the current; but if R be large, the corresponding diminution in h_1 will be much smaller*.

7. If the elevated area be a parallelogram, of which the length is much greater than the breadth, two waves will proceed in directions perpendicular to the longer sides of the area, to which sides the fronts of the wave (except near to its extremities) will be parallel. The breadth of the wave will depend on that of the elevated area. It is important to remark that the diminution in the height of the wave, and consequently in the velocity of the attendant current, will be much less rapid than in the case above considered of the circular wave. Instances of circular waves would necessarily present themselves in the elevatory movements of such a district as that of the Cumbrian mountains, while wholly or partially beneath the sea; and examples of the other kind, in the simultaneous elevation of the whole of such a range as the great mountain limestone ridge of the northern part of this kingdom.

8. In the case first considered the wave was supposed to be propagated along a canal of uniform width and depth. If, on the contrary, the depth or width decrease, the velocity of the current will be increased, as appears from the expression for v_1, (Arts. 3 and 4). Thus, if a portion of a great wave pass into the mouth of a channel which gradually contracts, the velocity

* For example, let $R = 20$ miles, and let the elevation be instantaneous. The depth of the ocean might be such that it should require 15 or 20 minutes for the surface of the water above the elevated area to be reduced again to the level of the general surface. In such cases, the elevatory movement might occupy several minutes without reducing h_1 very materially. But if, on the contrary, R did not exceed a mile or two, then, under the same conditions, h_1 would be reduced to a very small quantity, and the transporting power of the wave would be almost annihilated.

of the attending current may become much greater than in the uncontracted wave. Such must have been the case with respect to the portion of a wave diverging from the district of the Cumbrian mountains, and received into the strait which must have been formed by the pass of Stainmoor previously to its emergence from the ocean, but subsequently to that of the higher mountains to the north and south of it.

We may now proceed to investigate the transporting power of currents originating in the manner above explained.

SECTION II.

Transporting Power of Currents.

9. Whatever be the specific gravity of a body, if its dimensions be sufficiently small, it can never acquire more than a small velocity in descending by gravity in any fluid of which the density is not extremely small. Such a body may therefore be held in suspension in water for a considerable time, and when placed in running water, soon acquires a horizontal velocity indefinitely nearly equal to that of the current. It may therefore be transported to considerable distances before it descend to the bottom; or when once deposited on the bed of the stream, it may easily be again disturbed, and carried onward as before. When the body is not however of very small dimensions it can only be transported along the bottom by the impelling force of the current, its motion being retarded by friction, or the resistance of solid obstacles. In this latter case it is necessary to ascertain the relation between the velocity of the current and the dimensions and weight of the largest mass it is capable of moving. This relation depends not only on the volume and specific gravity of the mass, but also on its form; and therefore, in order to ascertain whether certain given bodies could be moved by a given current, a separate investigation would, in strictness, be necessary for each, supposing their forms to be different, though they might in all other respects be the same. To render our results immediately applicable however, with sufficient accuracy for our general purpose, it will be sufficient to investigate the above-mentioned relation for a few certain forms, and then to refer any proposed mass to that particular form to which it most nearly approximates, among those for which the above investigation has been made.

10. A body acted on by a current may be moved by *sliding* or by *rolling*. In the former case, a very uncertain element, the friction of the surface on which the body rests, is necessarily introduced into our calculations. It will depend on the nature of the surface over which the transport takes place, and on the force with which the body presses on that surface, and this force will depend very much on the extent of that portion of the surface of the body which may be in such close contact with the surface on which the body reposes as to exclude the water from penetrating between them, and exercising there its upward pressure. In those cases, however, in which the motion takes place by *rolling*, the uncertainty depending on friction is entirely removed, for such motion is independent of the magnitude of the friction. Also, during a rolling motion the body must be revolving round one *edge* as an instantaneous axis, so that the fluid pressure will act on all points of the surface except those very near to that axis. The abstraction, therefore, of the pressures on these latter points will have no material effect on the body's rolling motion, and may be neglected in our calculations. When the body passes from one edge to another, as a new instantaneous axis, the whole intervening surface might come in close contact with that over which the body moves; but if these edges be not too far apart (as will generally be the case in those bodies which tend to move by rolling rather than sliding) the body will necessarily *begin*, by its momentum to move round the second axis, and will consequently admit the fluid to exert its pressure on the lower surface of the body, after it has passed to a new axis of instantaneous rotation.

11. Hence if a body once begin to roll, and we would calculate the force of the current just sufficient to keep it in motion, we may consider the fluid pressures as acting on every part of its surface, and our results will be very approximately true, independently of the nature of the surface over which the motion takes place, provided that surface be sufficiently firm to give the requisite support to the rolling body. The force, however, thus determined might be insufficient to make the body *begin* to move, since it might rest in such a position as to exclude the fluid action from its lower surface. But here it should be carefully observed, that a current is not to be deemed inefficient in moving blocks of given weight and form, unless it be capable of moving all such blocks; on the contrary, it is to be considered efficient for that purpose, if it be sufficient to move such of them as may exist under conditions most favorable for transport. In many cases the incipient motion might be due to accidental causes, as, for example, an impulsive blow from another mass already in motion; and, moreover, it is probable that all blocks which may have been transported by this agency to considerable distances, have been carried on by currents of considerably greater force than that just sufficient to keep them in motion, and which may have been sufficient without accidental causes to move them from rest, even under conditions not the most favorable for their movement.

The preceding remarks are of the first importance as removing all doubt and uncertainty with respect to the applicability of our calculated results to actual cases of transport by the agency of currents, whenever those results involve the hypothesis of the rolling motion of the transported mass. Transported bodies may have moved by rolling or by sliding; but in the latter case, the retarding action of friction and local obstacles introduces so uncertain an element as to render calculation comparatively useless; but if in calculating the force necessary to move a block of considerable magnitude, we assume it to have moved by rolling, we avoid in a great degree the uncertainty arising from the above causes, and are in no danger of assigning its transport to a force much less than that which has been actually required for that purpose.

We may now proceed to investigate the force which a current is capable of exerting on bodies of particular forms. It will be sufficient for our purpose to take a few prismatic bodies, of which the sections perpendicular to their axes are triangles, rectangular parallelograms, pentagons or hexagons. These cases will shew how the transporting power of a current, as estimated by the mass it is capable of moving, depends on the form of the mass; and will enable us to estimate, to a sufficient degree of approximation, the velocity of a current capable of moving any proposed erratic block.

12. If a plain surface, whose area $= S$ be placed at rest in a fluid, whose density is ρ_1, moving with a velocity v, in a direction making an angle θ with the plane, we shall have

$$R = \phi(\theta) \cdot \frac{v^2}{2} \rho_1 S \sin^2\theta,$$

R being the moving force of the current on the plane estimated in the direction perpendicular to the plane; and if R' be the resolved part of this force in the direction of the current,

$$R' = \phi(\theta) \cdot \frac{v^2}{2} \rho_1 S \sin^3\theta,$$

which will be the whole force in this direction, if we neglect the friction between the fluid and the plane.

When $\theta = \frac{\pi}{2}$, numerous experiments, made by different persons, shew that

$$R' = \frac{v^2}{2} \rho_1 S$$

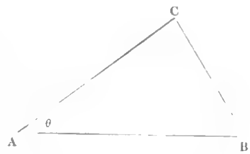
very approximately. The experiments have been made with different velocities up to 11 or 12 miles an hour, and we are justified in concluding by induction, that the expression will hold for still greater velocities. Hence $\phi\left(\frac{\pi}{2}\right) = 1$. It also results from experiment that the value of $\phi(\theta)$ is very nearly unity for all values of θ not exceeding 45°*, and therefore, since the applications I shall make of the above expressions are in cases where θ is less than that value, we may assume generally

$$R = \frac{v^2}{2} \rho_1 S \sin^2\theta,$$

and

$$R' = \frac{v^2}{2} \rho_1 S \sin^3\theta.$$

13. Let us first take the case of a prism, of which the axis is perpendicular to the current, and the section a triangle ABC.



If this section bisect the prism, it is manifest that the resultant of the whole pressure upon it produced by the current will pass through the middle point of AC. If therefore a perpendicular to AC through this middle point meet AB in B, or between A and B, it is manifest that the force of the current can have no tendency to make the prism turn over about the edge through B. Suppose the triangle equilateral; then on whichever side the prism may rest, the above perpendicular will pass through the opposite angular point, and the prism will not roll; and if the triangle be not equilateral, it is easily seen that there must necessarily be one side which, when the prism rests on it, will be met by the perpendicular. Consequently no triangular prism can continue to roll by the force of a current round each edge in succession.

To find under what conditions the prism will slide, I shall assume, as the most favorable condition for such motion, that the water has access to the lower side of the prism. In this case, taking ρ for the specific gravity of the prism, and ρ_1 for that of water, we shall have the weight of the body in water

$$= (\rho - \rho_1)\, g U,$$

U = volume of the prism, and g = accelerating force of gravity. Let $AB = a$, $AC = c$, the length of the prism $= b$, and $CAB = \theta$. Then if R = the normal force on the side of which AC is the section due to the current, and R' the horizontal force, we have (supposing θ not much less than 45°)

$$R = \frac{v^2}{2} \rho_1 S \sin^2\theta,$$

$$R' = \frac{v^2}{2} \rho_1 S \sin^3\theta;$$

or, since $S = bc$,

$$R = \frac{v^2}{2} \rho_1 bc \sin^2\theta,$$

* The most detailed experiments I have seen on this point are contained in a work, entitled *Nouvelles Experiences sur la Resistance des Fluides, par MM. D'Alembert, le Marquis de Condorcet, et l'Abbé Bossut, Membre de l'Academie des Sciences, &c. &c.* Par M. Bossut, Rapporteur, 1777. It was intended to appear in the Transactions of the Academy; but, on account of its length it was deemed better to publish it separately. When $\theta = 45^\circ$, these experiments give $\phi(\theta) = 1{,}08$, and values approximating to unity as their limit, for smaller values of θ. For greater values of θ, unity is no longer a near approximation to the value of $\phi(\theta)$.

$$R' = \frac{v^2}{2}\rho_1 bc \sin^3\theta.$$

Therefore, the vertical pressure on the base

$$= (\rho - \rho_1) gU + R\cos\theta$$

$$= \tfrac{1}{2}(\rho - \rho_1) g abc \sin\theta + \frac{v^2}{2}\rho_1 bc \sin^2\theta\cos\theta$$

$$= \tfrac{1}{2} bc \sin\theta \left\{(\rho - \rho_1) ga + v^2\rho_1 \sin\theta\cos\theta\right\}.$$

If we suppose the force (F) opposing the body's sliding to follow the ordinary law of friction, we shall have

$$F = \mu\ .\ \text{vertical pressure on the bottom,}$$

$$= \mu\left\{(\rho - \rho_1) gU + R\cos\theta\right\},$$

(where μ = coefficient of friction); and the condition of the prism being on the point of moving will be

$$R' = F.$$

Hence we obtain

$$\frac{v^2}{g}\rho_1 \sin^2\theta = \mu\left\{(\rho - \rho_1)a + \frac{v^2}{g}\rho_1\sin\theta\cos\theta\right\},$$

or,

$$\frac{v^2}{2g} = \frac{\frac{\mu}{2}\left(\frac{\rho}{\rho_1} - 1\right)a}{\sin\theta(\sin\theta - \mu\cos\theta)}.$$

This shews that a triangular prism with its axis perpendicular to the current cannot be moved by sliding unless $\tan\theta$ be $> \mu$, whatever be the velocity of the current.

If the section ABC be equilateral, $\theta = 60^\circ$, and we shall have

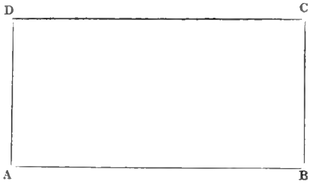
$$\frac{v^2}{2g} = \frac{2\mu\left(\frac{\rho}{\rho_1} - 1\right)}{\sqrt{3}(\sqrt{3} - \mu)}a.$$

If we take $\frac{\rho}{\rho_1} = 2{,}5$, which may be assumed as a mean value of that ratio, we shall have

$$a = \left(1 - \frac{\mu}{\sqrt{3}}\right)\frac{v^2}{2g}.$$

14. Let us now take the rectangular parallelopiped, of which $ABCD$ is the transverse section. Let $AB = a$, $AD = c$, and the length $= b$. Then

$$R' = \frac{v^2}{2}\rho_1 . bc;$$



and in order that the body may be on the point of rolling round the edge perpendicular to the plane of the paper through B, we must have

$$R'\frac{AD}{2} = (\rho - \rho_1) gU\frac{AB}{2};$$

$$\therefore\ \frac{v^2}{2}\rho_1 bc . \frac{c}{2} = (\rho - \rho_1) gabc\frac{a}{2},$$

$$\text{and } \frac{v^2}{2g} = \left(\frac{\rho}{\rho_1} - 1\right)\frac{a^2}{c}.$$

Let $c = na$, then

$$a = \frac{n}{\frac{\rho}{\rho_1} - 1} \cdot \frac{v^2}{2g}.$$

$$= \frac{n}{1,5} \cdot \frac{v^2}{2g}.$$

If the section be square, $n = 1$, and

$$a = \frac{1}{1,5} \cdot \frac{v^2}{2g}.$$

If the body be on the point of sliding, we must have

$$R' = \mu(\rho - \rho_1)gU,$$

or,

$$\frac{v^2}{2} \cdot \rho_1 \cdot bc = \mu(\rho - \rho_1)gabc,$$

$$\therefore a = \frac{1}{\mu\left(\frac{\rho}{\rho_1} - 1\right)} \cdot \frac{v^2}{2g}.$$

This value of a is less than the former one if

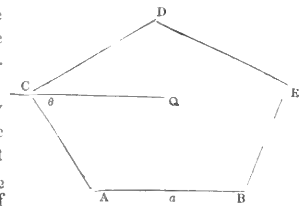
$$\mu > \frac{1}{n}.$$

If $n = 1$, or the section be a square, the condition becomes

$$\mu > 1.$$

If this hold, the body will roll rather than slide.

15. Let us next take a prism of which the transverse section is a pentagon.



Let the side of the pentagon $= a$, and the length of the prism $= b$; $ACQ = \theta$, CQ being horizontal; and let $R_1 R_2$ be the normal forces due to the current on AC, CD respectively.

To find the tendency of these forces to turn the body round an axis through B and perpendicular to the section, we observe that a perpendicular to CD through its middle point would pass through B, and consequently the moment of R_2 round the proposed axis will $= 0$. Also, since the direction of R_1 will bisect the angle at E, we shall have, when the prism is on the point of turning about the edge through B,

$$R_1\, a \sin\frac{BED}{2} = (\rho - \rho^1)Ug.\frac{a}{2},$$

or, substituting for R_1 and U,

$$\frac{v^2}{2}\rho_1 a^2 b \sin^2\theta \cos 36^{\circ} = \frac{5}{4}(\rho - \rho_1)\, g\, \frac{a^3 b}{2} \cot 36^{\circ}.$$

Therefore (since $\theta = 72^\circ$)

$$\frac{v^2}{2g} = \frac{5}{8}\left(\frac{\rho}{\rho_1} - 1\right)\frac{a}{\sin^2 72^0 \sin 36^0};$$

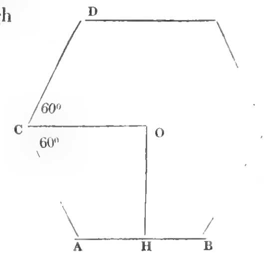
$$\therefore a = \frac{\sin^2 72^0 \sin 36^0}{\frac{5}{8}\left(\frac{\rho}{\rho_1} - 1\right)} \cdot \frac{v^2}{2g}$$

$$= ,567 \frac{v^2}{2g},$$

putting $\frac{\rho}{\rho_1} = 2,5$.

If we suppose the body on the point of sliding we find the value of a nearly equal to that just given, supposing $\mu = 1$.

16. Again, let the section of the prism be hexagonal. Let $AB = a$, and R' be the horizontal force of the current on the side AC = that on the side CD. Then when the body is on the point of turning about the side through B, we shall have



$$2 R'. HO = (\rho - \rho_1) g U \frac{a}{2}.$$

But

$$R' = \frac{v^2}{2} \rho_1 ab \sin^3 60,$$

and

$$U = 3 a . HO . b;$$

$$\therefore 2 \frac{v^2}{2} \rho_1 ab \sin^3 60 = \frac{3}{2}(\rho - \rho_1) g a^2 b,$$

and

$$a = \frac{4}{3} \cdot \frac{\sin^3 60^0}{\frac{\rho}{\rho_1} - 1} \cdot \frac{v^2}{2g}$$

$$= \frac{\sqrt{3}}{2\left(\frac{\rho}{\rho_1} - 1\right)} \cdot \frac{v^2}{2g}$$

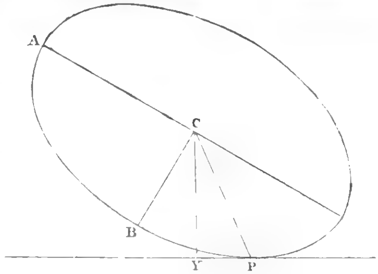
$$= ,57 . \frac{v^2}{2g} \quad \text{nearly.}$$

It will be observed that in all the preceding cases the results are independent of the lengths of the prisms, as they manifestly ought to be, since by changing the length of a prismatic body situated as above supposed, the mass and the force upon it are changed in the same ratio.

The tendency to roll as compared with that to slide is easily shewn to be greater in this than in the preceding cases; and if we take a prism of which the section should be a regular polygon of a still greater number of sides, the tendency to roll would be still greater. It is unnecessary to increase the number of examples of this kind; but there is another case somewhat different from the above which is deserving of notice.

17. Many of the erratic blocks which may be referred to the agency of currents are so rounded as to approximate more or less to the spherical form. Let APB represent the locus of those points on the surface of the body which come consecutively in contact with the ground in the

rolling movement, and which may be considered, for simplicity, as forming a plain curve. Moreover, since the greater and smaller axes of this curve will not differ much in magnitude, we may suppose it approximately to be an ellipse. Let its semi-major axis $AC = a$, and $BC = b$; $CP = r$, $CY = p$, CY being vertical, and therefore perpendicular to the horizontal tangent at P, the point of contact. The horizontal force of the current (R') will be approximately equal to that on a sphere whose radius $= a$, and its direction will pass nearly through C, which will also approximately coincide with the centre of gravity of the body. Hence when the body is in equilibrium in the position above represented, we shall have



$$R'p = (\rho - \rho_1) g U \sqrt{r^2 - p^2};$$

or,
$$R' = (\rho - \rho_1) g U \sqrt{\frac{r^2}{p^2} - 1};$$

and in order that R' may be just sufficient to make the body roll over, this equation must hold when the angle PCY is a maximum, *i.e.* when $\frac{p}{r}$ is a minimum. Now

$$p^2 = \frac{a^2 b^2}{(a^2 + b^2) - r^2},$$

$$\therefore \frac{p^2}{r^2} = \frac{a^2 b^2}{(a^2 + b^2) r^2 - r^4} = \text{min.}$$

which gives
$$r^2 = \frac{a^2 + b^2}{2},$$

and
$$p^2 = \frac{2a^2 b^2}{a^2 + b^2};$$

$$\therefore \frac{r^2}{p^2} - 1 = \frac{(a^2 + b^2)^2}{4a^2 b^2} - 1$$
$$= \frac{(a^2 - b^2)}{4a^2 b^2}.$$

Also
$$U = \frac{4}{3} \pi a^3 \text{ nearly,}$$

and
$$R' = \tfrac{1}{3} \pi a^2 . \frac{v^2}{2} \rho_1 \text{ nearly,}$$

as determined by experiment on the resistance on a sphere.

Hence the above equation becomes

$$\frac{1}{3} \pi a^2 . \frac{v^2}{2} \rho_1 = (\rho - \rho_1) g . \frac{4}{3} \pi a^3 . \frac{a^2 - b}{2ab},$$

$$\therefore \frac{v^2}{2g} = 2 \left(\frac{\rho}{\rho_1} - 1\right) a . \frac{(a + b)(a - b)}{ab}$$
$$= 4 \left(\frac{\rho}{\rho_1} - 1\right) (a - b) \text{ nearly;}$$

or if
$$a - b = \frac{a}{n},$$

$$a = \frac{n}{4\left(\frac{\rho}{\rho_1} - 1\right)} \cdot \frac{v^2}{2g}$$

$$= \frac{n}{6} \cdot \frac{v^2}{2g}.$$

18. To estimate numerically the dimensions and weights of the blocks which may be moved in each of the preceding cases by a current of given velocity, let us take $\frac{v^2}{2g} = 4$, which gives a velocity of about 16 feet per second, or about $10\frac{1}{2}$ miles an hour.

The *triangular prism* will never roll; when it is just on the point of sliding we shall have (the section being equilateral)

$$a = 4\left(1 - \frac{\mu}{\sqrt{3}}\right).$$

This involves the unknown quantity μ. Suppose the friction such that the body would just slide down the surface on which it rests, if that surface were inclined at an angle of 45^0 to the horizon; then $\mu = 1$, and

$$a = 1{,}68 \text{ feet}.$$

If $\mu = {,}5$

$$a = 2{,}84 \text{ feet}.$$

In the *parallelopiped* of which the section is a square, we have, when it is on the point of rolling,

$$a = 2\tfrac{2}{3} \text{ feet}.$$

If the length $= a$ the body will be a cube containing nearly 19 cubic feet, and weighing nearly $1\frac{1}{2}$ ton.

For the *pentagonal prism* on the point of rolling,

$$a = 2{,}268 \text{ feet}.$$

If the length of the prism be $2a$, the volume will be about 40 cubic feet, and the weight nearly 3 tons.

In the *hexagonal prism* on the point of rolling,

$$a = 2{,}28 \text{ feet}.$$

If the length of the prism $= 2a$, and therefore do not differ much from its height, its volume will be upwards of 60 cubic feet, and its weight between 4 and 5 tons.

When the body is approximately spherical, let $n = 4$; then

$$a = \tfrac{8}{3} \text{ feet}.$$

If we estimate the volume as equal to that of a sphere whose radius a' is a mean between a and b, we find

$$a' = \tfrac{7}{3} \text{ feet},$$

and the volume about 52 cubic feet. The weight will be about 4 tons.

If $n = 6$,

$$a = 4 \text{ feet},$$

$$a' = \tfrac{11}{12}\, a,$$

and the volume may be taken at nearly 200 cubic feet. Its weight will be 14 or 15 tons.

19. It will be observed in the expressions above given that the lines denoted by a vary, in every case, as v^2, and consequently the weight of the mass in each case, which varies as a^3, varies as v^6. Therefore *the moving force of a current estimated by the volume or weight of the mass of any proposed form which it is just capable of moving, varies as the sixth power of the velocity.*

This proposition may be easily proved independently of induction from particular cases.

Let a denote the length of any parameter in a proposed body of given form. Then, when v is given, the force (F) of the current, estimated as above, varies as the surface of the body, varies as a^2; and when the surface is given, the force varies as v^2. Therefore

$$F \propto a^2 v^2,$$

and the moment of F to make the body roll

$$\propto a^3 v^2$$

$$= C a^3 v^2 \quad (C = \text{constant}).$$

Also, the weight of the body $\propto a^3$, and its moment tending to keep the body at rest

$$\propto a^4$$

$$= C' a^4.$$

Hence, when the body is on the point of moving, we must have

$$C a^3 v^2 = C' a^4,$$

$$\therefore\ a = \frac{C}{C'} v^2,$$

$$\propto v^2,$$

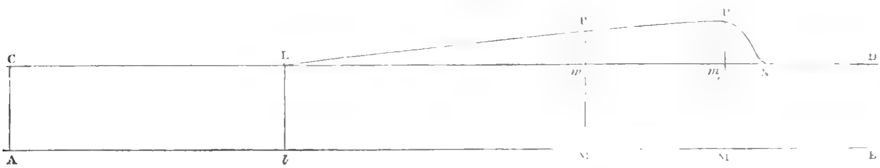
and the weight $\propto a^3 \propto v^6$; which proves the proposition.

This result shews how excessively erroneous an opinion we might form of the transporting power of rapid currents from that of the ordinary currents subjected to our observation. Thus if a stream of 10 miles an hour would just move a block of a certain form of 5 tons weight, a current of 15 miles an hour would move a block of similar form of upwards of 55 tons; and a current of 20 miles an hour would, according to the same law, move a block of 320 tons.

Again, according to the same law, a current of two miles an hour would move a pebble of similar form of only a few ounces in weight. And here it should also be remarked, that minute inequalities, or a want of perfect hardness in the bed of a current, which would produce little effect on the motion of a large block, would entirely destroy that of a small pebble; so that the circumstance of the transporting power of a stream of 2 or 3 miles an hour being inappreciable is perfectly consistent with the enormous power of rapid currents.

20. Let us now investigate the space through which a block might be conveyed by the current attending a single wave of elevation.

Let V be the velocity with which the wave is propagated.



v_1 the greatest velocity of the current, or its velocity in the transverse section through the crest or highest point of the wave, which will be very near the front of the wave, assuming it to have the character of a *bore*, as will necessarily be the case if the elevation producing it be paroxismal.

v the velocity of the current in any other section of the wave;

v_3 the velocity of a current just sufficient to move the block.

Let AB represent the surface on which the block rests, CD the general surface of the water and LPP_1N the wave, M the block at the time t, and P the point in which a vertical through M meets the surface of the wave.

MP and M_1P_1 are the sections in which the velocities are v and v_1 respectively.

Let $AM = x$, $Al = CL = x'$; then will

$$\frac{dx}{dt} = \text{vel. of the block,}$$

and

$$\frac{dx'}{dt} = \text{vel. of the wave} = V.$$

Also let v_2 be the velocity of a current just sufficient to move the block. Then, when the velocity of the current at the point where the block is situated becomes $= v_2$, the block will begin to move; and as the velocity of the current increases, that of the block will always very nearly = difference between the velocity of the current and that just necessary to move the block; so that we may consider the instantaneous velocity of the block as approximately $= v - v_2$. We shall then have

$$\frac{dx}{dt} = v - v_2,$$

or, substituting for v its value given by equation (1), (Art. 3.)

$$\frac{dx}{dt} = \frac{h}{H+h} \cdot V - v_2 \quad \text{.........(1).}$$

Also

$$\frac{dx'}{dt} = V \quad \text{.........(2);}$$

$$\therefore \frac{dx}{dx'} = \frac{h}{H+h} - \frac{v_2}{V} \quad \text{.........(3)}$$

h will be a function of $x - x'$ depending on the form of the wave. This form is not known, but as an approximation we may assume LP_1 to be a straight line; we shall then have

$$\frac{h}{h_1} = \frac{mP}{m_1P_1} = \frac{Lm}{Lm_1} = \frac{x - x'}{l},$$

l being the length of the wave to which Lm is very nearly equal. Therefore

$$\frac{l}{h_1} h = x - x' \quad \text{.........(4),}$$

and

$$\frac{dx}{dx'} = \frac{l}{h_1} \cdot \frac{dh}{dx'} + 1.$$

Hence, by substitution in (3) and reduction, we obtain

$$\frac{h_1}{l} \cdot \frac{dx'}{dh} = -\frac{V}{v_2} + \frac{V}{V+v_2} \cdot \frac{V}{v_2} \cdot \frac{H}{\frac{v_2}{V+v_2} . h + H};$$

and integrating,

$$\frac{h_1}{l} . x' = C - \frac{V}{v_2} h + \frac{V^2}{v_2^2} H . \log_e . \left(\frac{v_2}{V+v_2} . h + H\right).$$

Let a = the original distance of the block, and l = the length of the wave; then when $x' = a - l$, we shall have $h = h_1$. Therefore

$$\frac{x' - a + l}{l} = \frac{V}{v_2} \cdot \frac{(h_1 - h)}{h_1} - \frac{V^2}{v_2^2} \cdot \frac{H}{h_1} \cdot \log_e \frac{\dfrac{v_2}{V + v_2} h_1 + H}{\dfrac{v_2}{V + {}_2} h + H} \dots\dots\dots (5).$$

Equations (4) and (5) give the two relations between x, x' and h. Our object is to determine the value of x when the motion of the block ceases, when we have the condition $\frac{dx}{dt} = 0$, which gives from (1)

$$\frac{h}{H + h} V - v_2 = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (6).$$

From (4) (5) and (6) the required value of x can be determined, and thence $x - a$, the space through which the block will be transported, will be known.

Equation (6) gives

$$h = \frac{v_2}{V - v_2} . H;$$

and we have from Art. 3,

$$h_1 = \frac{v_1}{V - v_1} . H.$$

Also from (4),

$$x' = x - \frac{h}{h_1} . l.$$

Substituting this value of x' in (5) we obtain

$$\frac{x - a}{l} = \frac{V - v_2}{v_2} \left(1 - \frac{h}{h_1}\right) - \frac{V^2}{v_2^2} \cdot \frac{H}{h_1} \log_e \frac{1 + \dfrac{v_2}{V + v_2} \cdot \dfrac{h_1}{H}}{1 - \dfrac{v_2}{V + v_2} \cdot \dfrac{h}{H}}.$$

Since $\frac{h_1}{H}$ will always be less than unity and $\frac{v_2}{V}$ will generally be a small fraction, we shall obtain a near approximate value of $\frac{x - a}{l}$ if we expand the logarithm. We shall thus have, preserving terms of the second order,

$$\frac{x - a}{l} = - \frac{v_2}{V + v_2} \left(1 - \frac{h}{h_1}\right) + \frac{1}{2} \cdot \frac{V^2}{(V + v_2)^2} \cdot \frac{h_1^2 - h^2}{H h_1}$$

$$= - \frac{v_2}{V} \left(1 - \frac{h}{h_1}\right) + \frac{1}{2} \cdot \frac{h_1 + h}{H} \cdot \frac{h_1 - h}{h_1}.$$

Omitting $\frac{v_2}{V}$; and substituting the above values of h and h_1, we obtain finally

$$s = x - a = \frac{1}{2} \frac{(v_1 - v_2)^2}{(V + v_1) v_1} . l, \dots\dots\dots\dots\dots\dots\dots\dots\dots (7),$$

which gives the space through which the block will be transported.

If we put $v_2 = 0$ we have

$$s_0 = x - a = \frac{1}{2} \frac{v_1}{V + v_1} . l \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (8),$$

which gives the whole space through which each particle of the fluid is carried by the wave from its original position.

If $\frac{v_1}{V}$ be sufficiently small,

$$s = \frac{1}{2}\frac{(v_1 - v_2)^2}{V v_1} \cdot l, \qquad (9),$$

and

$$s_0 = \frac{1}{2} \cdot \frac{v_1}{V} \cdot l \quad \text{nearly}, \qquad (10)$$

$$= \frac{1}{2} \cdot \frac{h_1}{H} \cdot l \quad \text{approximately}.$$

I have supposed the section ($L\ P_1$) of the surface of the wave to be a straight line. It will generally be some curved line having its convexity turned upwards or downwards according to the nature of the disturbance in which the wave originates. In the former case, the value of s would be greater, and in the latter less than that here determined, which may therefore be considered as an approximation to the *mean* of the values of s for different waves, in which $v_1\ v_2\ V$ and l should be the same, but the original mode of disturbance, and therefore the form of the wave, different.

21. The following table exhibits numerical values of the velocity (V) with which the wave is propagated, of the maximum velocity (v_1) of the attendant current, and of the space (s) through which a block may be transported, for certain values of the original depth (H) of the water, of the height (h_1) of the wave, and of the velocity (v_2) of the current just sufficient to move the block. The values of H and h_1 are given in feet, those of V, v_1 and v_2 in miles, the velocities being estimated by the number of miles described in an hour; s is given in terms of l the breadth of the wave. The values of s are calculated from equation (9). The last column contains s_0 calculated from (10).

H	h_1	V	v_1	v_2	s	s_0
Feet.	Feet.	Miles.	Miles.	Miles.		
200	50	62	12	5	$\frac{l}{30}$ nearly	$\frac{l}{10}$ nearly
				10	$\frac{l}{372}$ ...	
300	50	73	10,4	5	$\frac{l}{52}$ ...	$\frac{l}{14}$ ...
300	100	77	19,4	5	$\frac{l}{14}$ ...	$\frac{l}{8}$ ...
				10	$\frac{l}{34}$ ...	
400	100	86	17	5	$\frac{l}{20}$ ...	$\frac{l}{10}$ ...
				10	$\frac{l}{60}$ ...	
450	150	95	23,5	10	$\frac{l}{24}$ ...	$\frac{l}{8}$ ...
450	150	106	20,5	10	$\frac{l}{39}$ ...	$\frac{l}{10}$ ...
500	200	150	28	10	$\frac{l}{26}$ ...	$\frac{l}{11}$ ...

SECTION III.

Application of the preceding Theory.—Comparison of different Modes of Transport.

22. In estimating the magnitude of a block which may be moved by a given current, the transport has been supposed to take place over a horizontal surface, sufficiently hard and even for the block to roll upon it without impediment. If the surface be otherwise constituted, the motion may be impeded or destroyed. The softness of a clayey surface would probably be most unfavourable to the motion; while the want of cohesion of a sandy bottom, from its opposing a less effective resistance to a motion rather by sliding than rolling, might be highly favourable to the transport of the block. In any case a constant action of denuding causes will be highly favourable to it, by the successive removal of temporary and local impediments. Abrupt inequalities, such for instance as those presented by ravines and steep escarpments, would present insuperable impediments to this mode of transport. It is important however to observe, that regular ascents, without rugged inequalities of surface, would offer no such serious impediment.

The difficulty in this theory arising from the presumed inequalities of the surface over which the blocks must have been transported, has been, I conceive, in many instances, far too much insisted on; for it has been made to rest on the assumption that the inequalities of surface between the present and original sites of erratic blocks were the same, or nearly so, at the time of transport as at present; an assumption which I regard as totally untenable. There are three obvious causes of inequality of surface—elevation and disruption, denudation during gradual emergence from beneath the ocean, and erosion after emergence. So far as sudden, abrupt inequalities can be traced to the first cause operating previously to the transport, the difficulty alluded to must be admitted; but in many cases existing inequalities have been produced by post-tertiary elevations, which we have no right to assume to have been entirely anterior to the transport of erratic blocks. Again, such great inequalities as those presented by the oolitic and chalk escarpments, have doubtless been due in a great measure to denudation, during the period of gradual emergence of the land, the higher levels being raised above the sphere of denuding action, while the lower levels remained exposed to it. Minor local irregularities of surface are also due in a great degree to erosion. All superficial inequalities, therefore, which are referrible to these causes, must have been posterior to the removal of erratic blocks transported by currents, and form no objection to that mode of transport. The only other causes which can materially affect the configuration of the terrestrial surface, are the deposition of new sedimentary beds, and denudation produced by ocean currents previously to any partial emergence of the surface. But it is manifest that both these causes, instead of creating those abrupt superficial inequalities, which alone would form a serious impediment to the transport we are considering, must constantly tend to destroy them wherever they may exist from other causes. For these reasons, I believe that there is no validity in the objection above stated to the theory of transporting currents. Those greater superficial inequalities which now exist, and are obviously referrible to denuding agencies, could not, I repeat, be the consequences of superficial denudation, while the whole surface was submerged beneath the ocean; and minor abrupt inequalities could not then have continued to exist, even if they had originally existed, for they would have been destroyed by the action of transporting currents themselves, though no other cause should have operated to produce that effect.

23. These currents, in addition to their transport of larger blocks, must manifestly tend to spread out the smaller detritus in a layer over the bottom of the ocean, supposed, for the reasons above stated, to form an even surface*. As the bottom rises in the process of slow elevation,

* In the sense for instance, in which the bottom of the German Ocean or English Channel is an even surface.

it will become exposed to all the action of denuding agents, which however will, in many instances, make less impression on those parts where the covering of detritus is thickest, or is composed of the coarsest materials. Such parts will therefore, *cæteris paribus*, emerge first from beneath the surface of the ocean; and thus, in the first instance, will form islands, and subsequently, when the whole shall have risen above the level of the sea, the summits of hills. Such summits may consequently be expected to be capped with transported materials, of which all traces may have been destroyed by denuding agents in the surrounding valleys. This phenomenon, of such frequent occurrence, is thus simply accounted for according to this theory.

24. It appears by the table above given, (Art. 21) that a wave of between 50 and 100 feet in height, (in an ocean of the original depth there supposed), would be accompanied with a current of which the velocity would be from 10 to 20 miles an hour; and it is demonstrated in the first section, that (under conditions which I conceive to be entirely admissible) currents of that velocity would possess a motive power abundantly sufficient to move the largest blocks, the transport of which it would be deemed necessary to refer to this cause. But I would particularly direct the attention of the reader to the fact, as exhibited in the values of s, in the table just referred to, that the space through which a block may be transported by a single wave, is equal only to a small fraction of the breadth of the wave. Consequently, a great number of waves might be necessary for the transport of blocks to distances to which they frequently have been transported. It must also be recollected, that sudden or paroxismal elevations only will produce waves of elevation of considerable transporting power. Hence it follows that this theory of transport is essentially and necessarily associated with that theory which regards the phenomena of elevation as the consequences of a series of paroxismal movements, the movements by which, in my opinion, those phenomena can be most satisfactorily accounted for. The instantaneous elevation of a determinate portion of the bottom of the sea would produce a wave whose height would be equal to that of the elevation itself, so that it may be asserted in general terms, that the theory of transport by elevation currents, in its application to existing phenomena of transport, involves the hypothesis of a succession of paroxismal movements beneath the ocean, the height of many of which must have varied from 50 to 100 feet at least.

25. If we allow the efficiency of each of the three recognized means of transport of erratic blocks—glaciers, floating ice, and currents—the difficulty which remains is that of separating the effects produced by these causes respectively. In some cases it is probable that doubt will always remain from insufficiency of evidence, but in others, I conceive, our conclusions may involve but little uncertainty. The distinctive characters in the transported materials must be sought in the magnitude and form of the blocks, the state of their surfaces, and the distribution of the general mass of the transported materials. The magnitude of a block can hardly be considered to increase the difficulty of its transport by ice, while it increases in a great degree the difficulty of transport by water. Again, blocks cannot generally be rounded by attrition when floated on icebergs or carried on the upper surface of a glacier. A small portion of those brought down by glaciers are rounded by being rolled between the ice and sides or bottom of the glacial valley; but this is a rough grinding, and all the specimens I recollect to have examined immediately at the termination of a glacier, wanted that more perfect smoothness of surface which distinguishes a water-worn boulder. It might be contended that blocks floated on icebergs might be rounded and polished before being taken up by the ice or after being deposited by it. If such were the case, the effects must be produced either on beaches by the action of breakers, or at the bottom of the sea by that of currents. The action of breakers, on large blocks, however, as far as my observation has extended, rarely tends to give to them a rounded form, but, on the contrary, to wear them into very irregular shapes, till they are so reduced in magnitude as to be rolled about by the force of the waves; the most prominent

points then become subject to the greatest attrition, and the surface afterwards assumes that form and polish which distinguish a water-worn boulder. I do not recollect, however, to have observed on any beach instances of this perfect rounding and polishing except in *pebbles*, much too small to afford any explanation of the cases of many of the erratic blocks which have been subjected to some similar and equally effective process of that kind. Moreover, should the efficiency of this cause be allowed, it must be recollected that the sphere of its operation is limited to the comparatively small area over which the waves *break*, for it is there alone that they can exert any effective power. How then shall we thus account for the water-worn appearance of innumerable blocks existing in the detritus spread out over a wide area, or in cases where the transported materials exist in layers of great thickness? If it should be contended that the water-worn appearance may be due to the other cause above alluded to—the action of water remote from shallow coasts—it must be replied, that that force which is capable of rolling a block is unquestionably sufficient to transport it, and therefore, that the solution does, in fact, admit the existence of transporting currents.

There is also another important point to be remarked with respect to the transport by ice, whether on land or by water—it affords no reason why the transported blocks should diminish in size, and become more generally rounded and polished, the more distant they are from their original localities. Such would necessarily be the consequences of transport by currents, but it must be a matter of indifference whether a block has been floated on an iceberg or carried by a glacier one mile or one hundred miles, so far as regards the form and dimensions of the block when ultimately deposited by the ice which conveyed it. If the great majority of the blocks transported from a given locality be rounded and polished, there is a strong presumption that water has been the transporting agent; if, moreover, the blocks do not exceed a weight of a few tons, the probability of that mode of transport is increased; and, finally, if we find that the magnitude of the blocks generally diminishes as their distance from their original site increases, till at length they degenerate into rounded pebbles, the previous probability appears to me to approximate as nearly to certainty as we can reasonably expect.

On the other hand, when erratic blocks are extremely large, the presumption is in favour of their having been transported by ice; and if, moreover, they retain sharp angular points and edges on their apparently unworn surfaces, and their magnitude bears no relation to the distance of transport, we may confidently conclude that the transporting agent has been ice, assuming always that the transport is attributable to one of the causes we have mentioned.

The main distinction between the cases of transport by glaciers and by floating ice, must be sought for, I conceive, in the distance which the blocks have travelled, and the nature of the surface over which the transport has taken place, and not in the character of the blocks themselves. If the motion of glaciers be due to gravity, as I have endeavoured to shew in a recent memoir, it would be an absurdity to attribute to their agency the transport of the blocks disseminated over the extensive flat plains of northern Germany and Russia. In such cases I should not hesitate to refer the removal of large angular blocks to the agency of floating ice. On the other hand, the transport of numerous blocks on the flanks of the Alpine chain can hardly be referred to any agency but that of glaciers of greater extent than those now existing. In other cases the transport may have been effected by a combination of these means. Blocks may have been brought down by glaciers from the mountains, and then floated on icebergs to distant localities. This process has been recently observed, on a magnificent scale, in a high northern latitude, and appears to me the simplest mode of accounting, in certain cases, for the transport of blocks now far above the level of the sea. If Switzerland were depressed 1600 or 1700 feet below its present level, the enormous angular block of *Pierre a bot* above Neuchatêl would be on the margin of an arm of the sea, occupying the present valley of Switzerland, while on the opposite margin there would be rocks bearing the strongest marks of glacial action. Under this hypothesis, and without assuming any material change in the general configuration of the surface, there remains no difficulty in accounting for the transport of the prodigious block above mentioned from the Alps to the Jura; a fact which on any other hypothesis hitherto made, presents, in my opinion, mechanical difficulties totally insurmountable. The supposition of an elevation of 1600 or 1700 feet since the period of transport offers, as I conceive, no *à priori* difficulty, when we recollect the evidences of recent elevation in other places. With conclusive evidence that Snowdon has been elevated 1200 or 1300 feet within a period which we have no reason for supposing more remote than that of the transport of erratic blocks, there can be little hesitation in admitting the elevation above supposed in the region of the Alps within the same period, as an hypothesis as probable at least as any other which might be adopted.

W. HOPKINS.

Cambridge,
April 29, 1844.

XVIII. *On the Foundation of Algebra*, No. IV., *on Triple Algebra*. *By* Augustus De Morgan, V.P.R.A.S., F.C.P.S., *of Trinity College; Professor of Mathematics in University College, London.*

[Read, *October* 28, 1844.]

In the *Philosophical Magazine* for July 1844, Sir William Rowan Hamilton has published the first part of a paper read before the Royal Irish Academy in November 1843, headed 'On Quaternions, or on a new System of Imaginaries in Algebra.' To this paper I am indebted for the idea of inventing a distinct system of unit-symbols, and investigating or assigning relations which define their mode of action on each other. The systems which I shall examine differ entirely from that of Sir William Hamilton, both as being triple instead of quadruple, and as preserving, in their laws of operation, a greater resemblance to those of ordinary Algebra.

§ 1. *Description of triple systems.* A system of Algebra of the n^{th} character is one in which there are n distinct symbols, $\xi_1, \xi_2, \ldots \xi_n$, each of which is a unit of its kind, of a difference from all other kinds such that $a_1\xi_1 + a_2\xi_2 + \ldots$ cannot be equivalent to $b_1\xi_1 + b_2\xi_2 + \ldots$ unless $a_1 = b_1$, $a_2 = b_2$, &c. This condition however is connected with the interpretation: a perfect symbolical system might very well exist without it. Having assumed a system, and also the ordinary laws of addition and subtraction, the introduction of the operation of multiplication requires that meanings should be assigned to $\xi_1\xi_2$, $\xi_1\xi_3$, &c., so that each of them may be regarded as coincident with such a form as $a_1\xi_1 + a_2\xi_2 + \ldots$. On the manner of assigning this form the properties of the system entirely depend; and if we are to preserve the ordinary rule of the convertibility of multiplications and divisions, we must not only provide that $\xi_1\xi_2 = \xi_2\xi_1$, &c., but also that $\xi_1^m\xi_2^n\xi_3^p \ldots$ shall give the same result in whatever order the operations are performed. This rule relative to multiplication may be reduced to two simple rules, $AB = BA$, and $A(BC) = (AB)C$. It is exactly the same thing as to additions, the convertibility of which is contained in the rules $A + B = B + A$ and $(A + B) + C = A + (B + C)$. This second rule is generally concealed in the common rule of signs, according to which $A + (B + C)$ or $A + (0 + B + C)$ is, by the assumed distributive character of the sign +, allowed to be transformed into $A + (+B) + (+C)$ which again by the rule of like signs, becomes $A + B + C$, a symbol identical in meaning with $(A + B) + C$. We might also use the signs × and ÷ in the same absolute manner, and assume a corresponding distributive character, and rule of like and unlike signs: considering $\times a$ and $\div a$ as abbreviations of $1 \times a$ and $1 \div a$. But it will be enough for my present purpose to note that the complete convertibility of multiplications will be secured if every triple combination, as $\xi_1\xi_2\xi_3$, $\xi_1^2\xi_2$, &c. has a meaning which is independent of the order of the operations.

Having settled the system, it must next be inquired, for the sake of the interpretation, what is the *modulus of multiplication*, namely, what function of a_1, a_2, &c. is it which, in the product, has the same value as the product of the functions of the factors. If, agreeably to the laws of the system, the product of $a_1\xi_1 + a_2\xi_2 + \ldots$ and $a'_1\xi_1 + a'_2\xi_2 + \ldots$ be $A_1\xi_1 + A_2\xi_2 + \ldots$, A_1, A_2, &c. being definite functions of a_1, a'_1, a_2, a'_2, &c., the modulus is to be found from the solution of the functional equation

$$\phi(a_1, a_2, \ldots) \times \phi(a'_1, a'_2, \ldots) = \phi(A_1, A_2, \ldots),$$

on which it is only to be observed that any powers, or products, or products of powers, of solutions, are themselves solutions. The most convenient modulus is that which, in one or more definite cases, reduces the system to the simple single or double Algebra already in use. In this common Algebra, in its widest form, there are two unit-symbols, say ξ and η, usually (not necessarily) representing units of length taken off on the rectangular axes of x and y; and the laws of combination are $\xi^2 = \xi$, $\eta^2 = -\xi$, $\xi\eta = \eta\xi = \eta$, which give $\xi\eta^2 = -\xi = (\xi\eta)\eta$, &c. The modulus of multiplication of $a\xi + b\eta$ is $\sqrt{(a^2+b^2)}$. Sir William Hamilton seems to have passed over triple Algebra altogether on the supposition that the modulus, if any, of $a\xi + b\eta + c\zeta$ must be $\sqrt{(a^2+b^2+c^2)}$. It is certain* that there cannot be a system of triple Algebra with such a modulus; but it is by no means requisite that the modulus should be a symmetrical function of a, b, and c. I should also notice that in Sir W. Hamilton's quadruple Algebra there is a complete departure from the ordinary symbolical rules: *AB* and *BA* have different meanings.

§ 2. *One mode of derivation of systems of triple Algebra.* Let $a\xi$, $b\eta$, $c\zeta$, represent lines of a, b, and c units measured on the axes of x, y and z. Let it be a condition that $b = 0$, $c = 0$, reduces the Algebra to the common single system; which might be worded thus: let the Algebra of the axis of x be the common single Algebra of positive and negative quantities. Also let η and ζ be interchangeable, and related in the same manner to ξ. We have then, for the forms which define the *actions* of the unit-symbols on each other,

$$\begin{array}{llll} \xi^2 & \text{means } \xi, & \eta\zeta & \text{means } p\xi + q\eta + q\zeta, \\ \eta^2 & \ldots\ldots\ a\xi + b\eta + c\zeta, & \zeta\xi & \ldots\ldots\ l\xi + m\eta + n\zeta, \\ \zeta^2 & \ldots\ldots\ a\xi + c\eta + b\zeta. & \xi\eta & \ldots\ldots\ l\xi + n\eta + m\zeta; \end{array}$$

and it will be found upon examination that the equations $\xi^2\eta = \xi(\xi\eta)$, $\xi\eta^2 = \eta(\xi\eta)$, $\eta^2\zeta = \eta(\eta\zeta)$, $\eta\zeta^2 = \zeta(\eta\zeta)$, $\zeta^2\xi = \xi(\zeta\xi)$, $\zeta\xi^2 = \xi(\zeta\xi)$, $\xi(\eta\zeta) = \eta(\zeta\xi) = \zeta(\xi\eta)$, will be satisfied by the following conditions; in using which care must be taken not to form new ones by introduction of subsequently vanishing factors without recurring to the original forms. Some of these conditions are included in the others, but it is nevertheless desirable to be reminded of them.

$$\begin{array}{llll} (1.) & a(q-c) + p(q-b) = l(a-p). & (4.) & l(m+n) = 0. \\ (2.) & l^2 + mp + na = a + (b+c)l. & (5.) & 2mn = m. \\ (3.) & l^2 + ma + np = p + 2ql. & (6.) & m^2 + n^2 = n. \end{array}$$

$$\begin{array}{llll} (7,\ 8.) & ln = (q-b)m = (c-q)m. & (11.) & (q+c)(q-c) = am - pn. \\ (9,\ 10.) & lm = (q-c)m = (b-q)m. & (12.) & (q+c)(q-b) = an - pm. \end{array}$$

From (5.) and (6.) we have either

$$m = 0, \quad n = 0; \quad m = 0, \quad n = 1; \quad m = \tfrac{1}{2}, \quad n = \tfrac{1}{2}; \quad m = -\tfrac{1}{2}, \quad n = \tfrac{1}{2}.$$

Proceeding by analogy, we might expect the triple Algebra which is the proper extension of the common double one to give $\eta^3 = -\xi$, $\zeta^3 = -\xi$, the necessary conditions of which are

$$(13.)\quad al + ab + cp = -1. \qquad (14.)\quad an + b^2 + cq = 0. \qquad (15.)\quad am + bc + cq = 0.$$

* Any one who will try to get three squares in which accented and unaccented letters enter symmetrically, and of which the sum is equal to the product of $a^2+b^2+c^2$ and $a'^2+b'^2+c'^2$ is engaged, whether he know it or not, upon the following problem;—To find three points of a sphere, each of which is opposite to both of the other two; also three other points each distant by a quadrant from each of the first three.

But at the same time it is desirable to examine the case of $\eta^2 = -\xi$, $\zeta^2 = -\xi$, the conditions of which are $a = -1$, $b = 0$, $c = 0$. These two systems may be called the simple cubic and quadratic systems, both being *triple*. I now proceed to a mere enumeration of cases to be presently discussed.

Case A. Let $m = 0$, $n = 0$; which gives either of the following

$$\begin{array}{ll|ll}
(A_1) & m = 0,\ n = 0,\ q = -c, & (A_2) & m = 0,\ n = 0,\ q = c = b, \\
 & -2ac - p(c+b) = l(a-p), & & a = p, \\
 & l^2 = a + (b+c)l, & & l^2 = a + 2bl. \\
 & l^2 = p - 2cl. & &
\end{array}$$

Neither gives a simple quadratic form, unless $l^2 = -1$, which is inadmissible.

Simple cubic forms are only such as are contained in

$$b = c = -q, \quad m = 0, \quad n = 0,$$

$$al + b(a+p) = -1, \quad 2b(a+p) = -l(a-p),$$

$$l^2 = a + 2bl = p - 2bl,$$

which give $p = a = 1$, $l = -1$, $b = 0$.

Case B. Let $m = 0$, $n = 1$. We have then

$$\begin{array}{ll}
\xi^2 = \xi, & \eta\zeta = -(q^2 - c^2)\xi + q(\eta + \zeta), \\
\eta^2 = (q+c)(q-b)\xi + b\eta + c\zeta, & \zeta\xi = \zeta, \\
\zeta^2 = (q+c)(q-b)\xi + c\eta + b\zeta. & \xi\eta = \eta.
\end{array}$$

This is the case, and the only one, in which the action of ξ upon both of the others is imperceptible. The following cases will be considered, the first of which is a species of simple quadratic form, the second a simple cubic, the only one which the case yields.

$$\begin{array}{ll|ll}
\xi^2 = \xi, & \eta\zeta = \xi, & \xi^2 = \xi, & \eta\zeta = \xi, \\
\eta^2 = -\xi + \eta + \zeta, & \zeta\xi = \zeta, & \eta^2 = -\zeta, & \zeta\xi = \zeta, \\
\zeta^2 = -\xi + \eta + \zeta. & \xi\eta = \eta. & \zeta^2 = -\eta. & \xi\eta = \eta.
\end{array}$$

Case C. Let $m = \tfrac{1}{2}$, $n = \tfrac{1}{2}$. This gives $l = 0$, $q = b = c$, $a = p$. The only simple quadratic and cubic forms are as follows:

$$\begin{array}{ll|ll}
\xi^2 = \xi, & \eta\zeta = -\xi, & \xi^2 = \xi, & \eta\zeta = -1 + \tfrac{1}{2}\eta + \tfrac{1}{2}\zeta. \\
\eta^2 = -\xi, & \zeta\xi = \tfrac{1}{2}\eta + \tfrac{1}{2}\zeta, & \eta^2 = -\xi + \tfrac{1}{2}\eta + \tfrac{1}{2}\zeta, & \zeta\xi = \tfrac{1}{2}\eta + \tfrac{1}{2}\zeta. \\
\zeta^2 = -\xi. & \xi\eta = \tfrac{1}{2}\eta + \tfrac{1}{2}\zeta. & \zeta^2 = -\xi + \tfrac{1}{2}\eta + \tfrac{1}{2}\zeta. & \xi\eta = \tfrac{1}{2}\eta + \tfrac{1}{2}\zeta.
\end{array}$$

Case D. Let $m = -\tfrac{1}{2}$, $n = \tfrac{1}{2}$. The equations of condition are reducible to

$$q = \frac{b+c}{2}, \qquad l = \frac{b-c}{2}, \qquad (b+3c)(b-c) = -2(a+p).$$

The simple quadratic and cubic forms are

$$\begin{array}{ll|ll}
\xi^2 = \xi, & \eta\zeta = \xi, & \xi^2 = \xi, & \eta\zeta = \xi, \\
\eta^2 = -\xi, & \zeta\xi = -\tfrac{1}{2}\eta + \tfrac{1}{2}\zeta, & \eta^2 = -\tfrac{1}{2}\xi + \tfrac{1}{2}\eta - \tfrac{1}{2}\zeta. & \zeta\xi = \tfrac{1}{2}\xi - \tfrac{1}{2}\eta + \tfrac{1}{2}\zeta, \\
\zeta^2 = -\xi. & \xi\eta = \tfrac{1}{2}\eta - \tfrac{1}{2}\zeta. & \zeta^2 = -\tfrac{1}{2}\xi - \tfrac{1}{2}\eta + \tfrac{1}{2}\zeta. & \xi\eta = \tfrac{1}{2}\xi + \tfrac{1}{2}\eta - \tfrac{1}{2}\zeta.
\end{array}$$

§ 3. *Simple and perfect cubic form.* I now proceed to consider the simple cubic form in case B. The equations of signification* are (dropping the distinctive symbol ξ, which is inoperative),

$$\eta^2 = -\zeta, \quad \zeta^2 = -\eta, \quad \eta\zeta = 1.$$

And the product of $a + b\eta + c\zeta$ and $a' + b'\eta + c'\zeta$ is

$$bc' + cb' + aa' + (ab' + ba' - cc')\,\eta + (ac' + ca' - bb')\,\zeta.$$

If the equations of signification be also consistently algebraical, and if $\eta = \mu$ and $\zeta = \nu$ satisfy them, then $a + b\mu + c\nu$ is a modulus of multiplication. Accordingly in the present instance, it is sufficient that μ and ν should be severally equal to -1, or else that they should be the imaginary cube roots of -1. Let them be the latter: then $a - b - c$, $a + \mu b + \nu c$, $a + \nu b + \mu c$, are moduli, and since any product of roots of moduli is a modulus, we have, taking such roots as are required by the condition that the Algebra is to become single if b and c always vanish, the following possible moduli,

$$a - b - c,$$

$$\sqrt{(a^2 + b^2 + c^2 + ab + ac - bc)},$$

$$\sqrt[3]{(a^3 - b^3 - c^3 - 3abc)}.$$

These expressions are connected with the third degree in the same manner as $a^2 + b^2$ with the second. Changing the signs of b and c &c., their modular character gives the following equations. Let

$$A = bc' + cb' + aa', \quad B = ab' + ba' + cc', \quad C = ac' + ca' + bb'.$$

Then $(a + b + c)(a' + b' + c') = A + B + C$

$$(a^2 + b^2 + c^2 - ab - bc - ca)(a'^2 + b'^2 + c'^2 - a'b' - b'c' - c'a') = A^2 + B^2 + C^2 - AB - BC - CA$$

$$(a^3 + b^3 + c^3 - 3abc)(a'^3 + b'^3 + c'^3 - 3a'b'c') = A^3 + B^3 + C^3 - 3ABC.$$

These might, I think, be made of the same sort of use in the theory of numbers with the equation $(a^2 + b^2)(a'^2 + b'^2) = (aa' - bb')^2 + (ab' + ba')^2$, which is the modular equation of the common Algebra. Thus of either of the forms $a^2 + b^2 + c^2 - ab - bc - ca$ and $a^3 + b^3 + c^3 - 3abc$ we may say that the product of two instances must be a third instance.

It appears that this cubic form of triple algebra may involve three cases, according to the modulus which we employ. Now we know that in common Algebra, $a + b\sqrt{-1}$ is made to depend upon a length and an angle, in such a manner that the length is represented by the modulus, and the product of two expressions has the product of the lengths for a length, and the sum of the angles for an angle. Suppose that we make $a + b\eta + c\zeta$ to depend upon the modulus and two angles, each having the same property as the angle of the former case: it is required to express $a + b\eta + c\zeta$ by $[l, \theta, \phi]$ in such manner that the following equation may be identically true,

$$[l, \theta, \phi] \cdot [l', \theta', \phi'] = [ll', \theta + \theta', \phi + \phi'].$$

Without as yet specifying which modulus we are to take, we must examine into the conditions of a species of *triple trigonometry*, in which two angles form the base of every expression. Looking at the form of the product of $a + b\eta + c\zeta$ and $a' + b'\eta + c'\zeta$, it is obvious that the problem is solved if we can assign

$$A_{\theta,\phi} = \frac{a}{l}, \quad B_{\theta,\phi} = \frac{b}{l}, \quad C_{\theta,\phi} = \frac{c}{l},$$

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* In this sense it ought to be remembered that they more resemble $- \times - = +$ than $ab = c$.

in such manner as to satisfy

$$\begin{aligned} A_{\theta+\mu,\phi+\nu} &= B_{\theta,\phi}C_{\mu,\nu} + B_{\mu,\nu}C_{\theta,\phi} + A_{\theta,\phi}A_{\mu,\nu}, \\ B_{\theta+\mu,\phi+\nu} &= B_{\theta,\phi}A_{\mu,\nu} + B_{\mu,\nu}A_{\theta,\phi} - C_{\theta,\phi}C_{\mu,\nu}, \ldots\ldots\ldots\ldots\ldots\ldots(M). \\ C_{\theta+\mu,\phi+\nu} &= C_{\theta,\phi}A_{\mu,\nu} + C_{\mu,\nu}A_{\theta,\phi} - B_{\theta,\phi}B_{\mu,\nu}. \end{aligned}$$

Here $A_{\theta,\phi}$ is a species of cosine of (θ, ϕ), and $B_{\theta,\phi}$ and $C_{\theta,\phi}$ are two different species of sines. The second sides of (M) must admit the interchange of θ and μ, and also of ϕ and ν. That this and all other conditions of self-consistence are satisfied, will appear as follows. We have

$$\begin{aligned} A_{\theta,\nu} &= B_{\theta,0}C_{0,\nu} + B_{0,\nu}C_{\theta,0} + A_{\theta,0}A_{0,\nu}, \\ B_{\theta,\nu} &= B_{\theta,0}A_{0,\nu} + B_{0,\nu}A_{\theta,0} - C_{\theta,0}C_{0,\nu}, \\ C_{\theta,\nu} &= C_{\theta,0}A_{0,\nu} + C_{0,\nu}A_{\theta,0} - B_{\theta,0}B_{0,\nu}. \end{aligned}$$

Again, $A_{\theta+\mu,\phi+\nu} = B_{\theta+\mu,0}C_{0,\phi+\nu} + B_{0,\phi+\nu}C_{\theta+\mu,0} + A_{\theta+\mu,0}A_{0,\phi+\nu}$

$$\begin{aligned} &= (B_{\theta,0}A_{\mu,0} + B_{\mu,0}A_{\theta,0} - C_{\theta,0}C_{\mu,0})(C_{0,\phi}A_{0,\nu} + C_{0,\nu}A_{0,\phi} - B_{\theta,\phi}B_{0,\nu}) \\ &+ (B_{0,\phi}A_{0,\nu} + B_{0,\nu}A_{0,\phi} - C_{0,\phi}C_{0,\nu})(C_{\theta,0}A_{\mu,0} + C_{\mu,0}A_{\theta,0} - B_{\theta,0}B_{\mu,0}) \\ &+ (B_{\theta,0}C_{\mu,0} + B_{\mu,0}C_{\theta,0} + A_{\theta,0}A_{\mu,0})(B_{0,\phi}C_{0,\nu} + B_{0,\nu}C_{0,\phi} + A_{0,\phi}A_{0,\nu}). \end{aligned}$$

Develope these products, and the results will be seen to be identical with

$$\begin{aligned} &(B_{\theta,0}A_{0,\phi} + B_{0,\phi}A_{\theta,0} - C_{\theta,0}C_{0,\phi})(C_{\mu,0}A_{0,\nu} + C_{0,\nu}A_{\mu,0} - B_{\mu,0}B_{0,\nu}) \\ &+ (B_{\mu,0}A_{0,\nu} + B_{0,\nu}A_{\mu,0} - C_{\mu,0}C_{0,\nu})(C_{\theta,0}A_{0,\phi} + C_{0,\phi}A_{\theta,0} - B_{\theta,0}B_{0,\phi}) \\ &+ (B_{\theta,0}C_{0,\phi} + B_{0,\phi}C_{\theta,0} + A_{\theta,0}A_{0,\phi})(B_{\mu,0}C_{0,\nu} + B_{0,\nu}C_{\mu,0} + A_{\mu,0}A_{0,\nu}), \end{aligned}$$

which is $\qquad B_{\theta,\phi}C_{\mu,\nu} + B_{\mu,\nu}C_{\theta,\phi} + A_{\theta,\phi}A_{\mu,\nu}$.

The other equations may be treated in the same way.

I am able to find the solutions of all three varieties of this system by means of that in which the modulus is $\sqrt{(a^2 + b^2 + c^2 + ab + ac - bc)}$; in which case the equation answering to $\sin^2\theta + \cos^2\theta = 1$ in common trigonometry is

$$A^2_{\theta,\phi} + B^2_{\theta,\phi} + C^2_{\theta,\phi} + A_{\theta,\phi}B_{\theta,\phi} + A_{\theta,\phi}C_{\theta,\phi} - B_{\theta,\phi}C_{\theta,\phi} = 1.$$

We have

$$a^2 + b^2 + c^2 + ab + ac - bc = \left(a + \frac{b+c}{2}\right)^2 + 3\left(\frac{b-c}{2}\right)^2.$$

Assume $\qquad A_{\theta,\phi} + \tfrac{1}{2}(B_{\theta,\phi} + C_{\theta,\phi}) = \cos\theta, \qquad \tfrac{1}{2}(B_{\theta,\phi} - C_{\theta,\phi}) = \dfrac{\sin\theta}{\sqrt{3}}.$

Then we must have equations of the following form

$$\begin{aligned} A_{\theta,\phi} &= \cos\theta + L_{\theta,\phi}, \\ B_{\theta,\phi} &= \frac{\sin\theta}{\sqrt{3}} - L_{\theta,\phi}, \\ C_{\theta,\phi} &= -\frac{\sin\theta}{\sqrt{3}} - L_{\theta,\phi}. \end{aligned}$$

Substitute these values in the first of equations (M), and we have

$$L_{\theta+\mu,\phi+\nu} = 3L_{\theta,\phi}L_{\mu,\nu} + \cos\theta\, L_{\mu,\nu} + \cos\mu\, L_{\theta,\phi} + \tfrac{1}{3}\sin\theta\sin\mu.$$

Assume $L_{\theta,\phi} = \frac{1}{3}(P_{\theta,\phi} - \cos\theta)$ which gives $P_{\theta+\phi,\phi+\nu} = P_{\theta,\phi}P_{\mu,\nu}$ the only solution of which is $P_{\theta,\phi} = \epsilon^{\alpha\theta+\beta\phi}$, giving

$$A_{\theta,\phi} = \tfrac{2}{3}\cos\theta + \tfrac{1}{3}\epsilon^{\alpha\theta+\beta\phi},$$

$$B_{\theta,\phi} = \tfrac{1}{3}\cos\theta + \frac{1}{\sqrt{3}}\sin\theta - \tfrac{1}{3}\epsilon^{\alpha\theta+\beta\phi},$$

$$C_{\theta,\phi} = \tfrac{1}{3}\cos\theta - \frac{1}{\sqrt{3}}\sin\theta - \tfrac{1}{3}\epsilon^{\alpha\theta+\beta\phi}.$$

This gives

$$A_{\theta,\phi} - B_{\theta,\phi} - C_{\theta,\phi} = \epsilon^{\alpha\theta+\beta\phi},$$

$$A^3_{\theta,\phi} - B^3_{\theta,\phi} - C^3_{\theta,\phi} - 3A_{\theta,\phi}B_{\theta,\phi}C_{\theta,\phi} = \epsilon^{\alpha\theta+\beta\phi}.$$

We can now get solutions on the supposition that the other moduli are used. If we take $l = a - b - c$, we have

$$A_{\theta,\phi} = \tfrac{2}{3}\cos\theta\,.\,\epsilon^{-(\alpha\theta+\beta\phi)} + \tfrac{1}{8},$$

$$B_{\theta,\phi} = \left(\frac{\cos\theta}{3} + \frac{\sin\theta}{\sqrt{3}}\right)\epsilon^{-(\alpha\theta+\beta\phi)} - \tfrac{1}{3},$$

$$C_{\theta,\phi} = \left(\frac{\cos\theta}{3} - \frac{\sin\theta}{\sqrt{3}}\right)\epsilon^{-(\alpha\theta+\beta\phi)} - \tfrac{1}{3}.$$

But if we use $\sqrt[3]{(a^3 - b^3 - c^3 - 3abc)}$, we have

$$A_{\theta,\phi} = \tfrac{2}{3}\cos\theta\,.\,\epsilon^{-\frac{1}{3}(\alpha\theta+\beta\phi)} + \tfrac{1}{3}\epsilon^{\frac{2}{3}(\alpha\theta+\beta\phi)},$$

$$B_{\theta,\phi} = \left(\frac{\cos\theta}{3} + \frac{\sin\theta}{\sqrt{3}}\right)\epsilon^{-\frac{1}{3}(\alpha\theta+\beta\phi)} - \tfrac{1}{3}\epsilon^{\frac{2}{3}(\alpha\theta+\beta\phi)},$$

$$C_{\theta,\phi} = \left(\frac{\cos\theta}{3} - \frac{\sin\theta}{\sqrt{3}}\right)\epsilon^{-\frac{1}{3}(\alpha\theta+\beta\phi)} - \tfrac{1}{3}\epsilon^{\frac{2}{3}(\alpha\theta+\beta\phi)}.$$

We must remember that, of any two solutions of (M), either must be the other multiplied by a solution of $P_{\theta+\mu,\phi+\nu} = P_{\theta,\phi}P_{\mu,\nu}$; and any solution of ($M$) multiplied by one of the last is also a solution of (M). And the form of the solutions might be generalized, but in appearance only, by writing $\cos(\alpha'\theta + \beta'\phi)$ and $\sin(\alpha'\theta + \beta'\phi)$ for $\cos\theta$ and $\sin\theta$. But by the same consideration it appears that the system is not less complete if we write ϕ for $\alpha\theta + \beta\phi$. Adopting this simplification, the equations of connexion between a &c. and l &c., are at full length as follows:

$$l = \sqrt{(a^2 + b^2 + c^2 + ab + ac - bc)},$$

$$a = l\left\{\tfrac{2}{3}\cos\theta + \tfrac{1}{3}\epsilon^{\phi}\right\}, \qquad a + \tfrac{1}{2}(b + c) = l\cos\theta,$$

$$b = l\left\{\tfrac{1}{3}\cos\theta + \frac{1}{\sqrt{3}}\sin\theta - \tfrac{1}{3}\epsilon^{\phi}\right\}, \qquad \frac{\sqrt{3}}{2}(b - c) = l\sin\theta,$$

$$c = l\left\{\tfrac{1}{3}\cos\theta - \frac{1}{\sqrt{3}}\sin\theta - \tfrac{1}{3}\epsilon^{\phi}\right\}, \qquad a - (b + c) = l\epsilon^{\phi}.$$

From these premises it follows that the product of $a + b\eta + c\zeta$ and $a' + b'\eta + c'\zeta$, or of $[l, \theta, \phi]$ and $[l', \theta', \phi']$ is $[ll', \theta + \theta', \phi + \phi']$. And it is certain that this is the only simple cubic system, except that noted under case A, which as will afterwards be seen, is deceptive: also that this is the only case of that system in which $l = \sqrt{(a^2 + \&c.)}$, the equations (M) admitting no other solution with that modulus.

We now come to the question of geometrical interpretation, the most difficult part of the question in one sense, the easiest in another. Every system of Algebra admits of an infinite number of geometrical interpretations. Take the common one, and instead of supposing $x + y\sqrt{-1}$ to stand for a line $r = \sqrt{(x^2 + y^2)}$ inclined to the axis of x at an angle $\theta = \tan^{-1}(y : x)$, let it stand for any line r_1 inclined at an angle θ_1, where r_1 and θ_1 are unambiguous functions of r and θ. Then the sign $+$ in $[r_1, \theta_1] + [r'_1, \theta'_1]$ must be defined in such a way that the preceding symbol may stand for the line determined by $r = \sqrt{\{(x + x')^2 + (y + y')^2\}}$ and $\tan\theta = (y + y') : (x + x')$; and similarly with the other signs. There is no question about the superior convenience and primary character of the usual interpretation: but others are not therefore absolutely excluded.

Analogy would lead us to infer that a, b, c should represent lines on the axes of x, y, z; and even if we took them to represent areas on the planes of yz, zx, and xy, we should be able to determine an area on the plane of yz (its form not being in question) by a line on the axis of x. Again, the same analogy would lead us to take l for the absolute length of $a + b\eta + c\zeta$: but all that is necessary is that l, θ, and ϕ should be sufficient determinants of that length. For instance, we may say, let $a + b\eta + c\zeta$ represent a length $r = \sqrt{(a^2 + b^2 + c^2)}$ inclined to the axes at angles having cosines λ, μ, ν, proportional to a, b, c: but then we give up the convenient property of the modulus of multiplication, and must form $(R, \Lambda, \mathrm{M}, \mathrm{N})$ the product of (r, λ, μ, ν) and $(r', \lambda', \mu', \nu')$ from the conditions

$$R\cos\Lambda = rr'(\mu\nu' + \nu\mu' + \lambda\lambda'),$$

$$R\cos\mathrm{M} = rr'(\lambda\mu' + \lambda'\mu - \nu\nu'),$$

$$R\cos\mathrm{N} = rr'(\lambda\nu' + \lambda'\nu - \mu\mu'),$$

so that R must depend on the angles of the factors as well as on their lengths. The systems I have given are the only ones in which the moduli represent the absolute magnitude of the symbols.

I am not able to present any striking geometrical interpretation. The symbols of the triple trigonometry on which it must be founded are mixed functions of circular and hyperbolic sines and cosines. If we take the equilateral hyperbola $x^2 - y^2 = 1$, and let x and y be called the hyperbolic sine and cosine of ϕ, the double of the sectorial area included between the axis of x, the radius vector, and the curve (for analogy, the angle must be replaced by the double of the area of a circular sector of radius unity), we have $\epsilon^\phi = \mathrm{COS}\,\phi + \mathrm{SIN}\,\phi$, using capital letters for distinction. We might very easily invent interpretations: but I see none which I think worth presenting. The transformation

$$\frac{\cos\theta}{3} \pm \frac{1}{\sqrt{3}}\sin\theta = \tfrac{2}{3}\cos(60^\circ \mp \theta)$$

will of course not be forgotten by any one who makes an attempt. This entrance of both species of sines and cosines is, both in this and other cases, the consequence of the determination to have what may be called a doubly logarithmic system, or one in which both angles, or magnitudes corresponding to them, have their sums in the product.

We may, if we like, consider the system as one in which there is a double modulus of multiplication; let $l \cdot \epsilon^\phi = m$, and we have

$$l = \sqrt{(a^2 + b^2 + c^2 + ab + ac - bc)}, \qquad m = a - b - c,$$

$$a = \tfrac{2}{3}l\cos\theta + \tfrac{1}{3}m, \qquad a + \tfrac{1}{2}(b + c) = l\cos\theta,$$

$$b = \tfrac{2}{3}l\cos(60^\circ - \theta) - \tfrac{1}{3}m, \qquad \tfrac{1}{2}\sqrt{3}\,.\,(b - c) = l\sin\theta.$$

$$c = \tfrac{2}{3}l\cos(60^\circ + \theta) - \tfrac{1}{3}m.$$

The product of $[l, m, \theta)$ and $[l', m', \theta']$ is now $[ll', mm', \theta + \theta']$.

The three axes on which a, b, c, are laid down, ought not to be rectangular axes, but those of y and z should be each inclined at 60^0 to the axis of x, so that units laid down on them may be cube roots of -1. The planes of xy and xz being at right angles, and Δ being the diagonal of the parallelepiped on a, b, c, we have $l^2 = \Delta^2 - \frac{2}{3}bc$.

Should a simple interpretation be obtained, the ancient difficulty of the imaginary quantity will immediately occur; for $\sqrt{m}$ must take the place of m in $\sqrt{[l, m, \theta]}$, and m may be negative. This system therefore will never be completely explained until it is interpreted on the supposition that a, b, &c. have the forms $a + a_{,}\sqrt{-1}$, $b + b_{,}\sqrt{-1}$, and also θ, l, &c. By analogy we might have expected this, in the following manner. As soon as pure arithmetic is converted into single Algebra by the extended definitions of $+$ and $-$, and the new symbol $\sqrt{-1}$ occurs, it occurs in conjunction with both the forms $+1$ and -1; and at the same time the vehicle of explanation takes two dimensions. If new distinct symbols be added, such as will require space of three dimensions, it is therefore natural to suppose that each of those new symbols will combine with the complete system of the double Algebra. By this, since $a + a_{,}\sqrt{-1}$ may mean any line in the plane of xy, it is reasonable to suppose that two new symbols will be required, to express removal into the planes of yz and zx, and that

$$(a + a_{,}\sqrt{-1}) + (b + b_{,}\sqrt{-1})\eta + (c + c_{,}\sqrt{-1})\zeta,$$

will signify some line in space, determined by three lines in the three co-ordinates planes.

§ 4. *Redundant biquadratic form.* The last remark suggests an examination of the method by which systems have hitherto proceeded, with a view to ascertain whether the hints which analogy might give are exhausted. If we look at the series $+1$, -1, $\sqrt{-1}$, we see that *one* new unit-symbol is introduced at each step, represented by a square root of the preceding. What then is the system in which *one* more unit-symbol is introduced, whose action resembles that of $\sqrt[4]{-1}$, the combination with preceding symbols being of the complete character just described.

Let the fundamental symbol be

$$[a, p, b, q] = a + p\sqrt{-1} + (b + q\sqrt{-1})\zeta,$$

where ζ^2 means $\sqrt{-1}$. Accordingly, the product of $[a, p, b, q]$ and $[a', p', b', q']$ is $[A, P, B, Q]$ where

$$A = aa' - pp' - bq' - b'q, \qquad B = ab' + a'b - pq' - p'q,$$

$$P = ap' + a'p + bb' - qq', \qquad Q = aq' + a'q + bp' + pb'.$$

The modulus of multiplication is found to be

$$l = \sqrt{\left\{\left(a + \frac{b-q}{\sqrt{2}}\right)^2 + \left(p + \frac{b+q}{\sqrt{2}}\right)^2\right\}}.$$

Now it is evident that, a line in space being determined by three data, we have here one to spare, since a, b, p and q must all be given before the fundamental symbol is completely determined. It would be in our power for instance, to consider the symbol as meaning a line of given length drawn from the origin in a given direction *at a given time;* or as determining a point which has a given position at a given instant. Let $a + p\sqrt{-1}$ represent in the usual manner a line in the plane of xy, and let ζ represent a unit somewhere in the plane of xz; we may easily see that it must be at 45^0 to the positive axis of x, if the rule of angles in multiplication is to be preserved. To satisfy this last condition, let $[a, p, b, q]$ represent a length l making an angle with the axis of x determined by

$$l\cos\theta = a + \frac{b-q}{\sqrt{2}}, \qquad l\sin\theta = p + \frac{b+q}{\sqrt{2}}.$$

Let ζ signify revolution through 45^0 in the plane of xz, so that if $a+p\sqrt{1} = r\epsilon^{a\sqrt{-1}}$, $b+q\sqrt{-1} = s\epsilon^{\beta\sqrt{-1}}$, we have $(b+q\sqrt{-1})\zeta$ signifying a line s at an angle $\beta+\frac{1}{4}\pi$ in the plane of xz. Moreover we have

$$l\cos\theta = r\cos a + s\cos(\beta+\tfrac{1}{4}\pi),\quad l\sin\theta = r\sin a + s\sin(\beta+\tfrac{1}{4}\pi),$$

so that the way to find l and θ geometrically is as follows. In any plane, say that of xy, set off r and s at angles a and $\beta+\frac{1}{4}\pi$: the diagonal of the parallelogram on these lines represents the length l inclined at the angle θ to the positive axis of x. In various systems I find that when $l\sin\theta$ has the form $\text{M}\pm\text{N}$, one of the simplest interpretations consists in making $\text{N} = \text{M}\tan\omega$, where ω is the angle which the plane of the line and the axis of x makes with the positive side of the plane of xy. In the present instance, this will give

$$\tan\omega = \frac{b+q}{p\sqrt{2}},\quad l\cos\theta = a + \frac{b-q}{\sqrt{2}},\quad \frac{l\sin\theta}{1+\tan\omega} = p,\quad \frac{l\sin\theta\tan\omega}{1+\tan\omega} = \frac{b+q}{\sqrt{2}}.$$

Here p, b, q can be found so as to give $[l, \theta, \omega]$ for any given value of a. The system is now complete, all the rules of Algebra are true of it, and it only remains to give the results their easiest geometrical form. The most natural mode of proceeding is to examine the mode of escaping redundancy, which consists in assigning one relation between a, b, p, and q. The case of $b = q$ will appear exceedingly remarkable, when viewed in connexion with the imperfect system which I shall describe in the next section.

According to our conventions, $a+p\sqrt{-1}+b(1+\sqrt{-1})\zeta$ represents a line of the length $l = \sqrt{\{a^2+(p+b\sqrt{2})^2\}}$ inclined at an angle having $a : l$ and $(p+b\sqrt{2}) : l$ for its cosine and sine, with a projection on the plane of yz which makes the angle $\tan^{-1}\{b\sqrt{2} : p\}$ with the positive axis of y. But the relation $B = Q$ does not obtain in the product; and if we bring it about by a proper use of our redundant letters, so as to represent the product $[L, \Theta, \Omega]$ under the form $V + W\sqrt{-1} + X(1+\sqrt{-1})\zeta$, we shall find that we have sacrificed the equation $A(BC) = (AB)C$, which is no longer a formula of the Algebra. Owing to the redundant letter, two lines may be identical in position, but must not therefore be considered as identical. Now the introduction of an equation of condition between a, b, p, q, and the alteration of the product in such a manner as to satisfy this same condition, is, in point of fact, the substitution for the product of a line equivalent in position only.

I shall resume this subject in the next section: but in the first place, observe that the modulus admits of resolution into the square root of the sum of two other squares, namely

$$l = \sqrt{\left\{\left(b+\frac{p+a}{\sqrt{2}}\right)^2 + \left(q+\frac{p-a}{\sqrt{2}}\right)^2\right\}}.$$

Take another angle κ such that

$$l\cos\kappa = b + \frac{p+a}{\sqrt{2}},\qquad l\sin\kappa = q + \frac{p-a}{\sqrt{2}}.$$

This angle κ is not a new directing angle, being in fact $\theta - \frac{1}{4}\pi$; and ζ^2 is $-\sqrt{-1}$.

The modes of interpretation will be better seen, so far as they are easily practicable, in the next section.

§ 5. *Imperfect form, derived from the preceding.* The first system of triple Algebra which I obtained was that in which $P = a + b\eta + c\zeta$, where ζ^2, η^2, and $\eta\zeta$ severally represent -1. I did not at first see that though this will give $PP' = P'P$, it will not give $P''(P'P) = (P''P')P$, except in particular cases; though it should have been obvious that $\eta^2\zeta$, for instance, is not the same thing as $(\eta\zeta)\eta$. Now this is precisely the case of the redundant system already noticed, in which $b = q$. If we multiply together $a+p\sqrt{-1}+b(1+\sqrt{-1})\zeta$ and $a'+p'\sqrt{-1}+b'(1+\sqrt{-1})\zeta$, under the condition that ζ^2 means $\sqrt{-1}$, and if we then reduce the result to a line of the same value of l, θ, ω, in which also $b = q$, we have

$$aa' - pp' - 2bb' - (pb' + p'b)\sqrt{2} + (ap' + a'p)\sqrt{-1} + (ab' + a'b)(1 + \sqrt{-1})\zeta.$$

Now for $b\sqrt{2}$ write b, and let $(1 + \sqrt{-1})\zeta \div \sqrt{2}$ be an independent unit symbol (it will be found by our conventions to be a unit on the axis of z), and for it write ζ; also for $\sqrt{-1}$, a unit on the axis of y, write η. Then it appears that the product of $a + b\eta + c\zeta$ (write c for p and then interchange it with b in the preceding), and $a' + b'\eta + c'\zeta$ is

$$aa' - (b + c)(b' + c') + (ab' + ba')\eta + (ac' + ca')\zeta,$$

which is here produced, and can only be produced, from $\eta^2 = -1$, $\zeta^2 = -1$, $\eta\zeta = -1$.

I shall give the interpretation of this synthetically, and with some minuteness, since the leading features of it belong to most of the other imperfect quadratic systems which I have tried.

Let every line drawn through the origin be considered as having for its plane that plane which also passes through the axis of x; and let the line in which that plane cuts the plane of yz be called the *imaginary axis* of that plane and of all lines in it (except the axis of x itself). Let a line ($z = -y$) which bisects the second and fourth right angles in the plane of yz be called the *neutral axis*, and one perpendicular to it, which therefore bisects the first and third right angles, the *primary axis*. Let every imaginary axis have for its sign the sign of the parts of y and z which lie on the same side of the neutral axis as itself: and let angles be measured positively in every plane by revolution from the positive axis of x towards the positive imaginary axis.

Let $a + b\eta + c\zeta$ represent a line of the length $l = \sqrt{\{a^2 + (b + c)^2\}}$ in a plane whose imaginary axis make with the positive axis of y the angle $= \tan^{-1}(c : b)$ having for projections on the real axis (the axis of x), and its own imaginary axis severally a and $b + c$; or making with the axis of x an angle θ whose sine is $b + c : l$ and whose cosine is $a : l$.

For addition, subtraction, multiplication and division, of two lines, make them both revolve round the axis of x into, say the plane of xy, taking care to bring the positive part of each imaginary axis into contact with the positive part of y. Then add, subtract, multiply and divide as in common double Algebra, and find the plane into which the results are to be finally transferred by the following rules.

In addition, set off on the primary axis lines equal to the projections of the given lines on their imaginary axes; or transfer the imaginary projections by revolution to their proper sides of the primary axis. From the extremities of the lines so drawn, draw lines perpendicular to the primary axis, meeting the imaginary axes of the two lines, so as to cut off two hypothenuses. On these hypothenuses describe a parallelogram; its diagonal from the origin is in the imaginary axis of the sum. And similarly for the subtraction, or the addition of the equal and opposite line.

In multiplication, first lay down on the primary axis lines proportional to the tangents of the angles which the factors make with the axis of x, and then proceed (exactly as in addition) to determine the imaginary axis of the product from the diagonal of the hypothenuses. And similarly for division.

In every plane, as long as lines are taken in that plane only, there is one complete system of double Algebra, admitting every rule of ordinary Algebra to its full extent. When lines from another plane are introduced, we lose the equation $A(BC) = (AB)C$, unless A and B be in one plane.

The theory of powers and roots is absolutely identical with that of common double Algebra for every line which is not on the axis of x, the plane of each line being the locus of all its powers. And $+1$ has only two square roots, as usual; but -1 has an infinite number of square roots, every imaginary axis of a unit in length being one of them. Also both $+1$ and -1 have an infinite number of third, fourth, &c. roots, one set of three, four, &c. in every plane.

For $a + b\eta + c\zeta$, or $[l, \theta, \omega]$, we may write

$$l\cos\theta + \frac{l\sin\theta}{1+\tan\omega}\eta + \frac{l\sin\theta \, . \tan\omega}{1+\tan\omega} . \zeta,$$

and if $\sqrt{}_\omega - 1$ denote the square root of -1 which is at an angle ω to the axis of y, we have

$$\sqrt{}_\omega - 1 = \frac{1}{1+\tan\omega}\eta + \frac{\tan\omega}{1+\tan\omega}\zeta, \qquad \eta = \sqrt{}_0 - 1, \qquad \zeta = \sqrt{}_{\frac{1}{2}\pi} - 1.$$

Call these last $\sqrt{-1}$ and $\sqrt{}_1 - 1$; we have then

$$[l, \theta, \omega] = l(\cos\theta + \sin\theta\sqrt{}_\omega - 1) = l\left(\cos\theta + \frac{\sin\theta}{1+\tan\omega} . \sqrt{-1} + \frac{\sin\theta\tan\omega}{1+\tan\omega}\sqrt{}_1 - 1\right).$$

The product of *any* two *positive* square roots of -1 is -1, and the product of a positive and negative square root is $+1$.

The Algebra of the *neutral plane*, which passes through the neutral axis and the axis of x is of a very peculiar character. In the first place, neither side of the neutral axis is necessarily positive or negative by our conventions, and the signs of this axis must be determined (like that of $\tan\frac{1}{2}\pi$) by the manner in which we come upon it. But this is not the chief peculiarity. If we call the point whose co-ordinates are a, b, c, the *subsidiary point* of L or $[l, \theta, \omega]$, the point and its subsidiary point are always in the same plane: but if the subsidiary point be on the neutral plane $(b + c = 0)$, the angle θ is 0 or π, and L is on the axis of x. But if on the other hand L be on the neutral plane, but not on the axis of x, then b and c are infinite (with contrary signs): and in this case, whatever line A may be, $L \pm A$, $A \pm L$, $A \times L$, $A \div L$, $L \div A$, are all on the neutral plane.

Hence 'a unit, situated on the positive side of the axis of x', is not a complete description of any line: for under that description comes every case of $1 + m(\eta - \zeta)$ in which m is finite. The fundamental unit 1 or $1 + 0\eta + 0\zeta$ is the line which requires that the preceding should be augmented by 'having its subsidiary point at its extremity.' It is true that no alteration could, in any case, be produced in l or θ, by substituting one case of $1 + m(\eta - \zeta)$ for another; but the effect would be seen in the value of ω. The rules of addition and multiplication, as above given, fail when one of the lines is of the form $a + m\eta - m\zeta$; we must replace them by others drawn from the use of the projections themselves.

I look upon the preceding system, as the one which has most general resemblance to the common system, from which I derived it, before I considered the subject generally.

It is demonstrably impossible that any system can give the convertibility of three factors, in which a line of a unit in length is represented by $\cos\theta + \sin\theta \, . P_\omega$, where $P_\omega P_{\omega'} = -1$. Calling this A, it will be found that $A''A'A$ and $A'A''A$ are not identical unless $\sin\theta \, . \sin\theta' . \sin\theta'' . P_{\omega'} = \sin\theta \, . \sin\theta' . \sin\theta'' . P_{\omega''}$, which, to be universal, requires $P_{\omega'} = P_{\omega''}$.

§ 6. *Second imperfect system deduced from the redundant system.* It is natural to examine that particular mode of getting rid of redundancy, which consists in reducing the modulus of multiplication to the form $\sqrt{(a^2 + p^2 + b^2 + q^2)}$. This is obviously

$$a(b - q) + p(b + q) = 0, \quad \text{or} \quad b(p + a) + q(p - a) = 0.$$

Now if we examine the corresponding function in the product, we find*

$$A(B - Q) + (B + Q)$$

$$= \{a(b-q) + p(b+q)\}\ \{a'^2 + b'^2 + p'^2 + q'^2\} + \{a'(b' - q') + p'(b' + q')\}\ \{a^2 + b^2 + p^2 + q^2\},$$

* Most easily seen thus: since

$$A^2 + B^2 + P^2 + Q^2 + \sqrt{2} \, . \{A(B - Q) + P(B + Q)\}$$

is identical with the product of the corresponding functions of a, b, &c. and a', b', &c., the parts affected with $\sqrt{2}$ are identical; whence follows the equation in the text, and also

$$A^2 + B^2 + P^2 + Q^2 = (a^2 + b^2 + p^2 + q^2)(a'^2 + b'^2 + p'^2 + q'^2)$$
$$+ 2\{a(b-q) + p(b+q)\}\{a'(b'-q')\} + \{p'(b'+q')\}.$$

So that if this condition be true of the factors, it is true of the product. Now if as before, $a + p\sqrt{-1} = r\epsilon^{a\sqrt{-1}}$, $b + p\sqrt{-1} = s\epsilon^{\beta\sqrt{-1}}$, we have, for the expression of the condition, $\tan(\beta - a) = 1$. This gives either

$$\beta + \tfrac{1}{4}\pi = a + \tfrac{1}{2}\pi, \qquad l\cos\theta = r\cos a - s\sin a, \qquad l\sin\theta = r\sin a + s\cos a,$$

$$\text{or} \qquad \beta + \tfrac{1}{4}\pi = a + \tfrac{3}{2}\pi, \qquad l\cos\theta = r\cos a + s\sin a, \qquad l\sin\theta = r\sin a - s\cos a.$$

The first will be the most convenient.

But though this condition may be satisfied for the product, when it is so for the factors, the same is not true of the components and the sum, unless $a : a' :: p : p'$. This system then would be perfect for multiplication, division, and all its consequences, as the former one is for addition and subtraction.

If we endeavour to find the system in which the sum of two lines is the diagonal of the parallelogram formed on them as they stand, at angles a and $\beta + \frac{1}{4}\pi$ to the axis of x in the two planes; we find the condition to be $p(b+q) = 0$. Now $b = -q$ satisfies this for additions, and $p = 0$ and $b = -q$ for both additions and multiplications: but an examination of this last case will shew that it gives nothing more than the common double Algebra; no line lying out of the plane of xy.

If there can be a perfect non-redundant system formed out of the redundant system, there must be some function $f(a, b, p, q)$ such that $f(A, B, P, Q)$ and $f(a + a', b + b', p + p', q + q')$ both vanish when $f(a, b, p, q)$ and $f(a', b', p', q')$ both vanish. The second condition cannot be satisfied unless $f(a, b, p, q)$ be of the first degree with respect to the letters specified, in which case the first condition cannot be satisfied.

§ 7. *Imperfect system, independent of all that precede.* Let the laws of combination of the symbols, ξ, η, ζ, in the expression $a\xi + b\eta + c\zeta$, be

$$\xi\eta = \eta\xi, \text{ \&c.}, \qquad \xi^2 = \xi, \qquad \eta^2 = -\xi, \qquad \zeta^2 = -\xi,$$

$$\eta\zeta = -\xi, \qquad \zeta\xi = \eta, \qquad \xi\eta = \zeta.$$

The product of $a\xi + b\eta + c\zeta$ and $a'\xi + b'\eta + c'\zeta$ is

$$\{aa' - (b+c)(b'+c')\}\xi + \{ac' + ca'\}\eta + \{ab' + ba'\}\zeta.$$

In this system, the properties of the neutral and primary axes, the conventions of sign connected with them, the modulus of multiplication, the rule of addition and subtraction, and the meaning of the angles θ and ω, are precisely as in the system described in § 5. But the product of two lines in this system differs from that in the preceding one as follows; the angle made by its imaginary axis with the axis of y is the complement of that made in § 5. Or, signifying by ϕ the angle made by $[l, \theta, \phi]$ or $a\xi + b\eta + c\zeta$ with the *primary axis*, then if $[l, \theta, \phi]$ and $[l', \theta', \phi']$ have the product $[L, \Theta, \Phi]$ in § 5, their product is $[L, \Theta, -\Phi]$ in the present system. Let two imaginary axes be called opposite which are equally inclined to the primary axis on opposite sides of it, and let the planes passing through them and the axis of x be called opposite planes. Then A^{2m} is in the plane opposite to that of A^m; A^4, A^{16}, A^{64}, &c. are in the plane of A; A^2, A^8, A^{32}, &c. are in the opposite plane. Generally speaking A^{2m+1} is in a new plane for every new value of m. But the character of the square roots of $-\xi$ resembles that in § 5, and we have

$$[l, \theta, \phi] = l\{\cos\theta . \xi + \sin\theta \sqrt{_\phi - \xi}\}$$

$$= l\cos\theta . \xi + l\sin\theta . \frac{1 - \tan\phi}{2} . \eta + l\sin\theta \frac{1 + \tan\phi}{2}\zeta.$$

The imperfection of this system, as in the former case, consists in the want of the equation

$$A(BC) = (AB)C.$$

There is a remarkable new consideration, which presents itself in these systems of inverted multiplication, as we might call them. When ξ is an inoperative symbol, that is, when $\xi\eta$ means η and $\xi\zeta$ means ζ, the abstract number of common arithmetic, m, may be represented by a line $\xi + 0\eta + 0\zeta$. But, in the case before us, the multiplier m and the multiplier $m\xi$ are very distinct things. The former has only the effect of multiplying the length by m, without altering angles. But there is still a line which has the effect of the abstract multiplier m, upon $a\xi + b\eta + c\zeta$: it is

$$m\xi + m\frac{c-b}{a}\eta + m\frac{b-c}{a}\zeta.$$

The product of these two lines is $ma\xi + mb\eta + mc\zeta$. Now the second line represents a line of the length m, on the axis of x; not having its subsidiary point at its extremity, but at a finite distance on the neutral plane. And thus it appears that every such line of the form $\xi + p\eta - p\zeta$ plays the part of the abstract multiplier 1 to every line of the form $a\xi + b\eta + (b + ap)\zeta$.

§ 8. On looking back to § 2, we see under case A, a perfect cubic form with the equations of signification

$$\xi^2 = \xi, \qquad \eta^2 = \xi, \qquad \zeta^2 = \xi, \qquad \eta\zeta = \xi, \qquad \zeta\xi = -\xi, \qquad \xi\eta = -\xi.$$

Accordingly every product is of the form $m\xi$, or according to our usual interpretation, must be laid down on the axis of x. Look at the quadratic and cubic cases that come under C and D. and it will be equally apparent that all products take the form $m\xi + n(\eta + \xi)$ or $m\xi + n(\eta - \zeta)$, according to the system: consequently all products come into one plane. It would be easy enough to make any number of triple systems, under such a condition.

The perfect quadratic system under B may be readily developed. Its modulus of multiplication is $\sqrt{\{a^2 + (b-c)^2\}}$ which will require that, in an explanation resembling that of § 5, the neutral and primary axes should change places. The line $n(\eta + \zeta)$ is one of no length in such a system, and if $n(\eta + \zeta)$ be added to $a\xi + b\eta + c\zeta$, nothing is changed except the position of the imaginary axis. Let all the explanations be as in § 5, after interchanging the neutral and primary axis: then the system before us is complete when we add to the explanations in § 5, thus altered, the condition that the product of $a\xi + b\eta + c\zeta$ and $a'\xi + b'\eta + c'\zeta$ is to have the addition $(bb' + cc')(\eta + \zeta)$, giving a certain alteration in its imaginary plane.

I should have liked to have delayed the present communication until I could have examined these and other cases in more detail. But as, owing to the approach of other occupations, any such delay must have lasted a year, I determined to send my thoughts just as they are, in the hope that others may be induced to pursue the subject. One great point of the interest which attaches to it. is the hope that the generalized notions of interpretation which it gives, will be found applicable to the common double Algebra, which is at present restricted to systems of linear co-ordinates: and as to which, though the restriction is clearly unnecessary, the proper direction of generalization is not seen.

A. DE MORGAN.

UNIVERSITY COLLEGE, LONDON,
October 9, 1844.

ADDITION.

IN single Algebra, we use no angles, and, so far as geometrical interpretation is concerned, only one dimension of space. In double Algebra, we use two dimensions of space, and the rectilinear angle. It might be supposed that in triple Algebra we should use three dimensions of space, and solid angles, considered as proportional to the areas of their subtending equi-radial spherical triangles. I can make no use of these solid angles; but others may be inclined to try them: I accordingly give the following results, connecting the solid angles of a system of co-ordinates with the plane ones.

Let the positive sides of the rectangular axis of x, y, z, meet the sphere in X, Y, Z; let P be any point on the sphere, and let the *cosines* of the angles PX, PY, PZ, be λ, μ, ν. Let the spherical excesses of the triangles PYZ, PZX, PXY, be α, β, γ: their signs being taken so that the equation $\alpha + \beta + \gamma = \frac{1}{2}\pi$, which obviously exists when P is inside the triangle XYZ, may be permanent. We then easily obtain

$$\cos\alpha = 1 - \frac{\lambda^2}{(1+\mu)(1+\nu)}, \quad \sin\alpha = \frac{\lambda(1+\mu+\nu)}{(1+\mu)(1+\nu)}, \text{ \&c.}$$

$$\frac{1+\lambda}{1+\sin\alpha} = \frac{1+\mu}{1+\sin\beta} = \frac{1+\nu}{1+\sin\gamma} = \frac{(1+\lambda)(1+\mu)(1+\nu)}{(1+\lambda)(1+\mu)(1+\nu) - \lambda\mu\nu},$$

which, since $\lambda^2 + \mu^2 + \nu^2 = 1$
$$= \frac{2(1+\lambda)(1+\mu)(1+\nu)}{(1+\lambda+\mu+\nu)^2}.$$

Also,
$$1 - \cos\alpha + \sin\alpha = \frac{\lambda(1+\lambda)(1+\lambda+\mu+\nu)}{(1+\lambda)(1+\mu)(1+\nu)},$$

$$1 + \lambda = \frac{2(1+\sin\alpha)}{(1-\cos\alpha+\sin\alpha) + (1-\cos\beta+\sin\beta) + (1-\cos\gamma+\sin\gamma)}, \text{ \&c.}$$

Having since I read this paper in proof, examined Sir W. Hamilton's system of quaternions, I may state that, in my view of the subject, it is not *quadruple*, but *triple*, since every symbol is explicable by a line drawn in space. His object has been, to secure interpretation, though it should cost the loss of some of the symbolic forms of Algebra; and his success has been of a most remarkable character. My object has been to detect systems in which the symbolic forms of common Algebra are true, without making any sacrifice to interpretation. The redundant biquadratic system in § 4 of this paper has a resemblance to Sir W. Hamilton's quaternion system in some of its formulæ, and a still greater one in its redundant character. It yet remains to be seen what systems exist in which the axes of y and z are *not* symmetrically related to that of x.

December 17, 1844.

XIX. *On the Values of the Sine and Cosine of an Infinite Angle.* By S. Earnshaw, M.A., *of St. John's College, Cambridge.*

[Read *December* 9, 1844.]

The usage of Mathematicians in reference to the symbols Sin ∞ and Cos ∞ does not seem to be in accordance with their expressed opinions. It does not appear to be questioned either by English or Foreign writers, that when x becomes infinite Sin x and Cos x cannot be said to be in one part of their periodicity rather than another. If this mean any thing, it must be understood to signify that Sin ∞ and Cos ∞ are indefinite. Yet this is not borne out in the usage of these symbols which we find in the writings of any author. Indeed, an opinion has been expressed that their indeterminateness is only apparent, and therefore not real: and that analysis has furnished definite equivalents for them by legitimate processes of investigation on principles which are allowed: and though some writers on Definite Integrals have abstained from stating in direct terms what are the values which analysis assigns to Sin ∞ and Cos ∞, all agree in practically affirming "that both the Sine and Cosine of an infinite angle are equal to zero." But while we find these values used wherever Sin ∞ and Cos ∞ occur in investigations, we do occasionally meet with expressions of doubtfulness respecting their universal truth. This seems to indicate that in the opinion of such writers the values of Sin ∞ and Cos ∞ depend on the circumstances under which they occur; but what those circumstances are which have this power over Sin ∞ and Cos ∞ I do not find any where pointed out. In fact, upon tracing the origin of this doubt respecting the universal truth of the equations Sin $\infty = 0$, Cos $\infty = 0$, I find that it has arisen from the occurrence of certain results of a character so obviously suspicious, perhaps I might say, erroneous and contradictory of evident truths, as to create a reasonable doubt of the propriety of writing zero for Sin ∞ and Cos ∞ in those cases. But though results have thus forced some writers to doubt respecting the general truth of the equations Sin $\infty = 0$ and Cos $\infty = 0$, it does not appear that they have any where admitted the demonstrations of the truth of these equations to be defective. We find ourselves then in this difficult position;—we have certain investigations presented to us in which there occur no doubted steps, and these investigations present us with certain absolute results;—but the certainty of these results thus established by a process of mathematical reasoning, the accuracy of which is no where called in question, we are afterwards required to look upon with suspicion;—and that sort of suspicion which while it throws doubt upon every thing affords us no clue for ascertaining what are the cases to which alone it ought to be attached. It is obviously desirable that some effort should be made to remove this uncertainty. Now some light may be thrown upon this difficulty by considering that Sin nx and Cos nx go through a whole period of values while x increases by $\frac{2\pi}{n}$. As long as n is finite $\frac{2\pi}{n}$ is finite, and all the values included in a period are therefore *consecutive*. But what happens when n increases in value? We easily see that as n increases the whole period becomes condensed so as to occupy a shorter and shorter portion of the current variable: and that when n approaches ∞, the values are no longer *consecutive* but *simultaneous*:—hence as n increases towards ∞ a whole period of values of Sin x or Cos x tends to *become* simultaneous, and in the limits *are* simultaneous: i.e., Sin ∞ has at once all values from -1 to $+1$: and the same property belongs to Cos ∞. Consequently according to this view it is not true that Sin ∞ and Cos ∞ have each a single value, or any finite set of values *definitely;* but they each have all possible values from -1 to $+1$ in such a manner and sense that not one of these values is pre-eminent above another, and no one has a claim to be put forward above its fellows, but all stand in exactly the same relation to the function Sin ∞ (or Cos ∞) so that at one and the same moment Sin ∞ (or Cos ∞) is equal to every one of them but not more properly equal to any one than any other of them. From this reasoning and kindred reasons of an equally general character, I satisfied myself that Sin ∞ and Cos ∞ cannot be replaced by zero, unless under some special hypothesis, and that when taken in a general sense they cannot justly be supposed to have definite values at all. I shall now proceed to some considerations which are preliminary to a more formal proof that they have not the value zero, even when considered as the limits of more general forms.

In conducting my inquiry into the values of the symbols Sin ∞, Cos ∞, I am unavoidably brought upon the confines of the much controverted subject of divergent series. In a certain sense which will be explained, I agree with Professor De Morgan that all non-convergent series stand on the same basis, though I cannot subscribe to the train of reasoning by which this is usually maintained, involving as it appears to me some disputable positions. Much of the obscurity which attaches itself to the subject of divergent series may be traced to the discordant and strange significations applied to the symbol =, when used in connection with infinite series. The presumption is that when this symbol stands between two quantities it indicates, that either may be used for the other in algebraical processes. A very eminent author states that it "may be rendered by the phrase *gives as its result*, when it is placed between two expressions, one of which is the result of an operation which in the other is indicated and not performed;"—an explanation which agrees exactly with what Woodhouse states in his *Principles of Analytical Calculations*, who insists upon this definition of it at intervals through his work with an earnestness which indicates the confidence with which he regarded it as true. Now if this definition be closely examined it cannot be understood to denote that the expressions connected by = differ in any thing but *form;* for one side denotes that an operation is *to be* performed, and the other is the *result* of the actual operation: if then the operation has been correctly and *completely* performed, there is no difference except in form between quantities connected by =. But an examination of the *Principles of Analytical Calculations*, will not fail to satisfy us, that in giving this definition the author must have understood it in some modified sense which he has not expressed in the definition itself. For when it is said that "= is a symbol which serves *merely* to connect an involved expression and the result of an operation," it is evident that "numerical equality" could not then be, what the author affirms it is, a contingent result. But whatever was the sense which the author mentally attached to the symbol, it involved a principle which necessitated the making distinctions where by ordinary minds the difference cannot easily be grasped: for it was found impossible to be consistent without demanding a license to consider $\frac{1}{2}$ and $\dfrac{1}{1+1}$ $\left(\text{as also } \dfrac{1}{1+x} \text{ and } \dfrac{1}{x+1}\right)$ as essentially distinct. Now what difference is there between 2 and $1+1$, except in form? Is not $1+1$ an expression in which an operation is to be performed the result of which is rightly denoted by 2? and if so, then by his own definition 2 and $1+1$ are algebraically equivalent. I must confess that I cannot consent to such distinctions as are here demanded without being satisfied that there is no means of avoiding them; and I cannot but suspect that in the present case there is no other necessity for them, than what arises from a misapplication of the definition which the author has given of the symbol =. For if this symbol serve merely to connect an involved expression and *the result* of an operation, it is clearly a misuse of it to employ it in connecting an involved expression and a *part* only of the result of an operation. Let me explain by an example. Professor Woodhouse writes $\dfrac{1}{1+x} = 1 - x + x^2 - \ldots\ldots$ Now the operation *denoted* on the left-hand is the division of 1 by $1+x$, and according to the definition of $=$, the other member is or ought to be *the result* of that operation. But we observe that $1 - x + x^2 - \ldots\ldots$ is a series of terms following the *same* law throughout, and shewing no indication of any terms which are not included in this law; yet it may be asked, have we any just ground for knowing that all the terms resulting from the division of 1 by $1 + x$ *do follow the same law throughout?* Let us examine; if we stop after one term of the quotient we find $\frac{1}{1+x} = 1 - \frac{x}{1+x}$; if we pursue the division a step further we find $\frac{1}{1+x} = 1 - x + \frac{x^2}{1+x}$; another step gives $\frac{1}{1+x} = 1 - x + x^2 - \frac{x^3}{1+x}$, and so on. In all these partial operations we observe that one term of the quotient is an exception to the law followed by the others. It is true, by continuing the process, we may push this anomalous term to any conceivable distance from the beginning of the series, but there is not the slightest indication that by so pushing it it will at length cease to be, or become zero: on the contrary, as Professor De Morgan justly remarks, by the prolongation of the operation it is *removed* farther off but not *destroyed*. Consequently, the operation represented by $\frac{1}{1+x}$ is of a character which can never be *completely* comprehended in any series of terms which follow one law: and therefore, strictly speaking, there is no such quantity as the definition requires which can be joined with it by the symbol $=$. Shall we then join it with as much of the quotient as does follow a fixed law? It is clear we cannot without violating the terms of the definition. When therefore we find $\frac{1}{1+x} = 1 - x + x^2 - \ldots$ *ad inf.* without an implied remainder, we are at a loss to understand in what way this use of it is reconciled with the meaning attached to the sign $=$ in the definition. Yet it is certain, that most eminent writers do use the symbol $=$ to connect a function with a series *every* term of which is supposed to follow a fixed law, as though the operation denoted by the function were capable of being represented by such a series of terms. Still, though it is thus rendered evident that the usage has not been sanctioned by the definition, the discrepancy is not very important in itself, seeing that an alteration may be admitted into the definition which shall make it agree with usage. The definition may then stand thus;—*the sign $=$ is used to connect an involved expression with the result of an operation as far as it is expressible in terms which follow a fixed law.* The really important point now to be examined is, whether that portion of a result herein included will in all cases represent, for algebraical purposes, the properties of the expression from which it was derived. If it will so represent the expression, then for algebraical purposes series of all kinds, whether convergent, periodic, or divergent, will stand on the same basis, and their use in all cases be equally safe. I need hardly say that this is a much disputed point, which has been warmly attacked and defended. I am induced to venture into the field on the side of the assailants from having observed that its advocates have defended the use of non-convergent series on grounds some of which are capable of being easily shewn to be fallacious: and though I cannot bind myself to the justness of all the arguments which have been opposed to them even by the most eminent and skilful analysts, I yet think there are sufficient reasons left to justify us in rejecting non-convergent series when in accordance with the above definition their remainders are thrown away.

Now according to the definition above proposed, it is evident that an involopment and its series are not *equal*, (they differ by the remainder) the question is, are they *equivalent?* does the series embody all the algebraical properties of the involopment, and no more? The discussions which have been so earnestly carried on with the view of arriving at a satisfactory settlement of this difficulty have not yet elicited any unanswerable arguments on either side: at any rate they have not been of such a character as to set the question at rest. Though I do not presume to hope that what is here brought forward will have the effect of satisfying those who entertain the opposite views, yet something may perchance be said which will in abler hands be made useful in settling some of the difficulties which beset the consideration of this perplexing subject.

1. The ground on which I would reject the use of non-convergent series is a conviction that such series may have some algebraical properties which their invelopments possess not, and may lack others which the invelopments have. For series of ordinary forms I think I shall be able to prove the truth of this as satisfactorily as such an intractable subject as an infinite non-converging series admits of.

2. Let us notice first, that there is a presumptive ground of suspicion of the truth of this (viz., that the algebraical properties of a non-convergent series are identical with those of its invelopment) in the rejection of the anomalous term (the remainder) which if preserved would certainly render their (numerical as well as) algebraical properties identical. Has the remainder *no* algebraical properties? If it has, then it will hardly be believed without proof, that in throwing it (and with it its properties) away we have not destroyed the algebraical equivalence which by its means existed between the invelopment and the series. I will endeavour to illustrate my meaning by instances.

3. It admits of no doubt that *including* the remainder the equation $\frac{1}{2} = 1 - 1 + 1 - 1 + \dots\dots$ *ad inf.* is strictly true. We are to examine whether this is algebraically true if the series be taken without its remainder. Denote the sum of n terms of the series by S_n; then it will be found that for all values of n, $S_n = S_n^2$. This equation being strictly true may be made use of in any algebraical operation: and as it is true however large be the value of n, it is impossible to refuse to admit that $S_\infty = S_\infty^2$ is a property of the infinite series. Hence $\frac{1}{2}$, not being a root of this equation, does not enjoy *this* property which the sum of the infinite series does enjoy, viz., that it is not altered in value by being squared. $\frac{1}{2}$ is the sum of the series inclusive of the remainder, and S_∞ is the sum of the same series exclusive of the remainder. Hence the rejection of the remainder has altered the algebraical properties of the symbol by which the series is represented.

4. But the algebraical importance of the remainder may be rendered still more striking, and the impropriety of rejecting it put in a stronger view. For if *any proper* fraction $\frac{a}{b}$ be put in the form $\frac{1+1+1+\dots \text{ to } a \text{ terms}}{1+1+1+\dots \text{ to } b \text{ terms}}$, it will be found by the ordinary process of algebraical division that

$$\frac{1+1+1+\dots \text{ to } a \text{ terms}}{1+1+1+\dots \text{ to } b \text{ terms}} = 1 - 1 + 1 - 1 + \dots\dots$$

Now many persons have found it difficult to reject $\frac{1}{2}$ as the algebraical equivalent of $1 - 1 + 1 - \dots$ because by ordinary algebraical development this series *ad infinitum* can be obtained from $\frac{1}{2}$. It is here shewn however that the very same process which elicits the series from $\frac{1}{2}$ would serve to elicit it from any proper fraction whatever: and this being so, by what distinguishing property are we to be guided, so as to be able to select amongst all proper fractions some one particular value as *the* equivalent, the *unique* equivalent of the infinite series? If $\frac{1}{2}$ be selected as embodying all the algebraical properties of the series, surely we must admit that for as good a reason $\frac{a}{b}$ embodies the whole of its properties; and thence we cannot avoid allowing that $\frac{1}{2}$ and $\frac{a}{b}$ are in an algebraical sense equivalent fractions.

5. But it is said in special favor of $\frac{1}{2}$ that from whatsoever more general series $1 - 1 + 1 - \dots$ be deduced the symbolical equivalent is always found to be $\frac{1}{2}$. If deduced, for example, from

$1 - x + x^2 - \dots$ by writing 1 for x the sum is $\frac{1}{2}$. Now let us turn the fraction $\dfrac{1 + x + x^2 + \dots x^{a-1}}{1 + x + x^2 + \dots x^{b-1}}$ into a series by the ordinary process of division; the result is, $(b > a)$

$$\frac{1 + x + \dots a \text{ terms}}{1 + x + \dots b \text{ terms}} = 1 - x^a + x^b - x^{a+b} + x^{2b} - \dots\dots$$

This series differs from $1 - x + x^2 - \dots$ only in being more general, for it includes it as a particular case (viz., when $a = 1$, and $b = 2$). If then it be lawful to write 1 for x in $1 - x + x^2 - \dots$ it is equally lawful to do so in the more general case: which being done we have $\dfrac{a}{b} = 1 - 1 + 1 - \dots$ *ad infinitum*. Here then is "a well-established instance in which $1 - 1 + 1 - \dots$ means other than $\frac{1}{2}$;" shall we say with Professor De Morgan, one such instance throws "doubt on all that Poisson and Fourier have written?"

6. It will hardly be considered necessary to defend a system which requires us to receive as a legitimate consequent that all proper fractions are algebraical equivalents. I apprehend therefore the last article but one will be sufficient to shew that in numerical forms of series the ability of an expression to furnish by legitimate expansion a proposed series is no presumption that the two are algebraically equivalent. Here then is fair ground for suspecting the existence of some grievous violation of just reasoning in depriving an infinite series of its remainder, i. e. in supposing that by pushing an expansion *in infinitum* the anomalous terms may be disregarded. In converting the expression $\dfrac{1 + 1 + 1 + \dots a \text{ terms}}{1 + 1 + 1 + \dots b \text{ terms}}$ into a series we observe that for all values of a and b $(a < b)$ the series of *quotients* are the same, and the various cases are distinguishable *only by their remainders*. The *distinctive* properties then of these proper fractions by the process of development are not thrown into the *quotients*, but are preserved in the *remainders*. How then shall we reject the remainders in any equation which professes to exhibit the equivalence of its members?

But there is yet another proof, which I shall now offer, that neither $\frac{1}{2}$ nor even any proper fraction whatever can be the proper equivalent of the series $1 - 1 + 1 - \dots\dots$

7. In perusing what has been written upon this series, we cannot but perceive that some authors, setting out with $\dfrac{1}{1 + x} = 1 - x + x^2 \dots\dots$ as an equation admitted on all hands to be true when x is less than 1, have argued that, being true when x is less than 1, however small $1 - x$ may be, it must needs be allowed in the limit. If the premises are true, I do not see how we can refuse to allow the conclusion. But it is obvious the premises assume that the series *is* convergent towards $1 - 1 + 1 - \dots$ when $1 - x$ is indefinitely small; *is this true?* If it is, I admit that $\frac{1}{2}$ is the equivalent of the series $1 - 1 + 1 - \dots$ in as good a sense as $\dfrac{1}{1 + x}$ is the equivalent of the converging series $1 - x + x^2 - \dots$ Mr. De Morgan questions this; but I see no objection in it which would not, if admitted here, overturn the whole fabric of the Differential Calculus. But we have to answer the question asked above, is it true that the series $1 - x + x^2 - x^3 + \dots$ is convergent towards $1 - 1 + 1 - \dots$ as its limiting form when $1 - x$ is indefinitely small?

8. Let y be any *finite* quantity, and assume $1 - x = \pm \dfrac{y}{n}$: then when n approaches infinity, $1 - x$ will be indefinitely small; but then limit of x^n = limit of $\left(1 \mp \dfrac{y}{n}\right)^n = e^{\mp y}$, the upper or lower sign being used according as x approaches 1 from inferior or superior values. Here then is a proof

that the terms of the series $1 - x + x^2 - \ldots$ at an infinite distance from the beginning do not converge towards 1 as their limit, but to one of the indeterminate quantities e^{-y} or e^{+y}; the values of these depending upon the law under which x approaches unity. Who shall prescribe this law? Surely it is (and must be left) arbitrary in the fullest sense of the word. It is not true then that the converging and diverging forms of $1 - x + x^2 - \ldots$ approach the same form, viz. $1 - 1 + 1 - 1 + \ldots$ *ad infinitum*, as their common limit. For the limiting forms both of convergency and divergency are arbitrary, yet so restricted that they never can mutually approach so near as to be separable by only a single form: for e^{+y} never can approach so near to e^{-y} that only unity lies between them, because y is necessarily *finite*, i. e. neither indefinitely large nor indefinitely small.

9. The unavoidable inference from the last article is that $1 - 1 + 1 - \ldots$ is an *isolated* form of $1 - x + x^2 - \ldots\ldots$ and separated from the limits of continuity on either side by a *finite* interval. For the same reason it is an isolated form of $1 - x^a + x^b - x^{a+b} + x^{2b} - \ldots$ Let it now be admitted that $\frac{1}{1+x}$ is the equivalent of $1 - x + x^2 - \ldots$ *ad infinitum*, then it will follow that

$$\text{limit of } \frac{1}{1+x} = \text{limit of } (1 - x + x^2 - \ldots)$$

$$\text{But limit of } (1 - x + x^2 - \ldots) \text{ is not } = 1 - 1 + 1 - \ldots\ldots$$

$$\therefore \text{ limit of } \frac{1}{1+x} \text{ is not } = 1 - 1 + 1 - \ldots$$

This then is the proof that $\frac{1}{2}$ is not, (and in a similar way it would follow that $\frac{a}{b}$ is not) the proper equivalent of $1 - 1 + 1 - \ldots\ldots$ even assuming $\frac{1}{1+x}$ to be the proper equivalent of $1 - x + x^2 - \ldots\ldots$ It is easily shewn, since $\frac{1 \pm x^n}{1+x} = 1 - x + x^2 - \ldots$ n terms, that $\frac{1}{2}(1 \pm e^{\pm y})$ is $=$ limit $(1 - x + x^2 - x^3 + \ldots$ *ad infinitum*$)$, which is therefore indefinite.

10. In a paper "On Divergent Series" by Mr. De Morgan, there is a remark which shews the important bearing of the results obtained in the preceding articles. "It is clear enough," writes the Professor, "from the manner in which Fourier, Poisson, Cauchy, &c. use the limiting form $1 - 1 + 1 - \ldots$ that they intend it to signify $\frac{1}{2}$ in an absolute manner. The whole fabric of periodic series and integrals, which all have had so much share in erecting, would fall instantly if it were shewn to be possible that $1 - 1 + 1 - \ldots$ might be one quantity as a limiting form of $A_0 - A_1 + A_2 - \ldots$ and another as a limiting form of $B_0 - B_1 + B_2 - \ldots$". I object, of course, to the assumption that $1 - 1 + 1 - \ldots$ is a *limiting* form of the series alluded to; but passing over that, it is shewn above that $1 - 1 + 1 - \ldots$ when taken as a form of $1 - x^a + x^b - \ldots$, which it certainly is, may be one thing or another, according to the values arbitrarily assigned to a and b. Indeed it is stated in Woodhouse's *Anal. Calc.* p. 61, that $1 - 1 + 1 - 1 + \ldots = \frac{1}{1+1+1}$, as well as $= \frac{1}{1+1}$. But Woodhouse either did not observe this evident contradiction, or must have got over it by the mystical maxim that $\frac{1}{1+1+1}$ is not $= \frac{1}{3}$, and $\frac{1}{1+1}$ is not $= \frac{1}{2}$; which is perhaps the case, for in a note he considers that Euler, Leibnitz, and Waring had fallen into a mistake by making $\frac{1}{1+1}$, $\frac{1}{1+1+1}$ &c. $= \frac{1}{2}, \frac{1}{3}$, &c. However, passing by this doctrine, it serves the purpose for which I quote it, for it exhibits Woodhouse as testifying to the propriety of taking $1 - 1 + 1 - \ldots$ to be a form of the series $1 - x + x^3 - x^4 + \ldots$ which arises from the expansion of $\frac{1}{1+x+x^2}$. In fact,

to this also Mr. De Morgan has given assent where he assumes that $1 - 1 + 1 - \dots$ is a form of $1 - x^1 + x^4 - x^9 + x^{16} - \dots\dots$ I have brought forward these testimonies, because it is not very unusual to cast a mantle of mystery over this subject, by introducing zeros into the expansion of $\frac{1}{1+1+1}$. But such a device, however much it may serve to satisfy the eye, cannot satisfy the head: for $\frac{1}{1+x+x^2}$ gives $1 - x + x^3 - x^4 + \dots$, there being no terms between x and x^3, x^4 and x^6, &c., in this, which is the general form of the series; and consequently it is not allowable to write $\frac{1}{1+1+1} = 1 - 1 + 0 + 1 - 1 + 0 + \dots$, if it be intended to insinuate thereby that the zeros make any difference in the sum of the infinite series: and if they make no difference, why introduce them?

11. On principles therefore which are allowed, and used by the writers quoted, it is established that $1 - 1 + 1 - \dots$ has no definite equivalent, in the sense in which this word is generally understood. I think also it is proved, that $\frac{1}{1+x}$ is in no proper sense the equivalent of $1 - x + x^2 - \dots$, except when this series is convergent. For that the two expressions may be equivalent to each other, it is essential that each should exhibit the same degree of indeterminateness of value in particular cases, and the same kind of discontinuity: but, as we have seen, there is no such agreement: on the contrary, while it is admitted that, as x converges towards 1, $\frac{1}{1+x}$ approaches towards $\frac{1}{2}$ as its unique limit, it is here shewn that the other member of the assumed equivalence approaches towards an indeterminate form of an ambiguous character, and absolutely refuses to approach in any case to $1 - 1 + 1 - \dots$ as a limit of continuity.

12. It is not the purpose of this paper to treat of Diverging Series in general, but only of the recurring form $1 - 1 + 1 - \dots$, and of this only because it has been connected with the values of Sin ∞ and Cos ∞, yet as the method above employed is applicable to the general form $\phi x = a_1 x^\alpha + a_2 x^\beta + \dots + a_n x^\nu + \dots$ I may state that the same mode of reasoning when applied to this, shews that ϕx does not embody the algebraical properties of the series, unless the value of x, and the form of the coefficients, be such as to make $a_n x^\nu$ tend to zero as its limit when n and ν approach ∞. Series which satisfy this test I call *convergent series*, whether the arithmetical sum thereof be finite or infinite: and all such series are distinguished by this property, that their involopes may be safely used as equivalent to them in every sense both algebraical and arithmetical.

13. From this it is evident, that the operation of integration performed upon a series will often (not always) have the effect of removing its discontinuity, and establishing a real equivalence though none existed before. And so the operation of differentiation will not unfrequently have the effect of introducing discontinuity, and destroying equivalence.

Hence we see why we *may* put 1 for x in $\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$, though we *may not* write 1 for x in $\frac{1}{1+x} = 1 - x + x^2 - \dots$ from which it was derived by integration.

14. But, in pursuance of the object of this memoir, it is time now to turn to the series $1 - \text{Cos}\,\theta + \text{Cos}\,2\theta - \text{Cos}\,3\theta + \dots$ which has been assumed to be a form which can approach $1 - 1 + 1 - \dots\dots$ as a limit by diminishing θ towards zero. Now assume y to be any arbitrary *finite* angle, and put $\theta = \pm \frac{y}{n}$ which will be indefinitely small for the terms where n is infinite. Hence in such terms $\text{Cos}\,n\theta = \text{Cos} \pm y = \text{Cos}\,y =$ a *finite* quantity, not equal to unity, because y

cannot be *equal to zero*. Hence the terms of this series at an infinite distance from the beginning are subject to discontinuity, and cannot be made to approach 1 as their limit; because if θ differ ever so little from zero there will always be a term so distant from the beginning as that $n\theta$ is *finite;* that term and all following ones will not approach 1 as their limit. Consequently $1 - 1 + 1 - \ldots$ is an *isolated* form of $1 - \operatorname{Cos}\theta + \operatorname{Cos} 2\theta - \ldots$

15. It is not necessary to repeat, in reference to this series, what has been already said upon $1 - x + x^2 - \ldots$; it is sufficient to remark that all results are nugatory which have been obtained upon the supposition of $1 - \operatorname{Cos}\theta + \operatorname{Cos} 2\theta - \ldots$ approaching $1 - 1 + 1 - \ldots$ as its limit as θ changes continuously towards 0. I might here add remarks in reference to the series $a_0 + a_1 \operatorname{Cos}\theta + a_2 \operatorname{Cos} 2\theta + \ldots + a_n \operatorname{Cos} \nu\theta + \ldots$ parallel to the remarks in (12) and (13).

16. Since it often happens that by integration as remarked in (13) a real equivalency is established, it is not unusual to find such series cited as confirmations and verifications of the propriety of the equivalency assumed to exist *before* integration. From what has been proved above however it is evident that such verifications are of no value, and do not in any degree justify the inference sought to be drawn from them.

17. I come now to examine the limiting values (if such there be) of Sin x and Cos x when x approaches ∞. As a preliminary step it is proper to remark, that ∞ is an indefinite symbol: and when it is said that x approaches ∞ as its limit, we are not to understand that x approaches towards some definite value, but merely that it approaches to a value of which we have no other property than this, that it is greater than any finite quantity. Yet there is such a thing as a restricted ∞. Thus, if x be an *odd* multiple of π by the nature of its definition, this restriction will not hinder its becoming infinite; yet then the symbol ∞ will be specific; and accordingly it is possible that under such a condition definite results in certain cases may be obtainable.

18. The above remarks respecting the essential indefiniteness of the symbol ∞ will enable us at once to reply to some questions which have been found perplexing. The question has been asked, is the series $P_1 - P_2 + P_3 - P_4 + \ldots\ldots$ *ad infinitum* equivalent to the series $(P_1 + A) - (P_2 + A) + (P_3 + B) - (P_4 + B) + \ldots$ *ad infinitum?* This has been rightly answered in the negative; but on erroneous grounds. The true reason is this: the terms $A, B, C \ldots$ are introduced in such a manner as *necessarily* involves the notion that ∞ is an *even* number, and therefore it creates an error unless it have been stipulated that ∞ *is* an *even* number. As from the nature of an infinite series no stipulation of this kind can be allowed, we are justified in saying that the two series are not equivalent.

19. If x be defined to be a term of the series 0, 2, 4, 6 ..., then $\operatorname{Cos} x\pi = \operatorname{Cos} 0^0$ when $x = \infty$; but if x be a term of the series 1, 3, 5, 7 ..., then $\operatorname{Cos} x\pi = \operatorname{Cos}\pi$ when $x = \infty$; but if x be defined to be a term of the series 0, 1, 2, 3, 4 ..., then it cannot be affirmed that x is an odd number, nor yet that it is an even number. To say only that x is a whole number, is to express oneself in a way that requires the result to leave the question as to whether x is odd or even undecided. Hence in this case we cannot say that $\operatorname{Cos}\infty = \operatorname{Cos} 0^0$, nor yet that it $= \operatorname{Cos}\pi$; but we must express the result in such terms as leave undecided which of these two is the value of $\operatorname{Cos}\infty$; for to select one of them and reject the other would narrow the restriction laid upon x by its definition, by deciding that it is not only an integer, but that it is a specific integer. Hence then in this case $\operatorname{Cos}\infty = \operatorname{Cos} 0^0$ or $\operatorname{Cos}\pi$ *indeterminably.*

This mode of reasoning can be extended without difficulty to the case where x is a continuous variable, and it leads us to this result, that on this hypothesis respecting the nature of x, $\operatorname{Cos}\infty$ (derived from $\operatorname{Cos} x$ by supposing x to approach towards ∞) is equal to the Cosine of any angle from 0^0 to 2π *indeterminably.* When I say indeterminably, I mean to say that we cannot fix on one of these angles and reject the others without violating the generality of the hypothesis: should we for instance say that $\text{Cos}\,\infty = 0$, the selection of this particular value would be equivalent to narrowing the hypothesis respecting x, as it would restrict x to be an odd multiple of $\frac{\pi}{2}$, and confine its variation to the terms of the series $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\ldots$; similar observations may be made respecting $\text{Sin}\,\infty$.

20. It is also very important to remark that $\text{Sin}\,ax$ and $\text{Cos}\,ax$ do not cease to be functions of a when x approaches ∞.

For since $\text{Cos}\,x = \pm\left(\frac{1+\text{Cos}\,2x}{2}\right)^{\frac{1}{2}}$, it appears that $\text{Cos}\,x$ has an ambiguity of value of which $\text{Cos}\,2x$ does not partake. We may follow out this mode of reasoning to shew that $\text{Cos}\,x$ and $\text{Cos}\,ax$ have not the same number of corresponding values, and that if the value of one of these were given the other would not be determinable from it except in an ambiguous form. Whatever indeterminableness attaches itself then to $\text{Cos}\,x$ when x approaches ∞, the same, and also another kind of, indeterminableness belongs to $\text{Cos}\,ax$ at the limit. We are then particularly to take notice that *Cos ∞ derived from Cos x may not be written for Cos ∞ derived from Cos ax.* Much error has arisen from want of attention to this caution. Also $\text{Cos}\,ax$ cannot be considered independent of a at the limit $x = \infty$, inasmuch as it is subject to two causes of indeterminateness which are distinct from each other.

21. Having thus given my reasons for considering that $\text{Cos}\,\infty$ and $\text{Sin}\,\infty$ have not definite values, it may be proper to examine the proofs which have been brought forward by those who have used definite values for $\text{Sin}\,\infty$ and $\text{Cos}\,\infty$. The following is the most direct proof I know of:

$$\therefore\ \int_0^\infty \text{Sin}\,x\,dx = \left(\int_0^\pi + \int_\pi^{2\pi} + \int_{2\pi}^{3\pi} + \ldots\ \textit{ad infinitum}\right)\text{Sin}\,x\,dx;$$

$$\therefore\ 1 - \text{Cos}\,\infty = 2 - 2 + 2 - \ldots\ \textit{ad infinitum} = 1;$$

$$\therefore\ \text{Cos}\,\infty = 0.$$

To this proof there are two objections, either of which is fatal to it. In the very first step it is assumed that ∞ *is an integer multiple of* π. For this assumption there is certainly no authority, neither is it compatible with the indeterminate nature of the symbol ∞ in the left-hand member of the equation. The next error is made in the summation of the infinite periodic series $2 - 2 + 2 - \ldots$, which I have shewn in the previous articles of this memoir cannot be equal to 1.

22. As the reader may wish to have a further proof of the error of principle involved in the first step of the above investigation, let him see the effect of a different distribution of ∞ into parts in the following process of reasoning, in which the question of summation of series is avoided.

$$\int_0^\infty \text{Sin}\,x\,dx = \left(\int_0^{\frac{2\pi}{3}} + \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} + \int_{\frac{4\pi}{3}}^{2\pi} + \ldots\ldots\ \textit{ad infinitum}\right)\text{Sin}\,x\,dx$$

$$= \frac{3}{2}(1 + 0 - 1 + 1 + 0 - 1 + \ldots\ldots)$$

$$= \frac{3}{2}\left(\int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + \int_{\frac{3\pi}{2}}^{2\pi} + \ldots\ldots\ \textit{ad infinitum}\right)\text{Sin}\,x\,dx$$

$$= \frac{3}{2}\int_0^\infty \text{Sin}\,x\,dx;$$

$$\therefore\ \int_0^\infty \text{Sin}\,x\,dx = 1 - \text{Cos}\,\infty = 0, \quad \therefore\ \text{Cos}\,\infty = 1.$$

It is for those mathematicians to reconcile these conflicting results, who maintain that providing the last limit of x be ∞ it is no matter whether it be a specific ∞ or a general ∞. The distinction is of first-rate importance in periodic functions. I think I am fairly entitled to affirm that specific values for Sin ∞ and Cos ∞ are obtained by such processes as that in (21), only because those very processes assume at the outset a specific form of ∞.

23. The next proof which I shall examine depends upon the principle of continuity "that what is true up to the limit is true at the limit." It is as follows:

$$\text{Since } \int e^{-ax} \operatorname{Sin} x\, dx = -\frac{e^{-ax}}{1+a^2}(\operatorname{Cos} x + a \operatorname{Sin} x);$$

$$\therefore \int_0^\infty e^{-ax} \operatorname{Sin} x\, dx = \frac{1}{1+a^2}.$$

This being true for all positive values of a, no matter how small, is taken to be true in the limit when $a = 0$, which gives (since e^{-ax} then $= 1$ for all values of x)

$$\int_0^\infty e^{-ax} \operatorname{Sin} x\, dx = \int_0^\infty \operatorname{Sin} x\, dx = 1;$$

$$\therefore \operatorname{Cos} \infty = 0.$$

24. To this investigation I have two objections to bring forward. The step which assumes that $e^{-ax} = 1$ for *all* values of x is not true at the limit $x = \infty$, for however small a become ax will be finite and arbitrary or infinite when $x = \infty$. Hence as we diminish a towards zero e^{-ax} approaches, not to 1 as its limit when $x = \infty$, but to e^{-y} an arbitrary value depending upon the relative laws with which x approaches ∞, and a zero. Now it is absolutely necessary in the above proof that for *all* values of x between zero and ∞, e^{-ax} should be equal to 1; and as this is not a true hypothesis, the proof fails.

Again, it is essential to the above investigation that $\frac{1}{1+a^2}$ should be the value of $-\frac{e^{-ax}}{1+a^2}$ $(\operatorname{Cos} x + a \operatorname{Sin} x)$ between the limits $x = 0$, $x = \infty$. But this will not be the case unless e^{-ax} *vanish* when $x = \infty$. Now I have just shewn that when a is made to approach zero e^{-ax} become e^{-y} at the limit $x = \infty$. This step therefore of the investigation is erroneous, and the proof fails.

Let us look at the first written equation in (23), and endeavour to answer these questions; can e^{-ax} in the left-hand member be *always* $= 1$, and yet in the right-hand member $= 0$, when $x = \infty$? If $x = \infty$ make $e^{-ax} = 0$ in the right-hand side, what can prevent the same being true in the left-hand side, seeing that the values of x are simultaneous in both members? Here is a plain contradiction of hypothesis in the two members of the fundamental equation the consequences of which no explanation can remove: and as both hypotheses are required to be true *together* to enable us to obtain the final result $\operatorname{Cos} \infty = 0$, I conclude that this result is not proved to be true. I think upon examination of the steps of the proof in (23) the reader will admit, that it is conducted upon the supposition that, as x varies from zero to ∞, e^{-ax} remains *constant* on the left-hand, and *decreases* from 1 to 0 on the right-hand.

25. These are the usual proofs that Cos $\infty = 0$; and it is not necessary for me to examine more, as all that I have met with involve erroneous reasoning of a character similar to that noticed in the two above given. Before concluding I wish however to notice one or two other cases in which great caution is necessary in managing the symbols Sin ∞ and Cos ∞.

26. The first which I shall notice is $\int_0^\infty \frac{\operatorname{Sin} ax}{x} dx$, which has been said to exhibit some

singular anomalies. It has been asserted to be equal to $\frac{\pi}{2}$, a result which is manifestly symbolically erroneous, seeing that it does not change sign with a, a property which the expression to be integrated shews *must* belong to the true integral. Such an objection as this would be held to be fatal to a result in other branches of analysis, and I am at loss to conceive why it has not been allowed the same force in this. It is true a proof has been offered that the integral ought to be independent of a; but if any thing can be inferred from that proof it is that the integral ought to be *indefinite* in every case. The proof alluded to is as follows:

$$\text{Since} \qquad \frac{\text{Sin}\, a x}{x} dx = \frac{\text{Sin}\, a x}{a x} d(a x) = \frac{\text{Sin}\, z}{z} dz,$$

$$\therefore \int_0^\infty \frac{\text{Sin}\, a x}{x} dx = \int_0^\infty \frac{\text{Sin}\, z}{z} dz = \int_0^\infty \frac{\text{Sin}\, x}{x} dx:$$

whence it is stated that the value of the integral is in every case the same as when $a = 1$: yet as I have said before, this inference is evidently erroneous when $-a$ is written for a. The probability is that the true integral is such a function of a as is constant for ordinary values of a, and changes sign with a; I say ordinary values, because it is easy to shew that the transformation fails as a approaches zero. For since the equation $ax = z$ must be respected, by means of which the transformation is effected, this shews that were a to become indefinitely small, z would not be ∞ when x approached ∞; but in that case the limits for z would be 0 and y (y being an arbitrary finite quantity). Consequently as a approaches towards zero, the integral approaches towards an indeterminate form as its limit.

The value of the integral when $a = 0$, would therefore seem to be isolated: and cannot be inferred from the above transformation. Expressed in a series the required expression for the integral is

$$a\infty - \tfrac{1}{3}\cdot\frac{(a\infty)^3}{1.2.3} + \tfrac{1}{5}\cdot\frac{(a\infty)^5}{1.2.3.4.5} - \dots\dots$$

which confirms the preceding reasoning in the case when a approaches zero.

27. The next case which I shall consider is $\int_0^\infty \frac{\text{Cos}\, bx}{a^2 + x^2} dx$, which has been stated to be equal to $\frac{\pi}{2a} e^{-ba}$ when b is positive, and to $\frac{\pi}{2a} e^{ba}$ when b is negative. As in the preceding case, so here, the symbolical inaccuracy of the integral brought forward is sufficiently indicated by the acknowledged necessity of empirically changing the form of it. As the erroneous principle by which this result is obtained has found its way into a great number of other integrals which, as well as this, are vitiated and rendered erroneous by it, I shall endeavour fully to expose it.

28. Denoting the required integral by P, we find

$$d_b^2 P - a^2 P = -\int_0^\infty \text{Cos}\, bx\, dx = \frac{\text{Sin}\,(b.0)}{b} - \frac{\text{Sin}\,(b.\infty)}{b}.$$

In the usual process, the last member of this equation is assumed to be zero: and with regard to the first term of it that assumption may be allowed; but the last term of it, it has been the object of this paper to prove, is *indeterminate*. It is also to be remarked, that this term forbids us to make b approach towards zero, because when b is indefinitely small the right-hand member approximates to ∞. Yet regardless of these cautions the right-hand member has been put equal to zero, and the value of P has been then found by integration to be

$$P = Ce^{ab} + C'e^{-ab}.$$

The first term of this integral has been put equal to zero on the ground that $b=\infty$ would make $P=\infty$ were this term allowed to remain. (I shall shew presently that it is not allowable to put $b=\infty$). The value of C' is then found by putting $b=0$, the very supposition which must necessarily render the result erroneous, seeing that $d_b^2 P - a^2 P$ is then equal to ∞. I infer therefore that there is no certain ground for writing $\frac{\pi}{2a}$ for C'; as little indeed as there is for rejecting the term Ce^{ab}. In fact, the given function being unchanged when $-b$ is written for b, the integral must possess the same property, which gives $C=C'$, and therefore we ought rather to write

$$P=C(e^{ab}+e^{-ab}).$$

29. I shall now endeavour to shew that we may not put $b=\infty$ in the value of P.

It is easy to shew that

$$d_a^2(aP)-b^2(aP)=\left\{\frac{2ax\operatorname{Cos}bx}{(a^2+x^2)^2}-\frac{ab\operatorname{Sin}bx}{a^2+x^2}\right\}\begin{matrix}x=\infty\\ x=0\end{matrix}.$$

For all *finite* values of b the right-hand member of this equation vanishes: but when $b=\infty$ the term $\frac{b}{a}\operatorname{Sin}(b\,.\,0)$ cannot be put equal to zero; this term corresponds to the limit $x=0$. Also the intermediate steps by which this equation is obtained from $P=\int_0^\infty \frac{\operatorname{Cos}bx}{a^2+x^2}dx$ forbid us to put $b=0$. Hence if we put the right-hand member of the equation equal to zero, we are to keep in mind that that step involves a prohibition against putting b either equal to zero or ∞. Exclusive then of these values of b, we have

$$d_a^2(aP)-b^2(aP)=0;$$

$$\text{and }\therefore\ aP=Be^{ab}+B'e^{-ab}.$$

For the same reason as before, $B'=B$; and by comparison of this with the value of P (admitting that value to be correct for the present), found in the last article we learn that B is independent both of a and b,

$$\therefore\ P=\frac{B}{a}(e^{ab}+e^{-ab}).$$

How B is to be determined, I know not, seeing that it is not allowable to put $b=0$, which is the usual plan.

30. There is great advantage in forming two distinct differential equations for P, as we may learn from one of them something which may assist us in managing the other. In Art. 29, we have seen that, subject to the condition of b being finite, we have strictly $d_a^2(aP)-b^2(aP)=0$: but this condition will not allow us to strike out the right-hand member of the equation in (28). This shews that B and B' in (29) are functions of b; and (29) shews that C, C' in (28) are functions of a.

In strictness then we ought to integrate the equation

$$d_b^2(aP)-b^2(aP)=-\frac{a}{b}\operatorname{Sin}(b\infty).$$

$$P=\frac{C}{a}(e^{ab}+e^{-ab})+\frac{e^{-ab}}{a}\int\frac{e^{ab}\operatorname{Sin}(b\infty)db}{b}-\frac{e^{ab}}{a}\int\frac{e^{-ab}\operatorname{Sin}(b\infty)db}{b}.$$

C being an absolute constant, the value of which I know no means of determining.

31. It is not necessary to examine other instances of definite integrals the values of which, as they have hitherto been obtained, I believe are not to be relied upon. They involve either the notions that $\text{Sin}\,\infty = \text{Sin}\,(a\,.\,\infty) = 0$, $\text{Cos}\,\infty = \text{Cos}\,(a\,.\,\infty) = 0$; or else depend upon the sum of the series $1 - 1 + 1 - \ldots\ldots$ being $= \frac{1}{2}$. The classes of definite integrals free from one or other of these errors are very few in number, not including some of those which analysts have evidently regarded with especial favor. It will be evident, if what has been written in the preceding pages be allowed, that nothing could be more troublesome than the very general adoption of 0 and ∞ as limits of integration when trigonometrical quantities are involved. The expansion also of functions in the form of series of multiple angles seems in very many instances to be attended with much uncertainty, on account of the fact that $\text{Sin}\,nx$ and $\text{Cos}\,nx$ become discontinuous when n is ∞: and Fourier's celebrated theorem, that any function whatever can be developed in a series of Sines and Cosines of multiple arcs, I regard as being fallacious in all cases where the coefficients do not converge to zero as n becomes ∞. As an instance, I have no doubt that $\frac{1}{2}$ is not equal to $1 + \text{Cos}\,x + \text{Cos}\,2x + \text{Cos}\,3x + \ldots\ldots$ for any value of x whatever. But this is too wide a field to enter upon in this paper, the object of which is to shew that $\text{Sin}\,\infty$ and $\text{Cos}\,\infty$ are not definite quantities, and that $\text{Sin}\,(a\infty)$, $\text{Cos}\,(a\infty)$ are functions of a.

32. Perhaps it may be proper to add something in explanation of what is said in (26), respecting the integral $\int_0^\infty \frac{\text{Sin}\,ax}{x}\,dx$, that it is such a function of a as is constant for ordinary values of a, and changes sign with a. This requires that a distinction should be allowed between arithmetical values and symbolical forms; and such a distinction must be allowed, if any operation with respect to a is to be performed on the expression $\int_0^\infty \frac{\text{Sin}\,ax}{x}\,dx$. An example will best explain what is meant.

In Fourier's *Theory of Heat*, we find the equation

$$\frac{\pi}{4} = \text{Cos}\,y - \tfrac{1}{3}\,\text{Cos}\,3y + \tfrac{1}{5}\,\text{Cos}\,5y - \ldots\ldots$$

This equality is established (pp. 167—174) by a method which is remarkable for its exhibiting no symptoms of the existence of failing cases: and hence it is with surprise we read soon after, that the left-hand member changes its value when y is comprised between certain limits. Guided by the investigation which Fourier gives of the sum of the series $\text{Cos}\,y - \frac{1}{3}\,\text{Cos}\,3y + \ldots\ldots$ we could have had no suspicion that the result is erroneous in any case; yet it is manifestly erroneous when y lies between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Hence the inference is plain, that the value $\frac{\pi}{4}$ is not *symbolically* correct, because it does not contain y, of which the proper form is obviously a function. The author, at page 208, proves that

$$\tfrac{1}{2}\tan^{-1}\left(\frac{2\,\text{Cos}\,y}{e^{x} - e^{-x}}\right) = e^{-x}\,\text{Cos}\,y - \tfrac{1}{3}e^{-3x}\,\text{Cos}\,3y + \tfrac{1}{5}e^{-5x}\,\text{Cos}\,5y - \ldots\ldots$$

And consequently, admitting the propriety of putting $x = 0$, we obtain

$$\tfrac{1}{2}\tan^{-1}(2\infty\;\text{Cos}\,y) = \text{Cos}\,y - \tfrac{1}{3}\,\text{Cos}\,3y + \tfrac{1}{5}\,\text{Cos}\,5y - \ldots\ldots$$

Now from this it is obvious that $\frac{1}{2}\tan^{-1}(2\infty\;\text{Cos}\,y)$ is numerically $= \frac{\pi}{4}$ for non-critical values of y, whenever $\text{Cos}\,y$ is positive; and equal to $-\frac{\pi}{4}$ numerically, whenever $\text{Cos}\,y$ is negative.

It appears from this example that, as has been before remarked, $a\,\infty$, which for distinction I call a symbolical ∞, is not to be confounded with ∞, a mere arithmetical infinity: for the former ceases not to be a function of a.

In (Art. 26.) then when it is said that $\int_0^\infty \frac{\sin a x}{x}\,dx$ is not symbolically equal to $\int_0^\infty \frac{\sin x}{x}\,dx$, the assertion is grounded upon this distinction between $a\,\infty$ and ∞; and it is manifest that in this case, supported as it is by the example quoted from Fourier, $\int_0^\infty \frac{\sin a x}{x}\,dx$ is symbolically a function of a, while $\int_0^\infty \frac{\sin x}{x}\,dx$ is not a function of a. This distinction between $a\,\infty$ and ∞ is of great importance in all definite integrals where the results are understood to be symbolically exact; as they are always supposed to be when they are made use of in obtaining from them other definite integrals by differentiation or integration with regard to parameters. It will be very obvious to any one who examines the definite integrals which have been published, that many of them have been obtained without sufficiently observing this caution with respect to symbolical exactness.

S. EARNSHAW.

CAMBRIDGE,
November 9, 1844.

XX. *On the Connexion between the Sciences of Mechanics and Geometry. By the Rev.* H. Goodwin, *Fellow of Caius College, and of the Cambridge Philosophical Society.*

[Read *February* 10, 1845.]

1. IT is well known, that the first step in proving the elementary propositions of Mechanics is usually to explain that for the purposes of demonstration forces are represented by straight lines, and so simple a step does this appear to be, that it has been complained that students frequently do not perceive that they have passed a distinct boundary-line in their transition from Geometry to Mechanics. It becomes therefore a matter of interesting inquiry, what is the ground of the connexion between the two sciences? is it merely conventional? or only partly so? or not at all? Is the substitution of lines for forces to be looked upon as a mere ingenious device, or has it such a natural basis in the reality of things, as to force itself in one form or another on the mind of every one capable of appreciating the subject? This is the question which I propose to examine.

2. Let it be observed then, that an indefinite straight line is merely the expression of the idea of *direction:* the idea of *direction* is a pure idea capable of no simpler expression, and, as I think, obviously not acquired from experience: no child ever walked from one point to another by a roundabout path, until it discovered that one path was shorter than any other; there might be a difficulty about understanding what was meant by a straight line lying *evenly* between its two extreme points, but about the fact that you would go in one determinate *direction* from one point if you wished to go to the other, there could be no doubt at all. I hold, therefore, that the idea of *direction* is a pure idea, independent of all experience, and that all definitions of a straight line are attempts, accompanied with more or less success, to give verbal expression to this idea*.

And so when I draw a mark on paper which I call a straight line, this is a method of representing rudely to the eye a certain direction, it enables me to speak of that direction intelligibly and to reason about it, the reasoning of course referring not to the mark on the paper, but to the ideal line or direction of which that mark is the visible memorandum.

When we speak of a *finite* straight line, we limit the idea of mere *direction* by introducing the new one of *magnitude.* The idea of magnitude is merely that of comparison of one quantity with another, and a straight line of certain magnitude is represented by taking two points on a given indefinite straight line, such that the distance between them is so many times greater than the distance between two standard points.

Thus a finite straight line given in position is the expression of the combined pure ideas of *direction* and *magnitude;* and a mark on paper standing for such a line is the exhibition to the eye of these two ideas.

And hence, further, we may say that all propositions concerning indefinite straight lines are deductions from the pure idea of direction; all propositions concerning finite straight lines not given in position are deductions from the pure idea of magnitude; and all those concerning finite straight lines given in position are deductions from these two pure ideas combined.

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* See Note (A).

We may put what has been said in other words by asserting that all properties of straight lines are functions either of their direction, or their magnitude, or both; a straight line has no other elements than these, and therefore every thing which is predicated of a straight line is predicated simply in consequence of that straight line having a certain direction and a certain magnitude.

3. Now from this point, I think we can see a simple road into Mechanical Science; for if there be anything physical which depends on no other elements, than those of direction and magnitude, there is no reason why a mark on a piece of paper should not stand for this physical embodiment of the two ideas as well as for the geometrical: and further, if there be anything physical, of which it can be predicated that it has no other elements than direction and magnitude, then all propositions which have been proved for straight lines will have their corresponding propositions, in fact will be true with a change of phraseology, in physics.

In devising a method therefore for representing to the eye the forces on which we reason in Statics, the question is not whether a force can be conveniently represented by an ideal straight line, but whether a force has such qualities that the same representation which serves for demonstrations respecting straight lines, will also serve for demonstrations respecting forces.

4. Now when we come to examine a Statical force, we find that it does involve, or rather it is a physical expression of, those two ideas of *direction* and *magnitude*, and of no others. For we measure a Statical force by the pressure which will counteract it; and what are the questions as to the counteracting force? these two—in what direction it must be applied, and with what intensity; it is clear that neither of these is sufficient without the other; for a particle left to itself under the action of a force will move off in a certain determinate direction, and it is a truth which requires no proof, but is purely axiomatic, that a force, however great, applied in any other but the exactly reverse direction will not prevent motion; and so likewise it is a self-evident fact, that the counteracting pressure must be of a certain determinate magnitude and no other. Thus, to a person who understands what I mean by the term *Force*, it will be apparent that the only ideas involved, are those of *direction* and *magnitude;* any cause tending to produce motion which involves any other element for its complete determination is not a *Force*, it may be called so popularly, but it is not included in the mathematical definition.

And it may be observed here, that as in Euclid, the definition given of a straight line, viz. "that it lies evenly between its extreme points," is virtually superseded by the axiom, that "two straight lines cannot inclose a space," so in elementary books of Mechanics, although the definition is given of a force that it is "any cause which produces or tends to produce motion," yet the fundamental proposition is usually made to depend on the axiom or fact (or whatever it is to be called) that a force may be supposed to act at any point in its direction, which is the same thing as saying, that if the magnitude be given the force depends on direction only.

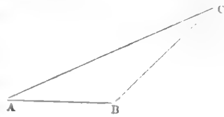
When the science of Mechanics was first studied, the simple view of force which I have given would, of course, not be immediately taken; the effect of force would probably be supposed to depend on other circumstances; but this is a matter of no consequence: the question is merely what we mean by *force* now, and what it is supposed to mean in all mechanical treatises; and it signifies not whether we start with the idea of a cause of tendency to motion involving the ideas of direction and magnitude only, and call the embodiment of that idea *force* by definition, or whether we examine the world we live in, and shew that such are the elements and the only elements of *force*.

5. Let it be granted then that the only ideas involved in that of force, are those of direction and magnitude, and we come to the case (already spoken of by anticipation) of a thing physical, involving exactly the same ideas as the straight line in Geometry; and we therefore lay down this proposition, that every theorem regarding straight lines will have its fellow in Mechanics, that

the theorems of the one science can be translated into the language of the other, and that the demonstration belonging to figures in which the marks represent straight lines will apply precisely as well to similar figures in which the marks represent forces; for in both cases the representation must be conventional: no inkmark can be a straight line, and no proposition concerning straight lines can be true of the inkmarks which represent them, and though it requires a greater abstraction of the mind to speak of an inkmark as a force, yet the speaking of it as a straight line is certainly as really conventional, and the proper utility of the figures in both cases is that they assist the mind artificially in drawing deductions from the pure ideas of direction and magnitude.

Velocity is another instance of a thing physical involving the ideas of direction and magnitude only, and of which therefore it may at once be predicated that the propositions respecting the straight line refer to it *mutatis mutandis*.

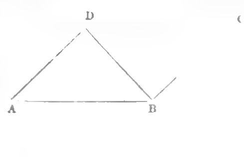
6. When it is said that every proposition respecting the straight line will have its fellow respecting force, it is of course equally true that each proposition in Mechanics will have its fellow in Geometry, and it will be asked, what proposition in Geometry corresponds to the parallelogram, or rather the triangle of forces: to which I reply, that when two lines AB, BC are given in position and magnitude, the straight line joining the points A and C will be as strictly their geometrical resultant, as the force represented by AC will be the resultant of the forces represented by AB, BC: for by speaking of the resultant of two lines we necessarily imply that the two lines are given to determine some third object, and that object must be a straight line, since the resultant of two things of the same kind must be of the same kind with those which produce it, and if there be any line which is to be considered as the resultant of AB, BC it must be AC, since this is the only new line whose position and magnitude is in any way whatever determined by the positions and magnitudes of AB and BC. If therefore we mean by the resultant of two straight lines given in position the straight line which is determined in magnitude and position by those straight lines, and this seems the most obvious meaning to give to the term resultant, then AC is the *resultant* of AB and BC.



The proposition of finding the resultant of two straight lines given in position may be generalized into that of finding the resultant of any number of straight lines forming an imperfect polygon. For if all the sides of a polygon be given except one, then that one will be the resultant of all the rest, inasmuch as it is the only new line whose position and magnitude becomes determinate in virtue of the other sides being given. It may be said that the extremity of one of the last sides may be joined with one of the angular points, and that thus some other line will be determined, but the obvious answer is, that this will not employ *all* the data, and that the line so determined will be the resultant of all those which are really made use of. In fact, a straight line may be given just as really, though not so directly, by giving in position all the other sides of a polygon of which this straight line forms the last; to give those other sides is, I say, precisely the same thing in fact as to give the line itself.

Conversely, a straight line may be considered as the resultant of any system of straight lines which with it form a polygon; and also in such polygon any one side may be called the resultant of all the rest; if two be missing, they cannot be replaced; but if one only, then is that missing one just as fixed and determinate as if it were represented as part of the polygon.

In speaking of the *direction* of lines, it is of course necessary to distinguish between a line AB, and a line BA, the direction of the one being considered exactly the reverse of that of the other. Thus, in the preceding investigation AC is the resultant of AB, BC, not of BA, BC: the resultant of those latter lines would be found by taking AD parallel and equal to BC: then BD would be the resultant of BA and BC.



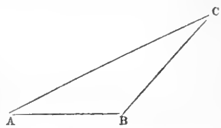
7. The principle of the third side of a triangle being the resultant of the other two may be applied to demonstrate certain propositions in plane Geometry, which I here introduce for illustration's sake.

It may be shewn from this principle, that the lines drawn from the bisections of the sides of a triangle perpendicular to the sides will pass through the same point. For suppose we bisect two of the sides, and draw lines perpendicular to them, (it is of course necessary to *bisect* the sides, because the middle point of a line is the only one which is similarly related to the two extremities), then these indefinite lines determine a new point, viz. the point of intersection; now if we perform the same operation on the third side, the result must be such that no new geometrical element is determined, since everything which is functional of the third side is already implicitly involved in the knowledge of the other two; therefore this third line must pass through the point of intersection of the other two, since if it did not it would determine two new points, which, by what has just been said, is impossible.

The same reasoning applies to the propositions that the lines bisecting the angles of a triangle pass through the same point; and that the lines joining the angular points with the bisections of the sides pass through the same point.

And, I may remark, that we have here the explanation of the fact, that some propositions in pure Geometry admit of simpler proof by referring to mechanical considerations than by the ordinary geometrical methods; as for example the last proposition of those first cited finds its solution at once in the property of the center of gravity of a plane triangle.

8. Taking the view which I have endeavoured to explain of these resultants, it will be obvious how close the analogy is between this case and that of forces; for if AB and BC represent two forces, then AC we know represents their resultant, and in general if two sides of a triangle represent two forces their resultant is given by the third, and still more generally if the sides of an imperfect polygon represent forces their resultant is given by the last side. Now the same thing holds in this case which was true in the case of Geometry, viz. that if AB, BC be given in position and magnitude, the only third term determined is AC; and therefore if AB, BC represent two forces, the magnitude and direction of the force AC is at once determined, but this can be asserted of no other. Now I do not say that this could be considered as a proper proof of the triangle of forces; but I do think that it is a way of considering the subject which, by careful thought, will lead to the intuitive perception of the truth of the proposition. It would be impossible to admit this as the only proof that the force AC would *balance* the two AB, BC, but at least it shews that AC is related to AB and BC in a manner in which no other force is related, that it is at once determined by them, so that to give *them* is to give *it*, and that this can be predicated in the same sense of no other force; and from this it seems possible by degrees to arrive at an intuitive perception of the truth that AC is in fact the resultant of AB, BC. And after all this is the point at which we should endeavour to arrive; the fundamental proposition in mechanics ought not to have a merely artificial basis, and to be such that the mind rather *concedes* it because it cannot deny it, than *sees* it to be true; and I cannot feel a doubt but that there must be some method of viewing the subject, which if we adopt, the fundamental propositions of Mechanics will gradually grow into as perfect axiomatic clearness as do the simple propositions of Geometry*.



9. To illustrate this point by contrast, let us for a moment consider the proof which is frequently given in elementary treatises of the triangle of forces, I mean that which is due to

* See Note (B).

Duchayla. Now this proof is certainly convincing; that is to say, it is not possible to point out any flaw in the steps of demonstration, but for *persuading* the intellect it seems to have no kind of fitness. The proof is essentially artificial, and is based on a simple case of composition of forces which seems very insufficient to suggest, as it is pretended that it does, the result sought. The character of the proof seems, if I may so express myself, to be that of *cunning* rather than honest argument; and yet I think that, however unsatisfactory the proof may appear in this light, we must feel convinced that, supposing it accurate as we do, there must be a meaning and principle about it at bottom, and that these are only smothered and obscured by the artificial contrivances of the demonstration. This I think we shall find to be really the case if we examine the proof in the light of the preceding observations. The first part of the proof seems to involve very faintly the idea of force; the only principle introduced being this—that a force may be applied at any point in its direction; and thus the distinctness of the proposition as a *mechanical* one seems rather obscured, but this difficulty of course vanishes if this first part of the proposition be what I should call a proposition in the science of *Pure Direction;* the proof involves the idea of force only indirectly, and this is exactly what ought to be the case if the proposition be true of several things, of which force is one: it is equally true of velocity, for example: force is an embodiment of the pure idea of *direction*, and therefore all theorems of pure direction will belong to force, not singly, but to it in common with all other embodiments of the same idea. In fact, the first portion of Duchayla's proof appears to be simply this, given two straight lines in position to ascertain the direction which will be determined by them.

But direction is not the only idea involved in *force:* there is magnitude as well, and therefore there is a second portion of the proof we are considering, in which it is shewn that, allowing the triangle of forces so far as direction is concerned, that part which regards magnitude necessarily follows; the extreme simplicity of this part of the proof shews how intimate the connexion must be between the two parts of the proposition, a connexion which I think we should not have been led to expect from anything occurring in the proof itself, for, although the fact that the direction of the resultant of two equal forces will bisect the angle between them is taken as suggestive of the general law of direction, there is not a shadow of a hint that in this simple case the law will hold as respects magnitude: so that a very remarkable proposition is proved by a mere artifice without apparently the least reason in the nature of things why we should anticipate the result. But if we consider the proposition from the same point of view as that from which we regarded the question of the resultant of two straight lines, we shall see that there is a necessary connexion between the two propositions, I mean those respecting *direction* and *magnitude;* for when we had two lines AB, BC given, the resultant AC became at once known both in *direction* and *magnitude;* the two things were co-ordinate, in fact, as this word suggests, they were merely two new *co-ordinates* of C which became known from the two given co-ordinates AB, and BC.

10. On the whole, therefore, I would urge that the proposition which we call the triangle of forces is a result of the combination of the pure ideas of *direction* and *magnitude*, and will therefore be true in some sense of all concrete existences which are embodiments of these two ideas and no other: and therefore I explain the fact of the unmechanical character of the proof we have been considering by observing, that the proposition is more general than the merely mechanical one, includes in fact the triangle of forces, the triangle of lines, the triangle of velocities, the triangle of couples, and perhaps other cognate propositions.

11. This subject will, I think, receive further elucidation as follows:

If l represent any quantity in magnitude only; then if the quantity depend on direction also, it will be necessary to assign the direction in which l is to be measured; but if this be done, it is possible to affect l by a symbol or sign of affection, which shall indicate for itself the direction in which it is measured. This symbol it is well known is $e^{\theta\sqrt{-1}}$, which is such that if l represent a line as to magnitude only, then $le^{\theta\sqrt{-1}}$ will represent the same line measured in a direction making an angle θ with some fixed line.

Now if ABC be a right-angled triangle and $BAC = \theta$, and $AB = l$ in magnitude,

$$\text{then } AB = le^{\theta\sqrt{-1}} = l\cos\theta + \sqrt{-1}\, l\sin\theta$$
$$= AC + \sqrt{-1}\,.\,BC;$$

or, if we omit $\sqrt{-1}$ which is a sign of affection,

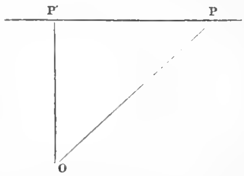
$$AB = AC + BC.$$

We may therefore say, that regard being had to *direction* as well as *magnitude*, AB the hypothenuse is the *sum* of the two sides AC and BC, or perhaps it would be more distinct to say that the hypothenuse is the *equivalent* of the sides, that is to say, that considered as embodiments of the ideas of direction and magnitude one is equivalent to the other; if the direction be disregarded it would be absurd to say that $AB = AC + BC$, and in like manner, if direction only be considered, there is no equivalence between the hypothenuse and sides, but combining the two there is an equivalence, and one may be substituted for the other in all sciences which are developements of these two fundamental ideas.

I may remark further, that we may consider the symbol $le^{\theta\sqrt{-1}}$ as the type of the sciences depending on these ideas, or rather one may say, that the symbol $le^{\theta\sqrt{-1}}$ is the germ from which may be evolved the fundamental principles of these sciences.

12. One more observation may be made on this symbolical representation. The symbol $le^{\theta\sqrt{-1}}$ is as we know equivalent to the expression $l\cos\theta + \sqrt{-1}\, l\sin\theta$, and therefore if this symbol were given to a person as the representation of force, it must at once strike him that the fundamental property of force was that of being made up of two other forces, which we will call as usual its *resolved parts*. Now what I would wish to observe, is, that this connexion supposed to be suggested by the symbolical formula is precisely that which would probably be suggested to the mind when it first began to engage itself with mechanical studies.

For suppose we have a force tending to draw a particle P in any direction OP; then if we wish to examine the nature of this force, and determine its laws, the obvious artifice would seem to be to constrain the particle in various ways, and reason as to the result. Suppose, for instance, a plane drawn perpendicular to OP and indefinitely near the particle P, then it is manifest that the particle will not move at all, this is a point which no one will question, and therefore we arrive at one property of force, namely, that it produces no effect in a direction perpendicular to its own. But, suppose we incline the plane at some angle θ to OP, then motion will ensue if not checked, and the question is, what force acting along the plane will be just sufficient to check motion? To determine this, take any point O in the direction OP and draw OP' perpendicular to the constraining plane, then it is easy to see that whatever relation the line OP has to the original force, the same relation has $P'P$ to the resolved part in the direction PP'; to make this apparent, I shall call a plane drawn through a point in the direction in which a particle has a tendency to move and perpendicular to that direction the *impossible* plane, and then the definition of OP will be, that it is the distance between the impossible planes corresponding to P and O: now suppose any two other parallel planes to be drawn through P and O, and let them be perpendicular to PP', then $P'P$ is the distance between these impossible planes, as OP was between the two former. This being the case, it will be allowed that if OP represents the original force, $P'P$ will represent the resolved part in the direction PP', that is, the resolved part will



be the original force reduced in the proportion of $PO : PP'$ or of $1 : \cos\theta$. The question remaining would be, whether a force could properly be represented by the distance between two impossible planes, a question which might perhaps be answered satisfactorily and without much difficulty if we consider that a finite line taken in the direction of a force will represent the two fundamental properties of force, namely, magnitude and direction. But if it appear in this way that PP' represents the *effective* part of the force acting on P it will be seen that in like manner $P'O$ represents the *ineffective* or *destroyed* part. And therefore the result of the artifice of constraining the particle P would be that when a force l acts on a particle, which is constrained by a plane inclined to the direction of the force at an angle θ, the force is equivalent to a force $l\cos\theta$ which is *effective* and a force $l\sin\theta$ which is *destroyed*, or a force $l\cos\theta$ in the direction in which motion is *possible* and a force $l\sin\theta$ in the *impossible* plane. And this is exactly what would result from considering force under the light of the formula

$$l e^{\theta\sqrt{-1}} = l\cos\theta + \sqrt{-1}\, l\sin\theta.$$

13. This symbolical representation*, though depending on refined principles, is nevertheless, I apprehend, valuable in the discussion of the question before us, because it is generally admitted as a complete method of geometrical representation, and those who study the question must perceive that its complete character is founded on something much deeper than a mere symbolical artifice, inasmuch as it expresses the equivalence between a line, considered in its direction and magnitude, and the two rectangular projections of that line. Now it has been the intention of what immediately precedes to point out the corresponding necessary connexion between a force and its resolved parts, and the perfect applicability of the same symbolical method to the two cases tends, it is presumed, to strengthen the characteristic view of this paper, viz., the essential identity of the Geometrical and Mechanical Sciences, considered as developements of the same combined fundamental ideas.

14. The preceding remarks have been wholly devoted to the consideration of force as acting on a single particle; it was my intention to have attempted a discussion of the case of a system of forces acting on a rigid body, and to have shewn how the science of Mechanics diverges from that of Geometry, by the introduction of this new idea of *Rigidity;* but perhaps what has been already said will be sufficient to put in a clear light the fundamental views which it is my desire to explain: my belief is, that these views contain the shadows at least of important truth, and that they will be seen to do so by any one who will devote attention to the subject. The great question is, what are the fundamental ideas of Elementary Mechanics, and what of Geometry? Are they the same, or are they cognate, or are they altogether distinct? If the last, then the resemblance between certain demonstrations and propositions in the two sciences is a curious and unexplained fact; but if the second or the first, then the explanation is obvious. And if the relation of the two sciences be such as I have represented it, then it seems to me to be most important that it should be recognized, and that for more reasons than one; first, this view connects two streams of truth, usually I believe considered distinct, and traces them to one fountain head, and this is an important simplification, in the same sense and for the same reason that it is an important simplification to trace two phenomena to the same physical cause; but, again, the foundation of geometrical truth is a matter of less question in general than that of mechanical; it is I suppose universally allowed that the propositions in pure Geometry are as they are, because they could not be otherwise, that they are necessary truths in every sense in which truths can be necessary, but there is not, I apprehend, such clearness of thought prevalent respecting mechanical truth. It is difficult to make out from the ordinary books on the subject

* See Note (C).

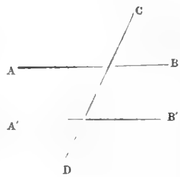
what the writer's belief is respecting the nature of the truths which he is developing; now this point is entirely resolved if it is shewn that the principles of Mechanics are identical with those of Geometry, that the two sciences not only have certain analogies, but are in essence identical as being two developements of the selfsame ideas, and hence, if this be true, we see at once the necessary character of the truths of Mechanics, or at least we shew them to stand on the same ground as others which are supposed to be admitted as necessary. I will close this paper by saying, that although I am well aware that what I have said in favour of the views propounded may not with many appear to amount to demonstration, and indeed perhaps demonstration in such a subject is not altogether possible, yet I am persuaded of their fundamental correctness by this consideration as much as by any, viz., they do seem to point out the road to absolute intuition of truth, they seem to mark out a method of thought according to which the elementary truths of Mechanics will present themselves gradually with axiomatic clearness. And certainly, whether this method be true or not, it cannot I think be doubted by any one who has reflected on the foundations of truth, that this is the natural course, viz., that all demonstrations gradually merge in intuition, and that all human knowledge converges towards that absolute intuition which is the attribute of the divine mind.

NOTES.

Note (A), page 269.

The word *direction* appears to be the best abstract word for expressing the idea which is intended to be embodied in the concrete form of a *straight line;* the evil of concrete terms is that they appear to assign physical existence to that which can have none, and by thus leading away the mind from the true idea tend to prevent the intuition of geometrical truth. If the idea intended to be embodied in the terms *point*, *straight line*, and *angle* be conceived in their abstract form, the simple propositions respecting them at once assume the character of axiomatic truth. I will here put down what appear to me to be the best abstract terms for expressing these three geometrical elements—

1. A point = Position.
2. A straight line = Direction.
3. An angle = Inclination of directions.



I will illustrate the intuitizing force of these terms by applying them to the doctrine of parallels. The idea of parallelism is that of *identity of direction without identity of position;* and from this it is evident that a straight line CD falling on two parallel lines AB. $A'B'$ makes the alternate angles equal; for since the question is one of *direction* only, whatever is predicated of the line AB may be predicated of the line $A'B'$, since they differ in *position* only and not in *direction*.

Note (B), page 272.

The proposition in pure Geometry which seems more than any other connected with Mechanics is Euclid I. 32, and it will be worth while to point out the self-evidence of this proposition both for its own sake, and also from the assistance it will afford in the intuition of the cognate mechanical proposition.

I observe first, that Euclid's last corollary seems to be the easiest proposition to grasp, and will be admitted without formal demonstration as soon as its meaning is apprehended. For if we consider the changes of direction which a straight line undergoes by successively coinciding with the sides of the polygon it is clear that when it has been made to coincide with all in succession, it will at last come into its original position, but a line which has revolved and come into its original position must have described four right angles; whence the proposition is manifest. From this of course the first corollary and the proposition itself immediately follow.

It appears therefore that Euclid I. 32. is only a form of the self-evident proposition, that a straight line being made to deviate from its original direction cannot assume it again until it has deviated through four right angles.

Now the condition of forces being such as will produce equilibrium, is simply that the lines respecting them shall form a polygon. And this proposition is I believe only an expression of the fact, that two fores cannot counteract each other unless they act in the same straight line, or, to express myself more in conformity with the geometrical proposition, that if a force has been made to change its direction it cannot produce the same effect as before unless its deviation has been through four right angles; but this thought I have not yet fully developed.

Note (C), page 275.

It may be well to remark here that the symbol $re^{\theta\sqrt{-1}}$ being the complete expression of magnitude and direction is also the complete expression of linear and angular distance, if r be measured from a fixed point, and θ from a fixed straight line: consequently $re^{\theta\sqrt{-1}}$ may be taken as the complete position-index of a point, or of a physical particle, in a given plane.

If we consider a particle whose position is changing with the time (t), then the symbol $\frac{d}{dt}.re^{\theta\sqrt{-1}}$ will express the complete variation of the position-index, and will therefore give the magnitude and direction of its velocity.

In like manner the symbol $\frac{d^2}{dt^2}.re^{\theta\sqrt{-1}}$ will give the complete variation of the velocity, and will therefore be the symbol for the accelerating force in both magnitude and direction.

Now suppose the particle to be in motion under the action of any forces, the complete expressions for which are $P.e^{\phi\sqrt{-1}}$, $P'.e^{\phi'\sqrt{-1}}$, &c.: then if M be the mass of the particle we shall have for its equation of motion

$$M\frac{d^2}{dt^2}.re^{\theta\sqrt{-1}} = Pe^{\phi\sqrt{-1}} + P'e^{\phi'\sqrt{-1}} + \ldots\ldots$$

This equation I have given merely in illustration of the principles of the preceding memoir, but in the *Cambridge Mathematical Journal*, (Vol. IV. p. 177.) I have shewn that the symbol $re^{\theta\sqrt{-1}}$ may be applied to mechanical investigations with considerable practical convenience.

H. GOODWIN.

XXI. *On the Pure Science of Magnitude and Direction.* *By the* REV. H. GOODWIN, *Fellow of Caius College, and of the Cambridge Philosophical Society.*

[Read *May* 12, 1845.]

IN a former Memoir, I have endeavoured to point out the *à priori* and necessary character of the fundamental proposition in Mechanics, by connecting it with the propositions of Geometry, and so bringing the demonstrative character of the two sciences into one and the same point of view. I there pointed out that the only elements of Force are *Magnitude* and *Direction*, and therefore that the only simple ideas of which the term Force is the expression are those of Magnitude and Direction, and hence, that all propositions respecting Force ought to follow demonstratively and perhaps intuitively from the possession of those two ideas combined, even as the propositions respecting straight lines arise necessarily from the same two ideas. In the course of that Memoir, I spoke of such a Science as that of *pure direction*, which should include within itself the Sciences of Geometry or rather of Position, of Kinematics, of Mechanics, and possibly other Sciences; it is the design of the present Memoir, to attempt to establish the fundamental Proposition of such a Science, or, as perhaps it may be more properly called, the *pure Science of Magnitude and Direction*.

1. The fundamental problem will be, to determine the combined effect of any number of causes, the magnitude and direction of each of which is given. It will be seen that this statement is perfectly general; for a line given in a certain direction may be looked upon as *cause*, the point in space determined by its extremity as *effect*, or if two lines be given, having an extremity in common, the line joining the other extremities which is thus determined may be regarded as *effect;* so likewise, if a particle be animated by two simultaneous velocities, they may be looked upon as *causes*, the resultant velocity as *effect;* and if a particle be acted upon by two forces, the resultant pressure will be the *effect* which results from the two given forces as *causes*, and hence it will appear, that the fundamental problem is to find the combined effect of any number of given causes.

2. Now, if the direction in which all the causes acted were the same, it is clear that the combined effect would be found by mere addition of the quantitative symbols which measure their respective effects; the only postulate here involved is, that two causes do not modify each other's effects, a postulate which is of the nature of an axiom, and which merely expresses such truths as these, that if a point be taken in a straight line at a distance (a) from a fixed origin, and another point at a distance (b) from the former, then the distance of the point last determined from the origin will be $a + b$, or again, that if there be two forces, one of which can lift a weight P, and another a weight Q, then the two together can lift a weight $P + Q$.

3. Hence, when the direction of a number of causes is the same, the process of addition serves to determine their combined effects, but when the directions are different, it will be necessary to determine according to what law variation of direction modifies the effect of a cause; in other words, suppose we take P as the quantitative symbol of the effect of a certain cause in a given direction, what will be the symbol for the effect of a cause of equal intensity whose direction makes an angle θ with the given direction?

Now, it may be assumed, that the effect of the oblique cause can be measured by a symbol of the form $Pf(\theta)$, where $f(\theta)$ is a modifying function, which would be 1 if θ were zero and whose general form must be determined. This assumption appears admissible, because if there be a symbolical expression for the effect of an oblique cause, it can be of no form more general than that assumed, and if there be no such symbol, this will appear by the impossibility of determining the form of $f(\theta)$.

4. To determine the form of the function f, I observe, that the fundamental law of such causes as we are considering is, that the exact reverse of any cause whose magnitude and direction are given is one of equal magnitude and exactly opposite direction; so that, if we denote opposite affections by + and −, then $+P$ must change into $-P$, while θ increases from 0 to π: moreover, the change of P, $\{$or rather of $Pf(\theta)\}$, as θ varies continuously, must manifestly be continuous, and not only continuous but *uniform*, that is, the rate at which the affection changes from + to − must be the same at all stages of the change, since there is no reason why the change should be more rapid for one value of θ than for another:—speaking symbolically, it may be said that $\dfrac{f(\theta+a)-f(\theta)}{f(\theta)}$ must be the same for all values of θ if a be a given quantity. This law, to which $f(\theta)$ is subject, and which flows at once from the pure idea of *direction*, is sufficient to determine the form of f. For suppose the angle π to be divided into n equal parts; then, if the direction vary through one of them, the symbol representing the effect of the oblique cause will be $Pf\left(\dfrac{\pi}{n}\right)$; if it vary through two such divisions, the symbol becomes $Pf\left(\dfrac{\pi}{n}\right)f\left(\dfrac{\pi}{n}\right)$, or $P\left\{f\left(\dfrac{\pi}{n}\right)\right\}^2$, (it also becomes $Pf\left\{\dfrac{2\pi}{n}\right\}$), and so on; and when the direction has varied through n angles each equal to $\dfrac{\pi}{n}$, the symbol becomes $P\left\{f\left(\dfrac{\pi}{n}\right)\right\}^n$, but by what has just been said this symbol must represent the exact reverse of $+P$, and must therefore be $= -P$: hence we have

$$P\left\{f\left(\frac{\pi}{n}\right)\right\}^n = -P,$$

$$\left\{f\left(\frac{\pi}{n}\right)\right\}^n = -1,$$

$$f\left(\frac{\pi}{n}\right) = (-1)^{\frac{1}{n}} = \cos\frac{\pi}{n} + (-1)^{\frac{1}{2}}\sin\frac{\pi}{n},$$

and if we put $\dfrac{\pi}{n} = \theta$,

$$f(\theta) = (-1)^{\frac{\theta}{\pi}} = \cos\theta + (-1)^{\frac{1}{2}}\sin\theta \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

It will be observed, that n may be made as large as we please, and therefore, the condition will be satisfied of θ varying continuously from 0 to π; also the change of f is not only continuous but uniform, for it is clear that the expression $(-1)^{\frac{\theta}{\pi}}$ satisfies the condition that $\dfrac{f(\theta+a)-f(b)}{f(\theta)}$ shall be the same for all values of θ; and this follows necessarily from the mode of determining f, without assuming that $f(\pi) = -1$, for the fundamental law of variation of f is expressed by the equation

$$\left\{f\left(\frac{\beta}{n}\right)\right\}^n = f(\beta),$$

$$\text{or,} \qquad f(\theta) = \{f(\beta)\}^{\frac{\theta}{\beta}},$$

constructed, can hardly be considered as affording us any material insight into the laws of nature; nor will they enable us to pass from the consideration of the phenomena from which they were derived to that of others of a different class, although depending on the same causes.

In reflecting on the principles according to which the motion of a fluid ought to be calculated when account is taken of the tangential force, and consequently the pressure not supposed the same in all directions, I was led to construct the theory explained in the first section of this paper, or at least the main part of it, which consists of equations (13), and of the principles on which they are formed. I afterwards found that Poisson had written a memoir on the same subject, and on referring to it I found that he had arrived at the same equations. The method which he employed was however so different from mine that I feel justified in laying the latter before this Society*. The leading principles of my theory will be found in the hypotheses of Art. 1, and in Art. 3.

The second section forms a digression from the main object of this paper, and at first sight may appear to have little connexion with it. In this section I have, I think, succeeded in shewing that Lagrange's proof of an important theorem in the ordinary theory of Hydrodynamics is untenable. The theorem to which I refer is the one of which the object is to show that $udx + vdy + wdz$, (using the common notation,) is always an exact differential when it is so at one instant. I have mentioned the principles of M. Cauchy's proof, a proof, I think, liable to no sort of objection. I have also given a new proof of the theorem, which would have served to establish it had M. Cauchy not been so fortunate as to obtain three first integrals of the general equations of motion. As it is, this proof may possibly be not altogether useless.

Poisson, in the memoir to which I have referred, begins with establishing, according to his theory, the equations of equilibrium and motion of elastic solids, and makes the equations of motion of fluids depend on this theory. On reading his memoir, I was led to apply to the theory of elastic solids principles precisely analogous to those which I have employed in the case of fluids. The formation of the equations, according to these principles, forms the subject of Sect. III.

The equations at which I have thus arrived contain two arbitrary constants, whereas Poisson's equations contain but one. In Sect. IV. I have explained the principles of Poisson's theories of elastic solids, and of the motion of fluids, and pointed out what appear to me serious objections against the truth of one of the hypotheses which he employs in the former. This theory seems to be very generally received, and in consequence it is usual to deduce the measure of the cubical compressibility of elastic solids from that of their extensibility, when formed into rods or wires, or from some quantity of the same nature. If the views which I have explained in this section be correct, the cubical compressibility deduced in this manner is too great, much too great in the case of the softer substances, and even the softer metals. The equations of Sect. III. have, I find, been already obtained by M. Cauchy in his *Exercises Mathématiques*, except that he has not considered the effect of the heat developed by sudden compression. The method which I have employed is different from his, although in some respects it much resembles it.

The equations of motion of elastic solids given in Sect. III. are the same as those to which different authors have been led, as being the equations of motion of the luminiferous ether in vacuum. It may seem strange that the same equations should have been arrived at for cases so different; and I believe this has appeared to some a serious objection to the employment of those equations in the case of light. I think the reflections which I have made at the end of Sect. IV., where I have examined the consequences of the law of continuity, a law which seems to pervade nature, may tend to remove the difficulty.

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* The same equations have also been obtained by Navier in the case of an incompressible fluid, (*Mém. de l'Institut*, T. VI.) but his principles differ from mine still more than do Poisson's.

SECTION I.

Explanation of the Theory of Fluid Motion proposed. Formation of the Differential Equations. Application of these Equations to a few simple cases.

1. Before entering on the explanation of this theory, it will be necessary to define, or fix the precise meaning of a few terms which I shall have occasion to employ.

In the first place, the expression "the velocity of a fluid at any particular point" will require some notice. If we suppose a fluid to be made up of ultimate molecules, it is easy to see that these molecules must, in general, move among one another in an irregular manner, through spaces comparable with the distances between them, when the fluid is in motion. But since there is no doubt that the distance between two adjacent molecules is quite insensible, we may neglect the irregular part of the velocity, compared with the common velocity with which all the molecules in the neighbourhood of the one considered are moving. Or, we may consider the mean velocity of the molecules in the neighbourhood of the one considered, apart from the velocity due to the irregular motion. It is this regular velocity which I shall understand by the *velocity of a fluid at any point*, and I shall accordingly regard it as varying continuously with the co-ordinates of the point.

Let P be any material point in the fluid, and consider the instantaneous motion of a very small element E of the fluid about P. This motion is compounded of a motion of translation, the same as that of P, and of the motion of the several points of E relatively to P. If we conceive a velocity equal and opposite to that of P impressed on the whole element, the remaining velocities form what I shall call the *relative velocities* of the points of the fluid about P; and the motion expressed by these velocities is what I shall call the *relative motion* in the neighbourhood of P.

It is an undoubted result of observation that the molecular forces, whether in solids, liquids, or gases, are forces of enormous intensity, but which are sensible at only insensible distances. Let E' be a very small element of the fluid circumscribing E, and of a thickness greater than the distance to which the molecular forces are sensible. The forces acting on the element E are the external forces, and the pressures arising from the molecular action of E'. If the molecules of E were in positions in which they could remain at rest if E were acted on by no external force and the molecules of E' were held in their actual positions, they would be in what I shall call a state of *relative equilibrium*. Of course they may be far from being in a state of actual equilibrium. Thus, an element of fluid at the top of a wave may be sensibly in a state of relative equilibrium, although far removed from its position of equilibrium. Now, in consequence of the intensity of the molecular forces, the pressures arising from the molecular action on E will be very great compared with the external moving forces acting on E. Consequently the state of relative equilibrium, or of relative motion, of the molecules of E will not be sensibly affected by the external forces acting on E. But the pressures in different directions about the point P depend on that state of relative equilibrium or motion, and consequently will not be sensibly affected by the external moving forces acting on E. For the same reason they will not be sensibly affected by any motion of rotation common to all the points of E; and it is a direct consequence of the second law of motion, that they will not be affected by any motion of translation common to the whole element. If the molecules of E were in a state of relative equilibrium, the pressure would be equal in all directions about P, as in the case of fluids at rest. Hence I shall assume the following principle:—

That the difference between the pressure on a plane in a given direction passing through any point P *of a fluid in motion and the pressure which would exist in all directions about* P *if the fluid in its neighbourhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about* P; *and that the relative motion* the other in the plane perpendicular to it or the impossible plane, and this being the case, all that is done by equation (B) is to assign the relative magnitudes of the two components. We have, in fact, these two things known respecting the oblique cause which we denote by $Pf(\theta)$, first, that

$$P.f(\theta) = P\cos\theta.f(0) + P\sin\theta.f\left(\frac{\pi}{2}\right);\ldots\ldots(B)$$

and, secondly, that the oblique cause may be supposed to result from two component causes, for one of which $\theta = 0$ and for the other $\theta = \frac{\pi}{2}$, and putting these two things together, there can, I conceive, be no doubt as to the conclusion that these components are represented by $P\cos\theta$ and $P\sin\theta$ respectively. I am not saying that the auxiliary consideration just used is really necessary for the interpretation of the equation (B), for I am inclined to believe that the generality of symbolical interpretation would justify us at once in construing the equation thus:—the effect of P acting at an angle θ = the effect of $P\cos\theta$ acting directly, *combined with* the effect of $P\sin\theta$ acting at right angles to the original direction; but at least the objection, if there be one, seems removed by the consideration adduced, and that is my reason for adducing it.

9. On the whole, I would submit that the preceding investigation not only is free from solid objection, but is in fact the true mode of viewing the subject; because it rests upon the leading idea of a uniform continuous passage of a cause from + to −, while its direction varies continuously. And if it be objected, that physical laws cannot be conceived of as the results of symbolical equations, it is to be answered that this is exactly the advantage of this mode of viewing the subject, that it shews that such laws as that of the composition of forces are not physical laws, in the sense of being laws known by experience or by induction from observation, but are necessary laws in the most exact sense of the word: there is nothing more incredible in the fact of Demoivre's formula containing the laws of Mechanics, than in that of its containing the laws of Space, and it is as credible that it should be capable of proof from that formula, that three forces are in equilibrium when they are each proportional to the sine of the angle contained between the other two, as that the sides of a triangle are proportional to the opposite sides. The fact of our making these conclusions depend on the interpretation of symbols is in the present state of analysis no objection at all, and it may well be supposed that some such method would be necessary in order to bring into one investigation subjects at first sight apparently so distinct as the laws of space and the laws of equilibrium of forces.

10. And I think it cannot be said that the method adopted in this paper is so artificial as those which are sometimes applied to what are called *functional* proofs of the rule for the composition of forces; for although the quantity $(-1)^{\frac{1}{2}}$ or $\sqrt{-1}$ enters into the investigation, still it is to be remembered, (and I wish to lay great stress upon this,) that this quantity is not introduced by any principles peculiar to this paper, *it enters by mathematical necessity and must be interpreted*; the only equation which it is incumbent upon me to prove is the equation $f(\theta) = (-1)^{\frac{\theta}{\pi}}$, and if this be established, the remainder necessarily follows. Indeed, so far from this method being of an artificial and therefore incomplete kind, I would venture to question whether the unsatisfactory character of some functional proofs of the law of composition of forces, and the extremely complicated nature of all, may not arise from the oversight of the fact that a function exists, which will express an oblique force in its totality and not only so far as it is effective. On this subject, however, I will not enlarge, but only remark that it seems to me a point of great beauty that a symbol, of such peculiar form as $\cos\theta + (-1)^{\frac{1}{2}}\sin\theta$, which meets us at every turn in analysis, should be the complete expression of a law by which all nature is governed.

11. I have now concluded all that I wish to say on the principal subject of this paper, but before bringing it to an end I am desirous of making some observations on the general question of the transition of quantities from the + to the − affection, which will, I believe, illustrate my general design.

That design may be stated to have been, to shew that there is in the nature of things one law according to which causes, which depend solely on magnitude and direction, vary with their obliquity from a given direction to the exact opposite, and according to which the cause P varies till it becomes $-P$. Now in considering the general case of the passage of a quantity from the + to the − affection, it is to be observed, that if the quantity be necessarily of one dimension, as *time* for example, then future time being denoted by $+t$, past time will be denoted by $-t$, and t will vary from + to − by simple diminution and passage through zero; in this case the sign $\sqrt{-1}$ can have no place as a sign of affection; I believe there is no conceivable interpretation to be put upon $t\sqrt{-1}$. And in like manner if distance be measured along a fixed axis the variation from + to − is by simple diminution; but, if space be considered in two dimensions so that a line may assume all oblique positions in varying from + to −, then the symbol $(-1)^{\frac{\theta}{\pi}}$ indicates the law according to which the change from + to − takes place. Here then are two laws according to which the affection of a quantity may be reversed, and there may possibly be others, and probably instances might be found in which such change is abrupt not continuous; for instance, Dr. Peacock illustrates the properties of $\sqrt{-1}$ by saying*, that supposing $+a$ to represent property *possessed*, and $-a$ to represent *debt*, then $\sqrt{-1}.a$ may represent property *deposited* "which admits of similar relations when considered as property possessed and property owed by another person;" it must however, I think, be admitted, that the use of the symbol $\sqrt{-1}$ in this case is conventional in a sense in which it is not when applied to lines or forces, and it may be doubted whether the symbol so used can be applied to the practical purposes of investigation; and indeed, if I might hazard an opinion, I should hold it probable that the symbol $\sqrt{-1}$ can only then be successfully used when it expresses a particular stage in the continuous change of affection from + to −.

12. It is not difficult to devise laws, according to which a quantity may change from + to −, other than those which have been specified. Suppose for example $f(\theta)$ to represent the sign of affection for a quantity P, and suppose

$$Pf(\theta) = P(-1)^{\sin\frac{\theta}{2}};$$

this form satisfies the condition that $f(\theta) = 1$ and $f(\pi) = -1$, and therefore so far agrees with the actual law of lines, forces, &c., as that the affection of P passes from + to − while θ varies from 0 to π. But if we examine this case, we find that it is widely diverse from the actual law just mentioned; for we have

$$f(\theta) = \cos\left(\pi \sin\frac{\theta}{2}\right) + (-1)^{\frac{1}{2}} \sin\left(\pi \sin\frac{\theta}{2}\right).$$

Now if $\theta = 0$ $f(\theta) = 1$

$$\theta = \frac{\pi}{3} \quad f(\theta) = (-1)^{\frac{1}{2}};$$

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* *Algebra*, 1st Edition, page 366.

$$\therefore\ f(\theta) = \cos\left(\pi \sin\frac{\theta}{2}\right) f(0) + \sin\left(\pi \sin\frac{\theta}{2}\right) f\left(\frac{\pi}{3}\right),$$

$$\text{and } Pf(\theta) = P\cos\left(\pi \sin\frac{\theta}{2}\right) f(0) + P\sin\left(\pi \sin\frac{\theta}{2}\right) f\left(\frac{\pi}{3}\right).$$

In this case, therefore, it would seem that the oblique quantity P would be equivalent to two components, one in the original direction, the other inclined at an angle of 60^0, the magnitude of the former being $P\cos\left(\pi \sin\frac{\theta}{2}\right)$, that of the latter $P\sin\left(\pi \sin\frac{\theta}{2}\right)$. But there will be a distinction between this and that of the ordinary formula $\cos\theta + (-1)^{\frac{1}{2}}\sin\theta$ which is to be observed; for in this latter case the impossible plane determined by $f(\theta) = (-1)^{\frac{1}{2}}$ coincides with that determined by $f(\theta) = (-1)^{\frac{1}{2}}$, but in the present hypothetical case, we have two impossible directions, one corresponding to $\theta = \frac{\pi}{3}$ or $f(\theta) = (-1)^{\frac{1}{2}}$, the other to $\theta = 2\pi - \frac{\pi}{3}$ or $f(\theta) = -(-1)^{\frac{1}{2}}$. And therefore that which is analogous to the impossible plane in the ordinary case is an *impossible cone* whose semi-vertical angle is an angle of 60^0.

There will be two impossible cones in like manner belonging to the formula

$$f(\theta) = (-1)^{\frac{\theta^2}{\pi^2}} = \cos\frac{\theta^2}{\pi} + (-1)^{\frac{1}{2}}\sin\frac{\theta^2}{\pi}$$

which, together with the formula (A), are particular instances of the general form,

$$f(\theta) = (-1)^{\left(\frac{\theta}{\pi}\right)^n} = \cos\frac{\theta^n}{\pi^{n-1}} + (-1)^{\frac{1}{2}}\sin\frac{\theta^n}{\pi^{n-1}} \dots\dots\dots\dots\dots (C).$$

13. The preceding cases are examples of the general formula $f(\theta) = (-1)^{\Theta}$, where Θ is some function of θ which $= 0$ when $\theta = 0$, and $= 1$ when $\theta = \pi$, the direction or directions for which $f(\theta) = (-1)^{\frac{1}{2}}$ are given by $\Theta = \frac{2k+1}{2}$. It may be observed, that all examples must have this property in common; that if we suppose a quantity P to be composed of two others whose directions are the line for which $\theta = 0$, and that for which $f(\theta) = (-1)^{\frac{1}{2}}$, and if we call these components x and y respectively, then x and y satisfy the condition

$$x^2 + y^2 = P^2;$$

and therefore if x and y be regarded as oblique co-ordinates of a point, the locus of that point is an ellipse; in the case of $f(\theta) = (-1)^{\frac{\theta}{\pi}}$ this locus becomes a circle.

14. It is evidently possible to vary indefinitely the law according to which $f(\theta)$ shall vary from $+1$ to -1, while θ increases from 0 to π, even though we confine ourselves to the form $f(\theta) = (-1)^{\Theta}$; and all these laws will express modes in which the affection of a quantity may be diametrically reversed; I am disposed to look upon most of them as fictitious generalizations which can have no type in the nature of things, just as we might construct a system of geometry of four dimensions which could have nothing real corresponding to it. It may be possible, however, to find some which have not this fictitious character, and which express physical laws. We shall obtain a distinct conception of the manner in which the law expressed by the formula $f(\theta) = (-1)^{\frac{\theta}{\pi}}$ differs from all others, by observing that if $(-1)^{\Theta}$ expresses the law of change from $+$ to $-$, the gradual change of affection, as compared with a change in the value of θ, will be

expressed by $\frac{d\Theta}{d\theta}$; now if $\Theta = \frac{\theta}{\pi}$, $\frac{d\Theta}{d\theta} = \frac{1}{\pi}$ a constant quantity, which can be true of no other law satisfying the conditions $f(0) = 1$ and $f(\pi) = -1$. If, for instance, $\Theta = \frac{\theta^2}{\pi^2}$, a form which satisfies the conditions $f(0) = 1$ and $f(\pi) = -1$, we have $\frac{d\Theta}{d\theta} = 2\frac{\theta}{\pi^2}$, and therefore the intensity of the minus affection which is measured by Θ would increase more rapidly as the angle θ approached the value π. And this also shews distinctly what is meant by saying that the change of affection, in such causes as we have been considering, is *uniform*, for this is, in fact, saying that $\frac{d\Theta}{d\theta}$ must be constant, a condition which must manifestly be satisfied when the case is one of pure direction, and when, therefore, there is no reason why Θ should increase more rapidly for one value of θ than another. Whether there be real cases of change of affection coming under the general type represented by $(-1)^{\Theta}$, in which this condition of uniformity is not satisfied, remains to be seen.

15. The condition of *uniform* change of affection is satisfied by the function $\Theta = \frac{\theta}{2a}$, where a is some constant angle, which, in the actual case of pure direction, is a right angle. If Θ have this value, we have

$$f(\theta) = \cos\frac{\pi\theta}{2a} + (-1)^{\frac{1}{2}} \sin\frac{\pi\theta}{2a};$$

and the impossible direction is given by

$$\theta = a.$$

For example, let $a = \frac{\pi}{4}$, then

$$f(\theta) = \cos 2\theta + (-1)^{\frac{1}{2}} \sin 2\theta,$$

a formula which represent the variation of a cause which changes uniformly, and produces exactly opposite effects in directions at right angles to each other. It seems not improbable that this formula may be found to represent something real: may it not represent the following case? Suppose a disturbing cause in an elastic medium which propagates simultaneously a condensed wave in two opposite directions, and a rarified wave in the direction perpendicular to them: then if ϕ be the condensation which would exist at a given time and a given distance from the centre, on the supposition of the condensing cause only acting, may not the complete expression for the condensation in the direction determined by the angle θ be

$$\phi \cos 2\theta + (-1)^{\frac{1}{2}} \phi \sin 2\theta?$$

A rough approximation to this case would be that of a tuning-fork.

16. A more general law than that expressed by the formula $f(\theta) = (-1)^{\Theta}$ is given by

$$f(\theta) = m(-1)^{\Theta} + m'(-1)^{\Theta'} + \&c.$$

There is only one example of this formula which I shall notice:

Suppose we have a quantity P determined by the equation

$$Pf(\theta) = a\cos\theta + (-1)^{\frac{1}{2}} b \sin\theta \ldots\ldots (D),$$

which comes under the above form of $f(\theta)$ for the equation, may be written

$$Pf(\theta) = \frac{a+b}{2}(-1)^{\frac{\theta}{\pi}} + \frac{a-b}{2}(-1)^{-\frac{\theta}{\pi}}.$$

We have here an effect which results from two causes, which separately vary uniformly. Considering equation (D), we may observe that the difference between it and the equation

$$Pf(\theta) = P\cos\theta + (-1)^{\frac{1}{2}} P\sin\theta \ldots\ldots (E)$$

is this, that, although both give $\theta = 0$ and $\theta = \frac{\pi}{2}$ as the directions of the components, yet the values of $Pf(0)$ and $Pf\left(\frac{\pi}{2}\right)$ which are the same in the latter case (omitting the sign of affection) are different in the former: for in equation (D)

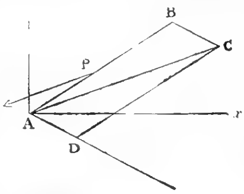
$$Pf(0) = a,$$

$$Pf\left(\frac{\pi}{2}\right) = b(-1)^{\frac{1}{2}}.$$

In fact, considering the equations (D) and (E) geometrically, the former represents an *ellipse*, the latter a *circle:* the angle θ will manifestly be the *eccentric anomaly.*

My reason for introducing the formula (D) is that I may remark that, whereas the formula (E) represents force considered in the light of *pure direction*, the formula (D) corresponds to a case of *polarity*. Pure force must, of course, be free from any polarity, that is to say, its absolute magnitude must be the same in whatever direction it acts, the direction will *modify* the effect, not *diminish* or *increase* it; but there are complicated instances of force in which this is not the case, but in which there is polarity; for example, in the case of an elastic medium under constraint from the action of the particles of a crystallized body which contains it. Now the formula (D) appears to be exactly calculated to express this kind of force; to fix our conceptions let an elastic medium have the same properties in all sections parallel to the plane of xy, and have polarity in that plane; consider any one section, and let the properties of this section of the medium be symmetrical about the axes of x and y, then the origin will be a position of rest for a particle, and if it be disturbed, the force of restitution may be represented by such a formula as (D).

In examining that formula we find that there are two directions perpendicular to each other, for which the force of restitution is in the direction of displacement; for all other displacements the force of restitution is not in the same direction, but will have to be determined thus; let AP be the direction of displacement: take APB proportional to $a + b$, and AD, making the same angle with Ax as AB, proportional to $a - b$; complete the parallelogram $ABCD$, and draw through P a line parallel to the diagonal AC; this will be the direction of the force of restitution. Hence then it appears, that the formula (D) will represent the kind of law which determines the force of restitution on a disturbed particle in the case of uniaxal crystals.



H. GOODWIN.

XXII. *On the Theories of the Internal Friction of Fluids in Motion, and of the Equilibrium and Motion of Elastic Solids. By* G. G. Stokes, M.A., *Fellow of Pembroke College.*

[Read *April* 14, 1845.]

The equations of Fluid Motion commonly employed depend upon the fundamental hypothesis that the mutual action of two adjacent elements of the fluid is normal to the surface which separates them. From this assumption the equality of pressure in all directions is easily deduced, and then the equations of motion are formed according to D'Alembert's principle. This appears to me the most natural light in which to view the subject; for the two principles of the absence of tangential action, and of the equality of pressure in all directions ought not to be assumed as independent hypotheses, as is sometimes done, inasmuch as the latter is a necessary consequence of the former*. The equations of motion so formed are very complicated, but yet they admit of solution in some instances, especially in the case of small oscillations. The results of the theory agree on the whole with observation, so far as the time of oscillation is concerned. But there is a whole class of motions of which the common theory takes no cognizance whatever, namely, those which depend on the tangential action called into play by the sliding of one portion of a fluid along another, or of a fluid along the surface of a solid, or of a different fluid, that action in fact which performs the same part with fluids that friction does with solids.

Thus, when a ball pendulum oscillates in an indefinitely extended fluid, the common theory gives the arc of oscillation constant. Observation however shows that it diminishes very rapidly in the case of a liquid, and diminishes, but less rapidly, in the case of an elastic fluid. It has indeed been attempted to explain this diminution by supposing a friction to act on the ball, and this hypothesis may be approximately true, but the imperfection of the theory is shown from the circumstance that no account is taken of the equal and opposite friction of the ball on the fluid.

Again, suppose that water is flowing down a straight aqueduct of uniform slope, what will be the discharge corresponding to a given slope, and a given form of the bed? Of what magnitude must an aqueduct be, in order to supply a given place with a given quantity of water? Of what form must it be, in order to ensure a given supply of water with the least expense of materials in the construction? These, and similar questions are wholly out of the reach of the common theory of Fluid Motion, since they entirely depend on the laws of the transmission of that tangential action which in it is wholly neglected. In fact, according to the common theory the water ought to flow on with uniformly accelerated velocity; for even the supposition of a certain friction against the bed would be of no avail, for such friction could not be transmitted through the mass. The practical importance of such questions as those above mentioned has made them the object of numerous experiments, from which empirical formulæ have been constructed. But such formulæ, although fulfilling well enough the purposes for which they were

* This may be easily shown by the consideration of a tetrahedron of the fluid, as in Art. 4.

constructed, can hardly be considered as affording us any material insight into the laws of nature; nor will they enable us to pass from the consideration of the phenomena from which they were derived to that of others of a different class, although depending on the same causes.

In reflecting on the principles according to which the motion of a fluid ought to be calculated when account is taken of the tangential force, and consequently the pressure not supposed the same in all directions, I was led to construct the theory explained in the first section of this paper, or at least the main part of it, which consists of equations (13), and of the principles on which they are formed. I afterwards found that Poisson had written a memoir on the same subject, and on referring to it I found that he had arrived at the same equations. The method which he employed was however so different from mine that I feel justified in laying the latter before this Society*. The leading principles of my theory will be found in the hypotheses of Art. 1, and in Art. 3.

The second section forms a digression from the main object of this paper, and at first sight may appear to have little connexion with it. In this section I have, I think, succeeded in shewing that Lagrange's proof of an important theorem in the ordinary theory of Hydrodynamics is untenable. The theorem to which I refer is the one of which the object is to show that $udx + vdy + wdz$, (using the common notation,) is always an exact differential when it is so at one instant. I have mentioned the principles of M. Cauchy's proof, a proof, I think, liable to no sort of objection. I have also given a new proof of the theorem, which would have served to establish it had M. Cauchy not been so fortunate as to obtain three first integrals of the general equations of motion. As it is, this proof may possibly be not altogether useless.

Poisson, in the memoir to which I have referred, begins with establishing, according to his theory, the equations of equilibrium and motion of elastic solids, and makes the equations of motion of fluids depend on this theory. On reading his memoir, I was led to apply to the theory of elastic solids principles precisely analogous to those which I have employed in the case of fluids. The formation of the equations, according to these principles, forms the subject of Sect. III.

The equations at which I have thus arrived contain two arbitrary constants, whereas Poisson's equations contain but one. In Sect. IV. I have explained the principles of Poisson's theories of elastic solids, and of the motion of fluids, and pointed out what appear to me serious objections against the truth of one of the hypotheses which he employs in the former. This theory seems to be very generally received, and in consequence it is usual to deduce the measure of the cubical compressibility of elastic solids from that of their extensibility, when formed into rods or wires, or from some quantity of the same nature. If the views which I have explained in this section be correct, the cubical compressibility deduced in this manner is too great, much too great in the case of the softer substances, and even the softer metals. The equations of Sect. III. have, I find, been already obtained by M. Cauchy in his *Exercises Mathématiques*, except that he has not considered the effect of the heat developed by sudden compression. The method which I have employed is different from his, although in some respects it much resembles it.

The equations of motion of elastic solids given in Sect. III. are the same as those to which different authors have been led, as being the equations of motion of the luminiferous ether in vacuum. It may seem strange that the same equations should have been arrived at for cases so different; and I believe this has appeared to some a serious objection to the employment of those equations in the case of light. I think the reflections which I have made at the end of Sect. IV., where I have examined the consequences of the law of continuity, a law which seems to pervade nature, may tend to remove the difficulty.

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* The same equations have also been obtained by Navier in the case of an incompressible fluid, (*Mém. de l'Institut*, T. VI.) but his principles differ from mine still more than do Poisson's.

SECTION I.

Explanation of the Theory of Fluid Motion proposed. Formation of the Differential Equations. Application of these Equations to a few simple cases.

1. Before entering on the explanation of this theory, it will be necessary to define, or fix the precise meaning of a few terms which I shall have occasion to employ.

In the first place, the expression "the velocity of a fluid at any particular point" will require some notice. If we suppose a fluid to be made up of ultimate molecules, it is easy to see that these molecules must, in general, move among one another in an irregular manner, through spaces comparable with the distances between them, when the fluid is in motion. But since there is no doubt that the distance between two adjacent molecules is quite insensible, we may neglect the irregular part of the velocity, compared with the common velocity with which all the molecules in the neighbourhood of the one considered are moving. Or, we may consider the mean velocity of the molecules in the neighbourhood of the one considered, apart from the velocity due to the irregular motion. It is this regular velocity which I shall understand by the *velocity of a fluid at any point*, and I shall accordingly regard it as varying continuously with the co-ordinates of the point.

Let P be any material point in the fluid, and consider the instantaneous motion of a very small element E of the fluid about P. This motion is compounded of a motion of translation, the same as that of P, and of the motion of the several points of E relatively to P. If we conceive a velocity equal and opposite to that of P impressed on the whole element, the remaining velocities form what I shall call the *relative velocities* of the points of the fluid about P; and the motion expressed by these velocities is what I shall call the *relative motion* in the neighbourhood of P.

It is an undoubted result of observation that the molecular forces, whether in solids, liquids, or gases, are forces of enormous intensity, but which are sensible at only insensible distances. Let E' be a very small element of the fluid circumscribing E, and of a thickness greater than the distance to which the molecular forces are sensible. The forces acting on the element E are the external forces, and the pressures arising from the molecular action of E'. If the molecules of E were in positions in which they could remain at rest if E were acted on by no external force and the molecules of E' were held in their actual positions, they would be in what I shall call a state of *relative equilibrium*. Of course they may be far from being in a state of actual equilibrium. Thus, an element of fluid at the top of a wave may be sensibly in a state of relative equilibrium, although far removed from its position of equilibrium. Now, in consequence of the intensity of the molecular forces, the pressures arising from the molecular action on E will be very great compared with the external moving forces acting on E. Consequently the state of relative equilibrium, or of relative motion, of the molecules of E will not be sensibly affected by the external forces acting on E. But the pressures in different directions about the point P depend on that state of relative equilibrium or motion, and consequently will not be sensibly affected by the external moving forces acting on E. For the same reason they will not be sensibly affected by any motion of rotation common to all the points of E; and it is a direct consequence of the second law of motion, that they will not be affected by any motion of translation common to the whole element. If the molecules of E were in a state of relative equilibrium, the pressure would be equal in all directions about P, as in the case of fluids at rest. Hence I shall assume the following principle:—

That the difference between the pressure on a plane in a given direction passing through any point P *of a fluid in motion and the pressure which would exist in all directions about* P *if the fluid in its neighbourhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about* P; *and that the relative motion* *due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned.*

Let us see how far this principle will lead us when it is carried out.

2. It will be necessary now to examine the nature of the most general instantaneous motion of an element of a fluid. The proposition in this article is however purely geometrical, and may be thus enunciated:—"Supposing space, or any portion of space, to be filled with an infinite number of points which move in any continuous manner, retaining their identity, to examine the nature of the instantaneous motion of any elementary portion of these points."

Let u, v, w be the resolved parts, parallel to the rectangular axes Ox, Oy, Oz, of the velocity of the point P, whose co-ordinates at the instant considered are x, y, z. Then the relative velocities at the point P', whose co-ordinates are $x + x'$, $y + y'$, $z + z'$, will be

$$\frac{du}{dx}x' + \frac{du}{dy}y' + \frac{du}{dz}z' \text{ parallel to } x,$$

$$\frac{dv}{dx}x' + \frac{dv}{dy}y' + \frac{dv}{dz}z' \ldots\ldots\ldots\ldots y,$$

$$\frac{dw}{dx}x' + \frac{dw}{dy}y' + \frac{dw}{dz}z' \ldots\ldots\ldots\ldots z,$$

neglecting squares and products of x', y', z'. Let these velocities be compounded of those due to the angular velocities ω', ω'', ω''' about the axes of x, y, z, and of the velocities U, V, W parallel to x, y, z. The linear velocities due to the angular velocities being $\omega''z' - \omega'''y'$, $\omega'''x' - \omega'z'$, $\omega'y' - \omega''x'$ parallel to the axes of x, y, z, we shall therefore have

$$U = \frac{du}{dx}x' + \left(\frac{du}{dy} + \omega'''\right)y' + \left(\frac{du}{dz} - \omega''\right)z',$$

$$V = \left(\frac{dv}{dx} - \omega'''\right)x' + \frac{dv}{dy}y' + \left(\frac{dv}{dz} + \omega'\right)z',$$

$$W = \left(\frac{dw}{dx} + \omega''\right)x' + \left(\frac{dw}{dy} - \omega'\right)y' + \frac{dw}{dz}z'.$$

Since ω', ω'', ω''' are arbitrary, let them be so assumed that

$$\frac{dU}{dy'} = \frac{dV}{dx'}, \quad \frac{dV}{dz'} = \frac{dW}{dy'}, \quad \frac{dW}{dx'} = \frac{dU}{dz'},$$

which gives

$$\omega' = \tfrac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right), \quad \omega'' = \tfrac{1}{2}\left(\frac{du}{dz} - \frac{dw}{dx}\right), \quad \omega''' = \tfrac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right), \quad \ldots\ldots\ldots(1)$$

$$\left.\begin{aligned} U &= \frac{du}{dx}x' + \tfrac{1}{2}\left(\frac{du}{dy} + \frac{dv}{dx}\right)y' + \tfrac{1}{2}\left(\frac{du}{dz} + \frac{dw}{dx}\right)z', \\ V &= \tfrac{1}{2}\left(\frac{dv}{dx} + \frac{du}{dy}\right)x' + \frac{dv}{dy}y' + \tfrac{1}{2}\left(\frac{dv}{dz} + \frac{dw}{dy}\right)z', \\ W &= \tfrac{1}{2}\left(\frac{dw}{dx} + \frac{du}{dz}\right)x' + \tfrac{1}{2}\left(\frac{dw}{dy} + \frac{dv}{dz}\right)y' + \frac{dw}{dz}z'. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The quantities ω', ω'', ω''' are what I shall call the *angular velocities of the fluid* at the point considered. This is evidently an allowable definition, since, in the particular case in which the element considered moves as a solid might do, these quantities coincide with the angular velocities considered in rigid dynamics. A further reason for this definition will appear in Sect. III.

Let us now investigate whether it is possible to determine x', y', z' so that, considering only the relative velocities U, V, W, the line joining the points P, P' shall have no angular motion. The conditions to be satisfied, in order that this may be the case, are evidently that the increments of the relative co-ordinates x', y', z' of the second point shall be ultimately proportional to those co-ordinates. If e be the rate of extension of the line joining the two points considered, we shall therefore have

$$\left.\begin{aligned} Fx' + hy' + gz' &= ex', \\ hx' + Gy' + fz' &= ey', \\ gx' + fy' + Hz' &= ez'; \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where $F = \frac{du}{dx}$, $G = \frac{dv}{dy}$, $H = \frac{dw}{dz}$, $2f = \frac{dv}{dz} + \frac{dw}{dy}$, $2g = \frac{dw}{dx} + \frac{du}{dz}$, $2h = \frac{du}{dy} + \frac{dv}{dx}$.

If we eliminate from equations (3) the two ratios which exist between the three quantities x', y', z', we get the well known cubic equation

$$(e - F)(e - G)(e - H) - f^2(e - F) - g^2(e - G) - h^2(e - H) - 2fgh = 0, \ldots\ldots(4)$$

which occurs in the investigation of the principal axes of a rigid body, and in various others. As in these investigations, it may be shewn that there are in general three directions, at right angles to each other, in which the point P' may be situated so as to satisfy the required conditions. If two of the roots of (4) are equal, there is one such direction corresponding to the third root, and an infinite number of others situated in a plane perpendicular to the former; and if the three roots of (4) are equal, a line drawn in any direction will satisfy the required conditions.

The three directions which have just been determined I shall call *axes of extension*. They will in general vary from one point to another, and from one instant of time to another. If we denote the three roots of (4) by e', e'', e''', and if we take new rectangular axes $Ox_{,}$, $Oy_{,}$, $Oz_{,}$, parallel to the axes of extension, and denote by $u_{,}$, $U_{,}$, &c. the quantities referred to these axes corresponding to u, U, &c., equations (3) must be satisfied by $y_{,}' = 0$, $z_{,}' = 0$, $e = e'$, by $x_{,}' = 0$, $z_{,}' = 0$, $e = e''$, and by $x_{,}' = 0$, $y_{,}' = 0$, $e = e'''$, which requires that $f_{,} = 0$, $g_{,} = 0$, $h_{,} = 0$, and we have

$$e' = F_{,} = \frac{du_{,}}{dx_{,}}, \quad e'' = G_{,} = \frac{dv_{,}}{dy_{,}}, \quad e''' = H_{,} = \frac{dw_{,}}{dz_{,}}.$$

The values of $U_{,}$, $V_{,}$, $W_{,}$, which correspond to the residual motion after the elimination of the motion of rotation corresponding to ω', ω'' and ω''', are

$$U_{,} = e'x_{,}', \quad V_{,} = e''y_{,}', \quad W_{,} = e'''z_{,}'.$$

The angular velocity of which ω', ω'', ω''' are the components is independent of the arbitrary directions of the co-ordinate axes: the same is true of the directions of the axes of extension, and of the values of the roots of equation (4). This might be proved in various ways; perhaps the following is the simplest. The conditions by which ω', ω'', ω''' are determined are those which express that the relative velocities U, V, W, which remain after eliminating a certain angular velocity, are such that $Udx' + Vdy' + Wdz'$ is ultimately an exact differential, that is to say when squares and products of x', y' and z' are neglected. It appears moreover from the solution that there is only one way in which these conditions can be satisfied for a given system of co-ordinate axes. Let us take new rectangular axes $O\mathrm{x}$, $O\mathrm{y}$, $O\mathrm{z}$, and let U, V, W be the resolved parts along these axes of the velocities U, V, W, and x', y', z', the relative co-ordinates of P'; then

$$U = \mathrm{U} \cos x\mathrm{x} + \mathrm{V} \cos x\mathrm{y} + \mathrm{W} \cos x\mathrm{z},$$
$$dx' = \cos x\mathrm{x}\, d\mathrm{x}' + \cos x\mathrm{y}\, d\mathrm{y}' + \cos x\mathrm{z}\, d\mathrm{z}', \text{ \&c.};$$

whence, taking account of the well known relations between the cosines involved in these equations, we easily find

$$U dx' + V dy' + W dz' = \mathrm{U}\, d\mathrm{x}' + \mathrm{V}\, d\mathrm{y}' + \mathrm{W}\, d\mathrm{z}'.$$

It appears therefore that the relative velocities U, V, W, which remain after eliminating a certain angular velocity, are such that $\mathrm{U}d\mathrm{x}' + \mathrm{V}d\mathrm{y}' + \mathrm{W}d\mathrm{z}'$ is ultimately an exact differential. Hence the values of U, V, W are the same as would have been obtained from equations (2) applied directly to the new axes, whence the truth of the proposition enunciated at the head of this paragraph is manifest.

The motion corresponding to the velocities $U_{,}$, $V_{,}$, $W_{,}$ may be further decomposed into a motion of dilatation, positive or negative, which is alike in all directions, and two motions which I shall call *motions of shifting*, each of the latter being in two dimensions, and not affecting the density. For let δ be the rate of linear extension corresponding to a uniform dilatation; let $\sigma x_{,}'$, $-\sigma y_{,}'$ be the velocities parallel to $x_{,}$, $y_{,}$, corresponding to a motion of shifting parallel to the the plane $x_{,}y_{,}$, and let $\sigma' x_{,}'$, $-\sigma' z_{,}'$ be the velocities parallel to $x_{,}$, $z_{,}$, corresponding to a similar motion of shifting parallel to the plane $x_{,}z_{,}$. The velocities parallel to $x_{,}$, $y_{,}$, $z_{,}$ respectively corresponding to the quantities δ, σ and σ' will be $(\delta + \sigma + \sigma')\, x_{,}'$, $(\delta - \sigma)\, y_{,}'$, $(\delta - \sigma')\, z_{,}'$, and equating these to $U_{,}$, $V_{,}$, $W_{,}$ we shall get

$$\delta = \tfrac{1}{3}(e' + e'' + e'''), \quad \sigma = \tfrac{1}{3}(e' + e''' - 2e''), \quad \sigma' = \tfrac{1}{3}(e' + e'' - 2e''').$$

Hence the most general instantaneous motion of an elementary portion of a fluid is compounded of a motion of translation, a motion of rotation, a motion of uniform dilatation, and two motions of shifting of the kind just mentioned.

3. Having determined the nature of the most general instantaneous motion of an element of a fluid, we are now prepared to consider the normal pressures and tangential forces called into play by the relative displacements of the particles. Let p be the pressure which would exist about the point P if the neighbouring molecules were in a state of relative equilibrium: let $p + p_{,}$ be the normal pressure, and $t_{,}$ the tangential action, both referred to a unit of surface, on a plane passing through P and having a given direction. By the hypotheses of Art. 1. the quantities $p_{,}$, $t_{,}$ will be independent of the angular velocities ω', ω'', ω''', depending only on the residual relative velocities U, V, W, or, which comes to the same, on e', e'' and e''', or on σ, σ' and δ. Since this residual motion is symmetrical with respect to the axes of extension, it follows that if the plane considered is perpendicular to any one of these axes the tangential action on it is zero, since there is no more reason why it should act in one direction rather than in the opposite; for by the hypotheses of Art. 1. the change of density and temperature about the point P is to be neglected, the constitution of the fluid being ultimately uniform about that point. Denoting then by $p + p'$, $p + p''$, $p + p'''$ the pressures on planes perpendicular to the axes of $x_{,}$, $y_{,}$, $z_{,}$, we must have

$$p' = \phi(e', e'', e'''), \quad p'' = \phi(e'', e''', e'), \quad p''' = \phi(e''', e', e''),$$

$\phi(e', e'', e''')$ denoting a function of e', e'' and e''' which is symmetrical with respect to the two latter quantities. The question is now to determine, on whatever may seem the most probable hypothesis, the form of the function ϕ.

Let us first take the simpler case in which there is no dilatation, and only one motion of shifting, or in which $e'' = -e'$, $e''' = 0$, and let us consider what would take place if the fluid consisted of smooth molecules acting on each other by actual contact. On this supposition, it is clear, considering the magnitude of the pressures acting on the molecules compared with their masses, that they would be sensibly in a position of relative equilibrium, except when the equilibrium of any one of them became impossible from the displacement of the adjoining ones, in which case the molecule in question would start into a new position of equilibrium. This start would cause a corresponding displacement in the molecules immediately about the one which started, and this disturbance would be propagated immediately in all directions, the nature of the displacement however being different in different directions, and would soon become insensible. During the continuance of this disturbance, the pressure on a small plane drawn through the element considered would not be the same in all directions, nor normal to the plane: or, which comes to the same, we may suppose a uniform normal pressure p to act, together with a normal pressure $p_{,,}$ and a tangential force $t_{,,}$, $p_{,,}$ and $t_{,,}$ being forces of great intensity and short duration, that is being of the nature of impulsive forces. As the number of molecules comprised in the element considered has been supposed extremely great, we may take a time τ so short that all summations with respect to such intervals of time may be replaced without sensible error by integrations, and yet so long that a very great number of starts shall take place in it. Consequently we have only to consider the average effect of such starts, and moreover we may without sensible error replace the impulsive forces such as $p_{,,}$ and $t_{,,}$, which succeed one another with great rapidity, by continuous forces. For planes perpendicular to the axes of extension these continuous forces will be the normal pressures p', p'', p'''.

Let us now consider a motion of shifting differing from the former only in having e' increased in the ratio of m to 1. Then, if we suppose each start completed before the starts which would be sensibly affected by it are begun, it is clear that the same series of starts will take place in the second case as in the first, but at intervals of time which are less in the ratio of 1 to m. Consequently the continuous pressures by which the impulsive actions due to these starts may be replaced must be increased in the ratio of m to 1. Hence the pressures p', p'', p''' must be proportional to e', or we must have

$$p' = Ce', \quad p'' = C'e', \quad p''' = C''e'.$$

It is natural to suppose that these formulæ held good for negative as well as positive values of e'. Assuming this to be true, let the sign of e' be changed. This comes to interchanging x and y, and consequently p''' must remain the same, and p' and p'' must be interchanged. We must therefore have $C'' = 0$, $C' = -C$. Putting then $C = -2\mu$ we have

$$p' = -2\mu e', \quad p'' = 2\mu e', \quad p''' = 0.$$

It has hitherto been supposed that the molecules of a fluid are in actual contact. We have every reason to suppose that this is not the case. But precisely the same reasoning will apply if they are separated by intervals as great as we please compared with their magnitudes, provided only we suppose the force of restitution called into play by a small displacement of *any one* molecule to be very great.

Let us now take the case of two motions of shifting which coexist, and let us suppose $e' = \sigma + \sigma'$, $e'' = -\sigma$, $e''' = -\sigma'$. Let the small time τ be divided into $2n$ equal portions, and let us suppose that in the first interval a shifting motion corresponding to $e' = 2\sigma$, $e'' = -2\sigma$ takes place parallel to the plane $x_{,}\, y_{,}$, and that in the second interval a shifting motion corresponding to $e' = 2\sigma'$, $e''' = -2\sigma'$ takes place parallel to the plane $x_{,}\, z_{,}$, and so on alternately. On this supposition it is clear that if we suppose the time $\frac{\tau}{2n}$ to be extremely small, the continuous forces by which the effect of the starts may be replaced will be $p' = -2\mu(\sigma + \sigma')$, $p'' = 2\mu\sigma$, $p''' = 2\mu\sigma'$. By supposing n indefinitely increased, we may make the motion considered approach as near as we please to that in which the two motions of shifting coexist; but we are not at liberty to do so, for in order to apply the above reasoning we must suppose the time $\frac{\tau}{2n}$ to be so large that the average effect of the starts which occur in it may be taken. Consequently it must be taken as an additional assumption, and not a matter of absolute demonstration, that the effects of the two motions of shifting are superimposed.

Hence if $\delta = 0$, *i.e.* if $e' + e'' + e''' = 0$, we shall have in general

$$p' = -2\mu e', \quad p'' = -2\mu e'', \quad p''' = -2\mu e''' \text{..................}(5)$$

It was by this hypothesis of starts that I first arrived at these equations, and the differential equations of motion which result from them. On reading Poisson's memoir however, to which I shall have occasion to refer in Section IV., I was led to reflect that however intense we may suppose the molecular forces to be, and however near we may suppose the molecules to be to their positions of relative equilibrium, we are not therefore at liberty to suppose them *in* those positions, and consequently not at liberty to suppose the pressure equal in all directions in the intervals of time between the starts. In fact, by supposing the molecular forces indefinitely increased, retaining the same ratios to each other, we may suppose the displacements of the molecules from their positions of relative equilibrium indefinitely diminished, but on the other hand the force of restitution called into action by a given displacement is indefinitely increased in the same proportion. But be these displacements what they may, we know that the forces of restitution make equilibrium with forces equal and opposite to the effective forces; and in calculating the effective forces we may neglect the above displacements, or suppose the molecules to move in the paths in which they would move if the shifting motion took place with indefinite slowness. Let us first consider a single motion of shifting, or one for which $e'' = -e'$, $e''' = 0$, and let $p_{,}$ and $t_{,}$ denote the same quantities as before. If we now suppose e' increased in the ratio of m to 1, all the effective forces will be increased in that ratio, and consequently $p_{,}$ and $t_{,}$ will be increased in the same ratio. We may deduce the values of p', p'' and p''' just as before, and then pass by the same reasoning to the case of two motions of shifting which coexist, only that in this case the reasoning will be demonstrative, since we *may* suppose the time $\dfrac{\tau}{2n}$ indefinitely diminished. If we suppose the state of things considered in this paragraph to exist along with the motions of starting already considered, it is easy to see that the expressions for p', p'' and p''' will still retain the same form.

There remains yet to be considered the effect of the dilatation. Let us first suppose the dilatation to exist without any shifting: then it is easily seen that the relative motion of the fluid at the point considered is the same in all directions. Consequently the only effect which such a dilatation could have would be to introduce a normal pressure $p_{,}$, alike in all directions, in addition to that due to the action of the molecules supposed to be in a state of relative equilibrium. Now the pressure $p_{,}$ could only arise from the aggregate of the molecular actions called into play by the displacements of the molecules from their positions of relative equilibrium; but since these displacements take place, on an average, indifferently in all directions, it follows that the actions of which $p_{,}$ is composed neutralize each other, so that $p_{,} = 0$. The same conclusion might be drawn from the hypothesis of starts, supposing, as it is natural to do, that each start calls into action as much increase of pressure in some directions as diminution of pressure in others.

If the motion of uniform dilatation coexists with two motions of shifting, I shall suppose, for the same reason as before, that the effects of these different motions are superimposed. Hence subtracting δ from each of the three quantities e', e'' and e''', and putting the remainders in the place of e', e'' and e''' in equations (5), we have

$$p' = \tfrac{2}{3}\mu(e'' + e''' - 2e'), \quad p'' = \tfrac{2}{3}\mu(e''' + e' - 2e''), \quad p''' = \tfrac{2}{3}\mu(e' + e'' - 2e''') \text{.........}(6)$$

If we had started with assuming $\phi(e', e'', e''')$ to be a linear function of e', e'' and e''', avoiding all speculation as to the molecular constitution of a fluid, we should have had at once $p' = Ce' + C'(e'' + e''')$, since p' is symmetrical with respect to e'' and e'''; or, changing the constants, $p' = \frac{2}{3}\mu(e'' + e''' - 2e') + \kappa(e' + e'' + e''')$. The expressions for p'' and p''' would be obtained by interchanging the requisite quantities. Of course we may at once put $\kappa = 0$ if we assume that in the case of a uniform motion of dilatation the pressure at any instant depends only on the actual density and temperature at that instant, and not on the rate at which the former changes with the time. In most cases to which it would be interesting to apply the theory of the friction of fluids the density of the fluid is either constant, or may without sensible error be regarded as constant, or else changes slowly with the time. In the first two cases the results would be the same, and in the third case nearly the same, whether κ were equal to zero or not. Consequently, if theory and experiment should in such cases agree, the experiments must not be regarded as confirming that part of the theory which relates to supposing κ to be equal to zero.

4. It will be easy now to determine the oblique pressure, or resultant of the normal pressure and tangential action, on any plane. Let us first consider a plane drawn through the point P parallel to the plane yz. Let $Ox_{,}$ make with the axes of x, y, z angles whose cosines are l', m', n': let l'', m'', n'' be the same for $Oy_{,}$, and l''', m''', n''' the same for $Oz_{,}$. Let P_1 be the pressure, and (xty), (xtz) the resolved parts, parallel to y, z respectively, of the tangential force on the plane considered, all referred to a unit of surface, (xty) being reckoned positive when the part of the fluid towards $-x$ urges that towards $+x$ in the positive direction of y, and similarly for (xtz). Consider the portion of the fluid comprised within a tetrahedron having its vertex in the point P, its base parallel to the plane yz, and its three sides parallel to the planes $x_{,}y_{,}$, $y_{,}z_{,}$, $z_{,}x_{,}$ respectively. Let A be the area of the base, and therefore $l'A$, $l''A$, $l'''A$ the areas of the faces perpendicular to the axes of $x_{,}$, $y_{,}$, $z_{,}$. By D'Alembert's principle, the pressures and tangential actions on the faces of this tetrahedron, the moving forces arising from the external attractions, not including the molecular forces, and forces equal and opposite to the effective moving forces will be in equilibrium, and therefore the sums of the resolved parts of these forces in the directions of x, y and z will each be zero. Suppose now the dimensions of the tetrahedron indefinitely diminished, then the resolved parts of the external, and of the effective moving forces will vary ultimately as the cubes, and those of the pressures and tangential forces on the sides as the squares of homologous lines. Dividing therefore the three equations arising from equating to zero the resolved parts of the above forces by A, and taking the limit, we have

$$P_1 = \Sigma l'^2 (p + p'),\quad (xty) = \Sigma l'm' (p + p'),\quad (xtz) = \Sigma l'n' (p + p'),$$

the sign Σ denoting the sum obtained by taking the quantities corresponding to the three axes of extension in succession. Putting for p', p'' and p''' their values given by (6), putting $e' + e'' + e''' = 3\delta$, and observing that $\Sigma l'^2 = 1$, $\Sigma l'm' = 0$, $\Sigma l'n' = 0$, the above equations become

$$P_1 = p - 2\mu\Sigma l'^2 e' + 2\mu\delta,\quad (xty) = -2\mu\Sigma l'm'e',\quad (xtz) = -2\mu\Sigma l'n'e'.$$

The method of determining the pressure on any plane from the pressures on three planes at right angles to each other, which has just been given, has already been employed by MM. Cauchy and Poisson.

The most direct way of obtaining the values of $\Sigma l'^2 e'$ &c. would be to express l', m' and n' in terms of e' by any two of equations (3), in which x', y', z' are proportional to l', m', n', together with the equation $l'^2 + m'^2 + n'^2 = 1$, and then to express the resulting symmetrical function of the roots of the cubic equation (4) in terms of the coefficients. But this method would be excessively laborious, and need not be resorted to. For after eliminating the angular motion of the element of fluid considered the remaining velocities are $e'x_{,}'$, $e''y_{,}'$, $e'''z_{,}'$, parallel to the axes of $x_{,}$, $y_{,}$, $z_{,}$. The sum of the resolved parts of these parallel to the axis of x is $l'e'x_{,}' + l''e''y_{,}' + l'''e'''z_{,}'$. Putting for $x_{,}'$, $y_{,}'$, $z_{,}'$ their values $l'x' + m'y' + n'z'$ &c., the above sum becomes

$$x'\Sigma l'^2 e' + y'\Sigma l'm'e' + z'\Sigma l'n'e';$$

but this sum is the same thing as the velocity U in equation (2), and therefore we have

$$\Sigma l'^2 e' = \frac{du}{dx},\quad \Sigma l'm'e' = \frac{1}{2}\left(\frac{du}{dy} + \frac{dv}{dx}\right),\quad \Sigma l'n'e' = \frac{1}{2}\left(\frac{du}{dz} + \frac{dw}{dx}\right).$$

It may also be very easily proved directly that the value of 3δ, the rate of cubical dilatation, satisfies the equation

$$3\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \quad \text{.......................... (7)}$$

Let P_2, (ytz), (ytx) be the quantities referring to the axis of y, and P_3, (ztx), (zty) those referring to the axis of z, which correspond to P_1 &c. referring to the axis of x. Then we see that $(ytz) = (zty)$, $(ztx) = (xtz)$, $(xty) = (ytx)$. Denoting these three quantities by T_1, T_2, T_3, and making the requisite substitutions and interchanges, we have

$$\left.\begin{aligned} P_1 &= p - 2\mu\left(\frac{du}{dx} - \delta\right), \\ P_2 &= p - 2\mu\left(\frac{dv}{dy} - \delta\right), \\ P_3 &= p - 2\mu\left(\frac{dw}{dz} - \delta\right), \\ T_1 = -\mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right),\ T_2 &= -\mu\left(\frac{dw}{dx} + \frac{du}{dz}\right),\ T_3 = -\mu\left(\frac{du}{dy} + \frac{dv}{dx}\right). \end{aligned}\right\} \quad \text{............(8)}$$

It may also be useful to know the components, parallel to x, y, z, of the oblique pressure on a plane passing through the point P, and having a given direction. Let l, m, n be the cosines of the angles which a normal to the given plane makes with the axes of x, y, z; let P, Q, R be the components, referred to a unit of surface, of the oblique pressure on this plane, P, Q, R being reckoned positive when the part of the fluid in which is situated the normal to which l, m and n refer is urged by the other part in the positive directions of x, y, z, when l, m and n are positive. Then considering as before a tetrahedron of which the base is parallel to the given plane, the vertex in the point P, and the sides parallel to the co-ordinate planes, we shall have

$$\left.\begin{aligned} P &= lP_1 + mT_3 + nT_2, \\ Q &= lT_3 + mP_2 + nT_1, \\ R &= lT_2 + mT_1 + nP_3. \end{aligned}\right\} \quad \text{................................ (9)}$$

In the simple case of a sliding motion for which $u = 0$, $v = f(x)$, $w = 0$, the only forces, besides the pressure p, which act on planes parallel to the co-ordinate planes are the two tangential forces T_3, the value of which in this case is $-\mu \dfrac{dv}{dx}$. In this case it is easy to show that the axes of extension are, one of them parallel to Oz, and the two others in a plane parallel to xy, and inclined at angles of 45^0 to Ox. We see also that it is necessary to suppose μ to be positive, since otherwise the tendency of the forces would be to increase the relative motion of the parts of the fluid, and the equilibrium of the fluid would be unstable.

5. Having found the pressures about the point P on planes parallel to the co-ordinate planes, it will be easy to form the equations of motion. Let X, Y, Z be the resolved parts, parallel to the axes, of the external force, not including the molecular force; let ρ be the density, t the time. Consider an elementary parallelepiped of the fluid, formed by planes parallel to the co-ordinate planes, and drawn through the point (x, y, z) and the point $(x + \Delta x, y + \Delta y, z + \Delta z)$. The mass of this element will be ultimately $\rho\Delta x\,\Delta y\,\Delta z$, and the moving force parallel to x arising from the external forces will be ultimately $\rho X\,\Delta x\,\Delta y\,\Delta z$; the effective moving force parallel to x will be ultimately $\rho\dfrac{Du}{Dt}\Delta x\,\Delta y\,\Delta z$, where D is used, as it will be in the rest of this paper,

to denote differentiation in which the independent variables are t and three parameters of the particle considered, (such for instance as its initial co-ordinates,) and not t, x, y, z. It is easy also to show that the moving force acting on the element considered arising from the oblique pressures on the faces is ultimately $\left(\frac{dP}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz}\right) \Delta x \, \Delta y \, \Delta z$, acting in the negative direction. Hence we have by D'Alembert's principle

$$\rho \left(\frac{Du}{Dt} - X\right) + \frac{dP_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz} = 0, \text{ \&c.,} \qquad (10)$$

in which equations we must put for $\frac{Du}{Dt}$ its value $\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}$, and similarly for $\frac{Dv}{dt}$ and $\frac{Dw}{Dt}$. In considering the general equations of motion it will be needless to write down more than one, since the other two may be at once derived from it by interchanging the requisite quantities. The equations (10), the ordinary equation of continuity, as it is called,

$$\frac{d\rho}{dt} + \frac{d\rho u}{dx} + \frac{d\rho v}{dy} + \frac{d\rho w}{dz} = 0, \qquad (11)$$

which expresses the condition that there is no generation or destruction of mass in the interior of a fluid, the equation connecting p and ρ, or in the case of an incompressible fluid the equivalent equation $\frac{D\rho}{Dt} = 0$, and the equation for the propagation of heat, if we choose to take account of that propagation, are the only equations to be satisfied at every point of the interior of the fluid mass.

As it is quite useless to consider cases of the utmost degree of generality, I shall suppose the fluid to be homogeneous, and of a uniform temperature throughout, except in so far as the temperature may be raised by sudden compression in the case of small vibrations. Hence in equations (10) μ may be supposed to be constant as far as regards the temperature; for, in the case of small vibrations, the terms introduced by supposing it to vary with the temperature would involve the square of the velocity, which is supposed to be neglected. If we suppose μ to be independent of the pressure also, and substitute in (10) the values of P_1 &c. given by (8), the former equations become

$$\rho \left(\frac{Du}{Dt} - X\right) + \frac{dp}{dx} - \mu \left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \text{ \&c.} \qquad (12)$$

Let us now consider in what cases it is allowable to suppose μ to be independent of the pressure. It has been concluded by Dubuat, from his experiments on the motion of water in pipes and canals, that the total retardation of the velocity due to friction is not increased by increasing the pressure. The total retardation depends, partly on the friction of the water against the sides of the pipe or canal, and partly on the mutual friction, or tangential action, of the different portions of the water. Now if these two parts of the whole retardation were separately variable with p, it is very unlikely that they should when combined give a result independent of p. The amount of the internal friction of the water depends on the value of μ. I shall therefore suppose that for water, and by analogy for other incompressible fluids, μ is independent of the pressure. On this supposition, we have from equations (11) and (12)

$$\rho \left(\frac{Du}{Dt} - X\right) + \frac{dp}{dx} - \mu \left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) = 0, \text{ \&c.} \qquad (13)$$

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

These equations are applicable to the determination of the motion of water in pipes and canals, to the calculation of the effect of friction on the motions of tides and waves, and such questions.

If the motion is very small, so that we may neglect the square of the velocity, we may put $\frac{Du}{Dt} = \frac{du}{dt}$, &c. in equations (13). The equations thus simplified are applicable to the determination of the motion of a pendulum oscillating in water, or of that of a vessel filled with water and made to oscillate. They are also applicable to the determination of the motion of a pendulum oscillating in air, for in this case we may, with hardly any error, neglect the compressibility of the air.

The case of the small vibrations by which sound is propagated in a fluid, whether a liquid or a gas, is another in which $\frac{d\mu}{dp}$ may be neglected. For in the case of a liquid reasons have been shown for supposing μ to be independent of p, and in the case of a gas we may neglect $\frac{d\mu}{dp}$, if we neglect the small change in the value of μ, arising from the small variation of pressure due to the forces X, Y, Z.

6. Besides the equations which must hold good at any point in the interior of the mass, it will be necessary to form also the equations which must be satisfied at its boundaries. Let M be a point in the boundary of the fluid. Let a normal to the surface at M, drawn on the outside of the fluid, make with the axes angles whose cosines are l, m, n. Let P', Q', R' be the components of the pressure of the fluid about M on the solid or fluid with which it is in contact, these quantities being reckoned positive when the fluid considered presses the solid or fluid outside it in the positive directions of x, y, z, supposing l, m and n positive. Let S be a very small element of the surface about M, which will be ultimately plane, S' a plane parallel and equal to S, and directly opposite to it, taken within the fluid. Let the distance between S and S' be supposed to vanish in the limit compared with the breadth of S, a supposition which may be made if we neglect the effect of the curvature of the surface at M; and let us consider the forces acting on the element of fluid comprised between S and S', and the motion of this element. If we suppose equations (8) to hold good to within an insensible distance from the surface of the fluid, we shall evidently have forces ultimately equal to PS, QS, RS, (P, Q and R being given by equations (9),) acting on the inner side of the element in the positive directions of the axes, and forces ultimately equal to $P'S$, $Q'S$, $R'S$ acting on the outer side in the negative directions. The moving forces arising from the external forces acting on the element, and the effective moving forces will vanish in the limit compared with the forces PS, &c.: the same will be true of the pressures acting about the edge of the element, if we neglect capillary attraction, and all forces of the same nature. Hence, taking the limit, we shall have

$$P' = P, \quad Q' = Q, \quad R' = R.$$

The method of proceeding will be different according as the bounding surface considered is a free surface, the surface of a solid, on the surface of separation of two fluids, and it will be necessary to consider these cases separately. Of course the surface of a liquid exposed to the air is really the surface of separation of two fluids, but it may in many cases be regarded as a free surface if we neglect the inertia of the air: it may always be so regarded if we neglect the friction of the air as well as its inertia.

Let us first take the case of a free surface exposed to a pressure Π, which is supposed to be the same at all points, but may vary with the time; and let $L = 0$ be the equation to the surface. In this case we shall have $P' = l\Pi$, $Q' = m\Pi$, $R' = n\Pi$; and putting for P, Q, R their values given by (9), and for P_1 &c. their values given by (8), and observing that in this case $\delta = 0$, we shall have

$$l(\Pi - p) + \mu \left\{ 2l \frac{du}{dx} + m \left(\frac{du}{dy} + \frac{dv}{dx} \right) + n \left(\frac{du}{dz} + \frac{dw}{dx} \right) \right\} = 0, \text{ \&c.,} \dots\dots\dots (14)$$

in which equations l, m, n will have to be replaced by $\frac{dL}{dx}$, $\frac{dL}{dy}$, $\frac{dL}{dz}$, to which they are proportional.

If we choose to take account of capillary attraction, we have only to diminish the pressure Π by the quantity $H\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$, where H is a positive constant depending on the nature of the fluid, and r_1, r_2 are the principal radii of curvature at the point considered, reckoned positive when the fluid is concave outwards. Equations (14) with the ordinary equation

$$\frac{dL}{dt} + u\frac{dL}{dx} + v\frac{dL}{dy} + w\frac{dL}{dz} = 0, \dots\dots\dots\dots\dots\dots\dots\dots (15)$$

are the conditions to be satisfied for points at the free surface. Equations (14) are for such points what the three equations of motion are for internal points, and (15) is for the former what (11) is for the latter, expressing in fact that there is no generation or destruction of fluid at the free surface.

The equations (14) admit of being differently expressed, in a way which may sometimes be useful. If we suppose the origin to be in the point considered, and the axis of z to be the external normal to the surface, we have $l = m = 0$, $n = 1$, and the equations become

$$\frac{dw}{dx} + \frac{du}{dz} = 0, \quad \frac{dw}{dy} + \frac{dv}{dz} = 0, \quad \Pi - p + 2\mu\frac{dw}{dz} = 0. \dots\dots\dots\dots\dots\dots (16)$$

The relative velocity parallel to z of a point $(x', y', 0)$ in the free surface, indefinitely near the origin, is $\frac{dw}{dx}x' + \frac{dw}{dy}y'$: hence we see that $\frac{dw}{dx}$, $\frac{dw}{dy}$ are the angular velocities, reckoned from x to z and from y to z respectively, of an element of the free surface. Subtracting the linear velocities due to these angular velocities from the relative velocities of the point (x', y', z'), and calling the remaining relative velocities U, V, W, we shall have

$$U = \frac{du}{dx}x' + \frac{du}{dy}y' + \left(\frac{du}{dz} + \frac{dw}{dx}\right)z',$$

$$V = \frac{dv}{dx}x' + \frac{dv}{dy}y' + \left(\frac{dv}{dz} + \frac{dw}{dy}\right)z',$$

$$W = \frac{dw}{dz}z'.$$

Hence we see that the first two of equations (16) express the conditions that $\frac{dU}{dz'} = 0$ and $\frac{dV}{dz'} = 0$, which are evidently the conditions to be satisfied in order that there may be no sliding motion in a direction parallel to the free surface. It would be easy to prove that these are the conditions to be satisfied in order that the axis of z may be an axis of extension.

The next case to consider is that of a fluid in contact with a solid. The condition which first occurred to me to assume for this case was, that the film of fluid immediately in contact with the solid did not move relatively to the surface of the solid. I was led to try this condition from the following considerations. According to the hypotheses adopted, if there was a very large relative motion of the fluid particles immediately about any imaginary surface dividing the fluid, the tangential forces called into action would be very large, so that the amount of relative motion would be rapidly diminished. Passing to the limit, we might suppose that if at any instant the velocities altered discontinuously in passing across any imaginary surface, the tangential force called into action would immediately destroy the finite relative motion of particles indefinitely close to each other, so as to render the motion continuous; and from analogy the same might be supposed to be true for the surface of junction of a fluid and solid. But having calculated, according to the conditions which I have mentioned, the discharge of long straight circular pipes and rectangular canals, and compared the resulting formulæ with some of the experiments of Bossut and Dubuat, I found that the formulæ did not at all agree with experiment. I then tried Poisson's conditions in the case of a circular pipe, but with no better success. In fact, it appears from experiment that the tangential force varies nearly as the square of the velocity with which the fluid flows past the surface of a solid, at least when the velocity is not very small. It appears however from experiments on pendulums that the total friction varies as the first power of the velocity, and consequently we may suppose that Poisson's conditions, which include as a particular case those which I first tried, hold good for very small velocities. I proceed therefore to deduce these conditions in a manner conformable with the views explained in this paper.

First, suppose the solid at rest, and let $L = 0$ be the equation to its surface. Let M' be a point within the fluid, at an insensible distance h from M. Let ϖ be the pressure which would exist about M if there were no motion of the particles in its neighbourhood, and let $p_{,}$ be the additional normal pressure, and $t_{,}$ the tangential force, due to the relative velocities of the particles, both with respect to one another and with respect to the surface of the solid. If the motion is so slow that the starts take place independently of each other, on the hypothesis of starts, or that the molecules are very nearly in their positions of relative equilibrium, and if we suppose as before that the effects of different relative velocities are superimposed, it is easy to show that $p_{,}$ and $t_{,}$ are linear functions of u, v, w and their differential coefficients with respect to x, y, and z; u, v, &c. denoting here the velocities of the fluid about the point M', in the expressions for which however the co-ordinates of M may be used for those of M', since h is neglected. Now the relative velocities about the points M and M' depending on $\frac{du}{dx}$ &c. are comparable with $\frac{du}{dx}h$, while those depending on u, v and w are comparable with these quantities, and therefore in considering the action of the fluid on the solid it is only necessary to consider the quantities u, v and w. Now since, neglecting h, the velocity at M' is tangential to the surface at M, u, v, and w are the components of a certain velocity V tangential to the surface. The pressure $p_{,}$ must be zero; for changing the signs of u, v, and w the circumstances concerned in its production remain the same, whereas its analytical expression changes sign. The tangential force at M will be in the direction of V, and proportional to it, and consequently its components along the axes of x, y, z will be proportional to u, v, w. Reckoning the tangential force positive when, l, m, and n being positive, the solid is urged in the positive directions of x, y, z, the resolved parts of the tangential force will therefore be νu, νv, νw, where ν must evidently be positive, since the effect of the forces must be to check the relative motion of the fluid and solid. The normal pressure of the fluid on the solid being equal to ϖ, its components will be evidently $l\varpi$, $m\varpi$, $n\varpi$.

Suppose now the solid to be in motion, and let u', v', w' be the resolved parts of the velocity of the point M of the solid, and ω', ω'', ω''' the angular velocities of the solid. By hypothesis, the forces by which the pressure at any point differs from the normal pressure due to the action of the molecules supposed to be in a state of relative equilibrium about that point are independent of any velocity of translation or rotation. Supposing then linear and angular velocities equal and opposite to those of the solid impressed both on the solid and on the fluid, the former will be for an instant at rest, and we have only to treat the resulting velocities of the fluid as in the first case.

Hence $P' = l\varpi + \nu(u - u')$, &c.; and in the equations (8) we may employ the actual velocities u, v, w, since the pressures P, Q, R are independent of any motion of translation and rotation common to the whole fluid. Hence the equations $P' = P$, &c. give us

$$l(\varpi - p) + \nu(u - u') + \mu\left\{2l\left(\frac{du}{dx} - \delta\right) + m\left(\frac{du}{dy} + \frac{dv}{dx}\right) + n\left(\frac{du}{dz} + \frac{dw}{dx}\right)\right\} = 0, \text{ \&c.,} \ldots\ldots(17)$$

which three equations with (15) are those which must be satisfied at the surface of a solid, together with the equation $L = 0$. It will be observed that in the case of a free surface the pressures P', Q', R' are given, whereas in the case of the surface of a solid they are known only by the solution of the problem. But on the other hand the form of the surface of the solid is given, whereas the form of the free surface is known only by the solution of the problem.

Dubuat found by experiment that when the mean velocity of water flowing through a pipe is less than about one inch in a second, the water near the inner surface of the pipe is at rest. If these experiments may be trusted, the conditions to be satisfied in the case of small velocities are those which first occurred to me, and which are included in those just given by supposing $\nu = \infty$.

I have said that when the velocity is not very small the tangential force called into action by the sliding of water over the inner surface of a pipe varies nearly as the square of the velocity. This fact appears to admit of a natural explanation. When a current of water flows past an obstacle, it produces a resistance varying nearly as the square of the velocity. Now even if the inner surface of a pipe is polished we may suppose that little irregularities exist, forming so many obstacles to the current. Each little protuberance will experience a resistance varying nearly as the square of the velocity, from whence there will result a tangential action of the fluid on the surface of the pipe, which will vary nearly as the square of the velocity; and the same will be true of the equal and opposite reaction of the pipe on the fluid. The tangential force due to this cause will be combined with that by which the fluid close to the pipe is kept at rest when the velocity is sufficiently small.

There remains to be considered the case of two fluids having a common surface. Let u', v', w', μ', δ' denote the quantities belonging to the second fluid corresponding to u, &c. belonging to the first. Together with the two equations $L = 0$ and (15) we shall have in this case the equation derived from (15) by putting u', v', w' for u, v, w; or, which comes to the same, we shall have the two former equations with

$$l(u - u') + m(v - v') + n(w - w') = 0 \ldots\ldots\ldots\ldots\ldots(18)$$

If we consider the principles on which equations (17) were formed to be applicable to the present case, we shall have six more equations to be satisfied, namely (17), and the three equations derived from (17) by interchanging the quantities referring to the two fluids, and changing the signs of l, m, n. These equations give the value of ϖ, and leave five equations of condition. If we must suppose $\nu = \infty$, as appears most probable, the six equations above mentioned must be replaced by the six $u' = u$, $v' = v$, $w' = w$, and

$$lp - \mu f(u, v, w) = lp' - \mu' f(u', v', w'), \text{ \&c.,}$$

$f(u, v, w)$ denoting the coefficient of μ in the first of equations (17). We have here six equations of condition instead of five, but then the equation (18) becomes identical.

7. The most interesting questions connected with this subject require for their solution a knowledge of the conditions which must be satisfied at the surface of a solid in contact with the fluid, which, except perhaps in case of very small motions, are unknown. It may be well however to give some applications of the preceding equations which are independent of these conditions. Let us then in the first place consider in what manner the transmission of sound in a fluid is affected by the tangential action. To take the simplest case, suppose that no forces act on the fluid, so that the pressure and density are constant in the state of

equilibrium, and conceive a series of plane waves to be propagated in the direction of the axis of x, so that $u = f(x, t)$, $v = 0$, $w = 0$. Let $p_{,}$ be the pressure, and $\rho_{,}$ the density of the fluid when it is in equilibrium, and put $p = p_{,} + p'$. Then we have from equations (11) and (12), omitting the square of the disturbance,

$$\frac{1}{\rho_{,}}\frac{d\rho}{dt} + \frac{du}{dx} = 0, \quad \rho_{,}\frac{du}{dt} + \frac{dp'}{dx} - \frac{4}{3}\mu\frac{d^2u}{dx^2} = 0 \ldots\ldots\ldots\ldots(19)$$

Let $A\Delta\rho$ be the increment of pressure due to a very small increment $\Delta\rho$ of density, the temperature being unaltered, and let m be the ratio of the specific heat of the fluid when the pressure is constant to its specific heat when the volume is constant; then the relation between p' and ρ will be

$$p' = mA(\rho - \rho_{,}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

Eliminating p' and ρ from (19) and (20) we get

$$\frac{d^2u}{dt^2} - mA\frac{d^2u}{dx^2} - \frac{4\mu}{3\rho_{,}}\frac{d^3u}{dt\,dx^2} = 0.$$

To obtain a particular solution of this equation, let $u = \phi(t)\cos\frac{2\pi x}{\lambda} + \psi(t)\sin\frac{2\pi x}{\lambda}$. Substituting in the above equation, we see that $\phi(t)$ and $\psi(t)$ must satisfy the same equation, namely,

$$\phi''(t) + \frac{4\pi^2}{\lambda^2}mA\phi(t) + \frac{16\pi^2\mu}{3\lambda^2\rho_{,}}\phi'(t) = 0,$$

the integral of which is

$$\phi(t) = \epsilon^{-ct}\left(C\cos\frac{2\pi bt}{\lambda} + C'\sin\frac{2\pi bt}{\lambda}\right),$$

where $c = \frac{8\pi^2\mu}{3\lambda^2\rho_{,}}$, $b^2 = mA - \frac{16\pi^2\mu^2}{9\lambda^2\rho_{,}^2}$, C and C' being arbitrary constants. Taking the same expression with different arbitrary constants for $\psi(t)$, replacing products of sines and cosines by sums and differences, and combining the resulting sines and cosines two and two, we see that the resulting value of u represents two series of waves propagated in opposite directions. Considering only those waves which are propagated in the positive direction of x, we have

$$u = C_1\epsilon^{-ct}\cos\left\{\frac{2\pi}{\lambda}(bt - x) + C_2\right\} \ldots\ldots\ldots\ldots(21)$$

We see then that the effect of the tangential force is to make the intensity of the sound diminish as the time increases, and to render the velocity of propagation less than what it would otherwise be. Both effects are greater for high, than for low notes; but the former depends on the first power of μ, while the latter depends only on μ^2. It appears from the experiments of M. Biot, made on empty water pipes in Paris, that the velocity of propagation of sound is sensibly the same whatever be its pitch. Hence it is necessary to suppose that for air $\frac{\mu^2}{\lambda^2\rho_{,}^2}$ is insensible compared with A or $\frac{p_{,}}{\rho_{,}}$. I am not aware of any similar experiments made on water, but the ratio of $\left(\frac{\mu}{\lambda\rho_{,}}\right)^2$ to A would probably be insensible for water also. The diminution of intensity as the time increases is, in the case of plane waves, due *entirely* to friction; but as we do not possess any means of measuring the intensity of sound the theory cannot be tested, nor the numerical value of μ determined, in this way.

The velocity of sound in air, deduced from the note given by a known tube, is sensibly less than that determined by direct observation. Poisson thought that this might be due to the retardation of the air by friction against the sides of the tube. But from the above investigation it seems unlikely that the effect produced by that cause would be sensible.

The equation (21) may be considered as expressing in all cases the effect of friction; for we may represent an arbitrary disturbance of the medium as the aggregate of series of plane waves propagated in all directions.

8. Let us now consider the motion of a mass of uniform inelastic fluid comprised between two cylinders having a common axis, the cylinders revolving uniformly about their axis, and the fluid being supposed to have attained its permanent state of motion. Let the axis of the cylinders be taken for that of z, and let q be the actual velocity of any particle, so that $u = -q\sin\theta$, $v = q\cos\theta$, $w = 0$, r and θ being polar co-ordinates in a plane parallel to xy.

Observing that $\dfrac{d^2f}{dx^2} + \dfrac{d^2f}{dy^2} = \dfrac{d^2f}{dr^2} + \dfrac{1}{r}\dfrac{df}{dr} + \dfrac{1}{r^2}\dfrac{d^2f}{d\theta^2}$, where f is any function of x and y, and that $\dfrac{dp}{d\theta} = 0$, we have from equations (13), supposing after differentiation that the axis of x coincides with the radius vector of the point considered, and omitting the forces, and the part of the pressure due to them,

$$\frac{dp}{dr} - \rho\frac{q}{r} = 0,$$

$$\frac{d^2q}{dr^2} + \frac{1}{r}\frac{dq}{dr} - \frac{q}{r^2} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

and the equation of continuity is satisfied identically.

The integral of (22) is $\qquad q = \dfrac{C}{r} + C'r.$

If a is the radius of the inner, and b that of the outer cylinder, and if q_1, q_2 are the velocities of points close to these cylinders respectively, we must have $q = q_1$ when $r = a$, and $q = q_2$ when $r = b$, whence

$$q = \frac{1}{b^2 - a^2}\left\{(bq_1 - aq_2)\frac{ab}{r} + (bq_2 - aq_1)r\right\} \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

If the fluid is infinitely extended, $b = \infty$, and

$$\frac{q}{q_1} = \frac{a}{r}.$$

These cases of motion were considered by Newton, (*Principia*, Lib. II. Prop. 51.) The hypothesis which I have made agrees in this case with his, but he arrives at the result that the velocity is constant, not, that it varies inversely as the distance. This arises from his having taken, as the condition of there being no acceleration or retardation of the motion of an annulus, that the force tending to turn it in one direction must be equal to that tending to turn it in the opposite, whereas the true condition is that the moment of the force tending to turn it one way must be equal to the moment of the force tending to turn it the other. Of course, making this alteration, it is easy to arrive at the above result by Newton's reasoning. The error just mentioned vitiates the result of Prop. 52. It may be shown from the general equations that in this case a permanent motion in annuli is impossible, and that, whatever may be the law of friction between the solid sphere and the fluid. Hence it appears that it is necessary to suppose that the particles move in planes passing through the axis of rotation, while they at the same time move round it. In fact, it is easy to see that from the excess of centrifugal force in the neighbourhood of the equator of the revolving sphere the particles in that part will recede from the sphere, and approach it again in the neighbourhood of the poles, and this circulating motion will be combined with a motion about the axis. If however we leave the centrifugal force out of consideration, as Newton has done, the motion in annuli becomes possible, but the solution is different from Newton's, as might have been expected.

The case of motion considered in this article may perhaps admit of being compared with experiment, without knowing the conditions which must be satisfied at the surface of a solid. A hollow, and a solid cylinder might be so mounted as to admit of being turned with different uniform angular velocities round their common axis, which is supposed to be vertical. If both cylinders are turned, they ought to be turned in opposite directions, if only one, it ought to be the outer one; for if the inner were made to revolve too fast, the fluid near it would have a tendency to fly outwards in consequence of the centrifugal force, and eddies would be produced. As long as the angular velocities are not great, so that the surface of the liquid is very nearly plane, it is not of much importance that the fluid is there terminated; for the conditions which must be satisfied at a free surface are satisfied for any section of the fluid made by a horizontal plane, so long as the motion about that section is supposed to be the same as it would be if the cylinders were infinite. The principal difficulty would probably be to measure accurately the time of revolution, and distance from the axis, of the different annuli. This would probably be best done by observing motes in the fluid. It might be possible also to discover in this way the conditions to be satisfied at the surface of the cylinders; or at least a law might be suggested, which could be afterwards compared more accurately with experiment by means of the discharge of pipes and canals.

If the rotations of the cylinders are in opposite directions, there will be a certain distance from the axis at which the fluid will not revolve at all. Writing $-q_1$ for q_1 in equation (23), we have for this distance $\sqrt{\dfrac{ab(bq_1+aq_2)}{bq_2+aq_1}}$.

9. Although the discharge of a liquid through a long straight pipe or canal, under given circumstances, cannot be calculated without knowing the conditions to be satisfied at the surface of contact of the fluid and solid, it may be well to go a certain way towards the solution.

Let the axis of z be parallel to the generating lines of the pipe or canal, and inclined at an angle α to the horizon; let the plane yz be vertical, and let y and z be measured downwards. The motion being uniform, we shall have $u=0$, $v=0$, $w=f(x,y)$, and we have from equations (13)

$$\frac{dp}{dx}=0,\quad \frac{dp}{dy}=g\rho\cos\alpha,\quad \frac{dp}{dz}=g\rho\sin\alpha+\mu\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}\right).$$

In the case of a canal $\dfrac{dp}{dz}=0$; and the calculation of the motion in a pipe may always be reduced to that of the motion in the same pipe when $\dfrac{dp}{dz}$ is supposed to be zero, as may be shown by reasoning similar to Dubuat's. Moreover the motion in a canal is a particular case of the motion in a pipe. For consider a pipe for which $\dfrac{dp}{dz}=0$, and which is divided symmetrically by the plane xz. From the symmetry of the motion, it is clear that we must have $\dfrac{dw}{dy}=0$ when $z=0$;

but this is precisely the condition which would have to be satisfied if the fluid had a free surface coinciding with the plane xz; hence we may suppose the upper half of the fluid removed, without affecting the motion of the rest, and thus we pass to the case of a canal. Hence it is the same thing to determine the motion in a canal, as to determine that in the pipe formed by completing the canal symmetrically with respect to the surface of the fluid.

We have then, to determine the motion, the equation

$$\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{g\rho \sin a}{\mu} = 0.$$

In the case of a rectangular pipe, it would not be difficult to express the value of w at any point in terms of its values at the several points of the perimeter of a section of the pipe. In the case of a cylindrical pipe the solution is extremely easy: for if we take the axis of the pipe for that of z, and take polar co-ordinates r, θ in a plane parallel to xy, and observe that $\frac{dw}{d\theta} = 0$, since the motion is supposed to be symmetrical with respect to the axis, the above equation becomes

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \frac{g\rho \sin a}{\mu} = 0.$$

Let a be the radius of the pipe, and U the velocity of the fluid close to the surface; then, integrating the above equation, and determining the abitrary constants by the conditions that w shall be finite when $r = 0$, and $w = U$ when $r = a$, we have

$$w = \frac{g\rho \sin a}{4\mu}(a^2 - r^2) + U.$$

SECTION II.

Objections to Lagrange's proof of the theorem that if $udx + vdy + wdz$ *is an exact differential at any one instant it is always so, the pressure being supposed equal in all directions. Principles of M. Cauchy's proof. A new proof of the theorem. A physical interpretation of the circumstance of the above expression being an exact differential.*

10. THE proof of this theorem given by Lagrange depends on the legitimacy of supposing u, v and w capable of expansion according to positive integral powers of t, for a sufficiently small finite value of t. It is clear that the expansion cannot contain negative powers of t, since u, v and w are supposed to be finite when $t = 0$; but it may be objected to Lagrange's proof that there are functions of t of which the expansion contains fractional powers of t, and that we do not know but that u, v and w may be such functions. This objection has been considered by Mr. Power*, who has shown that the theorem is true if we suppose u, v and w capable of expansion according to any powers of t. Still the proof remains unsatisfactory, in fact inconclusive, for these are functions of t, (for instance $e^{-\frac{1}{t^2}}$, $t \log t$,) which do not admit of expansion according

* *Cambridge Philosophical Transactions,* Vol. VII. Part 3

to powers of t, integral or fractional, and we do not know but that u, v and w may be functions of this nature. I do not here mention the proof which Poisson has given of the theorem in his *Traité de Méchanique*, because it appears to me liable to an objection to which I shall presently have occasion to refer: in fact, Poisson himself did not think the theorem generally true.

It is remarkable that Mr. Power's proof, if it were legitimate, would establish the theorem even when account is taken of the variation of pressure in different directions, according to the theory explained in Section 1, if we suppose that $\frac{d\mu}{dp} = 0$. To show this we have only got to treat equations (12) as Mr. Power has treated the three equations of fluid motion formed on the ordinary hypothesis. Yet in this case the theorem is evidently untrue. Thus, conceive a mass of fluid which is bounded by a solid plane coinciding with the plane yz, and which extends infinitely in every direction on the positive side of the axis of x, and suppose the fluid at first to be at rest. Suppose now the solid plane to be moved in any manner parallel to the axis of y; then, unless the solid plane exerts no tangential force on the fluid, (and we may suppose that it does exert some,) it is clear that at a given time we shall have $u = 0$, $v = f(x)$, $w = 0$, and therefore $udx + vdy + wdz$ will not be an exact differential. It will be interesting then to examine in this case the nature of the function of t which expresses the value of v.

Supposing X, Y, Z to be zero in equations (12), and observing that in the case considered we have $\frac{dp}{dy} = 0$, we get

$$\frac{dv}{dt} = \frac{\mu}{\rho}\frac{d^2v}{dx^2} \qquad (24)$$

Differentiating this equation $n - 1$ times with respect to t, we easily get

$$\frac{d^nv}{dt^n} = \left(\frac{\mu}{\rho}\right)^n \frac{d^{2n}v}{dx^{2n}}:$$

but when $t = 0$, $v = 0$ when $x > 0$, and therefore for a given value of x all the differential coefficients of v with respect to t are zero. Hence for indefinitely small values of t the value of v at a given point increases more slowly than if it varied ultimately as any power of t, however great; hence v cannot be expanded in a series according to powers of t. This result is independent of the condition to be satisfied at the surface of the solid plane.

I think what has just been proved shows clearly that Lagrange's proof of the theorem considered, even with Mr. Power's improvement of it, is inadmissible.

11. The theorem is however true, and a proof of it has been given by M. Cauchy*, which appears to me perfectly free from objection, and which is very simple in principle, although it depends on rather long equations. M. Cauchy first eliminates p from the three equations of motion by means of the conditions that $\frac{d^2p}{dxdy} = \frac{d^2p}{dydx}$ &c., he then changes the independent variables from x, y, z, t to a, b, c, t, where a, b, c are the initial co-ordinates of the particles. The three transformed equations admit each of being once integrated with respect to t; and determining the arbitrary functions of a, b, c by the initial values of u, v and w, the three integrals have the form

$$\omega_0' = F\omega' + G\omega'' + H\omega''', \text{ \&c.},$$

* *Mémoire sur la Théorie des Ondes*, in the first volume of the *Mémoires présentés à l'Institut*. M. Cauchy has not had occasion to enunciate the theorem, but it is contained in his equations (16). This equation may be obtained in the same manner in the more general case in which p is supposed to be a function of ρ.

ω', ω'' and ω''' denoting here the same as in Art. 2, and ω_0' &c. denoting the initial values of ω', &c. for the same particle. Solving the above equations with respect to ω', ω'' and ω''', the resulting equations are

$$\omega' = \frac{1}{S}\left(\frac{dx}{da}\omega_0' + \frac{dx}{db}\omega_0'' + \frac{dx}{dc}\omega_0'''\right), \text{ \&c.},$$

where S is a function of the differential coefficients of x, y and z with respect to a, b and c, which by the condition of continuity is shown to be equal to $\frac{\rho}{\rho_0}$, ρ_0 being the initial density about the particle whose density at the time considered is ρ. Since $\frac{dx}{da}$ &c. are finite, (for to suppose them infinite would be equivalent to supposing a discontinuity to exist in the fluid,) it follows at once from the preceding equations that if $\omega_0' = 0$, $\omega_0'' = 0$, $\omega_0''' = 0$, that is if $u_0 da + v_0 db + w_0 dc$ be an exact differential, either for the whole fluid or for any portion of it, then shall $\omega' = 0$, $\omega'' = 0$, $\omega''' = 0$, i.e. $u dx + v dy + w dz$ will be an exact differential, at any subsequent time, either for the whole mass or for the above portion of it.

12. It is not from seeing the smallest flaw in M. Cauchy's proof that I propose a new one, but because it is well to view the subject in different lights, and because the proof which I am about to give does not require such long equations. It will be necessary in the first place to prove the following lemma.

LEMMA. If ω_1, $\omega_2 \ldots \omega_n$ are n functions of t, which satisfy the n differential equations

$$\left.\begin{aligned} \frac{d\omega_1}{dt} &= P_1\omega_1 + Q_1\omega_2 \ldots + V_1\omega_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{d\omega_n}{dt} &= P_n\omega_1 + Q_n\omega_2 \ldots + V_n\omega_n, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

where P_1, $Q_1 \ldots V_n$ may be functions of t, $\omega_1 \ldots \omega_n$, and if when $\omega_1 = 0$, $\omega_2 = 0 \ldots \omega_n = 0$, none of the quantities P_1, $\ldots V_n$ is infinite for any value of t from 0 to T, and if $\omega_1 \ldots \omega_n$ are each zero when $t = 0$, then shall each of these quantities remain zero for all values of t from 0 to T.

DEMONSTRATION. Let τ be a finite value of t, then by hypothesis τ may be taken so small that the values of $\omega_1 \ldots \omega_n$ are sufficiently small to exclude all values which might render any one of the quantities $P_1 \ldots V_n$ infinite. Let L be a superior limit to the numerical values of the several quantities $P_1 \ldots V_n$ for all values of t from 0 to τ; then it is evident that $\omega_1 \ldots \omega_n$ cannot increase faster than if they satisfied the equations

$$\left.\begin{aligned} \frac{d\omega_1}{dt} &= L(\omega_1 + \omega_2 \ldots + \omega_n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{d\omega_n}{dt} &= L(\omega_1 + \omega_2 \ldots + \omega_n), \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

vanishing in this case also when $t = 0$. But if $\omega_1 + \omega_2 \ldots + \omega_n = \Omega$, we have by adding together the above equations

$$\frac{d\Omega}{dt} = nL\Omega:$$

if now Ω be not equal to zero, dividing this equation by Ω and integrating, we have

$$\Omega = Ce^{nLt};$$

but no value of C different from zero will allow Ω to vanish when $t = 0$, whereas by hypothesis it does vanish; hence $\Omega = 0$; but Ω is the sum of n quantities which evidently cannot be negative, and therefore each of these must be zero. Since then $\omega_1 \ldots \omega_n$ would have to be equal to zero for all values of t from 0 to τ even if they satisfied equations (26), they must *à fortiori* be equal to zero in the actual case, since they satisfy equations (25). Hence there is no value of t from 0 to T at which any one of the quantities $\omega_1 \ldots \omega_n$ can begin to differ from zero, and therefore these quantities must remain equal to zero for all values of t from 0 to T.

This lemma might be extended to the case in which $n = \infty$, with certain restrictions as to the convergency of the series. We may also, instead of the integers $1, 2 \ldots n$, have a continuous variable α which varies from 0 to a, so that ω is a function of the independent variables α and t, satisfying the differential equation

$$\frac{d\omega}{dt} = \int_0^a \psi(\alpha, \omega, t)\,\omega\, d\alpha,$$

where $\psi(\alpha, 0, t)$ does not become infinite for any value of α from 0 to a combined with any value of t from 0 to T. It may be shown, just as before, that if $\omega = 0$ when $t = 0$ for all values of α from 0 to a, then must $\omega = 0$ for all values of t from 0 to T. The proposition might be further extended to the case in which $a = \infty$, with a certain restriction as to the convergency of the integral, but equations (25) are already more general than I shall have occasion to employ.

It appears to me to be sometimes assumed as a principle that two variables, functions of another, t, are proved to be equal for all values of t when it is shown that they are equal for a certain value of t, and that whenever they are equal for the same value of t their increments for the same increment of t are *ultimately* equal. But according to this principle, if two curves could be shown always to touch when they meet they must always coincide, a conclusion manifestly false. I confess I cannot see that Newton in his *Principia*, Lib. I. Prop. 40 has proved more than that if the velocities of the two bodies are equal at equal distances, the increments of those velocities for equal increments of the distances are ultimately equal: at least something additional seems required to put the proof quite out of the reach of objection. Again it is usual to speak of the condition, that the motion of a particle of fluid in contact with the surface of a solid at rest is tangential to the surface, as the same thing as the condition that the particle shall always remain in contact with the surface. That it is the same thing might be shown by means of the lemma in this article, supposing the motion continuous; but independently of proof I do not see why a particle should not move in a curve not coinciding with the surface, but touching it where it meets it. The same remark will apply to the condition that a particle which at one instant lies in a free surface, or is in contact with a solid, shall ultimately lie in the free surface, or be in contact with the solid, at the consecutive instant. I refer here to the more general case in which the solid is at rest or in motion. For similar reasons Poisson's proof of the Hydrodynamical theorem which forms the principal subject of this section has always appeared to me unsatisfactory, in fact far less satisfactory than Lagrange's. I may add that Poisson's proof, as well as Lagrange's, would apply to the case in which friction is taken into account, in which case the theorem is not true.

13. Supposing ρ to be a function of p, $\dfrac{1}{f'(p)}$, the ordinary equations of Hydrodynamics are

$$\frac{df(p)}{dx} = X - \frac{Du}{Dt}, \quad \frac{df(p)}{dy} = Y - \frac{Dv}{Dt}, \quad \frac{df(p)}{dz} = Z - \frac{Dw}{Dt} \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

The forces X, Y, Z will here be supposed to be such that $X dx + Y dy + Z dz$ is an exact differential, this being the case for any forces emanating from centres, and varying as any functions of the distances. Differentiating the first of equations (27) with respect to y, and the second with respect to x, subtracting, putting for $\frac{Du}{Dt}$ and $\frac{Dv}{Dt}$ their values, adding and subtracting $\frac{du}{dz}\frac{dv}{dz}$, and employing the notation of Art. 2, we obtain

$$\frac{D\omega'''}{Dt} = \frac{du}{dz}\omega' + \frac{dv}{dz}\omega'' - \left(\frac{du}{dx} + \frac{dv}{dy}\right)\omega''' \ldots\ldots\ldots\ldots\ldots\ldots(28)$$

By treating the first and third, and then the second and third of equations (27) in the same manner, we should obtain two more equations, which may be got at once from that which has just been found by interchanging the requisite quantities. Now for points in the interior of the mass the differential coefficients $\frac{du}{dz}$, &c. will not be infinite, on account of the continuity of the motion, and therefore the three equations just obtained are a particular case of equations (25). If then $u\,dx + v\,dy + w\,dz$ is an exact differential for any portion of the fluid when $t = 0$, that is, if ω', ω'' and ω''' are each zero when $t = 0$, it follows from the lemma of the last article that ω', ω'' and ω''' will be zero for any value of t, and therefore $u\,dx + v\,dy + w\,dz$ will always remain an exact differential. It will be observed that it is for the same portion of fluid, not for the fluid occupying the same portion of space, that this is true, since equations (28), &c. contain the differential coefficients $\frac{D\omega'}{Dt}$ &c., and not $\frac{d\omega'}{dt}$, &c.

14. The circumstance of $u\,dx + v\,dy + w\,dz$ being an exact differential admits of a physical interpretation which may be noticed, as it is well to view a subject of this nature in different lights.

Conceive an indefinitely small element of a fluid in motion to become suddenly solidified, and the fluid about it to be suddenly destroyed; let the form of the element be so taken that the resulting solid shall be that which is the simplest with respect to rotatory motion, namely, that which has its three principal moments about axes passing through the centre of gravity equal to each other, and therefore every axis passing through that point a principal axis, and let us enquire what will be the linear and angular motion of this element just after solidification.

By the instantaneous solidification, velocities will be suddenly generated or destroyed in the different portions of the element, and a set of mutual impulsive forces will be called into action. Let x, y, z be the co-ordinates of the centre of gravity G of the element at the instant of solidification, $x + x'$, $y + y'$, $z + z'$ those of any other point in it. Let u, v, w be the velocities of G along the three axes just before solidification, u', v', w' the relative velocities of the point whose relative co-ordinates are x', y', z'. Let $\bar{u}$, $\bar{v}$, $\bar{w}$ be the velocities of G, $u_{,}$, $v_{,}$, $w_{,}$ the relative velocities of the point above mentioned, and ω', ω'', ω''' the angular velocities just after solidification. Since all the impulsive forces are internal, we have

$$\bar{u} = u,\ \bar{v} = v,\ \bar{w} = w.$$

We have also, by the principle of the conservation of areas,

$$\Sigma m\,\{y'(w_{,} - w') - z'(v_{,} - v')\} = 0, \ \&c.,$$

m denoting an element of the mass of the element considered. But $u_{,} = \omega''z' - \omega'''y'$, u' is ultimately equal to $\frac{du}{dx}x' + \frac{du}{dy}y' + \frac{du}{dz}z'$, and similar expressions hold good for the other quantities. Substituting in the above equations, and observing that $\Sigma m y' z' = \Sigma m' z' x' = \Sigma m x' y' = 0$, and $\Sigma m x'^2 = \Sigma m y'^2 = \Sigma m z'^2$, we have

$$\omega' = \frac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right), \text{ \&c.}$$

We see then that an indefinitely small element of the fluid, of which the three principal moments about the centre of gravity are equal, if suddenly solidified and detached from the rest of the fluid will begin to move with a motion simply of translation, which may however vanish, or a motion of translation combined with one of rotation, according as $u\,dx + v\,dy + w\,dz$ is, or is not an exact differential, and in the latter case the angular velocities will be the same as in Art. 2.

The principle which forms the subject of this section might be proved, at least in the case of a homogeneous incompressible fluid, by considering the change in the motion of a spherical element of the fluid in the indefinitely small time dt. This method of proving the principle would show distinctly its intimate connexion with the hypothesis of normal pressure, or the equivalent hypothesis of the equality of pressure in all directions, since the proof depends on the impossibility of an angular velocity being generated in the element in the indefinitely small time dt by the pressure of the surrounding fluid, inasmuch as the direction of the pressure at any point of the surface ultimately passes through the centre of the sphere. The proof I speak of is however less simple than the one already given, and would lead me too far from my subject.

SECTION III.

Application of a method analogous to that of Sect. 1. *to the determination of the equations of equilibrium and motion of elastic solids.*

15. ALL solid bodies are more or less elastic, as is shown by the capability they possess of transmitting sound, and vibratory motions in general. The solids considered in this section are supposed to be homogeneous and uncrystallized, so that when in their natural state the average arrangement of their particles is the same at one point as at another, and the same in one direction as in another. The natural state will be taken to be that in which no forces act on them, from which it may be shown that the pressure in the interior is zero at all points and in all directions, neglecting the small pressure depending on attractions of the nature of capillary attraction.

Let x, y, z be the co-ordinates of any point P in the solid considered when in its natural state, α, β, γ the increments of those co-ordinates at the time considered, whether the body be in a state of constrained equilibrium or of motion. It will be supposed that α, β and γ are so small that their squares and products may be neglected. All the theorems proved in Art. 2. with reference to linear and angular velocities will be true here with reference to linear and angular displacements, since these two sets of quantities are resolved according to the same laws, as long as the angular displacements are supposed to be very small. Thus, the most general displacement of a very small element of the solid consists of a displacement of translation, an angular displacement, and three displacements of extension in the direction of three rectangular axes, which may be called in this case, with more propriety than in the former, *axes of extension*. The three displacements of extension may be resolved into two displacements of shifting, each in two dimensions, and a displacement of uniform dilatation, positive or negative. The pressures about the element considered will depend on the displacements of extension only;

there may also, in the case of motion, be a small part depending on the relative velocities, but this part may be neglected, unless we have occasion to consider the effect of the internal friction in causing the vibrations of solid bodies to subside. It has been shown (Art. 7.) that the effect of this cause is insensible in the case of sound propagated through air; and there is no reason to suppose it greater in the case of solids than in the case of fluids, but rather the contrary. The capability which solids possess of being put into a state of isochronous vibration shows that the pressures called into action by small displacements depend on homogeneous functions of those displacements of one dimension. I shall suppose moreover, according to the general principle of the superposition of small quantities, that the pressures due to different displacements are superimposed, and consequently that the pressures are linear functions of the displacements. Since squares of α, β and γ are neglected, these pressures may be referred to a unit of surface in the natural state or after displacement indifferently, and a pressure which is normal to any surface after displacement may be regarded as normal to the original position of that surface. Let $-A\delta$ be the pressure corresponding to a uniform linear dilatation δ when the solid is in equilibrium, and suppose that it becomes $-mA\delta$, in consequence of the heat developed, when the solid is in a state of rapid vibration. Suppose also that a displacement of shifting parallel to the plane xy, for which $\alpha = kx$, $\beta = -ky$, $\gamma = 0$, calls into action a pressure $-Bk$ on a plane perpendicular to the axis of x, and a pressure Bk on a plane perpendicular to that of y; the pressures on these planes being equal and of opposite signs, that on a plane perpendicular to the axis of z being zero, and the tangential forces on those planes being zero, for the same reasons as in Sect. 1. It may also be shown as before that it is necessary to suppose B positive, in order that the equilibrium of the solid medium may be stable, and it is easy to see that the same must be the case with A for the same reason.

It is clear that we shall obtain the expressions for the pressures from those already found for the case of a fluid by merely putting α, β, γ, B for u, v, w, μ and $-A\delta$ or $-mA\delta$ for p, according as we are considering the case of equilibrium or of vibratory motion, the body being in the latter case supposed to be constrained only in so far as depends on the motion.

For the case of equilibrium then we have from equations (8)

$$P_1 = -A\delta - 2B\left(\frac{d\alpha}{dx} - \delta\right), \quad T_1 = -B\left(\frac{d\beta}{dz} + \frac{d\gamma}{dy}\right), \text{ \&c.,} \qquad (29)$$

δ being here $= \frac{1}{3}\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right)$; and the equations of equilibrium will be obtained from (12) by putting $\frac{Du}{Dt} = 0$, $p = -A\delta$, making the same substitution as before for u, v, w and μ. We have therefore, for the equations of equilibrium,

$$\rho X + \frac{1}{3}(A + B)\frac{d}{dx}\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right) + B\left(\frac{d^2\alpha}{dx^2} + \frac{d^2\alpha}{dy^2} + \frac{d^2\alpha}{dz^2}\right) = 0, \text{ \&c.} \qquad (30)$$

In the case of a vibratory motion, when the body is in its natural state except so far as depends on the motion, we have from equations (8)

$$P_1 = -mA\delta - 2B\left(\frac{d\alpha}{dx} - \delta\right), \quad T_1 = -B\left(\frac{d\beta}{dz} + \frac{d\gamma}{dy}\right), \text{ \&c.,} \qquad (31)$$

and the equations of motion will be derived from (12) as before, only $\frac{Du}{Dt}$ &c. must be replaced by $\frac{d^2\alpha}{dt^2}$ &c., and X, Y, Z put equal to zero. The equations of motion, then, are

$$\rho\frac{d^2\alpha}{dt^2} = \frac{1}{3}(mA + B)\frac{d}{dx}\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right) + B\left(\frac{d^2\alpha}{dx^2} + \frac{d^2\alpha}{dy^2} + \frac{d^2\alpha}{dz^2}\right), \text{ \&c.} \qquad (32)$$

16. The conditions to be satisfied at the surface of the solid may be easily deduced from the analogous conditions in the case of a fluid with a free surface, only it will be necessary to replace the normal pressure Π by an oblique pressure, of which the components will be denoted by X_1, Y_1, Z_1. We have then, making the necessary changes in the quantities involved in (14),

$$X_1 + lA\delta + B\left\{2l\frac{d\alpha}{dx} + m\left(\frac{d\alpha}{dy} + \frac{d\beta}{dx}\right) + n\left(\frac{d\alpha}{dz} + \frac{d\gamma}{dx}\right)\right\} = 0, \text{ \&c.}$$

for the case of equilibrium, and for the case of motion such as that just considered it will only be necessary to replace A by mA in these equations. If we measure the angles of which l, m, n are the cosines from the external normal, the forces X_1, Y_1, Z_1 must be reckoned positive when l, m and n being positive, the surface of the solid is urged in the negative directions of x, y, z, and in other cases the signs must be taken conformably.

If the solid considered is in a state of constraint when at rest, and is moreover put into a state of vibration, the pressures and displacements due to these two causes must be calculated separately and added together. If m were equal to 1, they could be calculated together from the same equations.

SECTION IV.

Principles of Poisson's theory of elastic solids, and of the oblique pressures existing in fluids in motion. Objections to one of his hypotheses. Reflections on the constitution, and equations of motion of the luminiferous ether in vacuum.

17. In the twentieth *Cahier* of the *Journal de l'Ecole Polytechnique* may be found a memoir by Poisson, entitled *Mémoire sur les Equations générales de l'Equilibre et du Mouvement des Corps solides élastiques et des Fluides*, which contains the substance of two memoirs presented by him to the Academy, brought together with some additions. In this memoir the author treats principally of the equations of equilibrium and motion of elastic solids, of the equations of equilibrium of fluids, with reference especially to capillary attraction, and of the equations of motion of fluids supposing the pressure not to be equal in all directions.

It is supposed by Poisson that all bodies, whether solid or fluid, are composed of ultimate molecules, separated from each other by vacant spaces. In the cases of an uncrystallized solid in its natural state, and of a fluid in equilibrium, he supposes that the molecules are arranged irregularly, and that the average arrangement is the same in all directions. These molecules he supposes to act on each other with forces, of which the main part is a force in the direction of the line joining the centres of gravity, and varying as some function of the distance of these points, and the remainder a secondary force, or it may be two secondary forces, depending on the molecules not being mathematical points. He supposes that it is on these secondary forces that the solidity of solid bodies depends. He supposes however that in calculating the pressures these secondary forces may be neglected, partly because they become insensible at much smaller distances than the main part of the forces, and partly because they act, on the average, alike in all directions. He supposes that the molecular force decreases very rapidly as the distance increases, yet not so rapidly but that the sphere in which the molecular action is sensible contains an immense number of molecules. He supposes consequently that in estimating the resultant force of a hemisphere of the medium on a molecule in the centre of its base the action of the neighbouring molecules, which are situated irregularly, may be neglected compared with the action of those more remote, of which the average may be taken. The consequence of this supposition of course is that the total action is normal to the base of the hemisphere, and sensibly the same for one molecule as for an adjacent one.

The rest of the reasoning by which Poisson establishes the equations of motion and equilibrium of elastic solids is purely mathematical, sufficient data having been already assumed. It might appear that the reasoning in Art. 16 of his memoir, by which the expression for N is simplified, required the fresh hypothesis of a symmetrical arrangement of the molecules; but it really does not, being admissible according to the principle of averages. Taking for the natural state of the body that in which the pressure is zero, the equations at which Poisson arrives contain only *one* unknown constant k, whereas the equations of Sect. III. of this paper contain *two*, A or mA and B. This difference depends on the assumption made by Poisson that the irregular part of the force exerted by a hemisphere of the medium on a molecule in the centre of its base may be neglected in comparison with the whole force. As a result of this hypothesis, Poisson finds that the change in direction, and the proportionate change in length, of a line joining two molecules are continuous functions of the co-ordinates of one of the molecules and the angles which determine the direction of the line; whereas in Sect. III., if we adopt the hypothesis of ultimate molecules at all, it is allowable to suppose that these quantities vary irregularly in passing from one pair of molecules to an adjacent pair. Of course the equations of Sect. III. ought to reduce themselves to Poisson's equations for a particular relation between A and B. Neglecting the heat developed by compression, as Poisson has done, and therefore putting $m = 1$, this relation is $A = 5B$.

18. Poisson's theory of fluid motion is as follows. The time t is supposed to be divided into a number n of equal parts, each equal to τ. In the first of these the fluid is supposed to be displaced as an elastic solid would be, according to Poisson's previous theory, and therefore the pressures are given by the same equations. If the causes producing the displacement were now to cease, the fluid would re-arrange itself, so that the average arrangement about each point should be the same in all directions after a very short time. During this time, the pressures would have altered, in an unknown manner, from those corresponding to a displaced solid to a normal pressure equal to $p + \frac{Dp}{Dt}\tau$, the pressures during the alteration involving an unknown function of the time elapsed since the end of the interval τ. Another displacement and another re-arrangement may now be supposed to take place, and so on. But since these very small relative motions will take place independently of each other, we may suppose each displacement to begin at the expiration of the time during which the preceding one is supposed to remain, and we may suppose each re-arrangement to be going on during the succeeding displacements. Supposing now n to become infinite, we pass to the case in which the fluid is supposed to be continually beginning to be displaced as a solid would, and continually re-arranging itself so as to make the average arrangement about each point the same in all directions.

Poisson's equations (9), page 152, which are applicable to the motion of a liquid, or of an elastic fluid in which the change of density is small, agree with equations (12) of this paper. For the quantity ψt is the pressure p which would exist at any instant if the motion were then to cease, and the increment, $\frac{d\psi t}{dt}\tau$ or $\frac{Dp}{Dt}\tau$, of this quantity in the very small time τ will depend only on the increment, $\frac{d\chi t}{dt}\tau$ or $\frac{D\rho}{Dt}\tau$, of the density χt or ρ. Consequently the value of $\frac{d\psi t}{dt}\tau$ will be the same as if the density of the particle considered passed from χt to $\chi t + \frac{d\chi t}{dt}\tau$ in the time τ by a uniform motion of dilatation. I suppose that according to Poisson's views such a motion would not require a re-arrangement of the molecules, since the pressure remains equal in all directions. On this supposition we shall get the value of $\frac{d\psi t}{dt}$ from that of $R'_{,} - K$ in the equations of page 140 by putting $\frac{du}{dx} = \frac{dv}{dy} = \frac{dw}{dz} = -\frac{1}{3\chi t}\frac{d\chi t}{dt}$. We have therefore

$$a\frac{d\psi t}{dt} = \frac{a}{3}(K - 5k)\frac{d\chi t}{\chi t dt}.$$

Putting now for $\beta + \beta'$ its value $2ak$, and for $\frac{1}{\chi t}\frac{d\chi t}{dt}$ its value given by equation (2), the expression for ϖ, page 152, becomes

$$\varpi = p + \frac{a}{3}(K + k)\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right).$$

Observing that $a(K + k) = \beta$, this value of ϖ reduces Poisson's equations (9) to the equations (12) of this paper.

Poisson himself has not made this reduction of his equations, nor any equivalent one, so that his equations, as he has left them, involve two arbitrary constants. The reduction of these two to one depends on the assumption that a uniform expansion of any particle does not require a re-arrangement of the molecules, as it leaves the pressure still equal in all directions. If we do not make this assumption, but retain the two arbitrary constants, the equations will be the same as those which would be obtained by the method of this paper, supposing the quantity κ of Art. 3 not to be zero.

19. There is one hypothesis made in the common theory of elastic solids, the truth of which appears to me very questionable. That hypothesis is the one to which I have already alluded in Art. 17, respecting the legitimacy of neglecting the irregular part of the action of the molecules in the immediate neighbourhood of the one considered, in comparison with the total action of those more remote, which is regular. It is from this hypothesis that it follows as a result that the molecules are not displaced among one another in an irregular manner, in consequence of the directive action of neighbouring molecules. Now it is obvious that the molecules of a fluid admit of being displaced among one another with great readiness. The molecules of solids, or of most solids at any rate, must admit of new arrangements, for most solids admit of being bent, permanently, without being broken. Are we then to suppose that when a solid is constrained it has no *tendency* to relieve itself from the state of constraint, in consequence of its molecules *tending towards* new relative positions, provided the amount of constraint be very small? It appears to me to be much more natural to suppose *á priori* that there should be some such tendency.

In the case of a uniform dilatation or contraction of a particle, a re-arrangement of its molecules would be of little or no avail towards relieving it from constraint, and therefore it is natural to suppose that in this case there is little or no tendency towards such a re-arrangement. It is quite otherwise, however, in the case of what I have called a displacement of shifting. Consequently B will be less than if there were no tendency to a re-arrangement. On the hypothesis mentioned in this article, of which the absence of such tendency is a consequence, I have said that a relation has been found between A and B, namely $A = 5B$. It is natural then to expect to find the ratio of A to B greater than 5, approaching more nearly to 5 as the solid considered is more hard and brittle, but differing materially from 5 for the softer solids, especially such as Indian rubber, or, to take an extreme case, jelly. According to this view the relation $A = 5B$ belongs only to an ideal elastic solid, of which the solidity, or whatever we please to call the property considered, is absolutely perfect.

To show how implicitly the common theory of elasticity seems to be received by some, I may mention that MM. Lamé and Clapeyron mention Indian rubber among the substances to which it would seem they consider their theory applicable *. I do not know whether the coefficient of elasticity, according to that theory, has been determined experimentally for Indian rubber, but one would fancy that the cubical compressibility thence deduced, by a method which will be seen in the next article, would turn out comparable with that of a gas.

20. I am not going to enter into the solution of equations (30), but I wish to make a few remarks on the results in some simple cases.

If k be the cubical contraction due to a uniform pressure P, then will

$$k = \frac{3P}{A}.$$

If a wire or rod, of which the boundary is any cylindrical surface, be pulled in the direction of its length by a force of which the value, referred to a unit of surface of a section of the rod, in P, the rod will extend itself uniformly in the direction of its length, and contract uniformly in the perpendicular direction; and if e be the extension in the direction of the length, and c the contraction in any perpendicular direction, both referred to a unit of length, we shall have

$$e = \frac{A+B}{3AB}P, \quad c = \frac{A-2B}{6AB}P:$$

also, the cubical dilatation $= e - 2c = \dfrac{P}{A}$.

If a cylindrical wire of radius r be twisted by a couple of which the moment is M, and if θ be the angle of torsion for a length z of the wire, we shall have

$$\theta = \frac{2Mz}{\pi B r^4}.$$

The expressions for k, c, e and θ, and of course all expressions of the same nature, depend on the reciprocals of A and B. Suppose now the value of e, or θ, or any similar quantity not depending on A alone, be given as the result of observation. It will easily be conceived that we might find very nearly the same value for B whether we supposed $A = 5B$ or $A = nB$, where n may be considerably greater than 5, or even infinite. Consequently the observation of two such quantities, giving very nearly the same value of B, might be regarded as confirming the common equations.

If we denote by E the coefficient of elasticity when A is supposed to be equal to $5B$ we have, neglecting the atmospheric pressure†,

$$e = \frac{2P}{5E}, \quad \theta = \frac{2Mz}{\pi E r^4}.$$

If now we denote by E_1 the value of E deduced from observation of the value of e, and by E_2 the value of E obtained by observing the value of θ, or else, which comes to the same, by observing the time of oscillation of a known body oscillating by torsion, we shall have

$$\frac{2}{5E_1} = \tfrac{1}{3}\left(\frac{1}{A} + \frac{1}{B}\right), \quad E_2 = B, \quad \text{whence } \frac{1}{A} = \frac{6}{5E_1} - \frac{1}{E_2}.$$

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* *Mémoires présentés à l'Institut*, Tom. IV. p. 469.

† Lamé, *Cours de Physique*, Tom. I.

If A be greater than $5B$, E_1 ought to be a little greater than E_2. This appears to agree with observation. Thus the following numbers are given by M. Lamé* $E_1 = 8000$, $E_2 = 7500$ for iron; $E_1 = 2510$, $E_2 = 2250$ for brass†. The difference between the values of E_1 and E_2 is attributed by M. Lamé to the errors to which the observation of the small quantity e is liable. If the above numbers may be trusted, we shall have

$$A = 60000,\ \ B = 7500,\ \ \frac{A}{B} = 8 \text{ for iron;}$$

$$A = 29724,\ \ B = 2250,\ \ \frac{A}{B} = 13{\cdot}21 \text{ for brass.}$$

The cubical contraction k is almost too small to be made the subject of direct observation‡, it is therefore usually deduced from the value of e, or from the coefficient of elasticity E found in some other way. On the supposition of a single coefficient E, we have $\frac{k}{e} = \frac{3}{2}$, but retaining the two, A and B, we have $\frac{k}{e} = \frac{9B}{A+B} = 9\left(1 + \frac{B}{A}\right)^{-1}\frac{B}{A}$, which will differ greatly from $\frac{3}{2}$ if $\frac{A}{B}$ be much greater than 5. The whole subject therefore requires, I think, a careful examination, before we can set down the values of the coefficients of cubical contraction of different substances in the list of well ascertained physical data. The result, which is generally admitted, that the ratio of the velocity of propagation of normal, to that of tangential vibrations in a solid is equal to $\sqrt{3}$, is another which depends entirely on the supposition that $A = 5B$. The value of m, again, as deduced from observation, will depend upon the ratio of A to B; and it would be highly desirable to have an accurate list of the values of m for different substances, in hopes of thereby discovering in what manner the action of heat on those substances is related to the physical constants belonging to them, such as their densities, atomic weights, &c.

The observations usually made on elastic solids are made on slender pieces, such as wires, rods, and thin plates. In such pieces, all the particles being at no great distance from the surface, it is easy to see that when any small portion is squeezed in one direction it has considerable liberty of expanding itself in a direction perpendicular to this, and consequently the results must depend mainly on the value of B, being not very different from what they would be if A were infinite. This is not so much the case with thick, stout pieces. If therefore such pieces could be put into a state of isochronous vibration, so that the musical notes and nodal lines could be observed, they would probably be better adapted than slender pieces for determining the value of mA. The value of m might be determined by comparing the value of mA, deduced from the observation of vibrations, with the value of A, deduced from observations made in cases of equilibrium, or, perhaps, of very slow motion.

21. The equations (32) are the same as those which have been obtained by different authors as the equations of motion of the luminiferous ether in vacuum. Assuming for the present that the equations of motion of this medium ought to be determined on the same principles as the equations of motion of an elastic solid, it will be necessary to consider whether the equations (32) are altered by introducing the consideration of a uniform pressure Π existing in the medium

* Lamé, *Cours de Physique*, Tom. I.

† These numbers refer to the French units of length and weight.

‡ I find however that direct experiments have been made by Prof. Oersted. According to these experiments the cubical compressibility of solids which would be obtained from Poisson's theory is in some cases as much as 20 or 30 times too great. See the *Report of the British Association* for 1833, p. 353, or *Archives des découvertes, &c.* for 1834, p. 94.

when in equilibrium; for we have evidently no right to assume, either that no such pressure exists, or, supposing it to exist, that the medium would expand itself but very slightly if it were removed. It will now no longer be allowable to confound the pressure referred to a unit of surface as it was, in the position of equilibrium of the medium, with the pressure referred to a unit of surface as it actually is. The latter mode of referring the pressure is more natural, and will be more convenient. Let the pressure, referred to a unit of surface as it is, be resolved into a normal pressure $\Pi + p_1$ and a tangential pressure t_1. All the reasoning of Sect. III. will apply to the small forces p_1 and t_1; only it must be remembered that in estimating the whole oblique pressure a normal pressure Π must be compounded with the pressures given by equations (31). In forming the equations of motion, the pressure Π will not appear, because the resultant force due to it acting on the element of the medium which is considered is zero. The equations (32) will therefore be the equations of motion required.

If we had chosen to refer the pressure to a unit of surface in the original state of the surface, and had resolved the whole pressure into a pressure $\Pi + p_1$ normal to the original position of the surface, and a pressure t_1 tangential to that position, the reasoning of Sect. III. would still have applied, and we should have obtained the same expressions as in (31) for the pressures P_1, T_1, &c., but the numerical value of A would have been different. According to this method, the pressure Π would have appeared in the equations of motion. It is when the pressures are measured according to the method which I have adopted that it is true that the equilibrium of the medium would be unstable if either A or B were negative. I must here mention that from some oversight the right-hand sides of Poisson's equations, at page 68 of the memoir to which I have referred, are wrong. The first of these equations ought to contain $\frac{\Pi}{\rho}\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right)$, instead of $\frac{\Pi}{\rho}\frac{d^2u}{dx^2}$, and similar changes must be made in the other two equations.

It is sometimes brought as an objection to the equations of motion of the luminiferous ether, that they are the same as those employed for the motion of solid bodies, and that it seems unnatural to employ the same equations for substances which must be so differently constituted. It was, perhaps, in consequence of this objection that Poisson proposes, at page 147 of the memoir which I have cited, to apply to the calculation of the motion of the luminiferous ether the same principles, with a certain modification, as those which he employed in arriving at his equations (9) page 152, *i. e.* the equations (12) of this paper. That modification consists in supposing that a certain function of the time $\phi(t)$ does not vary very rapidly compared with the variation of the pressure. Now the law of the transmission of a motion transversal to the direction of propagation depending on equations (12) of this paper is expressed, in the simplest case, by the equation (24); and we see that this law is the same as that of the transmission of heat, a law extremely different from that of the transmission of vibratory motions. It seems therefore unlikely that these principles are applicable to the calculation of the motion of light, unless the modification which I have mentioned be so great as wholly to alter the character of the motion, that is, unless we suppose the pressure to vary extremely fast compared with the function $\phi(t)$, whereas in ordinary cases of the motion of fluids the function $\phi(t)$ is supposed to vary extremely fast compared with the pressure.

Another view of the subject may be taken which I think deserves notice. Before explaining this view however it will be necessary to define what I mean in this paragraph by the word *elasticity*. There are two distinct kinds of elasticity; one, that by which a body which is uniformly compressed tends to regain its original volume, the other, that by which a body which is constrained in a manner independent of compression tends to assume its original form. The constants A and B of Sect. III. may be taken as measures of these two kinds of elasticity. In the present paragraph, the word will be used to denote the second kind. Now many highly elastic substances, as iron, copper, &c., are yet to a very sensible degree plastic. The plasticity of lead is greater than that of iron or copper, and, as appears from experiment, its elasticity less. On the whole it is probable that the greater the plasticity of a substance the less its elasticity, and *vice versâ*, although this rule is probably far from being without exception. When the plasticity of the substance is still further increased, and its elasticity diminished, it passes into a viscous fluid. There seems no line of demarcation between a solid and a viscous fluid. In fact, the practical distinction between these two classes of bodies seems to depend on the intensity of the *extraneous* force of gravity, compared with the intensity of the forces by which the parts of the substance are held together. Thus, what on the Earth is a soft solid might, if carried to the Sun, and retained at the same temperature, be a viscous fluid, the force of gravity at the surface of the Sun being sufficient to make the substance spread out and become level at the top: while what on the Earth is a viscous fluid might on the surface of Pallas be a soft solid. The gradation of viscous, into what are called perfect fluids seems to present as little abruptness as that of solids into viscous fluids; and some experiments which have been made on the sudden conversion of water and ether into vapour, when enclosed in strong vessels and exposed to high temperatures, go towards breaking down the distinction between liquids and gases.

According to the law of continuity, then, we should expect the property of elasticity to run through the whole series, only, it may become insensible, or else may be masked by some other more conspicuous property. It must be remembered that the elasticity here spoken of is that which consists in the tangential force called into action by a displacement of continuous sliding: the displacements also which will be spoken of in this paragraph must be understood of such displacements as are independent of condensation or rarefaction. Now the distinguishing property of fluids is the extreme mobility of their parts. According to the views explained in this article, this mobility is merely an extremely great plasticity, so that a fluid admits of a *finite*, but exceedingly small amount of constraint before it will be relieved from its state of tension by its molecules assuming new positions of equilibrium. Consequently the same oblique pressures can be called into action in a fluid as in a solid, provided the amount of relative displacement of the parts be exceedingly small. All we know for certain is that the effect of elasticity in fluids, (elasticity of the second kind be it remembered,) is quite insensible in cases of equilibrium, and it is probably insensible in all ordinary cases of fluid motion. Should it be otherwise, equations (8) and (12) will not be true, or only approximately true. But a little consideration will show that the property of elasticity may be quite insensible in ordinary cases of fluid motion, and may yet be that on which the phenomena of light entirely depend. When we find a vibrating string, the small extent of vibration of which can be actually seen, filling a whole room with sound, and remember how rapidly the intensity of the vibrations of the air must diminish as the distance from the string increases, we may easily conceive how small in general must be the amount of the relative motion of adjacent particles of air in the case of sound. Now the extent of the vibration of the ether, in the case of light, may be as small compared with the length of a wave of light as that of the air is compared with the length of a wave of sound: we have no reason to suppose it otherwise. When we remember then that the length of a wave of sound in air varies from some inches to several feet, while the greatest length of a wave of light is about .00003 of an inch, it is easy to imagine that the *relative* displacement of the particles of ether may be so small as not to reach, nor even come near to the greatest relative displacement which could exist without the molecules of the medium assuming new positions of equilibrium, or, to keep clear of the idea of molecules, without the medium assuming a new arrangement which might be permanent.

It has been supposed by some that air, like the luminiferous ether, ought to admit of transversal vibrations. According to the views of this article such would, mathematically speaking, be the case; but the extent of such vibrations would be necessarily so very small as to render them utterly insensible, unless we had organs with a delicacy equal to that of the retina adapted to receive them.

It has been shown to be highly probable that the ratio of A to B increases rapidly according as the medium considered is softer and more plastic. For fluids therefore, and among them for the luminiferous ether, we should expect the ratio of A to B to be extremely great. Now if N be the velocity of propagation of normal vibrations in the medium considered in Sect. III., and T that of transversal vibrations, it may be shown from equations (32) that

$$N^2 = \frac{mA + 4B}{3\rho}, \quad T^2 = \frac{B}{\rho}.$$

This is very easily shown in the simplest case of plane waves: for if $\beta = \gamma = 0$, $\alpha = f(x)$, the equations (32) give $\rho \dfrac{d^2\alpha}{dt^2} = \frac{1}{3}(mA + 4B)\dfrac{d^2\alpha}{dx^2}$, whence $\alpha = \phi(Nt - x) + \psi(Nt + x)$; and if $\alpha = \gamma = 0$, $\beta = f(x)$, the same equations give $\rho \dfrac{d^2\beta}{dt^2} = B\dfrac{d^2\beta}{dx^2}$, whence $\beta = \zeta(Tt - x) + \xi(Tt + x)$. Consequently we should expect to find the ratio of N to T extremely great. This agrees with a conclusion of the late Mr. Green's*. Since the equilibrium of any medium would be unstable if either A or B were negative, the least possible value of the ratio of N^2 to T^2 is $\frac{4}{3}$, a result at which Mr. Green also arrived. As however it has been shown to be highly probable that $A > 5B$ even for the hardest solids, while for the softer ones $\dfrac{A}{B}$ is much greater than 5, it is probable that $\dfrac{N}{T}$ is greater than $\sqrt{3}$ for the hardest solids, and much greater for the softer ones.

If we suppose that in the luminiferous ether $\dfrac{A}{B}$ may be considered infinite, the equations of motion admit of a simplification. For if we put $mA\left(\dfrac{d\alpha}{dx} + \dfrac{d\beta}{dy} + \dfrac{d\gamma}{dz}\right) = -p$ in equations (32), and suppose mA to become infinite while p remains finite, the equations become

$$\left.\begin{aligned} \rho\frac{d^2\alpha}{dt^2} &= -\frac{dp}{dx} + B\left(\frac{d^2\alpha}{dx^2} + \frac{d^2\alpha}{dy^2} + \frac{d^2\alpha}{dz^2}\right), \text{ \&c.} \\ \text{and} \quad \frac{d\alpha}{dx} &+ \frac{d\beta}{dy} + \frac{d\gamma}{dz} = 0. \end{aligned}\right\} \quad \cdots\cdots\cdots (33)$$

When a vibratory motion is propagated in a medium of which (33) are the equations of motion, it may be shown that $p = \psi(t)$ if the medium be indefinitely extended, or else if there be no motion at its boundaries. In considering therefore the transmission of light in an uninterrupted vacuum the terms involving p will disappear from equations (33); but these terms are, I believe, important in explaining Diffraction, which is the principal phenomenon the laws of which depend only on the equations of motion of the luminiferous ether in vacuum. It will be observed that putting $A = \infty$ comes to the same thing as regarding the ether as incompressible with respect to those motions which constitute Light.

G. G. STOKES.

* *Cambridge Philosophical Transactions*, Vol. VII. Part I. p. 2.

XXIII. *Calculations of the Heights of the Auroræ Boreales, of the 17th September and 12th October, 1833; with Observations upon the Locality of the Meteor. By* RICHARD POTTER, M.A., *late Fellow of Queens' College, Cambridge, and Professor of Natural Philosophy and Astronomy, University College, London.*

[Read *December* 8, 1845.]

THE data I have employed for the calculations of the heights of the arches of the Auroræ Boreales, which were seen on the nights of September 17th and October 12th, 1833, are chiefly contained in the Conspectuses of the observations printed and distributed, together with various recommendations, to members of the British Association for the advancement of Science.

In one instance, additional information is used from the Yorkshire Gazette; where Mr. Phillips gave the breadth of an arch which he had omitted in the Conspectus.

In consequence of the attention of scientific men having been drawn to the subject, the observations on these displays of the Aurora Borealis, were much more complete than had ever been obtained before. The time in the various observations was reduced to Greenwich time, by Mr. Phillips the Secretary of the Association, which thus facilitates the comparison of the phenomena noted by different observers: nevertheless they have never before been carefully discussed. The partial discussion communicated by Professor Airy to this Society in November 1833, and published in the *Philosophical Magazine* for December of that year, is the only previous discussion of them, that I am aware of; and the height was investigated only by a graphical method, which appears to have given results very inaccurate for many of the observations.

Regular observations on the Aurora of September 17th were taken by Mr. J. Phillips at York, by Mr. Clare, Mr. Hadfield and myself, at or near Manchester, by Professor Airy, at Cambridge, and by the Hon. C. Harris, near Gosport.

On the 12th October, regular observations were obtained by Professor Sedgwick, at Dent, near Sedbergh, by Mr. W. L. Wharton, near Guisborough, by Mr. J. Phillips, at York, by Mr. Clare, Mr. Hadfield and myself, at or near Manchester, by Dr. Robinson, at Armagh, by Professor Airy, at Cambridge, and by the Hon. Charles Harris, at Heron Court, near Christchurch, Hants.

The arches being perpendicular (or very nearly so) to the magnetic meridians of the places of observations, a base for trigonometical calculation is more certainly obtained with respect to them, than any other parts of the appearances. In the following calculations, I have accordingly used observations on the arches only.

In the Conspectus for the 17th September, I find only two sets of contemporaneous observations, the one for Cambridge and Manchester, at $8^{h}.25^{m}$ Greenwich time; the other for York and Gosport, at $11^{h}.0^{m}$. Manchester and York are too nearly of the same magnetic latitude to furnish an adequate base. To these I may add an observation of my own, of the altitude of an arch and its extent on the horizon, for calculating the height from an observation at one place only, by means of a subsidiary hypothesis that the arches are portions of small circles round the magnetic axis.

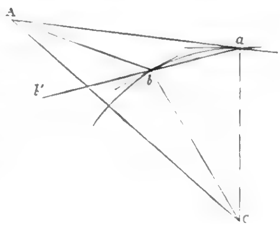
The Conspectus for the 12th October, furnishes more sets of contemporaneous observations, namely, Cambridge and York at $7^h.55^m$ Greenwich time; Guisborough and Heron Court at $8^h.20^m$; Dent and Manchester at $8^h.55^m$; Armagh and Manchester at $9^h.0^m$; and about 12 to 14 minutes later; Dent and Heron Court at $10^h.40^m$. Observations at Dent and Armagh, might have been taken, but with a much diminished base line; and Armagh is situated on so distant a magnetic meridian from that of Dent or Manchester, that the calculations have a greater value with respect to the law of terrestrial magnetism, than as giving very accurately the height of the Aurora.

The regular and perfect arches have their highest points so nearly in the magnetic meridian, that if there be any determinable deviation from this, more accurate methods of observation must be employed in order to measure it. If two places be situated on the same magnetic meridian, the point in the arch which has the greatest altitude above the horizon at the one place, will be the same as the point which has the greatest altitude at the other. If the places are not situated on the same magnetic meridian, this will not be the case; and in order to calculate the height of the arch above the earth's surface, from observations of the altitudes of the highest points, we must obtain our base by projecting the places on an intermediate magnetic meridian.

Let A and B be the two places, draw Aa, Bb perpendiculars on the magnetic meridian, then ab will be the base to be used in the trigonometrical calculations; and putting v = the magnetic variation, we have the formula in English miles,

$$ab = \{\text{difference of latitudes in degrees} \times \cos v \pm \text{difference of longitudes in degrees} \times \cos \text{latitude} \times \sin v\}\ 69.$$

The lower sign to be used when the place having the greater latitude, has the less West longitude. The arc of the magnetic meridian thus found and its chord, will not sensibly differ for any two of the places of observation; but the observed altitudes will require correction for the curvature of the meridian, in order to reduce the calculation to the case of a rectilineal triangle.



If C be the centre of the earth, A the point of the arch supposed to be observed at a and b, the projections as in the last figure. Then to solve the triangle Aab, we increase the observed altitude at a by half the angle aCb, and diminish the observed altitude at b by the same quantity, for the angles Abb', and Aab. Having found the distance Ab, we find the distance of A from the earth's centre by solving the triangle AbC; and therefore know the height above the earth's surface.

In this way I have calculated the following observations:

When the altitude of the arch was referred to a given star, I have calculated the altitude of the star from the Right Ascension and Declination given in the *Nautical Almanac*, for 1833.

In such case there was no correction for refraction to be applied, as the star and arch were equally affected.

In the observations on the 17th September, we have the following: the time in all cases being Greenwich time.

From Professor Airy's observations at Cambridge. " $8^h.25^m$.—The Aurora appeared in the form of a large bright cloud, bounded on the lower side by the horizon, and on the upper side by an arch of a small circle (not differing much from a great circle). The extremities of the arch were in the N. E. and W. N. W. or nearly W. The upper boundary was lower than β Ursæ Majoris by $\frac{3}{2}$ × distance from α Ursæ Majoris to β Ursæ Majoris," &c.

From Mr. Clare's observations at Manchester. "$8^h. 24^m$.—The arch 7° broad, includes Dubhe, Arcturus, and Capella, so that Capella is on the extreme upper edge; Dubhe rather above the middle of the breadth, and Arcturus rather below the middle, centre of the arch a little E. of δ Ursæ Majoris. Extent of the arch 130°."

Now the altitude of β Ursæ Majoris at $8^h. 25^m$. was 24°. 17′ and $\frac{3}{2} \times$ dist of α and $\beta = 8°. 4'$, therefore altitude of summit of arch = 16°. 13′ at Cambridge.

The altitude of α Ursæ Majoris (Dubhe) at Manchester at $8^h. 24^m$. was 31°. 14′, and azimuth 22°. 34′ N. towards W. about $2\frac{3}{4}°$. from the magnetic meridian. Therefore the altitude of the summit of the arch $= 31°. 14' + 3°. 15' = 34°. 29'$ nearly.

The distance of Cambridge and Manchester projected on the magnetic meridian whose variation is 24°. 30′ is 119.42 English miles.

These data give the distance of the arch from Manchester 123.27 miles, and the height above the earth's surface, of the upper edge, 71 miles. The breadth subtending 7° at Manchester, we find it to be 15 miles. Therefore the height of the lower edge was 56 miles.

The above arch having disappeared, and the Streamers and Auroral light having diminished, the appearances were subject to slight changes until $10^h. 49\frac{1}{2}^m$.; when another arch was seen at York by Mr. Phillips, and near Gosport by the Hon. C. Harris.

From Mr. J. Phillips's observations at York.

"$10^h. 49^m$. to $11^h. 19^m$.} A low faint arch stationary, its upper edge nearly reaching to η and γ Ursæ Majoris; its vertex under Mizar (alt. about 18° in the middle)."

In the *Yorkshire Gazette* for 21st September, 1833, Mr. Phillips states its breadth to be 4°; therefore the altitude of the under edge was 16°.

From the Hon. C. Harris's observations, at 1 mile W.N.W. of Gosport.

"$10^h. 49\frac{1}{2}^m$. to $11^h. 4\frac{1}{2}^m$.} Arch from N.W. to N.N.E. Its vertex under ζ Ursæ Majoris, and the edge of its base half way between that star and the horizon."

Now the altitude of ζ Ursæ Majoris at Gosport at $10^h. 57^m$. was 21°. 32′, and therefore the altitude of the lower edge was 10°. 46′.

The distance of York and Gosport projected on the magnetic meridian whose variation is 24°. 30′ is 197.66 miles.

These data give the distance from York 1011.53 miles, and the height above the earth's surface 389 miles.

In the Conspectus for the Aurora of October 12th, we have from Mr. Phillip's observations at York.

"$7^h. 56^m$.—The summit of the arch was now 3° below the stars β and γ Ursæ Maj. &c.

"$7^h. 57^m$.—Suddenly it appeared double, in consequence of the production of a very narrow faint arch above that seen before, and separated from it by a dark band.

"$7^h. 58^m$.—This upper arch rose, so as to include β and γ Ursæ Maj., in its midddle.

"$8^h. 2^m$.—It had vanished away, after rising still higher."

From Professor Airy's observations at Cambridge.

"$7^h. 54^m$.—The upper boundary of the bright cloud was extremely sharp; it began to the left of Arcturus, passed a very little above Arcturus, below γ Ursæ Maj. at exactly half the elevation of γ Ursæ Maj. (which was its highest point) and terminated E. of the N. at about half the azimuth of β Aurigæ. &c.

"$7^h. 59^m$.—A black line was discoverable very near the upper boundary and parallel to it. The upper part rose and the lower fell a little, thus widening the black line. About Arcturus the upper part rose most.

"$8^h. 2^m$.—The upper part after rising considerably had wholly disappeared, &c."

We have here one of the rare cases which fix the identity of the phenomenon seen; the arch appearing double at places so distant as Cambridge and York at the same time.

The distance of the projections of York and Cambridge on the magnetic meridian whose variation is 24°. 30′ is 129.97 miles. The altitude of γ Ursæ Majoris at York at $7^h. 56^m$. was 22°. 50′; therefore the altitude of the summit of the arch was 19°. 50′.

The altitude of γ Ursæ Majoris at Cambridge at $7^h. 54^m$. was 21°. 3′; consequently the altitude of the highest point of the arch was $10°. 31\frac{1}{2}′$.

These data give the summit of the arch 199.93 miles distant from York, and its height above the earth's surface 72.2 miles.

From Mr. W. L. Wharton's observations at Guisborough.

"$8^h. 20^m$.—Well defined arch, passing between α and β Ursæ Majoris its summit somewhat above ζ Ursæ Majoris, no radiations."

From the Hon. Charles Harris's observations at Heron Court, 4 miles N.W. of Christchurch, Hants.

"$8^h. 22^m$.—Bright, irregular arch, like a luminous bank of fog, about 8° above the horizon."

The distance of the projections of Guisborough and Heron Court on the magnetic meridian whose variation is 24°. 30′ is 225.1 miles.

The altitude of ζ Ursæ Majoris at Guisborough at $8^h. 20^m$. was 28°. 47′; therefore the summit of the arch would have an altitude of about 29°. The breadth of the arch passing between α and β Ursæ Majoris would be 5°; therefore the altitude of the lower edge would be 24°.

In the Hon. Mr. Harris's observation we have the altitude of the lower edge 8° − refraction = 7°. 53′.

From these data we find the distance from Guisborough to have been 167.34 miles, and the height of the under edge to have been 70.9 miles. The breadth being 14.6 miles, the height of the upper edge was 85.5 miles.

From the observations of Professor Sedgwick at Dent, near Sedbergh, Yorkshire.

"$8^h. 55^m$.—The upper part of the arch, better defined than before, passed between α and β Ursæ Maj. and very near ζ Ursæ Maj. Its vertex in or near the magnetic meridian. &c."

From my own observation near Manchester.

"$8^h. 53\frac{1}{2}^m$.—The arch has its vertex under ζ Ursæ Maj. and its upper edge touches γ Ursæ Maj., altitude about 19°. 30′."

At $8^h. 54^m$. Mr. Hadfield found, near Manchester, but on the opposite side, that the altitude was 20°, and the extent on the horizon 120°.

The distance of the projections of Dent and Manchester on the magnetic meridian with variation 25°. 30′ is 52.56 miles.

The altitude of ζ Ursæ Majoris at Dent at $8^h. 55^m$. was 26°. 32′, therefore the altitude of the arch passing near it we may call 26°. 26′. Mr. Hadfield's observation corrected for refraction gives the altitude at $8^h. 54^m$. as 19°. 57′.

With these data we find the arch to have been 183.38 miles from Dent, and the height of the upper edge to have been 84.97 miles.

The arch, or rather arches, appear to have been stationary from about $8^h. 54^m$. to $9^h. 10^m$., for from Professor Sedgwick's observations we have,

"$9^h. 10^m$.—Arch nearly as before."

From Mr. Clare's observations at Manchester, who has recorded the arch as double at $8^h. 54^m$., we have

"$9^h. 9^m$.—The two arches remain in the same position."

From Dr. Robinson's observations at Armagh.

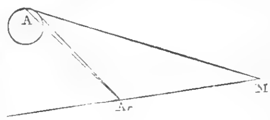
"$9^h. 1^m$.—Three parallel arches, the principal one has its upper edge on Polaris, and midway between Capella and β Aurigæ; its lower a little above β and γ Ursæ Majoris.

"$9^h. 6^m$.—Arch in the same place place, &c."

The distance of the projections of Armagh and Manchester on the magnetic meridian with variation 25°. 30′ is 128.76 miles.

By a careful examination of the course of the arch as seen at Armagh on a cœlestial globe, the altitude of the summit must have been about 60°, and the altitude at Manchester as above was 19°. 57′.

These data give the distance from Armagh 74.25 miles, and the height above the earth's surface 64.47 miles.



If we took the altitude at Armagh as 59°, and allowed 1°. 30′ for the point of the arch which appeared the highest at Armagh, not corresponding with that which appeared the highest at Manchester, on account of the elevation being so great at Armagh, as shewn by the figure, we should have the height of the arch above the earth's surface 66.5 miles, and the distance from Armagh 78.69 miles.

From Dr. Robinson's observations at Armagh.

"$9^h. 11^m$.—Upper edge of arch has risen to Lyra and Capella, and a new arch has risen beneath it, &c."

From my own observations near Manchester.

"$9^h. 14\frac{1}{2}^m$.—η Ursæ Majoris in the upper edge of the arch, the height of which by measure = 21°. 10′."

The altitude of η Ursæ Majoris at Manchester at $9^h. 14\frac{1}{2}^m$. was 21°. 6′ confirming the altitude I obtained by an instrument made purposely for observing the Aurora; as however there is a discrepancy between the height above the earth's surface deduced from these observations and the previous ones, I will suppose the extreme upper edge had an altitude of 22°, that we may be certain the discrepancy does not arise from an under valuing of the altitude at Manchester, but must be sought in other causes.

From Dr. Robinson's observation, the altitude of the upper edge must have been $71\frac{1}{2}°$, from which we may deduct $1\frac{1}{2}°$ for parallactic effect.

These data give the distance from Armagh 69.59 miles, and the height above the earth's surface 65.4 miles.

These results are remarkably in accordance with the others for the same places, but considerably different from the calculations for other places for nearly the same time; so that probably the method of projecting places of which the magnetic meridians are so distant as Armagh and Manchester upon an intermediate magnetic meridian to obtain a base line, is only approximately correct, from the course of the arch over the earth's surface, rather than for geometrical reasons.

Another arch was observed from $10^h. 34^m$. to $10^h. 45^m$. at Dent, Guisborough, York, Manchester, and Heron Court.

From Professor Sedgwick's observations we have,

"$10^h. 40^m$.—The bright space arranges itself into an arch, commencing nearly N., passing through η Ursæ Major.; about 25° high near the magnetic meridian (measured only by a geological clinometer)."

From the Hon. Charles Harris's observation.

"$10^h. 37^m$.—A low arch again formed, its base scarcely 5° above the horizon, extending to about 7°, &c."

The distance of the projections of Dent and Heron Court on the magnetic meridian with variation 25°, is 232.52 miles.

The altitudes of the summits of the arch being observed 25° at Dent and 7° at Heron Court, correcting these for refraction, we find the distance of the arch from Dent to have been 136.33 miles, and its height above the earth's surface 59.4 miles.

I some years ago shewed, in the *Edinburgh Journal of Science*, that the locality of an arch of an Aurora Borealis might be determined from observations at one place, by the help of the hypothesis that the arch is a small circle round the magnetic axis. This hypothesis cannot be accurate, from the change of the variation on the earth's surface, and we must conclude that, strictly, the regular arches are only perpendicular to a series of magnetic meridians; which for localities exterior to the earth's atmosphere, may be found, when the meteor has been more accurately observed, to differ from any assignable series on the earth's surface.

As an approximation this method gives the height sufficiently in accordance with the trigonometrical method, to induce us to attempt more accurate observations, when the theory of terrestrial magnetism shall be sufficiently advanced to enable us to profit by them.

The required observations are the altitude of the summit of the arch, and its extent on any given plane perpendicular to the magnetic meridian.

When the given plane is the horizon, the formula takes the following simple form:

$$R = 2r\sqrt{\frac{e^2 l p^2}{(p^2 l - m)^2} + \frac{1}{4}},$$

where r = earth's radius,

R = distance of the arch from the earth's centre,

e = trig. tang. of altitude of the summit,

m = (secant same angle)^2,

l = (secant $\frac{1}{2}$ extent on horizon)^2,

$p = 1 + eg$, where g = trig. tang. of magnetic polar distance of the place of observation.

In the Aurora of the 17th September, I obtained the following observation with the view to its being used with the above formula.

"$8^h.40\frac{1}{2}^m$. Arch 38° or 39° high, and extending about 160° on the horizon."

Taking the altitude 39°, and r = 3954 miles, the formula gives R = 4007.9 miles; whence

$R - r$ = height above the earth's surface = 53.9 miles.

We saw that the height of the under edge was 56 miles, and of the upper edge 71 miles at $8^h.24^m$.

From the preceeding results, we must conclude that the meteor occurs immediately beyond the ordinary limits assigned to the earth's atmosphere, and from that to very great altitudes; which is in accordance with the results of many previous calculations.

I shall conclude my paper with expressing my conviction that the Aurora Borealis will, in some future time, from its connection with the earth's magnetism, be subjected to much more accurate methods of observation than have hitherto been attempted.

R. POTTER.

XXIV. *The Mathematical Theory of the two great Solitary Waves of the First Order.* *By* S. Earnshaw, M.A., *of St. John's College, Cambridge.*

[Read *December* 8, 1845.]

Though it is now about a hundred years since the general equations of fluid motion, expressed in partial differential coefficients, were first given to the world, I am not aware that any important case of fluid motion has hitherto been rigorously extracted from them. This however has not arisen from want of effort, for the subject on account of its importance has successively occupied the attention of the first mathematicians from the days of D'Alembert to the present time; but rather from the peculiarly rebellious character of the equations themselves, which resist every attack, except it have reference to some case of a very simple and uninteresting nature.

This want of success I am inclined to attribute chiefly to our experimental ignorance of the peculiar and distinctive characters of different species of fluid motion. In this matter indeed there was a tendency to ignorance produced by that little success which had attended mathematical research; for as it was found that fluid motions of every sort, providing they are continuous, are all expressible by the same partial differential equations, it was thought that those equations ought to admit of being integrated in some general forms which should consequently include the properties of every possible kind of continuous fluid motion. The natural consequence of this idea has been that much effort has been unsuccessfully expended in attempts to obtain general integrals. Two ways of approximation however are open to research;—the one, in which the approximations are made by neglecting certain terms on account of their supposed smallness in comparison with the terms retained; and the other, in which ab initio hypotheses are made as to the paths or velocities or some other character of the motions of the particles. With regard to both these methods, it is evident that they must first be authorized by experiment, before they are used in verifying or predicting results. The former however is peculiarly liable to error, from our being uncertain in many cases, whether with the neglected terms, we may not have discarded some of the peculiar and essential properties of the motion we are investigating. And with respect to the latter method, recourse must be had to experiment to ascertain what are the really distinctive characters of the various kinds of fluid motion. Hence nothing seemed more likely to conduce to the advancement of the Theory of Hydrodynamics than the appointment of a Commission, by the British Association for the Advancement of Science, the object of which was the discovery of the "*Varieties*, *Phænomena*, and *Laws* of Waves:" for if there be *varieties* of waves differing in their phænomena and laws, it was too much to expect the mathematician (considering the exceedingly intractable nature of the equations with which he has to deal) to discover what are the precise hypotheses which lead to each variety. He must at least be allowed to know something of the peculiar phænomena of each variety, before he proceeds to the integration of his equations; and there is no way in which he could gain this knowledge except through the medium of experiments such as the Commission, just alluded to, were directed to institute. The differential equations of motion are too comprehensive to admit of general management. An hypothesis is in fact necessary to be made before we can advance a single step towards their integration; and by the aid of it we may only advance to a certain point, and no farther. If it be asked why we are thus stopped, the answer seems to be this; the results obtained up to that point are still of too general a character, embracing every variety of fluid motion which is compatible with the hypothesis on which we started. Now among the large class of such motions, there may be some varieties which cannot be analytically expressed by the same final formulæ; and consequently these require to be sifted from the others and from one another, by additional hypotheses; each hypothesis pointing at the variety or subdivision to which it belongs, and to no one else. Nothing in fact can be more clear than this, that if there be varieties of fluid motion the laws of which do not admit of being expressed in the same analytical forms, those varieties *must* be separately treated by the mathematician; and to the oversight of this necessity I attribute the insignificance of the progress which has hitherto been made in this subject.

I have thought it necessary to introduce these remarks, because some persons, especially among such as have not made Hydrodynamics a special object of study, are apt to depreciate investigations which set out upon a set of hypotheses which manifestly limit the range of the results obtained. They prefer investigations which set out with fewer and broader hypotheses, because they have the appearance of greater generality; and this character they continue to ascribe to such investigations, though it is found that in carrying them out it may have been found necessary to introduce a system of approximations by the neglect of certain terms. I am persuaded that this view is utterly fallacious in the majority of cases of any importance in nature: and that the wiser and better course when possible is, to consult experiment and thence obtain authority for a set of hypotheses to start with, and to carry out these hypotheses to the end without the introduction of analytical approximations. Our results will then be as comprehensive as our hypotheses, and as far as they go may be relied upon with unlimited confidence. This is the course which has been adopted in the following investigations. The experiments which I have taken as a guide in framing my hypotheses are those of Mr. Scott Russell which are printed in his "*Report on Waves*" in the "*Report of the Fourteenth Meeting of the British Association*." These experiments were conducted with well-contrived apparatus and great care, and are as worthy of confidence as experiments on wave motion can be: and there seems to be but one circumstance in them to be regretted, which is, that Mr. Russell having been led by his results to adopt a certain empirical formula for the velocity of transmission of a wave, his experiments seem in a great measure to have degenerated into an effort to establish the truth of that formula, in which he appears to have overlooked or forgotten the probability that after all it might only be an approximate result, and that the exact mathematical form might contain elements not recorded in his tables, because not required in his formula. The consequence of this oversight is that he has not recorded one element, very easy of observation, and of essential importance; viz. the distance through which each particle was transferred in space by a wave in passing it. Had this element been recorded, the experiments would have been much more complete: and without it they are certainly defective as accurate tests of theory. It is true Mr. Russell has given a rule for calculating this element; but he has not furnished us with the requisite data. These are the volume of the fluid which is elevated above the general level, and the breadth and depth of his canal. The last two are given, but the first is not given in any one instance. He has indeed stated the volume of fluid originally put in motion, and seems to have supposed that this would supply all that was wanted; entirely overlooking a fact, which must have forced itself upon his attention in the very first stages of his experimental researches, viz. that a single wave could never be generated alone, and that consequently all the fluid originally displaced did not go to form the single wave of observation; which besides, as the experiments themselves shewed, and as we shall prove theoretically, was continually wasting away, and thereby rendering the data still more inaccurate as the experiment proceeded.

And in fact Mr. Russell tells us he found it necessary to wait awhile after the completion of the process of generating a wave till the main wave had separated itself from the residuary waves, which always accompanied its genesis. To generate a single wave required, as we shall see, the exertion of a peculiar law of pressure; and as no attempt was made to secure the observance of this law in Mr. Russell's experiments, the inevitable consequence was the genesis of residuary waves. We shall also see from our theory, that the nature of the motions given to the particles of the fluid in this kind of wave produces a natural tendency in the wave to generate and cast off irregular disturbances from itself, working its own destruction as it proceeds. While therefore I look upon these experiments as very valuable additions to our knowledge, I still regard them as imperfect even to the extent to which they profess to have been carried. It is impossible indeed to read the Synopsis which Mr. Russell has given in page 343 of his *Report* without perceiving that he was too eager to adopt as results of experiments certain geometrical analogies, of which there seemed to be some faint shadowings indicated in his observations.

In his *Report* Mr. Russell conceives that his observations authorized him to consider waves as divisible into four *distinct* species: the first of which he has denominated "The great solitary wave." It is found to comprehend two varieties, the *positive* and the *negative* wave, which though agreeing in some general characters differ in others. The object of the present paper is to furnish the mathematical theory of this species. But how are we to sift this from the other species? I have examined the phænomena which Mr. Russell has recorded, and fixed upon such as belonged to this species alone; and these I have made the basis of my calculations. But it is obviously desirable that the phænomena thus selected should be of such a character as admitted of easy and accurate observation. That the reader may judge in this matter I will here propound them with Mr. Russell's statement of the method by which he obtained the one on which there might possibly be a doubt: merely premising that I suppose the wave to be transmitted in a horizontal canal of uniform breadth and depth, and that the fluid is incompressible.

1st. The velocity of transmission of a wave is uniform.

2nd. The horizontal velocity of all particles, which are situated in a vertical plane, intersecting the axis of the canal at right angles, is the same.

By a contrivance of peculiar ingenuity Mr. Russell was enabled to obtain the velocity of transmission with great exactness; and the result at which he arrived, and which we shall assume to be accurately true is, that abstracting from friction and the cohesion of particles, the velocity of transmission is uniform and the wave is permanent. We shall in the end shew that this hypothesis is not strictly accurate.

With respect to the verification of the other principle which I have assumed, Mr. Russell thus writes:—"The methods I had employed for such observations were the observation of the motion of small particles visible in the water of the same, or nearly the same specific gravity with water, or small globules of wax connected to very slender stems, so as to float at required depths. The motions of these were observed, from above on a minutely divided surface on the bottom of the channel; and from the side, through glass windows, themselves accurately graduated, the side of the channel opposite the windows being covered with lines at distances precisely equal to those on the window, and similarly situated. These methods are the only methods of observation I have found it useful to employ, but I have now increased the number and variety of the observations sufficiently to enable me to adduce the conclusions hereinafter following, as representing the phænomena as far as their nature will admit of accurate observation." "If the floating spherules before mentioned be arranged in repose in one vertical plane at right angles to the direction of transmission of a wave, and *carefully* observed during transmission, it will be noticed that *the particles remain in the same plane during the transmission, and repose in the same plane after transmission.* It is further found, as might be anticipated from the foregoing observations, that *a thin solid plane transverse to the direction of transmission, and so poised as to float in that position does not sensibly interfere with the motion of translation or of transmission.*"

From this statement it would appear that we may safely assume, as an experimental fact, the second principle which I have proposed to assume as the basis of calculations. The observations required to be made in establishing it are such as admitted of very accurate verification; and seem also to have been made with care, and therefore the principle must be either accurately true or very nearly so. By reference to the *Report* itself the reader will find that this property of the solitary wave is not shared by any of the other three species of waves, and is therefore very proper to serve as a distinctive assumption to sift this species from the general equations of fluid motion. The investigations which follow will therefore contain the Mathematical Theory of Waves of the First Species, *i.e.* of the *Positive and Negative Solitary Waves.*

PROBLEM.

A QUANTITY of incompressible fluid is in a state of repose in a straight horizontal canal, the sides of which are vertical and parallel, and the bottom horizontal. A single wave is generated by pushing in one end of the canal in a proper manner: to determine the subsequent motion of the fluid, on the two hypotheses before mentioned, viz.

1st. That the velocity of transmission of the wave is uniform.

And 2nd. That the horizontal velocity of every particle, in a transversal section of the canal, is the same.

Let a horizontal line drawn along the bottom of the canal, parallel to the sides, be taken for the axis of x; let the axis of y be vertical.

h = equilibrium depth of the fluid;

k = the depth from the top of a wave to the bottom of the fluid;

c = the velocity of transmission of the wave.

As the motion of each particle is manifestly in a vertical plane, it will not be necessary to take account of the breadth of the canal, nor of the third co-ordinate of any particle; let therefore xy be the co-ordinates and uv the velocities of any particle at the time t; and suppose p the pressure of the fluid at the same point; the density of the fluid being taken as unity.

Then by our second hypothesis u is a function of x and not of y; consequently the equations of motion are in this case,

$$d_x p = -d_t u - u d_x u \quad \text{.............................(1)},$$

$$d_y p = -g - d_t v - u d_x v - v d_y v \quad \text{..............(2)};$$

and the equation of continuity is,

$$0 = d_x u + d_y v \quad \text{...................................(3)},$$

and our first hypothesis gives,

$$0 = d_t u + c d_x u \quad \text{.............................(4)}.$$

From these four equations we are to obtain our results.

Integrating (3) with regard to y, remembering that u and therefore also $d_x u$, is independent of y, we find

$$v = -y d_x u \text{..........................}(5),$$

no arbitrary function of x being added to this integral, because manifestly $v = 0$ when $y = 0$, whatever be the value of x; and no function of t is added because from (4) t enters with x only.

By means of this result eliminating v from (2) it becomes

$$d_y p = -g + \{d_t d_x u + u d_x^2 u - (d_x u)^2\} y \text{..................}(6).$$

Now $d_y . d_x p = d_x . d_y p$; and as appears from (1) $d_x p$ being independent of y, $d_y . d_x p = 0$, consequently $d_y p$ must be independent of x; from which it follows that the coefficient of y in (6) though a function of u is not a function of x, and therefore not of t by (4); and of course it is not a function of y, consequently it is constant both with respect to x, y, and t;

$$\therefore \text{ constant} = d_t d_x u + u d_x^2 u - (d_x u)^2 \text{..................}(7).$$

Before proceeding farther it is necessary to ascertain whether this constant have a positive or negative sign. We may ascertain this as follows.

Let us use the letter δ as the symbol of differentiation, taking x and y to belong to the same particle through the time δt; then it is well known that instead of the equation (2) we may use the following which is exactly equivalent to it, viz.

$$d_y p = -g - \delta_t^2 y,$$

which being compared with (6) gives,

$$\delta_t^2 y = -\{d_t d_x u + u d_x^2 u - (d_x u)^2\} y,$$
$$= -(\text{constant}) y.$$

Hence the force which urges the vertical motion of any particle varies as the distance of the particle from the bottom of the canal, and has always the same sign. Consequently when the original displacement of the fluid is such that any particle attains thereby a higher position than it had when in equilibrium, the above force must act so as to bring it down to its original level; *i.e.* the force must then be negative. Hence for what Mr. Russell calls the positive wave the above constant is positive. In a similar way it appears that for the negative wave the constant has a negative sign. It is therefore now necessary to separate our investigation into two branches, treating separately of these two varieties of the solitary wave.

OF THE POSITIVE SOLITARY WAVE.

IN this case, representing the constant by n^2, we have for discussion the equations

$$n^2 = d_t d_x u + u d_x^2 u - (d_x u)^2; \text{ ....} (8),$$

$$\delta_t^2 y = -n^2 y \text{..}(9)$$

which belong only to the variety of wave we are now considering. The latter will furnish us with the law of the vertical motion of each particle; and it shews that it is expressible in the form of a sine or cosine of an angle the variable part of whose argument is nt.

$$\therefore\ y = A \cos (nt - a) \dots\dots (10),$$

$$\text{and } v = \delta_t y = -nA \sin (nt - a) \dots\dots (11).$$

$$\text{Also } d_x u = -\frac{v}{y} = n \tan (nt - a) \dots\dots (12).$$

If we knew the greatest and least values of y for any particle we should be able to deduce results from these equations. Now for a particle in the surface, k and h are the greatest and least values of y. If we call t_k, t_h the values of t when the particle has these values for its y; then $v = 0$, when $t = t_k$ and $y = k$;

$$\therefore\ nt_k - a = 0 \text{ from (11)} \dots\dots (13),$$

$$\text{and } \therefore\ k = A \text{ from (10)};$$

$$\therefore\ h = k \cos (nt_h - a) \text{ from (10)}$$

$$= k \cos (nt_h - nt_k) \text{ from (13)};$$

$$\therefore\ t_k \sim t_h = \frac{1}{n} \cos^{-1} \frac{h}{k} \dots\dots (14).$$

Since v is positive or negative according as a particle is in its ascending or descending phase, it appears from (11) that nt is less than a as long as the vertex of a wave is behind a particle; and equal to a when the vertex is passing it; and greater than a when the vertex has passed it. Hence the functions on the right-hand side of the equations (10) (11) (12) are to be treated discontinuously, *i.e.* their variation is to be confined within certain limits; between these limits however their variation is continuous. Since, from the nature of the case y cannot be zero for a particle not originally at the bottom of the canal, it appears from (10) that $nt \sim a$ must always be less than $\frac{\pi}{2}$. Equation (11) shews that the vertical velocity does not begin from zero; but that it suddenly has a finite value, which gradually decreases till it is all lost; at which moment the particle begins to descend, gradually regaining the lost velocity, which being accomplished it is as suddenly lost, as it was suddenly generated. All this agrees exactly with the recorded observations of Mr. Russell (see *Report*, p. 342). Equation (14) gives half the time during which the vertical motion of any particle lasts. Consequently the time a wave takes to pass a particle is $\frac{2}{n} \cos^{-1} \frac{h}{k} \dots\dots (15)$. The quantity n is unknown at the present stage of our investigation.

We must now proceed to integrate the equation (8). For this purpose we must remember that (4) gives us

$$u = \phi (ct - x),$$

which being written in (8), using ϕ for $\phi (ct - x)$ for brevity, we have

$$n^2 = -c d_x^2 \phi + \phi d_x^2 \phi - (d_x \phi)^2,$$

$$\text{and } \therefore\ \frac{d_x (c - \phi)}{c - \phi} = \frac{d_x \phi\, d_x^2 \phi}{n^2 + (d_x \phi)^2}:$$

from which by integration we find

$$c - \phi = C \sqrt{n^2 + (d_x \phi)^2};$$

C being an arbitrary constant, not containing t because t enters ϕ with x only in the form $ct - x$.

The last equation being again integrated gives

$$c - \phi + \sqrt{(c - \phi)^2 - C^2n^2} = De^{\frac{x}{c}}.$$

D being another arbitrary constant, a function of t such as makes the right-hand member a function of $ct - x$;

$$\therefore\ 2(c - \phi) = Cn\left(\frac{D}{Cn}e^{\frac{x}{c}} + \frac{Cn}{D}e^{-\frac{x}{c}}\right).$$

But since ϕ is a function of $ct - x$, this equation by introducing t, and properly assuming the origin of t, may be written

$$2(c - u) = Cn.\left(e^{\frac{ct-x}{c}} + e^{-\frac{ct-x}{c}}\right) \dots\dots (16);$$

$$\text{and } \therefore\ 2d_x u = n\left(e^{\frac{ct-x}{c}} - e^{-\frac{ct-x}{c}}\right) \dots\dots (17).$$

The last equation enables us to connect x and a; for comparing it with (12) we have

$$2\tan(nt - a) = e^{\frac{ct-x}{c}} - e^{-\frac{ct-x}{c}} \dots\dots (18).$$

For a given particle a is constant, and consequently for that particle x so varies with t as to preserve the truth of this equation.

Eliminating x between (16) and (18), we get

$$c - u = Cn\sec(nt - a).$$

Now for a particle in the surface $u = 0$ when $t = t_h$;

$$\therefore\ c = Cn\sec(nt_h - a)$$

$$= Cn\sec(nt_h - nt_k);$$

$$\therefore\ \frac{Cn}{c} = \cos(nt_h - nt_k) = \frac{h}{k} \text{ from (14)};$$

$$\therefore\ Cn = \frac{ch}{k}:$$

$$\text{consequently } c - u = \frac{ch}{k}\sec(nt - a) \dots\dots (19),$$

which gives the law of the horizontal velocity, as (11) gives the law of the vertical velocity of a particle: and it is worthy of remark that neither of these is represented by a sine or a cosine. An assumption therefore that they might be so represented would be improper: and from this assumption we may date in some degree the erroneousness of the results which have been obtained by some writers who have adopted methods of analytical approximation. We have seen also that the argument $nt - a$ does not vary from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ as some have supposed. Equation (19) shews that the horizontal unlike the vertical motion of a particle is wholly in one direction, and is a maximum when the particle has reached its greatest vertical displacement; after which it decreases to zero.

Let now x_h, x_k be the values of x for a given particle at the times t_h, t_k. Then (18) gives

$$2\tan(nt_h - a) = e^{\frac{ct_h - x_h}{C}} - e^{-\frac{ct_h - x_h}{C}},$$

$$\text{and}\quad 0 = 2\tan(nt_k - a) = e^{\frac{ct_k - x_k}{C}} - e^{-\frac{ct_k - x_k}{C}}.$$

The last line shews that $ct_k = x_k$; and the preceding line gives, remembering that $a = nt_k$,

$$e^{\frac{ct_h - x_h}{C}} = \tan\left(\frac{\pi}{4} - \frac{n}{2}.\overline{t_k - t_h}\right);$$

$$\text{and}\quad \therefore\ ct_h - x_h = C\log_e\tan\left(\frac{\pi}{4} - \tfrac{1}{2}\cos^{-1}\frac{h}{k}\right).$$

$$\text{But}\quad ct_k - x_k = 0;$$

$$\therefore\ x_k - x_h - c(t_k - t_h) = C\log_e\tan\left(\frac{\pi}{4} - \tfrac{1}{2}\cos^{-1}\frac{h}{k}\right).$$

Now $2(x_k - x_h)$ is the distance through which a particle is horizontally transferred by the transit of a wave; as this is an observable element we will denote it by β;

$$\therefore\ \beta = 2(x_k - x_h).$$

Also the wave has travelled over the space $c(t_k - t_h)$ in the time $t_k - t_h$.

Now if λ be the length of a wave, the wave in the time $2(t_k - t_h)$ has travelled over the space $\lambda + \beta$,

$$\therefore\ \lambda + \beta = 2c(t_k - t_h),$$

$$\text{and}\quad \therefore\ \lambda = 2C\log_e\tan\left(\frac{\pi}{4} + \tfrac{1}{2}\cos^{-1}\frac{h}{k}\right).$$

$$\text{Consequently}\quad n = \frac{ch}{Ck} = \frac{2ch}{\lambda k}\log_e\left(\frac{\pi}{4} + \tfrac{1}{2}\cos^{-1}\frac{h}{k}\right)\ldots\ldots\ldots\ldots(20).$$

$$\text{Also}\quad \lambda + \beta = 2c(t_k - t_h) = \frac{2c}{n}\cos^{-1}\frac{h}{k};$$

$$\therefore\ \frac{\beta}{\lambda} + 1 = \frac{\dfrac{k}{h}\cos^{-1}\dfrac{h}{k}}{\log_e\tan\left(\dfrac{\pi}{4} + \tfrac{1}{2}\cos^{-1}\dfrac{h}{k}\right)}\ \ldots\ldots\ldots\ (21).$$

We may consider this equation as giving the value of the length of a wave; and then (20) gives the value of n in terms of c.

If we expand the terms of equation (21) we find,

$$\frac{\beta}{\lambda} = \frac{k-h}{h} - \frac{k}{6h}\left(\cos^{-1}\frac{h}{k}\right)^2 - \frac{k}{72h}\left(\cos^{-1}\frac{h}{k}\right)^4 + \&c.\ \ldots\ldots\ldots\ldots(22).$$

which shews that as k diminishes, β diminishes compared with λ.

We may now proceed to determine the velocity of transmission; and the equation of a wave surface.

The equations for the pressure are,

$$d_x p = -d_t u - u d_x u = (c-u)\, d_x u,$$

$$\text{and} \quad d_y p = -g + n^2 y;$$

$$\therefore\ p = -\tfrac{1}{2}(c-u)^2 - gy + \tfrac{1}{2}n^2y^2 + \text{constant}.$$

Now for a particle in the surface of the fluid p is constant; and if z be the value of y for such a particle, then

$$\text{constant} = (c-u)^2 + 2gz - n^2z^2 \ldots\ldots\ldots\ldots (23).$$

But the value of $c - u$ is known in terms of x from (16), and consequently,

$$\text{constant} = \frac{c^2h^2}{4k^2}\left(e^{\frac{ct-x}{C}} + e^{-\frac{ct-x}{C}}\right)^2 + 2gz - n^2z^2 \ldots\ldots\ldots\ldots (24)$$

is the equation which gives the form of a wave. t is here to be considered constant.

Again, when $z = h$, $u = 0$,

$$\therefore\ \text{constant} = c^2 + 2gh - n^2h^2 \text{ from (23)}.$$

Also when $z = k$, $c - u = \dfrac{ch}{k}$ from 19, and consequently,

$$\text{constant} = \frac{c^2h^2}{k^2} + 2gk - n^2k^2 \text{ from (23)};$$

$$\therefore\ 0 = c^2\left(1 - \frac{h^2}{k^2}\right) + 2g(h-k) - n^2(h^2-k^2);$$

$$\therefore\ c^2\left(1 - \frac{h^2}{k^2}\right) = 2gk\left(1 - \frac{h}{k}\right) - n^2k^2\left(1 - \frac{h^2}{k^2}\right),$$

$$\text{and} \quad \therefore\ c^2 + n^2k^2 = \frac{2gk^2}{h+k}.$$

And if in this equation we write the value of n from (20), we obtain the following final equation for the velocity of transmission,

$$c^2 = \frac{\left(\dfrac{2gk^2}{h+k}\right)}{1 + \dfrac{4h^2}{\lambda^2}\left\{\log_e \tan\left(\dfrac{\pi}{4} + \dfrac{1}{2}\cos^{-1}\dfrac{h}{k}\right)\right\}^2} \ldots\ldots\ldots\ldots (25).$$

It is to be remarked, that if h be very nearly equal to k, the denominator of the fraction on the right-hand side of this equation becomes equal to 1; and the numerator equal to gk, so that $c = \sqrt{gk}$ in that case; which is the empirical formula used by Mr. Russell. If h be much less than k, then $\dfrac{2gk^2}{h+k}\left(= gk \cdot \dfrac{k+k}{h+k}\right)$ is greater than gk; but in that case the denominator is greater than 1, and consequently there is a tendency to compensation which causes the value of c to lean sensibly towards the value $\sqrt{gk}$; which accounts for the near agreement of Mr. Russell's formula with experiment; and shews that he was mistaken in imagining the velocity of transmission to be entirely independent of the length of a wave.

Equation (21) shews that waves which give the ratio between h and k the same, have their lengths exactly proportional to the spaces through which they respectively transfer a particle by transit past it.

Equation (25) shews that in waves which have the same values of h and k, those will be transmitted with the greatest velocity which are the longest; and those with the least velocity which are the shortest.

We may conclude this portion of our investigations with the determination of the exact path of each particle. The materials for this purpose are supplied by equations (10) and (19). In both of them a is constant for our present purpose. The former gives,

$$y = A \cos(nt - a) \ldots\ldots\ldots\ldots(26),$$

in which A is the maximum value of y for that particular particle.

Equation (19) gives

$$c - \delta_t x = \frac{ch}{k} \sec(nt - a);$$

$$\therefore\ x = ct - \frac{ch}{k}\int_t \sec(nt - a)$$

$$= \frac{c}{n}(nt - a) - \frac{ch}{nk}\log_e\{\tan(nt - a) + \sec(nt - a)\} + \text{constant};$$

t being eliminated between this and (26), we shall have the equation required; which is manifestly not that of an ellipse as has been found by approximate methods; though as far as the eye can judge in an experiment, it may not be distinguishable therefrom.

It is very easy to shew from (24) that the surface of a wave meets the level surface of the quiescent fluid in a finite angle; and that under certain conditions it may have a point of contrary flexure. The actual wave surface is only a symmetrical portion of the whole curve represented by the equation (24). When a wave first reaches a particle $d_t x = 0$, and $d_t y =$ a finite quantity; consequently the initial motion of each particle is vertically upwards with a finite velocity. When it has described half its path $d_t x = c\left(1 - \dfrac{h}{k}\right)$, and $d_t y = 0$; consequently its motion is then horizontal. At the termination of its motion $d_t x = 0$, and $d_t y = -$ (the initial velocity), so that the final velocity is vertically downwards, and is finite; which indicates that the motion ceases as suddenly as it began. This seems to coincide either accurately, or very nearly so, with the account Mr. Russell has given (*Report*, p. 342) of the observations he made on the motions of individual particles in his experiments.

Before we proceed to compare the formula (20) with the results of experiment, it is necessary to advert again to a circumstance which has been already alluded to. The formula of (20) involves λ. The value of this quantity not having been recorded in Mr. Russell's tables, I have been under the necessity, as the best substitute for exact measures, of having recourse to the rule, which he has given in page 343 of his *Report*, for computing its approximate value. In the notation of this paper, that rule may with sufficient accuracy be represented by the equation $\lambda = 8h - 2k$; for it is not necessary in computing the value of c that λ should be known with extreme accuracy, as the term in (25) into which it enters is very small, and has but little effect upon the value of c. With these premises we give the following table, exhibiting a comparison of theory and experiment.

k	h	Velocity by Theory.	Velocity by Experiment.	Proportionate Error.	$\frac{\lambda}{\beta}$.
Inches.	Inches.	Feet.	Feet.		
1.30	1.15	1.89	1.84	$+\frac{1}{38}$	11.6
3.23	3.08	2.96	2.99	$-\frac{1}{99}$	34.5
4.20	4.07	3.40	3.33	$+\frac{1}{49}$	50.0
5.61	5.04	3.95	4.05	$-\frac{1}{40}$	13.5
7.82	7.04	4.64	4.53	$+\frac{1}{42}$	13.9
42.00	39.00	10.71	10.61	$+\frac{1}{107}$	20.0
75.00	66.00	14.42	14.23	$+\frac{1}{76}$	11.2

A mere inspection of this table, the fifth column of which gives the proportion of the error of theory to the whole velocity, will enable the reader to judge whether the theory advanced in the preceding pages is borne out by experiment. I am not aware what degree of accuracy Mr. Russell is disposed to ascribe to his observations, but I imagine he will hardly maintain that the velocity of a wave could in any case be observed with greater accuracy than the fortieth part of the whole. I am therefore inclined to pronounce that the coincidence of theory and experiment is exact.

The last column of the above table, though not necessary for the comparison of theory with experiment, is added for the purpose of shewing that there was a considerable degree of variation in the circumstances which characterized the several waves that are here selected as tests of theory. It may also serve as a further test of theory, if ever the experiment should be repeated.

It is worthy of remark that the ratio $\frac{\lambda}{\beta}$ depends entirely on the value of the ratio $\frac{h}{k}$, and not at all on the absolute value of either h or k.

It also appears that there is no means of determining the absolute values of λ and β, from those of h and k: consequently we must consider either λ or β a necessary element in the experimental determination of a wave of the kind we have been considering.

If h k and β are observed, then all the circumstances of the wave can be calculated; *i.e.*, the path, velocity, and position at a given moment, of each particle; and the place, form and velocity of the wave.

We must now advert to a circumstance of considerable importance. The equation which we have found for the pressure at any point within the moving portion of fluid is

$$p = -\tfrac{1}{2}(c-u)^2 - gy + \tfrac{1}{2}n^2y^2 + \text{constant}.$$

Now for those particles of the wave which are immediately in contact with the quiescent portion of the fluid, $u = 0$; and consequently $p = \text{constant} - gy + \frac{1}{2}n^2y^2$; which varies partly as the depth and partly as the square of the depth of any particle below the quiescent surface.

But the pressure in the quiescent part varies with the depth only; and depends not at all on the square of the depth; consequently there is a discontinuity of the law of pressure in passing from the wave to the quiescent fluid. This is of course an impossibility; and therefore our equations, though they may represent the properties of the wave with as much accuracy as the experimental observations, cannot be regarded as the exact representatives of a possible wave motion. But as they are rigidly deduced from the two hypotheses which Mr. Russell considered to be experimentally justified, it follows as a necessary and indisputable consequence that it is impossible for the particles of a permanent wave to move in the manner here assumed, viz., so that those which are in a vertical plane at right angles to the axis of the canal should always continue in a vertical plane during the transit of the whole wave. This hypothesis, as we have seen, leads us to an impossible result; and it is of importance to notice that this impossibility could not have been affirmed to be a necessary consequence of our hypotheses had methods of approximation been followed in our investigations, because it obviously depends on quantities which are small.

It appears then that the pressure at the junction of the moving fluid with the quiescent fluid cannot practically be such as our two hypotheses require it should be, yet as the hypothesis respecting the continuance of particles in the same vertical plane is certainly known to be very nearly true, as nearly true indeed as observation has been able to discriminate, we may expect that it is the other hypothesis which deviates more sensibly from experiment. To the want of permanency of the wave therefore we must look for the experimental confirmation of the impossibility we have just discovered. We will therefore now turn to Mr. Russell's experiments for evidence upon this point.

At page 327 of *The Report on Waves*, we find what the author has designated the *History of a Solitary Wave of the First Order, from observation.* A wave such as we have been investigating was generated in a canal such as we have supposed. The depth of the level fluid was 5.1 inches; and $k - h$ or the altitude of the crest of the wave above the general level was at first 1.34 inches. An inspection of the table shews that the crest of the wave gradually fell, with so rapid a degree of degradation, that in five minutes it was reduced to ·08 inches, the wave having in that time described 1160 feet. The velocity of the wave in the same time fell from 4.21 feet per second to 3.61 feet per second; the difference being ·6 or one-seventh part of the whole original velocity. It is evident from this statement that the degradation of the wave was a rapid process, and that the consequent effect upon the velocity was considerable.

These effects, which are much greater than could have been caused by imperfect fluidity or friction against the sides and bottom of the canal, I consider are fully accounted for by the circumstance above-mentioned, viz. the impossibility there is that the pressure should be continuous and the wave at the same time permanent if the motions of the particles are such as we supposed them to be, and which experiment shews they very nearly are. We have certainly proved the truth of these two alternatives;—if particles continue in a vertical plane while a wave passes them, then the wave cannot be permanent;—and, if the wave be permanent then the motions of particles once in a vertical plane cannot preserve them in a vertical plane while the wave passes them.

In proportion as one of our two hypotheses is more nearly true the other is farther from being accurately true. Degradation of the wave is therefore the natural consequence of the law which we have assumed for the motions of the fluid particles; and if that law be an experimental truth, as we believe it is to a close degree of approximation, then the gradual destruction of the wave is a necessary consequence, resulting not from friction alone, nor from imperfect fluidity, but chiefly from the manner in which motion is initially communicated to the fluid particles.

Strictly speaking, our investigations have been conducted on two hypotheses which are incompatible with each other; but experiment shews that, though they may not be accurately true, they are approximately correct and compatible: and we claim for the results of our investigation the same degree of accuracy as belongs to the hypotheses, because we have no where infringed those hypotheses by analytical approximations. It is easy to shew that we cannot regard our second hypothesis as being strictly correct. For if it were a possible hypothesis, then as the first cannot be at the same time true, the quantity denoted by n^2 in equation (9) must be regarded as a slowly varying function of t. The equation for p then assumes the form $p = F(x,t) - gy + \frac{1}{2}n^2y^2$; which involves the same impossibility as before, because at any given moment, at the junction of the wave with the quiescent fluid, the pressure depends on y^2 as well as on y, which cannot be the case. Hence our second hypothesis is certainly not mathematically correct. u must therefore depend on y as well as on x.

We come now to the consideration

OF THE NEGATIVE SOLITARY WAVE.

In this case, we are to represent the constant of equation (7) by $-n^2$; the equations therefore which are peculiar to the wave we are now investigating are,

$$-n^2 = d_t d_x u + u d_x^2 u - (d_x u)^2 \ldots\ldots\ldots\ldots\ldots (8'),$$

$$\delta_t^2 y = n^2 y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9').$$

From the last we obtain

$$y = A(e^{nt-a} \pm e^{-nt+a}),$$

$$\text{and} \quad \therefore \delta_t y = An(e^{nt-a} \mp e^{-nt+a}).$$

Now by the nature of the case $\delta_t y = 0$ when the particle has gained its lowest position; but $\delta_t y$ can never become $= 0$, unless we use the upper sign; the upper sign must therefore be used; and consequently we obtain

$$y = A(e^{nt-a} + e^{-nt+a}) \ldots\ldots\ldots\ldots\ldots\ldots (10'),$$

$$v = \delta_t y = nA(e^{nt-a} - e^{-nt+a}) \ldots\ldots\ldots\ldots\ldots\ldots (11').$$

$$\text{Also} \quad d_x u = -\frac{v}{y} = -n \cdot \frac{e^{nt-a} - e^{-nt+a}}{e^{nt-a} + e^{-nt+a}} \ldots\ldots\ldots\ldots\ldots\ldots (12').$$

The form of (9′) shews that the force which regulates the vertical motion of each particle acts upwards, and consequently if the particle oscillate (which it must do if it be part of a wave) its motion at first must be downwards; it then comes to a minimum altitude above the bottom of the canal and then rises again to its original level. Let h be as before, and k the altitude of the lowest point of a wave above the bottom of the canal; then proceeding as in the corresponding part of the investigation for the positive wave, we obtain

$$nt_k - a = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13'),$$

$$k = 2A,$$

$$h = \frac{k}{2}\left(e^{nt_h - nt_k} + e^{nt_k - nt_h}\right) \ldots\ldots\ldots (14');$$

$$\text{and} \quad \therefore t_k - t_h = \frac{1}{n}\log_e \tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{k}{h}\right).$$

Since v is negative in the fore part of the wave, and positive in the hinder part, nt is less than a as long the particle is situated in the fore part, greater than a when it is in the hinder part, and equal to a when the vertex of the wave passes it.

We must now integrate equation (8′). Proceeding on the same plan as before, we obtain

$$n^2 - (d_x\phi)^2 = (c - \phi)d_x^{\,2}\phi\,;$$

$$\therefore\ c - \phi = C\sqrt{n^2 - (d_x\phi)^2}.$$

And integrating this equation, and assuming a convenient epoch for the commencement of t, we find

$$c - u = Cn\cos\frac{x - ct}{C} \quad \ldots\ldots\ldots\ldots\ldots\ldots(16')\,;$$

$$\therefore\ d_x u = n\sin\frac{x - ct}{C} \quad \ldots\ldots\ldots\ldots\ldots\ldots(17')\,;$$

$$\therefore\ \frac{e^{-nt+a} - e^{nt-a}}{e^{-nt+a} + e^{nt-a}} = \sin\frac{x - ct}{C}\,;$$

$$\text{and}\quad \therefore\ e^{nt-a} + e^{-nt+a} = 2\sec\frac{x - ct}{C} \quad \ldots\ldots(18').$$

Let x_h, x_k be the values of x for a given particle at the times t_h, t_k. Then (18′) gives $x_k = ct_k$, and

$$2\sec\frac{x_h - ct_h}{C} = e^{nt_k - nt_h} + e^{nt_h - nt_k}$$

$$= \frac{2h}{k} \text{ from } (14')\,;$$

$$\therefore\ x_h - ct_h = C\cos^{-1}\frac{k}{h}.$$

Now while a particle is transferred by the wave through the space $2(x_k - x_h)$ $(= \beta)$ the wave itself has travelled its own length $(= \lambda)$ in addition to this space; and the time occupied is $2(t_k - t_h)$,

$$\therefore\ c(t_k - t_h) = \frac{\lambda}{2} + x_k - x_h\,;$$

$$\therefore\ x_h - ct_h = \frac{\lambda}{2}, \text{ because } ct_k = x_k\,;$$

$$\therefore\ \lambda = 2C\cos^{-1}\frac{k}{h}.$$

But when $x = x_h$ and $t = t_h$, $u = 0$, and consequently from (16′)

$$c = Cn\cos\frac{x_h - ct_h}{C} = Cn\,.\,\frac{k}{h}\,;$$

$$\therefore\ Cn = \frac{ch}{k}\,;$$

$$\therefore\ n\lambda = \frac{2ch}{k}\cos^{-1}\frac{k}{h} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20').$$

Again, $\lambda + \beta = 2c(t_k - t_h)$

$$= \frac{2c}{n}\log_e\tan\left(\frac{\pi}{4} + \tfrac{1}{2}\cos^{-1}\frac{k}{h}\right)\,;$$

$$\therefore \frac{\beta}{\lambda} + 1 = \frac{\log_e \tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{k}{h}\right)}{\frac{h}{k}\cos^{-1}\frac{k}{h}} \text{ .... } (21').$$

To find the velocity of transmission we must refer to the equations for the pressure. These are

$$d_x p = -d_t u - u d_x u = (c-u)\, d_x u,$$

$$\text{and} \quad d_y p = \quad -g - n^2 y\,;$$

$$\therefore\ p = -\tfrac{1}{2}(c-u)^2 - gy - \tfrac{1}{2}n^2y^2 + \text{constant}.$$

For a particle at the surface p is constant; and therefore

$$\text{constant} = (c-u)^2 + 2gz + n^2z^2 \text{ .... } (23')$$

is the equation of the form of a wave: or restoring the value of u in terms of x, the equation of the curve of the wave is

$$\frac{c^2h^2}{k^2}\cos^2\frac{x-ct}{C} + 2gz + n^2z^2 = \text{constant} \text{ .... } (24'),$$

in which t is supposed constant.

When $z = h$, $u = 0$;

$$\therefore \text{ constant} = c^2 + 2gh + n^2h^2 \text{ from } (23').$$

Also when $z = k$, $c - u = \frac{ch}{k}$;

$$\therefore \text{ constant} = \frac{c^2h^2}{k^2} + 2gk + n^2k^2 \text{ from } (23');$$

$$\therefore\ 0 = c^2\left(\frac{h^2}{k^2} - 1\right) - 2g(h-k) - n^2(h^2-k^2);$$

$$\therefore\ c^2 - n^2k^2 = \frac{2gk^2}{h+k};$$

$$\therefore\ c^2 = \frac{\left(\frac{2gk^2}{h+k}\right)}{1 - \frac{4h^2}{\lambda^2}\left(\cos^{-1}\frac{k}{h}\right)^2} \text{ ... } (25').$$

Before submitting this formula to calculation a few words may be said respecting the experiments of Mr. Russell on negative waves, which without questioning his experimental accuracy in the least degree, I cannot but consider far less satisfactory than those which were made on positive waves. For to generate a perfect solitary negative wave it was necessary that a peculiar law of pressure should have been observed. Unless this law were observed it was a necessary consequence that residuary and superfluous waves would be formed. Now the mode of genesis which Mr. Russell employed seems to have been so little suitable to the nature of the negative wave, that throughout its whole course it seems to have been continually casting off superfluous (or, as Mr. R. calls them, companion) waves. This must have produced a direct effect upon the wave itself, and an indirect effect in keeping the surface of the fluid in a state of agitation till the return of the wave after reflection at the end of the canal; by which the difficulty of accurately observing the exact time of transit would be greatly increased. Without the aid of some supposition of this kind, I cannot account for the manifest irregularities exhibited in Mr. Russell's table of the observed velocities of negative waves. (*Report*, page 349).

k	h	Velocity by Theory.	Velocity by Experiment.	Proportionate Error.
Inches.	Inches.	Feet.	Feet.	
·96	1.00	1.59	1.38	$\frac{2}{15}$
3.30	4.10	2.88	2.65	$\frac{2}{25}$
3.40	4.10	2.93	2.43	$\frac{2}{12}$
4.60	5.10	3.46	3.37	$\frac{2}{77}$

An inspection of the fifth column will at once acquaint the reader, that the errors here are far larger than in case of the positive wave. Instances however might have been selected from Mr. Russell's table which would have exhibited a much closer agreement between theory and observation. It is not necessary to repeat the remarks before made respecting the discontinuity of the pressure, and the consequent destruction of the wave.

I will now conclude this paper with a few general remarks.

Mr. Russell states, "that the positive and negative waves do not stand to each other in the relation of companion phænomena. They cannot be considered in any case as the positive and negative portions of the same phænomena." This is completely borne out by the foregoing theory; which shews that the two waves are distinguished in our investigations by a circumstance which prevents their coexistence; a certain constant being positive for one, and negative for the other, thereby making it impossible for p to be the same for both at the junction of the two parts, supposing them to be portions of one wave.

If it were possible for both waves to coexist at the same place, by meeting each other, or by one overtaking the other, then we should have for the vertical motion of a particle in the compound wave,

$$\delta_t^2 y = (n'^2 - n^2) y.$$

This is obtained by uniting equations (9) and (9′). Hence if n' be greater than n, the result would be a negative wave; but if n' be less than n, the result would be a postive wave; and if $n' = n$ the result would be that y would be constant, or there would be no wave at all.

This explains the following phænomena observed by Mr. Russell.

"If a positive and negative wave of equal volume meet in opposite directions, they neutralize each other and both cease to exist."

"If a positive wave overtake a negative wave of equal volume, they also neutralize each other and cease to exist."

"If either be larger, the remainder is propagated as a wave of the larger class." (*Report*, p. 351)

S. EARNSHAW.

XXV. *On the Geometrical Representation of the Roots of Algebraic Equations. By the Rev.* H. GOODWIN, *late Fellow of Caius College, and Fellow of the Cambridge Philosophical Society.*

[Read *April* 27, 1846.]

1. IT is usual to distinguish the roots of Algebraic Equations into three classes, viz, positive, negative, and imaginary or impossible. Roots of all kinds may however be included under one head, by considering them as composed of a modulus and a sign of affection, that sign of affection being some power of -1: thus if a be the modulus, positive roots will be expressed by $(-1)^0 . a$, negative by $(-1)^1 . a$, and imaginary by $(-1)^{\frac{\theta}{\pi}} . a$, and thus we may take $(-1)^{\frac{\theta}{\pi}} . a$ as the general expression for the root of an algebraic equation, and if reasoning could be conducted by means of such a symbol it would not be necessary to distinguish between real and imaginary roots, but all would come under the same view; and speaking quite generally we may say, that the root of an algebraic equation is a quantity with the negative affection developed in any degree between zero and actual minus.

This mode of considering roots of course coincides with the ordinary mode of representing the root of an equation by $a(\cos\theta + \sqrt{-1}\sin\theta)$, which symbol will be real and positive if $\theta = 0$, real and negative if $\theta = \pi$, and imaginary in other cases; but what has been said appears to point out more clearly the true connexion between the different species of roots, and to remove in some degree the artificial character which at first sight attaches to the representation of real roots under an imaginary form.

2. We may also bring the roots of an equation under one view geometrically; for considering the positive and negative roots only, we should represent them by setting off distances in opposite directions from a given point along a given line: now instead of a line passing through the point which we take as origin conceive a plane drawn through it, then all the roots will be represented by lines in this plane; for the root $(-1)^{\frac{\theta}{\pi}} . a$ or $a(\cos\theta + \sqrt{-1}\sin\theta)$ will correspond to a line of length a and which is inclined at an angle θ to the line along which positive roots are measured; the conjugate root $a(\cos\theta - \sqrt{-1}\sin\theta)$ will be a line similarly situated on the opposite side of the positive line.

This is no new remark, but it has not, so far as I am aware, been followed into any of its consequences; reflection upon it has led me to consider whether it might not be developed into a theory which should throw some light on the nature of Algebraic Equations, that is, whether it would not be possible so to represent geometrically the changes of value of a function of x, as to throw light upon the existence of the roots of the equation $f(x) = 0$.

With this view I have composed the following Memoir, and though I am not aware of any practical step in the Theory of Equations which can result from my investigations, yet I think they tend to throw considerable light upon existing knowledge, and to give us as it were the rationale of some familiar theorems.

3. If we wish to represent the changes of value of $f(x)$ taking into account only real values of x, the mode adopted would be to construct the curve defined by the equation

$$z = f(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

but if we wish to give a general representation of the changes of the function, taking into account both real and imaginary values of x, we must construct the locus of the equation

$$z = f(x + y\sqrt{-1}) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

where $x\ y$ and z are to be considered as co-ordinates of a point in space as is usual. Now if we restrict ourselves to values of z which are real, equation (2) will divide itself into two equations, which will be the equations of a curve of double curvature, and the points in which this curve meets the plane of xy will determine by their distances from the origin the roots of the equation $f(x) = 0$.

I will observe here that $f(x)$ will be considered throughout this paper (unless the contrary is stated) as the representation of the quantity

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots\ldots\ldots + p_n$$

where $p_1\, p_2 \ldots\ldots\ldots\, p_n$ are real and either positive or negative.

4. The two equations to which (2) corresponds may be expressed in several ways, which I shall here put down together.

By direct expansion, equating real and imaginary parts, and dividing the second equation by y, we have

$$\left.\begin{aligned} z &= f(x) - f''(x)\frac{y^2}{\lfloor 2} + f^{\text{IV}}(x)\frac{y^4}{\lfloor 4} - \&c. \\ 0 &= f'(x) - f'''(x)\frac{y^2}{\lfloor 3} + f^{\text{V}}(x)\frac{y^4}{\lfloor 5} - \&c. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (3)$$

If n be even and $= 2m$, these equations become

$$\left.\begin{aligned} z &= f(x) - f''(x)\frac{y^2}{\lfloor 2} + f^{\text{IV}}(x)\frac{y^4}{\lfloor 4} - \ldots\ldots\ldots + (-1)^m y^{2m} \\ 0 &= f'(x) - f'''(x)\frac{y^2}{\lfloor 3} + f^{\text{V}}(x)\frac{y^4}{\lfloor 5} - \ldots\ldots\ldots + (-1)^{m-1}(2mx + p_1)\, y^{2m-2} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4):$$

and if n be odd and $= 2m + 1$, they become

$$\left.\begin{aligned} z &= f(x) - f''(x)\frac{y^2}{\lfloor 2} + f^{\text{IV}}(x)\frac{y^4}{\lfloor 4} - \ldots\ldots\ldots + (-1)^m \left(\overline{2m+1}\,.\,x + p_1\right) y^{2m} \\ 0 &= f'(x) - f'''(x)\frac{y^2}{\lfloor 3} + f^{\text{V}}(x)\frac{y^4}{\lfloor 5} - \ldots\ldots\ldots + (-1)^m y^{2m} \end{aligned}\right\} \quad \ldots\ldots \quad (5).$$

The equations also admit of a very neat symbolical expression, thus*:

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* The method which I have given of representing the locus of the equation $z = f(x)$ taking into account values of x not lying in the real plane, is applicable *mutatis mutandis* to curves defined by an implicit relation between the co-ordinates. Thus, let the equation be

$$f(x\,.\,z) = 0 \quad \ldots\ldots\ldots\ldots \quad (A),$$

then putting for $x\ x + y\sqrt{-1}$, this becomes

$$0 = f(x + y\sqrt{-1}\,.\,z)$$

$$= \epsilon^{y\sqrt{-1}\frac{d}{dx}} f(x\,.\,z)$$

$$= \left\{\cos y\frac{d}{dx} + (-1)^{\frac{1}{2}} \sin y\frac{d}{dx}\right\} f(x\,.\,z),$$

which is equivalent to the two following.

$$z = f(x + y\sqrt{-1})$$
$$= e^{y\sqrt{-1}\frac{d}{dx}} f(x),$$
$$= \left(\cos y \frac{d}{dx} + \sqrt{-1} \sin y \frac{d}{dx}\right) f(x),$$

which equation divides itself into these two

$$\left. \begin{aligned} z &= \left(\cos y \frac{d}{dx}\right) f(x) \\ 0 &= \left(\sin y \frac{d}{dx}\right) f(x) \end{aligned} \right\} \dots\dots\dots (6).$$

If the symbols in these expressions be expanded it is evident that equations (3) and (6) will coincide.

There is another mode of expressing the equations in question which will be found very useful in the sequel, and that is by polar co-ordinates.

$$\text{Put} \quad x = \rho \cos\theta, \quad y = \rho \sin\theta, \quad \text{then}$$
$$z = f(\rho\cos\theta + \sqrt{-1}\,\rho\sin\theta),$$

which divides itself into two

$$\left. \begin{aligned} z &= f(0) + f'(0)\,\rho\cos\theta + \frac{f''(0)}{\lfloor 2}\rho^2\cos 2\theta + \dots + \frac{f^n(0)}{\lfloor n}\rho^n \cos n\theta \\ 0 &= \qquad f'(0)\ \sin\theta + \frac{f''(0)}{\lfloor 2}\rho\sin 2\theta + \dots + \frac{f^n(0)}{\lfloor n}\rho^{n-1}\sin n\theta \end{aligned} \right\} \dots\dots\dots (7);$$

$$\left. \begin{aligned} \left(\cos y \frac{d}{dx}\right) f(x.z) = 0 \\ \left(\sin y \frac{d}{dx}\right) f(x.z) = 0 \end{aligned} \right\} \dots (B),$$

the differentiation indicated being partial with respect to x. Of course we might have treated z in equation (A) in the same manner as x, and this would have given the following

$$\left. \begin{aligned} \left(\cos y \frac{d}{dz}\right) f(x.z) = 0 \\ \left(\sin y \frac{d}{dz}\right) f(x.z) = 0 \end{aligned} \right\} \dots (C).$$

The equations (B) and (C) may be considered as the complete representation of the locus of (A).

For example, suppose

$$f(x.z) = \frac{x^2}{a^2} + \frac{z^2}{b^2} - 1,$$

$$\text{then} \quad \frac{df(x.z)}{dx} = \frac{2x}{a^2} \quad \frac{df(xz)}{dz} = \frac{2z}{b^2},$$

$$\frac{d^2 f(x.z)}{dx^2} = \frac{2}{a^2} \quad \frac{d^2 f(x.z)}{dz^2} = \frac{2}{b^2},$$

and equations (B) become

$$\left. \begin{aligned} \frac{x^2}{a^2} + \frac{z^2}{b^2} - \frac{y^2}{a^2} &= 1 \\ xy &= 0 \end{aligned} \right\} \dots\dots (B')$$

and equations (C) become

$$\left. \begin{aligned} \frac{x^2}{a^2} + \frac{z^2}{b^2} - \frac{y^2}{b^2} &= 1 \\ yz &= 0 \end{aligned} \right\} \dots\dots (C'),$$

and it will be seen that the systems (B') (C') are equivalent to these three,

$$\left. \begin{aligned} \frac{x^2}{a^2} + \frac{z^2}{b^2} &= 1 \\ y &= 0 \end{aligned} \right\}, \quad \left. \begin{aligned} \frac{z^2}{b^2} - \frac{y^2}{a^2} &= 1 \\ x &= 0 \end{aligned} \right\}, \quad \left. \begin{aligned} \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \\ z &= 0 \end{aligned} \right\}.$$

Or the locus of the ordinary equation of the ellipse, thus considered, comprehends an ellipse and two hyperbolas, the two hyperbolas setting off in planes perpendicular to that of the ellipse from the extremities of its axes.

I would refer here to two papers in the *Cambridge Mathematical Journal*, by Mr. Walton, of Trinity College, (Vol. II. p. 103 and p. 155) in the first of which the complete representation of the curve corresponding to a given equation between two variables is considered, and in the second the real nature of a maximum or minimum as being in fact a multiple point is noticed.

or we may write these,

$$\left.\begin{aligned} z &= p_n + p_{n-1}\rho\cos\theta + p_{n-2}\rho^2\cos 2\theta + \dots\dots + \rho^n\cos n\theta \\ 0 &= \quad p_{n-1}\sin\theta + p_{n-2}\rho\sin 2\theta + \dots\dots + \rho^{n-1}\sin n\theta \end{aligned}\right\} \dots\dots\dots\dots (8).$$

5. I now proceed to discuss these equations, and shall consider first the equation of the projection of the curve on the plane of xy.

This equation is in polar co-ordinates,

$$\rho^{n-1}\sin n\theta + p_1\rho^{n-2}\sin(n-1)\theta + \dots\dots + p_{n-1}\sin\theta = 0\dots\dots\dots\dots\dots(9).$$

To find the asymptotes, I observe that ρ will be infinite when $\sin n\theta = 0$, except for $\theta = 0$ and $\theta = \pi$; hence there will be infinite values of ρ for $\theta = \frac{\pi}{n}, \frac{2\pi}{n}, \dots\dots \frac{(n-1)\pi}{n}$.

Again,

$$\sin n\theta + \frac{p_1}{\rho}\sin(n-1)\theta + \dots\dots + \frac{p_{n-1}}{\rho^{n-1}}\sin\theta = 0;$$

$$\therefore\ n\cos n\theta - \sin(n-1)\theta.\frac{p_1}{\rho^2}\frac{d\rho}{d\theta} = 0 \text{ when } \rho = \infty,$$

$$\text{or,}\quad \rho^2\frac{d\theta}{d\rho} = \frac{p_1}{n}\frac{\sin(n-1)\theta}{\cos n\theta};$$

And if we put $\theta = k\frac{\pi}{n}$, k having any value from 1 up to $n-1$,

$$\rho^2\frac{d\theta}{d\rho} = -\sin\frac{k\pi}{n}.\frac{p_1}{n}.$$

Hence there will be an asymptote corresponding to each infinite value of ρ, and these will lie on the left of the corresponding infinite radius vectors looking from the pole. If however we suppose the given equation deprived of its second term, that is, if $p_1 = 0$, then the polar subtangent vanishes and the asymptotes pass through the origin and coincide with the radius vectors; and since this condition may always be fulfilled, I shall generally suppose that $p_1 = 0$, and then it may be stated that the projection of the imaginary branches of the curve on the plane of xy has $n - 1$ asymptotes, which pass through the origin, are equidistant from each other, and make the same angle with each other as the first of them makes with the axis of x.

The symmetry of these infinite branches with respect to the origin when $p_1 = 0$ seems to me to point out a kind of geometrical explanation of the great simplicity introduced in the solution of equations by first depriving them of their second terms.

6. To determine where ρ is a minimum, we have by differentiating (9) and putting $\frac{d\rho}{d\theta} = 0$,

$$n\rho^{n-1}\cos n\theta + (n-1)p_1\rho^{n-2}\cos(n-1)\theta + \dots\dots = 0\dots\dots\dots\dots(10),$$

which equation together with (9) will give the required values of ρ and θ. Now if we make $\theta = 0$, which satisfies (9), (10) becomes

$$n\rho^{n-1} + (n-1)p_1\rho^{n-2} + \dots\dots = 0,$$

$$\text{or,}\quad f'(\rho) = 0,$$

which (if x be written for ρ) is the equation for determining the maxima and minima of the real branch of the curve; hence ρ is a minimum for the projection of such points. Besides these there may be other minimum values of ρ lying between the different pairs of asymptotes.

7. Corresponding to the asymptotes of the curve in the plane of xy there will be infinite branches in space, and it is easy to shew that these go off alternately to positive and negative infinity. For from equations (8) we have, when $\theta = \frac{k\pi}{n}$ and ρ is consequently very large,

$$z = \rho^n \cos k\pi = (-1)^k \rho^n ;$$

therefore for odd values of k the limiting form of the curve is given by

$$z = -\rho^n,$$

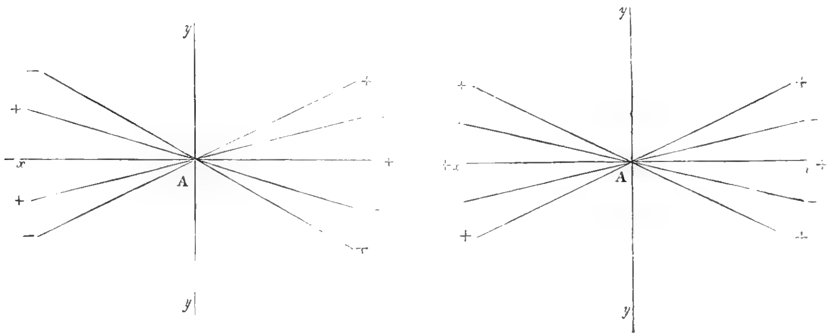
which represents a parabolic branch going off to negative infinity for positive values of ρ, and *vice versâ* if n is odd, and going off to negative infinity on both sides of the origin if n is even. And for even values of k the form is given by

$$z = \rho^n,$$

which represents a branch going off to positive infinity for positive values of ρ, and negative infinity for negative values of ρ if n be odd, and to positive infinity in both cases if n is even.

This proposition it is easily seen includes the real branch of the curve, and hence if we indicate by the mark + or − on an asymptote that the corresponding branch of the curve goes off to positive or negative infinity respectively, the arrangement of the infinite branches will be represented by the accompanying diagram.

n odd. n even.



8. I shall next prove the following theorem:

At points in the real branch of the curve for which the first p differential coefficients of $f(x)$ vanish, there are p imaginary branches going off on each side of the real plane or plane of xz, and these are curved alternately in opposite senses, the one nearest the real branch being curved in the opposite sense to that real branch.

Suppose the origin of co-ordinates such that the axis of z passes through the point in question, which may be done without in any way affecting the generality of the proof, then we shall have

$$f'(0) = 0, \quad f''(0) = 0 \ldots\ldots f^p(0) = 0,$$

and the equations (7) become

$$z = f(0) + \frac{f^{p+1}(0)}{\lfloor p+1} \rho^{p+1} \cos (p+1)\theta + \ldots\ldots$$

$$0 = \frac{f^{p+1}(0)}{\lfloor p+1} \rho^{p} \sin (p+1)\theta \quad + \ldots\ldots$$

The form of the curve very near the point in question will be given by taking only the terms of the series above set down, and therefore we shall have

$$\sin (p+1)\theta = 0,$$

$$(p+1)\theta = k\pi;$$

$$\therefore \theta = \frac{\pi}{p+1}, \quad \frac{2\pi}{p+1}, \quad \frac{3\pi}{p+1}, \quad \ldots\ldots \quad \frac{p\pi}{p+1};$$

omitting the values 0 and $p+1$ of k which correspond to the real branch. Hence there are p imaginary branches going off from this point; and to determine the nature of the curvature we have

$$z = f(0) + \frac{f^{p+1}(0)}{\lfloor p+1} \cos k\pi \,.\, \rho^{p+1}$$

$$= f(0) + (-1)^k \frac{f^{p+1}(0)}{\lfloor p+1} \rho^{p+1}.$$

The second term will be alternately positive and negative as k assumes successive value and when it is further observed that the equation of the real branch is given by putting $k =$ · the whole of the theorem will be seen to be true.

In the case of simple maxima and minima, for which $f'(x) = 0$ and $f''(x)$ does not vanisl this proposition admits of more simple and obvious proof; for we have from equations (3),

$$z = f(x) - f''(x) \frac{y^2}{\lfloor 2} + \ldots\ldots$$

$$0 = f'(x) - f'''(x) \frac{y^2}{\lfloor 3} + \ldots\ldots$$

and it is clear from these equations that when $y = 0$ $f'(x) = 0$, that is, where there is a maximum or minimum there is an imaginary branch; it is also evident that the imaginary branches can never cross the real plane except at points for which $f'(x) = 0$, that is, either at maximum or minimum points or points of inflexion: this last is an important consideration, because it shews that, in tracing the general form of a curve, after having traced the real branch and those imaginary branches which start from points at which $f'(x) = 0$ we may be quite sure that all the remainder of the curve lies in isolated infinite branches situated symmetrically with respect to the real plane.

If we suppose as before the axis of z to pass through the maximum or minimum point. we have,

$$z = f(0) - f''(0) \frac{y^2}{2},$$

which shews that the form of the imaginary branch is that of a parabola curved in the opposite sense to the real branch, or we may say that when there is a maximum in the real branch there is a minimum in the imaginary, and *vice versâ*.

It would not be difficult from this particular case in which only the first differential coefficient vanishes, to derive the other more general proposition in which the first and any number of subsequent differential coefficients vanish; at least we could conclude the existence of imaginary branches curved in opposite senses though perhaps not their directions. For we may consider a point for which $f'(x)\ f''(x) \ldots\ldots f^p(x)$ each $= 0$, as the case of p successive maxima and minima degenerating into one point, and since these maxima and minima must necessarily occur alternately there will be p imaginary branches curved alternately in opposite senses.

9. Let us now examine whether the ordinate z admits of any maximum or minimum values besides those which it has in the real branch of the curve.

The general equation of the curve is

$$z = f(\rho e^{\theta\sqrt{-1}}),$$

and the equation for finding the maxima and minima is

$$f'(\rho e^{\theta\sqrt{-1}}) = 0,$$

which is equivalent to these two

$$f'(0) + f''(0)\rho\cos\theta + f'''(0)\frac{\rho^2}{\underline{|2}}\cos 2\theta + \ldots\ldots + f^n(0)\frac{\rho^{n-1}}{\underline{|n-1}}\cos(n-1)\theta = 0,$$

$$f''(0)\sin\theta + f'''(0)\frac{\rho}{\underline{|2}}\sin 2\theta + \ldots\ldots + f^n(0)\frac{\rho^{n-2}}{\underline{|n-1}}\sin(n-1)\theta = 0;$$

and we have also the condition of z being real, which is,

$$f'(0)\sin\theta + f''(0)\frac{\rho}{\underline{|2}}\sin 2\theta + \ldots\ldots + f^n(0)\frac{\rho^{n-1}}{\underline{|n}}\sin n\theta = 0:$$

or these may be written

$$\left.\begin{array}{r} p_{n-1} + 2p_{n-2}\rho\cos\theta + \ldots\ldots + n\rho^{n-1}\cos(n-1)\theta = 0 \\ 2p_{n-2}\sin\theta + \ldots\ldots + n\rho^{n-2}\sin(n-1)\theta = 0 \\ p_{n-1}\sin\theta + p_{n-2}\rho\sin 2\theta + \ldots\ldots + \rho^{n-1}\sin n\theta = 0 \end{array}\right\} \ldots\ldots\ldots(11).$$

These three equations involving only two unknown quantities cannot be generally satisfied; I have not been able to shew directly that they never can be satisfied, though it seems possible that such may be the case; I can however give a complete solution of the question so far as the purpose of this memoir is concerned by proving that a maximum or minimum point is never unaccompanied by a branch curved in the opposite sense, in fact, by extending to all branches of the curve the proposition which has been proved above for the real branch.

10. The proof is as follows:

We have in general

$$z = f(x + y\sqrt{-1})$$
$$= P + Q\sqrt{-1}, \quad \text{suppose},$$

where $$P = \left(\cos y\frac{d}{dx}\right)f(x),$$

$$Q = \left(\sin y\frac{d}{dx}\right)f(x).$$

Now it will be easily seen, that if $\frac{dP}{dx}$ represent the partial differential coefficients of P with respect to x, then

$$\frac{dP}{dx} = \left(\cos y \frac{d}{dx}\right) f'(x),$$

in like manner,

$$\frac{dQ}{dy} = \left(\cos y \frac{d}{dx}\right) f'(x);$$

$$\therefore \frac{dP}{dx} = \frac{dQ}{dy} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12);$$

and similarly it may be shewn* that

$$\frac{dP}{dy} = -\frac{dQ}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

In order that δz may vanish when x and y vary, we must have

$$\frac{dP}{dx}\delta x + \frac{dP}{dy}\delta y = 0,$$

$$\frac{dQ}{dx}\delta x + \frac{dQ}{dy}\delta y = 0.$$

Multiplying these equations by $\frac{dP}{dy}$ and $\frac{dQ}{dy}$, and adding, we have, observing the relations (12) (13),

$$\left(\frac{dP}{dy}\right)^2 + \left(\frac{dQ}{dy}\right)^2 = 0;$$

$$\therefore \frac{dP}{dy} = 0 \quad \frac{dQ}{dy} = 0.$$

Hence also, $$\frac{dP}{dx} = 0 \quad \frac{dQ}{dx} = 0.$$

If the values of x and y which satisfy these equations also satisfy the equation $Q = 0$, this will indicate a *singular point* in the curve, and we must determine the nature of this point: to do this we have for the increment of z, supposing the terms of the first order to vanish.

$$2\delta^2 z = \frac{d^2P}{dx^2}\delta x^2 + \frac{d^2P}{dx\,dy} 2\delta x \delta y + \frac{d^2P}{dy^2}\delta y^2 \ldots\ldots\ldots\ldots\ldots\ldots(14);$$

(there is no term involving $\delta^2 y$ because its coefficient would be $\frac{dP}{dy}$ which in this case vanishes):

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* The roots of the equation $f(x) = 0$ may be considered as determined by the intersections of the curves $P = 0$ and $Q = 0$. These curves have the property of intersecting each other at right angles; for the equations of the tangents to the two curves at a common point (x, y) are

$$(x_1 - x)\frac{dP}{dx} + (y_1 - y)\frac{dP}{dy} = 0,$$

$$(x_1 - x)\frac{dQ}{dx} + (y_1 - y)\frac{dQ}{dy} = 0,$$

which in virtue of the relations $\frac{dP}{dx} = \frac{dQ}{dy}$ and $\frac{dP}{dy} = -\frac{dQ}{dx}$ represent two lines perpendicular to each other.

and we have also the relation,

$$0 = \frac{d^2 Q}{dx^2}\delta x^2 + \frac{d^2 Q}{dx\,dy} 2\delta x \delta y + \frac{d^2 Q}{dy^2}\delta y^2 \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

Now we have $P = \left(\cos y \dfrac{d}{dx}\right) f(x)$;

$$\therefore \frac{dP}{dx} = \left(\cos y \frac{d}{dx}\right) f'(x), \quad \frac{dP}{dy} = -\left(\sin y \frac{d}{dx}\right) f'(x),$$

$$\frac{d^2 P}{dx^2} = \left(\cos y \frac{d}{dx}\right) f''(x), \quad \frac{d^2 P}{dx\,dy} = -\left(\sin y \frac{d}{dx}\right) f''(x), \quad \frac{d^2 P}{dy^2} = -\left(\cos y \frac{d}{dx}\right) f''(x),$$

also $Q = \left(\sin y \dfrac{d}{dx}\right) f(x)$;

$$\therefore \frac{dQ}{dx} = \left(\sin y \frac{d}{dx}\right) f'(x), \quad \frac{dQ}{dy} = \left(\cos y \frac{d}{dx}\right) f'(x),$$

$$\frac{d^2 Q}{dx^2} = \left(\sin y \frac{d}{dx}\right) f''(x), \quad \frac{d^2 Q}{dx\,dy} = \left(\cos y \frac{d}{dx}\right) f''(x), \quad \frac{d^2 Q}{dy^2} = -\left(\sin y \frac{d}{dx}\right) f''(x).$$

Hence, if we call the values assumed by $\left(\sin y \dfrac{d}{dx}\right) f''(x)$ and $\left(\cos y \dfrac{d}{dx}\right) f''(x)$ at the point under consideration A and B respectively, (14) and (15) may be written thus:

$$2\,\delta^2 z = B\,\delta x^2 - 2A\,\delta x\,\delta y - B\,\delta y^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (16),$$

$$0 = A\,\delta x^2 + 2B\,\delta x\,\delta y - A\,\delta y^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (17).$$

Let $\delta x = \delta s \cos\phi$, $\delta y = \delta s \sin\phi$, then

$$2\frac{\delta^2 z}{\delta s^2} = B\cos 2\phi - A\sin 2\phi,$$

$$0 = A\cos 2\phi + B\sin 2\phi;$$

or
$$2\frac{\delta^2 z}{\delta s^2} = C\cos(2\phi + a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (18),$$

$$0 = \sin(2\phi + a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (19),$$

by putting $A = B \tan a$.

Equation (19) determines two values for $2\phi + a$, and one of these will make $\delta^2 z$ positive, the other negative; hence at the point in question there will be two branches curved in opposite senses, one will be a minimum, the other a maximum.

I have proved this proposition for simplicity's sake in the case of a double point, but the same mode of investigation may be applied to that in which the increments of z all vanish up to any given order; this I proceed to do.

It is not difficult to see that if the coefficients of the powers of δx and δy in the increments $\delta z, \delta^2 z \ldots\ldots \delta^{m-1} z$ all vanish, then the value of $\delta^m z$ may be written thus,

$$1.2 \ldots\ldots m\,\delta^m z = \left(\delta x \frac{d}{dx} + \delta y \frac{d}{dy}\right)^m P \ldots\ldots\ldots\ldots\ldots\ldots (20),$$

with the condition,

$$0 = \left(\delta x \frac{d}{dx} + \delta y \frac{d}{dy}\right)^m Q \text{ .... } (21).$$

And it will easily appear that

$$\frac{d^m P}{dx^m} = \left(\cos y \frac{d}{dx}\right) f^m(x), \quad \frac{d^m P}{dx^{m-1}dy} = -\left(\sin y \frac{d}{dx}\right) f^m(x), \quad \frac{d^m P}{dx^{m-2}dy^2} = -\left(\cos y \frac{d}{dx}\right) f^m(x) \ldots$$

$$\frac{d^m Q}{dx^m} = \left(\sin y \frac{d}{dx}\right) f^m(x), \quad \frac{d^m Q}{dx^{m-1}dy} = \left(\cos y \frac{d}{dx}\right) f^m(x), \quad \frac{d^m Q}{dx^{m-2}dy^2} = -\left(\sin y \frac{d}{dx}\right) f^m(x) \ldots$$

Therefore calling the values assumed by $\sin\left(y \frac{d}{dx}\right) f^m(x)$ and $\cos\left(y \frac{d}{dx}\right) f^m(x)$ at the point in question A and B respectively, equations (20) (21) may be written,

$$1.2 \ldots\ldots m\, \delta^m z = B\,\delta x^m - mA\,\delta x^{m-1}\,\delta y - \frac{m(m-1)}{1.2} B\,\delta x^{m-2}\,\delta y^2 + \&c.$$

$$0 = A\,\delta x^m + mB\,\delta x^{m-1}\,\delta y - \frac{m(m-1)}{1.2} A\,\delta x^{m-2}\,\delta y^2 - \&c.$$

Let $\delta x = \delta s \cos \phi$, $\delta y = \delta s \sin \phi$,

$$\therefore\ 1.2 \ldots\ldots m \frac{\delta^m z}{\delta s^m} = B\left(\cos^m \phi - \frac{m(m-1)}{1.2} \cos^{m-2}\phi \sin^2\phi + \ldots\ldots\right) - A\,(m \cos^{m-1}\phi \sin\phi - \ldots)$$

$$0 = A\left(\cos^m \phi - \frac{m(m-1)}{1.2} \cos^{m-2}\phi \sin^2\phi + \ldots\ldots\right) + B\,(m \cos^{m-1}\phi \sin\phi - \ldots)$$

or $1.2 \ldots\ldots m \dfrac{\delta^m z}{\delta s^m} = B \cos m\phi - A \sin m\phi,$

$$0 = A \cos m\phi + B \sin m\phi,$$

and lastly these expressions may be put under the form

$$1.2 \ldots\ldots m \frac{\delta^m z}{\delta s^m} = C \cos(m\phi + a) \text{ .... } (22),$$

$$0 = \sin(m\phi + a) \text{ .... } (23).$$

The last equation gives us

$$m\phi + a = k\pi,$$

where k may have any one of the values $0, 1, 2 \ldots (m-1)$; hence there will be m branches: also the sign of $\delta^m z$ depends upon that of $\cos k\pi$, or of $(-1)^k$, and will therefore be alternately positive and negative; hence the m branches will be curved alternately in opposite senses.

Hence, therefore, if values of x and y can be found which will make $\delta z = 0$ and $Q = 0$, there will not be a maximum or minimum point properly speaking, but a multiple point in which two or more branches of the curve meet, and these branches being, as has been proved, curved in opposite senses, there cannot be an absolute maximum or minimum, that is, a maximum for every branch or a minimum for every branch.

11. This proposition completes the theory of the roots of the equation $f(x) = 0$; for it has been shewn that the curve of double curvature corresponding to the equation $z = f(x)$ admits of no maxima or minima, and that it consists of n branches going off alternately to positive and negative infinity, hence the plane of xy or any plane parallel to it must necessarily cut the curve in n points, and the distances of these n points from the origin will be the n roots of the equation.

It may be observed that the preceding investigation applies to multiple points in the real plane by making $A = 0$.

12. A less general application of what has preceded presents itself in the case of an equation of an even degree having its last term positive: in this case it is well known that there is some difficulty in proving the existence of a root. But I observe that if $z = f(x)$, where $f(x)$ is of even dimensions, z has necessarily a minimum value, and from the minimum point an imaginary branch starts off on each side of the real plane, which will stretch out to negative infinity and therefore cut the plane of xy in two points which will correspond to imaginary roots. Hence we see as it were the rationale of such an equation having at least two roots, for $f(x)$ must admit of a minimum, and if this be negative the curve cuts the axis of x twice, if positive imaginary branches go off from the minimum and these take us down to the plane of xy.

13. The roots which are thus determined by the intersection with the plane of xy of imaginary branches starting from points of the real curve for which $f'(x) = 0$ are so related to the real roots, that it has seemed to me to be desirable to denote them by a distinct name; I therefore, for want of a better name, call such roots *connected* roots, and those which are determined by the intersection with the plane of xy of other infinite branches which, as I have shewn, never cross the real plane, I call *isolated* roots. Thus I should say of an equation of even dimensions, that it must have two roots either real or *connected*.

14. But more generally we may distribute the n roots of an equation into *real connected* and *isolated* roots. For suppose the real branch of the curve traced, and suppose that it has p points for which $f'(x) = 0$ and $f''(x)$ does not vanish, then it is easy to see from what has been said that there will be $p + 1$ roots either real or connected; from the p maxima and minima there go off $2p$ infinite branches which occupy $2p$ out of the $2n - 2$ asymptotes*, leaving $2n - 2p - 2$ asymptotes; between each pair of asymptotes there is an infinite branch which cutting the plane of xy gives a root, therefore there are $n - p - 1$ isolated roots; and thus we make up the whole number of roots n. I will just observe that $n - p - 1$ is obviously even, because if n is even p is necessarily odd, and *vice versâ*. The same proposition may be extended to the case in which other derived functions besides $f'(x)$ vanish at any point, by the reasoning used in Art. (8): for we may consider such a point to be the degeneration of a number of contiguous maxima and minima, for each of which the proposition is true. It may therefore be stated generally, that if there are p real values of x, whether all unequal or not, which make $f'(x)$ vanish, then the equation $f(x) = 0$ has $p + 1$ roots either real or connected.

15. It may be observed, that a pair of connected roots may be changed into a pair of real ones by altering the position of the plane of xy, or speaking algebraically by changing the value of the last term of the equation; and this fact points out the propriety of distinguishing between connected and isolated roots, which latter are necessarily imaginary wherever the plane of xy cuts the axis of z, since they are determined by the intersection of that plane with branches of the curve, which, as we have seen, never cross the real plane.

16. The number of real and connected roots evidently depends upon the number of real roots of the equation $f'(x) = 0$, and (as has been already in fact proved) if the number of real roots of this derived equation be p, then the number of real and connected roots of the original equation will be $p + 1$; consequently the number of isolated roots of the original equation is equal to the number of imaginary roots of the derived.

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* In Art. (5) I have spoken of $n - 1$ asymptotes, here of $2n - 2$; the difference consists merely in this, that in the former case I have considered the indefinite straight line through the origin as *one* asymptote, here for convenience I have considered the same line as *two* stretching out to infinity on opposite sides of the origin.

17. Hence also we see the truth of a theorem, of which I shall presently make use, namely, that an equation has at least as many imaginary roots as any one of its derivatives; for the equation $f(x) = 0$ has as many isolated roots as there are imaginary roots in $f'(x) = 0$, and therefore has at least as many imaginary roots; $f'(x) = 0$ has in like manner at least as many imaginary roots as $f''(x) = 0$, and so on: whence the truth of the proposition is clear.

18. If the plane of xy should happen to pass through a real maximum or minimum, which is as we have seen properly speaking a multiple point, there will be several equal roots. The condition of equal roots will be therefore that the plane of xy shall pass through a point for which one or more of the differential coefficients of $f(x)$ vanish, or which is the same thing, that $f(x) = 0$ and $f'(x) = 0$ shall have one or more roots in common; which as is well known is the test of equal roots. Or we may shew directly that at a point for which there are m equal roots there are m branches curved in opposite senses; for let $x = 0$ for simplicity's sake be the root which occurs m times, then

$$f(x) = x^m \{p_n + p_{n-1}x + \dots\dots + x^{n-m}\},$$

and the equations to the curve will be

$$z = p_n \rho^m \cos m\theta + \dots\dots,$$
$$0 = p_n \sin m\theta + \dots\dots;$$

therefore near the origin, $\sin m\theta = 0$,

$$\therefore\ m\theta = k\pi \text{ where } k \text{ may } = 0,\ 1\dots(m-1),$$

$$\text{and}\quad z = p_n \rho^m \cos k\pi$$
$$= (-1)^k p_n \rho^m;$$

therefore there will be m branches curved in alternately opposite senses.

19. It will be seen, that a pair of equal real roots in the equation $f'(0)$ implies a pair of imaginary roots in the equation $f(x) = 0$, since $f''(x)$ will also vanish for the same value of x as that which makes $f'(x) = 0$. And generally, if x be even, r equal roots of the equation $f'(x) = 0$ imply r imaginary roots in the equation $f(x) = 0$; if r be odd, there will be $r + 1$ or $r - 1$ imaginary roots according as $f(x)$ and $f^{r+1}(x)$, which is the first derived function which does not vanish, have the same or different signs.

This theorem requires no demonstration, as its truth will be seen at once on examination. By means of it I am able to prove the ordinary proposition relative to the number of imaginary roots belonging to an equation defective in any of its terms; the proof is as follows:

Suppose,

$$f(x) = p_n + p_{n-1}x + \dots\dots + Mx^\mu + Nx^{\mu+\nu+1} + \dots\dots + x^n,$$

where ν terms are wanting between the terms Mx^μ and $Nx^{\mu+\nu+1}$;

differentiating μ times, we have

$$f^\mu(x) = \mu(\mu-1)\dots\dots 2 \cdot 1\, M + (\mu+\nu+1)(\mu+\nu)\dots(\nu+2)Nx^{\nu+1} + \dots\dots$$
$$+ n(n-1)\dots(n-\mu+1)x^{n-\mu};$$

differentiating again,

$$f^{\mu+1}(x) = (\mu+\nu+1)\cdot(\mu+\nu)\dots(\nu+1)Nx^\nu + \dots\dots + n\cdot(n-1)\dots(n-\mu)x^{n-\mu-1}.$$

Hence the equation

$$f^{\mu+1}(x) = 0$$

has ν roots equal to 0, and therefore the equation $f^\mu(x) = 0$ has ν imaginary roots if ν be even, and if ν be odd, it has $\nu + 1$ or $\nu - 1$, according as $f^\mu(0)$ and $f^{\mu+\nu+1}(0)$ have the same or opposite signs, that is, according as M and N have the same or opposite signs. But, by a theorem cited and proved in Art. (17), $f(x) = 0$ has at least as many imaginary roots as any one of its derived equations; hence it will have at least ν imaginary roots if ν be even, and at least $\nu + 1$ or $\nu - 1$, according as M and N have the same or opposite signs, if ν be odd.

20. I will now illustrate what precedes by discussing some actual cases and tracing the corresponding curves.

Let the equation be a quadratic, that is, let

$$f(x) = x^2 - ax + b = 0 \quad \text{.........................} \quad (24),$$

$$\therefore f'(x) = 2x - a,$$

$$f''(x) = 2,$$

and the equations of the curve of double curvature are

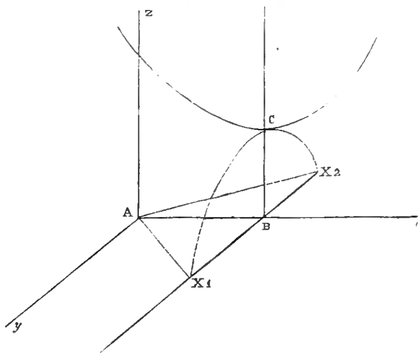
$$\left.\begin{array}{l} z = x^2 - ax + b - y^2 \\ 0 = 2x - a \end{array}\right\} \quad \text{.............................} \quad (25),$$

if we eliminate x by means of the second of these equations, we have

$$z = b - \frac{a^2}{4} - y^2,$$

$$x = \frac{a}{2}.$$

Hence the complete locus of the equation $z = f(x)$ will be in this case two parabolas in planes perpendicular to each other, with their vertices coincident and their curvatures in opposite senses: the height of the vertex above the plane of xy will be $b - \frac{a^2}{4}$, if this be positive the roots of the given equation are imaginary, if negative they are real, because in the former case the plane of xy cuts the imaginary branch, in the latter the real. We see in this simple instance what has already been proved generally, namely, that z does not admit of a maximum or minimum value properly speaking, because at the minimum point an imaginary branch goes off along which z still decreases.



The figure represents the curve corresponding to a quadratic equation,

$AB = \frac{a}{2}$, $BC = b - \frac{a^2}{4}$, which in the figure is supposed positive: $X_1\ CX_2$ is the imaginary branch cutting the plane of xy in X_1 and X_2, so that AX_1, AX_2 are the roots of the equation.

I may observe, that the mode of viewing the subject which is explained in this paper, though rather complicated when considered generally, is of very easy application in the case of a quadratic: for the ordinary solution gives us the roots

$$x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}, \text{ if } b \text{ be less than } \frac{a^2}{4},$$

$$\text{and} \quad x = \frac{a}{2} \pm \sqrt{-1}\ \sqrt{b - \frac{a^2}{4}}, \text{ if } b \text{ be greater than } \frac{a^2}{4}.$$

Now $x = \frac{a}{2}$ corresponds to the minimum value of $x^2 - ax + b$, and therefore the usual mode of interpreting the symbol $\sqrt{-1}$ would lead us to consider the preceding expressions as the distance of the minimum point from the origin $\pm$ a distance measured along the axis of x or perpendicular to it, according as b is less or greater than $\frac{a^2}{4}$.

21. Let us take the case of a cubic, which I shall suppose to be deprived of its second term for reasons heretofore assigned. We have then

$$f(x) = x^3 - qx + r = 0 \quad \ldots\ldots\ldots\ldots (26).$$
$$f'(x) = 3x^2 - q,$$
$$f''(x) = 6x,$$
$$f'''(x) = 6.$$

Hence the equations of the curve will be

$$\left.\begin{aligned} z &= x^3 - qx + r - 3xy^2 \\ 0 &= 3x^2 - q - y^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (27).$$

The curve will assume different forms according to the nature of the parameters q and r. Let us consider the real branch of the curve; then the condition $dz = 0$ gives us

$$3x^2 - q = 0, \quad \text{or } x = \pm \sqrt{\frac{q}{3}};$$

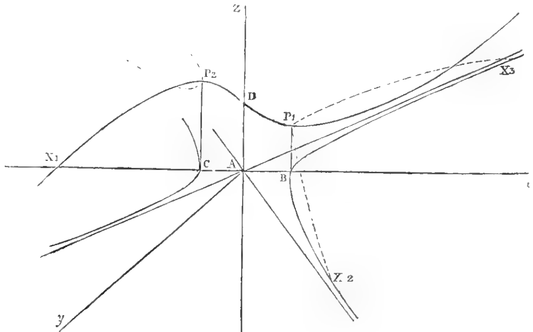
hence in order that there may be a maximum or minimum point q must be positive; suppose this to be the case, then there will be one maximum and one minimum, and for the value of z we have

$$z = r \mp 2\sqrt{\frac{q^3}{27}}:$$

I shall suppose r to be positive, and $\frac{r^2}{4} > \frac{q^3}{27}$, so that both values of z may be positive.

The curve in the plane of xy is evidently an hyperbola, the asymptotes to which are inclined at an angle of 60° to the axis of x, and in the case here supposed of q being positive the real principal axis will be the axis of x.

These indications are sufficient to shew the whole course of the curve which is represented in the annexed figure:



P_1, P_2, are respectively the minimum and maximum point; the real branch of the curve necessarily cuts the axis of x to the left of the origin; from the minimum point P_1 goes off an imaginary branch which meets the plane of xy in X_2, X_3, thus giving two imaginary roots. It will be remembered that the conventions which have been made are that q shall be positive, r positive, and $\frac{r^2}{4} > \frac{q^3}{27}$; it will be easily seen that the form of the curve will remain essentially the same so long as the first condition is fulfilled, and the changes introduced by varying the latter conditions may be represented by supposing the plane of xy shifted into different positions. Suppose for instance the plane of xy to cut the real branch between P_1 and D (the point of intersection of the curve with the axis of z); this will correspond to r positive, and $\frac{r^2}{4} < \frac{q^3}{27}$, then there are three real roots, two positive and one negative; if the plane of xy cuts the real branch between D and P_2, we have the case of r negative, and $\frac{r^2}{4} < \frac{q^3}{27}$, and there are one positive and two negative roots; lastly, if the plane of xy cuts the real branch above P_2, we have the case of r negative, and $\frac{r^2}{4} > \frac{q^3}{27}$, and we have one positive real root and two imaginary. I may just observe that all the imaginary roots here spoken of are of the class which I have termed *connected*.

If we suppose q negative we have an entirely different form of curve, for in this case the real branch has no maximum or minimum point, and therefore it is clear that one of the roots will be real and the other two *isolated* and imaginary. Also the real principal axis of the hyperbola in the plane of xy will be the axis of y, and not the axis of x as in the preceding instance. It is not necessary to trace the curve, as its form is easy to imagine and it presents no varieties.

The preceding discussion includes every case of cubic equations.

22. We may discuss in like manner the general biquadratic equation. In this case,

$$f(x) = x^4 + qx^2 + rx + s = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (28),$$

$$f'(x) = 4x^3 + 2qx + r,$$

$$f''(x) = 12x^2 + 2q,$$
$$f'''(x) = 24x,$$
$$f^{\text{iv}}(x) = 24;$$

and the two equations to the curve are therefore,

$$\left.\begin{aligned} z &= x^4 + qx^2 + rx + s - y^2(6x^2 + q) + y^4 \\ 0 &= 4x^3 + 2qx + r - 4y^2 x \end{aligned}\right\} \ldots\ldots\ldots\ldots (29).$$

Now the sign of s need not be considered, since (as has been observed before) a change in its sign will only correspond to a change in position of the plane of xy, the figure of the curve remaining the same; the combinations of sign of q and r will be as under,

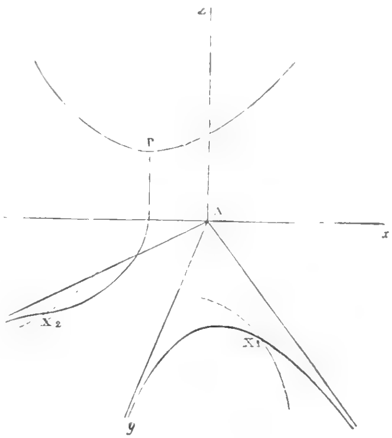
+	+
+	−
−	+
−	−

and these different cases must be considered.

The equation for determining the maxima and minima of the real branch of the curve is,

$$x^3 + \frac{q}{2}x + \frac{r}{4} = 0,$$

which has one real root if q is positive, and if q is negative it has one or three, according as $\frac{r^2}{8}$ is > or < than $\frac{q^3}{27}$.

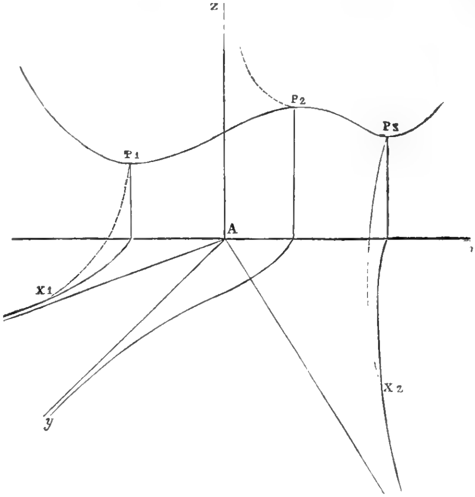


First then, let q be positive and let also r be positive, then it will be found that the curve will be such as is represented in the annexed figure. P is the minimum point of the real branch; the dotted lines represent the imaginary branches, which cut the plane of xy in the points X_1, X_2, and in two other similarly situated points on the other side of the plane of xz which are not represented for fear of complicating the figure.

If r be negative, the figure will be essentially the same, but must be supposed to revolve through two right angles about the axis of z.

Secondly, let q be negative; then if $\frac{r^2}{8}$ be $> \frac{q^3}{27}$, there will be no difference in the figure but this, that the curve of projection on the plane of xy will lie nearer to the axis of x than the asymptotes, instead of lying further away, as in the last case.

But if $\frac{r^2}{8}$ be $< \frac{q^3}{27}$, the form of the curve will be essentially different, and will be as in the annexed figure. If we suppose the figure to correspond to the case of r positive, then the figure for r negative will be found as before by supposing everything turned through two right angles about the axis of z.



23. The curve corresponding to the equation $x^n - 1 = 0$ is easily traced, and furnishes a good illustration of what precedes. I shall trace this curve with polar co-ordinates.

We have, $$z = \rho^n (\cos n\theta + \sqrt{-1} \sin n\theta) - 1 \quad\quad (30),$$
which divides itself into the two equations,

$$\left.\begin{array}{l} z = \rho^n \cos n\theta - 1 \\ 0 = \sin n\theta \end{array}\right\} \quad\quad (31);$$

from the latter of these $$n\theta = k\pi \text{ where } k = 0,\ 1,\ 2 \ldots\ldots (n-1);$$
$$\therefore z = (-1)^k \rho^n - 1.$$

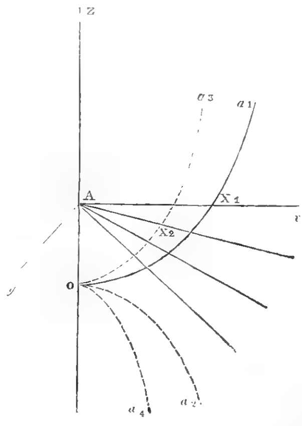
Hence the complete curve will consist of a series of parabolic curves defined by the equations $z = \rho^n - 1$ and $z = -\rho^n - 1$ alternately, and lying in planes passing through the axis of z and making with each other an angle $\frac{\pi}{n}$.

The figure represents the curve; Oa_1, Oa_3 ...... are the branches stretching up to positive infinity, Oa_2, Oa_4 ...... those to negative infinity: the plane of xy intersects the former set of branches but not the latter, and gives for the roots AX_1, AX_2 ....

If we suppose the plane of xy to intersect the branches Oa_2, Oa_4 ...... we should have the case of the equation

$$x^n + 1 = 0;$$

and if the plane were to pass through O, we should have the curve corresponding to $x^n = 0$, in which case the roots would be all equal to 0.



24. The investigations of this paper have been restricted to ordinary algebraic equations, nevertheless some of the results are of a more general character and need not be so restricted. The proposition contained in Art. (8) is, I believe, perfectly general, as also is the proposition of Art. (10) which is an extension of the former. In fact the theorems about maxima and minima will be true for all such points as do not involve a failure of Taylor's Theorem, which never occurs in the case of a rational algebraical function. The propositions concerning the number and position of infinite branches are of course applicable only to algebraic equations. I will just notice one instance of an equation not algebraic: suppose

$$f(x) = \sin x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (32),$$

then
$$z = \left(\cos y \frac{d}{dx}\right) \sin x$$

$$= \sin x \left\{1 + \frac{y^2}{\lfloor 2} + \frac{y^4}{\lfloor 4} + \ldots\ldots\right\}$$

$$= \sin x \, \frac{e^y + e^{-y}}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33).$$

and
$$0 = \left(\sin y \frac{d}{dx}\right) \sin x$$

$$= \cos x \left\{y + \frac{y^3}{\lfloor 3} + \frac{y^5}{\lfloor 5} + \ldots\ldots\ldots\right\}$$

$$= \cos x \left\{e^y - e^{-y}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (34).$$

In equation (34) the variables x and y are entirely separated; the factor $e^y - e^{-y}$ when equated to zero gives, as will easily be seen, only one real value of y, namely $y = 0$: this corresponds to the real plane, and if we make $y = 0$ in (33), that equation becomes

$$z = \sin x,$$

and we have the ordinary figure of sines in the real plane.

If we consider the factor $\cos x$ in (34), we have an infinite number of real roots for the equation, namely $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$, &c., and substituting these in (33), that equation becomes

$$z = \pm\frac{e^y + e^{-y}}{2},$$

which shews that from the maximum and minimum points of the real branch of the curve imaginary branches set off in planes at right angles to the real plane, which are in fact common catenaries, the directrices of which are in the plane of xy, and which go off alternately to positive and negative infinity.

25. In concluding this paper I will observe that I am not sufficiently well acquainted with the literature of the subject to be certain as to how far the idea of it has been anticipated. I will observe, however, that in the late Mr. Murphy's *Treatise on the Theory of Equations*, (published under the direction of the Society for the Diffusion of Useful Knowledge,) the existence of the roots of Algebraic Equations is demonstrated upon principles similar to those which I have adopted: it

is there proved, first, that after a rational function of n dimensions has attained a minimum value corresponding to a real value of x, it is possible to diminish the function still further by assigning to x an increment of the form $h + k\sqrt{-1}$, and then it is shewn that by assigning to x a value of like form, it is possible to give to a rational function of x of *even* dimensions a series of continually increasing or diminishing values, which propositions are akin to, but far less general than, those which I have proved in Arts. (8) and (10). Nevertheless the mode of viewing the subject is the same as that which I have adopted, and indeed suggested to me the possibility of illustrating the theory of equations by reference to the curve of double curvature, which represents the succession of real values of a function of x corresponding to values of the form $x + y\sqrt{-1}$: apart from which geometrical illustration, the theory of the roots of equations which depends upon the demonstrated impossibility of a maximum or minimum value of $f(x)$, when the values of x are of the form $x + y\sqrt{-1}$, appears to me to be more luminous than any other which I have seen.

H. GOODWIN.

XXVI. *On a Change in the State of an Eye affected with a Mal-formation.* *By* G. B. AIRY, *Esq., Astronomer Royal.*

[Read May 25, 1846.]

TWENTY years ago, I had the honour of submitting to this Society a statement of the effects of a mal-formation in my own left eye. The nature of the effect was this: that the rays of light coming from a luminous point and falling upon the whole surface of the pupil do not converge to a point at any position within the eye, but converge in such a manner as to pass through two lines at right angles to each other, (a geometrical phenomenon, to which the term *astigmatism* was very happily affixed by the present Master of Trinity College), and that these lines, in the ordinary position of the head, are both inclined to the vertical in the manner described in my paper (*Cambridge Philosophical Transactions*, Vol. II.) The evidence of this astigmatism, and the measure of it, are given by the simple observation of bringing the luminous point nearer and nearer to the eye; the lines of focal convergence, according to the usual rules of focal position, move in the same direction in the interior of the eye; and thus one line and the other line are successively brought upon the retina; or the image of the point becomes successively a line in one direction or in the other direction, these directions being at right angles to each other. It was found in 1825 that the distances at which the luminous point must be placed to give linear images were 3·5 and 6·0 inches; and the difference of the reciprocals of these numbers, or 0·119, is a proper measure of the astigmatism. The fault of the eye was corrected, as regards the production of distinct vision, by the use of a lens of which one surface was spherically concave, and the other surface cylindrically concave, and the radius of the cylindrical surface was such as to give a *power* 0·119, or, in combination with a plane surface, to give a focal length $\frac{1}{0{\cdot}119}$ inch, or it was in inches $\frac{n-1}{0{\cdot}119}$.

Some years since, I found, from some unrecorded observations, that the general short-sightedness of the eye had sensibly altered, but that the measure of astigmatism remained nearly the same as at first.

Lately, having found that the spectacles constructed for me in 1825 do not very well suit the present condition of the eye, I have made observations in precisely the same manner as in 1825, by viewing a very fine hole pricked in a card, and causing that card to slide upon a scale whose end rests upon the orbital bone of the eye, and measuring the distances at which the card is placed when the point appears as a line. I have been careful to hold the body and head in the same general position as before: the accuracy of the measures being sensibly affected by these circumstances.

As far as I can remember, the indication of the focal lines to the horizon, their length, and their sharpness, are not in the smallest degree changed. But the distances of the luminous point which produce them are sensibly changed. They were formerly 3·5 and 6·0 inches: they are now 4·7 and 8·9 inches. The eye therefore has become generally less short-sighted than it was formerly.

But the measure of the astigmatism, which was formerly

$$\frac{1}{3{\cdot}5} - \frac{1}{6{\cdot}0} = 0{\cdot}119, \text{ is now } \frac{1}{4{\cdot}7} - \frac{1}{8{\cdot}9} = 0{\cdot}100.$$

On examining the slightly discordant observations, I am inclined to think that a distance somewhat less than 4·7 is the true one, and this would increase the measure of astigmatism above 0·100, and would make it approach more nearly to the ancient value. It seems therefore that while the short-sightedness of the eye has materially diminished, the fault which produces the astigmatism has undergone very little or no alteration.

Upon examining the right eye in the same manner, I find no perceptible fault. The image of a fine hole is a luminous point very sharply defined. The distance of accurate definition is as nearly as possible 4·7 inches, the same as the nearest distance at which the left eye forms a well defined line for the image of a point. It would seem therefore that the normal formation of the two eyes is the same, and that the abnormal alteration in the left eye is of the nature of a refraction through a dense medium cylindrically concave, or through a rare medium cylindrically convex, superadded to the normal refraction.

G. B. AIRY.

Royal Observatory, Greenwich,
January 14, 1846.

XXVII. *A Theory of Luminous Rays on the Hypothesis of Undulations. By the Rev.* J. CHALLIS, M.A., *Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge.*

[Read *May* 11, 1846.]

IF a beam of Sun-light pass through a narrow aperture, about one-thirtieth of an inch in breadth and be received on a glass prism the edges of which are parallel to the borders of the aperture, a *spectrum* is formed by the transmitted light, which, when magnified and properly looked at, exhibits, as is well known, a large number of dark lines parallel to the refracting edge of the prism. If instead of passing through an aperture with parallel borders, the light passed through a circular aperture, one-thirtieth of an inch in diameter, a spectrum of diminished width would be seen, but of the same length as before and crossed by the same dark lines. The transmission through the prism has produced no change on the light: it has only brought into view the parts of which the incident beam is composed. Taking, for instance, a portion of light immediately contiguous to any one of the dark lines, the prism informs us that the incident beam contains light of that particular refrangibility, abruptly terminated in a plane passing through the axis of the beam perpendicular to the edge of the prism. The existence of this abrupt termination is owing to the cause, whatever it may be, which produces the dark line, and has nothing whatever to do with the transmission of the light through a small aperture. Let now the prism be turned about the axis of the beam to any other position. The spectrum will present exactly the same appearance as before, and light of the same refrangibility (not necessarily the same light) as in the former case, will still be bounded by a dark line. And so for every position the prism be made to take by being turned about the axis of the incident beam. This experiment proves that every beam of white light contains portions of light of a definite refrangibility, the *sides* of which are turned in all directions from the axis of the beam. This fact is at once explained by supposing light to consist of *rays;* and it does not appear possible to give any other explanation of it. Although the experimental evidence applies immediately only to the portions of light contiguous to dark lines, yet a very strong presumption is afforded by it that all light is in the form of rays. The existence of the dark lines themselves is most simply accounted for by supposing that certain *rays* of Sun-light are in some manner extinguished.

Admitting it to be a legitimate deduction from the facts of the Solar Spectrum, that light is composed of rays, it is clear that no Theory of Light can be complete which does not take account of this distinctive character. The facts are perfectly consistent with the Theory of Emission, and the advocates of that theory might justly appeal to them as evidence in its favour. My object in this communication will be to shew that rays of light are also to be accounted for on the Undulatory Theory.

It must here be premised that it is not my intention to treat the Undulatory Theory, as most optical writers of the present day have done, by a particular consideration of the molecular constitution of the æther. Not having been able to form the slightest conception how this view of Undulations can be reconciled with the existence of rays of light, I propose to regard the æther as a continuous fluid substance, such that small increments of its pressure are proportional to small increments of density, and to apply to it the usual hydrodynamical equations. The pressure being p and density ρ at the time t at any point whose co-ordinates are x, y, z, it will be assumed that $p = a^2\rho$, a^2 being a certain constant.

In a former communication which I made to this Society, I gave the proof of a new fundamental equation in Hydrodynamics, by the combination of which with the ordinary equation of continuity, an equation results which is indispensable in the present investigation. The process for deducing this last equation is given in the *Cambridge Philosophical Transactions*, (Vol. VII. Part III. pp. 385 and 386): it is also obtained (p. 387) by independent elementary considerations. Let V be the velocity and ρ the density at any time t, at a point where the principal radii of curvature of the surface cutting the directions of motion at right angles are R and R', and let ds be the increment of a line coincident with the directions at the time t of the motions of the particles through which it passes. Then the resulting equation I speak of is,

$$\frac{d\rho}{dt} + \frac{d \, . \, \rho V}{ds} + \rho V \left(\frac{1}{R} + \frac{1}{R'}\right) = 0, \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

the variation with respect to space being from point to point along the line s. Now the new fundamental equation above mentioned, combined with the two other fundamental equations, gives the means of obtaining a resulting equation, in which the variables are ψ, x, y, z and t, the principal variable ψ being such a function of the others that $\psi = 0$ is the equation of a surface normal to the directions of motion, in whatever way the motion of the fluid may have originated. It follows that the function ψ, since it is given by a partial differential equation, contains arbitrary functions of x, y, z and t, and that the normal surface is consequently arbitrary. The partial differential equations applicable to the Undulatory Theory of Light are linear with constant coefficients. For our present purpose, we have to enquire how far ψ is arbitrary when the equations are of this nature: whether, for instance, the normal surface must necessarily be either a plane or a spherical surface. The general equation which gives ψ by integration is too complicated to be employed in this investigation. We may, however, dispense with the use of it by combining equation (1) with the following general equation, which is obtained in p. 383 of the communication already referred to:

$$a^2 \, \text{Nap.} \log \rho + \int \frac{dV}{dt} ds + \frac{V^2}{2} = F(t), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

the variation with respect to space being, as before, from point to point of the line of motion. By differentiating this equation with respect to s and t successively we get,

$$\frac{a^2 d\rho}{\rho ds} + \frac{dV}{dt} + V\frac{dV}{ds} = 0, \quad \text{and} \quad \frac{a^2 d\rho}{\rho dt} + \int \frac{d^2 V}{dt^2} ds + V\frac{dV}{dt} = F'(t).$$

Also equation (1) may be put under the form,

$$\frac{d\rho}{\rho dt} + \frac{V \, . \, d\rho}{\rho ds} + \frac{dV}{ds} + V\left(\frac{1}{R} + \frac{1}{R'}\right) = 0.$$

Hence substituting for $\frac{d\rho}{\rho dt}$ and $\frac{d\rho}{\rho ds}$ from the preceding equations, and differentiating with respect to s, the result is,

$$\frac{d^2 V}{dt^2} - (a^2 - V^2)\frac{d^2 V}{ds^2} + 2V . \frac{d^2 V}{ds dt} + 2\frac{dV}{ds} . \frac{dV}{dt} + 2V\frac{dV}{ds}\left(\frac{1}{R} + \frac{1}{R'}\right) + a^2 V\left(\frac{1}{R^2} + \frac{1}{R'^2}\right) = 0;$$

for $dR = dR' = ds$. Now putting $\frac{dq}{ds}$ for V, and integrating with respect to s after the substitution, it will be found that

$$\frac{d^2 q}{dt^2} - \left(a^2 - \frac{dq^2}{ds^2}\right)\frac{d^2 q}{ds^2} + 2\frac{dq}{ds}\cdot\frac{d^2 q}{ds\,dt} - a^2\frac{dq}{ds}\left(\frac{1}{R} + \frac{1}{R'}\right) = F(t).$$

Lastly, for q put $\phi + \chi(t)$, the function $\chi(t)$ being such that $F(t) - \chi''(t) = 0$.

Then $\frac{d\phi}{ds} = \frac{dq}{ds} = V$, and

$$\frac{d^2 \phi}{dt^2} - \left(a^2 - \frac{d\phi^2}{ds^2}\right)\frac{d^2 \phi}{ds^2} + 2\frac{d\phi}{ds}\cdot\frac{d^2 \phi}{ds\,dt} - a^2\frac{d\phi}{ds}\left(\frac{1}{R} + \frac{1}{R'}\right) = 0 \ldots\ldots\ldots\ldots (3).$$

If the surface normal to the directions of motion be a plane, R and R' are each infinitely great, and the equation strictly applying to this case of motion, is

$$\frac{d^2 \phi}{dt^2} - \left(a^2 - \frac{d\phi^2}{ds^2}\right)\frac{d^2 \phi}{ds^2} + 2\frac{d\phi}{ds}\cdot\frac{d^2 \phi}{ds\,dt} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

It is well known that this equation is exactly satisfied by a particular integral applying to motion propagated in a single direction, namely, $\frac{d\phi}{ds} = F\left\{\left(a + \frac{d\phi}{ds}\right)t - s\right\}$; and that at the same time $\frac{d\phi}{ds} = a\,.\,\text{Nap. log}\,\rho$. From these two equations it follows that a given state of density and velocity is carried through space by the propagation and by the motion of the particles, with the velocity $a + \frac{d\phi}{ds}$. The rate of propagation is therefore strictly a, whatever be the velocity and density of the particles. Unless this were the case the velocity and arrangement of density in a given wave would change by propagation, however small the motion of the particles might be. Hence, in order that equation (3), in which R and R' are not supposed to be indefinitely great, may apply to motion in which the *type* of the waves remains altogether unchanged by propagation, it must be of the same form as equation (4). This will be the case if

$$a^2\frac{d\phi}{ds}\left(\frac{1}{R} + \frac{1}{R'}\right) = (a'^2 - a^2)\frac{d^2\phi}{ds^2}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5);$$

the resulting equation being the same as (4) with the difference of having a' in the place of a. Also $\frac{d\phi}{ds} = \frac{a^2}{a'}\,.\,\text{Nap. log}\,\rho$.

It is now important to remark that the general partial differential equation having ψ for its principal variable, to which I have already referred, is of the *third* order, and consequently its integral, supposing it could be obtained, would involve three arbitrary functions of the co-ordinates and the time. Hence the function ψ may be made to satisfy three arbitrary conditions. The first I shall suppose it to satisfy is, that the propagation of the motion be in a single direction: the next, that the motion of the particles situated in a fixed straight line, which I shall call the axis of z, be entirely in that line; the third condition I shall assume is, that for the motion along the axis of z the equation (5) is satisfied. It will appear from the reasoning that follows, that a form of ψ may be found consistent with these conditions.

Let $\phi_{,}(z, t)$ be the condensation at the time t at any point of the axis of z distant by z from the origin, and let the condensation, for a reason that will appear afterwards, be assumed to be $\phi_{,}(z, t) \times f(x, y)$ at any point whose co-ordinates are x, y, z. For shortness sake I shall write $\phi_{,}$ and f for these functions, treating $\phi_{,}$ as a function of z and t only, and f as a function of x and y only. Let ρ be the density, and u, v, w, the components of the velocity in the directions of the axes of co-ordinates, at the point xyz, and at the time t. It will be assumed that u, v, and w are always small velocities, and their first powers only will be taken account of. This being premised, we have

$$\rho - 1 = \phi_{,} f; \quad -\frac{a^2 d\rho}{\rho dx} = -\frac{a^2\phi_{,}}{\rho} \cdot \frac{df}{dx} = \left(\frac{du}{dt}\right);$$

and to the first approximation, $\dfrac{du}{dt} = -a^2\phi_{,}\dfrac{df}{dx}$. Hence

$$u = -a^2 \frac{df}{dx} \int \phi_{,} dt + c,$$

the arbitrary quantity c being in general a function of x, y, and z. In the same manner,

$$v = -a^2 \frac{df}{dy} \int \phi_{,} dt + c'.$$

Also since $-\dfrac{a^2 d\rho}{\rho dz} = -a^2 f \dfrac{d\phi_{,}}{\rho dz} = \left(\dfrac{dw}{dt}\right)$, we have to the same degree of approximation, $\dfrac{dw}{dt} = -a^2 f \cdot \dfrac{d\phi_{,}}{dz}$, and $w = -a f \int \dfrac{d\phi_{,}}{dz} dt + c'' = -a^2 f \dfrac{d \cdot \int \phi_{,} dt}{dz} + c''$. But it is evident from the assumed law of the condensation in any plane perpendicular to the axis of z, that the accelerative force parallel to z at any point of this plane must to the first degree of approximation be equal to $f \times$ the accelerative force at the point of intersection with the axis, and that the corresponding velocities must be in the same proportion. Hence, $\dfrac{d\phi}{dz}$ being the velocity at the point of the axis of z, we shall have $w = f\dfrac{d\phi}{dz}$. It follows that $\phi = -a^2 \int \phi_{,} dt$, and that $c'' = 0$.

For reasons which will appear hereafter I shall also suppose that $c = 0$, and $c' = 0$. Thus we shall have,

$$u = \phi \frac{df}{dx}; \quad v = \phi \frac{df}{dy}; \quad w = f \frac{d\phi}{dz}.$$

It is to be remarked that these equations are the more exact, the smaller the ratios of u and v to w.

From the foregoing reasoning it follows that

$$u dx + v dy + w dz = \phi \frac{df}{dx} dx + \phi \frac{df}{dy} dy + f \frac{d\phi}{dz} dz = d \cdot f\phi.$$

Hence $u dx + v dy + w dz$ is an exact differential; and it is well known that in a case of fluid motion in which the first power of the velocity is alone retained, this condition must be fulfilled. The assumed law of the distribution of the density consequently satisfies a necessary analytical condition, and on this principle is justified. It follows also that $d\psi = d \cdot f\phi$, and by integration that $\psi = f\phi$ + a function of $t = 0$. Thus the equation of the surface normal to the directions of motion is to a certain extent determined, and we may now proceed to obtain an expression for $\dfrac{1}{R} + \dfrac{1}{R'}$.

The known general expression for $\dfrac{1}{R} + \dfrac{1}{R'}$ is,

$$\left(\frac{d\psi^2}{dx^2} + \frac{d\psi^2}{dy^2} + \frac{d\psi^2}{dz^2}\right)^{-\frac{3}{2}} \times \left\{ \begin{array}{l} \left(\dfrac{d^2\psi}{dx^2} + \dfrac{d^2\psi}{dy^2} + \dfrac{d^2\psi}{dz^2}\right)\left(\dfrac{d\psi^2}{dx^2} + \dfrac{d\psi^2}{dy^2} + \dfrac{d\psi^2}{dz^2}\right) - \dfrac{d^2\psi}{dx^2} \cdot \dfrac{d\psi^2}{dx^2} - \dfrac{d^2\psi}{dy^2} \cdot \dfrac{d\psi^2}{dy^2} \\ - \dfrac{d^2\psi}{dz^2} \cdot \dfrac{d\psi^2}{dz^2} - 2\dfrac{d^2\psi}{dx dy} \cdot \dfrac{d\psi}{dx} \cdot \dfrac{d\psi}{dy} - 2\dfrac{d^2\psi}{dx dz} \cdot \dfrac{d\psi}{dx} \cdot \dfrac{d\psi}{dz} - 2\dfrac{d^2\psi}{dy dz} \cdot \dfrac{d\psi}{dy} \cdot \dfrac{d\psi}{dz} \end{array} \right\}.$$

also, $\frac{d\psi}{dx} = \phi \frac{df}{dx}$ $\quad \frac{d^2\psi}{dx^2} = \phi \frac{d^2f}{dx^2}$ $\quad \frac{d^2\psi}{dxdy} = \phi \frac{d^2f}{dxdy}$,

$$\frac{d\psi}{dy} = \phi \frac{df}{dy} \qquad \frac{d^2\psi}{dy^2} = \phi \frac{d^2f}{dy^2} \qquad \frac{d^2\psi}{dxdz} = \frac{d\phi}{dz} \cdot \frac{df}{dx},$$

$$\frac{d\psi}{dz} = f \frac{d\phi}{dz} \qquad \frac{d^2\psi}{dz^2} = f \frac{d^2\phi}{dz^2} \qquad \frac{d^2\psi}{dydz} = \frac{d\phi}{dz} \cdot \frac{df}{dy}.$$

Hence by substituting and reducing, it will be found that

$$\left(\frac{1}{R} + \frac{1}{R'}\right)\left(\frac{df^2}{dx^2} + \frac{df^2}{dy^2} + \frac{f^2}{\phi^2} \cdot \frac{d\phi^2}{dz^2}\right)^{\frac{3}{2}} = \frac{d^2f}{dy^2} \cdot \frac{df^2}{dx^2} + \frac{d^2f}{dx^2} \cdot \frac{df^2}{dy^2} - 2\frac{d^2f}{dxdy} \cdot \frac{df}{dx} \cdot \frac{df}{dy} + \frac{f^2}{\phi^2} \cdot \frac{d\phi^2}{dz^2}\left(\frac{d^2f}{dx^2} + \frac{d^2f}{dy^2}\right)$$

$$+ \left(\frac{f}{\phi} \cdot \frac{d^2\phi}{dz^2} - \frac{2f}{\phi^2} \cdot \frac{d\phi^2}{dz^2}\right)\left(\frac{df^2}{dx^2} + \frac{df^2}{dy^2}\right).$$

For our purpose we require an expression for $\frac{1}{R} + \frac{1}{R'}$ for any point on the axis of z. Now since by hypothesis $u = 0$ and $v = 0$ at all times for all points on the axis of z, it follows from the equations $u = \phi \frac{df}{dx}$ and $v = \phi \frac{df}{dy}$, that $\frac{df}{dx} = 0$ and $\frac{df}{dy} = 0$ for these points. Hence the foregoing equation gives,

$$\left(\frac{1}{R} + \frac{1}{R'}\right) \frac{f}{\phi} \cdot \frac{d\phi}{dz} = \frac{d^2f}{dx^2} + \frac{d^2f}{dy^2} \quad \text{.......................... (6).}$$

But equation (5) becomes for motion along the axis of z,

$$a^2 \frac{d\phi}{dz} \cdot \left(\frac{1}{R} + \frac{1}{R'}\right) = (a'^2 - a^2) \frac{d^2\phi}{dz^2}.$$

Consequently putting $a^2(1 + k)$ for a'^2, and substituting from equation (6),

$$\frac{d^2\phi}{dz^2} - \frac{1}{kf}\left(\frac{d^2f}{dx^2} + \frac{d^2f}{dy^2}\right)\phi = 0 \quad \text{.................... (7).}$$

In this equation the coefficient of ϕ, not containing x and y, is a constant, and we may assume it equal to $-n^2$. It hence follows that

$$\frac{d^2\phi}{dz^2} + n^2\phi = 0 \quad \text{................................ (8).}$$

I shall here take occasion to remark that since $\frac{d^2\phi}{dz^2} = 0$ when the velocity $\frac{d\phi}{dz}$ is a maximum, it appears by equation (8) that $\phi = 0$ in the same case. Hence also $u = 0$, and $v = 0$. Consequently the assumption made heretofore that the arbitrary quantities c and c' each $= 0$, was equivalent to assuming that the *transverse* velocity vanishes when the velocity is a maximum along the axis of z. It appears also from the expressions for u, v, and w, that, when the velocity $\frac{d\phi}{dz} = 0$, and ϕ is consequently a maximum, u and v are each a maximum.

At the same time that the equation (8) is true, we have by equation (3) neglecting the small terms, and by what has now been proved,

$$\frac{d^2\phi}{dt^2} - a'^2 \cdot \frac{d^2\phi}{dz^2} = 0 \quad \text{.................................. (9).}$$

The equation (8) is satisfied by $\phi = \psi(t)\cos\{nz + \chi(t)\}$; and the equation (9) by $\phi = \phi(nz - na't)$. Hence $\psi(t) = \text{constant } \frac{m}{n}$, and $\chi(t) = -na't$. Consequently $\phi = \frac{m}{n}\cos n(z - a't)$, and the velocity $\frac{d\phi}{dz} = m \sin n(a't - z) = m \sin \frac{2\pi}{\lambda}(a't - z)$ suppose.

It results from the foregoing reasoning that if the small vibrations of the æther in the direction of propagation follow the law expressed by the equation last obtained, the condensation in any plane perpendicular to an axis of rectilinear propagation may vary at a given time from point to point, and at the same time the propagation be uniform. A consequence of this result is that a very slender cylindrical portion of the æther may continue in agitation while the contiguous portions are at rest; and since the law above obtained is that which has been found by experience to apply to the phænomena of light, the existence of *rays* of light, which was proved experimentally at the commencement of this paper, is accounted for theoretically.

As far as we have hitherto proceeded, the function f has remained indeterminate. The considerations I am now about to enter upon will serve to ascertain its form. Take a plane perpendicular to the axis of z, in which the velocity parallel to the axis of z is a maximum, and in which consequently $u = 0$, and $v = 0$. As the motion at any point of this plane is parallel to the direction of propagation, and as the velocity of propagation is uniform, it follows that an equation like (5), applicable to this point, is obtained by simply substituting $f\phi$ for ϕ. Substituting also $a^2(1+k)$ for a'^2, we have,

$$f\frac{d\phi}{dz}\left(\frac{1}{R} + \frac{1}{R'}\right) = kf\frac{d^2\phi}{dz^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

At the same time the general expression for $\frac{1}{R} + \frac{1}{R'}$ gives,

$$\left(\frac{1}{R} + \frac{1}{R'}\right)\frac{f}{\phi}\cdot\frac{d\phi}{dz} = \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} - \frac{2}{f}\cdot\left(\frac{df^2}{dx^2} + \frac{df^2}{dy^2}\right)$$

$$= -f^2\left\{\frac{d^2 . \frac{1}{f}}{dx^2} + \frac{d^2 . \frac{1}{f}}{dy^2}\right\}.$$

Hence by substitution in equation (10),

$$\frac{d^2\phi}{dz^2} + \frac{f}{k}\cdot\left(\frac{d^2 . \frac{1}{f}}{dx^2} + \frac{d^2 . \frac{1}{f}}{dy^2}\right)\phi = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (11).$$

Consequently, by comparison with equation (8),

$$\frac{f}{k}\cdot\left(\frac{d^2 . \frac{1}{f}}{dx^2} + \frac{d^2 . \frac{1}{f}}{dy^2}\right) = n^2;$$

$$\text{or} \quad \frac{d^2 . \frac{1}{f}}{dx^2} + \frac{d^2 . \frac{1}{f}}{dy^2} - kn^2 . \frac{1}{f} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (12).$$

The function f must consequently be such as to satisfy this equation.

Again, as the phænomena of light shew that a ray of *common* light has similar relations to space in all directions perpendicular to its axis, the function f, which is arbitrary, to apply to this kind of light, must be assumed to be a function of the distance from the axis. That is, if $r^2 = x^2 + y^2$,

f is a function of r. And the equation (12) is quite consistent with this assumption. In fact, for this case it becomes,

$$\frac{d^2 \cdot \frac{1}{f}}{dr^2} + \frac{1}{r} \cdot \frac{d \cdot \frac{1}{f}}{dr} - kn^2 \cdot \frac{1}{f} = 0,$$

which equation determines the particular form of f applicable to common light. This equation does not appear to be exactly integrable. By putting it under the form,

$$\frac{d^2 f}{dr^2} - 2 \frac{df^2}{fdr^2} + \frac{df}{rdr} + kn^2 f = 0 \dots\dots\dots\dots\dots\dots (13),$$

it will be seen that the equation $f = \cos \frac{nr\sqrt{k}}{\sqrt{2}}$ satisfies it, when r is very small. By multiplying equation (13) by f, and supposing $\frac{df}{dr} = 0$, we shall have either $f = 0$, or $\frac{d^2 f}{dr^2} + kn^2 f = 0$. The latter equation is satisfied at the axis of the ray: the other by a certain value l of r, which may be called the radius of the ray. If S equal the condensation at the axis, and s the condensation at a point distant by r from the axis, by what has been already shewn, $s = Sf$. Hence where $r = l$, both $s = 0$, and $\frac{ds}{dr} = 0$; that is, at this distance there is neither condensation nor variation of condensation. Thus the parts of the fluid more distant from the axis than l may remain at rest, while those at less distance continue in agitation. As $a'^2 = a^2 (1 + k)$, and as it is not probable that a' differs much from a, k may be considered a very small numerical quantity. Hence the three first terms of equation (13) will be small, since each would vanish if k vanished. Consequently l the radius of the ray must be large compared to λ the breadth of an undulation. Because $\frac{df}{dr}$ is very small for all values of r, and f and $\frac{df}{dr}$ vanish together where $r = l$, it follows that the second term of equation (13) is very small compared to the others at all distances from the axis. By neglecting this term the equation becomes,

$$\frac{d^2 f}{dr^2} + \frac{df}{rdr} + kn^2 f = 0 \dots\dots\dots\dots\dots\dots (14).$$

which determines with sufficient approximation the function f.

By neglecting in the general equation (12) the terms containing $\frac{df^2}{fdx^2}$ and $\frac{df^2}{fdy^2}$, which are quantities of the same order as the neglected term of equation (13), we obtain,

$$\frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} + kn^2 f = 0 \dots\dots\dots\dots\dots\dots (15),$$

which is a general equation, applying to a ray of light of any kind, and including as a particular case equation (14). Since by hypothesis $s = Sf$, we immediately derive from (15),

$$\frac{d^2 s}{dx^2} + \frac{d^2 s}{dy^2} + kn^2 s = 0 \dots\dots\dots\dots\dots\dots (16),$$

a linear equation with constant coefficients, in which the principal variable is the condensation.

The velocity (u) in the direction of x we find to be $\phi \frac{df}{dx}$, which, since $s = Sf$, becomes $\frac{\phi}{S} \cdot \frac{ds}{dx}$.

So $v = \frac{\phi}{S} \cdot \frac{ds}{dy}$. But we have seen that $\phi = \frac{m\lambda}{2\pi} \cos \frac{2\pi}{\lambda} (a't - z)$. Hence $\frac{d\phi}{dt} = - m a' \sin \frac{2\pi}{\lambda} (a't - z)$ $= - a^2 S$. Therefore $S = \frac{m a'}{a^2} \sin \frac{2\pi}{\lambda} (a't - z)$, and $\frac{\phi}{S} = \frac{\lambda a^2}{2\pi a'} \cdot \cot \frac{2\pi}{\lambda} (a't - z)$. Call this quantity Φ. Then $u = \Phi \frac{ds}{dx}$, and $v = \Phi \frac{ds}{dy}$. Let now $s = \sigma_1 + \sigma_2$. Then

$$\frac{d^2 . (\sigma_1 + \sigma_2)}{dx^2} + \frac{d^2 . (\sigma_1 + \sigma_2)}{dy^2} + kn^2 (\sigma_1 + \sigma_2) = 0\,;$$

and as s in equation (16) is arbitrary, we may have separately,

$$\frac{d^2 \sigma_1}{dx^2} + \frac{d^2 \sigma}{dy^2} + kn^2 \sigma_1 = 0,$$

$$\text{and} \quad \frac{d^2 \sigma_2}{dx^2} + \frac{d^2 \sigma_2}{dy} + kn^2 \sigma_2 = 0,$$

and consider these equations to apply to two distinct rays. At the same time, since $s = \sigma_1 + \sigma_2$, $\Phi \frac{ds}{dx} = \Phi \frac{d\sigma_1}{dx} + \Phi \frac{d\sigma_2}{dx}$, and $\Phi \frac{ds}{dy} = \Phi \frac{d\sigma_1}{dy} + \Phi \frac{d\sigma_2}{dy}$; that is, the sums of the velocities in the two rays resolved in the directions of the axes of co-ordinates are equal to the resolved parts of the velocity of the original ray in the same direction. Similar reasoning would have applied if we had assumed $s = \sigma_1 + \sigma_2 + \sigma_3 + \&c.$ The general conclusion we may now draw is, that a ray may be conceived to be composed of two or more rays in the same phase of vibration, and that if, after a ray has been separated into distinct rays, the parts be put together in the same phase of vibration, they will compose the original ray.

The foregoing Theory of Luminous Rays, conducts to a very simple and satisfactory explanation of the phænomena of Polarized Light, which I propose to bring before the notice of the Society at a future opportunity.

Cambridge Observatory,
May 11, 1846.

XXVIII. *A Theory of the Polarization of Light on the Hypothesis of Undulations. By the Rev.* J. Challis, M.A., *Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge.*

[Read May 25, 1846.]

The Theory of Polarization contained in this Paper is founded on the Theory of Luminous Rays, given in my last communication, of which the present may be regarded as a continuation. I shall, therefore, use the same symbols as in the former Paper, and suppose their signification to be known, and for the sake of convenience, the reference numbers attached to the equations will be in continuation of those of the other Paper.

Conceive a ray of common light to be submitted to some action which is not symmetrical with respect to its axis, and which divides it into rays subsequently pursuing different paths. In general the arrangement of the condensation in neither of these rays will be symmetrical about its axis: but each may be supposed to consist of a symmetrical part having the properties of common light, and a part which has a different arrangement. The unsymmetrical part is considered to be *polarized.* A difference in the arrangement of the condensation in different directions transverse to the axis, corresponds in this Theory to Polarization. By experiment it appears that a polarized ray has a certain definite character, which is quite independent of the particular action producing the bifurcation of the original ray, being the same under modes of separation of very different kinds. The explanation of the phænomena of polarization is therefore to be sought for in the modifications of which the vibrations of a ray of common light are susceptible according to Hydrodynamical principles. In the phænomena of common light there is nothing to decide whether the sensation of light is due to the *direct* or the *transverse* vibrations. The phænomena of polarized light shew that it is to be attributed to the *transverse* vibrations, and our attention must therefore be directed to the modifications which these may undergo. The direct vibrations very probably are productive of heat.

In the Theory of Luminous Rays it was shewn that a ray in which the condensation at any point is s, may be supposed to be compounded of two rays in the same phase of vibration whose condensations are σ_1 and σ_2, if $s = \sigma_1 + \sigma_2$, independently of any relation between σ_1 and σ_2. We are therefore at liberty to assume another condition which these quantities shall satisfy. The assumption I shall make is, that the bifurcation of a ray takes place so that the transverse velocity at each point is converted into two velocities at right angles to each other, and that these are respectively the velocities at the corresponding points of the two polarized rays. This law is most probably deducible from purely Hydrodynamical principles; but in the present state of Hydrodynamics it must be regarded as an hypothesis. By the reasoning and notation of the former Paper, the component velocities in one ray are $\Phi \frac{d\sigma_1}{dx}$, and $\Phi \frac{d\sigma_1}{dy}$; and in the other, $\Phi \frac{d\sigma_2}{dx}$, and $\Phi \frac{d\sigma_2}{dy}$; and the hypothesis we make is, that

$$\frac{\frac{d\sigma_1}{dy}}{\frac{d\sigma_1}{dx}} = -\frac{\frac{d\sigma_2}{dx}}{\frac{d\sigma_2}{dy}}, \quad \text{or} \quad \frac{d\sigma_1}{dx} \cdot \frac{d\sigma_2}{dx} + \frac{d\sigma_1}{dy} \cdot \frac{d\sigma_2}{dy} = 0 \, \ldots\ldots\ldots \, (16).$$

Let us now consider by itself the polarized ray in which the condensation is σ_1. Since $s = \sigma_1 + \sigma_2$, we have $\frac{d\sigma_2}{dx} = \frac{ds}{dx} - \frac{d\sigma_1}{dx}$, and $\frac{d\sigma_2}{dy} = \frac{ds}{dy} - \frac{d\sigma_1}{dy}$. Substituting these values in (16), we obtain,

$$\frac{d\sigma_1^2}{dx^2} - \frac{ds}{dx} \cdot \frac{d\sigma_1}{dx} + \frac{d\sigma_1^2}{dy^2} - \frac{ds}{dy} \cdot \frac{d\sigma_1}{dy} = 0 \dots\dots\dots\dots (17).$$

Also σ_1 must satisfy equation (15). Consequently

$$\frac{d^2\sigma_1}{dx^2} + \frac{d^2\sigma_1}{dy^2} + n^2 k \sigma_1 = 0 \dots\dots\dots\dots (18).$$

The equations (17) and (18) determine the function that σ_1 is of x and y. For by eliminating $\frac{d^2\sigma_1}{dx^2}$ from (18) by means of (17), an equation results, which, as it contains only partial differential coefficients of σ_1 with respect to y, determines the form in which y enters into this function. The form in which x enters is similarly determined. The function expressing the value of σ_2 is determined by equations exactly the same as (17) and (18), having only σ_2 in the place of σ_1. In fact, putting equation (17) under the form $\frac{d\sigma_1^2}{dx^2} - A\frac{d\sigma_1}{dx} + B = 0$, we have $A = \frac{ds}{dx} = \frac{d\sigma_1}{dx} + \frac{d\sigma_2}{dx}$, and $B = -\frac{d\sigma_1}{dy}\left(\frac{ds}{dy} - \frac{d\sigma_1}{dy}\right) = -\frac{d\sigma_1}{dy} \cdot \frac{d\sigma_2}{dy} = \frac{d\sigma_1}{dx} \cdot \frac{d\sigma_2}{dx}$. The two roots of that equation are therefore $\frac{d\sigma_1}{dx}$ and $\frac{d\sigma_2}{dx}$, and hence the process indicated above which determines σ_1 determines σ_2 also.

Since the original ray is supposed to be one of common light, s is a function of r, and $\frac{ds}{dx} = f'(r)\frac{x}{r}$, $\frac{ds}{dy} = f'(r) \cdot \frac{y}{r}$. Substituting these values in equation (17), we have,

$$\frac{d\sigma_1^2}{dx^2} + \frac{d\sigma_1^2}{dy^2} - f'(r)\left(\frac{d\sigma_1}{dx} \cdot \frac{x}{r} + \frac{d\sigma_1}{dy} \cdot \frac{y}{r}\right) = 0 \dots\dots\dots (19).$$

If now the direction of the axes of co-ordinates be changed through 90° by putting $-x'$ for y and y' for x, neither this equation nor equation (18) will be altered in any respect, so that the same solution of the equations will result as before. Consequently by this transformation the function that σ_1 is of x and y is converted into the function that σ_2 is of the same variables, and *vice versâ*. It follows that the original ray is divided into two *equal* polarized rays, such that if one be turned about the axis of z through 90° it becomes identical with the other. Since also equations (18) and (19) are not altered by changing the axes of co-ordinates through 180°, that is, by altering the signs of x and y, it appears that σ_1 and σ_2 are symmetrical about planes passing through the axis of z. These planes, from what has just been proved, must be at right angles to each other. Also it is evident that a plane of symmetry of one ray must coincide with a plane of symmetry of the other. Hence each ray will have two planes of symmetry at right angles to each other.

The above results would be more properly derived from the functions of x and y expressing the values of σ_1 and σ_2, if the integrations could be performed by which these functions would result from equations (18) and (19). This it does not appear possible to do generally; but values of σ_1 and σ_2 applicable to small distances from the axis of z may be obtained as follows. We have seen that the solution of equation (13) for small values of r is $f = \cos n\sqrt{\frac{k}{2}}\,r$. Hence, as $s = fS$, we have to the same approximation, $s = S \cos n \sqrt{\frac{k}{2}} r$; $\frac{ds}{dx} = -\frac{S n \sqrt{k}}{\sqrt{2}} \cdot \sin n \sqrt{\frac{k}{2}} r \cdot \frac{x}{r}$; and $\frac{ds}{dy}$ $= -\frac{S n \sqrt{k}}{\sqrt{2}} \sin n \sqrt{\frac{k}{2}} r \cdot \frac{y}{r}$. Consequently substituting in (17), and putting the arc for the sine,

$$\frac{d\sigma_1^2}{dx^2} + \frac{d\sigma_1^2}{dy^2} + \frac{S n^2 k}{2}\left(x \frac{d\sigma_1}{dx} + y \frac{d\sigma_1}{dy}\right) = 0 \ldots\ldots\ldots\ldots\ldots (20).$$

Such a value of σ_1 is now to be found as will satisfy the equations (18) and (20) for small values of x and y. The equation $\sigma_1 = m \cos(gx + hy)$ will be found to answer. For substituting in (18) we get the equation of condition,

$$g^2 + h^2 - n^2 k = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21).$$

and by substituting in (20) we have,

$$m^2 (g^2 + h^2) \sin^2 (gx + hy) - \frac{S n^2 k m}{2} (gx + hy) \sin (gx + hy) = 0.$$

Hence, putting the arc for the sine,

$$m (g^2 + h^2) - \frac{S n^2 k}{2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22).$$

Comparing this equation with (21), it follows that $m = \frac{S}{2}$, and consequently that $\sigma_1 = \frac{S}{2} \cos (gx + hy)$. Similarly we should find that $\sigma_2 = \frac{S}{2} \cos (g'x + h'y)$. But since $s = \sigma_1 + \sigma_2$, we must have,

$$S \cos n \sqrt{\frac{k}{2}} r = \frac{S}{2} \cos (gx + hy) + \frac{S}{2} \cos (g'x + h'y).$$

Expanding to terms involving the squares of the small quantities,

$$n^2 k r^2 = (gx + hy)^2 + (g'x + h'y)^2$$
$$= (g^2 + g'^2) x^2 + (h^2 + h'^2) y^2 + 2 (gh + g'h') xy.$$

This equation accords with (21) if $g' = h$ and $h' = -g$. Thus we have

$$\sigma_1 = \frac{S}{2} \cos (gx + hy), \qquad \sigma_2 = \frac{S}{2} \cos (hx - gy).$$

It hence appears that σ_2 becomes identical with σ_1 by changing the directions of the axes of co-ordinates through 90°. Since $g^2 + h^2 = n^2 k$, we may assume that $g = n \sqrt{k} \cos \theta$, and $h = n \sqrt{k} \sin \theta$. Then,

$$\sigma_1 = \frac{S}{2} \cos \{ n \sqrt{k} (x \cos \theta + y \sin \theta) \},$$

$$\text{and } \sigma_2 = \frac{S}{2} \cos \{ n \sqrt{k} (x \sin \theta - y \cos \theta) \}.$$

As θ is quite arbitrary, let it equal 90°. Then $\sigma_1 = \frac{S}{2} \cos n \sqrt{k} y$, and $\sigma_2 = \frac{S}{2} \cos n \sqrt{k} x$. The axes of x and y are now evidently in the planes of symmetry, and these last values of σ and

σ_2 shew that at small distances from the axis of z the motion in one polarized ray is parallel to the plane zy, and in the other parallel to the plane zx. They may be said to be polarized in these planes.

The foregoing reasoning proves that a ray of common light is divisible into two rays polarized in planes at right angles to each other, and that these rays are necessarily *equal*. We have next to shew that they are each of *half* the intensity of the original ray. Since

$$\frac{ds}{dx} = \frac{d\sigma_1}{dx} + \frac{d\sigma_2}{dx}, \qquad \text{and} \quad \frac{ds}{dy} = \frac{d\sigma_1}{dy} + \frac{d\sigma_2}{dy},$$

by squaring,

$$\frac{ds^2}{dx^2} + \frac{ds^2}{dy^2} = \frac{d\sigma_1^2}{dx^2} + \frac{d\sigma_2^2}{dy^2} + 2\frac{d\sigma_1}{dx} \cdot \frac{d\sigma_2}{dx}$$

$$+ \frac{d\sigma_1^2}{dy^2} + \frac{d\sigma_2^2}{dy^2} + 2\frac{d\sigma_1}{dy} \cdot \frac{d\sigma_2}{dy}$$

$$= \left(\frac{d\sigma_1^2}{dx^2} + \frac{d\sigma_1^2}{dy^2}\right) + \left(\frac{d\sigma_2^2}{dx^2} + \frac{d\sigma_2^2}{dy^2}\right) \text{ by equation (16).}$$

Let u_1, v_1 be the velocities in x and y in one ray, u_2, v_2 in the other, and u, v in the original ray. Then since $u = \Phi \dfrac{ds}{dx}$, $\quad v = \Phi \dfrac{ds}{dy}$, $\quad u_1 = \Phi \dfrac{d\sigma_1}{dx}$, &c.

$$u^2 + v^2 = (u_1^2 + v_1^2) + (u_2^2 + v_2^2).$$

Hence the square of the velocity at any point of the undivided ray is equal to the sum of the squares of the velocities at the corresponding points of the polarized rays. This is true of the velocities in any tranverse section, and therefore true of the maximum transverse velocities. Measuring, therefore, the intensity of a ray by the sum of the squares of the maximum transverse velocities, it follows that the sum of the intensities of the polarized rays is equal to the intensity of the undivided ray, and, their intensities being equal, that each is of half the intensity of the undivided ray. This is conformable with experience.

Let us now proceed to estimate the intensity of a ray compounded of two rays polarized in opposite planes, *but not in the same phase.* As above we shall consider the intensity to depend entirely on the transverse velocity. In general for any ray not compounded,

$$u = \phi \frac{df}{dx}, \quad v = \phi \frac{df}{dy}, \quad \text{and} \quad \phi = \frac{m\lambda}{2\pi} \cos \frac{2\pi}{\lambda}(a't - z + c).$$

Now since the ratio $\dfrac{u}{v}$ is a function of x and y independent of z and t, the direction of the transverse velocity is independent of the phase of the direct velocity. Hence the transverse velocities in two rays polarized in opposite planes, which by hypothesis are at right angles to each other when the rays have the same phase, will be at right angles to each other whatever be the difference of phase. Let therefore for one ray

$$u_1 = \phi_1 \frac{df_1}{dx}, \quad v_1 = \phi_1 \frac{df_1}{dy}; \quad \text{and for the other,} \quad u_2 = \phi_2 \frac{df_2}{dx}, \quad v_2 = \phi_2 \frac{df_2}{dy}.$$

Then since they are polarized in opposite planes, $\dfrac{u_1}{v_1} = -\dfrac{v_2}{u_2}$, or $u_1 u_2 + v_1 v_2 = 0$. Consequently,

$$\frac{df_1}{dx} \cdot \frac{df_2}{dx} + \frac{df_1}{dy} \cdot \frac{df_2}{dy} = 0 \text{ .... (23).}$$

Suppose now that $u_1 = \frac{m\lambda}{2\pi}\cos\frac{2\pi}{\lambda}(a't - z)\frac{df_1}{dx}$,

$$v_1 = \frac{m\lambda}{2\pi}\cos\frac{2\pi}{\lambda}(a't - z)\frac{df_1}{dy},$$

$$u_2 = \frac{m'\lambda}{2\pi}\cos\frac{2\pi}{\lambda}(a't - z + c)\frac{df_2}{dx},$$

$$v_2 = \frac{m'\lambda}{2\pi}\cos\frac{2\pi}{\lambda}(a't - z + c)\frac{df_2}{dy}.$$

Then $u_1 + u_2 = \frac{\lambda}{2\pi}\cos\frac{2\pi}{\lambda}(a't - z)\left(m\frac{df_1}{dx} + m'\cos c\frac{df_2}{dx}\right) - \frac{\lambda}{2\pi}\sin\frac{2\pi}{\lambda}(a't - z)\, m'\sin c\frac{df_2}{dx}$, and if

$$\tan\frac{2\pi\theta'}{\lambda} = \frac{m'\sin c\frac{df_2}{dx}}{m\frac{df_1}{dx} + m'\cos c\frac{df_2}{dx}}.$$

$$U \text{ or } u_1 + u_2 = \frac{\lambda}{2\pi}\left\{m^2\frac{df_1^2}{dx^2} + 2mm'\cos c\frac{df_1}{dx}\cdot\frac{df_2}{dx} + m'^2\frac{df_2^2}{dx^2}\right\}^{\frac{1}{2}}\cos\frac{2\pi}{\lambda}(a't - z + \theta').$$

$$\text{So if } \tan\frac{2\pi\theta''}{\lambda} = \frac{m'\sin c\frac{df_2}{dy}}{m\frac{df_1}{dy} + m'\cos c\frac{df_2}{dy}},$$

$$V \text{ or } v_1 + v_2 = \frac{\lambda}{2\pi}\left\{m^2\frac{df_1^2}{dy^2} + 2mm'\cos c\frac{df_1}{dy}\cdot\frac{df_2}{dy} + m'^2\frac{df_2^2}{dy^2}\right\}^{\frac{1}{2}}\cos\frac{2\pi}{\lambda}(a't - z + \theta'').$$

The two velocities U and V are not in this case in the same phase, and consequently the transverse motion of a given particle, instead of being in a straight line, is in an ellipse or a circle. The effects of the resolved parts of the velocities in the directions of the axes of x and y may be assumed to be independent of each other, and the intensity of the compound ray will consequently be as the sum of the squares of the maximum values of U and V; that is, as

$$m^2\left(\frac{df_1^2}{dx^2} + \frac{df_1^2}{dy^2}\right) + 2mm'\cos c\left(\frac{df_1}{dx}\cdot\frac{df_2}{dx} + \frac{df_1}{dy}\cdot\frac{df_2}{dy}\right) + m'^2\left(\frac{df_2^2}{dx^2} + \frac{df_2^2}{dy^2}\right),$$

which on account of the equation (23) is independent of $\cos c$. Hence the intensity is the same whatever be the difference of phase, and therefore the same as when the two polarized rays have the same phase. This agrees with experience.

Let us now proceed to consider the bifurcation of a *polarized* ray; for instance, the ray whose condensation is σ_1. Suppose it separated by any means into two rays whose condensations are τ_1 and τ_2. We shall assume, as in the case of a ray of common light, that the sum of the condensations at corresponding points of the divided rays is equal to the condensation at the corresponding point of the original ray, and that the velocities at these points of the divided rays are in directions at right angles to each other. We have then, by what has gone before,

$$\sigma_1 = \tau_1 + \tau_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24).$$

$$\frac{d\tau_1}{dx}\cdot\frac{d\tau_2}{dx} + \frac{d\tau_1}{dy}\cdot\frac{d\tau_2}{dy} = 0 \ldots\ldots\ldots\ldots (25).$$

and τ_1, τ_2 must respectively satisfy the equations,

$$\frac{d^2\tau_1}{dx^2} + \frac{d^2\tau_1}{dy^2} + n^2 k \tau_1 = 0 \text{ .... } (26),$$

$$\frac{d^2\tau_2}{dx^2} + \frac{d^2\tau_2}{dy^2} + n^2 k \tau_2 = 0 \text{ .... } (27).$$

From the system of equations (24), (25), (26), and (27), it is required to determine the forms of the functions expressing the values of τ_1 and τ_2, that expressing the value of σ_1 being supposed to be known. It does not appear that this can be done generally; but as before, approximate solutions may be obtained applicable to parts of the rays contiguous to the axis. The process for this purpose will be analogous to that applied to the ray of common light.

Let $\tau_1 = m \cos(gx + hy)$, and $\tau_2 = m' \cos(g'x + h'y)$.

Then equations (26) and (27) are satisfied if

$$g^2 + h^2 - n^2 k = 0, \quad \text{and } g'^2 + h'^2 - n^2 k = 0.$$

Also equation (25) is satisfied if $gg' + hh' = 0$. And these three equations of condition are satisfied if $g = n\sqrt{k}\cos\theta$, $h = -n\sqrt{k}\sin\theta$, $g' = n\sqrt{k}\sin\theta$, $h' = n\sqrt{k}\cos\theta$. Hence since we have shewn that when the approximation is carried to the second powers of x and y, $\sigma_1 = \frac{S}{2}\cos n\sqrt{k}\,y$, we shall have by equation (24),

$$\frac{S}{2}\cos n\sqrt{k}\,y = m\cos(nx\sqrt{k}\cos\theta - ny\sqrt{k}\sin\theta) + m'\cos(nx\sqrt{k}\sin\theta + ny\sqrt{k}\cos\theta) \text{ .... } (28).$$

Hence, expanding to the second powers of x and y,

$$\frac{S}{2}\left(1 - \frac{n^2 k y^2}{2}\right) = m + m' - \frac{mn^2k}{2}(x\cos\theta - y\sin\theta)^2 - \frac{m'n^2k}{2}(x\sin\theta + y\cos\theta)^2.$$

Therefore $\frac{S}{2} = m + m'$,

and $\frac{S}{2}y^2 = (m\cos^2\theta + m'\sin^2\theta)\,x^2 - 2xy\sin\theta\cos\theta\,(m - m') + (m\sin^2\theta + m'\cos^2\theta)\,y^2$,

or, substituting $m + m'$ for $\frac{S}{2}$,

$$(m\cos^2\theta + m'\sin^2\theta)(y^2 - x^2) + 2xy\sin\theta\cos\theta\,(m - m') = 0 \text{ .... } (29).$$

It appears, therefore, that equation (28) is not satisfied to second powers of x and y for general values of these variables, and the functions assumed for τ_1 and τ_2 are consequently true in general only to first powers of x and y. It is however important to remark that the equation (29), being put under the following form,

$$\frac{y^2}{x^2} + \frac{2y}{x}\cdot\frac{\sin\theta\cos\theta\,(m - m')}{m\cos^2\theta + m'\sin^2\theta} - 1 = 0 \text{ .... } (30),$$

shews that for two directions at right angles to each other, the assumed values of τ_1 and τ_2 are true to the second powers of x and y. These two directions may be presumed to be the directions of the planes of polarization of the two rays. But because

$$\tau_1 = m \cos n\sqrt{k}\,(x\cos\theta - y\sin\theta), \text{ and } \tau_2 = m'\cos n\sqrt{k}\,(x\sin\theta + y\cos\theta),$$

these two planes evidently make angles θ and $\frac{\pi}{2} - \theta$ with the plane of polarization of the original ray. Hence putting $\cot\theta$ for $\frac{y}{x}$ in equation (30), we find

$$\frac{m}{m'} = \tan^2\theta,$$

and we also have $$\frac{S}{2} = m + m'.$$

Therefore $$m = \frac{S}{2}\sin^2\theta, \text{ and } m' = \frac{S}{2}\cos^2\theta.$$

The polarized ray is consequently divided in general into two unequal rays, the values of which are assigned by these equations. If $\theta = 45^0$, the two rays are equal; which accords with experience.

Suppose a polarized ray to be incident at the angle of complete polarization on a reflecting surface, and let θ be the angle which the plane of incidence makes with the plane of polarization of the incident ray. Then A being the portion of the ray transmitted without bifurcation, which we will suppose to be independent of θ, and I the portion bifurcated, the transmitted ray will be $A + I\sin^2\theta$, and the reflected ray $I\cos^2\theta$. If another equal ray, polarized in a plane at right angles to the plane of polarization of the former be incident in the same direction, the transmitted portion will be, $A + I\cos^2\theta$, and the reflected portion $I\sin^2\theta$. These two incident rays make up, according to our Theory, a ray of common light, the transmitted portion of which is $2A$, and the reflected portion $I\cos^2\theta + I\sin^2\theta$, or I, which is independent of θ, as we know from experience it should be. Respecting the law above found for the intensities of the two rays into which a polarized ray is separated, Sir John Herschel remarks in his Treatise on Light in the *Encyclopædia Metropolitana*, (Art. 850), "We must receive it as an empirical law at present, for which any good theory of polarization ought to be capable of assigning a reason *a priori*." Such a reason is given by the Theory I am advocating.

Two polarized rays formed by the separation into two parts of a polarized ray derived immediately from common light, possess in some respects the properties of polarized rays of the latter kind, for instance, the two rays pursuing the same paths will not interfere whatever be the difference of phase. This may be proved by the very same reasoning by which it has been already proved that two rays of first polarization do not interfere, the reasoning being purposely adapted to the case when m and m' are unequal. At the same time the rays of second polarization differ in this respect, that if they meet in the same phase they compose a plane polarized ray. When $\theta = 45^0$, we found that the two rays were equal. Yet their composition would form a polarized ray, whilst two equal rays of the first polarization meeting in the same phase would compose a ray of common light.

According to this Theory circularly and elliptically polarized light consists of two oppositely polarized rays differing in phase, the two rays when in the same phase constituting a polarized ray of the first kind. The reason Fresnel's Rhomb does not produce elliptically polarized light, when common light is used, is that common light may be supposed to consist of two rays in opposite polarizations, which produce exactly complementary effects. For the same reason common light produces no coloured rings by transmission through a thin plate of a uniaxal or biaxal crystal cut nearly perpendicularly to its axis. Each of the polarized rays, of which common light may be supposed to be composed, does in fact produce coloured rings, but the two sets being exactly complementary, the colours disappear.

Circularly and elliptically polarized light is capable of reflexion at the analyzing plate (in the experiment above alluded to) because it consists of two rays polarized in opposite planes, which cannot therefore both coincide at the same time with the plane of incidence. The analyzing plate is necessary for the production of the colours, because the rays come out of the crystal in opposite polarizations, and therefore not interfering. Those that fall on the plate in the same phase constitute a single ray polarized in the plane of original polarization, and are therefore incapable of reflexion when the plane of incidence on the plate is perpendicular to the plane of original polarization. The rest of the rays fall on the plate in the form of circularly or elliptically polarized light, and consequently from what we have already seen, are capable of reflexion. This explanation does not require the supposition of the loss of half an undulation.

The Theory might be compared with experiment in many other instances; but perhaps those I have adduced may suffice to gain for it the favourable consideration of mathematicians. I will only add, that having applied it in some degree to the phænomena of Double Refraction, I find that it leaves the mathematical Theory of Fresnel unaltered, while it offers in several respects a different physical explanation of the facts. Before I conclude it may also be proper to remark, that I have argued on the supposition that the quantity k which enters into this Theory is a constant. The reasoning would remain the same if k were a function of λ, provided it did not vary with the intensity of the ray.

Cambridge Observatory,
May 25, 1846.

XXIX. *On the Structure of the Syllogism, and on the Application of the Theory of Probabilities to Questions of Argument and Authority.* By Augustus De Morgan, *Sec.* R.A.S., *of Trinity College, Cambridge, Professor of Mathematics in University College, London.*

[Read Nov. 9, 1846.]

Since the time when the Aristotelian syllogism ceased to be regarded as an all-sufficient instrument of inquiry, it has remained precisely in the state in which those who are called the *schoolmen* left it. I have never heard* of any attempt to ascertain whether the forms which his followers derived from the writings of the great master were the perfection of system and simplicity which they were supposed to be. The uninquiring adherence of all writers on logic to the model of the middle ages, proves one of two things: either that the model is human perfection, or that the authority of the ancients has been followed as of course in the forms of logic long after it has been abandoned in every other part. With such an alternative, it is not presumption to venture upon the examination: and this is the more apparent when we consider that the general impression among writers seems to be that there cannot exist *any* other theory of the syllogism except that derived from Aristotle. If another can be produced, which is but self-consistent, true, and comprehensive, the tacit assertion of all writers is overthrown, whether that system be or be not judged superior to the one handed down.

I here venture to propose a derivation and classification of the forms of the syllogism, differing very widely from that in use.

Section I. *On the meaning of the simple term.*

A term, or name, is merely the word which it is lawful to apply to any one of a collection of objects of thought: and, in the language of Aristotle, that name may be *predicated* of each of those objects. He uses this word predicate only as "that which *can* be said of." When in later times the negative proposition "X is not Y" was said to have Y for its *predicate*, the word ought to have been *non-predicate*, or some equivalent. The proper predicate is not-Y, which I shall call the *contrary* of Y.

When we use a term, such as "man," we predicate, in Aristotle's sense of the word, of every individual *which the notion can suggest*, of John, Thomas, William, &c. If we extend the word, and allow Y to be called the predicate of "X is *not* Y," we must then affirm that the word "man" predicates of *every* object of thought, either affirmatively or negatively: affirmatively of John or Thomas, negatively of a certain tree, or quadruped, or book. Every name then, in this sense, predicates of every thing: "X is either Y or not-Y" is a proposition of universal identity.

The express introduction and consideration of contraries ought, I think, to have followed the extension of the word predicate. Aristotle rejects and then admits: not-man, he says, is not a name; and then he calls it an *aorist* name, which can be predicated both of existent and non-existent things. I deny the justice of this distinction, for two reasons.

Names in logic are used subjectively; they are the representations of the notion in the mind. Now *man* and *not-man* are equally the names of things which, objectively speaking, are non-existent. *Not-man*, Aristotle would say, is a name which can be predicated of the speaking bird

* See the Addition at the end of this Paper.

and the singing tree in the eastern fable: but surely, with as much justice, *man* may be predicated of the Shakspeare who wrote Paradise Lost, or the Cæsar who conquered Darius.

In the next place, it is not true that the aorist or indefinite character of the mere contrary actually exists in the use which we make of language. Writers on logic, it is true, do not find elbow-room enough in anything less than the whole universe of possible conceptions: but the universe of a particular assertion or argument may be limited in any matter expressed or understood. And this without limitation or alteration of any one rule of logic. Let every one of the possible points of space have one or more of the names X, Y, &c.: then if we can say, "No X is Y," of course we can say "No Y is X." But this is equally true if, by an understanding to that effect, the universe of our proposition be one square described in a certain plane. Divide the points of this square into Xs and not-Xs, and the not-X is no more an aorist term than the X.

By not dwelling upon this power of making what we may properly (inventing a new technical name) call the *universe* of a proposition, or of a name, matter of express definition, all rules remaining the same, writers on logic deprive themselves of much useful illustration. And, more than this, they give an indefinite negative character to the *contrary*, as Aristotle did when he said that not-man was not the name of anything. Let the universe in question be "man:" then *Briton* and *alien* are simple contraries; alien has no meaning of definition except not-Briton. But we cannot say that either term is positive or negative, except correlatively. As to a claim of right to be considered a prisoner of war, for instance, alien is the positive term, and Briton the negative one. We separate formal logic from language, if we refuse to admit this. In many cases, the language has the term which signifies the contrary, and wants the direct term: as in the word *parallels*, for example. To this day the word *intersectors* has not found its way into the idiom of geometry. In one case we give a name to the thing of course, and define the exception by means of a contrary: in another we find it more convenient to reverse the process. I hold that the system of formal logic is not well fitted to our mode of using language, until the rules of direct and contrary terms are associated: the words direct and contrary being merely correlative. Those who teach Algebra know how difficult it is to make the student fully aware that a may be the negative quantity, and $-a$ the positive one. There is a want of the similar perception in regard to direct and contrary terms.

Throughout this paper, I shall use the small letters x, y, z, &c. for names contrary to those represented by the capitals X, Y, Z, &c. Thus "every thing *in the universe* is either X or x," "No X is x," &c. are identical propositions.

SECTION II. *On the simple proposition.*

THERE is no need to dwell on the usual matters given as to the distinction of universal and particular, or of positive and negative. But, I think it cannot be denied, that the distinctions may, for logical purposes, be considered as accidents of language. Any proposition which is either of the four in one language, may be either of the others in another. Our language has, say the names X and Y, and suppose that "Every X is Y" is true. Another language translates X by X', but has no term for Y, but only y' for its contrary; the proposition is then "No X' is y'." In a third language Xs have no specific name; they appear but as individuals of the name X'': the proposition is then "Some X''s are Y''s." But if the last language had only possessed the name y'', it would have been "Some X''s are not y''s."

Very often, having established such a proposition as "Some Xs are Ys," we, for that reason, distinguish *those* Xs by a separate name, Z: and then we have "*Every* Z is Y." If language were copious enough, particular propositions would seldom occur: and the idioms of every tongue are probably influenced by its power of supplying universal terms, or of converting particulars into the form of universals.

I shall use, as is usual, the letters A and E to signify positive and negative universals: and I and O for the corresponding particulars: but with a modification presently noticed. I shall also use the following notation, without which I should hardly have had patience for the many hundreds of cases which this paper has required.

$P\,)\,Q$	signifies	Every P is Q.
$P\,.\,Q$	....	No P is Q.
PQ	....	Some Ps are Qs.
$P:Q$	....	Some Ps are not Qs.

I have taken for the convertible propositions, the symbols $P\,.\,Q$ and PQ, which the algebraist is accustomed to consider as identical with $Q\,.\,P$ and QP: the same thing is true under these meanings. But $P\,)\,Q$ and $P:Q$, which are also used in arithmetic and algebra, convey no idea of convertibility.

All expressions that have any meaning can of course be reduced to one of these forms. Aristotle denies this, and divides all expression into significative and enunciative, meaning by the latter that in which there is truth or falsehood. Thus prayer, he says, is speech, but neither true nor false. This is surely not correct;—*Deliver us from evil* may be either "To be delivered from evil is our prayer," or "We are of those who pray to be delivered from evil," or "Evil is a thing we pray to be delivered from." Or, as the text, it would be "Deliver us from evil, is the passage on which I mean to comment:" and the sermon would probably give all the enunciations above. In a request, command, inquiry, or announcement, the tone* of voice predicates.

In classifying all possible predications by means of two names Y and X, their contraries must be included. We must therefore consider all the relations that may exist between Y and X, X and y, y and x, x and Y. Between each of these there are six modes of enunciation: thus between P and Q we have

$$\underset{1}{P\,)\,Q},\quad \underset{2}{Q\,)\,P},\quad \underset{3}{P\,.\,Q = Q\,.\,P},\quad \underset{4}{PQ = QP},\quad \underset{5}{P:Q},\quad \underset{6}{Q:P}.$$

But it will be best to arrange these by contradictories, or propositions one of which must be true and the other false: as $P\,)\,Q$ and $P:Q$, $Q\,)\,P$ and $Q:P$, $P\,.\,Q$ and PQ. These six modes applied to each of the four variations of subjects, give twenty-four varieties, which are reducible to eight, being identical three and three, as follows:

$$X\,)\,Y = X\,.\,y = y\,)\,x \qquad X\,.\,Y = X\,)\,y = Y\,)\,x.$$
$$X:Y = Xy \;\; = y:x. \qquad XY = X:y = Y:x.$$
$$Y\,)\,X = Y\,.\,x = x\,)\,y. \qquad x\,.\,y = x\,)\,Y = y\,)\,X.$$
$$Y:X = Yx \;\; = x:y. \qquad xy = x:Y = y:X.$$

Though the use of the great and small letters may suit the eye, these lines should be read thus: "Every X is Y" is identical with "no X is not-Y," and "every not-Y is not-X," and so on. These eight modes may all be derived from the four Aristotelian modes by changing both terms into contraries; which suggests the following notation:

(A)	$X\,)\,Y$	$x\,)\,y = Y\,)\,X$	(a).
(O)	$X:Y$	$x:y = Y:X$	(o).
(E)	$X\,.\,Y$	$x\,.\,y$	(e).
(I)	XY	xy	(i).

* To call a person by his name is a proposition, perhaps more. There is certainly the full meaning of a syllogism in it. When a person calls—John! no one can say that any part of the following is not implied: "John is the person I want to speak to; you are John; therefore, you are the person I want to speak to." The least that can be said is, that he states the premises, and leaves John to draw the conclusion.

This notation is established with reference to the order XY: the inversion of the order interchanges small and large letters.

The propositions (A) and (O) are the assertion and denial of complete inclusion of the first name in the second. And (a) and (o), the assertion and denial of complete inclusion of contrary in contrary, are, as appears, equivalent to the assertion and denial of complete inclusion of the second in the first. Again, (E) and (I) are the assertion and denial of complete non-interference, or that each name is wholly contained in the contrary of the other. But (e) and (i), the propositions which I propose to add to those commonly received, may be explained as follows:

The proposition (i) or xy, affirming that there are individuals in the universe of the proposition which are neither Xs or Ys, merely affirms that X and Y are *not* contraries, and do not between them contain the universe. The contradiction of this, (e), or $x\,.\,y$, affirming that it is false that there are any individuals which are neither X or Y, might seem at first sight to declare that X and Y ***are*** contraries. But it is not so, since the preceding is perfectly consistent with there being individuals which are both Xs and Ys. In fact, to express that X and Y are contraries we must have both $x\,.\,y$ and $X\,.\,Y$.

The following tables show the relations of these propositions.

	Denies	Contains or is contained in	Can exist with or without		Denies	Contains or is contained in	Can exist with or without
A	OEe	Ii	ao	E	IAa	Oo	ei
O	A	Ee	$Iaio$	I	E	Aa	$Oeio$
a	oeE	Ii	AO	e	iaA	Oo	EI
o	a	Ee	$iAIO$	i	e	Aa	$oEIO$

Or each universal proposition denies, besides its own contradictory, the two universals of a different name; contains both particulars of the same name; and is independent of the other universal of the same name and its contradictory. Each particular proposition denies only its own contradictory; is contained in both the universals of the same name; and is independent of either of the other three particulars, as well as of the other universal (not its own contradictory) of a contrary name.

It is usual in modern works to say that a term which is universally spoken of is *distributed*. But in truth every proposition distributes, wholly or partially, among the individuals of the predicate, or of its contrary. It will be sufficient to call a term universal or particular, according to the manner in which it is spoken of. It will then be found that every proposition speaks in different ways of each term and its contrary; making one particular or universal, according as the other is universal or particular. The manner in which the subject is spoken of is expressed; as to the predicate, it is universal in negatives but particular in affirmatives. And of the two terms and their contraries, each proposition speaks universally of two, and particularly of two.

Let S signify that the subject is changed into its contrary, P, the same of the predicate. Let C signify that the copula is changed, from positive to negative, or *vice versâ*. Let T denote transformation or interchange of subject and predicate: to avoid confusion, either this must be done last, or the original subject and predicate are to retain those names after the transformation. Then we have the following tables, L standing for letting the proposition remain unaltered

L	T	SP	SPT	L	P	PT	S	ST	P
PC	SCT	SC	PCT	PC	C	$SPCT$	SPC	CT	C

Changes and combinations of changes that are written under one another, are *in all cases* of the same effect: thus by writing PT and $SPCT$ under one another, I mean that change of subject, predicate, copula, and order, are always of the same effect as change of predicate and order only. Thus the operation $SPTC$ performed upon $X\,)\,Y$ gives $y\,.\,x$. But PT only gives $y\,)\,X$ and $y\,.\,x = y\,)\,X$, as appears above. Further, when a *single* line separates two vertical pairs, the two pairs are identical when performed upon *inconvertible* propositions: when a *double* line, the same with respect to *convertible* propositions. Thus, as to convertible propositions, L, PC, T, SCT are all of the same effect: as to inconvertible propositions, T, SCT, SP and SC.

There is a point which developes itself very strongly when we come to consider the transformations upon instances; namely, the distinction between the assertion of a proposition subjectively and objectively. The former mode is that which is always presumed: but in actual use of logic the distinction must be drawn.

When we say, Every X is Y, as a proposition with meaning, and with or without truth as the case may be, we treat neither X nor Y as having any other existence except that which our minds give them: but we imply that if X have any such other existence, so has Y. But the syllogism "X) Y and Y) Z therefore X) Z" is not valid merely by understanding X and Z to be taken in the conclusion as in the premises. The middle term *must exist:* not necessarily objectively, but it must have a positive existence. It is no syllogism to say that X is Y, if there be such a thing, and Y (if &c.) is Z; therefore X is Z. And yet there is no offence against any of the ordinary rules of logic: the middle term is strictly *middle;* it is "Y, on the condition that Y exists" in both. Thus "Homer was a perfect poet (if ever there were one); a perfect poet (if &c.) is faultless in morals: therefore Homer was &c." The premises will sometimes be admitted; but they do not prove the conclusion: the proper conclusion is a dilemma, "Either Homer was faultless in morals, or there never was a perfect poet." The existence here spoken of is objective: but the same thing applies to purely subjective cases. The terms of the conclusion may be conditional: but inference requires that the middle term should be unconditional. Every X (if ever X existed) is Y; every Y is Z (if ever Z existed): therefore every X (if ever X existed) is Z (if ever Z existed). This is a good syllogism: but Y is here absolute.

When the syllogism can be converted into another, having for its middle term the contrary of the first middle term, the same absolute existence must be claimed for the contrary. And here again I remind the reader that the absolute existence spoken of is existence within the *universe* of the propositions. Thus X) Y and Y) Z give X) Z, or y) x and z) y give z) x. A positive existence is then required both for Y and y. There is an extreme case; y may not exist, that is, Y may contain the universe; but then Y and Z are identical, and the conclusion X) Z is identical with X) Y and z) x contains nothing.

Whatever sort of existence is spoken of is tacitly claimed for the terms of a proposition by the proposition itself: the refusal of this claim, or the denial by assertion of non-existence, being a distinct thing from denial by contradiction. A certain meadow (the universe of the proposition) is flooded during the hay-harvest: the proposition "No part of the crop that was not flooded was not saved" (of the form $x . y$) means logically that all which was not flooded || was saved, that all which was not saved || was flooded, and that part may have been both flooded and saved. Some reflexion (for want of habit of dealing with triple negatives makes the proposition rather complicated) will shew that a person who is apt to think objectively of propositions, as all do who are not trained in logical considerations, is much more likely to require the insertion of the words (*if any*) in two places (||) than he would be if the proposition were presented in the more simple form, "All the dry crop was saved." Probably such a person would not require the conditional words here, merely because he would take it that the proposition asserts that some *was* dry: reserving the right to deny by non-existence if there were none.

I suppose it is hardly necessary to remark that, in propositions, asserted as true, the same sort of existence is claimed for both terms: for instance, that there is no objective first term with a subjective second one. In such a proposition as "he is good" we may certainly say that "good" by itself is a purely subjective notion; a state of the mind in regard to an external object. But good is not the term of the proposition; it is he (an external object) is one of these external objects to which the mind attaches the idea of good. I can conceive opposition to this: what I say is that the opposition is not to me, but to the universal maxims of technical logic. For all writers admit that YX necessarily follows from XY: which cannot be if Y be a name of the state of the mind and X of an external object. Most of the Romans were brave; therefore some brave [men] were Romans. No hint is ever given by writers on logic of the necessity, previously to conversion, of attaching the subjective notion to an object.

SECTION III. *On the quantity of propositions.*

THE logical use of the word *some*, as merely "more than none," needs no further explanation. Exact knowledge of the extent of a proposition would consist in knowing, for instance in "some Xs are not Ys", both what proportion of the Xs are spoken of, and what proportion exists between the whole number of Xs and of Ys. The want of this information compels us to divide the exponents of our proportions into 0, more than 0 not necessarily 1, and 1. An algebraist learns to consider the distinction between 0 and quantity as identical, for many purposes, with that between one quantity and another: the logician must (all writers imply) keep the distinction between 0 and a, however small a may be, as sacred as that between 0 and $1 - a$: there being but the same form for the two cases. We shall now see that this matter has not been fully examined.

Inference must arise from bringing each two things which are to be compared into comparison with a third. Many comparisons may be made at once, but there must be this process in every one. When the comparison is that of identity, of *is* or *is not*, it can only be, in its ultimate or individual case, one of the two following;—"This X is a Y, this Z is the very same Y, therefore this X is this Z; or else "This X is a Y, this Z is not the very same Y, therefore this X is not this Z." And collectively, it must be either "Each of these Xs is a Y; each of these Ys is a Z; therefore each of these Xs is a Z;" or else "Each of these Xs is a Y, no one of these Ys is a Z, therefore no one of these Xs is a Z."

All that is essential then to a syllogism is that its premises shall mention a number of Ys, of each of which they shall affirm either that it is both X and Z, or that it is one and is not the other. The premises may mention more: but it is enough that this much can be picked out; and it is in this last process that inference consists.

Aristotle noticed but one way of being sure that the same Ys are spoken of in both premises: namely, by speaking of all of them in one at least. But this is only a case of the rule: for all that is necessary is *that more* Ys *in number than there exist separate* Ys *shall be spoken of in both premises together*. Having to make $m + n$ greater than unity, when neither m nor n is so, he admitted only that case in which one of the two m or n, *is* unity and the other is anything except 0. Here then are two syllogisms which ought to have appeared, but do not; and there are others;—

Most of the Ys are Xs	Most of the Ys are Xs
Most of the Ys are Zs	Most of the Ys are not Zs
∴ Some Xs are Zs	∴ Some of the Xs are not Zs.

And instead of most, or $\frac{1}{2} + a$, of the Ys, may be substituted any two fractions which have a sum greater than unity. If these fractions be m and n, then the *real* middle term is *at least* the fraction $m + n - 1$ of the Ys. It is not really even necessary that each Y should enter in one premiss or the other: for more than the fraction $m + n - 1$ of the whole may be found in each.

And in truth it is this mode of syllogizing that we are frequently obliged to have recourse to; perhaps more often than not in our universal syllogisms. "*All* men are capable of some instruction; all who are capable of any instruction can learn to distinguish their right and left hands by name; therefore all men can learn to do so." Let the word *all* in these two cases mean only *all but one*, and the books on logic tell us with one voice that the syllogism has particular premises, and *no conclusion can be drawn*. But in fact, idiots are capable of no instruction, many are deaf and dumb, some are without hands: and yet *a* conclusion is admissible. Here m and n are each very near to unity, and $m + n - 1$ is therefore near to unity. Some will say that this is a probable conclusion: that in the case of any one person it means there is the chance m that he can receive instruction, and

n that one so gifted can be made to name his right and left hand: therefore $m \times n$ (very near unity) is the chance that this man can learn so much.

But I cannot see how in this instance the probability is anything but another sort of inference from the demonstrable conclusion of the syllogism, which must exist, under the premises given. Besides which, even if we admit the syllogism as only probable with regard to any one man, it is absolute and demonstrative in regard to the whole number of men with which it concludes.

This is not the only case in which the middle term need not enter universally: this however is matter for the next Section: see also the Addition at the end. I now go on to another point.

Mathematicians, as such, are supposed to have a tendency to admit nothing but demonstration, and to become insensible to ordinary evidence. Instances of this there may be, though whether the temperament led them to mathematics, or mathematics brought on the temperament, has certainly not been inquired into by those who make the charge. But to me it seems very clear, that if *ordinary logic* do not produce this temperament in those who study it, there must be correctives elsewhere. It is the only science I ever came in contact with, in which the want of demonstration is formally made to amount to absolute rejection without further consideration. The mathematician, having a given formula on hand, can and does satisfy himself not only that it is true, if it be true, but that it is false, if it be false. But the young logician, when his premises do not yield their inference legitimately, drops that inference as a fallacy: and few indeed are the books which speak of the distinction between an invalid inference and a false conclusion in terms which shew that the same distinction is a well recognized topic of the subject. It is, I think, for the mathematician to try to correct the habit arising out of this omission, namely, the confusion between paralogism and falsehood: and also to introduce his notions of probability, so as to establish some little power of discriminating between the various degrees of fallacy which are all called by one name, whether that name be falsehood or not.

If some Ys be Xs and some Ys be Zs we have no right to draw any inference: at least so says many a one who thinks that mathematics would render him insensible to the evidence of high probability.

It will become of importance to reflect what the difference may be between the habit of not looking for high probability when it exists, and that of not acknowledging it when it ought to be seen—as soon as the following case is considered.

Let the whole number of Ys be s, the numbers which are Xs and Zs being severally m and n. Nothing is known or suspected as to whether a Y being X is favourable or unfavourable to its being also Z. It is required to ascertain what chance there is that there are Ys which are both Xs and Zs, $m + n$ not being so great as s. That is, when from "some Ys are Xs and some Ys are Zs" we decline to admit that some Xs are Zs, what is the chance that we reject a truth?

Let p_q signify the number of combinations of p out of q. If we pick out any m Ys to be Xs, there are n_{s-m} ways in which the Zs may be found among the rest. Consequently $m_s \times n_{s-m}$ is the whole number of ways in which "Some Xs are Zs" is false. But the whole number of possible cases is $m_s \times n_s$; whence the chance of the falsehood is

$$\frac{n_{s-m}}{n_s}, \quad \text{or} \quad \frac{[s-m]\,[s-n]}{[s]\,[s-m-n]}$$

where $[p]$ means $1.2.3 \ldots p$. If $s - m - n$ be not inconsiderable the substitution of

$$\sqrt{2\pi}\,.\,p^{p+\frac{1}{2}}\,\epsilon^{-p} \text{ for } [p] \text{ gives}$$

$$\sqrt{\frac{(s-m)\,(s-n)}{s\,(s-m-n)}}\,.\,\frac{(s-m)^{s-m}(s-n)^{s-n}}{s^s\,(s-m-n)^{s-m-n}}, \quad \text{or} \quad \sqrt{\frac{(1-\mu)\,(1-\nu)}{1-\mu-\nu}}\,.\left\{\frac{(1-\mu)^{1-\mu}\,(1-\nu)^{1-\nu}}{(1-\mu-\nu)^{1-\mu-\nu}}\right\}^s,$$

if μ and ν be the fractions which m and n are of s. For the calculation of this we have

$$\text{comm. log} \frac{(1-\mu)^{1-\mu}(1-\nu)^{1-\nu}}{(1-\mu-\nu)^{1-\mu-\nu}} = -\cdot 4342945 \left\{ \frac{(\mu+\nu)^2-\mu^2-\nu^2}{1.2} + \frac{(\mu+\nu)^3-\mu^3-\nu^3}{2.3} + \ldots \right\},$$

with which series we are to proceed until the term last obtained gives a sufficiently small product after multiplication by s.

Now first observe, that since the base of the s^{th} power is less than unity, s may, for any given values of μ and ν, be made great enough to make the probability that "Some Xs are Zs" is false as small as we please. Hence we have a right to assert the following:—

If, to our knowledge, a perceptible fraction of the Ys be Xs, and a perceptible fraction be Zs, and if the number of Ys be great beyond perception; and if moreover we know nothing, except what has just been stated, for or against a Y which is X being or not being Z,—we ought to treat it as a moral certainty that some one or more of those Xs which are Ys are also Zs.

I do not say that the above case is a fair statement of the usual conditions under which the syllogism with particular premises appears: nor does it matter to my argument whether it be or not. What I say is, that it *is* a fair statement of the circumstances under which the rejection of the conclusion "some Xs are Zs" is ordered to be made in books of logic.

If μ and ν be small, the *number of places of figures* in x, x to 1 being the odds in favour of one or more Xs being Zs, may be stated as the integer next above $\frac{43}{99}\mu\nu s$ *at least*. If s were 1000, and μ and ν each $\frac{1}{10}$, this would be five; or the odds 10,000 to 1. Calculate more strictly, and it will come out nearly 70,000 to 1. If a person then should distribute 100 sovereigns and 100 shillings at hazard among a crowd of 1000 persons, not giving any one more than one coin of either sort, it is about 70,000 to 1 that he gives *one or more* of them a guinea.

But to shew how wide the cases may be, which are *equally* rejected, let us take the following supposition, which perhaps more nearly represents, in many cases, the rationale of the argument. Representing all the Ys by aliquot parts of a certain line, it may be supposed that the Xs have some connexion of contiguity in time, place, or other circumstance: let it then be a collection of successively contiguous Ys which are Xs: and the same of the Zs. The state of the case is now as follows.

There is a line of given length, which we shall take for our unit. Two given lines, each less than the first line, are laid down in it at hazard, any one position of either being as likely as any other. Let the lengths of the lines be μ and μ': it is required to find the probability that μ and μ' shall not have a part exceeding ν in common.

First, let $\mu + \mu'$ be less than 1, so that the lesser lines can be quite clear of one another. We are to investigate the probability that they shall be so. Let μ be on the left and μ' on the right; and let x and x' be the distances of their left and right extremities from the corresponding ends of the unit. We must then have $x + x' + \mu + \mu'$ less than unity, in order that the lines may be clear of one another. Now since x may be anything less than $1 - \mu$, and x' anything less than $1 - \mu'$, and all possible positions are equally likely, it will follow that the chances of the lines called x and x' lying between x and $x + dx$, and x' and $x' + dx'$, will be $dx \div (1-\mu)$ and $dx' \div (1-\mu')$, and the chance of the joint event is

$$\frac{dx \,.\, dx'}{(1-\mu)(1-\mu')}.$$

If we integrate this over all positive values of x and x' in which $x + x'$ is less than $1 - \mu - \mu'$, we shall have the probability in favour of the two lesser lines having no point in common when μ is on the left, and μ' on the right. The result is easily shown to be the half of

$$\frac{(1-\mu-\mu')^2}{(1-\mu)(1-\mu')} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Consequently, since there is the same probability that they shall have nothing in common when μ is on the right and μ' on the left, the expression just written is the probability that μ and μ' shall be quite clear of one another.

The condition that x and x' shall not have so much as ν in common, is expressed by saying that $x + x'$ less than $1 - \mu - \mu' + \nu$: ν being less than μ and than μ'. Hence, by similar reasoning

$$\frac{(1 - \mu - \mu' + \nu)^2}{(1 - \mu)(1 - \mu')}$$

is the chance that μ and μ' have not so much as ν in common.

Under these new circumstances, if μ and μ' be each $= \frac{1}{10}$, the chance that they are clear of one another is $\frac{64}{81}$, or it is 64 to 17 in favour of it. That is, if a thousand persons were placed in a row, and two being selected at hazard, 100 sovereigns were given successively, beginning with the first, and 100 shillings successively, beginning with the second, it would now be *about* 64 to 17 that *no one* received a guinea.

SECTION IV. *On the Syllogism.*

THERE is much that is elegant and instructive about the theory of the four figures of the syllogism, three of which belong to Aristotle. And the magic words *Barbara*, *Camestres*, *&c.* are models of notation, almost every letter of the moods in the three latter figures being a rule of direction. The following old epitaph on a schoolman selects, I think, one of the best parts of the system for ridicule:

Hic jacet magister noster
Qui disputavit bis aut ter
In *Barbara* et *Celarent*
Ita ut omnes admirarent
In *Fapesmo* et *Frisesomorum*
Orate pro animis eorum!

In proposing another system of classification, in connexion with the use of contraries, I remark, first, that the ordinary method has two points of redundancy. The distinct use of the two forms of a convertible proposition, $X . Y$ and $Y . X$, XY and YX, is made for the system of figures, rather than the figures for it. It is desirable I think to confound them as much as possible; so that each may never fail to suggest the other. In the next place, if the use of contraries be introduced, every one of the twenty-four modes of predicating would claim admission into a system of figures, and their number would be increased to thirty-two.

Again, the first followers of Aristotle, in adopting the rule that no syllogism should be admitted in which the conclusion was not the strongest the premises would allow—in rejecting for instance "$X)Y$ and $Y)Z$ therefore XZ," because $X)Z$ also follows—did not adopt the equally obvious rule of admitting no syllogism in which a weaker premiss would lead to as strong a conclusion. They retained, for instance, "$Y)X$ and $Y)Z$ therefore XZ", though $Y)X$ and YZ would produce the same conclusion. Now I think it desirable to adopt the rule of producing the strongest conclusion with the weakest premises, not only because it will turn out that by so doing the number of forms is diminished, even when contraries are considered, but also because a better and clearer distinction is drawn between the necessary and the contingent.

I also drop the distinction of *minor* and *major* terms and premises. Aristotle meant them to apply only to affirmative propositions, in which the predicate includes the subject. But the use of them was extended, to the utter destruction of the meaning in negative propositions, or worse, to the danger of carrying a false meaning to the last named. The consequence of the distinction is that these four syllogisms,

	$X\,)\,Y$	$Z\,.\,Y$	$Y\,.\,Z$	$X\,)\,Y$
	$Z\,.\,Y$	$X\,)\,Y$	$X\,)\,Y$	$Y\,.\,Z$
	$Z\,.\,X$	$X\,.\,Z$	$X\,.\,Z$	$Z\,.\,X$
Figure	2	2	1	4
Name	*Camestres*	*Cesare*	*Celarent*	*Camenes*

which are identical in sense and effect, are made separate objects of study. In fact, as is known, all the figures are in the first, the fourth being occasionally brought into it by a reduction which deserves the name of *Barbara* in every case, and which the simplest use of contraries avoids.

For syllogisms I shall adopt such notation as

$$X\,)\,Y + Y\,)\,Z = X\,)\,Z.$$

Or if a weaker conclusion be taken,

$$X\,)\,Y + Y\,)\,Z > XZ.$$

As there are eight modes of predicating between X and Y, and between Y and Z, it follows that there are sixty-four combinations which may give conclusions. Of these, all but eight are distributable two and two into counterparts, in which X and Z are interchanged, everything else remaining the same; giving eight single, and twenty-eight pairs of counterparts. Of these, exactly half, (four single, and fourteen pairs) are wholly inconclusive. Of the conclusive cases, two single ones and two pairs are rejected, because as strong a conclusion can be obtained from a weaker premiss. There remain two single ones and twelve pairs, to which a systematic classification is to be given. Instead of enumerating, I shall state a mode of deriving all the cases from a common principle.

Since every proposition is, but for accidents of language, a universal affirmative, as before noticed, it will follow that there are really no forms of syllogism except those in which the premises and conclusion are universal affirmatives, or can be made so by use of contraries and invention of subgeneric terms. Now the only universal affirmative syllogism is

$$X\,)\,Y + Y\,)\,Z = X\,)\,Z$$

considering the counterpart $Z\,)\,Y + Y\,)\,X = Z\,)\,X$ as identical in form. If we take universal affirmative premises only, we have one which will have a particular conclusion, with respect to the names X and Z,

$$Y\,)\,X + Y\,)\,Z = XZ,$$

which must be used in discovery of forms, (and will in fact give the two single syllogisms of this system) though it will only ultimately enter as $Y\,)\,X + YZ = XZ$. Now if we change one or more of the terms X, Y, Z into their contraries, we have eight modes of transformation, according as we use

$$XZY, \quad XZy, \quad xzy, \quad xzY, \quad XzY, \quad Xzy, \quad xZY, \quad xZy.$$

First take the syllogisms

$$X\,)\,Y + Y\,)\,Z = X\,)\,Z \text{ and } XY + Y\,)\,Z = XZ;$$

the latter of which is only the first, with the premiss $X\,)\,Y$ weakened, and would be reduced to the first form by inventing a subgeneric name for the Xs there spoken of. Apply each of these to the eight varieties just named, transforming premises and conclusion, when necessary, to one of the eight standard forms of predication:

$$X\,)\,Z \quad X:Z \quad Z\,)\,X \quad Z:X \quad X\,.\,Z \quad XZ \quad x\,.\,y \quad xz.$$

To each result I attach a letter of notation, derived from the nature of the conclusion, with letters subscript indicative of the premises. Thus A_{Aa} signifies a syllogism with a conclusion A derived from premises of the forms A and a. The order of reference is XY, ZY, XZ, the middle term being the predicate of both premises *in the references.*

Universal Syllogisms.

First Form.	Transformation.	Description.
$X) Y + Y) Z = X) Z$	$X) Y + Y) Z = X) Z$	A_{Aa}
$X) y + y) Z = X) Z$	$X . Y + z . y = X) Z$	A_{Ee}
$x) y + y) z = x) z$	$Y) X + Z) Y = Z) X$	a_{aA}
$x) Y + Y) z = x) z$	$x . y + Z . Y = Z) X$	a_{eE}
$X) Y + Y) z = X) z$	$X) Y + Z . Y = X . Z$	E_{AE}
$X) y + y) z = X) z$	$X . Y + Z) Y = X . Z$	E_{EA}
$x) Y + Y) Z = x) Z$	$x . y + Y) Z = x . z$	e_{ea}
$x) y + y) Z = x) Z$	$Y) X + z . y = x . z$	e_{ae}

Of these forms four are distinct and the others are their counterparts. Writing each with its counterpart, we have

$$A_{Aa} \text{ and } a_{aA}, \quad A_{Ee} \text{ and } a_{eE}, \quad E_{AE} \text{ and } E_{EA}, \quad e_{ea} \text{ and } e_{ae}.$$

Particular Syllogisms.

First Form.	Transformation.	Description.
$XY + Y) Z = XZ$	$XY + Y) Z = XZ$	I_{Ia}
$Xy + y) Z = XZ$	$X : Y + z . y = XZ$	I_{Oe}
$xy + y) z = xz$	$xy + Z) Y = xz$	i_{iA}
$xY + Y) z = xz$	$Y . X + Z . Y = xz$	i_{oE}
$XY + Y) z = Xz$	$XY + Z . Y = X : Z$	O_{IE}
$Xy + y) z = Xz$	$X : Y + Z) Y = X : Z$	O_{OA}
$xY + Y) Z = xZ$	$Y : X + Y) Z = Z : X$	o_{oa}
$xy + y) Z = xZ$	$xy + z . y = Z : X$	o_{ie}

But though the eight universal syllogisms are counterparts, two and two, they admit of another division into pairs, in each of which the terms are the *contraries* of those of the other. Thus A_{Aa} is connected with a_{aA} in this manner, and A_{Ee} with a_{eE}: and these are counterparts. And E_{AE} and e_{ae} and E_{EA} and e_{ea}, which are not counterparts, have the same connexion. The eight particular syllogisms, which contain no counterparts, are divisible into four pairs with the same connexion. Thus I_{Ia} is changed into i_{iA}, I_{Oe} into i_{oE}, O_{IE} into o_{ie}, and O_{OA} into o_{oa}. It is also worth notice that when the conclusion is negative, the premises are always of contradictory forms, and when positive, of consistent ones: and that the substitution of the contradictory *forms* in the premises is equivalent to that of contrary *terms* in the conclusion. Thus from $XY + Y) Z = XZ$; if we substitute contradictories in the premises, we have $X . Y + Y : Z$, the conclusion of which is xz.

Before proceeding further it may be worth while to endeavour to impress the notation upon the reader. The letters A, E, I, O, have the well-known meanings, thus restricted, that they belong to a particular order of the terms, or a particular choice of subject and predicate. Thus, X and Y being the terms, in the order XY, or X being the subject and Y the predicate, the four capitals stand for $X)Y$, $X.Y\,(=Y.X)$, $XY\,(=YX)$ and $X:Y$. And a, e, i, o stand for the same propositions when both subject and predicate are changed into contraries: that is, they stand for $x)y$, $x.y\,(=y.x)$, $xy\,(=yx)$, $x:y$. In these last, system is sacrificed to simplicity in using $Y)X$ and $Y:X$ for $x)y$ and $x:y$. The following table shows what transformation takes place when the terms and orders are successively XY, Xy, xy, xY, YX, yX, yx, Yx.

XY	Xy	xy	xY	YX	yX	yx	Yx
A	E	a	e	a	E	A	e
O	I	o	i	o	I	O	i
a	e	A	E	A	e	a	E
o	i	O	I	O	i	o	I
E	A	e	a	E	a	e	A
I	O	i	o	I	o	i	O
e	a	E	A	e	A	E	a
i	o	I	O	i	O	I	o

Here we mean, for instance, that E of the order (X, Y) is the same thing as the A of $(X, y,)$ or the e of (x, y) &c,; or that $X.Y$, $X)y$ are the same. As to the e of (x, y) it is identically $X.Y$. The eight operations by which the transformations of headings are made are those which I have denoted by L, P, SP, S, T, PT, SPT, and ST, of which the simplest readings are

For inconvertibles $L \quad P \quad T \quad S \quad T \quad S \quad L \quad P$,
For convertibles $L \quad P \quad SP \quad S \quad L \quad P \quad SP \quad P$.

Taking the terms XY, YZ, ZX, for the premises and conclusion, and the order of reference XY, ZY, XZ, the notation given defines the syllogism. Thus, valid or not, the syllogism O_{eA} can be nothing but $x.y+Z)Y=X:Z$. In turning the fundamental syllogism into the form $X)Y+Y)Z=X)Z$, I have altered Aristotle's order of the premises, which would give $Y)Z+X)Y=X)Z$. Reasoning direct from his *dictum de omni et nullo*, namely, that what is true or false of all is true or false of every some, it would seem natural first to ascertain the fact relating to the whole, and then to introduce the part which is to be considered. But in another point of view it may be more natural to reverse this order. If there be three boxes P, Q, and R, of which I want to ascertain by means of Q whether P will go into R, it seems to me more natural to try first whether P will go into Q, and then whether Q will go into R. But if the question be whether R will hold P, then perhaps it may be more natural to try first whether R will hold Q, and then whether Q will hold P. It must be mere matter of opinion which should be taken; and the idioms of the language which people speak produce the associations on which they will decide.

The syllogisms which we have got as yet, four universal and eight particular, contain all those of Aristotle. Six of them indeed are enough for this purpose: that is to say, every syllogism of Aristotle is either one of these six, or one of them with a premiss converted or strengthened, or both. The six really distinct syllogisms of the old system, with the order XZ established in the conclusion, are as follows, with the scholastic names of the forms which they have or can be made to have:

A_{Aa}	$X)Y+Y)Z=X)Z$	*Barbara*,
I_{Ia}	$XY+Y)Z=XZ$	*Darii*, *Darapti*, *Disamis*, *Datisi*, *Bramantip*, *Dimaris*.
E_{AE}	$X)Y+Z.Y=X.Z$	*Celarent*, *Cesare*, *Camestres*, *Camenes*.
O_{IE}	$XY+Z.Y=X:Z$	*Ferio*, *Festino*, *Felapton*, *Feriso*, *Fesapo*, *Fresison*.
O_{OA}	$X:Y+Z)Y=X:Z$	*Baroko*.
O_{ao}	$Y)X+Y:Z=X:Z$	*Bokardo*.

If we wish to have a notation which neglects the premises, we may call these A, I, E, O_1, O_2, O_3, which may be separated into two connected sets, thus. The contradiction of a conclusion coupled with either premiss must give the contradiction of the other premiss. It will be found that if we call A, O_2, and O_3 *opponents*, and also E, I, and O_1, each syllogism can be produced from either of its two opponents, by coupling the denial of their conclusions with the affirmations of their premises.

The six new syllogisms, reduced to the same order, will be

$$\begin{array}{l|l}
A_{Ee} & X . Y + z . y = X) Z \\
i_{iA} & xy + Z) Y = xz \\
I_{Oe} & X : Y + z . y = XZ \\
i_{oE} & Y : X + Z . Y = xz \\
e_{ea} & x . y + Y) Z = x . z \\
O_{ei} & x . y + zy = X : Z.
\end{array}$$

The correlation of these two sets is by no means simple. Before examining it, observe that an interchange of X and Z, though it alters A into a and O into o, does not alter E and I, nor e and i. The counterpart of a syllogism, made by this interchange, is represented by simply inverting the letters of the premises, and interchanging A and a, O and o, in the letters of the conclusion. Thus the counterpart of i_{oE} is i_{Eo}: that of A_{Ee} is a_{eE}. Now if we take the six Aristotelian syllogisms, and make all the changes, and tabulate the results, we shall have as follows:

XYZ	xYZ	xyZ	xyz	xYz	XYz	Xyz	XyZ
A_{Aa}	e_{ea}	e_{ae}	a_{aA}	a_{eE}	E_{AE}	E_{EA}	A_{Ee}
I_{Ia}	o_{oa}	o_{ie}	i_{iA}	i_{oE}	O_{IE}	O_{OA}	I_{Oe}
E_{AE}	a_{eE}	a_{aA}	e_{ae}	e_{ea}	A_{Aa}	A_{Ee}	E_{EA}
O_{IE}	i_{oE}	i_{iA}	o_{ie}	o_{oa}	I_{Ia}	I_{Oe}	O_{OA}
O_{OA}	i_{iA}	i_{oE}	o_{oa}	o_{ie}	I_{Oe}	I_{Ia}	O_{IE}
O_{ao}	i_{Eo}	i_{Ai}	o_{AO}	o_{EI}	I_{aI}	I_{eO}	O_{ei}

The syllogisms written under each heading* are those which that written under the first becomes, when the variation shown in the heading is made. Thus if X and Z be changed into x and z, O_{IE} becomes o_{oa}, or

$$XY + Z . Y = X : Z \text{ becomes } xY + z . Y = x : z, \text{ or } Y : X + Y) Z = Z : X.$$

The new syllogisms have their letters in Italics. Each form, old or new, or its counterpart, occurs four times: but though the first column contains old syllogisms only, there is no column which contains none but new ones. So that it cannot be said that the new syllogisms are, on any one hypothesis, views of the old ones: though, in the column xyZ, five of them are so.

The following are the sets of opponents in the old and new system.

Old system	A_{Aa}	O_{OA}	O_{ao}	E_{AE}	O_{IE}	I_{Ia}
New system	A_{Ee}	I_{Oe}	i_{oE}	e_{ea}	O_{ei}	i_{iA}

* The order of the headings follows a recurring law, the next step of which would give XYZ again. If there be any *odd* number, n, of assertions, any one or more of which may be changed into its contrary, giving 2^n varieties, all the varieties may be gained as follows: write them in order, change the first, in that the second, in that the third, and so on to the end. Then go backwards with the same process, and then forwards and so on until 2^n have been made. Thus, if there were five, the changes would be made thus, 0 indicating no change, 0, 1, 2, 3, 4, 5, 4, 3, 2, 1, 2, 3, 4, 5, 4, 3, 2, 1, 2, &c. In the case of three, it is

0	1	2	3	2	1	2	3	2
XYZ	xYZ	xyZ	xyz	xYz	XYz	Xyz	XyZ	XYZ

But we have not yet closed our investigation; for we have to examine the remaining syllogism of universal premises, or $Y)X + Y)Z = XZ$. If we go through the cases, in the last order of headings, we shall find as follows.

First Form.	Transformation.	Description.	Remarks.
$Y)X + Y)Z = XZ$	$Y)X + Y)Z = XZ$	I_{aa}	Derived from I_{Ia}
$Y)x + Y)Z = xZ$	$Y.X + Y)Z = Z:X$	o_{Ea}	.... o_{oa}
$y)x + y)Z = xZ$	$X)Y + y.z = Z:X$	o_{Ae}	.... o_{ie}
$y)x + y)z = xz$	$X)Y + Z)Y = xz$	i_{AA}	Not yet obtained
$Y)x + Y)z = xz$	$Y.X + Y.Z = xz$	i_{EE}	Derived from i_{oE}
$Y)X + Y)z = Xz$	$Y)X + Y.Z = X:Z$	O_{aE}	.... O_{IE}
$y)X + y)z = Xz$	$y.x + Z)Y = X:Z$	O_{eA}	.... O_{ei}
$y)X + y)Z = XZ$	$y.x + y.z = XZ$	I_{ee}	Not yet obtained.

The derivation here mentioned is merely strengthening a premiss. We thus obtain the only two remaining forms

$$i_{AA} \quad X)Y + Z)Y = xz \quad \Big| \quad I_{ee} \quad y.x + y.z = XZ.$$

These cannot be derived from the twelve previously established by strengthening a premiss, though their equivalents (the other six) can. These two last syllogisms differ from all the rest in having no counterparts, and may therefore be called *single* syllogisms.

The old rules are of course true as to the old syllogisms: but most of them are inapplicable to the new ones. Particular premises, indeed, never gave a conclusion, as yet: but premises both negative may, and in the case of i_{AA}, the middle term is universal in neither premiss. Again, both premises may be negative, and may give a positive form of conclusion. The following rules, however, will be found to hold good.

From premises both particular, nothing follows. The middle term cannot be particular in both, except in i_{AA}: nor can its contrary be universal in both, except in I_{ee}. One negative premiss always yields a negative conclusion, and two negative premises an affirmative. When one premiss is particular, the conclusion is particular. When e is in the premises the conclusion is never in i.

I now take the two cases in which particular premises may give a conclusion: namely

$$I_{II} \quad XY + ZY = XZ \qquad\qquad XY + Y:Z = X:Z \quad O_{Io}$$

on the suppositions that the Ys mentioned in both premises are in number more than all the Ys. If Y_1 and Y_2 stand for the fractions of the whole number of Ys mentioned or implied in the two premises, and y_1 and y_2 for the fractions of the ys implied or mentioned, we shall by a repetition of the process on $YX + YZ = XZ$ (the other being obtained in the course of the process) arrive at the following results or their counterparts: remembering that $Y_1 + Y_2$ is greater or less than 1, according as $y_1 + y_2$ is less or greater. (See the Addition at the end of this paper.)

Designation.	Syllogism.	Condition of its validity.
I_{II}	$YX + YZ = XZ$	$Y_1 + Y_2$ greater than 1
O_{Io}	$YX + Y:Z = X:Z$	....
i_{oo}	$Y:X + Y:Z = xz$	....
O_{oi}	$X:Y + yz = X:Z$	$Y_1 + Y_2$ less than 1
i_{ii}	$yx + yz = xz$	....
O_{Oi}	$X:Y + yz = X:Z$	....
I_{OO}	$X:Y + Z:Y = XZ$	....

There are many remarks to be made on the demonstrative connexion of the parts of this system with one another, and on the explanations in general language of the new varieties of syllogism. The length of this paper, however, is a sufficient reason for stopping here with the formal part of the subject, and proceeding to the consideration of the probabilities of argument and authority.

SECTION V. *On the Application of the theory of Probabilities to questions of Argument and Authority.*

WRITERS on logic have made no effort to apply the mathematical theory of probabilities to the balance of arguments; for we can hardly call by that name the simple statement that the probability of the conclusion of a syllogism is the product of the probabilities of the premises. How far this is correct will appear in the course of the present section, which is intended to investigate the manner in which the probability of a conclusion is to be inferred from opposing arguments and authorities, of which the several probabilities are given.

Conclusions which are not absolutely demonstrated are established in our minds on two distinct bases, *argument* and *authority*. Even if there be appeal to authority in establishing the premises of an argument, the distinction is in no degree lost. This we shall see as soon as the terms are defined.

Argument is an offer of proof, and its failure is only a failure of proof: the conclusion may yet be true. *Authority* is an offer of testimony, and its failure is a failure of truth: nothing can furnish absolute reason for distrusting the authority on future occasions except the proof that the conclusion asserted is false. A person who had made a hundred assertions, all supported by inconclusive arguments, but all of which turned out to be *true*, would give a very high authority to his hundred and first assertion.

We have an unfortunate use of language in the mathematical application of the word *probability*. We say that small probability and great improbability are identical terms; which is not true in their common meaning. In fact, a being what we call the probability of an event, $a - \frac{1}{2}$ is what we ought to call by that name: and if $a - \frac{1}{2}$ be negative, we ought to call $\frac{1}{2} - a$ the *improbability* of the event. It would not be wise to introduce the same inaccuracy in the use of the word *authority*: accordingly, μ being the chance that an assertion of an individual, made on the best of his knowledge and belief, is true, I shall call μ the value of his *testimony*. When μ exceeds $\frac{1}{2}$, I shall say that he is authority for the conclusion. And, measuring absolute authority by unity, I shall take $2\mu - 1$ as the measure of his authority, which is against the conclusion, if $2\mu - 1$ be negative. Again, if ρ be the number of times his testimony is given to a truth for once which it is given to a falsehood (which we may call his *relative testimony*), and if a denote his authority, we shall have the following equations, which will all be useful:

$$a = 2\mu - 1 = \frac{\rho - 1}{\rho + 1},$$

$$\rho = \frac{\mu}{1 - \mu} = \frac{1 + a}{1 - a},$$

$$\mu = \frac{1 + a}{2} = \frac{\rho}{\rho + 1}.$$

In forming our opinions upon argument, we are told to leave authority altogether out of sight, and to consider only what is said, not who says it. It was Bacon, I believe, who first said that assertion is like the shot from the long bow, the force of which depends upon the arm which draws it; while argument is like the shot from a cross bow, which a child can discharge as well as a man. But the simile is as inapt as the recommendation it contains is unwise; for the endeavour is to hit the mark, not merely to fire a shot: and the bow which most often succeeds in doing that is the best. Closely examined, this direction to dispense with authority amounts to requiring us to suppose that the proposer of an argument is as often right as wrong, and wrong as right, in his conclusions. But what can be the wisdom of making believe that a person tells us ten truths to ten falsehoods, if we know it for a fact that he tells us nineteen truths to one falsehood? If absolute demonstration be given, no rule is necessary, for we cannot attend to authority. If something very near to demonstration be given, no rule is practically necessary, for we have what is called moral certainty.

But, it may be said, why not throw away authority altogether? I answer that it is impossible: and that any one who forms an undemonstrated conclusion independent of the authority of others, can only do it by assuming some value for his own. All arguments, and all balance of arguments, will leave three possible cases. Either one or more of the arguments for the conclusion will prove it, or one or more of the arguments against will refute it, or all the arguments are inconclusive. The conclusion is proved, disproved, or left neither proved nor disproved. But it is not one of the three, true, false, or neither true nor false: it must be either true or false. And the mind must come to some conclusion upon this point: it must, so to speak, distribute the inconclusiveness of the arguments, in some way or other, between belief and disbelief. In whatever way this is done, it amounts, as we shall see, to some assumption as to the authority either of the proposer or of the receiver, or of some third person, or of all together.

There is but one way in which we can really deprive the proposer of an argument of any authority; and that is, by depriving him of any *peculiar* authority. If Newton propose an argument, to the conclusion of which Halley assents without knowledge of the argument, we have a right to allow it to be reasonable that the argument should lend the same force to the conclusion as if Halley had proposed it, and Newton had assented, also without knowledge. Admit this, so far as the premises do not depend on the authority of the proposer, and we admit all the separation of argument and authority which is practicable.

A conclusion is usually opposed, in argument, to what logicians call the *contradictory*, which must be true if the conclusion be false, and *vice versâ*. It is not often that it is opposed to the *contrary*, which must be false when it is true, but not *vice versâ*. I shall first consider the proposition and its contradictory, as to authority, as to argument, and then as to the two in combination.

PROB. 1. *Required the joint value of authorities the separate values of which are given.*

Let the first authority be one of the testimony μ, or of m truths to n errors, μ being $m \div (m+n)$. Let μ', m', n', take the place of μ, m, n in the second authority: and so on. Now since the conclusion asserted cannot be true on one authority and false on another, our position with respect to the conclusion is as follows: We have an urn of m white and n black balls, another of m' white and n' black, &c. from each of which we have to draw. The balls however are not free, but are connected by such mechanism that no ball will leave its urn unless a simultaneous effort be made upon one of the same colour in every urn. Now the number of ways of choosing one white ball out of each urn is $m\,m'\,m''\ldots$; and of choosing one black ball $n\,n'\,n''\ldots$. Hence the united testimony for the conclusion is

$$\frac{m\,m'\,m''\ldots}{m\,m'\,m''\ldots + n\,n'\,n''\ldots} \text{ and against it } \frac{n\,n'\,n''\ldots}{m\,m'\,m''\ldots + n\,n'\,n''\ldots},$$

$$\text{or } \frac{\mu\mu'\mu''\ldots}{\mu\mu'\mu''\ldots + (1-\mu)(1-\mu')(1-\mu'')\ldots} \text{ and } \frac{(1-\mu)(1-\mu')(1-\mu'')\ldots}{\mu\mu'\mu''\ldots + (1-\mu)(1-\mu')(1-\mu'')\ldots}.$$

If $a = 2\mu - 1$, &c. we have for the joint anthority expressed in terms of the separate authorities,

$$\frac{(1+a)(1+a')(1+a'')\ldots-(1-a)(1-a')(1-a'')\ldots}{(1+a)(1+a')(1+a'')\ldots+(1-a)(1-a')(1-a'')\ldots}.$$

If $\rho = \mu \div (1-\mu)$ &c., we find that the joint relative testimony is the product of the separate relative testimonies; which is the easiest way of expressing the result. Thus two authorities of 3 and 4 truths to one error, amount to one authority of 12 truths to one error.

I need hardly say that the preceding conclusions are verified by their giving such results as the following;—that if one of the authorities be absolute, the joint authority is the same; that any number of testimonies, each without authority either way, gives no authority either way; that inauthoritative testimonies do not affect the authority of the rest; and so on.

Problems of the preceding character are usually solved by the inverse method; or by the determination of the probabilities of precedent states from an observed event. Others have noted, I suppose, what has often struck me, namely, that the arrangement of conditions into an observed event and its precedents, is sometimes made in a very indirect and unnatural manner. There are however two classes of problems which give the same results: each inverse problem has a direct problem of the other class connected with it. For instance, there are m and m' white balls, and n and n' black balls, in two urns. A white ball has been drawn; what is the probability that the first urn was that which held it? The answer is well known to be

$$m(m'+n') \text{ divided by } m(m'+n') + m'(m+n).$$

Now take the following problem. The black balls are absolutely fixed in the urns; and the white balls are so connected that one will come out of neither, except when a white ball is touched in both, which will only set free one, say the one which was touched first. With one hand in each urn, not knowing one from the other, the chance of bringing out a white ball from the first urn (if we try until a ball comes from one or the other) is the same as that above, namely, that a ball drawn white *was* in the first urn. These two problems are really the same; the first says that a white ball has been drawn, the second that a white ball must be drawn. And precisely the same sort and amount of reflexion which must be employed to make this sameness apparent, must also be employed before the problems above alluded to will lose that indirect and unnatural appearance to which I have referred. It should also be noticed, that any problem on an event to come may, by supposing the event to have happened, not being yet known, be made a problem of inverse probabilities.

Prob. 2. *Supposing the authorities to bias one another, required the method of allowing for the bias.*

When one authority expressly cites and defers to another, he does not thereby diminish his own authority. For what we want to know of him is simply the value of his assent, which, unless we have some specific reason, we have no more right to suppose less than his average when he judges of another, than we have to suppose it greater. And, in fact, there are men who are better authorities as to their judgment of others, than as to what they propose themselves. Neither, for a similar reason, does it diminish the value of the second authority, that the conclusion asserted never would have been known to him had it not been for the first. What we want to account for here is *undue bias*, which I define to exist when there is a proportion of the conclusions of the second authority which are no better for his testimony than they would have been if the first alone had asserted them. The case of a number of authorities would lead to a complicated result. Suppose three, the values of whose testimonies are μ, μ', μ''; and let λ' and λ'' be the probabilities that the second and third are unduly biassed by the first. Then the value of the joint testimony is

$$\lambda'\lambda''\mu + (1-\lambda')\lambda''\frac{\mu\mu'}{\mu\mu'+(1-\mu)(1-\mu')} + \lambda'(1-\lambda'')\frac{\mu\mu''}{\mu\mu''+(1-\mu)(1-\mu'')}$$
$$+ (1-\lambda')(1-\lambda'')\frac{\mu\mu'\mu''}{\mu\mu'\mu''+(1-\mu)(1-\mu')(1-\mu'')}.$$

If there be only two authorities, the formula is reduced to

$$\lambda\mu + (1-\lambda)\frac{\mu\mu'}{\mu\mu' + (1-\mu)(1-\mu')},$$

for the joint testimony, and the joint authority is

$$\frac{a + a' - 2\lambda a'(1-a)}{1 + aa'}.$$

Had it not been for the bias asserted, the authority would have been $(a + a') \div (1 + aa')$. When $a - a'$ is positive and a' negative, the joint authority is the greater for the correction of the bias. This is as it should be; for the bias is then that of contradiction, and tends, until corrected, to lessen the joint authority. I have only entered thus much into this part of the subject, merely to show that the results of the preceding mode of treating the problem are confirmed by those of common sense.

PROB. 3. *To determine the joint effect of a number of arguments, the validities of which are given, some for a conclusion, and some for its contradictory.*

By the validity of an argument, I mean the probability that it proves its conclusion. The argument being of a conclusion which is legitimately inferred from the premises, it is absolutely valid, if all the premises be true: and what is here called its validity therefore means the product of the probabilities of all the premises. Let a, a', a'', &c. be the validities of the several arguments for the conclusion, and b, b', b'', &c. those of the arguments for the contradiction. If one argument on either side be valid the conclusion of that argument is established. Hence the joint validity of the arguments for is that of an argument whose validity is

$$1 - (1-a)(1-a')(1-a'')\ldots \quad \text{or} \quad \Sigma a - \Sigma aa' + \Sigma aa'a'' - \ldots$$

which is the probability that one or more of the arguments *for* proves its conclusion. Similarly the arguments against amount to an argument the validity of which is

$$1 - (1-b)(1-b')(1-b'')\ldots \quad \text{or} \quad \Sigma' b - \Sigma' bb' + \Sigma' bb'b'' - \ldots$$

And having thus shown how to reduce several arguments of the same kind to one, we may now proceed as with one of each sort. If the process now coming be applied to several arguments of each kind, the result obtained will, as we might predict, verify the correctness of the preceding compositions.

Let there be then one argument of the validity a *for*, and one of the validity b for the contradiction, or *against*. Let the argument for, be as a drawing from an urn in which there are M valid and N invalid cases: let that against, be as from another in which there are P valid and Q invalid cases. Of course $M : N :: a : 1-a$ and $P : Q :: b : 1-b$. If either argument be valid the other must be invalid. Now it does not follow that if the argument for be valid, and be the case marked, say 1, the invalid argument against may be any one of the cases 1, 2, 3 ... up to Q. For it may happen that each particular mode of succeeding in one argument must be necessarily connected with some particular mode or modes of failing in the other. To represent this, let us separate the three cases, and assume as follows:

1. When the argument for is valid and that against invalid, let it be that $M = m_1 + m_2 + \ldots$, $Q = q_1 + q_2 + \ldots$, and that when the first succeeds in one of the m_1 ways, the second must fail in one of the q_1 ways; and the same of m_2 and q_2, m_3 and q_3, &c.

2. When the argument against is valid, and that for invalid, let $N = n_1 + n_2 + \ldots$, $P = p_1 + p_2 + \ldots$ with the same connexion.

3. When both arguments are invalid, let $N = n_1' + n_2' + \dots$, $Q = q_1' + q_2' + \dots$ with the same connexion.

It is now clear that the number of compatible cases, in which the argument for is valid, must be $m_1 q_1 + m_2 q_2 + \dots$ or Σmq. Similarly, Σnp and $\Sigma n'q'$ are the numbers of cases in which the argument against is valid, and in which both arguments are invalid. Hence we have for the probabilities of the three cases, namely, that the conclusion is established, that the contradictory is established, and that the arguments are inconclusive, the following expressions:

$$\frac{\Sigma mq}{\Sigma mq + \Sigma np + \Sigma n'q'}, \quad \frac{\Sigma np}{\Sigma mq + \Sigma np + \Sigma n'q'}, \quad \frac{\Sigma n'q'}{\Sigma mq + \Sigma np + \Sigma n'q'}.$$

To solve the question in the most general manner, would require that we should combine the preceding results in all cases, that is, for all values, and all subdivisions, of M, N, P, Q. Without attempting such generality, I may make the following observations. From what takes place in other similar questions, it is highly probable we should find the result of this combination either to agree with that in which any of the M cases may occur with any one of the Q cases, &c. or to approximate to such an agreement as M, &c. are increased without limit. Next, that this agreement actually takes place, when all the subdivisions are the same aliquot parts of their wholes. With these presumptions, I content myself with their result, which amounts to supposing that any one of the M cases may enter with any one of the Q cases, and so on. The probabilities then are, for the three cases above-mentioned,

$$\frac{MQ}{MQ + NP + NQ}, \quad \frac{NP}{MQ + NP + NQ}, \quad \frac{NQ}{MQ + NP + NQ};$$

$$\text{or} \quad \frac{a(1-b)}{1-ab}, \quad \frac{b(1-a)}{1-ab}, \quad \frac{(1-a)(1-b)}{1-ab}.$$

The third term is the chance of inconclusiveness, which necessarily renders this case indefinite: and all we can say is, that the chance of the truth of the conclusion is

$$\frac{1-b}{1-ab}\{a + \lambda(1-a)\},$$

where the value of λ cannot be determined from argument (for all the arguments are used in determining a and b).

When the arguments are of equal force, or $a = b$, we have

$$\frac{a}{1+a}, \quad \frac{a}{1+a}, \quad \frac{1-a}{1+a}.$$

Hence $a \div (1+a)$, which represents the probability that a verified conclusion was derived from an argument of the validity a rather than from demonstration (when it must have been one or the other), also represents the success of an argument of the validity a against an argument of equal force on the other side.

So far as an argument is not demonstrative, it must rest on authority, including under that word the authority of the recipient himself. Now a is in fact the testimony to the validity of the argument on one side, and b to that on the other. If these were testimonies to the truth or falsehood of the conclusion, the joint testimonies to the truth and falsehood of the conclusion would then be

$$\frac{a(1-b)}{a(1-b) + b(1-a)}, \quad \frac{b(1-a)}{a(1-b) + b(1-a)};$$

which, since $a + b - ab$ must be less than unity, are necessarily greater than the two first of the three expressions. Or, if we attempt to consider argument entirely without reference to any authority except that for the premises, the absolute testimony to the truth or falsehood of the conclusion thus obtained is not so great as would be obtained from testimonies to the conclusion as strong as those to the validities of the arguments.

Prob. 4. *Given a number of arguments for a conclusion and for its contradictory, and also a number of authorities, all of given probabilities; required the resulting probabilities for the conclusion and for its contradictory.*

Let a and b have the meaning of the last problem, and let μ be the testimony which the joint authorities give for the conclusion and against its contradictory. Let a and b be represented by urns of m and p valid cases, and n and q invalid ones; and let μ be represented by an urn of v truths and w falsehoods. Then there are mqv cases in which the argument for is valid and the conclusion true; npw cases in which the argument against is valid and the conclusion false; nqv cases in which both arguments are invalid and the conclusion true; nqw in which both arguments are invalid and the conclusion false. And these are all the possible combinations. Hence the probability that the conclusion is true must be

$$\frac{(m+n)qv}{(m+n)qv+(p+q)nw} \quad \text{or} \quad \frac{(1-b)\mu}{(1-b)\mu+(1-a)(1-\mu)},$$

and the probability that the conclusion is false must be

$$\frac{(p+q)nw}{(m+n)qv+(p+q)nw} \quad \text{or} \quad \frac{(1-a)(1-\mu)}{(1-b)\mu+(1-a)(1-\mu)}.$$

To show the accordance of these formulæ with common notions, observe that they give the first four of the following results:

1. In an impossible conclusion (or when $\mu=0$) the first expression vanishes: or no argument, however strong, can give any probability to an impossibility.

If $\mu=0$ and $a=1$, we have incompatible hypotheses, and the expressions take the form $\frac{0}{0}$.

2. If $a=1$, the conclusion is certain; or absolute demonstration establishes its result, in spite of any amount of authority against it.

3. If there be no authority, or if $\mu=\frac{1}{2}$, then the probability of the conclusion is

$$\frac{1-b}{1-b+1-a},$$

and hence counter-arguments of equal strength, applied with no authority, give no *authority* to the conclusion.

4. If $a=b$, the probability of the conclusion is μ; or counter-arguments of equal strength leave previous authority unaffected.

5. If $a+b=1$, the effect of the arguments is simply that of one more authority: and that independently of their inconclusiveness, which still remains.

6. If there be no argument against, or if $b=0$, the probability of the conclusion is not a (as stated by writers on logic*, who confound it with the conclusion made valid *by the argument*) but $\frac{\mu}{\mu+(1-a)(1-\mu)}$: or $\frac{1}{2-a}$, when there is no authority.

7. When there is no opposition, and no previous authority, any unopposed argument, however weak, gives some authority to the conclusion; and every argument, however weak, increases the probability derived from previous authority.

* Myself among the rest.

Authority apart, the odds for the conclusion are $1-b$ to $1-a$. When both arguments are of great force, or b and a both near to unity, the ratio of the small quantities $1-b$ and $1-a$, which determines the probability for the conclusion, cannot be distinctly apprehended. When, then, there is something as near to demonstration on both sides as can be found in a subject which does not admit of absolute demonstration, the mind ought not to arrive at any conclusion more favourable to one side than the other. We constantly see the refusal of human nature to acquiesce in this reasonable rule, and always with a determination to find out weakness in the argument on one side or the other. It must be sometimes true that false conclusions shall be the exceptional cases, in which arguments of the highest probability fail.

It also appears that moral demonstration on one side is not enough, if there be anything resembling it on the other. All controversialists admit this in fact, by the stress which they lay on answering the arguments of the opposite side. But they frequently do this as if it were a kind of surplusage, a charitable (but not in any other sense necessary) allowance for the weakness of those who do not see the force brought forward on their side of the question. Whereas it appears that it may be perfectly necessary to answer an opponent who admits all they say to the full extent which is demanded for it, supposing that to be anything short of absolute demonstration.

Prob. 5. *To ascertain the manner in which the inconclusiveness of the arguments is divided by the authorities between the probabilities of the truth and falsehood of the conclusion.*

If we find λ from the equation,

$$\frac{a(1-b)}{1-ab}+\lambda\frac{(1-a)(1-b)}{1-ab}=\frac{(1-b)\mu}{(1-b)\mu+(1-a)(1-\mu)};$$

$$\text{we find}\quad \lambda=\frac{(1+a)\mu-a}{(1-b)\mu+(1-a)(1-\mu)},$$

$$1-\lambda=\frac{(1+b)(1-\mu)-b}{(1-b)\mu+(1-a)(1-\mu)}.$$

From this it appears that λ is negative only when μ is less than $\frac{a}{1+a}$, and $1-\lambda$ when μ is greater than $\frac{1}{1+b}$. In the former case we see that unless the testimony of authority to the conclusion be greater than the success of the argument for the conclusion against a counter-argument of equal strength, the probability of the conclusion is less than that of the validity of the joint arguments.

If there be no authority, or if $\mu=\frac{1}{2}$, we have

$$\lambda=\frac{1-a}{1-b+1-a},\qquad 1-\lambda=\frac{1-b}{1-b+1-a},$$

a result which demonstrates the unmeaning character of the result of Problem 3. For the inconclusiveness is divided between the truth and falsehood of the conclusion in the proportion of the final probability of its falsehood to that of its truth. Or the more likely the conclusion is to be *false*, the larger proportion of the inconclusiveness does its *truth* get.

But we find

$$\frac{(1-b)\mu}{(1-b)\mu+(1-a)(1-\mu)}-\frac{1-b}{1-b+1-a}=\frac{(1-a)(1-b)}{1-b+1-a}\cdot\frac{2\mu-1}{(1-b)\mu+(1-a)(1-\mu)},$$

which shews the addition made to the probability of the conclusion in passing from the case of arguments without authority to that of arguments backed by the authority $2\mu-1$. In the case of arguments of equal strength, this is $\mu-\frac{1}{2}$, as it ought to be. When $\frac{1-b}{1-a}=\frac{1-\mu}{\mu}$, or when the *invalidities* of the arguments for and against are in the proportion of the testimonies of authority for and against, the same thing occurs; or the alteration of testimony in the above transition is exactly transferred to the probability of the conclusion. When $\frac{1-b}{1-a}$ lies between 1 and $\frac{1-\mu}{\mu}$, more than the alteration of testimony is transferred; in other cases less. The greatest transference is when $x = \sqrt{\left(\frac{1-\mu}{\mu}\right)}$, in which case the amount of probability transferred is

$$\frac{2}{1+2\sqrt{(\mu-\mu^2)}}\left(\mu-\tfrac{1}{2}\right).$$

It appears from what precedes that in the formulæ, the *invalidity* of the argument against, $1-b$, enters for the conclusion, and the invalidity of the argument for, $1-a$, enters against the conclusion, precisely in the same manner as the testimony for it, μ, and that against it, $1-\mu$. If then we call $\frac{1-b}{1-b+1-a}$ the *testimony of argument for* the conclusion, and $\frac{1-a}{1-b+1-a}$ that against it, just as we call μ and $1-\mu$ the *testimonies of authority* for and against: and if also we call $\frac{1-b}{1-a}$ the *relative testimony* of the arguments: then we may express the result of Problem 4 by saying that the joint relative testimony of the combined arguments and authorities is the product of all the separate relative testimonies, both of arguments and authorities.

It must be observed that the mode of entrance of the testimonies of argument makes it follow that if, after obtaining a result from certain arguments and authorities, we use the probability obtained as a new authority, in combination with additional data,—the final result will be the same as if we had collected all the arguments separately and all the authorities, and then proceeded as in Problem 4. This follows from the property of the functions $p \div (p+p')$ and $p' \div (p+p')$, which contain a mode of composition in which the order of the processes is indifferent, and their partial collection allowable. If we denote the preceding functions by $[p]$ and $[p']$, we have

$$[[p][q]] = [pq], \quad [[pq]r] = [pqr] \text{ \&c.}$$

When there are any number of arguments for, of validities a, a', a'', &c., the chance that one or more are valid is $1-(1-a)(1-a')(1-a'')\ldots$, and the testimony of argument against the conclusion is $(1-a)(1-a')(1-a'')$ divided by $(1-a)(1-a')\ldots+(1-b)(1-b')+\ldots$ Hence, the arguments against having the validities b, b', &c., and the authorities for and against being μ, μ', &c., and $1-\mu$, $1-\mu'$, &c., and A being the probability for the conclusion derived from the whole of the data, the *principle of relative testimonies* may be expressed thus:

$$\frac{A}{1-A} = \frac{1-b}{1-a}\cdot\frac{1-b'}{1-a'}\cdot\frac{1-b''}{1-a''}\ldots\ldots \times \frac{\mu}{1-\mu}\cdot\frac{\mu'}{1-\mu'}\cdot\frac{\mu''}{1-\mu''}\ldots\ldots$$

or as follows;—let the probabilities of the conclusion, derived from the several arguments backed by no authority, be considered as testimonies of authority to the conclusion, and used as in Problem 1.

It may happen that, besides the validity a, obtained directly from the premises, there is separate testimony of authority to the validity of an argument. Let it be ξ: then instead of a must be used $\frac{a\xi}{a\xi+(1-a)(1-\xi)}$.

I now return to the question of the dismissal of authority, which was partially entered on at the beginning of this paper. I assume that the mind will form an opinion upon any proposition which is laid before it. Even if the assertion were in a sealed packet, with no reason whatever to suppose it one rather than another of all that could possibly be made, an opinion would be formed as to its truth namely, that it is an even chance whether it be true or false. And this opinion is a just one;

for, since every assertion has its contradictory, and one of these two must be true and one false, it follows that the numbers of possible truths and falsehoods must be equal. When the packet is opened, this opinion will probably change: duly, in a manner depending upon previous associations of knowledge, or unduly, from what are then properly called *prejudices*. That every mind must form some opinion, may almost be concluded from the notorious fact that most minds, indeed nearly all uneducated ones, have little power except of absolute belief, or absolute unbelief. Their reasoning power is a spirit-level in awkward hands; the bulb is always at one end of the instrument or at the other. Now when it is recommended to dismiss authority, or to allow no authority, I apprehend that the advisers are not aware that they are promoting the specific plan of assuming that the proposer of the argument is a person of ten truths to ten errors. They rather wish to dismiss *testimony*, which it is clear, if it be that a conclusion must be formed, cannot be done.

Nor is it by any means true that the proper way of doing without authority is to assume the measure of authority = 0. If we wish to find the value of an argument, be the authority what it may, or as if the authority be unknown, we must allow for the effect of any possible authority, putting every value on equal terms with the rest. Let $d\mu$ be the chance that the testimony of authority lies between μ and $\mu + d\mu$, then the chance of the conclusion being true concomitantly with the authority lying between μ and $\mu + d\mu$ is

$$\frac{(1-b)\,\mu\,d\mu}{(1-b)\,\mu + (1-a)(1-\mu)},$$

which, integrated from $\mu = 0$ to $\mu = 1$, gives for the probable truth of the conclusion

$$\frac{r}{r-1}\left\{1 - \frac{\log r}{r-1}\right\} \quad \text{where } r = \frac{1-b}{1-a}.$$

If we assume that the chance of the testimony lying between μ and $\mu + d\mu$ is $M\phi\mu d\mu$, where M is the reciprocal of $\int_0^1 \phi\mu\,d\mu$, we have for the probable truth of the conclusion

$$M\int_0^1 \frac{r\mu\phi\mu\,d\mu}{r\mu + 1 - \mu},$$

aud some other supposition except $\phi\mu = 1$, is absolutely necessary: it is absurd to suppose equal chances for all values of the authority; to take the unknown proposer for instance, to be just as likely to be infallible as to be of no authority at all. What form should be assumed for $\phi\mu$ must be matter of opinion. If it be desired to try it on the supposition that μ is most likely near to some specific value λ, then, m and n being two integers in the proportion of λ and $1 - \lambda$, the assumption $\phi\mu = \mu^m (1-\mu)^n$ will represent the hypothesis, if m and n be considerable. And the greater m and n are taken, the smaller the chance that the testimony differs from λ by so much as a given quantity.

To give a case somewhat more like the proper notion of human authority than that in which all values of the testimony are equally probable, let us take $\phi\mu = \mu(1-\mu)$, $M = 6$. The above integral then becomes (after multiplication by M),

$$\frac{r}{(r-1)^3}\left\{6r\log r + 2 + 3r - 6r^2 + r^3\right\}.$$

If $r = 1$ this becomes $\frac{1}{2}$, as we might expect.

In the above conclusions, r is the relative testimony of the argument, on the supposition of no authority. If ρ be that of the authority, the joint relative testimony to the conclusion is $r\rho$: let us now see how far this is affected in the case of a *moral certainty* by the supposition that the chance of the testimony of authority lying between μ and $\mu + d\mu$ is $\mu^m (1-\mu)^n d\mu$, where $m \div (m+n)$ is the previous fixed value of μ. Now we have

$$\frac{r\mu\phi\mu}{r\mu + 1 - \mu} = \mu^m (1-\mu)^n - \frac{\mu^{m-1}(1-\mu)^{n+1}}{r} + \frac{\mu^{m-2}(1-\mu)^{n+2}}{r^2} - \frac{1}{r^3}\frac{r\mu^{m-2}(1-\mu)^{n+3}}{r\mu + 1 - \mu}.$$

And M is $[m+n+1] \div [m] \cdot [n]$, where $[m]$ means $1.2.3 \dots m$. If the probability required be denoted by $P_{m,n}$, we have, multiplying by $M\,d\mu$ and integrating from $\mu = 0$ to $\mu = 1$,

$$P_{m,n} = 1 - \frac{n+1}{m}\frac{1}{r} + \frac{(n+2)(n+1)}{m(m-1)}\frac{1}{r^2} - \frac{(n+3)(n+2)(n+1)}{m(m-1)(m-2)}\frac{1}{r^3}P_{m-3,\,n+3}.$$

Now $P_{m-3,\,n+3}$ is less than unity, so that if r be considerable, any degree of approximation may be obtained by this method, carried to more terms if necessary, and if the value of m will permit. Take the first three terms: then if the testimony of authority were given $= m \div (m+n)$, instead of being most likely to have something near that value, the approximation to $P_{m,n}$ would then be

$$1 - \frac{n}{m}\frac{1}{r} + \frac{n^2}{m^2}\frac{1}{r^2}.$$

Subtract the second from the first, and we have

$$-\frac{1}{mr} + \frac{3mn + 2m + n^2}{m^2(m-1)}\frac{1}{r^2}.$$

Write $(m+n)\mu$ and $(m+n)(1-\mu)$ for m and n, and we have, supposing m and $m+n$ considerable numbers,

$$-\frac{1}{m+n}\left\{\frac{1}{\mu r} - \frac{(2\mu+1)(1-\mu)}{\mu^3 r^2}\right\} \text{ nearly.}$$

Except then when μ is very small, the principle of relative testimonies is sufficiently accurate, in the case above supposed, taking for the testimony of authority the most probable value of that testimony.

Prob. 6. *Given arguments and authorities for a proposition and for its contrary, required the probability for the truth of each proposition, and for the falsehood of both.*

The contrary is thus distinguished from the contradictory: both the proposition and the contrary may be false, though both cannot be true: while either the proposition or its contradictory must be true. As far as the arguments alone are concerned, the problem is that of Problem 3: for either one of the arguments is valid and the other invalid, or else both are invalid. But there is a difference in the meaning of authorities; for, μ being the testimony to a proposition, $1-\mu$ is not necessarily the testimony to its contradictory. Let μ and ν be the testimonies of authority to the conclusion and its contradictory, and a and b the probable validities of the arguments. There are then five cases, two favourable to the truth of the proposition, two to that of the contrary, and one to that of the falsehood of both; 1. The argument for may be valid, in which case the proposition is true, the contrary false, and the argument against invalid. 2. The argument against may be valid, in which case the contrary is true, the proposition false, and the argument for invalid. 3. Both arguments may be invalid, and the proposition true. 4. Both arguments may be invalid and the contrary true. 5. Both arguments may be invalid and the proposition and contrary both false. Treating these in the manner in which the preceding problems have been solved, and which it is now unnecessary to repeat, we have the following expressions for the probability of the proposition, of its contrary, and of both being false,

$$\frac{(1-b)(1-\nu)\mu}{(1-b)(1-\nu)\mu+(1-a)(1-\mu)\nu+(1-a)(1-b)(1-\mu)(1-\nu)} \quad \frac{(1-a)(1-\mu)\nu}{(1-b)(1-\nu)\mu+(1-a)(1-\mu)\nu+(1-a)(1-b)(1-\mu)(1-\nu)} \quad \frac{(1-a)(1-b)(1-\mu)(1-\nu)}{(1-b)(1-\nu)\mu+(1-a)(1-\mu)\nu+(1-a)(1-b)(1-\mu)(1-\nu)}.$$

If there be no authorities, or if $\mu = \nu = \frac{1}{2}$, these become

$$\frac{1-b}{1-b+1-a+(1-a)(1-b)} \quad \frac{1-a}{1-b+1-a+(1-a)(1-b)} \quad \frac{(1-a)(1-b)}{1-b+1-a+(1-a)(1-b)}.$$

If the arguments be of equal strength these become

$$\frac{1}{3-a}, \quad \frac{1}{3-a}, \quad \frac{1-a}{3-a},$$

except when $a = 1$ (an absurdity in this case), in which they take the form $\frac{0}{0}$. But we get this result, that if $(1 - b) \div (1 - a) = \rho$, then the more nearly the arguments become demonstration, the more nearly is it certain that either the proposition or its contrary must be true, the probabilities for one and the other being as $\rho (1 - \nu) \mu$ and $(1 - \mu) \nu$. This is a singular result: for, since of two exceedingly strong arguments, one on each side, one must be invalid, it is not easy to explain from *à priori* notions why there is so great a probability that one or the other must be valid. That it is so appears from the probabilities of the validities* of the two arguments, and of the invalidity of both, namely,

$$\frac{a(1-b)}{a(1-b)+b(1-a)+(1-a)(1-b)},\quad \frac{b(1-a)}{a(1-b)+b(1-a)+(1-a)(1-b)},\quad \frac{(1-a)(1-b)}{a(1-b)+b(1-a)+(1-a)(1-b)}\cdot$$

in which $\rho : 1$ is the limiting ratio for the probabilities of the two validities. The same remark may be made with reference to the authorities: when two very high authorities affirm contraries, the higher the authorities the more likely is it that one or the other is right.

When there is no argument for the contrary, or $b = 0$, the three expressions become

$$\frac{(1-\nu)\mu}{(1-\nu)\mu+(1-a)(1-\mu)\nu+(1-a)(1-\mu)(1-\nu)},\quad \frac{(1-a)(1-\mu)\nu}{(1-\nu)\mu+(1-a)(1-\mu)\nu+(1-a)(1-\mu)(1-\nu)},\quad \frac{(1-a)(1-\mu)(1-\nu)}{(1-\nu)\mu+(1-a)(1-\mu)\nu+(1-a)(1-\mu)(1-\nu)}\cdot$$

when there are no authorities these become

$$\frac{1}{3-2a},\quad \frac{1-a}{3-2a},\quad \frac{1-a}{3-2a},$$

or when an argument is proposed, simply, the chance thereby given to the contrary is the same as that of neither being true.

It will seem strange at first, that the probability for the conclusion is not $\frac{1}{2-a}$: for it will be said, an argument and none for the contrary, is precisely the same position as an argument and none for the contradictory. But the suppositions as to authority are different. Looking to authorities only the chances of the three cases are

$$\frac{\mu(1-\nu)}{\mu(1-\nu)+\nu(1-\mu)+(1-\mu)(1-\nu)},\quad \frac{\nu(1-\mu)}{\mu(1-\nu)+\nu(1-\mu)+(1-\mu)(1-\nu)},\quad \frac{(1-\mu)(1-\nu)}{\mu(1-\nu)+\nu(1-\mu)+(1-\mu)(1-\nu)};$$

and in the case of no authorities, there is the chance $\frac{1}{3}$ for each of the cases. Now in treating the contradictory the testimony of *no* authority is $\frac{1}{2}$.

Let us now suppose that there is authority for each of the three cases, and also argument, or generally, let us take the following problem:

Prob. 7. *Let there be a dilemma of any number of horns, one or other of which, but only one, must be true; required the probabilities of the several horns, arguments and authorities being given for each.*

Let a, b, c, &c. be the probable validities of the several arguments, μ, ν, ξ, &c. the testimonies of authority. This problem, treated as before, gives the following result. Let

$$\Sigma = (1-b)(1-c)\ldots\mu(1-\nu)(1-\xi) + (1-a)(1-c)\ldots(1-\mu)\nu(1-\xi)\ldots$$
$$+ (1-a)(1-b)\ldots(1-\mu)(1-\nu)\xi\ldots$$

then the probabilities of the several horns containing the truth are

* I should have made this remark before, in regard to the *contradictories*, but for having written the denominator in the transformed shape $1 - ab$. I have always found the best rule to be, never perform operations in denominators.

$$\frac{(1-b)(1-c)\ldots\mu(1-\nu)(1-\xi)\ldots}{\Sigma}\quad\frac{(1-a)(1-c)\ldots(1-\mu)\nu(1-\xi)\ldots}{\Sigma}\quad\frac{(1-a)(1-b)\ldots(1-\mu)(1-\nu)\xi\ldots}{\Sigma}.$$

This problem contains all that have gone before, except the second. But this may not be apparent at first. In fact, if I had commenced this paper with the general case now in hand, and had then descended to the particular cases, the method of descending might have appeared exceptionable, requiring the authority of an independent consideration of the particular results arrived at. Suppose a dilemma of two horns, such as a proposition and its contradiction. If the testimony of authority for the proposition be μ, there is in this case the testimony $1-\mu$ *implied* for the contradiction. But this does not enter the formula: it is only the form belonging to the case of what is virtually represented in the general formula above, namely, that there is the testimony $1-\mu$ implied in favour of one or other of the horns following the first, because there is the testimony μ given for the first. No express testimony is given to the contradiction: so that it enters with the testimony $\frac{1}{2}$. And if there be only two horns, and the testimonies be μ and $\frac{1}{2}$, it will be found that the preceding expressions agree with the answer to Problem 4. There was no need in that case to suppose testimonies μ and ν, because, as the testimony to each horn is a definite testimony to the other, they would but have amounted to a joint testimony for the proposition.

If we want the case of the last problem, we have to take three horns, making $c=0$ and $\xi=\frac{1}{2}$. Or we may if we like suppose argument and testimony offered for the third case, namely, that both the proposition and its contrary are false.

If we wish to construct the general case upon the supposition that no one need be true, all we have to do is to add one more horn with an argument 0 and a testimony $\frac{1}{2}$.

The easiest way of representing the result of the general case is as follows. Let A_m represent the probability of the m^{th} horn from argument only, and M_m the same from authority only. We have then (using $a_1\ a_2$ &c. and $\mu_1\ \mu_2$ &c.),

$$A_m=\frac{\dfrac{1}{1-a_m}}{\Sigma\dfrac{1}{1-a_m}},\qquad M_m=\frac{\dfrac{\mu_m}{1-\mu_m}}{\Sigma\dfrac{\mu_m}{1-\mu_m}}$$

and the probabillty of the m^{th} horn is $\dfrac{A_m M_m}{\Sigma(A_m M_m)}$.

The term $A_m M_m$ or $\dfrac{1}{1-a_m}\dfrac{\mu_m}{1-\mu_m}$ may be called the *exponent of probability* of the m^{th} case: and the probability of that case is its exponent divided by the sum of all the exponents. This exponent is proportional to the number of balls in the urn the exits of which are favourable to the case. It is the product of two *relative testimonies*, that of the authority, and that of the argument alone, to establish the conclusion against its contradictory, that is against everything opposed to it.

Now suppose a complex dilemma of this kind, namely, that m of the horns, neither more nor less, must be true, and the rest false. An examination of this problem leads to the following result. The product of the exponents of any m cases, divided by the sum of the products of all the exponents, m and m together, is the probability that the m cases chosen are the true ones. Hence can be readily found the probability that any one case is among the true ones. If there be four cases, for instance, of which two must be true, and if e_1, e_2, e_3, e_4, be the exponents, the probability that the first case is true is

$$\frac{e_1(e_2+e_3+e_4)}{e_1e_2+e_1e_3+e_1e_4+e_2e_3+e_2e_4+e_3e_4}.$$

If it should be that m cases or fewer, but not more, may be true, then the probability that any $m-p$ cases and no others, shall be true, is the product of the exponents of those $m-p$ cases, divided by $1 + \Sigma e_1 + \Sigma e_1 e_2 + \Sigma e_1 e_2 + \ldots + \Sigma e_1 e_2 \ldots e_m$, the 1 being omitted if all cannot be false. The various restrictions which might be imposed, as that only an even number can be true, that no two, three, or any number of contiguous cases can be true together, &c. &c. may be easily contained under this one rule. In every set of cases that can be true together, multiply together the exponents of those cases; the product is the numerator of the probability that those cases only are true, and the sum of all the products is the denominator.

This rule applies to one case which we have not yet considered. When several arguments were proposed together, all for, or all against, a conclusion, it was supposed that they were perfectly independent. But it may happen that two or more arguments are so connected that some must be valid together and invalid together, or that some are valid when others are invalid, and *vice versâ*, or that the validity of one makes another valid, but the invalidity of the first has no influence on the validity of the second. All these cases, and a great number of others, including in fact, under one view or another, any question that may be proposed, may all be solved by the following RULE.

There is any number of events, each of which may happen in any number of ways, the separate probabilities of which are given, but so connected that there are specific necessary coincidences, or failures of coincidence. Take all the combinations which can happen, and compute the probability of each combination, as if its events were entirely unconnected. The resulting products are *proportional* to the probabilities of the several cases arising.

Thus, if there were three urns, the first giving white, black, or red (with chances w, b, r); the second white or black (with chances w', b'); the third white or black (with chances w'', b''), but so connected that black cannot be drawn from the first, nor white from all three, nor red from the first except when different colours come from the second and third, and it be required to find the chance of having a red ball, we proceed thus. Enumerate the possible cases, which are *WWB*, *WBW*, *WBB*, *RBW*, *RWB*, and the probability of a red ball is

$$\frac{r(b'w'' + w'b'')}{w(w'b'' + b'w'' + b'b'') + r(b'w'' + w'b'')}.$$

I have taken such an example, because it seems as if the condition that a black ball cannot be drawn from the first is equivalent to taking away those black balls, in which case the chances of the others cannot be w and r. But if the black balls be previously removed, then for w and r we must write $\frac{w}{1-b}$ and $\frac{r}{1-b}$, which will not affect the formula. In the same way any addition of other coloured balls, with the condition that they cannot be drawn, though it will affect the probabilities of the *independent* events made use of in the solution of the problem, will not affect the ratio which expresses the final result.

I have given so many proofs of particular cases of this principle that it is not necessary to say any thing on the general proof. But I shall observe that the circumstance noticed in combining argument and testimony, namely, that instead of the *validity* of an argument entering *for* the conclusion, the *invalidity* enters *against*,—is an immediate application of the preceding rule. For it is not the validity of an argument which is necessary to the truth of a conclusion, but the invalidity of it which is necessary to its falsehood. Thus, in Problem 4, the necessary cases are, either 1. Argument against invalid, and testimony for true, giving $(1-b)\mu$; or 2. Argument for invalid, and testimony against true, giving $(1-a)(1-\mu)$.

The application of the principles on which the preceding rule is established, would, I suspect, give much clearer views of many problems than the ordinary method of employing inverse considerations.

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October 3, 1846.

ADDITION.

SINCE this paper was written, I found that the whole theory of the syllogism might be deduced from the consideration of propositions in a form in which *definite quantity* of assertion is given both to the subject and the predicate of a proposition. I had committed this view to paper, when I learned from Sir William Hamilton of Edinburgh, that he had for some time past publickly taught a theory of the syllogism differing in detail and extent from that of Aristotle. From the prospectus of an intended work on logic, which Sir William Hamilton has recently issued, at the end of his edition of Reid, as well as from information conveyed to me by himself in general terms, I should suppose it will be found that I have been more or less anticipated in the view just alluded to. To what extent this has been the case, I cannot now ascertain: but the book of which the prospectus just named is an announcement, will settle that question. From the extraordinary extent of its author's learning in the history of philosophy, and the acuteness of his written articles on the subject, all who are interested in logic will look for its appearance with more than common interest.

The footing upon which we should be glad to put propositions, if our knowledge were minute enough, is the following. We should state how many individuals there are under the names which are the subject, and predicate, and of how many of each we mean to speak. Thus, instead of "Some *X*s are *Y*s," it would be, "Every one of *a* specified *X*s is one or other of *b* specified *Y*s." And the negative form would be as in "No one of *a* specified *X*s is any one of *b* specified *Y*s." If propositions be stated in this way, the conditions of inference are as follows. Let the *effective number* of a proposition be the number of mentioned cases of the *subject*, if it be an affirmative proposition, or of the *middle term*, if it be a negative proposition. Thus, in "Each one of 50 *X*s is one or other of 70 *Y*s," is a proposition, the effective number of which is always 50. But "No one of 50 *X*s is any one of 70 *Y*s" is a proposition, the effective number of which is 50 or 70, according as *X* or *Y* is the middle term of the syllogism in which it is to be used. Then two propositions, each of two terms, and having one term in common, admit an inference when 1. They are not both negative. 2. The sum of the effective numbers of the two premises is greater than the whole number of existing cases of the middle term. And the excess of that sum above the number of cases of the middle term is the number of the cases in the affirmative premiss which are the subjects of inference. Thus, if there be 100 *Y*s, and we can say that each of 50 *X*s is one or other of 80 *Y*s, and that no one of 20 *Z*s is any one of 60 *Y*s;—the effective numbers are 50 and 60. And $50 + 60$ exceeding 100 by 10, there are 10 *X*s of which we may affirm that no one of them is any one of the 20 *Z*s mentioned.

The following brief summary will enable the reader to observe the complete deduction of all the Aristotelian forms, and the various modes of inference from *specific particulars*, of which a short account has already been given.

Let a be the whole number of *X*s; and t the number specified in the premiss. Let c be the whole number of *Z*s; and w the number specified in the premiss. Let b be the whole number of *Y*s; and u and v the numbers specified in the premises of x and z. Let $X_t Y_u$ denote that each of t *X*s is affirmed to be one out of u *Y*s: and $X_t : Y_u$ that each of t *X*s is denied to be any one out of u *Y*s. Let $X_{m,n}$ signify m *X*s taken out of a larger specified number n: and so on. Then the five possible syllogisms, on the condition that no contraries are to enter either premises or conclusion, are as follows:—

1. $X_t Y_u + Z_w Y_v = X_{t+w-b,\,t}\ Z_w = Z_{t+w-b,\,w} X_t.$

2. $X_t Y_u + Y_v Z_w = X_{t+v-b,\,t}\ Z_w = Z_{t+v-b,\,w} X_t.$

3. $Y_u X_t + Y_v Z_w = X_{u+v-b,\,t}\ Z_w = Z_{u+v-b,\,w} X_t.$

4. $X_t Y_u + Z_w : Y_v = X_{t+v-b,\,t} : Z_w.$

5. $Y_u X_t + Z_w : Y_v = X_{u+v-b,\,t} : Z_w.$

The condition of inference expresses itself; in the $X_{m,\,t}$ of the conclusion, m must neither be 0 nor negative. The first case gives no Aristotelian syllogism; the middle term never entering universally (of necessity) into any of its forms, under any degree of specification which the usual modes of speaking allow. The other cases divide the old syllogisms among themselves in the following manner: they are written so as to show that there is sometimes a little difference of amount of specification between the results of different figures, which amount may change in the reduction from one figure to another. The Roman numerals mark the figures.

2.	$t=a,\ v=b$	$Y)Z_w + X)Y_u = X)Z_{u,\,w}$	*Barbara* I.
	$t=a,\ v=b$	$X)Y_u + Y)Z_w = Z_{a,\,w}X$	*Bramantip* IV.
	$t<a,\ v=b$	$Y)Z_w + X_tY_u = X_tZ_{u,\,w}$	*Darii* I.
	$t<a,\ v=b$	$X_tY_u + Y)Z_w = Z_{t,\,w}X_t$	*Dimaris* IV.
3.	$u=b,\ v=b$	$Y)X_t + Y)Z_w = Z_{b,\,w}X_{b,\,t}$	*Darapti* III.
	$u<b,\ v=b$	$Y_uX_t + Y)Z_w = Z_{u,\,w}X_{u,\,t}$	*Disamis* III.
	$u=b,\ v<b$	$Y)X_t + Y_vZ_w = Z_{v,\,w}X_{v,\,t}$	*Datisi* III.
4.	$t=a,\ v=b,\ w=c$	$Y.Z + X)Y_u = X.Z$	*Celarent* I.
	$t=a,\ v=b,\ w=c$	$Z.Y + X)Y_u = X.Z$	*Cesare* II.
	$t=a,\ v=b,\ w=c$	$X)Y_u + Z.Y = Z.X$	*Camestres* II.
	$t=a,\ v=b,\ w=c$	$X)Y_u + Y.Z = Z.X$	*Camenes* IV.
	$v=b,\ w=c$	$Y.Z + X_tY_u = X_t : Z$	*Ferio* I.
	$v=b,\ w=c$	$Z.Y + X_tY_u = X_t : Z$	*Festino* II.
	$t=a,\ v=b,$	$X)Y_u + Z_w : Y = Z_w : X$	*Baroko* II.
5.	$u=b,\ v=b,\ w=c$	$Y.Z + Y)X_t = X_{b,\,t} : Z$	*Felapton* III.
	$u=b,\ v=b,\ w=c$	$Z.Y + Y)X_t = X_{b,\,t} : Z$	*Fesapo* IV.
	$v=b,\ w=c$	$Y.Z + Y_uX_t = X_{u,\,t} : Z$	*Feriso* III.
	$v=b,\ w=c$	$Z.Y + Y_uX_t = X_{u,\,t} : Z$	*Fresison* IV.
	$u=b,\ \ w=c$	$Y_v : Z + Y)X_t = X_{v,\,t} : Z$	*Bokardo* III.

This system is complete in itself, if contraries be excluded. That in the body of this paper is also complete, if all specification be excluded, except which is contained in the usual words *some* and *all*. An attempt to combine the two systems would be useless, because its forms of expression would not be those of common language. For instance, the following must be one form of an affirmative proposition in the combined system "Of t Xs and t' xs every one is one or other of u Ys and u' ys." It would be useless to investigate the conditions of inference as to forms which are not those of speech in any language.

But at the same time there is a certain approach to the preceding forms, if we take in not merely the logical force of our common propositions, but also what is usually implied. He who says, "Some Xs are Ys," is generally held to mean that the other Xs are not Ys. The complex syllogisms, in which the alternatives left by the common forms are supposed to be definitely settled, are worthy of attention: and their theory is as follows.

With respect to the name Y, the name X may be of seven different kinds, distinguishable without numerical specification. These are as follows: neither term containing the whole universe.

(1.) The two terms may be *identical*, or $Y)X$ and $X)Y$. Let this be denoted by D. Taking the order XY, we have, to constitute D, the proposition A, a. And denoting coexistence by $+$, as before, we may write $D = A + a$.

(2.) X may be entirely contained in, but not repletive of, Y, or we may have $X)Y$ and $Y:X$. Let X be now called a *subidentical* of Y, and let $D_{\prime}$ denote this form. We have then $D_{\prime} = A + o$.

(3.) X may entirely contain Y, and more; or $Y)X$ and $X:Y$. Let X be now called a *superidentical* of Y, and let D' denote this form. We have then $D' = a + O$.

(4.) X may be the *contrary* of Y, both together filling up the universe of the proposition without anything in common; or $X.Y$ and $x.y$. Let this form be called C: we have then $C = E + e$.

(5.) X and Y may have nothing in common, but may not together fill up the universe of the proposition; or $X.Y$ and xy. Let them be called *subcontraries*, and let $C_{\prime}$ denote this form. We have then $C_{\prime} = E + i$.

(6.) X and Y may have something in common, and may together fill up the universe; or XY and $x.y$. Let these be called *supercontraries*, and let C' denote the form. We have then $C' = e + I$.

(7.) Each of the two may have something in common with the other and something not in common, both together not filling up the universe; or XY, xy, $X:Y$, $Y:X$. I cannot propose any name for this case with which I am in any degree satisfied: but as all the particular forms are here concerned, I will for the present call X and Y in this case *complete particulars* each of the other. Let P represent this form; we have then $P = I + O + i + o$.

In arranging for a syllogism, let the order be XY, ZY, XZ, the conclusion being described by what X is as related to Z, X coming from the first premiss; and both terms of the conclusion being described with respect to the middle term, Y. On examining the cases in which complete premises give a complete conclusion, I find as follows.

1. If one of the concluding terms be a complete particular of the middle term, there is no complete conclusion except when the other concluding term is either identical with or contrary to the middle term. And then each concluding term is a complete particular of the other.

2. The following table shows the result of all the other cases.

		Z					
		$D_{\prime}$	D	D'	$C_{\prime}$	C	C'
X	$D_{\prime}$	(CC')	$D_{\prime}$	$D_{\prime}$	$C_{\prime}$	$C_{\prime}$	(DD')
	D	D'	D	$D_{\prime}$	$C_{\prime}$	C	C'
	D'	D'	D'	$(C_{\prime}C)$	$(D_{\prime}D)$	C'	C'
	$C_{\prime}$	$C_{\prime}$	$C_{\prime}$	(DD')	(CC')	$D_{\prime}$	$D_{\prime}$
	C	$C_{\prime}$	C	C'	D'	D	$D_{\prime}$
	C'	$(D_{\prime}D)$	C'	C'	D'	D'	$(C_{\prime}C)$

This is a table of double entry, in which from the description of X and Z with respect to Y, we see set down that of X with respect to Z, when one can be affirmed: and, when nothing can be affirmed, all that can be denied, in parentheses. Thus, if X be a supercontrary of Y, and Z a subcontrary, X must be a superidentical of Z. But if X and Z be both subidenticals of Y, it may be denied that X is either the contrary or a supercontrary of Z.

I will not lengthen this addition by putting down in words all the rules which are expressed in the preceding table.

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University College, London,
February 27, 1847.

XXX. *Supplement to a Memoir On some Cases of Fluid Motion.* By George G. Stokes, M.A., *Fellow of Pembroke College.*

[Read Nov. 3, 1846.]

In a memoir which the Society did me the honour to publish in their Transactions*, I showed that when a box whose interior is of the form of a rectangular parallelepiped is filled with fluid and made to perform small oscillations the motion of the box will be the same as if the fluid were replaced by a solid having the same mass, centre of gravity, and principal axes as the solidified fluid, but different moments of inertia about those axes. The box is supposed to be closed on all sides, and it is also supposed that the box itself and the fluid within it were both at rest at the beginning of the motion. The investigation was founded upon the ordinary equations of Hydrodynamics, which depend upon the hypothesis of the absence of any tangential force exerted between two adjacent portions of a fluid in motion, an hypothesis which entails as a necessary consequence the equality of pressure in all directions. The particular case of motion under consideration appears to be of some importance, because it affords an accurate means of comparing with experiment the common theory of fluid motion, which depends upon the hypothesis just mentioned. In my former paper, I gave a series by means of which the numerical values of the principal moments of the solid which may be substituted for the fluid might be calculated with facility. The present supplement contains a different series for the same purpose, which is more easy of numerical calculation than the former. The comparison of the two series may also be of some interest in an analytical point of view, since they appear under very different forms. I have taken the present opportunity of mentioning the results of some experiments which I have performed on the oscillations of a box, such as that under consideration. The experiments were not performed with sufficient accuracy to entitle them to be described in detail.

The calculation of the motion of fluid in a rectangular box is given in the 13th article of my former paper. I shall not however in the first instance restrict myself to a rectangular parallelepiped, since the simplification which I am about to give applies more generally. Suppose then the problem to be solved to be the following. A vessel whose interior surface is composed of any cylindrical surface and of two planes perpendicular to the generating lines of the cylinder is filled with a homogeneous, incompressible fluid; the vessel and the fluid within it having been at first at rest, the former is then moved in any manner; required to determine the motion of the fluid at any instant, supposing that at that instant the vessel has no motion of rotation about an axis parallel to the generating lines of the cylinder.

I shall adopt the notation of my former paper. u, v, w are the resolved parts of the velocity at any point along the rectangular axes of x, y, z. Since the motion begins from rest we shall have $u\,dx + v\,dy + w\,dz$ an exact differential $d\phi$. Let the rectangular axes to which the fluid is referred

* Vol. VIII., Part I., p. 105.

be fixed relatively to the vessel, and let the axis of x be parallel to the generating lines of the cylindrical surface. The instantaneous motion of the vessel may be decomposed into a motion of translation, and two motions of rotation about the axes of y and z respectively; for by hypothesis there is no motion of rotation about the axis of x. According to the principles of my former paper, the instantaneous motion of the fluid will be the same as if it had been produced directly by impact, the impact being such as to give the vessel the velocity which it has at the instant considered. We may also consider separately the motion of translation of the vessel, and each of the motions of rotation; the actual motion of the fluid will be compounded of those which correspond to each of the separate motions of the vessel. For my present purpose it will be sufficient to consider one of the motions of rotation, that which takes place round the axis of z for instance. Let ω be the angular velocity about the axis of z, ω being considered positive when the vessel turns from the axis of x to that of y. It is easy to see that the instantaneous motion of the cylindrical surface is such as not to alter the volume of the interior of the vessel, supposing the plane ends fixed, and that the same is true of the instantaneous motion of the ends. Consequently we may consider separately the motion of the fluid due to the motion of the cylindrical surface, and to that of the ends. Let ϕ_c be the part of ϕ due to the motion of the cylindrical surface, ϕ_e the part due to the motion of the ends. Then we shall have

$$\phi = \phi_c + \phi_e \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Consider now the motion corresponding to a value of ϕ, ωxy. It will be observed that ωxy satisfies the equation, ((36) of my former paper,) which ϕ is to satisfy. Corresponding to this value of ϕ we have

$$u = \omega y, \quad v = \omega x, \quad w = 0.$$

Hence the velocity, corresponding to this motion, of a particle of fluid in contact with the cylindrical surface of the vessel, resolved in a direction perpendicular to the surface, is the same as the velocity of the surface itself resolved in the same direction, and therefore the fluid does not penetrate into, nor separate from the cylindrical surface. The velocity of a particle in contact with either of the plane ends, resolved in a direction perpendicular to the surface, is equal and opposite to the velocity of the surface itself resolved in the same direction. Hence we shall get the complete value of ϕ by adding the part already found, namely ωxy, to *twice* the part due to the motion of the plane ends. We have therefore,

$$\phi = \omega xy + 2\phi_e = 2\phi_c - \omega xy, \text{ by (1)} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\text{and} \quad \phi_c - \phi_e = \omega xy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Hence whenever either ϕ_c or ϕ_e can be found, the complete solution of the problem will be given by (2). And even when both these functions can be obtained independently, (2) will enable us to dispense with the use of one of them, and (3) will give a relation between them. In this case (3) will express a theorem in pure analysis, a theorem which will sometimes be very curious, since the analytical expressions for ϕ_c and ϕ_e will generally be totally different in form. The problem admits of solution in the case of a circular cylinder terminated by planes perpendicular to its axis, and in the case of a rectangular parallelepiped. In the former case, the numerical calculation of the moments of inertia of the solid by which the fluid may be replaced would probably be troublesome, in the latter it is extremely easy. I proceed to consider this case in particular.

Let the rectangular axes to which the fluid is referred coincide with three adjacent edges of the parallelepiped, and let a, b, c be the lengths of the edges. The motion which it is proposed to calculate is that which arises from a motion of rotation of the box about an axis parallel to that of z and passing through the centre of the parallelepiped. Consequently in applying (2) we must for a moment conceive the axis of z to pass through the centre of the parallelepiped, and then transfer the origin to the corner, and we must therefore write $\omega\left(x-\frac{a}{2}\right)\left(y-\frac{b}{2}\right)$ for ωxy. In the present case the cylindrical surface consists of the four faces which are parallel to the axis of x, and the remaining faces form the plane ends. The motion of the face xy and the opposite face has evidently no effect on the fluid, so that ϕ_c will be the part of ϕ due to the motion of the face xz and the opposite face. The value of this quantity is given near the top of page 133 in my former paper. We have then by the second of the formulæ (2)

$$\phi=\frac{8\omega a^2}{\pi^3}\Sigma_0\frac{1}{n^3}\frac{(\epsilon^{\frac{n\pi b}{a}}-1)\epsilon^{-\frac{n\pi y}{a}}+(\epsilon^{-\frac{n\pi b}{a}}-1)\epsilon^{\frac{n\pi y}{a}}}{\epsilon^{\frac{n\pi b}{a}}-\epsilon^{-\frac{n\pi b}{a}}}\cos\frac{n\pi x}{a}-\omega\left(x-\frac{a}{2}\right)\left(y-\frac{b}{2}\right)\ldots\ldots(4),$$

the sign Σ_0 denoting the sum corresponding to all odd integral values of n from 1 to ∞. This value of ϕ expresses completely the motion of the fluid due to a motion of rotation of the box about an axis parallel to that of z, and passing through the centre of its interior.

Suppose now the motion to be very small, so that the square of the velocity may be neglected. Then, p denoting the part of the pressure due to the motion, we shall have $p=-\rho\frac{d\phi}{dt}$. Also in finding $\frac{d\phi}{dt}$ we may suppose the axes to be fixed in space, since by taking account of their motion we should only introduce terms depending on the square of the velocity. In fact, if for the sake of distinction we denote the co-ordinates of a fluid particle referred to the moveable axes by x', y', while x, y denote its co-ordinates referred to axes fixed in space, which after differentiation with respect to t we may suppose to coincide with the moveable axes at the instant considered, and if we denote the differential coefficient of ϕ with respect to t by $\left(\frac{d\phi}{dt}\right)$ when x, y, t are the independent variables, and by $\frac{d\phi}{dt}$ when x', y', t are the independent variables, we shall have

$$\left(\frac{d\phi}{dt}\right)=\frac{d\phi}{dt}+\frac{d\phi}{dx'}\frac{dx'}{dt}+\frac{d\phi}{dy'}\frac{dy'}{dt}=\frac{d\phi}{dt}+u\frac{dx'}{dt}+v\frac{dy'}{dt}\,{}^{*};$$

for $\frac{d\phi}{dx'}$, $\frac{d\phi}{dy'}$ mean absolutely the same as $\frac{d\phi}{dx}$, $\frac{d\phi}{dy}$, and are therefore equal to u, v respectively. Now $\frac{dx'}{dt}$, $\frac{dy'}{dt}$, depending on the motion of the axes, are small quantities of the order ω; their values are in fact ωy, $-\omega x$; so that, omitting small quantities of the order ω^2, we have

$$\left(\frac{d\phi}{dt}\right)=\frac{d\phi}{dt}.$$

We shall therefore find the value of p from that of ϕ by merely writing $-\rho\frac{d\omega}{dt}$ for ω. In order

* It may be very easily proved by means of this equation, combined with the general equation which determines p, that whether the velocity be great or small the fluid will have the same effect on the motion of the box as the solid of which the moment of inertia is determined in this paper on the supposition that the motion is small.

to determine the motion of the box it will be necessary to find the resultant of the fluid pressures on its several faces. As shown in my former paper, these pressures will have no resultant force, but only a resultant couple, of which the axis will evidently be parallel to that of z. In calculating this couple, it is immaterial whether we take the moments about the axis of z, or about a line parallel to it passing through the centre of the parallelepiped: suppose that we adopt the latter plan. If we reckon the couple positive when it tends to turn the box from the axis of x to that of y we shall evidently have $-\int_0^a\int_0^c p_{y=0}\left(x-\frac{a}{2}\right)dx\,dz$ for the part arising from the pressure on the face xz, and $\int_0^b\int_0^c p_{x=0}\left(y-\frac{b}{2}\right)dy\,dz$ for the part arising from the pressure on the face yz. It is easily seen from (4) that $p_{x=a}=-p_{x=0}$, and $p_{y=b}=-p_{y=0}$, so that the couples due to the pressures on the faces xz, yz are equal to the couples due to the pressures on the opposite faces respectively. In order, therefore, to find the whole couple we have only got to double the part already found. As the integrations do not present the slightest difficulty, it will be sufficient to write down the result. It will be found that the whole couple is equal to $-C\frac{d\omega}{dt}$, where

$$C=\frac{\rho abc}{12}(b^2-3a^2)+\frac{64\rho a^4c}{\pi^5}\Sigma_0\frac{1}{n^5}\frac{1-\epsilon^{-\frac{n\pi b}{a}}}{1+\epsilon^{-\frac{n\pi b}{a}}}\quad\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

This expression has been simplified after integration by putting for $\Sigma_0\frac{1}{n^4}$ its value $\frac{\pi^4}{96}$.

It appears then that the effect of the inertia of the fluid is to increase the moment of inertia of the box about an axis passing through its centre and parallel to the edge c by the quantity C. In equation (40) of my former paper, there is given an expression for C which is apparently very different from that given by (5), but the numerical values of the two expressions are necessarily the same. If we denote the moment of inertia of the fluid supposed to be solidified by $C_{\prime}$, we shall have $C_{\prime}=\frac{\rho abc}{12}(a^2+b^2)$; and if we put

$$\frac{a}{b}=r,\qquad\frac{C}{C_{\prime}}=f(r),$$

and treat (5) as equation (40) of my former paper was treated, we shall find

$$f(r)=(1+r^2)^{-1}\left\{1-3r^2+2r^3\left(1.260497-1.254821\,\Sigma_0\frac{1}{n^5}\text{ versin }2\theta_n\right)\right\}\quad\ldots\ldots\ldots(6),$$

where,
$$\text{tab. log }\tan\theta_n=10-.6821882\,\frac{n}{r}.$$

The equation (6) is true, (except as regards the decimals omitted,) whatever be the value of r; but for convenience of calculation it will be proper to take r less than 1, that is, to choose for a the smaller of the two a, b. The value of $f(r)$ given by (6) is apparently very different from that given at the bottom of page 134 of my former paper, but any one may easily satisfy himself as to the equivalence of the two expressions by assigning to r a value at random, and calculating the value of $f(r)$ from the two expressions separately. The expression (6) is however preferable to the other, especially when we have to calculate the value of $f(r)$ for small values of r. The infinite series contained in (6) converges with such rapidity that in the most unfavourable case, that is, when $r=1$ nearly, the omission of all terms after the first would only introduce an error of about .000003 in the value of $f(r)$.

For the sake of showing the manner in which $f(r)$ alters with r, I have calculated the following values of the function. The expression (6) shows that $f'(r) = 0$, when $r = 0$; and $f'(r)$ is also $= 0$ when $r = 1$, since $f\left(\frac{1}{r}\right) = f(r)$.

r	$f(r)$	r	$f(r)$
0.0	1	0.6	0.3374
0.1	0.9629	0.7	0.2521
0.2	0.8655	0.8	0.1958
0.3	0.7322	0.9	0.1655
0.4	0.5873	1	0.1565
0.5	0.4512		

The experiments to which I have alluded were made with a wooden box measuring inside 8 inches by 4 square. The box weighed not quite 1 lb., and contained about $4\frac{1}{2}$ lbs. of water, so that the inertia of the water which had to be overcome was by no means small compared with that of the box. The box was suspended by two parallel threads 3 inches apart and between 4 and 5 feet long: it was twisted a little, and then left to itself, so that it oscillated about a vertical axis midway between the threads. The points of attachment of the threads were in a line drawn through the centre of the upper face parallel to one of its sides, and were equidistant from the centre. The weight of the box when empty, the length and distance of the threads, the time of oscillation, and the known length of the seconds' pendulum are data sufficient for determining the moment of inertia of the box about a vertical axis passing through its centre. When the box is filled with water the same quantities determine the moment of inertia of the box and the water it contains, whence the moment of inertia of the water alone is obtained by subtraction. It is suposed here that the centre of gravity of the box coincides with the centre of gravity of its interior volume. In the following experiments a different face of the box was uppermost each time. In Nos. 1 and 2 the long edges of the box were vertical, in Nos. 3 and 4 they were horizontal. In all cases the inertia determined by experiment was a little greater than that resulting from theory: the difference will be given in fractional parts of the latter. The difference was $\frac{1}{21}$ in No. 1, $\frac{1}{13}$ in No. 2, $\frac{1}{17}$ in No. 3, and $\frac{1}{21}$ in No. 4. On referring to the table at the end of the last paragraph, it will be seen that the ratio of the moment of inertia of the fluid to what it would be if the fluid were solid is about three times as great in the last two experiments as in the first two.

I had expected beforehand to find the inertia determined by experiment a little greater than that given by theory, for this reason. In the theory, it is supposed that both the fluid itself and the surface of the box are perfectly smooth. This however is not strictly true. The box by its roughness exerts a tangential force on the fluid immediately in contact with it, and this force produces an effect on the fluid at a small distance from the surface of the box, in consequence of the internal friction of the fluid itself. We may conceive the effect of this force on the time of oscillation in a general way by supposing a thin film of fluid close to the surface of the box to be dragged along with it. Consequently, the moment of inertia determined by experiment will be a little greater than it would have been had the fluid and the surface of the box been perfectly smooth.

These experiments are sufficient to show that in the case of a vessel of about the size and shape of the one I used, filled with water, and performing small oscillations of the duration of about one second, (as was the case in my experiments,) the time of oscillation is not much increased by friction; at least, if we suppose, as there is reason for supposing, that the effect of friction does not depend on the nature of the surface of the box. They are not however sufficiently exact to allow us to place any reliance on the accuracy of the small differences between the results of experiment, and of the common theory of fluid motion, and consequently they are useless as tests of any theory of friction.

G. G. STOKES.

XXXI. *On a New Notation for expressing various Conditions and Equations in Geometry, Mechanics, and Astronomy.* *By the Rev.* M. O'Brien, *late Fellow of Caius College, Professor of Natural Philosophy and Astronomy in King's College, London.*

[Read *November* 23, 1846.]

The notation $Du'.u$, the meaning and use of which is explained in the following pages, denotes a line of a certain length perpendicular to the lines denoted by the symbols u and u'. It is derived from the consideration of the rotation of a rigid body, in which the line u is fixed, about the line u', being, in fact, the differential coefficient of u with respect to the directions of the axes of co-ordinates, the line u' being constant, as will be explained.

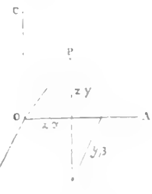
It will be found, that this notation and a corresponding notation, $\Delta u'.u$, have several important properties, that they express with great simplicity several conditions and equations in various parts of Mathematics, and especially in Mechanics, and that they simplify in a remarkable manner several complicated investigations.

The present paper contains an explanation of the meaning of the notation, and its application to Statics, and to the determination of the Rotation of a rigid body about its centre of gravity.

Of the Notation D u′. u.

1. Let us assume the symbols α, β, γ to denote the lines OA, OB, OC, each a unit of length, drawn from an origin O at right angles to each other, so forming a system of three rectangular axes. Let x, y, z denote any three abstract numbers; then $x\alpha$, $y\beta$, $z\gamma$ will denote three lines, drawn along (or parallel to) the three axes, and numerically equal to x, y, z respectively.

(Fig. 1.)



Let OP be any line drawn from O, and let us assume the symbol u to denote OP in magnitude and direction; then, if $x\alpha$, $y\beta$, $z\gamma$ be the co-ordinates of P, we have, according to well-known principles,

$$u = x\alpha + y\beta + z\gamma \qquad (1).$$

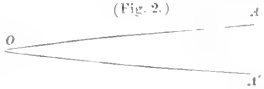
We shall now suppose that the axes OA, OB, OC are capable of motion about the point O, always however remaining at right angles to each other. We shall also suppose that x, y, z are not affected by this motion, or, in other words, that the position of P relatively to OA, OB, OC, does not alter. In fact, we assume that the point P and the axes OA, OB, OC are fixed in a rigid body which is capable of motion about the point O.

Let δ denote any indefinitely small displacement arising from a motion of this kind; then from (1) we have

$$\delta u = x\delta\alpha + y\delta\beta + z\delta\gamma \qquad (2).$$

Now, since α is invariable in length, $\delta\alpha$ denotes a displacement of the point A at right angles to OA: for, let OA' be the line denoted by $\alpha + \delta\alpha$; then, since $OA' = OA + AA'$, we have $\alpha + \delta\alpha = \alpha + AA'$, and therefore $\delta\alpha = AA'$. But, since $OA' = OA$ (α being invariable in length), and since the angle O is indefinitely small, AA' is perpendicular to OA. Hence $\delta\alpha$ denotes a displacement of A at right angles to OA.

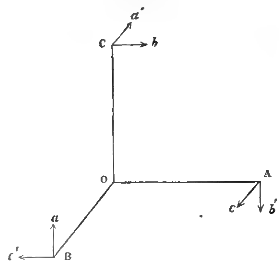
(Fig. 2.)



In like manner $\delta\beta$ and $\delta\gamma$ denote displacements of B and C at right angles to OB and OC respectively.

2. Let the displacement $\delta\alpha$ be resolved into two others, Ac and Ab', of which Ac is parallel to OB, and Ab' to CO. In like manner let $\delta\beta$ be resolved into Ba parallel to OC, and Bc' to AO; and let $\delta\gamma$ be resolved into Cb parallel to OA, and Ca' to BO.

(Fig. 3.)



Also let us denote the numerical magnitudes of these resolved displacements by c, b', a, c', b, a', respectively.

Then, since OA, OB, OC always remain at right angles to each other, it is evident that $a = a'$, $b = b'$, and $c = c'$. Hence, giving these displacements their proper signs of direction, namely β, $-\gamma$, γ, $-\alpha$, α, $-\beta$, respectively, we have,

$$\left.\begin{aligned}\delta\alpha &= Ac + Ab' = \beta c - \gamma b\\ \delta\beta &= Ba + Bc' = \gamma a - \alpha c\\ \delta\gamma &= Cb + Ca' = \alpha b - \beta a\end{aligned}\right\} \quad \text{......................... (3).}$$

The quantities a, b, c here denote any arbitrary numerical differentials.

Making these substitutions in equation (2), we find,

$$\delta u = (zb - yc)\,\alpha + (xc - za)\,\beta + (ya - xb)\,\gamma \quad \text{......... (4).}$$

3. Now it is evident from the nature of the motion which δ denotes, that δu represents an indefinitely small line at right angles to u; therefore, if λ be any numerical arbitrary quantity, $\lambda\delta u$ will represent any line (not necessarily small) at right angles to u. The sign $\lambda\delta$ therefore, written before u, changes u into the symbol of a line at right angles to u, and therefore has somewhat the same effect as the sign $\sqrt{-1}$, or $(-)^{\frac{1}{2}}$. Since however there may be an infinite number of different perpendiculars to u, it remains to put the sign $\lambda\delta$ in such a form as shall indicate what particular perpendicular $\lambda\delta u$ represents. We shall do this in the following manner.

4. Multiplying (4) by λ, and putting $\lambda a = x'$, $\lambda b = y'$, $\lambda c = z'$, we find

$$\lambda\delta u = (zy' - z'y)\,\alpha + (xz' - x'z)\,\beta + (yx' - y'x)\,\gamma \quad \text{...... (5).}$$

Now it is evident from this expression, that $\lambda\delta u$ vanishes when $x = x'$, $y = y'$, $z = z'$; in other words, if we assume

$$u' = x'\alpha + y'\beta + z'\gamma,$$

it follows, that $\lambda\delta u = 0$, when $u = u'$. Therefore $\lambda\delta u$ denotes a differential* of u taken on the supposition that u' is invariable.

On this account we shall replace $\lambda\delta$ by the sign $D_{u'}$, defining $D_{u'}$ to denote a differential taken on the supposition that u' is invariable. We have then,

$$D_{u'}u = (zy' - z'y)\,\alpha + (xz' - x'z)\,\beta + (yx' - y'x)\,\gamma.$$

If we interchange x, y, z, and x', y', z' respectively, this equation becomes

$$D_u u' = (z'y - zy')\,\alpha + (x'z - xz')\,\beta + (y'x - yx')\gamma.$$

Hence we find, that

$$D_u u' = -D_{u'}u.$$

From this equation we may shew that the operation $D_{u'}$ is *distributive* with respect to u'; that is to say, that

* Meaning by the word *differential* here *any* quantity proportional to an indefinitely small difference.

$$D_{u'+u''}(u) = D_{u'}u + D_{u''}u;$$

for we have

$$D_{u'+u''}(u) = -D_u(u'+u''),$$
$$-D_u u' - D_u u''$$
$$= D_{u'}u + D_{u''}u.$$

The operation $D_{u'}$ is therefore distributive with respect to u'.

To indicate that $D_{u'}$ is distributive with respect to u', we shall elevate the subscript index u', and write it in the same line as D, putting a dot between u' and the symbol on which the operation is performed; that is to say, we shall write

$$Du'.u \text{ instead of } D_{u'}u.$$

5. Having thus settled the form of the notation, we shall now interpret the meaning of the expression for $Du'.u$, namely,

$$Du'.u = (zy' - z'y)\,\alpha + (xz' - x'z)\,\beta + (yx' - y'x)\,\gamma \ldots\ldots\ldots (6),$$

from which, as we have seen, immediately follow the two equations

$$Du.u' = -Du'.u \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$
$$D(u'+u'').u = Du'.u + Du''.u \ldots\ldots\ldots (8).$$

1st. To determine the direction of the line $Du'.u$, let

$$Du'.u = x_{,}\alpha + y_{,}\beta + z_{,}\gamma,$$

and therefore, by (6),

$$\left.\begin{array}{l} x_{,} = zy' - z'y \\ y_{,} = xz' - x'z \\ z_{,} = yx' - y'x \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

From these equations we have immediately

$$x_{,}x + y_{,}y + z_{,}z = 0,$$
$$x_{,}x' + y_{,}y' + z_{,}z' = 0.$$

Whence it appears that the line drawn to the point $(x_{,}y_{,}z_{,})$ from O, is at right angles to the line drawn to (xyz) and the line drawn to $(x'y'z')$; in other words, $Du'.u$ is at right angles both to u and u'. This determines the direction of the line $Du'.u$.

2ndly. To determine the magnitude of $Du'.u$, let $r_{,}$, r, and r' denote the magnitudes of $Du'.u$, u, and u' respectively, and let θ be the angle made by u and u': then, by the equations (9),

$$x_{,}^2 + y_{,}^2 + z_{,}^2 = (x^2 + y^2 + z^2)(x'^2 + y'^2 + z'^2) - (xx' + yy' + zz')^2,$$

$$\text{or} \quad r_{,}^2 = r^2 r'^2 - (rr'\cos\theta)^2,$$

$$\text{and therefore} \quad r_{,} = rr'\sin\theta \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

Hence the numerical value of the line $Du'.u$ is the product of the numerical values of the lines u and u' multiplied by the sine of the angle they make with each other.

6. Since $rr'\sin\theta$ is the area of the parallelogram formed upon the lines u and u' as sides, it follows, that $Du'.u$ is a line numerically equal to the area of the parallelogram formed upon u and u', and perpendicular to its plane.

It follows from (7) that $Du.u'$ denotes a line equal in magnitude to $Du'.u$, but opposite in direction.

7. If μ be any numerical quantity, we have

$$D\mu u' . u = - Du . \mu u' = - \mu Du . u' = \mu Du' . u.$$

Hence we have

$$D\mu u' . u = \mu Du' . u \text{ .... } (11).$$

From which it appears that a numerical coefficient of u' may always be brought outside the sign D.

Hence $Du' . u = D(x'a + y'\beta + z'\gamma) . u,$

$= Dx'a . u + Dy'\beta . u + Dz'\gamma . u,$ by (8);

and therefore by (11) $Du' . u = x'Da . u + y'D\beta . u + z'D\gamma . u$ .... (12).

8. In the equation (6) putting all the co-ordinates, except x' and y, equal to zero, we find $Dx'a . y\beta = x'y\gamma$, and $\therefore Da . \beta = \gamma$: and in the same way we may shew that $D\beta . \gamma = a$, and $D\gamma . a = \beta$. We have therefore

$$Da . \beta = \gamma, \quad D\beta . \gamma = a, \quad D\gamma . a = \beta \text{ .... } (13).$$

From these equations we find by (7),

$$D\beta . a = -\gamma, \quad D\gamma . \beta = -a, \quad Da . \gamma = -\beta \text{ .... } (14).$$

Also we evidently have,

$$Du . u = 0 \text{ .... } (15).$$

And therefore

$$Da . a = 0, \quad D\beta . \beta = 0, \quad D\gamma . \gamma = 0 \text{ .... } (16).$$

9. $Du' . u$ is a line proportional to, and drawn in the same direction as the small displacement δu, which displacement takes place on the supposition that u' is invariable: in other words, the displacement δu results from giving a small angular motion, round the axis u', to the rigid body in which OA, OB, OC and P are fixed. From this consideration we may easily see that $Du' . u$ is at right angles to u' and u, and is proportional to $r \sin \theta$*.

It is plain from figure (3), that the rotation by which the displacement δu is generated is *right-handed*, supposing that we look along the axis of rotation (u') towards the origin. We may say, therefore, that $Du' . u$ is generated by right-handed rotation round the axis u'.

10. Since $Da . \beta = \gamma$, and $Da . \gamma = -\beta$, it follows that $(Da)^2 . \beta = -\beta$: and in the same way we may shew that $(Da)^2 . \gamma = -\gamma$; but, since $Da . a = 0$, we have $(Da)^2 . a = 0$, instead of $-a$. Hence $(Da)^2$ written before β or γ is equivalent to the sign $-$, and therefore $Da .$ is equivalent to the sign $(-)^{\frac{1}{2}}$, or $\sqrt{-1}$; but this is not true of $Da .$ written before a. Similar remarks may be made respecting $D\beta$, and $D\gamma$.

In general, we may see from what has been said above, that $(Du')^2 . u = -u$ when the numerical value of u' is unity, and u' is perpendicular to u: in this case, therefore, Du'. is equivalent to $(-)^{\frac{1}{2}}$, or $\sqrt{-1}$.

In this case, therefore, a line numerically equal to u, drawn at an angle θ to u, and at right angles to u', is expressed by the formula

$$u \cos \theta + (Du' . u) \sin \theta, \text{ or } \epsilon^{\theta Du'} \cdot u.$$

11. When two ore more of the symbols $Da .$, $D\beta .$, $D\gamma .$ come together, the order in which they are written must not be changed: thus $D\beta . Da . \beta = a$, but $Da . D\beta . \beta = 0$.

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* The ratio of $Du' . u$ to $r \sin \theta$ is arbitrary; we may therefore assume it to be r', and then we have $Du' . u = rr' \sin \theta$. This is equivalent to the assumption that, $\lambda a = x'$, $\lambda b = y'$, $\lambda c = z'$, in Article 4.

Of the Notation $\Delta u'.u$.

12. In obtaining the notation $Du'.u$ we supposed the axes α, β, γ to be varied in position, but not in length, always remaining at right angles to each other; we shall now obtain another notation by supposing the axes to undergo a different kind of variation.

Let δ denote any variation (whether in length or position) of the axes α, β, γ, xyz being supposed invariable: then

$$\delta u = x\delta\alpha + y\delta\beta + z\delta\gamma.$$

Let us assume that

$$\delta\alpha = x'\delta h, \quad \delta\beta = y'\delta h, \quad \delta\gamma = z'\delta h,$$

where δh is a small displacement in the direction of the line u', or $x'\alpha + y'\beta + z'\gamma$.

Thus we have

$$\delta u = (xx' + yy' + zz')\delta h.$$

$xx' + yy' + zz'$ is therefore the differential coefficient of u, when the axes α, β, γ suffer the variations $x'\delta h$, $y'\delta h$, $z'\delta h$ respectively, *i.e.* when the points A, B, C (fig 1.) receive displacements proportional to x', y', z' respectively in the direction of the line u'. We may therefore represent this differential coefficient by the notation $\Delta_{u'} u$, since the magnitude and direction of the variation of u depends upon u', or is, so to speak, a function of u'. We have therefore

$$\Delta_{u'} u = xx' + yy' + zz'.$$

It is evident from this expression that we may interchange u and u'. Also the operation $\Delta_{u'}$ is clearly distributive, and we shall therefore, as before, write $\Delta u'.u$ instead of $\Delta_{u'} u$. Hence we have,

$$\Delta u'.u = xx' + yy' + zz' \ldots\ldots (17).$$

$$\text{or } \Delta u'.u = rr' \cos\theta \ldots\ldots (18).$$

$$\Delta u'.u = \Delta u.u' \ldots\ldots (19).$$

$$\text{and } \Delta(u' + u'').u = \Delta u'.u + \Delta u''.u \ldots\ldots (20).$$

13. The following formulæ are also evident, namely,

$$\Delta u.u = r^2 \ldots\ldots (21).$$

If u' be at right angles to u, then

$$\Delta u'.u = 0 \ldots\ldots (22).$$

Hence it follows that, whatever u' be,

$$\Delta u'.(Du'.u) = 0 \ldots\ldots (23).$$

14. We may express xyz and u by the following formulæ,

$$\Delta\alpha.u = x, \quad \Delta\beta.u = y, \quad \Delta\gamma.u = z \ldots\ldots (24),$$

$$u = \alpha\Delta\alpha.u + \beta\Delta\beta.u + \gamma\Delta\gamma.u \ldots\ldots (25).$$

(25) may be expressed by saying that

$$\alpha\Delta\alpha + \beta\Delta\beta + \gamma\Delta\gamma = 1.$$

15. Hence we may easily shew that

$$\Delta\alpha . (D\beta . u) = \Delta\gamma . u, \quad \Delta\beta . (D\gamma . u) = \Delta\alpha . u, \text{ \&c. \&c.,}$$

or, omitting u,

$$\left.\begin{array}{lll} \Delta\alpha . D\beta . = \Delta\gamma ., & \Delta\beta . D\gamma . = \Delta\alpha ., & \Delta\gamma . D\alpha . = \Delta\beta . \\ \Delta\beta . D\alpha . = -\Delta\gamma ., & \Delta\gamma . D\beta . = -\Delta\alpha ., & \Delta\alpha . D\gamma . = -\Delta\beta . \end{array}\right\} \ldots (26).$$

Also [or from (23)] it follows, that

$$\Delta\alpha . D\alpha . = 0, \quad \Delta\beta . D\beta . = 0, \quad \Delta\gamma . D\gamma . = 0 \ldots (27).$$

16. It is easy to see that the displacement which gives rise to the differential coefficient $\Delta u' . u$, is caused by a uniform *expansion* of the rigid body (in which the axes and the point P are fixed) in the direction of the line u', the modulus of expansion being proportional to the numerical magnitude of u'. That plane containing the origin which is perpendicular to u' is unaffected by this expansion.

Instances of the application of the Notation Du'. u *and* Δu'. u *to Statics.*

17. The expression u, or

$$x\alpha + y\beta + z\gamma,$$

determines completely the position of the point P; on this account we shall call u the *symbol* of the point P.

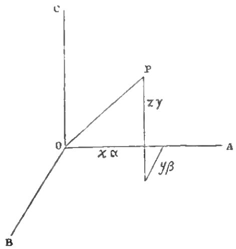
In like manner, if X, Y, Z be the three components of any force, and if

$$U = X\alpha + Y\beta + Z\gamma,$$

U is the symbolical expression for the force, representing it completely in magnitude and direction. We shall therefore call U the symbol of the force whose components are X, Y, and Z.

For brevity we shall generally say, "*the force* U" instead of, "*the force whose symbol is* U;" and, in like manner, "*the point* u," instead of, "*the point whose symbol is* u."

(Fig. 4.)



(I).

18. If the forces U, U', U'', &c. keep a rigid body at rest, the six equations of equilibrium are contained in the following equations, viz.

$$\Sigma U = 0 \ldots\ldots\ldots\ldots (28),$$

$$\Sigma Du . U = 0 \ldots\ldots\ldots\ldots (29).$$

$$\text{For } \Sigma U = \alpha\Sigma X + \beta\Sigma Y + \gamma\Sigma Z,$$

and therefore (28) is equivalent to the three equations

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0.$$

Again, by equation (6) we have,

$$Du . U = (Zy - Yz)\alpha + (Xz - Zx)\beta + (Yx - Xy)\gamma,$$

and therefore (29) is equivalent to the three equations

$$\Sigma(Zy - Yz) = 0, \quad \Sigma(Xz - Zx) = 0, \quad \Sigma(Yx - Xy) = 0.$$

(II).

19. To deduce the equations (28) and (29) immediately from the parallelogram of forces. We must premise the following Lemmas.

20. Lemma 1. If u and u' be any two points situated on the line of direction of the force U, then $Du \,.\, U = Du' \,.\, U$.

For the line $(u' - u)$* coincides in direction with the line U (forces being supposed to be represented by lines); and it is therefore evident from Art. 5, that $D(u' - u) \,.\, U = 0$, *i.e.* $Du' \,.\, U = Du \,.\, U$.

21. Lemma 2. If three forces P, Q, R, applied to a rigid body at the points p, q, r respectively, balance each other, then the conditions of equilibrium are

$$P + Q + R = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(30).$$

$$Dp \,.\, P + Dq \,.\, Q + Dr \,.\, R = 0 \;\ldots\ldots\ldots\ldots \;(31).$$

For P, Q, and R must meet in the same point; let u be that point: also $-R$ must be the resultant of P and Q, and therefore, expressing the parallelogram of forces symbolically, we have. $-R = P + Q$, or,

$$P + Q + R = 0.$$

Now performing the operation $Du \,.$ on this equation, we have

$$Du \,.\, P + Du \,.\, Q + Du \,.\, R = 0,$$

and therefore, by Lemma 1,

$$Dp \,.\, P + Dq \,.\, Q + Dr \,.\, R = 0.$$

Hence the conditions (30) and (31) must hold if P, Q, and R balance each other.

And, conversely, if (30) and (31) be true, the forces P, Q, and R will balance each other. For let u be the point of intersection of P and Q; then, by Lemma 1, we have $Du \,.\, P = Dp \,.\, P$, and $Du \,.\, Q = Dq \,.\, Q$; and therefore by (31), we have

$$Dr \,.\, R = -Du \,.\, (P + Q) = Du \,.\, R, \text{ by (30).}$$

Hence $D(r - u) \,.\, R = 0$, and therefore the line $r - u$ coincides with R in direction, *i.e.* u is a point in the line of direction of R. Hence P, Q, and R meet in the same point u. Also by (30), $-R = P + Q$, *i.e.* $-R$ is the resultant of P and Q. Hence P, Q, and R balance each other if the conditions (30) and (31) be satisfied. These conditions therefore are necessary and sufficient for equilibrium.

22. From these Lemmas we may now prove that the equations (28) and (29) are the necessary and sufficient conditions of equilibrium of a rigid body, acted upon by the forces U, U', U''. &c. at the points u, u', u'', &c.

Choose any three points†, p, q, r, in the rigid body; resolve U into three forces acting along the lines $u - p$, $u - q$, $u - r$, (*i.e.* the lines drawn from p, q, and r to u); let P, Q, R denote these forces respectively; in like manner resolve U' into P', Q', R', acting respectively along the lines $u' - p$, $u' - q$, $u' - r$: treat U'' similarly, and so on.

Then the forces U, U', U'', &c. are reduced to the three sets of forces,

P, P', P'', &c. acting at the point p,

Q, Q', Q'', &c. .... q,

R, R', R'', &c .... r,

* $(u' - u)$ expresses in magnitude and direction the line drawn from the point u to the point u'.

† These points are supposed not to lie in the same right line.

And, by the parallelogram (or rather, the polygon) of forces, these are equivalent to the three forces,

$$\Sigma P \text{ at } p, \quad \Sigma Q \text{ at } q, \quad \Sigma R \text{ at } r.$$

Hence the conditions of equilibrium of these three forces are the conditions of equilibrium of the forces U, U', U'', &c. Therefore, by Lemma 2, the conditions of equilibrium of the forces U, U', U'', &c. are

$$\Sigma P + \Sigma Q + \Sigma R = 0 \text{ .... } (32).$$

$$Dp \, . \, \Sigma P + Dq \, . \, \Sigma Q + Dr \, . \, \Sigma R = 0 \text{ ...... } (33).$$

Now, since U is the resultant of P, Q, and R, we have

$$P + Q + R = U,$$

and therefore (32) becomes, $\Sigma U = 0$.

Also we have

$$Du \, . \, P + Du \, . \, Q + Du \, . \, R = Du \, . \, U,$$

and therefore, by Lemma 1,

$$Dp \, . \, P + Dq \, . \, Q + Dr \, . \, R = Du \, . \, U.$$

Hence (33) becomes $\Sigma Du \, . \, U = 0$.

It appears therefore that the necessary and sufficient conditions of equilibrium of the forces U, U', U'', &c. acting at the points u, u', u'' &c. of a rigid body, are

$$\Sigma U = 0 \text{ .... } (28).$$

$$*\Sigma Du \, . \, U = 0 \text{ .... } (29).$$

(III.)

23 The equation (29) includes the whole theory of couples.

For, suppose the forces U, U', U'', &c. to constitute a set of couples, in other words, suppose, that

$$U' = -U, \quad U''' = -U'', \text{ \&c. \&c.}$$

Then the equation (29) evidently becomes

$$D(u' - u) \, . \, U + D(u''' - u'') \, . \, U'' + \text{\&c.} = 0 \text{ ...... } (34).$$

Now, by Art. (5), if r and R be the numerical magnitudes of $u' - u$ and U, and θ the angle contained by $u' - u$ and U, then the numerical magnitude of $D(u' - u) \, . \, U$ is $Rr \sin \theta$; which is the moment of the couple consisting of U and U'; for $r \sin \theta$ is evidently the perpendicular distance between U and U'. Also $D(u' - u) \, . \, U$ is a line perpendicular to $u' - u$ and U, and therefore to the plane of the couple (U, U'). Hence $D(u' - u) \, . \, U$ is the *axis* of the couple (U, U').

The equation (34) therefore indicates, that the symbolical sum of the axes of a set of couples which balance each other must be zero. Which includes all the propositions of the theory of couples.

(IV.)

24. When the forces U, U', U'', &c. do not balance each other, to find the condition of their having a single resultant.

Suppose that R is the resultant, and r its point of application; then since $-R$, U, U', &c. balance each other, we have, by (28) and (29),

$$\Sigma U - R = 0, \quad \Sigma Du \, . \, U - Dr \, . \, R = 0,$$

* Respecting this equation, we should have remarked, that $Du \, . \, U$ is the symbol of the *axis* of the couple which transfers the force U from the point u to the origin. See Article 24, page 423.

or, putting $\Sigma U = V$, and $\Sigma Du . U = W$, for brevity,

$$R = V, \quad Dr . R = W,$$

$$\text{and therefore, } Dr . V = W \ldots\ldots\ldots\ldots(35).$$

Which equation indicates that V and W are at right angles.

V is evidently the resultant of all the forces, supposing them transferred to the origin in their proper directions; and W is the axis of the resultant of the couples introduced by transferring the forces; for $Du . U$ is evidently the axis of the couple consisting of U acting at u, and $-U$ acting at the origin; and therefore $\Sigma Du . U$ is the sum of the axes of all such couples, and therefore the axis of the resultant couple. Hence the condition of the forces having a single resultant is, that the resultant force (V) shall be at right angles to the axis (W) of the resultant couple.

This condition is simply expressed by the equation,

$$0 = \Delta V . W,$$

which is got immediately by performing the operation ΔV on (35). See Article (13).

25. If we transfer the forces U, U', U'', &c. to any point v, instead of the origin, the resultant couple will be $\Sigma D(u-v) . U$ instead of $\Sigma Du . U$. Now $\Sigma D(u-v) . U = \Sigma Du . U - Dv . \Sigma U = W - Dv . V$. Hence, if we assume $W_{,}$ to denote the resultant couple when the forces are transferred to v, we have

$$W_{,} = W - Dv . V.$$

We may determine the minimum numerical value of $W_{,}$ as follows:

Let λV be the projection of the line W on the line V; then $W - \lambda V$ is perpendicular to V, and is therefore expressed by a symbol of the form $Dv' . V$, where v' denotes a line which we do not require to know.

Hence, we have $W = \lambda V + Dv' . V$, and therefore

$$W_{,} = \lambda V + D(v' - v) . V.$$

Since v is arbitrary, $D(v'-v) . V$ denotes any line whatever at right angles to V: hence the numerical value of $W_{,}$ is least when $D(v'-v) . V = 0$; and therefore $W_{,} = \lambda V$. To determine λ, since $W - \lambda V$ is at right angles to V, we have

$$\Delta V . (W - \lambda V) = 0, \text{ and } \therefore \lambda = \frac{\Delta V . W}{\Delta V . V}.$$

Hence the axis of the couple of minimum moment is

$$\frac{\Delta V . W}{\Delta V . V} . V.$$

We may observe that the equation $W_{,} = \lambda V$ indicates that the axis of the couple of minimum moment ($W_{,}$) is parallel to the resultant force (V).

These instances suffice to shew the application of the notation $Du' . u$, and $\Delta u' . u$ to Statics.

Application of the Notation Du′ . u *and* Δu′ . u *to the Calculation of the Motion of a Rigid Body about its Centre of Gravity.*

26. Let u' be the symbol of the position of any particle (δm) of a rigid body at any time (t). and U the accelerating force which acts upon δm: then, since $\delta m \dfrac{d^2 u'}{dt^2}$ is evidently the symbol of the effective force on δm, the forces $U \delta m$, and $-\dfrac{d^2 u'}{dt^2} \delta m$ applied to δm, and similar forces to the

other particles, must satisfy the conditions of equilibrium. We have therefore by equations (28) and (29),

$$\Sigma\left(U - \frac{d^2u'}{dt^2}\right)\delta m = 0, \qquad \Sigma Du' . \left(U - \frac{d^2u'}{dt^2}\right)\delta m = 0.$$

Let $\bar{u}$ be the symbol of the centre of gravity of the body, and assume $u' = \bar{u} + u$; then these equations become (observing that $\Sigma u \delta m = 0$),

$$m\frac{d^2\bar{u}}{dt^2} = \Sigma U\delta m, \qquad \Sigma Du . \frac{d^2u}{dt^2}\delta m = \Sigma Du . U\delta m.$$

Which equations are equivalent to the six equations of motion of a rigid body.

Since u is the symbol of δm with respect to the centre of gravity as origin, the second of these equations determines the motion of rotation of the rigid body about its centre of gravity, and, as far as this equation is concerned, the centre of gravity may be regarded as a fixed point.

Also, since $\frac{d}{dt}Du . \frac{du}{dt} = D\frac{du}{dt} . \frac{du}{dt} + Du . \frac{d^2u}{dt^2} = Du . \frac{d^2u}{dt^2}$,

this equation may be written in the following form,

$$\frac{d}{dt}\left\{\Sigma Du . \frac{du}{dt}\delta m\right\} = \Sigma Du . U\delta m \;\ldots\ldots(36).$$

27. To effect the integration denoted by Σ in the first member of equation (36).

Take the principal axes through the centre of gravity as the co-ordinate axes, and let x, y, z, be the co-ordinates of δm: then we have

$$u = x\alpha + y\beta + z\gamma,$$

and therefore, since x, y, z are independent of t,

$$\frac{du}{dt} = x\frac{d\alpha}{dt} + y\frac{d\beta}{dt} + z\frac{d\gamma}{dt} \;\ldots\ldots\ldots\ldots(37).$$

Now, referring to Art. 2, we may see immediately, that, if ω_1 denote the velocity of the point B parallel to OC, ω_2 the velocity of C parallel to OA, and ω_3 the velocity of A parallel to OB (in other words, ω_1, ω_2, ω_3, are the angular velocities about the axes OA, OB, OC, of the planes BOC, COA, AOB respectively), then we have

$$a = \omega_1 dt, \qquad b = \omega_2 dt, \qquad c = \omega_3 dt,$$

and therefore the equations (3) become

$$\left.\begin{aligned} \frac{d\alpha}{dt} &= \omega_3\beta - \omega_2\gamma\ ^* \\ \frac{d\beta}{dt} &= \omega_1\gamma - \omega_3\alpha \\ \frac{d\gamma}{dt} &= \omega_2\alpha - \omega_1\beta \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots(38).$$

We may here observe in passing, that, if we assume

$$\omega = \omega_1\alpha + \omega_2\beta + \omega_3\gamma \;\ldots\ldots\ldots\ldots\ldots(39),$$

* If we put these values in (37) the coefficients of α, β, γ are $\omega_2 z - \omega_3 y$, $\omega_3 x - \omega_1 z$, $\omega_1 y - \omega_2 x$; which are the well known expressions of the velocities of any point of a rigid body moving about a fixed point.

the equations (38) become

$$\frac{d\alpha}{dt} = D\omega \,.\, \alpha, \qquad \frac{d\beta}{dt} = D\omega \,.\, \beta, \qquad \frac{d\gamma}{dt} = D\omega \,.\, \gamma \ldots (40),$$

and therefore (37) becomes

$$\frac{du}{dt} = D\omega \,.\, u \ldots\ldots\ldots\ldots\ldots\ldots (41).$$

Now, referring to Art. 5, $D\omega \,.\, u$ is a line drawn perpendicular to u and ω, whose numerical magnitude is $n r \sin \theta$, where n and r are the numerical magnitudes of ω and u respectively, and θ the angle made by ω and u. Hence, the equation (41) indicates that the velocity $\frac{du}{dt}$ is due to the rotation of the rigid body about the axis ω with the angular velocity n. In other words, the symbol ω represents completely the motion of the rigid body; for ω represents, in direction, the axis of instantaneous rotation, and, in numerical magnitude, the angular velocity of the body about that axis.

Returning to equation (36), we find by (37), observing the properties of principal axes,

$$\Sigma D u \,.\, \frac{du}{dt} \delta m = \Sigma \delta m \, D(x\alpha + y\beta + z\gamma) \,.\, \left(x \frac{d\alpha}{dt} + y \frac{d\beta}{dt} + z \frac{d\gamma}{dt}\right)$$

$$= D\alpha \,.\, \frac{d\alpha}{dt} \Sigma \delta m x^2 + D\beta \,.\, \frac{d\beta}{dt} \Sigma \delta m y^2 + D\gamma \,.\, \frac{d\gamma}{dt} \Sigma \delta m z^2.$$

Now, by Art. 8, and by equations (38), we have

$$D\alpha \,.\, \frac{d\alpha}{dt} = \omega_3 \gamma + \omega_2 \beta, \qquad D\beta \,.\, \frac{d\beta}{dt} = \omega_1 \alpha + \omega_3 \gamma, \qquad D\gamma \,.\, \frac{d\gamma}{dt} = \omega_2 \beta + \omega_1 \alpha.$$

Hence we find,

$$\Sigma D u \,.\, \frac{du}{dt} \delta m = \omega_1 \alpha \Sigma (y^2 + z^2) \delta m + \omega_2 \beta \Sigma (z^2 + x^2) \delta m + \omega_3 \gamma \Sigma (x^2 + y^2) \delta m$$

$$= A \omega_1 \alpha + B \omega_2 \beta + C \omega_3 \gamma.$$

Hence the equation (36), cleared of the sign Σ, becomes

$$\frac{d}{dt} \{A \omega_1 \alpha + \beta \omega_2 \beta + C \omega_3 \gamma\} = \Sigma D u \,.\, U \delta m^{*} \ldots (42).$$

---

28. We shall now apply this equation to the problem of Precession and Nutation.

To effect the integration Σ in the second member of (42) when the force U arises from the attraction of a very distant body, which may be supposed to be collected into its centre of gravity.

Let u' be the symbol of the centre of gravity of the distant body, and let m' denote its absolute attractive force; then since $u' - u$ denotes the line drawn from δm to m', the attraction of m' on δm is

$$U = \frac{m'(u' - u)}{\{\Delta (u' - u) \,.\, (u' - u)\}^{\frac{3}{2}}};$$

---

* If we perform the operation $\frac{d}{dt}$ in the first member of this, by means of equations (38), it becomes

$$\left\{A \frac{d\omega_1}{dt} - (B - C)\omega_2 \omega_3\right\} \alpha + \left\{B \frac{d\omega_2}{dt} - (C - A)\omega_3 \omega_1\right\} \beta + \left\{C \frac{d\omega_3}{dt} - (A - B)\omega_1 \omega_2\right\} \gamma.$$

Whence it follows that the first three of Euler's six equations follow immediately from (42).

The last three of Euler's equations follow immediately from the equations (38), in the same manner as I have shown in my Mathematical Tracts.

for this represents a line drawn in the same direction as $u' - u$, and having its numerical magnitude equal to

$$\frac{m'}{R^2},$$

where R is the numerical magnitude of $u' - u$ [observing that $\Delta(u' - u) . (u' - u) = R^2$ (Art. 13, equation 21)].

Now $$\Delta(u' - u) . (u' - u) = \Delta u' . u' - 2\Delta u . u' + \Delta u . u$$
$$= r'^2 - 2\Delta u . u' + r^2, \text{ (Arts. 12 and 13).}$$

Therefore, since $\frac{r}{r'}$ is very small, we have

$$\{\Delta(u' - u) . (u' - u)\}^{\frac{3}{2}} = r'^{-3}\left(1 + 3\frac{\Delta u . u'}{r'^2}\right) \text{ very nearly.}$$

Hence $$U = \frac{m'}{r'^3}\left(1 + 3\frac{\Delta u . u'}{r'^2}\right)(u' - u);$$

and therefore, since $Du . u = 0$, and $Du' . u = -Du . u'$,

$$\Sigma Du . U\delta m = -\frac{m'}{r'^3}Du' . (\Sigma u\delta m + \frac{3}{r'^2}\Sigma u\Delta u . u'\delta m).$$

Now $\Sigma u\delta m = 0$, since the origin is centre of gravity: also, by Art. 12, equation (17), we have, observing the properties of principal axes,

$$\Sigma u\Delta u . u'\delta m = \Sigma(x\alpha + y\beta + z\gamma)(xx' + yy' + zz')\delta m$$
$$= x'\alpha\Sigma x^2\delta m + y'\beta\Sigma y^2\delta m + z'\gamma\Sigma z^2\delta m$$
$$= u'\Sigma r^2\delta m - (Ax'\alpha + By'\beta + Cz'\gamma),$$

since $$\Sigma x^2\delta m = \Sigma r^2\delta m - \Sigma(y^2 + z^2)\delta m, \text{ \&c. \&c.}$$

Hence, since $Du' . u' = 0$, we have $\Sigma Du . U\delta m = \frac{3m'}{r'^5}Du' . (Ax'\alpha + By'\beta + Cz'\gamma)$*.

Thus (42), cleared of the sign Σ, becomes,

$$\frac{d}{dt}(A\omega_1\alpha + B\omega_2\beta + C\omega_3\gamma) = \frac{3m'}{r'^5}Du' . (Ax'\alpha + By'\beta + Cz'\gamma) \text{ ... (44).}$$

---

29. To find the Solar Precession and Nutation by means of this equation.

Let γ be the north polar axis of the Earth; then $B = A$, and C exceeds A by a small quantity, λA suppose, and therefore $C = A(1 + \lambda)$. Hence, observing that $\omega_1\alpha + \omega_2\beta + \omega_3\gamma = \omega$, $x'\alpha + y'\beta + z'\gamma = u'$, and $D\omega . \omega = 0$, $Du' . u' = 0$, (44) becomes

$$\frac{d\omega}{dt} + \lambda\frac{d(\omega_3\gamma)}{dt} = \frac{3m'}{r'^5}\lambda z'Du' . \gamma \text{ .... (45).}$$

In the parts of this equation multiplied by the small quantity λ, we shall suppose that the Earth revolves about its polar axis with a uniform angular velocity, and that the Earth moves round the

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* Performing the operation Du' (i. e. $x'D\alpha . + y'D\beta . + z'D\gamma .$) the second member of (43) becomes

$$\frac{3m'}{r'^5}\{(B - C)y'z'\alpha + (C - A)z'x'\beta + (A - B)x'y'\gamma\}.$$

The coefficients of α, β, γ here are the well-known expressions for the moments of the attraction of Sun or Moon about the principal axes of the Earth.

Sun in a circle uniformly; in other words, we shall suppose that ω_3*, γ and r' are constant. Hence, observing that $z' = \Delta\gamma \, . \, u'$, (45) becomes

$$\frac{d\omega}{dt} = \frac{3m'}{r'^5}\lambda(\Delta u' . \gamma)(Du' . \gamma) \ldots\ldots\ldots\ldots (46).$$

Now let α' and β' be two unit axes at right angles to γ and to each other, one of which (α') points to the first point of Aries: also let β'' be a unit axis pointing to the north solstice: then, if we assume ϖ to denote the obliquity of the ecliptic, and $n't$ the Sun's longitude, we evidently have

$$\beta'' = \beta' \cos\varpi + \gamma \sin\varpi, \qquad u' = r'(\alpha' \cos n't + \beta'' \sin n't),$$

$$\text{and } \therefore \; u' = r'\{\alpha' \cos n't + \beta' \cos\varpi \sin n't + \gamma \sin\varpi \sin n't\}.$$

Hence we have

$$\Delta u' . \gamma = r' \sin\varpi \sin n't,$$

$$Du' . \gamma = - D\gamma \, . \, u' = r'(\alpha' \cos\varpi \sin n't - \beta' \cos n't),$$

and therefore, observing that $n'^2 = \frac{m'}{r'^3}$, (46) becomes

$$\frac{d\omega}{dt} = 3n'^2\lambda \sin\varpi \{\alpha' \cos\varpi \sin^2 n't - \beta' \cos n't \sin n't\} \ldots\ldots\ldots (47).$$

By integrating this equation we find ω, *i.e.*, $\omega_1\alpha + \omega_2\beta + \omega_3\gamma$; and therefore, by equating the coefficients of α, β, γ, we find ω_1, ω_2, ω_3; from which it appears that ω_3 is constant (as has been shewn before), and ω_1, ω_2 are small quantities.

Now, if n denote the numerical magnitude of ω, we have

$$n = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}.$$

Also the sine of the angle which the axis γ makes with the axis ω is

$$\frac{\sqrt{\omega_1^2 + \omega_2^2}}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}}.$$

But, since $n't$ varies but little in one revolution of the Earth, it follows from (47), that we may regard ω_1, ω_2, ω_3, as invariable for one day in quantities multiplied by λ.

Hence it follows, that in a day the axis γ describes a conical surface round the axis ω (*i.e.* the instantaneous axis) with a uniform angular velocity n; and therefore the *mean* daily motion of the axis γ must be the same as the motion of the axis ω; or, in other words, observing that the numerical magnitudes of γ and ω are 1 and n respectively, we have, as far as the mean daily motion of γ is concerned,

$$\frac{d\gamma}{dt} = \frac{d\left(\frac{\omega}{n}\right)}{dt} = \frac{1}{n}\frac{d\omega}{dt}.$$

Hence by (47) we find

$$\frac{d\gamma}{dt} = \frac{3n'^2}{n}\lambda \sin\varpi \{\alpha' \cos\varpi \sin n't - \beta' \sin n't \cos n't\} \ldots (48).$$

Which equation completely determines the motion of γ the Earth's north polar axis.

* It is so easy to see that the coefficient of γ in the first member of (45) is $(1+\lambda)\frac{d\omega_3}{dt}$, and that in the second member it is zero therefore ω_3 is constant, whether λ be small or not.

It appears from this equation that the north pole has two velocities, namely

$\frac{3n'^2}{n}\lambda \sin \varpi \cos \varpi \sin^2 n't$, parallel to α', *i.e.* perpendicular to the solstitial colure;

and $-\frac{3n'^2}{n}\lambda \sin \varpi \sin n't \cos n't$, parallel to β', *i.e.* in the plane of the solstitial colure, and parallel to the equatorial plane.

Hence the length of the path described parallel to α' in any time t is

$$\frac{3n'}{2n}\lambda \sin \varpi \cos \varpi \left(n't - \tfrac{1}{2}\cos 2n't\right),$$

and the path parallel to β' is

$$\frac{3n'}{4n}\lambda \sin \varpi \cos 2n't.$$

Which are the well-known values of the solar precession and nutation of the pole.

The calculation of Lunar Nutation may be effected very simply by the above method; in fact the equation

$$\frac{dy}{dt} = \frac{3m'}{nr'^5}\lambda (\Delta u' . \gamma)(Du' . \gamma);$$

still holds, and we have only to make the proper substitution for u' to suit the Moon's motions, and then integrate as above.

M. O'BRIEN.

Upper Norwood, Surrey,
Nov. 1846.

(NOTE.) In a series of papers on Symbolical Geometry by Sir W. Hamilton, which are at present being published in the *Cambridge and Dublin Mathematical Journal*, a very remarkable interpretation is given to the product of two symbols. According to this interpretation $\frac{1}{2}(uu' + u'u)$ means the same thing as $\Delta u . u'$ in the present paper, and $\frac{1}{2}(uu' - u'u)$ means the same thing as $Du . u'$.

XXXII. *On the Principle of Continuity, in reference to certain Results of Analysis. By* J. R. Young, *Professor of Mathematics in Belfast College.*

[Read *December* 7, 1846.]

The mathematical axiom that "what is true *up to* the limit is true *at* the limit," is necessarily implied in the general principle of Continuity. The recognition of this truth is essential to the very conception of continuity; of which indeed a sufficiently clear idea may be conveyed by the simple enunciation of the axiom itself. In Geometry the continuity here mentioned refers to magnitude only, irrespective of shape: in Analysis it refers simply to value. And in both, the limit spoken of is that, whatever it may be, at which the continuous series of individual cases terminates; or, if the expression be preferred, at which it commences.

It is plain that different continuous series may start from, or terminate in a common boundary: or the terminal limit of one series may be the commencement of another; each series being governed throughout by its own independent law. But there is a liability to suppose the limit unique when it is in reality multiple, or ambiguous; and indeed to confound the true limits with some unique isolated form, having no connexion whatever with either series.

Thus:—the tangent of x, when x commences in the first quadrant and continuously increases, arrives at its limit when x reaches 90^0. In like manner, the tangent of x, when x commences in the second quadrant and continuously diminishes, arrives at its limit when x reaches 90^0. But the two limits (which are very liable to be confounded) are perfectly distinct. In the former case the limit is, $\tan 90^0 = +\infty$: in the latter case, $\tan 90^0 = -\infty$. And, viewing the tangent independently,—that is, as altogether unconnected with a continuous series, and therefore as uncontrolled by any law of continuity,—the tangent of 90^0 is ambiguously $\pm\infty$: and we cannot select one of these values, to the exclusion of the other, without destroying the independence here supposed, and subjecting the tangent to the operation of a law binding it in connexion with a continuous series of tangents.

Again: the limit or extreme case of the continuous series of values of the progression

$$1 - x + x^2 - x^3 + x^4 - x^5 + \&c. \ ad\ inf. \ldots\ldots\ldots\ldots (1),$$

furnished by the continuous variation of x from some *inferior* value up to $x = 1$, or from some *superior* value down to $x = 1$, has been supposed in each case to be properly represented by

$$1 - 1 + 1 - 1 + 1 - 1 + \&c. \ ad\ inf. \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

But it has already been shown by the writer of these remarks*, that so far from this being the common limit, the two limits are totally distinct:—the one having for value $\frac{1}{2}$, and the other *infinity*: whilst the series (2) is not comprehended at all among the continuous cases of (1), but is entirely unconnected with, and independent of, those cases: its value is ambiguously 1 or 0.

In order that the influence of the law of continuity, which connects together all the individual cases of (1), may not be overlooked or evaded in the extreme one of those cases, it will be desirable to change the notation: writing $1 - \frac{1}{z}$ for x, when the limit 1 is to be reached through continuous *ascending* values of x, and $1 + \frac{1}{z}$ when it is to be reached through continuous *descending* values of x.

* *Philosophical Magazine* for November and December 1845.

It will then only be necessary to suppose x to approach infinity as its limit; the connexion of which limit, with the continuous set of values that it terminates, being preserved by the actual exhibition of x in its final state, or under the form ∞; a symbol which, it will be observed, thus *spontaneously* presents itself; and is not arbitrarily *introduced* to effect a purpose.

Hence, when the limit 1 is reached through continuous *ascending* values of x, the extreme case of the series (1) is

$$1-\left(1-\frac{1}{\infty}\right)+\left(1-\frac{1}{\infty}\right)^2-\left(1-\frac{1}{\infty}\right)^3+\&\text{c. } ad\ inf. \dots (3);$$

and when it is reached through descending values, the extreme case of the series is

$$1-\left(1+\frac{1}{\infty}\right)+\left(1+\frac{1}{\infty}\right)^2-\left(1+\frac{1}{\infty}\right)^3+\&\text{c. } ad\ inf. \dots\dots (4).$$

And the values of these, as shown in the publication referred to, are respectively $\frac{1}{2}$ and infinite.

For any *finite* number of terms, these series do not differ sensibly from one another, nor from the neutral or independent series (2). But since we know that $\left(1-\frac{1}{\infty}\right)^{\infty}=\frac{1}{e}$, and $\left(1+\frac{1}{\infty}\right)^{\infty}=e$, it follows that, after a finite number of terms, the three series are totally distinct: and we thus see that in such extreme cases as those we are now considering, it is not allowable,—as generally supposed,—to neglect the terms infinitely remote from the commencement of the series: for it is only in the infinitely remote region that the distinguishing peculiarities of the series become fully developed. And it is because of this, that in contemplating these extreme or limiting cases, *different orders of infinity* become unavoidably forced upon our attention. Thus, in the infinitely remote region of the series (3), it is obvious that there are places for the terms

$$\left(1-\frac{1}{\infty}\right)^{\infty},\quad \left(1-\frac{1}{\infty}\right)^{2\infty},\quad \left(1-\frac{1}{\infty}\right)^{3\infty}\dots\dots\left(1-\frac{1}{\infty}\right)^{\infty'\infty},$$

of which the numerical values are

$$\frac{1}{e},\quad \frac{1}{e^2},\quad \frac{1}{e^3},\ \dots\dots\ \frac{1}{e^{\infty'}}=0.$$

And all these terms, as far as the zero-term, being significant, necessarily affect the numerical expression for the sum of the whole; and cannot be neglected with impunity in a correct estimate of the value of the altogether boundless series (3).

The theorems proposed by Cauchy, for testing the convergency of infinite series, do not apply to the limiting cases, such as those here noticed. These theorems have in fact been the occasion of error in the treatment of those cases; and it is one object of the present communication to invite attention to this circumstance.

In discussing the series

$$\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\&\text{c. } ad\ inf. \dots\dots\dots\dots (5).$$

Cauchy observes* that it will be convergent, or divergent, according as the numerical value of x is inferior, or superior to unity; but that when the limits $x=1$, $x=-1$ are actually reached, the series will be divergent in the first case, and convergent in the second†. This is not a correct account of what happens at the limits: if x ascend from an inferior numerical value (that is from a fractional value, either positive or negative) up to $x=1$, or $x=-1$, the limiting cases will be *convergent*, like all the preceding cases: but if the same limits be reached through descending

* *Cours d'Analyse*, p. 153. † *Ibid.*, p. 155.

values of x, the extreme cases will then, on the contrary, be *divergent*. The truth of this will appear by writing these extreme cases with the proper *symbol or indication of continuity*, introduced, or rather preserved, as in the instances above. For we thus get the converging series

$$\pm\left(1-\frac{1}{\infty}\right)+\tfrac{1}{2}\left(1-\frac{1}{\infty}\right)^2\pm\tfrac{1}{3}\left(1-\frac{1}{\infty}\right)^3+\text{\&c. } ad\ inf.,$$

and the diverging series

$$\pm\left(1+\frac{1}{\infty}\right)+\tfrac{1}{2}\left(1+\frac{1}{\infty}\right)^2\pm\tfrac{1}{3}\left(1+\frac{1}{\infty}\right)^3+\text{\&c. } ad\ inf.,$$

That the former of these is convergent is obvious: and that the latter becomes divergent, in its infinitely remote terms, will be seen from the following considerations:—

As noticed above, $\left(1+\frac{1}{\infty}\right)^{\infty}=e$; so that, in the infinitely remote region, there occur the terms

$$\frac{e}{\infty},\ \frac{e^2}{2\infty},\ \frac{e^3}{3\infty},\ \ldots\ldots\frac{e^{\infty\prime}}{\infty\,.\,\infty'},\ \frac{e^{2\infty'}}{2\infty\,.\,\infty'},\ \frac{e^{3\infty'}}{3\infty\,.\,\infty'},\ \ldots\ldots$$

which evidently diverge after the term $\frac{e^{\infty\prime}}{\infty\,.\,\infty'}$, and, in fact, after $\frac{e}{\infty}$.

Similar reasoning applied to

$$1+x+2x^2+2\,.\,3x^3+2\,.\,3\,.\,4x^4+\text{\&c. } ad\ inf.\ \ldots\ldots\ (6),$$

another of the series considered by Cauchy, and which he affirms to be equal to 1 when x becomes zero, will show, that instead of 1, the value is infinite. For, writing the zero in the allowable form $\frac{1}{\infty}$, we find among the terms infinitely remote, the following: viz.

$$\frac{2\,.\,3\,.\,4\ldots\infty}{\infty^{\infty}},\quad \frac{2\,.\,3\,.\,4\ldots\infty\ldots\infty'}{\infty^{\infty\prime}},\ \text{\&c.}$$

in which, as ∞' may exceed ∞ in any ratio, the numerator may exceed the denominator in any ratio; so that the terms at length become infinitely great; that is to say, the extreme case, corresponding to $x=0$, is like all the other cases, divergent.

The preceding reasonings, in which terms infinitely remote, and infinites of different orders, are considered, may perhaps be regarded as too vague and subtil to justify an unhesitating reception of the conclusions to which they lead: and although they do not appear to me to be fairly chargeable with this objection, yet I wish them to be regarded—less as demonstrations of the truth of these conclusions, than as confirmations, supplied by the laws of analysis—when these are allowed to have their full and unrestricted scope—of the general axiom which stands at the head of this paper; and in virtue of which, if it be demonstrated, that an assigned analytical formula correctly expresses the sum of an infinite series for all cases short of a certain extreme case—however closely to this case we approach,—then we may safely *infer* that it equally, and as correctly, expresses the sum in the extreme case also: a fact which is as necessarily true as any of the axioms of Euclid; and which I think can be questioned only by those who overlook the controlling influence of the law of continuity over these terminal cases. It would be very wrong, in utter neglect of this law, to confound the series

$$1^2-2^2+3^2-4^2+\text{\&c.},$$

for instance, with what

$$1^2-2^2x+3^2x^2-4^2x^3+\text{\&c.}$$

becomes in the extreme case of $x=1$; and thence to assert, as indeed has been done, that its sum is zero, when in reality the sum is $\pm \infty$. The erroneous sums assigned to divergent series, will be found in many other instances besides this, to belong, not to the independent series themselves, but to the extreme cases of certain general forms. Yet the errors adverted to, and which formed the subject of a communication submitted to the British Association in 1844*, are not *always* of this character: the value ·596347..., for instance, assigned by Euler, and many succeeding writers, to the series

$$1 - 1 + 2 - 2 \,.\, 3 + 2 \,.\, 3 \,.\, 4 - \&c.,$$

neither belongs to this series nor yet to the extreme case of the general series (6), in which $x = 1 - \frac{1}{\infty}$; since we have seen that when x becomes $\frac{1}{\infty}$ even, the infinitely remote terms must still diverge.

In the Mémoires on Series and Definite Integrals, which Poisson has published in different Cahiers of the *Journal de l'Ecole Polytechnique*, a fault analogous to that above noticed is very frequently committed†. It is the common practice of this distinguished analyst arbitrarily to introduce the ascending powers of a foreign variable, in connexion with the terms of an isolated and independent series, and then to employ the extreme case of the general form thus obtained, when 1, or rather $1 - \frac{1}{\infty}$ is put for the new variable, instead of the original series. In this way he converts the neutral series $1 - 1 + 1 - 1 + \&c.$ into a convergent series, and thus gets $\frac{1}{2}$ for the sum; which is of course erroneous. He applies the same process to periodic series in general; thus, in fact, destroying their periodicity—at least in the infinitely remote terms—and thence obtains summations that are palpably wrong. Thus, in referring to a particular series of this kind, in his last great work, he says, "Elle est de l'espèce des séries périodiques, qui ne sont ni convergents ni divergents, mais qu'on peut néanmoins employer en les considérant comme les limites de séries convergentes, c'est-à-dire en multipliant leurs termes par les puissances ascendantes d'une quantité infiniment peu différent de l'unité"‡: the inaccuracy of which principle I have, I think, sufficiently discussed elsewhere||.

It is of importance to observe, however, that there is one class of series in reference to which the adoption of this principle is allowable, as its application will be unattended with error:—I mean convergent series. For since, as already shown, the foreign multiplier $1 - \frac{1}{\infty}$, becomes effective only in the terms infinitely remote, and as all these in converging series are themselves zero, these multipliers produce no modification of the character of the series, nor any change in its sum. In *periodic* series however error must of necessity arise from replacing them by the limits of converging series; inasmuch as these latter always tend to some determinate value—either finite or infinite: whereas an infinite periodic series, from its very nature, tends to *indeterminateness*. To attribute a unique value to such a series is therefore absurd.

I have here spoken of the sums of converging series as sometimes tending to *infinity*, which tendency some may suppose to be opposed to convergency: a simple reference however to the series $1 + x + x^2 + \&c.$ will I think correct this supposition, since it will be admitted that this continues convergent for all values of x from $x = \frac{1}{\infty}$ up to $x = 1 - \frac{1}{\infty}$: for which extreme value the sum is infinite §. I have also ventured to call the infinites, to which the extreme cases of certain convergent

* See also *Proceedings of the Royal Irish Academy*, 1845, No. 49, where the communication referred to is printed at length.

† *Journal de l'Ecole Polytechnique*, Cahiers 17, 18, and 19.

‡ *Théorie de la Chaleur*, p. 199.

|| *Philosophical Magazine*, Dec. 1845.

§ The series $1 + \frac{x}{1} + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \&c.$ also, is convergent for all real values of x, and tends to infinity as x does.

series thus tend, determinate: because if we reflect upon the peculiar character of a strictly divergent infinite series, we shall perceive, that however remotely into the region of infinity its terms be considered to extend, yet we can never, even in imagination, reach a stage beyond which the series ceases to be accumulative, and may be rejected as zero: the portion so rejected would, on the contrary, still be infinite; and this is a peculiarity which sufficiently distinguishes a divergent series from a convergent series with an infinite sum. It has place even in those slowly diverging series of which the individual terms continually tend to zero, as, for example, in the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c.$$

$$\text{and} \quad \frac{1}{1\,.\,3} + \frac{2}{3\,.\,5} + \frac{3}{5\,.\,7} + \frac{4}{7\,.\,9} + \&c.$$

for however remote the n^{th} term may be, n terms more of the first of these series will be

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots\ldots \frac{1}{2n},$$

and n terms more of the second,

$$\frac{n+1}{4(n+1)^2 - 1} + \frac{n+2}{4(n+2)^2 - 1} + \ldots\ldots \frac{2n}{4(2n)^2 - 1},$$

and these additional n terms will, in the first case, exceed

$$n \times \frac{1}{2n} \quad \text{or} \quad \frac{1}{2},$$

and in the second case,

$$n \times \frac{1}{8n - \dfrac{1}{2n}} \quad \text{or} \quad \frac{1}{8}.$$

A diverging infinite series therefore tends to no *limit*, either finite or infinite; and this consideration is perhaps sufficient to justify the language of the continental analysts, who say that such series have *no sum*.

It would seem desirable however to divide series into other classes besides convergent, divergent, and periodic; in order to distinguish those which come under the influence of continuity, from those which, like the series just considered, are entirely isolated and independent. The latter class might be called *independent* or *neutral* series; and the former *dependent* series. Hutton* appears to have called the series $1 - 1 + 1 - 1 + 1 - \&c.$ a *neutral* series, simply because it is neither convergent nor divergent. In the sense in which it is here proposed to use the term, no reference is made either to convergency or divergency: but merely to the fact of the series not being united to a set of others by the bond of continuity. A neutral series may therefore be either convergent, divergent, or periodic: the series

$$1 + \frac{1}{1} + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \&c.$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$$

$$1^2 - 2^2 + 3^2 - 4^2 + \&c.$$

$$1 - 1 + 1 - 1 + \&c.$$

* *Mathematical Tracts*, Vol. I. p. 178.

are all neutral. But, as already remarked, since the first of these is *convergent*, its sum does not differ from that of the corresponding dependent series*.

It is not in reference to *series* only that this distinction between neutrality and dependence has been overlooked. It has been improperly neglected in the treatment of an extensive class of definite integrals; all those, namely, that are analogous to periodic series, in respect to the indeterminateness which they involve. It has already been shown what contrivance Poisson resorts to, in order to get rid of this indeterminateness in the series: he destroys the indeterminateness of the integrals by a similar artifice. The series were rendered determinate by multiplying their terms by the ascending powers of a foreign factor; thus bringing them under a law of continuity from which originally they were wholly free. The integrals are rendered, in like manner, determinate by introducing, under the sign of integration, a new variable:—an exponential multiplier, in virtue of the variation of which, a bond of continuity is, as before, imposed upon the expression, and its indeterminateness thus overruled. The following definite integrals quoted from Poisson, and those who have espoused his principles, are all essentially indeterminate:—

$$\int_0^\infty dx \sin rx, \quad \int_0^\infty dx \cos rx, \quad \int_0^\infty dx\, x^{n-1} \sin rx, \quad \int_0^\infty dx\, x^{n-1} \cos rx,$$

$$\int_0^\infty dx\, x^{n-2} \sin rx, \quad \int_0^\infty dx\, x^{n-2} \cos rx, \text{ \&c., \&c.,}$$

the exponents of x in the last four being positive. A very little consideration will suffice to convince us of this: we need only revert to the ordinary ideas involved in the method of quadratures: for if in any of these forms the expression under the integral sign—omitting the dx—represent the ordinate of a curve, we at once see that for $x = \infty$—one of the proposed limits—that ordinate, and therefore the area, or the entire integral, must be indeterminate. By introducing the factor e^{-ax}, for which there is of course not the slightest warranty, these forms become changed into the following:—

$$\int_0^\infty dx\, e^{-ax} \sin rx, \quad \int_0^\infty dx\, e^{-ax} \cos rx, \quad \int_0^\infty dx\, e^{-ax} x^{n-1} \sin rx,$$

$$\int_0^\infty dx\, e^{-ax} x^{n-1} \cos rx, \quad \int_0^\infty dx\, e^{-ax} x^{n-2} \sin rx, \quad \int_0^\infty dx\, e^{-ax} x^{n-2} \cos rx,$$

in reference to which the ordinates, at the limit $x = \infty$, all vanish, irrespective of the value of a. If the integrations be now executed, each result will be a general expression involving a; and if we seek what this expression becomes when a, by continuous variation, arrives at zero, we shall truly obtain the limit of the integral; that is to say, we shall obtain the last of the continuous series of values which the integral passes through as a diminishes continuously, from some superior value, down to zero. These results therefore are all valid, as limits of the changed integrals; but have, in strictness, nothing to do with the integrals originally proposed; these latter being neutral. or independent; and therefore not included in the continuous series of values adverted to.

The impossibility of reconciling some of the erroneous, but prevalent conclusions that have been arrived at respecting the foregoing integrals, with certain known elementary truths, has led one or two recent writers to pass too sweeping a condemnation on integrals of this kind; and to reject, as false, integrations that may easily be proved to be true. I shall advert to some of these presently. But it may not be altogether out of place previously to remark, that much needless ingenuity seems of late to have been expended in proving that sin ∞ and cos ∞ cannot be zero; although such is unhesitatingly affirmed to be the case by the late Mr. Gregory†, and—with misgivings how-

* See Note (B), at the end of this Paper.

† "Both the sine and the cosine of an infinite angle are equal to zero." Gregory's *Examples*, p. 477.

ever—suspected by Mr. De Morgan. But it is proper to state that Poisson nowhere countenances this notion; nor is it implied in his principles, though it has been thought to follow from them.

It is true that Poisson makes

$$\int_0^\infty dx \sin x = 1, \quad \text{and} \int_0^\infty dx \cos x = 0.$$

It is also true that these integrals are respectively $1 - \cos \infty$, and $\sin \infty$: but it does not follow that we have any right to equate Poisson's results with these. Poisson virtually selects a particular value out of the innumerable values of $\cos \infty$; and a particular value out of the innumerable values of $\sin \infty$; these selected values are each *zero*. He does not deny the existence of the *other* values, nor say that $\sin \infty$ and $\cos \infty$ are zero only, as others have said: he expressly declares that he takes that particular value of $\cos \infty$ which unites in continuity with the values of

$$\int_0^\infty dx\, e^{-ax} \sin x;$$

and that particular value of $\sin x$ which unites in continuity with the values of

$$\int_0^\infty dx\, e^{-ax} \cos x,$$

it being understood that a varies from some superior value down to zero; and his doctrine is that, by taking the extreme limit thus reached, he gets, in each case, "une valeur unique qu'on peut employer dans l'analyse." The fault of Poisson consists solely in his bringing indeterminate expressions under the control of arbitrary conditions, in virtue of which that indeterminateness is destroyed, and unique values deduced; and in consequence of which these unique values—as in the instance of the series $1 - 1 + 1 - 1 +$ &c.—are frequently not even among the indeterminate set: but this great man must not be charged with the palpable error of making the sine and cosine of an infinite arc zero*. It should also, in justice to the same illustrious analyst, be observed further, that some English authors, under the impression that they have been carrying out Poisson's views, have also, on other points, employed reasonings, and arrived at conclusions, which those views do not justify. The results which Poisson assigns to the integrations noticed in this paper are all *true as far as they go.* He chooses one out of an infinite variety of equally admissible values, and disregards all the others:—a fault which appears to me to be analogous to that which would be committed by arbitrarily selecting *one* of the n roots of an equation of the n^{th} degree, to be employed in physical applications, and rejecting all the others. But, from a pretty careful examination of Poisson's different Memoires on Series and Definite Integrals, I can find no foundation for the statement recently made, that "Poisson would admit $1^2 - 2^2 + 3^2 - 4^2 + \ldots\ldots = 0$." He *rejects* diverging series: and in applying his principles to cases where divergency might be suspected, he takes care, in order to justify his mode of proceeding, to remove the suspicion, by showing that the series must be convergent. (See *Théorie de la Chaleur*, p. 188.)

Resuming now the consideration of the definite integrals, I have to remark, that among those that have been rejected are

$$\int_0^\infty \frac{\sin ax}{x}\, dx \quad \text{and} \quad \int_0^\infty \frac{\cos ax}{1 + x^2}\, dx:$$

the grounds of this rejection being that these integrals have not the values hitherto assigned

* "Les sinus et cosinus d'un arc infini sont évidemment des quantités indeterminées." Poisson: *Journal de l'Ecole Polytech.* Cah. XIX. p. 407.

"La manière dont nous avons considéré les séries périodique infinies, s'applique également aux intégrals définies de quantités périodique, que s'etendent à l'infini: ces integrals n'ont aussi des valeurs déterminées, que quand on les regarde comme les limites d'autres intégrales, dont les élémens convergent vers zero, et sont nuls a l'infini." *Ibid.*, p. 431.

to them, but are, on the contrary, indeterminate—like those already noticed *. That they are not indeterminate however will be obvious from again adverting to the notion of quadratures: the ordinates of the curves are evidently determinate throughout the whole extent of the integration,—that at the superior limit ∞ being zero. The first of these integrals has been proved—by what appears to me to be perfectly valid reasoning, though it has recently been objected to—to be altogether independent of the *value* of the constant a, and to be equal to $\frac{\pi}{2}$, or $-\frac{\pi}{2}$, according as the sign of this constant is positive or negative. Poisson indeed, following Euler and others, says that the values of the integral are $\frac{\pi}{2}$, 0, or $-\frac{\pi}{2}$, according as the constant is positive, zero, or negative †. But it should be remembered, in obedience to the law of continuity, that if a become zero, by passing through neighbouring values, and vanish *positively*, the value of the integral is still $\frac{\pi}{2}$; and if it vanish *negatively*, the value of the integral is still $-\frac{\pi}{2}$, as in all the continuous series of cases which these terminate.

The integration of the second of the preceding forms has however been effected by methods which are really objectionable, notwithstanding the accuracy of the results obtained by them: and it may not be uninstructive briefly to direct attention to this circumstance.

Legendre commences his process by at once destroying the generality of the proposed integral—taking for the limits, not $x = 0$, and $x = \infty$, but $x = 0$, and $x = \frac{2k\pi}{a}$, k being a whole number; and then, at a convenient stage of the investigation, making k infinite. By means of this artifice, the indeterminateness, which the method employed would otherwise have introduced at the limit $x = \infty$, is overruled by an arbitrary condition ‡. The true result however necessarily comes out; because that result is independent of *all* condition as to how the limit ∞ is reached.

In the other method of integration, the indeterminateness adverted to is not evaded, but is allowed to enter into the process: it is however wholly disregarded; and thus, by a sort of compensation of errors, the true result is again obtained. This, I presume, is the method to which Sir. W. R. Hamilton alludes, at page 16 of his profound and remarkable paper on Fluctuating Functions §, where an accurate investigation of this integral is given ||.

It may be proper to add, that when by applying differentiation to a determinate form, whether an infinite series or a definite integral, we are led to indeterminateness, the step must be regarded as inadmissible, and unless corrected, as leading to a false result. It is not difficult to see the reason of this. In each case a certain *constant* is considered to be *infinite;* for which extreme value a particular function of the variable, that for all other values of the constant would have entered the original expression, disappears; but which function if preserved, instead of being obliterated as zero, would reappear in an indeterminate form, after differentiation. The suppression however of the evanescent function in the original, precludes this reappearance; and thus leads to a defective result ¶. This, I think, is rather an interesting fact: it shows that the *differentials* of certain forms of analysis require indeterminate corrections, in a manner somewhat analogous to that by which the ordinary determinate corrections are introduced into *integrals;* and the omission of which indeterminate corrections has led to so many erroneous summations of certain trigonometrical series. From

* *Transactions of the Society*, Vol. VIII. Part III. Earnshaw's Paper on sin ∞ and cos ∞. It may be remarked here, in reference to the two integrals in the text, that the function under the sign of integration becomes in each case zero at the superior limit ∞: and that therefore, as was before observed of periodic series, the foreign factor, e^{-ex}, which Poisson introduces merely to destroy indeterminateness at this limit, is inoperative, and may therefore be admitted without incurring error: and the same remark applies whenever the subject of integration, in integrals of this kind, becomes zero for $x = \infty$.

† *Chaleur*, p. 288.

‡ Legendre: *Exercises de Calcul Intégral*, Tome I. p. 357.

§ *Transactions of the Royal Irish Academy*, Vol. XIX. Pt. II.

|| For the faulty process, see Gregory's *Examples*, p. 481.

¶ See a Paper by the author in the *Phil. Mag.* Vol. XXVIII. p. 213.

this omission, too, we further see how it happens that, in enquiries of this kind, we may be led from premisses absolutely *wrong*, and that by a train of correct reasoning, to conclusions absolutely *right*. We have only to take the results of differentiation here noticed, each with the indeterminate constant suppressed, and which are thus erroneous, and to apply the reverse process of integration, in order to arrive at correct forms. Thus Poisson, starting from the false equation

$$\tfrac{1}{2} = \cos\theta - \cos 2\theta + \cos 3\theta - \cos 4\theta + \&c.^{*},$$

in which he supposes $\theta < \pi$, multiplies by $d\theta$ and integrates; thus obtaining the true equation

$$\frac{\theta}{2} = \sin\theta - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \frac{\sin 4\theta}{4} + \&c. \ldots (A),$$

and from a second integration, the other true equation

$$\frac{\pi}{12} - \frac{\theta^2}{4} = \cos\theta - \frac{\cos 2\theta}{4} + \frac{\cos 3\theta}{9} - \frac{\cos 4\theta}{16} + \&c.$$

Again: proceeding from the false equation,

$$0 = \sin\theta - \sin 3\theta + \sin 5\theta - \sin 7\theta + \&c.,$$

he arrives, in a similar manner, at the true results

$$\frac{\pi}{4} = \cos\theta - \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5} - \frac{\cos 7\theta}{7} + \&c. \ldots (B),$$

$$\text{and} \quad \frac{\pi\theta}{4} = \sin\theta - \frac{\sin 3\theta}{9} + \frac{\sin 5\theta}{25} - \&c.;$$

in reference to which however, from neglecting the principle of continuity, he commits the error of supposing (A) to fail when $\theta = \pi$, and (B) to fail when $\theta = \frac{\pi}{2}$; although, in virtue of that principle, both must necessarily hold†.

As supplementary to the foregoing observations on the principle of continuity, I would wish to add a remark or two in reference to what has been called *discontinuity*:—a term which, I think, is sometimes injudiciously employed in analysis. Many expressions called discontinuous, should rather be considered as composed of different continuous groups united together under one general form. Distinct continuities, so to speak, may be comprehended in one and the same function; and it is obvious that these may be separately discussed, and the aggregate of the entire group estimated, without at all introducing the idea of discontinuity. For instance, certain functions, submitted to integration, become infinite between assigned limits of x:—would it not be better, and indeed more accurate, to say, of such functions, that each consists of two continuous series of values, within the proposed limits, both series terminating at the same absolute value of x, than to say that the function becomes discontinuous for that value? To obtain the definite integral in such a case, we should only have first to integrate over one of the continuous series of values, then to integrate over the other continuous series, and to unite the results, taking special care that the terminal or initial value of x, which unites the two series, obeys the law of continuity impressed upon each. And in this way may the integration be correctly executed, however often infinity may occur between the proposed limits. The definite integral $\int_{-n}^{+n} x^{-1}dx$ may serve for illustration. The function

* That this equation is false, has already been shown by the author in the *Phil. Mag.* for December, 1845. But it is sufficient to observe, both with respect to this equation and that next quoted, that it is impossible, from the character of sin ∞ and cos ∞, that the series-side of either can be a determinate quantity. I suppose Poisson considers the powers of his arbitrary multiplier, "infiniment peu différente de l'unité," to be virtually present in these series, to destroy their periodic character. But this does not interfere with the principle in the text.

† See *Journal de l'Ecole Polytech.*, Cahier XVIII. pp. 313–5.

x^{-1} becomes infinite for $x = 0$; so that we have two continuous series of values, each terminating, or each commencing at $x = 0$; which value of x however is united to one series by the sign *plus*, and to the other by the sign *minus*. Hence, integrating over the former series, we have $\log n - \log 0$; and integrating over the latter, we have $\log(-m) - \log(-0)$. Consequently

$$\int_{-m}^{+n} x^{-1}\,dx = \{\log n - \log 0\} - \{\log(-m) - \log(-0)\}$$

$$= \log\frac{n}{0} - \log\frac{m}{0} = \log\frac{n}{m}.$$

There is of course nothing new in thus dividing a definite integral into portions: but the treating of these portions, when their boundaries are infinite, as distinct continuities, allowing the influence of each continuous law to operate throughout the entire range, the limits included:—this mode, I say, of treating what are called discontinuous functions, is not that generally adopted; though the neglect of it has occasioned a difficulty that has appeared to interfere with the clearness of the idea of a definite integral when considered as the limit of a summation. Moreover, from this same neglect, Poisson and others have been led to very erroneous values for the definite integrals included in the form $\int_{-m}^{+n} x^{-p}\,dx$. Thus Poisson affirms that

$$\int_{-1}^{+1} x^{-1}\,dx = -(2n+1)\,\pi\sqrt{-1}^{*},$$

an imaginary quantity, instead of zero as above: and the value of $\int_{-1}^{+1} x^{-2}\,dx$ he states to be -2, instead of infinite, as it is found to be by the method here proposed, which gives

$$\int_{-1}^{+1} x^{-2}\,dx = (+\infty - 1) - (-\infty + 1) = 2\infty - 2,$$

and many other such errors might, if necessary, be adduced from his writings.

But the examples already given of the influence of the principle of continuity in extreme or limiting cases of general forms, and of the mistakes committed by analysts from disregarding this influence, will, I think, be considered as sufficient to invite more general attention to this matter: and I shall rejoice if the brief and imperfect sketch I have here attempted to give of the views and principles, by conforming to which such mistakes may be avoided, meet with acceptance from the Cambridge Philosophical Society. I have been induced to submit it to the indulgent consideration of that distinguished body, chiefly because the topics embraced in it have already furnished matter for two Papers printed in the *Cambridge Transactions*:—one by Professor De Morgan, and the other by the Rev. Mr. Earnshaw. I have ventured to entertain the opinion that the views and investigations of these excellent analysts do not preclude the necessity for a further consideration of the interesting and somewhat delicate points of analysis which they have discussed: an opinion which is strengthened by the fact, that the Papers referred to are in a considerable degree opposed to each other, both in principle and in result. It is scarcely necessary to add, that in the present communication I have contemplated the subject under an aspect more or less different from that in which it has been considered either by Mr. De Morgan, or by Mr. Earnshaw; and I think it probable, from the study of the three Papers, that the truth may be elicited; and something like consistency and stability be at length given to a portion of analytical science which has long been affected with much uncertainty, vagueness, and perplexity.

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* *Journal de l'Ecole Polytechnique*, Cah. XVIII. p. 318.

Of the two NOTES which follow, the second has already been referred to, by anticipation, in the text: the first is intended further to confirm and establish the accuracy of the general theory which pervades this Paper.

NOTE (A.)

IT appears from the preceding observations that in certain infinite series involving a quantity subject to continuous variation we are presented in the extreme or limiting cases, with instances of what may be called *insensible convergency* and *insensible divergency*. The peculiarity of such cases consists in this:—that, within a finite extent of a certain infinite range of terms, the convergency or divergency of the series is insensible; so that for such a finite extent the series does not sensibly differ from what I have proposed to call a neutral or independent series. When however we pass beyond this finite range, and in imagination contemplate the terms infinitely remote, we at once recognize the accumulated effect of these insensible variations; and the convergency or divergency of the series becomes abundantly apparent. The infinitely remote term at which this fact discovers itself, is alike the termination of one infinite range of terms and the commencement of another: the completion of which, if the expression may be allowed, shows the effect of the insensible variations through a second infinite range, and so on.

We are thus unavoidably led to the contemplation of different orders of infinites and different orders of zeros—things altogether beyond the reach of actual ocular examination. But those who take that comprehensive view of the scope and powers of analysis, which its own well-established results, and the practice of those most deeply imbued with its spirit so fully justify, will not, I think, found any objection to the reasonings in the foregoing Paper on this circumstance. In fact, in the common doctrine of vanishing fractions, the very same principles are virtually recognized: the symbols $\frac{0}{0}$ and $\frac{\infty}{\infty}$, which ought perhaps rather to be written $\frac{0}{0'}$ and $\frac{\infty}{\infty'}$, may each represent any ratio whatever:—even infinity: so that the reasonings adverted to involve in them nothing repugnant to generally received conclusions. The symbols here noticed, when really determinate, are so solely in consequence of their being governed by the principle of continuity. This is pretty generally admitted: but there are certain other results of analysis, which the same principle equally controls, but over which its influence is little suspected. Every one admits the truth of the equation $a^0 = 1$, whatever be the value of a, without any reference to the law of continuity: yet if we reason from this equation—still keeping the conditions of continuity out of sight—we shall speedily be led to conclusions of a very startling character, as follows:—

$$a^0 = 1\,; \quad \therefore\ a = 1^{\frac{1}{0}} = 1^{\infty},$$

that is to say, unity raised to the power infinity is equal to any quantity whatever!

NOTE (B.)

IT was observed at page 432 that every neutral converging series might, without error, be replaced by the corresponding dependent series. This observation might have been rendered more comprehensive; for diverging series, whose terms continually tend to zero, might also have been included, since the dependent series, corresponding to these, have infinite sums, as well as the independent diverging series themselves. These infinites however are not strictly identical in the two cases: and by saying that the one series may replace the other, nothing more is meant than that the sum in either case will be infinite. Poisson, Abel, and others, have shown that

$$\frac{1}{2}\log(1 + 2a\cos\phi + a^2) = a\cos\phi - \frac{1}{2}a^2\cos 2\phi + \frac{1}{3}a^3\cos 3\phi - \&c.$$

whenever the second member is a converging series. Abel says, "Pour avoir les sommes de ces séries lorsque $\alpha = +1$ ou -1, il faut seulement faire α converger vers cette limite *:" and he then writes

$$[A] \begin{cases} \frac{1}{2}\log(2+2\cos\phi) = \cos\phi - \frac{1}{2}\cos 2\phi + \frac{1}{3}\cos 3\phi - \&c. \\ \frac{1}{2}\log(2-2\cos\phi) = -\cos\phi - \frac{1}{2}\cos 2\phi - \frac{1}{3}\cos 3\phi - \&c. \end{cases}$$

which he says, "a lieu pour toute valeur de ϕ excepté pour $\phi = (2\mu + 1)\pi$ dans la première expression, et pour $\phi = 2\mu\pi$ dans la seconde." Now the second members of these equations are not the limits of the proposed general form, any more than $1 - 1 + 1 - \&c.$ is the limit of $1 - x + x^2 - \&c.$ A limit always implies continuity, and is never exempt from the control of that principle: putting therefore the condition of continuity in evidence, the preceding expressions should be written

$$\frac{1}{2}\log(2+2\cos\phi) = \left(1 - \frac{1}{\infty}\right)\cos\phi - \frac{1}{2}\left(1 - \frac{1}{\infty}\right)^2\cos 2\phi + \frac{1}{3}\left(1 - \frac{1}{\infty}\right)^3\cos 3\phi - \&c.$$

$$\frac{1}{2}\log(2-2\cos\phi) = -\left(1 - \frac{1}{\infty}\right)\cos\phi - \frac{1}{2}\left(1 - \frac{1}{\infty}\right)^2\cos 2\phi - \frac{1}{3}\left(1 - \frac{1}{\infty}\right)^3\cos 3\phi - \&c.$$

which are true, whatever be the value of ϕ, for the series are always convergent. As ϕ tends to the values excepted to by Abel, the series tend to infinity; which they actually attain when these excepted values are reached, as the first members sufficiently show. We thus see that

$$1 + \frac{1}{2}\left(1 - \frac{1}{\infty}\right) + \frac{1}{3}\left(1 - \frac{1}{\infty}\right)^2 + \frac{1}{4}\left(1 - \frac{1}{\infty}\right)^3 + \&c.$$

is infinite as well as

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$$

In like manner, from the development of $\log \frac{1+n}{1-n}$, we should infer that

$$1 + \frac{1}{3}\left(1 - \frac{1}{\infty}\right) + \frac{1}{5}\left(1 - \frac{1}{\infty}\right)^2 + \frac{1}{7}\left(1 - \frac{1}{\infty}\right)^3 + \&c.$$

is infinite as well as

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \&c.,$$

and thence that

$$\frac{1}{1\,.\,3} + \frac{2}{3\,.\,5}\left(1 - \frac{1}{\infty}\right) + \frac{3}{5\,.\,7}\left(1 - \frac{1}{\infty}\right) + \&c.$$

is infinite as well as

$$\frac{1}{1\,.\,3} + \frac{2}{3\,.\,5} + \frac{3}{5\,.\,7} + \&c.$$

so that any of these diverging infinite series may be replaced by the corresponding dependent converging series, and *vice versâ*, without numerical error. And *à priori* considerations, in reference to this class of diverging series, would lead us to the same conclusion. The equations [A] are thus universally true without any exception whatever.

* *Œuvres Complètes.* Tome I. p. 89.

Belfast, *September*, 1846.

XXXIII. *On the Theory of Oscillatory Waves.* *By* G. G. STOKES, M.A., *Fellow of Pembroke College.*

[Read *March* 1, 1847.]

IN the Report of the Fourteenth Meeting of the British Association for the Advancement of Science it is stated by Mr. Russell, as a result of his experiments, that the velocity of propagation of a series of oscillatory waves does not depend on the height of the waves*. A series of oscillatory waves, such as that observed by Mr. Russell, does not exactly agree with what it is most convenient, as regards theory, to take as the type of oscillatory waves. The extreme waves of such a series partake in some measure of the character of solitary waves, and their height decreases as they proceed. In fact it will presently appear that it is only an indefinite series of waves which possesses the property of being propagated with a uniform velocity, and without change of form: at least this is the case when the waves are such as can be propagated along the surface of a fluid which was previously at rest. The middle waves, however, of a series such as that observed by Mr. Russell agree very nearly with oscillatory waves of the standard form. Consequently, the velocity of propagation determined by the observation of a number of waves, according to Mr. Russell's method, must be very nearly the same as the velocity of propagation of a series of oscillatory waves of the standard form, and whose length is equal to the mean length of the waves observed, which are supposed to differ from each other but slightly in length.

On this account I was induced to investigate the motion of oscillatory waves of the above form to a second approximation, that is, supposing the height of the waves finite, though small. I find that the expression for the velocity of propagation is independent of the height of the waves to a second approximation. With respect to the form of the waves, the elevations are no longer similar to the depressions, as is the case to a first approximation, but the elevations are narrower than the hollows, and the height of the former exceeds the depth of the latter. This is in accordance with Mr. Russell's remarks at page 448 of his first Report†. I have proceeded to a third approximation in the particular case in which the depth of the fluid is very great, so as to find in this case the most important term, depending on the height of the waves, in the expression for the velocity of propagation. This term gives an increase in the velocity of propagation depending on the square of the ratio of the height of the waves to their length.

There is one result of a second approximation which may possibly be of practical importance. It appears that the forward motion of the particles is not altogether compensated by their backward motion; so that, in addition to their motion of oscillation, the particles have a progressive motion in the direction of propagation of the waves. In the case in which the depth of the fluid is very great, this progressive motion decreases rapidly as the depth of the particle considered increases. Now when a ship at sea is overtaken by a storm, and the sky remains overcast, so as to prevent astronomical observations, there is nothing to trust to for finding the ship's place but the dead reckoning. But the estimated velocity and direction of motion of the ship are her velocity and direction of motion relatively to the water. If then the whole of the water near the surface be moving in the direction of the waves, it is evident that the ship's estimated place will be erroneous. If, however, the velocity of the water can be expressed in terms of the length and height of the waves, both which can be observed approximately from the ship, the motion of the water can be allowed for in the dead reckoning.

* Page 369 (note), and page 370.

† *Reports of the British Association*, Vol. VI.

As connected with this subject, I have also considered the motion of oscillatory waves propagated along the common surface of two liquids, of which one rests on the other, or along the upper surface of the upper liquid. In this investigation there is no object in going beyond a first approximation. When the specific gravities of the two fluids are nearly equal, the waves at their common surface are propagated so slowly that there is time to observe the motions of the individual particles. The second case affords a means of comparing with theory the velocity of propagation of oscillatory waves in extremely shallow water. For by pouring a little water on the top of the mercury in a trough we can easily procure a sheet of water of a small, and strictly uniform depth, a depth, too, which can be measured with great accuracy by means of the area of the surface and the quantity of water poured in. Of course, the common formula for the velocity of propagation will not apply to this case, since the motion of the mercury must be taken into account.

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1. IN the investigations which immediately follow, the fluid is supposed to be homogeneous and incompressible, and its depth uniform. The inertia of the air, and the pressure due to a column of air whose height is comparable with that of the waves are also neglected, so that the pressure at the upper surface of the fluid may be supposed to be zero, provided we afterwards add the atmospheric pressure to the pressure so determined. The waves which it is proposed to investigate are those for which the motion is in two dimensions, and which are propagated with a constant velocity, and without change of form. It will also be supposed that the waves are such as admit of being excited, independently of friction, in a fluid which was previously at rest. It is by these characters of the waves that the problem will be rendered determinate, and not by the initial disturbance of the fluid, supposed to be given. The common theory of fluid motion, in which the pressure is supposed equal in all directions, will also be employed.

Let the fluid be referred to the rectangular axes of x, y, z, the plane xz being horizontal, and coinciding with the surface of the fluid when in equilibrium, the axis of y being directed downwards, and that of x taken in the direction of propagation of the waves, so that the expressions for the pressure, &c. do not contain z. Let p be the pressure, ρ the density, t the time, u, v the resolved parts of the velocity in the directions of the axes of x, y; g the force of gravity, h the depth of the fluid when in equilibrium. From the character of the waves which was mentioned last, it follows by a known theorem that $u\,dx + v\,dy$ is an exact differential $d\phi$. The equations by which the motion is to be determined are well known. They are

$$p = g\rho y - \rho \frac{d\phi}{dt} - \frac{\rho}{2}\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right\}, \quad \ldots\ldots (1);$$

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0, \quad \ldots\ldots (2);$$

$$\frac{d\phi}{dy} = 0, \text{ when } y = h, \quad \ldots\ldots (3);$$

$$\frac{dp}{dt} + \frac{d\phi}{dx}\frac{dp}{dx} + \frac{d\phi}{dy}\frac{dp}{dy} = 0, \text{ when } p = 0, \quad \ldots\ldots (4);$$

where (3) expresses the condition that the particles in contact with the rigid plane on which the fluid rests remain in contact with it, and (4) expresses the condition that the same surface of particles continues to be the free surface throughout the motion, or, in other words, that there is no generation or destruction of fluid at the free surface.

If c be the velocity of propagation, u, v and p will be by hypothesis functions of $x - ct$ and y. It follows then from the equations $u = \frac{d\phi}{dx}$, $v = \frac{d\phi}{dy}$ and (1), that the differential coefficients of ϕ with respect to x, y and t will be functions of $x - ct$ and y; and therefore ϕ itself must be of the form $f(x - ct,\ y) + Ct$. The last term will introduce a constant into (1); and if this constant be expressed, we may suppose ϕ to be a function of $x - ct$ and y. Denoting $x - ct$ by x', we have

$$\frac{dp}{dx} = \frac{dp}{dx'},\quad \frac{dp}{dt} = -c\frac{dp}{dx'},$$

and similar equations hold good for ϕ. On making these substitutions in (1) and (4), omitting the accent of x, and writing $-gk$ for C, we have

$$p = g\rho(y + k) + c\rho\frac{d\phi}{dx} - \frac{\rho}{2}\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right\}, \ldots\ldots\ldots\ldots (5),$$

$$\left(\frac{d\phi}{dx} - c\right)\frac{dp}{dx} + \frac{d\phi}{dy}\frac{dp}{dy} = 0, \text{ when } p = 0. \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Substituting in (6) the value of p given by (5), we have

$$g\frac{d\phi}{dy} - c^2\frac{d^2\phi}{dx^2} + 2c\left(\frac{d\phi}{dx}\frac{d^2\phi}{dx^2} + \frac{d\phi}{dy}\frac{d^2\phi}{dx\,dy}\right)$$

$$-\left(\frac{d\phi}{dx}\right)^2\frac{d^2\phi}{dx^2} - 2\frac{d\phi}{dx}\frac{d\phi}{dy}\frac{d^2\phi}{dx\,dy} - \left(\frac{d\phi}{dy}\right)^2\frac{d^2\phi}{dy^2} = 0, \ldots\ldots\ldots (7).$$

when $$g(y + k) + c\frac{d\phi}{dx} - \frac{1}{2}\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right\} = 0. \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

The equations (7) and (8) are exact; but if we suppose the motion small, and proceed to the second order only of approximation, we may neglect the last three terms in (7), and we may easily eliminate y between (7) and (8). For putting ϕ', $\phi_{,}$, &c. for the values of $\frac{d\phi}{dx}$, $\frac{d\phi}{dy}$, &c. when $y = 0$, the number of accents above marking the order of the differential coefficient with respect to x, and the number below its order with respect to y, and observing that k is a small quantity of the first order at least, we have from (8)

$$g(y + k) + c(\phi' + \phi_{,}'y) - \tfrac{1}{2}(\phi'^2 + \phi_{,}^2) = 0.$$

whence $$y = -k - \frac{c}{g}\phi' + \frac{c}{g}\phi_{,}'\left(k + \frac{c}{g}\phi'\right) + \frac{1}{2g}(\phi'^2 + \phi_{,}^2).^* \ldots\ldots (9).$$

Substituting the first approximate value of y in the first two terms of (7), putting $y = 0$ in the next two, and reducing, we have

$$g\phi_{,} - c^2\phi'' - (g\phi_{,,} - c^2\phi_{,}'')\left(k + \frac{c}{g}\phi'\right) + 2c(\phi'\phi'' + \phi_{,}\phi_{,}') = 0. \ldots (10).$$

ϕ will now have to be determined from the general equation (2) with the particular conditions (3) and (10). When ϕ is known, y, the ordinate of the surface, will be got from (9), and k will then be determined by the condition that the mean value of y shall be zero. The value of p, if required, may then be obtained from (5).

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* The reader will observe that the y in this equation is the ordinate of the surface, whereas the y in (1) and (2) is the ordinate of any point in the fluid. The context will always show in which sense y is employed.

2. In proceeding to a first approximation we have the equations (2), (3) and the equation obtained by omitting the small terms in (10), namely,

$$g\frac{d\phi}{dy} - c^2\frac{d^2\phi}{dx^2} = 0, \text{ when } y = 0. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11).$$

The general integral of (2) is

$$\phi = \Sigma A\epsilon^{mx+ny},$$

the sign Σ extending to all values of A, m and n, real or imaginary, for which $m^2 + n^2 = 0$: the particular values of ϕ, $Cx + C'$, $Dy + D'$, corresponding respectively to $n = 0$, $m = 0$, must also be included, but the constants C', D' may be omitted. In the present case, the expression for ϕ must not contain real exponentials in x, since a term containing such an exponential would become infinite either for $x = -\infty$, or for $x = +\infty$, as well as its differential coefficients which would appear in the expressions for u and v; so that m must be wholly imaginary. Replacing then the exponentials in x by circular functions, we shall have for the part of ϕ corresponding to any one value of m,

$$(A\epsilon^{my} + A'\epsilon^{-my}) \sin mx + (B\epsilon^{my} + B'\epsilon^{-my}) \cos mx,$$

and the complete value of ϕ will be found by taking the sum of all possible particular values of the above form and of the particular value $Cx + Dy$. When the value so formed is substituted in (3), which has to hold good for all values of x, the coefficients of the several sines and cosines, and the constant term must be separately equated to zero. We have therefore

$$D = 0, \quad A' = \epsilon^{2mh}A, \quad B' = \epsilon^{2mh}B;$$

so that if we change the constants we shall have

$$\phi = Cx + \Sigma\,(\epsilon^{m(h-y)} + \epsilon^{-m(h-y)})\,(A \sin mx + B \cos mx), \ldots \quad (12),$$

the sign Σ extending to all real values of m, A and B, of which m may be supposed positive.

3. To the term Cx in (12) corresponds a uniform velocity parallel to x, which may be supposed to be impressed on the fluid in addition to its other motions. If the velocity of propagation be defined merely as the velocity with which the wave form is propagated, it is evident that the velocity of propagation is perfectly arbitrary. For, for a given state of relative motion of the parts of the fluid, the velocity of propagation, as so defined, can be altered by altering the value of C. And in proceeding to the higher orders of approximation it becomes a question what we shall define the velocity of propagation to be. Thus, we might define it to be the velocity with which the wave form is propagated when the mean horizontal velocity of a particle in the upper surface is zero, or the velocity of propagation of the wave form when the mean horizontal velocity of a particle at the bottom is zero, or in various other ways. The following two definitions appear chiefly to deserve attention.

First, we may define the velocity of propagation to be the velocity with which the wave form is propagated in space, when the mean horizontal velocity *at each point of space occupied by the fluid* is zero. The term mean here refers to the variation of the time. This is the definition which it will be most convenient to employ in the investigation. I shall accordingly suppose $C = 0$ in (12), and c will represent the velocity of propagation according to the above definition.

Secondly, we may define the velocity of propagation to be the velocity of propagation of the wave form in space, when the mean horizontal velocity of the mass of fluid comprised between two very distant planes perpendicular to the axis of x is zero. The mean horizontal velocity of the mass means here the same thing as the horizontal velocity of its centre of gravity. This appears to be the most natural definition of the velocity of propagation, since in the case considered there is no current in the mass of fluid, taken as a whole. I shall denote the velocity of propagation according to this definition by c'. In the most important case to consider, namely, that in which the depth is infinite, it is easy to see that $c' = c$, whatever be the order of approximation. For when the depth becomes infinite, the velocity of the centre of gravity of the mass comprised between any two planes parallel to the plane yz vanishes, provided the expression for u contain no constant term.

4. We must now substitute in (11) the value of ϕ.

$$\phi = \Sigma\,(\epsilon^{m(h-y)} + \epsilon^{-m(h-y)})\,(A \sin mx + B \cos mx) \;\ldots\ldots\ldots\ldots\; (13);$$

but since (11) has to hold good for all values of x, the coefficients of the several sines and cosines must be separately equal to zero: at least this must be true, provided the series contained in (11) are convergent. The coefficients will vanish for any one value of m, provided

$$c^2 = \frac{g}{m}\,\frac{\epsilon^{mh} - \epsilon^{-mh}}{\epsilon^{mh} + \epsilon^{-mh}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (14).$$

Putting for shortness $2mh = \mu$, we have

$$\frac{d \log c^2}{d\mu} = -\frac{1}{\mu} + \frac{2}{\epsilon^{\mu} - \epsilon^{-\mu}},$$

which is positive or negative, μ being supposed positive, according as

$$2\mu >< \epsilon^{\mu} - \epsilon^{-\mu} >< 2\left(\mu + \frac{\mu^3}{1.2.3} + \ldots\ldots\right),$$

and is therefore necessarily negative. Hence the value of c given by (14) decreases as μ or m increases, and therefore (11) cannot be satisfied, for a given value of c, by more than one positive value of m. Hence the expression for ϕ must contain only one value of m. Either of the terms $A \cos mx$, $B \sin mx$ may be got rid of by altering the origin of x. We may therefore take, for the most general value of ϕ,

$$\phi = A\,(\epsilon^{m(h-y)} + \epsilon^{-m(h-y)}) \sin mx \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (15).$$

Substituting in (8), we have for the ordinate of the surface

$$y = -\frac{mAc}{g}\,(\epsilon^{mh} + \epsilon^{-mh}) \cos mx \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (16).$$

k being $= 0$, since the mean value of y must be zero. Thus everything is known in the result except A and m, which are arbitrary.

5. It appears from the above, that of all waves for which the motion is in two dimensions, which are propagated in a fluid of uniform depth, and which are such as could be propagated into fluid previously at rest, so that $udx + vdy$ is an exact differential, there is only one particular kind, namely, that just considered, which possesses the property of being propagated with a constant velocity, and without change of form; so that a solitary wave cannot be propagated in this manner. Thus the degradation in the height of such waves, which Mr. Russell observed, is not to be attributed wholly, (nor I believe chiefly,) to the imperfect fluidity of the fluid, and its adhesion to the sides and bottom of the canal, but it is an essential characteristic of a solitary wave. It is true that this conclusion depends on an investigation which applies strictly to indefinitely small motions only: but if it were true in general that a solitary wave could be propagated uniformly, without degradation, it would be true in the limiting case of indefinitely small motions; and to disprove a general proposition it is sufficient to disprove a particular case.

6. In proceeding to a second approximation we must substitute the first approximate value of ϕ, given by (15), in the small terms of (10). Observing that $k = 0$ to a first approximation, and eliminating g from the small terms by means of (14), we find

$$g\phi_{,} - c^2\phi'' - 6A^2m^3c \sin 2mx = 0 \;\ldots\ldots\ldots\; (17).$$

The general value of ϕ given by (13), which is derived from (2) and (3), must now be restricted to satisfy (17). It is evident that no new terms in ϕ involving $\sin mx$ or $\cos mx$ need be introduced, since such terms may be included in the first approximate value, and the only other term which can enter is one of the form $B\,(\epsilon^{2m(h-y)} + \epsilon^{-2m(h-y)}) \sin 2mx$. Substituting this term in (17), and simplifying by means of (14), we find

$$B = \frac{3mA^2}{c\,(\epsilon^{mh} - \epsilon^{-mh})^2}.$$

Moreover since the term in ϕ containing $\sin mx$ must disappear from (17), the equation (14) will give c to a second approximation.

If we denote the coefficient of $\cos mx$ in the first approximate value of y, the ordinate of the surface, by a, we shall have

$$A = -\frac{ga}{mc\,(\epsilon^{mh} + \epsilon^{-mh})} = -\frac{ca}{\epsilon^{mh} - \epsilon^{-mh}};$$

and substituting this value of A in that of ϕ, we have

$$\phi = -ac\,\frac{\epsilon^{m(h-y)} + \epsilon^{-m(h-y)}}{\epsilon^{mh} - \epsilon^{-mh}} \sin mx + 3ma^2c\,\frac{\epsilon^{2m(h-y)} + \epsilon^{-2m(h-y)}}{(\epsilon^{mh} - \epsilon^{-mh})^4} \sin 2mx \;\ldots\; (18).$$

The ordinate of the surface is given to a second approximation by (9). It will be found that

$$y = a\cos mx - ma^2\,\frac{(\epsilon^{mh} + \epsilon^{-mh})(\epsilon^{2mh} + \epsilon^{-2mh} + 4)}{2\,(\epsilon^{mh} - \epsilon^{-mh})^3} \cos 2mx \;\ldots\ldots\ldots\ldots\ldots\; (19),$$

$$k = \frac{ma^2}{\epsilon^{2mh} - \epsilon^{-2mh}}.$$

7. The equation to the surface is of the form

$$y = a\cos mx - Ka^2 \cos 2mx \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (20),$$

where K is necessarily positive, and a may be supposed to be positive, since the case in which it is negative may be reduced to that in which it is positive by altering the origin of x by the quantity $\frac{\pi}{m}$ or $\frac{\lambda}{2}$, λ being the length of the waves. On referring to (20) we see that the waves are symmetrical with respect to vertical planes drawn through their ridges, and also with respect to vertical planes drawn through their lowest lines. The greatest depression of the fluid occurs when $x = 0$ or $= \pm\lambda$, &c., and is equal to $a - a^2K$: the greatest elevation occurs when $x = \pm\frac{\lambda}{2}$ or $= \pm\frac{3\lambda}{2}$, &c., and is equal to $a + a^2K$. Thus the greatest elevation exceeds the greatest depression by $2a^2K$. When the surface cuts the plane of mean level, $\cos mx - aK\cos 2mx = 0$. Putting in the small term in this equation the approximate value $mx = \frac{\pi}{2}$, we have $\cos mx = -aK = \cos\left(\frac{\pi}{2} + aK\right)$ whence $x = \pm\left(\frac{\lambda}{4} + \frac{aK\lambda}{2\pi}\right)$, $= \pm\left(\frac{5\lambda}{4} + \frac{aK\lambda}{2\pi}\right)$, &c. We see then that the breadth of each hollow, measured at the height of the plane of mean level, is $\frac{\lambda}{2} + \frac{aK\lambda}{\pi}$, while the breadth of each elevated portion of the fluid is $\frac{\lambda}{2} - \frac{aK\lambda}{\pi}$.

It is easy to prove from the expression for K, which is given in (19), that for a given value of λ or of m, K increases as h decreases. Hence the difference in form of the elevated and depressed portions of the fluid is more conspicuous in the case in which the fluid is moderately shallow than in the case in which its depth is very great compared with the length of the waves.

8. When the depth of the fluid is very great compared with the length of a wave, we may without sensible error suppose h to be infinite. This supposition greatly simplifies the expressions already obtained. We have in this case

$$\phi = -ac\epsilon^{-my} \sin mx \quad \ldots\ldots (21),$$

$$y = a \cos mx - \tfrac{1}{2} ma^2 \cos 2mx \quad \ldots\ldots (22),$$

$$k = 0, \quad K = \frac{m}{2} = \frac{\pi}{\lambda}, \quad c^2 = \frac{g\lambda}{2\pi},$$

the y in (22) being the ordinate of the surface.

It is hardly necessary to remark that the state of the fluid at any time will be expressed by merely writing $x - ct$ in place of x in all the preceding expressions.

9. To find the nature of the motion of the individual particles, let $x + \xi$ be written for x, $y + \eta$ for y, and suppose x and y to be independent of t, so that they alter only in passing from one particle to another, while ξ and η are small quantities depending on the motion. Then taking the case in which the depth is infinite, we have

$$\frac{d\xi}{dt} = u = -mac\epsilon^{-m(y+\eta)} \cos m(x + \xi - ct) = -mac\epsilon^{-my} \cos m(x - ct) + m^2ac\epsilon^{-my} \sin m(x - ct) \,.\, \xi$$
$$+ m^2ac\epsilon^{-my} \cos m(x - ct) \,.\, \eta, \text{ nearly},$$

$$\frac{d\eta}{dt} = v = mac\epsilon^{-m(y+\eta)} \sin m(x + \xi - ct) = mac\epsilon^{-my} \sin m(x - ct) + m^2ac\epsilon^{-my} \cos m(x - ct) \,.\, \xi$$
$$- m^2ac\epsilon^{-my} \sin m(x - ct) \,.\, \eta, \text{ nearly}.$$

To a first approximation

$$\xi = a\epsilon^{-my} \sin m(x - ct), \quad \eta = a\epsilon^{-my} \cos m(x - ct),$$

the arbitrary constants being omitted. Substituting these values in the small terms of the preceding equations, and integrating again, we have

$$\xi = a\epsilon^{-my} \sin m(x - ct) + m^2a^2ct\epsilon^{-2my},$$

$$\eta = a\epsilon^{-my} \cos m(x - ct).$$

Hence the motion of the particles is the same as to a first approximation, with one important difference, which is that in addition to the motion of oscillation the particles are transferred forwards, that is, in the direction of propagation, with a constant velocity depending on the depth, and decreasing rapidly as the depth increases. If U be this velocity for a particle whose depth below the surface in equilibrium is y, we have

$$U = m^2a^2c\epsilon^{-2my} = a^2\left(\frac{2\pi}{\lambda}\right)^{\frac{3}{2}} g^{\frac{1}{2}} \epsilon^{-\frac{4\pi y}{\lambda}} \quad \ldots\ldots (23).$$

The motion of the individual particles may be determined in a similar manner when the depth is finite from (18). In this case the values of ξ and η contain terms of the second order, involving respectively $\sin 2m(x - ct)$ and $\cos 2m(x - ct)$, besides the term in ξ which is multiplied by t. The most important thing to consider is the value of U, which is

$$U = m^2a^2c \frac{\epsilon^{2m(y-h)} + \epsilon^{-2m(y-h)}}{(\epsilon^{mh} - \epsilon^{-mh})^2} \quad \ldots\ldots (24).$$

Since U is a small quantity of the order a^2, and in proceeding to a second approximation the velocity of propagation is given to the order a only, it is immaterial which of the definitions of velocity of propagation mentioned in Art. 3, we please to adopt.

10. The waves produced by the action of the wind on the surface of the sea do not probably differ very widely from those which have just been considered, and which may be regarded as the typical form of oscillatory waves. On this supposition the particles, in addition to their motion of oscillation, will have a progressive motion in the direction of propagation of the waves, and consequently in the direction of the wind, supposing it not to have recently shifted, and this progressive motion will decrease rapidly as the depth of the particle considered increases. If the pressure of the air on the posterior parts of the waves is greater than on the anterior parts, in consequence of the wind, as unquestionably it must be, it is easy to see that some such progressive motion must be produced. If then the waves are not breaking, it is probable that equation (23), which is applicable to deep water, may give approximately the mean horizontal velocity of the particles; but it is difficult to say how far the result may be modified by friction. If then we regard a ship as a mere particle, in the first instance, for the sake of simplicity, and put U_0 for the value of U when $y = 0$, it is easy to see that after sailing for a time t, the ship must be a distance $U_0 t$ to the lee of her estimated place. It will not however be sufficient to regard the ship as a mere particle, on account of the variation of the factor ϵ^{-2my}, as y varies from 0 to the greatest depth of the ship below the surface of the water. Let δ be this depth, or rather a depth something less, in order to allow for the narrowing of the ship towards the keel, and suppose the effect of the progressive motion of the water on the motion of the ship to be the same as if the water were moving with a velocity the same at all depths, and equal to the mean value of the velocity U from $y = 0$ to $y = \delta$. If U_1 be this mean velocity,

$$U_1 = \frac{1}{\delta}\int_0^\delta U dy = \frac{ma^2c}{2\delta}\left(1 - \epsilon^{-2m\delta}\right).$$

On this supposition, if a ship be steered so as to sail in a direction making an angle θ with the direction of the wind, supposing the water to have no current, and if V be the velocity with which the ship moves through the water, her actual velocity will be the resultant of a velocity V in the direction just mentioned, which, for shortness, I shall call the direction of steering, and of a velocity U_1 in the direction of the wind. But the ship's velocity as estimated by the log-line is her velocity relatively to the water at the surface, and is therefore the resultant of a velocity V in the direction of steering, and a velocity $U_0 - U_1$ in a direction opposite to that in which the wind is blowing. If then E be the estimated velocity, and if we neglect U^2,

$$E = V - (U_0 - U_1)\cos\theta.$$

But the ship's velocity is really the resultant of a velocity $V + U_1 \cos\theta$ in the direction of steering, and a velocity $U_1 \sin\theta$ in the perpendicular direction, while her estimated velocity is E in the direction of steering. Hence, after a time t, the ship will be a distance $U_0 t \cos\theta$ ahead of her estimated place, and a distance $U_1 t \sin\theta$ aside of it, the latter distance being measured in a direction perpendicular to the direction of steering, and on the side towards which the wind is blowing.

I do not suppose that the preceding formula can be employed in practice; but I think it may not be altogether useless to call attention to the importance of having regard to the magnitude and direction of propagation of the waves, as well as to the wind, in making the allowance for lee-way.

11. The formulæ of Art. 6 are perfectly general as regards the ratio of the length of the waves to the depth of the fluid, the only restriction being that the height of the waves must be sufficiently small to allow the series to be rapidly convergent. Consequently, they must apply to the limiting case, in which the waves are supposed to be extremely long. Hence long waves, of the kind considered, are propagated without change of form, and the velocity of propagation is independent of the height of the waves to a second approximation. These conclusions might seem, at first sight,

at variance with the results obtained by Mr. Airy for the case of long waves *. On proceeding to a second approximation, Mr. Airy finds that the form of long waves alters as they proceed, and that the expression for the velocity of propagation contains a term depending on the height of the waves. But a little attention will remove this apparent discrepancy. If we suppose mh very small in (19), and expand, retaining only the most important terms, we shall find for the equation to the surface

$$y = a \cos mx - \frac{3a^2}{4m^2h^3} \cos 2mx.$$

Now, in order that the method of approximation adopted may be legitimate, it is necessary that the coefficient of $\cos 2mx$ in this equation be small compared with a. Hence $\frac{a}{m^2h^3}$, and therefore $\frac{\lambda^2 a}{h^3}$ must be small, and therefore $\frac{a}{h}$ must be small compared with $\left(\frac{h}{\lambda}\right)^2$. But the investigation of Mr. Airy is applicable to the case in which $\frac{\lambda}{h}$ is very large; so that in that investigation $\frac{a}{h}$ is large compared with $\left(\frac{h}{\lambda}\right)^2$. Thus the difference in the results obtained corresponds to a difference in the physical circumstances of the motion.

12. There is no difficulty in proceeding to the higher orders of approximation, except what arises from the length of the formulæ. In the particular case in which the depth is considered infinite, the formulæ are very much simpler than in the general case. I shall proceed to the third order in the case of an infinite depth, so as to find in that case the most important term, depending on the height of the waves, in the expression for the velocity of propagation.

For this purpose it will be necessary to retain the terms of the third order in the expansion of (7). Expanding this equation according to powers of y, and neglecting terms of the fourth, &c. orders, we have

$$g\phi_{,} - c^2\phi'' + (g\phi_{,,} - c^2\phi_{,}'')\,y + (g\phi_{,,,} - c^2\phi_{,,}'')\,\frac{y^2}{2} + 2c\,(\phi'\phi'' + \phi_{,}\phi_{,}')$$

$$+\, 2c\,(\phi_{,}'\phi'' + \phi'\phi_{,}'' + \phi_{,,}\phi_{,}' + \phi_{,}\phi_{,,}')y - \phi'^2\phi'' - 2\phi'\phi_{,}\phi_{,}' - \phi_{,}^2\phi_{,,} = 0. \;\ldots\ldots\; (25).$$

In the small terms of this equation we must put for ϕ and y their values given by (21) and (22) respectively. Now since the value of ϕ to a second approximation is the same as its value to a first approximation, the equation $g\phi_{,} - c^2\phi'' = 0$ is satisfied to terms of the second order. But the coefficients of y and $\frac{y^2}{2}$, in the first line of (25), are derived from the left-hand member of the preceding equation by inserting the factor ϵ^{-my}, differentiating either once or twice with respect to y, and then putting $y = 0$. Consequently these coefficients contain no terms of the second order, and therefore the terms involving y in the first line of (25) are to be neglected. The next two terms are together equal to $c\frac{d}{dx}(\phi'^2 + \phi_{,}^2)$. But

$$\phi'^2 + \phi_{,}^2 = m^2a^2c^2,$$

which does not contain x, so that these two terms disappear. The coefficient of y in the second line of (25) may be derived from the two terms last considered in the manner already indicated, and therefore the terms containing y will disappear from (25). The only small terms

* *Encyclopædia Metropolitana, Tides and Waves*, Articles 198, &c.

remaining are the last three, and it will easily be found that their sum is equal to $m^4 a^3 c^3 \sin mx$, so that (25) becomes

$$g\,\phi_{,} - c^2 \phi'' + m^4 a^3 c^3 \sin mx = 0 \ldots\ldots\ldots\ldots\ldots\ldots (26).$$

The value of ϕ will evidently be of the form $A\epsilon^{-my} \sin mx$. Substituting this value in (26), we have

$$(m^2 c^2 - mg)\,A + m^4 a^3 c^3 = 0.$$

Dividing by mA, and putting for A and c^2 their approximate values $-ac$, $\dfrac{g}{m}$ respectively in the small term, we have

$$mc^2 = g + m^2 a^2 g,$$

whence

$$c = \left(\frac{g}{m}\right)^{\frac{1}{2}}\left(1 + \tfrac{1}{2} m^2 a^2\right) = \left(\frac{g\lambda}{2\pi}\right)^{\frac{1}{2}}\left(1 + \frac{2\pi^2 a^2}{\lambda^2}\right).$$

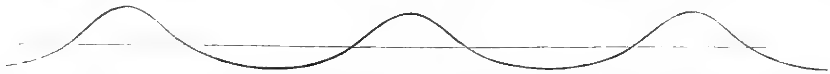
The equation to the surface may be found without difficulty. It is

$$y = a \cos mx - \tfrac{1}{2} m a^2 \cos 2mx + \tfrac{3}{8} m^2 a^3 \cos 3mx^*, \ldots\ldots\ldots\ldots\ldots (27):$$

we have also

$$k = 0, \quad \phi = -ac\,(1 - \tfrac{5}{8} m^2 a^2)\,\epsilon^{-my} \sin mx.$$

The following figure represents a vertical section of the waves propagated along the surface of deep water. The figure is drawn for the case in which $a = \dfrac{7\lambda}{80}$. The term of the third order in (27) is retained, but it is almost insensible. The straight line represents a section of the plane of mean level.



13. If we consider the manner in which the terms introduced by each successive approximation enter into equations (7) and (8), we shall see that, whatever be the order of approximation, the series expressing the ordinate of the surface will contain only cosines of mx and its multiples, while the expression for ϕ will contain only sines. The manner in which y enters into the coefficient of $\cos rmx$ in the expression for ϕ is determined in the case of a finite depth by equations (2) and (3). Moreover, the principal part of the coefficient of $\cos rmx$ or $\sin rmx$ will be of the order a^r at least. We may therefore assume

$$\phi = \Sigma_1^{\infty}\, a^r A_r (\epsilon^{rm(h-y)} + \epsilon^{-rm(h-y)}) \sin rmx,$$

$$y = a \cos mx + \Sigma_2^{\infty}\, a^r B_r \cos rmx,$$

and determine the arbitrary coefficients by means of equations (7) and (8), having previously expanded these equations according to ascending powers of y. The value of c^2 will be determined by equating to zero the coefficient of $\sin mx$ in (7).

Since changing the sign of a comes to the same thing as altering the origin of x by $\frac{1}{2}\lambda$, it is plain that the expressions for A_r, B_r and c^2 will contain only even powers of a. Thus the values of each of these quantities will be of the form $C_0' + C_1 a^2 + C_2 a^4 + \ldots$

It appears also that, whatever be the order of approximation, the waves will be symmetrical with respect to vertical planes passing through their ridges, as also with respect to vertical planes passing through their lowest lines.

* It is remarkable that this equation coincides with that of the prolate cycloid, if the latter equation be expanded according to ascending powers of the distance of the tracing point from the centre of the rolling circle, and the terms of the fourth order be omitted. The prolate cycloid is the form assigned by Mr. Russell to waves of the kind here considered. *Reports of the British Association*, Vol. VI. p. 448. When the depth of the fluid is not great compared with the length of a wave, the form of the surface does not agree with the prolate cycloid even to a second approximation.

14. Let us consider now the case of waves propagated at the common surface of two liquids, of which one rests on the other. Suppose as before that the motion is in two dimensions, that the fluids extend indefinitely in all horizontal directions, or else that they are bounded by two vertical planes parallel to the direction of propagation of the waves, that the waves are propagated with a constant velocity, and without change of form, and that they are such as can be propagated into, or excited in the fluids supposed to have been previously at rest. Suppose first that the fluids are bounded by two horizontal rigid planes. Then taking the common surface of the fluids when at rest for the plane xz, and employing the same notation as before, we have for the under fluid

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0, \quad \text{(28)},$$

$$\frac{d\phi}{dy} = 0 \text{ when } y = h, \quad \text{(29)},$$

$$p = C + g\rho y + c\rho \frac{d\phi}{dx},$$

neglecting the squares of small quantities. Let $h_,$ be the depth of the upper fluid when in equilibrium, and let $p_,$, $\rho_,$, $\phi_,$, $C_,$ be the quantities referring to the upper fluid which correspond to p, ρ, ϕ, C referring to the under: then we have for the upper fluid

$$\frac{d^2\phi_,}{dx^2} + \frac{d^2\phi_,}{dy^2} = 0 \quad \text{(30)},$$

$$\frac{d\phi_,}{dy} = 0 \text{ when } y = -h_, \quad \text{(31)},$$

$$p_, = C_, + g\rho_, y + c\rho_, \frac{d\phi_,}{dx}.$$

We have also, for the condition that the two fluids shall not penetrate into, nor separate from each other,

$$\frac{d\phi}{dy} = \frac{d\phi_,}{dy}, \text{ when } y = 0 \quad \text{(32)}.$$

Lastly, the condition answering to (11) is

$$g\left(\rho\frac{d\phi}{dy} - \rho_,\frac{d\phi_,}{dy}\right) - c^2\left(\rho\frac{d^2\phi}{dx^2} - \rho_,\frac{d^2\phi_,}{dx^2}\right) = 0 \quad \text{(33)},$$

when $$C - C_, + g(\rho - \rho_,)\,y + c\left(\rho\frac{d\phi}{dx} - \rho_,\frac{d\phi_,}{dx}\right) = 0 \quad \text{(34)}.$$

Since $C - C'$ is evidently a small quantity of the first order at least, the condition is that (33) shall be satisfied when $y = 0$. Equation (34) will then give the ordinate of the common surface of the two liquids when y is put $= 0$ in the last two terms.

The general value of ϕ suitable to the present case, which is derived from (28) subject to the condition (29), is given by (13) if we suppose that the fluid is free from a uniform horizontal motion compounded with the oscillatory motion expressed by (13). Since the equations of the present investigation are linear, in consequence of the omission of the squares of small quantities, it will be sufficient to consider one of the terms in (13). Let then

$$\phi = A\left(\epsilon^{m(h-y)} + \epsilon^{-m(h-y)}\right)\sin mx \quad \text{(35)}.$$

The general value of $\phi_{,}$ will be derived from (13) by merely writing $-h_{,}$ for h. But in order that (32) may be satisfied, the value of $\phi_{,}$ must reduce itself to a single term of the same form as the second side of (35). We may take then for the value of $\phi_{,}$

$$\phi_{,} = A_{,}\left(\epsilon^{m(h_{,}+y)} + \epsilon^{-m(h_{,}+y)}\right)\sin mx \ldots\ldots (36).$$

Putting for shortness

$$\epsilon^{mh} + \epsilon^{-mh} = S, \quad \epsilon^{mh} - \epsilon^{-mh} = D,$$

and taking $S_{,}$, $D_{,}$ to denote the quantities derived from S, D by writing $h_{,}$ for h, we have from (32)

$$DA + D_{,}A_{,} = 0 \ldots\ldots (37),$$

and from (33)

$$\rho\,(gD - mc^2 S)\,A + \rho_{,}\,(gD_{,} + mc^2 S_{,})\,A_{,} = 0 \ldots\ldots (38)$$

Eliminating A and $A_{,}$ from (37) and (38), we have

$$c^2 = \frac{g}{m}\,\frac{(\rho - \rho_{,})\,DD_{,}}{\rho SD_{,} + \rho_{,}S_{,}D} \ldots\ldots (39).$$

The equation to the common surface of the liquids will be obtained from (34). Since the mean value of y is zero, we have in the first place

$$C_{,} = C \ldots\ldots (40).$$

We have then, for the value of y,

$$y = a\cos mx \ldots\ldots (41),$$

where

$$a = \frac{mc}{g}\,\frac{\rho_{,}A_{,}S_{,} - \rho AS}{\rho - \rho_{,}} = \frac{DD_{,}}{c}\,\frac{\rho_{,}A_{,}S_{,} - \rho AS}{\rho SD_{,} + \rho_{,}S_{,}D} \ldots\ldots (42).$$

Substituting in (35) and (36) the values of A and $A_{,}$ derived from (37) and (42), we have

$$\phi = -\frac{ac}{D}\left(\epsilon^{m(h-y)} + \epsilon^{-m(h-y)}\right)\sin mx \ldots\ldots (43),$$

$$\phi_{,} = \frac{ac}{D_{,}}\left(\epsilon^{m(h_{,}+y)} + \epsilon^{-m(h_{,}+y)}\right)\sin mx \ldots\ldots (44).$$

Equations (39), (40), (41), (43) and (44) contain the solution of the problem. It is evident that C remains arbitrary. The values of p and $p_{,}$ may be easily found if required.

If we differentiate the logarithm of c^2 with respect to m, and multiply the result by the product of the denominators, which are necessarily positive, we shall find a quantity of the form $P\rho + P_{,}\rho_{,}$, where P and $P_{,}$ do not contain ρ or $\rho_{,}$. It may be proved in nearly the same manner as in Art. 4, that each of the quantities P, $P_{,}$ is necessarily negative. Consequently c will decrease as m increases, or will increase with λ. It follows from this that the value of ϕ cannot contain more than two terms, one of the form (35), and the other derived from (35) by replacing $\sin mx$ by $\cos mx$, and changing the constant A: but the latter term may be got rid of by altering the origin of x.

The simplest case to consider is that in which both h and h' are regarded as infinite compared with λ. In this case we have

$$\phi = -ac\epsilon^{-my}\sin mx, \quad \phi_{,} = ac\epsilon^{my}\sin mx, \quad c^2 = \frac{\rho - \rho_{,}}{\rho + \rho_{,}}\,\frac{g}{m}, \; y = a\cos mx,$$

the latter being the equation to the surface.

15. The preceding investigation applies to two incompressible fluids, but the results are applicable to the case of the waves propagated along the surface of a liquid exposed to the air, provided that in considering the effect of the air we neglect terms which, in comparison with those retained, are of the order of the ratio of the length of the waves considered to the length of a wave of sound of the same period in air. Taking then ρ for the density of the liquid, $\rho_{\prime}$ for that of the air at the time, and supposing $h_{\prime} = \infty$, we have

$$c^2 = \frac{g}{m}\frac{(\rho - \rho_{\prime}) D}{\rho S + \rho_{\prime} D} = \frac{gD}{mS}\left\{1 - \left(1 + \frac{D}{S}\right)\frac{\rho_{\prime}}{\rho}\right\}, \text{ nearly.}$$

If we had considered the buoyancy only of the air, we should have had to replace g in the formula (14) by $\frac{\rho - \rho_{\prime}}{\rho_{\prime}} g$. We should have obtained in this manner

$$c^2 = \frac{g}{m}\frac{(\rho - \rho_{\prime}) D}{\rho S} = \frac{gD}{mS}\left(1 - \frac{\rho_{\prime}}{\rho}\right).$$

Hence, in order to allow for the inertia of the air, the correction for buoyancy must be increased in the ratio of 1 to $1 + \frac{D}{S}$. The whole correction therefore increases as the ratio of the length of a wave to the depth of the fluid decreases. For very long waves the correction is that due to buoyancy alone, while in the case of very short waves the correction for buoyancy is doubled. Even in this case the velocity of propagation is altered by only the fractional part $\frac{\rho_{\prime}}{\rho}$ of the whole, and as this quantity is much less than the unavoidable errors of observation, the effect of the air in altering the velocity of propagation may be neglected.

16. There is a discontinuity in the density of the fluid mass considered in Art. 14, in passing from one fluid into the other; and it is easy to show that there is a corresponding discontinuity in the velocity. If we consider two fluid particles in contact with each other, and situated on opposite sides of the surface of junction of the two fluids, we see that the velocities of these particles resolved in a direction normal to that surface are the same; but their velocities resolved in a direction tangential to the surface are different. These velocities are, to the order of approximation employed in the investigation, the values of $\frac{d\phi}{dx}$ and $\frac{d\phi_{\prime}}{dx}$ when $y = 0$. We have then from (43) and (44), for the velocity with which the upper fluid slides along the under,

$$mac\left(\frac{S_{\prime}}{D_{\prime}} + \frac{S}{D}\right)\cos mx.$$

17. When the upper surface of the upper fluid is free, the equations by which the problem is to be solved are the same as those of Art. 14, except that the condition (31) is replaced by

$$g\frac{d\phi_{\prime}}{dy} - c^2\frac{d^2\phi_{\prime}}{dx^2} = 0, \text{ when } y = -h_{\prime}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (45);$$

and to determine the ordinate of the upper surface, we have

$$C_{\prime} + g\rho_{\prime} y + c\rho_{\prime}\frac{d\phi_{\prime}}{dx} = 0,$$

where y is to be replaced by $-h_{\prime}$ in the last term. Let us consider the motion corresponding to the value of ϕ given by (35). We must evidently have

$$\phi_{\prime} = (A_{\prime}\epsilon^{my} + B_{\prime}\epsilon^{-my})\sin mx.$$

where $A_,$ and $B_,$ have to be determined. The conditions (32), (33) and (45) give

$$DA + A_, - B_, = 0,$$

$$\rho\,(gD - mc^2 S)\,A + \rho_,(g + mc^2)\,A_, - \rho_,(g - mc^2)\,B_, = 0,$$

$$(g + mc^2)\,\epsilon^{-mh_,}A_, - (g - mc^2)\,\epsilon^{mh_,}B_, = 0.$$

Eliminating A, $A_,$ and $B_,$ from these equations, and putting

$$c^2 = \frac{g\zeta}{m},$$

we find

$$(\rho SS_, + \rho_, DD_,)\,\zeta^2 - \rho\,(SD_, + S_,D)\,\zeta + (\rho - \rho_,)\,DD_, = 0. \ldots (46).$$

The equilibrium of the fluid being supposed to be stable, we must have $\rho_, < \rho$. This being the case, it is easy to prove that the two roots of (46) are real and positive. These two roots correspond to two systems of waves of the same length, which are propagated with the same velocity.

In the limiting case in which $\dfrac{\rho}{\rho_,} = \infty$, (46) becomes

$$SS_,\zeta^2 - (SD_, + S_,D)\,\zeta + DD_, = 0,$$

the roots of which are $\dfrac{D}{S}$ and $\dfrac{D_,}{S_,}$, as they evidently ought to be, since in this case the motion of the under fluid will not be affected by that of the upper, and the upper fluid can be in motion by itself.

When $\rho_, = \rho$ one root of (46) vanishes, and the other becomes $\dfrac{SD_, + S_,D}{SS_, + DD_,}$ or $\dfrac{\epsilon^{m(h+h_,)} - \epsilon^{-m(h+h_,)}}{\epsilon^{m(h+h_,)} + \epsilon^{-m(h+h_,)}}$. The former of these roots corresponds to the waves propagated at the common surface of the fluids, while the latter gives the velocity of propagation belonging to a single fluid having a depth equal to the sum of the depths of the two considered.

When the depth of the upper fluid is considered infinite, we must put $\dfrac{D_,}{S_,} = 1$ in (46). The two roots of the equation so transformed are 1 and $\dfrac{(\rho - \rho_,)\,D}{\rho S + \rho_, D}$, the former corresponding to waves propagated at the upper surface of the upper fluid, and the latter agreeing with Art. 15.

When the depth of the under fluid is considered infinite, and that of the upper finite, we must put $\dfrac{D}{S} = 1$ in (46). The two roots will then become 1 and $\dfrac{(\rho - \rho_,)\,D_,}{\rho S_, + \rho_, D_,}$. The value of the former root shows that whatever be the depth of the upper fluid, one of the two systems of waves will always be propagated with the same velocity as waves of the same length at the surface of a single fluid of infinite depth. This result is true even when the motion is in three dimensions, and the form of the waves changes with the time, the waves being still supposed to be such as could be excited in the fluids, supposed to have been previously at rest, by means of forces applied at the upper surface. For the most general small motion of the fluids in this case may be regarded as the resultant of an infinite number of systems of waves of the kind considered in this paper. It is remarkable that when the depth of the upper fluid is very great, the root $\zeta = 1$ is that which corresponds to the waves for which the upper fluid is disturbed, while the under is sensibly at rest; whereas, when the depth of the upper fluid is very small, it is the other root which corresponds to those waves which are analogous to the waves which would be propagated in the upper fluid if it rested on a rigid plane.

When the depth of the upper fluid is very small compared with the length of a wave, one of the roots of (46) will be very small; and if we neglect square and products of $mh_{,}$ and ζ, the equation becomes $2\rho D\zeta - 2(\rho - \rho_{,})mh_{,}D = 0$, whence

$$\zeta = \frac{\rho - \rho_{,}}{\rho} mh_{,}, \quad c^2 = \frac{\rho - \rho_{,}}{\rho} gh_{,} \ldots\ldots\ldots\ldots\ldots\ldots \text{(47)}.$$

These formulæ will not hold good if mh be very small as well as $mh_{,}$, and comparable with it, since in that case all the terms of (46) will be small quantities of the second order, $mh_{,}$ being regarded as a small quantity of the first order. In this case, if we neglect small quantities of the third order in (46), it becomes

$$4\rho\zeta^2 - 4m\rho(h + h_{,})\zeta + 4(\rho - \rho_{,})m^2hh_{,} = 0,$$

whence $$c^2 = \frac{g}{2}\left\{h + h_{,} \pm \sqrt{(h - h_{,})^2 + \frac{4\rho_{,}}{\rho}hh_{,}}\right\} \ldots\ldots\ldots\ldots \text{(48)}.$$

Of these values of c^2, that in which the radical has the negative sign belongs to that system of waves to which the formulæ (47) apply when $h_{,}$ is very small compared with h.

If the two fluids are water and mercury, $\dfrac{\rho}{\rho_{,}}$ is equal to about 13.57. If the depth of the water be very small compared both with the length of the waves and with the depth of the mercury, it appears from (47) that the velocity of propagation will be less than it would have been, if the water had rested on a rigid plane, in the ratio of .9624 to 1, or 26 to 27 nearly.

G. G. STOKES.

XXXIV. *On the Internal Pressure to which Rock Masses may be subjected, and its possible Influence in the Production of the Laminated Structure.* *By* W. Hopkins, Esq., M.A., F.R.S., &c.

[Read *May* 3, 1847.]

One of the most curious phenomena in the constitution of rock masses, consists in the laminated structure which pervades so large a portion of the older sedimentary formations, producing what is called their slaty cleavage. In some cases, this lamination is comparatively coarse and ill-defined, but in others (as in the roofing slates) it is so fine and regular as to leave no doubt of its being the result of some kind of molecular action of the constituent particles on each other, analogous to that of crystallization, and not the direct and immediate mechanical effect of external forces acting on the mass. But still it would seem very possible, that these external forces may maintain the mass in a state of internal constraint which may possibly be a condition favourable to the production of the laminated structure, and observations have lately been made which seem to afford some confirmation of this notion. Professor Phillips, some years ago, and Mr. Sharpe, more recently, have recognized some curious and interesting facts respecting the frequent distortion of fossil shells, and other organic remains, from their original well-known forms; and these distortions appear to indicate certain relations between the positions of the cleavage planes and the directions of the internal pressures which must have produced the distortions in question. These distortions of determinate organic forms indicate, in fact, corresponding distortions in those elements of the mass in which they are respectively comprized. To explain the nature of the tensions or pressures acting on any such element and the distortion produced by them, let us denote by s a small plane surface passing through any point P. Generally, there will be an action between the particles (M) on one side of this small plane, and M', those in contact with them on the opposite side. If s be sufficiently small, we may represent the whole action of M on M' by ps, a force having a determinate direction, which we may suppose to make an angle δ with the normal to s. Then will

$$ps\cos\delta, \quad \text{and} \quad ps\sin\delta,$$

be the normal and tangential parts of the whole action of M on M', and

$$-ps\cos\delta, \quad \text{and} \quad -ps\sin\delta,$$

will manifestly be the same parts of the reaction of M' on M. If the normal force be a *pressure*, it will only tend to preserve the contact of the particles immediately on opposite sides of s; but if that force be a *tension*, then will $ps\cos\delta$ tend to separate these particles by motions normal to s, and in opposite directions. In all cases there will be likewise forces equal to $ps\sin\delta$, and $-ps\sin\delta$, tending to separate any one particle immediately on one side of s, from the particle originally in contact with it on the other side of s, by communicating to these particles, motions in opposite directions parallel to the plane of s. If this plane assume different positions by moving about P as a fixed point, the normal and tangential forces acting on it will have different values, assuming maxima or minima values for certain determinate positions of s, and it is on these particular positions of s that the distortion of a small portion of the mass about P, and that of any organic form contained in it, will depend. Generally, The linear dimensions of the element will be altered by extension or compression, and it will also be *twisted*, so that if it were originally a rectangular parallelopiped it will become an oblique-angled one, and these changes of form will be indicated by the corresponding distortions of the organic remains. Now, if the directions of the cleavage planes were originally determined by the state of internal tension and pressure of the mass, it would seem probable that they would be perpendicular to the directions of greatest, or to those of least normal pressure, or that they would coincide with the planes of greatest tangential action. These hypotheses must be tested by the evidence derived from the organic forms, as will be explained in the sequel, but for that purpose it will be necessary in the first place, to investigate the relative positions of the lines and planes just mentioned. This investigation will form the first Section of this memoir.

SECTION I.

Relative positions of the lines of maximum and minimum tension, and planes of maximum tangential force in the interior of a continuous mass.

1. TAKING any point P of the mass, let it be made the origin of co-ordinates xyz. Let the small plane s be conceived as before to pass through P, and let the forces upon it in the positions specified be denoted as follows, all being referred to a unit of surface.

(1.) When a perpendicular to the plane coincides with the axis of x, let

The *normal force* $= A$; The *tangential force* $= \begin{cases} B' \text{ parallel to } y, \\ C' \ldots\ldots\ldots\ldots z. \end{cases}$

(2.) When a perpendicular to the plane coincides with the axis of y, let

The *normal force* $= B$; The *tangential force* $= \begin{cases} C'' \text{ parallel to } z, \\ A' \ldots\ldots\ldots\ldots x. \end{cases}$

(3.) When a perpendicular to the plane coincides with the axis of z, let

The *normal force* $= C$; The *tangential force* $= \begin{cases} A'' \text{ parallel to } x, \\ B'' \ldots\ldots\ldots\ldots y. \end{cases}$

Between the six accented quantities there are three essential relations, which are easily found. On the three co-ordinate axes at P, construct an indefinitely small parallelopiped whose edges are δx, δy, and δz. The six equations of equilibrium of this element will express the conditions that the sums of all the resolved parts of the forces parallel to the co-ordinate axes shall respectively be equal to zero; and that the moments of the forces with reference to three axes, shall also severally be equal to zero. Let us take the three latter conditions, lines through the center of gravity of the element and parallel to the co-ordinate axes being taken for the axes of the component couples. The tangential force parallel to the axis of x on the side $\delta x . \delta z$ being A', that on the opposite side will be $-(A' + \frac{dA'}{dy}\delta y)$; and the couple resulting from these forces about the axis parallel to z, will be

$$A'\delta x\delta z . \frac{\delta y}{2} + (A' + \frac{dA'}{dy}\delta y)\,\delta x\delta z . \frac{\delta y}{2};$$

or, omitting small terms of the fourth order,

$$A'\delta x\delta y\delta z.$$

Similarly, the couple arising from the forces B' and $B' + \frac{dB'}{dx}\delta x$ about the same axis parallel to z, will be

$$-B'\delta x\delta y\delta z.$$

Also the moment of the normal forces A, B, C, with reference to the above-mentioned axes, will be zero, always omitting small quantities of the fourth order. Consequently the whole moment of the forces on the parallelopiped with reference to the axis parallel to that of z, will be

$$(A' - B')\,\delta x\delta y\delta z\,;$$

which must = zero by the conditions of equilibrium; and therefore

$$A' = B'.$$

In exactly the same way we find, by taking the moments with reference to the axes parallel respectively to those of y and x,

$$A'' = C',$$
$$B'' = C''.$$

By means of these three relations the six accented quantities are reduced to three independent quantities.

2. Let us now conceive a plane to meet the three co-ordinate planes so as to form with them a tetrahedron, whose vertex is at the origin P. Suppose the exterior normals to the three faces formed by the co-ordinate planes to point respectively towards the positive directions of x y and z; and let α β and γ be the angles which the normal to the base of the tretrahedron makes with the co-ordinate axes of x y and z. Also let s denote the area of the base, and s' s'' and s''' the areas of the sides of the tetrahedron perpendicular respectively to the axes of x y and z, all these quantities being indefinitely small.

Again, let ps denote the whole resultant force acting on s, and let λ μ and ν be the angles which its direction makes with lines parallel to the co-ordinate axes of x y and z, this direction being exterior to the tetrahedron. Then, in order that the tetrahedron may be in equilibrium, we must have

$$ps\,.\cos\lambda = As' + A's'' + A''s''',$$
$$ps\,.\cos\mu = Bs'' + B's' + B''s''',$$
$$ps\,.\cos\nu = Cs''' + C's' + C''s'':$$

but

$$\frac{s'}{s} = \cos\alpha,\quad \frac{s''}{s} = \cos\beta,\quad \frac{s'''}{s} = \cos\gamma\,;$$

making these substitutions, and also putting

$$B'' = C'' = D,$$
$$A'' = C' = E,$$
$$A' = B' = F,$$

we shall have

$$\left.\begin{aligned} p\,.\cos\lambda &= A\cos\alpha + F\cos\beta + E\cos\gamma,\\ p\,.\cos\mu &= B\cos\beta + F\cos\alpha + D\cos\gamma,\\ p\,.\cos\nu &= C\cos\gamma + E\cos\alpha + D\cos\beta\,;\end{aligned}\right\}\dots\dots\dots\dots\dots\dots (a),$$

formulæ in which the notation agrees with that of M. Cauchy (*Exercises de Mathématique*, Vol. II. p. 48).

If δ denote the angle between the direction of p and the normal to s, we shall have $p \cos \delta$ for the whole *normal* force acting on the area s in a direction exterior to the tetrahedron, and $p \sin \delta$ the whole *tangential* force acting on the same area. Our first object will be to determine a β and γ, or the position of the base (s) of the tetrahedron, so that the normal action upon it, $p \cos \delta$, shall be a maximum. We shall afterwards have a similar investigation with reference to the tangential force $p \sin \delta$.

We have $$\cos \delta = \cos \lambda \cos a + \cos \mu \cos \beta + \cos \nu \cos \gamma,$$

whence we immed ately obtain

$$p \cos \delta = A \cos^2 a + B \cos^2 \beta + C \cos^2 \gamma + 2D \cos \beta \cos \gamma + 2E \cos a \cos \gamma + 2F \cos a \cos \beta, \ldots (1),$$

and since $$\cos^2 a + \cos^2 \beta + \cos^2 \gamma = 1, \ldots\ldots (2),$$

we have (L being an arbitrary multiplier),

$$(A + L) \cos^2 a + (B + L) \cos^2 \beta + (C + L) \cos^2 \gamma + 2D \cos \beta \cos \gamma + 2E \cos a \cos \gamma + 2F \cos a \cos \beta = max.$$

Hence,

$$\left.\begin{array}{l} \{(A + L) \cos a + E \cos \gamma + F \cos \beta\} \sin a = 0 \\ \{(B + L) \cos \beta + D \cos \gamma + F \cos a\} \sin \beta = 0 \\ \{(C + L) \cos \gamma + D \cos \beta + E \cos a\} \sin \gamma = 0 \end{array}\right\} \ldots\ldots (b).$$

To satisfy these equations together with

$$\cos^2 a + \cos^2 \beta + \cos^2 \gamma = 1,$$

we must equate the first brackets to zero. We thus have four equations from which L may be eliminated, and a β and γ determined.

If we multiply the first factors on the left-hand sides of equations (b) by $\cos a$, $\cos \beta$ and $\cos \gamma$ respectively, and add them together, we have by virtue of equations (a),

$$L = - P \cos \delta,$$

and substituting for L in equations (b), we have

$$p \cos \delta \cos a = A \cos a + F \cos \beta + E \cos \gamma,$$
$$= p \cos \lambda;$$
$$\therefore \quad \cos \delta \cos a = \cos \lambda.$$

Similarly, $$\cos \delta \cos \beta = \cos \mu,$$
$$\cos \delta \cos \gamma = \cos \nu;$$

whence $$\cos^2 \delta = 1,$$
$$\delta = 0,$$

which shews that when the resultant force p is a maximum or minimum, its direction coincides with that of the normal to the plane s. Consequently, also, the tangential force $p \sin \delta$ then becomes = zero.

This value of δ gives, $$L = - p,$$

and substituting for L in equations (b) we have,

$$\left.\begin{array}{l} (A - p) \cos a + F \cos \beta + E \cos \gamma = 0 \\ F \cos a + (B - p) \cos \beta + D \cos \gamma = 0 \\ E \cos a + D \cos \beta + (C - p) \cos \gamma = 0 \end{array}\right\} \ldots\ldots (c),$$

and eliminating $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ by cross multiplication, we obtain

$$(A-p)(B-p)(C-p) - D^2(A-p) - E^2(B-p) - F^2(C-p) + 2DEF = 0.$$

If we take the three values of p deducible from this equation and substitute them successively in equations (c), those equations combined with (2) will give three distinct systems of values for $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, belonging (as is well-known) to three lines perpendicular to each other.

Hence, it follows, that there is at every point (P) of a continuous solid mass under extension or compression, a system of three rectangular axes, such, that if the small plane (s) at P be so placed that its normal shall coincide with one of those axes the whole resultant action on s shall be *normal* to it, the tangential action upon it being then equal to zero. These three axes are called the *axes of principal pressure* or *tension* with reference to the point P.

3. M. Cauchy, in the Memoir above referred to, converts equation (1) into the equation to a surface of the second order, by putting

$$p\cos\delta = \pm\frac{1}{r^2}, \quad r\cos\alpha = x, \quad r\cos\beta = y, \quad r\cos\gamma = z.$$

The inverse of the square of any radius vector will manifestly be a measure of the normal action on a small plane through P perpendicular to the radius vector, the axes of principal pressure or tension coinciding with the axes of this surface of the second order. We may remark, that of the three principal pressures or tensions above determined, one will be a maximum and another a minimum, while that of an intermediate value will be neither, though it satisfies the conditions $\frac{d\,.\,p\cos\delta}{d\alpha} = 0$, and $\frac{d\,.\,p\cos\delta}{d\beta} = 0$. It is, in fact, that value of $p\cos\delta$ which is represented by the inverse of the square of the mean axis of the surface, and that mean axis, considered as a particular radius vector, is a maximum with reference to one principal section, and a minimum with reference to the other to which it belongs, so that though $\frac{d}{d\alpha}\left(\frac{1}{r^2}\right) = 0$, and $\frac{d}{d\beta}\left(\frac{1}{r^2}\right) = 0$ when r = mean axis, all the conditions of a maximum or minimum are not satisfied.

I make these remarks here because a similar mode of geometrical representation may be found useful in explaining the results obtained in the succeeding part of the investigation.

4. I shall now proceed to investigate the positions of the plane s passing through P, when the tangential action upon it is greatest, *i. e.* when $p\sin\delta = max.$

To simplify our formulæ, we may here take the axes of principal pressure or tension as the co-ordinate axes. In this case there will be no tangential force on the plane s when it is perpendicular to any of these axes, and consequently, we must have

$$D = 0, \quad E = 0, \quad F = 0,$$

and, therefore, equations (a) give

$$p^2 = A^2\cos^2\alpha + B^2\cos^2\beta + C^2\cos^2\gamma,$$

and equation (1) gives,

$$p\cos\delta = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma.$$

Hence we have

$$p^2\sin^2\delta = A^2\cos^2\alpha + B^2\cos^2\beta + C^2\cos^2\gamma - (A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma)^2,$$

the quantity which is to be made a maximum subject to the condition

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

By virtue of the last equation, we have

$$p^2\sin^2\delta = (\cos^2\alpha + \cos^2\beta + \cos^2\gamma)(A^2\cos^2\alpha + B^2\cos^2\beta + C^2\cos^2\gamma) - (A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma)^2,$$

which by reduction gives

$$p^2 \sin^2 \delta = (A - B)^2 \cos^2 \alpha \cos^2 \beta + (A - C)^2 \cos^2 \alpha \cos^2 \gamma + (B - C)^2 \cos^2 \beta \cos^2 \gamma = max \ldots(3).$$

Hence, if L be an arbitrary multiplier, we obtain

$$\left.\begin{aligned} \{(A - B)^2 \cos^2 \beta + (A - C)^2 \cos^2 \gamma + L\} \cos \alpha \sin \alpha = 0 \\ \{(A - B)^2 \cos^2 \alpha + (B - C)^2 \cos^2 \gamma + L\} \cos \beta \sin \beta = 0 \\ \{(A - C)^2 \cos^2 \alpha + (B - C)^2 \cos^2 \beta + L\} \cos \gamma \sin \gamma = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(d).$$

Let us first suppose these equations satisfied by equating, in each case, their first factors to zero; and for brevity put

$$P = A - B, \quad Q = A - C, \quad R = B - C;$$

$$\therefore P = Q - R.$$

Now, substituting $1 - \cos^2 \alpha - \cos^2 \beta$ for $\cos^2 \gamma$, and eliminating L between the first and third, and the second and third equations, we obtain

$$(P^2 - Q^2 - R^2) \cos^2 \beta - 2 Q^2 \cos^2 \alpha + Q^2 = 0,$$

$$(P^2 - Q^2 - R^2) \cos^2 \alpha - 2 R^2 \cos^2 \beta + R^2 = 0;$$

or since

$$P^2 = (Q - R)^2,$$

$$P^2 - Q^2 - R^2 = -2QR;$$

$$\therefore 2R \cos^2 \beta + 2Q \cos^2 \alpha - Q = 0,$$

$$2Q \cos^2 \alpha + 2R \cos^2 \beta - R = 0,$$

which cannot hold simultaneously unless $Q = R$, and $\therefore P = 0$; or $A = B$. This mode, therefore, of satisfying equations (d) is not admissible.

Again, we may satisfy those equations by

$$\sin \alpha = 0, \quad \cos \beta = 0, \quad \cos \gamma = 0,$$

a system of equations which also satisfy (2). In this case the normal to the small plane s will coincide with the axis of x, *i.e.* with an axis of principal pressure, and therefore, these values ought to give the tangential force = zero, as they do. Zero is in fact a *minimum* value of that force.

Similar conclusions hold with reference to the axes of y and z for the following systems of values.

$$\cos \alpha = 0, \quad \sin \beta = 0, \quad \cos \gamma = 0;$$

$$\cos \alpha = 0, \quad \cos \beta = 0, \quad \sin \gamma = 0.$$

Finally, we may satisfy equations (d) by

$$\cos \alpha = 0,$$

$$P^2 \cos^2 \alpha + R^2 \cos^2 \gamma + L = 0,$$

$$Q^2 \cos^2 \alpha + R^2 \cos^2 \beta + L = 0.$$

Eliminating L, we have

$$\cos^2 \beta = \cos^2 \gamma;$$

$$\therefore 2 \cos^2 \beta = 1;$$

$$\therefore \beta = \gamma = \pm 45^\circ.$$

Two other systems of values may evidently be obtained in a similar manner, and thus equations (d) and (2) are satisfied by the three following systems of values:

$$\left.\begin{aligned} \alpha = 90^\circ, \quad & \beta = \gamma = \pm 45^\circ, \\ \beta = 90^\circ, \quad & \gamma = \alpha = \pm 45^\circ, \\ \gamma = 90^\circ, \quad & \alpha = \beta = \pm 45^\circ. \end{aligned}\right\} \ldots\ldots\ldots\ldots(e).$$

5. If T_1, T_2, and T_3 be the corresponding values of the tangential force, we have

$$T_1^2 = \tfrac{1}{4}(B-C)^2, \quad T_2^2 = \tfrac{1}{4}(A-C)^2, \quad T_3^2 = \tfrac{1}{4}(A-B)^2.$$

If A, B, C be taken as they always may be, so that A shall be the greatest and C the least, T_2 will be the greatest of these values, and I shall shew that it in fact is the only one which satisfies the conditions of being a maximum. To do this, and to explain the relations of these particular values of the tangential force to its general values, it will be convenient to have recourse to a geometrical representation, analogous to that before spoken of with reference to the normal forces. For this purpose assume

$$p \sin \delta = T_0 \frac{c^2}{r^2},$$

$$x = r\cos\alpha, \quad y = r\cos\beta, \quad z = r\cos\gamma;$$

where T_0 denotes a constant force, and c a constant line. Then equation (3) becomes

$$T_0^2 c^4 = P^2x^2y^2 + Q^2x^2z^2 + R^2y^2z^2 \quad \ldots\ldots\ldots\ldots\ldots(4),$$

the equation to a surface such that the inverse of the square of the radius vector from the point P, will be proportional to the tangential force on the plane s when perpendicular to that radius vector.

To find the intersections of the surface and the co-ordinate planes, put $x = 0$, $y = 0$, and $z = 0$, consecutively; we thus have

$$yz = \pm \frac{T_0}{R} c^2,$$

$$xz = \pm \frac{T_0}{Q} c^2,$$

$$xy = \pm \frac{T_0}{P} c^2,$$

as the equations to the intersections, each of which consists of two equal hyperbolas referred to the asymptotes as axes of co-ordinates, as represented in the annexed diagram. PA, making equal angles with the two co-ordinate axes in the plane of the paper, is the semi-axis major, and minimum radius vector in the hyperbola whose vertex is A. Its position and that of PA' correspond to the first, second, or third of the systems of values (e) of α β and γ, according as the plane in which the hyperbolas lie is that of yz, xz, or xy. Also the values of $\frac{1}{AP^2}$ in these cases respectively are

$$\frac{R}{2T_0}\cdot\frac{1}{c^2}, \quad \frac{Q}{2T_0}\cdot\frac{1}{c^2}, \quad \frac{P}{2T_0}\cdot\frac{1}{c^2},$$

which are proportional to

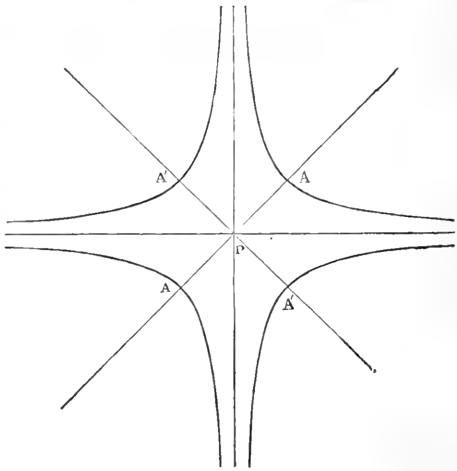
$$R, \quad Q, \quad P;$$

or to $\quad B-C, \quad A-C, \quad A-B,$

or to the three tangential forces previously designated by

$$T_1, \quad T_2, \quad T_3.$$

(Fig. 1.)



In each case PA is a *minimum* value of the radius vector with reference to the hyperbola of which it is the semi-axis major, and consequently PA is the position of the normal to the plane s, when the tangential force upon it is, in the same relative sense, a *maximum*. It still remains to be determined whether PA is also a *minimum* value of the radius vector in a section of the surface made by a plane through PA and the co-ordinate axis perpendicular to the plane of the paper. For this purpose, let this last-mentioned axis be first taken as that of x, and let

$$x = r\cos\theta, \qquad y = r\sin\theta\sin\phi, \qquad z = r\sin\theta\,.\cos\phi.$$

r θ and ϕ being the ordinary polar co-ordinates. Substituting these values in the equation (4) to the surface, and putting $\phi = 45°$, we obtain

$$T_0^2\frac{c^4}{r^4} = \frac{P^2+Q^2}{2}\sin^2\theta - \left(\frac{P^2+Q^2}{2} - \frac{R^2}{4}\right)\sin^4\theta \ldots\ldots\ldots\ldots(5),$$

the polar equation to the section through the axis of x and the axis of either hyperbola in the plane of yz.

Similarly, putting

$$y = r\cos\theta, \qquad z = r\sin\theta\sin\phi, \qquad x = r\sin\theta\,.\cos\phi.$$

we obtain

$$T_0^2\frac{c^4}{r^4} = \frac{P^2+R^2}{2}\sin^2\theta - \left(\frac{P^2+R^2}{2} - \frac{Q^2}{4}\right)\sin^4\theta \ldots\ldots\ldots\ldots(6),$$

the polar equation to the section of the surface by a plane through the axis of y and the axis of either hyperbola in the plane of xz.

Again, putting

$$z = r\cos\theta, \qquad y = r\sin\theta\sin\phi, \qquad x = r\sin\theta\cos\phi,$$

we have

$$T_0^2\frac{c^2}{r^4} = \frac{Q^2+R^2}{2}\sin^2\theta - \left(\frac{Q^2+R^2}{2} - \frac{P^2}{4}\right)\sin^4\theta \ldots\ldots\ldots\ldots(7),$$

the equation to the section through the axis of z, and the axis of either hyperbola in the plane of xy.

Differentiating (5), (6), and (7), and putting $\frac{dr}{d\theta} = 0$, we obtain in the several cases,

$$\left.\begin{aligned}\{(P^2+Q^2) - \{2(P^2+Q^2) - R^2\}\sin^2\theta\}\sin\theta\cos\theta = 0,\\ \{(P^2+R^2) - \{2(P^2+R^2) - Q^2\}\sin^2\theta\}\sin\theta\cos\theta = 0,\\ \{(Q^2+R^2) - \{2(Q^2+R^2) - P^2\}\sin^2\theta\}\sin\theta\cos\theta = 0,\end{aligned}\right\}\ldots\ldots\ldots\ldots(f).$$

Each of these equations may be satisfied by

$$\sin\theta = 0, \qquad \cos\theta = 0.$$

The first corresponds to $r = \infty$, the axis from which θ is measured being an asymptote to the curve. The second gives $\theta = 90°$, and therefore $r = AP$, which is consequently either a maximum or a minimum value of r with respect to the curve in which r and θ are the variable co-ordinates. Now since $r = \infty$ when $\theta = 0$, or $180°$, and $r = AP$ when $\theta = 90°$, it is manifest that AP must be a *minimum* value of r, provided $\frac{dr}{d\theta}$ is not rendered zero by any value of θ between 0 and $90°$; but if, on the contrary, $\frac{dr}{d\theta}$ become zero for some value of θ between those limits, the corresponding value of r must be a *minimum*, in which case PA will be a *maximum* value, since maxima and minima occur alternately. To ascertain whether any value of θ between 0 and 90° does render $\frac{dr}{d\theta} = 0$, we must see whether such value can be derived in any of the three cases above, by equating to zero the expressions within the brackets in equations (f). Taking the first of those equations, we have

$$\sin^2\theta = \frac{P^2 + Q^2}{2(P^2 + Q^2) - R^2},$$

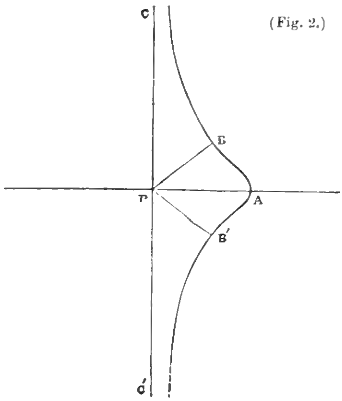
which will give a value of θ between 0 and 90°, provided the fraction be positive and less than unity. Now the difference between the numerator and denominator

$$= P^2 + Q^2 - R^2,$$

and A, B, and C being taken in order of magnitude, and A the greatest, $Q\ (= A - C)$ is greater than $R\ (= B - C)$. Consequently $P^2 + Q^2 - R^2$ is positive, and the denominator of the above fraction is positive and greater than the numerator, and $\sin\theta$ is possible. The value of r corresponding to the value of θ thus obtained, will be a *minimum*, and therefore PA will in this case be a *maximum*. Hence it appears that PA is a *maximum* value of the radius vector with reference to the section of the surface by a plane through the axis of x, while it is a *minimum* with reference to the section, perpendicular to the former, made by the plane of yz. In this case then PA is neither a maximum nor minimum value of the radius vector of the surface.

Exactly the same conclusion may be drawn from the third of equations (f), in which case the two sections to which PA is common, and one through the axis of z, and that made by the plane of xy.

The annexed figure (2) represents the curve in each of the above cases referred to r and θ, CPC' being in the first case the axis of x, and in the second the axis of z. PB and PB' represent the two minima radii vectores in these sections.



(Fig. 2.)

It remains to consider the second of equations (f), which gives

$$\sin^2\theta = \frac{P^2 + R^2}{2(P^2 + R^2) - Q^2}.$$

Here, the denominator − the numerator $= P^2 + R^2 - Q^2$.

$$\text{Now} \qquad P = Q - R, \quad (\text{Art. } 4);$$

$$\therefore\ P^2 + R^2 - Q^2 = 2R^2 - 2RQ$$

$$= -2R(Q - R),$$

which, since Q is greater than R, shews that the denominator is less than the numerator. Consequently there is no value of θ between 0 and 90°, in this case, which renders $\frac{dr}{d\theta} = 0$, and PA is here a *minimum* value of the radius vector, *i.e.* in the section made by a plane through the axis of y. PA is also a minimum for the section made by the plane of xz. Consequently if figure (1) represent the plane of xz, each of the four equal lines PA is an absolute minimum value of the radius vector of the surface, and $\frac{1}{PA^2}$ represents the absolute maximum value of the tangential force. The positions of these lines correspond to the following system of values of α β and γ,

$$\beta = 90^\circ, \qquad \gamma = \alpha = \pm 45^\circ,$$

the second of the systems (c), (Art. 4), and the corresponding value of the tangential force is

$$T_2 = \tfrac{1}{2}(A - C),$$

the absolute maximum value of the tangential force acting on a plane of indefinitely small and constant area (s), passing through any assigned point P of the solid body, and capable of assuming any angular position about that point. For this maximum value, the normal to the small plane, maylie in two positions, both in the plane of xz, and bisecting the angle between the axes of x and z, the one above and the other below the axis of x, those axes being so taken as to coincide, the former with the direction of the greatest *principal tension* at P, and the latter with that of the least. If one of the principal tensions be changed into a *pressure*, it must be regarded as a *negative tension*, and therefore as the *least principal tension*, and its direction taken as the axis of z. In this case we shall have $T_2 = \frac{1}{2}(A + C)$. If there be two pressures, the greatest will $= -C$. If all the principal forces be *pressures*, the least pressure will be $-A$, and the greatest pressure $-C$, and therefore $T_2 = \frac{1}{2}(C - A)$. Thus T_2 will in all cases be the algebraical difference of the greatest and least principal tensions, considering pressures as negative tensions.*

6. As an elucidation of the subject, I shall consider a few particular cases.

(1) Suppose $B = C$; the normal tensions will be the same for all positions of the plane s, in which its normal lies in the plane yz, and there will be an infinite number of positions of the plane s corresponding to the maximum tangential action, such that the locus to the normals of s will be a conical surface whose axis is that of x, the semi-vertical angle of the cone being 45^0.

(2) If the mass at any proposed point (P) be acted upon only by two tensions acting as principal tensions, these must be considered as the axes of x and y, the axis of z, that of least principal tension (supposed here = zero) being perpendicular to the plane of the two tensions.

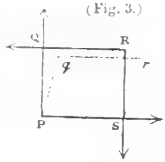
(3) If there be only two principal tensions, as in the last case, but one of them become a pressure, the direction of this latter must be taken as the axis of z, that of least tension.

(4) If both these principal tensions become pressures, the line perpendicular to the plane in which they act, must be taken for the axis of x, (the axis of greatest principal tension), and the direction of the greatest pressure for the axis of z.

The axes of x and z, those of greatest and least principal tensions being known, the two positions of the plane of maximum tangential action are immediately known.

(5) Let $PQRS$ represent a plane section of an elementary parallelopiped of the body parallel to two opposite sides, and suppose $PQRS$ a square. Let the forces on the element be entirely tangential and parallel to the plane of the paper, there being no force perpendicular to that plane. Then (Art. 1) the tangential force on each side of the element will be the same; let it $= f$ and act on each side in the directions indicated by the arrows. Also let $PqrS$ be the section of the same element, supposing the forces f not to act; then it is manifest that these forces produce an *extension* $= SQ - Sq$ in the direction SQ, and a *compression* $= Pr - PR$ in the direction RP perpendicular to SQ. In fact the forces f may be resolved into $f \,.\, \cos 45^0$ parallel to SQ, extending each particle in that direction, and an equal force compressing the particles perpendicular to SQ. The former will act as a principal tension, the latter as a principal pressure. If A and $-C$ be their values referred to a unit of surface, we must have

(Fig. 3.)



$$A \,.\, QR \sin 45^0 = f \,.\, QR \cos 45^0,$$

and $$C \,.\, QR \cos 45^0 = f \,.\, QR \sin 45^0;$$

$$\therefore\ A = f, \text{ and } C = f;$$

* Since $T_2^2 = (A - C)^2$, $T_2 = \pm(A - C)$. All notice of the negative sign is omitted in the text, as altogether unessential.

and therefore the greatest tangential force

$$= \tfrac{1}{2}(A + C) = f,$$

and it acts in the planes PQ and PS; *i. e.* it is the impressed tangential force, as it is sufficiently manifest it ought to be.

(6) It is of some importance with reference to the particular application of these investigations which is here contemplated, to remark that when the general mass is so acted on by external forces that its different elementary portions are subjected in very different degrees to the kind of distortion represented in fig. 3., there may be a great extension or compression at particular points without a correspondent increase or decrease on the same scale in the general dimensions of the mass. Indications of such local extension and compression seem to be frequently indicated by the distortion of organic remains.

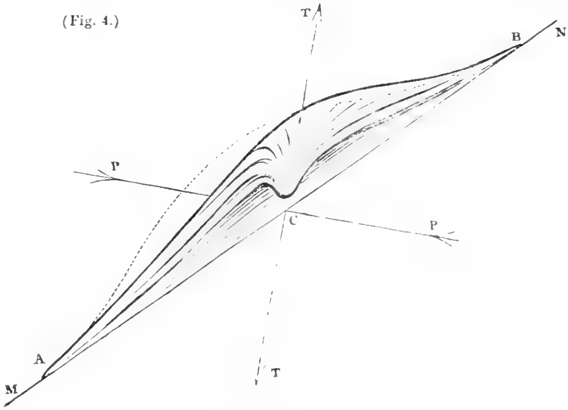
SECTION II.

7. ORGANIC remains, more especially shells, are usually found in greatest abundance along those surfaces within a fossiliferous mass, which we recognize as planes or surfaces of separation between contiguous beds. These shells, especially the flatter ones, will generally be found with their flatter surfaces parallel to the surface of the bed on which they lie, and such may also be expected to be the case as a general rule, with respect to shells contained within a bed instead of being between two contiguous beds. The first pressure to which these shells was subjected must have been that due to the weight of the superincumbent beds deposited upon them, while the whole remained undisturbed. If the shell yielded to this pressure it would become flattened, and frequently also extended in length or breadth, or in all directions according to the nature of the shell. It would seem probable that the proportions of the linear horizontal dimensions would not be much altered by this vertical compression, but the possibility of its being otherwise should not be forgotten by the observer. It may also be remarked, that should any horizontal elongation take place from this cause in one direction more than another, that direction can only have reference to the shell itself, and not to any fixed lines in space, unless it can be shown that the position in which the shell was originally imbedded bore some relation to such lines, as for instance, that the median lines from the beak to the margin in different shells should have been parallel to some common direction. Any such law, however, would seem to carry with it the highest degree of *à priori* improbability.

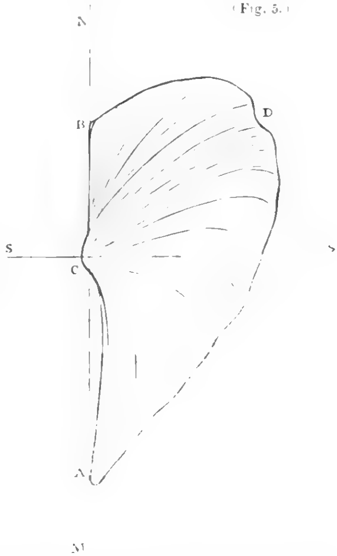
When the mass became elevated and dislocated, especially in the degree in which such has been the case in most of the ancient fossiliferous rocks, it would generally be subjected to great pressures and tensions; but it is of the first importance to remark, that none but comparatively small pressures or tensions could be called into action in the direction of the *strike* of the beds, by their elevation into straight, or approximately straight anticlinal ridges; and that, consequently, two of the directions of principal tension or pressure must lie in a vertical plane perpendicular to the direction of the anticlinal line and strike of the beds, with which the third axis of principal tension must coincide. Now in this elevation, it is highly probable that the mass will generally be *extended* in some directions, and I consider it almost certain that it must, in most cases, be *compressed* in other directions, these compressions and extensions taking place in the above mentioned vertical plane perpendicular to the strike of the beds. Hence, we may conclude that generally the *minimum tension* will be a pressure, as in (3) of last Article. The axes of greatest and least tension through any point will lie in a vertical plane perpendicular to the strike of the beds, and consequently the intersections of the planes of greatest tangential action with the planes of the beds will be horizontal lines. Through every point there will be two planes of maximum tangential action perpendicular to each other, and therefore, *dipping* one of them in the same direction as the beds, and the other in exactly the opposite direction, the *strike* of all these planes being the same.

8. Let us now consider how the distorted forms of organic remains may indicate the directions which must have been those of maximum and minimum tension or pressure, and the position of the planes of maximum tangential action at some former epoch, posterior to the elevation which raised the general mass into anticlinal ridges. In the first place, suppose the planes of maximum tangential action to coincide, at least approximately, with those of stratification. Let MN represent one of these planes on which, between two beds, the fossil shell AB is found, the undistorted form of the shell being known. MN is supposed to coincide with the dip of the beds, and the median line of the shell to lie in the direction of their strike, the plane of the paper being vertical. Also, let CT and CP be the directions of maximum and minimum tension respectively, each inclined at an angle of 45° to MN. Then the continuous line will represent the distorted form of the shell, of which the original form is indicated by the dotted line. Fig. 5, in which the plane of the paper represents the plane of a bed, will represent the distorted form of the upper valve of the same shell. It is important to remark, that this *angular* distortion will take place in *the direction of the dip* (MN) of the beds, and perpendicular to their strike (SCS).

(Fig. 4.)



(Fig. 5.)



Again, let the planes of stratification be perpendicular to one of the directions of principal tension, then will MN the direction of the dip, be a direction of maximum tension or of maximum pressure In the former case an imbedded fossil will be *elongated*, and in the latter case *compressed*, in the direction MN, but without any of that *angular* distortion represented in the previous case (Fig. 4), unless it should be accidentally produced by direct compression, in which case, however, it will have no such necessary reference to the directions of dip and strike as above mentioned.

Conversely, if it be observed that the organic forms lying between two contiguous beds, have undergone great angular distortion, we may conclude that the planes of stratification must have coincided more or less, approximately, with those of maximum tangential action at the time when the distortions were produced; but if the observed distortions indicate only direct compression or extension, unaccompanied by angular distortion, we may conclude, that the planes of stratification, at the time just mentioned, must have coincided at least approximately, with the directions of maximum or of minimum pressure.

9. The application of these conclusions to the leading object of this communication, the possible influence of internal pressure in producing cleavage structure, may be made very briefly. If we recognize the probability of this influence, if not as a primary cause, yet as effective in determining the *positions* of the planes of cleavage, we must, I think, almost necessarily suppose, as I have before remarked, that those planes must coincide more or less accurately either with planes perpendicular to the directions of maximum pressure, or with those perpendicular to the directions of minimum pressure, or with the planes of greatest tangential action. Now, let us suppose the organic forms lying on the surface of a bed to have suffered great angular distortion, and therefore the planes of stratification and of greatest tangential action to have been at least approximately coincident; then, if the planes of cleavage nearly coincide with those of stratification, we may conclude that the tangential action and not the direct pressure or tension has been the effective agency in determining the position of the cleavage planes; and the conclusion will be strengthened if we find that, as a general rule, the angular distortion is greater the more nearly the planes of stratification and of cleavage are coincident. Again, suppose the observed distortions to consist in direct compression or extension, without considerable angular distortion, and therefore the planes of stratification to have been perpendicular either to the directions of greatest pressure or to those of greatest tension, and consequently inclined at an angle of 45^0 to the planes of greatest tangential action; then, if the cleavage planes be also inclined at an angle of nearly 45^0 to the planes of stratification, we shall be again led to the same conclusion as above. If, on the contrary, it should be found that when the cleavage planes and the planes of stratification are nearly coincident, the distortion consists only in direct compression; or if, with great angular distortion, the cleavage planes should be inclined at about 45^0 to those of stratification (cases exactly opposite to those previously supposed,) we must conclude that direct pressure has been the influential cause in determining the position of the planes of cleavage.

In the memoir already referred to, Mr. Sharpe has collected, I believe, nearly all the evidence which has hitherto been obtained on this subject, consisting principally of observations made by himself and Professor Phillips, and has given drawings of several characteristic distortions, principally of *spirifer disjunctus*, a frequent and well-known shell in some of the older formations in which the cleavage structure is very distinctly developed. In the most remarkable specimens of Mr. Sharpe's collection (for the inspection of which I am indebted to him) the distortions are very striking, and, for the most part, of that kind which I have termed *angular distortion*. Now all the most remarkable instances of this kind, as Mr. Sharpe has stated in his memoir, are those in which the planes of stratification and those of cleavage are approximately coincident, the angles between them varying from one or two to ten or fifteen degrees; whence I should conclude that the cleavage planes must have approximately coincided with the *planes of greatest tangential action*, and consequently that it is to this kind of mechanical action, and not to direct pressure, that the influence in the production of the cleavage structure must be attributed. Mr. Sharpe has also described and figured other specimens taken from beds in which the planes of stratification are inclined to those of cleavage at angles varying from forty to sixty degrees, and in these cases the distortions (as described in his memoir) consist in a shortening of the axes of the shells in directions perpendicular to the intersections of the planes of stratification with those of cleavage, such as would result from *direct pressure* in that direction. So far this evidence is perfectly in accordance with that previously cited, for it indicates that the direction of maximum pressure must have approximately coincided with the planes of stratification, and therefore that these planes must have been inclined approximately at an angle of forty-five degrees to those of *maximum tangential action*. Consequently these latter planes must have approximately coincided with the cleavage planes in this case as well as in the former one. This latter evidence, however, furnished by Mr. Sharpe's specimens is not, probably, nearly so complete with respect either to the number of distorted shells or the distinctness of their distortions, as that furnished by the shells first mentioned as so curiously and distinctly characterized by great angular distortion. Still, the evidence hitherto adduced appears to be, on the whole, consistent with itself and strongly in favour of the conclusion that whatever may have been the influence of pressure in producing a laminated structure, that influence must have been due to the tangential action parallel to those planes, and not to direct pressure perpendicular to them. In fact, I regard the specimens above mentioned, in which there is great angular distortion combined with an approximate coincidence of the planes of cleavage and of stratification, as almost decisive against the latter conclusion.

In the search of further evidence, the observer should direct his attention especially to those cases in which the inclination of the cleavage planes to the bedding is either small or nearly 45°. In the former case, according to the above inferences, he may expect to find great angular distortion of the fossils lying (as they will very generally be found to lie,) with the plane of separation of the two valves parallel to the surfaces of the beds; and in the latter case he may expect to find the shells characterized more especially by direct compression or extension (more probably the former,) in the plane of the bed, and in directions perpendicular to the intersections of that plane with the planes of cleavage. At the same time it should be remarked that the angular distortion may be accompanied by a lengthening or shortening of the shell, (more probably the former.) in the direction of the dip, and also that a considerable direct compression is not likely to be produced without some degree of angular distortion; but still, if the above conclusions be true, angular distortion in the one case, and direct compression or extension in the other, ought especially to characterize the actual forms of the organic remains.

It might be objected against the theory to which the preceding conclusions tend to lead us, that if tangential action has been an effective cause in the production of the laminated structure, there ought to be two systems of cleavage planes at right angles to each other, since there are two such systems of parallel planes in which the tangential action is a maximum; and this might, I think, be regarded as a valid objection to a theory which should assign the mechanical action here considered as the primary cause of the laminated structure; but the objection may probably be obviated in a great degree, if we regard this kind of mechanical action only as a secondary cause, for it is very conceivable that it might have greater effect in aiding the development of the structure in question along one of the systems of planes of greatest tangential action than along the other. Whatever may be the apparent force of this objection, however, the discussion of it may be regarded, perhaps, somewhat premature till further observation shall have ascertained more distinctly what indications may be found of the existence of a second set of cleavage planes less developed than those which more immediately attract our notice. The point is deserving of the attention of the geologist.

The adoption of the opinion, that the mechanical agency above described has been one efficient cause of the laminated structure, necessarily involves the conclusion of that structure having originated at some epoch posterior to the great movements which have determined the general configuration of the external surface, and the geological structure of large portions of the earth's crust, which are observed to possess this laminated character. It is also a necessary inference that the line of strike of the planes of lamination must coincide with that of the planes of stratification (Art. 7). The results of observation undoubtedly corroborate this latter inference, for it would appear that we may state as a general fact, that the strike of the planes of cleavage is parallel to the directions of the anticlinal lines of the district. The amount and character of the local deviations from this law are not yet in any case, I believe, accurately determined. Observed facts appear, also, to corroborate the above conclusion respecting the epoch at which the laminated structure was superinduced; for the persistency with which the strike and dip of the cleavage planes are frequently maintained through disrupted and contorted strata, distinctly implies that the lamination must have been produced after the elevation and disruption of the general mass. These general facts are in harmony with the theoretical view of the subject which has been here presented; how far the more detailed results of observation will be found so remains to be determined.

Geologists are well aware that currents of electricity have been assigned as a probable cause of the laminated structure, and that this hypothesis has received great support from the results of experiments made in the first instance by Mr. Fox, and afterwards repeated by Mr. Hunt, as described by the latter gentleman in an interesting memoir, contained in the *Memoirs of the Geological Survey of Great Britain*, Vol. I., on the influence of magnetism on crystallization. It would be foreign to my object to enter into any discussion of a theory founded on these experimental results; and indeed detailed discussion of any theory on the subject would be, I conceive, at present entirely premature; but I would remark that views of the subject founded on these results, and those founded on the observed facts respecting the distortion of organic forms, are by no means to be considered as opposed to each other. On the contrary, it is very possible tion we may hereafter be better able to account for the phenomena of lamination by the joint operathat of the causes to which they would be referred respectively according to these two views of the subject, than by the independent operation of one of those causes only.

In concluding this communication, I would especially remark that the advocacy of any particular theory on the subject of cleavage structure has formed no part of my object. Our ignorance of the physical causes of crystallization, or the manner in which such supposed causes may operate, is too great to admit of our forming at present any theory on the subject which might not be deemed altogether premature. All I would here insist upon is this—that the facts observed by Professor Phillips and Mr. Sharpe indicate certain determinate relations between the distortions of organic forms and the positions of the planes of lamination of the beds in which those forms are discovered, relations which seem to imply that the forces which produced the distortion had also their influence in determining the planes of lamination. My object has been to point out the accurate mechanical conditions of the problem, and thus to indicate the points to which the attention of future observers should be especially directed in order that their observations may afford conclusive tests of the truth of any theory which may hereafter recognize the efficiency of the mechanical agency explained in this paper, as one of the causes of the laminated structure.

W. HOPKINS.

CAMBRIDGE, *May* 3, 1847.

XXXV. *On the Partition of Numbers, and on Combinations and Permutations.* *By* HENRY WARBURTON, M.A., M.P., F.R.S., F.G.S., *formerly of Trinity College.*

[Read *March* 1, 1847.]

Introduction.

THE researches of which an account is here presented, had their origin in the following manner. In the Autumn of 1846, having communicated a Theorem (which will be found in the sequel) on the Partition of Numbers to Professor A. De Morgan, I received from him an obliging reply, wherein he intimated a wish that I would turn my attention to Combinations, as a department in Mathematics, which, he thought, much needed cultivation. I acted upon this suggestion, and shortly afterwards sent to Mr. De M. results, and subsequently from time to time further results, which he wished me to render public. These I placed at his disposal; and, with my concurrence, he drew up an account of my Researches, in a Paper which was read before the Society on the 1st of March, 1847.

After the reading of this Paper, further suggestions presented themselves to me, of which I drew up an account, and this was laid before the Society by way of Supplement to the former Paper of Professor De Morgan. Still further improvements again occurred to me; and it then seemed to me desirable that both Papers should be withdrawn, to give me an opportunity of revising my own researches, and of incorporating the revision in one Paper to be communicated to the Society.

Many important original observations on the same heads of inquiry, proceeding from Professor De Morgan himself, were contained in the Paper which he drew up; and I should much regret if, in consequence of the course which I have suggested of withdrawing that communication, those observations were to be lost to the Society and the public.

It was as impossible for me, as for any other person, to hold communication with that gentleman on Mathematical questions, and avoid deriving great advantage from his sagacity and erudition in Mathematics. I have not, I trust, abused those advantages by appropriating to myself anything which belongs to him; but I have endeavoured, while possessing those advantages, to carry on my researches with originality and independence.

SECTION I.

On the Partitions of Numbers.

1. FROM a recollection of the important application made by Waring of the Partitions of Numbers to the developement of the power of a Polynome, I was led to investigate their properties, in the hope of discovering some ready method of determining in how many different ways a given Number can be resolved into a given number of parts.

Assuming the Unit to be the lower limit of the magnitude of the parts, I found that if the Number to be partitioned, N, were expressed in terms of a certain Modulus, m, so that N was

$= mt + r$, the number of the ways of resolving N into p parts could be expressed in the form of a rational and integral function of the factor, t. Thus, in the case of Bi-partition, 2 being the Modulus, and the Number being $2t$, or $2t+1$, t is the number of the partitions. In the case of Tri-partition, 6 being the Modulus,

For $N = 6t$, the number of the partitions is $3t^2$,
... $= 6t \pm 1$ .... $t[3t \pm 1]$,
... $= 6t \pm 2$ .... $t[3t \pm 2]$,
... $= 6t + 3$ .... $3t^2 + 3t + 1$,

and in the case of Quadri-partition, when the Modulus is 12, and t becomes of 3 dimensions, I also ascertained the formulæ. But perceiving that, since the modulus and the dimensions would increase with the number of the parts, the functions obtained would be so many, and of such complexity as to be of little or no practical utility, I abandoned that method, and sought for some other. Having at last discovered the method here proposed, (Arts. 7 and 8, Sect. 1,) I communicated the same to Professor A. De Morgan, trusting to his known Mathematical erudition for obtaining the information I required—whether the method was novel. By his reply, I was made aware that the Partitions of Numbers had received a share of his attention, and that he had written a paper on the subject, which was published anonymously in the 4th Volume of the *Cambridge Mathematical Journal*. He further stated that, after the date of that publication, he had also discovered the Theorem which I communicated to him; though he had not announced it; and since I have no doubt of the entire accuracy* of that statement, he must participate fully in any credit that may attach to the discovery of the formula in question.

In this Section of my present Paper, I have limited myself, as regards these partitions, to what I considered necessary for the proof and illustration of the Theorem in question. Other matters bearing on the question of Partitions occur in the Section on Combinations.

2. The number of the different ways in which a Number, N, can be resolved into p parts, when no number is admitted as a part, but such as is either equal to, or greater than, the arbitrary number, η, may be denoted by $[N, p_\eta]$. We may term η the lower limit of the parts, or partition, or, for brevity, the lower limit. By a p – partition of N, I mean any set of p numbers, having N for their sum.

A partition included among those, the number of which is denoted by $[N, p_\eta]$, may consist of parts exclusively equal to, or exclusively greater than η; or it may contain some parts equal to, and some parts greater than η.

$[N, p_\eta]$ includes the whole of $[N, p_{\eta+1}]$;
$[N, p_{\eta+1}]$ includes the whole of $[N, p_{\eta+2}]$;

and, generally, the partitions which have η for their lower limit, include all those partitions in which the lower limit is greater than η.

3. If to or from each part in every partition of N whose lower limit is η, a given number θ be added or subtracted, N will be increased or diminished by the amount $p\theta$; but the number of the partitions, and the number of the parts in every partition, will remain unchanged: *i. e.*

$$[N, p_\eta] = [N \pm p\theta, p_{\eta\pm\theta}] \text{........................(1).}$$

This involves the conclusion, that we recognize 0, and negative numbers also, among the admissible parts; unless we expressly assume that they are to be excluded. It also involves the recognition of negative numbers, as the subjects of partition, unless their exclusion be expressly stipulated.

4. Since $N = \eta + (N - \eta)$, therefore take all such $(p - 1)$ – partitions of $N - \eta$, as have η for their lower limit; and with every such set of parts let η be conjoined as an additional, or p^{th} part. We shall thus obtain all such p – partitions of N, as, besides having η for their lower limit, agree also in containing at least one part equal to η. From these partitions, therefore, are excluded all those p – partitions of N which have for their lower limit $\eta + 1$. Hence,

$$[N,\ p_\eta] - [N,\ p_{\eta+1}] = [N - \eta,\ p - 1_\eta] \ldots\ldots\ldots\ldots (\text{II});$$

$$\therefore \text{ also } [N + \eta,\ p + 1_\eta] - [N + \eta,\ p + 1_{\eta+1}] = [N,\ p_\eta] \ldots\ldots\ldots\ldots (\text{II}^*).$$

5. In $[N,\ p_\eta]$, let $N < p\eta$. If negatives be admitted, then let N be the greater negative; and by the addition to η of a positive quantity greater than η, and to N of a positive quantity greater than $p\eta$, let the lower limit and the number to be partitioned be rendered positive. Since no positive integer less than $p\eta$ can be resolved into p parts having a positive integer, η, for their lower limit, no partition of the kind indicated by the notation can be effected. Therefore,

$$\text{when } \quad N < p\eta,\ [N,\ p_\eta] = 0 \ldots\ldots\ldots\ldots (\text{III}).$$

Then if η be positive, and p is also a positive integer, $[0,\ p_\eta] = 0$.

The two following extreme cases, $[N,\ 0_\eta]$, and $[0,\ 0_\eta]$ require explanation.

$$\text{By (II.)} \quad [N + \eta,\ 1_\eta] - [N + \eta,\ 1_{\eta+1}] = [N,\ 0_\eta].$$

But, when N and η are positive integers, $[N + \eta,\ 1_\eta] = 1$,

$$\text{and} \quad [N + \eta,\ 1_{\eta+1}] = 1;$$

$$\therefore [N,\ 0_\eta] = 1 - 1 = 0 \ldots\ldots\ldots\ldots (\text{IV}).$$

$$\text{Also, by (II.)} \quad [N,\ 1_N] - [N,\ 1_{N+1}] = [0,\ 0_N].$$

$$\text{But} \quad [N,\ 1_N] = 1; \quad \text{and, by (III.)} \quad [N,\ 1_{N+1}] = 0;$$

$$\therefore [0,\ 0_N] = 1 - 0 = 1 \ldots\ldots\ldots\ldots (\text{V}).$$

Hence also, if $N = p\eta$, $[N,\ p_\eta] = [p\eta,\ p_\eta] = [0,\ p_0] = [p,\ p_1] = 1$.

6. Professor De Morgan (as he informs me) has, in the *Cambridge Mathematical Journal*, traced Equation (II.) to its consequences, in the case where the number of parts is preserved constant, and the variation is thrown on the number to be partitioned.

In this case, $[N,\ p_\eta] - [N,\ p_{\eta+1}] = \ldots\ldots\ldots\ldots = [N - \eta,\ p - 1_\eta]$,

$$[N,\ p_{\eta+1}] - [N,\ p_{\eta+2}] = [N - \eta - 1,\ p - 1_{\eta+1}] = [N - \eta - p,\ p - 1_\eta].$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$[N,\ p_{\eta+\theta}] - [N,\ p_{\eta+\theta+1}] = [N - \eta - \theta,\ p - 1_{\eta+\theta}] = [N - \eta - p\theta,\ p - 1_\eta];$$

$$\therefore [N,\ p_\eta] - [N,\ p_{\eta+\theta+1}] = \underset{z}{S}{}_0^\theta [N - \eta - pz,\ p - 1_\eta].$$

Here η is the lower limit of the parts admitted, and $\eta + \theta + 1$ is the lower limit of the parts excluded; that is to say, all those partitions are excluded which have every part greater than $\eta + \theta$. Write Y' for $\eta + \theta$. Then let $N = pY + r$, r being a remainder less than p, and Y the integer nearest to, but not exceeding $\frac{N}{p}$. When Y' becomes Y, $[N,\ p_{Y'-1}] = 0$;

$$\therefore [N,\ p_\eta] = [N,\ p_{Y'+1}] + \underset{z}{S}{}_0^{Y'-\eta} [N - \eta - pz,\ p - 1_\eta]$$

$$= \underset{z}{S}{}_0^{Y'-\eta} [N - \eta - pz,\ p - 1_\eta]$$

$$= [N,\ p_{Y'+1}] + \underset{z}{S}{}_0^{Y'-\eta} [N - (1 + p[\eta - 1]) - pz,\ p \underset{1}{-} 1] \ldots\ldots\ldots (\text{VI.})$$

$$= \underset{z}{S}{}_0^{Y'-\eta} [N - (1 + p[\eta - 1]) - pz,\ p \underset{1}{-} 1] \ldots\ldots\ldots (\text{VII.})$$

Thus, $[31, 5_3] = \underset{z}{S_0^3}[20 - 5z, 4_1],$

$= [20, 4_1] + [15, 4_1] + [10, 4_1] + [5, 4_1];$

i.e. 101 = 64 + 27 + 9 + 1.

7. By a different transformation of the equation of differences, (II), we arrive at a different summation; in which the number remains constant, while the parts vary. In that equation, if we write q for p, we have,

$$[N+\eta, \underset{\eta}{q+1}] - [N, q_\eta] = [N+\eta, \underset{\eta+1}{q+1}] = [N - q\eta, \underset{1}{q+1}];$$

$$[N+2\eta, \underset{\eta}{q+2}] - [N+\eta, \underset{\eta}{q+1}] = [N+2\eta, \underset{\eta+1}{q+2}] = [N - q\eta, \underset{1}{q+2}].$$

....

$$[N+p\eta, \underset{\eta}{q+p}] - [N+(p-1)\eta, \underset{\eta}{q+(p-1)}] = [N+p\eta, \underset{\eta+1}{q+p}] = [N - q\eta, \underset{1}{q+p}];$$

$$\therefore\ [N+p\eta, \underset{\eta}{q+p}] - [N, q_\eta] = \underset{z}{S_1^p}[N - q\eta, \underset{1}{q+z}].$$

Now $[N, q_\eta]$ vanishes, either when $N < q\eta$, or when $q = 0$; the exception to the latter case being when $N = q\eta$. If $N < q\eta$; then, since

$$[N+p\eta, \underset{\eta}{q+p}] = [N - q\eta + q + p, \underset{1}{q+p}] = 0,$$

$$\text{and } [N, q_\eta] = [N - q\eta + q, q_1] = 0,$$

$$\text{it follows that } \underset{z}{S_1^p}[N - q\eta, \underset{1}{q+z}] = 0.$$

And we have only $0 + 0 = 0$.

But if $q = 0$,

$$[N+p, p_1] - [N, 0_1] = \underset{z}{S_1^p}[N, z_1];$$

$$\therefore\ [N+p, p_1] = \underset{z}{S_0^p}[N, z_1] \text{ .... (VIII)},$$

$$\text{or } [N, p_1] = \underset{z}{S_0^p}[N-p, z_1] \text{ .... (VIII*)};$$

$$\therefore \text{ also } [N, p_\eta] = \underset{z}{S_0^p}[N - p\eta, z_1] \text{ .... (IX)}.$$

Thus $[31, 5_3] = [16, 0] + [16, 1] + [16, 2] + [16, 3] + [16, 4] + [16, 5]$.

Or, 101 = 0 + 1 + 8 + 21 + 34 + 37.

The following very elementary proof of this proposition has also suggested itself to me.

We shall exhaust all the ways of resolving N into p parts, having 1 for their lower limit, if we take

1st, $p - 1$ units, and the remainder $N - (p - 1)$ entire, not less than 2.

2d, $p - 2$ units, and the Bi-partitions of the remainder $N - p + 2$, not less than 2.

....

$m$thly, $p - m$ units, and the m – partitions of $N - p + m$, not less than 2.

Lastly, $p - p = 0$ units, and the p – partitions of $N - p + p = N$, not less than 2*.

* When $p > \frac{N}{2}$, the greatest value of m is $N - p$; and the partition of N, corresponding to that value, is [$(2p - N)$ units, and $(N - p)$ repetitions of the number 2]. As regards the *number* of the partitions, the two cases of p not $> \frac{N}{2}$, and $p > \frac{N}{2}$, are both comprehended in formula (VIII*).

From each of the parts, in every one of these partitions, deduct 1.
Then we shall have

$$[N, p_1] = [N - p, 1_1] + [N - p, 2_1] + \dots\dots\dots [N - p, m_1] + \dots\dots\dots [N - p_1];$$

Also, $[N + p, p_1] = [N, 1_1] + [N, 2_1] + \dots\dots\dots [N, p_1]$;

$$= [N, p_0].$$

$N =$	0	1	2	3	4	5	6	7	8	9	10	11	12
p = 0	1	0	0	0	0	0	0	0	0	0	0	0	0
... = 1	0	1	1	1	1	1	1	1	1	1	1	1	1
... = 2	0	0	1	1	2	2	3	3	4	4	5	5	6
... = 3	0	0	0	1	1	2	3	4	5	7	8	10	12
... = 4	0	0	0	0	1	1	2	3	5	6	9	11	15
... = 5	0	0	0	0	0	1	1	2	3	5	7	10	13
... = 6	0	0	0	0	0	0	1	1	2	3	5	7	11
... = 7	0	0	0	0	0	0	0	1	1	2	3	5	7
... = 8	0	0	0	0	0	0	0	0	1	1	2	3	5
... = 9	0	0	0	0	0	0	0	0	0	1	1	2	3
... = 10	0	0	0	0	0	0	0	0	0	0	1	1	2
... = 11	0	0	0	0	0	0	0	0	0	0	0	1	1
... = 12	0	0	0	0	0	0	0	0	0	0	0	0	1
Sum of Columns.	1	1	2	3	5	7	11	15	22	30	42	56	77

8. I shall proceed to shew the application of these latter formulæ to the construction of a Table of the Partitions of Numbers, and point out the leading properties of such a table: and since all partitions, whatever may be their lower limit, are reducible to partitions whose lower limit is 1, I shall confine my observations to a table whose lower limit is the Unit.

To the Equation of Differences, 11, we may give the following forms:

$$[N, p_1] \quad - [N - 1, \underset{1}{p - 1}] \quad = [N - p, p_1]; \dots\dots\dots\dots\dots (\text{x}).$$

$$\text{or } [N + p, p_1] - [N + p - 1, \underset{1}{p - 1}] = [N, p_1] \dots\dots\dots\dots\dots\dots\dots (\text{x}^*).$$

and these will best serve for the construction of the table.

The annexed table is one of double entry, N being the index of the columns, and p of the lines. $[N, p_1]$ is the term in column N, line p. In formula x, the change to $N - 1$, and $p - 1$, marks that we are to recede simultaneously one column and one line, that is, diagonally. The diagonal will cut line 0 at the head of column $N - p$, and $[N - p, p_1]$ is the term on the p^{th} line of that column. Thus the term on the p^{th} line in the vertical column is the difference between the terms on the p^{th} and $(p - 1)^{\text{th}}$ lines on the diagonal. Suppose that all the terms are known in the vertical column N, and that we have determined all the terms on the diagonal, proceeding from the head of that column to the $(p - 1)^{\text{th}}$ line inclusive. Then the term in the diagonal on the p^{th} line, that is, $[N + p, p_1]$, is equal to $[N + p - 1, \underset{1}{p - 1}] + [N, p_1]$; and in the same way the term on the $(p+1)^{\text{th}}$ line of the same diagonal may be found; and so in succession, to any required extent, until they become constant: (vide § 9).

The consequence of the preceding equation is, that any term in the table, say that on the p^{th} line, in column N, is equal to the sum of all the terms from line 0 to line p inclusive in column $N-p$; which is the column at the same distance backwards from column N, that the line 0 is from the line p.

9. If in the table we draw a zig-zag line from [0, 0] to [12, 6], it will be seen that all the terms below that line are of constant recurrence, and are identical with the numbers 1, 1, 2, 3, 5, 7, 11, 15, &c., which arise from the summation respectively of all the terms in the columns ...... 0, 1, 2, 3, 4, 5, 6, 7, &c. I proceed to explain this. Let any diagonal line proceed from the head, say of column A, advancing simultaneously one column and one line. When that diagonal cuts the line p, N will be equal to $A+p$.

$$\text{Now } [A+p,\ p] = S_0^p [A,\ z],$$

and when $p = A$, or $> A$, its value becomes constant, and is $[2A,\ A] = [A,\ 0] + [A,\ 1] + \ldots [A,\ A]$, that is, it becomes equal to the sum of all the terms in column A. Thus one-half of the whole table is occupied by terms $= 0$; and an additional fourth of it by these constants; and were it thought requisite to compute a table of the partitions of numbers, it is only the terms that occupy the remaining fourth of the whole space of the table, that would actually require computation by the method of differences: and of this fourth the three first lines are so obvious, as merely to require being transcribed.

SECTION II.

On Combinations.

1. THE well-known Theorem in Combinations enables us to determine in how many different ways u elements can be taken at a time out of s elements, all dissimilar. It is the coefficient of x^u in the developed power of the binome, $[1+x]^s$, which, in this case, affords the solution of the problem.

2. In the first case of combinations which I now propose to investigate, the combining elements are also of s different kinds; but there may be more than one element of the same kind: for instance, α of the elements A, β of the elements B, and so on; and the question proposed is,—In how many different ways u of the said $[\alpha+\beta+\text{\&c.}] = \sigma$ elements can be taken at a time, on condition that those which are *plural* in their respective kinds, may be repeated in the same combination?

3. Combining elements of the form proposed are found in the s geometrically progressing polynomes,

$$[1 + Ax + A^2x^2 + \ldots\ldots A^\alpha x^\alpha] \times [1 \times Bx + \ldots\ldots B^\beta x^\beta] \times \text{\&c.},$$

and all the possible combinations of these elements, taken 0, 1, 2, ..., u, ... σ, at a time, are respectively found aggregated, each with a positive sign, in the coefficients we obtain of

$$x^0,\ x^1,\ x^2,\ \ldots\ldots\ x^u,\ \ldots\ldots,\ x^\sigma,$$

when the product of the said polynomes is developed according to the powers of x. That developement, supposing all the coefficients to be complete, is of the form

$$1 + S[A]\,x + S[A^2 + AB]\,x^2 + S[A^3 + A^2B + ABC]\,x^3 + \ldots\ldots$$
$$+ S[A^u + A^{u-1}B + A^{u-2}(B^2 + BC) + \text{\&c.}]\,x^u + \ldots\ldots$$

If 1 be now substituted for each of the elements A, B, C, &c., the polynomes will respectively, become $[1 + x + x^2 + \ldots\ldots x^{a}]$, $[1 + x + x^2 + \ldots\ldots x^{\beta}]$, &c., and the coefficient of x^u in the developed product, now that 1 has become the value also of each term of the form $A^p B^q C^r$ in that product, will represent, not only the sum, but also the number of all such terms: that is to say, of the different combinations which can be formed with the σ elements, taken u at a time.

4. That coefficient is an explicit function of u, which I now proceed to determine.

The product of these geometrical polynomes, is

$$\frac{1 - x^{a+1}}{1 - x} \cdot \frac{1 - x^{\beta+1}}{1 - x} \cdot \&c.;$$

$$\text{that is, } (1 - x)^{-s}\left[1 - x^{a+1}\right]\left[1 - x^{\beta+1}\right].\&c. \ldots\ldots\ldots \text{(XI.)}$$

$$\text{But } [1 - x]^{-s} = 1 + sx + \frac{s^{2|1}}{1^{2|1}}x^2 + \ldots\ldots \frac{s^{u|1}*}{1^{u|1}}x^u +$$

$$= \frac{1}{1^{s-1|1}}\left[1^{s-1|1} + 2^{s-1|1}x + \ldots\ldots [u+1]^{s-1|1}x^u +\right].$$

For the sake of brevity,

Let $u + 1$ be represented by $u_{,}$;

$a + 1$ .... by $a_{,}$;

$\beta + 1$ .... by $\beta_{,}$; &c.

1st. When each of the s kinds of elements, A, B, C, &c. admits of unlimited repetition, the required coefficient of x^u, will be

$$\frac{u_{,}^{s-1|1}}{1^{s-1|1}}, \quad \text{or} \quad \frac{s^{u|1}}{1^{u|1}}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(XI*)};$$

and in this case, of plural elements, all kinds admitting of unlimited repetition, a solution of the combination problem, to the same effect as the preceding, has, as Mr. De Morgan informs me, been given by Hirsch.

2dly. When the elements of one kind, A, are limited in number to a, but the elements of the other $(s - 1)$ kinds may be repeated without limit, the required coefficient, (which is that of $(1 - x)^{-s}[1 - x^{a_{,}}]$), will manifestly be

$$\frac{1}{1^{s-1|1}}\left[u_{,}^{s-1|1} - [u_{,} - a_{,}]^{s-1|1}\right], \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(XII)},$$

from which expression however, the second term is to be excluded, in case $[u_{,} - a_{,}]$ should be negative.

3dly. When the elements of two kinds, A and B, are limited in number to a and β respectively, but the elements of the other $s - 2$ kinds may be repeated without limit, the required coefficient, (which is that of x^u in the developement of $(1 - x)^{-s}[1 - x^{a_{,}}][1 - x^{\beta_{,}}]$), will be obtained by performing on formula (XII) with $\beta_{,}$ the same operation that was before performed on formula (XI*) with $a_{,}$. The result will manifestly be

$$\frac{1}{1^{s-1|1}}\left\{\begin{matrix} u_{,}^{s-1|1} - [u_{,} - a_{,}]^{s-1|1} + [u_{,} - a_{,} - \beta_{,}]^{s-1|1} \\ - [u_{,} - \beta_{,}]^{s-1|1} \end{matrix}\right\} \ldots\ldots\ldots\ldots \text{(XIII)},$$

---

* I use the factorial notation, in which

$s^{u|1}$ represents $s(s+1)(s+2)\ldots\ldots\ldots\ldots[s+(u-1)]$,

and $s^{u|-1}$ .... $s(s-1)(s-2)\ldots\ldots\ldots\ldots[s-(u-1)]$.

from which expression, however, every factorial expression is to be excluded which has a negative quantity for its first factor.

4thly. By operating on formula (XIII), with $\gamma_{,}$ in a similar manner, and on the result of that operation with $\delta_{,}$, and so on in succession, until there remain no factors undisposed of, we shall obtain for the coefficient of the developement of $[1-x]^{-s} \times [1-x^{\alpha_{,}}][1-x^{\beta_{,}}]$, &c. the following expression, subject to the same rule as before, of omitting every factorial which has a negative quantity for its first factor:

$$\frac{1}{1^{s-1|1}}\left\{\begin{array}{l} u_{,}^{s-1|1} - [u_{,}-\alpha_{,}]^{s-1|1} + [u_{,}-\alpha_{,}-\beta_{,}]^{s-1|1} - \&c. \\ \qquad - [u_{,}-\beta_{,}]^{s-1|1} + [u_{,}-\alpha_{,}-\gamma_{,}]^{s-1|1} - \&c. \\ \qquad - [u_{,}-\gamma_{,}]^{s-1|1} + [u_{,}-\beta_{,}-\gamma_{,}]^{s-1|1} - \&c. \\ \qquad - \&c. \qquad\qquad + \&c. \end{array}\right\} \ldots\ldots (\text{XIV}).$$

5. Now if α, β, γ, &c., are all equal, that is to say, if the required coefficient is that of x^{u} in the developement of $\left(\frac{1-x^{\alpha_{,}}}{1-x}\right)^{s}$, formula (XIV) will become

$$\frac{1}{1^{s-1|1}}\left\{\begin{array}{l} u_{,}^{s-1|1} - s\,[u_{,}-\alpha_{,}]^{s-1|1} + \frac{s^{2|-1}}{1^{2|1}}[u_{,}-2\,\alpha_{,}]^{s-1|1} - \&c. \\ \qquad\qquad + (-1)^{\theta}\,\frac{s^{\theta|-1}}{1^{\theta|1}}\cdot[u_{,}-\theta\alpha_{,}]^{s-1|1} + \&c. \end{array}\right\} \ldots\ldots (\text{XV});$$

where, for any determinate value of u, the maximum of θ is the integer nearest to, and not exceeding $\frac{u+1}{\alpha+1}$; but if u attain its maximum (which is $s\alpha$), then the maximum of θ is the integer nearest to, and not exceeding $\frac{s\alpha+1}{\alpha+1}$.

Example of formula (XIV).

How many different combinations can be formed by taking 2, or 8, at a time, of the 10 elements, of 4 different kinds,

A, BB, CCC, DDDD?

Answer, for $u = 2$; $\frac{1}{1.2.3}[3.4.5 - 1.2.3] = 9.$

Answer for $u = 8$.

$$\frac{1}{1.2.3}\left\{\begin{array}{r} 9.10.11 - 7.8.9 + 4.5.6 \\ - 6.7.8 + 3.4.5 \\ - 5.6.7 + 2.3.4 \\ - 4.5.6 + 2.3.4 \\ + 1.2.3 \end{array}\right\} = 9.$$

Example of formula (XV).

How many different combinations can be formed by taking 2, or 8, at a time, of the 10 elements, belonging to 5 different kinds,

AA, BB, CC, DD, EE?

Answer for 2; $\frac{1}{1.2.3.4}.3.4.5.6 = 15.$

.... 8; $\frac{1}{1.2.3.4}[9.10.11.12 - 5.6.7.8.9 + 10.3.4.5.6] = 15.$

6. In cases of Combination, such as those to which formulas (XIV.) and (XV.) apply, when it is required to determine the number of Combinations corresponding, not merely to one or two powers of x, but to the entire range of the values of u, from 0 to $[\alpha + \beta + \gamma, \&c.] = \sigma$ in the former case, and from 0 to $s\alpha = \sigma$ in the latter, the expression (XI.) for the product of the s Polynomes suggests the following method for determining arithmetically the entire series of the Coefficients. The method will be best explained by an example.

How many Combinations can be formed from the Six Elements A, BB, CCC, taking 0, 1, 2, 3, 4, 5, 6 of them at a time.

Different Values of u	0	1	2	3	4	5	6	7
Coefficients of $(1-x)^{-3}$	1	3	6	10	15	21	28	36
Subtract			1	3	6	10	15	21
Coefficients of $(1-x)^{-3} \times [1-x^2]$	1	3	5	7	9	11	13	15
Subtract				1	3	5	7	9
Coefficients of $(1-x)^{-3}[1-x^2][1-x^3]$	1	3	5	6	6	6	6	6
Subtract					1	3	5	6
Coefficients of $(1-x)^{-3}[1-x^2][1-x^3][1-x^4]$	1	3	5	6	5	3	1	0

The law of the terms in the last line, which contains the answer, deserves notice: viz. that the terms corresponding to the indices u and $6-u$, are equal.

How many different Combinations can be formed from the Four Elements AA, BB, taking 0, 1, 2, 3, 4 at a time?

Different values of u	0	1	2	3	4	5
Coefficients of $(1-x)^{-2}$	1	2	3	4	5	6
Subtract				1	2	3
Coefficients of $(1-x)^{-2}[1-x^3]$	1	2	3	3	3	3
Subtract				1	2	3
Coefficients of $[1-x]^{-2}[1-x^3]^2$	1	2	3	2	1	0

7. From the given numbers of the Combinations formed by τ elements of t different kinds which combine v at a time, and by $\sigma - \tau$ elements, of $s - t$ different kinds, which combine $u - v$ at a time, it is required to determine the numbers of the Combinations formed by those elements

conjoined; that is, by σ elements, of s different kinds, which combine u at a time; assuming the t kinds to be different from the $s-t$ kinds.

Let the Combinations which can be formed by

taking u* at a time of the σ elements, be $\{u, \sigma\}$.
... v .... τ .... $\{v, \tau\}$.
... $u-v$ .... $\sigma-\tau$ .... $\{u-v, \sigma-\tau\}$.

Imagine some determinate values given to the variables u and v. Every Combination $\{u-v, \sigma-\tau\}$ may be paired with every Combination $\{v, \tau\}$; and thence will arise $\{u-v, \sigma-\tau\} \times \{v, \tau\}$ different pairs, each containing u elements of the s kinds. If u remains constant, while v varies, there will be a pair for every value of v from 0 to u, if $u < \tau$; and from 0 to τ, if $u > \tau$.

Thus we have, for $u < \tau$,

$$\{u, \sigma\} = \{u, \sigma-\tau\}\{0, \tau\} + \{u-1, \sigma-\tau\} \times \{1, \tau\} + \ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots + \{1, \sigma-\tau\}\{u-1, \tau\} + \{0, \sigma-\tau\}\{u, \tau\} \ldots\ldots (\text{XVI}).$$

For $u > \tau$,

$$\{u, \sigma\} = \{u, \sigma-\tau\}\{0, \tau\} + \{u-1, \sigma-\tau\}\{1, \tau\} + \ldots\ldots\ldots\ldots (\text{XVII}).$$
$$\ldots\ldots\ldots\ldots + \{u-\tau+1, \sigma-\tau\}\{\tau-1, \tau\} + \{u-\tau, \sigma-\tau\}\{\tau, \tau\}\dagger.$$

Now in each of these expressions write $(\sigma-u)$ for u; and we shall have in the former $\sigma-u > \sigma-\tau$; therefore some of the terms at the commencement of this formula fail; in the second expression we shall have $\sigma-u < \sigma-\tau$.

For $\sigma-u > \sigma-\tau$,

$$\{\sigma-u, \sigma\} = \{\sigma-\tau, \sigma-\tau\}\{\tau-u, \tau\} + \ldots\ldots\ldots \{\sigma-\tau-u, \sigma-\tau\}\{\tau, \tau\} \ldots\ldots\ldots (\text{XVIII}).$$

For $\sigma-u < \sigma-\tau$,

$$\{\sigma-u, \sigma\} = \{\sigma-u, \sigma-\tau\}\{0, \tau\} + \ldots\ldots\ldots\ldots \{\sigma-\tau-u, \sigma-\tau\}\{\tau, \tau\} \ldots\ldots\ldots\ldots (\text{XIX}).$$

I shall apply this method to the two examples before given, where we have for the Combinations of

A, *BB*, *CCC*, taken 0, 1, 2, 3, 4, 5, 6 at a time,
1, 3, 5, 6, 5, 3, 1 Combinations;

and for the Combinations of

DD, *EE*, taken 0, 1, 2, 3, 4 at a time,
1, 2, 3, 2, 1 Combinations.

From these numbers, we have to determine the number of the Combinations of all the Elements *A*, *BB*, *CCC*, *DD*, *EE*, taken together 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 at a time.

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* I write $\{u, \sigma\}$ to denote the number of Combinations formed by σ elements, plural or singular, of any kind or numbers of kinds, when those elements are taken u at a time: { } is used instead of [ ], in order not to confound Combinations with Partitions.

† The formula is perhaps better given in the condensed form,

$$\{u, \sigma\} = S_{0}^{\tau}[\{v, \tau\} \times \{u-v, \sigma-\tau\}];$$

and the terms of the table itself are given by the equation $[u, v] = [v, v]\,[u-v, 0]$; which means that the term on line u, column v, is equal to the product of the term on line v, column v, by the term on line $(u-v)$, column 0.

$v =$	0	1	2	3	4	
$\{v, \tau\} =$	1	2	3	2	1	9 the total.
u =	$\{u - v, \sigma - \tau\}$ =					$\{u, \sigma\}$
0	1					$1 = \{0, \sigma\}$
1	3	2				$5 = \{1, \sigma\}$
2	5	6	3			$14 = \{2, \sigma\}$
3	6	10	9	2		$27 = \{3, \sigma\}$
4	5	12	15	6	1	$39 = \{4, \sigma\}$
5	3	10	18	10	3	$44 = \{5, \sigma\}$
6	1	6	15	12	5	$39 = \{6, \sigma\}$
7		2	9	10	6	$27 = \{7, \sigma\}$
8			3	6	5	$14 = \{8, \sigma\}$
9				2	3	$5 = \{9, \sigma\}$
10					1	$1 = \{10, \sigma\}$
	24	48	72	48	24	$216 = 9 \times 24$

8. From the foregoing investigation, I deduce the following important Corollary; that σ Elements of various kinds, and singular or plural in their several kinds, will form the same number of different Combinations, whether they combine u, or $\sigma - u$, at a time.

For if we compare the 1st, 2nd, 3rd, &c. terms in formula (XVI.) with the last, penultimate, and ante-penultimate, &c. terms respectively in formula (XVIII.); and if we make a like comparison, term by term, of the (XVII.) with the (XIX.) formula, we shall find that the first term and the last, the 2nd term and the penultimate, the 3rd and the ante-penultimate, and so on, are identical, in the series $\{0, \sigma\}$, $\{1, \sigma\}$, ..., $\{\sigma - 1, \sigma\}$, $\{\sigma, \sigma\}$, provided it can be shewn that the same law applies to the terms of each of the two component series,

$$\{0, \tau\}, \{1, \tau\}, \ldots\ldots\ldots\ldots \{\tau - 1, \tau\}, \{\tau, \tau\},$$

and $$\{0, \sigma - \tau\}, \{1, \sigma - \tau\}, \ldots\ldots \{\sigma - \tau - 1, \sigma - \tau\}, \{\sigma - \tau, \sigma - \tau\}.$$

But when the τ, and the $(\sigma - \tau)$ elements each consist of only one kind, the number of the Combinations that can be formed by taking 0, 1, 2, 3, &c. of these elements at a time, is invariably 1, 1, 1, 1, &c. and this series is identical, whether it be taken in direct, or in reverse order. Therefore the law will apply to the series formed by the elements of two single kinds conjoined; and therefore to the elements of three kinds conjoined; and therefore universally, of whatever number of different kinds the elements may consist.

Hence it appears that, to diminish the labour of computation in the application of formulas (XIV.) and (XV.) to particular cases, we ought always to make a selection of the least of the two numbers u and $\sigma - u$, before substituting one of them for the variable in either of these formulas.

The theorem just established may also be enunciated in the following terms:

If the product of any number of geometrically progressing Polynomes, each of which has a limited number of terms, and x for the common ratio of the terms, be developed according to the powers of x; then, assuming σ to be the sum of the dimensions of all the Polynome factors, the Coefficient of x^u, in the product, will be equal to the Coefficient of $x^{\sigma - u}$.

9. Hitherto, the Combinations I have been considering, have been subject only to the condition, that they all contained u of the given Elements. But we may impose the further one, that the

number of the kinds from which the u Elements are taken, shall be z; or this additional limitation, that

m of the z kinds shall each contain v elements;
m' .... v' ....;
m'' .... v'' ....;

and so on; and the Elements from which such Combinations are to be formed, may admit either of limited, or unlimited repetition.

10. If the given Elements are of s kinds, and may be repeated in each kind without limit, the Coefficient of x^u, in the product of the geometrically progressing Polynomes, will consist of terms in which there are u elements of one kind, ... of 2 kinds, ... of z kinds, ... z never exceeding u, and finally, when u becomes equal to, or greater than s, becoming equal to s. Consequently, the models or types, after which these several terms in the Coefficient of x^u are formed, will depend altogether on the partitions of the number u into 1, 2, 3, ... z parts. If $u < s$, the number of these terms will depend on the number of the partitions of u enumerated in the expression,

$$[u, 1_1] + [u, 2_1] + \ldots\ldots [u, u_1] = [2\,u, u_1].$$

When u becomes equal to s, the number of these partitions will be $[2\,s, s]$. When $u > s$, the number of the partitions will be

$$[u, 1_1] + [u + 2_1] + \ldots\ldots [u, s_1] = [u + s, s_1].$$

See Article 7, Section I., of the present Paper.

Thus, if the Elements are of 6 kinds, and they are to be combined together 7 at a time, there will be in all $[13, 6] = 14$ types, in accordance with which all the Combinations, containing 7 Elements each, will have to be constructed; and these types are the following partitions of the number 7.

Number of kinds.	Partitions of 7.	Corresponding Type.
1	7	A^7
2	6, 1	A^6B
	5, 2	A^5B^2
	4, 3	A^4B^3
3	5, 1, 1	A^5BC
	4, 2, 1	A^4B^2C
	3, 3, 1	A^3B^3C
	3, 2, 2	$A^3B^2C^2$
4	4, 1, 1, 1,	A^4BCD
	3, 2, 1, 1,	A^3B^2CD
	2, 2, 2, 1,	$A^2B^2C^2D$
5	3, 1, 1, 1, 1,	A^3BCDE
	2, 2, 1, 1, 1,	A^2B^2CDE
6	2, 1, 1, 1, 1, 1	A^2BCDEF

10.* Let one of the $z-$ partitions of u be

$$v,\ v,\ v, \ldots\ldots (m)\ v',\ v',\ v', \ldots\ldots (m')\ v'',\ v'',\ v'', \ldots\ldots (m''), \text{ \&c.},$$

so that $m + m' + m'' + \ldots\ldots\ldots\ldots = z$,

and $m\,v + m'\,v' + m''\,v'' + \ldots\ldots\ldots = u$.

Since the s kinds of Elements can be combined z at a time in $\frac{s^{z|-1}}{1^{z|1}}$ different ways, and since the different z parts of the above partition admit of being permuted, and in that way differently distributed among the z kinds of Elements, in $\frac{1^{z|1}}{1^{m|1} \cdot 1^{m'|1} \cdot 1^{m''|1} \ldots}$ different ways, the number of the different Combinations of the proposed form, in case of unlimited repetition, will be

$$\frac{s^{z|-1}}{1^{z|1}} \times \frac{1^{z|1}}{1^{m|1} \cdot 1^{m'|1} \cdot 1^{m''|1} \ldots} = \frac{s^{z|-1}}{1^{m|1} \cdot 1^{m'|1}\, 1^{m''|1} \ldots} \ldots\ldots\ldots\ldots \text{(XX)}.$$

and if corresponding to every different z – partition of u, we construct a similar expression, the sum of these will give the total number of the Combinations which can be formed from the s kinds of Elements, when in each Combination there are u Elements of z kinds.

11. In the case of unlimited repetition, the aggregate of all the terms, containing u Elements of z kinds, admits of Summation. For, if in each of the z – partitions of the number u, the parts be permuted one with another, the number of all these permutations will be

$$\frac{[u-1]^{z-1|-1}}{1^{z-1|1}} = \frac{[u-1]^{u-z|-1}}{1^{u-z|1}} \ldots\ldots\ldots\ldots\ldots \text{(XXI)},$$

equivalent terms in the developement of the Binomial $[1+1]^{u-1}$. This will appear from the following consideration. In the case of $\left.\begin{matrix} u - \\ \text{or } 1 - \end{matrix}\right\}$ partition, the parts can be permuted in one way. In the case of $\left.\begin{matrix} (u-1) - \\ \text{or } Bi - \end{matrix}\right\}$ partition, the parts can be permuted in $(u-1)$ ways. Integrate the factorial successively up to $\Sigma^{z-1}[1]$, or $\Sigma^{z-2}[u-1]$; and the formula (XXI.) will be the Integral.

Consequently, the number of the different Combinations, containing u Elements of z kinds, will be

$$\frac{s^{z|-1}}{1^{z|1}} \times \frac{(u-1)^{z-1|-1}}{1^{z-1|1}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(XXII)}.$$

Example. How many Combinations, containing eight Elements of three kinds each, can be formed from four kinds of Elements, unlimited in number. Answer $\frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} \times \frac{7 \cdot 6}{1 \cdot 2} = 84$.

Now the sum of all the terms of the form (XXII.), from $z = 1$ to $z = u$, ought to be equal to the Coefficient of x^u, or to $\frac{s^{u|1}}{1^{u|1}}$: and accordingly, if we give to the product (XXII.) the form $\{z, s\} \times \{z-1, u-1\}$, it will appear to be a particular case of the general theorem, Article 7 of the present Section, last demonstrated; so that $\underset{z}{S_0^u}\left[\{z, s\} \times \{z-1, u-1\}\right] = \{u, s+u-1\} = \frac{s^{u|1}}{1^{u|1}}$, the first term in the developement being 0.

12. Suppose the Elements in the s given kinds to be limited in point of number. Let it be required to form, from these elements, Combinations, each containing (u) elements of z kinds, with this further limitation, that

m of the z kinds shall contain v elements, each :
m' .... v' .... :
m'' .... v'' .... : and so on.

1st. If none of the given kinds contain as many elements as are denoted by any one of the numbers v, v', v'', ..., no such Combinations as are required, can be formed from the given elements.

2ndly. If any of the given kinds contain fewer elements than are denoted by the least of the numbers v, v', v'' ... , such kinds, it is manifest, may be wholly omitted from consideration.

3rdly. If any of the given kinds contain more elements than are denoted by the greatest of the numbers v, v', v'' ... , the excess above such greatest number may be wholly omitted from consideration; and, in the same manner, if any of the given kinds contain a number of elements intermediate between two of the numbers v, v', v'', ... , the excess above the least of these two numbers may, in the course of the operation hereinafter directed, be wholly omitted from consideration.

Thus the given set of elements admits of reduction to t kinds, containing at least v elements each:

+ T' kinds, none of which contain v elements, but each of which contains at least v' elements:

+ T'' kinds, none of which contain v' elements, but each of which contain at least v'' elements, &c.

Thus there will be t kinds to supply m kinds in each Combination with v elements each: $t - m + T' = t'$ kinds, to supply m' kinds in each Combination with v' elements, each: $t' - m' + T'' = t''$ kinds, to supply m'' kinds in each combination with v'' elements each: and so on.

Therefore since m kinds have been chosen out of t kinds;

m' .... out of t' ... ;

m'' .... out of t'' ... ; &c.

the number of the Combinations of kinds that will be formed, in which the several kinds will contain the requisite number of elements, will be

$$\frac{t^{m|-1}}{1^{m|1}} \times \frac{t'^{m'|-1}}{1^{m'|1}} \times \frac{t''^{m''|-1}}{1^{m''|1}} \times \&c. \ldots\ldots \text{(XXIII)}.$$

EXAMPLE.—From the elements F^6, E^5, D^4, C^3, B^2, A, how many Combinations of the form or type,

3, 2, 1, 1

can be constructed?

Since 3 is the highest number in the type, reduce the given elements from

	6,	5,	4,	3,	2,	1,
to	3,	3,	3,	3,	2,	1;
then to		2,	2,	2,	2,	1;
then to			1,	1,	1,	1.

Then since $m = 1$; $m' = 1$; $m'' = 2$;

$t = 4$; $t' = 4$; $t'' = 4$.

$$\therefore \frac{t^{m|-1}}{1^{m|1}} \times \frac{t'^{m'|-1}}{1^{m'|1}} \times \frac{t''^{m''|-1}}{1^{m''|1}} = \frac{4}{1} \times \frac{4}{1} \times \frac{4 \cdot 3}{1 \cdot 2} = 96.$$

13. When, however, after previous reduction, if requisite, the limited number of elements is the same in each of the given kinds, and it is required to determine how many Combinations can be formed from those elements in accordance with a given type, either all or none of the kinds will contain the number of elements requisite to form any required Combination: and the formula applicable to the case of unlimited repetition, viz. (xx.)

$$\frac{s^{z|-1}}{1^{m|1} \cdot 1^{m'|1} \cdot 1^{m''|1}} \ldots\ldots,$$

is to be applied.

EXAMPLE.—Given the elements A^5, B^5, C^5, D^5.

and the type 4, 2, 2.

How many Combinations can be formed in accordance with that type?

Here $s = 4;\quad z = 3;\quad m = 1;\quad m' = 2.$

And the number of Combinations $= \dfrac{4 \,.\, 3 \,.\, 2}{1^1|^1 \,.\, 1^2|^1} = 12.$

Given the same elements.—How many Combinations can be formed in accordance with the type 6, 1, 1?

Since 6 is greater than the given limit 5, the Answer is 0.

14. Before closing this Section on Combinations, I shall beg to notice that all of the theorems it contains, admit of an important application, and that is, to the properties of Composite Numbers.

It is known, for instance, that, if the elements A, B, C, &c. represent primes, a composite number, of the form $A^{\alpha}.\, B^{\beta}.\, C^{\gamma}.$ &c., will have the total number of its divisors represented by $(\alpha + 1)(\beta + 1)(\gamma + 1)$ &c.; but if the question be, how many divisors such a number has that are of u dimensions, the answer to that question will be obtained by means of formula (XIV). But it will suffice to have hinted at these analogies.

SECTION III.

On Permutations.

1. When there are s different kinds, each containing only a single element, these elements, taken u at a time, will form $s^u|^{-1}$ different Permutations; where $s^u|^{-1} = 1^u|^1 \times$ the coefficient of x^u in $[1 + x]^s$ developed. But when any of the s kinds contain more than one element, and the plurality of the elements is short of infinity, it is only in the particular case where u is equal to the united number of all the elements belonging to the s kinds, that the number of the permutations has hitherto been determined. In this case, if there be α elements of one kind, β of a second kind, γ of a third kind, &c., and $\alpha + \beta + \gamma +$ &c. $= \sigma$, the number of the permutations formed by the σ things taken all at a time, is $\dfrac{1^{\sigma}|^1}{1^{\alpha}|^1 \,.\, 1^{\beta}|^1 \,.\, 1^{\gamma}|^1 \,.\, \&c.}$, according to the well-known theorem.

2. The latter formula denotes the number of permutations which the α elements of the kind A, the β elements of the kind B, the γ elements of the kind C, &c. are capable of forming, when, instead of being permuted indiscriminately, the A's, the B's, the C's, &c. change their order of sequence in respect of one another, but in respect of the elements of their own several kinds, preserve an immutable order of sequence. If the α elements A, not to the full extent of $1^{\alpha}|^1$, but to some limited extent, undergo the permutations $P(\alpha)$; and in like manner the β elements B to the limited extent $P(\beta)$; and the γ elements C to the limited extent $P(\gamma)$, &c. the number of the permutations which the σ things will then together form, will be

$$\frac{1^{\sigma}|^1}{1^{\alpha}|^1 . 1^{\beta}|^1 . 1^{\gamma}|^1,\ \&c.} \times P(\alpha),\ P(\beta),\ P(\gamma),\ \&c. \ldots\ldots\ldots\ldots\ldots\ldots (\text{XXIV}).$$

3. To determine generally the number of permutations which can be formed from any given set of elements, taken u at a time.

Let any partition of u be $p + q + r +$ &c. $= u$.

It has been shewn, in Articles 12 and 13 of the preceding Section, how to determine the number of all the Combinations which can be formed from a given set of elements, when each Combination is to consist of u elements of z kinds, and is to accord with any particular partition of u, or type. If that partition be p, q, r, then, in the number of combinations so determined, are included, not only those of the form $A^p B^q C^r$, $D^p E^q F^r$, &c. (in which the *kinds* of elements are changed), but also those of the form $A^r B^p C^q$, $D^q E^r F^p$, &c. (in which the kinds remaining the same, the *order of sequence* in the numbers p, q, r is altered). Let the total number of the combinations corresponding to one such partition of u be denoted by Q. Then since every such combination will give rise to $\frac{1^{p+q+r|1}}{1^{p|1}.1^{q|1}.1^{r|1}.} = \frac{1^{u|1}}{1^{p|1}.1^{q|1}.1^{r|1}}$ different permutations, if we denote $\frac{1^{u|1}}{1^{p|1}.1^{q|1}.1^{r|1}}$ by P, every different partition of u, or type, will give rise to $Q \times P$ permutations. We must therefore determine by Articles 13 and 14, the number of the combinations corresponding to all the different partitions of u, and also the corresponding permutation factors, and take the product; and the sum of all these particular products, or $S[Q \times P]$, will give the total of the permutations which can be formed from the elements taken u at a time.

1st EXAMPLE. Given the set of Elements A^4, B^3, C^2. Required the number of all the Combinations and Permutations of those Elements, when 7 are taken at a time.

Here $u = 7$; $s = 3$; and, since all parts are to be excluded which exceed 4, z in this case varies only from 2 to 3.

Given Elements.	4, 3, 2	m	t	m'	t'	m''	t''	$\frac{t^{m\|-1}}{1^{m\|1}}$	$\frac{t'^{m'\|-1}}{1^{m'\|1}}$	$\frac{''t^{m''\|-1}}{1^{m''\|1}}$	Combination factor.	Permutation factor.	No. of Permutations.
Partitions of 7	4, 3,	1	1	1	1			1	1		1	$\frac{5.6.7}{1.2.3} = 35$	35
	4, 2, 1	1	1	1	2	1	1	1	2	1	2	$\frac{5.6.7}{1.2} = 105$	210
	3, 3, 1	2	2	1	1			$\frac{2.1}{1.2}$	1		1	$\frac{4.5.6.7}{1.2.3} = 140$	140
	3, 2, 2	1	2	2	2			2	$\frac{2.1}{1.2}$		2	$\frac{4.5.6.7}{1.2.1.2} = 210$	420
											6		805

2nd EXAMPLE. Given the set of Elements A^5, B^5, C^5; required the number of all the Combinations and Permutations of these Elements, when 8 are taken at a time.

Here $u = 8$; $s = 3$; and, since all parts are to be excluded which exceed 5, z in this case varies only from 2 to 3. The combinations are here obtained by the formula

$$\frac{s^{z|-1}}{1^{m|1}.1^{m'|1}.1^{m''|1}}.$$

Given Elements.	z	5, 5, 5	m	m'	m''	Combinations.	Permutation Factor.	No. of Permutations.
Partitions of 8	2	5, 3,	1	1		$\frac{3.2}{1.1} = 6$	$\frac{6.7.8}{1.2.3} = 56$	336
		4, 4,	2			$\frac{3.2}{1.2} = 3$	$\frac{5.6.7.8}{1.2.3.4} = 70$	210
	3	5, 2, 1	1	1	1	$\frac{3.2.1}{1.1.1} = 6$	$\frac{6.7.8}{1.2.1} = 168$	1008
		4, 3, 1	1	1	1	$\frac{3.2.1}{1.1.1} = 6$	$\frac{5.6.7.8}{1.2.3.1} = 280$	1680
		4, 2, 2	1	1	1	$\frac{3.2.1}{1.1.2} = 3$	$\frac{5.6.7.8}{1.2.1.2} = 420$	1260
		3, 3, 2	2	1		$\frac{3.2.1}{1.2.1} = 3$	$\frac{4.5.6.7.8}{1.2.3.1.2} = 560$	1680
						27		6174

4. The method I have just described, of determining in succession the permutations corresponding to the different partitions of the number u, must, in cases of limited repetition, have been adopted to determine also the number of the *combinations*, when u elements are taken at a

time, had not the formulæ XIV and XV, or the method described in Article (6) of the former Section, afforded a readier method of attaining the same object. In the case of Permutations, I have not succeeded, except in certain cases, in readily determining by means of an explicit function of u, the number of the permutations formed by u elements of s kinds.

5. It is a known formula, that when the elements in all the s kinds admit of unlimited repetition, the number of the permutations which can be formed by taking u elements at a time, is expressed by s^u.

If we take the form I have assumed, in Articles (10) and (12) of the former Section, for the resolution of u into z parts, where these parts are represented by

$$v, v, v, \ldots (m)\ v', v', v', \ldots (m')\ v'', v'', v'', \ldots (m''),$$

$$\text{and } m + m' + m'' + \ldots\ldots\ldots\ldots = z,$$

$$\text{and } mv + m'v' + m''v'' + \ldots\ldots = u;$$

we shall, in the case of unlimited repetition, have the combination factor,

$$Q, = \frac{1^z|^1}{1^m|^1 . 1^{m'}|^1 . 1^{m''}|^1};$$

and the permutation factor,

$$P, = \frac{1^u|^1}{[1^v|^1]^m [1^{v'}|^1]^{m'} [1^{v''}|^1]^{m''}} \ldots\ldots\ldots\ldots\ldots (\text{XXV}).$$

the partitions being those in which the lower limit of the parts is 1; and z extending from 1 to u, when $u < s$; and from 1 to s; when $u =$ or $> s$.

But, by Art. 7, Sect. I, p. 5, $[u, 1_1] + [u, 2_1] + \ldots\ldots [u, z_1] = [u, z_0]$;

and the product $Q \times P$ therefore coincides with the expression given by Lagrange, in his demonstration of the Polynome theorem, for the terminus generalis of the expansion of

$$\epsilon^{[1+1+1 \ldots (s)]x}, \quad \text{or } \epsilon^x \times \epsilon^x \times \epsilon^x \ldots\ldots\ldots\ldots\ldots (s),$$

when multiplied by the factor $1^u|^1$. The terminus generalis, so multiplied, is

$$[1 + 1 + 1 + \ldots (s)]^u = 1^u|^1 S \left(\frac{1^p . 1^q . 1^r . \ldots}{1^p|^1 . 1^q|^1 . 1^r|^1 . \ldots}\right),$$

where p, q, r, &c., are all the different parts obtained by s – partitioning u, the lower limit of the parts being 0; and for every determinate set of values assigned to p, q, r, &c., these letters receiving every different order of sequence possible.

In the case, therefore, of unlimited repetition, the number of permutations which can be formed by taking u elements of s kinds at a time, is the coefficient of x^u in the product of the s infinite series,

$$\left[1 + x + \frac{x^2}{1 . 2} + \&\text{c.}\right] \left[1 + x + \frac{x^2}{1 . 2} + \&\text{c.}\right] \ldots\ldots\ldots (s); \text{ multiplied by } 1^u|^1.$$

6. It is manifest, therefore, that if with respect to the elements of any one kind, A, we restrict the number of elements to a; and in another kind, B, to β; and so on, we must make a corresponding restriction in the terms of 1, 2, or more, of the above Polynomes. And this leads to the following theorem: viz. that the number of the permutations which can be formed by the elements of s kinds, whose respective limits are a, β, γ, &c., when those elements are taken u at a time, is the coefficient of x^u in the product of the s Polynomes,

$$\left(1 + x + \frac{x}{1 . 2} \cdots \frac{x^a}{1^a|^1}\right) \left(1 + x + \frac{x}{1 . 2} + \cdots \frac{x^\beta}{1^\beta|^1}\right), \&\text{c., multiplied by } 1^u|^1 \ldots\ldots\ldots\ldots (\text{XXVI}).$$

This theorem, I fear, is not likely to facilitate much the practical computation of the permutations of plural elements, though perhaps it may lead to curious Algebraic results. Professor De Morgan, since I made known to him this Theorem, has done much to remove the difficulties which beset the computing of permutations by means of it. But I doubt whether the method will be rendered more simple than that derived from a direct consideration of the problem of permutations, given in Article 2 of the present Section. Thoroughly examined, the two methods must in the end prove identical.

I had some expectation that by giving to the Polynomes the form

$$\begin{aligned}(\epsilon^x - a)\,(\epsilon^x - b)\,(\epsilon^x - c),\ \&c. = \epsilon^{sx} &- a\,\epsilon^{sx-1} + a\,b\,\epsilon^{sx-2} - \&c.\\ &- b \qquad\; + a\,c\\ &- c \qquad\; + b\,c,\end{aligned}$$

some facilities might be afforded to the computing of permutations in certain cases; but I do not at present believe that any such results are to be anticipated.

7. The theorem, XXVI, has led me to the determination, in one particular case, of an explicit function of u, for expressing the number of the permutations formed by s kinds of elements taken u at a time: the case is that where, in all the kinds, the elements are dual. If we develope

$$[A_0 + A_1 x + A_2 x^2 + \&c.]^s,$$

by Arbogast's method, we obtain for the coefficient of x^u,

$$\underset{c}{D}^u\,[A_0^s] = \underset{\lambda}{S}{}_0^{u-1}\left(\frac{s^{u-\lambda}|^{-1}}{1^{u-\lambda}|^1}\,A_0^{s+\lambda-u}\,\underset{c}{D}^\lambda\,(A_1)^{u-\lambda}\right),$$

where $\underset{c}{D}^\lambda\,(A_1)^{u-\lambda}$ is the coefficient of x^λ in

$$[A_1 + A_2 x + A_3 x^2 + \&c.]^{u-\lambda};$$

and a second developement leads to a double series, in which, if A_0 and A_1 are made equal to 1, and A_2 to $\frac{1}{2}$, and all the other terms A_3, A_4, &c., are made equal to 0, we obtain terms expressing the coefficient of x^u in $\left(1 + x + \dfrac{x^2}{2}\right)^s$; and that series multiplied by $1^u|^1$ gives the number of the permutations in the case stated*.

---

* I here transcribe the coefficient of x^u, in $[1 + x + A_2 x^2 + A_3 x^3 + \&c.]^s$ obtained by Arbogast's method, slightly modified.

0. $\dfrac{1}{1^u|^1}\,s^u|^{-1}.$

1. $\dfrac{1}{1^{u-2}|^1}\,s^{u-1}|^{-1}\,A_2.$

2. $\dfrac{1}{1^{u-3}|^1}\,s^{u-2}|^{-1}\,A_3.$

3. $\dfrac{1}{1^{u-4}|^1}\left[s^{u-3}|^{-1}\,A_4 + s^{u-2}|^{-1}\,\dfrac{(A_2)^2}{1^2|^1}\right].$

4. $\dfrac{1}{1^{u-5}|^1}\,[s^{u-6}|^{-1}A_5 + s^{u-3}|^{-1}A_2\,A_3].$

5. $\dfrac{1}{1^{u-6}|^1}\left[s^{u-5}|^{-1}A_6 + s^{u-4}|^{-1}\left[A_2 A_4 + \dfrac{A_3{}^2}{1^2|^1}\right] + s^{u-3}|^{-1}\,\dfrac{A_2{}^3}{1^3|^1}\right].$

6. $\dfrac{1}{1^{u-7}|^1}$ [&c.] (the law of the terms is obvious).

The series may be expressed by

$$\underset{\theta}{S}\left(\frac{u^{2\theta|-1} \, . \, s^{u-\theta|-1}}{2^{\theta|2}}\right), \; \ldots\ldots\ldots\ldots\ldots \text{(XXVII.)}$$

the limits of θ, when u is even, being 0 and $\frac{u}{2}$;

.... odd, ... 0 and $\frac{u-1}{2}$.

This gives the number of permutations formed by s dual sets of elements, when taken u at a time.

Thus when $u = 6$, we have

$$s^{6|-1} + \frac{6^{2|-1} s^{5|-1}}{2} + \frac{6^{4|-1} \, . \, s^{4|-1}}{2 \, . \, 4} + \frac{6^{6|-1} s^{3|-1}}{2 \, . \, 4 \, . \, 6}.$$

And when $u = 7$,

$$s^{7|-1} + \frac{7^{2|-1} s^{6|-1}}{2} + \frac{7^{4|-1} \, . \, s^{5|-1}}{2 \, . \, 4} + \frac{7^{6|-1} \, . \, s^{4|-1}}{2 \, . \, 4 \, . \, 6}.$$

When $u = 6$, let $s = 3$.

$$\text{The number of the permutations} = \frac{6 \, . \, 5 \, . \, 4 \, . \, 3 \, . \, 2 \, . \, 1 \times 3 \, . \, 2 \, . \, 1}{2 \, . \, 4 \, . \, 6}$$

$$= 5 \times 3 \times 3 \times 2$$

$$= 90.$$

When $u = 7$, let $s = 4$.

$$\text{The number of permutations is then} = \frac{7 \, . \, 6 \, . \, 5 \, . \, 4 \, . \, 3 \, . \, 2 \, . \, 4 \, . \, 3 \, . \, 2 \, . \, 1}{2 \, . \, 4 \, . \, 6} = 7 \, . \, 6 \, . \, 5 \, . \, 4 \, . \, 3 = 2520.$$

It is not improbable that a further developement of the series obtained by Arbogast's method, and a subsequent equating of particular terms to 0, might lead to other symmetrical and curious results.

8. In Art. 7 of Section II., I have given a method, from the known combinations of two independent and separate sets of elements, differing from one another in kind, to determine the combinations of the two sets, when united. I proceed to apply a precisely similar method to Permutations.

If v of τ elements form $\{v, \tau\}$ combinations, and $(u - v)$ of $(\sigma - \tau)$ elements form $\{u - v, \sigma - \tau\}$ combinations, they will together form

$$\{v, \tau\} \times \{u - v, \sigma - \tau\} \text{ combinations.}$$

Take some one of the $\{v, \tau\}$ and some one of the $\{u - v, \sigma - \tau\}$ combinations. If it be a condition that the elements belonging to these two sets, separately considered, shall preserve their original order of sequence, but that those of one set of elements, as compared with those of the other, may change their places, the permutations formed by the two sets of this particular combination united, will be

$$\frac{1^{u|1}}{1^{v|1} \, . \, 1^{u-v|1}};$$

But, if the v elements, and the $u - v$ elements, considered separately, may change their places, and the former may undergo $P(v)$ changes, and the latter $P(u - v)$ changes, and if the same may be predicated of the elements contained in each different pair of combinations, $\{v, \tau\}$ and $\{u - v, \sigma - \tau\}$, then the number of the permutations, corresponding to every pair of determinate values given to v and $u-v$, will be

$$*P\{u,\ \sigma\} = \{v,\ \tau\}\ \{u-v,\ \sigma-\tau\}\ P(v)\,P(u-v),$$
$$\times\ \frac{1^{u}|^{1}}{1^{v}|^{1}\,.\,1^{u-v}|^{1}}.$$

And for the line u of the table, u remaining constant, and v only varying, we have

$$P\{u,\ \sigma\} = \underset{v}{S^{\tau}_{0}}\left[P\{v,\ \tau\}\,P\{u-v,\ \sigma-\tau\}\,\frac{1^{u}|^{1}}{1^{v}|^{1}\,.\,1^{u-v}|^{1}}\right].$$

And the equation of the terms of the table of double entry, in which u is the index of the line, and v of the column, is

$$[u,\ v] = [v,\ v] \times [u-v,\ 0]\,\frac{[u+1-v]^{v}|^{1}}{1^{v}|^{1}} \ldots\ldots\ldots\ldots (\text{XXVIII.}),$$

I give an example of such a table, applied to the case of the two sets of elements,

A, *BB*, *CCC*; and *DDDD*, *EEEEE*.

Permutations of A, B^2, C^3; whose Combinations are 1, 3, 5, 6, 5, 3, 1
and of D^4, E^5 ; .... 1, 2, 3, 4, 5, 5, 4, 3, 2, 1.

v =	0	1	2	3	4	5	6	
$P\{v,\tau\}$ =	1	3	8	19	38	60	60	
u =	$P\{u-v,\ \sigma-\tau\}$ =							$P\{u,\ \sigma\}$ =
0	1							1
1	2	3						5
2	4	12	8					24
3	8	36	48	19				111
4	16	96	192	152	38			494
5	31	240	640	760	380	60		2111
6	56	558	1920	3040	2280	720	60	8634
7	91	1176	5208	10,640	10,440	5040	840	33,435
8	126	2184	12,544	32,984	35,560	26,880	6720	116,998
9	126	3402	26,208	89,376	148,428	107,520	40,320	415,380
10		3780	45,360	207,480	446,880	468,720	201,600	1,373,820
11			55,440	395,010	1,141,140	1,552,320	859,320	4,003,230
12				526,680	2,370,060	4,324,320	3,104,640	10,325,700
13					3,423,420	9,729,720	9,369,360	22,522,500
14						15,135,120	22,702,680	37,837.800†
15							37,837,800	37,837,800†

* $P\{u,\ \sigma\}$ means the permutations of the elements contained in the combination $\{u,\ \sigma\}$.

† The truth of Article 9, Section 3, is here exemplified.

9. The following corollary results from the preceding Article.

The formula $\dfrac{1^{\sigma|1}}{1^{\alpha|1}\,1^{\beta|1}\,1^{\gamma|1}\,\&c.}$ expresses the number of the permutations, not only of the $\alpha+\beta+\gamma+\&c.$ elements, when they are all permuted together, but also of $[\alpha+\beta+\gamma+\&c.]-1$ elements, when permuted together: of which the following is the demonstration.

Since, by XXVIII,
$$P\{\sigma-1,\ \sigma\} = P\{\tau-1,\ \tau\}\,P\{\sigma-\tau,\ \sigma-\tau\}\,\frac{1^{\sigma-1|1}}{1^{\tau-1|1}\,1^{\sigma-\tau|1}},$$
$$+\,P[\tau,\ \tau]\,P\{\sigma-\tau-1,\ \sigma-\tau\}\,\frac{1^{\sigma-1|1}}{1^{\tau|1}\,1^{\sigma-\tau-1|1}};$$

assume, for a moment, that
$$P\{\tau,\ \tau\} = P\{\tau-1,\ \tau\};\ \text{and let each} = J,$$
and that $P\{\sigma-\tau,\ \sigma-\tau\} = P\{\sigma-\tau-1,\ \sigma-\tau\}$; and let each $= K$.

Then,
$$P\{\sigma-1,\ \sigma\} = J\,.\,K\,.\,1^{\sigma-1|1}\,\frac{\tau+\sigma-\tau}{1^{\tau|1}\,.\,1^{\sigma-\tau|1}}$$
$$= J\,.\,K\,\frac{1^{\sigma|1}}{1^{\tau|1}\,.\,1^{\sigma-\tau|1}},$$

But,
$$P\{\sigma,\ \sigma\} = P\{\tau,\ \tau\}\,P\{\sigma-\tau,\ \sigma-\tau\}\,\frac{1^{\sigma|1}}{1^{\tau|1}\,.\,1^{\sigma-\tau|1}}$$
$$= J\,.\,K\,.\,\frac{1^{\sigma|1}}{1^{\tau|1}\,.\,1^{\sigma-\tau|1}}.$$

Hence the law enunciated will be true of the two sets of elements conjoined, if it be ever true of each of the two sets separately. But it is true of two separate sets, when each consists of elements of only one kind; for then, whatever may be the number of the elements permuted at a time, the number of the permutations is constantly one. Consequently, the law holds true when there are two kinds of elements conjoined; consequently, when there are three kinds; and therefore universally.

Hence, in the product of the Polynomes,
$$\left[1+x+\frac{x^2}{1\,.\,2}+\ldots\frac{x^\alpha}{1^{\alpha|1}}\right]\left[1+x+\ldots\frac{x^\beta}{1^{\beta|1}}\right]\times\&c.$$
the penultimate coefficient $=\sigma\times$ ultimate coefficient.

HENRY WARBURTON.

May, 1847.

ADDENDUM.

SINCE this Paper was corrected for publication, a member of the Society, distinguished for his mathematical erudition, has caused the Author's attention to be drawn to the work of Bézout* on Elimination, as containing a formula similar in structure to the Author's formula XIV.

In the Author's researches in Combinations, his concern has been exclusively with such of the terms of a polynome function of the s quantities, A, B, C, &c., as were of some one, say, the u^{th}, dimension. By such modes of investigation as occurred to him, he obtained an expression representing the number of such terms.

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* *Théorie Générale des Equations Algébriques,* par M. Bézout, 4to, 471 pp. Paris, 1779.

with the special object of applying it to denote the number of the combinations which can be formed with plural elements.

Bézout's object, at least the sole use to which he applies his formulæ throughout his work, is elimination. His concern is with all the terms, in the aggregate, of how many dimensions soever, belonging to such a polynome function as is above described. By a mode of investigation, entirely different from that of the Author, he obtains a formula expressing the number of the terms which, in a polynome, complete in all its terms of every dimension from o to u, are not divisible by any of the factors, A^{α}, B^{β}, C^{λ}, &c.* He finds that, in the complete polynome, the number of all the terms is represented by $\frac{[u+1]^{s}|^{1}}{1^{s}|^{1}}$; and the number of the terms not divisible by any of the said factors, by

$$\frac{1}{1^{s}|^{1}}\left(\begin{array}{l}[u+1]^{s}|^{1}-[u+1-\alpha]^{s}|^{1}+[u+1-\alpha-\beta]^{s}|^{1}-\&c.\\ \qquad\qquad -[u+1-\beta]^{s}|^{1}+\&c.\\ \qquad\qquad -\&c.\end{array}\right)\dots\dots\dots\dots (1.)$$

and this is the formula to which the Author's attention has been directed. It agrees in its general structure with the Author's formula XIV: the points in which it differs will presently come under notice.

In his 4th problem, Bézout considers a particular case of an incomplete polynome, meaning thereby a polynome in which the highest dimension of one of the s quantities, A, is α, of another B, is β, and so on; α, β, &c., belng less than u, the highest dimension of the polynome itself: and he here makes the observation, that there are as many terms in such a polynome as there would be of terms not divisible by any of the factors $A^{\alpha+1}$, $B^{\beta+1}$, &c., in the polynome, supposing it to be complete; but he gives no formula coextensive with the generality of that observation. By following out that observation, we may, by two steps, deduce the Author's formula XIV. from that of Bézout.

The first step is the following. The terms which in the polynome, if complete, would be non-divisible by any of the factors $A^{\alpha+1}$, $B^{\beta+1}$, &c., amount in point of number to

$$\frac{1}{1^{s}|^{1}}\left(\begin{array}{l}[u-1]^{s}|^{1}-[u+1-(\alpha+1)]^{s}|^{1}+[u+1-(\alpha+1)-(\beta+1)]^{s}|^{1}-\&c.\\ \qquad\qquad -[u+1-(\beta+1)]^{s}|^{1}\ \&c.\\ \qquad\qquad -\&c.\end{array}\right)\dots\dots (2.)$$

and such, therefore, is the number of the terms in the incomplete polynome function of s quantities, where α, β, γ, &c., are the limits of the dimensions of A, B, C, &c., respectively; the highest dimension of the polynome itself being u.

The second step is the following. If from a polynome whose highest dimension is u, all the terms of the dimensions not exceeding $(u-1)$ be deducted, the remainder will be the terms which the polynome contains of the u^{th} dimension. Hence the number of the terms of the u^{th} dimension in the incomplete polynome will be obtained, if in (2) we substitute u for $(u+1)$, and deduct the result from (2). That is to say, the required number of terms will be $\Delta\,(2)$, meaning, by $\Delta\,(2)$, $[1-E^{-1}]\,(2)$; i.e.,

$$\frac{1}{1^{s-1}|^{1}}\left(\begin{array}{l}[u+1]^{s-1}|^{1}-[u+1-(\alpha+1)]^{s-1}|^{1}+[u+1-(\alpha+1)-(\beta+1)]^{s-1}|^{1}-\&c.\\ \qquad\qquad -[u+1-(\beta+1)]^{s-1}|^{1}+\&c.\\ \qquad\qquad -\&c.\end{array}\right)\dots (3.)$$

which agrees with the Author's formula XIV.

Considering that Bézout's work has now been published nearly seventy years, it will no doubt excite the surprise of many members of the Society, that a deduction from Bézout's formula so easy as the foregoing, should not have been made long ago, and applied to the solution of the problem of the combinations of plural elements.

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* The complexity of Bézout's notation rendered it inexpedient to retain it in its original form. To facilitate comparison, the letters have been assimilated to those used by the Author.

XXXVI. *On a Peculiar Defect of Vision.* *By* Henry Goode, M.B., *of Pembroke College.*

[Read *November* 9, 1846, and *May* 17, 1847.]

The following details of a case of defective vision may not be uninteresting.

About ten years ago I first perceived a defect of vision in the right eye, the extent of which, before that period, I believe to have been inconsiderable: the defect being that small objects, when viewed at the distance of greatest distinctness, appear as two. My attention having been called to Professor Airy's Paper on his own eye, I find that my eye, tested in the manner he proposes, exhibits a similar defect. This method is to view with the defective eye a pinhole in a card, which slides along a graduated scale, one extremity of the scale being applied to the cheek-bone, and the other directed towards an illuminated sheet of paper.

The following are the appearances observed:

1. When the card is quite close to the eye, the image of the pinhole is perfectly circular.

2. As the card is removed to a greater distance, the image becomes gradually elongated in the form of an ellipse, with a sharp dark line in the long diameter, most distinct at the distance of 4.5 inches, and best visible in a minute hole.

3. At 6.13 inches the image has become extended into a bright well-defined line, of the breadth of the pinhole as estimated by the sound eye, and crossed in the centre by a dark line perpendicular to the former dark line which has disappeared: if several pinholes be pricked near one another, the dark band holds the same relative position in all of them.

4. As the card is removed to a further distance, the bright line becomes gradually shortened, and at the distance of more than a foot appears as two bright spots only, situated one on each side of the dark band; but, at the same time, in the direction of, and as it were overlying the dark band, a bright line gradually appears, short at first, and becoming elongated with the removal of the card, so that at about 10 inches or more the appearance is that of a cross, most strongly illuminated in the position of the two bright spots before described. At 12.2 inches this cross appears as a regular quadrangular figure with concave sides, the two spots being most strongly illuminated. If a dark spot on a sheet of white paper be viewed in the same manner, the appearance is necessarily the same; but owing to the greater distinctness of the two spots, the remainder of the figure is easily overlooked, and the appearance is that of a double spot; consequently, if a page of small type be viewed at this distance, the print appears double.

5. When the card is at 25 inches, and all greater distances, the image is a bright line perpendicular to that seen at 6.13 inches, the two spots representing that line having almost coalesced into one, causing the bright line to be brightest in the centre.

Distant luminous objects with clear defined outlines, such as the Moon, appear as a succession of well-defined images overlying one another with their centres in this line.

6. The more distant line is inclined to the mesial plane of the body at an angle of 21°, and the upper part falls inwards towards this plane.

It appears in the above, that a short distance within the nearer focus a dark line occupies the position of the bright line seen at that focus, while beyond the focus at all distances the line continues illuminated. The same holds with regard to the second focus.

7. On viewing two dark or two bright lines drawn in the form of a cross, and held in the position of the lines above indicated, the vertical line appears broad and very faint at the distance of 6.13 inches, the horizontal line appearing clear and well defined; while the reverse is the case beyond 25 inches.

There is no apparent defect in the left eye. If several holes be pricked near one another, and viewed by this eye at the distance of 5 inches, which is somewhat within the range of distinct vision, a dark central spot is observable in the centre of each: also a narrow luminous slit appears traversed in the direction of its length by two central parallel dark lines.

It is probable that the defect of the eye is inherited, as my mother has a defect of a similar nature in both eyes. A circular pinhole viewed with either eye at the distance of 7.5 inches appears as an ellipse with the major axis parallel to the mesial plane of the body, while at the distance of 5 inches the image is an ellipse with the major axis perpendicular to the former.

SINCE the period when the above measurements were taken, I have made frequent use of the eye; owing, most probably, to this circumstance, a very considerable amelioration has taken place in it; the first focus, which in the month of June last was at 6.13 inches, in the month of December was at about 10: the second focus was readily ascertained in the month of June to be at between 24 and 25 inches; but in December it was impossible to determine the exact position of it by the simple observation of a pinhole; because, instead of appearing as a sharp distinct line, as before, the image was always confused by the presence of the luminous square above described. The image was, in fact, a rhomb, with the longer diagonal, distinguished by its brightness, in the direction of the further line, while the line seen at the nearer focus never disappeared, but became shortened, remaining always the brightest part of the image, and forming the shorter axis of the rhomb. However, by means of the instrument described below, the second focus was ascertained to be at a distance of between 27 and 28 inches. Since December, up to the month of March of this year, no change whatever has taken place in the eye, notwithstanding the constant use of it.

The length of the line, as observed at either focus, is, of course, dependent on the aperture of the pupil, and the distance from the retina, before or behind it, of the line of convergence of the rays refracted from the other focus.

The differences in the eye observed in June and December, are exactly such as occur, when similar observations are made on a sound eye, to which is applied, in one case, a cylindrical convex lens of short focal length, and in the other a lens of weaker power.

The instrument above alluded to as serving to determine the distance of the foci is simply that of Scheiner. Let a tube which slides within another in the manner of a telescope be closed at its extremity by a card pierced by a single pinhole, while the other extremity of the apparatus is closed by a card pierced by two holes, the distance of which from one another is less than the diameter of the pupil of the eye of the observer. When the extremity pierced by the single hole is presented towards a luminous surface, and the other is applied close to a sound eye, if the distance of the single hole is equal to the most convenient distance of distinct vision, free from any exertion, the hole will appear single; but if the distance be greater or less than this, the hole will appear as two; as is well known. This instrument may be applied to the determination of the two foci of a defective eye, by observing that, in order to ascertain the distance of either focus, the line passing through the two pinholes must be perpendicular to the direction of the line, which forms the image of the point at that focus.

There are, however, two inconveniences attending the use of this instrument; namely, firstly, that if the eye, on which the observation is made, be at all long-sighted, so that the pinhole requires to be placed at a considerable distance, the two pencils of light falling on the two pinholes are nearly parallel, and therefore the pinhole may be moved through a considerable space backwards and forwards, without much affecting the position of the resulting images. The other inconvenience is, that the eye is naturally endowed with a power of adapting itself to different distances, and that this power is very little under command in an eye which is not habitually used, a circumstance, perhaps, frequent with those who have one eye defective: such an eye, when tested by any instrument, will at one instant appear to have one focal length, and at another instant another. When a cross drawn on paper is held at a distance between the two foci, I find that I can at will discern either the horizontal line, or the perpendicular, without altering the position of the paper.

There are, therefore, no means of attaining the requisite measurements beyond an approximation, and the rest must be ascertained by direct experiment with a series of cylindrical lenses.

Having calculated an approximation of the glass I required, I applied to M. Chamblant, a working optician at Paris, who occupies himself solely with the construction of lenses and spectacle glasses with cylindrical surfaces, and after several trials I succeeded in obtaining a glass, which gives me distinct vision of objects both far and near alike, thus shewing that the error of malformation is independent of the state of adaptation of the eye. The glass I use is plano-cylindrical, the cylindrical surface concave, with a radius of curvature of nine French inches. The axis of the cylinder when presented to the eye, coincides of course with the direction of the line at the nearer focus.

A plano-convex glass also, with the axis perpendicular to the direction of the line at the first focus, and the curvature of which is the same, gives distinct vision, provided that the object is placed sufficiently near to the eye; or even a glass much stronger, when the object is very close to the eye.

Considering that the inclination of the lines at the foci might have a physiological importance, I devised the following method of determining it accurately. If a number of pinholes be pricked in a card, in a straight line, and the card be fixed in such a manner that it may be made to revolve, and have an illuminated surface behind it, when a defective eye is placed at the proper distance, it readily recognizes the position of the card in which all the lines representing the images of the pinholes lie in one straight line, being the line in which the holes are pricked: care must be taken that the body is held perpendicularly. It is easy now to determine the inclination of this line to a hair stretched vertically by a weight.

Within the last few months I have met with three or four cases of defective vision similar to my own; only two of which are of sufficient magnitude to be worthy of mention.

One is that of Mr. Parry, who has served many years as a medical officer in the army. This gentleman's left eye is perfect, except in being somewhat presbyopic, but from the time of his earliest recollection he has never had distinct vision with his right eye; he has never been able to read with it, though he has an indistinct vision of objects at all distances.

His eye, tested by a pinhole in a card, perceived the hole as a horizontal line at the distance of 37 centimetres (about $14\frac{1}{2}$ inches); the line is inclined at an angle of 87 degrees to the mesial plane of the body, and meets this plane produced inwards and upwards. At some distance beyond this the hole appeared enlarged, and of a rhomboidal figure, but never as a line.

When he viewed two lines drawn in the form of a cross, he saw well enough the horizontal line at $14\frac{1}{2}$ inches, and for some distance beyond, but at no distance could he discern the vertical line. The error therefore seemed to consist in an exceedingly feeble refractive power in horizontal planes: I therefore tested his eye with plano-cylindrical convex glasses, in order to obtain data for calculating the forms of glasses to be used for viewing objects at different distances; and we found that with a glass of $2\frac{1}{2}$ French inches radius, the two lines of the cross, at 12 or 14 inches distance, appeared of nearly equal brightness. This glass was rather too strong, while 3 inches gave a glass rather too weak. To view distant objects, therefore, I caused to be made a glass cylindrical concave on one side, with a radius of $7\frac{1}{2}$ French inches, cylindrical convex on the other, with a radius of $4\frac{1}{2}$; the axes of the cylinders of course crossing at right angles, and the axis of the convex surface being in the vertical direction. This glass appears to fulfil the required conditions: it enables Mr. Parry to read inscriptions at a few yards distance, and also to have a distinct perception of very distant and minute objects, such as are presented in an extensive landscape.

In order to ascertain if it were possible to detect any error of curvature on the surface of the cornea, I observed the appearance of the reflection of a small luminous square held a few inches from the eye; but in the central part of this structure the reflected image was perfectly square, while the distortions produced at the circumference were equally produced in the sound eye; and there was no reason to conclude that the defect of vision arises from any defect in the cornea.

Mr. Parry finds his sight considerably improved by looking through a small hole in a card, so as to admit pencils only to fall on the central parts of the cornea; or, still better, by looking through a narrow vertical slit, provided that the illumination of the object be sufficient to compensate for the smallness of the pencils admitted. He finds that a very slight pressure on the eyeball, applied at the outer angle of the eye, improves the vision. I also find the same when gentle pressure is made at the upper and outer part of the ball. It is to be observed, that the application of a narrow slit to a sound eye produces an effect nearly analogous to that produced by a plano-cylindrical lens.

The second case is that of a student, who stated, that in observing small objects at 20 or 30 yards distance, he saw a second image of the objects, one image, however, being much fainter than the other. He considered that his sight had become impaired by too intense application to books, having only observed that his eyes were defective after several years close study.

On testing his eyes by a pinhole in a card, he saw the hole as a horizontal line most distinct at about 35 centimetres distance; beyond this the hole appeared indistinct. Also when he viewed two lines in the form of a cross, when they were held at 35 centimetres distance, he perceived most distinctly the horizontal line, and at some distance beyond this the vertical line. The line seen at the nearer focus was exactly perpendicular to the mesial plane of the body. I ascertained that the distinctness of his vision was considerably improved by applying to the eye a plano-cylindrical concave glass, of about 16 French inches radius.

SINCE the above paper was read, I have met with three gentlemen in the University, all of whom have one of their eyes affected with a malformation similar to my own; or with "astigmatism," as it has been called. The amount of the "astigmatism" in all of them appears to be corrected by a plano-cylindrical glass, the curvature of which is 12 inches radius.

In one of these gentlemen it is the more perfect eye that is thus affected. This eye, as observed in some other cases, gives diplopic vision of objects at a certain distance. Another stated that the vision of his eye was perfect until a few years since.

HENRY GOODE.

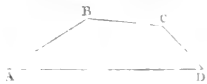
XXXVII. *Contributions towards a System of Symbolical Geometry and Mechanics. By the Rev.* M. O'Brien, *Professor of Natural Philosophy and Astronomy in King's College, London, and late Fellow of Caius College, Cambridge.*

[Read *March* 15, 1847.]

1. The important distinction which has been made by an eminent Authority in Mathematics between *Arithmetical* and *Symbolical* Algebra, may be extended to most of the Sciences which call in the aid of Algebra. Thus we may distinguish between *Symbolical Geometry* and *Arithmetical Geometry*, *Symbolical Mechanics* and *Arithmetical Mechanics*. This distinction does not imply, that in one division numbers only are used, and in the other symbols, for symbols are equally used in both, but it relates to the degree of generality of the symbolization. In the *Arithmetical* Science the symbols have a purely numerical signification, but in the *Symbolical* they represent, not only abstract quantity, but all the circumstances which, as it is usually expressed, *affect* quantity. The Arithmetical Science is, in fact, the first step of generalization, and the Symbolical the complete generalization.

In this view of the case, I have ventured to entitle the following Paper "Contributions towards a System of *Symbolical* Geometry and Mechanics." The Geometrical System about to be proposed consists, first, in representing curves and surfaces by *symbolical formulæ*, and secondly, in using the *Differential Notation* to denote *Perpendicularity*, according to the principles explained in a Paper read a few months since at a Meeting of the Society. The proposed Mechanical System is analogous in many respects to the Geometrical: examples of it have already been given in the Paper just quoted.

2. The following well-known principles are those upon which the Geometrical System is based.



1st. If $ABCD$ be any polygon, then $AD = AB + BC + CD$.

This may be regarded as the definition of +.

2ndly. Giving the usual definition of − it follows, that, in the triangle ABC,

$$AC - AB = BC.$$

3rdly. Where it follows that, if a denote any right line, $-a$ denotes an equal right line measured in an opposite direction.

4thly. If m denote any number, ma denotes a line m times the length of a drawn in the same direction as a. This follows immediately from the first principle.

These principles, with some others which we need not specify here, form the basis of the Geometrical System about to be proposed.

3. It will be convenient to consider that every line is traced by the *motion* of a point, and this will lead us to distinguish between the *beginning* and *end* of a line, the *beginning* being the extremity from which the tracing point starts, and the *end* the other extremity.

4. When we say that a symbol, a for instance, represents a straight line, we mean that a defines the *magnitude* and *direction* of the line, but not its *beginning;* in other words, the line is supposed to be drawn of a given length and in a given direction, but not from a given point.

However, if the contrary be not specified or implied, we shall always suppose the line to *begin* at the *origin, i. e.* at a certain point chosen for the purpose of reference.

5. We shall use the term *Direction Unit* to denote a straight line of a unity of length drawn in any particular direction. We shall always use the letters α, β, γ to denote *direction units,* and, unless the contrary be stated, we shall also suppose these three directions to be at right angles to each other: in other words, we shall assume α, β, γ to represent three straight lines drawn at right angles to each other, and each a unity of length.

6. We shall divide symbols into two classes, *Number Symbols* and *Line Symbols,* the former representing numerical quantities positive or negative, the latter straight lines in magnitude and direction.

7. We shall define the *position of a point in space* by the Line Symbol representing its *distance* from the origin: thus, whenever we speak of the point a, we mean the point whose distance from the origin is represented in magnitude and direction by the symbol a.

In our idea of *distance* here we suppose *direction,* as well as magnitude, to be included.

8. If a, b, c be any line symbols, it follows, from the first principle above stated, that $a+b+c$ represents the distance of the *end* of the line c from the *beginning* of the line a; the end of a being supposed to coincide with the beginning of b, and the end of b with the beginning of c.

In like manner $a-b$ denotes the distance of the end of a from the end of b, a and b being supposed to have the same beginning.

Hence, if a and b be the symbols of any two points A and B, $a-b$ is the symbol of the right line drawn from B to A, and $b-a$ the symbol of the line drawn from A to B.

9. If x be any number symbol, and α any direction unit, $x\alpha$ represents a straight line of the length x drawn in the direction α.

Hence, if r be the length of a right line drawn from the origin, x y z the lengths of the co-ordinates of the end of that line, and α β γ the direction units of the three co-ordinate axes, the three co-ordinates will be represented by the symbols $x\alpha$, $y\beta$, $z\gamma$, and the line by the symbol

$$x\alpha + y\beta + z\gamma.$$

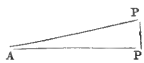
This symbol also defines the position of the point whose co-ordinates are x y z.

If a b c be the direction cosines of the line, its symbol becomes

$$r(a\alpha + b\beta + c\gamma).$$

The coefficient of r is evidently the direction unit of the line.

10. Let r and r' be the lengths of any two lines AP and AP' drawn from a point A, and let ϵ and ϵ' represent their direction units; then the symbols of these lines will be $r\epsilon$ and $r'\epsilon'$, and therefore the symbol of the line PP' will be



$$r'\epsilon' - r\epsilon.$$

If $r'=r$ and $\epsilon'-\epsilon$ is indefinitely small, this expression becomes

$$r\,d\epsilon.$$

Now in this case PP' is at right angles to AP, and therefore it follows that $r\,d\epsilon$ is the symbol of an indefinitely small line perpendicular to the line $r\epsilon$.

The length of this small line is $rd\theta$, assuming $d\theta$ = angle PAP'; but the direction unit of a line is expressed by dividing the symbol of the line by the symbol of its length; hence the direction unit of the small line is

$$\frac{rd\epsilon}{rd\theta} \text{ or } \frac{d\epsilon}{d\theta}.$$

Hence the direction unit of a perpendicular to a line $r\epsilon$ is $\frac{d\epsilon}{d\theta}$.

In the Paper already referred to, which was read before the Society some months since, the reader will find this method of representing *perpendicularity* by the differential notation fully developed, and the notation $Du \,.\, u'$, thence derived, explained, together with an auxiliary notation. $\Delta u \,.\, u'$; both of which we shall have occasion to make use of hereafter.

11. The following is the method we shall adopt of representing curves and surfaces symbolically.

To represent a curve or line we shall suppose a *variable parameter* to be involved in the symbol of a point, in which case it is clear, that the point will be indeterminate in position, but restricted so far, that it will always be found upon some curve or line. The symbol of a point therefore, when it involves a variable parameter and is thereby made indeterminate, becomes a *symbolical formula* defining some line or curve, and may be called the *formula of* that line or curve.

In like manner the symbol of a point, when it involves *two* variable parameters, becomes a *symbolical formula* defining some surface, and may be called the *formula* of that surface. This virtually amounts to defining lines and surfaces by symbolical polar equations.

It is important, however, to observe that we suppose the variable parameters here spoken of to be *number symbols.* If the variable parameter be a *direction unit*, it must be regarded as equivalent to two number symbols.

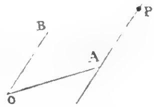
12. The following are examples of this method of representing curves and surfaces.

The general *symbolical formula* of a straight line in space is

$$u + r\epsilon,$$

where u is the symbol of a given point, r a *numerical* variable parameter, and ϵ a given direction unit.

For take $OA = u$ (O being the origin) $OB = \epsilon$, draw a line through A parallel to OB, taking upon it AP equal in length to r. Then AP is represented by the symbol $r\epsilon$, and therefore $u + r\epsilon$ is the symbol of the point P, which, since r is indeterminate, may be *any point* of the line drawn through A parallel to OB.



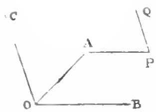
It appears, therefore, that $u + r\epsilon$ is the *formula* a straight line drawn through the point whose symbol is u, in the direction represented by ϵ.

13. In like manner the general *symbolical formula* of a plane is

$$u + r\epsilon + r'\epsilon'$$

r and r' being numerical variable parameters.

For take $OA = u$, $OB = \epsilon$, $OC = \epsilon'$, draw AP parallel to OB and equal in length to r, PQ parallel to OC and equal in length to r'. Then, it is evident, that $u + r\epsilon + r'\epsilon'$ is the symbol of the point Q; and that, since r and r' are indeterminate, Q is *any* point of the plane which contains the point A and is parallel to OB and OC.



Hence $u + r\epsilon + r'\epsilon'$ is the *formula* of a plane which is parallel to the directions represented by ϵ and ϵ', and contains the point whose symbol is u.

14. The following is an example of the case where the variable parameter is a direction unit.

The *formula* of a sphere is

$$u + r\epsilon,$$

where ϵ is the variable parameter, and r determinate.

For $u + r\epsilon$ represents a point whose distance from the point u is indeterminate in direction but determinate in length, being always equal to r. Therefore the *formula* $u + r\epsilon$ defines a sphere whose centre is u and radius r.

We shall now illustrate this method of Symbolical Geometry by the following propositions, without attempting any systematic arrangement, as our only object is to shew the nature and use of the method.

15. To deduce the equation of the plane from the *formula* of the plane, namely, $u + r\epsilon + r'\epsilon'$.

Let $x\ y\ z$ be the co-ordinates of the point represented by $u + r\epsilon + r'\epsilon'$, $x_,\ y_,\ z_,$ of the point represented by u, let $\alpha\ \beta\ \gamma$ be the direction units of the three co-ordinate axes, and let

$$\epsilon = a\alpha + b\beta + c\gamma, \quad \epsilon' = a'\alpha + b'\beta + c'\gamma.$$

Then we have

$$x\alpha + y\beta + z\gamma = u + r\epsilon + r'\epsilon'$$
$$= x_,\alpha + y_,\beta + z_,\gamma + r(a\alpha + b\beta + c\gamma) + r'(a'\alpha + b'\beta + c'\gamma);$$

and $\therefore$, equating coefficients of α, β, γ,
$$x = x_, + ra + r'a',$$
$$y = y_, + rb + r'b',$$
$$z = z_, + rc + r'c',$$

whence eliminating r and r' we find an equation of the form

$$Ax + By + Cz = D.$$

16. To express the *formula* of the plane by means of the symbol D.

If v be an indeterminate line symbol, and ϵ a determinate direction unit, $Dv.\epsilon$ denotes a line of any length drawn at right angles to ϵ in any direction. Hence it is evident that the *formula*

$$r\epsilon + Dv.\epsilon, \quad \text{or} \quad (r + Dv.)\epsilon,$$

represents the plane whose perpendicular distance from the origin is $r\epsilon$.

17. The *formula* of the right line drawn through the two points represented by u and u' is evidently

$$u + m(u' - u)$$

where m is a numerical variable parameter.

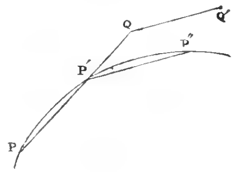
18. Hence if u be the *formula* of any curve the *formula* of the tangent at the point u is

$$u + m\,du.$$

19. To shew that the *formula* of the osculating plane of the same curve, at the point u, is

$$u + m\,du + n\,d^2u.$$

Let PP' and $P'P''$ be two consecutive chords of the curve; produce PP' to any point Q, and draw QQ' of any length parallel to $P'P''$: then Q' is any point of the plane containing the two chords, which plane, when the chords are indefinitely small, becomes the osculating plane.



Let $u\ u'\ u''$ be respectively the symbols of the points $P\ P'\ P''$; then the symbols of PP' and $P'P''$ are respectively $u' - u$ and $u'' - u'$, and therefore the symbols of PQ and QQ' are $m(u' - u)$ and $n(u'' - u')$, m and n being arbitrary numbers. Hence the symbol of the point Q', and therefore the *formula* of the plane containing the two chords, is

$$u + m(u' - u) + n(u'' - u').$$

Now in the limit we may put

$$u' - u = du, \quad u'' - u' = du + d^2u.$$

Hence the *formula* of the osculating plane is of the form

$$u + m\,du + n\,d^2u,$$

putting m in place of $m + n$.

20. From this *formula* to deduce the ordinary equation of the osculating plane.

Let $x\ y\ z$ be the co-ordinates of P, then

$$u = x\alpha + y\beta + z\gamma,$$

and therefore the *formula* of the osculating plane is

$$(x + m\,dx + n\,d^2x)\,\alpha + (y + m\,dy + n\,d^2y)\,\beta + (z + m\,dz + n\,d^2z)\,\gamma,$$

whence, if $x_{,}\ y_{,}\ z_{,}$ be the co-ordinates of any point of the osculating plane, we find

$$x_{,} = x + m\,dx + n\,d^2x, \quad y_{,} = y + m\,dy + n\,d^2y, \quad z_{,} = z + m\,dz + n\,d^2z,$$

from these equations, eliminating the variable parameters m and n, we find the common equation of the osculating plane.

21. Respecting the geometrical meaning of the symbols du and d^2u it is worth observing, that du represents in magnitude and direction the element (ds) of the arc of the curve defined by the *formula* u, and d^2u represents what is called a *double sagitta*, as we may prove very easily; for, let $P\ P'\ P''$ be three consecutive points of the curve indefinitely near each other; complete the parallelogram PP'' and draw the diagonal $P'Q$. Let $u\ u'\ u''$ be the symbols of the points $P\ P'\ P''$, then $u'' - u'$ represents the line $P'\ P''$, and therefore the line PQ, and $u' - u$ represents the line PP'; hence, it follows, that the line $P'Q$ is represented in magnitude and direction by $(u'' - u') - (u' - u)$, or, passing to differentials, by d^2u. $P'Q$ is a *double sagitta* of the arc PP'' *.

22. From this we may derive the following remarkable theorem.

If u be the *formula* of any curve in space, s the numerical length of the arc of the curve measured from any fixed point to the point u, v the numerical magnitude of d^2u, and ϵ the direction unit of d^2u, so that $d^2u = v\epsilon$; then $2\dfrac{ds^2}{v}$ expresses the numerical length of the chord of curvature drawn in the direction ϵ.

The direction ϵ is perfectly arbitrary, depending on what the independent variable in the differential d^2u is supposed to be. If we consider s to be the independent variable, it is evident that PP' and $P'P''$ are equal in magnitude, and therefore the chord of curvature becomes the diameter of curvature.

23. Another remarkable theorem is the following:

The symbol $d\left(\dfrac{du}{ds}\right)$ represents a line drawn from the point of contact towards the centre of curvature, and numerically equal to the *angle of contingence*.

This may be proved as follows. du represents a line whose length is ds drawn in the direction of the tangent at the point u, therefore $\dfrac{du}{ds}$ represents the direction unit of the tangent. Hence, if we draw two direction units *from the same point* parallel to two consecutive tangents, the symbol

* If we take s as the independent variable, in which case $PP' = P'P''$, $P'Q$ will be perpendicular to PP'', and d^2u will represent the double sagitta pointing towards the centre of curvature.

of the elementary line joining the ends of these two direction units will be $d\left(\frac{du}{ds}\right)$. Now this elementary line is evidently parallel to the normal drawn to the centre of curvature, and numerically equal to the angle of contingence (as the angle made by two consecutive tangents is commonly called).

24. Let v be the numerical magnitude of $d\left(\frac{du}{ds}\right)$, ϵ its direction unit, and ρ the radius of curvature: then, according to a well-known theorem,

$$\rho = \frac{ds}{v}.$$

Hence, we may immediately deduce the well-known expressions for ρ.

We have $u = x\alpha + y\beta + z\gamma$,

$$\text{and } \therefore d\left(\frac{du}{ds}\right) = \alpha d\left(\frac{dx}{ds}\right) + \beta d\left(\frac{dy}{ds}\right) + \gamma d\left(\frac{dz}{ds}\right) = d\left(\frac{du}{ds}\right) = v\epsilon;$$

$$\text{and } \therefore v^2 = \left\{d\left(\frac{dx}{ds}\right)\right\}^2 + \left\{d\left(\frac{dy}{ds}\right)\right\}^2 + \left\{d\left(\frac{dz}{ds}\right)\right\}^2.$$

Hence

$$\rho = \frac{ds}{\sqrt{\left\{d\left(\frac{dx}{ds}\right)\right\}^2 + \left\{d\left(\frac{dy}{ds}\right)\right\}^2 + \left\{d\left(\frac{dz}{ds}\right)\right\}^2}},$$

which is the well-known expression for ρ, the independent variable being arbitrary.

25. If m be any number, it is clear that $md\left(\frac{du}{ds}\right)$ represents in magnitude and direction any line drawn from the point of contact through the centre of curvature. Hence, the *formula* of that normal which lies in the osculating plane is

$$u + md\left(\frac{du}{ds}\right),$$

m being the variable parameter.

26. The symbol of the centre of curvature is evidently,

$$u + \frac{ds}{v}\epsilon,$$

ϵ being the direction unit, and v the numerical value of $d\left(\frac{du}{ds}\right)$.

27. If δ denote any arbitrary variation (as in the Calculus of Variations), then $\delta\left(\frac{du}{ds}\right)$ denotes *any* small line at right angles to the direction unit $\frac{du}{ds}$, *i. e.* to the tangent. Hence, the *formula* of *any* normal at the point u is

$$u + m\delta\left(\frac{du}{ds}\right).$$

It is obvious that this is also the *formula* of the normal plane, for it is the symbol of *any* point in the normal plane.

To deduce from this expression the common equation of the normal plane.

Let $x_{,}y_{,}z_{,}$ be any point in the normal plane; then

$$x_{,}\alpha + y_{,}\beta + z_{,}\gamma = u + m\delta\left(\frac{du}{ds}\right)$$

$$= \left\{x + m\delta\left(\frac{dx}{ds}\right)\right\}\alpha + \left\{y + m\delta\left(\frac{dy}{ds}\right)\right\}\beta + \left\{z + m\delta\left(\frac{dz}{ds}\right)\right\}\gamma.$$

Hence we have

$$x_{,} - x = m\delta\frac{dx}{ds}, \quad y_{,} - y = m\delta\frac{dy}{ds}, \quad z_{,} - z = m\delta\frac{dz}{ds},$$

$$\text{and } \therefore\ (x_{,} - x)\frac{dx}{ds} + (y_{,} - y)\frac{dy}{ds} + (z_{,} - z)\frac{dz}{ds} = \frac{m}{2}\delta\left\{\frac{(ds)^2}{(ds)^2}\right\} = 0,$$

which is the common equation.

28. It is however much more convenient to use the symbol D in expressing perpendicularity. $Dv\,.\,du$ denotes a line of any length perpendicular to du, supposing v to be any arbitrary line symbol. Hence the *formula* of the normal plane is

$$u + Dv\,.\,du.$$

29. The *formula* of the normal perpendicular to the osculating plane is

$$u + mDd^2u\,.\,du,$$

because du and d^2u both respresent lines lying in the osculating plane.

30. We shall now give a few examples of the application of this method to surfaces and to some common geometrical problems.

If u be the *formula* of a surface, the *formula* of the tangent plane at the point u, is

$$u + mdu,$$

m being a numerical variable parameter.

For du represents the elementary line joining any two contiguous points of the surface, and therefore mdu represents a line of *any* length touching the surface at the point u.

31. The *formula* of any normal plane (*i. e.* any plane containing the normal at the point u) is evidently

$$u + Dv\,.\,du,$$

v being any arbitrary line symbol.

u, being the *formula* of a surface, must involve two variable parameters: let them be m and n (both numerical), and let $d_m u$ and $d_n u$ represent the respective partial differential coefficients of u with respect to m and n: then the *formula* of the normal at the point u, is

$$u + pDd_m u\,.\,d_n u,$$

p being a numerical variable parameter.

32. The *formula* of a plane containing the three points u, u', u'', is

$$u + m(u' - u) + n(u'' - u),$$

or what is the same thing,

$$mu + m'u' + m''u''.$$

Where m, m', m'' are numerical parameters subject to the condition $m + m' + m'' = 1$.

33. If u be the *formula* of a right line (involving of course one variable parameter), the *formula* of a plane containing that line is evidently

$$u + mv,$$

m being a numerical variable parameter, and v any determinate line symbol.

If the plane be also restricted to contain a given point u', its *formula* is

$$u + m(u' - u), \quad \text{or} \quad mu + m'u',$$

where $m + m' = 1$.

34. Let the symbols of the angular points of a triangle be u, u', u''; then the symbol of the point mid-way between u' and u'' is $\frac{1}{2}(u' + u'')$, and the *formula* of the line drawn through this point and u is

$$u + m\left(\frac{u' + u''}{2} - u\right),$$

$$\text{or} \quad \left(1 - \frac{3m}{2}\right)u + \frac{m}{2}(u + u' + u'').$$

Now if we put $m = \frac{2}{3}$, this *formula* becomes symmetrical with respect to u, u', u''; which shews that the point whose symbol is $\frac{1}{3}(u + u' + u'')$ is common to the three bisectors of the sides of a triangle drawn from the opposite angles.

35. *We shall now give a few examples of this method applied to Mechanics.* We have already (in the Paper read a few months since) shewn how the fundamental principles of Statics may be proved and expressed with great simplicity by means of the symbol D. We have also shewn how the motion of a rigid body about its centre of gravity may be investigated by means of this notation, and exemplified its use in the problem of Precession and Nutation.

36. We may investigate the equations for finding the motion of a planet in the plane of its orbit, and the motion of that plane, as follows.

Let u be the symbol of the position of the planet at any time t, then the symbol of the force acting on the planet will be

$$\frac{d^2u}{dt^2}.$$

Let r be the radius vector of the planet, α, β, γ, three direction units at right angles to each other, α being the direction unit of u (and $\therefore u = r\alpha$), and γ being perpendicular to the plane of the orbit: let ω_1 denote the angular velocity of β and γ about α, ω_2 that of γ and α about β, ω_3 that of α and β about γ; then ω_3 is the angular velocity of the planet in its orbit, ω_1 is the angular velocity of the plane of the orbit about the radius vector, and ω_2 is evidently zero. Hence, (see Equations 38, former Paper,) we have

$$\frac{d\alpha}{dt} = \omega_3\beta, \quad \frac{d\beta}{dt} = \omega_1\gamma - \omega_3\alpha, \quad \frac{d\gamma}{dt} = -\omega_1\beta.$$

Now $u = r\alpha$; wherefore differentiating and substituting for $\frac{d\alpha}{dt}$, $\frac{d\beta}{dt}$, and $\frac{d\gamma}{dt}$ we have,

$$\frac{du}{dt} = \frac{dr}{dt}\alpha + r\frac{d\alpha}{dt}$$

$$= \frac{dr}{dt}\alpha + r\omega_3\beta;$$

$$\therefore \frac{d^2u}{dt^2} = \frac{d^2r}{dt^2}\alpha + \frac{dr}{dt}\frac{d\alpha}{dt} + \frac{d(r\omega_3)}{dt}\beta + r\omega_3\frac{d\beta}{dt}$$

$$= \left(\frac{d^2 r}{dt^2} - r\omega_3^2\right) a + \left\{\frac{dr}{dt}\omega_3 + \frac{d(r\omega_3)}{dt}\right\} \beta + r\omega_3\omega_1 \cdot \gamma.$$

This is the general symbolical expression for the force acting on the planet, and it consists of three parts whose direction units are a, β, γ, that is, which act, along the radius vector, perpendicular to it, and perpendicular to the plane of the orbit. Hence, if P, Q, S be the forces which act on the planet in these three directions respectively, we have,

$$P = \frac{d^2 r}{dt^2} - r\omega_3^2,$$

$$Q = \frac{dr}{dt}\omega_3 + \frac{d(r\omega_3)}{dt} = \frac{1}{r}\,\frac{d(r^2\omega_3)}{dt},$$

$$S = r\omega_3\omega_1,$$

which are the general equations for determining the motion of the planet, and of the plane of the orbit.

37. To determine the motion of a particle acted on by a central force varying inversely as the square of the distance.

Using the same notation as in the preceding Article, it is clear that the symbol of the force is

$$-\frac{\mu a}{r^2},$$

and therefore we have

$$\frac{d^2 u}{dt^2} = -\frac{\mu a}{r^2} \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1).$$

Performing the operation Du on each member of this equation, and observing that

$$Du \,.\, a = rDa \,.\, a = 0, \text{ we have}$$

$$Du \,.\, \frac{d^2 u}{dt^2} = 0;$$

$$\text{and therefore } Du\frac{du}{dt} = \text{constant} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2);$$

for the former equation is evidently the differential of the latter, observing that $D\frac{du}{dt} \,.\, \frac{du}{dt} = 0$*.

Now $u = ra$, $\frac{du}{dt} = \frac{dr}{dt} a + r\omega\beta$ (writing ω instead of ω_3), and therefore, since $Da \,.\, a = 0$, and $Da \,.\, \beta = \gamma$, the equation (2) becomes

$$r^2\omega\gamma = \text{constant}.$$

Hence, γ is an invariable direction (*i. e.*, the motion is in one plane) and $r^2\omega$ is constant, equal to h suppose.

Now $\frac{d\beta}{dt} = -\omega a$, and therefore (1) becomes

$$\frac{d^2 u}{dt^2} = \frac{\mu}{r^2\omega}\,\frac{d\beta}{dt} = \frac{\mu}{h}\,\frac{d\beta}{dt},$$

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* It is obvious that $d(Du \,.\, v) = Ddu \,.\, v + Du \,.\, dv$.

and therefore

$$\frac{du}{dt} = \frac{\mu}{h}\beta + \text{constant} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

and therefore putting for $\frac{du}{dt}$ its value, observing that $r\omega = \frac{h}{r}$, and assuming ϵ to be the direction unit of the constant, and e its numerical magnitude, we have

$$\frac{dr}{dt}\alpha + \frac{h}{r}\beta = \frac{\mu}{h}\beta + e\epsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

performing on this the operation $\Delta\beta$, observing that $\Delta\beta . \alpha = 0$, $\Delta\beta . \beta = 1$, we find

$$\frac{h}{r} = \frac{\mu}{h} + e\Delta\beta . \epsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5),$$

which is the polar equation of a conic section, the origin being focus, e being the eccentricity, and ϵ perpendicular to the axis major; for $\Delta\beta . \epsilon$ is the cosine of the angle which β makes with ϵ, *i. e.* the cosine of the angle which the radius vector makes with a perpendicular to ϵ.

If we perform the operation $\Delta\alpha$ upon (4), we obtain

$$\frac{dr}{dt} = e\Delta\alpha . \epsilon;$$

$\Delta\alpha . \epsilon$ denoting the cosine of the angle which the radius vector makes with a perpendicular to the axis major.

38. To determine the motion of the particle when it is acted upon by any disturbing force U in addition to the central force.

In this case instead of the equation (1), we have

$$\frac{d^2u}{dt^2} = -\frac{\mu\alpha}{r^2} + U \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6).$$

Treating this equation as we did (1), we find

$$Du . \frac{d^2u}{dt^2} = Du . U;$$

$$\text{and } \therefore \frac{d(h\gamma)}{dt} = Du . U \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7),$$

for $Du . \frac{du}{dt} = r^2\omega\gamma = h\gamma$, using h to denote $r^2\omega$.

By integrating equation (7), we find h and γ, and thus by integrating one equation we determine three elements of the orbit, for γ, being perpendicular to the plane of the orbit, determines both the inclination and the position of the node.

If we integrate (7), after having performed the operation $\Delta\gamma$ on each side, we find

$$h = \int\Delta\gamma . (Du . U)\, dt.$$

$$\text{Now } \Delta\gamma . (D\alpha . U) = \Delta\beta . U^*,$$

hence, since $u = r\alpha$, we have

$$h = \int r\Delta\beta . U dt \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8),$$

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* For $U = \alpha(\Delta\alpha . U) + \beta(\Delta\beta . U) + \gamma(\Delta\gamma . U)$,
and therefore performing successively the operations $D\alpha$ and $\Delta\gamma$, we find
$\Delta\gamma (D\alpha . U) = \Delta\beta . U$.

and from (7), we have

$$\gamma = \frac{1}{h} \int Du \,.\, U \,.\, dt \text{ .... } (9).$$

(8) and (9) give h and γ separately.

We may observe respecting these formulæ for h and γ, that $\Delta\beta \,.\, U$ expresses the resolved part of the disturbing force U in the direction β, *i. e.* perpendicular to the radius vector and in the plane of the orbit; and $Du \,.\, U$ is the symbol representing in magnitude and direction the moment of the Couple which transfers the force U from the point u to the origin. (See former Paper.)

39. To integrate the equation (6) directly as we did the equation (1), we have only to take the same steps, (observing that h is now variable,) as follows, (6) becomes

$$\frac{d^2u}{dt^2} = \frac{\mu}{h} \frac{d\beta}{dt} + U,$$

hence, (integrating $\frac{\mu}{h} \frac{d\beta}{dt}$ by parts), we have

$$\frac{du}{dt} = \frac{\mu}{h} \beta + \int \frac{\mu}{h^2} \frac{dh}{dt} \beta dt + \int U dt,$$

and therefore by (8),

$$\frac{du}{dt} = \frac{\mu}{h} \beta + \int \frac{\mu r}{h^2} (\Delta\beta \,.\, U) \beta dt + \int U dt \text{ .... } (10).$$

This is the symbolical expression for the velocity of the disturbed body. To find the parallax, put in (10),

$$\frac{du}{dt} = \frac{dr}{dt} \alpha + \frac{h}{r} \beta,$$

and then, performing the operation $\Delta\beta$ on both sides, we find

$$\frac{1}{r} = \frac{\mu}{h^2} + \Delta\beta \,.\, \int \left\{ \frac{\mu r}{h^2} (\Delta\beta \,.\, U) \beta + U \right\} dt,$$

which determines the parallax.

40. To determine the eccentricity and position of the axis major.

We have seen, that when there is no disturbance,

$$\frac{du}{dt} = \frac{\mu}{h} \beta + e\epsilon,$$

assuming this equation to be still true, e and ϵ being now variables, and comparing it with (10), we find

$$e\epsilon = \int \left\{ \frac{\mu r}{h^2} (\Delta\beta \,.\, U) \beta + U \right\} dt \text{ .... } (11);$$

$$\text{or } \frac{d(e\epsilon)}{dt} = \frac{\mu r}{h^2} (\Delta\beta \,.\, U) \beta + U \text{ .... } (12).$$

Now e is the eccentricity, and ϵ is the direction unit of a line at right angles to the axis major in the plane of the orbit: hence, (12) or (11) determines at the same time the eccentricity and the position of the axis major.

The Dynamical investigations just given are good instances of the nature of the Symbolical method here proposed.

M. O'BRIEN.

Upper Norwood, Surrey.
January, 1847.

XXXVIII. *On the Symbolical Equation of Vibratory Motion of an Elastic Medium, whether Crystallized or Uncrystallized. By the Rev.* M. O'BRIEN, *late Fellow of Caius College, Professor of Natural Philosophy and Astronomy in King's College, London.*

[Read *March* 15, 1847.]

Preliminary Observations.

THE object of the following Paper is twofold; *first*, to shew that the equations of vibratory motion of a crystallized or uncrystallized medium may be obtained in their most general form, and very simply, without making any assumption as to the nature of the molecular forces; and, *secondly*, to exemplify the use of the symbolical method and notation explained in two Papers read before the Society during the present academical year.

First, with regard to the Method of obtaining the Equations of Vibratory Motion.

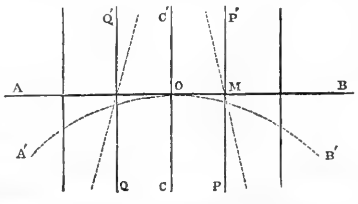
This method consists, first, in representing the *disarrangement* (or state of relative displacement) of the medium in the vicinity of any point xyz by the equation

$$\delta v = \frac{dv}{dx}\delta x + \frac{dv}{dy}\delta y + \frac{dv}{dz}\delta z + \frac{1}{2}\frac{d^2 v}{dx^2}\delta x^2 + \frac{d^2 v}{dx\,dy}\delta x \delta y + \&c.,$$

(where $v = \xi a + \eta \beta + \zeta \gamma$, $\xi\ \eta\ \zeta$ denoting, as usual, the displacements at the point xyz, and $a\ \beta\ \gamma$ the *direction units** of the three co-ordinate axes), and, secondly, in finding the *whole* force brought into play at the point xyz (in consequence of this disarrangement) by the *symbolical addition* of the different forces brought into play by the several terms of δv, *each considered separately*. It is easy to see that these different forces may be found with great facility, without assuming anything respecting the constitution of the medium more than this, that it possesses *direct* and *lateral elasticity*. By *direct elasticity* we mean that elasticity in virtue of which *direct* or *normal* vibrations take place, and by *lateral* that in virtue of which *lateral* or *transverse* vibrations take place.

The forces due to the several terms of δv are obtained by means of the following simple considerations:—

Let AB be any line in a perfectly uniform medium, and conceive the medium to be divided into elementary slices by planes perpendicular to AB; let OM $(= x)$ be the distance of any slice PP' from any particular point O of AB, and suppose this slice to suffer a displacement equal to $\frac{1}{2}cx^2$ (c being a constant) *in* the direction AB, and the other slices to be similarly displaced. Then it is evident that the medium suffers by these displacements a uniformly increasing *expansion* in the direction OB, and a uniformly increasing condensation in the direction OA, the rate of increase both of the expansion and condensation being c. Now in all known substances, whether solid, fluid, or



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* *i.e.* Three lines, each a unit of length, drawn parallel to the three axes.

gaseous, a disarrangement of this kind would bring into play on the slice O a force along the line AB proportional to the rate of increase c, i.e. a force Ac, A being a constant depending upon what we may call the *direct elasticity* of the substance.

Again, suppose that the slice PP' receives a displacement $\frac{1}{2}cx^2$ in the direction OC *perpendicular* to AB, and the other slices similar displacements. Then the line AB will become curved into a parabola $A'OB'$, and all the lines of the medium parallel to AB will be similarly curved, the radius of curvature being equal to $\frac{1}{c}$, and perpendicular to AB. Now in all known substances* a disarrangement of this kind would bring into play upon the slice O a force in the direction OC proportional to the curvature c, i. e. a force Bc, B being a constant depending upon what we may call the *lateral elasticity* of the substance.

Lastly, suppose that $MP = y$, and that the point P of the medium receives a displacement cxy parallel to AB, and the other points similar displacements. Then the slice PP' will, in consequence of this kind of displacement, turn through an angle $\tan^{-1}(cx)$ into the dotted position, and the other slices will suffer similar rotations, those on the other side of O, such as QQ', turning the opposite way. Now it is easy to see that a disarrangement of this kind produces a uniformly ncreasing expansion in the direction OC, and a uniformly increasing condensation in the direction OC', the rate of increase both of the expansion and condensation being c. But the expansion and condensation here described are quite different from those previously noticed, since they are produced, not by displacements parallel to $C'C$, but by *lateral* displacements, *i.e. perpendicular* to $C'C$. On this account all that we can assert without further investigation is, that the force brought into play upon an element at O by this disarrangement acts along the line $C'C$, and is proportional to c, i. e. equal to Cc, where C is a constant evidently depending in some way both upon the *direct* and *lateral* elasticity of the medium.

There is, however, a very simple way of finding the precise value of the force brought into play by a disarrangement of this kind; for, if we turn the axes of x and y in the plane of the paper through an angle of 45°, it will be found, that this disarrangement is nothing but a combination of the two kinds of disarrangement previously noticed, and from this it immediately follows, in the case of an uncrystallized medium, that the force brought into play at O is $(A - B)c$; in other words, the coefficient C, which must be multiplied into c in order to give the force brought into play by the disarrangement cxy, is equal to the coefficient of direct elasticity (A) minus the coefficient of lateral elasticity (B).

In the case of a crystallized medium it may be shewn that *six relations*, corresponding to the relation $C = A - B$, are most probably true, and are *essential* to Fresnel's Theory of Transverse Vibrations; that is to say, the medium is capable of propagating waves of transverse vibrations, if these six conditions hold, but otherwise it is not.

In employing the above considerations to determine the equations of vibratory motion, the directions AB and $C'C$ are always taken so as to coincide with some two of the three co-ordinate axes, and it is this circumstance that makes the method peculiarly applicable to crystallized media. Indeed, if it were necessary to take the lines AB and CC' in any directions but those of the axes of symmetry, the above considerations would not apply without considerable modification.

The equations of vibratory motion obtained by this method for an uncrystallized medium are the well-known equations involving the two constants A and B. The equations obtained for a crystallized medium are perfectly free from any restriction of any kind, are applicable to all kinds of substance, whether we suppose its structure to be analogous to that of a solid, fluid, or gas, and hold for all kinds of disarrangement, whether consisting of normal, or transverse displacements, or both.

* Fluids and gases possess lateral elasticity as well as solids, only in a comparatively feeble degree.

When we introduce the six relations between the constants above alluded to, and moreover assume that the vibrations constituting a polarized ray are *in* the plane of polarization, we arrive at Professor Mac Cullagh's equations *. If, on the contrary, we suppose the vibrations to be *perpendicular* to the plane of polarization, we arrive at equations which agree exactly with Fresnel's Theory in every particular †.

If we introduce these six relations into the equations for crystallized media deduced from M. Cauchy's hypothesis, that the molecular forces act along the lines joining the different particles of the medium, it will be found that these equations are immediately reduced to the equations for an uncrystallized medium. From this it follows that M. Cauchy's hypothesis cannot be applied to any but uncrystallized media. In fact, it may be easily proved, that, if the equations derived from M. Cauchy's hypothesis be true, a crystallized medium is incapable of propagating transverse vibrations.

Secondly, respecting the use of the Symbolical Method and Notation above alluded to.

THE application of the *Symbolical Method and Notation* to the subject of vibratory motion is very remarkable, and leads to equations of great simplicity. In the case of an uncrystallized medium, the three ordinary equations of motion are included in the single symbolical equation.

$$\frac{d^2 v}{dt^2} = B\left\{\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right\} v + (A - B)\left(\alpha\frac{d}{dx} + \beta\frac{d}{dy} + \gamma\frac{d}{dz}\right)\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right).$$

If we employ the notation $\Delta u'.u$, and assume the symbol $\mathfrak{D}$ to represent the operation

$$\alpha\frac{d}{dx} + \beta\frac{d}{dy} + \gamma\frac{d}{dz},$$

the equation of motion becomes

$$\frac{d^2 v}{dt^2} = B(\Delta\mathfrak{D}.\mathfrak{D})v + (A - B)\mathfrak{D}\Delta\mathfrak{D}.v;$$

or, by using the notation $Du'.u$ also, it may be put in the form

$$\frac{d^2 v}{dt^2} = \{A\mathfrak{D}\Delta\mathfrak{D} - B(D\mathfrak{D}.)^2\}v.$$

The symbol $\mathfrak{D}$ written before any quantity U, which is a function of xyz, has a very remarkable signification; the *direction unit* of the symbol $\mathfrak{D}U$ is that direction *perpendicular* to which there is no variation of U at the point xyz, and the *numerical magnitude* of $\mathfrak{D}U$ is the *rate of variation* of U when we pass from point to point *in that direction.*

The symbols $\Delta\mathfrak{D}.v$ and $D\mathfrak{D}.v$ have also remarkable significations. $\Delta\mathfrak{D}.v$ is a numerical quantity, representing the *degree of expansion*, or, what is called the *rarefaction* of the medium at the point xyz. $D\mathfrak{D}.v$ represents, in magnitude, the degree of *lateral disarrangement* of the medium at the point xyz, and, in direction, the *axis* about which that displacement takes place.

These two symbols may be found separately by the integration of an equation of the form

$$\frac{d^2 U}{dt^2} = C\left(\frac{d^2 U}{dx^2} + \frac{d^2 U}{dy^2} + \frac{d^2 U}{dz^2}\right).$$

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* Given in a Paper read to the Royal Irish Academy, December 9, 1839, p. 14.

† On this subject see a Paper by the late Mr. Greene in the seventh Volume of the *Cambridge Transactions*, p. 121.

When the six conditions above alluded to are introduced, the equation of motion for a crystallized medium becomes

$$\frac{d^2 v}{dt^2} = \left(A_1 a \frac{d}{dx} + A_2 \beta \frac{d}{dy} + A_3 \gamma \frac{d}{dz}\right) \Delta \mathfrak{D}.v$$

$$+ D\mathfrak{D}.\left\{\left(B_2 \frac{d\eta}{dz} - B'_3 \frac{d\zeta}{dy}\right) a + \left(B_3 \frac{d\zeta}{dx} - B'_1 \frac{d\xi}{dz}\right)\beta + \left(B_1 \frac{d\xi}{dy} - B'_2 \frac{d\eta}{dx}\right)\gamma\right\}.$$

Where A_1 A_2 A_3 are the three coefficients of *direct* elasticity with reference to the three axes of symmetry, and B_1 B_1' B_2 B_2' B_3 B_3' the six coefficients of *lateral* elasticity with reference to the same axes.

If the vibrations be transverse, this equation is reducible to the form

$$\frac{d^2 v}{dt^2} = -(D\mathfrak{D}.)^2 (a^2 \xi a + b^2 \eta \beta + c^2 \zeta \gamma)$$

$$\text{or } \frac{d^2 v}{dt^2} = -(D\mathfrak{D}.)^2 (a^2 a \Delta a + b^2 \beta \Delta \beta + c^2 \gamma \Delta \gamma) v, \; \ldots\ldots\ldots\ldots (A),$$

assuming the vibrations of a polarized ray to be *perpendicular* to the plane of polarization.

The well-known condition that a plane polarized ray may be transmissible without subdivision, and the expression for the velocity of propagation, may be immediately deduced from this equation.

If we assume the vibrations of a polarized ray to be *in* the plane of polarization, the equation becomes

$$\frac{d^2 v}{dt^2} = -D\mathfrak{D}.(a^2 a \Delta a + b^2 \beta \Delta \beta + c^2 \gamma \Delta \gamma) D\mathfrak{D}.v. \; \ldots\ldots\ldots\ldots (B).$$

The equation (*A*) agrees in all respects with Fresnel's Theory, and the equation (*B*) includes Professor Mac Cullagh's three equations. It is curious that (*A*) and (*B*) should differ from each other only in the *order* of the operations performed on v in the second member.

Investigation of the Symbolical Equation of Vibratory Motion of an Uncrystallized Medium.

1. Let $u\,(= ax + \beta y + \gamma z)$* be the symbol of any particle (P) of an elastic medium in a state of equilibrium, $v\,(= a\xi + \beta\eta + \gamma\zeta)$ the symbol of the displacement of the particle at any time t, $u + \delta u$ ($\delta u = a\delta x + \beta \delta y + \gamma \delta z$) the symbol of the equilibrium position of a contiguous particle (P'), and $v + \delta v$ ($\delta v = a\delta\xi + \beta\delta\eta + \gamma\delta\zeta$) the symbol of the displacement of P' at the time t; then we have

$$\delta v = \frac{dv}{dx}\delta x + \frac{dv}{dy}\delta y + \frac{dv}{dz}\delta z + \tfrac{1}{2}\frac{d^2 v}{dx^2}\delta x^2 + \frac{d^2 v}{dx\,dy}\delta x \delta y + \&c. \ldots\ldots\ldots (1).$$

This equation expresses the *disarrangement*, or state of displacement, of the medium in the immediate vicinity of P, for δv is the *relative* displacement of P' with reference to P, and by giving different values to δx δy δz in (1), corresponding to the different particles near P, we find the displacements of those particles relatively to P.

2. In consequence of the disarrangement of the medium in the vicinity of P, represented by (1), a force will be brought into play upon P; *our object is to find this force.*

Now, by a well-known principle, the force on P resulting from the disarrangement

$$\delta v = \frac{dv}{dx}\delta x + \frac{dv}{dy}\delta y + \&c.,$$

is the resultant (or symbolical sum) of the forces due to the separate disarrangements

$$\delta v = \frac{dv}{dx}\delta x, \quad \delta v = \frac{dv}{dy}\delta y, \quad \delta v = \frac{dv}{dz}\delta z, \quad \&c.$$

Hence, if we find the forces due to the several terms of the expression (1), and add them together, the resulting sum will express, in magnitude and direction, the whole force brought into play upon P by the disarrangement (1). This we now proceed to do.

3. To find the force brought into play on P by the disarrangement,

$$\delta v = \frac{dv}{dx}\delta x = \alpha\frac{d\xi}{dx}\delta x + \beta\frac{d\eta}{dx}\delta x + \gamma\frac{d\zeta}{dx}\delta x.$$

$\alpha\frac{d\xi}{dx}\delta x$ represents a small line, proportional to δx, drawn in the direction α; therefore the disarrangement indicated by

$$\delta v = \alpha\frac{d\xi}{dx}\delta x$$

is a uniform expansion of the medium in the direction α. This brings no force into play upon P.

$\beta\frac{d\eta}{dx}\delta x$ represents a small line, proportional to δx, drawn in the direction β; therefore the disarrangement indicated by

$$\delta v = \beta\frac{d\eta}{dx}\delta x$$

takes place as follows: Suppose the medium when at rest to be divided into physical lines parallel to the direction α, let MN be any one of these lines, M being the point when it meets the plane perpendicular to α containing P, and let MN' be a line parallel to the plane of xy, making an angle $\tan^{-1}\left(\frac{d\eta}{dx}\right)$ with MN. Then the disarrangement consists in the displacement of the line MN into the position MN', and a similar displacement of all the other physical lines. This disarrangement evidently brings no force into play upon P.

The same reasoning applies to the remaining term $\gamma\frac{d\zeta}{dx}\delta x$.

4. Reasoning therefore in this way it is clear, that the disarrangement represented by the first three terms of the expression (1) brings no force into play upon P.

5. To find the force brought into play on P by the disarrangement represented by

$$\delta v = \tfrac{1}{2}\frac{d^2 v}{dx^2}\delta x^2 = \tfrac{1}{2}\frac{dv^2}{dx^2}(\alpha\xi + \beta\eta + \gamma\zeta)\delta x^2.$$

$\frac{1}{2}\alpha\frac{d^2\xi}{dx^2}\delta x^2$ represents a small line, proportional to δx^2, drawn in the direction α; therefore the disarrangements indicated by

$$\delta v = \tfrac{1}{2}\alpha\frac{d^2\xi}{dx^2}\delta x^2$$

is a uniformly increasing expansion of the medium in the direction α, the rate of increase of the expansion being $\dfrac{d^2\xi}{dx^2}$. Hence, according to well-known principles, this disarrangement brings into play on P, a force proportional to $\dfrac{d^2\xi}{dx^2}$ in the direction α, that is, a force whose symbol is

$$A\alpha\frac{d^2\xi}{dx^2},\ A \text{ being some constant.}$$

Again, $\frac{1}{2}\beta\dfrac{d^2\eta}{dx^2}\delta x^2$ represents a small line, proportional to δx^2, drawn in the direction β; therefore the disarrangement indicated by

$$\delta v = \tfrac{1}{2}\beta\frac{d^2\eta}{dx^2}\delta x^2$$

is a *curvature* of the physical line MN (see Art. 3.), and a similar curvature of all the other physical lines, the symbol of the *index of curvature* (*i.e.* a line equal to the reciprocal of the radius of curvature drawn towards the centre of curvature) being

$$\beta\frac{d^2\eta}{dx^2}.$$

Hence, according to well-known principles, this disarrangement brings into play upon P a force proportional to $\dfrac{d^2\eta}{dx^2}$ in the direction β, that is, a force whose symbol is

$$B\beta\frac{d^2\eta}{dx^2},\ B \text{ being some constant.}$$

In the same manner we may shew that the force brought into play by the disarrangement

$$\delta v = \tfrac{1}{2}\gamma\frac{d^2\zeta}{dx^2}\delta x^2$$

is represented by the symbol

$$B\gamma\frac{d^2\zeta}{dx^2}.$$

Hence the whole force brought into play by the disarrangement

$$\delta v = \frac{1}{2}\frac{d^2v}{dx^2}\delta x^2,$$

is represented in magnitude and direction by the symbol

$$\left(\frac{d}{dx}\right)^2\{A\alpha\xi + B(\beta\eta + \gamma\zeta)\},$$

$$\text{or}\quad \left(\frac{d}{dx}\right)^2\{(A - B)\,\alpha\xi + Bv\}.$$

6. To find the force brought into play by the disarrangement represented by

$$\delta v = \frac{d^2v}{dx\,dy}\delta x\delta y = \frac{d^2}{dx\,dy}(\alpha\xi + \beta\eta + \gamma\zeta)\,\delta x\delta y.$$

Let $x'y'$ be co-ordinates referred to two new axes $(\alpha'\beta')$ in the plane of xy, making respectively angles 45^0 and $90^0 + 45^0$ with the axis of x; then

$$\delta x = \frac{1}{\sqrt{2}}(\delta x' - \delta y'), \quad \delta y = \frac{1}{\sqrt{2}}(\delta x' + \delta y'),$$

$$\alpha = \frac{1}{\sqrt{2}}(\alpha' - \beta'), \qquad \beta = \frac{1}{\sqrt{2}}(\alpha' + \beta').$$

Making these substitutions, we find

$$\delta v = \tfrac{1}{2}\frac{d^2}{dxdy}\left\{\alpha'\frac{1}{\sqrt{2}}(\xi + \eta) + \beta'\frac{1}{\sqrt{2}}(\eta - \xi) + \gamma\zeta\right\}(\delta x'^2 - \delta y'^2).$$

Hence, by what has been already proved, the force brought into play will be

$$\frac{d^2}{dxdy}\left\{(A - B)\,\alpha'\frac{1}{\sqrt{2}}(\xi + \eta) + (B - A)\,\beta'\frac{1}{\sqrt{2}}(\eta - \xi) + (B - B)\,\gamma\zeta\right\}$$

$$= (A - B)\frac{d^2}{dxdy}(\beta\xi + \alpha\eta).$$

7. We may now write down the symbol of the whole force brought into play by the disarrangement represented by the expression (1), neglecting terms beyond those of the second order. It will be as follows,

$$F = B\left\{\left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2\right\}v$$

$$+ (A - B)\left\{\begin{array}{l}\alpha\dfrac{d^2\xi}{dx^2} + \beta\dfrac{d^2\eta}{dy^2} + \gamma\dfrac{d^2\zeta}{dz^2} \\ + \dfrac{d^2}{dxdy}(\beta\xi + \alpha\eta) + \dfrac{d^2}{dydz}(\gamma\eta + \beta\zeta) + \dfrac{d^2}{dzdx}(\alpha\zeta + \gamma\xi)\end{array}\right\}.$$

The coefficients of α, β, γ in this expression are the well-known differential formulæ for the three forces (parallel to the three axes) brought into play by the displacements ξ, η, ζ.

The part of F which is multiplied by $A - B$, may be put in the form

$$\left(\alpha\frac{d}{dx} + \beta\frac{d}{dy} + \gamma\frac{d}{dz}\right)\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right).$$

Hence, the equation of motion of the medium (which includes the three ordinary equations) assumes the following form,

$$\frac{d^2v}{dt^2} = B\left\{\left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2\right\}v + (A - B)\left(\alpha\frac{d}{dx} + \beta\frac{d}{dy} + \gamma\frac{d}{dz}\right)\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right).$$

8. This equation may be put in a remarkably simple form by the use of the notation $\Delta u'.u$.

Let us assume the symbol $\mathfrak{D}$ to represent the operation

$$\alpha\frac{d}{dx} + \beta\frac{d}{dy} + \gamma\frac{d}{dz},$$

then, since $v = \alpha\xi + \beta\eta + \gamma\zeta$, we have

$$\Delta\mathfrak{D}.v = \frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}.$$

Also $$\Delta \mathfrak{D} \,.\, \mathfrak{D} = \left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2.$$

Hence the equation of motion becomes

$$\frac{d^2 v}{dt^2} = B\,(\Delta \mathfrak{D} \,.\, \mathfrak{D})\, v + (A - B)\, \mathfrak{D} \Delta \mathfrak{D} \,.\, v \,\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}.$$

We may also put it in somewhat a different form by using the notation $Du'.u$; for

$$(\Delta \mathfrak{D} \,.\, \mathfrak{D})\, v - \mathfrak{D} \Delta \mathfrak{D} \,.\, v = -\,(D \mathfrak{D})^2 .\, v^*,$$

therefore (3) becomes

$$\frac{d^2 v}{dt^2} = \{ A \mathfrak{D} \Delta \mathfrak{D} \,.\, - B\,(D \mathfrak{D}\,.)^2 \}\, v \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(4)}.$$

9. The symbol $\mathfrak{D}$ has a very remarkable meaning which we shall now proceed to explain.

$\frac{d}{dx}$ denotes the rate of variation when x alone is varied, that is, the rate of variation *in the direction* α. To indicate this, we shall employ the notation d_α instead of $\frac{d}{dx}$; *i.e.*, if U be any quantity which is a function of $x\ y\ z$, and which therefore varies when we pass from one point to another of the medium, then $d_\alpha U$ denotes the *rate of variation* of U, when we pass from point to point *in the direction* α.

Now this rate of variation may be *affected*, like an ordinary velocity, with a sign of direction: and it may be *resolved* or *compounded* in the same manner, and by the same rules, as an ordinary velocity.

Hence, we may see immediately the meaning of the expression

$$\mathfrak{D} U, \quad \text{or} \quad \alpha d_\alpha U + \beta d_\beta U + \gamma d_\gamma U;$$

for $\alpha d_\alpha U$ is the rate of variation of U in the direction α, affected with its proper sign of direction α, $\beta d_\beta U$ is the rate of variation in the direction β, and $\gamma d_\gamma U$ in the direction γ, each affected with its proper sign of direction. Hence, compounding these rates of variation as if they were ordinary velocities, it follows, that the symbolical sum

$$\alpha d_\alpha U + \beta d_\beta U + \gamma d_\gamma U$$

expresses, in magnitude and direction, *the complete rate of variation* of the quantity U.

10. We may shew this differently as follows.

Let $\alpha_,\ \beta_,\ \gamma_,$ be any three direction units at right angles to each other; then it is easy to prove, that

$$\alpha_, d_{\alpha_,} + \beta_, d_{\beta_,} + \gamma_, d_{\gamma_,} = \alpha d_\alpha + \beta d_\beta + \gamma d_\gamma, \,\ldots\ldots\ldots\ldots\ldots\ldots \text{(5)}.$$

Let us now choose $\alpha_,\ \beta_,\ \gamma_,$, so that $\alpha_,$ shall be in the direction of the normal to the surface

$$dU = 0,$$

at the point $x\ y\ z$; in other words, supposing U to denote some disturbance or displacement of the medium, $\alpha_,$ is chosen so as to be perpendicular to the surface called the *front of the wave*, for $dU = 0$ is evidently the differential equation of that surface.

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* For let u and u' be any two lines, and let α represent the direction unit of u'; then, if $u' = r'\alpha$, and $u = a\alpha + b\beta + c\gamma$, we have $Du'.u = r'(b\gamma - c\beta)$, and $\therefore\ Du'.(Du'.u) = r'^2(-b\beta - c\gamma)$.

Now $\Delta u'.u' = r'^2$, and $u'\Delta u'.u = r'^2 a\alpha$; therefore $Du'.(Du'.u)$ or $(Du')^2.u = u'\Delta u'.u - (\Delta u'.u')\,u$.

$\alpha_{,}$ being thus assumed, we have $d_{\beta_{,}} U = 0$, $d_{\gamma_{,}} U = 0$, and therefore by (5),

$$\alpha_{,} d_{\alpha_{,}} U = \alpha d_{\alpha} U + \beta d_{\beta} U + \gamma d_{\gamma} U = \mathfrak{D} U.$$

Now $\alpha_{,}$ is the *direction of propagation* of the disturbance U, therefore $d_{\alpha_{,}} U$ is the rate of variation of U in the direction of propagation, and $\alpha_{,} d_{\alpha_{,}} U$ is that rate affected with its proper sign of direction.

Hence, the symbol $\mathfrak{D} U$ expresses, in magnitude and direction, the *complete rate of variation* of the quantity U, that is to say, the direction of $\mathfrak{D} U$ is that direction *perpendicular* to which there is *no variation* of U at the point xyz, and the magnitude of $\mathfrak{D} U$ is the *rate of variation* of U in that direction.

It is manifest, therefore, that the symbol $\mathfrak{D}$ has a very important signification, especially in investigations relating to the propagation of waves.

11. Returning now to the equation (4) we shall, in the first place, interpret the meaning of the symbols $\Delta \mathfrak{D} . v$, and $D \mathfrak{D} . v$.

Let $\xi_{,}$ $\eta_{,}$ $\zeta_{,}$ be the resolved parts of the displacement v in the directions $\alpha_{,}$ $\beta_{,}$ $\gamma_{,}$ respectively; then, choosing (as we may do)* the direction $\gamma_{,}$ so that $\zeta_{,} = 0$, we have

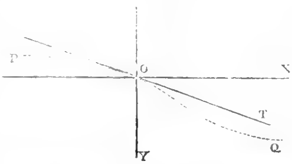
$$v = \xi_{,} \alpha_{,} + \eta_{,} \beta_{,},$$

and therefore, since $\mathfrak{D} = \alpha_{,} d_{\alpha_{,}}$, we find

$$\Delta \mathfrak{D} . v = d_{\alpha_{,}} \xi_{,},$$

$$D \mathfrak{D} . v = \gamma_{,} d_{\alpha_{,}} \eta_{,}.$$

Let OX and OY be the directions $\alpha_{,}$ and $\beta_{,}$, $\gamma_{,}$ being perpendicular to the plane of the paper; let O be the point (xyz) of the medium, POQ the line of particles which, in a state of equilibrium, lie in the direction $\alpha_{,}$, and OT the tangent to POQ at O. Then since OX is the direction of propagation, and since the disturbance (v) consists of two parts, namely $\xi_{,}$ in the direction OX, and $\eta_{,}$ in the direction OY, it is evident, that $d_{\alpha_{,}} \xi_{,}$ is the *expansion* (*i. e.* the *degree of expansion*, or, what is called the *rarefaction*) of the medium at the point O; also $d_{\alpha_{,}} \eta_{,}$ is the tangent of the angle TOX, and therefore measures the *degree of lateral displacement* of the medium at the point O.



This lateral displacement consists in the *rotation* of the line OT about the axis $\gamma_{,}$, and a corresponding rotation of all the other lines of particles which constitute the medium in the immediate vicinity of the point O, these lines being supposed to be parallel to OX in a state of equilibrium. Hence it follows that the symbol $D \mathfrak{D} . v$ represents, in direction, the *axis* round which the lateral displacement takes place, and in magnitude, the *degree* of lateral displacement.

Thus it appears that the symbols $\Delta \mathfrak{D} . v$ and $\alpha D \mathfrak{D} . v$ have a very important signification in investigations relating to the propagation of waves, the former expressing the *degree of expansion* of the medium at the point xyz, and the latter representing, in magnitude, the *degree of lateral displacement* at the point xyz, and, in direction, the *axis about which that displacement takes place*.

12. Hence it is evident that the symbol $\Delta \mathfrak{D} . v$ defines the kind of disturbance which constitutes *normal waves*, and $D \mathfrak{D} . v$ that which constitutes *transverse waves*.

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* $\alpha_{,}$ is the direction of *propagation*, as in the preceding article, and $\gamma_{,}$ is chosen at right angles, not only to $\alpha_{,}$, but also to the direction of *vibration* at the point xyz. In this case $\zeta_{,}$ is clearly zero.

13. $\Delta\mathfrak{D}.v$, and $D\mathfrak{D}.v$ may be found separately by the integration of a differential equation of the form

$$\frac{d^2U}{dt^2} = C\left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

For performing the operation $\Delta\mathfrak{D}.$ on both sides of (3), and putting $\Delta\mathfrak{D}.v = U$, we find

$$\frac{d^2}{dt^2}(\Delta\mathfrak{D}.v) = B(\Delta\mathfrak{D}.\mathfrak{D})\,\Delta\mathfrak{D}.v + (A-B)(\Delta\mathfrak{D}.\mathfrak{D})\,\Delta\mathfrak{D}.v;$$

$$\text{or,} \quad \frac{d^2U}{dt^2} = A(\Delta\mathfrak{D}.\mathfrak{D})\,U;$$

and performing the operation $D\mathfrak{D}.$, and putting $D\mathfrak{D}.v = U$, we find (observing that $D\mathfrak{D}.\mathfrak{D} = 0$),

$$\frac{d^2U}{dt^2} = B(\Delta\mathfrak{D}.\mathfrak{D})\,U:$$

Now
$$\Delta\mathfrak{D}.\mathfrak{D} = \left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2;$$

hence, $\Delta\mathfrak{D}.v$ and $D\mathfrak{D}.v$ may be found separately by the integration of an equation of the form (6), C being equal to A in one case, and to B in the other.

Investigation of the Equation of Vibratory Motion of a Crystallized Medium.

14. When the constitution of the vibrating medium is crystalline, we may obtain a differential equation similar to that we have found for an uncrystallized medium, and by exactly the same method; the only difference will be in the constants introduced, in regard to which we must bear in mind, that the elasticity of the medium is no longer the same in all directions, and therefore the constants A and B, which we may call the *coefficients of direct and lateral elasticity* respectively, will be different with respect to different directions. The method we have employed to find the force brought into play by a disarrangement of the medium requires us to consider this difference of elasticity only with respect to the three directions α, β, γ, assuming that the medium is still symmetrical with respect to these three axes, *i.e.* supposing them to be the *axes of elasticity*.

15. Hence, reasoning as in article (5), the force brought into play by the disarrangement,

$$\delta v = \tfrac{1}{2}\left\{\alpha\frac{d^2\xi}{dx^2}\delta x^2 + \beta\frac{d^2\eta}{dy^2}\delta y^2 + \gamma\frac{d^2\zeta}{dz^2}\delta z^2\right\},$$

will be
$$A_1\alpha\frac{d^2\xi}{dx^2} + A_2\beta\frac{d^2\eta}{dy^2} + A_3\gamma\frac{d^2\zeta}{dz^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (U),$$

A_1, A_2, A_3, being the coefficients of *direct elasticity* for the three directions α, β, γ.

Again, reasoning as in the same article, the force brought into play by the disarrangement,

$$\delta v = \tfrac{1}{2}\left\{\alpha\left(\frac{d^2\xi}{dy^2}\delta y^2 + \frac{d^2\xi}{dz^2}\delta z^2\right) + \beta\left(\frac{d^2\eta}{dz^2}\delta z^2 + \frac{d^2\eta}{dx^2}\delta x^2\right) + \gamma\left(\frac{d^2\zeta}{dx^2}\delta x^2 + \frac{d^2\zeta}{dy^2}\delta y^2\right)\right\},$$

will be

$$\alpha\left(B_1\frac{d^2\xi}{dy^2} + B_1'\frac{d^2\xi}{dz^2}\right) + \beta\left(B_2\frac{d^2\eta}{dz^2} + B_2'\frac{d^2\eta}{dx^2}\right) + \gamma\left(B_3\frac{d^2\zeta}{dx^2} + B_3'\frac{d^2\zeta}{dy^2}\right), \ldots (V),$$

B_1, B_1', B_2, B_2', &c., denoting the coefficients of *lateral elasticity* for the three directions α, β, γ.

We make a difference here between B_1 and B_1', because the disarrangements,

$$\tfrac{1}{2}\,a\,\frac{d^2\xi}{dy^2}\,\delta y^2, \quad \text{and} \quad \tfrac{1}{2}\,a\,\frac{d^2\xi}{dz^2}\,\delta z^2,$$

are of a different nature, though they consist of displacements in the same direction α; for the former disarrangement consists in the rotation (*i.e.* the curvature) of physical lines parallel to β about the axis γ, and the latter of physical lines parallel to γ about the axis β. The same remarks apply to B_2 and B_2', B_3 and B_3'. Fresnel virtually assumed that $B_1 = B_1'$, $B_2 = B_2'$, $B_3 = B_3'$.

16. By reasoning as in article (6), we might easily shew, that the force brought into play by the disarrangement,

$$\delta v = \frac{d^2 v}{dx\,dy}\,\delta x\,\delta y = \frac{d^2}{dx\,dy}(\alpha\xi + \beta\eta + \gamma\zeta)\,\delta x\,\delta y,$$

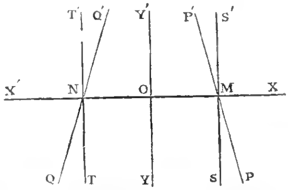
is

$$\beta C\,\frac{d^2\xi}{dx\,dy} + \alpha C'\,\frac{d^2\eta}{dx\,dy}.$$

But we shall shew this somewhat differently, in order to find out what relation subsists (if any) between C C' and the constants already introduced.

The disarrangement $\alpha\,\dfrac{d^2\xi}{dx\,dy}\,\delta x\,\delta y$, is of the following nature.

Let OX and OY represent the directions α and β, and O the point (xyz); take $OM = \delta x$, draw SMS' parallel to YY', and PMP' making the tangent of the angle PMS equal to $\dfrac{d^2\xi}{dx\,dy}\,\delta x$. Then it is clear that the disarrangement $\alpha\,\dfrac{d^2\xi}{dx\,dy}\,\delta x\,\delta y$ causes the physical line SMS' to assume the position PMP'. In like manner, if $ON = -\delta x$, and TNT' is parallel to YY', the physical line TNT' will, in consequence of the disarrangement, assume the position QNQ', the angles QNT and PMS being equal. The physical lines (taken parallel to YY') between SS' and TT' will suffer similar deviations, the tangent of the angle of deviation being proportional to δx.



17. The effect of a disarrangement of this kind is obvious; for it produces a uniformly increasing expansion of the medium as we go along the line OY, and a uniformly increasing condensation as we go along the line OY', the rate of increase both of the expansion and condensation being, as it is easy to see,

$$\frac{d^2\xi}{dx\,dy}.$$

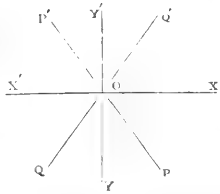
The effect of this will be to bring into play upon the particle O a force in the direction OY proportional to this rate of increase, *i.e.* a force whose symbol is,

$$\beta C\,\frac{d^2\xi}{dx\,dy}, \quad C \text{ being some constant.}$$

In like manner we may shew that the disarrangement $\beta\,\dfrac{d^2\eta}{dx\,dy}\,\delta x\,\delta y$ brings into play upon O a force whose symbol is,

$$\alpha C'\,\frac{d^2\eta}{dx\,dy}, \quad C' \text{ being some constant.}$$

18. We may easily shew that the disarrangement $\gamma \frac{d^2\zeta}{dxdy}\delta x\delta y$ brings no force into play upon O; for it is perpendicular to the plane of the paper, and its nature is as follows. Draw two physical lines QQ' and PP' through O equally inclined to XX'; then in consequence of the disarrangement the lines OP and OP' will become bent *upwards* (*i.e.* upwards considering the plane of the paper to be horizontal), and the lines OQ and OQ' will be bent *downwards*; also the curvature of POP' will be exactly the same as that of QOQ', only opposite in directions. Hence the two forces brought into play on O by the curvature of the two physical lines PP' and QQ' will be equal and opposite; and the same may be said of every other pair of physical lines drawn through O equally inclined to XX'. It is therefore manifest that no force will be brought into play on O by this disarrangement.



19. Thus it appears that the force brought into play by the disarrangement,

$$\frac{d^2v}{dxdy}\delta x\delta y, \quad \text{or} \quad \frac{d^2}{dxdy}(a\xi + \beta\eta + \gamma\zeta)\delta x\delta y,$$

$$\text{will be} \quad \beta C\frac{d^2\xi}{dxdy} + aC'\frac{d^2\eta}{dxdy}.$$

Hence the force brought into play by the disarrangement,

$$\delta v = \frac{d^2v}{dxdy}\delta x\delta y + \frac{d^2v}{dydz}\delta y\delta z + \frac{d^2v}{dzdx}\delta z\delta x$$

will be expressed by a symbol of the form

$$\frac{d^2}{dydz}(C_1\eta\gamma + C_1'\zeta\beta) + \frac{d^2}{dz\,dx}(C_2\zeta a + C_2'\xi\gamma) + \frac{d^2}{dxdy}(C_3\xi\beta + C_3'\eta a) \quad \ldots\ldots\ldots\ldots\ldots (U'').$$

20. Hence, collecting these three results, the general equation of vibratory motion will be

$$\frac{d^2v}{dt^2} = U + U' + U'' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

21. We have seen that, in the case of an uncrystallized medium, the constant C (*i.e.* the constant to which the different C's in U'' become equal when the medium becomes uncrystallized) is equal to $A - B$; in other words, C is the difference between the coefficients of *direct* and *lateral* elasticity; and it is easy to explain how this is on simple mechanical principles, which appear to apply to a crystallized medium as well as to an uncrystallized, and which therefore will furnish us with certain probable relations between the coefficients involved in equation (8). These relations, as we shall presently shew, have a very important physical signification.

22. Referring to the figure in p. 11, we may explain the physical meaning of the relation, $C = A - B$ as follows:—

The disarrangement represented in this figure consists of an increasing expansion of the medium as we go along the line YY', caused, not by *direct* displacements (*i. e.* displacements parallel to $Y'Y$), but by *lateral* displacements (*i.e.* displacements perpendicular to $Y'Y$). Consequently the force brought into play upon O by this increase of expansion will be modified by the lateral elasticity of the medium, which tends to restore the physical lines PP', QQ', &c. to their equilibrium positions SS', TT', &c. In fact the unequal expansion caused by the disarrangement is resisted, and, to a certain extent, balanced (so to speak) by the lateral elasticity, and therefore the unequal expansion has not its *full effect* in producing force upon O, but a certain part is spent upon the lateral elasticity.

If there was no lateral elasticity the force on O would be the same as if the displacements were *direct* (*i.e.* parallel to $Y'Y$), for then the unequal expansion would produce its full effect; in other words the force brought into play on O would be

$$A\frac{d^2\xi}{dxdy}\beta,$$

observing that the rate of increase of the expansion of the medium as we go along $Y'Y$ is $\frac{d^2\xi}{dxdy}$.

To find the force actually brought into play upon O, allowing for the lateral elasticity, we must diminish this force by a certain quantity depending upon the lateral elasticity, which quantity must of course be proportional to $\frac{d^2\xi}{dxdy}$. It is clear therefore that the force actually brought into play upon O is expressed by a symbol of the form

$$(A - P)\frac{d^2\xi}{dxdy}\beta,$$

P being a certain constant depending upon the lateral elasticity. Art. 6 shows that $P = B$.

This evidently explains the physical meaning of the relation, $C = A - B$, for this relation indicates that the force brought into play by the disarrangement $a\frac{d^2\xi}{dxdy}\partial x\partial y$ is, not the force $A\frac{d^2\xi}{dxdy}\beta$, which is the force due to the full effect of the unequal expansion, but the force $(A - B)\frac{d^2\xi}{dxdy}\beta$, which is equal to the former force diminished by a quantity depending on the lateral elasticity, and proportional to the rate of increase of the expansion.

23. From this explanation of the meaning of the relation $C = A - B$, it is very probable, I think, that a similar relation holds when the medium is crystallized; for it does not seem essential to this explanation that the medium shall be perfectly uniform in all directions; all that seems really necessary is, that the medium shall be symmetrically arranged with reference to the two axes XX' and YY'. We must take care, however, in applying this explanation to a crystallized medium, to give A and B their proper values, namely A_2 and B_1; for by A is to be understood the coefficient of *direct* elasticity *in the direction* OY, that is A_2, and by B the coefficient of *lateral* elasticity brought into action by the unequal rotation of physical lines *parallel* to OY *about* the axis of z, that is B_1 (for B_1 is the coefficient of lateral elasticity for the *curvature* of such lines about the axis of z). Hence the relation, $C = A - B$, transferred to a crystallized medium, is $C_3 = A_2 - B_1$, and therefore, writing down this relation for the six C's in the expression U'', we have the following six relations, viz.:—

$$\left.\begin{array}{lll} C_3 = A_2 - B_1 & C_1 = A_3 - B_2 & C_2 = A_1 - B_3 \\ C_3' = A_1 - B_2' & C_1' = A_2 - B_3' & C_2' = A_3 - B_1' \end{array}\right\} \dots\dots\dots\dots\dots (9).$$

24. We shall now shew, that, if these six relations hold, the forces brought into play by any system of transverse vibrations constituting a single wave, are always perpendicular to the direction of propagation of the wave; but if these relations do not hold, the forces will not be perpendicular to that direction. In other words, we shall shew that *these six relations are essential to Fresnel's Theory of Transverse Vibrations.*

If we substitute in (8) the values of the C's given in (9), we find the following value of U'', viz.,

$$A_1 \alpha \frac{d}{dx}\left(\frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) + A_2\beta \frac{d}{dy}\left(\frac{d\zeta}{dz} + \frac{d\xi}{dx}\right) + A_3\gamma \frac{d}{dz}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right)$$

$$- \frac{d^2}{dydz}(B_2\eta\gamma + B_3'\zeta\beta) - \frac{d^2}{dzdx}(B_3\zeta\alpha + B_1'\xi\gamma) - \frac{d^2}{dxdy}(B_1\xi\beta + B_2'\eta\alpha),$$

and therefore (8) becomes

$$\left.\begin{aligned} \frac{d^2 v}{dt^2} = &\left(A_1\alpha\frac{d}{dx} + A_2\beta\frac{d}{dy} + A_3\gamma\frac{d}{dz}\right)\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) \\ &+ B_1\left(\alpha\frac{d}{dy} - \beta\frac{d}{dx}\right)\frac{d\xi}{dy} + B_1'\left(\alpha\frac{d}{dz} - \gamma\frac{d}{dx}\right)\frac{d\xi}{dz} \\ &+ B_2\left(\beta\frac{d}{dz} - \gamma\frac{d}{dy}\right)\frac{d\eta}{dz} + B_2'\left(\beta\frac{d}{dx} - \alpha\frac{d}{dy}\right)\frac{d\eta}{dx} \\ &+ B_3\left(\gamma\frac{d}{dx} - \alpha\frac{d}{dz}\right)\frac{d\zeta}{dx} + B_3'\left(\gamma\frac{d}{dy} - \beta\frac{d}{dx}\right)\frac{d\zeta}{dy} \end{aligned}\right\} \text{..................(10).}$$

By using the notation in Art. 8, &c., and observing that $\alpha\frac{d}{dy} - \beta\frac{d}{dx} = D\mathfrak{D}.\gamma$, $\alpha\frac{d}{dz} - \gamma\frac{d}{dx} = - D\mathfrak{D}.\beta$, &c. &c. the equation (10) becomes

$$\left.\begin{aligned} \frac{d^2 v}{dt^2} = &\left(A_1\alpha\frac{d}{dx} + A_2\beta\frac{d}{dy} + A_3\gamma\frac{d}{dz}\right)\Delta\mathfrak{D}.v \\ + D\mathfrak{D}.&\left\{\left(B_2\frac{d\eta}{dz} - B_3'\frac{d\zeta}{dy}\right)\alpha + \left(B_3\frac{d\zeta}{dx} - B_1'\frac{d\xi}{dz}\right)\beta + \left(B_1\frac{d\xi}{dy} - B_2'\frac{d\eta}{dx}\right)\gamma\right\} \end{aligned}\right\} \text{...(11).}$$

For transverse vibrations we have $\Delta\mathfrak{D}.v = 0$, and therefore,

$$\frac{d^2 v}{dt^2} = D\mathfrak{D}.\left\{\left(B_2\frac{d\eta}{dz} - B_3'\frac{d\zeta}{dy}\right)\alpha + \left(B_3\frac{d\zeta}{dx} - B_1'\frac{d\xi}{dz}\right)\beta + \left(B_1\frac{d\xi}{dy} - B_2'\frac{d\eta}{dx}\right)\gamma\right\} \text{...(12).}$$

Now $Du'.u$ is the symbol of a line perpendicular to u' and u; hence (12) indicates that the force $\frac{d^2 v}{dt^2}$ is perpendicular to the direction of $\mathfrak{D}$, and that direction, as we have seen, is the direction of propagation. It follows, therefore, that if the relations (9) hold, the forces brought into play by transverse vibrations are always perpendicular to the direction of propagation.

25. We shall now shew that this cannot be the case except the conditions (9) hold.

If the conditions (9) do not hold we must add to the second member of (12) an expression of the form

$$\frac{d^2}{dydz}(E_1\eta\gamma + E_1'\zeta\beta) + \frac{d^2}{dzdx}(E_2\zeta\alpha + E_2'\xi\gamma) + \frac{d^2}{dxdy}(E_3\xi\beta + E_3'\eta\alpha) = V, \text{ suppose}$$

$E_1\ E_1'\ E_2\ E_2'\ E_3\ E_3'$ being the unknown *corrections* to be made in the second members of the relations (9).

Performing on V the operation $\Delta\mathfrak{D}.$, we find,

$$\Delta\mathfrak{D}.V = \frac{d^2}{dydz}\left(E_1\frac{d\eta}{dz} + E_1'\frac{d\zeta}{dy}\right) + \frac{d^2}{dzdx}\left(E_2\frac{d\zeta}{dx} + E_2'\frac{d\xi}{dz}\right) + \frac{d^2}{dxdy}\left(E_3\frac{d\xi}{dy} + E_3'\frac{d\eta}{dx}\right).$$

Now, if the second member of (12) + V is perpendicular to the direction of $\mathfrak{D}$, the same must be true of V, and therefore $\Delta\mathfrak{D}\,.\,V$ must be *always* zero. Consequently $\Delta\mathfrak{D}\,.\,V$ must be zero in the *particular case* where $\zeta = 0$, and ξ and η are functions of x and y only, in which case we have

$$\Delta\mathfrak{D}\,.\,V = \frac{d^2}{dx\,dy}\left(E_3\frac{d\xi}{dy} + E_3'\frac{d\eta}{dx}\right)$$

$$= \left\{E_3\left(\frac{d}{dy}\right)^2 - E_3'\left(\frac{d}{dx}\right)^2\right\}\frac{d\xi}{dx}$$

observing that $\dfrac{d\xi}{dx} + \dfrac{d\eta}{dy} + \dfrac{d\zeta}{dz}(=0) = 0.$

That this expression should be always zero evidently requires E_3 and E_3' to be each zero; and in the same way we may shew that the other E's are each zero.

Hence, it appears, that the forces brought into play by transverse vibrations are not perpendicular to the direction of propagation, except the conditions (9) hold.

These conditions are therefore *essential* to the truth of Fresnel's *Theory of Transverse Vibrations.*

26. Hence it follows that (12) is the most general form of the equation of vibratory motion, when the transmission of a wave of transverse vibrations through the medium in every direction is possible.

27. Experiment shews that the six constants involved in (12) are reducible to three in the case of ordinary Biaxal Crystals; for it appears, that, when the plane of polarization of a ray coincides with the plane $\alpha\beta$, the velocity of transmission in the direction α is the same as that in the direction β. Now, first, let us assume with Fresnel, that the vibrations are perpendicular to the plane of polarization ($\alpha\beta$); then, for the directions α and β the equation (12) becomes, in each case respectively,

$$\frac{d^2(\gamma\zeta)}{dt^2} = D\alpha\,\frac{d}{dx}\,.\left(B_3\frac{d\zeta}{dx}\beta\right), \quad \text{or} \quad \frac{d^2\zeta}{dt^2} = B_3\frac{d^2\zeta}{dx^2},$$

$$\text{and} \quad \frac{d^2(\gamma\zeta)}{dt^2} = D\beta\,\frac{d}{dy}\,.\left(-B_3'\frac{d\zeta}{dy}\alpha\right), \quad \text{or} \quad \frac{d^2\zeta}{dt^2} = B_3'\frac{d^2\zeta}{dy^2}.$$

Hence, if c denote the common velocity of transmission in the two directions, we have

$$B_3 = B_3' = c^2.$$

In exactly the same way we may shew, that

$$B_1 = B_1' = a^2,$$

$$B_2 = B_2' = b^2,$$

where a is the velocity of transmission in the directions β and γ of a ray polarized in the plane $\beta\gamma$, and b the velocity in the directions γ and α of a ray polarized in the plane $\gamma\alpha$.

Hence the equation (12) becomes

$$\frac{d^2v}{dt^2} = D\mathfrak{D}\,.\left\{a^2\left(\gamma\frac{d}{dy} - \beta\frac{d}{dz}\right)\xi + b^2\left(\alpha\frac{d}{dz} - \gamma\frac{d}{dx}\right)\eta + c^2\left(\beta\frac{d}{dx} - \alpha\frac{d}{dy}\right)\zeta\right\},$$

$$\left.\begin{aligned} \text{or} \quad \frac{d^2v}{dt^2} &= -(D\mathfrak{D}\,.)^2(a^2\xi\alpha + b^2\eta\beta + c^2\zeta\gamma), \\ \text{or} \quad \frac{d^2v}{dt^2} &= -(D\mathfrak{D}\,.)^2(a^2\alpha\Delta\alpha + b^2\beta\Delta\beta + c^2\gamma\Delta\gamma)v, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(13).$$

If, however, we suppose that the vibrations of a polarized ray are *in* the plane of polarization. we may shew as above, that

$$B_2 = B_3' = a^2,$$
$$B_3 = B_1' = b^2,$$
$$B_1 = B_2' = c^2,$$

and therefore (12) becomes,

$$\frac{d^2 v}{dt^2} = D\mathfrak{D} . \{a^2 \left(\frac{d\eta}{dz} - \frac{d\zeta}{dy}\right) a + b^2 \left(\frac{d\zeta}{dx} - \frac{d\xi}{dz}\right) \beta + c^2 \left(\frac{d\xi}{dy} - \frac{d\eta}{dx}\right) \gamma\},$$

or $$\frac{d^2 v}{dt^2} = - D\mathfrak{D} . (a^2 a \Delta a + b^2 \beta \Delta \beta + c^2 \gamma \Delta \gamma) D\mathfrak{D} . v \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

28. Taking the equation (13), we shall now find under what circumstances the force $\frac{d^2 v}{dt^2}$ is in the direction of vibration.

Let us choose $a_{\prime}\ \beta_{\prime}\ \gamma_{\prime}$ as in Art. 10, $a_{\prime}$ being the direction of propagation, and $\beta_{\prime}$ that of vibration; and let $v = \eta_{\prime} \beta_{\prime}$. Then, as in the article just referred to, we have $\mathfrak{D} = a_{\prime} d_{a_{\prime}}$.

Now, the condition that the force $\frac{d^2 v}{dt^2}$ may be in the direction $\beta_{\prime}$ is $\Delta \gamma_{\prime} . \frac{d^2 v}{dt^2} = 0$ (for $\frac{d^2 v}{dt^2}$ is already perpendicular to $a_{\prime}$, and this condition makes it perpendicular to $\gamma_{\prime}$ likewise), or by (13).

$$\Delta \gamma_{\prime} . (D a_{\prime})^2 . (a^2 a \Delta a + b^2 \beta \Delta \beta + c^2 \gamma \Delta \gamma) \beta_{\prime} = 0.$$

But, by the general proportions of the notation D and Δ, we have $\Delta \gamma_{\prime} . D a_{\prime} . = \Delta \beta_{\prime} .$, and therefore $\Delta \gamma_{\prime} . (D a_{\prime} .)^2 = \Delta \beta_{\prime} . D a_{\prime} . = - \Delta \gamma_{\prime} .$. Hence this condition becomes

$$a^2 (\Delta a . \gamma_{\prime}) (\Delta a . \beta_{\prime}) + b^2 (\Delta \beta . \gamma_{\prime}) (\Delta \beta . \beta_{\prime}) + c^2 (\Delta \gamma . \gamma_{\prime}) (\Delta \gamma . \beta_{\prime}) = 0.$$

This is the well-known condition of Fresnel that the force brought into play by a transverse vibration may be in the direction of that vibration; for

$$\Delta a . \gamma_{\prime} = \cos (a \gamma_{\prime}) \quad \Delta a . \beta_{\prime} = \cos (a \beta_{\prime}) \text{ \&c. \&c.}$$

To find the velocity of propagation in this case, we have, performing the operation $\Delta \beta_{\prime}$ on both sides of (13),

$$\frac{d^2 \eta_{\prime}}{dt^2} = \{a^2 (\Delta a . \beta_{\prime})^2 + b^2 (\Delta \beta . \beta_{\prime})^2 + c^2 (\Delta \gamma . \beta_{\prime})^2\} d_{a_{\prime}}^{\ 2} \eta_{\prime},$$

and therefore the square of the velocity of propagation is

$$a^2 (\Delta a . \beta_{\prime})^2 + b^2 (\Delta \beta . \beta_{\prime})^2 + c^2 (\Delta \gamma . \beta_{\prime})^2,$$

which is Fresnel's expression.

29. We may treat the equation (14) in exactly the same way.

M. O'BRIEN.

UPPER NORWOOD, *April*, 1847.

XXXIX. *A Theory of the Transmission of Light through Transparent Media, and of Double Refraction, on the Hypothesis of Undulations. By the Rev.* J. CHALLIS, M.A., *Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge.*

[Read *May* 17, 1847.]

IN a former communication to this Society, I ventured to advance a new Theory of the Polarization of Light, founded on a Mathematical Theory of Luminous Rays. (*Cambridge Philosophical Transactions,* Vol. VIII. Part III. pp. 361, and 371.) As the Theory was not then applied to the phænomena of Double Refraction, I propose in this Paper to attempt to give it that extension. The course of the reasoning will require a general consideration of the transmission of light through transparent media. I shall therefore commence with this part of the subject.

1. It will be assumed that the æther is of the same uniform density and elasticity within any transparent medium as it is without; and that the diminished rate of propagation in the medium is owing to the obstacle which its atoms oppose to the free motion of the ætherial particles. Considering the proximity of the atoms to each other, and that the retarding effect of each atom at a given instant, extends through many multiples of its linear dimensions, it is presumed that the mean retardation, though resulting from the presence of discrete atoms, may be regarded as continuous. It will also be supposed that the mean effect of the presence of the atoms is to produce an apparent diminution of the elasticity of the æther, the motion in all other respects being the same as in free space. Let a = the velocity of propagation without the medium, and $\frac{a}{\mu}$ = that within. Then, ρ being the density in a line of rectilinear propagation, at a point distant by x from the origin, the effective accelerative force $= -\frac{a^2}{\mu^2}\cdot\frac{d\rho}{\rho dx}$. If there were no retarding effect of the atoms, the accelerative force would be $-\frac{a^2 d\rho}{\rho dx}$. Hence, the accelerative force of the retardation (R) is equal to $a^2\left(1-\frac{1}{\mu^2}\right)\frac{d\rho}{\rho dx}$. For this force another expression may be obtained by the following considerations. If v be the velocity of the æther at the time t at the point whose co-ordinate is x, we have by known equations,

$$v = \frac{a}{\mu}\,\text{Nap. log. } \rho = \phi\left(\frac{a}{\mu}t - x\right).$$

Now the accelerative force of the retardation at a given point must vary conjointly as the number of atoms in a given space, that is, as the density of the medium, and as the effective accelerative force of the æther at that point. Hence, K being a certain constant, and δ the density of the medium,

$$R = -K\delta\left(\frac{dv}{dt}\right) = -K\delta\frac{dv}{dt} \text{ very nearly.}$$

Consequently by the foregoing equations,

$$R = K\delta \cdot \frac{a}{\mu} \cdot \frac{dv}{dx} = K\delta \cdot \frac{a^2}{\mu^2} \cdot \frac{d\rho}{\rho dx}.$$

Comparing this expression for R with the former, we have

$$a^2\left(1 - \frac{1}{\mu^2}\right)\frac{d\rho}{\rho dx} = K\delta \cdot \frac{a^2}{\mu^2} \cdot \frac{d\rho}{\rho dx}; \text{ or } \mu^2 - 1 = K\delta.$$

2. Hitherto we have supposed the atoms of the medium to be absolutely fixed. If, as it is reasonable to suppose, they are moveable by the mechanical action of the ætherial vibrations, the retardation produced by them will differ from that obtained above. Assuming the mean effect of the presence of the atoms in this case also to be an apparent diminution of the elasticity of the æther, the accelerative force of the retardation will vary as the density of the medium and the difference of the effective accelerative forces of the æther and the atoms at a given position. That is, if v' be the velocity of an atom, where the velocity of the vibrating æther is v, we shall have $R = -K\delta\left(\frac{dv}{dt} - \frac{dv'}{dt}\right)$, very nearly. And, as before, $R = a^2\left(1 - \frac{1}{\mu^2}\right)\frac{d\rho}{\rho dx} = -(\mu^2 - 1)\frac{dv}{dt}$. Hence, putting q for the ratio of $\frac{dv'}{dt}$ to $\frac{dv}{dt}$, it follows that $\mu^2 - 1 = K\delta(1 - q)$.

3. Since the retardation will be less and the velocity of propagation greater when the atoms are moved than when they are fixed, μ will be less in the former case than in the latter, and consequently q is a positive quantity. As it is known from experience that the rate of propagation of light in a given direction in a medium, is uniform and independent of the intensity of the light, the ratio of $\frac{dv'}{dt}$ to $\frac{dv}{dt}$ must be the same at different points of the same wave, and the same also for vibrations of different magnitudes, if the breadths of the waves be given. But to account for the phænomenon of dispersion, q must be a function of λ the breadth of the wave. For our present enquiry it is not necessary to ascertain the form of this function. It is only necessary to assume that in crystallized media q is different for different *directions*. The theoretical reason for this probably is, that the retardation depends on the *elasticity* of the medium, and that the elasticity of crystallized media, and consequently the mobility of their particles, depends on the direction.

4. What has been said above respecting the transmission of light through transparent media, will suffice for the consideration of the theory of Double Refraction, on which I am now about to enter. It will be assumed that in any medium which does not retard the progression of the luminous rays equally in all directions, there are at least three directions at right angles to each other, in which the retardation will take place in the manner hitherto supposed. Let $a_{\prime}^2$, $b_{\prime}^2$, $c_{\prime}^2$ be the constants of elasticity for *plane* waves in these three directions, and let a be the velocity of the waves in free space. Then, q_1, q_2, q_3, being the values of q for the same directions, the time of vibration being given, we have,

$$\frac{a^2}{a_{\prime}^2} = 1 + K\delta(1 - q_1), \quad \frac{a^2}{b_{\prime}^2} = 1 + K\delta(1 - q_2), \quad \frac{a^2}{c_{\prime}^2} = 1 + K\delta(1 - q_3).$$

5. When an atom of the medium is displaced in one of the three rectangular directions above mentioned, the direction of displacement coincides by hypothesis with the line of propagation of the waves. Although in general this will not be the case, waves may still be propagated in all directions in the medium. For supposing plane waves of given breadth to be propagated simultaneously in the three rectangular directions, (which may be called the axes of elasticity,) the resulting effect on a

given particle of the æther, according to the principle of the coexistence of small vibrations, may be a vibration in a certain resulting direction, of the same period as that of the component vibrations. Consequently waves which would produce the same vibration of the ætherial particle may be propagated in that direction. But the displacement of the *atoms of the medium* does not necessarily take place in the same direction. If this displacement be resolved in two directions, one coinciding with the direction of vibration of the ætherial particles, and the other perpendicular to this, the resolved part of the displacement in the latter direction, will give rise to ætherial vibrations which will be propagated laterally and produce no sensation of light. With reference to phænomena of light, the other part alone requires to be taken account of. The above considerations will enable us to determine the effective elasticity in any direction in the medium, in terms of the elasticities in the directions of the axes.

6. Let v be the velocity of a particle of the æther, the vibrations of which are due to waves propagated in a direction making angles α, β, γ, with the axes of elasticity; and let v' be the resolved part in that direction of the velocity of an atom of the medium situated where the velocity of the æther is v. Then by Art. 2, the accelerative force of the retardation is equal to

$$-K\delta\left(\frac{dv}{dt}-\frac{dv'}{dt}\right),\quad \text{or}\quad -K\delta(1-q)\frac{dv}{dt}.$$

If now the velocity v be resolved in the directions of the axes, the accelerative forces of retardation corresponding to the resolved parts of the velocity will be,

$$-K\delta(1-q_1)\cos\alpha\frac{dv}{dt},\quad -K\delta(1-q_2)\cos\beta\frac{dv}{dt},\quad -K\delta(1-q_3)\cos\gamma\frac{dv}{dt}.$$

And by the considerations in Art. 5, the accelerative force of the retardation in the given direction of propagation, is equal to the resultant of these forces. Hence

$$-K\delta(1-q)\frac{dv}{dt}=-K\delta\frac{dv}{dt}\cdot\{(1-q_1)\cos^2\alpha+(1-q_2)\cos^2\beta+(1-q_3)\cos^2\gamma\}.$$

Let now r^2 be the constant of elasticity in the direction of propagation. Then by the equations in Art. 4, we have,

$$\frac{a^2}{r^2}-1=K\delta(1-q),\quad \frac{a^2}{a_{,}^2}-1=K\delta(1-q_1),\quad \frac{a^2}{b_{,}^2}-1=K\delta(1-q_2),\quad \frac{a^2}{c_{,}^2}-1=K\delta(1-q_3),$$

Hence, by substitution in the foregoing equation,

$$\frac{a^2}{r^2}-1=\left(\frac{a^2}{a_{,}^2}-1\right)\cos^2\alpha+\left(\frac{a^2}{b_{,}^2}-1\right)\cos^2\beta+\left(\frac{a^2}{c_{,}^2}-1\right)\cos^2\gamma.$$

Consequently,

$$\frac{1}{r^2}=\frac{\cos^2\alpha}{a_{,}^2}+\frac{\cos^2\beta}{b_{,}^2}+\frac{\cos^2\gamma}{c_{,}^2}.$$

The surface of which this is the equation in polar co-ordinates, may be called the surface of elasticity. It is evidently that of an ellipsoid. The radius vector r, represents the velocity of propagation of *plane* waves in any direction coinciding with that of r

7. We have now to find the velocity of propagation in a *filament* of the æther corresponding to a *ray* of light. In considering the motion in a filament of a medium the elasticity of which varies with the direction, I shall proceed in a method analogous to that employed in my Paper on Luminous Rays. (*Camb. Phil. Trans.* Vol. VIII. Part III. p. 365). It will be supposed that in the filament there is an axis of no transverse velocity. This is taken for the axis of z. The

condensation at any point of the filament is assumed to be $\phi_{,}(z, t) \times f(x, y)$, which for shortness sake, will be written $\phi_{,}f$, $\phi_{,}$ being treated as a function of z and t only, and f as a function of x and y only. Let ρ be the density, and u, v, w the components of the velocity in the directions of the axes of co-ordinates, at the point xyz, and at the time t. Also let a'^2, b'^2, c'^2 be the coefficients of elasticity in the directions of the axes of x, y, z respectively. First powers only of the velocities u, v, w, and of the condensation $\rho - 1$ will be taken account of. This being premised, we have,

$$\left(\frac{du}{dt}\right) = -\frac{a'^2 d\rho}{\rho dx} = -\frac{a'^2\phi_{,}}{\rho} \cdot \frac{df}{dx};$$

and to the first approximation, $\dfrac{du}{dt} = -a'^2\phi_{,}\dfrac{df}{dx}$. Hence,

$$u = -a'^2 \frac{df}{dx} \int \phi_{,} dt + C,$$

the arbitrary quantity c being in general a function of x, y, and z. So also

$$v = -b'^2 \frac{df}{dy} \int \phi\, dt + C'.$$

Again, since $\left(\dfrac{dw}{dt}\right) = -\dfrac{c'^2 d\rho}{\rho dz}$, we have to the same degree of approximation,

$$\frac{dw}{dt} = -c'^2 f \cdot \frac{d\phi_{,}}{dz}, \quad \text{and } w = -c'^2 f \int \frac{d\phi_{,}}{dz} dt + C'' = -c'^2 f \frac{d\int\phi_{,}dt}{dz} + C''.$$

But from the supposed law of condensation in any plane perpendicular to the axis of z, it follows, that the accelerative force parallel to this axis at any point of the plane, must to the first degree of approximation, be equal to $f \times$ the accelerative force at the point of intersection with the axis, and the corresponding velocities must be in the same proportion. Hence, $\dfrac{d\phi}{dz}$ being the velocity at the point of intersection with the axis, we shall have

$$w = f\frac{d\phi}{dz}. \quad \text{Consequently } -c'^2 \int \phi_{,} dt = \phi, \text{ and } C'' = 0.$$

Assuming now that C and C' are each equal to zero when $\phi = 0$, we obtain,

$$u = \frac{a'^2}{c'^2} \cdot \phi \frac{df}{dx}, \quad \text{and } v = \frac{b'^2}{c'^2} \cdot \phi \frac{df}{dy}. \quad \text{Hence,}$$

$$u\,dx + v\,dy + w\,dz = \frac{a'^2}{c'^2} \cdot \phi \frac{df}{dx} dx + \frac{b'^2}{c'^2} \cdot \phi \frac{df}{dy} dy + f\frac{d\phi}{dz} dz.$$

In this case $u\,dx + v\,dy + w\,dz$ is not an exact differential. Let $\dfrac{a'^2}{c'^2} = h$, and $\dfrac{b'^2}{c'^2} = l$, and suppose that $f = F_1^{\frac{1}{h}} \cdot F_2^{\frac{1}{l}}$, the function F_1 containing x only, and the function F_2 containing y only. By this supposition, a factor which will render the above quantity an exact differential may be found, which, though not the most general, will suffice for our present purpose. By differentiating,

$$\frac{df}{fdx} = \frac{1}{hF_1} \cdot \frac{dF_1}{dx}, \quad \frac{df}{fdy} = \frac{1}{lF_2} \cdot \frac{dF_2}{dy}.$$

Hence, $u\,dx + v\,dy + w\,dz = \dfrac{\phi f}{F_1}\cdot\dfrac{dF_1}{dx}dx + \dfrac{\phi f}{F_2}\cdot\dfrac{dF_2}{dy}dy + f\dfrac{d\phi}{dz}dz$

$$= F_1^{\frac{1}{h}-1}\,F_2^{\frac{1}{l}-1}\left\{\left(F_2\frac{dF_1}{dx}dx + F_1\frac{dF_2}{dy}dy\right)\phi + F_1F_2\frac{d\phi}{dz}dz\right\}$$

$$= F_1^{\frac{1}{h}-1}\,F_2^{\frac{1}{l}-1}.d.F_1F_2\phi.$$

Consequently the required factor is $F_1^{1-\frac{1}{h}}.F_2^{1-\frac{1}{l}}$; and the differential equation of the surface cutting at right angles the directions of the motion, is $d.F_1F_2\phi = 0$. If $\psi = 0$, be the equation of this surface, we have $\psi = F_1F_2\phi +$ a function of t. We may now proceed to find a value of $\dfrac{1}{R}+\dfrac{1}{R'}$, the sum of the reciprocals of the principal radii of curvature of the surface at any point, by substituting in the general expression for $\dfrac{1}{R}+\dfrac{1}{R'}$, viz.,

$$\left\{\frac{d\psi^2}{dx^2}+\frac{d\psi^2}{dy^2}+\frac{d\psi^2}{dz^2}\right\}^{-\frac{3}{2}} \times \left\{\begin{array}{l}\left(\dfrac{d^2\psi}{dx^2}+\dfrac{d^2\psi}{dy^2}+\dfrac{d^2\psi}{dz^2}\right)\left(\dfrac{d\psi^2}{dx^2}+\dfrac{d\psi^2}{dy^2}+\dfrac{d\psi^2}{dz^2}\right) - \dfrac{d^2\psi}{dx^2}.\dfrac{d\psi^2}{dx^2} - \dfrac{d\psi}{dy}.\dfrac{d\psi^2}{dy^2} \\ - \dfrac{d^2\psi}{dz^2}.\dfrac{d\psi^2}{dz^2} - 2\dfrac{d^2\psi}{dxdy}.\dfrac{d\psi}{dx}\dfrac{d\psi}{dy} - 2\dfrac{d^2\psi}{dxdz}.\dfrac{d\psi}{dx}.\dfrac{d\psi}{dz} - 2.\dfrac{d^2\psi}{dydz}.\dfrac{d\psi}{dy}.\dfrac{d\psi}{dz}\end{array}\right\}.$$

Now $\dfrac{d\psi}{dx} = F_1^{1-\frac{1}{h}}.F_2^{1-\frac{1}{l}}.u$, and $\dfrac{d\psi}{dy} = F_1^{1-\frac{1}{h}}.F_2^{1-\frac{1}{l}}.v$; and therefore $\dfrac{d\psi}{dx}=0$ if $u=0$, and $\dfrac{d\psi}{dy}=0$ if $v=0$. As we shall require the value of $\dfrac{1}{R}+\dfrac{1}{R'}$ only for points where $u=0$ and $v=0$, we shall suppose in the general expression that $\dfrac{d\psi}{dx}=0$ and $\dfrac{d\psi}{dy}=0$. Hence

$$\frac{1}{R}+\frac{1}{R'} = \frac{\dfrac{d^2\psi}{dx^2}+\dfrac{d^2\psi}{dy^2}}{\dfrac{d\psi}{dz}} = \frac{\phi F_2.\dfrac{d^2F_1}{dx^2}+\phi F_1.\dfrac{d^2F_2}{dy^2}}{F_1F_2\dfrac{d\phi}{dz}}.$$

Taking now the equation (3) obtained in page 365 of the Paper on Luminous Rays, and supposing it to apply to any point of the plane perpendicular to the axis of z in which $u=0$ and $v=0$, we shall have, neglecting small terms,

$$\frac{d^2.f\phi}{dt^2} - c'^2.\frac{d^2.f\phi}{dz^2} - c'^2.\frac{d.f\phi}{dz}.\left(\frac{1}{R}+\frac{1}{R'}\right) = 0.$$

That this equation may be of the form $\dfrac{d^2.f\phi}{dt^2} - c''^2.\dfrac{d^2.f\phi}{dz^2} = 0$, which, for the reasons given in the Paper just cited, it is required to be, we must have

$$c'^2.\frac{d.f\phi}{dz}\left(\frac{1}{R}+\frac{1}{R'}\right) = (c''^2 - c'^2)\frac{d^2.f\phi}{dz^2}$$

$$= c'^2k\frac{d^2.f\phi}{dz^2},\ \text{if } c''^2 = c'^2(1+k).$$

Hence, $\frac{d\phi}{dz} \cdot \left(\frac{1}{R} + \frac{1}{R'}\right) = k \frac{d^2\phi}{dz^2}$: and by substituting the value of $\frac{1}{R} + \frac{1}{R'}$, obtained above,

$$k \cdot \frac{d^2\phi}{dz^2} = \phi \left(\frac{d^2 F_1}{F_1 dx^2} + \frac{d^2 F_2}{F_2 dy^2}\right).$$

For a point on the axis of z this equation becomes $\frac{d^2\phi}{dz^2} + n^2\phi = 0$, the constant n^2 being such that if λ be the breadth of the waves, $n = \frac{2\pi}{\lambda}$. Hence, substituting $-n^2\phi$ for $\frac{d^2\phi}{dz^2}$ in the foregoing equation, the result is

$$\frac{d^2 F_1}{F_1 dx^2} + \frac{d^2 F_2}{F_2 dy^2} + kn^2 = 0.$$

By taking account of the equality $f = F_1^{\frac{1}{h}} \cdot F_2^{\frac{1}{l}}$, we obtain by substitution in the above equation,

$$h \cdot \frac{d^2 f}{dx^2} + l \cdot \frac{d^2 f}{dy^2} + \frac{h \cdot (h-1)}{f} \cdot \frac{df^2}{dx^2} + \frac{l \cdot (l-1)}{f} \cdot \frac{df^2}{dy^2} + kn^2 f = 0.$$

If now for the same reasons as those given in p. 369 of the Paper on Luminous Rays, the terms involving $\frac{df^2}{f dx^2}$ and $\frac{df^2}{f dy^2}$ be neglected, we have, finally,

$$h \cdot \frac{d^2 f}{dx^2} + l \cdot \frac{d^2 f}{dy^2} + kn^2 f = 0.$$

The general result from this course of reasoning is, that a ray of which the condensation in the transverse direction is defined by a function of x and y, which satisfies this equation, may be propagated in a medium whose elasticity varies with the direction of propagation. The reasoning, however, only applies to a function of x and y, which is the product of a function of x and a function of y. It is evident that f cannot be a function of $x^2 + y^2$, and, consequently, that the ray cannot be one of common light.

8. It is found by experience that a *polarized* ray may be transmitted in certain transparent crystallized media. I shall assume that in these media the retardation of the propagation produced by the presence and inertia of the atoms, is such as corresponds to an apparent diminution of the elasticity of the æther, different in degree in different directions. I shall assume also that there are three rectangular axes of elasticity, and that, in accordance with the result contained in Art. 8. of this Paper, the surface of elasticity is an ellipsoid. On these suppositions the ray cannot be one of common light, because the effective elasticity is different in different directions transverse to the axis of the ray. But the suppositions are consistent with the transmission of a polarized ray. For according to the Theory of Polarization contained in my Paper in the *Camb. Phil. Transactions*, (Vol. VIII. Part III. p. 372), the condensations for a polarized ray must be disposed symmetrically with reference to two planes at right angles to each other passing through the axis of the ray. Consequently the force of retardation and the effective elasticity must act symmetrically with reference to two such planes. And this will evidently be the case: for any section through the centre of the surface of elasticity is an ellipse, the radii of which drawn from its centre, are symmetrically disposed with reference to its axes. It is possible that the function f for a polarized ray may be such as that supposed in the preceding Article, namely, the product of a function of x and a function of y. All, however, that can be affirmed respecting this function from the reasoning in the Paper above referred to is, that for small distances from the axis of the ray, it is a function of one co-ordinate only, the axis of x and y

being supposed to be in the planes of symmetry. Let, therefore, in the last obtained equation, f be a function of x only. Then,

$$\frac{d^2f}{dx^2} + \frac{kn}{h} f = 0.$$

Now in the Paper on the Polarization of Light, (p. 373), the particular value of f for a polarized ray was found to be $\cos n\sqrt{k}x$. By substituting this value in the equation above, we obtain the equation of condition $h = 1$, or $a'^2 = c'^2$. It would appear therefore that a polarized ray cannot be transmitted in the medium, the transverse elasticity being different from that in the direction of propagation, *if the velocity of propagation really be* c'', or $c'\sqrt{1+k}$. For the transmission of the polarized ray it is necessary to suppose an alteration of the rate of propagation. This may be conceived to take place as follows: First, suppose $h = 1$, and a polarized ray in which the breadth of the waves is λ, or $\frac{2\pi}{n}$, to be transmitted with the velocity $c'\sqrt{1+k}$. Then suppose the elasticity in the direction of the plane of polarization to be altered from c'^2 to a'^2, and a polarized ray to be still propagated. By hypothesis the nature of the medium is such as to allow of this taking place. Now as f, and consequently the transverse section of the ray, do not alter by the supposed change of elasticity, the only way in which the condensation can be altered is by a change of λ. The time of vibration of a given ætherial particle remaining constant, the rate of propagation will be altered in the same ratio. Let therefore $f = \cos n'\sqrt{k}x$, and let λ' be the new value of λ. By substitution in the foregoing equation, we obtain the equation of condition

$$n'^2 = \frac{n^2}{h} = \frac{n^2 c'^2}{a'^2}. \quad \text{Hence } \frac{n'}{n}, \text{ or } \frac{\lambda}{\lambda'} = \frac{c'}{a'};$$

and the velocity of propagation

$$= c'\sqrt{1+k} \times \frac{\lambda'}{\lambda} = c'\sqrt{1+k} \times \frac{a'}{c'} = a'\sqrt{1+k}.$$

The foregoing reasoning involves the inference that the rate of propagation of a ray in a medium is not solely due to the effective elasticity in the direction of its axis, but is affected also by the circumstance that the medium is incapable of transmitting any but a *polarized* ray, and that for such a ray k is a constant.

9. We are now prepared to find the equation of a surface, the radius-vector of which drawn in any direction from a fixed point, shall represent the velocity of propagation of a ray in that direction. As we found the velocity of propagation to be that due to the elasticity in the direction of a line drawn perpendicular to the axis of the ray in the plane of polarization, the process will evidently be the following. Cut the surface of the ellipsoid of elasticity by a plane perpendicular to the direction of propagation. The semi-axes of the section will be the radius-vectors in that direction of the surface required. Let a, b, c be the semi-axes of the ellipsoid. Its equation in rectangular co-ordinates referred to the axes and the centre will be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Let the directions of the rectangular axes be changed by substituting for x, y, and z the following values:

$$\begin{aligned} x &= \alpha x' + \beta y' + \gamma z', \\ y &= \alpha' x' + \beta' y' + \gamma' z', \\ z &= \alpha'' x' + \beta'' y' + \gamma'' z', \end{aligned}$$

and in the result make $z' = 0$, in order to obtain the equation of the section. This equation will thus become

$$\frac{(\alpha x' + \beta y')^2}{a^2} + \frac{(\alpha' x' + \beta' y')^2}{b^2} + \frac{(\alpha'' x' + \beta'' y')^2}{c^2} = 1.$$

Supposing in this equation x' and y' to be referred to the axes of the section, and r, r', to be the two semi-axes, we shall have

$$\frac{1}{r^2} = \frac{\alpha^2}{a^2} + \frac{\alpha'^2}{b^2} + \frac{\alpha''^2}{c^2},$$

$$\frac{1}{r'^2} = \frac{\beta^2}{a^2} + \frac{\beta'^2}{b^2} + \frac{\beta''^2}{c^2},$$

$$\text{and } 0 = \frac{\alpha\beta}{a^2} + \frac{\alpha'\beta'}{b^2} + \frac{\alpha''\beta''}{c^2}.$$

The equation of the surface, the radius-vectors of which in a given direction are r and r', is consequently the following:

$$\left(\frac{1}{r^2} - \frac{\alpha^2}{a^2} - \frac{\alpha'^2}{b^2} - \frac{\alpha''^2}{c^2}\right)\left(\frac{1}{r^2} - \frac{\beta^2}{a^2} - \frac{\beta'^2}{b^2} - \frac{\beta''^2}{c^2}\right) = 0,$$

$$\text{or } \frac{1}{r^4} - \frac{1}{r^2}\left(\frac{\alpha^2 + \beta^2}{a^2} + \frac{\alpha'^2 + \beta'^2}{b^2} + \frac{\alpha''^2 + \beta''^2}{c^2}\right) + \frac{\alpha^2\beta^2}{a^4} + \frac{\alpha'^2\beta'^2}{b^4} + \frac{\alpha''^2\beta''^2}{c^4} + \frac{\alpha^2\beta'^2 + \alpha'^2\beta^2}{a^2 b^2}$$
$$+ \frac{\alpha^2\beta''^2 + \alpha''^2\beta^2}{a^2c^2} + \frac{\alpha'^2\beta''^2 + \alpha''^2\beta'^2}{b^2c^2} = 0.$$

By combining with this the equations

$$\alpha^2 + \beta^2 + \gamma^2 = 1,$$
$$\alpha'^2 + \beta'^2 + \gamma'^2 = 1,$$
$$\alpha''^2 + \beta''^2 + \gamma''^2 = 1,$$
$$\left(\frac{\alpha\beta}{a^2} + \frac{\alpha'\beta'}{b^2} + \frac{\alpha''\beta''}{c^2}\right)^2 = 0,$$

we obtain,

$$\frac{1}{r^4} - \frac{1}{r^2}\left(\frac{1-\gamma^2}{a^2} + \frac{1-\gamma'^2}{b^2} + \frac{1-\gamma''^2}{c^2}\right) + \frac{(\alpha\beta' - \alpha'\beta)^2}{a^2b^2} + \frac{(\alpha\beta'' - \alpha''\beta)^2}{a^2c^2} + \frac{(\alpha'\beta'' - \alpha''\beta')^2}{b^2c^2} = 0.$$

Again, from the equations

$$\alpha^2 + \alpha'^2 + \alpha''^2 = 1,$$
$$\beta^2 + \beta'^2 + \beta''^2 = 1,$$
$$\gamma^2 + \gamma'^2 + \gamma''^2 = 1,$$
$$\alpha\beta + \alpha'\beta' + \alpha''\beta'' = 0,$$

we have,

$$\alpha^2\beta^2 + \alpha'^2\beta'^2 + 2\alpha\beta\alpha'\beta' = \alpha''^2\beta''^2 = (1 - \alpha^2 - \alpha'^2)(1 - \beta^2 - \beta'^2)$$
$$= 1 - \alpha^2 - \beta^2 - \alpha'^2 - \beta'^2 + \alpha^2\beta^2 + \alpha'^2\beta'^2 + \alpha^2\beta'^2 + \alpha'^2\beta^2.$$

Hence, $0 = \gamma^2 - \alpha'^2 - \beta'^2 + (\alpha\beta' - \beta\alpha')^2$;

or, $(\alpha\beta' - \beta\alpha')^2 = \alpha'^2 + \beta'^2 - \gamma^2 = 1 - \gamma'^2 - \gamma^2 = \gamma''^2$;

so $(\alpha\beta'' - \alpha''\beta)^2 = \gamma'^2$, and $(\alpha'\beta'' - \alpha''\beta')^2 = \gamma^2$.

The equation consequently becomes

$$\frac{1}{r^4} - \frac{1}{r^2}\left(\frac{1-\gamma^2}{a^2} + \frac{1-\gamma'^2}{b^2} + \frac{1-\gamma''^2}{c^2}\right) + \frac{\gamma''^2}{a^2b^2} + \frac{\gamma'^2}{a^2c^2} + \frac{\gamma^2}{b^2c^2} = 0.$$

Transforming it into rectangular co-ordinates by putting x^2 for $r^2\gamma^2$, y^2 for $r^2\gamma'^2$, z^2 for $r^2\gamma''^2$, and $x^2 + y^2 + z^2$ for r^2, there results

$$1 - \frac{y^2+z^2}{a^2} - \frac{x^2+z}{b^2} - \frac{x^2+y^2}{c^2} + (x^2+y^2+z^2)\left(\frac{z^2}{a^2b^2} + \frac{y^2}{a^2c^2} + \frac{x^2}{b^2c^2}\right) = 0;$$

or, $$(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2) - a^2b^2(x^2+y^2) - a^2c^2(x^2+z^2) - b^2c^2(y^2+z^2) + a^2b^2c^2 = 0.$$

This is the equation of the wave surface in Fresnel's Theory of Double Refraction. It is very remarkable that principles and reasoning so widely different from those of that Theory should have led to the same result. It is needless to go farther in the investigation, as all subsequent deductions may be made in the same manner as in the received Theory.

10. In conclusion, I beg leave to refer to an objection which may be raised against the Theory of Polarization which I have brought forward. It may be urged, that as a wave is conceived in this Theory to be composed of a vast number of rays in the same phase of vibration, the transverse vibrations of the different rays will mutually destroy each other, leaving only the direct vibrations, which by hypothesis do not produce the sensation of light. To this it may be replied, that it is only the *axis* of a ray which can be considered as subject to the law of refraction; for the motion of the ætherial particles along the axis is rectilinear, and coincident in direction with the line of propagation, while at every other part of the ray the direction of the motion of a given particle is continually varying, and is generally not coincident with the line of propagation. Admitting the independent motion of each ray, it is possible that by refraction through the eye, the directions of the axes of different rays may be brought to pass nearly through the same point of the retina, in obedience to the common law of refraction, while the separate rays, not being subject in other parts to this law, may not be altered as to the diameters of their transverse sections. The constancy of the transverse section is, in fact, a necessary consequence of a supposition already made in the course of this Theory, namely, that the quantity k is a fixed numerical quantity, the same for rays propagated in media as for rays propagated in free space. As, however, I am not at present provided with the means of ascertaining the nature and value of that quantity, this part of the subject must be considered as open to further inquiry.

J. CHALLIS.

Cambridge Observatory,
May 17, 1847.

XL. *On the Critical Values of the Sums of Periodic Series.* By G. G. STOKES, M.A., *Fellow of Pembroke College, Cambridge.*

[Read *December* 6, 1847.]

THERE are a great many problems in Heat, Electricity, Fluid Motion, &c., the solution of which is effected by developing an arbitrary function, either in a series or in an integral, by means of functions of known form. The first example of the systematic employment of this method is to be found in Fourier's *Theory of Heat.* The theory of such developements has since become an important branch of pure mathematics.

Among the various series by which an arbitrary function $f(x)$ can be expressed within certain limits, as 0 and a, of the variable x, may particularly be mentioned the series which proceeds according to sines of $\frac{\pi x}{a}$ and its multiples, and that which proceeds according to cosines of the same angles. It has been rigorously demonstrated that an arbitrary, but finite function of x, $f(x)$, may be expanded in either of these series. The function is not restricted to be continuous in the interval, that is to say, it may pass abruptly from one finite value to another; nor is either the function or its derivative restricted to vanish at the limits 0 and a. Although however the *possibility* of the expansion of an arbitrary function in a series of sines, for instance, when the function does not vanish at the limits 0 and a, cannot but have been contemplated, the *utility* of this form of expansion has hitherto, so far as I am aware, been considered to depend on the actual evanescence of the function at those limits. In fact, if the conditions of the problem require that $f(0)$ and $f(a)$ be equal to zero, it has been considered that these conditions were satisfied by selecting the form of expansion referred to. The chief object of the following paper is to develope the principles according to which the expansion of an arbitrary function is to be treated when the conditions at the limits which determine the particular form of the expansion are apparently violated; and to shew, by examples, the advantage that frequently results from the employment of the series in such cases.

In Section I. I have begun by proving the possibility of the expansion of an arbitrary function in a series of sines. Two methods have been principally employed, at least in the simpler cases, in demonstrating the possibility of such expansions. One, which is that employed by Poisson, consists in considering the series as the limit of another formed from it by multiplying its terms by the ascending powers of a quantity infinitely little less than 1; the other consists in summing the series to n terms, that is, expressing the sum by a definite integral, and then considering the limit to which the sum tends when n becomes infinite. The latter method certainly appears the more direct, whenever the summation to n terms can be effected, which however is not always the case; but the former has this in its favour, that it is thus that the series present themselves in physical problems. The former is the method which I have followed, as being that which I employed when I first began the following investigations, and accordingly that which best harmonizes with the rest of the paper. I should hardly have ventured to bring a somewhat modified proof of a well-known theorem before the notice of this Society, were it not for the doubts which some mathematicians seem to feel on this subject, and because there are some points which Poisson does not seem to have treated with sufficient detail.

I have next shewn how the existence and nature of the discontinuity of $f(x)$ and its derivatives may be ascertained merely from the series, whether of sines or cosines, in which $f(x)$ is developed, even though the summation of the series cannot be effected. I have also given formulæ for obtaining the developements of the derivatives of $f(x)$ from that of $f(x)$ itself. These developements cannot in general be obtained by the immediate differentiation of the several terms of the developement of $f(x)$, or in other words by differentiating under the sign of summation.

It is usual to restrict the expanded function to be finite. This restriction however is not necesssary, as is shewn towards the end of the section. It is sufficient that the integral of the function be finite.

Section II. contains formulæ applicable to the integrals which replace the series considered in Section I. when the extent a of the variable throughout which the function is considered is supposed to become infinite.

Section III. contains some general considerations respecting series and integrals, with reference especially to the discontinuity of the functions which they express. Some of the results obtained in this section are referred to by anticipation in Sections I. and II. They could not well be introduced in their place without too much interrupting the continuity of the subject.

Section IV. consists of examples of the application of the preceding results. These examples are all taken from physical problems, which in fact afford the best illustrations of the application of periodic series and integrals. Some of the problems considered are interesting on their own account, others, only as applications of mathematical processes. It would be unnecessary here to enumerate these problems, which will be found in their proper place. It will be sufficient to make one or two remarks.

The problem considered in Art. 52., which is that of determining the potential due to an electrical point in the interior of a hollow conducting rectangular parallelepiped, and to the electricity induced on the surface, is given more for the sake of the artifice by which it is solved than as illustrating the methods of this paper. The more obvious mode of solving this problem would lead to a very complicated result.

The problem solved in Art. 54. affords perhaps the best example of the utility of the methods given in this paper. The problem consists in determining the motion of a fluid within the sector of a cylinder, which is made to oscillate about its axis, or a line parallel to its axis. The expression for the moment of inertia of the fluid which would be obtained by the methods generally employed in the solution of such problems is a definite integral, the numerical calculation of which would be very laborious; whereas the expression obtained by the method of this paper is an infinite series, which may be summed, to a sufficient degree of approximation, without much trouble.

The series for the developement of an arbitrary function considered in this paper are two, a series of sines and a series of cosines, together with the corresponding integrals; but similar methods may be applied in other cases. I believe that the following statement will be found to embrace the cases to which the method will apply.

Let u be a continuous function of any number of independent variables, which is considered for values of the variables lying within certain limits. For facility of explanation, suppose u a function of the rectangular co-ordinates x, y, z, or of x, y, z and t, where t is the time, and suppose that u is considered for values of x, y, z, t lying between 0 and a, 0 and b, 0 and c, 0 and T, respectively. For such values suppose that u satisfies a linear partial differential equation, and suppose it to satisfy certain linear equations of condition for the limiting values of the variables. Let $U = 0$, $U' = 0$ be two of the equations of condition, corresponding to the two limiting values of one of the variables, as x. Then the expansion of u to which these equations lead may be applied to the more general problem which leads to the corresponding equations of condition $U = F$, $U' = F'$, where F and F' are any functions of all the variables except x, or of any number of them.

SECTION I.

Mode of ascertaining the nature of the discontinuity of a function which is expanded in a series of sines or cosines, and of obtaining the developements of the derived functions.

1. By the term *function* I understand in this paper a quantity whose value depends in any manner on the value of the variable, or on the values of the several variables of which it is composed. Thus the functions considered need not be such as admit of being expressed by any combination of algebraical symbols, even between limits of the variables ever so close. I shall assume the ordinary rules of the differential and integral calculus as applicable to such functions, supposing those rules to have been established by the method of limits, which does not in the least require the possibility of the algebraical expression of the functions considered.

The term *discontinuous*, as applied to a function of a single variable, has been used in two totally different senses. Sometimes a function is called discontinuous when its algebraical expression for values of the variable lying between certain limits is different from its algebraical expression for values of the variable lying between other limits. Sometimes a function of x, $f(x)$, is called continuous when, for all values of x, the difference between $f(x)$ and $f(x \pm h)$ can be made smaller than any assignable quantity by sufficiently diminishing h, and in the contrary case discontinuous. If $f(x)$ can become infinite for a finite value of x, it will be convenient to consider it as discontinuous according to the second definition. It is easy to see that a function may be discontinuous in the first sense and continuous in the second, and *vice versâ*. The second is the sense in which the term *discontinuous* is I believe generally employed in treatises on the differential calculus which proceed according to the method of limits, and is the sense in which I shall use the term in this paper. The terms continuous and discontinuous might be applied in either of the above senses to functions of two or more independent variables. If I have occasion to employ them as applied to such a function, I shall employ them in the second sense; but for the present I shall consider only functions of one independent variable.

In the case of the functions considered in this paper, the value of the variable is usually supposed to be restricted to lie within certain limits, as will presently appear. I exclude from consideration all functions which either become infinite themselves, or have any of their differential coefficients of the orders considered becoming infinite, within the limits of the variable within which the function is considered, or at the limits themselves, except when the contrary is expressly stated. Thus in an investigation into which $f(x)$ and its first n differential coefficients enter, and in which $f(x)$ is considered between the limits $x = 0$ and $x = a$, those functions are excluded, at least at first, which are such that any one of the quantities $f(x)$, $f'(x) \ldots f^{n}(x)$ is infinite for a value of x lying between 0 and a, or for $x = 0$ or $x = a$; but the differential coefficients of the higher orders may become infinite. The quantities $f(x)$, $f'(x) \ldots f^{n}(x)$ may however alter discontinuously between the limits $x = 0$ and $x = a$, but I exclude from consideration all functions which are such that any one of the above quantities alters discontinuously an infinite number of times between the limits within which x is supposed to lie.

The terms *convergent* and *divergent*, as applied to infinite series, will be used in this paper in their usual sense; that is to say, a series will be called convergent when the sum to n terms approaches a finite and unique limit as n increases beyond all limit, and divergent in the contrary case. Series such as $1 - 1 + 1 - \ldots$, $\sin x + \sin 2x + \sin 3x + \ldots$, (where x is supposed not to be 0 or a multiple of π,) will come under the class divergent; for, although the sum to n terms does not increase beyond all limit, it does not approach a unique limit as n increases beyond all limit. Of course the first n terms of a divergent series may be the limits of those of a convergent series: nor does it appear possible to invent a series so rapidly divergent that it shall not be possible to find a convergent series which shall have for the limits of its first n terms the first n terms respectively of the divergent series. Of course we may employ a divergent series merely as an abbreviated mode of expressing the limit of the sum of a convergent series. Whenever a divergent series is employed in this way in the present paper, it will be expressly stated that the series is so regarded.

Convergent series may be divided into two classes, according as the series resulting from taking all the terms of the given series positively is convergent or divergent. It will be convenient for the purposes of the present paper to have names for these two classes. I shall accordingly call series belonging to the first class *essentially convergent*, and series belonging to the second *accidentally convergent*, while the term *convergent*, simply, will be used to include both classes. Thus, according to the definitions which will be employed in this paper, the series

$$x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \dots$$

is essentially convergent so long as $x^2 < 1$; it is divergent when $x^2 > 1$, and when $x = 1$; and it is accidentally convergent when $x = -1$.

The same definitions may be applied to integrals, when one at least of the limits of integration is ∞. Thus, if $a > 0$, $\int_a^\infty x^{-2}dx$ is essentially convergent at the limit ∞, while $\int_a^\infty \frac{\sin x}{x}dx$ is only accidentally convergent, and $\int_a^\infty \sin x\, dx$, not being convergent, comes under the class of divergent integrals. These definitions may be applied also to integrals taken between finite limits, when the quantity under the integral sign becomes infinite within the limits of integration, or at one of the limits. Thus $\int_0^a \log x\, dx$ is convergent, but $\int_0^a \frac{dx}{x}$ divergent at the limit 0.

2. Let $f(x)$ be a function of x which is only considered between the limits $x = 0$ and $x = a$, and which can be expanded between those limits in a convergent series of sines of $\frac{\pi x}{a}$ and its multiples, so that

$$f(x) = A_1 \sin\frac{\pi x}{a} + A_2 \sin\frac{2\pi x}{a} \dots + A_n \sin\frac{n\pi x}{a} + \dots\dots\dots\dots (1).$$

To determine A_n, multiply both sides of (1) by $\sin\frac{n\pi x}{a}dx$, and integrate from $x = 0$ to $x = a$. Since the series in (1) is convergent, and $\sin\frac{n\pi x}{a}$ does not become infinite for any real value of x, we may first multiply each term by $\sin\frac{n\pi x}{a}dx$ and integrate, and then sum, instead of first summing and then integrating*. But each term of the series in (1) except the n^{th} will produce in the new series a term equal to zero, and the n^{th} will produce $\frac{1}{2}aA_n$. Hence

$$A_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx,$$

and therefore $$f(x) = \frac{2}{a}\,\Sigma \int_0^a f(x)\sin\frac{n\pi x}{a}\,dx \,.\, \sin\frac{n\pi x}{a} \,.\, \dots\dots\dots\dots (2).$$

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* Moigno, *Leçons de Calcul Différentiel*, &c. Tom. II. p. 70.

3. Hence, whenever $f(x)$ can be expanded in the convergent series which forms the right-hand side of (1), the value of A_n can be very readily found, and the expansion performed. But this leaves us quite in the dark as to the degree of generality that a function which can be so expanded admits of. In considering this question it will be convenient, instead of endeavouring to develope $f(x)$, to seek the value of the infinite series

$$\frac{2}{a}\Sigma\int_0^a f(x')\sin\frac{n\pi x'}{a}\,dx'.\sin\frac{n\pi x}{a}, \quad\ldots\ldots\ldots\ldots (3),$$

provided the series be convergent; for it is only in that case that we can, without further definition, speak of the sum of the series at all. Now if we had only a finite number n of terms in the series (3) we might of course replace the series by

$$\frac{2}{a}\int_0^a f(x')\left\{\sin\frac{\pi x'}{a}\sin\frac{\pi x}{a}+\sin\frac{2\pi x'}{a}\sin\frac{2\pi x}{a}\ldots+\sin\frac{n\pi x'}{a}\sin\frac{n\pi x}{a}\right\}dx'.\ \ldots\ (4).$$

As it is however this transformation cannot be made, because, the series within brackets in the expression which would replace (4) not being convergent, the expression would be a mere symbol without any meaning. If however the series (3) is essentially convergent, its sum is equal to the limit of the sum of the following essentially convergent series

$$\frac{2}{a}\Sigma g^n\int_0^a f(x')\sin\frac{n\pi x'}{a}\,dx'.\sin\frac{n\pi x}{a}, \quad\ldots\ldots\ldots\ldots (5),$$

when g from having been less than 1 becomes in the limit 1. It will be observed that if (3) were only accidentally convergent, we could not with certainty affirm the sum of (3) to be the limit of the sum of (5). For it is conceivable, or at least not at present proved to be impossible, that the mode of the mutual destruction of the terms of (3) in the infinitely remote part of the series should be altered by the introduction of the factor g^n, however little g might differ from 1. Let us now, instead of seeking the sum of (3) in those cases in which the series is convergent, seek the limit to which the sum of (5) approaches as g approaches to 1 as its limit.

4. The transformation already referred to, which could not be effected on the series (3), may be effected on (5), that is to say, instead of first integrating the several terms and then summing, we may first sum and then integrate. We have thus, for the value of the series,

$$\frac{2}{a}\int_0^a f(x')\left\{\Sigma g^n\sin\frac{n\pi x'}{a}\sin\frac{n\pi x}{a}\right\}dx'.\quad\ldots\ldots\ldots\ldots (6).$$

The convergent series within brackets can easily be summed. The expression (6) thus becomes

$$\frac{1}{2a}\int_0^a f(x')\left\{\frac{1-g^2}{1-2g\cos\dfrac{\pi(x'-x)}{a}+g^2}-\frac{1-g^2}{1-2g\cos\dfrac{\pi(x'+x)}{a}+g^2}\right\}dx'.\ \ldots.\ (7).$$

Now since the quantity under the integral sign vanishes when $g=1$, provided $\cos\dfrac{\pi(x'\pm x)}{a}$ be not $=1$, the limit of (7) when $g=1$ will not be altered if we replace the limits 0 and a of x' by any other limits or groups of limits as close as we please, provided they contain the values of x' which render $x'\pm x$ equal to zero or any multiple of $2a$. Let us first suppose that we are considering a value of x lying between 0 and a, and in the neighbourhood of which $f(x)$ alters continuously. Then, since $x'+x$ never becomes equal to zero or any multiple of $2a$ within the limits of integration, we may omit the second term within brackets in (7); and since $x'-x$ never becomes equal to any multiple of $2a$, and vanishes only when $x'=x$, we may take for the limits of x' two quantities lying as close as we please to x, and therefore so close as to exclude all values of x' for which $f(x')$ alters discontinuously. Let $g = 1 - h$, $x' = x + \xi$, expand $\cos \frac{\pi\xi}{a}$ by the ordinary formula, and put $f(x') = f(x) + R$. Then the limit of (7) will be the same as that of

$$\frac{2-h}{2a} \int \{f(x) + R\} \frac{h\,d\xi}{h^2 + g\left(\frac{\pi^2\xi^2}{a^2} - \ldots\right)}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(8)}.$$

the limits of ξ being as small as we please, the first negative and the second positive. Let now

$$g\left(\frac{\pi^2\xi^2}{a^2} - \ldots\right) = \xi'^2,$$

so that $\frac{d\xi}{d\xi'}$ is ultimately equal to $\frac{a}{\pi}$, that is to say when g is first made equal to 1, and then the limits of ξ, and therefore those of ξ', are made to coalesce. Let now G, L be respectively the greatest and least values of $(1 - \frac{1}{2}h) \frac{1}{a} \frac{d\xi}{d\xi'} \{f(x) + R\}$ within the limits of integration. Then if we observe that $\int \frac{h\,d\xi'}{h^2 + \xi'^2} = \tan^{-1} \frac{\xi'}{h} + C$, where $\tan^{-1}$ denotes an angle lying between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, putting $-\xi_1$, ξ_2 for the limits of ξ', we shall see that the value of the integral (8) lies between

$$G\left(\tan^{-1}\frac{\xi_1}{h} + \tan^{-1}\frac{\xi_2}{h}\right) \text{ and } L\left(\tan^{-1}\frac{\xi_1}{h} + \tan^{-1}\frac{\xi_2}{h}\right):$$

but in the the limit, that is to say, when we first suppose h to vanish and then ξ_1 and ξ_2, G and L become equal to each other and to $\frac{1}{\pi} f(x)$, and $\tan^{-1}\frac{\xi_1}{h} + \tan^{-1}\frac{\xi_2}{h}$ becomes equal to π. Hence, $f(x)$ is the limit of (7).

Next, suppose that the value of x which we are considering lies between 0 and a, and that as x' passes through it $f(x')$ alters suddenly from M to N. Then the reasoning will be exactly as before, except that we must integrate separately for positive and negative values of ξ', replacing $f(x) + R$ by $M + R$ in the latter case, and by $N + R'$ in the former. Hence, the limit of (7) will be $\frac{1}{2}(M + N)$.

Lastly, if we are considering the extreme values $x = 0$ and $x = a$, it follows at once from the form of (7) that its limiting value is zero.

Hence the limit to which the sum of the convergent series (5) tends as g tends to 1 as its limit is $f(x)$ for values of x lying between 0 and a, for which $f(x)$ alters continuously, it is $\frac{1}{2}(M + N)$ for values of x for which $f(x)$ alters suddenly from M to N, and it is zero for the extreme values 0 and a.

5. Of course the limiting value of the series (5) is $f(0)$ and not zero, if we suppose that g first becomes 1 and then x passes from a positive value to zero. In the same way, if $f(x)$ alters abruptly from M to N as x increases through x_1, the limiting value of (5) will be M if we suppose that g first becomes 1 and then x increases to x_1, and it will be N if we suppose that g first becomes 1 and then x decreases to x_1. It would be futile to argue that the limiting value of (5) for $x = 0$ is zero rather than $f(0)$, or $f(0)$ rather than zero, since that entirely depends on the sense in which we employ the expression *limiting value*. Whichever sense we please to adopt, no error can possibly result, provided we are only consistent, and do not in the course of the same investigation change the meaning of our words.

It is a principle of great importance in these investigations, that a function of two independent variables which becomes indeterminate for particular values of the variables may have different limiting values according to the order in which we suppose the variables to assume their particular values, or according to the nature of the arbitrary relation which we conceive imposed on them as they approach those values together.

I would here make one remark on the subject of consistency. We may speak of the sum of an infinite series which is not convergent, if we define it to mean the limit of the sum of a convergent series of which the first n terms become in the limit the same as those of the divergent series. According to this definition, it appears quite conceivable that the same divergent series should have a different sum according as it is regarded as the limit of one convergent series or of another. If however we are careful in the same investigation always to regard the same divergent series, and the series derived from it, as the limits of the same convergent series and the series derived from it, it does not appear possible to fall into error, assuming of course that we always reason correctly. For example, we may employ the series (3), and the series derived from it by differentiation, &c., without fear, provided we always regard these series when divergent, or only accidentally convergent, as the limits of the *particular* convergent series formed by multiplying their n^{th} terms by g^n.

6. We may now consider the convergency of the series (3), in order to find whether we may employ it directly, or whether we must regard it as the limit of (5).

By integrating by parts in the n^{th} term of (3), we have

$$\frac{2}{a}\int f(x')\sin\frac{n\pi x'}{a}\,dx' = -\frac{2}{n\pi}f(x')\cos\frac{n\pi x'}{a} + \frac{2a}{n^2\pi^2}f'(x')\sin\frac{n\pi x'}{a} - \frac{2a}{n^2\pi^2}\int f''(x')\sin\frac{n\pi x'}{a}\,dx' \ldots \text{ (9)}.$$

Suppose that $f(x)$ does not necessarily vanish at the limits $x=0$ and $x=a$, and that it alters discontinuously any finite number of times between those limits, passing abruptly from M_1 to N_1 when x increases through α_1, from M_2 to N_2 when x increases through α_2, and so on. Then, if we put S for the sign of summation referring to the discontinuous values of $f(x')$, on taking the integrals in (9) from $x=0$ to $x=a$, we shall get for the part of the integral corresponding to the first term at the right-hand side of the equation

$$\frac{2}{n\pi}\left\{f(0)-(-)^n f(a)+S(N-M)\cos\frac{n\pi\alpha}{a}\right\} \ldots\ldots\ldots\ldots \text{ (10)}.$$

It is easily seen that the last two terms in (9) will give a part of the integral taken from 0 to a, which is numerically inferior to $\frac{L}{n^2}$, where L is a constant properly chosen. As far as regards these terms therefore the series (3) will be essentially convergent, and its sum will therefore be the limit of the sum of (5).

Hence, in examining the convergency or divergency of the series (3), we have only got to consider the part of the coefficient of $\sin\frac{n\pi x}{a}$ of which (10) is the expression. The terms $f(0)$, $f(a)$ in this expression may be included under the sign S if we put for the first $\alpha=0$, $M=0$, $N=f(0)$, and for the second $\alpha=a$, $M=f(a)$, $N=0$. We have thus got a set of series to consider of which the type is

$$\frac{2}{\pi}(N-M)\,\Sigma\,\frac{1}{n}\cos\frac{n\pi\alpha}{a}\sin\frac{n\pi x}{a} \ldots\ldots\ldots\ldots \text{ (11)}.$$

If we replace the product of the sine and cosine in this expression by the sum of two sines, by means of the ordinary formula, and omit unnecessary constants, we shall have for the series to consider

$$\Sigma \frac{1}{n} \sin nz. \quad \ldots\ldots\ldots\ldots \quad (12).$$

Let now
$$u = \sin z + \tfrac{1}{2} \sin 2z \ldots + \frac{1}{n} \sin nz, \quad \ldots\ldots\ldots \quad (13),$$

then
$$\frac{du}{dz} = \cos z + \cos 2z \ldots + \cos nz = \frac{\sin (n + \frac{1}{2}) z}{2 \sin \frac{1}{2} z} - \tfrac{1}{2};$$

and since u vanishes with z, in which case $\dfrac{\sin (n + \frac{1}{2}) z}{\sin \frac{1}{2} z}$ is finite, we shall have, supposing z to lie between -2π and $+2\pi$, so that the quantity under the integral sign does not become infinite within the limits of integration,

$$u = \tfrac{1}{2} \int_0^z \frac{\sin (n + \frac{1}{2}) z}{\sin \frac{1}{2} z} dz - \frac{z}{2}; \quad \ldots\ldots\ldots \quad (14),$$

and we have to find whether the integral contained in this equation approaches a finite limit as n increases beyond all limit, and if so what that limit is. Since u changes sign with z, we need not consider the negative values of z.

First suppose the superior limit z to lie between 0 and 2π; and to simplify the integral write $2z$ for z, n for $2n + 1$, so that the superior limit of the new integral lies between 0 and π; then

$$\text{the integral} = \int_0^z \frac{\sin nz}{\sin z} dz = \int_0^z \frac{\sin nz}{z} \frac{z}{\sin z} dz = \int_0^z \frac{\sin nz}{z} (1 + Rz)\, dz,$$

where $R = \dfrac{z - \sin z}{z \sin z}$, a quantity which does not become infinite within the limits of integration. Hence, as is known, the limit of $\int_0^z \sin nz \,.\, R dz$ when n increases beyond all limit is zero. Hence, if I be the limit of the integral,

$$I = \text{limit of} \int_0^z \frac{\sin nz}{z} dz = \text{limit of} \int_0^{nz} \frac{\sin \zeta}{\zeta} d\zeta.$$

Now, z being given, the limit of nz is ∞, and therefore

$$I = \int_0^\infty \frac{\sin \zeta}{\zeta} d\zeta = \frac{\pi}{2}.$$

Secondly, suppose z in (14) to be equal to 0. Then it follows directly from this equation, or in fact at once from (13), that $u = 0$, and consequently the limit of $u = 0$.

The value of u in all other cases, if required, may be at once obtained from the consideration that the values of u recur when z is increased or diminished by 2π.

Hence, the series (12) is in all cases convergent, and has for its sum 0 when $z = 0$, and $\frac{1}{2}(\pi - z)$ when z lies between 0 and 2π.

Now, if in the theorem of Article 4. we write z for x, and put $a = \pi$, $f(z) = \frac{1}{2}(\pi - z)$, we find, for values of z lying between 0 and π, and for $z = \pi$,

$$\text{limit of } \Sigma \frac{1}{n} g^n \sin nz = \tfrac{1}{2}(\pi - z);$$

and evidently

$$\text{limit of } \Sigma \frac{1}{n} g^n \sin nz = 0, \text{ when } z = 0,$$

that is of course supposing z first to vanish and then g to become 1. Also the limit of $\Sigma \frac{1}{n} g^n \sin nz$ changes sign with z, and recurs when z is increased or diminished by 2π. Hence, the series (12), which has been proved to be convergent, is in all cases the limit to which the sum of the convergent series $\Sigma \frac{1}{n} g^n \sin nz$ tends as g tends to 1 as its limit. Now the series (11) may be decomposed into two series of the form just discussed, whence it follows that the series (3) is always convergent, and its sum for all values of x, critical as well as general, is the limit of the sum of the series (5), when g becomes equal to 1.

The examination of the convergency of the series (3) in the only doubtful case, that is to say, the case in which $f(x)$ is discontinuous, or does not vanish for $x = 0$ and for $x = a$, is more curious than important. For in the analytical applications of the series (3) it would be sufficient to regard it as the limit of the series (5); and in the case in which (3) is only accidentally convergent, we should hardly think of employing it in the numerical computation of $f(x)$ if we could possibly help it, and it will immediately appear that in all the cases which are most important to consider we can get rid of the troublesome terms without knowing the sum of the series.

The proof of the convergency of the series (3) which has just been given, though in some respects I believe new, is certainly rather circuitous, and it has the disadvantage of not applying to the case in which $f'(x)$ is infinite*, an objection which does not apply to the proof given by M. Dirichlet†. It has been supposed moreover that $f''(x)$ is not infinite. The latter restriction however may easily be removed, as in the end of the next article.

7. Let $f(x)$ be a function of x which is expanded between the limits $x = 0$ and $x = a$ in the series (3). Let the series be

$$A_1 \sin \frac{\pi x}{a} + A_2 \sin \frac{2\pi x}{a} \dots + A_n \sin \frac{n\pi x}{a} + \dots, \quad \dots\dots\dots\dots\dots (15),$$

and suppose that we have given the coefficients $A_1, A_2 \dots$, but do not know the sum of the series $f(x)$. We may for all that find the values of $f(0)$ and $f(a)$, and likewise the values of x for which $f(x)$ is discontinuous, and the quantity by which $f(x)$ is increased as x increases through each of these critical values.

For from (9) and (10)

$$nA_n = \frac{2}{\pi}\left\{f(0) - (-1)^n f(a) + S(N - M)\cos\frac{n\pi a}{a}\right\} + \frac{R}{n}.$$

R being a quantity which does not become infinite with n. If then we use the term *limit* in an extended sense, so as to include quantities of the form $C \cos n\gamma$, (of course $C(-1)^n$ is a particular case,) or the sum of any finite number of such quantities, we shall have for $n = \infty$,

$$\text{limit of } nA_n = \frac{2}{\pi}\left\{f(0) - (-1)^n f(a) + S(N - M)\cos\frac{n\pi a}{a}\right\}. \dots (16).$$

* This restriction may however be dispensed with by what is proved in Art. 20.

† Crelle's *Journal*, Tom. IV. p. 157.

Let then the limit of nA_n be found. It will appear under the form

$$C_0 + C_1(-1) + SC\cos n\gamma \quad \dots\dots\dots\dots\dots\dots\dots (17).$$

Comparing this expression with (16), we shall have

$$f(0) = \frac{\pi}{2} C_0, \quad f(a) = -\frac{\pi}{2} C_1;$$

and for each term of the series denoted by S we shall have

$$\alpha = \frac{a\gamma}{\pi}, \quad N - M = \frac{\pi}{2} C.$$

In particular, if $f(x)$ is continuous, and if the limit of nA_n is L_0 or L_e according as n is odd or even, we shall have

$$L_0 = \frac{2}{\pi}\{f(0) + f(a)\}, \quad L_e = \frac{2}{\pi}\{f(0) - f(a)\};$$

whence

$$f(0) = \frac{\pi}{4}(L_0 + L_e), \quad f(a) = \frac{\pi}{4}(L_0 - L_e). \quad \dots\dots\dots\dots\dots\dots\dots (18).$$

If $f(x)$ were discontinuous for an infinite number of values of x lying between 0 and a, it is conceivable that the infinite series coming under the sign S might be divergent, or if convergent might have a sum from which n might wholly or partially disappear, in which case the limit of nA_n might not come out under the form (17). It was for this reason among others, that in Art. 1, I excluded such functions from consideration.

If $f(x)$ be expressible algebraically between the limits $x = 0$ and $x = a$, or if it admit of different algebraical expressions within different portions into which that interval may be divided, A_n will be an algebraical function of n, and the limit of nA_n may be found by the ordinary methods. Under the term *algebraical function*, I here include transcendental functions, using the term *algebraical function* in opposition to what has been sometimes called an *empirical function*, or a *general function*, that is, a function in the sense in which the ordinate of a curve traced *liberâ manu* is a function of the abscissa. Of course, in applying the theorem in this article to general functions, it must be taken as a postulate that the limit of nA_n can be found, and put under the form (17).

The theorem in question has been proved by means of equation (9), in which it is supposed that $f''(x)$ does not become infinite within the limits of integration. The theorem is however true independently of this restriction. To prove it we have only got to integrate by parts once instead of twice, and we thus get for the quantity which replaces $\frac{R}{n}$ the integral

$$\frac{2}{\pi}\int_0^a f'(x') \cos \frac{n\pi x'}{a} dx',$$

which by the principle of fluctuation* vanishes when n becomes infinite. There is however this difference between the two cases. When the series (15) has been cleared of the part for which the

* I borrow this term from a paper by Sir William R. Hamilton *On Fluctuating Functions*. Transactions of the Royal Irish Academy, Vol. XIX. p. 264. Had I been earlier acquainted with this paper, and that of M. Dirichlet already referred to, I would probably have adopted the second of the methods mentioned in the introduction for establishing equation (2) for any function, or rather, would have begun with Art. 7. taking that equation as established. I have retained Arts. (2)—(6), first, because I thought the reader would enter more readily into the spirit of the paper if these articles were retained, and secondly, because I thought that Section III, which is adapted to this mode of viewing the subject, might be found useful.

limit of nA_n is finite, by the method which will be explained in the next article, the part which remains will be at least as convergent in the former case as the series $\frac{1}{1^2}+\frac{1}{2^2}\ldots+\frac{1}{n^2}+\ldots$, whereas we cannot affirm this to be true, and in fact it may be proved that it is not true, in the case in which $f''(x)$ becomes infinite. Observing that the same remark will apply when we come to consider the critical values of the differential coefficients of $f(x)$, I shall suppose the functions and derived functions employed in each investigation not to become infinite, according to what has been already stated in Art. 1.

8. After having found the several values of α, and the corresponding values of $N-M$, we may subtract the expression (10) from A_n, provided we subtract from the sum of the series (15) the sums of the several series such as (11). Now if X be the sum of the series (11),

$$X=\frac{1}{\pi}(N-M)\left\{\Sigma\frac{1}{n}\sin\frac{n\pi(x+\alpha)}{a}+\Sigma\frac{1}{n}\sin\frac{n\pi(x-\alpha)}{a}\right\}\ \ldots\ (19).$$

But it has been already shown that $\Sigma\frac{1}{n}\sin nz=\frac{1}{2}(\pi-z)$ when z lies between 0 and 2π, $=0$ when $z=0$, and $=-\frac{1}{2}(\pi+z)$ when z lies between 0 and -2π. Now when x lies between 0 and α, $\frac{\pi(x+\alpha)}{a}$ lies between 0 and 2π, and $\frac{\pi(x-\alpha)}{a}$ lies between -2π and 0; and when x lies between α and a, $\frac{\pi(x+\alpha)}{a}$ still lies between 0 and 2π, and $\frac{\pi(x-\alpha)}{a}$ now lies between the same limits. Hence

$$\left.\begin{aligned}X&=-(N-M)\frac{x}{a}, &&\text{when } x \text{ lies between } 0 \text{ and } \alpha\\ &=(N-M)\frac{a-x}{a}, &&\text{when } x \text{ lies between } \alpha \text{ and } a\end{aligned}\right\}\ \ldots\ldots\ (20).$$

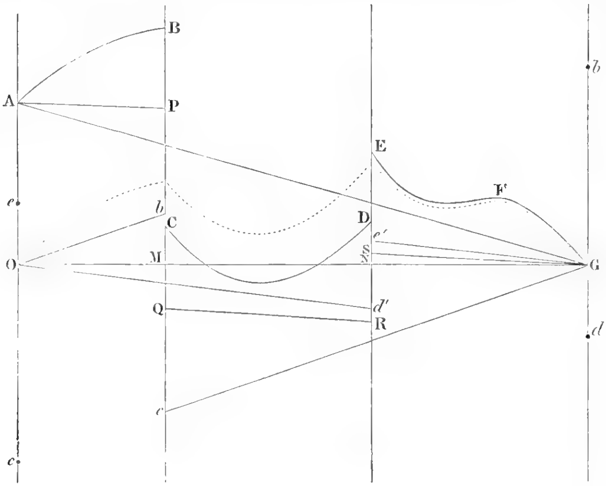
We need not trouble ourselves with the singular values of the sum of the series (15), since we have seen that a singular value is always the arithmetic mean of the values of the sum for values of x immediately above and below the critical value. This rule will apply to the extreme cases in which $x=0$ and $x=a$, if we consider the sum of the series for values of x lying beyond those limits. The rule applies to the series in (19), which is only a particular case of (15), and consequently will apply to any combination of series having this property, formed by way of addition or subtraction; since, when we increase or diminish any two quantities M_0, N_0 by any other two M, N respectively, we increase or diminish the arithmetic mean of the two former by the arithmetic mean of the two latter.

It has been already stated that we may, with a certain convention, include quantities referring to the limits $x=0$ and $x=a$ under the sign of summation S. If we do so, and put Ξ for the sum of the series (15), and B_n for the remainder arising from subtracting the expression (10) from A_n, we shall have

$$\Xi-SX=\Sigma B_n\sin\frac{n\pi x}{a},$$

and the sum of the series forming the right-hand side of this equation will be a continuous function of x. As to SX, the value of each series contained in it is given by equation (20).

To illustrate this, suppose Ξ the ordinate of a curve of which x is the abscissa. Let OG be the axis of x; OA, MB, ND, Gb right lines perpendicular to it, and let $OG=a$. Let the curve of which Ξ is the ordinate be the discontinuous curve AB, CD, EFG. Take Gb equal to BC,



and on the positive or negative side of the axis of x according as the ordinate decreases or increases as x increases through OM, and from O measure an equal length Oc on the opposite side of the axis. Take Gd, Oe, each equal to DE, and draw the right lines AG, $Ob'b$, $cc'G$, $Od'd$, $ee'G$. Then it will be easily seen that if X_0, X_1, X_2 be the values of X corresponding to the critical values of x, $x = 0$, $x = OM$, $x = ON$, respectively, X_0 will be represented by the right line AG; X_1 by the discontinuous right line Ob', $c'G$; and X_2 by the discontinuous right line Od', $e'G$. Take MP equal to the sum of the ordinates of the points in which the right lines lying between OA and $c'B$ cut the latter line; MQ equal to the sum of the ordinates of the points in which the right lines lying between $c'B$ and $d'E$ cut the former, and so on, the ordinates being taken with their proper signs. Let P, Q, R, S be the points thus found, and join AP, QR, SG. Then SX will be represented by the discontinuous right line AP, QR, SG. Let the ordinates of the discontinuous curve be diminished by those of the discontinuous right line last mentioned, and let the dotted curve be the result. Then $\Xi - SX$ will be represented by the continuous, dotted curve. It will be observed that the two portions of the dotted curve which meet in each of the ordinates MB, NE may meet at a finite angle. If there should be a point in one of the continuous portions, such as AB, of the discontinuous curve where two tangents meet at a finite angle, there will of course be a corresponding point in the dotted curve.

If we choose to take account of the conjugate points of the curve of which SX is the ordinate, it will be observed that they are situated at O, and midway between P and Q, and between R and S.

9. There are a great many series, similar to (3), in which $f(x)$ may be expanded within certain limits of x. I shall consider one other, which as well as (3) is of great use, observing that almost exactly the same methods and the same reasoning will apply in other cases.

The limit of the sum of the series

$$\frac{1}{a}\int_0^a f(x')\,dx' + \frac{2}{a}\Sigma g^n \int_0^a f(x')\cos\frac{n\pi x'}{a}\,dx' . \cos\frac{n\pi x}{a} \text{ ...... (21)},$$

when g from having been less than 1 becomes 1, is $f(x)$, x being supposed not to lie beyond the limits 0 and a. For values, however, of x for which $f(x)$ alters discontinuously, the limit of the sum is the arithmetic mean of the values of $f(x)$ for values of x immediately above and below the critical value. I assume this as being well known, observing that it may be demonstrated just as a similar theorem has been demonstrated in Art. 4.

10. Let us now consider the series

$$\frac{1}{a}\int_0^a f(x')\,dx' + \frac{2}{a}\Sigma\int_0^a f(x')\cos\frac{n\pi x'}{a}\,dx'.\cos\frac{n\pi x}{a}\ \ldots\ldots\ldots\ (22).$$

We have by integration by parts

$$\frac{2}{a}\int f(x')\cos\frac{n\pi x'}{a}\,dx' = \frac{2}{n\pi}f(x')\sin\frac{n\pi x'}{a} + \frac{2a}{n^2\pi^2}f'(x')\cos\frac{n\pi x'}{a} - \frac{2a}{n^2\pi^2}\int f''(x')\cos\frac{n\pi x'}{a}\,dx';$$

and now, taking the limits properly, and employing the letters M, N, α and S in the same sense as before, we have

$$\frac{2}{a}\int_0^a f(x')\cos\frac{n\pi x'}{a}\,dx' = -\frac{2}{n\pi}S(N-M)\sin\frac{n\pi\alpha}{a} + \frac{R}{n^2},\ \ldots\ (23),$$

R being a quantity which does not become infinite with n. It follows from (23), that the series (22) is in all cases convergent, and its sum for all values of x, critical as well as general, is the limit of the sum of (21).

It will be observed that if $f(x)$ is a continuous function the series (22) is at least as convergent as the series $\Sigma\frac{1}{n^2}$. This is not the case with the series (3), unless $f(0) = f(a) = 0$.

If the constant term and the coefficient of $\cos\frac{n\pi x}{a}$ in the general term of (22) are given, $f(x)$ itself not being known, except by its developement, we may as before find the values of x for which $f(x)$ is discontinuous, and the quantity by which $f(x)$ is suddenly increased as x increases through each critical value. We may also, if we please, clear the series (22) of the slowly convergent part corresponding to the discontinuous values of $f(x)$.

11. Since the series (3) is convergent, if we have occasion to integrate $f(x)$ we may, instead of first summing the series and then integrating, first integrate the general term and then sum. More generally, if $\phi(x)$ be any function of x which does not become infinite between the limits $x = 0$ and $x = a$, we shall have

$$\int_0^x f(x)\,\phi(x)\,dx = \frac{2}{a}\Sigma\int_0^a f(x')\sin\frac{n\pi x'}{a}\,dx'.\int_0^x \phi(x)\sin\frac{n\pi x}{a}\,dx.$$

the superior limit x of the integrals being supposed not to lie beyond the limits 0 and a; and the series at the second side of the above equation will be convergent. In fact, even in the case in which $f(x)$ is discontinuous the series will be as convergent as the series $\Sigma\frac{1}{n^2}$. A second integration would give a series still more rapidly convergent, and so on. Hence, the resulting series may be employed directly, and not merely when regarded as limits of converging series. The same remarks apply in all respects to the series (22) as to the series (3).

12. But the series resulting from differentiating (3) or (22) once, twice, or any number of times would not in general be convergent, and could not be employed directly, but only as limits of the convergent series which would be formed by inserting the factor g^n in the general term.

This mode of treating the subject however appears very inconvenient, except in the case in which the series are only temporarily divergent, being rendered convergent again by new integrations; and even then it requires great caution. The series in question may however be rendered convergent by means of transformations to which I now proceed, and which, with their applications, form the principal object of this paper.

The most important case to consider is that in which $f(x)$ and its derivatives are continuous, so that the divergency arises from what takes place at the limits 0 and a. I shall suppose then, for the present, that $f(x)$ and its derivatives of the orders considered are continuous, except the last, which will only appear under the sign of integration, and which may be discontinuous.

Consider first the series of sines. Suppose that $f(x)$ is not given in finite terms, but only by its developement

$$f(x) = \Sigma A_n \sin \frac{n\pi x}{a}, \quad \ldots\ldots\ldots\ldots (24),$$

where A_n is supposed to be given, and where the developement of $f(x)$ is supposed to be that which would result from the formula (3). I shall call the expansions of $f(x)$ which are obtained, or which are to be looked on as obtained from the formulæ (3) and (22) *direct* expansions, as distinguished from other expansions which may be obtained by differentiation, and which, being divergent, cannot be directly employed. Let us consider first the even differential coefficients of $f(x)$, and let A''_n, A^{4}_n ... be the coefficients of $\sin \frac{n\pi x}{a}$ in the direct expansions of $f''(x)$, $f^{4}(x)$ ... The coefficient of $\sin \frac{n\pi x}{a}$ in the series which would be obtained by differentiating twice the several terms in the series in (24) would be $-\frac{n^2\pi^2}{a^2} A_n$. Now

$$A_n = \frac{2}{a} \int_0^a f(x') \sin \frac{n\pi x'}{a}\, dx';$$

and we have by integrating by parts

$$-\frac{2n^2\pi^2}{a^3} \int f(x') \sin \frac{n\pi x'}{a}\, dx' = \frac{2n\pi}{a^2} f(x') \cos \frac{n\pi x'}{a} - \frac{2}{a} f'(x') \sin \frac{n\pi x'}{a}$$
$$+ \frac{2}{a} \int f''(x') \sin \frac{n\pi x'}{a}\, dx'.$$

Taking now the limits, remembering the expression for A''_n, and transposing, we get

$$A''_n = \frac{2n\pi}{a^2} \{f(0) - (-1)^n f(a)\} - \frac{n^2\pi^2}{a^2} A_n. \quad \ldots\ldots\ldots (25).$$

Any even differential coefficient may be treated in the same way. We thus get, μ being even,

$$(-1)^{\frac{\mu}{2}} A^{\mu}_n = \left(\frac{n\pi}{a}\right)^{\mu} A_n - \frac{2}{a}\left(\frac{n\pi}{a}\right)^{\mu-1} \{f(0) - (-1)^n f(a)\} + \frac{2}{a}\left(\frac{n\pi}{a}\right)^{\mu-3} \{f''(0) - (-1)^n f''(a)\} - \ldots$$
$$+ (-1)^{\frac{\mu}{2}} \frac{2}{a} \cdot \frac{n\pi}{a} \{f^{\mu-2}(0) - (-1)^n f^{\mu-2}(a)\}. \ldots (26).$$

13. In the applications of these equations which I have principally in view, $f(0)$, $f(a)$, $f''(0)$... are given, and A_1, A_2, A_3 ... are indeterminate coefficients. If however A_1, A_2 ... A_n ... are given, and $f(0)$, $f(a)$ ... unknown, we must first find $f(0)$, $f(a)$ ..., and then we shall be able to substitute in (25) and (26). This may be effected in the following manner.

We get by integrating by parts

$$\int f(x')\sin\frac{n\pi x'}{a}\,dx' = -\frac{a}{a\pi}f(x')\cos\frac{n\pi x'}{a} + \left(\frac{a}{n\pi}\right)^2 f'(x')\sin\frac{n\pi x'}{a} + \left(\frac{a}{n\pi}\right)^3 f''(x')\cos\frac{n\pi x'}{a} - \ldots$$

Multiplying now both sides by $\frac{2}{a}$, and taking the limits of the integrals, we get

$$A_n = \frac{2}{a}\cdot\frac{a}{n\pi}\{f(0) - (-1)^n f(a)\} - \frac{2}{a}\cdot\left(\frac{a}{n\pi}\right)^3\{f''(0) - (-1)^n f''(a)\} + \ldots, \quad \ldots (27).$$

Hence, if n be always odd or always even, A_n can be expanded, at least to a certain number of terms, in a series according to descending powers of n, the powers being odd, and the first of them -1. The number of terms to which the expansion in this form is possible will depend on the number of differential coefficients of $f(x)$ which remain finite and continuous between the limits $x = 0$ and $x = a$. Let the expansion be performed, and let the result be

$$\left.\begin{aligned} A_n &= O_0\frac{1}{n} + O_2\frac{1}{n^3} + O_4\frac{1}{n^5} + \ldots \text{ when } n \text{ is odd;} \\ A_n &= E_0\frac{1}{n} + E_2\frac{1}{n^3} + E_4\frac{1}{n^5} + \ldots \text{ when } n \text{ is even.}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (28).$$

Comparing (27) and (28), we shall have

$$\left.\begin{aligned} f(0) &= \frac{\pi}{4}(O_0 + E_0), & f(a) &= \frac{\pi}{4}(O_0 - E_0), \\ f''(0) &= -\frac{\pi^3}{4a^2}(O_2 + E_2), & f''(a) &= -\frac{\pi^3}{4a^2}(O_2 - E_2), \\ f^{\text{iv}}(0) &= \frac{\pi^5}{4a^4}(O_4 + E_4), & f^{\text{iv}}(a) &= \frac{\pi^5}{4a^4}(O_4 - E_4), \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (29),$$

and so on. The first two of these equtions agree with (18).

If we conceive the value of A_n given by (27) substituted in (26), we shall arrive at a very simple rule for finding the direct expansion of $f^{\mu}(x)$. It will only be necessary to expand A_n as far as $\frac{1}{n^{\mu-1}}$, admitting $(-1)^n$ into the expansion as if it were a constant coefficient, and then, subtracting from A_n the sum of the terms thus found, employ the series which would be obtained by differentiating the equation (24) μ times. It will be necessary to assure ourselves that the term in $\frac{1}{n^{\mu}}$ vanishes in the expansion of A_n, since otherwise $f^{\mu}(x)$ might be infinite, or $f^{\mu-1}(x)$ discontinuous without our being aware of it. It will be seen however presently (Art. 20) that the former circumstance would not vitiate the result, nor introduce a term involving $n^{-\mu}$.

Should A_n already appear under such a form as $\frac{1}{n} + c^n$; $(-1)^n\frac{1}{n^3} + n^2c^n$, &c., where $c^2 < 1$, it will be sufficient to differentiate equation (24) μ times, and leave out the part of the series which becomes divergent. For it will be observed that the terms c^n, n^2c^n, in the examples chosen, decrease with $\frac{1}{n}$ faster than any inverse power of n.

14. Let us now consider the odd differential coefficients of $f(x)$, supposing $f(x)$ to be expanded in a series of cosines, so that

$$f(x) = B_0 + \Sigma B_n \cos \frac{n\pi x}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (30).$$

Let A'_n, A'''_n ... be the coefficients of $\sin \frac{n\pi x}{a}$ in the direct expansions of $f'(x)$, $f'''(x)$ ... in series of sines. If we were to differentiate (30) once we should have $-\frac{n\pi x}{a} B_n$ for the coefficient of $\sin \frac{n\pi x}{a}$. Now

$$-\frac{n\pi}{a} \cdot \frac{2}{a} \int f(x') \cos \frac{n\pi x}{a} dx' = -\frac{2}{a} f(x) \sin \frac{n\pi x'}{a} + \frac{2}{a} \int f'(x') \sin \frac{n\pi x'}{a} dx';$$

and taking the limits of the integrals, and introducing B_n and A'_n, we get

$$A'_n = -\frac{n\pi}{a} B_n. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (31).$$

Hence, the series arising from differentiating (30) once gives the direct expansion of $f(x)$ in a series of sines.

The coefficient of $\sin \frac{n\pi x}{a}$ in the series which would be obtained by differentiating (30) μ times, μ being odd, would be $(-1)^{\frac{\mu+1}{2}} \left(\frac{n\pi}{a}\right)^\mu B_n$. By proceeding just as in the last article we obtain

$$(-1)^{\frac{\mu+1}{2}} A^\mu_n = \left(\frac{n\pi}{a}\right)^\mu B_n + \frac{2}{a} \left(\frac{n\pi}{a}\right)^{\mu-2} \{f'(0) - (-1)^n f'(a)\} - \frac{2}{a} \left(\frac{n\pi}{a}\right)^{\mu-4} \{f'''(0) - (-1)^n f'''(a)\} + \ldots$$
$$+ (-1)^{\frac{\mu+1}{2}} \frac{2}{a} \cdot \frac{n\pi}{a} \{f^{\mu-2}(0) - (-1)^n f^{\mu-2}(a)\}. \ldots\ldots \quad (32).$$

When $f'(0)$, $f'(a)$, &c., are known, this series enables us to develope $f^\mu(x)$ in a direct series of sines, the direct developement of $f(x)$ in a series of cosines being given.

15. If we treat the expression for B_n by integration by parts, just as the expression for A_n was treated, going on till we arrive at the integral which gives A^μ_n, and observe that the very same process is used in deducing the value of A^μ_n from that of B_n as in expanding the latter according to inverse powers of n, and that the index of n in the coefficient of A^μ_n is $-\mu$, and that A_n vanishes when n becomes infinite, we shall see that in order to obtain the direct expansion of $f^\mu(x)$ we have only got to expand B_n as far as $\frac{1}{n^\mu}$, (the coefficient of $\frac{1}{n^\mu}$ will vanish,) and subtract from B_n these terms of the expansion, and then differentiate (30) μ times.

The expansion of B_n, at least to a certain number of terms, will proceed according to even powers of $\frac{1}{n}$, beginning with $\frac{1}{n^2}$. If we suppose that

$$\left.\begin{aligned} B_n &= O_1 \frac{1}{n^2} + O_3 \frac{1}{n^4} + O_5 \frac{1}{n^6} + \ldots \text{ when } n \text{ is odd,} \\ B_n &= E_1 \frac{1}{n^2} + E_3 \frac{1}{n^4} + E_5 \frac{1}{n^6} + \ldots \text{ when } n \text{ is even,} \end{aligned}\right\} \ldots\ldots\ldots\ldots \quad (33).$$

and compare these expansions with that given by integration by parts, we shall have

$$\left.\begin{aligned} f'(0) &= -\frac{\pi^2}{4a}(O_1+E_1), & f'(a) &= -\frac{\pi^2}{4a}(O_1-E_1), \\ f'''(0) &= \frac{\pi^4}{4a^3}(O_3+E_3), & f'''(a) &= \frac{\pi^4}{4a^3}(O_3-E_3), \end{aligned}\right\} \quad \ldots\ldots\ldots (34),$$

and so on, the signs of the coefficients being alternately + and −, and the index of $\frac{\pi}{a}$ increasing by 2 each time.

16. The values of $f^\mu(0)$ and $f^\mu(a)$ when $f(x)$ is expanded in a series of sines and μ is odd, or when $f(x)$ is expanded in a series of cosines and μ is even, will be expressed by infinite series. To find these values we should first have to obtain the direct expansion of $f^\mu(x)$, which would be got by differentiating the equation (24) or (30) μ times, expanding A_n or B_n according to powers of $\frac{1}{n}$, and rejecting the terms which would render the series contained in the μ^{th} derived equation divergent. The reason of this is the same as before.

17. The direct expansions of the derivatives of $f(x)$ may be obtained in a similar manner in the cases in which $f(x)$ itself, or any one of its derivatives is discontinuous. In what follows, a will be taken to denote a value of x for which $f(x)$ or any one of its derivatives of the orders considered is discontinuous; $Q, Q_1, \ldots Q_\mu$ will denote the quantities by which $f(x), f'(x), \ldots f^\mu(x)$ are suddenly increased as x increases through a; S will be used for the sign of summation relative to the different values of a, and will be supposed to include the extreme values 0 and a, under the convention already mentioned in Art. 6. Of course $f(x)$ may be discontinuous for a particular value of x while $f^\mu(x)$ is continuous, and *vice versâ*. In this case one of the two Q, Q_μ will be zero while the other is finite.

The method of proceeding is precisely the same as before, except that each term such as $f(x)\cos\frac{n\pi x}{a}$ in the indefinite integral arising from the integration by parts will give rise to a series such as $-SQ\cos\frac{n\pi a}{a}$ in the integral taken between limits. We thus get in the case of the even derivatives of $f(x)$, when $f(x)$ is expanded in a series of sines,

$$(-1)^{\frac{\mu}{2}}A_n^\mu = \left(\frac{n\pi}{a}\right)^\mu A_n - \frac{2}{a}\cdot\left(\frac{n\pi}{a}\right)^{\mu-1} SQ\cos\frac{n\pi a}{a} + \frac{2}{a}\cdot\left(\frac{n\pi}{a}\right)^{\mu-2} SQ_1\sin\frac{n\pi a}{a}$$
$$+\frac{2}{a}\cdot\left(\frac{n\pi}{a}\right)^{\mu-3} SQ_2\cos\frac{n\pi a}{a} - \ldots + (-1)^{\frac{\mu}{2}+1}\cdot\frac{2}{a}\cdot SQ_{\mu-1}\sin\frac{n\pi a}{a} \quad \ldots\ldots (35).$$

In the case of the odd derivatives of $f(x)$, when $f(x)$ is expanded in a series of cosines, we get

$$(-1)^{\frac{\mu+1}{2}}A_n^\mu = \left(\frac{n\pi}{a}\right)^\mu B_n + \frac{2}{a}\left(\frac{n\pi}{a}\right)^{\mu-1} SQ\sin\frac{n\pi a}{a} + \frac{2}{a}\left(\frac{n\pi}{a}\right)^{\mu-2} SQ_1\cos\frac{n\pi a}{a} - \ldots$$
$$+(-1)^{\frac{\mu-1}{2}}\frac{2}{a}SQ_{\mu-1}\sin\frac{n\pi a}{a} \quad \ldots\ldots (36).$$

When the several values of a, Q, $Q_1 \ldots$ are given, these equations enable us to find the direct expansion of $f^\mu(x)$. The series corresponding to the odd derivatives in the first case and the even in the second might easily be found.

If we wish to find the direct expansion of $f^\mu(x)$ in the case in which A_n or B_n is given, we have only to expand A_n or B_n in a series according to descending powers of n, regarding $\cos n\gamma$ or $\sin n\gamma$, as well as $(-1)^n$, as constant coefficients, and then reject from the series obtained by the immediate differentiation of (24) or (30) those terms which would render it divergent. This readily follows as in Art. 15, from the consideration of the mode in which A_n^μ is obtained from A_n or B_n.

The equations (35) and (36) contain as particular cases (26) and (32) respectively. It was convenient however to have the latter equations, on account of their utility, expressed in a form which requires no transformation.

18. If we transform A_n and B_n by integration by parts, we get

$$A_n = \frac{2}{n\pi} SQ \cos \frac{n\pi a}{a} - \frac{2a}{n^2\pi^2} SQ_1 \sin \frac{n\pi a}{a} - \frac{2a^2}{n^3\pi^3} SQ_2 \cos \frac{n\pi a}{a} + \ldots, \quad \ldots\ldots (37),$$

$$B_n = -\frac{2}{n\pi} SQ \sin \frac{n\pi a}{a} - \frac{2a}{n^2\pi^2} SQ_1 \cos \frac{n\pi a}{a} + \frac{2a^2}{n^3\pi^3} SQ_2 \sin \frac{n\pi a}{a} + \ldots, \quad \ldots\ldots (38),$$

where the law of the series is evident, if we only observe that two signs of the same kind are always followed by two of the opposite kind. The equations (37), (38) may be at once obtained from (35), (36). The former equations give the true expansions of A_n and B_n according to powers of $\frac{1}{n}$; because when we stop after any number of integrations by parts the last integral with its proper coefficient always vanishes compared with the coefficient of the preceding term.

Hence A_n and B_n admit of expansion according to powers of $\frac{1}{n}$, if we regard $\cos n\gamma$ or $\sin n\gamma$ as a constant coefficient in the expansion. Moreover quantities such as $\cos n\gamma$, $\sin n\gamma$ will occur alternately in each expansion, the one kind going along with odd powers of $\frac{1}{n}$ and the other along with even. If we suppose the value of A_n or B_n, as the case may be, given, and the expansion performed, so that

$$A_n = SF \cos n\gamma \cdot \frac{1}{n} + SF_1 \sin n\gamma \cdot \frac{1}{n^2} + SF_2 \cos n\gamma \cdot \frac{1}{n^3} + \ldots, \quad \ldots\ldots (39),$$

$$B_n = SG \sin n\gamma \cdot \frac{1}{n} + SG_1 \cos n\gamma \cdot \frac{1}{n^2} + SG_2 \sin n\gamma \cdot \frac{1}{n^3} + \ldots, \quad \ldots\ldots (40),$$

and compare these expansions with (37) or (38), we shall get the several values of a, and the corresponding values of $Q, Q_1, Q_2 \ldots$ We may thus, without being able to sum the series in equation (24) or (30), find the values of x for which $f(x)$ itself or any one of its derivatives is discontinuous, and likewise the quantity by which the function or derivative is suddenly increased. This remark will apply to the extreme values 0 and a of x if we continue to denote the sum of the series by $f(x)$ when x is outside of the limits 0 and a.

19. Having found the values of a, Q, $Q_1 \ldots$, we may if we please clear the series in (24) or (30) of the terms which render $f(x)$ itself, or any one of it dervatives, discontinuous. If we wish the function which remains expressed by an infinite series and its first μ derivatives to be continuous, we have only to subtract from A_n or B_n the terms at the commencement of its expansion, ending with the term containing $\frac{1}{n^{\mu+1}}$, and from $f(x)$ itself the sums of the series corresponding to the terms subtracted from A_n or B_n. These sums will be obtained by transforming products of sines and cosines into sums or differences, and then employing known formulæ such as

$$\frac{\cos z}{1^2} + \frac{\cos 3z}{3^2} + \ldots = \frac{\pi^2}{8} - \frac{\pi z}{4}, \text{ from } z = 0 \text{ to } z = \pi, \quad \ldots\ldots (41),$$

which are obtained by integrating several times the equation

$$\sin z + \tfrac{1}{2}\sin 2z + \tfrac{1}{3}\sin 3z + \ldots = \tfrac{1}{2}(\pi - z), \text{ from } z = 0 \text{ to } z = 2\pi,$$

or the equation deduced from it by writing $\pi - z$ for z, and taking the semi-sum of the results. It will be observed that in the several series to be summed we shall always have sines coming with odd powers of n and cosines with even. Of course, by clearing the series in (24) or (30) in the way just mentioned we shall increase the convergency of the infinite series in which a part of $f(x)$ still remains developed.

When A_n or B_n decreases faster than any inverse power of n as n increases, (as is the case for instance when it is the n^{th} term of a geometric series with a ratio less than 1,) all the terms of its expansion in a series according to inverse powers of n vanish. In this case, then, $f(x)$ and its derivatives of all orders are continuous.

20. In establishing the several theorems contained in this Section, it has been supposed that none of the derivatives of $f(x)$ which enter into the investigation are infinite. It should be observed, however, that if $f^{\mu}(x)$ is the last derivative employed, which only appears under the sign of integration, it is allowable to suppose that $f^{\mu}(x)$ becomes infinite any finite number of times within the limits of integration. To show this, we have only got to prove that

$$\int_0^a f^{\mu}(x) \sin \nu x\, dx \text{ or } \int_0^a f^{\mu}(x) \cos \nu x\, dx$$

approaches zero as its limit as ν increases beyond all limit. Let us consider the former of these integrals, and suppose that $f^{\mu}(x)$ becomes infinite only once, namely, when $x = \alpha$, within the limits of integration. Let the interval from 0 to a be divided into these four intervals 0 to $\alpha - \zeta$, $\alpha - \zeta$ to α, α to $\alpha + \zeta'$, $\alpha + \zeta'$ to a, where ζ and ζ' are supposed to be taken sufficiently small to exclude all values of x lying between the limits $\alpha - \zeta$ and $\alpha + \zeta'$ for which $f^{\mu-1}(x)$ alters discontinuously, or for which $f^{\mu}(x)$ changes sign, unless it be the value α. For the first and fourth intervals $f^{\mu}(x)$ is not infinite, and therefore, as it is known, the corresponding parts of the integral vanish for $\nu = \infty$. Since $\sin \nu x$ cannot lie beyond the limits $+1$ and -1, and is only equal to either limit for particular values of x, it is evident that the second and third portions of the integral are together numerically inferior to I, where

$$I = \{f^{\mu-1}(\alpha - \epsilon) \sim f^{\mu-1}(\alpha - \zeta)\} + \{f^{\mu-1}(\alpha + \zeta') \sim f^{\mu-1}(\alpha + \epsilon)\},$$

the symbol $A \sim B$ denoting the arithmetical difference of A and B, and ϵ being an infinitely small quantity, so that $f(\alpha - \epsilon)$, $f(\alpha + \epsilon)$ denote the limits to which $f(x)$ tends as x tends to the limit α by increasing and decreasing respectively. Hence the limit of the integral first considered, for $\nu = \infty$, must be less than I. But I may be made as small as we please by diminishing ζ and ζ', and therefore the limit required is zero.

The same proof applies to the integral containing $\cos \nu x$, and there is no difficulty in extending it to the case in which $f^{\mu}(x)$ is infinite more than once within the limits of integration, or at one of the limits.

21. It has hitherto been supposed that the function expanded in the series (3) or (22) does not become infinite; but the expansions will still be correct even if $f(x)$ becomes infinite any finite number of times, provided that $\int f(x)\, dx$ be essentially convergent. Suppose that $f(x)$ becomes infinite only when $x = \alpha$. Then it is evident that we may find a function of x, $F(x)$, which shall be equal to $f(x)$ except when x lies between the limits $\alpha - \zeta$ and $\alpha + \zeta'$, which shall remain

finite from $x = a - \zeta$ to $x = a + \zeta'$, and which shall be such that $\int_{a-\zeta}^{a+\zeta'} F(x)\,dx = \int_{a-\zeta}^{a+\zeta'} f(x)\,dx$. Suppose that we are considering the series (3). Then, if C_n be the coefficient of $\sin \frac{n\pi x}{a}$ in the expansion of $F(x)$ in a series of the form (3), it is evident that C_n will approach the finite limit A_n when ζ and ζ' vanish, where $A_n = \frac{2}{a}\int_0^a f(x) \sin \frac{n\pi x}{a}\,dx$. But so long as ζ and ζ' differ from zero the series $\Sigma C_n \sin \frac{n\pi x}{a}$ is convergent, and has $F(x)$ for its sum, and $F(x)$ becomes equal to $f(x)$ when ζ and ζ' vanish, for any value of x except a. We might therefore be disposed to conclude at once that the series (3) is convergent, and has $f(x)$ for its sum, unless it be for the particular value $x = a$; but this point will require examination, since we might conceive that the series (3) became divergent, or if it remained convergent that it had a sum different from $f(x)$, when ζ and ζ' were made to vanish before the summation was performed. If we agree not to consider the series (3) directly, but only the limit of the series (5) when g becomes 1, it follows at once from (7) that for values of x different from a that limit is the same as in Art. 4. For $x = a$ the limit required is that of $\frac{1}{2}\{f(a-\epsilon) + f(a+\epsilon)\}$ when ϵ vanishes. If $f(x)$ does not change sign as x passes through a the limit required is therefore positive or negative infinity, according as $f(x)$ is positive or negative; but if $f(x)$ changes sign in passing through ∞ the limit required may be zero, a finite quantity, or infinity. The expression just given for the limit may be proved without difficulty. In fact, according to the method of Art. 4, we are led to examine an integral of the form

$$\frac{1}{\pi}\int_0^{\zeta} \{f(a-\xi) + f(a+\xi)\} \frac{h\,d\xi}{h^2+\xi^2},$$

where ζ is a constant quantity which may be taken as small as we please, and supposed to vanish after h. Now by a known property of integrals the above integral is equal to

$$\frac{1}{\pi}\{f(a-\xi_1) + f(a+\xi_1)\}\int_0^{\zeta} \frac{h\,d\xi}{h^2+\xi^2}, \text{ where } \xi_1 \text{ lies between 0 and } \zeta.$$

But $\int_0^{\zeta} \frac{h\,d\xi}{h^2+\xi^2}$, which is equal to $\tan^{-1}\frac{\zeta}{h}$, becomes equal to $\frac{\pi}{2}$ when h vanishes, and the limit of ξ_1 when h vanishes must be zero, since it cannot be greater than ζ, and ζ may be made to vanish after h.

22. The same thing may be proved by the method which consists in summing the series $\Sigma \sin \frac{n\pi x}{a} \sin \frac{n\pi x'}{a}$ to n terms. If we adopt this method, then so long as we are considering a value of x different from a it will be found that the only peculiarity in the investigation is, that the quantity under the integral sign in the integrals we have to consider becomes infinite for one value of the variable; and it may be proved just as in Art. 20, that this circumstance has no effect on the result. If we are considering the value $x = a$, it will be found that the integral we shall have to consider will be

$$\frac{1}{\pi}\int_0^{\zeta} \frac{\sin \nu\xi}{\xi}\left\{f\left(a + \frac{2a}{\pi}\xi\right) + f\left(a - \frac{2a}{\pi}\xi\right)\right\} d\xi, \quad \dots\dots\dots\dots\dots\dots (42),$$

where ν is first to be made infinite, and then ζ may be supposed to vanish. If $f(a+\epsilon) + f(a-\epsilon)$ approaches a finite limit, or zero, when ϵ vanishes, as may be the case if $f(x)$ changes sign in passing through ∞, it may be proved, just as in the case in which $f(x)$ does not become infinite, that the above integral approaches the same limit as $\frac{1}{2}\{f(a+\epsilon) + f(a-\epsilon)\}$. In all cases however in which $f(x)$ does not change sign in passing through ∞, and in some cases in which it does change sign, $f(a+\epsilon)+f(a-\epsilon)$ becomes infinite when ϵ vanishes.

In such cases put for shortness

$$f(a+\frac{2a}{\pi}\xi)+f(a-\frac{2a}{\pi}\xi)=F(\xi),$$

and let the numerical values of the integral $\int\frac{\sin\nu\xi}{\xi}d\xi$ taken from 0 to $\frac{\pi}{\nu}$, from $\frac{\pi}{\nu}$ to $\frac{2\pi}{\nu}$ ... or which is the same those of $\int\frac{\sin\xi}{\xi}d\xi$ taken from 0 to π, from π to 2π ... be denoted by I_1, I_2 ... Then evidently $I_1>I_2>I_3$ ... Also, if ζ be sufficiently small, $F(\xi)$ will decrease from $\xi=0$ to $\xi=\zeta$, if we suppose, as we may, $F(\xi)$ to be positive. Hence the integral (42), which is equal to

$$\frac{1}{\pi}\{I_1F(\xi_1)-I_2F(\xi_2)+I_3F(\xi_3)-...\}, \quad\text{......................... (43)},$$

where ξ_1, ξ_2 ... are quantities lying between 0 and $\frac{\pi}{\nu}$, $\frac{\pi}{\nu}$ and $\frac{2\pi}{\nu}$ ... is greater than

$$\frac{1}{\pi}\{I_1F(\xi_1)-I_2F(\xi_2)\},$$

if we neglect the incomplete pair of terms which may occur at the end of the series (43), and which need not be considered, since they vanish when $\nu=\infty$. Hence, the integral (42) is *a fortiori* $>\frac{1}{\pi}(I_1-I_2)F(\xi_1)$. But ξ_1 vanishes and $F(\xi_1)$ becomes infinite when ν becomes infinite; and therefore for the particular value $x=a$ the sum of the first n terms of the series (3) increases indefinitely with n.

If a coincides with one of the extreme values 0 and a of x, the sum of the series (3) vanishes for $x=a$. This comes under the formula given above if we consider the sum of the series for values of x lying beyond the limits 0 and a. The same proof as that given in the present and last article will evidently apply if $f(x)$ become infinite for several values of x, or if the series considered be (22) instead of (3). In this case, the sum of the series becomes infinite for $x=a$ when $a=0$ or $=a$.

23. Hence it appears that $f(x)$ may be expanded in a series of the form (3) or (22), provided only $\int f(x)\,dx$ be continuous. It should be observed however that functions like $\left(\sin\frac{c}{x}\right)^{-\frac{2}{3}}$, which become infinite or discontinuous an infinite number of times within the limits of the variable within which they are considered, have been excluded from the previous reasoning.

Hence, we may employ the formulæ such as (26), (35), &c., to obtain the direct developement of $f^\mu(x)$, without enquiring whether it becomes infinite or not within the limits of the variable for which it is considered. All that is necessary is that $f(x)$ and its derivatives up to the $(\mu-1)^{\text{th}}$ inclusive should not be infinite within those limits, although they may be discontinuous.

24. In obtaining the formulæ of Arts. 7 and 13, and generally the formulæ which apply to the case in which A_n or B_n is given, and $f(x)$ is unknown, it has hitherto been supposed that we knew *a priori* that $f(x)$ was a function of the class proposed in Art. 1 for consideration, or at least of that class with the extension mentioned in the preceding article. Suppose now that we have simply presented to us the series (3) or (22), namely

$$\Sigma A_n\sin\frac{n\pi x}{a} \text{ or } B_0+\Sigma B_n\cos\frac{n\pi x}{a},$$

where A_n or B_n is supposed given, and want to know, *first*, whether the series is convergent, *secondly*, whether if it be convergent it is the direct developement of its sum $f(x)$, and *thirdly*, whether we may directly employ the formulæ already obtained, trusting to the formulæ themselves to give notice of the cases to which they do not apply by leading to processes which cannot be effected.

25. If the series ΣA_n or ΣB_n is essentially convergent, it is evident *a fortiori* that the series (3) or (22) is convergent.

If $A_n = S\dfrac{c}{n}\cos n\gamma + C_n$, or if $B_n = S\dfrac{c}{n}\sin n\gamma + C_n$, where ΣC_n is essentially convergent, the given series will be convergent, as is proved in Art. 6.

In either of these cases let $f(x)$ be the sum of the given series. Suppose that it is the series of sines which we are considering. Let E_n be the coefficient of $\sin\dfrac{n\pi x}{a}$ in the direct developement of $f(x)$. Then we have

$$f(x) = \Sigma A_n \sin\frac{n\pi x}{a} = \Sigma E_n \sin\frac{n\pi x}{a};$$

and since both series are convergent, if we multiply by any finite function of x, $\phi(x)$, and integrate, we may first integrate each term, and then sum, instead of first summing and then integrating. Taking $\phi(x) = \sin\dfrac{n\pi x}{a}$, and integrating from $x=0$ to $x=a$, we get $E_n = A_n$, so that the given series is the direct developement of its sum $f(x)$. The proof is the same for the series of cosines.

26. Consider now the more general case in which the series $\Sigma\dfrac{1}{n}A_n$ is essentially convergent. The reasoning which is about to be offered can hardly be regarded as absolutely rigorous; nevertheless the proposition which it is endeavoured to establish seems worthy of attention. Let u_n be the sum of the first n terms of the given series, and $F(n, x)$ the sum of the first n terms of the series $\Sigma -\dfrac{a}{n\pi}A_n\cos\dfrac{n\pi x}{a}$. Then we have

$$\int(u_{n+m} - u_n)\,dx = F(n+m, x) - F(n, x) = \psi(n, x), \text{ suppose. } \ldots\ldots (44).$$

Now by hypothesis the series $\Sigma\dfrac{1}{n}A_n$ is essentially convergent, and therefore *a fortiori* the series $\Sigma -\dfrac{a}{n\pi}A_n\cos\dfrac{n\pi x}{a}$ is convergent, and therefore $\psi(\infty, x) = 0$, whatever be the value of m. Let the limits of x in (44) be x and $x + \Delta x$, and divide by Δx, and we get

$$\frac{1}{\Delta x}\int_x^{x+\Delta x}(u_{n+m} - u_n)\,dx = \frac{\Delta\psi(n, x)}{\Delta x}:$$

and as we have seen the limit of the second side of this equation when we suppose n first to become infinite and then Δx to vanish is zero. But *for general values of* x the limit will remain the same if we first suppose Δx to vanish and then n to become infinite; and on this supposition we have

$$\text{limit of } (u_{n+m} - u_n) = 0, \text{ for } n = \infty;$$

so that for general values of x the series considered is convergent.

To illustrate the assumption here made that for general values of x the order in which n and Δx assume their limiting values is immaterial, let $\psi(y, x)$ be a continuous function of x which becomes equal to $\psi(n, x)$ when y is a positive integer; and consider the surface whose equation is $z = \psi(y, x)$. Since $\psi(\infty, x) = 0$ for integral values of y, the surface approaches indefinitely to the plane xy when y becomes infinite; or rather, among the infinite number of admissible forms of $\psi(y, x)$ we may evidently choose an infinite number for which that is the case. Now the assertion made comes to this; that if we cut the surface by a plane parallel to the plane xz, and at a distance n from it, the tangent at the point of the section corresponding to any given value of x will ultimately lie in the plane xy when n becomes infinite, except in the case of singular, isolated values of x, whose number is finite between $x = 0$ and $x = a$. For such values the sum $f(x)$ of the infinite series may become infinite, while $\int f(x)\,dx$ remains finite. The assumption just made appears evident unless A_n be a function of n whose complexity increases indefinitely with its rank, *i.e.* with the value of n.

Since the integral of $f(x)$ is continuous, $f(x)$ may be expanded by the formula in a series of sines. Let E_n be the coefficient of $\sin \frac{n\pi x}{a}$ in its direct expansion; so that,

$$\left.\begin{aligned} f(x) &= \Sigma A_n \sin \frac{n\pi x}{a}, \\ f(x) &= \Sigma E_n \sin \frac{n\pi x}{a}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(45),$$

where both series are convergent, except it be for isolated values of x. Consequently, we have, in a series which is convergent, at least for general values of x,

$$0 = \Sigma(A_n - E_n)\sin\frac{n\pi x}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(46).$$

The series (45) may become divergent for isolated values of x, and are in fact divergent for values of x which render $f(x)$ infinite. But the first side of (46) being constantly zero, and the series at the second side being convergent for general values of x, it does not seem that it can become divergent for isolated values. Hence according to the preceding article the second side of the equation is the direct developement of the first side, *i.e.* of zero; and therefore $E_n = A_n$, or the given series is the direct developement of its sum, which is what it was required to prove. The same reasoning applies to the series of cosines.

It may be observed that the well known series,

$$\tfrac{1}{2} + \cos x + \cos 2x + \cos 3x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(47),$$

forms no exception to the preceding observation. This series is in fact divergent for general values of x, that is to say not convergent, and in that respect it totally differs from the series in (46). When it is asserted that the sum of the series (47) is zero except for $x = 0$ or any multiple of 2π, when it is infinite, all that is meant is that the limit to which the sum of the convergent series $\frac{1}{2} + \Sigma g^n \cos nx$ approaches when g becomes 1 is zero, except for $x = 0$ or any multiple of 2π, in which case it is infinity.

27. It follows from the preceding article that even without knowing *a priori* the nature of the function $f(x)$ we may employ the formulæ such as (35), provided that if $n^{-\mu}$ be the highest power of $\frac{1}{n}$ required by the formula, and $n^{-\mu}C_n$ the remainder in the expansion of A_n, the series $\Sigma \frac{1}{n} C_n$ be essentially convergent. For let G_n be the sum of the terms as far as that containing $n^{-\mu}$ in the expansion of A_n, those terms having the form assigned by (35), that is to say cosines like $\cos n\gamma$ coming along with odd powers of $\frac{1}{n}$, and sines along with even powers. Then $A_n = G_n + n^{-\mu}C_n$.

Let
$$\Sigma G_n \sin \frac{n\pi x}{a} = F(x);$$

then
$$f(x) - F(x) = \Sigma n^{-\mu} C_n \sin \frac{n\pi x}{a} \quad \text{.........................(48)}.$$

Now if $\phi(x) = \Sigma u_n$, where the series Σu_n, $\Sigma \frac{du_n}{dx}$ are both convergent, we may find $\phi'(x)$ by differentiating under the sign of summation. This is evident, since by the theorem referred to in Art. 2 (note), we may find $\int \Sigma \frac{du_n}{dx} dx$ by integrating under the sign of summation. Consequently we have from (48)

$$f^{\mu-1}(x) - F^{\mu-1}(x) = \pm \left(\frac{\pi}{a}\right)^{\mu-1} \Sigma \frac{1}{n} C_n \begin{matrix}\sin\\ \cos\end{matrix} \frac{n\pi x}{a} \quad \text{.....................(49)};$$

and since the series $\Sigma \frac{1}{n} C_n$ is essentially convergent, the convergency of the series forming the right-hand side of (49) cannot become infinitely slow (see Sect. III.), and therefore, the n^{th} term being a continuous function of x, the sum is also a continuous function of x, and therefore $f^{\mu}(x) - F^{\mu}(x)$ is a function which by Art. 23 can be expanded in a series of sines or cosines. But $F^{\mu}(x)$ is also such a function, being in fact a constant, and therefore $f^{\mu}(x)$ is a function of the kind considered in Art. 23, which is what is assumed in obtaining the formula (35).

It may be observed that these results do not require the assumptions of Art. 26 in the case in which the series ΣC_n is essentially convergent, or composed of an essentially convergent series and of a series of the form $\Sigma S \frac{c}{n} \sin n\gamma$ or $\Sigma S \frac{c}{n} \cos n\gamma$, according as C_n is the coefficient of a cosine or of a sine.

SECTION II.

Mode of ascertaining the nature of the discontinuity of the integrals which are analogous to the series considered in Section I, and of obtaining the developements of the derivatives of the expanded functions.

28. LET us consider the following integral, which is analogous to the series in (1),

$$\int_0^\infty \phi(\beta) \sin \beta x \, d\beta \quad \text{..........................(50)},$$

where
$$\phi(\beta) = \frac{2}{\pi} \int_0^a f(x') \sin \beta x' \, dx' \quad \text{..............(51)}.$$

Although the integral (50) may be written as a double integral,

$$\frac{2}{\pi} \int_0^\infty \int_0^a f(x') \sin \beta x \sin \beta x' \, d\beta \, dx' \quad \text{..............(52)},$$

the integration with respect to x' must be performed first, because, the integral of $\sin\beta x \sin\beta x' d\beta$ not being convergent at the limit ∞, $\int_0^\infty \sin\beta x \sin\beta x' d\beta$ would have no meaning. Suppose, however, that instead of (52) we consider the integral,

$$\frac{2}{\pi}\int_0^\infty \int_0^a f(x')\epsilon^{-h\beta} \sin\beta x \sin\beta x' d\beta dx' \ldots\ldots\ldots\ldots (53),$$

where h is a positive constant, and ϵ is the base of the Napierian logarithms. It is easy to see that at least in the case in which the integral (50) is essentially convergent its value is also the limit to which the integral (53) tends when h tends to zero as its limit. It is well known that the limit of (53) when h vanishes is in general $f(x)$; but when $x=0$ the limit is zero; when $x=a$ the limit is $\frac{1}{2}f(a)$; and when $f(x)$ is discontinuous it is the arithmetic mean of the values of $f(x)$ for two values of x infinitely little greater and less respectively than the critical value. When $x > a$ it is zero, and in all cases it is the same, except as to sign, for negative as for positive values of x.

We may always speak of the limit of (53), but we cannot speak of the integral (50) till we assure ourselves that it is convergent. Now we get by integration by parts,

$$\int f(x')\sin\beta x' dx' = -\frac{1}{\beta}f(x')\cos\beta x' + \frac{1}{\beta^2}f'(x')\sin\beta x' - \frac{1}{\beta^2}\int f''(x')\sin\beta x' dx' \ldots\ldots (54).$$

When this integral is taken between limits, the first term will furnish a set of terms of the form $\frac{C}{\beta}\cos\beta\alpha$, where α may be zero, and the last two terms will give a result numerically less than $\frac{L}{\beta^2}$, where L is a constant properly chosen. Now whether α be zero or not, $\int\cos\beta\alpha\sin\beta x\frac{d\beta}{\beta}$ is convergent at the limit ∞, and moreover its value taken from any finite value of β to $\beta=\infty$ is the limit to which the integral deduced from it by inserting the factor $\epsilon^{-h\beta}$ tends when h vanishes. The remaining part of the integral (50) is essentially convergent at the limit ∞. Hence the integral (50) is convergent, and its value for all values of x, both critical and general, is the limit to which the value of the integral (53) tends when h vanishes.

29. Suppose that we want to find $f''(x)$, knowing nothing about $f(x)$, at least for general values of x, except that it is the value of the integral (50), and that it is not a function of the class excluded from consideration in Art. 1. We cannot differentiate under the integral sign, because the resulting integral would, usually at least, be divergent at the limit ∞. We may however find $f''(x)$ provided we know the values of x for which $f(x)$ and $f'(x)$ are discontinuous, and the quantities by which $f(x)$ and $f'(x)$ are suddenly increased as x increases through each critical value, supposing the extreme values included among those for which $f(x)$ or $f'(x)$ is discontinuous, under the same convention as in Art. 6. Let α be any one of the critical values of x; Q, Q_1 the quantities by which $f(x)$, $f'(x)$ are suddenly increased as x increases through α; S the sign of summation referring to the critical values of x; $\phi_2(\beta)$ the coefficient of $\sin\beta x$ in the direct developement of $f''(x)$ in a definite integral of the form (50). Then taking the integrals in (54) between limits, and applying the formula (51) to $f''(x)$, we get

$$\phi_2(\beta) = -\beta^2\phi(\beta) + \frac{2}{\pi}\beta SQ\cos\beta\alpha - \frac{2}{\pi}SQ_1\sin\beta\alpha.$$

We may find $\phi_\mu(\beta)$ in a similar manner. We get thus when μ is even

$$(-1)^{\frac{\mu}{2}}\phi_\mu(\beta) = \beta^\mu\phi(\beta) - \frac{2}{\pi}\beta^{\mu-1}SQ\cos\beta\alpha + \frac{2}{\pi}\beta^{\mu-2}SQ_1\sin\beta\alpha + \ldots$$

$$+ (-1)^{\frac{\mu}{2}+1}\frac{2}{\pi}SQ_{\mu-1}\sin\beta\alpha \ldots\ldots\ldots\ldots\ldots\ldots (55),$$

where sines and cosines occur alternately, and two signs of the same kind are always followed by two of the opposite. The expression for $\phi^{\mu}(\beta)$ when μ is odd might be found in a similar manner. These formulæ enable us to express $f^{\mu}(x)$ when $\phi(\beta)$ is an arbitrary function which has to be determined, and $f(0)$, &c. are given.

30. If however $\phi(\beta)$ should be given, and $f(0)$, &c. be unknown, $\phi(\beta)$ will admit of expansion according to powers of β^{-1}, beginning with the first, provided we treat $\sin\beta a$ or $\cos\beta a$ as if it were a constant coefficient; and $\sin\beta a$, $\cos\beta a$ will occur with even and odd powers of β respectively. The possibility of the expansion of $\phi(\beta)$ in this form depends of course on the circumstance that $\phi(x)$ is a function of the class which it is proposed in Art. 1, to consider, or at least with the extension mentioned in Art. 23. It appears from (55) that in order to express $f^{\mu}(x)$ as a definite integral of the form (50) we have only got to expand $\phi(\beta)$, to differentiate (50) μ times with respect to x, differentiating under the integral sign, and to reject those terms which appear under the integral sign with positive powers of β or with the power 0. The same rule applies whether μ be odd or even.

31. If we have given $\phi(a)$, but are not able to evaluate the integral (50), we may notwithstanding that find the values of x which render $f(x)$ or any of its derivatives discontinuous, and the quantities by which the function considered is suddenly increased. For this purpose it is only necessary to compare the expansion of $\phi(\beta)$ with the expansion

$$\phi(\beta) = \frac{2}{\pi\beta} SQ\cos\beta a - \frac{2}{\pi\beta^2} SQ_1 \sin\beta a - \dots \quad \dots\dots\dots\dots(56),$$

given by (55), just as in the case of series.

We may easily if we please clear the function $\phi(\beta)$ of the part for which $f(x)$ or any one of its derivatives is discontinuous, or does not vanish for $x = 0$ and $x = a$. For this purpose it will be sufficient to take any function $F(x)$ at pleasure, which as well as its derivatives of the orders considered has got the same discontinuity as $f(x)$ and its derivatives, to develope $F(x)$ in a definite integral of the form $\int_0^\infty \Phi(\beta)\sin\beta x\, d\beta$ by the formula (51), and to subtract $F(x)$ from $f(x)$ and $\Phi(\beta)$ from $\phi(\beta)$. It will be convenient to choose such simple functions as $l + mx + nx^2$; $l\sin x + m\cos x$; $l\epsilon^{-x} + m\epsilon^{-2x}$, &c. for the algebraical expressions of $F(x)$ for the several intervals throughout which it is continuous, the functions chosen being such as admit of easy integration when multiplied by $\sin\beta x dx$, and which furnish a sufficient number of indeterminate coefficients to allow of the requisite conditions as to discontinuity being satisfied. These conditions are that the several values of Q, Q_1, &c. shall be the same for $F(x)$ as for $f(x)$.

32. Whenever $\int_0^\infty f(x)\,dx$ is essentially convergent, we may at once put $a = \infty$ in the preceding formulæ. For, first, it may be easily proved that in this case, (though not in this case only,) the limit of (53) when h vanishes is $f(x)$; secondly, the limit of (53) is also the value of (52); and, lastly, all the derivatives of $f(x)$ have their integrals, (which are the preceding derivatives,) essentially convergent, and therefore ∞ may be put for a in the developements of the derivatives in definite integrals.

When $f(x)$ tends to zero as its limit as x becomes infinite, and moreover after a finite value of x does not change from decreasing to increasing nor from increasing to decreasing,

$$\int_0^\infty \epsilon^{-hx'} f(x')\sin\beta x' dx'$$

will be more convergent than $\int_0^\infty f(x')\sin\beta x'dx'$, and the latter integral will be convergent, and its convergency will remain finite* when β vanishes. In this case also we may put $a=\infty$.

Thus if $f(x)=\sin lx(b^2+x^2)^{-1}$, we may put $a=\infty$ because $f(x)$ has its integral essentially convergent: if $f(x)=(b+x)^{-\frac{1}{2}}$, we may put $a=\infty$ because $f(x)$ is always decreasing to zero as its limit. But if $f(x)=\sin lx\,(b+x)^{-\frac{1}{2}}$, the preceding rules will not apply, because $f(x)$, though it has zero for its limit, is sometimes increasing and sometimes decreasing. And in fact in this case the integral in equation (51) will be divergent when $\beta=l$, and $\phi(\beta)$ will become infinite for that value of β. It is true that $f(x)$ is still the limit to which the integral (53) tends when h vanishes; but I do not intend to enter into the consideration of such cases in this paper.

33. When ∞ may be put for a, and $f(x)$ is continuous, we get from (55)

$$(-1)^{\frac{\mu}{2}}\phi_\mu(\beta)=\beta^\mu\phi(\beta)-\frac{2}{\pi}\beta^{\mu-1}f(0)+\frac{2}{\pi}\beta^{\mu-3}f''(0)-\ldots+(-1)^{\frac{\mu}{2}}\frac{2}{\pi}\beta f^{\mu-2}(0).\quad\ldots\ldots(57).$$

In this case $\phi(\beta)$ will admit of expansion, at least to a certain number of terms, according to odd negative powers of β. If we suppose $\phi(\beta)$ known, and the expansion performed, so that

$$\phi(\beta)=H_0\beta^{-1}+H_2\beta^{-3}+H_4\beta^{-5}+\ldots$$

and compare the result with (49), we shall get

$$f(0)=\frac{\pi}{2}H_0;\quad f''(0)=-\frac{\pi}{2}H_2;\quad f^{\text{I}}(0)=\frac{\pi}{2}H_4;\ \&\text{c}.\quad\ldots\ldots(58).$$

34. The integral

$$\int_0^\infty\psi(\beta)\cos\beta x\,d\beta,\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (59),$$

where
$$\psi(\beta)=\frac{2}{\pi}\int_0^a f(x')\cos\beta x'dx',\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (60),$$

which is analogous to the series (22), is another in which it is sometimes useful to develope a function or conceive it developed. For positive values of x the value of (59) is the same as that of (50). When $x=0$ the value is $f(0)$; and for negative values of x it is the same as for positive. It is supposed here that the integral (59) is convergent, which it may be proved to be in the same manner as the integral (50) was proved to be convergent.

Suppose that we wish to find, in terms of $\psi(\beta)$, the developement of $f^\mu(x)$ in a definite integral of the form (50) or (59), according as μ is odd or even. We cannot differentiate under the integral sign, because the resulting integral would be divergent. We may however obtain the required developement by transforming the expression $\psi(\beta)$ by integration by parts, just as before. We thus get for the case in which μ is odd

$$(-1)^{\frac{\mu+1}{2}}\phi_\mu(\beta)=\beta^\mu\psi(\beta)+\frac{2}{\pi}\beta^{\mu-1}SQ\sin\beta a+\frac{2}{\pi}\beta^{\mu-2}SQ_1\cos\beta a-\ldots$$
$$+(-1)^{\frac{\mu-1}{2}}\frac{2}{\pi}SQ_{\mu-1}\sin\beta a,\ \ldots\ldots(61),$$

where $\phi_\mu(\beta)$ is the value of $\phi(\beta)$ in the direct developement of $f^\mu(x)$ in the integral (50). In the same way we may get the value of $\psi_\mu(\beta)$ when μ is even, $\psi_\mu(\beta)$ being the value of $\psi(\beta)$ in the direct developement of $f^\mu(x)$ by the formulæ (59), (60).

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* See next Section.

The equation (61) is applicable to the case in which $\psi(\beta)$ is an arbitrary function, and a, Q, &c., are given. If however $\psi(\beta)$ should be given, we may find $\phi_\mu(\beta)$ or $\psi_\mu(\beta)$ by the same rule as before.

In the case in which $\psi(\beta)$ is given, we may find the values of a, Q, &c., without being able to evaluate the integral (59). For this purpose it is sufficient to expand $\psi(\beta)$ according to negative powers of β, and compare the expansion with that furnished by equation (61).

35. The same remarks as to the cases in which we are at liberty to put ∞ for a apply to (60) as to (51), with one exception. In the case in which $f(x)$ approaches zero as its limit, and is at last always decreasing numerically, or at least never increasing, as x increases, while $\int f(x)\,dx$ is divergent at the limit ∞, it has been observed that $\phi(\beta)$ remains finite when β vanishes. This however is not the case with $\psi(\beta)$, at least in general. I say *in general*, because, although $\int_0^x f(x)\,dx$ increases indefinitely with its superior limit, we are not entitled at once to conclude from thence that $\int_0^\infty \cos\beta x f(x)\,dx$ becomes infinite when β vanishes, as will appear in Section III. It may be shown from the known value of $\int_0^\infty x^{-n}\cos\beta x\,dx$, where $1 > n > 0$, that if $f(x) = F(x) + Cx^{-n}$, where $F(x)$ is such that $\int F(x)\,dx$ is convergent at the limit ∞, $\psi(\beta)$ becomes infinite when β vanishes; and the same would be true if there were any finite number of terms of the form Cx^{-n}. There is no occasion however to enquire whether $\psi(\beta)$ *always* becomes infinite: the point to consider is whether the integral (59) is always convergent at the limit zero.

In considering this question, we may evidently begin the integration relative to x' at any value x_0 that we please. Suppose first that we integrate from $x' = x_0$ to $x' = X$, and let $\varpi(\beta)$ be the result, so that

$$\varpi(\beta) = \frac{2}{\pi}\int_{x_0}^{X} f(x')\cos\beta x'\,dx'.$$

Let $\varpi_{,}(\beta)$ be the indefinite integral of $\varpi(\beta)\,d\beta$: then, c being a positive quantity, we get from the above equation

$$\varpi_{,}(\beta) - \varpi_{,}(c) = \frac{2}{\pi}\int_{x_0}^{X} f(x')\,\{\sin\beta x' - \sin cx'\}\,\frac{dx'}{x'}.$$

Now put $X = \infty$. Then since $\int_{x_0}^{\infty} f(x')\,\dfrac{\sin\beta x'}{x'}\,dx'$ is a convergent integral, and its convergency remains finite (Art. 39.) when β vanishes, as may be proved without much difficulty, its value cannot become infinite, and therefore $\varpi_{,}(\beta)$ does not become infinite when β vanishes. Now

$$\int\varpi(\beta)\cos\beta x\,d\beta = \varpi_{,}(\beta)\cos\beta x + x\int\varpi_{,}(\beta)\sin\beta x\,d\beta, \quad \ldots\ldots\ldots (62),$$

when x is positive; and when $x = 0$,

$$\int\varpi(\beta)\,d(\beta) = \varpi_{,}(\beta):$$

hence in either case $\int\varpi(\beta)\cos\beta x\,d\beta$ is convergent at the limit zero. Now the quantity by which $\varpi(\beta)$ differs from $\psi(\beta)$ evidently cannot render (59) divergent, and therefore in the case considered the integral (59) is convergent at the limit zero.

By treating $\int_0^\infty \varpi(\beta)\,\epsilon^{-h\beta}\cos\beta x\,d\beta$ in the manner in which $\int\varpi(\beta)\cos\beta x\,d\beta$ is treated in (62), it may be shown that the convergency of the former integral remains finite when h vanishes. Hence, not only is the integral (59) convergent, but its value is the limit to which the integral similar to (53) tends when h vanishes.

When $f(x)$ is continuous, and ∞ may be put for a, we have from (61)

$$(-1)^{\frac{\mu+1}{2}}\phi_\mu(\beta) = \beta^\mu\psi(\mu) + \frac{2}{\pi}\beta^{\mu-2}f'(0) - \frac{2}{\pi}\beta^{\mu-4}f'''(0) + \dots + (-1)^{\frac{\mu+1}{2}}\frac{2}{\pi}\beta f^{\mu-2}(0). \dots (63).$$

If $\psi(\beta)$ be given we can find the values of $f'(0)$, $f'''(0)$ ... just as before.

36. The integral

$$\frac{1}{\pi}\int_0^\infty \int_{-a_,}^{a} \cos\beta(x'-x)f(x')\,d\beta\,dx', \dots\dots\dots\dots (64),$$

in which the integration with respect to x' is supposed to be performed before that with respect to β, so that the integral has the form

$$\int_0^\infty \chi(\beta)\cos\beta x\,d\beta + \int_0^\infty \sigma(\beta)\sin\beta x\,d\beta, \quad \dots\dots\dots (65),$$

may be treated just as the integral (59); and it may be shown that in the same circumstances we may replace the limits $-a_,$ and a by $-\infty$, $+\infty$ respectively. If we suppose $\chi(\beta)$ and $\sigma(\beta)$ known, we may find as before the values of x for which $f(x)$, $f'(x)$ ... are discontinuous, and the quantities by which those functions are suddenly increased. We may also find the direct developement of $f'(x)$, $f''(x)$ ... in two integrals of the form (65); and we may if we please clear the integrals (65) of the part which renders $f(x)$, $f'(x)$ ... discontinuous.

37. In the developement of $f(x)$ in an integral of the form (50) or (59), or in two integrals of the form (65), it has hitherto been supposed that $f(x)$ is not infinite. It may be observed however that it is allowable to suppose $f(x)$ to become infinite any finite number of times, provided $\int f(x)\,dx$ be essentially convergent about the values of x which render $f(x)$ infinite. This may be shown just as in the case of series. Hence, the formulæ such as (55) which give the developement of $f^\mu(x)$ are true even when $f^\mu(x)$ is infinite, $f^{\mu-1}(x)$ being finite.

SECTION III.

On the discontinuity of the sums of infinite series, and of the values of integrals taken between infinite limits.

38. LET $u_1 + u_2 \dots + u_n + \dots$ .... (66),

be a convergent infinite series having U for its sum. Let

$v_1 + v_2 \dots + v_n + \dots$ .... (67),

be another infinite series of which the general term v_n is a function of the positive variable h, and becomes equal to u_n when h vanishes. Suppose that for a sufficiently small value of h and all inferior values the series (67) is convergent, and has V for its sum. It might at first sight be supposed that the limit of V for $h = 0$ was necessarily equal to U. This however is not true. For let the sum to n terms of the series (67) be denoted by $f(n, h)$: then the limit of V is the limit of $f(n, h)$ when n first becomes infinite and then h vanishes, whereas U is the limit of $f(n, h)$ when h first vanishes and then n becomes infinite, and these limits may be different. Whenever a discontinuous function is developed in a periodic series like (15) or (30) we have an instance of this: but it is easy to form two series, having nothing to do with periodic series, in which the same happens. For this purpose it is only requisite to take for $f(n, h) - U_n$, (U_n being the sum of the first n terms of (66),) a quantity which has different limiting values according to the order in which n and h are supposed to assume their limiting values, and which has for its finite difference a quantity which vanishes when n becomes infinite, whether h be a positive quantity sufficiently small or be actually zero.

For example, let

$$f(n,h) - U_n = \frac{2nh}{nh+1}, \quad \ldots\ldots\ldots\ldots \quad (68),$$

which vanishes when $n = 0$. Then

$$\Delta\{f(n,h) - U_n\} = v_{n+1} - u_{n+1} = \frac{2h}{(nh+1)(nh+h+1)}.$$

Assume $$U_n = 1 - \frac{1}{n+1}, \text{ so that } u_n = \Delta U_{n-1} = \frac{1}{n(n+1)},$$

and we get the series

$$\frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} \ldots + \frac{1}{n(n+1)} + \ldots, \quad \ldots\ldots\ldots\ldots \quad (69),$$

$$\frac{1+5h}{2(1+h)} \ldots + \frac{h(h+2)n^2 + h(4-h)n + 1 - h}{n(n+1)\{(n-1)h+1\}(nh+1)} + \ldots, \quad \ldots\ldots \quad (70).$$

which are both convergent, and of which the general terms become the same when h vanishes. Yet the sum of the first is 1, whereas the sum of the second is 3.

If the numerator of the fraction on the right-hand side of (68) had been pnh instead of $2nh$, the sum of the series (70) would have been $p+1$, and therefore the limit to which the sum approaches when h vanishes would have been $p+1$. Hence we can form as many series as we please like (67) having different quantities for the limits of their sums when h vanishes, and yet all having their n^{th} terms becoming equal to u_n when h vanishes. This is equally true whether the series (66) be convergent or divergent, the series like (67) of course being always supposed to be convergent for all positive values of h however small.

39. It is important for the purposes of the present paper to have a ready mode of ascertaining in what cases we may replace the limit of (67) by (66). Now it follows from the following theorem that this substitution may at once be made in an extensive class of cases.

THEOREM. The limit of V can never differ from U unless the convergency of the series (67) become infinitely slow when h vanishes.

The convergency of the series is here said to become infinitely slow when, if n be the number of terms which must be taken in order to render the sum of the neglected terms numerically less than a given quantity e which may be as small as we please, n increases beyond all limit as h decreases beyond all limit.

DEMONSTRATION. If the convergency do not become infinitely slow, it will be possible to find a number n_1 so great that for the value of h we begin with and for all inferior values greater than zero the sum of the neglected terms shall be numerically less than e. Now the limit of the sum of the first n_1 terms of (67) when h vanishes is the sum of the first n_1 terms of (66). Hence if e' be the numerical value of the sum of the terms after the n_1^{th} of the series (66), U and the limit of V cannot differ by a quantity so great as $e + e'$. But e and e' may be made smaller than any assignable quantities, and therefore U is equal to the limit of V.

COR. 1. If the series (66) is essentially convergent, and if, either from the very beginning, or after a certain term whose rank does not depend upon h, the terms of (67) are numerically less than the corresponding terms of (66), the limit of V is equal to U.

For in this case the series (67) is more rapidly convergent than (66), and therefore its convergency remains finite.

Cor. 2. If the series (66) is essentially convergent, and if the terms of (67) are derived from those of (66) by multiplying them by the ascending powers of a quantity g which is less than 1, and which becomes 1 in the limit, the limit of V is equal to U.

It may be observed that when the convergency of (67) does not become infinitely slow when h vanishes there is no occasion to prove the convergency of (66), since it follows from that of (67). In fact, let V_n be the sum of the first n terms of (67), U_n the same for (66), V_0 the value of V for $h=0$. Then by hypothesis we may find a finite value of n such that $V-V_n$ shall be numerically less than e, however small h may be; so that

$$V = V_n + \text{ a quantity always numerically less than } e.$$

Now let h vanish: then V becomes V_0 and V_n becomes U_n. Also e may be made as small as we please by taking n sufficiently great. Hence U_n approaches a finite limit when n becomes infinite, and that limit is V_0.

Conversely, if (66) is convergent, and if $U=V_0$, the convergency of the series (67) cannot become infinitely slow when h vanishes.

For if U_n', V_n' represent the sums of the terms after the n^{th} in the series (66), (67) respectively, we have

$$V = V_n + V_n', \quad U = U_n + U_n';$$

whence

$$V_n' = V - U - (V_n - U_n) + U_n'.$$

Now $V-U$, V_n-U_n vanish with h, and U_n' vanishes when n becomes infinite. Hence for a sufficiently small value of h and all inferior values, together with a value of n sufficiently large, and independent of h, the value of V_n' may be made numerically less than any given quantity e however small; and therefore, by definition, the convergency of the series (67) does not become infinitely slow when h vanishes.

On the whole, then, when the convergency of the series (67) does not become infinitely slow when h vanishes, the series (66) is necessarily convergent, and has V_0 for its sum: but in the contrary case there must necessarily be a discontinuity of some kind. Either V must become infinite when h vanishes, or the series (66) must be divergent, or, if (66) is convergent as well as (67), U must be different from V_0.

When a finite function of x, $f(x)$, which passes suddenly from M to N as x increases through α, where $a > \alpha > 0$, is expanded in the series (15) or (30), we have seen that the series is always convergent, and its sum for all values of x except critical values is $f(x)$, and for $x=\alpha$ its sum is $\frac{1}{2}(M+N)$. Hence the convergency of the series necessarily becomes infinitely slow when $\alpha - x$ vanishes. In applying the preceding reasoning to this case it will be observed that h is $\alpha - x$, V_0 is M, and U is $\frac{1}{2}(M+N)$, if we are considering values of x a little less than α; but h is $x-\alpha$ and V_0 is N, if we are considering values of x a little greater than α.

When the series (66) is convergent, as well as (67), it may be easily proved that in all cases

$$U = V_0 - L,$$

where L is the limit of V_n' when h is first made to vanish and then n to become infinite.

40. Reasoning exactly similar to that contained in the preceding article may be applied to integrals, and the same definitions may be used. Thus if $\int_a^{\infty} F(x, h)\,dx$ is a convergent integral, we may say that the convergency becomes infinitely slow when h vanishes, when, if X be the superior limit to which we must integrate in order that the neglected part of the integral, or

$\int_X^\infty F(x, h)\, dx$, may be numerically less than a given constant e which may be as small as we please, X increases beyond all limit when h vanishes.

The reasoning of the preceding article leads to the following theorems.

If $V = \int_a^\infty F(x, h)\, dx$, if V_0 be the limit of V when $h = 0$, and if $F(x, 0) = f(x)$; then, if the convergency of the integral V do not become infinitely slow when h vanishes, $\int_a^\infty f(x)\, dx$ must be convergent, and its value must be V_0. But in the contrary case either V must become infinite when h vanishes, or the integral $\int_a^\infty f(x)\, dx$ must be divergent, or if it be convergent its value must differ from V_0.

When the integral $\int_a^\infty f(x)\, dx$ is convergent, if we denote its value by U, we shall have in all cases

$$U = V_0 - L,$$

where L is the limit to which $\int_X^\infty F(x, h)\, dx$ approaches when h is first made to vanish and then X to become infinite.

The same remarks which have been made with reference to the convergency of series such as (15) or (30) for values of x near critical values will apply to the convergency of integrals such as (50), (59) or (65).

The question of the convergency or divergency of an integral might arise, not from one of the limits of integration being ∞, but from the circumstance that the quantity under the integral sign becomes infinite within the limits of integration. The reasoning of the preceding article will apply, with no material alteration, to this case also.

41. It may not be uninteresting to consider the bearing of the reasoning contained in this Section and a method frequently given of determining the values of two definite integrals, more especially as the values assigned to the integrals have recently been called into question, on account of their discontinuity.

Consider first the integral

$$u = \int_0^\infty \frac{\sin a x}{x}\, dx, \quad \ldots\ldots\ldots\ldots\ldots\ldots (71),$$

where a is supposed positive. Consider also the integral

$$v = \int_0^\infty \epsilon^{-hx} \frac{\sin a x}{x}\, dx.$$

It is easy to prove that the integral v is convergent, and that its convergency does not become infinitely slow when h vanishes. Consequently the integral u is also convergent, (as might also be proved directly in the same way as in the case of v,) and its value is the limit of u for $h = 0$. But we have

$$\frac{dv}{dh} = -\int_0^\infty \epsilon^{-hx} \sin a x\, dx = -\frac{a}{a^2 + h^2};$$

whence

$$v = C - \tan^{-1}\frac{h}{a};$$

and since v evidently vanishes when $h = \infty$, we have $C = \frac{\pi}{2}$, whence

$$v = \frac{\pi}{2} - \tan^{-1}\frac{h}{a}, \quad u = \frac{\pi}{2}.$$

Also $u = 0$ when $a = 0$, and $u = -\frac{\pi}{2}$ when a is negative, since u changes sign with a. By the value of u for $a = 0$, which is asserted to be 0, is of course meant the limit of $\int_0^X \frac{\sin ax}{x}\,dx$ when a is *first* made to vanish and *then* X made infinite.

It is easily proved that the convergency of the integral u becomes infinitely slow when a vanishes. In fact if

$$u' = \int_X^\infty \frac{\sin ax}{x}\,dx,$$

we get by changing the independent variable

$$u' = \int_{aX}^\infty \frac{\sin x}{x}\,dx:$$

but for any given value of X, however great, the value of u' becomes when a vanishes $\int_0^\infty \frac{\sin x}{x}\,dx$, an integral which might have been very easily proved to be greater than zero even had we been unable to find its value. It readily follows from the above that if u' has to be less than e the value of X increases indefinitely as a approaches to zero.

42. Consider next the integrals

$$u = \int_0^\infty \frac{\cos ax\,dx}{1+x^2}, \quad v = \int_0^\infty \epsilon^{-hx}\frac{\cos ax\,dx}{1+x^2} \quad \text{.................. (72).}$$

It is easily proved that the convergency of the integral v does not become infinitely slow when h vanishes, whatever be the value of a. Consequently u is in all cases the limit of v for $h = 0$. Now v satisfies the equation

$$\frac{d^2v}{da^2} - v = -\int_0^\infty \epsilon^{-hx}\cos ax\,dx = -\frac{h}{h^2+a^2}. \quad \text{.................. (73).}$$

It is not however necessary to find the general value of v; for if we put $h = 0$ we see that u satisfies the equation

$$\frac{d^2u}{da^2} - u = 0, \quad \text{.................................. (74),}$$

so long as a is kept always positive or always negative: but we cannot pass from the value of u found for positive values of a to the value which belongs to negative values of a by merely writing $-a$ for a in the algebraical expression obtained. For although u is a continuous function of a, it readily follows from (73) that $\frac{du}{da}$ is discontinuous. In fact, we have from this equation

$$\left(\frac{dv}{da}\right)_{a=\lambda} - \left(\frac{dv}{da}\right)_{a=-\lambda} = \int_{-\lambda}^{\lambda} v\,da - 2\tan^{-1}\frac{\lambda}{h}.$$

Now let h first vanish and then λ. Then v becomes u, and $\int_{-\lambda}^{\lambda} v\,da$ vanishes, since v does not become infinite for $a = 0$, whether h be finite or be zero. Threfore $\frac{du}{da}$ is suddenly decreased by π as a increases through zero, as might have been easily proved from the expression for u by means of the known integral (71), even had we been unable to find the value of u in (72). The equation (74) gives, a being supposed positive,

$$u = C\epsilon^{-a} + C'\epsilon^{a}.$$

But u evidently does not increase indefinitely with a, and $u = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$ when $a = 0$; whence $C' = 0$, $C = \frac{\pi}{2}$, $u = \frac{\pi}{2}\epsilon^{-a}$. Also, since the numerical value of u is unaltered when the sign of a is changed, we have $u = \frac{\pi}{2}\epsilon^{a}$ when a is negative.

It may be observed that if the form of the integral u had been such that we could not have inferred its value for a negative from its value for a positive, nor even known that u is not infinite for $a = -\infty$, we might yet have found its value for a negative by means of the known continuity of u and discontinuity of $\frac{du}{da}$ when a vanishes. For it follows from (74) that $u = C_1\epsilon^{a} + C_2\epsilon^{-a}$ for a negative; and knowing already that $u = \frac{\pi}{2}\epsilon^{-a}$ for a positive, we have

$$\frac{\pi}{2} = C_1 + C_2, \quad -\frac{\pi}{2} = C_1 - C_2 - \pi;$$

whence $C_1 = \frac{\pi}{2}$, $C_2 = 0$, $u = \frac{\pi}{2}\epsilon^{a}$, for a negative.

Of course the easiest way of verifying the result $u = \frac{\pi}{2}\epsilon^{-a}$ for a positive is to develope ϵ^{-x} for x positive in a definite integral of the form (59), by means of the formula (60).

SECTION IV.

Examples of the application of the formulæ proved in the preceding Sections.

43. BEFORE proceeding with the consideration of particular examples, it will be convenient to write down the formulæ which will have to be employed. Some of these formulæ have been proved, and others only alluded to, in the preceding Sections.

In the following formulæ, when series are considered, $f(x)$ is supposed to be a function of x which, as well as each of its derivatives up to the $(\mu - 1)^{\text{th}}$ order inclusive, is continuous between the limits $x = 0$ and $x = a$, and which is expanded between those limits in a series either of sines or of cosines of $\frac{\pi x}{a}$ and its multiples. A_n denotes the coefficient of $\sin\frac{n\pi x}{a}$ when the series is one of sines, B_n the coefficient of $\cos\frac{n\pi x}{a}$ when the series is one of cosines, A_n^{μ} or B_n^{μ} the coefficient of $\sin\frac{n\pi x}{a}$ or $\cos\frac{n\pi x}{a}$ in the expansion of the μ^{th} derivative. When integrals are considered, $f(x)$

and its first $\mu - 1$ derivatives are supposed to be functions of the same nature as before, which are considered between the limits $x = 0$ and $x = \infty$; and it is moreover supposed that $f(x)$ decreases as x increases to ∞, sufficiently fast to allow $\int f(x)dx$ to be essentially convergent at the limit ∞, or else that $f(x)$ vanishes when $x = \infty$, and after a finite value of x never changes from increasing to decreasing nor from decreasing to increasing. $\phi(\beta)$ or $\psi(\beta)$ denotes the coefficient of $\sin \beta x$ or $\cos \beta x$ in the developement of $f(x)$ in a definite integral of the form $\int_0^\infty \phi\beta \sin \beta x dx$ or $\int_0^\infty \psi(\beta) \cos \beta x dx$, $\phi_\mu(\beta)$ or $\psi_\mu(\beta)$ denotes the coefficient of $\sin \beta x$ or $\cos \beta x$ in the developement of the μ^{th} derivative of $f(x)$. The formulæ are

$$(-1)^{\frac{\mu-1}{2}} B_n^\mu = \left(\frac{n\pi}{a}\right)^\mu A_n - \frac{2}{a}\left(\frac{n\pi}{a}\right)^{\mu-1} \{f(0) - (-1)^n f(a)\}$$
$$+ \frac{2}{a}\left(\frac{n\pi}{a}\right)^{\mu-3} \{f''(0) - (-1)^n f''(a)\} - \dots \ (\mu \text{ odd}) \dots\dots\dots\dots (A),$$

$$(-1)^{\frac{\mu}{2}} A_n^\mu = \left(\frac{n\pi}{a}\right)^\mu A_n - \frac{2}{a}\left(\frac{n\pi}{a}\right)^{\mu-1} \{f(0) - (-1)^n f(a)\} + \dots \ (\mu \text{ even}) \dots\dots\dots\dots (B),$$

$$(-1)^{\frac{\mu+1}{2}} A_n^\mu = \left(\frac{n\pi}{a}\right)^\mu B_n + \frac{2}{a}\left(\frac{n\pi}{a}\right)^{\mu-2} \{f'(0) - (-1)^n f'(a)\} - \dots \ (\mu \text{ odd}) \dots\dots\dots\dots (C),$$

$$(-1)^{\frac{\mu}{2}} B_n^\mu = \left(\frac{n\pi}{a}\right)^\mu B_n + \frac{2}{a}\left(\frac{n\pi}{a}\right)^{\mu-2} \{f'(0) - (-1)^n f'(a)\} - \dots \ (\mu \text{ even}) \dots\dots\dots\dots (D),$$

except when $n = 0$, in which case we have always

$$B_0^\mu = \frac{1}{a}\{f^{\mu-1}(a) - f^{\mu-1}(0)\},$$

B_0 being the constant term in the expansion of $f^\mu(x)$ in a series of cosines.

In the formulæ (A), (B), (C), (D) we must stop when we have written the term containing the power 1 or 0, (as the case may be,) of $\frac{n\pi}{a}$. The formulæ for integrals are

$$(-1)^{\frac{\mu-1}{2}} \psi_\mu(\beta) = \beta^\mu \phi(\beta) - \frac{2}{\pi}\beta^{\mu-1} f(0) + \frac{2}{\pi}\beta^{\mu-3} f''(0) - \dots \ (\mu \text{ odd}) \dots\dots\dots\dots\dots (a),$$

$$(-1)^{\frac{\mu}{2}} \phi_\mu(\beta) = \beta^\mu \phi(\beta) - \frac{2}{\pi}\beta^{\mu-1} f(0) + \frac{2}{\pi}\beta^{\mu-3} f''(0) - \dots \ (\mu \text{ even}) \dots\dots\dots\dots (b),$$

$$(-1)^{\frac{\mu+1}{2}} \phi_\mu(\beta) = \beta^\mu \psi(\beta) + \frac{2}{\pi}\beta^{\mu-2} f'(0) - \frac{2}{\pi}\beta^{\mu-4} f'''(0) + \dots \ (\mu \text{ odd}) \dots\dots\dots\dots (c),$$

$$(-1)^{\frac{\mu}{2}} \psi_\mu(\beta) = \beta^\mu \psi(\beta) + \frac{2}{\pi}\beta^{\mu-2} f'(0) - \frac{2}{\pi}\beta^{\mu-4} f'''(0) + \dots \ (\mu \text{ even}) \dots\dots\dots\dots (d),$$

where we must stop with the last term involving a positive power of β or the power zero.

44. As a first example of the application of the principles contained in Sections I. and II. suppose that we have to determine the value of ϕ for values of x lying between 0 and a, 0 and b respectively, from the equation

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (75),$$

with the particular conditions

$$\frac{d\phi}{dy} = \omega(x - \tfrac{1}{2}a), \quad \text{when } y = 0 \text{ or } = b \ldots\ldots\ldots\ldots(76),$$

$$\frac{d\phi}{dx} = -\omega(y - \tfrac{1}{2}b), \text{ when } x = 0 \text{ or } = a \ldots\ldots\ldots\ldots(77).$$

This is the problem in pure analysis to which we are led in seeking to determine the motion of a liquid within a closed rectangular box which is made to oscillate.

For a given value of y, the value of ϕ can be expanded in a convergent series of cosines of $\frac{\pi x}{a}$ and its multiples; for another value of y, ϕ can be expanded in a similar series with different coefficients, and so on. Hence, in general, ϕ can be expanded in a convergent series of the form

$$\Sigma Y_n \cos\frac{n\pi x}{a} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (78),$$

where Y_n is a certain function of y, which has to be determined.

In the first place the value of ϕ given by (78) must satisfy (75). Now the direct developement of $\frac{d^2\phi}{dy^2}$ in a series of cosines will be obtained from (78) by differentiating under the sign of summation; the direct developement of $\frac{d^2\phi}{dx^2}$ will be given by the formula (D). We thus get

$$\Sigma\left\{\frac{d^2Y_n}{dy^2} - \frac{n^2\pi^2}{a^2}Y_n + \frac{2\omega}{a}\{1-(-1)^n\}(y - \tfrac{1}{2}b)\right\}\cos\frac{n\pi x}{a} = 0;$$

and the left-hand member of this equation being the result of directly developing the right-hand member in a series of cosines, we have

$$\frac{d^2Y_n}{dy^2} - \frac{n^2\pi^2}{a^2}Y_n = -\frac{4\omega}{a}(y - \tfrac{1}{2}b) \text{ or } = 0,$$

according as n is odd or even. This equation is easily integrated, and the integral contains two arbitrary constants, C_n, D_n, suppose. It only remains to satisfy (76). Now the direct developement of $\frac{dY_n}{dy}$ will be obtained by differentiating under the sign of summation, and the direct developement of $\omega(x - \frac{1}{2}a)$ is easily found to be $-\Sigma_0 \frac{4\omega a}{\pi^2 n^2}\cos\frac{n\pi x}{a}$, the sign Σ_0 denoting that odd values only of n are to be taken. We have then, both for $y = 0$ and for $y = b$,

$$\frac{dY_n}{dy} = -\frac{4\omega a}{\pi^2 n^2} \text{ or } = 0,$$

according as n is odd or even, which determines C_n and D_n.

It is unnecessary to write down the result, because I have already given it in a former paper*, where it is obtained by considerations applicable to this particular problem. The result is contained in equation (4) of that paper. The only step of the process which I have just indicated which requires notice is, that the term containing $(x - \frac{1}{2}a)(y - \frac{1}{2}b)$ at first appears as an infinite

* Supplement to a Memoir 'On some Cases of Fluid Motion,' p. 409 of the present Volume.

series, which may be summed by the formula (41). The present example is a good one for showing the utility of the methods contained in the present paper, inasmuch as in the Supplement referred to I have pointed out the advantage of the formula contained in equation (6), with respect to facility of numerical calculation, over one which I had previously arrived at by using developements, in series of cosines, of functions whose derivatives vanish for the limiting values of the variable.

45. Let it be required to determine the permanent state of temperature in a rectangle which has two of its opposite edges kept up to given temperatures, varying from point to point, while the other edges radiate into a space at a temperature zero. The rectangle is understood to be a section of a rectangular bar of infinite length, which has all the points situated in the same line parallel to the axis at the same temperature, so that the propagation of heat takes place in two dimensions.

Let the rectangle be referred to the rectangular axes of x, y, the axis of y coinciding with one of the edges whose temperature is given, and the origin being in the middle point of the edge. Let the unit of length be so chosen that the length of either edge parallel to the axis of x shall be π, and let 2β be the length of each of the other edges. Let u be the temperature at the point (x, y), h the ratio of the exterior, to the interior conductivity. Then we have

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (79),$$

$$\frac{du}{dy} - hu = 0, \text{ when } y = -\beta \;\ldots\ldots\ldots\ldots\; (80),$$

$$\frac{du}{dy} + hu = 0, \text{ when } y = \beta \;\ldots\ldots\ldots\ldots\ldots\; (81),$$

$$u = f(y), \text{ when } x = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (82),$$

$$u = F(y), \text{ when } x = a \;\ldots\ldots\ldots\ldots\ldots\ldots\; (83),$$

$f(y)$, $F(y)$ being the given temperatures of two of the edges.

According to the method by which Fourier has solved a similar problem, we should first take a particular function $Y\epsilon^{\lambda x}$, where Y is a function of y, and restrict it to satisfy (79). This gives $Y = A\cos\lambda y + B\sin\lambda y$, A and B being arbitrary constants. We may of course take, still satisfying (79), the sum of any number of such functions. It will be convenient to take together the functions belonging to two values of λ which differ only in sign. We may therefore take, by altering the arbitrary constants,

$$u = \Sigma\left\{A\left(\epsilon^{\lambda(\pi-x)} - \epsilon^{-\lambda(\pi-x)}\right) + B\left(\epsilon^{\lambda x} - \epsilon^{-\lambda x}\right)\right\}\cos\lambda y,$$
$$+ \Sigma\left\{C\left(\epsilon^{\lambda(\pi-x)} - \epsilon^{-\lambda(\pi-x)}\right) + D\left(\epsilon^{\lambda x} - \epsilon^{-\lambda x}\right)\right\}\sin\lambda y \;\ldots\ldots\ldots\; (84),$$

in which expression it will be sufficient to take only one of two values of λ which differ only by sign, so that λ, if real, may be taken positive. Substituting now in (80) and (81) the value of u given by (84), we get either $C = 0$, $D = 0$, and

$$\lambda\beta \,.\, \tan\lambda\beta = h\beta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (85),$$

or else $A = 0$, $B = 0$, and

$$\lambda\beta \,.\, \cot\lambda\beta = -h\beta \;\ldots\ldots\ldots\ldots\ldots\ldots\; (86).$$

It is easy to prove that the equation (85), in which $\lambda\beta$ is regarded as the unknown quantity, has an infinite number of real positive roots lying between each even multiple of $\frac{\pi}{2}$, including zero, and the next odd multiple. The equation (86) has also an infinite number of real positive roots lying between each odd multiple of $\frac{\pi}{2}$ and the next even multiple. The negative roots of (85) and (86) need not be considered, since the several negative roots have their numerical values equal to those of the positive roots; and it may be proved that the equations do not admit of imaginary roots. The values of λ in (84) must now be restricted to be those given by (85) for the first line, and those given by (86) for the second. It remains to satisfy (82) and (83). Now let

$$f(y) + f(-y) = 2f_1(y), \quad f(y) - f(-y) = 2f_2(y),$$
$$F(y) + F(-y) = 2F_1(y), \quad F(y) - F(-y) = 2F_2(y):$$

then we must have for all values of y from 0 to β, and therefore for all values from $-\beta$ to 0,

$$\Sigma AL \cos\lambda y = f_1(y), \quad \Sigma BL\cos\lambda y = F_1(y) \ldots\ldots\ldots\ldots\ldots (87),$$
$$\Sigma CM \sin\mu y = f_2(y), \quad \Sigma DM \sin\mu y = F_2(y) \ldots\ldots\ldots\ldots (88),$$

where $$L = \epsilon^{\lambda\pi} - \epsilon^{-\lambda\pi}, \quad M = \epsilon^{\mu\pi} - \epsilon^{-\mu\pi},$$

μ denoting one of the roots of the equation

$$\mu\beta . \cot\mu\beta = -h\beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (89),$$

and the two signs Σ extending to all the positive roots of the equations (85), (89), respectively. To determine A and B, multiply both sides of each of the equations (87) by $\cos\lambda' y dy$, λ' being any root of (85), and integrate from $y = 0$ to $y = \beta$. The integral at the first side will vanish, by virtue of (85), except when $\lambda' = \lambda$, in which case it will become $\frac{1}{4\lambda}(2\lambda\beta + \sin 2\lambda\beta)$, whence A and B will be known. C and D may be determined in a similar manner by multiplying both sides of each of the equations (88) by $\sin\mu' y dy$, μ' being any root of (89), integrating from $y = 0$ to $y = \beta$, and employing (89). We shall thus have finally

$$u = 4\Sigma\lambda(2\lambda\beta + \sin 2\lambda\beta)^{-1}(\epsilon^{\lambda\pi} - \epsilon^{-\lambda\pi})^{-1}\{(\epsilon^{\lambda(\pi-x)} - \epsilon^{-\lambda(\pi-x)})\int_0^\beta f_1(y)\cos\lambda y\,dy$$
$$+ (\epsilon^{\lambda x} - \epsilon^{-\lambda x})\int_0^\beta F_1(y)\cos\lambda y\,dy\}\cos\lambda y,$$
$$+ 4\Sigma\mu(2\mu\beta - \sin 2\mu\beta)^{-1}(\epsilon^{\mu\pi} - \epsilon^{-\mu\pi})^{-1}\{(\epsilon^{\mu(\pi-x)} - \epsilon^{-\mu(\pi-x)})\int_0^\beta f_2(y)\sin\mu y\,dy$$
$$+ (\epsilon^{\mu x} - \epsilon^{-\mu x})\int_0^\beta F_2(y)\sin\mu y\,dy\}\sin\mu y \ldots\ldots\ldots (90).$$

46. Such is the solution obtained by a method similar to that employed by Fourier. A solution very different in appearance may be obtained by expanding u in a series $\Sigma Y\sin nx$, and employing the formula (B). We thus get from the equation (79)

$$\frac{d^2Y}{dy^2} - n^2Y + \frac{2n}{\pi}\{f(y) - (-1)^n F(y)\} = 0,$$

which gives

$$Y = A\epsilon^{ny} + B\epsilon^{-ny} - \frac{1}{\pi}\int_0^y \{f(y') - (-1)^n F(y')\}(\epsilon^{n(y-y')} - \epsilon^{-n(y-y')})\,dy';$$

whence, $\frac{du}{dy} = \Sigma Y' \sin nx$, where

$$Y' = nA\epsilon^{ny} - nB\epsilon^{-ny} - \frac{n}{\pi}\int_0^y \{f(y') - (-1)^n F(y')\} \left(\epsilon^{n(y-y')} + \epsilon^{-n(y-y')}\right) dy'.$$

The values of A and B are to be determined by (80) and (81), which require that

$$\frac{dY}{dy} \pm hY = 0 \text{ when } y = \pm\beta.$$

We thus get

$$(n+h)\epsilon^{n\beta}A - (n-h)\epsilon^{-n\beta}B - \frac{1}{\pi}\int_0^\beta \{f(y') - (-1)^n F(y')\}\{(n+h)\epsilon^{n(\beta-y')} + (n-h)\epsilon^{-n(\beta-y')}\} dy' = 0,$$

and the equation derived from this by changing the signs of h and β; whence the values of A and B may be found. We get finally

$$u = \Sigma Y \sin nx, \quad \text{.............................} \quad (91),$$

where

$$Y = \frac{1}{\pi}\{(n+h)\epsilon^{n\beta} - (n-h)\epsilon^{-n\beta}\}^{-1}(\epsilon^{ny} + \epsilon^{-ny})\int_0^\beta \{(n+h)\epsilon^{n(\beta-y')} + (n-h)\epsilon^{-n(\beta-y')}\}$$
$$\{f_1(y') - (-1)^n F_1(y')\} dy'$$
$$-\frac{1}{\pi}\int_0^y (\epsilon^{n(y-y')} - \epsilon^{-n(y-y')})\{f_1(y') - (-1)^n F_1(y')\} dy'$$
$$+\frac{1}{\pi}\{(n+h)\epsilon^{n\beta} + (n-h)\epsilon^{-n\beta}\}^{-1}(\epsilon^{ny} - \epsilon^{-ny})\int_0^\beta \{(n+h)\epsilon^{n(\beta-y')} + (n-h)\epsilon^{-n(\beta-y')}\}$$
$$\{f_2(y') - (-1)^n F_2(y')\} dy'$$
$$-\frac{1}{\pi}\int_0^y (\epsilon^{n(y-y')} - \epsilon^{-n(y-y')})\{f_2(y') - (-1)^n F_2(y')\} dy'. \quad \text{........................} \quad (92).$$

47. The two expressions for u given, one by (90), and the other by (91) and (92), are necessarily equal for values of x and y lying between the limits 0 and π, $-\beta$ and β respectively. They are also equal for the limiting values $y = -\beta$ and $y = \beta$, but not for the limiting values $x = 0$ and $x = \pi$, since for these values (91) fails; that is to say, in order to find from this series the value of u for $x = 0$ or $x = \pi$, we should have *first* to sum the series, and *then* put $x = 0$ or $x = \pi$.

The comparison of these expressions leads to two remarkable formulæ. In the first place it will be observed that the first and second lines in the right-hand side of (92) are unchanged when y changes sign, while the third and fourth lines change sign with y. This is obvious with respect to the first and third lines, and may be easily proved with respect to the second and fourth by taking $-y'$ instead of y' for the variable with respect to which the integration is performed, and remembering that $f_1(y)$, $F_1(y)$ are unchanged, and $f_2(y)$, $F_2(y)$ change sign, when y changes sign. Consequently the part of u corresponding to the first two lines of (92) is equal to the part expressed by the first two in (90), and the part corresponding to the last two lines of (92) equal to the part expressed by the last two in (90). Hence the equation obtained by equating the two expressions for u splits into two; and each of the new equations will again split into two in consequence of the independence of the functions f, F, which are arbitrary from $y = 0$ to $y = \beta$. As far however as anything peculiar in the transformations is concerned, it is evident that we may suppress one of the functions f, F, suppose F, and consider only an element of the integral by which f is developed, or, which is the same, suppose $f_1(y')$ or $f_2(y')$ to be zero except for values of the variable infinitely close to a particular value y', and divide both sides of the equation by

$$\int f_1(y')\,dy' \text{ or } \int f_2(y')\,dy'.$$

We get thus from the first two lines of (90) and the first two of (92), supposing y and y' positive, and y' the greater of the two,

$$\Sigma \frac{4\lambda}{2\lambda\beta + \sin 2\lambda\beta} \frac{\epsilon^{\lambda(\pi-x)} - \epsilon^{-\lambda(\pi-x)}}{\epsilon^{\lambda\pi} - \epsilon^{-\lambda\pi}} \cos\lambda y \cos\lambda y'$$

$$= \frac{1}{\pi} \Sigma \frac{(\epsilon^{ny} + \epsilon^{-ny})\left\{(n+h)\,\epsilon^{n(\beta-y')} + (n-h)\,\epsilon^{-n(\beta-y')}\right\}}{(n+h)\,\epsilon^{n\beta} - (n-h)\,\epsilon^{-n\beta}} \sin nx, \quad \ldots (93),$$

where the first Σ refers to the positive roots of (85), and the second to positive integral values of n from 1 to ∞.

Of course, if y become greater than y', y and y' will have to change places in the second side of (93). This is in accordance with the formula (92), since now the second line does not vanish; and it will easily be found that the first and second lines together give the same result as if we had at once made y and y' change places. Although y has been supposed positive in (93), it is easily seen that it may be supposed negative, provided it be numerically less than y'.

The other formula above alluded to is obtained in a manner exactly similar by comparing the last two lines in (92) with the last two in (90). It is

$$\Sigma \frac{4\mu}{2\mu\beta - \sin 2\mu\beta} \frac{\epsilon^{\mu(\pi-x)} - \epsilon^{-\mu(\pi-x)}}{\epsilon^{\mu\pi} - \epsilon^{-\mu\pi}} \sin\mu y \sin\mu y'$$

$$= \frac{1}{\pi} \Sigma \frac{(\epsilon^{ny} - \epsilon^{-ny})\left\{(n+h)\,\epsilon^{n(\beta-y')} + (n-h)\,\epsilon^{-n(\beta-y')}\right\}}{(n+h)\,\epsilon^{n\beta} + (n-h)\,\epsilon^{-n\beta}} \sin nx \ldots\ldots\ldots (94),$$

where the first Σ refers to the positive roots of (89), the second to positive integral values of n, and where x is supposed to lie between 0 and π, y' between 0 and β, y between 0 and y', or, it may be, between $-y'$ and y'. Although x has been supposed less than π, it may be observed that the formulæ (93), (94) hold good so long as x, being positive, is less than 2π.

48. Let it be required to determine the permanent state of temperature in a homogeneous rectangular parallelepiped, supposing the surface kept up to a given temperature, which varies from point to point.

Let the origin be in one corner of the parallelepiped, and let the adjacent edges be taken for the axes of x, y, z. Let a, b, c be the lengths of the edges; $f_1(y, z)$, $F_1(y, z)$, the given temperatures of the faces for which $x = 0$ and $x = a$ respectively; $f_2(z, x)$, $F_2(z, x)$ the same for the faces perpendicular to the axis of y; $f_3(x, y)$, $F_3(x, y)$ the same for those perpendicular to the axis of z. Then if we put for shortness ∇ to denote the operation otherwise denoted by

$$\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2},$$

as will be done in the rest of this paper, and write only the characteristics of the functions, we shall have, to determine the temperature u, the general equation $\nabla u = 0$ with the particular conditions

$$u = f_1, \text{ when } x = 0; \quad u = F_1, \text{ when } x = a \ldots\ldots\ldots\ldots\ldots\ldots (95);$$

$$u = f_2, \text{ when } y = 0; \quad u = F_2, \text{ when } y = b \ldots\ldots\ldots\ldots\ldots\ldots (96);$$

$$u = f_3, \text{ when } z = 0; \quad u = F_3, \text{ when } z = c \ldots\ldots\ldots\ldots\ldots\ldots (97);$$

It is evident that u is the sum of three temperatures u_1, u_2, u_3, where u_1 satisfies the conditions (95), and vanishes at the four remaining faces, and u_2, u_3 are related to the axes of y, z as u_1 is related to that of x, each of the quantities u_1, u_2, u_3 representing a possible permanent temperature. Now u_3 may be expanded in a double series $\Sigma\Sigma Z_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$, where Z_{mn} is a function of z which has to be determined. Let for shortness

$$\frac{m\pi}{a} = \mu, \quad \frac{n\pi}{b} = \nu, \quad \frac{p\pi}{c} = \varpi;$$

then the substitution of the above value of u_3 in the equation $\nabla u_3 = 0$ leads to the equation

$$\frac{d^2 Z_{mn}}{dz^2} - q^2 Z_{mn} = 0,$$

where $q^2 = \mu^2 + \nu^2$, which gives $Z_{mn} = A_{mn}\epsilon^{qz} + B_{mn}\epsilon^{-qz}$; and the constants A_{mn}, B_{mn} are easily determined by the condition (97). We may find u_1 and u_2 in a similar manner, and the sum of the results gives u. It is thus that such problems are usually solved.

We may, however, expand u in a series of the form $\Sigma\Sigma Z_{mn} \sin \mu x \sin \nu y$, even though it does not vanish for $x = 0$ and $x = a$, and for $y = 0$ and $y = b$; and the formulæ proved in Section I. enable us to make use of this expansion.

Let then $$u = \Sigma\Sigma Z \sin \mu x \sin \nu y,$$
the suffixes of Z being omitted for the sake of simplicity. We have by the formula (B)

$$\frac{d^2 u}{dx^2} = \Sigma\left\{ -\mu^2 \Sigma Z \sin \nu y + \frac{2\mu}{a}[f_1 - (-1)^m F_1]\right\} \sin \mu x.$$

Let $f_1(y, z) - (-1)^m F_1(y, z)$ be expanded in the series $\Sigma Q \sin \nu y$ by the formula (3), so that Q will be a known function of z, m, and n. Then

$$\frac{d^2 u}{dx^2} = \Sigma\Sigma\left\{ -\mu^2 Z + \frac{2\mu}{a} Q\right\} \sin \mu x \sin \nu y.$$

The value of $\frac{d^2 u}{dy^2}$ may be expressed in a similar manner, and that of $\frac{d^2 u}{dz^2}$ is found by direct differentiation. We have thus, for the direct developement of ∇u, the double series

$$\Sigma\Sigma\left\{\frac{d^2 Z}{dz^2} - (\mu^2 + \nu^2) Z + \frac{2\nu}{b} P + \frac{2\mu}{a} Q\right\} \sin \mu x \sin \nu y,$$

where P is for x what Q is for y. The above series being the direct developement of ∇u, and ∇u being equal to zero, each coefficient must be equal to zero, which gives

$$\frac{d^2 Z}{dz^2} - q^2 Z + \frac{2\nu}{b} P + \frac{2\mu}{a} Q = 0 \text{..........................} (98)$$

where q means the same as before. The integral of the equation (98) is

$$Z = A\epsilon^{qz} + B\epsilon^{-qz} - \frac{1}{q}\epsilon^{qz}\int_0^z \epsilon^{-qz} T dz + \frac{1}{q}\epsilon^{-qz}\int_0^z \epsilon^{qz} T dz,$$

$2T$ denoting the sum of the last two terms of (98). It only remains to satisfy (97). If the known functions $f_3(x, y)$, $F_3(x, y)$ be developed in the double series $\Sigma\Sigma G \sin \mu x \sin \nu y$, $\Sigma\Sigma H \sin \mu x \sin \nu y$, we shall have from (97)

$$A + B = G,$$

$$A\epsilon^{qc} + B\epsilon^{-qc} - \frac{1}{q}\epsilon^{qc}\int_0^c \epsilon^{-qz} T dz + \frac{1}{q}\epsilon^{-qc}\int_0^c \epsilon^{qz} T dz = H.$$

A and B may be easily found from these equations, and we shall have finally

$$(\epsilon^{qc} - \epsilon^{-qc}) Z = G (\epsilon^{q(c-z)} - \epsilon^{-q(c-z)}) + H (\epsilon^{qz} - \epsilon^{-qz}) + \frac{1}{q} (\epsilon^{q(c-z)} - \epsilon^{-q(c-z)}) \int_0^z (\epsilon^{qz'} - \epsilon^{-qz'}) T' dz'$$

$$+ \frac{1}{q} (\epsilon^{qz} - \epsilon^{-qz}) \int_z^c (\epsilon^{q(c-z')} - \epsilon^{-q(c-z')}) T' dz',$$

T' being the value of T when $z = z'$. It will be observed that the letters Z, P, Q, T, A, B, G, H ought properly to be affected with the double suffix mn. It would be useless to write down the expression for u in terms of the known quantities $f_1(y, z)$, &c.

It will be observed that u might equally have been expressed by means of the double series $\Sigma\Sigma X_{np} \sin \nu y \sin \varpi z$, or $\Sigma\Sigma Y_{mp} \sin \mu x \sin \varpi z$, where p is any integer. We should thus have three different expressions for the same quantity u within the limits $x = 0$ and $x = a$, $y = 0$ and $y = b$, $z = 0$ and $z = c$. The comparison of these three expressions when particular values are assigned to the known functions $f_1(y, z)$ &c. would lead to remarkable transformations. The expressions differ however in one respect which deserves notice. Their numerical values are the same for values of the variables lying within the limits 0 and a, 0 and b, 0 and c. The first expression holds good for the extreme values of z, but fails for those of x and y: in other words, in order to find from the series the value of u for the face considered, instead of first giving x or y its extreme value and then summing, which would lead to a result zero, we should first have to sum with respect to m or n, or conceive the summation performed, and then give x or y its extreme value. The same remarks apply, *mutatis mutandis*, to the second and third expressions; so that the three expressions are not equivalent if we take in the extreme values of the variables.

49. Many other remarkable transformations might be obtained from those already referred to by differentiation and integration. We might for instance compare the three expressions which would be obtained for $\int_0^a \int_0^b \int_0^c u\, dx\, dy\, dz$, and we should thus have three different expressions for the same function of the three independent variables a, b, c, which are supposed to be positive, but may be of any magnitudes. Some examples of the results of transformations of this kind may be seen by comparing the formulæ obtained in the Supplement alluded to in Art. 44 with the corresponding formulæ contained in the Memoir itself to which the Supplement has been added. Such transformations, however, when separated from physical problems, are more curious than useful. Nevertheless, it may be worth while to exhibit in its simplest shape the formula from which they all flow, so long as we restrict ourselves to a function u satisfying the equation $\nabla u = 0$, and expanded between the limits $x = 0$ and $x = a$, &c. in a double series of sines.

The functions $f_1(y, z)$ &c., which are supposed known, are arbitrary, and enter into the expression for u under the sign of double integration. Consequently we shall not lose generality, so far as anything peculiar in the transformations is concerned, by considering only one element of the integrals by which one of the functions is developed. Let then all the functions be zero except f_3; and since in the process f_3 has to be developed in the double series

$$\frac{4}{ab} \Sigma\Sigma \int_0^a \int_0^b f_3(x', y') \sin \mu x' \sin \nu y' dx' dy' . \sin \mu x \sin \nu y,$$

consider only the element $f_3(x', y') \sin \mu x' \sin \nu y' dx' dy'$ of the double integral, omit the $dx'dy'$, and put $f_3(x', y') = 1$ for the sake of simplicity. If we adopt the first expansion of u, and put q^2 for $\mu^2 + \nu^2$, we shall have

$$Z = A (\epsilon^{q(c-z)} - \epsilon^{-q(c-z)}),\quad (\epsilon^{qc} - \epsilon^{-qc}) A = \frac{4}{ab} \sin \mu x' \sin \nu y';$$

whence

$$u = \frac{4}{ab} \Sigma\Sigma \frac{\epsilon^{q(c-z)} - \epsilon^{-q(c-z)}}{\epsilon^{qc} - \epsilon^{-qc}} \sin \mu x' \sin \nu y' \sin \mu x \sin \nu y \ldots\ldots\ldots\ldots\ldots (99).$$

By expanding u in the double series $\Sigma\Sigma Y \sin \mu x \sin \varpi z$ we should get

$$u = \frac{2}{ac}\Sigma\Sigma\frac{\varpi}{s}\frac{(\epsilon^{sy} - \epsilon^{-sy})(\epsilon^{s(b-y')} - \epsilon^{-s(b-y')})}{\epsilon^{sb} - \epsilon^{-sb}} \sin \mu x' \sin \mu x \sin \varpi z \;\ldots\ldots\ldots(100),$$

where $s^2 = \mu^2 + \varpi^2$, and y' is the greater of the two y, y'. The third expansion would be derived from the second by interchanging the requisite quantities. In these formulæ z may have any positive value less than $2c$.

We should get in a similar manner in the case of two variables x, y

$$u = \frac{2}{b}\Sigma\frac{\epsilon^{\nu(a-x)} - \epsilon^{-\nu(a-x)}}{\epsilon^{\nu a} - \epsilon^{-\nu a}} \sin \nu y' \sin \nu y = \frac{1}{a}\Sigma\frac{(\epsilon^{\mu y} - \epsilon^{-\mu y})(\epsilon^{\mu(b-y')} - \epsilon^{-\mu(b-y')})}{\epsilon^{\mu b} - \epsilon^{-\mu b}} \sin \mu x, \;\ldots\; (101),$$

where x is supposed to lie between 0 and a, y' between 0 and b, and y between 0 and y'. This formula is however true so long as x lies between 0 and $2a$, and y between $-y'$ and y'.

If we compare the two expressions for $\int_0^b\int_0^b\int_0^a u\,dy\,dy'\,dx$ obtained from (101), taking Σ_0 for the sign of summation corresponding to odd values of n from 1 to ∞, putting $a = rb$, and replacing $\Sigma_0 \frac{1}{n^2}$ by its value $\frac{\pi^2}{8}$, we shall get the formula

$$\frac{1}{r}\Sigma_0\frac{1}{n^3}\frac{1 - \epsilon^{-n\pi r}}{1 + \epsilon^{-n\pi r}} + r\Sigma_0\frac{1}{n^3}\frac{1 - \epsilon^{-\frac{n\pi}{r}}}{1 + \epsilon^{-\frac{n\pi}{r}}} = \frac{\pi^3}{16}, \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (102),$$

which is true for all positive values of r, and likewise for all negative values, since the left-hand side of (102) is not changed when $-r$ is put for r. In integrating the second side of (101), supposing that we integrate for y before integrating for y', we must integrate separately from $y = 0$ to $y = y'$, and from $y = y'$ to $y = b$, since the algebraical expression of the quantity to be integrated changes when y passes the value y'.

It would be useless to go on with these transformations, which may be multiplied to any extent, and which cease to be useful when they are separated from physical problems to which they relate, and of which we wish to obtain solutions.

It may be observed that instead of supposing, in the case of the parallelepiped, the value of u known for all points of the surface, we might have supposed the value of the flux known, subject of course to the condition that the total flux shall be zero. This would correspond to the following problem in fluid motion, u taking the place of the quantity usually denoted by ϕ, "To determine the initial motion at any point of a homogeneous incompressible fluid contained in a closed vessel of the form of a rectangular parallelepiped, which it completely fills, supposing the several points of the surface of the vessel suddenly moved in any manner consistent with the condition that the volume be not changed." In this case we should expand u in a series of cosines instead of sines, and employ the fomula (D) instead of (B). We might, again, suppose the value of u known for the faces perpendicular to one or two of the axes, and the value of the flux known for the remaining faces. In this case we should employ sines involving the co-ordinates perpendicular to the first set of faces, and cosines involving the others.

The formulæ would also be modified by supposing some one or more of the faces to move off to an infinite distance. In this case some of the series would be replaced by integrals. Thus, in the case in which the value of u at the surface is known, if we supposed a to become infinite we should employ the integral (50) instead of the series (3), as far as relates to the variable x, and the formula (b) instead of (B). If we were considering a rectangular bar infinitely extended both ways we should employ the integral (65). Of course, if we had already obtained the result for the case of the parallelepiped, the shortest way would be thence to deduce the result for the case of the bar infinite in one or in both directions, but if we began with considering the bar it would be best to start with the integrals (50) or (65).

50. To give one example of transformations of this kind, let us suppose b to become infinite in (101). Observing that $\nu = \frac{n\pi}{b}$, $\Delta\nu = \frac{\pi}{b}$, we get on passing to the limit

$$\frac{2}{\pi}\int_0^\infty \frac{\epsilon^{\nu(a-x)} - \epsilon^{-\nu(a-x)}}{\epsilon^{\nu a} - \epsilon^{-\nu a}} \sin\nu y' \sin\nu y\, d\nu = \frac{1}{a}\Sigma\,(\epsilon^{\mu y} - \epsilon^{-\mu y})\,\epsilon^{-\mu y'} \sin\mu x. \quad \ldots (103).$$

Multiply both sides of this equation by $dx\,dy$, and integrate from $x = 0$ to $x = a$, and from $y = 0$ to $y = \infty$. With respect to the integration of the second side, it is only necessary to remark that when y becomes greater than y', y and y' must be made to change places in the expression written down in (103). As to the integration of the first side, if we first integrate from $y = 0$ to $y = Y$, we get, putting $f(\nu, x)$ for the fraction involving x,

$$\frac{2}{\pi}\int_0^\infty f(\nu, x) \sin\nu y'\,(1 - \cos\nu Y)\,\frac{d\nu}{\nu}.$$

Now let Y become infinite; then the term involving $\cos\nu Y$ may be omitted, not because $\cos\nu Y$ vanishes when Y becomes infinite, which is not true, but because, as may be rigorously proved, the integral in which it occurs vanishes when Y becomes infinite. If we write 1 for a, as we may without loss of generality, we get finally

$$\int_0^\infty \frac{1 - \epsilon^{-\nu}}{1 + \epsilon^{-\nu}} \sin\nu y' \frac{d\nu}{\nu^2} = \frac{2}{\pi}\Sigma_0 \frac{1}{n^2}\,(1 - \epsilon^{-n\pi y'}). \quad \ldots\ldots\ldots\ldots (104).$$

51. Hitherto in satisfying the general equation $\nabla u = 0$, together with the particular conditions at the surface, the value of u has been expanded in a double series involving two of the variables, and the functions of the third variable which enter as coefficients into the double series have been determined by an ordinary differential equation such as (98). We might however expand u in a triple series, and thus satisfy at the same time the equation $\nabla u = 0$ and the conditions at the surface, without using an ordinary differential equation at all, but simply by means of the terms introduced into the series by differentiation, which are given by the formulæ at the beginning of this Section; and then by summing the triple series once, which may be done in any one of three ways, we should arrive at the same results as if we had employed in succession three double series, involving circular functions of x and y, y and z, z and x respectively, and the corresponding ordinary differential equations. I am indebted for this method to my friend Prof. William Thomson, to whom I showed the method given in Art. 48.

Let us take the case of the permanent state of temperature in a rectangular parallelepiped, supposing the temperature at the several points of the surface known. For more simplicity suppose the temperature zero at the surface, except infinitely close to the point (x', y') in the face for which $z = 0$, so that all the functions f, &c. are zero, except $f_3(x, y)$, and $f_3(x, y)$ itself zero except for values of x, y infinitely close to x', y' respectively; and let $\iint f_3(x, y)\,dx\,dy = 1$, provided the limits of integration include the values $x = x'$, $y = y'$. Let u be expanded in the triple series

$$\Sigma\Sigma\Sigma A_{mnp} \sin\mu x \sin\nu y \sin\varpi z, \quad \ldots\ldots\ldots\ldots (105),$$

where μ, ν, ϖ mean the same as in Art. 48. Then

$$\frac{d^2u}{dz^2} = \Sigma_p \left\{ - \Sigma_m \Sigma_n \varpi^2 A_{mnp} \sin\mu x \sin\nu y + \frac{2\varpi}{c} f_3(x, y) \right\} \sin\varpi z. \quad \ldots\ldots (106).$$

Now the expansion of $f_3(x, y)$ in a double series is $\frac{4}{ab}\Sigma\Sigma \sin\mu x' \sin\nu y' \sin\mu x \sin\nu y$, that is to say with this understanding, that the result is to be substituted in (106); for it would be absurd to speak, except by way of abbreviation, of a quantity which is zero except for particular values of x and y, for which it is infinite. The values of $\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dy^2}$ will be obtained by direct differentiation. We have therefore for the direct developement of ∇u in a triple series

$$\nabla u = \Sigma\Sigma\Sigma \left\{ -(\mu^2 + \nu^2 + \varpi^2) A_{mnp} + \frac{8\varpi}{abc} \sin \mu x' \sin \nu y' \right\} \sin \mu x \sin \nu y \sin \varpi z.$$

But ∇u being equal to zero, each coefficient will have to zero, from whence we get A_{mnp}, and then

$$u = \frac{8}{abc} \Sigma\Sigma\Sigma \frac{\varpi}{\mu^2 + \nu^2 + \varpi^2} \sin \mu x' \sin \nu y' \sin \mu x \sin \nu y \sin \varpi z. \;\ldots\ldots\; (107).$$

One of the three summations, whichever we please, may be performed by means of the known formulæ

$$\Sigma \frac{\varpi \sin \varpi z}{\varpi^2 + k^2} = \frac{c}{2} \frac{\epsilon^{k(c-z)} - \epsilon^{-k(c-z)}}{\epsilon^{kc} - \epsilon^{-kc}}, \text{ if } 2c > z > 0, \;\ldots\ldots\ldots\ldots\; (108),$$

$$\frac{1}{2k} + \Sigma \frac{k \cos \nu y_{,}}{k^2 + \nu^2} = \frac{b}{2} \frac{\epsilon^{k(b-y_{,})} + \epsilon^{-k(b-y_{,})}}{\epsilon^{kb} - \epsilon^{-kb}}, \text{ if } 2b > y_{,} > 0, \;\ldots\ldots\; (109),$$

which may be obtained by developing the second members between the limits $z = 0$ and $z = c$, $y_{,} = 0$ and $y_{,} = b$ by the formulæ (2), (22), and observing that the expansions hold good within the limits written after the formulæ, since $\epsilon^{k(c-z)} - \epsilon^{-(c-z)}$ has the same magnitude and opposite signs for values of z equidistant from c, and $\epsilon^{k(b-y_{,})} + \epsilon^{-k(b-y_{,})}$ has the same magnitude and sign for values of $y_{,}$ equidistant from b. If in equation (107) we perform the summation with respect to p, by means of the formula (108), we get the equation (99): if we perform the summation with respect to n, by means of the formula (109), we get the equation (100).

52. The following problem will illustrate some of the ideas contained in this paper, although, in the mode of solution which will be adopted, the formulæ given at the beginning of this Section will not be required.

A hollow conducting rectangular parallelepiped is in communication with the ground: required to express the potential, at any point in the interior, due to a given interior electrical point and to the electricity induced on the surface.

Let the axes be taken as in Art. 48. Let x', y', z' be the co-ordinates of the electrical point, m the electrical mass, v the required potential. Then v is determined *first* by satisfying the equation $\nabla v = 0$, *secondly* by being equal to zero at the surface, *thirdly* by being equal to $\frac{m}{r}$ infinitely close to the electrical point, r being the distance of the points (x, y, z), (x', y', z'), and by being finite and continuous at all other points within the parallelepiped.

Let $v = \frac{m}{r} + v_1$, so that v_1 is the potential due to the electricity induced on the surface. Then v_1 is finite and continuous within the parallelepiped, and is determined by satisfying the general equation $\nabla v_1 = 0$, and by being equal to $-\frac{m}{r}$ at the surface. Consequently v_1 can be determined precisely as u in Arts. 48 or 51. This separation however of v into two parts seems to introduce a degree of complexity not inherent in the problem; for v itself vanishes at the surface; and it is when the function expanded vanishes at the limits that the application of the series (2) involves least complexity. On the other hand we cannot immediately expand v in a triple series of the form (105), on account of its becoming infinite at the point (x', y', z').

Suppose, therefore, for the present that the electricity is diffused over a finite space: then it is evident that we may suppose the electrical density, ρ, to change so gradually, and pass so gradually into zero, that the derivatives of v, of as many orders as we please, shall be continuous functions. We may now suppose v expanded in a triple series, so that

$$v = \Sigma\Sigma\Sigma A_{mnp} \sin \mu x \sin \nu y \sin \varpi z;$$

and we shall have

$$\nabla v = -\Sigma\Sigma\Sigma(\mu^2 + \nu^2 + \varpi^2) A_{mnp} \sin \mu x \sin \nu y \sin \varpi z.$$

But we have also, by a well-known theorem, $\nabla v = -4\pi\rho$; and

$$\rho = \Sigma\Sigma\Sigma R_{mnp} \sin \mu x \sin \nu y \sin \varpi z,$$

where
$$R_{mnp} = \frac{8}{abc}\int_0^a \int_0^b \int_0^c \rho' \sin \mu x' \sin \nu y' \sin \varpi z' \, dx' \, dy' \, dz',$$

ρ' being the same function of x', y', z' that ρ is of x, y, z. We get therefore by comparing the two expansions of ∇v

$$A_{mnp} = 4\pi (\mu^2 + \nu^2 + \varpi^2)^{-1} R_{mnp},$$

whence the value of v is known. We may now, if we like, suppose the electricity condensed into a point, which gives

$$R_{mnp} = \frac{8m}{abc} \sin \mu x' \sin \nu y' \sin \varpi z',$$

$$v = \frac{32\pi m}{abc} \Sigma\Sigma\Sigma (\mu^2 + \nu^2 + \varpi^2)^{-1} \sin \mu x' \sin \nu y' \sin \varpi z' \sin \mu x \sin \nu y \sin \varpi z \ldots\ldots\ldots (110).$$

One of the summations may be performed just as before. We thus get, by summing with respect to p,

$$v = \frac{8\pi m}{ab} \Sigma\Sigma \frac{1}{q} \frac{(\epsilon^{qz} - \epsilon^{-qz})(\epsilon^{q(c-z')} - \epsilon^{-q(c-z')})}{\epsilon^{qc} - \epsilon^{-qc}} \sin \mu x' \sin \nu y' \sin \mu x \sin \nu y \ldots\ldots (111),$$

where $q^2 = \mu^2 + \nu^2$, and z is supposed to be the smaller of the two z, z'. If z be greater than z', we have only to make z and z' change places in (111).

53. The equation (110) shows that the potential at the point (x, y, z) due to a unit of electricity at the point (x', y', z') and to the electricity induced on the surface of the parallelepiped is equal to the potential at the point (x', y', z') due to a unit of electricity at the point (x, y, z) and to the electricity induced on the surface. This however is only a particular case of a general theorem proved by Green *.

Of course the parallelepiped includes as particular cases two parallel infinite planes, two parallel infinite planes cut at right angles by a third infinite plane, &c. The value of v being known, the density of the induced electricity at any point of the surface is at once obtained, by means of a known theorem.

If we suppose a ball-pendulum to oscillate within a rectangular case, the value of ϕ belonging to the motion of the fluid which is due to the direct motion of the ball and to the motion reflected from the case can be found in nearly the same manner. The expression *reflected motion* is here used in the sense explained in Art. 6 of my paper, "On some Cases of Fluid Motion†." In the present instance we should expand ϕ in a triple series of cosines.

54. Let a hollow cylinder, containing one or more plane partitions reaching from the axis to the curved surface, be filled with homogeneous incompressible fluid, and made to oscillate about its

* *Essay on Electricity*, p. 19.

† See p. 111 of the present Volume.

axis, both ends being closed: required to determine the effect of the inertia of the fluid on the motion of the cylinder.

If there be more than one partition, it will evidently be sufficient to consider one of the sectors into which the cylinder is divided, since the solution obtained may be applied to the others. In the present case the motion is such that $udx + vdy + wdz$, (according to the usual notation,) is an exact differential $d\phi$. The motion considered is in two dimensions, taking place in planes perpendicular to the axis of the cylinder. Let the fluid be referred to polar co-ordinates r, θ in a plane perpendicular to the axis, r being measured from the axis, and θ from one of the bounding partitions of the sector considered, being reckoned positive when measured inwards. Let the radius of the cylinder be taken for the unit of length, and let α be the angle of the sector, and ω the angular velocity of the cylinder at the instant considered. It will be observed that $\alpha = 2\pi$ corresponds to the case of a single partition. Then to determine ϕ we have the general equation

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \frac{1}{r^2}\frac{d^2\phi}{d\theta^2} = 0 \ldots\ldots (112),$$

with the conditions

$$\frac{1}{r}\frac{d\phi}{d\theta} = \omega r, \text{ when } \theta = 0 \text{ or } = \alpha \ldots\ldots (113),$$

$$\frac{d\phi}{dr} = 0, \text{ when } r = 1 \ldots\ldots (114),$$

and, that ϕ shall not become infinite when r vanishes.

Let $r = \epsilon^{-\lambda}$, and take θ, λ for the independent variables; then (112), (113), (114) become

$$\frac{d^2\phi}{d\lambda^2} + \frac{d^2\phi}{d\theta^2} = 0 \ldots\ldots (115),$$

$$\frac{d\phi}{d\theta} = \omega\epsilon^{-2\lambda}, \text{ when } \theta = 0 \text{ or } = \alpha \ldots\ldots (116),$$

$$\frac{d\phi}{d\lambda} = 0, \text{ when } \lambda = 0 \ldots\ldots (117).$$

Let ϕ be expanded between the limits $\theta = 0$ and $\theta = \alpha$ in a series of cosines, so that

$$\phi = \Lambda_0 + \Sigma\Lambda_n \cos\frac{n\pi\theta}{\alpha} \ldots\ldots (118),$$

Λ_0, Λ_n being functions of λ. Then we have by the formula (D) and the condition (116) applied to the general equation (115)

$$\frac{d^2\Lambda_0}{d\lambda^2} = 0.$$

$$\frac{d^2\Lambda_n}{d\lambda^2} - \left(\frac{n\pi}{\alpha}\right)^2\Lambda_n - \frac{2\omega}{\alpha}\{1 - (-1)^n\}\epsilon^{-2\lambda} = 0;$$

whence

$$\Lambda_0 = A_0\lambda + B_0,$$

$$\Lambda_n = A_n\epsilon^{\frac{n\pi\lambda}{\alpha}} + B_n\epsilon^{-\frac{n\pi\lambda}{\alpha}} - \frac{2\omega\alpha\{1 - (-1)^n\}}{n^2\pi^2 - 4\alpha^2}\epsilon^{-2\lambda}.$$

Since ϕ is not to be infinite when r vanishes, that is when λ becomes infinite, we have in the first place $A_0 = 0$, $A_n = 0$. We have then by the condition (117)

$$B_n = \frac{8\omega\alpha^2}{n\pi(n^2\pi^3 - 4\alpha^2)},$$

when n is odd, and $B_n = 0$ when n is even. If then we omit B_0, which is useless, and put for λ its value, we get

$$\phi = 4\omega\alpha\Sigma_0 \frac{\frac{2\alpha}{n\pi} r^{\frac{n\pi}{\alpha}} - r^2}{n^2\pi^2 - 4\alpha^2} \cos\frac{n\pi\theta}{\alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots (119).$$

The series multiplied by r^2 may be summed. For if we expand $\sin 2(\theta - \frac{1}{2}\alpha)$ between the limits $\theta = 0$, $\theta = \alpha$ in a series of cosines, we get

$$\sin(2\theta - \alpha) = -\Sigma_0 \frac{8\alpha\cos\alpha}{n^2\pi^2 - 4\alpha^2}\cos\frac{n\pi\theta}{\alpha};$$

whence

$$\phi = 8\omega\alpha^2\Sigma_0 \frac{r^{\frac{n\pi}{\alpha}}\cos\frac{n\pi\theta}{\alpha}}{n\pi(n^2\pi^2 - 4\alpha^2)} + \frac{\omega}{2\cos\alpha} r^2\sin(2\theta - \alpha) \ldots\ldots\ldots (120).$$

In determining the motion of the cylinder, the only quantity we care to know is the moment of the fluid pressures about the axis. Now if the motion be so small that we may omit the square of the velocity we shall have, putting $\phi = -\omega f(r, \theta)$,

$$p = \psi(t) + \frac{d\omega}{dt} f(r, \theta),$$

where p is the pressure, $\psi(t)$ a function of the time t, whose value is not required, and where the density is supposed to be 1, and the pressure due to gravity is omitted, since it may be taken account of separately. The moment of the pressure on the curved surface is zero, since the direction of the pressure at any point passes through the axis. The expression (119) or (120) shows that the moments on the plane faces of the sector are equal, and act in the same direction; so that it will be sufficient to find the moment on one of these faces and double the result. If we consider a portion of the face for which $\theta = 0$ whose length in the direction of the axis is unity, we shall have for the pressure on an element dr of the surface $\frac{d\omega}{dt} f(r, 0)\,dr$; and if we denote the whole moment of the pressures by $-C\frac{d\omega}{dt}$, reckoned positive when it tends to make the cylinder move in the direction of θ positive, we shall have

$$C = 2\int_0^1 f(r, 0)\,r\,dr.$$

Taking now the value of $f(r, 0)$ from (120), and performing the integration, we shall have

$$C = \frac{1}{4}\tan\alpha - 16\alpha^3\Sigma_0 \frac{1}{(n\pi - 2\alpha)n\pi(n\pi + 2\alpha)^2} \ldots\ldots\ldots\ldots\ldots\ldots (121).$$

The mass of the portion of fluid considered is $\frac{1}{2}\alpha$; and if we put

$$C = \tfrac{1}{2}\alpha k'^2,$$

and write $\frac{s\pi}{2}$ for α, so that s may have any value from 0 to 4, we shall have

$$k'^2 = \frac{1}{s\pi}\tan\frac{s\pi}{2} - \frac{8s^2}{\pi^2}\Sigma_0 \frac{1}{(n - s)\,n\,(n + s)^2} \ldots\ldots\ldots\ldots\ldots\ldots (122).$$

When s is an odd integer, the expression for k'^2 takes the form $\infty - \infty$, and we shall easily find

$$k'^2 = \frac{4}{s^2\pi^2} - \frac{8s^2}{\pi^2}\Sigma_0 \frac{1}{(n-s)\,n\,(n+s)^2} \quad\text{......................(123)},$$

where all odd values of n except s are to be taken.

The quantity k' may be called the *radius of gyration* of the fluid about the axis. It would be easy to prove from general dynamical principles, without calculation, that if k be the corresponding quantity for a parallel axis passing through the centre of gravity of the fluid, h the distance of the axes

$$k'^2 = k^2 + h^2 \quad\text{..............................(124)},$$

in fact, in considering the motion of the cylinder, which is supposed to take place in two dimensions, the fluid may be replaced by a solid having the same mass and centre of gravity as the fluid, but a moment of inertia about an axis passing through the centre of gravity and parallel to the axis of the cylinder different from the moment of inertia of the fluid supposed to be solidified. If K', K be the radii of gyration of the solidified fluid about the axis of the cylinder and a parallel axis passing through the centre of gravity respectively, we shall have

$$K'^2 = \tfrac{1}{2} = K^2 + h^2,\quad h = \frac{4}{3}\frac{\sin\frac{1}{2}\alpha}{\alpha} = \frac{8}{3s\pi}\sin\frac{s\pi}{4} \quad\text{..............(125)}.$$

If we had restricted the application of the series and the integrals involving cosines to those cases in which the derivative of the expanded function vanishes at the limits, we should have expanded ϕ in the definite integral $\int_0^\infty \zeta(\theta,\beta)\cos\beta\lambda\,d\beta$, and the equation (115) would have given

$$\zeta(\theta,\beta) = \xi(\beta)\epsilon^{\beta\theta} + \chi(\beta)\epsilon^{-\beta\theta},$$

ξ, χ denoting arbitrary functions, which must be determined by the conditions (116). We should have obtained in this manner

$$\phi = \frac{4\omega}{\pi}\int_0^\infty \frac{\epsilon^{\beta(\theta-\frac{1}{2}\alpha)} - \epsilon^{-\beta(\theta-\frac{1}{2}\alpha)}}{\beta(\beta^2+4)(\epsilon^{\frac{1}{2}\beta\alpha}+\epsilon^{-\frac{1}{2}\beta\alpha})}\cos\left(\beta\log\frac{1}{r}\right)d\beta \quad\text{...........(126)},$$

$$k'^2 = \frac{32}{\pi\alpha}\int_0^\infty \frac{1-\epsilon^{-\beta\alpha}}{1+\epsilon^{-\beta\alpha}}\frac{d\beta}{\beta(\beta^2+4)^2} \quad\text{.................(127)}.$$

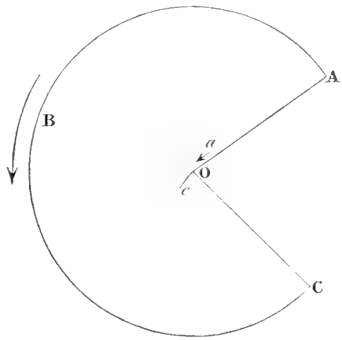
It will be seen at once that k'^2 is expressed in a much better form for numerical computation by the series in (122) than by the integral in (127). Although the nature of the problem restricts α to be at most equal to 2π, it will be observed that there is no such restriction in the analytical proof of the equivalence of the two expressions for ϕ, or for k'^2.

In the following table the first column gives the angle of the cylindrical sector, the second the square of the radius of gyration of the fluid about the axis of the cylinder, the radius of the cylinder being taken for the unit of length, the third the square of the radius of gyration of the fluid about a parallel axis passing through the centre of gravity, the fourth and fifth the ratios of the quantities in the second and third to the corresponding quantities for the solidified fluid.

a	k'^2	k^2	$\frac{k'^2}{K'^2}$	$\frac{k^2}{K^2}$
0°	·50000	·05556	1·0000	1·0000
45°	·45385	·03179	·9077	·4079
90°	·39518	·03492	·7904	·2499
135°	·34775	·07442	·6955	·3283
180°	·31057	·13044	·6211	·4078
225°	·28101	·18261	·5620	·4547
270°	·25703	·21700	·5141	·4718
315°	·23720	·22858	·4744	·4652
360°	·22051	·22051	·4410	·4410

55. When α is greater than π, it will be observed that the expression for the velocity which is obtained from (119) becomes infinite when r vanishes. Of course the velocity cannot really become infinite, but the expression (119) fails for points very near the axis. In fact, in obtaining this expression it has been assumed that the motion of the fluid is continuous, and that a fluid particle at the axis may be considered to belong to either of the plane faces indifferently, so that its velocity in a direction normal to either of the faces is zero. The velocity obtained from (119) satisfies this latter condition so long as α is not greater than π. For when $\alpha < \pi$ the velocity vanishes with r, and when $\alpha = \pi$ the velocity is finite when r vanishes, and is directed along the single plane face which is made up of the two plane faces before considered.

But when α is greater than π the motion which takes place appears to be as follows. Let $OABC$ be a section of the cylindrical sector made by a plane perpendicular to the axis, and cutting it in O. Suppose the cylinder to be turning round O in the direction indicated by the arrow at B. Then the fluid in contact with OA and near O will be flowing, relatively to OA,



towards O, as indicated by the arrow a. When it gets to O it will shoot past the face OC; so that there will be formed a surface of discontinuity Oe extending some way into the fluid, the fluid to the left of Oe and near O flowing in the direction AO, while the fluid to the right is nearly at rest. Of course, in the case of fluids such as they exist in nature, friction would prevent the velocity in a direction tangential to Oe from altering abruptly as we pass from a particle on one side of Oe to a particle on the other; but I have all along been going on the supposition that the fluid is perfectly smooth, as is usually supposed in Hydrodynamics. The extent of the surface of discontinuity Oe will be the less the smaller be the motion of the cylinder; and although the expression (119) fails for points very near O, that does not prevent it from being sensibly correct for the remainder of the fluid, so that we may calculate k'^2 from (122) without committing a sensible error. In fact, if γ be the angle through which the cylinder oscillates, since the extent of the surface of discontinuity depends upon the first power of γ, the error we should commit would depend upon γ^2. I expect, therefore, that the moment of inertia of the fluid which would be determined by experiment would agree with theory nearly, if not quite, as well when $\alpha > \pi$ as when $\alpha < \pi$, care being taken that the oscillations of the cylinder be very small.

As an instance of the employment of analytical expressions which give infinite values for physical quantities, I may allude to the distribution of electricity on the surfaces of conducting bodies which have sharp edges.

56. The preceding examples will be sufficient to show the utility of the methods contained in this paper. It may be observed that in all cases in which an arbitrary function is expanded between certain limits in a series of quantities whose form is determined by certain conditions to be satisfied at the limits, the expansion can be performed whether the conditions at the limits be satisfied or not, since the expanded function is supposed perfectly arbitrary. Analogy would lead us to conclude that the derivatives of the expanded functions could not be found by direct differentiation, but would have to be obtained from formulæ answering to those at the beginning of this Section. If such expansions should be found useful, the requisite formulæ would probably be obtained without difficulty by integration by parts. This is in fact the case with the only expansion of the kind which I have tried, which is that employed in Art. 45. By means of this expansion and the corresponding formulæ we might determine in a double series the permanent temperature in a homogeneous rectangular parallelepiped which radiates into a medium whose temperature varies in any given manner from point to point; or we might determine in a triple series the variable temperature in such a solid, supposing the temperature of the medium to vary in a given manner with the time as well as with the co-ordinates, and supposing the initial temperature of the parallelepiped given as a function of the co-ordinates. This problem, made a little more general by supposing the exterior conductivity different for the six faces, has been solved in another manner by M. Duhamel in the Fourteenth Volume of the *Journal de l'Ecole Polytechnique.* Of course such a problem is interesting only as an exercise of analysis.

G. G. STOKES.

ADDITIONAL NOTE.

If the series by which r^2 is multiplied in (119) had been left without summation, the series which would have been obtained for k'^2 would have been rather simpler in form than the series in (122), although more slowly convergent. One of these series may of course be obtained from the other by means of the developement of $\tan x$ in a harmonic series. When s is an integer, k'^2 can be expressed in finite terms. The result is

$$k'^2 = 8s^{-1}\pi^{-2}\log_\epsilon 2 + 8s^{-1}\pi^{-2}\{2^{-1}+4^{-1}\ldots+(s-1)^{-1}\} + 4\pi^{-2}\{2^{-2}+4^{-2}\ldots+(s-1)^{-2}\} - \tfrac{1}{6}, \quad (s \text{ odd}),$$

$$k'^2 = 8s^{-1}\pi^{-2}\{1^{-1}+3^{-1}\ldots+(s-1)^{-1}\} + 4\pi^{-2}\{1^{-2}+3^{-2}\ldots+(s-1)^{-2}\} - \tfrac{1}{2}. \quad (s \text{ even}).$$

Moreover when $2s$ is an odd integer, or when $\alpha = 45^0$, or $= 135^0$, &c., k'^2 can be expressed in finite terms if the sum of the series $1^{-2}+5^{-2}+9^{-2}+\ldots$ be calculated, and then be regarded as a known transcendental quantity.

XLI. *A Mathematical Theory of Luminous Vibrations. By the Rev.* J. CHALLIS, M.A., F.R.A.S., *Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge.*

[Read *March* 6, 1848.]

IN three preceding communications to this Society I endeavoured to explain some of the principal phænomena of Light on the Hypothesis of Undulations, regarding the æther as a continuous and elastic fluid, and applying to it the usual Hydrodynamical Equations. I propose now, on the same principles, to investigate the particular nature of the ætherial vibrations which produce light, and the laws of their propagation under given circumstances. As this communication is intended to be supplementary to the three former, I shall take occasion to advert to any reasoning they contain, to which I may be able to add elucidation or confirmation.

1. Let $a^2(1+s)$ be the pressure at any point xyz of the æther at any time t, s being a small numerical quantity, the powers of which above the first are neglected; and let u, v, w, be the resolved parts of the velocity at the same point and at the same time, in the directions of the axes of co-ordinates. Then, retaining only the first powers of u, v, w, we have, as is known,

$$a^2 \cdot \frac{ds}{dx} + \frac{du}{dt} = 0, \ (1) \qquad a^2 \cdot \frac{ds}{dy} + \frac{dv}{dt} = 0, \ (2) \qquad a^2 \cdot \frac{ds}{dz} + \frac{dw}{dt} = 0, \ (3)$$

$$\text{and } \frac{ds}{dt} + \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

The last of these equations gives by means of the other three,

$$\frac{d^2 s}{dt^2} - a^2 \cdot \left(\frac{d^2 s}{dx^2} + \frac{d^2 s}{dy^2} + \frac{d^2 s}{dz^2}\right) = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Suppose, for the moment, that s has been obtained from this equation by integration. Then for the velocities we have,

$$u = c - a^2 \int \frac{ds}{dx} \, dt = c - a^2 \cdot \frac{d \cdot \int s \, dt}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

$$v = c' - a^2 \int \frac{ds}{dy} \, dt = c' - a^2 \cdot \frac{d \int s \, dt}{dy} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$w = c'' - a^2 \int \frac{ds}{dz} \, dt = c'' - a^2 \cdot \frac{d \int s \, dt}{dz} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

wnere c, c', and c'' are functions of co-ordinates only. It is to be observed that these values of u, v, w are perfectly general, being obtained prior to any consideration of the way in which the fluid was put in motion, and consequently apply to all points of the fluid in every instance of motion in which powers of the velocity and condensation above the first may be neglected. Now the motions of the ætherial medium are *vibratory*, or, at least, not permanent. There is no known cause to produce motions in the æther, which either wholly or in part remain permanently the same at the same points of space for any length of time. And even if, from causes with which we are unacquainted, such motion should exist, provided it be small compared with the velocity a, we may abstract from it in considering vibratory motion. This will appear as follows. Since equation (5) is linear, we may suppose s to be composed of parts due to separate causes, among which may be included the cause that produces the permanent part of the motion. But the condensation due to this cause being represented by σ, we shall have,

$$a^2 \cdot \frac{d\sigma}{dx} = -\frac{dc}{dt} = 0, \qquad a^2 \cdot \frac{d\sigma}{dy} = -\frac{dc'}{dt} = 0, \qquad a^2 \cdot \frac{d\sigma}{dz} = -\frac{dc''}{dt} = 0.$$

That is, σ is either constant throughout the fluid, or is a quantity of an order already neglected. Hence the values of $\frac{ds}{dx}, \frac{ds}{dy}, \frac{ds}{dz}$ in equations (6), (7), (8), remain the same whatever be the permanent motion. Hence also $u - c$, $v - c'$, $w - c''$, or the parts of the motion which are not permanent, are the same whatever be c, c', c''. We may, therefore, either put $c = 0$, $c' = 0$, $c'' = 0$; or, suppose u, v, w, to stand respectively for $u - c$, $v - c'$, $w - c''$.

This being premised, let $\psi = -a^2 \int s\,dt$. Then

$$u = \frac{d\psi}{dx}, \quad v = \frac{d\psi}{dy}, \quad w = \frac{d\psi}{dz},$$

and $u\,dx + v\,dy + w\,dz = (d\psi)$, an exact differential. Also by means of equation (4), we derive,

$$\frac{d^2\psi}{dt^2} - a^2 \cdot \left(\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} + \frac{d^2\psi}{dz^2}\right) = 0 \quad \dots\dots\dots\dots\dots\dots \quad (9).$$

2. The motions of the æther which correspond to the phænomena of Light are vibratory. Hence in treating the Undulatory Theory of Light hydrodynamically, the quantity $u\,dx + v\,dy + w\,dz$ must be an exact differential, by what is shewn above, without reference to the manner in which the fluid was put in motion, the reasoning being prior to, and entirely independent of, any such considerations. The condition of integrability is to be satisfied *generally*. One obvious method of doing this, is to suppose the motion to consist of separate motions which tend to or from fixed centres, and are functions of the distances from the centres. But the phænomena of Light do not accord with this supposition, since, instead of spreading equally in all directions from a centre, it is generally propagated in the form of rays. Another way of satisfying the condition of integrability in a general manner, is to suppose ψ to be the product of two functions ϕ and f, such that ϕ does not contain x or y, and f does not contain z. For on these suppositions,

$$u = \phi \frac{df}{dx}, \quad v = \phi \frac{df}{dy}, \quad w = f \frac{d\phi}{dz}, \quad \text{and}$$

$$u\,dx + v\,dy + w\,dz = \phi \left(\frac{df}{dx}\,dx + \frac{df}{dy}\,dy\right) + f \frac{d\phi}{dz}\,dz,$$

which is an exact differential of $f\phi$ with respect to co-ordinates. The consequences of thus satisfying the condition of integrability, which are of a very remarkable kind, I now proceed to develope.

3. The above values of u, v, w give,

$$\frac{du}{dx} = \phi \frac{d^2 f}{dx^2}, \quad \frac{dv}{dy} = \phi \frac{d^2 f}{dy^2}, \quad \frac{dw}{dz} = f \frac{d^2\phi}{dz^2}.$$

Let us now, for a reason that will presently appear, suppose that f does not contain t. Then since $\psi = \phi f = -a^2 \int s\,dt$, it follows that $f\frac{d\phi}{dt} = -a^2 s$, and $\frac{ds}{dt} = -\frac{f}{a^2}\cdot\frac{d^2\phi}{dt^2}$. Hence substituting in the equation (4), we obtain,

$$\frac{d^2\phi}{dt^2} = a^2\frac{d^2\phi}{dz^2} + \frac{a^2}{f}\cdot\left(\frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2}\right)\phi \quad \ldots\ldots\ldots\ldots\ldots (10).$$

Now the nature of the question under consideration requires that this equation should be *linear*. Let therefore the coefficient of ϕ be equal to a constant $-b^2$. According to this supposition ϕ may be a function of z and t only, and f a function of x and y only; as, in fact, they ought to be, in consequence of suppositions already made on these quantities. Thus equation (10) resolves itself into the two following,

$$\frac{d^2\phi}{dt^2} - a^2\cdot\frac{d^2\phi}{dz^2} + b^2\phi = 0 \quad \ldots\ldots\ldots\ldots\ldots (11),$$

$$\frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} + \frac{b^2}{a^2} f = 0 \quad \ldots\ldots\ldots\ldots\ldots (12).$$

The equation (11) is transformable into the following:

$$\frac{d^2\phi}{du\,dv} - \frac{b^2}{4a^2}\phi = 0 \quad \ldots\ldots\ldots\ldots\ldots (13),$$

in which $u = z + at$ and $v = z - at$. (See Peacock's *Examples*, p. 466). For convenience sake put e for $\frac{b^2}{4a^2}$. Then, regarding e as a small quantity, the integral of (13) may be obtained in a series as follows.

Let $\frac{d^2\phi}{du\,dv} = 0$; then $\frac{d\phi}{du} = F'(u)$, and $\phi = F(u) + G(v)$.

Hence $\frac{d^2\phi}{du\,dv} = e\,\{F(u) + G(v)\}$ approximately.

$$\frac{d\phi}{dv} = G'(v) + e\,\{F_1(u) + u\,G(v)\}$$

$$\phi = F(u) + G(v) + e\,\{v\,F_1(u) + u\,G_1(v)\}; \text{ and so on.}$$

Thus $\phi = F(u) + G(v) + e\,\{v\,F_1(u) + u\,G_1(v)\} + \frac{e^2}{1\,.\,2}\{v^2 F_2(u) + u^2 G_2(v)\} + \&c.$

where $F_1(u) = \int F(u)\,du$, $F_2(u) = \int F_1(u)\,du$, $G_1(v) = \int G(v)\,dv$, &c. Each of the functions F and G separately satisfies the given equation. Let us, therefore, for the purpose of drawing some inferences from this integral, suppose that $F = 0$. Then,

$$\phi = G(v) + eu\,G_1(v) + \frac{e^2 u^2}{1\,.\,2}\,.\,G_2(v) + \frac{e^3 u^3}{1\,.\,2\,.\,3}\,.\,G_3(v) + \&c. \ldots\ldots\ldots (14).$$

4. It appears by this result that ϕ does not admit of being expressed exactly so long as the form of the function G is entirely arbitrary. No inference, therefore, can be drawn from the integral (14) in its general form. The nature of the series, however, suggests at once a particular form of G, which gives to ϕ an exact expression, and which, as we shall see, applies to our present enquiry; viz. the form $A\epsilon^{qv}$. As we have already introduced the condition that the velocity and condensation be small, and consequently that ϕ be small, whatever be z and t,

it is clear that G must be a circular function. Let therefore $G(v) = A\epsilon^{nv\sqrt{-1}} + B\epsilon^{-nv\sqrt{-1}}$; or, what is equivalent, let $G(v) = m\cos(nv + c)$. Then,

$$G_1(v) = \int m\cos(nv + c)\,dv = \frac{m}{n}\sin(nv + c) = -\frac{m}{n^2}\cdot\frac{d.\cos(nv + c)}{dv},$$

$$G_2(v) = \int \frac{m}{n}\sin(nv + c)\,dv = -\frac{m}{n^2}\cos(nv + c) = \frac{m}{n^4}\cdot\frac{d^2.\cos(nv + c)}{dv^2},$$

$$G_3(v) = \int -\frac{m}{n^2}\cos(nv + c)\,dv = -\frac{m}{n^3}\sin(nv + c) = -\frac{m}{n^6}\cdot\frac{d^3.\cos(nv + c)}{dv^3};$$

$$\&c. = \quad \&c.$$

Consequently,

$$\phi = m\cos(nv + c) - \frac{m}{n^2}\cdot\frac{d.\cos(nv + c)}{dv}\cdot eu + \frac{m}{n^4}\cdot\frac{d^2.\cos(nv + c)}{dv^2}\cdot\frac{e^2u^2}{1.2}$$
$$- \frac{m}{n^6}\cdot\frac{d^3.\cos(uv + c)}{dv^3}\cdot\frac{e^3u^3}{1.2.3} + \&c.$$

$$= m\cos\left\{n\left(v - \frac{eu}{n^2}\right) + c\right\}$$

$$= m\cos\left\{n(z - at) - \frac{e}{n}(z + at) + c\right\}$$

$$= m\cos\left\{\left(n - \frac{e}{n}\right)z - \left(n + \frac{e}{n}\right)at + c\right\}.$$

Let, now, $n - \frac{e}{n} = \frac{2\pi}{\lambda}$. Then $n + \frac{e}{n} = \sqrt{\frac{4\pi^2}{\lambda^2} + 4e}$. We have, therefore, finally,

$$f\phi = mf\cos\frac{2\pi}{\lambda}\left(z - at\sqrt{1 + \frac{e\lambda^2}{\pi^2}} + c'\right) = \psi.$$

The velocity in the direction of z is $f\frac{d\phi}{dz}$. Hence, if $m_{\prime} = -\frac{2\pi m}{\lambda}$.

$$w = m_{\prime}f\sin\frac{2\pi}{\lambda}\left(z - at\sqrt{1 + \frac{e\lambda^2}{\pi^2}} + c'\right) \text{ ....(15).}$$

Also, since (Art. 3) $f.\frac{d\phi}{dt} + a^2 s = 0$, it follows that

$$as = -\frac{f}{a}\frac{d\phi}{dt} = m_{\prime}f\sqrt{1 + \frac{e\lambda^2}{\pi^2}}\sin\frac{2\pi}{\lambda}\left(z - at\sqrt{1 + \frac{e\lambda^2}{\pi^2}} + c'\right) \text{......(16).}$$

It hence appears that the velocity of propagation of the wave whose breadth is λ, is $a\sqrt{1 + \frac{e\lambda^2}{\pi^2}}$.

The value of e depends on equation (12). If the velocity of propagation be independent of λ, we shall have $\frac{e\lambda^2}{\pi^2} = k$, a numerical constant, and consequently $e = \frac{k\pi^2}{\lambda^2}$.

5. Since equation (9) is linear with constant coefficients, it will be satisfied by the sum of any number of such values of ψ as that just obtained, f, e, m, λ, and c', being different for each. Hence we have generally,

$$\psi = \Sigma \left\{ fm \cos \frac{2\pi}{\lambda} \left(z - at\sqrt{1 + \frac{e\lambda^2}{\pi^2}} + c\right)\right\}$$

$$\frac{d\psi}{dz} = w = \Sigma \left\{ fm_{,} \sin \frac{2\pi}{\lambda} \left(z - at\sqrt{1 + \frac{e\lambda^2}{\pi^2}} + c\right)\right\}$$

$$-\frac{1}{a} \cdot \frac{d\psi}{dt} = as = \Sigma \left\{ fm_{,} \sqrt{1 + \frac{e\lambda^2}{\pi^2}} \sin \frac{2\pi}{\lambda} \left(z - at\sqrt{1 + \frac{e\lambda^2}{\pi^2}} + c\right)\right\}.$$

It follows, since in each of the terms under the sign Σ the quantities which are independent of z and t are at our disposal, that we may satisfy by this integral any state of the fluid in the direction of z, subject to the limitation that the condensation and velocity are at all times small. The course of the reasoning shews that the particular form of the function G which has conducted to the above results, has not been adopted as an analytical artifice, but is really the only form which determines the velocity of propagation, and gives a definite solution of the Problem. The particular kind of motion it represents, and the component character of the whole motion as consisting of an indefinite number of such motions, are accordingly to be regarded as physically true. These results explain the fact of the *composition* of light.

6. Before proceeding farther, it will be worth while to compare the foregoing investigation with that which I have given in my Paper on Luminous Rays. (*Cambridge Philosophical Transactions*, Vol. VIII. Part III. p. 363). It may be remarked, that the two investigations agree in their results, but differ in the course of the reasoning. In the Paper referred to, the velocity of propagation is *assumed* to be uniform (p. 365), and the form of the function expressing the nature of the vibrations is deduced from this assumption (p. 368). In the present communication the form of that function is first obtained by *à priori* reasoning from the Hydrodynamical Equations, and the uniformity of the rate of propagation is then strictly deduced. The inferences in the former Paper (p. 365), drawn from the supposition that the velocity of propagation is uniform when the motion is not small, still hold good. It may also here be remarked, that the considerations in p. 366 on which the arbitrary quantities c, c', c'' were made to vanish, are superseded by the more general reasoning in Art. I. of this Paper.

7. I proceed now to the consideration of equation (12), viz.

$$\frac{d^2f}{dx^2} + \frac{d^2f}{dy^2} + 4ef = 0.$$

As this equation does not contain t, there is no propagation of motion in any direction parallel to the plane of xy; or, the propagation in the direction of z takes place without lateral spreading. A value of f expressed in finite terms is not therefore required, as in the case of the integration of equation (11), for deducing velocity of propagation. It may however be argued, that as a particular form of ϕ was found, by which the vibrations in the direction of z were defined, prior to any consideration of the manner in which the fluid was put in motion, so a particular form of f exists by which the condensation and velocity in directions transverse to the axis of z are defined, and which is equally independent of the arbitrary disturbance. As this form may, or may not, be capable of expression in exact terms, I shall first apply to equation (12) the process already applied to equation (11), for the purpose of ascertaining whether any exact value of the integral satisfies the conditions of the Problem.

8. The equation (11) coincides in form with (12) by putting -1 for a^2, and $4e$ for b^2. That is, since $e = \frac{b^2}{4a^2}$, we shall have $-e$ in the place of e in the integral of (12). Hence

if $u = x + y\sqrt{-1}$ and $v = x - y\sqrt{-1}$, by integrating as before,

$$f = F(u) + G(v) - e\{v F_1(u) + u G_1(v)\} + \frac{e^2}{1.2}\{v^2 F_2(u) + u^2 G_2(v)\} - \&c. \quad \ldots\ldots (17).$$

The impossible quantities are got rid of by making G the same function as F. If then, to obtain an exact value of f, we suppose that $F(u) = A\epsilon^{ku}$ and $G(v) = A\epsilon^{kv}$, we shall have,

$$f = A(\epsilon^{ku} + \epsilon^{kv}) - \frac{eA}{k}(v\epsilon^{ku} + u\epsilon^{kv}) + \frac{e^2 A}{1.2.k^2}(v^2\epsilon^{ku} + u^2\epsilon^{kv}) - \&c.$$

$$= A\epsilon^{ku}\left(1 - \frac{ev}{k} + \frac{e^2v^2}{1.2.k^2} - \frac{e^3v^3}{1.2.3.k^3} + \&c.\right) + A\epsilon^{kv}\left(1 - \frac{eu}{k} + \frac{e^2u^2}{1.2.k^2} - \frac{e^3u^3}{1.2.3.k^3} + \&c.\right)$$

$$= A.\epsilon^{ku - \frac{ev}{k}} + A\epsilon^{kv - \frac{eu}{k}}$$

$$= A\epsilon^{k(x+y\sqrt{-1}) - \frac{e}{k}(x-y\sqrt{-1})} + A.\epsilon^{k(x-y\sqrt{-1}) - \frac{e}{k}(x+y\sqrt{-1})}$$

$$= A\epsilon^{\left(k-\frac{e}{k}\right)x}.\epsilon^{\left(k+\frac{e}{k}\right)y\sqrt{-1}} + A\epsilon^{\left(k-\frac{e}{k}\right)x}.\epsilon^{-\left(k+\frac{e}{k}\right)y\sqrt{-1}}$$

$$= 2A\epsilon^{\left(k-\frac{e}{k}\right)x}\cos\left(k + \frac{e}{k}\right)y.$$

Since, from the form of equation (12) x and y are interchangeable, we shall also have

$$f = 2A'\epsilon^{\left(k'-\frac{e}{k'}\right)y}.\cos\left(k' + \frac{e}{k'}\right)x.$$

Therefore generally,

$$f = 2A\epsilon^{\left(k-\frac{e}{k}\right)x}\cos\left(k + \frac{e}{k}\right)y + 2A'\epsilon^{\left(k'-\frac{e}{k'}\right)y}\cos\left(k' + \frac{e}{k'}\right)x.$$

As the quantities k and k' may be any whatever, this solution is so far indeterminate. But it is clear that the value of f must not, from the nature of the question, increase indefinitely with x and y, and that consequently the exponentials must be made to disappear. Hence we shall have $k = k' = \sqrt{e}$, and

$$f = 2A\cos 2\sqrt{e}\,y + 2A'\cos 2\sqrt{e}\,x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (18).$$

This then is the general form of f expressed in finite terms, and subject to the limitation of being free from exponentials. Other forms may be adduced, apparently, but not really, different from this, which equally satisfy the equation (12). For instance $f = A\cos qx\cos q'y$, provided $q^2 + q'^2 = 4e$. But this is reducible to the form of the terms of equation (18), by a change in the direction of the axes of x and y. (See *Theory of the Polarization of Light*, p. 373.)

I shall have occasion hereafter to advert to equation (18). At present I have only to remark that the above form of f does not correctly define the motion transverse to the axis of z, at least for *all* values of x and y, for this reason. At the boundary beyond which the motion does not extend in directions transverse to z there must be neither condensation nor variation of condensation, otherwise there will be transverse propagation. Hence f, $\frac{df}{dx}$, and $\frac{df}{dy}$ must vanish together. But plainly this is not the case with the value of f obtained above.

9. From the above reasoning we may conclude that the form of f we are seeking for, is not expressible in finite terms, and must consequently be obtained in an infinite series. The

only way in which a particular value of f is deducible from the general integral (17) without assigning arbitrary forms to the functions F and G, is to suppose $F(u)$ and $G(v)$ to be arbitrary *constants*. Let, therefore, $F(u) = c$, and $G(v) = c'$. Then

$$F_1(u) = cu, \qquad G_1(v) = c'v,$$

$$F_2(u) = \frac{cu^2}{2}, \qquad G_2(v) = \frac{c'v^2}{2},$$

$$\&c. = \&c. \qquad \&c. = \&c.$$

$$\text{Hence } f = (c + c')\left(1 - euv + \frac{e^2}{1^2 . 2^2} u^2 v^2 - \frac{e^3}{1^2 . 2^2 . 3^2} . u^3 v^3 + \&c.\right),$$

$$= (c + c')\left(1 - er^2 + \frac{e^2 r^4}{1^2 . 2^2} - \frac{e^3 r^6}{1^2 . 2^2 . 3^2} + \&c.\right), \ldots\ldots (19),$$

by putting r^2 for $x^2 + y^2$. Determining the arbitrary quantities so that $f = 1$ when $r = 0$, we have $c + c' = 1$. Also $\frac{df}{dr} = 0$ when $r = 0$, and $\frac{d^2 f}{dr^2} = -2e$. Hence f has a maximum value at the axis of z, and is a function of the distance from that axis.

10. It appears, therefore, that the required form of f is derived from equation (12), by supposing f to be a function of $x^2 + y^2$. That equation accordingly becomes,

$$\frac{d^2 f}{dr^2} + \frac{df}{rdr} + 4ef = 0. \ldots\ldots (20).$$

Equation (19) is the integral of this equation in a series, the only mode in which it appears to be expressible. By putting $f = 0$, we have for determining the corresponding values of r the equation,

$$0 = 1 - er^2 + \frac{e^2 r^4}{1^2 . 2^2} - \frac{e^3 r^6}{1^2 . 2^2 . 3^2} + \&c.,$$

from which it appears that there are an unlimited number of possible values of r for which f vanishes. Since there is no lateral propagation, the motion does not extend beyond a certain limiting distance from the axis, at which f and $\frac{df}{dr}$ both vanish. It is not, however, apparent from equation (20) that these quantities may vanish together, that being an approximate equation which does not give the exact value of $\frac{df}{dr}$ when $f = 0$. To ascertain whether this will be the case, recourse must be had to equation (12) in the Paper on Luminous Rays, (p. 368), which was obtained without neglecting small terms. On putting $4e$ for kn^2 that equation becomes,

$$\frac{d^2 . \frac{1}{f}}{dx^2} + \frac{d^2 . \frac{1}{f}}{dy^2} - 4e . \frac{1}{f} = 0. \ldots\ldots (21).$$

Assuming now that f is a function of r, we obtain,

$$f . \frac{d^2 f}{dr^2} - 2\frac{df^2}{dr^2} + \frac{f}{r} . \frac{df}{dr} + 4ef^2 = 0 \ \ldots\ldots \ (22),$$

whence it is clear that if $f = 0$, $\frac{df}{dr}$ also vanishes. Since $\frac{df}{dr} = 0$ both when $f = 1$ and $f = 0$, for some intermediate value $\frac{df}{dr}$ must be a maximum. Hence if a curve were described having for its equation $y = f(x)$, it would have a point of contrary flexure between the values of x corresponding to $y = 1$ and $y = 0$, and would resemble the trochoid. It is also to be remarked that the second term of equation (22) must be regarded as very small compared to the other terms, in order that that equation may be equivalent to a *linear* equation in x and y, excepting where f is very small. By the omission of the second term, equation (22) becomes identical with equation (20). Hence, with the exception just mentioned, the curves which represent the integrals of (20) and (22) coincide; and as we found that the curve corresponding to (20) cuts the axis of abscissæ in an unlimited number of points, the same must be the case with the curve corresponding to equation (22). But for the latter curve we have shewn that $f = 0$ and $\frac{df}{dr} = 0$ at a point of intersection. Hence the motion does not extend beyond the *least* value of r corresponding to $f = 0$.

11. The integral of equation (21) is derived from that of (12) by putting $\frac{1}{f}$ for f and $-e$ for e. Hence the integral of (21) in a series is,

$$\frac{1}{f} = 1 + er^2 + \frac{e^2 r^4}{1^2 \cdot 2^2} + \frac{e^3 r^6}{1^2 \cdot 2^2 \cdot 3^2} + \&c.$$

Whence $$f = 1 - er^2 + \frac{3e^2 r^4}{4} - \frac{19 e^3 r^6}{36} + \&c. \ldots\ldots\ldots\ldots (23).$$

This series diverges from the approximate series (19) after the second term. Let l be the least value of r corresponding to $f = 0$. Then,

$$0 = 1 - el^2 + \frac{3e^2 l^4}{4} - \frac{19 e^3 l^6}{36} + \&c.$$

Hence el^2 is a numerical quantity. Let $el^2 = q$. Then, as we have also $\frac{e\lambda^2}{\pi^2} = k$, it follows that $k = \frac{q\lambda^2}{\pi^2 l^2}$. Hence k is a constant quantity for all vibrations, if the ratio $\frac{\lambda}{l}$ be a constant. Now it may be thus argued that λ and l have a constant ratio to each other. These quantities must be related in some way, otherwise the motion is not defined. Let $F(\lambda, l, S) = 0$ express this relation, S being the maximum condensation corresponding to $f = 1$. As there are no other quantities concerned in this relation, and as λ and l are the only linear quantities, this equation is equivalent to $\frac{\lambda}{l} = F_1(S)$. And we have above, $\frac{\lambda}{l} = \pi \sqrt{\frac{k}{q}}$. Hence $\pi \sqrt{\frac{k}{q}} = F_1(S)$. But it has already been shewn (Art. 4) that k is independent of S. Hence $F_1(S)$ is a constant, the same for all vibrations. Hence also k is the same for all vibrations.

12. We have now found for f a particular value which satisfies the *hydrodynamical* conditions of the question, but does not admit of being definitely expressed. It can only be expressed in an infinite series, the terms of which do not necessarily converge. If, therefore, the phænomena of light be expounded by a definite form of f, this can agree with equation (23) only under certain limitations. Now, by equation (18), we have a definite form of f obtained in a general manner, without reference to the mode of disturbance. If in this equation $2A = 2A' = \frac{1}{2}$, we obtain,

$$f = \tfrac{1}{2}\left(1 - 2ex^2 + \frac{2e^2x^4}{3} - \&c.\right) + \tfrac{1}{2}\left(1 - 2ey^2 + \frac{2e^2y^4}{3} - \&c.\right)$$

$$= 1 - e(x^2 + y^2) + \frac{e^2}{3}(x^4 + y^4) - \&c.$$

This value of f agrees with that given by equation (23) only to two terms. Consequently the exact integral (18) may be employed only for small values of x and y. With this limitation, it gives a value of f definitely expressed, and at the same time satisfying the hydrodynamical conditions. These results point to the inference that the phænomena of light depend exclusively on the motions contiguous to the axis of z; for it may be presumed that so far as the motions correspond to the phænomena of light, they admit of being defined by exact expressions. The ratios $\frac{r}{\lambda}$ and $\frac{r}{l}$ as applied to the *luminous* ray, will each be very small.

13. It may here be remarked, that in my Paper on the *Polarization of Light*, the equation $f = \cos\sqrt{2er}$ corresponds to common light, and the equations $f = \cos 2\sqrt{ex}$, $f = \cos 2\sqrt{ey}$, to light polarized in the planes of xz and yz, subject to the limitation of taking r, x, and y, very small. The first equation was obtained by assuming f to be a function of r, because common light is *observed* to have the same relations to space in all directions perpendicular to the direction of its propagation, and the other two were deduced from the first, by *assuming* the bifurcation of a ray of common light to take place, so that the sum of the condensations at corresponding points of the two parts, is equal to the condensation at the corresponding point of the original ray, and the velocities are the parts of the original velocity resolved in directions at right angles to each other. Since in the present Paper the same values of f have been arrived at by *à priori* considerations, that particular property of common light, and its resolution in that particular manner, may be said to be accounted for on hydrodynamical principles.

14. The foregoing theoretical conclusions serve to explain some general phænomena of light. In Article 7. it was argued that the motion transverse to the axis of the fluid *filament*, must be defined by a particular form of f independent of the arbitrary disturbance of the fluid, and in Art. 9, a form of this function was found without assigning particular forms to the arbitrary functions, which in Art. 10. was proved to be consistent with the hydrodynamical conditions. As this form indicates that the condensation is arranged alike in all directions about an axis of propagation, it follows that light which comes directly to the eye from its origin, of whatever kind the disturbance may be, is common light, the distinctive property of which is, that it is alike affected in all directions perpendicular to the direction of propagation. This inference is confirmed by the fact that Light from the Sun, from Stars, from a lamp, from the electric spark, from lightning, &c. is common light. The dispersed light by which objects are rendered visible, which originates in the disturbances passively caused by the presence of the individual atoms of the medium on which any ray impinges, should according to the theory be common light: and such it is found to be. Moon-light and light from the Planets come under the same description.

Again, the form which the ray assumes at its origin determines it to have *direction*, for it is clear that the direction of its propagation must from the first be coincident with the axis about which the condensation is symmetrical. Hence as direction is determined without reference to the mode of disturbance, there may be an unlimited number of directions of propagation, as there may be an unlimited number of rays, (see Art. 5), due to the same disturbance. In fact, the state of the fluid at the first instant, whatever it may be, can be satisfied by having at disposal in the equations $V = aS = m \sin\frac{2\pi}{\lambda}(at - z + c)$, the quantities m, λ, c, and by an unlimited number of rays unlimited as to direction, notwithstanding that the functions ϕ and f are defined for each ray. This agrees with the fact, that light coming immediately from its origin, is seen in all directions.

15. Hitherto we have reasoned on the supposition that no extraneous *force* acted on the æther. It is quite possible that a ray, after taking its original form and direction, may be modified subsequently in both these respects, by the action of forces, and retain the new form and direction after the action of the forces has ceased. For instance, in the case of the ordinary reflexion of a ray, forces act upon it for a short time and through a short space at the surface of the reflecting medium, which, as they do not act symmetrically with reference to the axis of the ray, alter the form of f. The analytical fact that this function is given generally by the integration of a partial differential equation, and therefore not necessarily always of the same form, is quite consistent with such an alteration. But on the principle that the transverse motion in the modified ray, so far as it corresponds to phænomena of light, is still defined by an exact expression, the new form of f will be consistent with equation (18). Consequently, as A and A' in that equation are arbitrary, the new ray will either be completely polarized, or will consist partly of a common ray and partly of a polarized ray. We cannot however suppose any alteration of the function ϕ, unless the forces be such as to destroy the luminous character of the ray; for on the particular form of ϕ which we found in Art. 4, depends the uniformity of propagation, a property which a ray of light is supposed to retain under the modifications here contemplated. It is unnecessary to point out the accordance of the above theoretical inferences with observed facts.

16. A ray may also be modified by forces which act upon it continuously, as is the case on its intromittence into a transparent medium, the modifying forces being the retardations which the vibrations suffer by encountering the atoms of the medium. This kind of modification I have considered in my Paper on the "Transmission of Light through Transparent Media, and on Double Refraction." (*Cambridge Philosophical Transactions*, Vol. VIII. Part IV. p. 524.) I have seen no reason to correct the Theory therein contained, and have only to remark, that the approximate equation in p. 529, which determines f, may be arrived at by reasoning similar to that in Arts. 2 and 3 of this Paper, as follows. We have, as in Art. 7 of the Paper cited,

$$a'^2 \cdot \frac{ds}{dx} + \frac{du}{dt} = 0, \qquad b'^2 \cdot \frac{ds}{dy} + \frac{dv}{dt} = 0, \qquad c'^2 \cdot \frac{ds}{dz} + \frac{dw}{dt} = 0.$$

Hence, $$u = a'^2 \cdot \frac{d \cdot \int s\,dt}{dx}, \qquad v = b'^2 \cdot \frac{d \cdot \int s\,dt}{dy}, \qquad w = c'^2 \cdot \frac{d \cdot \int s\,dt}{dz};$$

no arbitrary function of co-ordinates being added for the reasons heretofore given in Art. 1. Consequently $\frac{u}{a'^2} dx + \frac{v}{b'^2} dy + \frac{w}{c'^2} dz$ must be an exact differential. This will be the case if $s = f\phi'$, f being a function of x and y, and ϕ' a function of z and t. For then,

$$\int s\,dt = \int f\phi' dt = f\phi; \text{ so that, } \frac{u}{a'^2} = \phi \frac{df}{dx}, \quad \frac{v}{b'^2} = \phi \frac{df}{dy},$$

$$\frac{w}{c'^2} = f \frac{d\phi}{dz}, \text{ and } \frac{u}{a'^2} dx + \frac{v}{b'^2} dy + \frac{w}{c'^2} dz = \phi \left(\frac{df}{dx} dx + \frac{df}{dy} dy \right) + f \frac{d\phi}{dz} dz = (d \cdot f\phi).$$

Also $$\frac{ds}{dt} = f \frac{d^2\phi}{dt^2}, \qquad \frac{du}{dx} = a'^2 \phi \frac{d^2 f}{dx^2}, \qquad \frac{dv}{dy} = b'^2 \phi \frac{d^2 f}{dy^2}, \qquad \frac{dw}{dz} = c'^2 f \frac{d^2\phi}{dz^2}.$$

Hence by equation (4),

$$\frac{d^2\phi}{dt^2} = c'^2 \cdot \frac{d^2\phi}{dz^2} + c'^2 \left(\frac{a'^2}{c'^2} \cdot \frac{d^2 f}{f dx^2} + \frac{b'^2}{c'^2} \cdot \frac{d^2 f}{f dy^2} \right) \phi,$$

or, if $$\frac{a'^2}{c'^2} = h \text{ and } \frac{b'^2}{c'^2} = l,$$

$$\frac{d^2\phi}{dt^2} = c'^2 \cdot \frac{d^2\phi}{dz^2} + c'^2 \left(\frac{h}{f} \cdot \frac{d^2 f}{dx^2} + \frac{l}{f} \cdot \frac{d^2 f}{dy^2}\right) \phi,$$

which equation resolves itself into the two following. (See Art. 3),

$$h \cdot \frac{d^2 f}{dx^2} + l \cdot \frac{d^2 f}{dy^2} + 4ef = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24),$$

$$\frac{d^2\phi}{dt^2} - c'^2 \cdot \frac{d^2\phi}{dz^2} + 4ec'^2\phi = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25).$$

The former of these equations is the one it was required to obtain. By reasoning like that by which equation (18) was derived, the analogous integral of equation (24) is,

$$f = A \cos 2\sqrt{\frac{e}{h}}\, x + A' \cos 2\sqrt{\frac{e}{l}}\, y.$$

Hence it appears that a ray of common light cannot be transmitted in the medium so long as h and l are different quantities. Hence also two rays of opposite polarizations cannot in general be transmitted in the same direction with the same velocity, for in that case they would, if they were equal, be equivalent to a ray of common light. But equation (25), integrated in the same manner as equation (11), gives for the velocity of propagation, $c'\sqrt{1 + \frac{e\lambda^2}{\pi^2}}$, which, if $\frac{e\lambda^2}{\pi^2}$ be equal to the constant k, is the same for rays of opposite polarizations. In explanation of this apparent contradiction, it is to be said that if $\frac{e\lambda^2}{\pi^2} = k$, and consequently $e = \frac{\pi^2 k}{\lambda^2}$ the value of f for a ray polarized in the plane of xz is $\cos \frac{2\pi}{\lambda}\sqrt{\frac{k}{h}}\, x$, which is not independent of h, and therefore not independent of the nature of the medium; whereas experience shews that a polarized ray remains the same under all circumstances, and is in no way affected by the medium through which it passes. That the value of f may be that which belongs to a polarized ray, we must have $\lambda\sqrt{h} = \lambda'$ the breadth of the wave; or, $\frac{\lambda'}{\lambda} = \frac{a'}{c'}$. But the velocity of propagation corresponding to λ is $c'\sqrt{1 + k}$. Hence the time of vibration of a given particle, or the *colour* of the light, remaining the same, the velocity of propagation must be altered in the ratio of λ' to λ, and consequently becomes $a'\sqrt{1 + k}$. This result was obtained by somewhat different considerations in Art. 8. of my Paper on *Double Refraction.*

J. CHALLIS.

CAMBRIDGE OBSERVATORY,
March 2, 1848.

XLI.* *Supplement to a Paper "On the Intensity of Light in the neighbourhood of a Caustic."* *By* GEORGE BIDDELL AIRY, ESQ., *Astronomer Royal.*

[Read *May* 8, 1848.]

IN a Paper "On the Intensity of Light in the neighbourhood of a Caustic" communicated to the Cambridge Philosophical Society about ten years ago, and printed in the 6th Volume of their *Transactions*, I shewed that the expression for the intensity of light near a caustic would depend on the infinite integral

$$\int_w \cos\frac{\pi}{2}(w^3 - m.w)^* \left\{ \text{from } w = 0 \text{ to } w = \frac{1}{0} \right\},$$

where m is a quantity proportional to the distance of a point from the geometrical caustic, measured in a direction perpendicular to the caustic, and estimated positive towards the bright side of the caustic: and I gave a detailed account of the method of quadratures by which I had computed the numerical value of this infinite integral for the values of $m - 4\cdot0$, $- 3\cdot8$, &c. as far as $+ 4\cdot0$; and I exhibited in a table the computed values of the integral.

The computation by quadratures was exceedingly laborious, and I did not resort to it without trying other methods of a more refined nature. But in every attempt at expansion of the formula I was met by the integral of a sine or cosine with infinite limits. The reasonings upon which several mathematicians have attempted to establish the value of such an integral appeared to me so little conclusive, that I preferred at once to abandon the expansions which introduced them, and to rely only on the infallible but laborious method of quadratures.

On my stating to Professor De Morgan, after terminating the calculations, the scruples which had led me to reject the expansions, he expressed himself so strongly confident of the correctness of the conclusions upon the point which I had considered doubtful, that I was induced to undertake the numerical computation of the series given by expansion of the formula. I proceeded at once as far as it was possible to go with 7-figure logarithms, when I was interrupted, and the computations were laid aside for some years. I have lately taken them up again, and have completed them as far as they can be carried with 10-figure logarithms. It is the result of this calculation, and the comparison of this result with that formerly obtained from quadratures, that I now beg leave to present to this Society.

Before entering upon the numerical investigations, I will transcribe a letter which Professor De Morgan at my request has written to me, and which he has permitted me to publish. It contains an explanation of his views upon the evidence for the numerical certainty of the results obtained by such integrals as those to which I have alluded.

* I retain this notation in preference to that which is commonly employed, partly because it is familiar to me, and because I have used it in the paper to which I refer, partly because I think that any notation which requires the expression of a differential at the end is for that reason objectionable.

"*University College, London, March* 11, 1848.

"IN reply to your request that I would send you a sketch of the method which I communicated to you some years ago, for finding the numerical value of $\int_0^\infty \cos(w^3 - mw)\,dw$, I send you the following. I am not aware that there is anything about it peculiarly my own, or other than what would suggest itself as a matter of course to any one familiar with the current methods in definite integrals.

"The series which I furnished depend ultimately upon the following formulæ:—

$$\int_0^\infty \epsilon^{-r\cos\theta . w} . \cos(r\sin\theta . w) . w^{n-1}\,dw = \Gamma_n \frac{\cos n\theta}{r^n},$$

$$\int_0^\infty \epsilon^{-r\cos\theta . w} . \sin(r\sin\theta . w) . w^{n-1}\,dw = \Gamma_n \frac{\sin n\theta}{r^n},$$

in which r and θ are independent of w, $r\cos\theta$ is positive, n is positive, and Γ_n stands for $\int_0^\infty \epsilon^{-x} x^{n-1}\,dx$, as usual. Under these conditions the theorems do not or need not rely upon any notion of algebraical as distinguished from numerical equality. Calling either of them $\int \phi w . dw$, common arithmetical calculation would establish any degree of approximation between the convergent series $\phi 0 . a + \phi a . a + \phi 2a . a + \ldots$ and the asserted value of the definite integral, if a were taken small enough. And this for any value of θ, from $\theta = 0$ to $\theta = \frac{\pi}{2} - \beta$, β being of any degree of smallness. But when $\theta = \frac{\pi}{2}$, the numerical character of the equivalence is lost, and the equations assume the same character as $1 - 1 + 1 - 1 + \ldots\ldots = \frac{1}{2}$, and are subject to the same discussion.

"The above equations were first obtained by substituting $a + b\sqrt{-1}$ for a in

$$\int_0^\infty \epsilon^{-aw} w^{n-1}\,dw = \frac{\Gamma_n}{a^n},$$

which is an equivalence of numerical character even after the substitution, if a be positive, and b (be it positive or negative) numerically less than a. For the use of the expansions of $\epsilon^{-bw\sqrt{-1}}$ and $(a + b\sqrt{-1})^{-n}$ in powers of b would produce an equivalence such as

$$A_0 + A_1 k + A_2 k^2 + \ldots\ldots = B_0 + B_1 k + B_2 k^2 + \ldots\ldots$$

where $k = \sqrt{-1}$, $A_n = B_n$ is a numerical equivalence, and ΣA_n is a convergent series. But, when b is numerically greater than a, a convergent series would be rendered divergent *in integration*: and, when this happens, I do not see any way to place the divergent series so obtained upon the same footing as those of ordinary algebra.

"It is not however necessary to depend upon this introduction of divergency. If we call the two integrals C_n and S_n, and differentiate both with respect to θ, we have

$$\frac{dC_n}{d\theta} = r\sin\theta . C_{n+1} - r\cos\theta . S_{n+1},$$

$$\frac{dS_n}{d\theta} = r\sin\theta . S_{n+1} + r\cos\theta . C_{n+1};$$

whence $$r C_{n+1} = \frac{dC_n}{d\theta} \sin\theta + \frac{dS_n}{d\theta} \cos\theta,$$

$$r S_{n+1} = \frac{dS_n}{d\theta} \sin\theta - \frac{dC_n}{d\theta} \cos\theta,$$

from which it easily appears that for what value soever of n the equations first given are true, they remain true when that value is increased by a unit. And that they are true when $n = 1$ is proved by common integration by parts.

"If instead of w we write w^k, k being positive, and then for kn write n, we have

$$\int_0^\infty \epsilon^{-r\cos\theta . w^k} . \cos(r \sin\theta . w^k) . w^{n-1}\, dw = \frac{1}{k} \Gamma_{\left(\frac{n}{k}\right)} . \cos\frac{n\theta}{k} . r^{-\frac{n}{k}},$$

$$\int_0^\infty \epsilon^{-r\cos\theta . w^k} . \sin(r \sin\theta . w^k) . w^{n-1}\, dw = \frac{1}{k} \Gamma_{\left(\frac{n}{k}\right)} . \sin\frac{n\theta}{k} . r^{-\frac{n}{k}}.$$

"If $r = 1$, and we call these integrals C_n and S_n, let us take

$$\cos(\sin\theta . w^3 - mw) = \cos(\sin\theta . w^3) . \left\{1 - \frac{m^2 w^2}{2} + \dots\dots\right\}$$

$$+ \sin(\sin\theta . w^3) . \left\{mw - \frac{m^3 w^3}{2 . 3} + \dots\dots\right\}.$$

Multiplying by $\epsilon^{-\cos\theta . w^3} . dw$, and integrating, we have

$$\int_0^\infty \epsilon^{-\cos\theta . w^3} . \cos(\sin\theta . w^3 - mw)\, dw = C_1 + S_2 . m - C_3 \frac{m^2}{2} - S_4 \frac{m^3}{2 . 3} + \dots\dots$$

"If we now make $\theta = \frac{\pi}{2}$, and observe that in this case C_n vanishes whenever n is an odd multiple of 3, and S_n whenever n is an even multiple, we obtain

$$\int_0^\infty \cos(w^3 - mw)\, dw = C_1 - S_4 \frac{m^3}{2 . 3} - C_7 \frac{m^6}{2 . 3 \dots\dots 6} + S_{10} \frac{m^9}{2 . 3 \dots\dots 9} - \dots\dots$$

$$+ S_2 m + C_5 \frac{m^4}{2 . 3 . 4} - S_8 \frac{m^7}{2 . 3 \dots\dots 7} - C_{11} \frac{m^{10}}{2 . 3 \dots\dots 10} + \dots\dots$$

$$= \frac{1}{3} \Gamma_{\frac{1}{3}} \cos\left(\frac{1}{3} . \frac{\pi}{2}\right) - \frac{1}{3} \Gamma_{\frac{4}{3}} \sin\left(\frac{4}{3} . \frac{\pi}{2}\right) . \frac{m^3}{2 . 3} - \frac{1}{3} \Gamma_{\frac{7}{3}} \cos\left(\frac{7}{3} . \frac{\pi}{2}\right) \frac{m^6}{2 . 3 \dots 6} + \dots\dots$$

$$+ \frac{1}{3} \Gamma_{\frac{2}{3}} \sin\left(\frac{2}{3} . \frac{\pi}{2}\right) m + \frac{1}{3} \Gamma_{\frac{5}{3}} \cos\left(\frac{5}{3} . \frac{\pi}{2}\right) . \frac{m^4}{2 . 3 . 4} - \frac{1}{3} \Gamma_{\frac{8}{3}} \sin\left(\frac{8}{3} . \frac{\pi}{2}\right) \frac{m^7}{2 . 3 \dots 7} - \dots\dots$$

$$= \frac{1}{3} \Gamma_{\frac{1}{3}} . \cos\frac{\pi}{6} . \left\{1 - \frac{1}{3} . \frac{m^3}{2 . 3} + \frac{4}{3} . \frac{1}{3} . \frac{m^6}{2 . 3 \dots 6} - \frac{7}{3} . \frac{4}{3} . \frac{1}{3} . \frac{m^9}{2 . 3 \dots 9} + \dots\right\}$$

$$+ \frac{1}{3} \Gamma_{\frac{2}{3}} . \cos\frac{\pi}{6} . \left\{m - \frac{2}{3} . \frac{m^4}{2 . 3 . 4} + \frac{5}{3} . \frac{2}{3} . \frac{m^7}{2 . 3 \dots 7} - \frac{8}{3} . \frac{5}{3} . \frac{2}{3} . \frac{m^{10}}{2 . 3 \dots 10} + \dots\right\}.$$

"I may observe that the precautions which I have taken, to shew that the *algebraical* cases are limits of arithmetical ones, are not absolutely necessary in this instance. For if we resolve

$\int_0^\infty \cos w^3 dw$ into its successive positive and negative portions, we have $A_0 - A_1 + A_3 - A_5 + \ldots$ in which by A_{2n+1} is meant the portion of the integral (taken positively) which occurs from $w^3 = (2n+1)\frac{\pi}{2}$ to $w^3 = (2n+3)\frac{\pi}{2}$. The greater n is made, the smaller is the interval of this partial integration; and these successive portions diminish, and diminish without limit, so that the series is convergent, and the error always less than the first term rejected. And $\int_0^\infty \sin w^3 dw$ may be treated in the same way.

"A. DE MORGAN."

The following numerical values occurring in the application of Professor De Morgan's final series may be conveniently placed here:—

$$\text{Log } \Gamma_{\frac{1}{3}} = 0{\cdot}4279627493.$$

$$\text{Log } \Gamma_{\frac{2}{3}} = 0{\cdot}1316564916.$$

With these series I have computed the values of $\int_w \cos \frac{\pi}{2}(w^3 - m \,.\, w)$ $\left(m = 0 \text{ to } m = \frac{1}{0}\right)$; for $w = -5\,.\,6$, $-5\,.\,4$, &c. as far as $+5\,.\,6$: and I now exhibit a table of the results, compared with those deduced from quadratures as far as the latter were carried. Each term of the series was computed to 6 decimals, and one figure was struck off in the sum.

Values of m.	Values of Integral by Quadratures.	Values of Integral by Series.	Values of m.	Values of Integral by Quadratures.	Values of Integral by Series.	Values of m.	Values of Integral by Quadratures.	Values of Integral by Series.
− 5·6		+ 0·00011	− 1·8	+ 0·10377	+ 0·10377	+ 2·0	+ 0·56490	+ 0·56490
− 5·4		+ 0·00018	− 1·6	+ 0·13461	+ 0·13462	+ 2·2	+ 0·35366	+ 0·35366
− 5·2		+ 0·00028	− 1·4	+ 0·17254	+ 0·17254	+ 2·4	+ 0·11722	+ 0·11722
− 5·0		+ 0·00041	− 1·2	+ 0·21839	+ 0·21839	+ 2·6	− 0·12815	− 0·12815
− 4·8		+ 0·00063	− 1·0	+ 0·27283	+ 0·27283	+ 2·8	− 0·36237	− 0·36237
− 4·6		+ 0·00093	− 0·8	+ 0·33621	+ 0·33622	+ 3·0	− 0·56322	− 0·56323
− 4·4		+ 0·00138	− 0·6	+ 0·40839	+ 0·40839	+ 3·2	− 0·70874	− 0·70876
− 4·2		+ 0·00204	− 0·4	+ 0·48856	+ 0·48856	+ 3·4	− 0·78018	− 0·78021
− 4·0	+ 0·00298	+ 0·00297	− 0·2	+ 0·57507	+ 0·57507	+ 3·6	− 0·76516	− 0·76516
− 3·8	+ 0·00431	+ 0·00429	0·0	+ 0·66527	+ 0·66527	+ 3·8	− 0·66054	− 0·66044
− 3·6	+ 0·00618	+ 0·00621	+ 0·2	+ 0·75537	+ 0·75537	+ 4·0	− 0·47446	− 0·47419
− 3·4	+ 0·00879	+ 0·00878	+ 0·4	+ 0·84040	+ 0·84040	+ 4·2		− 0·22645
− 3·2	+ 0·01239	+ 0·01239	+ 0·6	+ 0·91431	+ 0·91431	+ 4·4		+ 0·05193
− 3·0	+ 0·01730	+ 0·01730	+ 0·8	+ 0·97012	+ 0·97012	+ 4·6		+ 0·32258
− 2·8	+ 0·02393	+ 0·02393	+ 1·0	+ 1·00041	+ 1·00041	+ 4·8		+ 0·54475
− 2·6	+ 0·03277	+ 0·03277	+ 1·2	+ 0·99786	+ 0·99786	+ 5·0		+ 0·68182
− 2·4	+ 0·04442	+ 0·04442	+ 1·4	+ 0·95606	+ 0·95607	+ 5·2		+ 0·70818
− 2·2	+ 0·05959	+ 0·05959	+ 1·6	+ 0·87048	+ 0·87048	+ 5·4		+ 0·61515
− 2·0	+ 0·07908	+ 0·07908	+ 1·8	+ 0·73939	+ 0·73939	+ 5·6		+ 0·41460

It is impossible to make the calculation for larger values of m, positive or negative, even with 10-figure logarithms, on account of the divergence of the first terms of the series. For the values

± 5.6, the largest term in the series is 169·044826: and it is necessary to proceed as far as the 45^{th} power of m. The result + 0·000114 for $m = -5·6$ is obtained by combining the sum of positive terms + 614·149962 with the sum of negative terms − 614·149848: and the result + 0·414595 for $m = +5·6$ is obtained by combining the sum of positive terms + 614·357203 with the sum of negative terms − 613·942608. For values of m greater than ± 5·6, the calculation must be made in natural numbers.

The agreement of the values of the integral, computed by methods so totally different, is not a little remarkable. On the one hand, it may be received by some persons as a proof of the correctness of that part of the theory of the series which asserts the evanescence of the integral of a cosine when the limits are 0 and $\frac{1}{0}$: on the other hand it may be considered to afford evidence of the great care with which the quadrature computations had been made.

For the last two or three sets of numbers compared, there is a trifling discordance. It will be remarked that in my account of the computation by quadratures I have shewn that difficulties begin to arise in the accurate computation for the values of m approaching to 4·0, (unless the actual summation were carried to higher values of w than I carried it in those computations). That the source of the discordances is in these difficulties and the consequent inaccuracy of the quadratures, and not in the inaccuracy of the series, is evident from the following consideration. The numbers computed by the two methods agree well for the values of m − 4·0, − 3·8, − 3·6: and as the quadratures there present no difficulty, it is reasonable to suppose that both sets of numbers are accurate (within such limits as are possible for the sums of numerous figures). Now the terms of the series combined to form the value of the integral for $m = +4·0$, $+3·8$, $+3·6$, are exactly the same as those by which the value of the integral for $m = -4·0$, $-3·8$, $-3·6$, is formed: the only difference being that they are combined in a different manner, and therefore, from the evident accuracy of the series for $m = -4·0$, $-3·8$, $-3·6$, we are entitled to infer the accuracy of the series for $m = +4·0$, $+3·8$, $+3·6$.

G. B. AIRY.

ROYAL OBSERVATORY, GREENWICH,
March 24, 1848.

XLII. *Some Remarks on the Theory of Matter.* *By* ROBERT L. ELLIS, M. A., *Fellow of Trinity College, Cambridge.*

[Read *May* 22, 1848.]

IN the present state of Science, there are few subjects of greater interest than the enquiry whether all the phenomena of the universe are to be explained by the agency of mechanical force, and if not whether the new principles of causation, such as chemical affinity and vital action, are to be conceived of as wholly independent of mechanical force, or in some way not hitherto explained cognate and connected with it. One reason among many which makes this enquiry interesting is the circumstance that the application of mathematics to natural philosophy has, up to the present time, either been confined to phenomena, which were supposed to be explicable without assuming any other principle of causation than ordinary "push and pull" forces, or as in Fourier's theory of heat and Ohm's theory of the galvanic circuit, have been based on proximate empirical principles.

2. The intention of the remarks which I have the honour to offer to the Society is to suggest reasons for believing that while on the one hand it is impossible not merely from the short-comings of our analysis but from the nature of the case to reduce, as it appears that Laplace wished to do, all the phenomena of the universe to one great dynamical problem, we cannot recognise the existence of any principle of causation wholly disconnected with ordinary mechanical force, or of which the nature could be explained without a reference to local motion: in other words, that the idea of "qualitative action" in the sense which the phrase naturally suggests must be rejected. It will be seen from the explanations I am about to attempt that the objection which Leibnitz has opposed to the atomic, and in effect to any mechanical philosophy, namely, that on such principles a finite intelligence might be conceived to exist by which all the phenomena of the universe would be fully comprehended, does not (whatever may be thought of its validity) appear to apply to the views which I have been led to entertain. For these views essentially depend on the conception of what may be called a hierarchy of causes, to which we have no reason for assigning any finite limit. Of this series of principles of causation, ordinary mechanical force is the first term.

3. With respect to the first point, namely, the impossibility of explaining all phenomena mechanically, it may be remarked, that we are met, in the attempt to discuss it, by the difficulty which always attends the establishment of a negative proposition. It is clear that as in the present state of our knowledge we are far from being able to enumerate and classify the phenomena which are or which might be produced by the combined agency of conceivable mechanical forces, we are not in a position to decide *à priori* that any given phenomenon might not be thus produced. *Non constat*, but that the impossibility we find in the attempt to explain the causes of its existence may have no higher origin than the imperfect command which we have as yet obtained of the principles of mechanical causation. We meet, it may be said, with a multitude of ordinary dynamical problems which have as yet received no adequate solution—why then should we have recourse to new kinds of causes, while we have not as yet exhausted the resources, if the expression may thus be used, of those which we already recognise? To this enquiry no conclusive answer can be given, but the following considerations will I think naturally suggest themselves.

4. In the first place, no even moderately successful attempt has, I think, yet been made to explain any chemical phenomenon on mechanical principles. It is quite true that we are unable, to take a particular instance, fully to comprehend the mechanical constitution of the luminiferous ether; the determinations which have as yet been attempted of the law of attraction between its molecules cannot, I apprehend, be accepted as any thing more than hypothetical or provisional results, and there are other points involved in yet greater obscurity. Nevertheless the undulatory theory of light has, as we all know, given consistent and satisfactory explanations of a great variety of phenomena. Thus it appears, and the same remark might be educed from other though similar considerations, that we are by no means absolutely estopped by the imperfection of our mechanical philosophy, from explaining phenomena really due to mechanical forces, even when these phenomena are connected with subjects not as yet fully comprehended: why then cannot some progress be made in the mechanical explanation of chemical phenomena, or of those, to mention no other class, which we are in the habit of referring to vital action? In these cases, we see or seem to see that the action of mechanical laws is modified or suspended; and though it is not demonstrably impossible that this is not really the case, and that no other causes are at work beside the "push and pull" forces of ordinary mechanics, yet we are at least much tempted to believe, that the difficulties we meet with do not arise from what may be called the disguised action of mechanical forces but from the presence of an agency of a distinct nature. And to this view we find that most of those incline who have made themselves familiar with the science of chemistry or with that which has been called biology; and further that, (with reference to the latter science) the insufficiency not only of a mechanical but even of a chemical physiology has been generally admitted.

Secondly, it is to be observed that even if it be considered doubtful whether a mechanical philosophy be not after all sufficient for the explanation of all phenomena, it is at least certain that it has not been proved to be so: and that by rejecting other conceivable modes of action than those which are recognised by it, we unnecessarily and arbitrarily limit the problem which the universe presents to us; falling thereby into an error similar to that of the atomists, who starting from the assumption that the ἀρχάι, or first principles of all things, are atoms and a vacuum proceeded to construct an imaginary world, in accordance with this arbitrary hypothesis. At the same time it must be granted that a purely mechanical* system such as that of Boscovich is more self consistent and contains, so to speak, less that is discontinuous, than any which should recognise other principles, for instance chemical affinity, distinct from force without enquiring into the relation which subsists between them.

5. It may however be asserted that this enquiry is altogether superfluous—that the power of exerting attractive or repulsive force is one property of matter that chemical affinity, (and so in other cases,) is another—that the two are not merely distinct, but absolutely independent and heterogeneous. But to this view the arguments which seem to have led to the adoption of a purely mechanical system, appear to prevent our assenting. I shall therefore attempt to state what I conceive these arguments to have been.

6. It is a fundamental principle of the secondary mechanical sciences, for instance of the theory of light, that the secondary qualities of bodies are to be explained by means of the primary. Every substance, to use for a moment the language of Leibnitz, is essentially active; in other words it is to be conceived of as the formal cause of the sensible qualities which are referred to it. If we ask why gold is yellow and silver white, the answer at once presents itself that the difference

* The word *mechanical* is of course not used in antithesis to *dynamical*, in the sense in which the latter is commonly employed by the philosophical writers of Germany. The antithesis in question is foreign to the scope of the present essay, and I have accordingly elsewhere used the word dynamical in its ordinary acceptation.

of colour corresponds and is due to a difference between the essential constitution of the two substances. Now the essential constitution here spoken of, and consequently the differences which individuate it in different cases, may conceivably be something altogether incognisable to the human intellect. The notion that it is so was expressed scholastically by saying that substantial forms are not cognoscible. But if, setting aside this opinion, we affirm that the essential constitution of each substance is a matter of which the mind can take cognisance, we are led at once to the distinction between primary and secondary qualities. The first are ascribed to each substance as its essential attributes, in virtue of which it is that which it is—the second result from the primary*, (by which as we have said the essential or formal constitution of the substance in question is determined,) and have reference to the mind by which they are perceived, while the primary are ascribed to it independently of any reference to a percipient mind: and a distinction, analogous or identical with that between primary and secondary qualities, has accordingly been expressed by the antithesis between that which is *a parte hominis* and that which is *a parte universi*. That the distinction between primary and secondary qualities is necessary *on the hypothesis on which we are proceeding*, appears at once from the consideration that if we affirm that all the qualities of bodies of which we can form any conception are equally subjective and phenomenal, nothing will remain of which the mind can take cognisance, and by means of which our conception of the nature of any one substance can be discriminated from that of any other†. Let it be granted therefore that the distinction of primary and secondary qualities is a necessary element of physical science. It follows from this that the secondary qualities in a manner disappear when we look at the universe from the scientific point of view. Instead of colours we have vibrations of the luminiferous ether—instead of sounds vibrations of the ambient air, and so on. Now from hence it follows that all the phenomena which we see produced, of whatever nature they may be, are all in reality dependent on the primary qualities of matter. Furthermore, these primary qualities themselves all involve the idea of motion or of a tendency to motion. A body changes its form in virtue of the local motion (absolute or relative) of some of its parts; and when I press a stone between my hands, I find that I can produce no sensible change of form, while contrariwise the stone reacts against my hands, tending to make them move in opposite directions. I then say that the stone is hard as a mode of expressing this, viz. that when an attempt is made to produce relative local motion of its parts, it resists it in virtue of its reactive tendency to produce motion in that which acts upon it. Again, a body whose parts are readily susceptible of relative local motion is said to be soft or fluid, and when a sensible change of form is accompanied by a tendency to such motion as shall restore the original form, it is said to be elastic, and so on. We thus arrive at a point of view at which all secondary qualities having disappeared, and all primary ones‡ having been resolved into motion and tendency to motion, the sciences which relate to phenomena appear to be resolved into the general doctrine of motion. But if this be true the universe can it is said present to us nothing but one great dynamical problem. Motion, and force the cause of motion, belong essentially to the domain of mechanics: and if chemical affinity be a cause of local motion, that is, if in virtue of its action || a particle of matter finds itself at a given time in a position different from that which it would else have occupied, chemical affinity is not really distinct from mechanical force (which looked at from the dynamical point of view includes everything which is a cause of motion); whereas if it be not a cause of motion the enquiry at once presents itself of what is it? In illustration of this view we may refer to any chemical experiment. If an acid is dropped into a glass containing any vegetable blue, the colour is changed to red. But to say this is to say that the

* Or that which in its formation it was to be, τὸ τί ἦν εἶναι.

† The doctrine of the cognoscibility of substantial forms, which is intimately connected with this distinction, is as Leibnitz in effect remarks, as it were the common character of those who with more or less success attempted in the seventeenth century, the restoration of science. Vid. Leibnitz, *Epist. ad Thomas*, I.

‡ That is, all that are commonly enumerated as primary qualities.

|| As, for instance, in the phenomenon of crystallization.

liquid when the acid is introduced into it begins to act on the luminiferous vibrations which exist near it in a different manner from that in which it had previously acted. The whole change, whether we call it a chemical phenomenon or not, consists in the introduction of new forms of motion in virtue of the action of mechanical force.

7. From considerations of this kind it appears to follow that a complete explanation of all phenomena would introduce no principles beyond those with which the science of mechanics is conversant. And in truth if the conclusion drawn had been that all phenomena might, if our knowledge of nature were sufficiently extensive, be reduced to cinematical considerations (using the word cinematics in the large sense in which it is equivalent to the doctrine of motion), I do not see how on our fundamental hypothesis we could refuse to assent to it. But the conclusion drawn by the maintainers of the all-sufficiency of a mechanical philosophy is something different from this—and as I conceive the error they appear to have committed is to be sought for in this discrepancy. But before entering into the discussion of this point, I will make a few remarks on certain points in the history of what may be called the theory of matter.

8. If we suppose the maxim that secondary qualities are to be explained by means of the primary to have been accepted (either in that or in some equivalent form) or if not formally accepted, at least unconsciously assumed, at a time when the idea of mechanical force was as yet very imperfectly apprehended—the natural result of this state of things is the formation of an atomic theory. For in order to individuate the constitution of any given body, we could only have had recourse to the configuration or motion of its parts. Gold, to return to our previous example, was said to be yellow in virtue of such and such a configuration of its parts; since except configuration there appeared to be no disposable circumstance*, if I may so speak, whereby gold was in its intimate constitution to be distinguished from silver or from any thing else. But this configuration must be independent of the body's visible and external form, since changes of the latter do not affect the body's sensible qualities. Hence it must be a configuration of small parts, and we are thus at once led to the primitive form of the atomic theory. In this the atoms possess the primary qualities of larger bodies—they are of various forms and act if the expression may be used by their forms, not by being centres of attractive forces. Such was the atomistic system of the school of Democritus†—a system which as we know found no little favour among the scientific reformers of the seventeenth century‡. As an instance of the influence it exerted, I need only mention the great work of Cudworth, in which it is presented apart from the atheistical doctrines with which it had often been connected. Cudworth goes so far as to affirm that Democritus and his followers had corrupted and degraded the atomistic system which was originally altogether free from any irreligious tendency and which he sought to restore to its first estate.

But as the imperfections of the atomic system became manifest, and on the other hand mechanical conceptions came to be more developed a new form of this system arose. The atoms, retaining their forms and those which are commonly called their primary qualities, were now supposed to act as centres of attractive force, in other words, each atom was to the rest a cause of motion. But as the ordinary "primary qualities" of bodies may as we have seen be analysed into conceptions which involve nothing beside motion and force, this new form of the doctrine may clearly be considered merely as a state of transition to that which is now known by the title

* Specific differences of motion seem for more than one reason not to have been used in giving an account of the differences of bodies.

† See for a more favourable, and I think, a juster view of the philosophy of Democritus than that which we commonly meet with in the writings of modern historians of philosophy, Zeller's *Philosophie Der Griechen*, I. § 10.

‡ The physical theories of Des Cartes, though not properly atomistic, since he proceeded on the hypothesis of a plenum, yet in many respects are akin to those of which we are speaking.

of Boscovich's theory*. To Boscovich appears to belong the credit of having perceived that if the atoms were conceived of simply as unextended centres of force the primary qualities of bodies might sufficiently be accounted for without supposing them to result from the primary qualities of their constituent atoms—a mode of explanation of which, though there has been something like a return to it in some recent speculations, it may be observed that it explains nothing. Boscovich's theory seems to have been so completely in accordance with the direction in which mathematical physics have of late been moving, that it was adopted as it were unconsciously—almost all modern investigations on subjects connected with molecular action are in effect based on his views, though his name is, comparatively speaking, but seldom mentioned. And this theory, (whether or not the hypothesis of the existence of *discrete* centres of action be or be not essential to it, a question connected with that which in former times caused so much perplexity, namely, the nature of continuity, and which it is not necessary to my present purpose to consider), is in truth the highest developement which the mathematical theory of matter has as yet received — it is that on which the pretensions of mathematical physicists to vindicate for their own methods the right, so to speak, if not the power, to explain all phenomena mainly depend. Adopting for the sake of definite conception the received form of this theory, that namely in which the centres of force are discrete and at insensible distances from each other, I now shall attempt to show what ulterior developements it admits of, and how by means of these the error noticed at the close of the last Section, namely, the confounding the admission that all phenomena are to be explained *cinematically* with the assertion that they can all be explained *mechanically* may be met, and, as it seems to me, sufficiently refuted.

9. I begin by observing that though we speak and shall continue to do so of the *action* of matter on matter, yet that no part of the views I am about to state depends on the hypothesis we adopt touching the nature of causation. They would remain unchanged whether we accept a theory of pre-established harmony, or one of physical influence, or whether we abstain from all theories on the subject. This being understood, we may, I think, lay down the axiom that whatever property we ascribe to matter, we may also ascribe to it, the property of producing in other portions of matter the former property. Of this axiom the present state of Boscovich's theory affords a familiar illustration. Every portion of matter is locally moveable, therefore we may ascribe to any portion of matter the power of producing motion in any other, hereby giving rise to the whole doctrine of attractive and repulsive forces. At this point we have hitherto stopped, but for no satisfactory reason. We may proceed farther, and we are therefore bound, in constructing the most general possible hypothesis, to do so: we may ascribe to each portion of matter the power of engendering in any other that which we call force, in other words the power of producing the power of actuating the potential mobility of matter. It is not *a priori* at all more easy to conceive that A should have the power of setting B in motion, or of changing the velocity it already has, than that C should have the power of enabling A to act on B, or of changing the mode of action which A already possesses. And let it be observed, that the new power thus ascribed to C is as distinct from force, as force is from velocity. The two are related as cause and effect, but formally are wholly independent. Now unless this hypothetically possible mode of action can be shown to have no existence *in rerum naturâ*, it is clear that the inference from the conclusion that no phenomenon can be imagined not resoluble *en dernière analyse*, into local motion to the assertion that mechanical force is the only agency to be recognised in the

* It is, I believe, known that Boscovich's fundamental idea was deduced by a not unnatural filiation from the monadism of Leibnitz. Yet the scope and limits which he proposed to himself differ essentially from those of the German philosopher, inasmuch as they are essentially physical. Moreover, the latter would have objected on the principle of sufficient reason to the want of any thing to individuate the atoms of Boscovich; and, at least in the latter years of his life, to the "Ferne Wirkung," on which the whole theory depends.

material universe is altogether illusory. For matter may act on matter in a manner wholly distinct from force, and yet this kind of action shall, ultimately and indirectly, manifest itself in modifications of local motion. Furthermore, if for an instant we call this kind of action $(\text{force})^2$, we shall at once be led to recognise a hypothetically possible mode of action of matter on matter which in accordance with analogy we shall call $(\text{force})^3$, which consists in the power of modifying $(\text{force})^2$. And so on, *sine limite*.

10. If we compare the language in which the relation between mechanical force and chemical affinity is commonly spoken of, we shall I think perceive its analogy with that which I have used in describing the mode of action which we have called $(\text{force})^2$. Its chemical affinity is spoken of as something which suspends or modifies the action of force, as something distinct from it, but which yet interferes with its effects. Or again, if in physiological writings we observe the manner in which vital action* is described we recognise, or seem at least to do so, the possibility of referring its effects to that mode of action which we have called $(\text{force})^3$. I do not however wish to lay much stress on these similarities, because I think the kind of reasoning we have pursued shows more satisfactorily than they can do, that if chemical affinity and vital action are not resoluble into force, they must be referred to some of the modes of action we have pointed out.

It would be useless to remark on the many points of speculation which here present themselves. The expansion of bodies by heat may however be particularly mentioned, because notwithstanding what has been learnt with relation to the theory of heat, nothing like a mechanical explanation of this phenomenon has as yet been discovered. It seems to depend not on the introduction of new mechanical forces, but on a modification of those which already exist; such modification, in cases of ordinary conduction, being propagated from one part of the body to that which is next it.—It is easy to conceive that by an alteration in the function which expresses the mutual action of the molecules, the body may pass into a new state of equilibrium in which the average distance between adjacent molecules may be increased or diminished. If such an explanation could be established, we should have a case of the action of $(\text{force})^2$.

11. In conclusion, it may be well to remark that mathematical analysis is conceivably as applicable to these new modes of action of matter on matter as to ordinary questions in dynamics. It is, however, easily seen that as in these we deal chiefly with differential equations of the second order, and in merely cinematical questions with equations of the first only, so contrariwise when we introduce higher powers of force (so to call them) we shall correspondingly have to do with equations of higher orders. I venture to predict with a degree of confidence, which doubtless I shall not communicate to many, that if we ever succeed in establishing a mathematical theory of chemistry, it will be as much conversant with equations of the third or of a higher order, as physical astronomy is with equations of the second.

R. L. ELLIS.

May 1, 1848.

* I am, of course, not to be understood as suggesting a materialistic explanation of phenomena of thought or volition.

XLIII. *Methods of Integrating Partial Differential Equations.* *By* AUGUSTUS DE MORGAN, *of Trinity College, Cambridge, Secretary of the Royal Astronomical Society, and Professor of Mathematics in University College, London.*

[Read *June* 5, 1848.]

THE following methods for the treatment of certain cases of partial differential equations of two independent variables will be interesting, both as having something new, and as combining and bringing together some isolated instances given by different writers.

FIRST METHOD.

Let the differential equation be

$$\phi(x, y, p, q) = 0,$$

p and q meaning $\frac{dz}{dx}$ and $\frac{dz}{dy}$. Contrive that this equation, $\phi = 0$, shall be the result of elimination between two others, $A = 0$, $B = 0$, or, at full length,

$$A(x, y, p, q, v) = 0, \qquad B(x, y, p, q, v) = 0.$$

Accordingly, v is an implicit function of x and y. Let r, s, and t, as usual, be the second differential coefficients of z, and form the four additional equations

$$A_x + A_p r + A_q s + A_v \frac{dv}{dx} = 0, \qquad B_x + B_p r + B_q s + B_v \frac{dv}{dx} = 0,$$

$$A_y + A_p s + A_q t + A_v \frac{dv}{dy} = 0, \qquad B_y + B_p s + B_q t + B_v \frac{dv}{dy} = 0.$$

From the six equations* eliminate p, q, r, s, t; there will result an equation between $x, y, v, \frac{dv}{dx}, \frac{dv}{dy}$, which will often be more tractable than $\phi = 0$. When, after integration, v is found in terms of x and y, p and q can be found in the same terms from $A = 0$, $B = 0$, and then z from $dz = p\,dx + q\,dy$.

This method was derived from the suggestions afforded by a previous treatment of the equation

$$Apq + Bp + Cq + D = 0,$$

A, &c. being functions of x and y; which occurs in the process of developing any surface which admits it upon a plane. Reduce the preceding to the form

$$(p + P)(q + Q) = R.$$

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* With regard to the notation, I must state that by such a symbol as A_a I mean the partial differential coefficient of A with respect to a, as obtained from an equation in which A is explicitly given in the form $A = \phi(a, \ldots\ldots)$. I have found this notation, however useful it may be as an abbreviation, almost as useful in the way of distinction. It points out the ultimate and elementary process, on one or more of which the implicit differential coefficients depend.

$$\text{Let} \quad p + P = Mv, \qquad q + Q = \frac{N}{v}.$$

MN being any convenient resolution of R into two factors. We have then

$$s + P_y = M\frac{dv}{dy} + M_y v,$$

$$s + Q_x = -\frac{N}{v^2}\frac{dv}{dx} + \frac{N_x}{v}.$$

$$M\frac{dv}{dy} + \frac{N}{v^2}\frac{dv}{dx} = P_y - Q_x + \frac{N_x}{v} - M_y v,$$

which depends on ordinary differential equations. But it must be observed that the integration of this subsidiary equation frequently leads to a form from which v cannot be directly exhibited as a function of x and y. Where this happens, we must obtain a particular form which contains one arbitary constant; another will be introduced in the integration by which z is obtained; and Lagrange's process may then be applied to the primary form so obtained.

For example, let $pq = px + qy$, or, $(p - y)(q - x) = xy$.

$$\text{Let} \quad p - y = xv, \quad s - 1 = x\frac{dv}{dy},$$

$$q - x = \frac{y}{v}, \quad s - 1 = -\frac{y}{v^2}\frac{dv}{dx}, \qquad x\frac{dv}{dy} + \frac{y}{v^2}\frac{dv}{dx} = 0,$$

or $x^2v^2 - y^2 = f(v)$. Let $fv = av^2$, and we have

$$v = \frac{y}{\sqrt{(x^2 - a)}}, \quad p = y + \frac{xy}{\sqrt{x^2 - a}}, \qquad q = x + \sqrt{x^2 - a},$$

$$z = xy + y\sqrt{x^2 - a} + b.$$

Let $b = \phi a$: then the general solution of $pq = px + qy$ may be obtained by eliminating a from

$$z = xy + y\sqrt{(x^2 - a)} + \phi a,$$

$$0 = -\frac{y}{2\sqrt{(x^2 - a)}} + \phi' a.$$

But if we take $p - y = v$, we find

$$v^2 - y^2 = \phi\left(\frac{v}{x}\right),$$

and we ultimately obtain the same form.

We may also obtain as the primary solution

$$z = \sqrt{.(x^2 - a)(y^2 - \phi a)} + xy + \psi a.$$

If we apply the whole process to $pq = \phi x . \psi y$, we find for a primary solution

$$z = 2\sqrt{\{\phi_1 x(\psi_1 y - a)\}} + fa,$$

$$\text{where} \qquad \phi_1 x = \int \phi x\, dx, \quad \psi_1 y = \int \psi y\, dy.$$

Next, take the instance $(p + q)(px + qy) = 1$.

Let $$px + qy = v, \qquad p + q = \frac{1}{v}.$$

Form the four other equations and eliminate, which gives

$$(v^2 + x)\frac{dv}{dx} + (v^2 + y)\frac{dv}{dy} = v,$$

or $$\frac{v^2 - y}{v} = f\left(\frac{v^2 - x}{v}\right).$$

Let $$fv = av\,; \text{ then } v = \sqrt{\frac{y - ax}{1 - a}},$$

$$p = -\frac{a}{\sqrt{(1 - a)}}\cdot\frac{1}{\sqrt{(y - ax)}}, \qquad q = \frac{1}{\sqrt{(1 - a)}}\ \frac{1}{\sqrt{(y - ax)}},$$

$$z = 2\sqrt{\frac{y - ax}{1 - a}} + b.$$

For $-a\,(1 - a)^{-1}$ write a: then we deduce the general solution by eliminating a between

$$z = 2\sqrt{\{a(x - y) + y\}} + \phi a,$$

$$0 = \frac{x - y}{\sqrt{\{a\,(x - y) + y\}}} + \phi' a.$$

Let $$Ap^2 + Bpq + Cq^2 = D$$

which can be resolved into $(p + Kq)(p + Lq) = MN$.

Let $$p + Kq = Mv, \qquad p = \left(\frac{KN}{v} - LMv\right)(K - L)^{-1},$$

$$p + Lq = \frac{N}{v}, \qquad q = \left(Mv - \frac{N}{v}\right)(K - L)^{-1}.$$

The two values of s thence derived, equated to each other, give the equation for determining v. Accordingly, since A &c. may be any functions of x and y, the general equation of the second degree is reducible to ordinary differential equations, provided that z do not appear in it.

In these examples, I have chosen, merely for simplicity, cases in which p and q are explicitly found, and the values of s equated. This amounts to exhibiting $\phi = 0$ under the form of $A = 0$ and $B = 0$, and determining v so that $pdx + qdy$ may be a complete differential. And in like manner as every particular value of v leads to a particular value of z, so does each value of z lead to one of v. And in this way a particular solution of one partial differential equation may lead to a particular solution for another and a more difficult one. Thus, if $\phi = 0$ be derived from $A = 0$, $B = 0$, leading to the new partial equation $U = 0$; and if it also be derived from $A' = 0$, $B' = 0$, leading to $U' = 0$: by means of a solution of $U = 0$, leading to a solution of $\phi = 0$, one solution of $U' = 0$ may be found.

Take the instance $$\sqrt{p} + \sqrt{q} = 2x,$$

or $$p = (x - v)^2, \qquad q = (x + v)^2,$$

$$(x + v)\frac{dv}{dx} + (x - v)\frac{dv}{dy} = -(x + v),$$

$$y + \frac{vx}{v+x} = f(v+x), \text{ say } = a,$$

$$v = \frac{(a-y)\,x}{x+y-a},$$

$$dz = \left(\frac{x^2 + 2xy - 2ax}{x+y-a}\right)^2 dx + \frac{x^4}{(x+y-a)^2}\,dy,$$

$$z = \frac{4}{3}x^3 - \frac{x^4}{x+y-a} + b.$$

Make $b = \phi a$, and proceed as before.

Another form of solution is derived from $v + x = a$, or

$$dz = (2x-a)^2\,dx + a^2 dy,$$

$$z = \frac{1}{6}(2x-a)^3 + a^2 y + \phi a.$$

Resolve the same equation into

$$p = \left(\frac{2x}{1+v}\right)^2, \qquad q = \left(\frac{2xv}{1+v}\right)^2,$$

giving
$$xv\frac{dv}{dx} + x\frac{dv}{dy} = -v^2(1+v).$$

From the solution of the original take $q = \left(\dfrac{x^2}{x+y-a}\right)^2$ and make $\left(\dfrac{x^2}{z+y-a}\right)^2 = \left(\dfrac{2xv}{1+v}\right)^2$,

$$\text{or } v = \frac{x}{x+y-2a}.$$

This ought to be a solution of the differential equation last written, and it will be found to be so on trial.

SECOND METHOD.

Let there be given the equation

$$\phi(x, y, z, p, q, r, s, t) = 0.$$

Interchange p and x, q and y, z and $px + qy - z$, r and $\dfrac{t}{rt-s^2}$, s and $\dfrac{-s}{rt-s^2}$, t and $\dfrac{r}{rt-s^3}$,

giving
$$\phi\left(p, q, px+qy-z, x, y, \frac{t}{rt-s^2}, \frac{-s}{rt-s^2}, \frac{r}{rt-s^2}\right) = 0.$$

If either of these equations can be integrated, say by

$$Z = \psi(X, Y),$$

then the solution of the other is obtained by eliminating X and Y from

$$x = \frac{dZ}{dX}, \quad y = \frac{dZ}{dY},$$

$$z = xX + yY - Z.$$

The root of this theorem lies in the following, that the interchanges above mentioned do not alter the truth of the equations

$$dz = p\,dx + q\,dy, \qquad dp = r\,dx + s\,dy, \qquad dq = s\,dx + t\,dy,$$

so that we have

$$d(px + qy - z) = x\,dp + y\,dq,$$

$$dx = \frac{t}{rt - s^2}\,dp - \frac{s}{rt - s^2}\,dq,$$

$$dy = -\frac{s}{rt - s^2}\,dp + \frac{r}{rt - s^2}\,dq.$$

Let x, y, z be considered as functions of p and q, derived from the equations

$$f(x, y, z) = 0, \qquad f_x + f_z\,p = 0, \qquad f_y + f_z\,q = 0;$$

but remark that there is a case of exception, namely, when the second and third equations give simultaneous elimination of x, y, and z, or lead to $\psi(p, q) = 0$. Since

$$z = px + qy - \int(x\,dp + y\,dq),$$

$x\,dp + y\,dq$ must be a complete differential. Let it be dv, then we have, v being a function of p and q,

$$x = \frac{dv}{dp}, \qquad y = \frac{dv}{dq}, \qquad z = px + qy - v.$$

Let the second differential coefficients of v be ρ, σ, τ, we have then

$$dx = \rho\,dp + \sigma\,dq, \qquad dp = \frac{\tau}{\rho\tau - \sigma^2}\,dx - \frac{\sigma}{\rho\tau - \sigma^2}\,dy,$$

$$dy = \sigma\,dp + \tau\,dq, \qquad dq = -\frac{\sigma}{\rho\tau - \sigma^2}\,dx + \frac{\rho}{\rho\tau - \sigma^2}\,dy,$$

whence $$r = \frac{\tau}{\rho\tau - \sigma^2}, \qquad s = \frac{-\sigma}{\rho\tau - \sigma^2}, \qquad t = \frac{\rho}{\rho\tau - \sigma^2}.$$

Hence, in order to make p and q the independent variables instead of x and y, we must assume a function v, of p and q, such that

$$x = \frac{dv}{dp}, \qquad y = \frac{dv}{dq}, \qquad z = p\frac{dv}{dp} + q\frac{dv}{dq} - v,$$

and then we must find v by integrating

$$\phi\left(\frac{dv}{dp}, \frac{dv}{dq}, p\frac{dv}{dp} + q\frac{dv}{dq} - v, p, q, \frac{\tau}{\rho\tau - \sigma^2}, \frac{-\sigma}{\rho\tau - \sigma^2}, \frac{\rho}{\rho\tau - \sigma^3}\right) = 0.$$

The manner in which I first stated the theorem changes the meaning of the letters x and y without changing the letters themselves.

Of this method, I find one instance. Legendre (see *Lacroix*, Vol. II. p. 622) has employed it as a casual artifice for the reduction of

$$f_1(p, q) \,.\, r + f_2(p, q) \,.\, s + f_3(p, q) \,.\, t = 0,$$

to $$f_1(x, y) \,.\, r - f_2(x, y) \,.\, s + f_3(x, y) \,.\, t = 0.$$

But I am not able to find that it has ever been applied to any equation *of the first order*. *Lacroix* (Vol. II. p. 558) gives something as near to the whole method as can well be imagined. He sees everything except the completely interchangeable character of z and $px + qy - z$; that he did not see this last may be suspected from his making the restriction that z must only enter in $px + qy - z$.

It is to be noted that, so far as equations of the *first order* are concerned, the solution takes exactly the same form, even though we can only integrate the transformed equation by reducing it to

$$\psi(X, Y, Z, A) = 0, \qquad \frac{d\psi}{dA} = 0,$$

for the forms of $\frac{dZ}{dX}$ and $\frac{dZ}{dY}$ are unaltered. There is now one equation more, $\frac{d\psi}{dA} = 0$, and one more quantity, A, to eliminate.

Let the first instance be

$$Ax + By + Cz + D = 0,$$

where each of the four, A, &c. is any function whatever of p, q, and $px + qy - z$. The transformed equation is obviously of the form $Pp + Qq + R = 0$, where P, Q, R are all functions of x, y, z.

Lagrange has given a laborious method for the integration of $z = pq$, and Lacroix (Vol. II. p. 565) does not refer to p. 558, I suppose for the reason just given. The transformed equation is $px + qy - z = xy$, of which the integral is

$$z - xy = xf\left(\frac{y}{x}\right).$$

We may therefore find the general solution of $z = pq$ from

$$Z = XY + Xf\left(\frac{Y}{X}\right), \qquad x = Y - \frac{Y}{X}f'\left(\frac{Y}{X}\right) + f\left(\frac{Y}{X}\right),$$

$$y = X + f'\left(\frac{Y}{X}\right), \qquad z = XY.$$

Generally, however, the most convenient method is to select an appropriate primary solution, and then to use Lagrange's process. This may be done, if we please, from the common differential equations which integrate the transformed partial ones. These are, in the present case,

$$z = xy + bx, \qquad y = ax.$$

The retransformed equations are

$$px + qy - z = pq + bp, \qquad q = ap.$$

With these, and $z = pq$, eliminate p and q, which gives

$z = \dfrac{(x + ay - b)^2}{4a}$, so that we have the general solution by eliminating a from

$$z = \frac{(x + ay + \phi a)^2}{4a}, \text{ and } \frac{dz}{da} = 0.$$

But we may often, most often I think, procure the primary solution in an easier manner from the result of the complete method. Let $fz = az + b$, and we then have

$$x = Y + b, \qquad y = X + a, \qquad z = XY,$$

or $z = (x-b)(y-a)$, a very obvious solution. Hence we must eliminate a from

$$z = (x - \phi a)(y - a), \qquad 0 = (x - \phi a) + (y - a)\phi' a.$$

In my *Differential Calculus*, p. 717, I gave a method for the general case $z = \phi(p, q)$; but the following, derived from the present method, is preferable.

$$\text{Let} \qquad f(X, a) = \int \phi(X, aX) \,.\, X^{-2} dX.$$

$$\text{Then} \qquad Z = Xf\left(X, \frac{Y}{X}\right) + XF\left(\frac{Y}{X}\right),$$

with which proceed as before.

Of the instances which I have tried by the other method,

$pq = px + qy$ gives $px + qy = xy$, from which

$$Z = \tfrac{1}{2} XY + f\left(\frac{Y}{X}\right), \qquad x = \tfrac{1}{2} Y - \frac{Y}{X^2} f'\left(\frac{Y}{X}\right),$$

$$y = \tfrac{1}{2} X + \frac{1}{X} f'\left(\frac{Y}{X}\right), \qquad z = \tfrac{1}{2} XY - f\left(\frac{Y}{X}\right).$$

In this case we may conveniently take the retransformed equations

$$q = ap, \qquad px + qy - z = \tfrac{1}{2} pq + b, \quad \text{which with } pq = px + qy,$$

give $\qquad 2a(z + b) = (x + ay)^2$, say $2az + b = (x + ay)^2$.

Again, $\quad (p + q)(px + qy) = 1$ transforms into

$$(x + y)(px + qy) = 1, \quad \text{or} \quad z + \frac{1}{x + y} = f\left(\frac{y}{x}\right).$$

Treat this by the method, and assume $fz = \dfrac{a}{1 + z} + b$, which will show that the general solution can be obtained by eliminating a between

$$z = \frac{x - y}{a} + \frac{ay}{x - y} + \psi a,$$

$$0 = -\frac{x - y}{a^2} + \frac{y}{x - y} + \psi' a.$$

The equation $\sqrt{p} + \sqrt{q} = 2x$ transforms into

$$\sqrt{x} + \sqrt{y} = 2p, \quad \text{or} \quad z = \tfrac{1}{3} x^{\frac{3}{2}} + \tfrac{1}{2} x y^{\frac{1}{2}} + fy,$$

$$x = \tfrac{1}{2} X^{\frac{1}{2}} + \tfrac{1}{2} Y^{\frac{1}{2}},$$

$$y = \tfrac{1}{4} X Y^{-\frac{1}{2}} + f'Y,$$

$$z = \tfrac{1}{6} X^{\frac{3}{2}} - \tfrac{1}{4} X Y^{\frac{1}{2}} + Yf'Y - fY.$$

This is not an easy form. But if we take the retransformed equations

$$q = a^2, \qquad px + qy - z = \tfrac{1}{3} p^{\frac{3}{2}} + \tfrac{1}{2} p q^{\frac{1}{2}} + b,$$

and join $\qquad p^{\frac{1}{2}} + q^{\frac{1}{2}} = 2x$ with them, we find

$z = \frac{1}{6}(2x - a)^3 + a^2 y + b$, a primary solution, being the one already obtained.

The equation $$ar + bs + ct = (rt - s^2) . f(p, q)$$
transforms into $at - bs + cr = f(x, y)$, which, a, b, c, being constants, is integrable.

Since $rt - s^2$ transforms into $(rt - s^2)^{-1}$, the equations $rt - s^2 = f(p, q)$, and $rt - s^2 = \{f(x, y)\}^{-1}$ depend each upon the other.

The failure of this method in the case of developable surfaces may be illustrated geometrically, as follows. Let the equation $\phi = 0$ be that of a surface, and for each point (x, y, z) of that surface, take another point having for its co-ordinates p, q, $px + qy - z$. The surface which has the second point for its locus is conjugate with the first; that is, what properties soever connect the first with the second, the same connect the second with the first. This conjugation cannot exhibit any absolute geometrical properties, for the conjugate surface depends, as to what it shall be, not only on the primitive surface, but on the position of the axes of co-ordinates, and also on the linear unit chosen. Thus it will be found that the conjugate surface of a given sphere is a double hyperboloid of revolution, having for its real axis the diameter of the sphere which is parallel to the axis of z, and for its imaginary semiaxis the linear unit. Now when the first surface is developable, its conjugate surface becomes a cylinder described by a straight line parallel to z, guided by the curve $f(p, q) = 0$ on the plane of xy. There is then no relation which involves all the three co-ordinates.

It may be worth while to notice, that we can at pleasure obtain forms for elimination which reproduce the function originally given, by assuming an equation which is its own tranformation.

A. DE MORGAN.

UNIVERSITY COLLEGE, LONDON,
April 27, 1848.

June 1, 1847. I had finished the foregoing Paper, as here written and dated, and it was in the hands of a friend for transmission to the Society, when I happened to have occasion to turn over *all* the Notes of M. Chasles's *Aperçu Historique .... des méthodes en Géométrie*, that I might collect all that has reference to the history of Arithmetic. To my surprise, at Note XXX. p. 376, under the head *Sur les Courbes et Surfaces* réciproques *de Monge*, being an account of an *unpublished* memoir of Monge in possession of the Institute, I found the second of these methods fully described. But to judge from all elementary writings, as well as from the apparent resources of those who have had to use modes of integration, this method is not known: and therefore I do not abandon my intention of communicating it to the Society.

A. DE MORGAN.

XLIV. *Second Memoir on the Fundamental Antithesis of Philosophy.* By W. Whewell, D.D., *Master of Trinity College, and Professor of Moral Philosophy.*

[Read *November* 13, 1848.]

31. In the course of 1844 I had the honour of reading before the Philosophical Society a Memoir *On the Fundamental Antithesis of Philosophy;* and this Memoir has since been printed in the Society's *Transactions.* The Fundamental Antithesis of which I then treated, is that which is expressed in various ways:—for instance, by speaking of Things and Thoughts; of Sensations and Ideas; of Fact and Theory; of Experience and Necessary Truth; of the Objective and Subjective Elements of our Knowledge. I endeavoured to make it apparent that all these are, at bottom, the same antithesis, and that this antithesis is an antithesis of inseparable Elements;—so inseparable, that the opposed terms cannot, either of them, be applied absolutely and exclusively in any case.

32. To give value to the exposition of this antithesis, it must be used in the expression of philosophical truth. The antithesis may be looked upon in the light of a Definition by which we are to enunciate one or more Propositions. In this, as in other cases, the Definition gives meaning to the Proposition, the Proposition gives reality to the Definition. The Definition saves the Proposition from being vague or ambiguous; the Proposition saves the Definition from being arbitrary or empty.

In the Memoir just referred to, I have already used the fundamental antithesis in stating views respecting the reality and the developement of human knowledge. But I would wish to be allowed to pursue the subject a step further, and to express in a more general and distinct form than I have there done, a general truth in the history of science, which I have there stated in a partial and imperfect manner.

33. The general Truth of which I speak may be thus expressed:—that the Progress of Science consists in a perpetual reduction of Facts to Ideas. Portions are perpetually transferred from one side to another of the Fundamental Antithesis: namely, from the Objective to the Subjective side. The Center or Fulcrum of the Antithesis is shifted by every movement which is made in the advance of science, and is shifted so that the ideal side gains something from the real side.

34. I will proceed to illustrate this Proposition a little further. Necessary Truths belong to the Subjective, Observed Facts, to the Objective side of our knowledge. Now in the progress of that exact speculative knowledge which we call Science, Facts which were at a previous period merely Observed Facts, come to be known as Necessary Truths; and the attempts at new advances in science generally introduce the representation of known truths of fact, as included in higher and wider truths, and therefore, so far, necessary.

35. We may exemplify this progress in the history of the science of Mechanics. Thus the property of the lever, the inverse proportion of the weights and arms, was known as a fact before the time of Aristotle, and known as no more; for he gives many fantastical and inapplicable reasons

for the fact. But in the writings of Archimedes we find this fact brought within the domain of necessary truth. It was there transferred from the empirical to the ideal side of the Fundamental Antithesis; and thus a progressive step was made in science. In like manner, it was at first taken by Galileo as a mere fact of experience, that in a falling body, the velocity increases in proportion to the time; but his followers have seen in this the necessary effect of the uniform force of gravity. In like manner, Kepler's empirical Laws were shewn by Newton to be necessary results of a central force attracting inversely as the square of the distance. And if it be doubtful whether this is the necessary law of a central force, as some philosophers have maintained that it is, we cannot doubt that if those philosophers could establish their doctrine as certain, they would make an important step in science, in addition to those already made.

And thus, such steps in science are made, whenever empirical facts are discerned to be necessary laws; or, if I may be allowed to use a briefer expression, whenever *facts are idealized.*

36. In order to shew how widely this statement is applicable, I will exemplify it in some of the other sciences.

In Chemistry, not to speak of earlier steps in the science, which might be presented as instances of the same general process, we may remark that the analyses of various compounds into their elements, according to the quantity of the elements, form a vast multitude of facts, which were previously empirical only, but which are reduced to a law, and therefore to a certain kind of ideal necessity, by the discovery of their being compounded according to definite and multiple proportions. And again, this very law of definite proportions, which may at first be taken as a law given by experience only, it has been attempted to make into a necessary truth, by asserting that bodies must necessarily consist in atoms, and atoms must necessarily combine in definite small numbers. And however doubtful this Atomic Theory may at present be, it will not be questioned that any chemical philosopher who could establish it, or any other Theory which would produce an equivalent change in the aspect of the science, would make a great scientific advance. And thus, in this Science also, the Progress of Science consists in the transfer of facts from the empirical to the necessary side of the antithesis; or, as it was before expressed, in the idealization of facts.

37. We may illustrate the same process in the Natural History Sciences. The discovery of the principle of Morphology in plants, was the reduction of a vast mass of Facts to an Idea; as Schiller said to Göthe when he explained the discovery; although the latter, cherishing a horrour of the term *Idea*, which perhaps is quite as common in England as in Germany, was extremely vexed at being told that he possessed such furniture in his mind. The applications of this Principle to special cases, for instance, to Euphorbia by Brown, to Reseda by Lindley, have been attempts to idealize the facts of these special cases.

38. We may apply the same view to steps in Science which are still under discussion;—the question being, whether an advance has really been made in science or not. For instance, in Astronomy, the Nebular Hypothesis has been propounded, as an explanation of many of the observed phenomena of the Universe. If this Hypothesis could be conceived ever to be established as a true Theory, this must be done by its taking into itself, as necessary parts of the whole Idea, many Facts which have already been observed; such as the various form of nebulæ; many Facts which it must require a long course of years to observe, such as the changes of nebulæ from one form to another; and many facts which, so far as we can at present judge, are utterly at variance with the Idea, such as the motions of satellites, the relations of the elements of planets, the existence of vegetable and animal life upon their surfaces. But if all these Facts, when fully studied, should appear to be included in the general Idea of Nebular Condensation according to the Laws of Nature, the Facts so idealized would undoubtedly constitute a very remarkable advance in science. But then, we are to recollect that we are not to suppose that the Facts will agree with the Idea, merely because the Idea, considered by itself, and without carefully attending to the Facts, is a large and striking Idea. And we are also to recollect that the Facts may be compared with another Idea, no

less large and striking; and that if we take into our account, (as, in forming an Idea of the Course of the Universe, we must do,) not only vegetable and animal, but also human life, this other Idea appears likely to take into it a far larger portion of the known Facts, than the Idea of the Nebular Hypothesis. The other Idea which I speak of is the Idea of Man as the principal Object in the Creation; to whose sustenance and developement the other parts of the Universe are subservient as means to an end; and although, in our attempts to include all known Facts in this Idea, we again meet with many difficulties, and find many trains of Facts which have no apparent congruity with the Idea; yet we may say that, taking into account the Facts of man's intellectual and moral condition, and his history, as well as the mere Facts of the material world, the difficulties and apparent incongruities are far less when we attempt to idealize the Facts by reference to this Idea, of Man as the End of Creation, than according to the other Idea, of the World as the result of Nebular Condensation, without any conceivable End or Purpose. I am now, of course, merely comparing these two views of the Universe, as supposed steps in science, according to the general notion which I have just been endeavouring to explain, that a step in science is some Idealization of Facts.

39. Perhaps it will be objected, that what I have said of the Idealization of Facts, as the manner in which the progress of science goes on, amounts to no more than the usual expressions, that the progress of science consists in reducing Facts to Theories.. And to this I reply, that the advantage at which I aim, by the expression which I have used, is this, to remind the reader—that Fact and Theory, in every subject, are not marked by separate and prominent features of difference, but only by their present opposition, which is a transient relation. They are related to each other no otherwise than as the poles of the fundamental antithesis; the point which separate those poles shifts with every advance of science; and then, what was Theory becomes Fact. As I have already said, elsewhere, a true Theory is a Fact; a Fact is a familiar Theory. If we bear this in mind, we express the view on which I am now insisting when we say that the progress of science consists in reducing Facts to Theories. But I think that speaking of Ideas as opposed to Facts, we express more pointedly the original Antithesis, and the subsequent identification of the Facts with the Idea. The expression appears to be simple and apt, when we say, for instance, that the Facts of Geography are identified with the Idea of the globular Earth; the Facts of Planetary Astronomy with the Idea of the Heliocentric system; and ultimately, with the Idea of universal Gravitation.

40. We may further remark, that though by successive steps in science, successive Facts are reduced to Ideas, this process can never be complete. However the point may shift which separates the two poles, the two poles will always remain. However far the ideal element may extend, there will always be something beyond it. However far the phenomena may be idealized, there will always remain a portion which are not idealized, and which are mere phenomena. This also is implied by making our expressions refer to the fundamental antithesis: for because the antithesis *is* fundamental, its two elements will always be present; the objective as well as the subjective. And thus, in the contemplation of the universe, however much we understand, there must always be something which we do not understand; however far we may trace necessary truths, there must always be things which are to our apprehension arbitrary: however far we may extend the sphere of our internal world, in which we feel power and see light, it must always be surrounded by our external world, in which we see no light, and only feel resistance. Our subjective being is inclosed in an objective shell, which, though it seems to yield to our efforts, continues entire and impenetrable beyond our reach, and even enlarges in its extent while it appears to give up to us a portion of its substance.

ADDITIONAL NOTE TO TWO MEMOIRS "ON THE FUNDAMENTAL ANTITHESIS OF PHILOSOPHY."

Of certain Modern Systems of Philosophy.

I am desirous of adding, as a note to this and the preceding Memoir, some very brief remarks relative to certain philosophical systems which have been much spoken of in modern times, especially those of the celebrated German philosophers, Kant, Fichte, Schelling, and Hegel.

Every system of philosophy offers to us a special and characteristic mode of criticizing preceding systems: and since every new system aspires to be true, it includes that which was true in the preceding systems, and is therefore able to point out where the true part of each is. The doctrine which I have endeavoured to explain in the two preceding Memoirs is, that there is a Fundamental Antithesis of two elements, of which the union is involved in all knowledge, and of which the separation is the task of all philosophy. This doctrine naturally directs us to consider how far each preceding system of philosophy has performed this task; and the survey of such systems from this point of view, may enable us to characterize them by a few sentences, at least so far as they regard one leading point of such systems, the account which they give of the nature and foundations of human knowledge.

The doctrine of the Fundamental Antithesis, which I have endeavoured to expound in the above Memoirs, and in other places, is briefly this:

That in every act of knowledge (1) *there are two opposite elements which we may call Ideas and Perceptions; but of which the opposition appears in various other antitheses; as Thoughts and Things, Theories and Facts, Necessary Truths and Experiential Truths; and the like:* (2) *that our knowledge derives from the former of these elements, namely our Ideas, its form and character as knowledge, our Ideas of space and time being the necessary forms, for instance, of our geometrical and arithmetical knowledge;* (3) *and in like manner, all our other knowledge involving a developement of the ideal conditions of knowledge existing in our minds:* (4) *but that though ideas and perceptions are thus separate elements in our philosophy, they cannot, in fact, be distinguished and separated, but are different aspects of the same thing;* (5) *that the only way in which we can approach to truth is by gradually and successively, in one instance after another, advancing from the perception to the idea; from the fact to the theory; from the apprehension of truths as actual to the apprehension of them as necessary.* (6) *This successive and various progress from fact to theory constitutes the history of science;* (7) *and this progress, though always leading us nearer to that central unity of which both the idea and the fact are emanations, can never lead us to that point, nor to any measurable proximity to it, or definite comprehension of its place and nature.*

Now the doctrine of the Fundamental Antithesis being thus stated, the successive sentences of the statement contain the successive steps of German philosophy, as it has appeared in the series of great authors whom I have named.

Ideas, and Perceptions or Sensations, being regarded as the two elements of our knowledge, Locke, or at least the successors of Locke, had rejected the former element, Ideas, and professed to resolve all our knowledge into Sensation. After this philosophy had prevailed for a time, Kant exposed, to the entire conviction of the great body of German speculators, the untenable nature of this account of our knowledge. He taught (one of the first sentences of the above statement) that (2) *Our knowledge derives from our Ideas its form and character as knowledge; our Ideas of space and time being, for instance, the necessary forms of our geometrical and arithmetical knowledge.* Fichte carried still further this view of our knowledge, as derived from our Ideas, or from its nature as knowledge; and held that (3) *all our knowledge is a developement of the ideal conditions of knowledge existing in our minds*, (one of our next following sentences). But when the ideal element of our knowledge was thus exclusively dwelt upon, it was soon seen that this ideal system no more gave a complete explanation of the real nature of knowledge, than the old sensational doctrine had done. Both elements, Ideas and Sensations, must be taken into account. And this was attempted by Schelling, who, in his earlier works, taught (as we have also stated above) that (4) *Ideas and Facts are different aspects of the same thing*:—this thing, the original basis of truth in which both elements are involved and identified, being, in Schelling's language, the *Absolute*, while each of the separate elements is subjected to *conditions* arising from their union. But this Absolute, being a point inaccessible to us, and inconceivable by us, as *our* philosophy teaches (as above), cannot to any purpose be made the basis of our philosophy: and accordingly this *Philosophy of the Absolute* has not been more permanent than its predecessors. Yet the philosophy of Hegel, which still has a wide and powerful sway in Germany, is, in the main, a developement of the same principle as that of Schelling;—the identity of the idea and the fact; and Hegel's *Identity System*, is rather a more methodical and technical exposition of Schelling's Philosophy of the Absolute than a new system. But Hegel traces the manifestation of the identity of the idea and fact in the progress of human knowledge; and thus in some measure approaches to our doctrine (above stated), that (5) *the way in which we approach to truth is by gradually and successively, in one instance after another*, that is, *historically, advancing from the perception to the idea, from the fact to the theory*: while at the same time Hegel has not carried out this view in any comprehensive or complete manner, so as to show that (6) *this process constitutes the history of science*: and alike with Schelling, his system shews an entire want of the conviction (above expressed as part of our doctrine), (7) that *we can never, in our speculations reach or approach to the central unity of which both idea and fact are emanations.*

This view of the relation of the Sensational School, Kant, Fichte, Schelling, and Hegel, and of the fundamental defects of all, may be further illustrated. It will, of course, be understood that our illustration is given only as a slight and imperfect sketch of their philosophies; but their relation may perhaps become more apparent by the very brevity with which it is stated; and the object of the present note is not detailed criticism, but this very relation of systems to each other.

The actual and the ideal, the external and the internal elements of knowledge, were called by the Germans the *objective* and the *subjective* elements respectively. The forms of knowledge and especially space and time, were pronounced by Kant to be essentially *subjective*; and this view of the nature of knowledge more fully unfolded and extended to all knowledge, became the *subjective ideality* of Fichte. But the subjective and the objective are, as we have said, in their ultimate and supreme form, one; and hence we are told of the *subjective-objective*, a phrase which has also been employed by Mr. Coleridge. Fichte had spoken of the subjective element as the *Me*, (das Ich); and of the objective element as the *Not-me*, (das Nicht-Ich); and has deduced the *Not-me* from the *Me*. Schelling, on the contrary, laboured with great subtlety to deduce the *Me* from the *Absolute* which includes both. And this Absolute, or Subjective-objective, is spoken of by Schelling as unfolding itself into endless other antitheses. It was held that from the assumption of such a principle might be deduced and explained the oppositions which, in the contemplation of nature, present themselves at every step, as leading points of general philosophy:—for example, the opposition of matter as *passive* and *active*, as *dead* and *organized*, as *unconscious* or *conscious*; the opposition of *individual* and *species*, of *will* and *moral rule*. And this antithetical developement was carried further by Hegel, who taught that the absolute idea developes itself so as to assume qualities, limitations, and seeming oppositions, and thus completes the cycle of its developement by returning into unity.

That there is, in the history of Science, much which easily lends itself to such a formula, the views which I have endeavoured to expound, show and exemplify in detail. But yet the attempts to carry this view into detail by conjecture, by a sort of divination, with little or no attention to the historical progress and actual condition of knowledge, (and such are those which have been made by the philosophers whom I have mentioned,) have led to arbitrary and baseless views of almost every branch of knowledge. Such oppositions and differences as are found to exist in nature, are assumed as the representatives of the elements of necessary antitheses, in a manner in which scientific truth and inductive reasoning are altogether slighted. Thus, this peculiar and necessary antithetical character is assumed to be displayed in attraction and repulsion, in centripetal and centrifugal forces, in a supposed positive and negative electricity, in a supposed positive and negative magnetism; in still more doubtful positive and negative elements of light and heat; in the different elements of the atmosphere which are, quite groundlessly, assumed to have a peculiar antithetical character: in animal and vegetable life: in the two sexes: in gravity and light. These and many others, are given by Schelling, as instances of the radical opposition of forces and elements which necessarily pervades all nature. I conceive that the heterogeneous and erroneous principles involved in these views of the material world show us how unsafe and misleading is the philosophical assumption on which they rest. And the triads of Hegel, consisting of thesis, antithesis, and union, are still more at variance with all sound science. Thus we are told that matter and motion are determined as *inertia, impulsion, fall;* that absolute Mechanics determines itself as *centripetal force, centrifugal force, universal gravitation.* Light, it is taught, is a secondary determination of matter. Light is the most intimate element of nature, and might be called *the Me* of nature: it is limited by what we may call negative light, which is darkness.

In these rash and blind attempts to construct physical science *à priori*, we may see how imperfect the Hegelian doctrines are, as a complete philosophy. In the views of moral and political subjects the results of such a scheme are naturally less obviously absurd, and may often be for a moment striking and attractive, as is usually the case with attempts to reduce history to a formula. Thus we are told that *the State* appears under the following determinations:—first, as one, substantial, self-included: next, varied, individual, active, disengaging itself from the substantial and motionless unity: next, as two principles, altogether distinct, and placed front to front in a marked and active opposition: then, arising out of the ruins of the preceding, the idea appears afresh, one, identical, harmonious. And the East, Greece, Rome, Germany, are declared to be the historical forms of these successive determinations. Whatever amount of real historical colour there may be for this representation, it will hardly, I think, be accepted as evidence of a profound political philosophy; but on such parts of the subject I shall not here dwell.

I may observe that in the series of philosophical systems now described, the two elements of the Fundamental Antithesis are, alternately dwelt upon in an exaggerated degree, and then confounded. The Sensational School could see in human knowledge nothing but facts: Kant and Fichte fixed their attention almost entirely upon ideas: Schelling and Hegel assume the identity of the two, (a point which we never can reach,) as the origin of their philosophy. The external world in Locke's school was all in all. In the speculations of Kant this external world became a dim and unknown region. Things were acknowledged to be *something* in themselves, but *what*, the philosopher could not tell. Besides the *phænomenon* which we see, Kant acknowledged a *noumenon* which we think of; but this assumption, for such it is, exercises no influence upon his philosophy. Things in themselves, are in his Drama, merely a kind of mute personages, κωφὰ πρόσωπα, which stand on the stage to be pointed at and talked about, but which do not tell us anything, or enter into the action of the piece. Fichte carries this further, and if we go on with the same illustration, we may say that he makes the whole drama into a kind of monologue; in which the author tells the story, and merely names the persons who appear. If we would still carry on the image, we may say that Schelling, going upon the principle that the whole of the drama is merely a progress to the denoument, which denoument contains the result of all the preceding scenes and events, starts with the last scene of the piece, and bringing all the characters on the stage in their final attitudes, would elicit the story from this. While the true mode of proceeding is, to follow the drama scene by scene, learning as much as we can of the action and the characters, but knowing that we shall not be allowed to see the denoument, and that to do so is probably not the lot of our species on earth. So far as any philosopher has thus followed the historical progress of the grand spectacle offered to the eyes of speculative man, in which the Phenomena of Nature are the Scenes, and the Theory of them the Plot, he has taken the course by which knowledge really has made its advances. But those who have partially done this, have often, like Hegel, assumed that they had divined the whole course and end of the story, and have thus criticized the scenes and the characters in a spirit quite at variance with that by which any real insight into the import of the representation can be obtained*.

I will only offer one more illustration of the relative position of these successive philosophies. Kant compares the change which he introduced into philosophy to the change which Copernicus introduced into astronomical theory. When Copernicus found that nothing could be made of the phenomena of the heavens so long as everything was made to turn about the spectator, he tried whether the matter might not be better explained if he made the spectator turn, and left the stars at rest. So Kant conceives that our experience is regulated by our own faculties, as the phenomena of the heavens are regulated by our own motions. But accepting and carrying out this illustration, we may say that Kant, in explaining the phenomena of the heavens by means of the motions of the earth, has almost forgotten that the planets have their own proper motions, and has given us a system which hardly explains anything besides broadest appearances, such as the annual and daily motions of the sun; and that Fichte appears as if he wished to deduce all the motions of the planets, as well as of the sun, from the conditions of the spectator;—while Schelling goes to the origin of the system like Descartes, and is not content to shew how the bodies move, without also proving, that from some assumed original condition, also the movements and relations of the system must necessarily be what they are. It may be that a theory which explains how the planets with their orbits and accompaniments have come into being may offer itself to bold speculators, like those who have framed and produced the nebular hypothesis. But I need not here remind my hearers either how precarious such a hypothesis is, or that if it be capable of being considered probable, its proofs must gradually dawn upon us, step by step, age after age: and that a system of doctrine which requires such a scheme as a certain and fundamental truth, and deduces the whole of astronomy from it, must needs be arbitrary, and liable to the gravest error at every step. Such a precarious and premature philosophy, at best, is that of Schelling and Hegel; especially as applied to those sciences in which, by the past progress of all sure knowledge, we are taught what the real cause and progress of knowledge is: while at the same time we may allow that all these forms of philosophy, since they do recognize the condition and motion of the spectator, as a necessary element in the explanation of the phenomena, are a large advance upon the Ptolemaic scheme, the view of those who appeal to phenomena as the source of our knowledge, and say that the sun, the moon, and the planets move as we see them move, and that all further theory is imaginary and fantastical.

W. WHEWELL.

* If it be asked which position we can assign, in this dramatic illustration, to those who hold that all our knowledge is derived from facts only, and who reject the supposition of ideas; we may say that they look on with a belief that the drama has no plot, and that these scenes are improvised without connexion or purpose.

XLV. *Observations of the Aurora Borealis of November* 17, 1848, *made at the Cambridge Observatory. By the Rev.* J. CHALLIS, M.A., F.R.S., F.R.A.S., *Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge.*

[Read *November* 27, 1848.]

THE observations I am about to lay before the Society, relate principally to the position of the Corona in the splendid display of Aurora Borealis which occurred on the night of Nov. 17. During thirteen years that I have resided at the Cambridge Observatory, there certainly has not been so favourable an opportunity of observing the position of this critical point of the phænomenon as on the recent occasion: and as the observations I took have enabled me to make a comparison of the position of the Corona with the Magnetic Declination and Dip at the place of observation more accurately than in any former instance that I am acquainted with, I have thought them worthy, with their results, of being formally submitted to the notice of the Society.

The observations were made partly by estimation of the position of the Corona with reference to neighbouring stars, and partly by means of a small altitude and azimuth instrument, which was constructed by Mr. Simms (Fleet Street, London), according to my directions, expressly for taking observations of this kind. I call the instrument a *Meteoroscope*. It has a graduated azimuth circle of four inches radius, and a portion of an altitude circle of the same radius graduated from 0^0 to 120^0. An arm somewhat longer than the radius of the altitude circle, and turning about a horizontal axis passing through the centre of the graduation of that circle, carries a bar eighteen inches long, by means of which the observations are taken. To that extremity of the bar which is turned towards the object observed, a rectangular piece is attached having one side horizontal, and consequently the other movable in a vertical plane. The other end carries a plate in which is made an eye-let hole one-sixth of an inch in diameter. An altitude is taken by observing through the eye-let hole the coincidence of the object with the horizontal side of the rectangular piece, and an azimuth by observing its coincidence with the vertical side. Both are taken simultaneously by observing the coincidence with the angular point. The bar is set obliquely on the arm which carries it, for the purpose of observing altitudes a few degrees beyond the zenith, for which purpose also the graduation of the altitude circle extends beyond 90^0. When the object is near the zenith, for convenience it is looked at through another eye-let hole made in a small plate standing at right-angles to the larger plate, the object being seen by reflexion at a small mirror, the plane of which is inclined at an angle of 45^0 to the direction of the bar. In this case the other angular point of the rectangular piece is brought into coincidence with the object, care having been taken by the maker of the instrument that the direction of collimation should in the two cases be the same. The bar and altitude circle may be readily turned together about the vertical axis, and the bar separately about its horizontal axis of motion, and both may be quickly clamped as soon as the observation is taken. The graduations are read off by verniers to single minutes. The instrument has a tripod stand, furnished with adjusting screws for the purpose of placing the axis of motion vertical by means of a small spirit-level, which is carried round with the vertical circle. The feet of the screws rest in three angular grooves formed each by two plane faces, by applying the feet to which, the instrument is instantly in position, the vertical adjustment of the axis having been previously made. It is proper on each occasion of using it to determine the index errors by observations of stars.

I proceed now to give the observations just as they were set down in the memorandum-book, inclusive of those for index errors, premising only that the instrumental azimuths are reckoned from *East* towards *South*, and that the noted times were taken from a solar chronometer, which by a comparison with the transit clock immediately after the observations was found to be $1^m. 52^s.$ fast on Greenwich Mean Time.

(1)......$8^h. 49^m. 0^s$ Corona 1^0 South and $\frac{1}{2}^0$ East of β Andromedæ.
(2)......8 . 51 . 0 .... $1\frac{1}{4}^0$ South and $\frac{3}{4}^0$ ....
(3)......8 . 55 . 0 .... 1^0 South and $\frac{3}{4}^0$ ....
(4)......9 . 1 . 40 .... $2\frac{1}{2}^0$ South and $1\frac{1}{4}^0$ ....
(5)......9 . 4 . 25 .... 3^0 South and 2^0 ....
(6)......9 . 9 . 0 Altitude of Corona by Meteoroscope $68^0. 11'$ Azimuth $43^0. 44'$
(7)......9 . 10 . 23 .... β Andromedæ .... 73 . 15 .... 59 . 58
(8)......9 . 13 . 15 .... Corona .... 67 . 4 .... 37 . 56
(9)......9 . 19 . 20 .... Corona .... 71 . 47 .... 42 . 28
(10)......9 . 29 . 12 .... β Andromedæ .... 72 . 20 .... 77 . 10
(11)......9 . 32 . 0 Corona 1^0 South, and $\frac{1}{2}^0$ East of β Trianguli.
(12)......9 . 35 . 15 Altitude of Corona by Meteoroscope 69 . 20 Azimuth 37 . 24
(13)......9 . 44 . 10 .... Corona .... 68 . 55 .... 41 . 21
(14)......9 . 49 . 40 .... β Andromedæ .... 71 . 20 .... 84 . 47
(15)......9 . 56 . 0 Corona 2^0 South, and $1\frac{1}{2}^0$ East of β Trianguli
(16)......10 . 6 . 0 ...... 2^0 South, and 3^0 ....
(17)......10 . 10 . 11 Altitude of Corona by Meteoroscope 69 . 28 Azimuth 42 . 12
(18)......10 . 14 . 36 .... β Trianguli .... 72 . 5 .... 65 . 30
(19)......10 . 20 . 30 .... Corona .... 69 . 30 .... 51 . 0
(20)......10 . 23 . 5 .... Corona .... 70 . 28 .... 40 . 0
(21)......11 . 6 . 20 .... Corona .... 71 . 0 .... 62 . 35
(22)......11 . 10 . 56 .... Corona .... 69 . 5 .... 55 . 30
(23)......11 . 12 . 50 .... Corona .... 70 . 55 .... 52 . 4
(24)......11 . 14 . 10 .... Corona .... 69 . 50 .... 47 . 6
(25)......11 . 16 . 0 The star ζ Persei appears in the middle of the Corona.
(26)......11 . 17 . 30 Altitude of Corona by Meteoroscope 67 . 30 Azimuth 49 . 40
(27)......11 . 22 . 0 { .... ζ Persei .... 69 . 10 .... 46 . 14
The Corona seemed coincident with ζ Persei }
(28)......11 . 26 . 0 Altitude of Corona by Meteoroscope 70 . 10 Azimuth 43 . 6

Of the above observations Nos. (19) and (21) were marked 'doubtful.' Nos. (23) and (26) were reckoned good.

The position of the Corona was calculated from these observations in the following manner. When the observation was made by reference to a star, from the noted time corrected for error of the chronometer, and the known longitude of the place of observation (viz. $23^s,5$ East), the sidereal time was calculated in the usual way, and then from the known Right Ascension of the Star, the hour angle (h) Eastward was deduced. The co-latitude of the Observatory (viz. $37^0 . 47'$) being represented by λ, and δ being the North Polar distance of the Star, its distance (m) from the meridian, and its distance (z) from the astronomical zenith, were calculated by the following formulæ:

$$\sin m = \sin \delta \sin h, \quad \tan \phi = \tan \delta \cos h, \quad \cos z = \cos (\phi - \lambda) \cos m.$$

Let μ and ζ represent the estimated distances of the Corona from the star Eastward and Southward. Since these were made in the directions of the arcs m and z, the distance (M) of the Corona from the meridian is $m + \mu$, and its distance (Z) from the zenith is $z + \zeta$.

When the Meteoroscope was used, the recorded altitude A, and azimuth B from East towards South, were first corrected for index error by α and β respectively, and Z and M were then obtained by the formulæ,

$$Z = 90^0 - (A + \alpha), \quad \sin M = \cos (B + \beta) \sin Z.$$

The index corrections were deduced from Nos. (7), (10), (14), (18), and (27). The calculated altitudes and azimuths of the stars, compared with the instrumental readings, gave the following values of α and β.

No.	(α)	(β)
(7)	− 40′	+ 27°. 8′
(10)	− 2	+ 23 . 1
(14)	+ 5	+ 28 . 29
(18)	+ 9	+ 25 . 6
(27)	− 53	+ 22 . 48
Mean	− 16	+ 25 . 18

Respecting these values it is to be remarked, that the discordances between them are much greater than might have been expected. From subsequent trials of the Meteoroscope I have found that, without taking particular care in making the observation, the error in an arc of a great circle may amount to 12′. Whether the discordances above arose from unsteadiness in the support, the observations being made on the roof of a small out-building, on which several persons were standing; or from incautiously bending the collimating bar in the act of observing; or, in short, from inexperience in the use of the instrument, this being the first occasion of my using it in a series of observations, I am unable to say. On replacing the instrument (Nov. 24), I obtained from much more consistent values, the mean results − 51′ and + 23° . 34′. I have, however, considered it best to adopt the first determinations.

I have now to explain in what manner the point of the heavens to which the South end of the Dipping Needle was directed, which for the sake of brevity I call the *Magnetic Zenith*, was ascertained. As we have no Magnetic Observatory here, this was done *inferentially*. I have assumed that for any place in England, Scotland, and Ireland, the Westerly Declination of the needle (V) and the Dip (D) may be given approximately by the formulæ,

$$V = V_0 + a\lambda + bl$$
$$D = D_0 + a'\lambda + b'l,$$

V_0 and D_0 being the Declination and Dip at the Greenwich Observatory, λ the Longitude of the place *Westward* of Greenwich, l the excess of its Latitude above that of Greenwich, and a, b, a', b' certain constants, which may be calculated by knowing the simultaneous values of V and D at Greenwich and two other positions. From the published results of magnetic observations made in the year 1843 at Greenwich, and at the Observatory of Sir Thomas M. Brisbane (Makerstoun); and from a communication, kindly made to me by Professor Lloyd, of the mean Declination at Dublin for the same year as determined by 3600 observations, (viz. 27° . 9′, 87,) and the Dip at Dublin as determined by an elaborate series of observations in September of 1843, (viz. 70° . 41′,3), I have deduced very accurate contemporaneous values of V and D, which with the Latitudes and Longitudes of the three positions are here subjoined.

	Lat.	Long. West.	Declination.	Dip.
Greenwich	51° . 28′,6	0m . 0s,0	23° . 17′,59	69° . 1′,9
Makerstoun	55 . 34,7	10 . 3,5	25 . 22,85	71 . 25,0
Dublin	53 . 21,0	25 . 4,0	27 . 9,87	70 . 41,3

From these data, were derived the following formulæ, which probably may be applied at the present time and for several years to come, with considerable accuracy to any place in the United Kingdom:

$$V - V_0 = 0{,}142518\lambda + 0{,}159548l$$
$$D - D_0 = 0{,}027713\lambda + 0{,}513523l.$$

These formulæ give $V-V_0$ and $D-D_0$ in *minutes*, λ being expressed in *seconds of time*, and l in minutes.

For the Cambridge Observatory, $V - V_0 = +3',7$, and $D - D_0 = +22',0$.

In order to make the proposed comparison of the position of the Corona with the Magnetic Zenith, it is now only necessary to obtain the Magnetic Declination and Dip at the respective times of observation. These I have derived from observations made at the Greenwich Observatory during the prevalence of the Aurora, which, on my preferring a request, were promptly forwarded to me with all the requisite data, by James Glaisher, Esq., who is at the head of the Magnetical Department in that Institution, and which the Astronomer Royal has allowed me to publish with this communication. For this favour I beg here to express my thanks. The observations are given at length in Tables I, II, and III, at the end of this Paper, as well because they are used in the calculations, as because they present so striking an instance of great magnetic disturbances occurring simultaneously with an extraordinary display of the Aurora Borealis, that the connexion in some way of the two kinds of phænomena must be regarded as a physical fact.

The Westerly Declinations at Cambridge at the times of observation were inferred from those at Greenwich at the same times by merely applying the value of $V-V_0$ already obtained, viz. $+3',7$. The latter were deduced from the declinations recorded in Table I. by simple interpolation, it being understood that the motion of the magnet was uniform in the intervals between the times there given. The Greenwich observations were made by the admirable photographic process, which has been brought to so great perfection by C. Brooke, Esq., of St. John's College in this University. Between $9^h. 25^m$ and $9^h. 44^m$, the disturbance was so great that the magnet passed the limits of the photographic paper. The same thing took place in the contrary direction between $10^h. 10^m$ and $10^h. 40^m$. As Mr. Glaisher states that the motions at these times were smooth and without checks, I have ventured to deduce the maximum elongation between $9^h. 25^m$ and $9^h. 44^m$ on the supposition that the magnet continued to move after $9^h. 25^m$ in the same manner as from $9^h. 20^m$ to $9^h. 25^m$, till it attained the maximum, and then that it immediately returned by the same motion that it had from $9^h. 44^m$ to $10^h. 10^m$. The maximum elongation between $10^h. 10^m$ and $10^h. 40^m$ was inferred on the same principle.

Mr. Glaisher furnished me with the following values of the Dip at Greenwich :

			Dip.
1848. Nov.	12.	21^h.	$68^0. 54',0$
	16.	3	68 . 56 ,3
	19.	21	68 . 53 ,7
	23.	3	68 . 55 ,5

Hence it is inferred that the Dip, if undisturbed, would have been $68^0. 55',0$ during the Aurora. The disturbed Dip was calculated in the manner I am about to explain. In the Greenwich observations (Tables II. and III.), the readings for the horizontal force variations are given in terms of the whole horizontal force; but the vertical force readings are given in divisions of the scale, which require to be converted into parts of the whole vertical force. The factor for this purpose is 0,00067, which is the value of one division.

The scale reading of the vertical force magnet at Nov. 17, 0^h, was $21^{div.},7$, and at Nov. 18, 0^h, $21^{div.},5$, at which times there appears to have been no disturbance. The undisturbed reading is consequently assumed to be $21^{div.},6$.

The reading of the horizontal force magnet in parts of the whole horizontal force, was 0,1099 at November 17, 0^h, and 0,1074, at November 18, 0^h, the latter of which Mr. Glaisher states to be somewhat below the average value for the season and time of day. The undisturbed reading during the Aurora is assumed to be the mean between those two readings, viz. 0,1086.

The variation of horizontal force was not registered from $10^h.2^m$ to 12^h, the disturbance carrying the magnet out of the limits of the photographic paper. From 12^h the observations were made independently of the self-registering process. I have assumed that from $10^h.2^m$ to 12^h the disturbance followed the same law as from $12^h.39^m$ to $14^h.5^m$, when the phænomenon reappeared in a similar phase, and accordingly have taken 0,0890 to be the mean horizontal force reading in the former interval.

Let now X and Y be the undisturbed horizontal and vertical forces respectively, X', Y' their disturbed values at any given time, and x, y, the horizontal and vertical force readings at that time, deduced by interpolation from Tables II. and III, the former divided by 10,000. Then

$$X' = X - (0{,}1086 - x)\,X, \quad Y' = Y - (21{,}6 - y)\,0{,}00067\,Y.$$

Hence since $\frac{Y'}{X'}$ = tan of the actual Dip, and $\frac{Y}{X} = \tan 68^{\circ}.\,55'$, it is readily shewn that

$$\text{the actual Dip} = 70^{\circ}.\,43',6 - [3{,}06215]\,x + [9{,}88822]\,y,$$

the numbers in brackets being the Logs of the coefficients of x and y. The Dip at Cambridge is assumed to be the value given by this formula, increased by $D - D_0$, or $+\,22',0$.

From the Declination (V) and Dip (D), the distance Z' of the Magnetic Zenith from the Astronomical Zenith, and its distance M' from the meridian are given by the expressions,

$$Z' = 90^{\circ} - D, \qquad \sin M' = \sin V \cos D.$$

The following are the results of the calculations which have been now explained.

Greenwich Mean Time 1848. Nov. 17.	Zen. Dist. of Corona.	Zen. Dist. of Magr. Zenith.	$Z - Z'$.	Dist. of Corona from Merid.	Dist. of Magr. Zen. from Merid.	$M - M'$.	Mode of Observing.
$8^h.\ 47^m,1$	18°. 59′	20°. 35′	− 1°. 36′	6°. 3′	7°. 48′	− 1°. 45′	*s*
49,1	19 . 10	20 . 39	− 1 . 29	5 . 36	7 . 48	− 2 . 12	*s*
53,1	18 . 44	20 . 38	− 1 . 54	4 . 47	7 . 46	− 2 . 59	*s*
8 . 59,8	20 . 3	20 . 37	− 0 . 34	3 . 55	7 . 50	− 3 . 55	*s*
9 . 2,6	20 . 30	20 . 36	− 0 . 6	4 . 6	7 . 51	− 3 . 45	*s*
7,1	22 . 5	20 . 38	+ 1 . 27	7 . 44	7 . 49	− 0 . 5	*m*
11,4	23 . 12	20 . 36	+ 2 . 36	10 . 13	7 . 36	+ 2 . 37	*m*
17,5	18 . 29	20 . 42	− 2 . 13	6 . 53	7 . 44	− 0 . 51	*m*
30,1	20 . 27	20 . 26	+ 0 . 1	9 . 8	8 . 23	+ 0 . 45	*s*
33,4	20 . 56	20 . 26	+ 0 . 30	9 . 26	8 . 19	+ 1 . 7	*m*
42,3	21 . 21	20 . 32	+ 0 . 49	8 . 18	8 . 9	+ 0 . 9	*m*
9 . 54,1	20 . 14	20 . 42	− 0 . 28	5 . 10	7 . 55	− 2 . 45	*s*
10 . 4,1	20 . 0	20 . 34	− 0 . 34	4 . 36	7 . 38	− 3 . 2	*s*
8,3	20 . 48	20 . 28	+ 0 . 20	7 . 49	7 . 30	+ 0 . 19	*m*
18,6	20 . 46	20 . 30	+ 0 . 16	4 . 52	7 . 22	− 2 . 30	*m*
10 . 21,2	19 . 48	20 . 31	− 0 . 43	8 . 8	7 . 20	+ 0 . 48	*m*
11 . 4,6	(19 . 16	20 . 23	− 1 . 7	0 . 42	7 . 35	− 6 . 53)	*m*
9,1	21 . 11	20 . 24	+ 0 . 47	3 . 19	7 . 36	− 4 . 17	*m*
11,0	19 . 21	20 . 24	− 1 . 3	4 . 9	7 . 37	− 3 . 28	*m*
12,3	20 . 26	20 . 24	+ 0 . 2	6 . 4	7 . 37	− 1 . 33	*m*
14,1	22 . 7	20 . 24	+ 1 . 43	8 . 52	7 . 37	+ 1 . 15	*s*
15,6	22 . 46	20 . 23	+ 2 . 23	5 . 46	7 . 37	− 1 . 51	*m*
20,1	21 . 43	20 . 24	+ 1 . 19	7 . 35	7 . 39	− 0 . 4	*s*
	21 . 6	20 . 24	+ 0 . 42	6 . 33	7 . 39	− 1 . 6	*m*
11 . 24,1	20 . 6	20 . 24	− 0 . 18	7 . 16	7 . 40	− 0 . 24	*m*
Means	20 . 35,8	20 . 30,9	+ 0 . 4,9	6 . 30,8	7 . 44,6	− 1 . 13,8	

The mode of observing by reference to a star is indicated by the letter *s*, and that by the Meteoroscope by the letter *m*. In taking the means, the observation at $11^{h}.4^{m},6$ is excluded, the Corona at that time being seen very obscurely after an interval of total disappearance. The great westerly deviation given by that observation is, however, supported by the two that follow.

The observations by stars taken separately, give $-0^{0}.21',8$ for the value of $Z-Z'$, and $-1^{0}.50',7$ for that of $M-M'$. The observations by the Meteoroscope give $+23',9$ for the former, and $-0^{0}.47',5$ for the latter.

The discordances in the positions of the Corona deduced from observation, are no doubt partly owing to errors of estimation, or instrumental errors, and partly to the extreme difficulty of fixing with precision on the centre of convergence of the Auroral streamers. But if these were the only sources of discordance the distances from the zenith and from the meridian would be equally affected, whereas the latter appear to be the more discordant. The fact seems to be, *that the centre of the Corona is continually shifting its position.* This may be owing to several causes. The formation of the Corona is merely an effect of *perspective*, the apparent convergence of the streamers being due to the immense height to which they rise. If the streamers were all parallel to a fixed straight line, they would apparently converge to a fixed point. But the foregoing discussion, and facts that will be hereafter mentioned, shew that they take, at least very approximately, the direction of the dipping needle at the locality from which they ascend. Consequently the point of convergence will be different for streamers rising from different quarters. Again, the directions of the streamers may vary by the same causes which produce the disturbances of the position of the dipping needle: and this change of direction would of course alter the position of the Corona. Lastly, the course of the streamers may not be *rectilinear*. The foregoing comparison appears to prove that the Corona is decidedly more *Westward* than the Magnetic Zenith, being less distant from the meridian than the latter by $1^{0}.14'$. This is accounted for by saying that the streamers on rising from the Earth are bent in a westerly direction. The apparent point of convergence would thus depend on the *height* to which they rise, and would be continually varying. It is quite possible that streamers rising from different quarters and to different heights, might apparently cross each other, and so form a fictitious point of convergence. This explanation will, I think, sufficiently account for the discordances observable in the foregoing results, and will serve also to shew why they exhibit no decided agreement between the changes of position of the Corona and the changes of position of the Magnetic Zenith. Such agreement may very well be veiled by the causes just mentioned. It seems to me, however, that a general accordance of this nature is perceptible. As when the needle was most disturbed, a large Easterly deviation of the South End was succeeded by a large Westerly deviation, so a large deviation of the Corona to the East of its mean position was succeeded by a large Westerly deviation: and as the changes of M' are more marked than those of Z', so the changes of M are greater those of Z.

For the purpose of farther illustrating the subject, I propose to add a discussion of a few observations of the position of the Corona, made in the instance before us, and in one or two others, in different parts of England. I have selected those of which the data seemed to be most precise.

Mr. Boreham of Haverhill informed me by letter that he found the Right Ascension of the Corona of the Aurora Borealis of Nov. 17, to be $1^{h}.58^{m}.3^{s}$; and its declination $+31^{0}.18'$, at $9^{h}.15^{m}$ Greenwich mean time, the latitude of the place of observation being $52^{0}.5'$, and the longitude $1^{m}.46^{s}$. East. Hence I find by calculating as already described,

Z	Z'	$Z-Z'$	M	M'	$M-M'$
$22^{0}.56'$	$20^{0}.45'$	$+2^{0}.11'$	$11^{0}.18'$	$7^{0}.38'$	$+3^{0}.40'$

The differences in this instance are large, but not very different from those resulting from the observation made at Cambridge at $9^{h}.11^{m},4$.

An anonymous observer at Darlington, states in the Durham Advertiser of Nov. 24, 1848, that at Nov. 17, 11^h. 27^m, which I suppose to be Darlington time, ξ Persei was exactly in the centre of the Corona. The latitude of Darlington is 54^0. 32$'$, and the longitude 6^m. 12^s. West. Hence I have deduced,

Z	Z′	Z − Z′	M	M′	M − M′
20^0. 3′	18^0. 59′	+ 1^0. 4′	6^0. 51′	7^0. 34′	− 0^0. 43′

From observations of the Aurora of November 17, made at Lansdowne Crescent, Bath, by H. Lawson, Esq., and E. J. Lowe, Esq.; (1) "At 10^h. 20^m. (Bath time), the Corona was situated at 21 Persei." (2) At 11^h. 20^m. the centre of the Cupola was ζ Persei." The assumed latitude of Bath is 51^0. 22′, and the assumed longitude 9^m. 28^s. West; and the results of calculation are

	Z	Z′	Z − Z′	M	M′	M − M′
(1)	21^0. 20′	20^0. 36′	+ 0^0. 44′	8^0. 28′	7^0. 47′	+ 0^0. 41′
(2)	20 . 32	20 . 32	+ 0 . 0	5 . 41	8 . 9	− 2 . 28
Means	20 . 56	20 . 34	+ 0 . 22	7 . 5	7 . 58	− 0 . 53

A remarkable Aurora occurred on Oct. 18, 1848, which was not seen at Cambridge, on account of clouds. A description of it was sent to me by J. F. Miller, Esq., of Whitehaven. The most precise observations of the position of the Corona contained in the account are the following:—

(1). "10^h. 7^m. G. M. T., the centre of the Corona is about 5^0 above α Andromedæ, and nearly in a line with γ Pegasi." It had consequently nearly the same Right Ascension as α Andromedæ.

(2). "10^h. 24^m. G. M. T. The whole hemisphere is covered with streamers converging around π Andromedæ."

(3). "10^h. 52^m. G. M. T. β or μ Andromedæ, appears to be the centre of convergence." I have taken the mean position between the two stars.

(4). "11^h. 17^m. G. M. T. I have watched the Corona very attentively some time, and I think β Andromedæ as nearly as possible marks its centre."

(5). "11^h. 37^m. G. M. T. The coronal centre seems now to be about mid-way between γ Andromedæ, and β Trianguli."

I have compared these observations with the *mean* Magnetic Declination and Dip, deduced from those of Greenwich, which for Oct. 18, are assumed to be 22^0. 53′, and 68^0. 55′. The latitude of Whitehaven is 54^0. 33′, and the longitude 14^m. 12^s. West. The following are the results of the calculations.

	Z	Z′	Z − Z′	M	M′	M − M′
(1)	21^0. 30′	19^0. 7′	+ 2^0. 23′	3^0. 42′	8^0. 4′	− 4^0. 22′
(2)	22 . 14	19 . 7	+ 3 . 7	6 . 4	8 . 4	− 2 . 0
(3)	18 . 53	19 . 7	− 0 . 14	5 . 25	8 . 4	− 2 . 39
(4)	19 . 47	19 . 7	+ 0 . 40	1 . 42	8 . 4	− 6 . 22
(5)	18 . 16	19 . 7	− 0 . 51	8 . 46	8 . 4	+ 0 . 42
Means	20 . 8	19 . 7	+ 1 . 1	5 . 8	8 . 4	− 2 . 56

In the instance of the Aurora Borealis of Oct. 24, 1847, I observed that at 10^h. 10^m., Cambridge Mean Time, the centre of the Corona was at a point of less R. A. than β Andromedæ by 10^m, and of greater N. P. D. by 2^0. I am able to compare this observation with the actual Declination and Dip at the noted time, by means of Greenwich Magnetical Observations inserted in the published account of this Aurora drawn up by Mr. Morgan and Mr. Barber. From these data I find that the Declination at Cambridge was 23^0. 5′, and the Dip 69^0. 24′. Hence the result of the comparison is,

Z	Z′	Z − Z′	M	M′	M − M′
20^0. 10′	20^0. 36′	− 0^0. 26′	6^0. 24′	7^0. 56′	− 1^0. 32′

On this occasion the Auroral light descended but a few degrees southward of the Corona, and the streamers forming the Corona did not meet in a point, but left a circular dark space, which seemed to be constant in its position, and the centre of which it was easy to fix upon. On this account I consider the above results, though derived from a single observation, to be worthy of confidence.

From a consideration of all the results derived from the foregoing discussion of observations made on different occasions and at different places, the following conclusions seem to be established :—

First, that the Corona of an Aurora Borealis is formed near the Magnetic Zenith of the place of observation.

Secondly, that the observations, while they indicate no decided difference of altitude between the two points, shew with great probability that the Corona is situated between 1^0 and 2^0 more to the West than the Magnetic Zenith.

The Aurora Borealis which gave rise to the present communication, was more remarkable in its features and more extensively seen than any that have occurred for a long period, having been visible, as appears by authentic accounts, in France, Italy, Spain, Portugal, and the Azores. I have therefore thought it would not be out of place to add here a description of it which I derived from memoranda made very soon after its occurrence, and which was communicated to the Cambridge Chronicle of Nov. 25, 1848.

" Shortly after eight o'clock on the evening of Friday, Nov. 17, my attention was called to an unusual appearance of light stretching from N. to W., which gave indication of a coming Aurora. There was no arch, but the light was diffused and of considerable brilliancy. The maximum of the brightness was at a position a few degrees N. of W., at an altitude of about 20^0, which appeared to be a stationary centre of luminosity during the whole of the display. The diffused light increased by degrees in intensity, and spread upwards till it reached the Zenith ; but during this time there were no streamers. The principal features of the phænomenon were, frequent pulsations, and sudden appearances and disappearances of streaks and large patches of light, so much resembling white clouds that but for their rapid changes of form and brightness, it would have been difficult to distinguish them from the latter. The streaks darted in various quarters and different directions, waning as quickly as they formed, and auroral clouds of all imaginable shapes were continually bursting forth and vanishing, so as to present a spectacle of the utmost *bizarrerie*, till at length greater order began to prevail. Streamers of some degree of definiteness arose, and in a short time surrounded the magnetic Zenith. I then first observed the appearance of a corona or central point towards which the streamers converged, and estimated its position at $8^h. 47^m$ Greenwich mean time, to be one degree South and half a degree East of β Andromedæ.

" A large red patch due West and rising about 20^0, was observed to retain its position from $8^h. 35^m$ to $8^h. 51^m$. At $8^h. 56^m$ a broad red band stretched from the Corona through Capella, and in a few seconds changed to an auroral cloud of great brilliancy having Capella at its centre. At $8^h. 58^m$ an extraordinary red band of irregular width was formed extending across the heavens from a little S. of W. to N.E. These two azimuths were the prevailing positions of the red light during the whole of the phænomenon. The band seemed to be a kind of junction of two red clouds. Its general course was through α and β Andromedæ to the Corona, and from thence its axis passed through Capella.

" At $9^h. 15^m$ the phænomenon was at its greatest height of beauty and perfection. Streamers reached the Corona or Magnetic Zenith from all points of the Compass. The *tout ensemble* was a canopy of drapery, having the Corona for the point of divergence of the folds, and extending rather more Northward than Southward of the Astronomical Zenith ; while its boundary all round was considerably elevated above the horizon. The outline was very irregular, but sharply defined, giving irresistibly the idea of the lower boundary of a suspended curtain. This feature was in greatest perfection towards the N.W., where a broad space appeared so dark by contrast with the bright curtain above it, that it might have been mistaken for a cloud had not stars shone through it. The predominating colour of the streamers was white, but about W. S. W. and N. E., the peculiar ruddy tint of the Aurora was remarkably intense, and in other quarters the streamers were tinged with green and blue. Altogether, both as to form and colour, the spectacle at this time was so singular and so beautiful, that those who witnessed it here could not forbear giving repeated expression to their feelings of wonder and delight.

" The heavens were then partially covered with light clouds, through which the brightness of the Aurora seemed to penetrate. At $9^h. 58^m$, a red patch covered the constellation of Orion. At 10^h, when the clouds had dispersed, the general light resembled that of a night in midsummer, or the dawn of morning. Birds were heard to chirp in several quarters.

" At $10^h. 15^m$, I saw a meteor, as bright as a star of the second magnitude, move slowly in a westerly direction, and disappear at an altitude of about 53^0, and at an azimuth of about 28^0 from W. towards S. Flashes, supposed to be of lightning, were twice noticed. One occurred in the S. W. at $10^h. 23^m$. At this time the Aurora had much declined in brightness; but at 11^h it broke out afresh, and the Corona was again formed, not however with the same distinctness as before. At $11^h. 18^m$, a meteor, equal in brightness to a star of the second magnitude, was seen to cross the heavens slowly from E. to W. N. W., leaving a train behind it. Shortly after $11^h. 24^m$ the Corona became invisible, and the Aurora generally declined. I saw it, however, again between 14^h and 15^h in great brilliancy: a tolerably regular arch was formed in the N. W., from which very definite streamers rose, but did not reach the zenith: the red light also re-appeared in the West."

With reference to the above particulars I have two remarks to make. First, having in repeated instances of the Aurora observed the red light to prevail in the same azimuths, I made a comparison of the azimuths noted here in the instance of November 17 with statements respecting the prevailing direction of the red light given in descriptions of the same phænomenon as seen at other places, and it seems to me probable that the red auroral clouds are formed over the Atlantic and German oceans. Secondly, the occurrence of meteors during an Aurora has been so frequently remarked, that one can hardly avoid suspecting some connexion between the two kinds of phænomenon.

The following are the Tables of Magnetic Observations referred to in the foregoing communication.

TABLE I. The Westerly Declinations of the Declination Magnet about the time of the Aurora Borealis of November 17, 1848, as observed at the Royal Observatory of Greenwich.

Greenwich Mean Time, 1848.	Westerly Declination.	Greenwich Mean Time, 1848.	Westerly Declination.	Greenwich Mean Time, 1848.	Westerly Declination.
h. m.	° ′ ″	h. m.	° ′ ″	h. m.	° ′ ″
Nov. 17. 0 . 0	23 . 2 . 0	Nov. 17. 6 . 40	21 . 54 . 35	Nov. 17. 13 . 26	22 . 48 . 15
20	2 . 30	7 . 10	22 . 25 . 30	28	21 . 45
0 . 50	8 . 30	8 . 2	48 . 30	42	55 . 0
1 . 0	24 . 0	7	33 . 30	13 . 50	24 . 30
5	23 . 12 . 30	15	44 . 30	14 . 3	6 . 45
12	22 . 58 . 0	20	34 . 50	35	22 . 29 . 35
15	23 . 4 . 30	40	51 . 0	14 . 53	23 . 10 . 50
33	22 . 54 . 30	8 . 53	29 . 0	15 . 16	22 . 30 . 35
1 . 53	23 . 12 . 30	9 . 5	50 . 40	15 . 42	57 . 0
2 . 15	3 . 40	7	42 . 0	16 . 2	35 . 35
2 . 45	14 . 0	9	3 . 30	17	54 . 0
3 . 0	5 . 50	14	2 . 0	19	47 . 45
5	18 . 30	17	22 . 24 . 0	30	58 . 25
20	8 . 0	20	21 . 51 . 0	16 . 44	22 . 50 . 0
28	23 . 18 . 30	25	23 . 37 . 30 +	17 . 12	23 . 26 . 0
3 . 47	22 . 55 . 25	9 . 44	23 . 37 . 45	17 . 25	23 . 16 . 0
4 . 0	23 . 2 . 45	10 . 10	21 . 43 . 40 −	19 . 20	22 . 57 . 30
10	22 . 54 . 0	40	21 . 43 . 40	20 . 20	57 . 30
4 . 20	22 . 55 . 50	10 . 48	22 . 18 . 30	21 . 20	55 . 45
5 . 20	22 . 54 . 0	11 . 2	10 . 0	22 . 20	55 . 0
5 . 50	23 . 7 . 0	32	31 . 0	23 . 20	22 . 54 . 30
6 . 0	23 . 1 . 30	11 . 55	46 . 0		
6 . 15	22 . 34 . 0	12 . 20	54 . 0		

To these observations the following remarks were attached :

"The magnet began to be disturbed at $0^h . 50^m$."

"Rapid changes occurred from 6^h to 8^h."

"It is not known to what extent the Magnet moved between $9^h . 25^m$ and $9^h . 44^m$ on one side, and between $10^h . 10^m$ and $10^h . 40^m$ on the other. The motion at these times was smooth and certainly without any checked motion whatever."

TABLE II. Readings of the Horizontal Force Magnet in parts of the whole horizontal force, about the time of the Aurora Borealis of November 17, 1848, as observed at the Royal Observatory of Greenwich, the whole force being reckoned 10,000, and increasing numbers denoting an increase of force.

Greenwich Mean Time, 1848.	Horizonl. Force Reading.	Greenwich Mean Time, 1848.	Horizonl. Force Reading.	Greenwich Mean Time, 1848.	Horizonl. Force Reading.	Greenwich Mean Time, 1848.	Horizonl. Force Reading.
h. m.		*h. m.*		*h. m.*		*h. m*	
Nov. 17. 0 . 0	1099	Nov. 17. 8 . 40	958 –	Nov. 17. 14 . 9	939	Nov. 17. 15 .17	979
0 . 40	1119	45	958 –	11	950	19	1002
1 . 2	1155	8 . 48	1001	13	968	21	992
10	1083	9 . 50	967	15	991	23	989
40	1121	52	1032	17	989	25	985
1 . 53	1133	9 . 55	968	19	985	27	983
2 . 25	1111	10 . 2	958	21	971	29	990
45	1153	12 . 0	910	23	977	31	981
2 . 57	1109	29	937	25	977	33	991
3 . 0	1114	34	956	27	967	35	1000
5	1131	39	956	29	927	37	1011
18	1103	45	965	31	882	39	1025
26	1140	12 . 49	953	33	829	41	1044
3 . 45	1068	13 . 9	829	35	800	43	1042
4 . 3	1065	14	902	37	792	45	1019
4 . 35	1096	19	862	39	775	47	1002
5 . 7	1111	24	871	41	779	49	982
12	1079	29	877	43	805	51	969
5 . 21	1115	34	906	45	810	53	950
6 . 0	1096	39	750	47	816	15 . 55	969
5	1101	41	781	49	827	16 . 3	928
10	1084	43	846	51	831	16 . 5	926
37	1046	45	710	53	850	18 . 0	931
48	1100	47	885	55	831	6	924
6 . 58	1046	49	948	57	819	15	938
7 . 23	1017	51	948	14 . 59	827	21	993
7 . 40	1043	53	985	15 . 1	842	24	928
8 . 5	1096	55	937	3	851	27	960
8	1028	57	943	5	865	18 . 48	1076
14	958 –	13 . 59	903	7	884	21 . 9	1075
24	958 –	14 . 1	892	9	898	Nov. 18. 0 . 0	1074
25	997	3	900	11	923		
30	972	5	916	13	937		
8 . 35	992	14 . 7	935	15 . 15	952		

These observations were accompanied by the following remarks:

"Between $7^h . 23^m$ and $7^h . 40^m$ small changes were constant. Between $8^h . 14^m$ and $8^h . 24^m$, and again between $8^h . 40^m$ and $8^h . 45^m$, the registering pencil of light went beyond the photographic paper. The same thing happened at $10^h . 2^m$, and the light remaining off the paper, the remaining observations, commencing at $12^h . 0^m$, were made independently of the self-registering apparatus."

"At $13^h . 45^m$ the force was at its lowest value, being at that time below its usual value by about one twenty-fifth part of the whole horizontal force. The force at $14^h . 40^m$ was but little above the minimum, after which it increased very gradually till about noon of November 18 it nearly reached the average value for the season and time of day."

TABLE III. Approximate scale divisions of the Vertical Force Magnet about the time of the Aurora Borealis of November 17, 1848, as observed at the Royal Observatory of Greenwich.

Increasing readings denote an increase of force. One scale division is equal to the fractional part 0,00067 of the whole vertical force.

Greenwich Mean Time, 1848.	Vertical Force Scale Readings.	Greenwich Mean Time, 1848.	Vertical Force Scale Readings.	Greenwich Mean Time, 1848.	Vertical Force Scale Readings.	Greenwich Mean Time, 1848.	Vertical Force Scale Readings.
h. m.	div.	h. m.	div.	h. m.	div.	h. m.	div.
Nov. 17. 0. 0	21,7	Nov. 17. 9.14	11,3	Nov. 17. 10.37	17,5	Nov. 17. 13.49	13,2
1.20	20,3	16	12,5	39	18,8	51	17,3
2.20	20,7	20	0,4	40	17,2	13.59	6,0
3.20	25,5	27	27,0 +	45	20,9	14.28	18,0
4.20	24,6	37	27,0 +	53	17,8	37	14,8
5.20	22,2	45	{ 10,0	10.56	20,3	14.49	18,5
6.18	26,5		18,8 }	11.13	16,3	15. 0	13,0
30	25,7	48	5,1	15	18,1	20	19,0
35	26,8	49	11,8	19	15,8	32	16,8
38	26,2	50	6,2	20	17,0	44	20,5
6.40	26,8	51	11,5	21	15,8	16. 0	17,0
8. 0	16,8	53	6,6	34	21,1	3	18,3
7	4,0	9.57	11,8	43	16,0	7	16,0
30	14,3	10. 0	10,1	11.50	20,7	15	18,1
35	9,2	5	13,8	12. 0	19,0	22	15,2
40	15,1	8	10,2	12	20,8	16.37	19,4
44	13,2	10	13,6	48	{ 14,7	17.23	15,2
47	16,8	12	11,0		16,1 }	34	16,7
50	13,5	15	13,0	12.55	17,0	17.35	15,0
53	15,3	17	9,5	13.13	9,3	20. 5	23,0
8.57	14,0	20	{ 8,5	30	16,8	22.20	19,3
9. 2	16,3		1 0,7 }	38	17,0	23.20	20,0
8	14,0	22	7,0	44	11,0	Nov. 18. 0. 0	21,5
9.11	16,0	10.33	18,4	13.45	15,0		

"Between $9^h. 27^m$ and $9^h. 37^m$, the reading was somewhat greater, the light being off the paper. Where two readings are included by a bracket, it is to be understood that the motion of the Magnet was so rapid that both took place nearly simultaneously. Between $17^h . 35^m$ and $20^h . 5^m$ the reading increased very gradually."

A comparison of Tables II. and III. shews that the disturbance of the Horizontal Force was much more considerable than that of the Vertical Force, and that both were generally below their average values.

CAMBRIDGE OBSERVATORY,
November 27, 1848.

J. CHALLIS.

XLVI. *On Clock Escapements.* *By* Edmund Beckett Denison, Esq., M.A., *of Trinity College, Cambridge.*

[Read *November* 27, 1848.]

In the year 1827 the present Astronomer Royal wrote a paper in the Cambridge *Phil. Trans.*, Vol. III. p. 105, "On the Disturbances of Pendulums and the Theory of Escapements," in which he investigated the effects produced on a free pendulum by connecting it with each of the three classes of escapements; and from the amount of the disturbance in each case, he inferred the relative merits of the escapements. He added: "The Theory of Escapements is by no means complete, but I hope it will be found that the principal points have been touched on, and that enough has been said to enable any one else to pursue the subject as far as he may wish."

I know of no work in which the subject has been pursued further; and therefore I propose to exhibit a few of the results which are to be obtained by following up Mr. Airy's calculations, and which I arrived at in investigating the merits of an improved *remontoir* or *gravity* escapement, invented and constructed by a friend of mine*, avoiding certain mechanical objections to which such escapements have hitherto been liable; and it will be seen, from the following remarks, that they may be made, by a particular arrangement of the parts, free from the mathematical objection which Mr. Airy says renders them almost as bad as the common recoil escapement. Mr. Bloxam has had some communication respecting his clock with the Astronomer Royal; and I shall be glad if he is thereby induced to complete his Theory of Escapements. In the mean time the following remarks may be of some use. I shall take the mathematical results, though not the practical conclusions, of Mr. Airy's paper for granted, as they are sufficient for my purpose; and his method of obtaining them may be seen either in the volume referred to, or in Pratt's Mechanics, into which the substance of his paper has been copied.

I shall presume that every one who is at all acquainted with clocks understands the construction of the Dead Escapement, as it has superseded all others in clocks that are expected (as the clock-makers say) to perform correctly; though it does not appear to be generally known from what the accuracy of its performance really arises. I shall follow Mr. Airy in assuming the maintaining force to be constant, although it is not quite so, since the inclination of the tooth of the escape-wheel to the face of the pallet is greater at the end of the impulse than at the beginning, by nearly the angle which the wheel moves through in one beat. Let β be the angle which the faces of the pallets make with their dead or circular part; then, since the tooth ought to be a tangent to the dead part, β will also be the inclination of the tooth to the face of the pallet at the beginning of the impulse; and we shall assume it to remain the same throughout the impulse.

Let Pg be the moving force of the clock-weight referred to the extremity of the escape-wheel's teeth: p the length of the pallet measured from the axis of suspension of the pendulum: M the mass of the pendulum, and l its length: θ its angle with the vertical. Then the equation of motion is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\left(\theta + \frac{Pp}{Ml}\tan\beta\right) = -\frac{g}{l}\left(\theta + \phi\right)$$

say, (neglecting the moment of inertia of the wheel, and putting θ for $\sin\theta$ as usual).

* J. M. Bloxam, Esq., of Lincoln's Inn, Barrister-at-Law.

Then if a be the extreme value of θ, γ the angle at which the impulse begins, and γ' on the other side of zero at which it ends, Mr. Airy shews that Δ, the increase of the time of an oscillation due to the escapement,

$$= \frac{\phi}{\pi a^2} \{\sqrt{a^2 - \gamma^2} - \sqrt{a^2 - \gamma'^2}\}$$

$$= \frac{\phi}{2\pi a^3} (\gamma' + \gamma)(\gamma' - \gamma) \text{ nearly,}$$

if γ and γ' are so small, that $\frac{\gamma^4}{a^4}$ may be neglected.

Mr. Airy remarks: "This is a quantity extremely minute; for γ and γ' are generally small, and $\gamma' - \gamma$ may be made almost as small as we please. It cannot, however, be made absolutely 0; for the wheel must be so adapted to the pallets, that when it is disengaged from one it may strike the other not on the acting surface, but a little above it; therefore γ' must be greater than γ; but the difference may be made so small that the effect on the clock's rate shall be almost imperceptible. This escapement therefore approaches nearly to absolute perfection; and in this respect theory and practice are in exact agreement."

Since Δ is only the increase in the time of one vibration, and there are 86,400 vibrations in a day, (assuming the clock to have a second's pendulum,) and a second a day is a large error, it is worth while to see what Δ really is. If Wg be the clock-weight, and h its fall in a day; then, since $p(\gamma + \gamma')\tan\beta$ is the thickness of the pallets, or the drop of a tooth in one beat,

$$\frac{Pp}{Ml}(\gamma + \gamma')\tan\beta, \text{ or } \phi(\gamma + \gamma') = \frac{Wh}{Ml\,86400};$$

and this quantity (which we may call F), will be the same for all clocks of the same kind, whatever β or $\gamma + \gamma'$ may be; and

$$\Delta = \frac{F}{2\pi a^3}(\gamma' - \gamma).$$

Now a weight of 2lbs. falling 9 inches a day will keep a well-made clock of this kind vibrating 2° on each side of zero. l is 39 inches, and M is usually about 14lbs. Therefore (allowing nothing for the friction of the train),

$$F = \frac{2 \times 9}{14 \times 39 \times 86400} = \frac{.033}{86400}$$

$$\text{and } 86400\,\Delta = \frac{.005}{.001}\,\frac{\gamma' - \gamma}{a} \text{ since } a = 2^0 = .035.$$

I understand from clockmakers that $\gamma' - \gamma$ can hardly be made less than 20′, and is seldom so little; $\therefore \frac{\gamma' - \gamma}{a} = \frac{1}{6}$, and $86400\,\Delta = .8$ of a second, nearly. This is the amount of Δ in a day But it is *not* the error of the clock, being only the difference between the rate of a free pendulum and one disturbed by this escapement. The error, or, as it is called, the "*rate*," of the clock, with the sign changed from what it would naturally have, is the variation of Δ, which depends on the variation of a and of F, according to the friction of the train and the pallets.

Differentiating Δ with regard both to a and F,

$$d\Delta = \frac{\gamma' - \gamma}{2\pi a^3} dF - \frac{3}{2} \frac{F(\gamma' - \gamma)}{\pi a^4} da,$$

or the "daily rate" $= -.8^s \left(\frac{dF}{F} - \frac{3da}{a}\right)$.

We see, therefore, that the real merit of this escapement arises from the two causes of error tending to counteract each other; for, though no exact relation can be determined between the changes of the arc and of the force, since they depend on the changes in the friction of different parts of the clock, yet it is easy to see that a will diminish when F does, under the influence of increasing friction as the clock gets dirty. It appears that $\frac{da}{a}$ is not generally so much as $\frac{1}{3}\frac{dF}{F}$*, and therefore the clock gains as the arc diminishes. Moreover, the circular error, which is never completely corrected by the pendulum-spring, I understand, tends to make the clock gain as the arc diminishes; since $d\Delta$ for the circular error $= \frac{a\,da}{8}$, as may be seen from any book on pendulums. I have in one instance seen the contrary effect take place, where a church-clock, soon after it was put up, spontaneously increased its arc by more than a degree, from the pallets polishing themselves more perfectly than had been done by the maker, and at the same time it gained considerably, as we see it ought to have done. The tendency to gain as the arc diminishes has led to the practice of making turret-clocks, which are liable to great changes both in the force and the arc, with a slight recoil in the place of the dead part of the pallets, as the effect of the recoil is to diminish the time as the arc increases.

The principle of nearly all the gravity or remontoir escapements is this: There are two small arms three or four inches long on each side of the pendulum suspended separately on an axis coincident with that of the pendulum and moving in the same plane with it: these arms carry a small weight at their lower ends, and also a detent to stop a tooth of the escape-wheel, and a pallet of some kind by means of which the arms are alternately raised by the wheel at every beat. The pendulum in ascending, at an angle γ from the vertical, impinges on one of the arms, unlocks the wheel and carries the arm with it as far as it swings; the arm then descends with the pendulum, not only to γ, but farther to an angle β, less than γ. The maintaining force of the arms therefore acts on the pendulum through $\gamma - \beta$, and the work which the clock has to do is raising the arms from β to γ. This is the way in which these escapements have been usually made, I suppose with the view of keeping the pendulum free during as much of its arc as possible; but we shall see that it is much better to make $\beta = -\gamma$, or one arm to be taken up by the pendulum just when the other is left; and as it is also more simple, I shall consider that case first.

In order to find the errors of such an escapement, let p be the length of the arms supposed to be without weight; Pg the weight they carry at their lower end; δ the angle which the arm in contact makes with the pendulum when it is vertical. Then the equation of motion will be

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \frac{\left\{\theta + \frac{Pp}{Ml}\sin(\delta + \theta)\right\}}{1 + \frac{Pp^2}{Ml^2}} = 0.$$

We may, in considering the error in the going of the clock, neglect the denominator $1 + \frac{Pp^2}{Ml^2}$, not only because it very nearly $= 1$, but because it only causes a permanent change in the effective

* It must be remembered that experiments of altering the clock-weight, to find what effect is produced on the arc, do not represent what takes place in the clock naturally; for when the clock is left to itself the arc varies probably more from the varying friction and state of the oil on the pallets than from the change of force in the train.

length of the pendulum, since one of the arms is, in this form of escapement, always acting on the pendulum. And expanding $\sin(\delta + \theta)$ we may put 1 for $\cos\theta$, and θ for $\sin\theta$ as before,

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\left\{\left(1 + \frac{Pp\cos\delta}{Ml}\right)\theta + \frac{Pp\sin\delta}{Ml}\right\} = 0,$$

which is of the form $\frac{\delta^2\theta}{dt^2} + \frac{g}{l}(m\theta + \phi) = 0$;

$\therefore$ if ϕ were $= 0$ the time would $= \pi\sqrt{\frac{l}{gm}}$, or for a second's pendulum $\frac{g}{l}$ must $= \frac{\pi^2}{m} = \pi'^2$ say, which is a very little less than π^2. The only part of the force which produces an effect involving the arc is $\pi'^2\phi$, and it is a constant force. Therefore we may apply to it Mr. Airy's expression for the increase of time due to such a force acting from a down to $-\gamma$; and we have

$$\Delta = -\frac{1}{\pi a^2}\int_a^{-\gamma}\frac{\phi\theta d\theta}{\sqrt{a^2-\theta^2}} = -\frac{\phi}{\pi a^2}\sqrt{a^2-\gamma^2};$$

and it is the same from $-\gamma$ to the other extremity of the arc; therefore for the whole oscillation

$$\Delta = -\frac{2\phi}{\pi a^2}\sqrt{a^2-\gamma^2}.$$

This is, in fact, Mr. Airy's result for a recoil escapement; and if the pallets of a recoil escapement were made of any regular form, so that we could separate the force into one part varying as the arc and the other part constant, it would be the same thing as a gravity escapement, only with much greater friction, and the important difference, that the force depends upon the train, whereas in a gravity escapement it is independent, and therefore uniform. Mr. Airy proceeds to remark, that "the differential coefficient of this quantity with respect to a is

$$\frac{2\phi}{\pi}\,\frac{a^2-2\gamma^2}{a^3\sqrt{a^2-\gamma^2}} = \frac{d\Delta}{da}.$$

Hence it appears that the vibrations are quicker "than they would be without the maintaining force; but if the arc be increased while the maintaining force remains the same, the vibrations are slower. If while the arc remains the same the force be increased, the vibrations are quicker."

But something else appears also: viz. the important fact, that if γ be made $= \frac{a}{\sqrt{2}}$, $\frac{d\Delta}{da} = 0$, provided the force remains the same, as it does in a gravity escapement. And luckily this is a perfectly practicable value for γ, though it is larger than a clockmaker would probable make it without knowing anything of this result; for if $a = 120'$, γ will $= .7 \times 120' = 84'$, and $a - \gamma$, or the space in which the unlocking has to take place $= 36'$, which with $p = 4$ or 5 inches will do very well for a clock which is liable to such small changes of arc as these are. Therefore a gravity escapement may be made, in which the error will be nothing for a small alteration of the arc; and in such an escapement there is no such variation in either force or friction as can cause any material change in the arc.

In a recoil escapement we should have to differentiate Δ with respect both to a and ϕ;

$$\therefore d\Delta = \frac{2\phi}{\pi a^2}\left\{\frac{a^2-2\gamma^2}{\sqrt{a^2-\gamma^2}}\,\frac{da}{a} - \sqrt{a^2-\gamma^2}\,\frac{d\phi}{\phi}\right\}.$$

This is always negative; for it cannot be + unless

$$\frac{\frac{d\phi}{\phi}}{\frac{da}{a}} \text{ is } < \frac{a^2 - 2\gamma^2}{a^2 - \gamma^2},$$

which is impossible, since ϕ always increases faster than a. Therefore the recoil escapement always gains as the arc increases, as is well known; and the cause of its inferiority to either of the others is evident.

But still we want to ascertain what the error of a gravity escapement with γ of the proper value will amount to, for some definite value of da, which the clock is not likely to exceed. Therefore we must find the value of ϕ.

Now the work done by the clock-weight is raising the weight P through

$$p\{\cos(\delta - \gamma) - \cos(\delta + \gamma)\} = 2p \sin \delta \sin \gamma, \text{ 86400 times a day.}$$

Then assuming W and p the same as before (though this clock evidently does not require the same maintaining power as the dead escapement with its large amount of friction, I believe not half as much),

$$2Pp \sin \delta \sin \gamma = \frac{Wp}{86400} = \frac{2 \times 9}{86400};$$

$$\therefore \frac{2\phi}{\pi} = \frac{2Pp \sin \delta}{Ml\pi} = \frac{4 \times 9}{\pi 14 \times 39 \times 86400\gamma} = \frac{.01}{86400\gamma};$$

$$\therefore 86400\Delta = -\frac{.012}{a^2\gamma}\{\sqrt{a^2 - \gamma^2}\} = -20 \text{ sec., if } \frac{a}{\gamma} \text{ be made } = 2, \text{ and } a = 2^0$$

$$= 0 \text{ when } \gamma = a.$$

as a clockmaker would probably make it, in ignorance of the fact that $\frac{a}{\gamma}$ should $= \sqrt{2}$;

$$\therefore 86400\,\delta\Delta = \frac{.012da}{\gamma a^3}\frac{a^2 - 2\gamma^2}{\sqrt{a^2 - \gamma^2}} = .012\frac{\frac{a^2}{\gamma^2} - 2}{\sqrt{\frac{a^2}{\gamma^2} - 1}}\frac{da}{a}$$

$$= .577 \text{ sec., for the last mentioned value of } \frac{a}{\gamma}, \text{ if } da = 5'.$$

This then is the daily error of a gravity escapement made, as we may say, at random, for an increase of the arc of 5′, remembering that we have taken ϕ twice as large as it need be.

But if $\frac{\gamma}{a}$ is made of the proper magnitude, so as to make $\frac{d\Delta}{da} = 0$, we must differentiate again, and put $a^2 = 2\gamma^2$, in order to find the actual error for a small increase of a: then we have

$$86400\frac{d^2\Delta}{da^2} = -\frac{.012}{a^3\gamma}\frac{2a}{\sqrt{a^2 - \gamma^2}};$$

$$\text{or the daily rate} = \frac{.012}{a^2}\frac{2ada}{\gamma^2\sqrt{\frac{a^2}{\gamma^2} - 1}}\frac{da}{a} = \frac{1}{48} \text{ of a second, if } da = 5'.$$

And we see that the clock will gain if a be either increased or diminished from $\gamma\sqrt{2}$. Therefore if the pendulum be adjusted when the clock is clean to vibrate $2'$ or $3'$ more than $\gamma\sqrt{2}$, the arc may diminish (and it will never spontaneously increase) as much as $5'$ or $6'$ with even a much less error than that above deduced, which it is to be remembered was already too large in consequence of our assuming the same maintaining force as in the dead escapement.

I have omitted in this calculation the effect of the impact of the pendulum against the arms, and the small friction at unlocking, as I found in the calculations which I made retaining them, that they only introduced a very small term of a lower order than ϕ.

There is another form of the gravity escapement, in which, instead of one arm being taken up just as the other is left, the pendulum is free for some space in the middle of its arc. This is evidently inferior to the other, for if the arc varies, the proportion between the time during which the pendulum has only its own moment of inertia, and that during which it has that of one of the arms also, will vary. And it will be seen that the inferiority is still greater from another cause.

For if we put γ for the angle at which each arm is met by the pendulum, and β for that to which it descends; then for the two portions of the arc in which the pendulum is acted on by the arms, we may integrate the same expression as before, only from γ to a, and down again to β;

$$\therefore \Delta = \frac{-\phi}{\pi a^2}\{\sqrt{a^2-\beta^2}+\sqrt{a^2-\gamma^2}\},$$

$$\text{and } \frac{d\Delta}{da} = \frac{\phi}{\pi a^3}\left\{\frac{a^2-2\beta^2}{\sqrt{a^2-\beta^2}}+\frac{a^2-2\gamma^2}{\sqrt{a^2-\gamma^2}}\right\}.$$

One value of β and γ that will make this $= 0$ is evidently $\beta = \pm\gamma = \dfrac{a}{\sqrt{2}}$; but if $\beta = \gamma$ there is no maintaining power; and if $\beta = -\gamma$ it becomes the former kind of clock. In order to find other values,

$$\text{Let } a^2-\beta^2 = x^2; \quad \therefore a^2 - 2\beta^2 = 2x^2 - a^2,$$

$$a^2-\gamma^2 = y^2; \quad \therefore a^2 - 2\gamma^2 = 2y^2 - a^2,$$

then we shall find that

$$\frac{d\Delta}{da} = 0, \text{ if } \frac{a^2}{2} = xy = \sqrt{a^2-\beta^2}\sqrt{a^2-\gamma^2}.$$

Since the value of $\dfrac{a}{\sqrt{2}}$ for β and γ is useless, let us take the highest value for β that will leave $a-\beta$ of a sufficient size to secure the unlocking always taking place; which can hardly be less than $30'$ with arms of moderate length: then β will be $90'$ and $\gamma = 78'$. And this leaves only $12'$ for the maintaining power to act through. An escapement of this sort is therefore barely practicable; and in it the weight of the arms, and consequently the errors of the clock, must be much larger than in one where the action takes place through $84'$ on each side of zero. This kind of escapement, however, would do for such a clock in which the force acts on the bob of the pendulum for a short distance at each extremity of the arc—the worst possible place, unless the arc through which it acts satisfies the above condition. However, the object of this paper was to shew that, mathematically speaking, gravity escapements *may* be made very superior to the dead escapement with its large amount of friction and variation of arc, and to remove the cloud which has hitherto lain over them in consequence of it being supposed that whatever mechanical improvements might be made in them, they must remain liable to an insuperable mathematical objection.

E. B. DENISON.

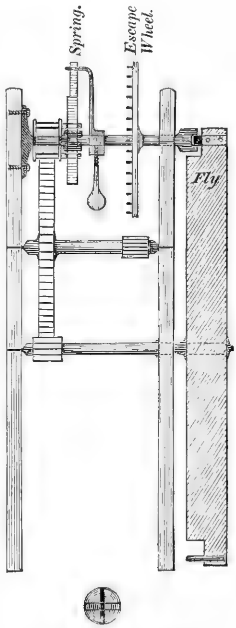
XLVII. (SUPPLEMENT.) *On Turret-Clock Remontoirs.* *By* E. B. DENISON, M.A., *of Trinity College, Cambridge.*

[Read *February* 26, 1849.]

I HAVE given above a general description of a remontoir escapement, and shewn its advantages when properly made. But a remontoir apparatus may be introduced below the escape-wheel of a common escapement, and will have the same effect as a remontoir escapement, except that it will not remove the variable friction of the pallets, and will generally introduce some friction of its own. For astronomical clocks, probably a remontoir escapement is the best construction. But there is another class of clocks on which some attention has at last begun to be bestowed in this country, and which, from the great length and weight that may be given to their pendulums, are capable, when properly made, of excelling the performance of most astronomical clocks: I mean turret-clocks. And these clocks require a remontoir more than all others, on account of the great inequality in the force of the train, arising from the varying friction of the very heavy machinery, and the occasional exposure of the oil to a freezing temperature, and the action of the wind on the hands.

Now any remontoir escapement, to satisfy the condition which I have shewn ought to be satisfied by them, will require great accuracy in its construction, and will be too expensive to have any chance of being generally adopted. Moreover, there are two other conditions which a turret-clock must satisfy, in order to be of any use as a public regulator of other clocks; viz. that of striking the first blow at exactly the proper second, and that of enabling people to distinguish every twentieth or thirtieth second by a quick and visible motion of the minute-hand only at those intervals. These conditions were laid down by the Astronomer-Royal for the Royal Exchange Clock, and are also proposed by him for the great Clock for the Houses of Parliament. And these conditions, especially the second of them, can only be satisfied by introducing a remontoir into the train somewhere between the dial-work and the escapement. In the Exchange Clock a small weight is raised by a wheel with internal teeth at every twentieth second: in some French turret-clocks the weight is raised in a somewhat similar manner by two bevelled wheels: in a clock put up in Edinburgh in the last century, mentioned by Reid in his book on Clock-making, there was an endless chain remontoir (which has also been attempted again in France); but it was removed on account of the rapid wearing both of the chain and the letting-off pins. But for this, and the variable friction of such a chain, that kind of remontoir is probably the most tempting, as it is the most simple, of the gravity remontoirs. Both the other constructions are complicated and expensive, and have a good deal of friction of their own; and though I think a different kind of gravity remontoir may still be made, more simple and quite as effective (which however I shall not stay to describe), I am inclined to propose a spring remontoir as superior to any gravity one, on account of the greater facility of its construction, and the unusual circumstance of its being possible totally to exclude friction in its application; and I may also mention, as an incidental advantage, that it possesses a sort of natural compensation, the spring being stronger in cold weather, when the oil on the pallets is less fluid, and therefore a greater maintaining force is required. I find indeed that a spring remontoir is not new, having been tried in France, but without success, from evident defects in its construction, the clock sometimes failing to wind it up, which of course need no more happen with a spring than a gravity remontoir. It seems also to have been applied to a peculiar kind of escapement; which was trying two experiments at once,—always an unscientific proceeding.

The obvious mode of applying a spring remontoir is to make the pinion of the escape-wheel ride upon the axis instead of being fixed to it, and to connect the pinion and the wheel by a spiral spring. Then if the pinion is turned (say) a quarter round by the train at intervals, the wheel will be driven through a quarter of a revolution by the force of the spring only. This is the plan I find described for the French spring remontoir, and a similar one has been proposed by Mr. Airy, only to be wound up at every beat by means of a double escape-wheel and pallets, and the principle of it was applied to a chronometer many years ago. But if nothing more than this is done, the escape-wheel axis will have to turn within the pinion, as in a socket with considerable pressure and friction upon it, which will probably be worse than the ordinary friction of the train. The method I propose therefore is, to make the pinion (a brass lantern-pinion, having the inner end of the spiral spring attached to it) ride upon a steel pin fixed to the frame in the same line as the axis of the escape-wheel, and having in its end (or rather in a piece of brass screwed on to its end) the pivot hole for the escape-wheel axis. The outer end of the spring is to take hold of what I believe is called a *dog* (the shape of which will be best described by the drawing), which screws on to the escape-wheel axis (the screw also acting as a connterpoise), so that the tension of the spring can be adjusted to make the pendulum swing as far as is required. It is evident that the wheel will thus be driven by the spring without any friction.



End view of Cylinder.

The mode of letting off the train at intervals, adopted in the Exchange Clock and the above-mentioned Clock by Reid, is by fixing two or more sets of long teeth on spikes in two or more planes on the broad rim of a wheel on the same axis as the wheel which drives the escape-wheel pinion; and notches are cut nearly half through the escape-wheel axis over each set of spikes, which will let a spike pass through whenever the corresponding notch is in its lowest position; and the driving wheel is then stopped by one of the other set of spikes coming against the axis in a place where the corresponding notch is not yet in a position to let that spike pass. The objection to this is, that the spikes strike the axis with considerable force, and also press on it pretty heavily when at rest, which causes additional friction and requires a stronger maintaining power than would otherwise be necessary. The blow against the axis it is already proposed to diminish by a fly, to restrain the velocity of the train when it is let off; and a fly is now used in the French remontoirs; where however it is much less needed, for they are let off by pins raising a lever just like a common striking part; and it does not signify how hard the lever is struck. But this plan also is objectionable because of the friction and loss of power in the escape-wheel in raising such a lever, which is much more than would be due to the pressure of the same lever, if only exerting a dead pressure on the axis until it slips through a notch.

I propose to use a fly, but more for the sake of diminishing the pressure on the escape-wheel axis than of diminishing the velocity of the train; which is immaterial, except so far as it effects the escape-wheel. The two letting-off pins are to be one at each end of the fly; and if the radius of the fly is equal to the diameter of the driving wheel, and the fly makes twelve revolutions for one of the driving wheel, the fly-pins will only exert $\frac{1}{24}$th of the pressure on the axis that the spikes on the driving wheel exert. The fly is very light and made of thin brass, and is of itself a spring; and so its axis will not be stopped with a sudden shock, and the impact of the end of the fly on the escape-wheel axis may be made inconsiderable. This axis must be prolonged beyond the frame, to allow a fly of larger radius than the driving wheel to be used, and must end in a cylinder about half an inch thick. If the remontoir is to be let off every thirty seconds, which is a better interval for observation than twenty, the projecting cylinder may have two notches cut in it as before described, if the escape-wheel revolves in a minute. But it is generally made to revolve in two minutes, in order to save a wheel in the train; and in that case the letting off may be done better than with a four-armed fly, by making two notches across the end of the cylinder, at right angles to each other, one broad, and the other narrow and deeper, so that a broad pin will pass through one of the notches only, and a narrow and long pin through the other only. These pins are of course to be parallel to the axis of the fly; and the fly pinion must have half the number of leaves that the escape-wheel pinion has, whatever may be the number of teeth of the driving wheel. A Church-clock is now making on this plan. I have added a drawing of the material parts, placed in the way most convenient for shewing their action.

E. B. DENISON.

XLVIII. *On the Formation of the Central Spot of Newton's Rings beyond the Critical Angle.* *By* G. G. STOKES, M.A., *Fellow of Pembroke College, Cambridge.*

[Read *December* 11, 1848.]

WHEN Newton's Rings are formed between the under surface of a prism and the upper surface of a lens, or of another prism with a slightly convex face, there is no difficulty in increasing the angle of incidence on the under surface of the first prism till it exceeds the critical angle. On viewing the rings formed in this manner, it is found that they disappear on passing the critical angle, but that the central black spot remains. The most obvious way of accounting for the formation of the spot under these circumstances is, perhaps, to suppose that the forces which the material particles exert on the ether extend to a small, but sensible distance from the surface of a refracting medium; so that in the case under consideration the two pieces of glass are, in the immediate neighbourhood of the point of contact, as good as a single uninterrupted medium, and therefore no reflection takes place at the surfaces. This mode of explanation is however liable to one serious objection. So long as the angle of incidence falls short of the critical angle, the central spot is perfectly explained, along with the rest of the system of which it forms a part, by ordinary reflection and refraction. As the angle of incidence gradually increases, passing through the critical angle, the appearance of the central spot changes gradually, and but slightly. To account then for the existence of this spot by ordinary reflection and refraction so long as the angle of incidence falls short of the critical angle, but by the finite extent of the sphere of action of the molecular forces when the angle of incidence exceeds the critical angle, would be to give a discontinuous explation to a continuous phenomenon. If we adopt the latter mode of explanation in the one case we must adopt it in the other, and thus separate the theory of the central spot from that of the rings, which to all appearance belong to the same system; although the admitted theory of the rings fully accounts likewise for the existence of the spot, and not only for its existence, but also for some remarkable modifications which it undergoes in certain circumstances*.

Accordingly the existence of the central spot beyond the critical angle has been attributed by Dr. Lloyd, without hesitation, to the disturbance in the second medium which takes the place of that which, when the angle of incidence is less than the critical angle, constitutes the refracted light†. The expression for the intensity of the light, whether reflected or transmitted, has not however been hitherto given, so far as I am aware. The object of the present paper is to supply this deficiency.

In explaining on dynamical principles the total internal reflection of light, mathematicians have been led to an expression for the disturbance in the second medium involving an exponential, which contains in its index the perpendicular distance of the point considered from the surface. It follows from this expression that the disturbance is insensible at the distance of a small multiple of the length of a wave from the surface. This circumstance is all that need be attended to, so far as the refracted light is concerned, in explaining total internal reflection; but in considering the theory of the central spot in Newton's Rings, it is precisely the superficial disturbance just mentioned that must be taken into account. In the present paper I have not adopted any special dynamical theory: I have preferred deducing my results from Fresnel's formulæ for the intensities of reflected and re-

* I allude especially to the phenomena described by Mr. Airy in a paper printed in the fourth Volume of the *Cambridge Philosophical Transactions*, p. 409.

† Report on the present state of Physical Optics. *Reports of the British Association.* Vol. III. p. 310.

fracted polarized light, which in the case considered became imaginary, interpreting these imaginary expressions, as has been done by Professor O'Brien*, in the way in which general dynamical considerations show that they ought to be interpreted.

By means of these expressions, it is easy to calculate the intensity of the central spot. I have only considered the case in which the first and third media are of the same nature: the more general case does not seem to be of any particular interest. Some conclusions follow from the expression for the intensity, relative to a slight tinge of colour about the edge of the spot, and to a difference in the size of the spot ascending as it is seen by light polarized in, or by light polarized perpendicularly to the plane of incidence, which agree with experiment.

1. Let a plane wave of light be incident, either externally or internally, on the surface of an ordinary refracting medium, suppose glass. Regard the surface as plane, and take it for the plane xy; and refer the media to the rectangular axes of x, y, z, the positive part of the latter being situated in the second medium, or that into which the refraction takes place. Let l, m, n be the cosines of the angles at which the normal to the incident wave, measured in the direction of propagation, is inclined to the axes; so that $m = 0$ if we take, as we are at liberty to do, the axis of y parallel to the trace of the incident wave on the reflecting surface. Let $V, V_{,}, V'$ denote the incident, reflected, and refracted vibrations, estimated either by displacements or by velocities, it does not signify which; and let $a, a_{,}, a'$ denote the coefficients of vibration. Then we have the following possible system of vibrations:

$$\left.\begin{aligned} V &= a \cos \frac{2\pi}{\lambda}(vt - lx - nz), \\ V_{,} &= a_{,} \cos \frac{2\pi}{\lambda}(vt - lx + nz), \\ V' &= a' \cos \frac{2\pi}{\lambda'}(v't - l'x - n'z), \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (A).$$

In these expressions v, v' are the velocities of propagation, and λ, λ' the lengths of wave, in the first and second media; so that v, v', and the velocity of propagation in vacuum, are proportional to λ, λ', and the length of wave in vacuum: l is the sine, and n the cosine of the angle of incidence, l' the sine, and n' the cosine of the angle of refraction, these quantities being connected by the equations

$$\frac{l}{v} = \frac{l'}{v'}, \quad n = \sqrt{1 - l^2}, \quad n' = \sqrt{1 - l'^2}. \quad \ldots\ldots\ldots\ldots (1).$$

2. The system of vibrations (A) is supposed to satisfy certain linear differential equations of motion belonging to the two media, and likewise certain linear equations of condition at the surface of separation, for which $z = 0$. These equations lead to certain relations between a, $a_{,}$, and a', by virtue of which the ratios of $a_{,}$ and a' to a are certain functions of l, v, and v', and it might be also of λ. The equations, being satisfied identically, will continue to be satisfied when l' becomes greater than 1, and consequently n' imaginary, which may happen, provided $v' > v$; but the interpretation before given to the equations (A) and (1) fails.

When n' becomes imaginary, and equal to $\nu'\sqrt{-1}$, ν' being equal to $\sqrt{l'^2 - 1}$, z, instead of appearing under a circular function in the third of equations (A), appears in one of the exponentials $\epsilon^{\pm k'\nu' z}$, k' being equal to $\frac{2\pi}{\lambda'}$. By changing the sign of $\sqrt{-1}$ we should get a second system of equations (A), satisfying, like the first system, all the equations of the problem; and we should

* *Cambridge Philosophical Transactions*, Vol. VIII. p. 20

get two new systems by writing $vt + \frac{\lambda}{4}$ for vt. By combining these four systems by addition and subtraction, which is allowable on account of the linearity of our equations, we should be able to get rid of the imaginary quantities, and likewise of the exponential $\epsilon^{+k'\nu'z}$, which does not correspond to the problem, inasmuch as it relates to a disturbance which increases indefinitely in going from the surface of separation into the second medium, and which could only be produced by a disturbing cause existing in the second medium, whereas none such is supposed to exist.

3. The analytical process will be a good deal simplified by replacing the expressions (A) by the following symbolical expressions for the disturbance, where k is put for $\frac{2\pi}{\lambda}$, so that $kv = k'v'$;

$$\left.\begin{aligned} V &= a\epsilon^{k(vt-lx-nz)\sqrt{-1}}, \\ V_{,} &= a_{,}\epsilon^{k(vt-lx+nz)\sqrt{-1}}, \\ V' &= a'\epsilon^{k'(v't-l'x-n'z)\sqrt{-1}}, \end{aligned}\right\} \quad \ldots\ldots\ldots (B).$$

In these expressions, if each exponential of the form $\epsilon^{P\sqrt{-1}}$ be replaced by $\cos P + \sqrt{-1}\sin P$, the real part of the expressions will agree with (A), and therefore will satisfy the equations of the problem. The coefficients of $\sqrt{-1}$ in the imaginary part will be derived from the real part by writing $t + \frac{\lambda}{4v}$ for t, and therefore will form a system satisfying the same equations, since the form of these equations is supposed in no way to depend on the origin of the time; and since the equations are linear they will be satisfied by the complete expressions (B).

Suppose now l' to become greater than 1, so that n' becomes $\pm\nu'\sqrt{-1}$. Whichever sign we take, the real and imaginary parts of the expressions (B), which must separately satisfy the equations of motion and the equations of condition, will represent two possible systems of waves; but the upper sign does not correspond to the problem, for the reason already mentioned, so that we must use the lower sign. At the same time that n' becomes $\nu'\sqrt{-1}$, let $a, a_{,}, a'$ become

$$\rho\epsilon^{\theta\sqrt{-1}}, \quad \rho_{,}\epsilon^{\theta_{,}\sqrt{-1}}, \quad \rho'\epsilon^{\theta'\sqrt{-1}}, \text{ respectively:}$$

then we have the symbolical system

$$\left.\begin{aligned} V &= \rho\epsilon^{-\theta\sqrt{-1}}.\epsilon^{k(vt-lx-nz)\sqrt{-1}}, \\ V_{,} &= \rho_{,}\epsilon^{-\theta_{,}\sqrt{-1}}.\epsilon^{k(vt-lx+nz)\sqrt{-1}}, \\ V' &= \rho'\epsilon^{-\theta'\sqrt{-1}}.\epsilon^{-k'\nu'z}.\epsilon^{k'(v't-l'x)\sqrt{-1}}, \end{aligned}\right\} \ldots (C),$$

of which the real part

$$\left.\begin{aligned} V &= \rho\cos\{k(vt - lx - nz) - \theta\} \\ V_{,} &= \rho_{,}\cos\{k(vt - lx + nz) - \theta_{,}\}, \\ V' &= \rho'\,\epsilon^{-k'\nu'z}\cos\{k'(v't - l'x) - \theta'\}, \end{aligned}\right\} \ldots (D).$$

forms the system required.

As I shall frequently have occasion to allude to a disturbance of the kind expressed by the last of equations (D), it will be convenient to have a name for it, and I shall accordingly call it a *superficial undulation*.

4. The interpretation of our results is not yet complete, inasmuch as it remains to consider what is meant by V'. When the vibrations are perpendicular to the plane of incidence there is no

difficulty. In this case, whether the angle of incidence be greater or less than the critical angle, V' denotes a displacement, or else a velocity, perpendicular to the plane of incidence. When the vibrations are in the plane of incidence, and the angle of incidence is less than the critical angle, V' denotes a displacement or velocity in the direction of a line lying in the plane xz, and inclined at angles $\pi - i'$, $-\left(\frac{\pi}{2} - i'\right)$ to the axes of x, z, i' being the angle of refraction. But when the angle of incidence exceeds the critical angle there is no such thing as an angle of refraction, and the preceding interpretation fails. Instead therefore of considering the whole vibration V', consider its resolved parts V_x', V_z' in the direction of the axes of x, z. Then when the angle of incidence is less than the critical angle, we have

$$V_x' = -n' V' = -\cos i' \,.\, V'; \quad V_z' = l' V' = \sin i' \,.\, V',$$

V' being given by (A), and being reckoned positive in that direction which makes an acute angle with the positive part of the axis of z. When the angle of incidence exceeds the critical angle, we must first replace the coefficient of V' in V_x', namely $-n'$, by $\nu' \epsilon^{\frac{\pi}{2}\sqrt{-1}}$, and then, retaining ν' for the coefficient, add $\frac{\pi}{2}$ to the phase, according to what was explained in the preceding article.

Hence, when the vibrations take place in the plane of incidence, and the angle of incidence exceeds the critical angle, V' in (D) must be interpreted to mean an expression from which the vibrations in the directions of x, z may be obtained by multiplying by ν', l', respectively, and increasing the phase in the former case by $\frac{\pi}{2}$. Consequently, so far as depends on the third of equations (D), the particles of ether in the second medium describe small ellipses lying in the plane of incidence, the semi-axes of the ellipses being in the directions of x, z, and being proportional to ν', l', and the direction of revolution being the same as that in which the incident ray would have to revolve in order to diminish the angle of incidence.

Although the elliptic paths of the particles lie in the plane of incidence, that does not prevent the superficial vibration just considered from being of the nature of transversal vibrations. For it is easy to see that the equation

$$\frac{dV_x'}{dx} + \frac{dV_z'}{dz} = 0$$

is satisfied; and this equation expresses the condition that there is no change of density, which is the distinguishing characteristic of transversal vibrations.

5. When the vibrations of the incident light take place in the plane of incidence, it appears from investigation that the equations of condition relative to the surface of separation of the two media cannot be satisfied by means of a system of incident, reflected, and refracted waves, in which the vibrations are transversal. If the media be capable of transmitting normal vibrations with velocities comparable with those of transversal vibrations, there will be produced, in addition to the waves already mentioned, a series of reflected and a series of refracted waves in which the vibrations are normal, provided the angle of incidence be less than either of the two critical angles corresponding to the reflected and refracted normal vibrations respectively. It has been shewn however by Green, in a most satisfactory manner, that it is necessary to suppose the velocities of propagation of normal vibrations to be incomparably greater than those of transversal vibrations, which comes to the same thing as regarding the ether as sensibly incompressible; so that the two critical angles mentioned above must be considered evanescent*. Consequently the reflected and

* *Cambridge Philosophical Transactions*, Vol. VII. p. 2.

refracted normal waves are replaced by undulations of the kind which I have called superficial. Now the existence of these superficial undulations does not affect the interpretation which has been given to the expressions (A) when the angle of incidence becomes greater than the critical angle corresponding to the refracted transversal wave; in fact, so far as regards that interpretation, it is immaterial whether the expressions (A) satisfy the linear equations of motion and condition alone, or in conjunction with other terms referring to the normal waves, or rather to the superficial undulations which are their representatives. The expressions (D) however will not represent the whole of the disturbance in the two media, but only that part of it which relates to the transversal waves, and to the superficial undulation which is the representative of the refracted transversal wave.

6. Suppose now that in the expressions (A) n becomes imaginary, n' remaining real, or that n and n' both become imaginary. The former case occurs in the theory of Newton's Rings when the angle of incidence on the surface of the second medium becomes greater than the critical angle, and we are considering the superficial undulation incident on the third medium: the latter case would occur if the third medium as well as the second were of lower refractive power than the first, and the angle of incidence on the surface of the second were greater than either of the critical angles corresponding to refraction out of the first into the second, or out of the first into the third. Consider the case in which n becomes imaginary, n' remaining real; and let $\sqrt{l^2-1}=\nu$. Then it may be shewn as before that we must put $-\nu\sqrt{-1}$, and not $\nu\sqrt{-1}$, for n; and using ρ, θ in the same sense as before, we get the symbolical system,

$$\left.\begin{aligned} V &= \rho\epsilon^{-\theta\sqrt{-1}}.\,\epsilon^{-k\nu z}.\,\epsilon^{k(vt-lx)\sqrt{-1}},\\ V_{,} &= \rho_{,}\epsilon^{-\theta_{,}\sqrt{-1}}.\,\epsilon^{k\nu z}.\,\epsilon^{k(vt-lx)\sqrt{-1}},\\ V' &= \rho'\epsilon^{-\theta'\sqrt{-1}}.\,\epsilon^{k'(v't-l'x-n'z)\sqrt{-1}}, \end{aligned}\right\} \ldots\ldots (E),$$

to which corresponds the real system

$$\left.\begin{aligned} V &= \rho\epsilon^{-k\nu z}\cos\{k(vt-lx)-\theta\},\\ V_{,} &= \rho_{,}\epsilon^{k\nu z}\cos\{k(vt-lx)-\theta_{,}\},\\ V' &= \rho'\cos\{k'(v't-l'x-n'z)-\theta'\}, \end{aligned}\right\} \ldots\ldots (F).$$

When the vibrations take place in the plane of incidence, V and $V_{,}$ in these expressions must be interpreted in the same way as before. As far as regards the incident and reflected superficial undulations, the particles of ether in the first medium will describe small ellipses lying in the plane of incidence. The ellipses will be similar and similarly situated in the two cases; but the direction of revolution will be in the case of the incident undulation the same as that in which the refracted ray would have to turn in order to diminish the angle of refraction, whereas in the reflected undulation it will be the opposite.

It is unnecessary to write down the formulæ which apply to the case in which n and n' both become imaginary.

7. If we choose to employ real expressions, such as (D) and (F), we have this general rule. When any one of the undulations, incident, reflected, or refracted, becomes superficial, remove z from under the circular function, and insert the exponential $\epsilon^{-k\nu z}$, $\epsilon^{k\nu z}$, or $\epsilon^{-k'\nu' z}$, according as the incident, reflected, or refracted undulation is considered. At the same time put the coefficients, which become imaginary, under the form ρ $(\cos\theta \pm \sqrt{-1}\sin\theta)$, the double sign corresponding to the substitution of $\pm\nu\sqrt{-1}$, or $\pm\nu'\sqrt{-1}$ for n or n', retain the modulus ρ for coefficient, and subtract θ from the phase.

It will however be far more convenient to employ symbolical expressions such as (B). These expressions will remain applicable without any change when n or n' becomes imaginary: it will only be necessary to observe to take $\pm \nu \sqrt{-1}$, or $\pm \nu' \sqrt{-1}$ with the negative sign. If we had chosen to employ the expressions (B) with the opposite sign in the index, which would have done equally well, it would then have been necessary to take the positive sign.

8. We are now prepared to enter on the regular calculation of the intensity of the central spot; but before doing so it will be proper to consider how far we are justified in omitting the consideration of the superficial undulations which, when the vibrations are in the plane of incidence, are the representatives of normal vibrations. These undulations may conveniently be called *normal superficial undulations*, to distinguish them from the superficial undulations expressed by the third of equations (D), or the first and second of equations (F), which may be called *transversal*. The former name however might, without warning, be calculated to carry a false impression; for the undulations spoken of are not propagated by way of condensation and rarefaction; the disturbance is in fact precisely the same as that which exists near the surface of deep water when a series of oscillatory waves is propagated along it, although the cause of the propagation is extremely different in the two cases.

Now in the ordinary theory of Newton's Rings, no account is taken of the normal superficial undulations which may be supposed to exist; and the result so obtained from theory agrees very well with observation. When the angle of incidence passes through the critical angle, although a material change takes place in the nature of the refracted transversal undulation, no such change takes place in the case of the normal superficial undulations: the critical angle is in fact nothing particular as regards these undulations. Consequently, we should expect the result obtained from theory when the normal superficial undulations are left out of consideration to agree as well with experiment beyond the critical angle as within it.

9. It is however one thing to show why we are justified in expecting a near accordance between the simplified theory and experiment, beyond the critical angle, in consequence of the observed accordance within that angle; it is another thing to show why a near accordance ought to be expected both in the one case and in the other. The following considerations will show that the effect of the normal superficial undulations on the observed phenomena is most probably very slight.

At the point of contact of the first and third media, the reflection and refraction will take place as if the second medium were removed, so that the first and third were in contact throughout. Now Fresnel's expressions satisfy the condition of giving the same intensity for the reflected and refracted light whether we suppose the refraction to take place directly out of the first medium into the third, or take into account the infinite number of reflections which take place when the second medium is interposed, and then suppose the thickness of the interposed medium to vanish. Consequently the expression we shall obtain for the intensity by neglecting the normal superficial undulations will be strictly correct for the point of contact, Fresnel's expressions being supposed correct, and of course will be sensibly correct for some distance round that point. Again, the expression for the refracted normal superficial undulation will contain in the index of the exponential $-k\,l\,z$, in place of $-k\sqrt{l^2 - \frac{v^2}{v'^2}}\,z$, which occurs in the expression for the refracted transversal superficial undulation; and therefore the former kind of undulation will decrease much more rapidly, in receding from the surface, than the latter, so that the effect of the former will be insensible at a distance from the point of contact at which the effect of the latter is still important. If we combine these two considerations, we can hardly suppose the effect of the normal superficial undulations at intermediate points to be of any material importance.

10. The phenomenon of Newton's Rings is the only one in which I see at present any chance of rendering these undulations sensible in experiment: for the only way in which I can conceive them to be rendered sensible is, by their again producing transversal vibrations; and in consequence of the rapid diminution of the disturbance on receding from the surface, this can only happen when

there exists a second reflecting surface in close proximity with the first. It is not my intention to pursue the subject further at present, but merely to do for angles of incidence greater than the critical angle what has long ago been done for smaller angles, in which case light is refracted in the ordinary way. Before quitting the subject however I would observe that, for the reasons already mentioned, the near accordance of observation with the expression for the intensity obtained when the normal superficial undulations are not taken into consideration cannot be regarded as any valid argument against the existence of such undulations.

11. Let Newton's Rings be formed between a prism and a lens, or a second prism, of the same kind of glass. Suppose the incident light polarized, either in the plane of incidence, or in a plane perpendicular to the plane of incidence. Let the coefficient of vibration in the incident light be taken for unity; and, according to the notation employed in Airy's *Tract*, let the coefficient be multiplied by b for reflection and by c for refraction when light passes from glass into air, and by e for reflection and f for refraction when light passes from air into glass. In the case contemplated b, c, e, f become imaginary, but that will be taken into account further on. Then the incident vibration will be represented symbolically by

$$\epsilon^{k(vt-lx-nz)\sqrt{-1}},$$

according to the notation already employed; and the reflected and refracted vibrations will be represented by

$$b\epsilon^{k(vt-lx+nz)\sqrt{-1}},$$

$$c\epsilon^{-k'v'z}\,.\,\epsilon^{k'(v't-l'x)\sqrt{-1}}.$$

Consider a point at which the distance of the pieces of glass is D; and, as in the usual investigation, regard the plate of air about that point as bounded by parallel planes. When the superficial undulation represented by the last of the preceding expressions is incident on the second surface, the coefficient of vibration will become cq, q being put for shortness in place of $\epsilon^{-k'v'D}$; and the reflected and refracted vibrations will be represented by

$$cqe\epsilon^{k'v'z}\,.\,\epsilon^{k'(v't-l'x)\sqrt{-1}},$$

$$cqf\epsilon^{k(vt-lx-nz)\sqrt{-1}},$$

z being now measured from the lower surface. It is evident that each time that the undulation passes from one surface to the other the coefficient of vibration will be multiplied by q, while the phase will remain the same. Taking account of the infinite series of reflections, we get for the symbolical expression for the reflected vibration

$$\{b + cefq^2(1 + e^2q^2 + e^4q^4 + \ldots)\}\,\epsilon^{k(vt-lx+nz)\sqrt{-1}}.$$

Summing the geometric series, we get for the coefficient of the exponential

$$b + \frac{cefq^2}{1 - e^2q^2}.$$

Now it follows from Fresnel's expressions that

$$b = -e, \quad cf = 1 - e^2 *.$$

These substitutions being made in the coefficient, we get for the symbolical expression for the reflected vibration

$$\frac{(1-q^2)\,b}{1-q^2b^2}\,\epsilon^{k(vt-lx+nz)\sqrt{-1}} \ldots\ldots\ldots\ldots\ldots (G).$$

* I have proved these equations in a very simple manner, without any reference to Fresnel's formulæ, in a paper which will appear in the next number of the *Cambridge and Dublin Mathematical Journal.*

Let the coefficient, which is imaginary, be put under the form $\rho(\cos\psi + \sqrt{-1}\sin\psi)$; then the real part of the whole expression, namely

$$\rho\cos\{k(vt - lx + nz) + \psi\},$$

will represent the vibration in the reflected light, so that ρ^2 is the intensity, and ψ the acceleration of phase.

12. Let i be the angle of incidence on the first surface of the plate of air, μ the refractive index of glass; and let λ now denote the length of wave in air. Then in the expression for q

$$k'v' = \frac{2\pi}{\lambda}\sqrt{\mu^2\sin^2 i - 1}.$$

In the expression for b we must, according to Art. 2, take the imaginary expression for $\cos i'$ with the negative sign. We thus get for light polarized in the plane of incidence (Airy's *Tract*, p. 362, 2nd edition*), changing the sign of $\sqrt{-1}$,

$$b = \cos 2\theta + \sqrt{-1}\sin 2\theta,$$

where

$$\tan\theta = \frac{\sqrt{\mu^2\sin^2 i - 1}}{\mu\cos i} \;\ldots\ldots\ldots\ldots\; (2).$$

Putting C for the coefficient in the expression (G), we have

$$C = \frac{1-q^2}{b^{-1} - q^2 b} = \frac{1-q^2}{(1-q^2)\cos 2\theta - \sqrt{-1}(1+q^2)\sin 2\theta}$$

$$= \frac{(1-q^2)\{(1-q^2)\cos 2\theta + \sqrt{-1}(1+q^2)\sin 2\theta\}}{(1-q^2)^2 + 4q^2\sin^2 2\theta};$$

whence

$$\tan\psi = \frac{1+q^2}{1-q^2}\tan 2\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\; (3),$$

$$\rho^2 = \frac{(1-q^2)^2}{(1-q^2)^2 + 4q^2\sin^2 2\theta} \;\ldots\ldots\ldots\ldots\; (4),$$

where

$$q = \epsilon^{-\frac{2\pi D}{\lambda}\sqrt{\mu^2\sin^2 i - 1}} \;\ldots\ldots\ldots\ldots\; (5).$$

If we take ρ positive, as it will be convenient to do, we must take ψ so that $\cos\psi$ and $\cos 2\theta$ may have the same sign. Hence from (3) $\sin\psi$ must be positive, since $\sin 2\theta$ is positive, inasmuch as θ lies between 0 and $\frac{\pi}{2}$. Hence, of the two angles lying between $-\pi$ and π which satisfy (2), we must take that which lies between 0 and π.

For light polarized perpendicularly to the plane of incidence, we have merely to substitute ϕ for θ in the equations (3) and (4), where

$$\tan\phi = \frac{\mu\sqrt{\mu^2\sin^2 i - 1}}{\cos i} \;\ldots\ldots\ldots\ldots\; (6).$$

The value of q does not depend on the nature of the polarization.

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* Mr. Airy speaks of "vibrations perpendicular to the plane of incidence," and "vibrations parallel to the plane of incidence," adopting the theory of Fresnel; but there is nothing in this paper which requires us to enter into the question whether the vibrations in plane polarized light are in or perpendicular to the plane of polarization.

13. For the transmitted light we have an expression similar to (G), with $-nz$ in place of nz, and a different coefficient $C_{,}$, where

$$C_{,} = cqf(1 + e^2q^2 + e^4q^4 + \ldots) = \frac{qcf}{1 - e^2q^2} = \frac{q(1-b^2)}{1-q^2b^2} = \frac{q(b^{-1}-b)}{b^{-1}-q^2b}.$$

When the light is polarized in the plane of incidence we have

$$C_{,} = \frac{-\sqrt{-1}\,.\,2q\sin 2\theta}{(1-q^2)\cos 2\theta - \sqrt{-1}\,(1+q^2)\sin 2\theta}$$

$$= \frac{2q\sin 2\theta\{(1+q^2)\sin 2\theta - \sqrt{-1}\,(1-q^2)\cos 2\theta\}}{(1-q^2)^2 + 4q^2\sin^2 2\theta} \ldots (7);$$

so that if $\psi_{,}$ and $\rho_{,}$ refer to the transmitted light we have

$$\tan\psi_{,} = -\frac{1-q^2}{1+q^2}\cot 2\theta \ldots\ldots\ldots\ldots\ldots (8),$$

$$\rho_{,}^2 = \frac{4q^2\sin^2 2\theta}{(1-q^2)^2 + 4q^2\sin^2 2\theta} \ldots\ldots\ldots (9).$$

If we take $\rho_{,}$ positive, as it will be supposed to be, we must take $\psi_{,}$ such that $\cos\psi_{,}$ may be positive; and therefore, of the two angles lying between $-\pi$ and π which satisfy (8), we must choose that which lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$. Hence, since from (3) and (8) $\psi_{,}$ is of the form $\psi + \frac{\pi}{2} + n\pi$, n being an integer, we must take $\psi_{,} = \psi - \frac{\pi}{2}$.

For light polarized perpendicularly to the plane of incidence we have only to put ϕ for θ. It follows from (4) and (9) that the sum of the intensities of the reflected and transmitted light is equal to unity, as of course ought to be the case. This renders it unnecessary to discuss the expression for the intensity of the transmitted light.

14. Taking the expression (4) for the intensity of the reflected light, consider first how it varies on receding from the point of contact.

As the point of contact $D = 0$, and therefore from (5) $q = 1$, and therefore $\rho^2 = 0$, or there is absolute darkness. On receding from the point of contact q decreases, but slowly at first, inasmuch as D varies as r^2, r being the distance from the point of contact. It follows from (4) that the intensity ρ^2 varies ultimately as r^4, so that it increases at first with extreme slowness. Consequently the darkness is, as far as sense can decide, perfect for some distance round the point of contact. Further on q decreases more rapidly, and soon becomes insensible. Consequently the intensity decreases, at first rapidly, and then slowly again as it approaches its limiting value 1, to which it soon becomes sensibly equal. All this agrees with observation.

15. Consider next the variation of intensity as depending on the colour. The change in θ and ϕ in passing from one colour to another is but small, and need not here be taken into account: the quantity whose variation it is important to consider is q. Now it follows from (5) that q changes the more rapidly in receding from the point of contact the smaller be λ. Consequently the spot must be smaller for blue light than for red; and therefore towards the edge of the spot seen by reflection, that is beyond the edge of the central portion of it, which is black, there is a predominance of the colours at the blue end of the spectrum; and towards the edge of the bright spot seen by transmission the colours at the red end predominate. The tint is more conspicuous in the trans-

mitted, than in the reflected light, in consequence of the quantity of white light reflected about the edge of the spot. The separation of colours is however but slight, compared with what takes place in dispersion or diffraction, for two reasons. First, the point of minimum intensity is the same for all the colours, and the only reason why there is any tint produced is, that the intensity approaches more rapidly to its limiting value 1 in the case of the blue than in the case of the red. Secondly, the same fraction of the incident light is reflected at points for which $D \propto \lambda$, and therefore $r \propto \sqrt{\lambda}$; and therefore, on this account also, the separation of colours is less than in diffraction, where the colours are arranged according to the values of λ, or in dispersion, where they are arranged according to values of λ^{-2} nearly. These conclusions agree with observation. A faint blueish tint may be perceived about the dark spot seen by reflection; and the fainter portions of the bright spot seen by transmission are of a decided reddish brown.

16. Let us now consider the dependance of the size of the spot on the nature of the polarization. Let s be the ratio of the intensity of the transmitted light to that of the reflected; s_1, s_2, the particular values of s belonging to light polarized in the plane of incidence and to light polarized perpendicularly to the plane of incidence respectively; then

$$s_1 = \frac{4q^2 \sin^2 2\theta}{(1-q^2)^2}, \quad s_2 = \frac{4q^2 \sin^2 2\phi}{(1-q^2)^2},$$

$$\frac{s_1}{s_2} = \left(\frac{\sin 2\theta}{\sin 2\phi}\right)^2 = \{(\mu^2+1)\sin^2 i - 1\}^2 \ldots\ldots (10).$$

Now according as s is greater or less, the spot is more or less conspicuous; that is, conspicuous in regard to extent, and intensity at some distance from the point of contact; for in the immediate neighbourhood of that point the light is in all cases wholly transmitted. Very near the critical angle we have from (10) $s_2 = \mu^4 s_1$, and therefore the spot is much more conspicuous for light polarized perpendicularly to the plane of incidence than for light polarized in that plane. As i increases the spots seen in the two cases become more and more nearly equal in magnitude: they become exactly alike when $i = \iota$, where

$$\sin^2 \iota = \frac{2}{1+\mu^2}.$$

When i becomes greater than ι the order of magnitude is reversed; and the spots become more and more unequal as i increases. When $i = 90^\circ$ we have $s_1 = \mu^4 s_2$, so that the inequality becomes very great. This however must be understood with reference to relative, not absolute magnitude; for when the angle of incidence becomes very great both spots become very small.

I have verified these conclusions by viewing the spot through a rhomb of Iceland spar, with its principal plane either parallel or perpendicular to the plane of incidence, as well as by using a doubly refracting prism; but I have not attempted to determine experimentally the angle of incidence at which the spots are exactly equal. Indeed, it could not be determined in this way with any precision, because the difference between the spots is insensible through a considerable range of incidence.

17. It is worthy of remark that the angle of incidence ι at which the spots are equal, is exactly that at which the difference of acceleration of phase of the oppositely polarized pencils, which arises from total internal reflection, is a maximum.

When $i = \iota$ we have

$$\sin 2\theta = \sin 2\phi = \frac{2\mu}{\mu^2+1}; \quad \text{whence } \cot\theta = \tan\phi = \mu \ldots (11);$$

$$\text{and } \rho^2 = \frac{(1+\mu^2)^2(1-q^2)^2}{(1+\mu^2)^2(1-q^2)^2 + 16\mu^2 q^2}, \quad \text{where } q = \epsilon^{-\frac{2\pi D}{\lambda}\sqrt{\frac{\mu^2-1}{\mu^2+1}}} \ldots (12).$$

If we determine in succession the angles θ, ζ, η from the equations $\cot\theta = \mu$, $\tan\zeta = q$, $\tan\eta = \sin 2\theta \tan 2\zeta$, we have $\rho_{,}^2 = 1 - \rho^2 = \frac{1}{2}$ versin 2η. The expression for the intensity may be adapted to numerical computation in the same way for any angle of incidence, except that θ or ϕ must be determined by (2) or (6) instead of (11), and q by (5) instead of (12).

18. When light is incident at the critical angle, which I shall denote by γ, the expression for the intensity takes the form $\frac{0}{0}$. Putting for shortness $\sqrt{\mu^2 \sin^2 i - 1} = w$, we have ultimately

$$q = 1 - \frac{2\pi D}{\lambda} w, \quad \tan\theta = \theta = \frac{w}{\mu \cos i} = \frac{w}{\sqrt{\mu^2 - 1}}, \quad \phi = \mu^2 \theta;$$

and we get in the limit

$$\rho^2 = \frac{\left(\frac{\pi D}{\lambda}\right)^2}{\left(\frac{\pi D}{\lambda}\right)^2 + \frac{1}{\mu^2 - 1}}, \quad \text{or} = \frac{\left(\frac{\pi D}{\lambda}\right)^2}{\left(\frac{\pi D}{\lambda}\right)^2 + \frac{\mu^4}{\mu^2 - 1}} \ldots\ldots (13),$$

according as the light is polarized in or perpendicularly to the plane of incidence. The same formulæ may be obtained from the expression given at page 304 of Airy's *Tract*, which gives the intensity when $i < \gamma$, and which like (4) takes the form $\frac{0}{0}$ when i becomes equal to γ, in which case e becomes equal to -1.

19. When i becomes equal to γ, the infinite series of Art. 11 ceases to be convergent: in fact, its several terms become ultimately equal to each other, while at the same time the coefficient by which the series is multiplied vanishes, so that the whole takes the form $0 \times \infty$. The same remark applies to the series at page 303 of Airy's *Tract*. If we had included the coefficient in each term of the series, we should have got series which ceased to be convergent at the same time that their several terms vanished. Now the sum of such a series may depend altogether on the point of view in which it is regarded as a limit. Take for example the convergent infinite series

$$f(x, y) = x \sin y + \tfrac{1}{3} x^3 \sin 3y + \tfrac{1}{5} x^5 \sin 5y + \ldots = \tfrac{1}{2} \tan^{-1} \frac{2x \sin y}{1 - x^2},$$

where x is less than 1, and may be supposed positive. When x becomes 1 and y vanishes $f(x, y)$ becomes indeterminate, and its limiting value depends altogether upon the order in which we suppose x and y to receive their limiting values, or more generally upon the arbitrary relation which we conceive imposed upon the otherwise independent variables x and y as they approach their limiting values together. Thus, if we suppose y first to vanish, and then x to become 1, we have $f(x,y) = 0$; but if we suppose x first to become 1, and then y to vanish, $f(x,y)$ becomes $\pm \frac{\pi}{4}$, + or − according as y vanishes positively or negatively. Hence in the case of such a series a mode of approximating to the value of x or y, which in general was pefectly legitimate, might become inadmissible in the extreme case in which $x = 1$, or nearly $= 1$. Consequently, in the case of Newton's Rings when $i \sim \gamma$ is extremely small, it is no longer safe to neglect the defect of parallelism of the surfaces. Nevertheless, inasmuch as the expression (4), which applies to the case in which $i > \gamma$, and the ordinary expression which applies when $i < \gamma$, alter continuously as i alters, and agree with (13) when $i = \gamma$, we may employ the latter expression in so far as the phenomenon to be explained alters continuously as i alters. Consequently we may apply the expression (13) to the central spot when $i = \gamma$, or nearly $= \gamma$, at least if we do not push the expression beyond values of D corresponding to the limits of the central spot as seen at other angles of incidence. To explain however the precise mode of disappearance of the rings, and to determine their greatest dilatation, we should have to enter on a special investigation in which the inclination of the surfaces should be taken into account.

20. I have calculated the following Table of the intensity of the transmitted light, taking the intensity of the incident light at 100. The Table is calculated for values of D increasing by $\frac{1}{4}\lambda$, and for three angles of incidence, namely, the critical angle, the angle ι before mentioned, and a considerable angle, for which I have taken 60^0. I have supposed $\mu = 1{\cdot}63$, which is about the refractive index for the brightest part of the spectrum in the case of flint glass. This value of μ gives $\gamma = 37^0\,51'$, $\iota = 42^0\,18'$. The numerals I., II. refer to light polarized in and perpendicularly to the plane of incidence respectively.

$\frac{4D}{\lambda}$	$i=\gamma$ I.	$i=\gamma$ II.	$i=\iota$ I. and II.	$i=60^0$ I.	$i=60^0$ II.
0	100	100	100	100	100
1	49	87	33	16	6
2	20	63	5	1	0
3	10	43	1	0	
4	6	30	0		
5	4	22			
6	3	16			
7	2	12			
8	2	10			
9	1	8			
10	1	6			
11	1	5			
12	1	5			
13	1	4			
⋮		⋮			
26		1			
27		0			

21. A Table such as this would enable us to draw the curve of intensity, or the curve in which the abscissa is proportional to the distance of the point considered from the point of contact, and the ordinate proportional to the intensity. For this purpose it would only be requisite to lay down on the axis of the abscissæ, on the positive and negative sides of the origin, distances proportional to the squares of the numbers in the first column, and to take for ordinates lengths proportional to the numbers in one of the succeeding columns. To draw the curve of intensity for $i=\iota$ or for $i = 60^0$, the table ought to have been calculated with smaller intervals between the values of D; but the law of the decrease of the intensity cannot be accurately observed.

22. From the expression (13) compared with (4), it will be seen that the intensity decreases much more rapidly, at some distance from the point of contact, when i is considerably greater than γ than when $i=\gamma$ nearly. This agrees with observation. What may be called the *ragged edge* of the bright spot seen by transmission is in fact much broader in the latter case than in the former.

When i becomes equal to 90^0 there is no particular change in the value of q, but the angles θ and ϕ become equal to 90^0, and therefore $\sin 2\theta$ and $\sin 2\phi$ vanish, so that the spot vanishes. Observation shows that the spot becomes very small when i becomes nearly equal to 90^0.

23. Suppose the incident light to be polarized in a plane making an angle α with the plane of incidence. Then at the point of contact the light, being transmitted as if the first and third media formed one uninterrupted medium, will be plane polarized, the plane of polarization being the same as at first. At a sufficient distance from the point of contact there is no sensible quantity of light transmitted. At intermediate distances the transmitted light is in general elliptically polarized, since it follows from (8) and the expression thence derived by writing ϕ for θ that the two streams of light, polarized in and perpendicularly to the plane of incidence respectively, into which the incident light may be conceived to be decomposed, are unequally accelerated or retarded. At the point of contact, where $q=1$, these two expressions agree in giving $\psi_{,}=0$. Suppose now that the transmitted light is analyzed, so as to extinguish the light which passes through close to the point of contact. Then the centre of the spot will be dark, and beyond a certain distance all round there will be darkness, because no sensible quantity of light was incident on the analyzer; but at intermediate distances a portion of the light incident on the analyzer will be visible. Consequently the appearance will be that of a luminous ring with a perfectly dark centre.

24. Let the coefficient of vibration in the incident light be taken for unity; then the incident vibration may be resolved into two, whose coefficients are $\cos\alpha$, $\sin\alpha$, belonging to light polarized in and perpendicularly to the plane of incidence respectively. The phases of vibration will be accelerated by the angles $\psi_{,}$, $\psi_{,,}$, and the coefficients of vibration will be multiplied by $\rho_{,}$, $\rho_{,,}$, if $\psi_{,,}$, $\rho_{,,}$ are what $\psi_{,}$, $\rho_{,}$ in Art. (13) become when ϕ is put for θ. Hence we may take

$$\rho_{,}\cos\alpha\,.\cos\left\{\frac{2\pi}{\lambda}(vt-\mu x')+\psi_{,}\right\},$$

$$\rho_{,,}\sin\alpha\,.\cos\left\{\frac{2\pi}{\lambda}(vt-\mu x')+\psi_{,,}\right\}$$

to represent the vibrations which compounded together make up the transmitted light, x' being measured in the direction of propagation. The light being analyzed in the way above mentioned, it is only the resolved parts of these vibrations in a direction perpendicular to that of the vibrations in the incident light which are preserved. We thus get, to express the vibration with which we are concerned,

$$\sin\alpha\cos\alpha\left\{\rho_{,}\cos\left(\frac{2\pi}{\lambda}(vt-\mu x')+\psi_{,}\right)-\rho_{,,}\cos\left(\frac{2\pi}{\lambda}(vt-\mu x')+\psi_{,,}\right)\right\},$$

which gives for the intensity (I) at any point of the ring

$$I=\tfrac{1}{4}\sin^2 2\alpha\left\{(\rho_{,}\cos\psi_{,}-\rho_{,,}\cos\psi_{,,})^2+(\rho_{,}\sin\psi_{,}-\rho_{,,}\sin\psi_{,,})^2\right\}\ \dots\ (14),$$

$$=\tfrac{1}{4}\sin^2 2\alpha\left\{\rho_{,}^2+\rho_{,,}^2-2\rho_{,}\rho_{,,}\cos(\psi_{,,}-\psi_{,})\right\}.$$

Let P_θ, Q_θ be respectively the real part of the expression at the second side of (7) and the coefficient of $\sqrt{-1}$, and let P_ϕ, Q_ϕ be what P_θ, Q_θ become when ϕ is put for θ. Then we may if we please replace (14) by

$$I=\tfrac{1}{4}\sin^2 2\alpha\left\{(P_\theta-P_\phi)^2+(Q_\theta-Q_\phi)^2\right\}.\ \dots\dots\dots\dots\dots\ (15).$$

The ring is brightest, for a given angle of incidence, when $\alpha=45°$. When $i=\iota$, the two kinds of polarized light are transmitted in the same proportion; but it does not therefore follow that the ring vanishes, inasmuch as the change of phase is different in the two cases. In fact, in this case the angles ϕ, θ are complementary; so that $\cot 2\phi$, $\cot 2\theta$ are equal in magnitude but opposite in sign, and therefore from (8) the phase in the one case is accelerated and in the other case retarded by the angle

$$\tan^{-1}\left(\frac{1-q^2}{1+q^2}\cot 2\theta\right),\quad\text{or}\quad\tan^{-1}\left(\frac{1-q^2}{1+q^2}\,\frac{\mu^2-1}{2\mu}\right).$$

It follows from (14) that the ring cannot vanish unless $\rho_{\prime} \cos\psi_{\prime} = \rho_{\prime\prime} \cos\psi_{\prime\prime}$, and $\rho_{\prime} \sin\psi_{\prime} = \rho_{\prime\prime} \sin\psi_{\prime\prime}$. This requires that $\rho_{\prime}^2 = \rho_{\prime\prime}^2$, which is satisfied only when $i = \iota$, in which case as we have seen the ring does not vanish. Consequently a ring is formed at all angles of incidence; but it should be remembered that the spot, and consequently the ring, vanishes when i becomes 90^0.

25. When $i = \gamma$, the expressions for P_θ, Q_θ, take the form $\frac{0}{0}$, and we find, putting for shortness $\frac{\pi D}{\lambda} = p$,

$$P_\theta = \frac{(\mu^2 - 1)^{-1}}{p^2 + (\mu^2 - 1)^{-1}}, \qquad P_\phi = \frac{\mu^4(\mu^2 - 1)^{-1}}{p^2 + \mu^4(\mu^2 - 1)^{-1}},$$

$$Q_\theta = -\frac{p(\mu^2 - 1)^{-\frac{1}{2}}}{p^2 + (\mu^2 - 1)^{-1}}, \qquad Q_\phi = -\frac{p\mu^2(\mu^2 - 1)^{-\frac{1}{2}}}{p^2 + \mu^4(\mu^2 - 1)^{-1}}.$$

If we take two subsidiary angles χ, ω, determined by the equations

$$\frac{\pi D}{\lambda}\sqrt{\mu^2 - 1} = \tan\chi = \mu^2 \tan\omega,$$

we get

$$P_\theta = \cos^2\chi, \quad P_\phi = \cos^2\omega, \quad Q_\theta = -\sin\chi\cos\chi, \quad Q_\phi = -\sin\omega\cos\omega.$$

Substituting in (15) and reducing we get, supposing $\alpha = 45^0$,

$$I = \tfrac{1}{8}\operatorname{versin}(2\chi - 2\omega) \ldots\ldots\ldots\ldots\ldots (16).$$

When $i = \iota$, $\cos 2\phi = -\cos 2\theta$, $\sin 2\phi = \sin 2\theta$; and therefore $P_\phi = P_\theta$, $Q_\phi = -Q_\theta$, which when $\alpha = 45^0$ reduces (15) to $I = Q_\theta^2$.

If we determine the angle ϖ from the equation

$$1 - q^2 = 2q\sin 2\theta\tan\varpi, \text{ or } \tan\varpi = \cot 2\zeta \,.\, \operatorname{cosec} 2\theta,$$

we get

$$I = \tfrac{1}{4}\sin^2 2\varpi \,.\, \cos^2 2\theta \ldots\ldots\ldots\ldots (17).$$

In these equations

$$\log_e \tan\zeta = -\frac{2\pi D}{\lambda}\sqrt{\frac{\mu^2 - 1}{\mu^2 + 1}}, \quad \cot\theta = \mu.$$

26. The following Table gives the intensity of the ring for the two angles of incidence $i = \gamma$ and $i = \iota$, and for values of D increasing by $\frac{1}{10}\lambda$. The intensity is calculated by the formulæ (16) and (17). The intensity of the incident polarized light is taken at 100, and μ is supposed equal to 1·68, as before.

$\frac{D}{\lambda}$	I $i=\gamma$	I $i=\iota$
·0	0·0	0·0
·1	1·3	3·2
·2	3·5	5·1
·3	4·8	3·6
·4	5·1	1·9
·5	4·9	·9
·6	4·5	·4
·7	4·0	·2
·8	3·6	·1
·9	3·1	·0
1·0	2·8	
1·1	2·4	
1·2	2·1	
1·3	1·9	
1·4	1·7	
1·5	1·5	
1·6	1·4	
1·7	1·2	
1·8	1·1	
1·9	1·0	
2·0	·9	
2·1	·9	
2·2	·8	
2·3	·7	
2·4	·7	
2·5	·6	
2·6	·6	
2·7	·5	
2·8	·5	
2·9	·5	
3·0	·4	

The column for $i=\gamma$ may be continued with sufficient accuracy, by taking I to vary inversely as the square of the number in the first column.

27. I have seen the ring very distinctly by viewing the light transmitted at an angle of incidence a little greater than the critical angle. In what follows, in speaking of angles of position, I shall consider those positive which are measured in the direction of motion of the hands of a watch, to a person looking at the light. The plane of incidence being about 45^0 to the positive side of the plane of primitive polarization, the appearance presented as the analyzer, (a Nicol's prism,) was turned, in the positive direction, through the position in which the light from the centre was extinguished, was as follows. On approaching that position, in addition to the general darkening of the spot, a dark ring was observed to separate itself from the dark field about the spot, and to move towards the centre, where it formed a broad dark patch, surrounded by a rather faint ring of light. On continuing to turn, the ring got brighter, and the central patch ceased to be quite black. The

light transmitted near the centre increased in intensity till the dark patch disappeared: the patch did not break up into a dark ring travelling outwards.

On making the analyzer revolve in the contrary direction, the same appearances were of course repeated in a reverse order: a dull central patch was seen, which became darker and darker till it appeared quite black, after which it broke up into a dark ring which travelled outwards till it was lost in the dark field surrounding the spot. The appearance was a good deal disturbed by the imperfect annealing of the prisms. When the plane of incidence was inclined at an angle of about -45^0 to the plane of primitive polarization, the same appearance as before was presented on reversing the direction of rotation of the analyzer.

28. Although the complete theoretical investigation of the moving dark ring would require a great deal of numerical calculation, a general explanation may very easily be given. At the point of contact the transmitted light is plane polarized, the plane of polarization being the same as at first*. At some distance from the point of contact, although strictly speaking the light is elliptically polarized, it may be represented in a general way by plane polarized light with its plane of polarization further removed than at first from the plane of incidence, in consequence of the larger proportion in which light polarized perpendicularly to the plane of incidence is transmitted, than light polarized in that plane. Consequently the transmitted light may be represented in a general way by plane polarized, with its plane of polarization receding from the plane of incidence on going from the centre outwards. If therefore we suppose the position of the plane of incidence, and the direction of rotation of the analyzer, to be those first mentioned, the plane of polarization of light transmitted by the analyzer will become perpendicular to the plane of polarization of the transmitted light of the spot sooner towards the edge of the spot than in the middle. The locus of the point where the two planes are perpendicular to each other will in fact be a circle, whose radius will contract as the analyzer turns round. When the analyzer has passed the position in which its plane of polarization is perpendicular to that of the light at the centre of the spot, the inclination of the planes of polarization of the analyzer and of the transmitted light of the spot decreases, for a given position of the analyzer, in passing from the centre outwards; and therefore there is formed, not a dark ring travelling outwards as the analyzer turns round, but a dark patch, darkest in the centre, and becoming brighter, and therefore less and less conspicuous, as the analyzer turns round. The appearance will of course be the same when the plane of incidence is turned through 90^0, so as to be equally inclined to the plane of polarization on the opposite side, provided the direction of rotation of the analyzer be reversed.

29. The investigation of the intensity of the spot formed beyond the critical angle when the third medium is of a different nature from the first, does not seem likely to lead to results of any particular interest. Perhaps the most remarkable case is that in which the second and third media are both of lower refractive power than the first, and the angle of incidence is greater than either of the critical angles for refraction out of the first medium into the second, or out of the first into the third. In this case the light must be wholly reflected; but the acceleration of phase due to the total internal reflection will alter in the neighbourhood of the point of contact. At that point it will be the same as if the third medium occupied the place of the second as well as its own; at a distance sufficient to render the influence of the third medium insensible, it will be the same as if the second medium occupied the place of the third as well as its own. The law of the variation of the acceleration from the one to the other of its extreme values, as the distance from the point of contact varies, would result from the investigation. This law could be put to the test of experiment by examining the nature of the elliptic polarization of the light reflected in the neighbourhood of the point of

* The rotation of the plane of polarization due to the refraction at the surfaces at which the light enters the first prism and quits the second is not here mentioned, as it has nothing to do with the phenomenon discussed.

contact when the incident light is polarized at an azimuth of 45^0, or thereabouts. The theoretical investigation does not present the slightest difficulty in principle, but would lead to rather long expressions; and as the experiment would be difficult, and is not likely to be performed, there is no occasion to go into the investigation.

30. In viewing the spot formed between a prism and a lens, I was struck with the sudden, or nearly sudden disappearance of the spot at a considerable angle of incidence. The cause of the disappearance no doubt was that the lens was of lower refractive power than the prism, and that the critical angle was reached which belongs to refraction out of the prism into the lens. Before disappearing, the spot became of a bright sky blue, which shows that the ratio of the refractive index of the prism to that of the lens was greater for the blue rays than for the red. As the disappearance of the spot can be observed with a good deal of precision, it may be possible to determine in this way the refractive index of a substance of which only a very minute quantity can be obtained. The examination of the refractive index of the globule obtained from a small fragment of a fusible mineral might afford the mineralogist a means of discriminating between one mineral and another. For this purpose a plate, which is what a prism becomes when each base angle becomes 90^0, would probably be more convenient than a prism. Of course the observation is possible only when the refractive index of the substance to be examined is less than that of the prism or plate.

G. G. STOKES.

XLIX. *Of the Intrinsic Equation of a Curve**, *and its Application.*
By W. Whewell, D.D., *Master of Trinity College, Cambridge.*

[Read *February* 12, 1849.]

1. Mathematicians are aware how complex and intractable are generally the expressions for the lengths of curves referred to rectilinear coordinates, and also the determinations of their involutes and evolutes. It appears a natural reflexion to make, that this complexity arises in a

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CORRECTIONS.

	line		*for*	*read*
Article 4,	2		if s	it,
—	4		$QB - BA$	$QB + BA$
31	3		for y	for $y = 0$
last page of the *additional Note*,				
	5		$m^{[illegible]}$	$m^{[illegible]}$
	6	in the denominators	m^2, $m^{[illegible]}$, m'	$m^{[illegible]}$, $m^{[illegible]}$, m'
	8		[illegible]	[illegible]

3. It is evident that $\dfrac{ds}{d\phi}$ is ρ, the radius of curvature. Hence $\rho = f'(\phi)$, and the curve may also be contracted approximately by taking small finite differences of ϕ, drawing a line perpendicular to the curve at first, setting off the value of ρ, drawing a circular arc to radius ρ for $\Delta\phi$, then setting off ρ_1, and drawing a circular arc with radius ρ_1 for $\Delta\phi_1$, and so on. See Fig. 2.

4. The evolute of a curve is easily found from this equation. For if (Fig. 3) AP be the curve, BQ the evolute, $AP = s$, $BQ = s'$, it is evident that ϕ is the same for both s and s', if s in BQ, ϕ be measured from BA, perpendicular to Ax.

And $QP = QB - BA$, or $\rho = s' + C$. Hence $s' = f'(\phi) - C = \dfrac{ds}{d\phi} - C$.

If the curves have the forms represented in Fig. 3*a*, the formulæ are nearly similar.

* After writing this paper, I found that Euler had, in the solution of a particular problem, expressed curves by means of an equation between the arc and the radius of curvature. This equation is, as is shown in the paper, the differential of my "intrinsic equation," and has an equally good right to the name. My equation being the integral of Euler's, has, of course, one more arbitrary constant than his. There may very possibly be other modes of expressing curves which may be fitly described as "intrinsic equations" to the curves. I was not able to find any other name for the equation which I have employed.

5. Hence s', s'', &c. indicating the successive evolutes, ρ, ρ', ρ'', &c. the successive radii of curvature, we have,

$$s' = \frac{ds}{d\phi} - C,\ s'' = \frac{ds'}{d\phi} - C',\ \frac{ds'''}{d\phi} - C''', \text{ \&c.}$$

$$\rho = \frac{ds}{d\phi},\ \rho' = \frac{ds'}{d\phi},\ \rho'' = \frac{ds''}{d\phi}, \text{ \&c.}$$

6. Also we may in like manner find the successive involutes s_1, s_2, s_3, &c. For we have

$$\frac{ds}{d\phi} = s' + C = f'(\phi) + C,\ s = f(\phi) + C\phi.$$

So $$\frac{ds_1}{d\phi} = s + C_1 = f(\phi) + C\phi + C_1$$

$$s_1 = f_1(\phi) + C\frac{\phi^2}{2} + C_1\,\phi.$$

Hence s is known in function of ϕ, and therefore the curve known. And in like manner s_2, s_3, &c., if these be the arcs of the successive involutes.

In Fig. 4, CR, BQ, AP are successive involutes of DS.

7. It is evident that the intrinsic equation to the circle is

$s = a\phi$, a being the radius.

Also for the equiangular spiral, since the curve from its origin is everywhere similar to itself, the radius of curvature is proportional to the whole arc. Hence

$$\frac{ds}{d\phi} = ms;$$ whence $s = a^{m\phi}$, if s be measured from the pole.

If s and ϕ vanish at the same time, $s = a\,(e^{m\phi} - 1)$.

We shall afterwards give general formulæ for obtaining the intrinsic equation from the ordinary coordinate equation, and reversely. But the operation of our method will be better seen by first taking some special cases.

Of Cycloids, Epicycloids, and Hypocycloids.

8. In the Cycloid, if VB, Fig. 5, be the diameter of the generating circle, rolling on the straight line DB from the initial position AD, when it is perpendicular to DB, and P the describing point at that time, PQ being the diameter, by the mode of description, the arc $BQ = BD$. But the curve at P is perpendicular to PB; and if ϕ be the angle of deflexion, $\phi = VBP$, and $2\phi = VCP$. Hence chord $VP = 2b \sin \phi$, if b be the radius of the circle. And the arc $AP = 2$ chord VP. Hence the intrinsic equation to the cycloid is

$$s = 4\,b \sin \phi.$$

When ϕ becomes a right angle, s becomes a maximum. At this point there is a cusp (Z), and the added part of s after this is negative; and so continues, till $\phi = 3$ right angles, where there is another cusp (Z), and the added part of s becomes positive; and so on.

9. In the Epicycloid, if we take Newton's construction (Princip. B. IV. Sect. 10, Prop. 49), Fig. 6, *CB* the radius of the *globe*, *VB* the diameter of the *wheel* when the describing point is at *P*, *E* its center, we have, by Newton's proposition, only measuring the arc from *A* the vertex, instead of *Z* the cusp, the arc at *P* perpendicular to the chord *BP*; and

$$CB : 2CE :: \text{chord } VP : \text{arc } AP.$$

Let a be the radius of the globe, b the radius of the wheel: θ the angle *DCB*, through which the wheel has rolled upon the globe. Then (*PQ* being a diameter) by the mode of description, arc $BQ = BD$. Therefore angle $BEQ = \frac{a\theta}{b}$: therefore chord $VP = 2b \sin \frac{a\theta}{2b}$; and

$$a : 2(a+b) :: 2b \sin \frac{a\theta}{2b} : s; \text{ whence } s = \frac{4(a+b)b}{a} \sin \frac{a\theta}{2b}.$$

But $VBP = \frac{a\theta}{2b}$, and $DCB = \theta$. Hence *BP* makes with *CA* an angle $= \frac{a\theta}{ab} + \theta$. And since the curve at *A* is perpendicular to *CD*, and at *P*, to *BP*, it is evident that if ϕ be the angle through which the curve has deflected at *P*, $\phi = \frac{a\theta}{2b} + \theta = \frac{a+2b}{2b}\theta$.

Hence $\theta = \frac{2b}{a+2b}\phi$; and the intrinsic equation to the epicycloid is

$$s = \frac{4(a+b)b}{a} \sin \frac{a}{a+2b}\phi.$$

This may coincide with any curve of which the equation is

$$s = l \sin m\phi, \text{ where } m \text{ is less than } 1.$$

In this case, $m = \frac{a}{a+2b}$, $l = \frac{4(a+b)b}{a}$;

$$\text{whence } \frac{b}{a} = \frac{1-m}{2m}, \quad l = \frac{2(1+m)b}{m}.$$

10. In the same manner we shall find that the intrinsic equation to the hypocycloid is

$$s = \frac{4(a-b)b}{a} \sin \frac{a}{a-2b}\phi.$$

And this may coincide with any curve of which the equation is

$$s = l \sin m\phi, \text{ where } m \text{ is greater than } 1, \text{ by making}$$

$$\frac{b}{a} = \frac{m-1}{2m}, \quad l = \frac{2(1+m)b}{m}.$$

11. It is evident from the equation $s = l \sin m\phi$, that the curves represented by that equation will be of such forms as are seen to result from the epicycloidal mode of description. Thus the equation $s = l \sin \frac{\phi}{2}$ gives a curve in which s continues to increase from *A*, where $\phi = 0$, till $\phi = \pi$, after which it decreases. Hence there will be a cusp when the curve has deflected through two right angles, as at *Z*, Fig. 7. After this point the curve goes on in an identical inverted course, till $\phi = 2\pi$, as at *A'*, when $s = 0$, the negative part having destroyed the positive

part. The negative value of s goes on increasing till $\phi = 3\pi$, at Z', when there is another cusp Afterwards the arc becomes positive, and the curve returns to A, having deflected through 4π*.

The curve is an epicycloid in which $b = \frac{1}{2} a$.

12. Again, if $s = l \sin 2\phi$, s increases from A, where $\phi = 0$, till $\frac{\phi}{4} = \pi$, when it is a maximum, and there is a cusp, Z, Fig. 8. After this the arc (from Z) is negative till $\phi = \frac{3\pi}{4}$, when there is a second cusp, Z'. Then the arc is positive, till $\phi = \frac{5\pi}{4}$ (at Z''). Then it is negative till $\phi = \frac{7\pi}{4}$ (at Z'''). When $\phi = 2\pi$, the curve returns to A.

The curve is a hypocycloid in which $b = \frac{1}{4} a$.

13. If $s = l \sin \frac{\phi}{3}$, $s = l \sin \frac{\phi}{4}$, $s = l \sin \frac{\phi}{5}$, &c.

we have epicycloids in which $\frac{b}{a}$ is respectively

$$1, \ \tfrac{3}{2}, \ 2, \ \&c.$$

The radius of the wheel in these latter cases is greater than that of the globe, and the curve is deflected through more than a whole circumference before it comes to a cusp. Thus in the case $s = l \sin \frac{\phi}{5}$, the curve deflects through $2\pi + \frac{\pi}{2}$ to come to a cusp. See Fig. 9.

14. In the same way, if we have

$$s = l \sin 3\phi, \quad s = l \sin 4\phi, \quad s = l \sin 5\phi, \ \&c.$$

we have a series of hypocycloids, in which $\frac{b}{a}$ is respectively

$$\frac{1}{3}, \ \frac{3}{8}, \ \frac{4}{9}, \ \&c.$$

As m becomes larger and larger, $\frac{b}{a}$ approaches more and more nearly to $\frac{1}{2}$, but never attains that magnitude. As is well known, for that supposition, the hypocycloid is a straight line.

15. It is evident that the ordinary properties of epicycloids and hypocycloids, as to their lengths, radii of curvature, involutes, evolutes, &c., all follow with great facility from the use of our equation. Thus the length of the epicycloid from the vertex A to the cusp Z is had by making the angle $\frac{a}{a+2b}\phi = \frac{\pi}{2}$, which gives the length of that half of the curve $= \frac{4(a+b)b}{a}$, and the whole length $\frac{8(a+b)b}{a}$ from cusp to cusp, the known values.

Also the radius of curvature of the epicycloid

$$\frac{ds}{d\phi} = \frac{4(a+b)b}{a+2b} \cos \frac{a}{a+2b}\phi, \text{ the known value.}$$

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* That there will be a cusp when s is a maximum, appears also by considering that in that case $\frac{ds}{d\phi} = 0$, that is, the radius of curvature vanishes.

16. Again, for the evolute of the epicycloid, let, in Fig. 6, $ZO = s'$ be the arc of the evolute. Therefore

$$s' = \frac{4(a+b)b}{a+2b} \cos \frac{a}{a+2b} \phi.$$

But at Z, where $s' = 0$, $\frac{a}{a+2b} \phi = \frac{\pi}{2}$, $\phi = \frac{\pi}{2} \cdot \frac{a+2b}{a}$.

And the deflexion of the evolute beginning from Z and going to O is the excess of this value of ϕ above the value at P, because at every point the evolute is perpendicular to the curve. Therefore if ϕ' be the deflection of s', $\phi' = \frac{\pi}{2} \cdot \frac{a+2b}{a} - \phi$; and $\frac{a}{a+2b} \phi = \frac{\pi}{2} - \frac{a}{a+2b} \phi'$.

Therefore $s' = \frac{4(a+b)b}{a+2b} \sin \frac{a}{a+2b} \phi'$. This is an epicycloid similar to the first; for the equation agrees with $s' = \frac{4(a'+b')b'}{a'} \sin \frac{a'}{a'+2b'} \phi'$, if $\frac{b'}{a'} = \frac{b}{a}$, and $\frac{a'}{a} = \frac{a}{a+2b}$.

Of Running-pattern Curves.

17. By *Running-pattern Curves* I mean curves in which a certain form of curve is repeated over and over again in the progress of the whole curve. For example, let $\phi = \sin s$; as s increases from 0 to infinity, it becomes successively $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}$, &c., and the corresponding values of ϕ are 0, 1, 0, -1, 0, 1, &c.: and the curve is evidently a sinuous curve, as represented in Fig. 10, in which the same form is constantly repeated every time that s goes through the value 2π.

The greatest angle which the curve makes with the original direction is 1 and -1; that is, the angle of which the arc $= 1$, to the one side and to the other.

18. If $\phi = m . \sin s$, we shall in like manner have a sinuous curve in which the greatest angles of deflexion to one side and the other are $= m$.

If $\phi = \frac{\pi}{2} \sin s$, these deflexions become right angles, and the curve is as represented in Fig. 11.

19. If $\phi = \pi \sin s$, the curve from $s = 0$ to $s = \frac{\pi}{2}$ is of the form CA, Fig. 12; A being behind C. For in this case, $\frac{ds}{d\phi} = \frac{1}{\pi \cos s}$. Hence the radius of curvature is $\frac{1}{\pi}$ at C, where $s = 0$, and increases to A, where it is infinite. The evolute is of the form BD, and has for its asymptote the line AE, perpendicular to the original direction. And hence the general form of the curve CA is manifest. At A there will be a point of inflexion; and after A the curve will be repeated in inverse position, as AC', and then continued reversely from C' to A', and so on, as in the Figure.

20. If $\phi = 2\pi \sin s$, it will be seen, in like manner, that the curve from $s = 0$ to $s = \frac{\pi}{2}$ is of the form CA, Fig. 13, and by the repetition of this, we have the curve as represented.

21. The *pattern* in the above curves is symmetrical with regard to a line transverse to the line $\phi = 0$. But we may have patterns which are not thus symmetrical.

Let $y = \frac{\sin x}{1 + m \cos x}$, whence $\frac{dy}{dx} = \frac{m + \cos x}{(1 + m \cos x)^2}$. $(m < 1)$.

If x, y, be ordinary coordinates, these equations represent a curve sinuous, but each *sinus* not symmetrical. The angles at which the curve cuts the axis are alternately those of which the tangents are

$$\frac{1}{1+m}, \; -\frac{1}{1-m}.$$

Hence the descending side is more inclined than the ascending.

We shall obviously have a curve nearly resembling this, if we take the intrinsic equation $\phi = \frac{m + \cos s}{(1 + m\cos s)^2}$; which differs from the former by putting ϕ for $\tan\phi$, and s for x.

The curve will be a sinuous curve, inclined to the original line $\phi = 0$, at maximum angles $\phi = \frac{1}{1+m}$, $\phi = \frac{1}{1-m}$, on one side, and on the other, when $\cos s = \pm 1$. And if a be the arc in the first quadrant for which $\cos a = m$, $\phi = 0$ when $s = \pi - a, \pi + a, 3\pi - a, 3\pi + a$, &c.; and the curve will be as represented in Fig. 14.

For example, if $m = \frac{1}{3}$, the curve deflects alternately on the positive side, so that the angle of deflexion is $\frac{3}{4}$, and on the negative side, so that the angle is $\frac{3}{2}$; that is, the angles are respectively 43° and 86^{0} nearly.

We have, in this case, $$\frac{ds}{d\phi} = \frac{(1 + m\cos s)^3}{(2m^2 - 1 + m\cos s)\sin s}.$$

This is the radius of curvature, which becomes infinite when $s = 0, \pi, 2\pi$, &c.; that is, at A, A', A'', &c., when there are points of inflexion.

22. If we have $\phi = p \cdot \frac{m + \cos s}{(1 + m\cos s)^2}$

we shall still have a sinuous curve, and the greatest deflexions will be $\frac{p}{1+m}$, and $-\frac{p}{1-m}$.

Thus if $p = \frac{\pi}{2}$, and $m = \frac{1}{2}$, the greatest angles are

$$\frac{2}{3}\,\frac{\pi}{2}, \text{ and } -2\cdot\frac{\pi}{2}; \text{ that is } 60^{0} \text{ and } -180^{0}.$$

Hence the curve will be of the form represented in Fig. 15, making at A an angle of 60^{0} with the line $\phi = 0$, and at B, where $\cos s = -1$, the curve being parallel to $\phi = 0$, but in the opposite direction.

The radius of curvature is infinite at A and at B, and has a minimum value at some intermediate point, nearer to B.

23. It is easy to construct running patterns curves of this kind which have any given angles for their extreme deflexions. Thus let Fig. 16 represent a pattern curve which sinuates between the angles 60^{0} one way and $3 \times 90^{0} = 270^{0}$ the other. Then,

$$\frac{\pi}{3} = \frac{p}{1+m}, \quad \frac{3\pi}{2} = \frac{p}{1-m}; \quad \frac{1+m}{1-m} = \frac{9}{2}, \quad m = \frac{7}{11}, \quad p = \frac{6\pi}{11}.$$

And the curve is $\phi = \frac{6\pi}{11} \cdot \frac{m + \cos s}{(l + m \cos s)^2}, \quad \left(m = \frac{7}{11}\right)$.

The curve $\phi = \frac{\pi}{2} \frac{m + \cos s}{(l + m \cos s)^2}, \quad \left(m = \frac{5}{8}\right)$

gives the angles of deflection $= \frac{8}{13} \frac{\pi}{2}$ and $\frac{8}{3} \frac{\pi}{2}$; that is, $55\frac{1}{2}^\circ$ and 240°, which nearly resembles the last, and may also be represented by Fig. 16.

The curve may deflect through more than a circumference. Thus if $\phi = \frac{16\pi}{27} \frac{m + \cos s}{(l + m \cos s)^2}$, $\left(m = \frac{7}{9}\right)$, the greatest deflexions are $\frac{\pi}{3}$ and $\frac{8\pi}{3}$; that is 60° positive, and $360^\circ + 60^\circ$ negative. Hence the curve at A and B, Fig. 17, is parallel, at both points making an angle of 60 with $\phi = 0$.

Such a curve has a loop; C being the place of minimum radius of curvature, the curve opens both ways from C.

Of Diminishing Running-pattern Curves.

24. If $\phi = \sin s^2 a$, where a is a quadrant, we shall have a sinuous curve;

And if we make $s = 1, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{9}, \sqrt{11}$, &c.,

we shall have a series of points of inflexion in the curve. And since these values of s have for their differences

$$\sqrt{3} - 1, \quad \sqrt{5} - \sqrt{3}, \quad \sqrt{7} - \sqrt{5}, \quad \sqrt{9} - \sqrt{7}, \quad \sqrt{11} - \sqrt{9}, \quad \&c.$$

which are a decreasing series, it is evident that we shall have such a curve as Fig. 18, in which the lengths of the curve between the points of inflexion, AA', $A''A''$, $A''A'''$, &c. constantly decrease.

The same will be the case if $\phi = p \sin s^2 a$.

If p be large enough, such curves will have loops, like those represented by $\phi = p \sin s$.

Thus if $\phi = \frac{2\pi}{3} \sin s^2 a$, we shall have a curve such as Fig. 19. (See Fig. 12, which represents $\phi = \frac{2\pi}{3} \sin s$, as to its general form).

The lengths of the alternate loops will constantly diminish, and the whole curve will occupy a triangular space, like a writing-master's flourish.

25. We may have a similar flourish, but unsymmetrical, by taking, instead of $\sin s^2 a$, the expression $\frac{m + \cos s^2 a}{(1 + m \cos s^2 a)^2}$.

This will give a figure like Fig. 20.

Of Circularly-running Pattern Curves.

26. If we take the equation $\phi = p \sin \frac{s}{b} + \frac{s}{a}$,

we shall have the figure in which a curve such as $\phi = p \sin \frac{s}{b}$, runs along the circumference of a circle.

And in the same manner, by adding to the value of ϕ, in any of the other cases previously given, a term $\frac{s}{a}$, we have the equation to the pattern curve there considered, made to run round the circumference of a circle.

Thus $$\phi = \sin \frac{s}{b} + \frac{s}{a} \text{ gives such a figure as Fig. 21;}$$

$$\phi = \frac{\pi}{2} \frac{m + \cos s}{(l + m \cos s)^2} + \frac{s}{a} \text{ such a figure as Fig. 22,}$$

m being about $\frac{1}{2}$, as in Fig. 16.

27. The radius of the circle round which the pattern runs is less than a. When ϕ has gone through all its values, so that $s = 2\pi a$, the curve has not been laid along the circumference of the circle, but has, besides, followed all the sinuosities of the pattern.

Of the Catenary and Tractrix.

28. The intrinsic equations simplify the properties of these curves.

Fig. 23. Let CO be any arc of a Catenary from C the lowest point; OS, CS, tangents, OV vertical, meeting CS; therefore OSV is the triangle of the forces which support the weight of CO; and if O be the tension at C, expressed in length of the curve,

$$\frac{s}{a} = \frac{OV}{SV} = \tan OSV, \text{ and if } OSV = \phi,$$

$$s = a \tan \phi,$$

the equation to the catenary.

29. For the Tractrix, let PT be the tangent, AT the fixed line, PN, perpendicular on $AT = x$, $\tan NPT = p$. Then $PT = x\sqrt{1 + p^2} = c$, a constant, by hypothesis.

$$\text{Hence } x = \frac{c}{\sqrt{1+p^2}}; \quad \frac{dx}{dp} = -\frac{cp}{(1+p^2)^{\frac{3}{2}}}; \text{ also } (s \text{ being now } CP,) \frac{ds}{dx} = -\sqrt{1+p^2}.$$

$$\text{Therefore } \frac{ds}{dp} = \frac{cp}{1+p^2}.$$

But if ϕ be the angle of deflexion, beginning when the curve is perpendicular to AT, $p = \tan \phi$; therefore $\frac{dp}{d\phi} = \sec^2 \phi = 1 + p^2$.

$$\text{Hence } \frac{ds}{d\phi} = cp = c \tan \phi = c \frac{\sin \phi}{\cos \phi}.$$

This is the equation to the evolute. Integrating

$$s = c\, l \frac{1}{\cos \phi}; \text{ or } \cos \phi = e^{-\frac{s}{c}},$$

the equation to the tractrix.

30. It appears in this investigation, that the evolute of the tractrix is the catenary, a well-known property.

General Properties of the Intrinsic Equation.

31. Given the equation of a curve to rectangular coordinates x, y, to find the intrinsic equation.

Let $y = f(x)$: hence, $f'(x) = \dfrac{dy}{dx} = \dfrac{1}{\tan\phi}$, ϕ being 0 for y.

Hence x is known in terms of $\tan\phi$. Let $x = F(\tan\phi)$,

$$\text{Then } \frac{dx}{d\phi} = F'(\tan\phi) \times \sec^2\phi.$$

$$\text{Also } \frac{ds}{dx} = \operatorname{cosec}\phi.$$

$$\text{Hence } \frac{ds}{d\phi} = F'(\tan\phi) \cdot \sec^2\phi \cdot \operatorname{cosec}\phi = \frac{F'(\tan\phi)}{\sin\phi \cdot \cos^2\phi}.$$

32. Examples. 1. *The Common Parabola.*

$$y^2 = 4ax; \quad \sqrt{\frac{a}{x}} = \frac{dy}{dx} = \frac{1}{\tan\phi}; \text{ hence } x = a\tan^2\phi.$$

$$\text{Hence } \frac{dx}{d\phi} = 2a\tan\phi \cdot \sec^2\phi = \frac{2a\sin\phi}{\cos^3\phi}.$$

$$\text{And } \frac{ds}{dx} = \frac{1}{\sin\phi}. \quad \text{Hence } \frac{ds}{d\phi} = \frac{2a}{\cos^3\phi}.$$

Hence the equation to the evolute of the parabola is $s' = \dfrac{2a}{\cos^3\phi}$.

The length of the parabola may be found by integrating $\dfrac{2a}{\cos^3\phi}$.

2. *The Semicubical Parabola.*

$$y^3 = ax^2, \quad \frac{dy}{dx} = \frac{2}{3} \cdot \frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}} = \frac{1}{\tan\phi}; \text{ hence } x = \frac{8a\tan^3\phi}{27}.$$

$$\frac{dx}{d\phi} = \frac{8a}{9}\tan^2\phi \cdot \sec^2\phi; \text{ and } \frac{ds}{dx} = \frac{1}{\sin\phi}; \text{ whence } \frac{ds}{d\phi} = \frac{8a}{9} \cdot \frac{\sin\phi}{\cos^4\phi};$$

the intrinsic equation to the curve.

Integrating, we have $s = \dfrac{8a}{27}\left\{\dfrac{1}{\cos^3\phi} - 1\right\}$.

3. *The Ellipse.*

$$y = \frac{b}{a}\sqrt{a^2 - x^2}, \quad \frac{dy}{dx} = -\frac{b}{a} \cdot \frac{x}{\sqrt{a^2 - x^2}} = \frac{1}{\tan\phi}. \quad \text{Hence } x = \frac{a^2}{\sqrt{a^2 + b^2\tan^2\phi}}.$$

$$\frac{dx}{d\phi} = -\frac{a^2b^2 \tan\phi . \sec^2\phi}{(a^2+b^2\tan^2\phi)^{\frac{3}{2}}} = -\frac{a^2 b^2 \sin\phi}{(a^2\cos^2\phi + b^2\sin^2\phi)^{\frac{3}{2}}}.$$

And $\frac{ds}{dx} = -\frac{1}{\sin\phi}$. Hence $\frac{ds}{d\phi} = \frac{a^2 b^2}{(a^2\cos^2\phi + b^2\sin^2\phi)^{\frac{3}{2}}}$, the intrinsic equation to the ellipse.

Hence the radius of curvature is $\frac{a^2 b^2}{(a^2\cos^2\phi + b^2\sin^2\phi)^{\frac{3}{2}}}$.

When $\phi = 0$, this radius is $\frac{b^2}{a}$; ϕ beginning at the extremity of the major axis, which was the supposition made.

The intrinsic equation to the evolute of the ellipse is

$$s' = \frac{a^2 b^2}{(a^2\cos^2\phi + b^2\sin^2\phi)^{\frac{3}{2}}} - \frac{b^2}{a},$$

if s' begin from a cusp, where $\phi = 0$.

33. Given the intrinsic equation, to find the equation to rectangular coordinates.

Let the coordinates x, y, be in the positions $\phi = 0$, $\phi = \frac{\pi}{2}$.

Then it is evident that $x = \int ds \,.\cos\phi$, $y = \int ds \,.\sin\phi$:

and the equation being given, these coordinates are found by integration.

Thus in the cycloid, $s = a \sin\phi$. Hence $ds = a\cos\phi \,. d\phi$;

$$x = \int a\cos^2\phi \,. d\phi, \quad y = \int a \sin\phi\cos\phi d\phi. \quad \text{Hence, integrating,}$$

$$x = \frac{a}{2}\sin\phi\cos\phi + \frac{a}{2}\phi, \quad y = \frac{a}{2}\sin^2\phi: \text{ the equation to the cycloid.}$$

34. In the running-pattern curves (Art. 17, &c.) of which the equation is

$$\phi = m\sin s, \text{ we have } \frac{d\phi}{ds} = m\cos s = \sqrt{(m^2-\phi^2)}: \frac{ds}{d\phi} = \frac{1}{\sqrt{(m^2-\phi^2)}}.$$

$$\text{Hence} \quad x = \int\frac{\cos\phi d\phi}{\sqrt{(m^2-\phi^2)}}; \quad y = \int\frac{\sin\phi d\phi}{\sqrt{(m^2-\phi^2)}}.$$

If these could be integrated, we could find the dimensions of the loops in Figures 11, 12, 13.

There is one case for which $\int\frac{\cos\phi d\phi}{\sqrt{(m^2-\phi^2)}}$ taken from $\phi = 0$ to $\phi = m$ is $= 0$. In this case the curve neither runs forward as in Fig. 11, nor backward as in Fig. 12, but is simply two loops. Fig. 24.

35. The following proposition, enunciated by John Bernoulli and proved by Euler, may easily be proved by means of the intrinsic equation.

If AB be any curve, AB' its involute beginning from A, $B'A'$ the involute of AB' beginning from B', $A'B''$ the involute of $A'B'$ beginning from A'; and so on alternately and indefinitely: the successive involutes approach indefinitely to the form of the common cycloid, provided the tangents at A and B in the original curve are perpendicular to each other.

(The proof of this is here omitted, being included in the proof of the extended propositions given in the *Additional Note.*)

36. The following proposition may be proved by the intrinsic equation.

Fig. 25. Let any curve be evolved, and its involute evolved, and the involute of that evolved, and so on, beginning always the evolution with a rectilinear *tail*, AA', extending beyond the curve, and all these tails being equal. The curve tends continually to the form of the equiangular spiral.

Let s, s', s'', s''', &c. be the successive curves, ϕ the angle, which is the same for all, beginning from 0 for each. And let each of the tails AA', $A'A''$, $A''A'''$, &c. $= a$.

Let $s = a_1\phi + a_2\phi^2 + a_3\phi^3 +$ &c., which may express any curve.

$$\text{Then}\quad s' = \int (a + s)\, d\phi = a\phi + \frac{a_1}{2}\phi^2 + \frac{a_2}{3}\phi^3 + \frac{a_3}{4}\phi^4 + \&c.$$

$$s'' = \int (a + s')\, d\phi = a\phi + \frac{a}{1\,.\,2}\phi^2 + \frac{a_1}{2\,.\,3}\phi^3 + \frac{a_2}{3\,.\,4}\phi^4 + \&c.$$

$$s''' = \int (a + s'')\, d\phi = a\phi + \frac{a}{1\,.\,2}\phi^2 + \frac{a}{1\,.\,2\,.\,3}\phi^3 + \frac{a_1\phi^4}{2\,.\,3\,.\,4} + \&c.$$

And as the operation goes on, the terms in a_1, a_2, a_3, &c. being divided by the factorials $2\,.\,3\,.\,4$, &c. indefinitely, may be neglected as to their influence on the curve. Therefore ultimately

$$s = a\phi + \frac{a^2}{1\,.\,2}\phi^2 + \frac{a^3}{1\,.\,2\,.\,3}\phi^3 + \&c. = a\left[\epsilon^{\phi} - 1\right],$$

which is (Art. 7) the equation to the equiangular spiral.

Of course, from the nature of the construction, the curvature of the original curve is throughout towards the same side*.

Additional Note to a Memoir on the Intrinsic Equation of Curves.

TRINITY COLLEGE, *April* 12, 1849.

In the Memoir on the Intrinsic Equation of Curves, I gave a proof of the following Proposition, which was enunciated by John Bernoulli, and demonstrated by Euler. (*Novi Comm. Petrop.* Tom. x.)

Fig. 26 and 27. If AB be any curve, AB' its involute beginning from A, $B'A'$ the involute of AB' beginning from B', $A'B''$ the involute of $A'B'$, beginning from A'; and so on, alternately and indefinitely: the successive involutes approach indefinitely to the form of the common cycloid. *provided the tangents at* A *and* B *in the original curve are perpendicular to each other.*

The question naturally offers itself, What is the curve to which the successive involutes tend, if the original curve do not conform to the condition above stated, that the total deflexion is a right angle?

I am now able to state that in that case the curve will be an *epicycloid* or a *hypocycloid* as the total deflexion is greater or less than a right angle.

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* Also it is necessary, as has been remarked to me, that the point where $\phi = 0$, is not a point of contrary flexure from the original curve, or any of its evolutes *in infinitum*. For if it were, some of the quantities a_1, a_2, &c. would be infinite.

The proof of this extension of Bernoulli's proposition easily follows from the mode of representing curves by their Intrinsic Equation, namely, the equation between the tangent and the deflexion.

Let AOB be any curve, and let the tangents at A and at B make with each other any angle $m\alpha$, α being a right angle.

Let APB', $B'O'A'$, $A'P'B''$, $B''O''A''$, &c. be the successive involutes, beginning alternately at opposite ends.

Let $AB' = b_1$, $A'B'' = b_2$, &c. the whole arcs of the alternate involutes.

Let the intrinsic equation to AOB be

$$s = a_1\phi + \frac{a_2}{1\,.\,2}\phi^2 + \frac{a_3}{1\,.\,2\,.\,3}\phi^3 + \text{\&c., which may express any curve.}$$

$$\text{Hence } AP = t_1 = \frac{a_1}{1\,.\,2}\phi^2 + \frac{a_2}{1\,.\,2\,.\,3}\phi^3 + \frac{a_3}{1\,.\,2\,.\,3\,.\,4}\phi^4 + \text{\&c.}$$

$$\text{And } B'P = b_1 - \frac{a_1}{1\,.\,2}\phi^2 - \frac{a_2}{1\,.\,2\,.\,3}\phi^3 - \frac{a_3}{1\,.\,2\,.\,3\,.\,4}\phi^4 - \text{\&c.};$$

$$\therefore\ A'O' = s_1 = \int PO'.\,d\phi = \int B'P.\,d\phi, \text{ beginning from } \phi = 0 \text{ at } A';$$

$$= b_1\phi - \frac{a_1\phi^3}{1\,.\,2\,.\,3} - \frac{a_2}{1\,.\,2}\phi^4 - \text{\&c.}$$

In like manner, if $A'P' = t_2$, $A''O'' = s_2$, &c.

$$t_2 = \frac{b_1}{1\,.\,2}\phi^2 + \frac{a_1}{1\,.\,2\,.\,3\,.\,4}\phi^4 + \text{\&c.}$$

$$s_2 = b_2\phi - \frac{b_1}{1\,.\,2\,.\,3}\phi^3 - \frac{a_1}{1\,\ldots\,5}\phi^5 + \text{\&c.}$$

$$t_3 = \frac{b_2}{1\,.\,2}\phi^2 - \frac{b_1}{1\,.\,2\,.\,3\,.\,4}\phi^4 - \frac{a_1}{1\,\ldots\,6}\phi^6 + \text{\&c.}$$

$$s_3 = b_3\phi - \frac{b_2}{1\,.\,2\,.\,3}\phi^3 + \frac{b_1}{1\,\ldots\,5}\phi^5 + \frac{a_1}{1\,\ldots\,7}\phi^7 + \text{\&c.}$$

......

$$s_n = b_n\phi - \frac{b_{n-1}}{1\,.\,2\,.\,3}\phi^3 + \frac{b_{n-2}}{1\,\ldots\,n}\phi^n - \text{\&c.} \pm \frac{a_1}{1\,\ldots\,(2n+1)}\phi^{2n+1} \pm \text{\&c.}$$

Now as n becomes larger, the terms in a_1, a_2, &c. which have for denominators the factorials $1\,.\,2\,.\,3\,\ldots\,(n-1)$ &c. become smaller and smaller, and thus the arc s_n depends less and less upon the form of the original arc AOB. Hence we may ultimately omit those terms.

Of the arcs $t_1, t_2, \ldots t_n$, each vanishes when $\phi = 0$; and when $\phi = m\alpha$, they become respectively $b_1, b_2, \ldots b_n$. Hence, by the expression for t_n, ultimately

$$b_n = b_{n-1}\frac{m^2\alpha^2}{1\,.\,2} - b_{n-2}\frac{m^4\alpha^4}{1\,.\,2\,.\,3\,.\,4} + b_{n-3}\frac{m^6\alpha^6}{1\,\ldots\,6} - \text{\&c.}$$

This expresses a relation among the successive arcs, b_1, b_2, $b_3 \ldots b_n$, which relation, it appears, is ultimately independent of the form of the curve AB. But since α is a quadrant, and $\cos\alpha = 0$, we have

$$0 = 1 - \frac{a^2}{1\,.\,2} + \frac{a^4}{1\,\ldots\,4} - \frac{a^6}{1\,\ldots\,6} + \&c.$$

whence

$$b_n = b_n \frac{a^2}{1\,.\,2} - b_n \frac{a^n}{1\,\ldots\,4} + b_n \frac{a^6}{1\,\ldots\,6} - \&c.$$

Hence the necessary relation among b_1, b_2, b_3, &c. is satisfied if

$$b_n = m^2 b_{n-1} = m^n b_{n-2} = m^6 b_{n-3} \ldots\ldots = m^{2r} b_{n-r},$$

that is, if $b_{n-1} = \frac{1}{m^2} b_n$, $b_{n-2} = \frac{1}{m^n} b_n$, $b_{n-3} = \frac{1}{m^6} b_n$, &c.

Hence we have, by the expression for s_n, ultimately,

$$s_n = m b_n \left\{ \frac{\phi}{m} - \frac{1}{1\,.\,2\,.\,3} \frac{\phi^3}{m^3} + \frac{1}{1\,\ldots\,5} \frac{\phi^n}{m^n} - \&c. \right\}$$

$$= m b_n \sin \frac{\phi}{m}.$$

This is the equation to an epicycloid, if $m > 1$; and to an hypocycloid, if $m < 1$.

If A and B be the radius of the *globe* and *wheel* of the epicycloid;

$$s = \frac{4B\,(A + B)}{A} \sin \frac{A}{A + 2B}\,\phi.$$

$$\text{Hence } m = \frac{A + 2B}{A}; \quad \frac{B}{A} = \frac{m - 1}{2}.$$

$$\text{For the hypocycloid, } s = \frac{4B\,(A - B)}{A} \sin \frac{A}{A - 2B}\,\phi.$$

$$\text{Hence } m = \frac{A - 2B}{A}; \quad \frac{B}{A} = \frac{1 - m}{2}.$$

In the figures, the angle $ACB = (m - 1)\,a$, when $m > 1$:

$ACB = (1 - m)\,a$, when $m < 1$.

L. *On the Variation of Gravity at the Surface of the Earth.* *By* G. G. Stokes, M.A., *Fellow of Pembroke College, Cambridge.*

[Read *April* 23, 1849.]

On adopting the hypothesis of the earth's original fluidity, it has been shewn that the surface ought to be perpendicular to the direction of gravity, that it ought to be of the form of an oblate spheroid of small ellipticity, having its axis of figure coincident with the axis of rotation, and that gravity ought to vary along the surface according to a simple law, leading to the numerical relation between the ellipticity and the ratio between polar and equatoreal gravity which is known by the name of Clairaut's Theorem. Without assuming the earth's original fluidity, but merely supposing that it consists of nearly spherical strata of equal density, and observing that its surface may be regarded as covered by a fluid, inasmuch as all observations relating to the earth's figure are reduced to the level of the sea, Laplace has established a connexion between the form of the surface and the variation of gravity, which in the particular case of an oblate spheroid agrees with the connexion which is found on the hypothesis of original fluidity. The object of the first portion of this paper is to establish this general connexion without making any hypothesis whatsoever respecting the distribution of matter in the interior of the earth, but merely assuming the theory of universal gravitation. It appears that if the form of the surface be given, gravity is determined throughout the whole surface, except so far as regards one arbitrary constant which is contained in its complete expression, and which may be determined by the value of gravity at one place. Moreover the attraction of the earth at all external points of space is determined at the same time; so that the earth's attraction on the moon, including that part of it which is due to the earth's oblateness, and the moments of the forces of the sun and moon tending to turn the earth about an equatoreal axis, are found quite independently of the distribution of matter within the earth.

The near coincidence between the numerical values of the earth's ellipticity deduced independently from measures of arcs, from the lunar inequalities which depend on the earth's oblateness, and, by means of Clairaut's Theorem, from pendulum experiments, is sometimes regarded as a confirmation of the hypothesis of original fluidity. It appears, however, that the form of the surface (which is supposed to be a surface of equilibrium,) suffices to determine both the variation of gravity and the attraction of the earth on an external particle*, and therefore the coincidence in question, being a result of the law of gravitation, is no confirmation of the hypothesis of original fluidity. The evidence in favour of this hypothesis which is derived from the figure and attraction of the earth consists in the perpendicularity of the surface to the direction of gravity, and in the circumstance that the surface is so nearly represented by an oblate spheroid having for its axis the axis of rotation. A certain degree of additional evidence is afforded by the near agreement between

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* It has been remarked by Professor O'Brien, (*Mathematical Tracts*, p. 56) that if we have given the form of the earth's surface and the variation of gravity, we have data for determining the attraction of the earth on an external particle, the earth being supposed to consist of nearly spherical strata of equal density; so that the motion of the moon furnishes no additional confirmation of the hypothesis of original fluidity.

If we have given the component of the attraction of any mass, however irregular as to its form and interior constitution, in a direction perpendicular to the surface, throughout the whole of the surface, we have data for determining the attraction at every external point, as well as the components of the attraction at the surface in two directions perpendicular to the normal. The corresponding proposition in Fluid Motion is self-evident.

the observed ellipticity and that calculated with an assumed law of density which is likely *a priori* to be not far from the truth, and which is confirmed, as to its general correctness, by leading to a value for the annual precession which does not much differ from the observed value.

Since the earth's actual surface is not strictly a surface of equilibrium, on account of the elevation of the continents and islands above the sea level, it is necessary to consider in the first instance in what manner observations would have to be reduced in order to render the preceding theory applicable. It is shewn in Art. 13 that the earth may be regarded as bounded by a surface of equilibrium, and therefore the expressions previously investigated may be applied, provided the sea level be regarded as the bounding surface, and observed gravity be reduced to the level of the sea by taking account only of the change of distance from the earth's centre. Gravity reduced in this manner would, however, be liable to vary irregularly from one place to another, in consequence of the attraction of the land between the station and the surface of the sea, supposed to be prolonged underground, since this attraction would be greater or less according to the height of the station above the sea level. In order therefore to render the observations taken at different places comparable with one another, it seems best to correct for this attraction in reducing to the level of the sea; but since this additional correction is introduced in violation of the theory in which the earth's surface is regarded as one of equilibrium, it is necessary to consider what effect the habitual neglect of the small attraction above mentioned produces on the values of mean gravity and of the ellipticity deduced from observations taken at a number of stations. These effects are considered in Arts. 17, 18.

Besides the consideration of the mode of determining the values of mean gravity, and thereby the mass of the earth, and of the ellipticity, and thereby the effect of the earth's oblateness on the motion of the moon, it is an interesting question to consider whether the observed anomalies in the variation of gravity may be attributed wholly or mainly to the irregular distribution of land and sea at the surface of the earth, or whether they must be referred to more deeply seated causes. In Arts. 19, 20, I have considered the effect of the excess of matter in islands and continents, consisting of the matter which is there situated above the actual sea level, and of the defect of matter in the sea, consisting of the difference between the mass of the sea, and the mass of an equal bulk of rock or clay. It appears that besides the attraction of the land lying immediately underneath a continental station, between it and the level of the sea, the more distant portions of the continent cause an increase in gravity, since the attraction which they exert is not wholly horizontal, on account of the curvature of the earth. But besides this direct effect, a continent produces an indirect effect on the magnitude of apparent gravity. For the horizontal attraction causes the verticals to point more inwards, that is, the zeniths to be situated further outwards, than if the continent did not exist; and since a level surface is everywhere perpendicular to the vertical, it follows that the sea level on a continent is higher than it would be at the same place if the continent did not exist. Hence, in reducing an observation taken at a continental station to the level of the sea, we reduce it to a point more distant from the centre of the earth than if the continent were away; and therefore, on this account alone, gravity is less on a continent than on an island. It appears that this latter effect more than counterbalances the former, so that on the whole, gravity is less on a continent than on an island, especially if the island be situated in the middle of an ocean. This circumstance has already been noticed as the result of observation. In consequence of the inequality to which gravity is subject, depending on the character of the station, it is probable that the value of the ellipticity which Mr. Airy has deduced from his discussion of pendulum observations is a little too great, on account of the decided preponderance of oceanic stations in low latitudes among the group of stations where the observations were taken.

The alteration of attraction produced by the excess and defect of matter mentioned in the preceding paragraph does not constitute the whole effect of the irregular distribution of land and sea, since if the continents were cut off at the actual sea level, and the sea were replaced by rock and clay, the surface so formed would no longer be a surface of equilibrium, in consequence of the change produced in the attraction. In Arts 25—27, I have investigated an expression for the reduction of observed gravity to what would be observed if the elevated solid portions of the earth were to become fluid, and to run down, so as to form a level bottom for the sea, which in that case would cover the whole earth. The expressions would be very laborious to work out numerically, and besides, they require data, such as the depth of the sea in a great many places, &c., which we do not at present possess; but from a consideration of the general character of the correction, and from the estimation given in Art. 21 of the magnitude which such corrections are likely to attain, it appears probable that the observed anomalies in the variation of gravity are mainly due to the irregular distribution of land and sea at the surface of the earth.

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1. Conceive a mass whose particles attract each other according to the law of gravitation, and are besides acted on by a given force f, which is such that if X, Y, Z be its components along three rectangular axes, $Xdx + Ydy + Zdz$ is the exact differential of a function U of the co-ordinates. Call the surface of the mass S, and let V be the potential of the attraction, that is to say, the function obtained by dividing the mass of each attracting particle by its distance from the point of space considered, and taking the sum of all such quotients. Suppose S to be a surface of equilibrium. The general equation to such surfaces is

$$V + U = c, \quad \dots\dots\dots\dots\dots (1)$$

where c is an arbitrary constant; and since S in included among these surfaces, equation (1) must be satisfied at all points of the surface S, when some one particular value is assigned to c. For any point external to S, the potential V satisfies, as is well known, the partial differential equation

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} = 0; \quad \dots\dots\dots (2)$$

and evidently V cannot become infinite at any such point, and must vanish at an infinite distance from S. Now these conditions are sufficient for the complete determination of the value of V for every point external to S, the quantities U and c being supposed known. The mathematical problem is exactly the same as that of determining the permanent temperature in a homogeneous solid, which extends infinitely around a closed space S, on the conditions, (1) that the temperature at the surface S shall be equal to $c - U$, (2) that it shall vanish at an infinite distance. This problem is evidently possible and determinate. The possibility has moreover been demonstrated mathematically.

If U alone be given, and not c, the general value of V will contain one arbitrary constant, which may be determined if we know the value of V, or of one of its differential coefficients, at one point situated either in the surface S or outside it. When V is known, the components of the force of attraction will be obtained by mere differentiation.

Nevertheless, although we know that the problem is always determinate, it is only for a very limited number of forms of the surface S that the solution has hitherto been effected. The most important of these forms is the sphere. When S has very nearly one of these forms the problem may be solved by approximation.

2. Let us pass now to the particular case of the earth. Although the earth is really revolving about its axis, so that the bodies on its surface are really describing circular orbits

about the axis of rotation, we know that the relative equilibrium of the earth itself, or at least its crust, and the bodies on its surface, would not be affected by supposing the crust at rest, provided that we introduce, in addition to the attraction, that fictitious force which we call the centrifugal force. The vertical at any place is determined by the plumb-line, or by the surface of standing fluid, and its determination is therefore strictly a question of relative equilibrium. The intensity of gravity is determined by the pendulum; but although the result is not mathematically the same as if the earth were at rest and acted on by the centrifugal force, the difference is altogether insensible. It is only in consequence of its influence on the direction and magnitude of the force of gravity that the earth's actual motion need be considered at all in this investigation: the mere question of attraction has nothing to do with motion; and the results arrived at will be equally true whether the earth be solid throughout or fluid towards the centre, even though, on the latter supposition, the fluid portions should be in motion relatively to the crust.

We know, as a matter of observation, that the earth's surface is a surface of equilibrium, if the elevation of islands and continents above the level of the sea be neglected. Consequently the law of the variation of gravity along the surface is determinate, if the form of the surface be given, the force f of Art. 1 being in this case the centrifugal force. The nearly spherical form of the surface renders the determination of the variation easy.

3. Let the earth be referred to polar co-ordinates, the origin being situated in the axis of rotation, and coinciding with the centre of a sphere which nearly represents the external surface. Let r be the radius vector of any point, θ the angle between the radius vector and the northern direction of the axis, ϕ the angle which the plane passing through these two lines makes with a plane fixed in the earth and passing through the axis. Then the equation (2) which V has to satisfy at any external point becomes by a common transformation

$$r\frac{d^2.rV}{dr^2}+\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right)+\frac{1}{\sin^2\theta}\frac{d^2V}{d\phi^2}=0. \quad\text{..............(3)}$$

Let ω be the angular velocity of the earth; then $U=\frac{\omega^2}{2}r^2\sin^2\theta$, and equation (1) becomes

$$V+\frac{\omega^2}{2}r^2\sin^2\theta=c, \quad\text{..................................(4)}$$

which has to be satisfied at the surface of the earth.

For a given value of r, greater than the radius of the least sphere which can be described about the origin as centre so as to lie wholly without the earth, V can be expanded in a series of Laplace's coefficients

$$V_0+V_1+V_2+\dots;$$

and therefore in general, provided r be greater than the radius of the sphere above mentioned, V can be expanded in such a series, but the general term V_n will be a function of r, as well as of θ and ϕ. Substituting the above series in equation (3), and observing that from the nature of Laplace's coefficients

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV_n}{d\theta}\right)+\frac{1}{\sin^2\theta}\frac{d^2V_n}{d\phi^2}=-n(n+1)V_n, \quad\text{..............(5)}$$

we get

$$\Sigma\left\{r\frac{d^2.rV_n}{dr^2}-n(n+1)V_n\right\}=0,$$

where all integral values of n from 0 to ∞ are to be taken.

Now the differential coefficients of V_n with respect to r are Laplace's coefficients of the n^{th} order as well as V_n itself; and since a series of Laplace's coefficients cannot be equal to zero unless the Laplace's coefficients of the same order are separately equal to zero, we must have

$$r\frac{d^2 . rV_n}{dr^2} - n(n+1)V_n = 0 \text{.....................} (6)$$

The integral of this equation is

$$V_n = \frac{Y_n}{r^{n+1}} + Z_n r^n,$$

where Y_n and Z_n are arbitrary constants so far as r is concerned, but contain θ and ϕ. Since these functions are multiplied by different powers of r, V_n cannot be a Laplace's coefficient of the n^{th} order unless the same be true of Y_n and Z_n. We have for the complete value of V

$$\frac{Y_0}{r} + \frac{Y_1}{r^2} + \frac{Y_2}{r^3} + \ldots + Z_0 + Z_1 r + \ldots\ldots$$

Now V vanishes when $r = \infty$, which requires that $Z_0 = 0$, $Z_1 = 0$, &c.; and therefore

$$V = \frac{Y_0}{r} + \frac{Y_1}{r^2} + \frac{Y_2}{r^3} + \text{.......................} (7)$$

4. The preceding equation will not give the value of the potential throughout the surface of a sphere which lies partly within the earth, because although V, as well as any arbitrary but finite function of θ and ϕ, can be expanded in a series of Laplace's coefficients, the second member of equation (3) is not equal to zero in the case of an internal particle, but to $-4\pi\rho r^2$, where ρ is the density. Nevertheless we may employ equation (7) for values of r corresponding to spheres which lie partly within the earth, provided that in speaking of an internal particle we slightly change the signification of V, and interpret it to mean, not the actual potential, but what would be the potential if the protuberant matter were distributed within the least sphere which cuts the surface, in such a manner as to leave the potential unchanged throughout the actual surface. The possibility of such a distribution will be justified by the result, provided the series to which we are led prove convergent. Indeed, it might easily be shewn that the potential at any internal point near the surface differs from what would be given by (7) by a small quantity of the second order only; but its differential coefficient with respect to r, which gives the component of the attraction along the radius vector, differs by a small quantity of the first order. We do not, however, want the potential at any point of the interior, and in fact it cannot be found without making some hypothesis as to the distribution of the matter within the earth.

5. It remains now to satisfy equation (4). Let $r = a(1 + u)$ be the equation to the earth's surface, where u is a small quantity of the first order, a function of θ and ϕ. Let u be expanded in a series of Laplace's coefficients $u_0 + u_1 + \ldots$. The term u_0 will vanish provided we take for a the mean radius, or the radius of a sphere of equal volume. We may, therefore, take for the equation to the surface

$$r = a(1 + u_1 + u_2 + \ldots) \text{.......................} (8)$$

If the surface were spherical, and the earth had no motion of rotation, V would be independent of θ and ϕ, and the second member of equation (7) would be reduced to its first term. Hence, since the centrifugal force is a small quantity of the first order, as well as u, the succeeding terms must be small quantities of the first order; so that in substituting in (7) the value of r given by (8) it will be sufficient to put $r = a$ in these terms. Since the second term in equation (4) is a small quantity of the first order, it will be sufficient in that term likewise to put $r = a$. We thus get from (4), (7), and (8), omitting the squares of small quantities,

$$\frac{Y_0}{a}(1 - u_1 - u_2 - u_3 \ldots) + \frac{Y_1}{a^2} + \frac{Y_2}{a^3} + \ldots + \frac{\omega^2 a^2}{2}\sin^2\theta = c. \ldots\ldots\ldots (9)$$

The most general Laplace's coefficient of the order 0 is a constant; and we have

$$\sin^2\theta = \tfrac{2}{3} + (\tfrac{1}{3} - \cos^2\theta),$$

of which expression the two parts are Laplace's coefficients of the orders 0, 2, respectively. We thus get from (9), by equating to zero Laplace's coefficients of the same order,

$$Y_0 = ac - \tfrac{1}{3}\omega^2 a^3,$$
$$Y_1 = a\, Y_0 u_1,$$
$$Y_2 = a^2 Y_0 u_2 - \tfrac{1}{2}\omega^2 a^5 (\tfrac{1}{3} - \cos^2\theta),$$
$$Y_3 = a^3 Y_0 u_3, \text{ \&c.}$$

The first of these equations merely gives a relation between the arbitrary constants Y_0 and c; the others determine Y_1, Y_2, &c.; and we get by substituting in (7)

$$V = Y_0\left(\frac{1}{r} + \frac{a}{r^2}u_1 + \frac{a^2}{r^3}u_2 + \ldots\right) - \frac{\omega^2 a^5}{2r^3}(\tfrac{1}{3} - \cos^2\theta) \ldots\ldots\ldots\ldots (10)$$

6. Let g be the force of gravity at any point of the surface of the earth, dn an element of the normal drawn outwards at that point; then $g = -\frac{d}{dn}(V + U)$. Let ψ be the angle between the normal and the radius vector; then $g\cos\psi$ is the resolved part of gravity along the radius vector, and this resolved part is equal to $-\frac{d}{dr}(V + U)$. Now ψ is a small quantity of the first order, and therefore we may put $\cos\psi = 1$, which gives

$$g = -\frac{d}{dr}(V + U),$$

where, after differentiation, r is to be replaced by the radius vector of the surface, which is given by (8). We thus get

$$g = \frac{Y_0}{a^2}(1 - 2u_1 - 2u_2 - 2u_3 \ldots) + \frac{Y_0}{a^2}(2u_1 + 3u_2 + 4u_3 \ldots) - \tfrac{3}{2}\omega^2 a (\tfrac{1}{3} - \cos^2\theta) - \omega^2 a (\tfrac{2}{3} + \tfrac{1}{3} - \cos^2\theta),$$

which gives, on putting

$$\frac{Y_0}{a^2} - \tfrac{2}{3}\omega^2 a = G, \quad \frac{\omega^2 a}{G} = m, \ldots\ldots\ldots\ldots\ldots (11)$$

and neglecting squares of small quantities,

$$g = G\{1 - \tfrac{5}{2}m(\tfrac{1}{3} - \cos^2\theta) + u_2 + 2u_3 + 3u_4 \ldots\}. \ldots\ldots\ldots (12)$$

In this equation G is the mean value of g taken throughout the whole surface, since we know that $\int_0^\pi \int_0^{2\pi} u_n \sin\theta\, d\theta d\phi = 0$, if n be different from zero. The second of equations (11) shews that m is the ratio of the centrifugal force at a distance from the axis equal to the mean distance to mean gravity, or, which is the same, since the squares of small quantities are neglected, the ratio of the centrifugal force to gravity at the equator. Equation (12) makes known the variation of gravity when the form of the surface is given, the surface being supposed to be one of equilibrium; and, conversely, equation (8) gives the form of the surface if the variation of gravity be known. It may be observed that on the latter supposition there is nothing to determine u_1. The most general form of u_1 is

$$\alpha \sin\theta\cos\phi + \beta\sin\theta\sin\phi + \gamma\cos\theta,$$

where α, β, γ are arbitrary constants; and it is very easy to prove that the co-ordinates of the centre of gravity of the volume are equal to $a\alpha$, $a\beta$, $a\gamma$ respectively, the line from which θ is measured being taken for the axis of z, and the plane from which ϕ is measured for the plane of xz. Hence the term u_1 in (8) may be made to disappear by taking for origin the centre of gravity of the volume. It is allowable to do this even should the centre of gravity fall a little out of the axis of rotation, because the term involving the centrifugal force, being already a small quantity of the first order, would not be affected by supposing the origin to be situated a little out of the axis.

Since the variation of gravity from one point of the surface to another is a small quantity of the first order, its expression will remain the same whether the earth be referred to one origin or another nearly coinciding with the centre, and therefore a knowledge of the variation will not inform us what point has been taken for the origin to which the surface has been referred.

7. Since the angle between the vertical at any point and the radius vector drawn from the origin is a small quantity of the first order, and the angles θ, ϕ occur in the small terms only of equations (8), (10), and (12), these angles may be taken to refer to the direction of the vertical, instead of the radius vector.

8. If E be the mass of the earth, the potential of its attraction at a very great distance r is ultimately equal to $\frac{E}{r}$. Comparing this with (10), we get $Y_0 = E$, and therefore, from the first of equations (11),

$$E = G a^2 + \tfrac{2}{3}\omega^2 a^3 = G a^2 (1 + \tfrac{2}{3} m), \ldots\ldots (13)$$

which determines the mass of the earth from the value of G determined by pendulum experiments.

9. If we suppose that the surface of the earth may be represented with sufficient accuracy by an oblate spheroid of small ellipticity, having its axis of figure coincident with the axis of rotation, equation (8) becomes

$$r = a \left\{1 + \epsilon \left(\tfrac{1}{3} - \cos^2\theta\right)\right\}, \ldots\ldots\ldots\ldots\ldots (14)$$

where ϵ is a constant which may be considered equal to the ellipticity. We have therefore in this case $u_1 = 0$, $u_2 = \frac{1}{3} - \cos^2\theta$, $u_n = 0$ when $n > 2$; so that (12) becomes

$$g = G \left\{1 - \left(\tfrac{5}{2} m - \epsilon\right)\left(\tfrac{1}{3} - \cos^2\theta\right)\right\}, \ldots\ldots\ldots\ldots\ldots (15)$$

which equation contains Clairaut's Theorem. It appears also from this equation that the value of G which must be employed in (13) is equal to gravity at a place the square of the sine of whose latitude is $\frac{1}{3}$.

10. Retaining the same supposition as to the form of the surface, we get from (10), on replacing Y_0 by E, and putting in the small term at the end $\omega^2 a^5 = m G a^4 = m E a^2$,

$$V = \frac{E}{r} + \left(\epsilon - \tfrac{1}{2} m\right) \frac{E a^2}{r^3} \left(\tfrac{1}{3} - \cos^2\theta\right) \ldots\ldots\ldots\ldots (16)$$

Consider now the effect of the earth's attraction on the moon. The attraction of any particle of the earth on the moon, and therefore the resultant attraction of the whole earth, will be very nearly the same as if the moon were collected at her centre. Let therefore r be the distance of the centre of the moon from that of the earth, θ the moon's North Polar Distance, P the accelerating force of the earth on the moon resolved along the radius vector, Q the force perpendicular to the radius vector, which acts evidently in a plane passing through the earth's axis; then

$$P = -\frac{dV}{dr}, \quad Q = \frac{dV}{r d\theta},$$

whence we get from (16)

$$P = \frac{E}{r^2} + 3\,(\epsilon - \tfrac{1}{2}m)\,\frac{Ea^2}{r^4}\,(\tfrac{1}{3} - \cos^2\theta),\ Q = 2\,(\epsilon - \tfrac{1}{2}m)\,\frac{Ea^2}{r^4}\sin\theta\cos\theta \ldots\ldots(17)$$

The moving forces arising from the attraction of the earth on the moon will be obtained by multiplying by M, where M denotes the mass of the moon; and these are equal and opposite to the moving forces arising from the attraction of the moon on the earth. The component MQ of the whole moving force is equivalent to an equal and parallel force acting at the centre of the earth and a couple. The accelerating forces acting on the earth will be obtained by dividing by E; and since we only want to determine the relative motions of the moon and earth, we may conceive equal and opposite accelerating forces applied both to the earth and to the moon, which comes to the same thing as replacing E by $E + M$ in (17). If K be the moment of the couple arising from the attraction of the moon, which tends to turn the earth about an equatoreal axis, $K = MQr$, whence

$$K = 2\,(\epsilon - \tfrac{1}{2}m)\,\frac{MEa^2}{r^3}\sin\theta\cos\theta \ldots\ldots(18)$$

The same formula will of course apply, *mutatis mutandis*, to the attraction of the sun.

11. The spheroidal form of the earth's surface, and the circumstance of its being a surface of equilibrium, will afford us some information respecting the distribution of matter in the interior. Denoting by x', y', z' the co-ordinates of an internal particle whose density is ρ', and by x, y, z those of the external point of space to which V refers, we have

$$V = \iiint \frac{\rho'\,dx'\,dy'\,dz'}{\{(x - x')^2 + (y - y')^2 + (z - z')^2\}^{\frac{1}{2}}},$$

the integrals extending throughout the interior of the earth. Writing dm' for $\rho'\,dx'\,dy'\,dz'$, putting λ, μ, ν for the direction-cosines of the radius vector drawn to the point (x, y, z), so that $x = \lambda r$, $y = \mu r$, $z = \nu r$, and expanding the radical according to inverse powers of r, we get

$$V = \frac{1}{r}\iiint dm' + \Sigma\frac{\lambda}{r^2}\iiint x'\,dm' + \frac{1}{2r^3}\,\Sigma\,(3\lambda^2 - 1)\iiint x'^2\,dm' + \frac{3}{r^3}\,\Sigma\lambda\mu\iiint x'\,y'\,dm' + \ldots(19)$$

Σ denoting the sum of the three expressions necessary to form a symmetrical function. Comparing this expression for V with that given by (10), which in the present case reduces itself to (16), we get $Y_0 = \iiint dm' = E$, as before remarked, and

$$\iiint x'\,dm' = 0,\ \iiint y'\,dm' = 0,\ \iiint z'\,dm' = 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

$$\tfrac{1}{2}\,\Sigma\,(3\lambda^2 - 1)\iiint x'^2\,dm' + 3\,\Sigma\lambda\mu\iiint x'y'dm' = (\epsilon - \tfrac{1}{2}m)\,Ea^2\,(\tfrac{1}{3} - \cos^2\theta);\ldots\ldots(21)$$

together with other equations, not written down, obtained by equating to zero the coefficients of $\frac{1}{r^4}$, $\frac{1}{r^5}$ &c. in (19).

Equations (20) shew that the centre of gravity of the mass coincides with the centre of gravity of the volume. In treating equation (21), it is to be remarked that λ, μ, ν are not independent, but connected by the equation $\lambda^2 + \mu^2 + \nu^2 = 1$. If now we insert $\lambda^2 + \mu^2 + \nu^2$ as a coefficient in each term of (21) which does not contain λ, μ, or ν, the equation will become homogeneous with respect to λ, μ, ν, and will therefore only involve the two independent ratios which exist between these three quantities, and consequently we shall have to equate to zero the coefficients of corresponding powers of λ, μ, ν. By the transformation just mentioned, equation (21) becomes, since $\cos\theta = \nu$,

$$\Sigma\,(\lambda^2 - \tfrac{1}{2}\mu^2 - \tfrac{1}{2}\nu^2)\iiint x'^2\,dm' + 3\,\Sigma\lambda\mu\iiint x'y'dm' = (\epsilon - \tfrac{1}{2}m)\,Ea^2\,(\tfrac{1}{3}\lambda^2 + \tfrac{1}{3}\mu^2 - \tfrac{2}{3}\nu^2);$$

and we get

$$\iiint x'y'dm' = 0,\ \iiint y'z'dm' = 0,\ \iiint z'x'dm' = 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

$$\left.\begin{aligned}&\iiint x'^2dm' - \tfrac{1}{2}\iiint y'^2dm' - \tfrac{1}{2}\iiint z'^2dm' = \iiint y'^2dm' - \tfrac{1}{2}\iiint z'^2dm' - \tfrac{1}{2}\iiint x'^2dm' \\ &= -\tfrac{1}{2}\iiint z'^2dm' + \tfrac{1}{4}\iiint x'^2dm' + \tfrac{1}{4}\iiint y'^2dm' = \tfrac{1}{3}(\epsilon - \tfrac{1}{2}m)\,Ea^2.\end{aligned}\right\}\ldots\ldots(23)$$

Equations (22) shew that the co-ordinate axes are principal axes. Equations (23) give in the first place

$$\iiint x'^2 dm' = \iiint y'^2 dm',$$

which shews that the moments of inertia about the axes of x and y are equal to each other, as might have been seen at once from (22), since the principal axes of x and y are any two rectangular axes in the plane of the equator. The two remaining equations of the system (23) reduce themselves to one, which is

$$\iiint x'^2 dm' - \iiint z'^2 dm' = \tfrac{2}{3}(\epsilon - \tfrac{1}{2}m) E a^2.$$

If we denote the principal moments of inertia by A, A, C, this equation becomes

$$C - A = \tfrac{2}{3}(\epsilon - \tfrac{1}{2}m) E a^2, \quad\dots\dots\dots\dots\dots\dots\dots\dots \quad (24)$$

which reconciles the expression for the couple K given by (18) with the expression usually given, which involves moments of inertia, and which, like (18), is independent of any hypothesis as to the distribution of the matter within the earth.

It should be observed that in case the earth be not solid to the centre the quantities A, C must be taken to mean what would be the moments of inertia if the several particles of which the earth is composed were rigidly connected.

12. In the preceding article the surface has been supposed spheroidal. In the general case of an arbitrary form we should have to compare the expressions for V given by (10) and (19). In the first place it may be observed that the term u_1 can always be got rid of by taking for origin the centre of gravity of the volume. Equations (20) shew that in the general case, as well as in the particular case considered in the last article, the centre of gravity of the mass coincides with the centre of gravity of the volume.

Now suppress the term u_1 in u, and let $u = u' + u''$, where $u'' = \frac{1}{2}m(\frac{1}{3} - \cos^2\theta)$. Then u' may be expanded in a series of Laplace's coefficients $u'_2 + u'_3 + \dots$; and since $Y_0 = E$, equation (10) will be reduced to

$$V = E\left(\frac{1}{r} + \frac{a^2}{r^3}u'_2 + \frac{a^3}{r^4}u'_3 \dots\right) \quad\dots\dots\dots\dots\dots\dots \quad (25)$$

If the mass were collected at the centre of gravity, the second member of this equation would be reduced to its first term, which requires that $u'_2 = 0$, $u'_3 = 0$, &c. Hence (8) would be reduced to $r = a(1 + u'')$, and therefore au'' is the alteration of the surface due to the centrifugal force, and au' the alteration due to the difference between the actual attraction and the attraction of a sphere composed of spherical strata. Consider at present only the term u'_2 of u'. From the general form of Laplace's coefficients it follows that au'_2 is the excess of the radius vector of an ellipsoid not much differing from a sphere over that of a sphere having a radius equal to the mean radius of the ellipsoid. If we take the principal axes of this ellipsoid for the axes of co-ordinates, we shall have

$$u'_2 = \epsilon'(\tfrac{1}{3} - \sin^2\theta\cos^2\phi) + \epsilon''(\tfrac{1}{3} - \sin^2\theta\sin^2\phi) + \epsilon'''(\tfrac{1}{3} - \cos^2\theta),$$

$\epsilon', \epsilon'', \epsilon'''$ being three arbitrary constants, and θ, ϕ denoting angles related to the new axes of x, y, z in the same way that the angles before denoted by θ, ϕ were related to the old axes. Substituting the preceding expression for u'_2 in (25), and comparing the result with (19), we shall again obtain equations (22). Consequently the principal axes of the mass passing through the centre of gravity coincide with the principal axes of the ellipsoid. It will be found that the three equations which replace (23) are equivalent to but two, which are

$$A - \tfrac{2}{3}\epsilon' E a^2 = B - \tfrac{2}{3}\epsilon'' E a^2 = C - \tfrac{2}{3}\epsilon''' E a^2,$$

where A, B, C denote the principal moments.

The permanence of the earth's axis of rotation shews however that one of the principal axes of the ellipsoid coincides, at least very nearly, with the axis of rotation; although, strictly speaking, this conclusion cannot be drawn without further consideration except on the supposition that the earth is solid to the centre. If we assume this coincidence, the term $\epsilon'''(\frac{1}{3} - \cos^2\theta)$ will unite with the term u'' due to the centrifugal force. Thus the most general value of u is that which belongs to an ellipsoid having one of its principal axes coincident with the axis of rotation, added to a quantity which, if expanded in a series of Laplace's coefficients, would furnish no terms of the order 0, 1, or 2.

It appears from this and the preceding article that the coincidence of the centres of gravity of the mass and volume, and that of the axis of rotation and one of the principal axes of the ellipsoid whose equation is $r = a(1 + u_2)$, which was established by Laplace on the supposition that the earth consists of nearly spherical strata of equal density, holds good whatever be the distribution of matter in the interior.

13. Hitherto the surface of the earth has been regarded as a surface of equilibrium. This we know is not strictly true, on account of the elevation of the land above the level of the sea. The question now arises, By what imaginary alteration shall we reduce the surface to one of equilibrium?

Now with respect to the greater portion of the earth's surface, which is covered with water, we have a surface of equilibrium ready formed. The expression *level of the sea* has a perfectly definite meaning as applied to a place in the middle of a continent, if it be defined to mean the level at which the sea-water would stand if introduced by a canal. The surface of the sea, supposed to be prolonged in the manner just considered, forms indeed a surface of equilibrium, but the preceding investigation does not apply directly to this surface, inasmuch as a portion of the attracting matter lies outside it. Conceive however the land which lies above the level of the sea to be depressed till it gets below it, or, which is the same, conceive the land cut off at the level of the sea produced, and suppose the density of the earth or rock which lies immediately below the sea-level to be increased, till the increase of mass immediately below each superficial element is equal to the mass which has been removed from above it. The whole of the attracting matter will thus be brought inside the original sea-level; and it is easy to see that the attraction at a point of space external to the earth, even though it be close to the surface, will not be sensibly affected. Neither will the sea-level be sensibly changed, even in the middle of a continent. For, suppose the sea-water introduced by a pipe, and conceive the land lying above the sea-level condensed into an infinitely thin layer coinciding with the sea-level. The attraction of an infinite plane on an external particle does not depend on the distance of the particle from the plane; and if a line be drawn through the particle inclined at an angle a to the perpendicular let fall on the plane, and be then made to revolve around the perpendicular, the resultant attraction of the portion of the plane contained within the cone thus formed will be to that of the whole plane as versin a to 1. Hence the attraction of a piece of table-land on a particle close to it will be sensibly the same as that of a solid of equal thickness and density comprised between two parallel infinite planes, and that, even though the lateral extent of the table-land be inconsiderable, only equal, suppose, to a small multiple of the length of a perpendicular let fall from the attracted particle on the further bounding plane. Hence the attraction of the land on the water in the tube will not be sensibly altered by the condensation we have supposed, and therefore we are fully justified in regarding the level of the sea as unchanged.

The surface of equilibrium which by the imaginary displacement of matter just considered has also become the bounding surface, is that surface which at the same time coincides with the surface of the actual sea, where the earth is covered by water, and belongs to the system of surfaces of equilibrium which lie wholly outside the earth. To reduce observed gravity to what would have been observed just above this imaginary surface, we must evidently increase it in the inverse ratio of the square of the distance from the centre of the earth, without taking account of the attraction of the table-land which lies between the level of the station and the level of the sea. The question now arises, How shall we best determine the numerical value of the earth's ellipticity, and how best compare the form which results from observation with the spheroid which results from theory on the hypothesis of original fluidity?

14. Before we consider how the numerical value of the earth's ellipticity is to be determined, it is absolutely necessary that we define what we mean by ellipticity; for, when the irregularities of the surface are taken into account, the term must be to a certain extent conventional.

Now the attraction of the earth on an external body, such as the moon, is determined by the function V, which is given by (10). In this equation, the term containing r^{-2} will disappear if r be measured from the centre of gravity; the term containing r^{-4}, and the succeeding terms, will be insensible in the case of the moon, or a more distant body. The only terms, therefore, after the first, which need be considered, are those which contain r^{-3}. Now the most general value of u_2 contains five terms, multiplied by as many arbitrary constants, and of these terms one is $\frac{1}{3} - \cos^2\theta$, and the others contain as a factor the sine or cosine of ϕ or of 2ϕ. The terms containing $\sin\phi$ or $\cos\phi$ will disappear for the reason mentioned in Art. 12; but even if they did not disappear their effect would be wholly insensible, inasmuch as the corresponding forces go through their period in a day, a lunar day if the moon be the body considered. These terms therefore, even if they existed, need not be considered; and for the same reason the terms containing $\sin 2\phi$ or $\cos 2\phi$ may be neglected; so that nothing remains but a term which unites with the last term in equation (10). Let ϵ be the coefficient of the term $\frac{1}{3} - \cos^2\theta$ in the expansion of u: then ϵ is the constant which determines the effect of the earth's oblateness on the motion of the moon, and which enters into the expression for the moment of the attractions of the sun and moon on the earth; and in the particular case in which the earth's surface is an oblate spheroid, having its axis coincident with the axis of rotation, ϵ is the ellipticity. Hence the constant ϵ seems of sufficient dignity to deserve a name, and it may be called in any case the *ellipticity.*

Let r be the radius vector of the earth's surface, regarded as coincident with the level of the sea; and take for shortness $\mathfrak{M}\{f(\theta, \phi)\}$ to denote the mean value of the function $f(\theta, \phi)$ throughout all angular space, or $\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} f(\theta, \phi)\sin\theta\, d\theta\, d\phi$. Then it follows from the theory of Laplace's coefficients that

$$\epsilon = \frac{45}{4a}\mathfrak{M}.\{(\tfrac{1}{3} - \sin^2 l)\,r\}, \ldots\ldots\ldots\ldots\ldots (26)$$

l being the latitude, or the compliment of θ. To obtain this equation it is sufficient to multiply both sides of (8) by $\frac{1}{4\pi}(\frac{1}{3} - \cos^2\theta)\sin\theta d\theta\, d\phi$, and to integrate from $\theta = 0$ to $\theta = \pi$, and from $\phi = 0$ to $\phi = 2\pi$. Since $\frac{1}{3} - \cos^2\theta$ is a Laplace's coefficient of the second order, none of the terms at the second side of (8) will furnish any result except u_2, and even in the case of u_2 the terms involving the sine or cosine of ϕ or of 2ϕ will disappear.

15. Let g be gravity reduced to the level of the sea by taking account only of the height of the station. Then this is the quantity to which equation (12) is applicable; and putting for u_2 its value we get by means of the properties of Laplace's coefficients

$$G = \mathfrak{M}(g),\ G(\tfrac{5}{2}m - \epsilon) = -\frac{45}{4}\mathfrak{M}\{(\tfrac{1}{3} - \sin^2 l)\,g\} \ldots\ldots\ldots\ldots\ldots (27)$$

If we were possessed of the values of g at an immense number of stations scattered over the surface of the whole earth, we might by combining the results of observation in the manner indicated by equations (27) obtain the numerical values of G and ϵ. We cannot, however, obtain by observation the values of g at the surface of the sea, and the stations on land where the observations have been made from which the results are to be obtained are not very numerous. We must consider therefore in what way the variations of gravity due to merely local causes are to be got rid of, when we know the causes of disturbance; for otherwise a local irregularity, which would be lost in the mean of an immense number of observations, would require undue importance in the result.

16. Now the most obvious cause of irregularity consists in the attraction of the land lying between the level of the station and the level of the sea. This attraction would render the values of g sensibly different, which would be obtained at two stations only a mile or two apart, but situated at different elevations. To render our observations comparable with one another, it seems best to correct for the attraction of the land which lies underneath the pendulum; but then we must consider whether the habitual neglect of this attraction may not affect the mean values from which G and ϵ are to be found.

Let $g = g_1 + g'$, where g' is the attraction just mentioned, so that g_1 is the result obtained by reducing the observed value of gravity to the level of the sea by means of Dr. Young's formula*. Let h be the height of the station above the level of the sea, σ the superficial density of the earth where not covered by water; then by the formula for the attraction of an infinite plane we have $g' = 2\pi\sigma h$. To make an observation, conceived to be taken at the surface of the sea, comparable with one taken on land, the correction for local attraction would be additive, instead of subtractive; we should have in fact to add the excess of the attraction of a layer of earth or rock, of a thickness equal to the depth of the sea at that place, over the attraction of so much water. The formula $g' = 2\pi\sigma h$ will evidently apply to the surface of the sea, provided we regard h as a negative quantity, equal to the depth of the sea, and replace σ by $\sigma - 1$, the density of water being taken for the unit of density; or we may retain σ as the coefficient, and diminish the depth in the ratio of σ to $\sigma - 1$.

Let ρ be the mean density of the earth, then

$$g' = 2\pi\sigma h = G\frac{2\pi\sigma h}{\frac{4}{3}\pi\rho a} = G\frac{3\sigma h}{2\rho a}.$$

If we suppose $\sigma = 2\frac{1}{2}$, $\rho = 5\frac{1}{2}$, $a = 4000$ miles, and suppose h expressed in miles, with the understanding that in the case of the sea h is a negative quantity equal to $\frac{3}{5}$ths of the actual depth, we have $g' = .00017\,Gh$ nearly.

17. Consider first the value of G. We have by the preceding formula, and the first of equations (27),

$$G = \mathfrak{m}(g_1) + G \times .00017\,\mathfrak{m}(h).$$

According to Professor Rigaud's determination, the quantity of land on the surface of the earth is to that of water as 100 to 276†. If we suppose the mean elevation of the land $\frac{1}{5}$th of a mile, and the mean depth of the sea $3\frac{1}{2}$ miles, we shall have

$$\mathfrak{m}(h) = -\frac{\frac{3}{5} \times 3\frac{1}{2} \times 276 - \frac{1}{5} \times 100}{376} = -1.49 \text{ nearly};$$

so that the value of G determined by g_1 would be too great by about .000253 of the whole. Hence the mass of the earth determined by the pendulum would be too great by about the one four-thousandth of the whole; and therefore the mass of the moon, obtained by subtracting from the sum of the masses of the earth and moon, as determined by means of the coefficient of lunar parallax, the mass of the earth alone, as determined by means of the pendulum, would be too small by about the one four-thousandth of the mass of the earth, or about the one fiftieth of the whole.

18. Consider next the value of ϵ. Let ϵ_1 be the value which would be determined by substituting g_1 for g in (27), and let

* *Phil. Trans.* for 1819. Dr. Young's formula is based on the principle of taking into account the attraction of the table-land existing between the station and the level of the sea, in reducing the observation to the sea level. On account of this attraction, the multiplier $\left(\frac{2h}{a}\right)$ which gives the correction for elevation alone must be reduced in the ratio of 1 to $1-\frac{3\sigma}{4\rho}$, or 1 to .66 nearly, if $\sigma = 2\frac{1}{2}$, $\rho = 5\frac{1}{2}$. Mr Airy, observing that the value $\sigma = 2\frac{1}{2}$ is a little too small, and $\rho = 5\frac{1}{2}$ a little too great, has employed the factor .6, instead of .66.

† *Cambridge Philosophical Transactions*, Vol. VI. p. 297.

$$\frac{45}{4}\,\mathfrak{m}\left\{(\sin^2 l - \tfrac{1}{3})g'\right\} = Gq.$$

In considering the value of q we may attend only to the land, provided we transfer the defect of density of the sea with an opposite sign to the land, because if g' were constant, q would vanish. This of course proceeds on the supposition that the depth of the sea is constant. Since $\epsilon = \epsilon_1 - q$, if q were positive, the ellipticity determined by the pendulum would appear too great in consequence of the omission of the force g'. I have made a sort of rough integration by means of a map of the world, by counting the quadrilaterals of land bounded each by two meridians distant 10^0, and by two parallels of latitude distant 10^0, estimating the fraction of a broken quadrilateral which was partly occupied by sea. The number of quadrilaterals of land between two consecutive parallels, as for example 50^0 and 60^0, was multiplied by $12\,(\frac{1}{3} - \sin^2 l)\cos l$, or $3\cos 3l + \cos l$, where for l was taken the mean latitude, (55^0 in the example,) the sum of the results was taken for the whole surface, and multiplied by the proper coefficient. The north pole was supposed to be surrounded by water, and the south pole by land, as far as latitude 80^0. It appeared that the land lying beyond the parallels for which $\sin^2 l = \frac{1}{3}$, that is, beyond the parallels 35^0 N. and 35^0 S. nearly, was almost exactly neutralized by that which lay within those parallels. On the whole, q appeared to have a very small positive value, which on the same suppositions as before respecting the height of the land and the depth of the sea, was .0000012. It appears, therefore, that the omission of the force g' will produce no sensible increase in the value of ϵ, unless the land be on the whole higher, or the sea shallower, in high latitudes than in low. If the land had been collected in a great circular continent around one pole, the value of q would have been .000268; if it had been collected in a belt about the equator, we should have had $q = -.000362$. The difference between these values of q is about one fifth of the whole ellipticity.

19. The attraction g' is not the only irregularity in the magnitude of the force of gravity which arises from the irregularity in the distribution of land and sea, and in the height of the land and depth of the sea, although it is the only irregularity, arising from that cause, which is liable to vary suddenly from one point at the surface to another not far off. The irregular coating of the earth will produce an irregular attraction besides that produced by the part of this coating which lies under and in the immediate neighbourhood of the station considered, and it will moreover cause an irregular elevation or depression in the level of the sea, and thereby cause a diminution or increase in the value of g_1.

Consider the attraction arising from the land which lies above the level of the sea, and from the defect of attracting matter in the sea. Call this excess or defect of matter the *coating* of the earth: conceive the coating condensed into a surface coinciding with the level of the sea, and let $A\delta$ be the mass contained in a small element A of this surface. Then $\delta = \sigma h$ in the case of the land, and $\delta = -(\sigma - 1)h$ in the case of the sea, h being in that case the depth of the sea. Let V_c be the potential of the coating, V', V'' the values of V_c outside and inside the surface respectively. Conceive δ expanded in a series of Laplace's coefficients $\delta_0 + \delta_1 + \ldots$, then it is easily proved that

$$V' = 4\pi a^2\left(\frac{1}{r}\delta_0 + \frac{a}{3r^2}\delta_1 + \frac{a^2}{5r^3}\delta_2 + \ldots\right),\quad V'' = 4\pi a^2\left(\frac{1}{a}\delta_0 + \frac{r}{3a^2}\delta_1 + \ldots\right), \ldots\ldots\ldots (28)$$

r being the distance of the point considered from the centre. These equations give

$$\frac{dV'}{dr} = -4\pi\Sigma\frac{i+1}{2i+1}\left(\frac{a}{r}\right)^{i+2}\delta_i,\quad \frac{dV''}{dr} = 4\pi\Sigma\frac{i}{2i+1}\left(\frac{r}{a}\right)^{i-1}\delta_i. \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

Consider two points, one external, and the other internal, situated along the same radius vector very close to the surface. Let E be an element of this surface lying around the radius vector, an element which for clear ideas we may suppose to be a small circle of radius s, and let s be at the same time infinitely small compared with a, and infinitely great compared with the distance between the points. Then the limiting values of $\frac{dV'}{dr}$ and $\frac{dV''}{dr}$ will differ by the attraction of the element E, an attraction which, as follows from what was observed in Art. 13, will be ultimately the same as that of an infinite plane of the same density, or $2\pi\delta$*. The mean of the values of $\frac{dV'}{dr}$ and $\frac{dV''}{dr}$ will express the attraction of the general coating in the direction of the radius vector, the general coating being understood to mean the whole coating, with the exception of a superficial element lying adjacent to the points where the attraction is considered. Denoting this mean by $\frac{dV_c}{dr}$, we get, on putting $r = a$,

$$\frac{dV_c}{dr} = -2\pi\Sigma\frac{\delta_i}{2i+1}.$$

This equation becomes by virtue of either of the equations (28)

$$\frac{dV_c}{dr} = -\frac{V_c}{2a}, \quad\text{.........................} \quad (30)$$

which is a known equation. Let either member of this equation be denoted by $-g''$. Then gravity will be increased by g'', in consequence of the attraction of the general coating.

20. But besides its direct effect, the attraction of the coating will produce an indirect effect by altering the sea-level. Since the potential at any place is increased by V_c in consequence of the coating, in passing from what would be a surface of equilibrium if the coating were removed, to the actual surface of equilibrium corresponding to the same parameter, {that is, the same value of the constant c in equation (1),} we must ascend till the labouring force expended in raising a unit of mass is equal to V_c, that is, we must ascend through a space $\frac{V_c}{g}$, or $\frac{V_c}{G}$ nearly. In consequence of this ascent, gravity will be diminished by the quantity corresponding to the height $G^{-1}V_c$, or h' suppose. If we take account only of the alteration of the distance from the centre of the earth, this diminution will be equal to $G.\frac{2h'}{a}$, or $\frac{2V_c}{a}$, or $4g''$, and therefore the combined direct and indirect effects of the general coating will be to diminish gravity by $3g''$.

But the attraction of that portion of the stratum whose thickness is h', which lies immediately about the station considered, will be a quantity which involves h' as a factor, and to include this attraction we must correct for the change of distance h' by Dr. Young's rule, instead of correcting merely according to the square of the distance. In this way we shall get for the diminution of gravity due to the general coating, not $3g''$, but only $4\left(1-\frac{3\sigma}{4\rho}\right)g'' - g''$, or kg'' suppose. If $\sigma : \rho :: 5 : 11$, we have $k = 1.64$ nearly.

* This result readily follows from equations (28), which give, on putting $r=a$, $\frac{dV''}{dr} - \frac{dV'}{dr} = 4\pi\Sigma\delta_i = 4\pi\delta$. This difference of attraction at points infinitely close can evidently only arise from the attraction of the interposed element of surface, which, being ultimately plane, will act equally at both points; and, therefore, the attraction will be in each case $2\pi\delta$, and will act outwards in the first case, and inwards in the second.

If we cared to leave the mean value of gravity unaltered, we should have to use, instead of δ, its excess over its mean value δ_0. In considering, however, only the variation of gravity from one place to another, this is a point of no consequence.

21. In order to estimate the magnitude which the quantity $3g''$ is likely to attain, conceive two stations, of which the first is surrounded by land, and the second by sea, to the distance of 1000 miles, the distribution of land and sea beyond that distance being on the average the same at the two stations. Then, by hypothesis, the potential due to the land and sea at a distance greater than 1000 miles is the same at the two stations; and as we only care for the difference between the values of the potential of the earth's coating at the two stations, we may transfer the potential due to the defect of density at the second station with an opposite sign to the first station. We shall thus have around the first station, taking h' for the depth of the sea around the second station, $\delta = \sigma h + (\sigma - 1)\, h'$. In finding the difference V of the potentials of the coating, it will be amply sufficient to regard the attracting matter as spread over a plane disk, with a radius s equal to 1000 miles. On this supposition we get

$$V = \int_0^s s^{-1} . 2\pi\delta s ds = 2\pi\delta s.$$

Now $G = \frac{4}{3}\pi\rho a$, and therefore $3g'' = \dfrac{3V}{2a} = \dfrac{9\delta s}{4\rho a^2} G = \dfrac{9}{4} . \dfrac{\sigma h + (\sigma - 1) h'}{\rho a} . \dfrac{s}{a} G$. Making the same suppositions as before with regard to the numerical values of σ, ρ, h, h', and a, we get $3g'' = \cdot 000147\, G$. This corresponds to a difference of 6·35 vibrations a day in a seconds' pendulum. Now a circle with a radius of 1000 miles looks but small on a map of the world, so that we may readily conceive that the difference depending on this cause between the number of vibrations observed at two stations might amount to 15 or 20, that is 7.5 or 10 on each side of the mean, or even more if the height of the land or the depth of the sea be under-estimated. This difference will however be much reduced by using kg'' in place of $3g''$*.

22. The value of V_c at any station is expressed by a double integral, which is known if δ be known, and which may be calculated numerically with sufficient accuracy by dividing the surface into small portions and performing a summation. Theoretically speaking, V_c could be expressed for the whole surface at once by means of a series of Laplace's coefficients; the constants in this series could be determined by integration, or at least the approximate integration obtained by summation, and then the value of V_c could be obtained by substituting in the series the latitude and longitude of the given station for the general latitude and longitude. But the number of terms which would have to be retained in order to represent with tolerable accuracy the actual state of the earth's surface would be so great that the method, I apprehend, would be practically useless; although the leading terms of the series would represent the effect of the actual distribution of land and sea in its broad features. It seems better to form directly the expression for V_c at any station. This expression may be calculated numerically for each station by using the value of δ most likely to be correct, if the result be thought worth the trouble; but even if it be not calculated numerically, it will enable us to form a good estimation of the variation of the quantity $3g''$ or kg'' from one place to another.

Let the surface be referred to polar co-ordinates originating at the centre, and let the angles ψ, χ be with reference to the station considered what θ, ϕ were with reference to the north pole. The mass of a superficial element is equal to $\delta a^2 \sin\psi d\psi d\chi$, and its distance from the station is $2a \sin\dfrac{\psi}{2}$. Hence we have

$$V_c = a \iint \delta \cos\frac{\psi}{2} d\psi d\chi. \quad \ldots\ldots (31)$$

* The effect of the irregularity of the earth's surface is greater than what is represented by kg'', for a reason which will be explained further on (Art. 25).

Let δ_m be the mean value of δ throughout a circle with an angular radius ψ, then the part of V_c which is due to an annulus having a given infinitely small angular breadth $d\psi$ is proportional to $\delta_m \cos \frac{\psi}{2}$, or to δ_m nearly when ψ is not large. If we regard the depth of the sea as uniform, we may suppose $\delta = 0$ for the sea, and transfer the defect of density of the sea with an opposite sign to the land. We have seen that if we set a circle of land $\frac{1}{5}$ mile high of 1000 miles radius surrounding one station against a circle of sea $3\frac{1}{2}$ miles deep, and of the same radius, surrounding another, we get a difference of about $\frac{1}{3} \times 1.64 \times 6.35$, or $3\frac{1}{2}$ nearly, in the number of vibrations performed in one day by a seconds' pendulum. It is hardly necessary to remark that high table-land will produce considerably more effect than land only just raised above the level of the sea, but it should be observed that the principal part of the correction is due to the depth of the sea. Thus it would require a uniform elevation of about 2.1 miles, in order that the land elevated above the level of the sea should produce as much effect as is produced by the difference between a stratum of land $3\frac{1}{2}$ miles thick and an equal stratum of water.

23. These considerations seem sufficient to account, at least in a great measure, for the apparent anomalies which Mr. Airy has noticed in his discussion of pendulum experiments*. The first table at p. 230 contains a comparison between the observations which Mr. Airy considers first-rate and theory. The column headed "Error in Vibrations" gives the number of vibrations *per diem* in a seconds' pendulum corresponding to the excess of observed gravity over calculated gravity. With respect to the errors Mr. Airy expressly remarks "upon scrutinizing the errors of the first-rate observations, it would seem that, *cæteris paribus*, gravity is greater on islands than on continents." This circumstance appears to be fully accounted for by the preceding theory. The greatest positive errors appear to belong to oceanic stations, which is just what might be expected. Thus the only errors with the sign + which amount to 5 are, Isle of France + 7.0; Marian Islands + 6.8; Sandwich Islands + 5.2; Pulo Gaunsah Lout (a small island near new Guinea and almost on the equator,) + 5.0. The largest negative errors are, California − 6.0; Maranham − 5.6; Trinidad − 5.2. These stations are to be regarded as continental, because generally speaking the stations which are the most continental in character are but on the coasts of continents, and Trinidad may be regarded as a coast station. That the negative errors just quoted are larger than those that stand opposite to more truly continental stations such as Clermont, Milan, &c. is no objection, because the errors in such different latitudes cannot be compared except on the supposition that the value of the ellipticity used in the comparison is correct.

Now if we divide the 49 stations compared into two groups, an equatoreal group containing the stations lying between latitudes 35°N. and 35°S., and a polar group containing the rest, it will be found that most if not all of the oceanic stations are contained in the former group, while the stations belonging to the latter are of a more continental character. Hence the observations will make gravity appear too great about the equator and too small towards the poles, that is, they will on the whole make gravity vary too little from the equator to the poles; and since the variation depends upon $\frac{5}{2}m - \epsilon$, the observations will be best satisfied by a value of ϵ which is too great. This is in fact precisely the result of the discussion, the value of ϵ which Mr. Airy has obtained from the pendulum experiments (.003535) being greater than that which resulted from the discussion of geodetic measures (.003352), or than any of the values (.003370, .003360, and .003407), obtained from the two lunar inequalities which depend upon the earth's oblateness.

Mr. Airy has remarked that in the high north latitudes the greater number of errors have the sign +, and that those about the latitude 45° have the sign −; those about the equator being

* *Encyclopædia Metropolitana.* Art. Figure of the Earth.

nearly balanced. To destroy the errors in high and mean latitudes without altering the others, he has proposed to add a term $- A \sin^2\lambda \cos^2\lambda$, where λ is the latitude. But a consideration of the character of the stations seems sufficient, with the aid of the previous theory, to account for the apparent anomaly. About latitude 45^0 the stations are all continental; in fact, ten consecutive stations including this latitude are Paris, Clermont, Milan, Padua, Fiume, Bordeaux, Figeac, Toulon, Barcelona, New York. These stations *ought*, as a group, to appear with considerable negative errors. Mr. Airy remarks "If we increased the multiplier of $\sin^2\lambda$," and consequently diminished the ellipticity, "we might make the errors at high latitudes as nearly balanced as those at the equator: but then those about latitude 45^0 would be still greater than at present."

The largeness of the ellipticity used in the comparison accounts for the circumstance that the stations California, Maranham, Trinidad, appear with larger negative errors than any of the stations about latitude 45^0, although some of the latter appear more truly continental than the former. On the whole it would seem that the best value of the ellipticity is one which, supposing it left the errors in high latitudes nearly balanced, would give a decided preponderance to the negative errors about latitude 45^0 N. and a certain preponderance to the positive errors about the equator, on account of the number of oceanic stations which occur in low latitudes.

If we follow a chain of stations from the sea inland, or from the interior to the coast, it is remarkable how the errors decrease algebraically from the sea inwards. The chain should not extend over too large a portion of the earth's surface, as otherwise a small error in the assumed ellipticity might affect the result. Thus for example, Spitzbergen + 4.3, Hammerfest − 0.4, Drontheim − 2.7. In comparing Hammerfest with Drontheim, we may regard the former as situated at the vertex of a slightly obtuse angle, and the latter as situated at the edge of a straight coast. Again, Dunkirk − 0.1, Paris − 1.9, Clermont − 3.9, Figeac − 3.8, Toulon − 0.1, Barcelona 0.0, Fomentera + 0.2. Again, Padua + 0.7, Milan − 2.8. Again, Jamaica − 0.8, Trinidad − 5.2.

24. Conceive the correction kg'' calculated, and suppose it applied, as well as the correction $-g'$, to observed gravity reduced to the level of the sea, or to g, and let the result be $g_{\prime\prime}$. Let $\epsilon_{\prime\prime}$ be the ellipticity which would be determined by means of $g_{\prime\prime}$, $\epsilon_{\prime\prime} + \Delta\epsilon_{\prime\prime}$ the true ellipticity. Since $g_{\prime\prime} = g - g' + k'g'$, and therefore $g = g_{\prime\prime} + g' - kg''$, we get by (27)

$$\Delta\epsilon_{\prime\prime} = \frac{45}{4G}\mathfrak{m}\left\{\left(\tfrac{1}{3} - \sin^2 l\right)(g' - kg'')\right\}. \quad\ldots\ldots\ldots\ldots (32)$$

Now $g' = 2\pi\sigma h = 2\pi\delta = 2\pi\Sigma\delta_i$; and we get from (30) and (28)

$$kg'' = -k\frac{dV_c}{dr} = \frac{kV_c}{2a} = 2k\pi\Sigma\frac{\delta_i}{2i+1}.$$

All the terms δ_i will disappear from the second side of (32) except δ_2, and we therefore get

$$\Delta\epsilon_{\prime\prime} = \frac{45}{4G}\mathfrak{m}\left\{\left(\tfrac{1}{3} - \sin^2 l\right)\left(1 - \frac{k}{5}\right)2\pi\delta_2\right\}.$$

Hence the correction $\Delta\epsilon_{\prime\prime}$ is less than that considered in Art. 18, in the ratio of $5 - k$ to 5, and is therefore probably insensible on account of the actual distribution of land and water at the surface of the earth.

25. Conceive the islands and continents cut off at the level of the sea, and the water of the sea replaced by matter having the same density as the land. Suppose gravity to be observed at the surface which would be thus formed, and to be reduced by Dr Young's rule to the level of what would in the altered state of the earth be a surface of equilibrium. It is evident that $g_{\prime\prime}$ expresses the gravity which would be thus obtained.

The irregularities of the earth's coating would still not be wholly allowed for, because the surface which would be formed in the manner just explained would no longer be a surface of equilibrium, in consequence of the fresh distribution of attracting matter. The surface would thus preserve traces of its original irregularity. A repetition of the same process would give a surface still more regular, and so on indefinitely. It is easy to see the general nature of the correction which still remains. Where a small island was cut off, there was previously no material elevation of the sea-level, and therefore the surface obtained by cutting off the island and replacing the surrounding sea by land will be very nearly a surface of equilibrium, except in so far as that may be prevented by alterations which take place on a large scale. But where a continent is cut off there was a considerable elevation in the sea-level, and therefore the surface which is left will be materially raised above the surface of equilibrium which most nearly represents the earth's surface in its altered state. Hence the general effect of the additional correction will be to increase that part of g'' which is due to causes which act on a larger scale, and to leave nearly unaffected that part which is due to causes which are more local.

The form of the surface of equilibrium which would be finally obtained depends on the new distribution of matter, and conversely, the necessary distribution of matter depends on the form of the final surface. The determination of this surface is however easy by means of Laplace's analysis.

26. Conceive the sea replaced by solid matter, of density σ, having a height from the bottom upwards which is to the depth of the sea as 1 to σ. Let h be the height of the land above the actual sea-level, h being negative in the case of the sea, and equal to the depth of the sea multiplied by $1-\sigma^{-1}$. Let x be the unknown thickness of the stratum which must be removed in order to leave the surface a surface of equilibrium, and suppose the mean value of x to be zero, so that on the whole matter is neither added nor taken away. The surface of equilibrium which would be thus obtained is evidently the same as that which would be formed if the elevated portions of the irregular surface were to become fluid and to run down.

Let V be the potential of the whole mass in its first state, V_x the potential of the stratum removed. The removal of this stratum will depress the surface of equilibrium by the space $G^{-1}V_x$; and the condition to be satisfied is, that this new surface of equilibrium, or else a surface of equilibrium belonging to the same system, and therefore derived from the former by further diminishing the radius vector by the small quantity c', shall coincide with the actual surface. We must therefore have

$$G^{-1}V_x + c' = x - h. \quad\ldots\ldots\ldots\ldots\ldots (33)$$

Let h and x be expanded in series of Laplace's coefficients $h_0 + h_1 + \ldots$ and $x_0 + x_1 + \ldots$ Then the value of V_x at the surface will be obtained from either of equations (28) by replacing $\hat{c}$ by σx and putting $r = a$. We have therefore

$$V_x = 4\pi\sigma a\left(x_0 + \tfrac{1}{3}x_1 + \tfrac{1}{5}x_2 + \ldots\right). \quad\ldots\ldots\ldots\ldots\ldots (34)$$

After substituting in (33) the preceding expressions for V_x, h, and x, we must equate to zero Laplace's coefficients of the same order. The condition that $x_0 = 0$ may be satisfied by means of the constant c', and we shall have

$$G^{-1}.\,4\pi\sigma a\,\frac{x_i}{2i+1} = x_i - h_i,$$

which gives, on replacing $G^{-1}.4\pi\sigma a$ by its equivalent $\dfrac{3\sigma}{\rho}$,

$$x_i = \frac{(2i+1)\,\rho}{(2i+1)\,\rho - 3\sigma}\,h_i = \left\{1 + \frac{3\sigma}{(2i+1)\,\rho - 3\sigma}\right\}h_i, \quad\ldots\ldots\ldots(35)$$

We see that for terms of a high order x_i is very nearly equal to h_i, but for terms of a low order, whereby the distribution of land and sea would be expressed as to its broad features, x_i is sensibly greater than h_i.

27. Let it be required to reduce gravity g to the gravity which would be observed, in the altered state of the surface, along what would then be a surface of equilibrium. Let the correction be denoted by $g' - 3g'''$, where g' is the same as before. The correction due to the alteration of the coating in the manner considered in Art. 20 has been shewn to be equal to

$$2\pi\delta - 6\pi\Sigma\frac{\delta_i}{2i+1},$$

and the required correction will evidently be obtained by replacing δ by σx. Putting for x_i its value got from (35) we have

$$g' - 3g''' = 2\pi\sigma\Sigma\frac{(2i-2)\rho}{(2i+1)\rho - 3\sigma}h_i = 2\pi\sigma\Sigma\left\{1 - \frac{3\rho - 3\sigma}{(2i+1)\rho - 3\sigma}\right\}h_i,$$

which gives, since $2\pi\sigma\Sigma h_i = 2\pi\sigma h = g'$, and $G = \frac{4}{3}\pi\rho a$,

$$3g''' = G\frac{3\sigma}{2\rho}\Sigma\frac{3\rho - 3\sigma}{(2i+1)\rho - 3\sigma}\frac{h_i}{a}, \text{ .... } (36)$$

If we put $\sigma = 2\frac{1}{2}$, $\rho = 5\frac{1}{2}$, $a = 4000$, and suppose h expressed in miles, we get

$$3g''' = G.\frac{15}{88000}\Sigma\frac{9h_i}{11i-2} = G \times .00017\,(-4.5h_0 + h_1 + .45h_2 + .290h_3 + .214h_4 + \ldots). \ldots (37)$$

Had we treated the approximate correction $3g''$ in the same manner we should have had

$$3g'' = G\frac{3\sigma}{2\rho a}\Sigma\frac{3h_i}{2i+1} = G \times .00017\,(3h_0 + h_1 + .6h_2 + .429h_3 + .333h_4 + \ldots)$$

whereas, since $k = 3\left(1 - \frac{\sigma}{\rho}\right)$, we get

$$kg'' = G\frac{3\sigma}{2\rho a}\Sigma\frac{(3\rho - 3\sigma)h_i}{(2i+1)\rho} = G \times .00017\,(1.636h_0 + .545h_1 + .327h_2 + .234h_3 + .182h_4 + \ldots) \ldots (38)$$

The general expressions for $3g'''$, $3g''$, and kg'' shew that the approximate correction kg'' agrees with the true correction $3g'''$ so far as regards terms of a high order, whereas the leading terms, beginning with the first variable term, are decidedly too small; so that, as far as regards these terms, $3g'''$ is better represented by $3g''$ than by kg''. This agrees with what has been already remarked in Art. 25.

If we put $g - g' + 3g''' = g_{,,,}$, and suppose G and ϵ determined by means of $g_{,,,}$, small corrections similar to those already investigated will have to be applied in consequence of the omission of the quantity $g' - 3g'''$ in the value of g. The correction to ϵ would probably be insensible for the reason mentioned in Art. 18. If we are considering only the variation of gravity, we may of course leave out the term h_0.

The series (37) would probably be too slowly convergent to be of much use. A more convergent series may be obtained by subtracting kg'' from $3g'''$, since the terms of a high order in $3g'''$ are ultimately equal to those in kg''. We thus get

$$3g''' = kg'' + G \times .00017\,(-6.136h_0 + .455h_1 + .123h_2 + .056h_3 + .032h_4 + \ldots) \ldots\ldots (39)$$

which gives g''' if g'' be known by quadratures for the station considered.

Although for facility of calculation it has been supposed that the sea was first replaced by a stratum of rock or earth of less thickness, and then that the elevated portions of the earth's surface became fluid and ran down, it may be readily seen that it would come to the same thing if we supposed the water to remain as it is, and the land to become fluid and run down, so as to form for the bottom of the sea a surface of equilibrium. The gravity $g_{,,,}$ would apply to the earth so altered.

28. Let us return to the quantity V_c of Art. 19, and consider how the attraction of the earth's irregular coating affects the direction of the vertical. Let l be the latitude of the station, which for the sake of clear ideas may be supposed to be situated in the northern hemisphere, ϖ its longitude west of a given place, ξ the displacement of the zenith towards the south produced by the attraction of the coating, η its displacement towards the east. Then

$$\xi = \frac{1}{Ga}\frac{dV_c}{dl}, \quad \eta = \frac{\sec l}{Ga}\frac{dV_c}{d\varpi},$$

because $\frac{1}{a}\frac{dV_c}{dl}$ and $\frac{\sec l}{a}\frac{dV_c}{d\varpi}$ are the horizontal components of the attraction towards the north and towards the west respectively, and G may be put for g on account of the smallness of the displacements.

Suppose the angle χ of Art. 22 measured from the meridian, so as to represent the north azimuth of the elementary mass $\delta a^2 \sin\psi d\psi d\chi$. On passing to a place on the same meridian whose latitude is $l + dl$, the angular distance of the elementary mass is shortened by $\cos\chi \,.\, dl$, and therefore its linear distance, which was a chord ψ, or $2a\sin\frac{\psi}{2}$, becomes $2a\sin\frac{\psi}{2} - a\cos\frac{\psi}{2}\cos\chi \,.\, dl$. Hence the reciprocal of the linear distance is increased by $\frac{1}{4a}\cos\frac{\psi}{2}\operatorname{cosec}^2\frac{\psi}{2}\cos\chi \,.\, dl$, and therefore the part of V_c due to this element is increased by $\frac{1}{2}\delta a\cos^2\frac{\psi}{2}\operatorname{cosec}\frac{\psi}{2}\cos\chi \,.\, d\psi d\chi dl$. Hence we have

$$\frac{dV_c}{dl} = \frac{a}{2}\iint \frac{\cos^2\frac{\psi}{2}\cos\chi}{\sin\frac{\psi}{2}}\delta d\psi d\chi \text{.} \;\ldots\ldots\ldots \;(40)$$

Although the quantity under the integral sign in this expression becomes infinite when ψ vanishes, the integral itself has a finite value, at least if we suppose δ to vary continuously in the immediate neighbourhood of the station. For if δ becomes δ' when χ becomes $\chi + \pi$, we may replace δ under the integral sign by $\delta - \delta'$, and integrate from $\chi = 0$ to $\chi = \pi$, instead of integrating from $\chi = 0$ to $\chi = 2\pi$, and the limiting value of $\frac{\delta - \delta'}{\sin\frac{\psi}{2}}$ when ψ vanishes is $4\frac{d\delta}{d\psi}$, which is finite.

To get the easterly displacement of the zenith, we have only to measure χ from the west instead of from the north, or, which comes to the same, to write $\chi + \frac{\pi}{2}$ for χ, and continue to measure χ from the north. We get

$$\sec l\frac{dV_c}{d\varpi} = -\frac{a}{2}\iint \cos^2\frac{\psi}{2}\operatorname{cosec}\frac{\psi}{2}\sin\chi \,.\, \delta d\psi d\chi \ldots\ldots\ldots (41)$$

29. The expressions (40) and (41) are not to be applied to points very near the station if δ vary abruptly, or even very rapidly, about such points. Recourse must in such a case be had to direct triple integration, because it is not allowable to consider the attracting matter as condensed into a surface. If however δ vary gradually in the neighbourhood of the station, the expression (40) or (41) may be used without further change. For if we modify (40) in the way explained in the preceding article, or else by putting the integral under the form $\int_0^\pi \int_0^{2\pi} \cos^2\frac{\psi}{2}\operatorname{cosec}\frac{\psi}{2}\cos\chi$ $(\delta - \delta_1)\, d\psi d\chi$, where δ_1 denotes the value of δ at the station, we see that the part of the integral

due to a very small area surrounding the station is very small. If δ vary abruptly, in consequence suppose of the occurrence of a cliff, we may employ the expressions (40), (41), provided the distance of the cliff from the station be as much as three or four times its height.

These expressions shew that the vertical is liable to very irregular deviations depending on attractions which are quite local. For it is only in consequence of the opposition of attractions in opposite quarters that the value of the integral is not considerable, and it is of course larger in proportion as that opposition is less complete. Since $\sin\frac{\psi}{2}$ is but small even at the distance of two or three hundred miles, a distant coast, or on the other hand a distant tract of high land of considerable extent, may produce a sensible effect; although of course in measuring an arc of the meridian those attractions may be neglected which arise from masses which are so distant as to affect both extremities of the arc in nearly the same way.

If we compare (40) or (41) with the expression for g'' or g''', we shall see that the direction of the vertical is liable to far more irregular fluctuations on account of the inequalities in the earth's coating than the force of gravity, except that part of the force which has been denoted by g', and which is easily allowed for. It has been supposed by some that the force of gravity alters irregularly along the earth's surface, and so it does, if we compare only distant stations. But it has been already remarked with what apparent regularity gravity when corrected for the inequality g' appears to alter, in the direction in which we should expect, in passing from one station to another in a chain of neighbouring stations.

30. There is one case in which the deviation of the vertical may become unusually large, which seems worthy of special consideration.

For simplicity, suppose δ to be constant for the land, and equal to zero for the sea, which comes to regarding the land as of constant height, the sea as of uniform depth, and transferring the defect of density of the sea with an opposite sign to the land. Apply the integral (40) to those parts only of the earth's surface which are at no great distance from the station considered, so that we may put $\cos\frac{\psi}{2}=1$, $\sin\frac{\psi}{2}=\frac{\psi}{2}=\frac{s}{2a}$, if s be the distance of the element, measured along a great circle. In going from the station in the direction determined by the angle χ, suppose that we pass from land to sea at distances s_1, s_3, s_5,...and from sea to land at the intermediate distances s_2, s_4....On going in the opposite direction suppose that we pass from land to sea at the distances s_{-1}, s_{-3}, s_{-5}, ... and from sea to land at the distances s_{-2}, s_{-4}....Then we get from (40),

$$\frac{dV_c}{dl}=a\delta\int\{\log s_1-\log s_{-1}-(\log s_2-\log s_{-2})+\log s_3-\log s_{-3}-\dots\}\cos\chi . d\chi.$$

If the station be near the coast, one of the terms $\log s_1$, $\log s_{-1}$ will be large, and the zenith will be sensibly displaced towards the sea by the irregular attraction. On account of the shelving of the coast, the preceding expression, which has been formed on the supposition that δ vanished suddenly, would give too great a displacement; but the object of this article is not to perform any precise calculation, but merely to shew how the analysis indicates a case in which there would be unusual disturbance. A cliff bounding a tract of table-land would have the same sort of effect as a coast, and indeed the effect might be greater, on account of the more sudden variation of δ. The effect would be nearly the same at equal distances from the edge above and below, that distance being supposed as great as a small multiple of the height of the cliff, in order to render the expression (40) applicable without modification.

31. Let us return now to the force of gravity, and leaving the consideration of the connexion between the irregularities of gravity and the irregularities of the earth's coating, and of the

possibility of destroying the former by making allowance for the latter, let us take the earth such as we find it, and consider further the connexion between the variations of gravity and the irregularities of the surface of equilibrium which constitutes the sea-level.

Equation (12) gives the variation of gravity if the form of the surface be known, and conversely, (8) gives the form of the surface if the variation of gravity be known. Suppose the variation of gravity known by means of pendulum-experiments performed at a great many stations scattered over the surface of the earth; and let it be required from the result of the observations to deduce the form of the surface. According to what has been already remarked, a series of Laplace's coefficients would most likely be practically useless for this purpose, unless we are content with merely the leading terms in the expression for the radius vector; and the leading character of those terms depends, not necessarily upon their magnitude, but only on the wide extent of the inequalities which they represent. We must endeavour therefore to reduce the determination of the radius vector to quadratures.

For the sake of having to deal with small terms, let g be represented, as well as may be, by the formula which applies to an oblate spheroid, and let the variable term in the radius vector be calculated by Clairaut's Theorem. Let g_c be calculated gravity, r_c the calculated radius vector, and put $g = g_c + \Delta g$, $r = r_c + a\Delta u$. Suppose Δg and Δu expanded in series of Laplace's coefficients. It follows from (12) that Δg will have no term of the order 1; indeed, if this were not the case, it might be shewn that the mutual forces of attraction of the earth's particles would have a resultant. Moreover the constant term in Δg may be got rid of by using a different value of G. No constant term need be taken in the expansion of Δu, because such a term might be got rid of by using a different value of a, and a of course cannot be determined by pendulum-experiments. The term of the first order will disappear if r be measured from the common centre of gravity of the mass and volume. The remaining terms in the expansion of Δu will be determined from those in the expansion of Δg by means of equations (8) and (12).

Let
$$\Delta g = G(v_2 + v_3 + v_4 + \ldots), \qquad (42)$$
and we shall have
$$\Delta u = v_2 + \tfrac{1}{2}v_3 + \tfrac{1}{3}v_4 + \ldots \qquad (43)$$

Suppose $\Delta g = GF(\theta, \phi)$. Let ψ be the angle between the directions determined by the angular co-ordinates θ, ϕ and θ', ϕ'. Let $(1 - 2\zeta\cos\psi + \zeta^2)^{\frac{1}{2}}$ be denoted by R, and let Q_i be the coefficient of ζ^i in the expansion of R^{-1} in a series according to ascending powers of ζ. Then

$$v_i = \frac{2i+1}{4\pi}\int_0^\pi\int_0^{2\pi} F(\theta', \phi')Q_i \sin\theta' d\theta' d\phi',$$

and therefore if ζ be supposed to be less than 1, and to become 1 in the limit, we shall have

$$4\pi\Delta u = \text{limit of } \int_0^\pi\int_0^{2\pi} F(\theta', \phi')\left(5\zeta Q_2 + \frac{7}{2}\zeta^2 Q_3 \ldots + \frac{2i+1}{i-1}\zeta^{i-1}Q_i + \ldots\right)\sin\theta' d\theta' d\phi'. \ \ldots \ (44)$$

Now assume

$$\gamma = 5\zeta Q_2 + \frac{7}{2}\zeta^2 Q_3 \ldots + \frac{2i+1}{i-1}\zeta^{i-1}Q_i + \ldots,$$

and we shall have

$$\frac{d\gamma}{d\zeta} = 5Q_2 + 7\zeta Q_3 \ldots + (2i+1)\zeta^{i-2}Q_i + \ldots;$$

$$\int_0^{\sqrt{\zeta}} \zeta^2 \frac{d\gamma}{d\zeta} d.\zeta^{\frac{1}{2}} = \zeta^{\frac{5}{2}}Q_2 + \zeta^{\frac{7}{2}}Q_3 \ldots + \zeta^{i+\frac{1}{2}}Q_i + \ldots = \zeta^{\frac{1}{2}}(R^{-1} - Q_0 - \zeta Q_1);$$

whence we get, putting Z for $R^{-1} - Q_0 - \zeta Q_1$, $\gamma = 2\int \zeta^{-\frac{3}{2}} d.\zeta^{\frac{1}{2}} Z$.

Integrating by parts, and observing that γ vanishes with ζ, we get

$$\gamma = 2\zeta^{-1}Z + 3\int_0^{\zeta}\zeta^{-2}Z\,d\zeta.$$

The last integral may be obtained by rationalization. If we assume $R = w - \zeta$, and observe that $Q_0 = 1$, $Q_1 = \cos\psi$, and that $w = 1$ when ζ vanishes, we shall find

$$\int_0^{\zeta}\zeta^{-2}Z\,d\zeta = \cos\psi\,.\log\frac{w-\cos\psi}{1-\cos\psi} - (1+\cos\psi)\frac{w-1}{w+1} - 2\cos\psi\,.\log\frac{w+1}{2}.$$

When $\zeta=1$ we have $Z = (2-2\cos\psi)^{-\frac{1}{2}} - (1+\cos\psi)$, $w = 1 + 2\sin\frac{\psi}{2}$, and

$$\int_0^{\zeta}\zeta^{-2}Z\,d\zeta = -2\sin\frac{\psi}{2}\left(1-\sin\frac{\psi}{2}\right) - \cos\psi\log\left\{\sin\frac{\psi}{2}\left(1+\sin\frac{\psi}{2}\right)\right\}.$$

Putting $f(\psi)$ for the value of γ when $\zeta = 1$, we have

$$f(\psi) = \operatorname{cosec}\frac{\psi}{2} + 1 - 6\sin\frac{\psi}{2} - 5\cos\psi - 3\cos\psi\log\left\{\sin\frac{\psi}{2}\left(1+\sin\frac{\psi}{2}\right)\right\}. \quad\ldots\ldots(45)$$

In the expression for Δu, we may suppose the line from which θ' is measured to be the radius vector of the station considered. We thus get, on replacing $F(\theta', \phi')$ by $G^{-1}\Delta g$, and employing the notation of Art. 22,

$$\Delta u = \frac{1}{4\pi G}\int_0^{\pi}\int_0^{2\pi}\Delta g\,.f(\psi)\sin\psi\,d\psi\,d\chi. \quad\ldots\ldots\ldots\ldots (46)$$

32. Let $\Delta g = g' + \Delta' g$. Then $\Delta' g$ is the excess of observed gravity reduced to the level of the sea by Dr. Young's rule over calculated gravity; and of the two parts g' and $\Delta' g$ of which Δg consists, the former is liable to vary irregularly and abruptly from one place to another, the latter varies gradually. Hence, for the sake of interpolating between the observations taken at different stations, it will be proper to separate Δg into these two parts, or, which comes to the same, to separate the whole integral into two parts, involving g' and $\Delta' g$ respectively, so as to get the part of Δu which is due to g' by our knowledge of the height of the land and the depth of the sea, and the part which depends on $\Delta' g$ by the result of pendulum-experiments. It may be observed that a constant error, or a slowly varying error, in the height of the land would be of no consequence, because it would enter with opposite signs into g' and $\Delta' g$.

It appears, then, that the results of pendulum-experiments furnish sufficient data for the determination of the variable part of the radius vector of the earth's surface, and consequently for the determination of the particular value which is to be employed at any observatory in correcting for the lunar parallax, subject however to a constant error depending on an error in the assumed value of a.

33. The expression for g''' in Art. 27 might be reduced to quadratures by the method of Art. 31, but in this case the integration with respect to ζ could not be performed in finite terms, and it would be necessary in the first instance to tabulate, once for all, an integral of the form $\int_0^1 f(\zeta, \cos\psi)\,d\zeta$ for values of ψ, which need not be numerous, from 0 to π. This table being made, the tabulated function would take the place of $f(\psi)$ in (46), and the rest of the process would be of the same degree of difficulty as the quadratures expressed by the equations (31) and (46).

34. Suppose Δu known approximately, either as to its general features, by means of the leading terms of the series (43), or in more detail from the formula (46), applied in succession to a great many points on the earth's surface. By interpolating between neighbouring places for which Δu has been calculated, find a number of points where Δu has one of the constant values

$-2\beta, -\beta, 0, \beta, 2\beta$ ..., mark these points on a map of the world, and join by a curve those which belong to the same value of Δu. We shall thus have a series of contour lines representing the elevation or depression of the actual sea-level above or below the surface of the oblate spheroid, which has been employed as most nearly representing it. If we suppose these lines traced on a globe, the reciprocal of the perpendicular distance between two consecutive contour lines will represent in magnitude, and the perpendicular itself in direction, the deviation of the vertical from the normal to the surface of the spheroid, or rather that part of the deviation which takes place on an extended scale: for sensible deviations may be produced by attractions which are merely local, and which would not produce a sensible elevation or depression of the sea-level; although of course, as to the merely mathematical question, if the contour lines could be drawn sufficiently close and exact, even local deviations of the vertical would be represented.

Similarly, by joining points at which the quantity denoted in Art. 19 by V has a constant value, contour lines would be formed representing the elevation of the actual sea-level above what would be a surface of equilibrium if the earth's irregular coating were removed. By treating V_s in the same way, contour lines would be formed corresponding to the elevation of the actual sea-level above what would be the sea-level if the solid portions of the earth's crust which are elevated were to become fluid and to run down, so as to form a level bottom for the sea, which would in that case cover the whole earth.

These points of the theory are noticed more for the sake of the ideas than on account of any application which is likely to be made of them; for the calculations indicated, though possible with a sufficient collection of data, would be very laborious, at least if we wished to get the results with any detail.

35. The squares of the ellipticity, and of quantities of the same order, have been neglected in the investigation. Mr. Airy, in the Treatise already quoted, has examined the consequence, on the hypothesis of fluidity, of retaining the square of the ellipticity, in the two extreme cases of a uniform density, and of a density infinitely great at the centre and evanescent elsewhere, and has found the correction to the form of the surface and the variation of gravity to be insensible, or all but insensible. As the connexion between the form of the surface and the variation of gravity follows independently of the hypothesis of fluidity, we may infer that the terms depending on the square of the ellipticity which would appear in the equations which express that connexion would be insensible. It may be worth while, however, just to indicate the mode of proceeding when the square of the ellipticity is retained.

By the result of the first approximation, equation (1) is satisfied at the surface of the earth, as far as regards quantities of the first order, but not necessarily further, so that the value of $V + U$ at the surface is not strictly constant, but only of the form $c + H$, where H is a small variable quantity of the second order. It is to be observed that V satisfies equation (3) exactly, not approximately only. Hence we have merely to add to V a potential V' which satisfies equation (3) outside the earth, vanishes at an infinite distance, and is equal to H at the surface. Now if we suppose V' to have the value H at the surface of a sphere whose radius is a, instead of the actual surface of the earth, we shall only commit an error which is a small quantity of the first order compared with H, and H is itself of the second order, and therefore the error will be only of the third order. But by this modification of one of the conditions which V' is to satisfy, we are enabled to find V' just as V was found, and we shall thus have a solution which is correct to the second order of approximation. A repetition of the same process would give a solution which would be correct to the third order, and so on. It need hardly be remarked that in going beyond the first order of approximation, we must distinguish in the small terms between the direction of the vertical, and that of the radius vector.

G. G. STOKES.

LI. *On Hegel's Criticism of Newton's Principia.* By W. WHEWELL, D.D., *Master of Trinity College, Cambridge.*

[Read *May* 21, 1849.]

THE Newtonian doctrine of universal gravitation, as the cause of the motions which take place in the solar system, is so entirely established in our minds, and the fallacy of all the ordinary arguments against it is so clearly understood among us, that it would undoubtedly be deemed a waste of time to argue such questions in this place, so far as physical truth is concerned. But since in other parts of Europe, there are teachers of philosophy whose reputation and influence are very great, and who are sometimes referred to among our own countrymen as the authors of new and valuable views of truth, and who yet reject the Newtonian opinions, and deny the validity of the proofs commonly given of them, it may be worth while to attend for a few minutes to the declarations of such teachers, as a feature in the present condition of European philosophy. I the more readily assume that the Cambridge Philosophical Society will not think a communication on such a subject devoid of interest, in consequence of the favourable reception which it has given to philosophical speculations still more abstract, which I have on previous occasions offered to it. I will therefore proceed to make some remarks on the opinions concerning the Newtonian doctrine of gravitation, delivered by the celebrated Hegel, of Berlin, than whom no philosopher in modern, and perhaps hardly any even in ancient times, has had his teaching received with more reverential submission by his disciples, or been followed by a more numerous and zealous band of scholars bent upon diffusing and applying his principles.

The passages to which I shall principally refer are taken from one of his works which is called *the Encyclopædia* (Encyklopädie), of which the First Part is *the Science of Logic,* the Second, the *Philosophy of Nature*, the Third, the *Philosophy of Spirit*. The Second Part, with which I am here concerned, has for an *aliter* title, *Lectures on Natural Philosophy* (Vorlesungen über Naturphilosophie), and would through its whole extent offer abundant material for criticism, by referring it to principles with which we are here familiar: but I shall for the present confine myself to that part which refers to the subject which I have mentioned, the Newtonian Doctrine of Gravitation, § 269, 270, of the work. Nor shall I, with regard to this part, think it necessary to give a continuous and complete criticism of all the passages bearing upon the subject; but only such specimens, and such remarks thereon, as may suffice to show in a general manner the value and the character of Hegel's declarations on such questions. I do not pretend to offer here any opinion upon the value and character of Hegel's philosophy in general: but I think it not unlikely that some impression on that head may be suggested by the examination, here offered, of some points in which we can have no doubt where the truth lies; and I am not at all persuaded that a like examination of many other parts of the Hegelian *Encyclopædia* would not confirm the impression which we shall receive from the parts now to be considered.

Hegel both criticises the Newtonian doctrines, or what he states as such; and also, not denying the truth of the laws of phenomena which he refers to, for instance Kepler's laws, offers his own proof of these laws. I shall make a few brief remarks on each of these portions of the pages before me. And I would beg it to be understood that where I may happen to put my remarks in a short, and what may seem a peremptory form, I do so for the sake of saving time; knowing that among us, upon subjects so familiar, a few words will suffice. For the same reason, I shall take passages from Hegel, not in the order in which they occur, but in the order in which they best illustrate what I have to say. I shall do Hegel no injustice by this mode of proceeding: for I will annex a faithful translation, so far as I can make one, of the whole of the passages referred to, with the context.

No one will be surprised that a German, or indeed any lover of science, should speak with admiration of the discovery of Kepler's laws, as a great event in the history of Astronomy, and a glorious distinction to the discoverer. But to say that the glory of the discovery of the proof of these laws has been unjustly transferred from Kepler to Newton, is quite another matter. This is what Hegel says (*a**). And we have to consider the reasons which he assigns for saying so.

He says (*b*) that "it is allowed by mathematicians that the Newtonian Formula may be derived from the Keplerian laws," and hence he seems to infer that the Newtonian law is not an additional truth. That is, he does not allow that the discovery of the cause which produces a certain phenomenal law is anything additional to the discovery of the law itself.

"The Newtonian formula may be derived from the Keplerian law." It was professedly so derived; but derived by introducing the Idea of *Force*, which Idea and its consequences were not introduced and developed till after Kepler's time.

"The Newtonian formula may be derived from the Keplerian law." And the Keplerian law may be derived, and was derived, from the observations of the Greek astronomers and their successors; but was not the less a new and great discovery on that account.

But let us see what he says further of this derivation of the Newtonian "formula" from the Keplerian Law. It is evident that by calling it a *formula*, he means to imply, what he also asserts, that it is no new law, but only a new form (and a bad one) of a previously known truth.

How is the Newtonian "formula," that is, the law of the inverse squares of the central force, derived from the Keplerian law of the cubes of the distances proportional to the squares of the times? This, says Hegel, is the "immediate derivation." (*c*).—By Kepler's law, A being the distance and T the periodic time, $\frac{A^3}{T^2}$ is constant. But Newton *calls* $\frac{A}{T^2}$ universal gravitation; whence it easily follows that gravitation is inversely as A^2.

This is Hegel's way of representing Newton's proof. Reading it, any one who had never read the *Principia* might suppose that Newton *defined* gravitation to be $\frac{A}{T^2}$. We, who have read the *Principia*, know that Newton *proves* that in circles, the *central force* (not the *universal gravitation*) is as $\frac{A}{T^2}$: that he proves this, by setting out from the idea of force, as that which deflects a body from the tangent, and makes it describe a curve line: and that in this way, he passes from Kepler's laws of mere motion to his own law of Force.

But Hegel does not see any value in this. Such a mode of treating the subject he says (*i*) "offers to us a tangled web, formed of the Lines of the mere geometrical construction, to which a physical meaning of independent forces is given." That a *measure* of forces is *found* in such lines as the sagitta of the arc described in a given time, (not such a *meaning* arbitrarily *given* to them,) is certainly true, and is very distinctly proved in Newton, and in all our elementary books.

But, says Hegel, as further shewing the artificial nature of the Newtonian formulæ, (*h*) "Analysis has long been able to derive the Newtonian expression and the laws therewith connected out of the Form of the Keplerian Laws;" an assertion, to verify which he refers to Poisson's *Mécanique*.

* These letters refer to passages in the Translation annexed to this Memoir.

This is apparently in order to shew that the "lines" of the Newtonian construction are superfluous. We know very well that analysis does not always refer to visible representations of such lines: but we know too, (and Francœur would testify to this also,) that the analytical proofs contain equivalents to the Newtonian lines. We, in this place, are too familiar with the substitution of analytical for geometrical proofs, to be led to suppose that such a substitution affects the substance of the truth proved. The conversion of Newton's geometrical proofs of his discoveries into analytical processes by succeeding writers, has not made them cease to be discoveries: and accordingly, those who have taken the most prominent share in such a conversion, have been the most ardent admirers of Newton's genius and good fortune.

So much for Newton's comparison of the Forces in different circular orbits, and for Hegel's power of understanding and criticising it. Now let us look at the motion in different parts of the same elliptical orbit, as a further illustration of the value of Hegel's criticism. In an elliptical orbit the velocity alternately increases and diminishes. This follows necessarily from Kepler's law of the equal description of the areas, and so Newton explains it. Hegel, however, treats of this acceleration and retardation as a separate fact, and talks of another explanation of it, founded upon Centripetal and Centrifugal Force (*o*). Where he finds this explanation, I know not; certainly not in Newton, who in the second and third section of the *Principia* explains the variation of the velocity in a quite different manner, as I have said; and nowhere, I think, employs centrifugal force in his explanations. However, the notion of centrifugal as acting along with centripetal force is introduced in some treatises, and may undoubtedly be used with perfect truth and propriety. How far Hegel can judge when it is so used, we may see from what he says of the confusion produced by such an explanation, which is, he says, a maximum. In the first place, he speaks of the motion being *uniformly* accelerated and retarded in an elliptical orbit, which, in any exact use of the word *uniformly*, it is not. But passing by this, he proceeds to criticise an explanation, not of the variable velocity of the body in its orbit, but of the alternate access and recess of the body to and from the center. Let us overlook this confusion also, and see what is the value of his criticism on the explanation. He says (*p*), "according to this explanation, in the motion of a planet from the aphelion to the perihelion, the centrifugal is less than the centripetal force; and in the perihelion itself the centripetal force is supposed suddenly to become greater than the centrifugal;" and so, of course, the body re-ascends to the aphelion.

Now I will not say that this explanation has never been given in a book professing to be scientific; but I have never seen it given; and it never can have been given but by a very ignorant and foolish person. It goes upon the utterly unmechanical supposition that the approach of a body to the center at any moment depends solely upon the excess of the centripetal over the centrifugal force; and reversely. But the most elementary knowledge of mechanics shews us that when a body is moving *obliquely* to the distance from the center, it approaches to or recedes from the center in virtue of this obliquity, even if no force at all act. And the total approach to the center is the approach due to this cause, *plus* the approach due to the centripetal force, *minus* the recess due to the centrifugal force. At the aphelion, the centripetal is greater than the centrifugal force; and *hence* the motion becomes oblique; and *then*, the body approaches to the center on *both* accounts, and approaches on account of the obliquity of the path even when the centrifugal has become greater than the centripetal force, which it becomes before the body reaches the perihelion. This reasoning is so elementary, that when a person who cannot see this, writes on the subject with an air of authority, I do not see what can be done but to point out the oversight and leave it.

But there is, says Hegel (*q*), another way of explaining the motion by means of centripetal and centrifugal forces. The two forces are supposed to increase and decrease gradually, according to different laws. In this case, there must be a point where they are equal, and in equilibrio; and this being the case, they will always continue equal, for there will be no reason for their going out of equilibrium.

This, which is put as *another* mode of explanation, is, in fact, the same mode; for, as I have already said, the centrifugal force, which is less than the centripetal at the aphelion, becomes the greater of the two before the perihelion; and there is an intermediate position, at which the two forces are equal. But at this point, is there no reason why, being equal, the forces should become unequal? Reason abundant: for the body, being there, moves in a line oblique to the distance, and so changes its distance; and the centripetal and centrifugal force, depending upon the distance by different laws, they forthwith become unequal.

But these modes of explanation, by means of the centripetal and centrifugal forces and their relation, are not necessary to Newton's doctrine, and are nowhere used by Newton; and undoubtedly much confusion has been produced in other minds, as well as Hegel's, by speaking of the centrifugal force, which is a mere intrinsic geometrical result of a body's curvilinear motion round a center, in conjunction with centripetal force, which is an extrinsic force, acting upon the body and urging it to the center. Neither Newton, nor any intelligent Newtonian, ever spoke of the centripetal and centrifugal force as two distinct forces both extrinsic to the motion, which Hegel accuses them of doing. (*n*)

I have spoken of the third and second of Kepler's laws; of Newton's explanations of them, and of Hegel's criticism. Let us now, in the same manner, consider the first law, that the planets move in ellipses. Newton's proof that this was the result of a central force varying inversely as the square of the distance, was the solution of a problem at which his contemporaries had laboured in vain, and is commonly looked upon as an important step. "But," says Hegel, (*d*) "the proof gives a conic section generally, whereas the main point which ought to be proved is, that the path of the body is an ellipse only, not a circle or any other conic section." Certainly if Newton *had* proved that a planet cannot move in a circle, (which Hegel says he ought to have done), his system would have perplexed astronomers, since there are planets which move in orbits hardly distinguishable from circles, and the variation of the extremity from planet to planet shews that there is nothing to prevent the excentricity vanishing and the orbit becoming a circle.

"But," says Hegel again, (*e*) "the conditions which make the path to be an ellipse rather than any other conic section, are empirical and extraneous;—the supposed casual strength of the impulsion originally received." Certainly the circumstances which determine the amount of excentricity of a planet's orbit are derived from experience, or rather, observation. It is not a part of Newton's system to determine *à priori* what the excentricity of a planet's orbit must be. A system that professes to do this will undoubtedly be one very different from his. And as our knowledge of the excentricity is derived from observation, it is, in that sense, empirical and casual. The strength of the original impulsion is a hypothetical and impartial way of expressing this result of observation. And as we see no reason why the excentricity should be of any certain magnitude, we see none why the fraction which expresses the excentricity should not become as large as unity, that is, why the orbit should not become a parabola; and accordingly, some of the bodies which revolve about the same appear to move in orbits of this form: so little is the motion in an ellipse, as Hegel says, (*f*) "the only thing to be proved."

But Hegel himself has offered proof of Kepler's laws, to which, considering his objections to Newton's proofs, we cannot help turning with some curiosity.

And first, let us look at the proof of the Proposition which we have been considering, that the path of a planet is necessarily an ellipse. I will translate Hegel's language as well as I can; but without answering for the correctness of my translation, since it does not appear to me to conform to the first condition of translation, of being intelligible. The translation however, such as it is, may help us to form some opinion of the validity and value of Hegel's proofs as compared with Newton's. (*r*)

"For absolutely uniform motion, the circle is the only path ... The circle is the line returning into itself in which all the radii are equal; there is, for it, only one determining quantity, the radius.

"But in free motion, the determination according to space and to time come into view with differences. There must be a difference in the spatial aspect in itself, and therefore the form requires two determining quantities. Hence the form of the path returning into itself is an ellipse."

Now even if we could regard this as reasoning, the conclusion does not in the smallest degree follow. A curve returning into itself and determined by two quantities, may have innumerable forms besides the ellipse; for instance, any *oval* form whatever, besides that of the conic section.

But why must the curve be a curve returning into itself? Hegel has professed to prove this previously (*m*) from "the determination of particularity and individuality of the bodies in general, so that they have partly a center in themselves, and partly at the same time their center in another." Without seeking to find any precise meaning in this, we may ask whether it proves the impossibility of the orbits with moveable apses, (which do not return into themselves,) such as the planets (affected by perturbations) really do describe, and such as we know that bodies must describe in all cases, except when the force varies exactly as the square of the distance? It appears to do so: and it proves this impossibility of known facts at least as much as it proves anything.

Let us now look at Hegel's proof of Kepler's second law, that the elliptical sectors swept by the radius vector are proportional to the time. It is this: (*s*).

"In the circle, the arc or angle which is included by the two radii is independent of them. But in the motion [of a planet] as determined by the conception, the distance from the center and the arc run over in a certain time must be compounded in one determination, and must make out a whole. This whole is the sector, a space of two dimensions. And hence the arc is essentially a Function of the radius vector; and the former (the arc) being unequal, brings with it the inequality of the radii."

As was said in the former case, if we could regard this as reasoning, it would not prove the conclusion, but only, that the arc is *some function or other* of the radii.

Hegel indeed offers (*t*) a reason why there must be an arc involved. This arises, he says, from "the determinateness [of the nature of motion], at one while as time in the root, at another while as space in the square. But here the quadratic character of the space is, by the returning of the line of motion into itself, limited to a sector."

Probably my readers have had a sufficient specimen of Hegel's mode of dealing with these matters. I will however add his proof of Kepler's third law, that the cubes of the distances are as the squares of the times.

Hegel's proof in this case (*u*) has a reference to a previous doctrine concerning falling bodies, in which time and space have, he says, a relation to each other as root and square. Falling bodies however are the case of only *half-free* motion, and the determination is incomplete.

"But in the case of absolute motion, the domain of *free* masses, the determination attains its totality. The time as the root is a mere empirical magnitude: but as a component of the developed Totality, it is a Totality in itself: it produces itself, and therein has a reference to itself. And in this process, Time, being itself the dimensionless element, only comes to a formal identity with itself and reaches the square: Space, on the other hand, as a positive external relation, comes to the full dimensions of the conception of space, that is, the cube. The Realization of the two conceptions (space and time) preserves their original difference. This is the third Keplerian law, the relation of the Cubes of the distances to the squares of the times."

"And this," he adds, (*v*) with remarkable complacency, "represents simply and immediately *the reason of the thing*:—while on the contrary, the Newtonian Formula, by means of which the Law is changed into a Law for the Force of Gravity, shews the distortion and inversion of *Reflexion*, which stops half-way."

I am not able to assign any precise meaning to the *Reflexion*, which is here used as a term of condemnation, applicable especially to the Newtonian doctrine. It is repeatedly applied in the same manner by Hegel. Thus he says, (*g*) "that what Kepler expresses in a simple and sublime manner in the form of Laws of the Celestial Motions, Newton has metamorphosed into the *Reflexion-Form* of the Force of Gravitation."

Though Hegel thus denies Newton all merit with regard to the explanation of Kepler's laws by means of the gravitation of the planets to the sun, he allows that to the Keplerian Laws Newton added the Principle of Perturbations (*k*). This Principle he accepts to a certain extent, transforming the expression of it after his peculiar fashion. "It lies," he says, (*l*) "in this: that matter in general assigns a center for itself: the collective bodies of the system recognize a reference to their sun, and all the individual bodies, according to the relative positions into which they are brought by their motions, form a momentary relation of their gravity towards each other."

This must appear to us a very loose and insufficient way of stating the Principle of Perturbations, but loose as it is, it recognises that the Perturbations depend upon the gravity of the planets one to another, and to the sun. And if the Perturbations depend upon these forces, one can hardly suppose that any one who allows this will deny that the primary undisturbed motions depend upon these forces, and must be explained by means of them; yet this is what Hegel denies.

It is evident, on looking at Hegel's mode of reasoning on such subjects, that his views approach towards those of Aristotle and the Aristotelians; according to which motions were divided into *natural* and *unnatural*;—the *celestial motions* were circular and uniform in their nature;—and the like. Perhaps it may be worth while to shew how completely Hegel adheres to these ancient views, by an extract from the additions to the Articles on Celestial Motions, made in the last edition of the Encyclopædia. He says (*w*),

"The motion of the heavenly bodies is not a being pulled this way and that, as is imagined (by the Newtonians). *They go along, as the ancients said, like blessed gods* The celestial conformity is not such a one as has the principle of rest or motion external to itself. It is not right to say because a stone is inert, and the whole earth consists of stones, and the other heavenly bodies are of the same nature as the earth, therefore the heavenly bodies are inert. This conclusion makes the properties of the whole the same as those of the part. Impulse, Pressure, Resistance, Friction, Pulling, and the like, are valid only for other than celestial matter."

There can be no doubt that this is a very different doctrine from that of Newton.

I will only add to these specimens of Hegel's physics, a specimen of the logic by which he refutes the Newtonian argument which has just been adduced; namely, that the celestial bodies are matter, and that matter, as we see in terrestrial matter, is inert. He says (*x*),

"Doubtless both are matter, as a good thought and a bad thought are both thoughts; but the bad one is not therefore good, because it is a thought."

TRINITY LODGE,
May 2, 1849.

Appendix to the Memoir on Hegel's Criticism of Newton's Principia.

HEGEL. *Encyclopædia* (2nd Ed. 1827) Part XI., p. 250.

C. *Absolute Mechanics.*

§ 269.

GRAVITATION is the true and determinate conception of material Corporeity, which (Conception) is realized to the Idea (zur Idee). *General* Corporeity is separable essentially into *particular* Bodies, and connects itself with the Element of *Individuality* or subjectivity, as apparent (phenomenal) presence in the *Motion*, which by this means is immediately a system of *several Bodies.*

Universal gravitation must, as to itself, be recognised as a profound thought, although it was principally as apprehended in the sphere of Reflexion that it eminently attracted notice and confidence on account of the quantitative determinations therewith connected, and was supposed to find its confirmation in *Experiments* (Erfahrung) pursued from the Solar System down to the phenomena of Capillary Tubes.—But Gravitation contradicts immediately the Law of Inertia, for in virtue of it (Gravitation) matter tends *out of itself* to the other (matter).—In the *Conception* of *Weight*, there are, as has been shewn, involved the two elements—Self-existence, and Continuity, which takes away self-existence. These elements of the Conception, however, experience a fate, as particular forces, corresponding to Attractive and Repulsive Force, and are thereby apprehended in nearer determination, as *Centripetal* and *Centrifugal Force*, which (Forces) like weight, *act upon Bodies*, independent of each other, and are supposed to come in contact accidentally in a third thing, Body. By this means, what there is of profound in the thought of universal weight is again reduced to nothing; and Conception and Reason cannot make their way into the doctrine of absolute motion, so long as the so highly-prized discoveries of Forces are dominant there. In the conclusion which contains the *Idea* of Weight, namely, [contains this Idea] as the Conception which, in the case of motion, enters into external Reality through the particularity of the Bodies, and at the same time into this [Reality] and into their Ideality and self-regarding Reflexion, (Reflexion-in-sich). the rational identity and inseparability of the elements is involved, which at other times are represented as independent. Motion itself, as such, has only its meaning and existence in a system of *several* bodies, and those, such as stand in relation to each other according to different determinations.

§ 270.

As to what concerns bodies in which the conception of gravity (weight) is realized free by itself, we say that they have for the determinations of their different nature the elements (momente) of their conception. One [conception of this kind] is the *universal* center of the abstract reference [of a body] to itself. Opposite to this [conception] stands the immediate, extrinsic, centreless *Individuality*, appearing as *Corporeity* similarly independent. Those [Bodies] however which are particular, which stand in the determination of extrinsic, and at the same time of intrinsic relation, are centers for themselves, and [also] have a reference to the first as to their essential unity.

The Planetary Bodies, as the immediately concrete, are in their existence the most complete. Men are accustomed to take the Sun as the most excellent, inasmuch as the understanding prefers the abstract to the concrete, and in like manner the Fixed stars are esteemed higher than the Bodies of the Solar System. Centreless Corporeity, as belonging to externality, naturally separates itself into the opposition of the lunar and the cometary Body. The laws of absolutely free motion, as is well known, were discovered by Kepler;—a discovery of

immortal fame. Kepler has proved these laws in this sense, that for the empirical data he found their general expression. Since then, it has become a common way of speaking to say
(a) that Newton first found out the proof of these Laws. It has rarely happened that fame has been more unjustly transferred from the first discoverer to another person. On this subject I make the following remarks.

1. That it is allowed by Mathematicians that the Newtonian Formulæ may be derived
(b) from the Keplerian Laws. The completely immediate derivation is this: In the third Kep-
(c) lerian Law, $\frac{A^3}{T^2}$ is the constant quantity. This being put as $\frac{A \cdot A^2}{T^2}$, and calling, with Newton, $\frac{A}{T^2}$ universal Gravitation, his expression of the effect of gravity in the reciprocal ratio of the square of the distances is obvious.

(d) 2. That the Newtonian proof of the Proposition that a body subjected to the Law of Gravitation moves about the central body in an *Ellipse*, gives a *Conic Section* generally, while the main Proposition which ought to be proved is that the fall of such a Body is *not* a *Circle or any other Conic Section*, but an *Ellipse only*. Moreover, there are objections which may be made against this proof in itself; (Princ. Math. l. 1. Sect. II. Prop. 1.) and although
(e) it is the foundation of the Newtonian Theory, analysis has no longer any need of it. The conditions which in the sequel make the path of the Body to a determinate Conic Section, are referred to an *empirical* circumstance, namely, a particular position of the Body at a determined moment of time, and the *casual* strength of an *impulsion* which it is supposed to have
(f) received originally; so that the circumstance which makes the Curve be an Ellipse, which alone ought to be the thing proved, is extraneous to the Formula.

3. That the Newtonian Law of the so-called Force of Gravitation is in like manner only proved from experience by Induction.

(g) The sum of the difference is this, that what Kepler expressed in a simple and sublime manner in the Form of *Laws of the Celestial Motions*, Newton has metamorphosed into the *Reflexion-Form* of the *Force of Gravitation*. If the Newtonian Form has not only its convenience but its necessity in reference to the analytical method, this is only a difference of the
(h) mathematical formulæ; Analysis has long been able to derive the Newtonian expression, and the Propositions therewith connected, out of the Form of the Keplerian Laws; (on this subject I refer to the elegant exposition in *Francœur's* Traité Elém. de Mécanique, Liv. II. Ch. xi.
(i) n. iv.)—The old method of so-called proof is conspicuous as offering to us a tangled web, formed of the *Lines* of the mere geometrical construction, to which a physical meaning of *independent Forces* is given; and of empty Reflexion-determinations of the already mentioned *Accelerating Force* and *Vis Inertiæ*, and especially of the relation of the so-called gravitation itself to the centripetal force and centrifugal force, and so on.

The remarks which are here made would undoubtedly have need of a further explication to shew how well founded they are: in a Compendium, propositions of this kind which do not agree with that which is assumed, can only have the shape of assertions. Indeed, since they contradict such high authorities, they must appear as something worse, as presumptuous assertions. I will not, on this subject, support myself by saying, by the bye, that an interest in these subjects has occupied me for 25 years; but it is more precisely to the purpose to remark, that the distinctions and determinations which Mathematical Analysis introduces, and the course which it must take according to its method, is altogether different from that which a physical reality must have. The Presuppositions, the Course, and the Results, which the Analysis necessarily has and gives, remain quite extraneous to the considerations which determine the physical value and the signification of those determinations and of that course. To this it is that attention should be directed. We have to do with a consciousness relative to the deluging of physical Mechanics with an *inconceivable* (unsäglichen) *Metaphysic*, which—contrary to experience and conception—has those mathematical determinations alone for its source.

It is recognized that what Newton—besides the foundation of the analytical treatment, the developement of which, by the bye, has of itself rendered superfluous, or indeed rejected much which belonged to Newton's essential Principles and glory—has added to the Keplerian Laws is the Principle of *Perturbations*,—a Principle whose importance we may here accept thus far; (hier in sofern anzufuhren ist); namely, so far as it rests upon the Proposition that
(*k*) the so-called attraction is an operation of all the individual parts of bodies, as being material. It
(*l*) lies in this, that matter in general assigns a center for itself, (sich das centrum setzt), and the figure of the body is an element in the determination of its place; that collective bodies of the system recognize a reference to their Sun, (sich ihre Sonne setzen) but also the individual bodies themselves, according to the relative position with regard to each other into which they come by their general motion, form a momentary relation of their gravity (schwere) *towards each other*, and are related to each other not only in abstract spatial relations, but at the same time assign to themselves a joint center, which however is again resolved [into the general center] in the universal system.

As to what concerns the features of the path, to shew how the fundamental determinations of Free Motion are connected *with the Conception*, cannot here be undertaken in a satisfactory and detailed manner, and must therefore be left to its fate. The proof from reason of the quantitative determinations of free motion can only rest upon the *determinations* of *Conceptions* of space and time, the elements whose relation (intrinsic not extrinsic) motion is.

(*m*) That, *in the first place*, the motion in general is a motion *returning into itself*, is founded on the determination of particularity and individuality of the bodies in general (§ 269), so that partly they have a center in themselves, and partly at the same time their center in another. These are the determinations of Conceptions which form the basis of the false representatives
(*n*) of Centripetal Force and Centrifugal Force, as if each of these were self-existing, extraneous to the other, and independent of it; and as if they only came in contact in their operations and consequently *externally*. They are, as has already been mentioned, the Lines which must be drawn for the mathematical determinations, transformed into physical realities.

Further, this motion is *uniformly accelerated*, (and—as returning into itself—in turn uniformly retarded). In motion as *free*, Time and Space enter as *different* things which are to make themselves effective in the determination of the motion, (§ 266, note.) In the so-
(*o*) called *Explanation* of the uniformly accelerated and retarded motion, by means of the alternate decrease and increase of the magnitude of the Centripetal Force and Centrifugal Force, the *confusion* which the assumption of such independent Forces produces is at its
(*p*) greatest height. According to this explanation, in the motion of a Planet from the Aphelion to the Perihelion, the centrifugal is *less* than the centripetal force, and on the contrary, in the Perihelion itself, the centrifugal force is supposed to become greater than the centripetal. For the motion from the Perihelion to the Aphelion, this representation makes the forces pass into the opposite relation in the same manner. It is apparent that such a sudden conversion of the preponderance which a force has obtained over another, into an inferiority to the other, cannot be anything taken out of the nature of Forces. On the contrary it must be concluded, that a preponderance which one Force has obtained over another must not only be preserved, but must go onwards to the complete annihilation of the other Force, and the motion must either, by the Preponderance of the Centripetal Force, proceed till it ends in rest, that is, in the Collision of the Planet with the Central Body, or till by the Preponderance of the Centri-

(q) fugal Force it ends in a straight line. But now, if in place of the suddenness of the conversion, we suppose a gradual increase of the Force in question, then, since rather the other Force ought to be assumed as increasing, we lose the opposition which is assumed for the sake of the explanation; and if the increase of the one is assumed to be different from that of the other, (which is the case in some representations,) then there is found at the mean distance between the apsides a point in which the Forces are *in equilibrio*. And the transition of the Forces out of Equilibrium is a thing just as little without any sufficient reason as the aforesaid suddenness of inversion. And in the whole of this kind of explanation, we see that the mode of remedying a bad mode of dealing with a subject leads to newer and greater confusion.—A similar confusion makes its appearance in the explanation of the phænomenon that the pendulum oscillates more slowly at the equator. This phænomenon is ascribed to the Centrifugal Force, which it is asserted must then be greater; but it is easy to see that we may just as well ascribe it to the augmented gravity, inasmuch as that holds the pendulum more strongly to the perpendicular line of rest.

§ 240.

(r) And now first, as to what concerns the *Form of the Path*, the *Circle* only can be conceived as the path of an *absolutely uniform* motion. *Conceivable*, as people express it, no doubt it is, that an increasing and diminishing motion should take place in a circle. But this conceivableness or possibility means only an abstract capability of being represented, which leaves out of sight that Determinate Thing on which the question turns.

The Circle is the line returning into itself in which all the radii are *equal*, that is, it is completely determined by means of the radius. There is only *one* Determination, and that is the *whole* Determination.

But in free motion, in which the Determinations according to space and according to time come into view with Differences, in a qualitative relation to each other, this Relation appears in the spatial aspect as a *Difference* thereof in itself, which therefore requires two Determinations. Hereby the Form of the path returning into itself is essentially an *Ellipse*.

(s) The abstract Determinateness which produces the circle appears also in this way, that the arc or angle which is included by two Radii is independent of them, a magnitude with regard to them completely empirical. But since in the motion as determined by the Conception, the distance from the center, and the arc which is run over in a certain time, must be comprehended in one determinateness, [*and*] make out a whole, this is the sector, a space-determination of two dimensions: in this way, the arc is essentially a Function of the Radius vector; and the former (the arc) being unequal, brings with it the inequality of the Radii. That the determination with regard to the space by means of the time appears as a Determination of two Dimensions,—as a Superficies-Determination,—agrees with what was said
(t) before (§ 266) respecting Falling Bodies, with regard to the exposition of the same Determinateness, at one while as Time in the root, at another while as Space in the square. Here, however, the Quadratic character of the space is, by the returning of the Line of motion into itself, limited to a Sector. These are, as may be seen, the general principles on which the Keplerian Law, that in equal times equal sectors are cut off, rests.

This Law becomes, as is clear, only the relation of the arc to the Radius Vector, and the Time enters there as the abstract Unity, in which the different Sectors are compared, because as Unity it is the Determining Element. But the further relation is that of the Time, not as Unity, but as a Quantity in general,—as the time of Revolution—to the magnitude of the Path, or, what is the same thing, the distance from the center. As Root and Square, we saw that Time and Space had a relation to each other, in the case of Falling Bodies, the case of half-free motion—because that [*motion*] is determined on one side by the conception, on the

other by external [*conditions*]. But in the case of absolute motion—the domain of *free*
(*u*) masses—the determination attains its Totality. The Time as the Root is a mere empirical magnitude; but as a component (moment) of the developed Totality, it is a Totality in itself,—it produces itself, and therein has a reference to itself;—as the Dimensionless Element in itself, it only comes to a formal identity with itself, the Square; Space, on the other hand, as the positive Distribution (aussereinander) [*comes*] to the Dimension of the Conception, *the* CUBE. Their Realization preserves their original difference. This is the
(*v*) third Keplerian Law, the relation of the *Cubes* of the *Distances* to the *Squares* of the *Times*;—a Law which is so great on this account, that it represents so simply and immediately *Reason as belonging to the thing*: while on the contrary the Newtonian Formula, by means of which the Law is changed into a Law for the Force of Gravity, shews the Distortion, Perversion and Inversion of *Reflexion* which stops half-way.

Additions to new Edition. § 269.

The center has no sense without the circumference, nor the circumference without the center. This makes all physical hypotheses vanish which sometimes proceed from the center, sometimes from the particular bodies, and sometimes assign this, sometimes that, as the original [cause of motion]...It is silly (läppisch) to suppose that the centrifugal force, as a tendency to fly off in a Tangent, has been produced by a lateral projection, a projectile force, an impulse which they have retained ever since they set out on their journey (von Haus aus). Such casualty of the motion produced by external causes belongs to inert matter; as when a stone fastened to a thread which is thrown transversely tries to fly from the thread. We are not to talk in this way of Forces. If we will speak of Force, there is one Force, whose elements
(*w*) do not draw bodies to different sides as if they were two Forces. The motion of the heavenly bodies is not a being pulled this way or that, such as is thus imagined; it is free motion: they go along, as the ancients said, as blessed Gods (sie gehen als selige Götter einher). The celestial corporeity is not such a one as has the principle of rest or motion external to itself. Because stone is inert, and all the earth consists of stones, and the other heavenly bodies are of the same nature,—is a conclusion which makes the properties of the whole the same as those of the part. Impulse, Pressure, Resistance, Friction, Pulling, and the like, are valid only for
(*x*) an existence of matter other than the celestial. Doubtless that which is common to the two is matter, as a good thought and a bad thought are both thoughts; but the bad one is not therefore good, because it is a thought.

LII. *Discussion of a Differential Equation relating to the breaking of Railway Bridges.* *By* G. G. STOKES, M.A., *Fellow of Pembroke College, Cambridge.*

[Read *May* 21, 1849.]

To explain the object of the following paper, it will be best to relate the circumstance which gave rise to it. Some time ago Professor Willis requested my consideration of a certain differential equation in which he was interested, at the same time explaining its object, and the mode of obtaining it. The equation will be found in the first article of this paper, which contains the substance of what he communicated to me. It relates to some experiments which have been performed by a Royal Commission, of which Professor Willis is a member, appointed on the 27th of August, 1847, "for the purpose of inquiring into the conditions to be observed by engineers in the application of iron in structures exposed to violent concussions and vibration." The object of the experiments was to examine the effect of the velocity of a train in increasing or decreasing the tendency of a girder bridge over which the train is passing to break under its weight. In order to increase the observed effect, the bridge was purposely made as slight as possible: it consisted in fact merely of a pair of cast or wrought iron bars, nine feet long, over which a carriage, variously loaded in different sets of experiments, was made to pass with different velocities. The remarkable result was obtained that the deflection of the bridge increased with the velocity of the carriage, at least up to a certain point, and that it amounted in some cases to two or three times the central statical deflection, or that which would be produced by the carriage placed at rest on the middle of the bridge. It seemed highly desirable to investigate the motion mathematically, more especially as the maximum deflection of the bridge, considered as depending on the velocity of the carriage, had not been reached in the experiments*, in some cases because it corresponded to a velocity greater than any at command, in others because the bridge gave way by the fracture of the bars on increasing the velocity of the carriage. The exact calculation of the motion, or rather a calculation in which none but really insignificant quantities should be omitted, would however be extremely difficult, and would require the solution of a partial differential equation with an ordinary differential equation for one of the equations of condition by which the arbitrary functions would have to be determined. In fact, the forces acting on the body and on any element of the bridge depend upon the positions and motions, or rather changes of motion, both of the body itself and of every other element of the bridge, so that the exact solution of the problem, even when the deflection is supposed to be small, as it is in fact, appears almost hopeless.

In order to render the problem more manageable, Professor Willis neglected the inertia of the bridge, and at the same time regarded the moving body as a heavy particle. Of course the masses of bridges such as are actually used must be considerable; but the mass of the bars in the experiments was small compared with that of the carriage, and it was reasonable to expect a near accordance between the theory so simplified and experiment. This simplification of the problem reduces the calculation to an ordinary differential equation, which is that which has been already mentioned; and it is to the discussion of this equation that the present paper is mainly devoted.

This equation cannot apparently be integrated in finite terms, except for an infinite number of particular values of a certain constant involved in it; but I have investigated rapidly convergent series whereby numerical results may be obtained. By merely altering the scale of the

* The details of the experiments will be found in the Report of the Commission, to which the reader is referred.

abscissæ and ordinates, the differential equation is reduced to one containing a single constant β, which is defined by equation (5). The meaning of the letters which appear in this equation will be seen on referring to the beginning of Art. 1. For the present it will be sufficient to observe that β varies inversely as the square of the horizontal velocity of the body, so that a small value of β corresponds to a high velocity, and a large value to a small velocity.

It appears from the solution of the differential equation that the trajectory of the body is unsymmetrical with respect to the centre of the bridge, the maximum depression of the body occurring beyond the centre. The character of the motion depends materially on the numerical value of β. When β is not greater than $\frac{1}{4}$, the tangent to the trajectory becomes more and more inclined to the horizontal beyond the maximum ordinate, till the body gets to the second extremity of the bridge, when the tangent becomes vertical. At the same time the expressions for the central deflection and for the tendency of the bridge to break become infinite. When β is greater than $\frac{1}{4}$, the analytical expression for the ordinate of the body at last becomes negative, and afterwards changes an infinite number of times from negative to positive, and from positive to negative. The expression for the reaction becomes negative at the same time with the ordinate, so that in fact the body leaps.

The occurrence of these infinite quantities indicates one of two things: either the deflection really becomes very large, after which of course we are no longer at liberty to neglect its square; or else the effect of the inertia of the bridge is really important. Since the deflection does not really become very great, as appears from experiment, we are led to conclude that the effect of the inertia is not insignificant, and in fact I have shewn that the value of the expression for the *vis viva* neglected at last becomes infinite. Hence, however light be the bridge, the mode of approximation adopted ceases to be legitimate before the body reaches the second extremity of the bridge, although it may be sufficiently accurate for the greater part of the body's course.

In consequence of the neglect of the inertia of the bridge, the differential equation here discussed fails to give the velocity for which T, the tendency to break, is a maximum. When β is a good deal greater than $\frac{1}{4}$, T is a maximum at a point not very near the second extremity of the bridge, so that we may apply the result obtained to a light bridge without very material error. Let T_1 be this maximum value. Since it is only the inertia of the bridge that keeps the tendency to break from becoming extremely great, it appears that the *general* effect of that inertia is to preserve the bridge, so that we cannot be far wrong in regarding T_1 as a superior limit to the actual tendency to break. When β is very large, T_1 may be calculated to a sufficient degree of accuracy with very little trouble.

Experiments of the nature of those which have been mentioned may be made with two distinct objects; the one, to analyze experimentally the laws of some particular phenomenon, the other, to apply practically on a large scale results obtained from experiments made on a small scale. With the former object in view, the experiments would naturally be made so as to render as conspicuous as possible, and isolate as far as might be, the effect which it was desired to investigate; with the latter, there are certain relations to be observed between the variations of the different quantities which are in any way concerned in the result. These relations, in the case of the particular problem to which the present paper refers, are considered at the end of the paper.

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1. It is required to determine, in a form adapted to numerical computation, the value of y' in terms of x', where y' is a function of x' defined by satisfying the differential equation

$$\frac{d^2y'}{dx'^2} = a - \frac{by'}{(2cx' - x'^2)^2}, \quad \ldots\ldots\ldots\ldots \quad (1)$$

with the particular conditions

$$y' = 0, \quad \frac{dy'}{dx'} = 0, \text{ when } x' = 0, \quad \ldots\ldots\ldots \quad (2)$$

the value of y' not being wanted beyond the limits 0 and $2c$ of x'. It will appear in the course of the solution that the first of the conditions (2) is satisfied by the complete integral of (1), while the second serves of itself to determine the two arbitrary constants which appear in that integral.

The equation (1) relates to the problem which has been explained in the introduction. It was obtained by Professor Willis in the following manner. In order to simplify to the very utmost the mathematical calculation of the motion, regard the carriage as a heavy particle, neglect the inertia of the bridge, and suppose the deflection very small. Let x', y' be the co-ordinates of the moving body, x' being measured horizontally from the beginning of the bridge, and y' vertically downwards. Let M be the mass of the body, V its velocity on entering the bridge, $2c$ the length of the bridge, g the force of gravity, S the deflection produced by the body placed at rest on the centre of the bridge, R the reaction between the moving body and the bridge. Since the deflection is very small, this reaction may be supposed to act vertically, so that the horizontal velocity of the body will remain constant, and therefore equal to V. The bridge being regarded as an elastic bar or plate, propped at the extremities, and supported by its own stiffness, the depth to which a weight will sink when placed in succession at different points of the bridge will vary as the weight multiplied by $(2cx' - x'^2)^2$, as may be proved by integration, on assuming that the curvature is proportional to the moment of the bending force. Now, since the inertia of the bridge is neglected, the relation between the depth y' to which the moving body has sunk at any instant and the reaction R will be the same as if R were a weight resting at a distance x' from the extremity of the bridge; and we shall therefore have

$$y' = CR\,(2cx' - x'^2)^2,$$

C being a constant, which may be determined by observing that we must have $y' = S$ when $R = Mg$ and $x' = c$; whence

$$C = \frac{S}{Mgc^4}.$$

We get therefore for the equation of motion of the body

$$\frac{d^2y'}{dt^2} = g - \frac{gc^4y'}{S\,(2cx' - x'^2)^2},$$

which becomes on observing that $\dfrac{dx'}{dt} = V$

$$\frac{d^2y'}{dx'^2} = \frac{g}{V^2} - \frac{gc^4}{V^2S}\,\frac{y'}{(2cx' - x'^2)^2},$$

which is the same as equation (1), a and b being defined by the equations

$$a = \frac{g}{V^2}, \qquad b = \frac{gc^4}{V^2S}\,. \;\;\ldots\ldots\ldots\ldots\; (3)$$

2. To simplify equation (1) put

$$x' = 2cx, \qquad y' = 16c^4ab^{-1}y, \qquad b = 4c^2\beta,$$

which gives

$$\frac{d^2y}{dx^2} = \beta - \frac{\beta y}{(x - x^2)^2} \;\;\ldots\ldots\ldots\ldots\ldots (4)$$

It is to be observed that x denotes the ratio of the distance of the body from the beginning of the bridge to the length of the bridge; y denotes a quantity from which the depth of the body below the horizontal plane in which it was at first moving may be obtained by multiplying by $16c^4ab^{-1}$ or $16S$; and β, on the value of which depends the form of the body's path, is a constant defined by the equation

$$\beta = \frac{gc^2}{4V^2S}\,. \;\;\ldots\ldots\ldots\ldots\ldots\ldots\; (5)$$

3. In order to lead to the required integral of (4), let us first suppose that x is very small. Then the equation reduces itself to

$$\frac{d^2y}{dx^2} = \beta - \frac{\beta y}{x^2}, \quad \text{.............. (6)}$$

of which the complete integral is

$$y = \frac{\beta x^2}{2+\beta} + A\,x^{\frac{1}{2}+\sqrt{\frac{1}{4}-\beta}} + Bx^{\frac{1}{2}-\sqrt{\frac{1}{4}-\beta}}, \quad \text{...... (7)}$$

and (7) is the approximate integral of (4) for very small values of x. Now the second of equations (2) requires that $A = 0$, $B = 0$,* so that the first term in the second member of equation (7) is the leading term in the required solution of (4).

4. Assuming in equation (4) $y = (x - x^2)^2 z$, we get

$$\frac{d^2}{dx^2}\left\{(x - x^2)^2 z\right\} + \beta z = \beta. \quad \text{......... (8)}$$

Since (4) gives $y = (x - x^2)^2$ when $\beta = \infty$, and (5) gives $\beta = \infty$ when $V = 0$, it follows that z is the ratio of the depression of the body to the equilibrium depression. It appears also from Art. 3, that for the particular integral of (8) which we are seeking, z is ultimately constant when x is very small.

To integrate (8) assume then

$$z = A_0 + A_1 x + A_2 x^2 + \ldots = \Sigma A_i x^i, \text{......(9)}$$

and we get

$$\Sigma\,(i+2)\,(i+1)\,A_i x^i - 2\Sigma\,(i+3)\,(i+2)\,A_i x^{i+1} + \Sigma\,(i+4)\,(i+3)\,A_i x^{i+2} + \beta\Sigma A_i x^i = \beta,$$

or

$$\Sigma\left\{[(i+1)(i+2)+\beta]\,A_i - 2\,(i+1)\,(i+2)\,A_{i-1} + (i+1)\,(i+2)\,A_{i-2}\right\} x^i = \beta, \ \ldots\ (10)$$

where it is to be observed that no coefficients A_i with negative suffixes are to be taken.

Equating to zero the coefficients of the powers 0, 1, 2...of x in (10), we get

$$(2+\beta)\,A_0 = \beta,$$
$$(6+\beta)\,A_1 - 12\,A_0 = 0, \text{ \&c.}$$

and generally

$$\left\{(i+1)\,(i+2)+\beta\right\} A_i - 2\,(i+1)\,(i+2)\,A_{i-1} + (i+1)\,(i+2)\,A_{i-2} = 0. \text{ ...... (11)}$$

The first of these equations gives for A_0 the same value which would have been got from (7). The general equation (11), which holds good from $i = 1$ to $i = \infty$, if we conventionally regard A_{-1} as equal to zero, determines the constants A_1, A_2, A_3... one after another by a simple and uniform arithmetical process. It will be rendered more convenient for numerical computation by putting it under the form

$$A_i = \left\{A_{i-1} + \Delta A_{i-2}\right\}\left\{1 - \frac{\beta}{(i+1)\,(i+2)+\beta}\right\}; \text{ .... (12)}$$

---

* When $\beta > \frac{1}{4}$, the last two terms in (7) take the form $x^{\frac{1}{2}}\{C\cos(q\log x) + D\sin(q\log x)\}$; and if y_1 denote this quantity we cannot in strictness speak of the limiting value of $\frac{dy_1}{dx}$ when $x = 0$. If we give x a small positive value, which we then suppose to decrease indefinitely, $\frac{dy_1}{dx}$ will fluctuate between the constantly increasing limits $\pm x^{-\frac{1}{2}}\sqrt{\{(\frac{1}{2}C + qD)^2 + (\frac{1}{2}D - qC)^2\}}$, or $\pm x^{-\frac{1}{2}}\sqrt{\{\beta(C^2 + D^2)\}}$, since $q = \sqrt{(\beta - \frac{1}{4})}$. But the body is supposed to enter the bridge horizontally, that is, in the direction of a tangent, since the bridge is supposed to be horizontal, so that we must clearly have $C^2 + D^2 = 0$, and therefore $C = 0$, $D = 0$. When $\beta = \frac{1}{4}$ the last two terms in (7) take the form $x^{\frac{1}{2}}(E + F\log x)$, and we must evidently have $E = 0$, $F = 0$.

for it is easy to form a table of differences as we go along; and when i becomes considerable, the quantity to be subtracted from $A_{i-1} + \Delta A_{i-2}$ will consist of only a few figures.

5. When i becomes indefinitely great, it follows from (11) or (12) that the relation between the coefficients A_i is given by the equation

$$A_i - 2A_{i-1} + A_{i-2} = 0, \quad \text{......} \quad (13)$$

of which the integral is

$$A_i = C + C'i. \quad \text{......} \quad (14)$$

Hence the ratio of consecutive coefficients is ultimately a ratio of equality, and therefore the ratio of the $(i+1)$th term of the series (9) to the ith is ultimately equal to x. Hence the series is convergent when x lies between the limits -1 and $+1$; and it is only between the limits 0 and 1 of x that the integral of (8) is wanted. The degree of convergency of the series will be ultimately the same as in a geometric series whose ratio is x.

6. When x is moderately small, the series (9) converges so rapidly as to give z with little trouble, the coefficients $A_1, A_2 \ldots$ being supposed to have been already calculated, as far as may be necessary, from the formula (12). For larger values, however, it would be necessary to keep in a good many terms, and the labour of calculation might be abridged in the following manner.

When i is very large, we have seen that equation (12) reduces itself to (13), or to $\Delta^2 A_{i-2} = 0$, or, which is the same, $\Delta^2 A_i = 0$. When i is large, $\Delta^2 A_i$ will be small; in fact, on substituting in the small term of (12) the value of A_i given by (14), we see that $\Delta^2 A_i$ is of the order i^{-1}. Hence $\Delta^3 A_i$, $\Delta^4 A_i \ldots$ will be of the orders $i^{-2}, i^{-3} \ldots$, so that the successive differences of A_i will rapidly decrease. Suppose i terms of the series (9) to have been calculated directly, and let it be required to find the remainder. We get by finite integration by parts

$$\Sigma A_i x^i = \text{const.} + A_i \frac{x^i}{x-1} - \Delta A_i \frac{x^{i+1}}{(x-1)^2} + \Delta^2 A_i \frac{x^{i+2}}{(x-1)^3} - \ldots,$$

and taking the sum between the limits i and ∞ we get

$$A_i x^i + A_{i+1} x^{i+1} + \ldots \text{ to inf.} = x^{i-1} \left\{ A_i \frac{x}{1-x} + \Delta A_i \left(\frac{x}{1-x}\right)^2 + \Delta^2 A_i \left(\frac{x}{1-x}\right)^3 + \ldots \right\}; \quad \ldots (15)$$

z will however presently be made to depend on series so rapidly convergent that it will hardly be worth while to employ the series (15), except in calculating the series (9) for the particular value $\frac{1}{2}$ of x, which will be found necessary in order to determine a certain arbitrary constant*.

7. If the constant term in equation (4) be omitted, the equation reduces itself to

$$\frac{d^2 y}{dx^2} + \frac{\beta y}{(x - x^2)^2} = 0. \quad \text{......} \quad (16)$$

The form of this equation suggests that there may be an integral of the form $y = x^m (1-x)^n$. Assuming this expression for trial, we get

$$(x - x^2)^2 \frac{d^2 y}{dx^2} = x^m (1-x)^n \{m(m-1)(1-x)^2 - 2mnx(1-x) + n(n-1)x^2\}$$

$$= y \{m(m-1) - 2m(m+n-1)x + (m+n)(m+n-1)x^2\}.$$

The second member of this equation will be proportional to y, if

$$m + n - 1 = 0, \quad \text{......} \quad (17)$$

---

* A mode of calculating the value of z for $x = \frac{1}{2}$ will presently be given, which is easier than that here mentioned, unless β be very large. See equation (42) at the end of this paper.

and will be moreover equal to $-\beta y$, if

$$m^2 - m + \beta = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

It appears from (17) that m, n are the two roots of the quadratic (18). We have for the complete integral of (16)

$$y = Ax^m (1-x)^n + Bx^n (1-x)^m. \quad \ldots\ldots\ldots\ldots\ldots \quad (19)$$

The complete integral of (4) may now be obtained by replacing the constants A, B by functions R, S of x, and employing the method of the variation of parameters. Putting for shortness

$$x^m (1-x)^n = u, \quad x^n (1-x)^m = v,$$

we get to determine R and S the equations

$$u \frac{dR}{dx} + v \frac{dS}{dx} = 0,$$

$$\frac{du}{dx}\frac{dR}{dx} + \frac{dv}{dx}\frac{dS}{dx} = \beta.$$

Since $v \dfrac{du}{dx} - u \dfrac{dv}{dx} = m - n$, we get from the above equations

$$\frac{dR}{dx} = \frac{\beta v}{m-n}, \quad \frac{dS}{dx} = -\frac{\beta u}{m-n},$$

whence we obtain for a particular integral of (4)

$$y = \frac{\beta}{m-n}\left\{x^m (1-x)^n \int_0^x x^n (1-x)^m dx - x^n (1-x)^m \int_0^x x^m (1-x)^n dx\right\}; \quad \ldots\ldots \quad (20)$$

and the complete integral will be got by adding together the second members of equations (19), (20). Now the second member of equation (20) varies ultimately as x^2, when x is very small, and therefore, as shewn in Art. 3, we must have $A = 0$, $B = 0$, so that (20) is the integral we want.

When the roots of the quadratic (18) are real and commensurable, the integrals in (20) satisfy the criterion of integrability, so that the integral of (4) can be expressed in finite terms without the aid of definite integrals. The form of the integral will, however, be complicated, and y may be readily calculated by the method which applies to general values of β.

8. Since $\int_0^x F(x)\, dx = \int_0^1 F(x)\, dx - \int_0^{1-x} F(1-x)\, dx$, we have from (20)

$$y = \frac{\beta}{m-n}\left\{x^m (1-x)^n \int_0^1 x^n (1-x)^m dx - x^n (1-x)^m \int_0^1 x^m (1-x)^n dx\right\},$$

$$+ \frac{\beta}{m-n}\left\{x^n (1-x)^m \int_0^{1-x} (1-x)^m x^n dx - x^m (1-x)^n \int_0^{1-x} (1-x)^n x^m dx\right\}.$$

If we put $f(x)$ for the second member of equation (20), the equation just written is equivalent to

$$f(x) = f(1-x) + \phi(x), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

where

$$\phi(x) = \frac{\beta}{m-n}\left\{x^m (1-x)^n \int_0^1 x^n (1-x)^m dx - x^n (1-x)^m \int_0^1 x^m (1-x)^n dx\right\} \ldots \quad (22)$$

Now since $m + n = 1$,

$$\int x^n (1-x)^m dx = \int x (x^{-1} - 1)^m dx = -\int w^{-1} (w-1)^m w^{-2} dw = -\int \frac{s^m ds}{(1+s)^3}.$$

At the limits $x = 0$ and $x = 1$, we have $w = \infty$ and $w = 1$, $s = \infty$ and $s = 0$, whence if I denote the definite integral,

$$I = \int_0^1 x^n (1 - x)^m \, dx = \int_0^\infty \frac{s^m \, ds}{(1 + s)^3}.$$

We get by integration by parts

$$\int \frac{s^m \, ds}{(1 + s)^3} = -\frac{s^m}{2(1 + s)^2} + \frac{m}{2} \int \frac{s^{m-1} \, ds}{(1 + s)^2},$$

and again by a formula of reduction

$$\int \frac{s^{m-1} \, ds}{(1 + s)^2} = \frac{s^m}{1 + s} + (1 - m) \int \frac{s^{m-1} \, ds}{1 + s}.$$

Now β being essentially positive, the roots of the quadratic (18) are either real, and comprised between 0 and 1, or else imaginary with a real part equal to $\frac{1}{2}$. In either case the expressions which are free from the integral sign vanish at the limits $s = 0$ and $s = \infty$, and we have therefore, on replacing $m(1 - m)$ by its value β,

$$I = \frac{\beta}{2} \int_0^\infty \frac{s^{m-1} \, ds}{1 + s}.$$

The function $\phi(x)$ will have different forms according as the roots of (18) are real or imaginary. First suppose the roots real, and let $m = \frac{1}{2} + r$, $n = \frac{1}{2} - r$, so that

$$r = \sqrt{\tfrac{1}{4} - \beta}. \quad \dots\dots\dots\dots (23)$$

In this case m is a real quantity lying between 0 and 1, and we have therefore by a known formula

$$\int_0^\infty \frac{s^{m-1} \, ds}{1 + s} = \frac{\pi}{\sin m\pi} = \frac{\pi}{\cos r\pi}, \quad \dots\dots\dots\dots (24)$$

whence we get from (22), observing that the two definite integrals in this equation are equal to each other,

$$\phi(x) = \frac{\beta^2 \pi}{4r \cos r\pi} \sqrt{x - x^2} \left\{ \left(\frac{x}{1 - x} \right)^r - \left(\frac{x}{1 - x} \right)^{-r} \right\}. \quad \dots\dots\dots\dots (25)$$

This result might have been obtained somewhat more readily by means of the properties of the first and second Eulerian integrals.

When β becomes equal to $\frac{1}{4}$, r vanishes, the expression for $\phi(x)$ takes the form $\frac{0}{0}$, and we easily find

$$\phi(x) = \frac{\pi}{32} \sqrt{x - x^2} \log \frac{x}{1 - x}. \quad \dots\dots\dots\dots (26)$$

When $\beta > \frac{1}{4}$, the roots of (18) become imaginary, and r becomes $\rho \sqrt{-1}$, where

$$\rho = \sqrt{\beta - \tfrac{1}{4}}. \quad \dots\dots\dots\dots (27)$$

The formula (25) becomes

$$\phi(x) = \frac{\beta^2 \pi}{\rho(\epsilon^{\rho\pi} + \epsilon^{-\rho\pi})} \sqrt{x - x^2} \sin \left(\rho \log \frac{x}{1 - x} \right). \quad \dots\dots\dots\dots (28)$$

If $f(x)$ be calculated from $x = 0$ to $x = \frac{1}{2}$, equation (21) will enable us to calculate it readily from $x = \frac{1}{2}$ to $x = 1$, since it is easy to calculate $\phi(x)$.

9. A series of a simple form, which is more rapidly convergent than (9) when x approaches the value $\frac{1}{2}$, may readily be investigated.

Let $x = \frac{1}{2}(1 + w)$; then substituting in equation (8) we get

$$\frac{1}{4}\frac{d^2}{dw^2}\{(1 - w^2)^2 z\} + \beta z = \beta. \quad \ldots\ldots\ldots\ldots (29)$$

Assume

$$z = B_0 + B_1 w^2 + B_2 w^4 \ldots = \Sigma B_i w^{2i}, \quad \ldots\ldots\ldots\ldots (30)$$

then substituting in (29) we get

$$\Sigma B_i\{2i(2i - 1)w^{2i-2} - 2(2i + 2)(2i + 1)w^{2i} + (2i + 4)(2i + 3)w^{2i+2} + 4\beta w^{2i}\} = 4\beta,$$

or,

$$\Sigma\{i(2i - 1)B_i - 2[i(2i - 1) - \beta]B_{i-1} + i(2i - 1)B_{i-2}\}w^{2i-2} = 2\beta.$$

This equation leaves B_0 arbitrary, and gives on dividing by $i(2i - 1)$, and putting in succession $i = 1$, $i = 2$, &c.,

$$B_1 - 2\left(1 - \frac{\beta}{1.1}\right)B_0 = 2\beta, \ldots\ldots\ldots\ldots (31).$$

$$B_2 - 2\left(1 - \frac{\beta}{2.3}\right)B_1 + B_0 = 0, \text{ \&c.};$$

and generally when $i > 1$,

$$B_i = B_{i-1} + \Delta B_{i-2} - \frac{2\beta}{i(2i - 1)}B_{i-1} \ldots\ldots\ldots\ldots (32).$$

The constants B_1, B_2,... being thus determined, the series (30) will be an integral of equation (29), containing one arbitrary constant. An integral of the equation derived from (29) by replacing the second member by zero may be obtained in just the same way by assuming $z = C_0 w + C_1 w^3 + \ldots$ when $C_1, C_2 \ldots$ will be determined in terms of C_1, which remains arbitrary. The series will both be convergent between the limits $w = -1$ and $w = 1$, that is, between the limits $x = 0$ and $x = 1$. The sum of the two series will be the complete integral of (29), and will be equal to $(x - x^2)^{-2}f(x)$ if the constants B_0, C_0 be properly determined. Denoting the sums of the two series by $F_e(w)$, $F_0(w)$ respectively, and writing $\sigma(x)$ for $(x - x^2)^{-2}f(x)$, so that $z = \sigma(x)$, we get

$$\sigma(x) = F_e(w) + F_0(w), \ \sigma(1 - x) = F_e(w) - F_0(w);$$

and since $2F_0(w) = \sigma(x) - \sigma(1 - x) = (x - x^2)^{-2}\phi(x)$ by (21), we get

$$\sigma(x) = F_e(w) + \tfrac{1}{2}(x - x^2)^{-2}\phi(x), \ \sigma(1 - x) = F_e(w) - \tfrac{1}{2}(x - x^2)^{-2}\phi(x) \ldots\ldots\ldots (33).$$

To determine B_0 we have

$$B_0 = \sigma(\tfrac{1}{2}), \ldots\ldots\ldots\ldots (34).$$

which may be calculated by the series (9).

10. The series (9), (30) will ultimately be geometric series with ratios x, w^2, or x, $(2x - 1)^2$, respectively. Equating these ratios, and taking the smaller root of the resulting quadratic, we get $x = \frac{1}{4}$. Hence if we use the series (9) for the calculation of $\sigma(x)$ from $x = 0$ to $x = \frac{1}{4}$, and (30) for the calculation of $\sigma(x)$ from $x = \frac{1}{4}$ to $x = \frac{1}{2}$, we shall have to calculate series which are ultimately geometric series with ratios ranging from 0 to $\frac{1}{4}$.

Suppose that we wish to calculate $\sigma(x)$ or z for values of x increasing by .02. The process of calculation will be as follows. From the equation $(2 + \beta)A_0 = \beta$ and the general formula (12), calculate the coefficients A_0, A_1, A_2,... as far as may be necessary. From the series (9), or else from the series (9) combined with the formula (15), calculate $\sigma(\frac{1}{2})$ or B_0, and then calculate B_1, B_2... from equations (31), (32). Next calculate $\sigma(x)$ from the series (9) for the values .02, .04,....26 of x, and $F_e(w)$ from (30) for the values .04, .08..., .44 of w, and lastly $(x - x^2)^{-2}\phi(x)$ for the values .52, .54..., .98 of x. Then we have $\sigma(x)$ calculated directly from $x = 0$ to $x = .26$; equations (33) will give $\sigma(x)$ from $x = .28$ to $x = .72$, and lastly the equation $\sigma(x) = \sigma(1-x) + (x - x^2)^{-2}\phi(x)$ will give $\sigma(x)$ from $x = .74$ to $x = 1$.

11. The equation (21) will enable us to express in finite terms the vertical velocity of the body at the centre of the bridge. For according to the notation of Art. 2, the horizontal and vertical coordinates of the body are respectively $2cx$ and $16Sy$, and we have also $\frac{d \, . \, 2cx}{dt} = V$, whence, if v be the vertical velocity, we get

$$v = \frac{d \, . \, 16\,Sy}{dx}\frac{dx}{dt} = \frac{8SV}{c}f'(x).$$

But (21) gives $f'(\frac{1}{2}) = \frac{1}{2}\phi'(\frac{1}{2})$, whence if v_c be the value of v at the centre, we get

$$v_c = \frac{4\pi SV\beta^2}{c\cos r\pi}, \text{ or } = \frac{8\pi SV\beta^2}{c(\epsilon^{\rho\pi} + \epsilon^{-\rho\pi})}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots(35)$$

according as $\beta <> \frac{1}{4}$.

In the extreme cases in which V is infinitely great and infinitely small respectively, it is evident that v_c must vanish, and therefore for some intermediate value of V, v_c must be a maximum. Since $V \propto \beta^{-\frac{1}{2}}$ when the same body is made to traverse the same bridge with different velocities, v_c will be a maximum when p or q is a minimum, where

$$p = 2\beta^{-\frac{3}{2}}\cos r\pi, \quad q = \beta^{-\frac{3}{2}}(\epsilon^{\rho\pi} + \epsilon^{-\rho\pi}).$$

Putting for $\cos r\pi$ its expression in a continued product, and replacing r by its expression in terms of β, we get

$$p = 8\beta^{-\frac{1}{2}}\left(1 - \frac{1-4\beta}{3^2}\right)\left(1 - \frac{1-4\beta}{5^2}\right)\ldots\ldots\ldots$$

whence

$$\frac{d\log p}{d\beta} = -\frac{1}{2\beta} + \frac{1}{1.2+\beta} + \frac{1}{2.3+\beta} + \ldots\ldots\ldots\ldots(36)$$

The same expression would have been obtained for $\frac{d\log q}{d\beta}$. Call the second member of equation (36) $F(\beta)$, and let $-N$, P be the negative and positive parts respectively of $F(\beta)$. When $\beta = 0$, $N = \infty$, and $P = \frac{1}{1.2} + \frac{1}{2.3}\ldots = 1$, and therefore $F(\beta)$ is negative. When β becomes infinite, the ratio of P to N becomes infinite, and therefore $F(\beta)$ is positive when β is sufficiently large; and $F(\beta)$ alters continuously with β. Hence the equation $F(\beta) = 0$ must have at least one positive root. But it cannot have more than one; for the rates of proportionate decrease of the quantities N, P, or $-\frac{1}{N}\frac{dN}{d\beta}$, $-\frac{1}{P}\frac{dP}{d\beta}$, are respectively

$$\frac{1}{\beta}, \frac{(1.2+\beta)^{-2} + (2.3+\beta)^{-2} + \ldots}{(1.2+\beta)^{-1} + (2.3+\beta)^{-1} + \ldots},$$

and the several terms of the denominator of the second of these expressions are equal to those of the numerator multiplied by $1.2+\beta$, $2.3+\beta$,... respectively, and therefore the denominator is equal to the numerator multiplied by a quantity greater than $2+\beta$, and therefore greater than β; so that the value of the expression is less than $\frac{1}{\beta}$. Hence for a given infinitely small increment of β the change $-dN$ in N bears to N a greater ratio than $-dP$ bears to P, so that when N is greater than or equal to P it is decreasing more rapidly than P, and therefore after having once become equal to P it must remain always less than P. Hence v_c admits of but one maximum or minimum value, and this must evidently be a maximum.

When $\beta = \frac{1}{4}$, $N = 2$, and $P < \frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} + \ldots$ or < 1, and therefore $F(\beta)$ has the same sign as when β is indefinitely small. Hence it is q and not p which becomes a minimum. Equating $\frac{dq}{d\beta}$ to zero, employing (27), and putting $2\pi\rho = \log_\epsilon \zeta$, we find

$$\frac{6(\zeta+1)}{\zeta-1} = \log_\epsilon \zeta + \pi^2 (\log_\epsilon \zeta)^{-1}.$$

The real positive root of this equation will be found by trial to be 36.3 nearly, which gives $\rho = .5717$, $\beta = \frac{1}{4} + \rho^2 = .5768$. If V_1 be the velocity which gives v_c a maximum, v_1 the maximum value of v_c, U the velocity due to the height S, we get

$$V_1 = \sqrt{\frac{gc^2}{4\beta S}} = \frac{c}{S}\frac{U}{\sqrt{8\beta}}, \text{ and } v_1 = \frac{8\pi\beta^2}{\zeta^{\frac{1}{2}} + \zeta^{-\frac{1}{2}}}\frac{S}{c} V_1, \text{ whence}$$

$$V_1 = .4655 \frac{c}{S} U, \; v_1 = .6288 U.$$

12. Conceive a weight W placed at rest on a point of the bridge whose distance from the first extremity is to the whole length as x to 1. The reaction at this extremity produced by W will be equal to $(1-x)W$, and the moment of this reaction about a point of the bridge whose abscissa $2cx_1$ is less than $2cx$ will be $2c(1-x)x_1W$. This moment measures the tendency of the bridge to break at the point considered, and it is evidently greatest when $x_1 = x$, in which case it becomes $2c(1-x)xW$. Now, if the inertia of the bridge be neglected, the pressure R produced by the moving body will be proportional to $(x-x^2)^{-2}y$, and the tendency to break under the action of a weight equal to R placed at rest on the bridge will be proportional to $(1-x)x \times (x-x^2)^{-2}y$, or to $(x-x^2)z$. Call this tendency T, and let T be so measured that it may be equal to 1 when the moving body is placed at rest on the centre of the bridge. Then $T = C(x-x^2)z$, and $1 = C(\frac{1}{2} - \frac{1}{4})$, whence

$$T = 4(x-x^2)z.$$

The tendency to break is actually liable to be somewhat greater than T, in consequence of the state of vibration into which the bridge is thrown, in consequence of which the curvature is alternately greater and less than the statical curvature due to the same pressure applied at the same point. In considering the motion of the body, the vibrations of the bridge were properly neglected, in conformity with the supposition that the inertia of the bridge is infinitely small compared with that of the body.

The quantities of which it will be most interesting to calculate the numerical values are z, which expresses the ratio of the depression of the moving body at any point to the statical depression, T, the meaning of which has just been explained, and y', the actual depression. When z has been calculated in the way explained in Art. 10, T will be obtained by multiplying by $4(x-x^2)$, and then $\frac{y'}{S}$ will be got by multiplying T by $4(x-x^2)$.

13. The following Table gives the values of these three quantities for each of four values of β, namely $\frac{5}{36}$, $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{5}{4}$, to which correspond $r = \frac{1}{3}$, $r = 0$, $\rho = \frac{1}{2}$, $\rho = 1$, respectively. In performing the calculations I have retained five decimal places in calculating the coefficients A_0, A_1, $A_2 \ldots$ and B_0, B_1, $B_2 \ldots$ and four in calculating the series (9) and (30). In calculating $\phi(x)$ I have used four-figure logarithms, and I have retained three figures in the result. The calculations have not been re-examined, except occasionally, when an irregularity in the numbers indicated an error.

TABLE I.

x	z				T				$\frac{y'}{S}$			
	$\beta=\frac{5}{36}$	$\beta=\frac{1}{4}$	$\beta=\frac{1}{2}$	$\beta=\frac{5}{4}$	$\beta=\frac{5}{36}$	$\beta=\frac{1}{4}$	$\beta=\frac{1}{2}$	$\beta=\frac{5}{4}$	$\beta=\frac{5}{36}$	$\beta=\frac{1}{4}$	$\beta=\frac{1}{2}$	$\beta=\frac{5}{4}$
.00	.065	.111	.200	.385	.000	.000	.000	.000	.000	.000	.000	.000
.02	.067	.115	.208	.398	.005	.009	.016	.031	.000	.001	.001	.002
.04	.070	.120	.216	.412	.011	.018	.033	.063	.002	.003	.005	.010
.06	.073	.125	.224	.426	.017	.028	.051	.096	.004	.006	.011	.022
.08	.076	.130	.233	.441	.022	.038	.069	.130	.007	.011	.020	.038
.10	.080	.136	.243	.457	.029	.049	.087	.165	.010	.018	.032	.059
.12	.083	.142	.253	.474	.035	.060	.107	.200	.015	.025	.045	.085
.14	.087	.148	.264	.493	.042	.071	.127	.237	.020	.034	.061	.114
.16	.091	.155	.276	.512	.049	.083	.148	.275	.026	.045	.080	.148
.18	.096	.162	.288	.532	.056	.096	.170	.314	.033	.057	.100	.185
.20	.100	.170	.302	.553	.064	.109	.193	.354	.041	.070	.124	.227
.22	.105	.179	.316	.576	.072	.123	.217	.396	.050	.084	.149	.272
.24	.111	.188	.331	.601	.081	.137	.242	.438	.059	.100	.176	.320
.26	.117	.198	.348	.627	.090	.152	.267	.482	.069	.117	.206	.371
.28	.122	.208	.367	.642	.099	.168	.296	.519	.080	.135	.239	.418
.30	.130	.220	.386	.676	.109	.185	.324	.568	.092	.155	.272	.477
.32	.138	.232	.406	.705	.120	.202	.354	.614	.104	.176	.308	.534
.34	.146	.246	.429	.751	.131	.221	.386	.674	.118	.198	.346	.605
.36	.155	.261	.454	.783	.143	.241	.419	.721	.132	.222	.386	.665
.38	.165	.277	.480	.829	.155	.261	.453	.781	.150	.240	.427	.736
.40	.176	.295	.509	.870	.169	.283	.489	.835	.162	.272	.470	.802
.42	.188	.314	.541	.916	.182	.306	.527	.892	.178	.298	.513	.869
.44	.201	.336	.576	.966	.198	.331	.568	.951	.195	.326	.560	.939
.46	.216	.360	.615	1.02	.214	.358	.611	1.01	.213	.347	.607	1.01
.48	.232	.386	.657	1.08	.231	.385	.656	1.08	.231	.385	.655	1.10
.50	.250	.416	.705	1.14	.250	.416	.705	1.14	.250	.416	.705	1.14
.52	.271	.449	.758	1.22	.270	.449	.757	1.21	.270	.448	.755	1.21
.54	.294	.487	.817	1.29	.292	.484	.812	1.28	.290	.481	.807	1.28
.56	.320	.529	.884	1.38	.316	.522	.871	1.36	.311	.514	.859	1.34
.58	.350	.578	.959	1.47	.342	.563	.935	1.44	.333	.548	.911	1.40
.60	.385	.633	1.05	1.58	.370	.608	1.00	1.52	.355	.584	.964	1.46
.62	.425	.697	1.14	1.70	.401	.657	1.08	1.60	.378	.619	1.02	1.51
.64	.472	.771	1.26	1.82	.435	.710	1.16	1.68	.401	.654	1.07	1.55
.66	.527	.858	1.39	1.98	.473	.771	1.25	1.78	.425	.692	1.12	1.59
.68	.592	.961	1.54	2.13	.516	.837	1.34	1.86	.449	.728	1.17	1.62
.70	.671	1.08	1.72	2.32	.563	.910	1.45	1.95	.473	.765	1.22	1.64
.72	.765	1.23	1.94	2.54	.617	.994	1.56	2.05	.498	.801	1.26	1.65
.74	.883	1.40	2.20	2.80	.680	1.08	1.69	2.15	.523	.830	1.30	1.66
.76	1.03	1.64	2.52	3.08	.751	1.20	1.84	2.25	.548	.874	1.34	1.64
.78	1.22	1.93	2.92	3.42	.835	1.32	2.00	2.35	.573	.908	1.38	1.61
.80	1.46	2.30	3.43	3.81	.935	1.46	2.19	2.44	.598	.933	1.40	1.56
.82	1.79	2.79	4.08	4.26	1.05	1.65	2.41	2.52	.623	.972	1.42	1.49
.84	2.24	3.46	4.96	4.79	1.20	1.86	2.67	2.58	.647	1.00	1.43	1.39
.86	2.88	4.41	6.17	5.41	1.39	2.13	2.97	2.61	.669	1.00	1.43	1.25
.88	3.87	5.84	7.91	6.10	1.63	2.47	3.34	2.58	.691	1.04	1.41	1.09
.90	5.47	8.12	10.6	6.81	1.97	2.92	3.80	2.45	.708	1.05	1.37	.883
.92	8.34	12.1	15.0	7.27	2.45	3.57	4.41	2.14	.723	1.05	1.30	.630
.94	14.3	20.3	23.2	6.44	3.25	4.58	5.23	1.45	.730	1.00	1.18	.318
.96	29.6	43.5	41.8	− .600	4.55	6.69	6.42	− .09	.699	1.00	.987	− .014
.98	112	139	106	−65.8	8.80	10.9	8.32	−5.16	.690	.857	.652	− .404
1.00	∞	∞	± ∞	± ∞	∞	∞	± ∞	± ∞	.000	.000	.000	.000

14. Let us first examine the progress of the numbers. For the first two values of β, z increases from a small positive quantity up to ∞ as x increases from 0 to 1. As far as the table goes, z is decidedly greater for the second of the two values of β than for the first. It is easily proved however that before x attains the value 1, z becomes greater for the first value of β than for the second. For if we suppose x very little less than 1, $f(1-x)$ will be extremely small compared with $\phi(x)$, or, in case $\phi(x)$ contain a sine, compared with the coefficient of the sine. Writing x_1 for $1-x$, and retaining only the most important term in $f(x)$, we get from (21), (25), (26), and (28)

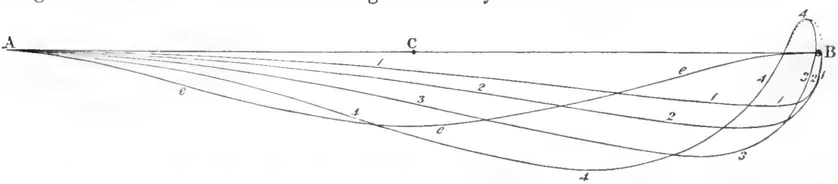
$$f(x) = \frac{\beta^2\pi}{4r\cos r\pi}x_1^{\frac{1}{2}-r}, \quad = \frac{\pi}{32}x_1^{\frac{1}{2}}\log\frac{1}{x_1}, \quad \text{or} = \frac{\beta^2\pi}{\rho(\epsilon^{\rho\pi}+\epsilon^{-\rho\pi})}x_1^{\frac{1}{2}}\sin\left(\rho\log\frac{1}{x_1}\right) \quad \text{.........} \quad (37)$$

according as $\beta<\frac{1}{4}$, $\beta=\frac{1}{4}$, or $\beta>\frac{1}{4}$; and z will be obtained by dividing $f(x)$ by x_1^2 nearly. Hence if $\frac{1}{4}>\beta_2>\beta_1>0$, z is ultimately incomparably greater when $\beta=\beta_1$ than when $\beta=\beta_2$, and when $\beta=\beta_2$ than when $\beta=\frac{1}{4}$. Since $f(0)=A_0=\beta(2+\beta)^{-1}=(2\beta^{-1}+1)^{-1}$, $f(0)$ increases with β, so that $f(x)$ is at first larger when $\beta=\beta_2$ than when $\beta=\beta_1$, and afterwards smaller.

When $\beta>\frac{1}{4}$, z vanishes for a certain value of x, after which it becomes negative, then vanishes again and becomes positive, and so on an infinite number of times. The same will be true of T. If ρ be small, $f(x)$ will not greatly differ, except when x is nearly equal to 1, from what it would be if ρ were equal to zero, and therefore $f(x)$ will not vanish till x is nearly equal to 1. On the other hand, if ρ be extremely large, which corresponds to a very slow velocity, z will be sensibly equal to 1 except when x is nearly equal to 1, so that in this case also $f(x)$ will not vanish till x is nearly equal to 1. The table shews that when $\beta=\frac{1}{2}$, $f(x)$ first vanishes between $x=.98$ and $x=1$, and when $\beta=\frac{5}{4}$ between $x=.94$ and $x=.96$. The first value of x for which $f(x)$ vanishes is probably never much less than 1, because as β increases from $\frac{5}{4}$ the denominator $\rho(\epsilon^{\rho\pi}+\epsilon^{-\rho\pi})$ in the expression for $\phi(x)$ becomes rapidly large.

15. Since when $\beta>\frac{1}{4}$, T vanishes when $x=0$, and again for a value of x less than 1, it must be a maximum for some intermediate value. When $\beta=\frac{1}{2}$ the table appears to indicate a maximum beyond $x=.98$. When $\beta=\frac{5}{4}$, the maximum value of T is about 2.61, and occurs when $x=.86$ nearly. As β increases indefinitely, the first maximum value of T approaches indefinitely to 1, and the corresponding value of x to $\frac{1}{2}$. Besides the first maximum, there are an infinite number of alternately negative and positive maxima; but these do not correspond to the problem, for a reason which will be considered presently.

16. The following curves represent the trajectory of the body for the four values of β contained in the preceding table. These curves, it must be remembered, correspond to the ideal limiting case in which the inertia of the bridge is infinitely small.



In this figure the right line AB represents the bridge in its position of equilibrium, and at the same time represents the trajectory of the body in the ideal limiting case in which $\beta=0$ or $V=\infty$. $AeeeB$ represents what may be called the *equilibrium trajectory*, or the curve the body would describe if it moved along the bridge with an infinitely small velocity. The trajectories corresponding to the four values of β contained in the above table are marked by 1, 1, 1, 1; 2, 2, 2; 3, 3, 3; 4, 4, 4, 4 respectively. The dotted curve near B is meant to represent the parabolic arc which the body really describes after it rises above the horizontal line AB*. C is the centre of the right line AB: the curve $AeeeB$ is symmetrical with respect to an ordinate drawn through C.

17. The inertia of the bridge being neglected, the reaction of the bridge against the body, as already observed, will be represented by $\dfrac{Cy}{(x-x^2)^2}$, where C depends on the length and stiffness of the bridge. Since this expression becomes negative with y, the preceding solution will not be applicable beyond the value of x for which y first vanishes, unless we suppose the body held down to the bridge by some contrivance. If it be not so held, which in fact is the case, it will quit the bridge when y becomes negative. More properly speaking, the bridge will follow the body, in consequence of its inertia, for at least a certain distance above the horizontal line AB, and will exert a positive pressure against the body: but this pressure must be neglected for the sake of consistency, in consequence of the simplification adopted in Art. 1, and therefore the body may be considered to quit the bridge as soon as it gets above the line AB. The preceding solution shews that when $\beta > \frac{1}{4}$ the body will inevitably leap before it gets to the end of the bridge. The leap need not be high; and in fact it is evident that it must be very small when β is very large. In consequence of the change of conditions, it is only the first maximum value of T which corresponds to the problem, as has been already observed.

18. According to the preceding investigation, when $\beta < \frac{1}{4}$ the body does not leap, the tangent to its path at last becomes vertical, and T becomes infinite. The occurrence of this infinite value indicates the failure, in some respect, of the system of approximation adopted. Now the inertia of the bridge has been neglected throughout; and, consequently, in the system of the bridge and the moving body, that amount of labouring force which is requisite to produce the *vis viva* of the bridge has been neglected. If ξ, η be the coordinates of any point of the bridge on the same scale on which x, y represent those of the body, and ξ be less than x, it may be proved on the supposition that the bridge may be regarded at any instant as in equilibrium, that

$$\frac{2\eta}{y} = \left(\frac{2}{x} + \frac{1}{1-x}\right)\xi - \frac{\xi^3}{x^2(1-x)} \;\ldots\ldots\ldots\ldots\ldots\ldots \quad (38)$$

When x becomes very nearly equal to 1, y varies ultimately as $(1-x)^{\frac{1}{2}-r}$, and therefore η contains terms involving $(1-x)^{-\frac{1}{2}-r}$, and $\left(\dfrac{d\eta}{dx}\right)^2$, and consequently $\left(\dfrac{d\eta}{dt}\right)^2$, contains terms involving $(1-x)^{-3-2r}$. Hence the expression for the *vis viva* neglected at last becomes infinite; and therefore however light the bridge may be, the mode of approximation adopted ceases to be legitimate before the body comes to the end of the bridge. The same result would have been arrived at if β had been supposed equal to or greater than $\frac{1}{4}$.

19. There is one practical result which seems to follow from the very imperfect solution of the problem which is obtained when the inertia of the bridge is neglected. Since this inertia is the main cause which prevents the tendency to break from becoming enormously great, it would seem that of two bridges of equal length and equal strength, but unequal mass, the lighter would

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* The dotted curve ought to have been drawn wholly outside the full curve. The two curves touch each other at the point where they are cut by the line ACB, as is represented in the figure.

be the more liable to break under the action of a heavy body moving swiftly over it. The effect of the inertia may possibly be thought worthy of experimental investigation.

20. The mass of a rail on a railroad must be so small compared with that of an engine, or rather with a quarter of the mass of an engine, if we suppose the engine to be a four-wheeled one, and the weight to be equally distributed between the four wheels, that the preceding investigation must be nearly applicable till the wheel is very near the end of the rail on which it was moving, except in so far as relates to regarding the wheel as a heavy point. Consider the motion of the fore wheels, and for simplicity suppose the hind wheels moving on a rigid horizontal plane. Then the fore wheels can only ascend or descend by the turning of the whole engine round the hind axle, or else the line of contact of the hind wheels with the rails, which comes to nearly the same thing. Let M be the mass of the whole engine, l the horizontal distance between the fore and hind axles, h the horizontal distance of the centre of gravity from the latter axle, k the radius of gyration about the hind axle, x, y the coordinates of the centre of one of the fore wheels, and let the rest of the notation be as in Art. 1. Then to determine the motion of this wheel we shall have

$$Mk^2 \frac{d^2}{dt^2}\left(\frac{y}{l}\right) = Mgh - \frac{2Cy}{(2cx - x^2)^2} \cdot l,$$

whereas to determine the motion of a single particle whose mass is $\frac{M}{4}$ we should have had

$$\frac{M}{4}\frac{d^2y}{dt^2} = \frac{M}{4}g - \frac{Cy}{(2cx - x^2)^2}.$$

Now h must be nearly equal to $\frac{l}{2}$, and k^2 must be a little greater than $\frac{1}{3}l^2$, say equal to $\frac{1}{2}l^2$, so that the two equations are very nearly the same.

Hence, β being the quantity defined by equation (5), where S denotes the central statical deflection due to a weight $\frac{Mg}{4}$, it appears that the rail ought to be made so strong, or else so short, as to render β a good deal larger than $\frac{1}{4}$. In practice, however, a rail does not rest merely on the chairs, but is supported throughout its whole length by ballast rammed underneath.

21. In the case of a long bridge, β would probably be large in practice. When β is so large that the coefficient $\frac{\beta^2\pi}{\rho(\epsilon^{\rho\pi} + \epsilon^{-\rho\pi})}$, or $\pi\beta^{\frac{3}{2}}\epsilon^{-\pi\beta^{\frac{1}{2}}}$ nearly, in $\phi(x)$ may be neglected, the motion of the body is sensibly symmetrical with respect to the centre of the bridge, and consequently T, as well as y, is a maximum when $x = \frac{1}{2}$. For this value of x we have $4(x - x^2) = 1$, and therefore $z = T = y$. Putting C_i for the $(i+1)^{\text{th}}$ term of the series (9), so that $C_i = A_i 2^{-i}$, we have for $x = \frac{1}{2}$

$$T = C_0 + C_1 + C_2 + \ldots\ldots \quad (39)$$

where

$$C_0 = \frac{\beta}{2+\beta}, \quad C_1 = \frac{6C_0}{6+\beta},$$

and generally,

$$C_i = \frac{(i+1)(i+2)}{(i+1)(i+2)+\beta}\left\{C_{i-1} - \tfrac{1}{4}C_{i-2}\right\},$$

whence T is easily calculated. Thus for $\beta = 5$ we have $\pi\beta^{\frac{3}{2}}\epsilon^{-\pi\beta^{\frac{1}{2}}} = .031$ nearly, which is not large, and we get from the series (39) $T = 1.27$ nearly. For $\beta = 10$, the approximate value of the coefficient in $\phi(x)$ is .0048, which is very small, and we get $T = 1.14$. In these calculations the inertia of the bridge has been neglected, but the effect of the inertia would probably be rather to diminish than to increase the greatest value of T.

22. The inertia of a bridge such as one of those actually in use must be considerable: the bridge and a carriage moving over it form a dynamical system in which the inertia of all the parts ought to be taken into account. Let it be required *to construct the same dynamical system on a different scale.* For this purpose it will be necessary to attend to the dimensions of the different constants on which the unknown quantities of the problem depend, with respect to each of the independent units involved in the problem. Now if the thickness of the bridge be regarded as very small compared with its length, and the moving body be regarded as a heavy particle, the only constants which enter into the problem are M, the mass of the body, M', the mass of the bridge, $2c$, the length of the bridge, S, the central statical deflection, V, the horizontal velocity of the body, and g, the force of gravity. The independent units employed in dynamics are three, the unit of length, the unit of time, and the unit of density, or, which is equivalent, and which will be somewhat more convenient in the present case, the unit of length, the unit of time, and the unit of mass. The dimensions of the several constants M, M', &c., with respect to each of these units are given in the following table.

	Unit of length.	Unit of time.	Unit of mass.
M and M'.	0	0	1
c and S.	1	0	0
V.	1	− 1	0
g.	1	− 2	0

Now any result whatsoever concerning the problem will consist of a relation between certain unknown quantities x', x''... and the six constants just written, a relation which may be expressed by

$$f(x', x'', \ldots M, M', c, S, V, g) = 0. \quad \ldots\ldots\ldots\ldots \quad (40)$$

But by the principle of homogeneity and by the preceding table this equation must be of the form

$$F\left\{\frac{x'}{(x')}, \frac{x''}{(x'')} \ldots, \frac{M'}{M}, \frac{S}{c}, \frac{V^2}{cg}\right\} = 0, \quad \ldots\ldots\ldots\ldots\ldots \quad (41)$$

where (x'), (x'') ..., denote any quantities made up of the six constants in such a manner as to have with respect to each of the independent units the same dimensions as x', x''..., respectively. Thus, if (40) be the equation which gives the maximum value $T_{,}$ of T in terms of the six constants, we shall have but one unknown quantity x', where $x' = T_{,}$, and we may take for (x'), Mcg, or else $M'V^2$. If (40) be the equation to the trajectory of the body, we shall have two unknown constants, x', x'', where x' is the same as in Art. 1, and $x'' = y'$, and we may take $(x') = c$, $(x'') = c$. The equation (41) shows that in order to keep to the same dynamical system, only on a different scale, we must alter the quantities M, M', &c. in such a manner that

$$M' \propto M, \; S \propto c, \; V^2 \propto cg,$$

and consequently, since g is not a quantity which we can alter at pleasure in our experiments, V must vary as $\sqrt{c}$. A small system constructed with attention to the above variations forms an *exact dynamical model* of a larger system with respect to which it may be desired to obtain certain results. It is not even necessary for the truth of this statement that the thickness of the large bridge be small in comparison with its length, provided that the same proportionate thickness be preserved in the model.

To take a numerical example, suppose that we wished, by means of a model bridge five feet long and weighing 100 ounces, to investigate the greatest central deflection produced by an engine weighing 20 tons, which passes with the successive velocities of 30, 40, and 50 miles an hour over a bridge 50 feet long weighing 100 tons, the central statical deflection produced by the engine being one inch. We must give to our model carriage a weight of 20 ounces, and make the small bridge of such a stiffness that a weight of 20 ounces placed on the centre shall cause a deflection of $\frac{1}{10}$th of an inch; and then we must give to the carriage the successive velocities of $3\sqrt{10}$, $4\sqrt{10}$, $5\sqrt{10}$, or 9.49, 12.65, 15.81 miles per hour, or 13.91, 18.55, 23.19 feet per second. If we suppose the observed central deflections in the model to be .12, .16, .18 of an inch, we may conclude that the central deflections in the large bridge corresponding to the velocities of 30, 40, and 50 miles per hour would be 1.2, 1.6, and 1.8 inch.

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Addition to the preceding Paper.

SINCE the above was written, Professor WILLIS has informed me that the values of β are much larger in practice than those which are contained in Table I, on which account it would be interesting to calculate the numerical values of the functions for a few larger values of β. I have accordingly performed the calculations for the values 3, 5, 8, 12, and 20. The results are contained in Table II. In calculating z from $x = 0$ to $x = .5$, I employed the formula (12), with the assistance occasionally of (15). I worked with 4 places of decimals, of which 3 only are retained. The values of z for $x = .5$, in which case the series are least convergent, have been verified by the formula (42) given below: the results agreed within two or three units in the fourth place of decimals. The remaining values of z were calculated from the expression for $(x - x^2)^{-2}\phi(x)$. The values of T and $\frac{y'}{S}$ were deduced from those of z by merely multiplying twice in succession by $4x(1 - x)$. Professor WILLIS has laid down in curves the numbers contained in the last five columns. In laying down these curves several errors were detected in the latter half of the Table, that is, from $x = .55$ to x .95. These errors were corrected by re-examining the calculation; so that I feel pretty confident that the table as it now stands contains no errors of importance.

TABLE II.

x	z					T					$\frac{y'}{S}$				
	$\beta=3$	$\beta=5$	$\beta=8$	$\beta=12$	$\beta=20$	$\beta=3$	$\beta=5$	$\beta=8$	$\beta=12$	$\beta=20$	β 3	$\beta=5$	$\beta=8$	$\beta=12$	$\beta=20$
.00	.600	.714	.800	.857	.909	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
.05	.640	.755	.835	.886	.931	.122	.143	.159	.168	.177	.023	.027	.030	.032	.034
.10	.689	.798	.872	.915	.950	.248	.287	.314	.330	.342	.089	.103	.113	.119	.123
.15	.751	.846	.910	.945	.970	.383	.431	.464	.482	.495	.195	.220	.237	.246	.252
.20	.799	.897	.950	.975	.989	.511	.574	.608	.624	.633	.327	.367	.389	.399	.405
.25	.863	.951	.991	1.004	1.016	.647	.714	.743	.753	.762	.486	.535	.558	.565	.572
.30	.936	1.010	1.023	1.032	1.023	.786	.849	.859	.867	.859	.661	.713	.722	.728	.721
.35	1.018	1.073	1.074	1.059	1.038	.926	.976	.977	.963	.944	.843	.888	.389	.877	.859
.40	1.110	1.138	1.114	1.081	1.049	1.066	1.092	1.069	1 038	1.007	1.023	1.049	1.026	.997	.966
.45	1.214	1.207	1.150	1.099	1.056	1.202	1.195	1.138	1.089	1.046	1.190	1.183	1.127	1.078	1.035
.50	1.331	1.274	1.180	1.111	1.060	1.331	1.274	1.180	1.111	1.060	1.331	1.274	1.180	1.111	1.060
.55	1.461	1.341	1.203	1.114	1.058	1.446	1.327	1.191	1.103	1.047	1.431	1.314	1.179	1.092	1.037
.60	1.602	1.390	1.202	1.105	1.051	1.538	1.334	1.154	1.060	1.009	1.486	1.281	1.108	1.018	.968
.65	1.748	1.417	1.179	1.081	1.038	1.590	1.289	1.072	.983	.945	1.446	1.173	.954	.895	.860
.70	1.891	1.393	1.107	1.039	1.021	1.588	1.170	.930	.873	.858	1.334	.983	.781	.733	.720
.75	1.974	1.273	1.003	.984	1.013	1.481	.955	.752	.738	.760	1.111	.716	.564	.554	.570
.80	1.885	.968	.832	.932	.989	1.206	.620	.532	.596	.633	.772	.396	.341	.382	.405
.85	1.286	.344	.660	.925	.976	.656	.176	.336	.472	.498	.335	.090	.172	.241	.254
.90	−.970	−.616	.802	1.013	.947	−.349	−.222	.289	.365	.341	−.126	−.080	.104	.131	.123
.95	−8.227	+1.248	1.884	.720	.943	−1.563	+.237	.358	.137	.179	−.297	+.045	.068	.026	.034

The form of the trajectory will be sufficiently perceived by comparing this table with the curves represented in the figure. As β increases, the first point of intersection of the trajectory with the equilibrium trajectory *eee* moves towards *A*. Since $z = 1$ at this point, we get from the part of the table headed "z," for the abscissa of the point of intersection, by taking proportional parts, .34, .29, .26, .24, and .22, corresponding to the respective values 3, 5, 8, 12, and 20 of β. Beyond this point of intersection the trajectory passes below the equilibrium trajectory, and remains below it during the greater part of the remaining course. As β increases, the trajectory becomes more and more nearly symmetrical with respect to C: when $\beta = 20$ the deviation from symmetry may be considered insensible, except close to the extremities A, B, where however the depression itself is insensible. The greatest depression of the body, as appears from the column which gives y', takes place a little beyond the centre; the point of greatest depression approaches indefinitely to the centre as β increases. This greatest depression of the body must be carefully distinguished from the greatest depression of the bridge, which is decidedly larger, and occurs in a different place, and at a different time. The numbers in the columns headed "T" shew that T is a maximum for a value of x greater than that which renders y' a maximum, as in fact immediately follows from a consideration of the mode in which y' is derived from T. The first maximum value of T, which according to what has been already remarked is the only such value that we need attend to, is about 1.59 for $\beta = 3$, 1.33 for $\beta = 5$, 1.19 for $\beta = 8$, 1.11 for $\beta = 12$, and 1.06 for $\beta = 20$.

When β is equal to or greater than 8, the maximum value of T occurs so nearly when $x = .5$

that it will be sufficient to suppose $x = .5$. The value of z, T, or $\frac{y'}{S}$ for $x = .5$ may be readily calculated by the method explained in Art. 21. I have also obtained the following expression for this particular value;

$$z = 2\beta - 4\beta^2 \left\{ \frac{1}{1.2+\beta} - \frac{1}{2.3+\beta} + \frac{1}{3.4+\beta} - \dots \right\} \dots\dots\dots (42)$$

When β is small, or only moderately large, the series (42) appears more convenient for numerical calculation, at least with the assistance of a table of reciprocals, than the series (39), but when β is very large the latter is more convenient than the former. In using the series (42), it will be best to sum the series within brackets directly to a few terms, and then find the remainder from the formula

$$u_x - u_{x+1} + u_{x+2} - \dots = \tfrac{1}{2} u_x - \tfrac{1}{4} \Delta u_x + \tfrac{1}{8} \Delta^2 u_x - \dots$$

The formula (42) was obtained from equation (20) by a transformation of the definite integral. In the transformation of Art. 8, the limits of s will be 1 and ∞, and the definite integral on which the result depends will be

$$\int_1^\infty \frac{s^{m-1} - s^{-m}}{1+s} \, ds.$$

The formula (42) may be obtained by expanding the denominator, integrating, and expressing m in terms of β.

In practice the values of β are very large, and it will be convenient to expand according to inverse powers of β. This may be easily effected by successive substitutions. Putting for shortness $x - x^2 = X$, equation (4) becomes by a slight transformation

$$y = X^2 - \beta^{-1} X^2 \frac{d^2 y}{dx^2},$$

and we have for a first approximation $y = X^2$, for a second $y = X^2 - \beta^{-1} X^2 \frac{d^2 X^2}{dx^2}$, and so on. The result of the successive substitutions may be expressed as follows:

$$y = X^2 - \beta^{-1} X^2 \frac{d^2}{dx^2} X^2 + \beta^{-2} X^2 \frac{d}{dx^2} X^2 \frac{d^2}{dx^2} X^2 - \&c., \dots\dots\dots\dots (43)$$

where each term, taken positively, is derived from the preceding by differentiating twice, and then multiplying by $\beta^{-1} X^2$.

For such large values of β, we need attend to nothing but the value of z for $x = \frac{1}{2}$, and this may be obtained from (43) by putting $x = \frac{1}{2}$, after differentiation, and multiplying by 16. It will however be more convenient to replace x by $\frac{1}{2}(1 + w)$, which gives $\frac{d^2}{dx^2} = 4 \frac{d^2}{dw^2}$; $X^2 = \frac{1}{16} W$, where $W = (1 - w^2)^2$. We thus get from (43)

$$z = W - (4\beta)^{-1} W \frac{d^2}{dw^2} W + (4\beta)^{-2} W \frac{d^2}{dw^2} W \frac{d^2}{dw^2} W - \dots,$$

where we must put $w = 0$ after differentiation, if we wish to get the value of z for $x = \frac{1}{2}$. This equation gives, on performing the differentiations and multiplications, and then putting $w = 0$,

$$z = 1 + \beta^{-1} + \tfrac{5}{2} \beta^{-2} + 13\beta^{-3} + \dots\dots\dots\dots\dots\dots (44)$$

In practical cases this series may be reduced to $1 + \beta^{-1}$. The latter term is the same as would be got by taking into account the centrifugal force, and substituting, in the small term involving that force, the radius of curvature of the equilibrium trajectory for the radius of curvature of the actual trajectory. The problem has already been considered in this manner by others by whom it has been attacked.

My attention has recently been directed by Professor Willis to an article by Mr. Cox *On the Dynamical Deflection and strain of Railway Girders*, which is printed in *The Civil Engineer and Architect's Journal* for September, 1848. In this article the subject is treated in a very original and striking manner. There is, however, one conclusion at which Mr. Cox has arrived which is so directly opposed to the conclusions to which I have been led, that I feel compelled to notice it. By reasoning founded on the principle of *vis viva*, Mr. Cox has arrived at the result that the moving body cannot in any case produce a deflection greater than double the central statical deflection, the elasticity of the bridge being supposed perfect. But among the sources of labouring force which can be employed in deflecting the bridge, Mr. Cox has omitted to consider the *vis viva* arising fiom the horizontal motion of the body. It is possible to conceive beforehand that a portion of this *vis viva* should be converted into labouring force, which is expended in deflecting the bridge. And this is, in fact, precisely what takes place. During the first part of the motion, the horizontal component of the reaction of the bridge against the body impels the body forwards, and therefore increases the *vis viva* due to the horizontal motion; and the labouring force which produces this increase being derived from the bridge, the bridge is less deflected than it would have been had the horizontal velocity of the body been unchanged. But during the latter part of the motion the horizontal component of the reaction acts backwards, and a portion of the *vis viva* due to the horizontal motion of the body is continually converted into labouring force, which is stored up in the bridge. Now, on account of the asymmetry of the motion, the direction of the reaction is more inclined to the vertical when the body is moving over the second half of the bridge than when it is moving over the first half, and moreover the reaction itself is greater, and therefore, on both accounts, more *vis viva* depending upon the horizontal motion is destroyed in the latter portion of the body's course than is generated in the former portion; and therefore, on the whole, the bridge is more deflected than it would have been ha the horizontal velocity of the body remained unchanged.

It is true that the change of horizontal velocity is small; but nevertheless, in this mode of treating the subject, it must be taken into account. For, in applying to the problem the principle of *vis viva*, we are concerned with the square of the vertical velocity, and we must not omit any quantities which are comparable with that square. Now the square of the absolute velocity of the body is equal to the sum of the squares of the horizontal and vertical velocities; and the change in the square of the horizontal velocity depends upon the product of the horizontal velocity and the change of horizontal velocity; but this product is not small in comparison with the square of the vertical velocity.

In Art. 22 I have investigated the changes which we are allowed by the general principle of homogeneous quantities to make in the parts of a system consisting of an elastic bridge and a travelling weight, without affecting the results, or altering anything but the scale of the system. These changes are the most general that we are at liberty to make by virtue merely of that general principle, and without examining the particular equations which relate to the particular problem here considered. But when we set down these equations, we shall see that there are some further changes which we may make without affecting our results, or at least without ceasing to be able to infer the results which would be obtained on one system from those actually obtained on another.

In an apparatus recently constructed by Professor Willis, which will be described in detail in the report of the commission, to which the reader has already been referred, the travelling weight moves over a single central trial bar, and is attached to a horizontal arm which is moveable, with as little friction as possible, about a fulcrum carried by the carriage. In this form of the experiment, the carriage serves merely to direct the weight, and moves on rails quite independent of the trial bar.

For the sake of greater generality I shall suppose the travelling weight, instead of being free, to be attached in this manner to a carriage.

Let M be the mass of the weight, including the arm, k the radius of gyration of the whole about the fulcrum, h the horizontal distance of the centre of gravity from the fulcrum, l the horizontal distance of the point of contact of the weight with the bridge, x, y the coordinates of that point at the time t, ξ, η those of any element of the bridge, R the reaction of the bridge against the weight, M' the mass of the bridge, R', R'' the vertical pressures of the bridge at its two extremities, diminished by the statical pressures due to the weight of the bridge alone. Suppose, as before, the deflection to be very small, and neglect its square.

By D'Alembert's principle the effective moving forces reversed will be in statical equilibrium with the impressed forces. Since the weight of the bridge is in equilibrium with the statical pressures at the extremities, these forces may be left out in the equations of equilibrium, and the only impressed forces we shall have to consider will be the weight of the travelling body and the reactions due to the motion. The mass of any element of the bridge will be $\frac{M'}{2c}d\xi$ very nearly; the horizontal effective force of this element will be insensible, and the vertical effective force will be $\frac{M'}{2c}\frac{d^2\eta}{dt^2}d\xi$, and this force, being reversed, must be supposed to act vertically upwards.

The curvature of the bridge being proportional to the moment of the bending forces, let the reciprocal of the radius of curvature be equal to K multiplied by that moment. Let A, B be the extremities of the bridge, P the point of contact of the bridge with the moving weight, Q any point of the bridge between A and P. Then by considering the portion AQ of the bridge we get, taking moments round Q,

$$-\frac{d^2\eta}{d\xi^2} = K\left\{R'\xi + \frac{M'}{2c}\int_0^\xi \frac{d^2\eta'}{dt^2}(\xi - \xi')\,d\xi'\right\}, \ldots\ldots\ldots\ldots (45)$$

η' being the same function of ξ' that η is of ξ. To determine K, let S be the central statical deflection produced by the weight Mg resting partly on the bridge and partly on the fulcrum, which is equivalent to a weight $\frac{h}{l}Mg$ resting on the centre of the bridge. In this case we should have

$$-\frac{d^2\eta}{d\xi^2} = K\frac{Mgh}{2l}\xi.$$

Integrating this equation twice, and observing that $\frac{d\eta}{d\xi} = 0$ when $\xi = c$, and $\eta = 0$ when $\xi = 0$, and that S is the value of η when $\xi = c$, we get

$$K = \frac{6lS}{Mghc^3}. \ldots\ldots\ldots\ldots\ldots (46)$$

Returning now to the bridge in its actual state, we get to determine R', by taking moments about B,

$$R'.2c - R(2c - x) + \frac{M'}{2c}\int_0^{2c}\frac{d^2\eta'}{dt^2}(2c - \xi')\,d\xi' = 0. \ldots\ldots\ldots\ldots\ldots (47)$$

Eliminating R' between (45) and (47), putting for K its value given by (46), and eliminating t by the equation $\frac{dx}{dt} = V$, we get

$$-\frac{d^2\eta}{d\xi^2} = \frac{3lS}{Mghc^4}\left\{(2c - x)\xi R - \frac{M'V^2}{2c}\left[(2c - \xi)\int_0^\xi \frac{d^2\eta'}{dx^2}\xi' d\xi' + \xi\int_\xi^{2c}\frac{d^2\eta'}{dx^2}(2c - \xi')\,d\xi'\right]\right\}. \ldots (48)$$

This equation applies to any point of the bridge between A and P. To get the equation which applies to any point between P and B, we should merely have to write $2c - \xi$ for ξ, $2c - x$ for x.

If we suppose the fulcrum to be very nearly in the same horizontal plane with the point of contact, the angle through which the travelling weight turns will be $\frac{y}{l}$ very nearly; and we shall have, to determine the motion of this weight,

$$Mk^2 V^2 \frac{d^2 y}{dx^2} = Mghl - Rl^2. \quad \text{......} \quad (49)$$

We have also the equations of condition,

$$\left.\begin{array}{l} \eta = 0 \text{ when } x = 0, \text{ for any value of } \xi \text{ from } 0 \text{ to } 2c; \\ \eta = y \text{ when } \xi = x, \text{ for any value of } x \text{ from } 0 \text{ to } 2c; \\ \eta = 0 \text{ when } \xi = 0 \text{ or } = 2c;\ y = 0 \text{ and } \dfrac{dy}{dx} = 0 \text{ when } x = 0. \end{array}\right\} \quad \text{......} \quad (50)$$

Now the general equations (48), (or the equation answering to it which applies to the portion PB of the bridge,) and (49), combined with the equations of condition (50), whether we can manage them or not, are sufficient for the complete determination of the motion, it being understood that η and $\frac{d\eta}{d\xi}$ vary continuously in passing from AP to PB, so that there is no occasion formally to set down the equations of condition which express this circumstance. Now the form of the equations shews that, being once satisfied, they will continue to be satisfied provided $\eta \propto y$, $\xi \propto x \propto c$, and

$$\frac{y}{c^3} \propto \frac{lSR}{Mghc^2} \propto \frac{lSM'V^2y}{Mghc^4},\ Mk^2 V^2 \frac{y}{c^2} \propto Mghl \propto Rl^2.$$

These variations give, on eliminating the variation of R,

$$y \propto S,\ \frac{gc^2}{V^2S} \propto \frac{k^2}{hl},\ \frac{M}{M'} \propto \frac{l^2}{k^2}. \quad \text{......} \quad (51)$$

Although g is of course practically constant, it has been retained in the variations because it may be conceived to vary, and it is by no means essential to the success of the method that it should be constant. The variations (51) shew that if we have any two systems in which the ratio of Mk^2 to $M'l^2$ is the same, and we conceive the travelling weights to move over the two bridges respectively, with velocities ranging from 0 to ∞, the trajectories described in the one case, and the deflections of the bridge, correspond exactly to the trajectories and deflections in the other case, so that to pass from the one to the other, it will be sufficient to alter all horizontal lines on the same scale as the length of the bridge, and all vertical lines on the same scale as the central statical deflection. The velocity in the one system which corresponds to a given velocity in the other is determined by the second of the variations (51).

We may pass at once to the case of a free weight by putting $h = k = l$, which gives

$$y \propto S,\ V^2 S \propto gc^2,\ M \propto M'. \quad \text{......} \quad (52)$$

The second of these variations shews that corresponding velocities in the two systems are those which give the same value to the constant β. When $S \propto c$ we get $V^2 \propto gc$, which agrees with Art. 22.

In consequence of some recent experiments of Professor Willis's, from which it appeared that the deflection produced by a given weight travelling over the trial bar with a given velocity was in some cases increased by connecting a balanced lever with the centre of the bar, so as to increase its inertia without increasing its weight, while in other cases the deflection was diminished, I have been induced to attempt an approximate solution of the problem, taking into account the inertia of the bridge. I find that when we replace each force acting on the bridge by a uniformly distributed force of such an amount as to produce the same mean deflection as would be produced by the actual force taken alone, which evidently cannot occasion any very material error, and when we moreover neglect the difference between the pressure exerted by the travelling mass on the bridge and its weight, the equation admits of integration in finite terms.

Let the notation be the same as in the investigation which immediately precedes; only, for simplicity's sake, take the length of the bridge for unity, and suppose the travelling weight a heavy particle. It will be easy in the end to restore the general unit of length if it should be desirable. It will be requisite in the first place to investigate the relation between a force acting at a given point of the bridge and the uniformly distributed force which would produce the same mean deflection.

Let a force F act vertically downwards at a point of the bridge whose abscissa is x, and let y be the deflection produced at that point. Then, ξ, η being the coordinates of any point of the bridge, we get from (38)

$$\int_0^x \eta \, d\xi = \frac{y}{2}\left\{x + \frac{x^2}{4(1-x)}\right\}.$$

To obtain $\int_x^1 \eta \, d\xi$, we have only got to write $1-x$ in place of x. Adding together the results, and observing that, according to a formula referred to in Art. 1, $y = 16\,S \, . \, \dfrac{F}{Mg} \, . \, x^2 (1-x)^2$, we obtain

$$\int_0^1 \eta \, d\xi = \frac{2\,SF}{Mg}\left\{x(1-x) + x^3(1-x)^2\right\}; \quad\ldots\ldots\ldots\ldots(53)$$

and this integral expresses the mean deflection produced by the force F, since the length of the bridge is unity.

Now suppose the bridge subject to the action of a uniformly distributed force F'. In this case we should have

$$-\frac{d^2\eta}{d\xi^2} = K\left\{\tfrac{1}{2}F'\xi - \int_0^\xi (\xi - \xi')\, F' d\xi'\right\} = \tfrac{1}{2} K F' (\xi - \xi^2).$$

Integrating this equation twice, and observing that $\dfrac{d\eta}{d\xi} = 0$ when $\xi = \tfrac{1}{2}$, and $\eta = 0$ when $\xi = 0$, and that (46) gives, on putting $l = h$ and $c = \tfrac{1}{2}$, $K = \dfrac{48\,S}{Mg}$, we obtain

$$\eta = \frac{2\,SF'}{Mg}(\xi - 2\xi^3 + \xi^4) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(54)$$

This equation gives for the mean deflection

$$\int_0^1 \eta \, d\xi = \frac{2\,SF'}{5\,Mg}; \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(55)$$

and equating the mean deflections produced by the force F acting at the point whose abscissa is x, and by the uniformly distributed force F', we get $F' = uF$, where

$$u = 5x(1-x) + 5x^2(1-x)^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(56)$$

Putting μ for the mean deflection, expressing F' in terms of μ, and slightly modifying the form of the quantity within parentheses in (54), we get for the equation to the bridge when at rest under the action of any uniformly distributed force

$$\eta = 5\mu\{\xi(1-\xi) + \xi^2(1-\xi)^2\} \ldots\ldots\ldots\ldots\ldots\ldots(57)$$

If D be the central deflection, $\eta = D$ when $\xi = \frac{1}{2}$; so that $D : \mu :: 25 : 16$.

Now suppose the bridge in motion, with the mass M travelling over it, and let x, y be the coordinates of M. As before, the bridge would be in equilibrium under the action of the force $M\left(g - \frac{d^2 y}{dt^2}\right)$ acting vertically downwards at the point whose abscissa is x, and the system of forces such as $M' d\xi \cdot \frac{d^2\eta}{dt^2}$ acting vertically upwards at the several elements of the bridge. According to the hypothesis adopted, the former force may be replaced by a uniformly distributed force the value of which will be obtained by multiplying by u, and each force of the latter system may be replaced by a uniformly distributed force obtained by multiplying by u', where u' is what u becomes when ξ is put for x. Hence if F_1 be the whole uniformly distributed force we have

$$F_1 = M\left(g - \frac{d^2 y}{dt^2}\right) u - M' \int_0^1 \frac{d^2\eta}{dt^2} u' \, d\xi \ldots\ldots\ldots\ldots(58)$$

Now according to our hypothesis the bridge must always have the form which it would assume under the action of a uniformly distributed force; and therefore, if μ be the mean deflection at the time t, (57) will be the equation to the bridge at that instant. Moreover, since the point (x, y) is a point in the bridge, we must have $\eta = y$ when $\xi = x$, whence $y = \mu u$. We have also

$$\eta = \mu u', \quad \frac{d^2\eta}{dt^2} = \frac{d^2\mu}{dt^2} u', \quad \int_0^1 \frac{d^2\eta}{dt^2} u' \, d\xi = \frac{d^2\mu}{dt^2} \int_0^1 u'^2 \, d\xi = \frac{155}{126} \frac{d^2\mu}{dt^2}.$$

We get from (55), $F_1 = \frac{5Mg}{2S}\mu$. Making these various substitutions in (58), and replacing $\frac{d}{dt}$ by $V\frac{d}{dx}$, we get for the differential equation of motion

$$\frac{5Mg}{2S}\mu = Mgu - MV^2 u \frac{d^2 \mu u}{dx^2} - \frac{155}{126} M' V^2 \frac{d^2\mu}{dx^2} \ldots\ldots\ldots\ldots(59)$$

Since μ is comparable with S, the several terms of this equation are comparable with

$$Mg, \quad Mg, \quad MV^2 S, \quad M'V^2 S,$$

respectively. If then $V^2 S$ be small compared with g, and likewise M small compared with M', we may neglect the third term, while we retain the others. This term, it is to be observed, expresses the difference between the pressure on the bridge and the weight of the travelling mass. Since $c = \frac{1}{2}$, we have $\frac{V^2 S}{g} = \frac{1}{16\beta}$, which will be small when β is large, or even moderately large. Hence the conditions under which we are at liberty to neglect the difference between the pressure on the bridge and the weight of the travelling mass are, *first*, that β be large, *secondly*, that the mass of the travelling body be small compared with the mass of the bridge. If β be large, but M be comparable with M', it is true that the third term in (59) will be small compared with the leading terms; but then it will be comparable with the fourth, and the approximation adopted in neglecting the third term alone would be faulty, in this way, that of two small terms comparable with each other, one would be retained while the other was neglected. Hence, although the absolute error of our results would be but small, it would be comparable with the difference between the results actually obtained and those which would be obtained on the supposition that the travelling mass moved with an infinitely small velocity.

Neglecting the third term in equation (59), and putting for u its value, we get

$$\frac{d^2\mu}{dx^2} + q^2\mu = 2q^2 S(x - 2x^3 + x^4), \quad \text{.....................(60)}$$

where

$$q^2 = \frac{63\,Mg}{31\,M'V^2S} = \frac{1008\,M\beta}{31\,M'} \quad \text{.....................(61)}$$

The linear equation (60) is easily integrated. Integrating, and determining the arbitrary constants by the conditions that $\mu = 0$, and $\frac{d\mu}{dx} = 0$, when $x = 0$, we get

$$\mu = 2S\left\{x^4 - 2x^3 - \frac{12x^2}{q^2} + \left(1 + \frac{12}{q^2}\right)\left(x - \frac{\sin qx}{q}\right) + \frac{24}{q^4}(1 - \cos qx)\right\}; \quad \text{... (62)}$$

and we have for the equation to the trajectory

$$y = 5\mu(x - 2x^3 + x^4) = 5\mu(X + X^2), \quad \text{...........(63)}$$

where as before $X = x(1 - x)$.

When $V = 0$, $q = \infty$, and we get from (62), (63), for the approximate equation to the equilibrium trajectory,

$$y = 10S(X + X^2)^2; \quad \text{.......................(64)}$$

whereas the true equation is

$$y = 16SX^2. \quad \text{..............................(65)}$$

Since the forms of these equations are very different, it will be proper to verify the assertion that (64) is in fact an approximation to (65). Since the curves represented by these equations are both symmetrical with respect to the centre of the bridge, it will be sufficient to consider values of x from 0 to $\frac{1}{2}$, to which correspond values of X ranging from 0 to $\frac{1}{4}$. Denoting the error of the formula (64), that is the excess of the y in (64) over the y in (65), by $S\delta$, we have

$$\delta = -6X^2 + 20X^3 + 10X^4,$$

$$\frac{d\delta}{dx} = 4(-3 + 15X + 10X^2)X\frac{dX}{dx}.$$

Equating $\frac{d\delta}{dx}$ to zero, we get $X = 0$, $x = 0$, $\delta = 0$, a maximum; $X = .1787$, $x = .233$, $\delta = -.067$, nearly, a minimum; and $x = \frac{1}{2}$, $\delta = -.023$, nearly, a maximum. Hence the greatest error in the approximate value of the ordinate of the equilibrium trajectory is equal to about the one-fifteenth of S.

Putting $\mu = \mu_0 + \mu_1$, $y = y_0 + y_1$, where μ_0, y_0 are the values of μ, y for $q = \infty$, we have

$$\mu_1 = 2S\left\{\frac{12}{q^2}x(1 - x) - \left(\frac{1}{q} + \frac{12}{q^3}\right)\sin qx + \frac{24}{q^4}(1 - \cos qx)\right\} \quad \text{...... (66)}$$

$$y_1 = 5x(1 - x)\{1 + x(1 - x)\}\mu_1 \quad \text{.................................(67)}$$

The values of μ_1 and y_1 may be calculated from these formulæ for different values of q, and they are then to be added to the values of μ_0, y_0, respectively, which have to be calculated once for all. If instead of the mean deflection μ we wish to employ the central deflection D, we have only got to multiply the second sides of equations (62), (66) by $\frac{25}{16}$, and those of (63), (67) by $\frac{16}{25}$, and to write D for μ. The following table contains the values of the ratios of D and y to S for ten different values of q, as well as for the limiting value $q = \infty$, which belongs to the equilibrium trajectory.

TABLE III.

x	Values of $\frac{D}{S}$ when $\frac{2q}{\pi}$ is equal to										
	1	2	3	4	5	6	8	10	12	16	∞
.00	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
.05	.004	.004	.005	.006	.007	.008	.014	.019	.025	.041	.156
.10	.009	.013	.022	.027	.037	.053	.081	.117	.158	.239	.307
.15	.017	.028	.048	.075	.108	.146	.234	.327	.412	.530	.449
.20	.025	.052	.099	.159	.231	.309	.469	.607	.696	.707	.580
.25	.041	.093	.177	.285	.406	.531	.746	.871	.884	.707	.696
.30	.056	.144	.282	.451	.626	.787	1.003	1.031	.915	.689	.794
.35	.070	.214	.418	.650	.871	1.045	1.180	1.052	.845	.814	.873
.40	.100	.300	.578	.870	1.115	1.265	1.238	.967	.796	1.017	.930
.45	.134	.399	.757	1.097	1.332	1.412	1.178	.859	.856	1.097	.965
.50	.169	.516	.947	1.310	1.492	1.460	1.036	.812	1.004	.991	.977
.55	.213	.640	1.139	1.491	1.574	1.403	.870	.860	1.127	.862	.965
.60	.256	.776	1.321	1.619	1.562	1.250	.739	.969	1.115	.872	.930
.65	.306	.913	1.482	1.681	1.454	1.027	.682	1.054	.918	.959	.873
.70	.359	1.050	1.609	1.663	1.257	.769	.695	1.031	.718	.924	.794
.75	.419	1.181	1.691	1.560	.990	.517	.746	.869	.549	.707	.696
.80	.475	1.296	1.717	1.371	.677	.303	.777	.604	.499	.472	.580
.85	.533	1.399	1.681	1.106	.350	.149	.733	.325	.516	.384	.449
.90	.586	1.476	1.588	.776	.037	.064	.579	.117	.477	.385	.307
.95	.646	1.525	1.402	.400	−.234	.025	.321	.021	.296	.276	.156
1.00	.699	1.540	1.158	.000	−.446	.019	.000	.001	−.001	.000	.000

x	Values of $\frac{y}{S}$ when $\frac{2q}{\pi}$ is equal to										
	1	2	3	4	5	6	8	10	12	16	∞
.00	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
.05	.001	.001	.001	.001	.001	.001	.002	.003	.004	.006	.025
.10	.003	.004	.007	.008	.012	.017	.025	.037	.050	.075	.096
.15	.008	013	.022	.034	.050	.067	.108	.150	.190	.244	.207
.20	.015	.031	.059	.095	.137	.184	.279	.360	.414	.420	.344
.25	.029	.056	.126	.203	.290	.378	.532	.621	.630	.504	.496
.30	.045	.117	.230	.366	.509	.610	.814	.839	.744	.560	.646
.35	.063	.191	.374	.581	.778	.934	1.054	.940	.755	.727	.780
.40	.096	.285	.550	.828	1.062	1.205	1.178	.921	.759	.969	.886
.45	.133	.394	.748	1.085	1.316	1.395	1.164	.849	.846	1.084	.954
.50	.169	.516	.947	1.310	1.492	1.460	1.036	.812	1.004	.991	.977
.55	.210	.632	1.126	1.473	1.555	1.387	.860	.850	1.114	.852	.954
.60	.244	.739	1.258	1.542	1.487	1.191	.704	.923	1.062	.830	.886
.65	.274	.816	1.325	1.502	1.300	.917	.609	.942	.848	.857	.780
.70	.292	.854	1.308	1.352	1.022	.626	.565	.839	.584	.752	.646
.75	.298	.842	1.205	1.111	.705	.369	.532	.619	.391	.488	.496
.80	.282	.770	1.020	.814	.402	.180	.462	.359	.297	.280	.344
.85	.245	.644	.774	.509	.161	.069	.337	.149	.237	.178	.207
.90	.184	.463	.498	.244	.012	.020	.182	.037	.150	.121	.096
.95	.103	.243	.224	.064	−.037	.004	.051	.003	.047	.044	.025
1.00	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000

The numerical results contained in Table III. are represented graphically in figs. 2 and 3 of the accompanying plate, where however some of the curves are left out, in order to prevent confusion in the figures. In these figures the numbers written against the several curves are the values of $\frac{2q}{\pi}$ to which the curves respectively belong, the symbol ∞ being written against the equilibrium curves. Fig. 2 represents the trajectory of the body for different values of q, and will be understood without further explanation. In the curves of fig. 3, the ordinate represents the deflection of the centre of the bridge when the moving body has travelled over a distance represented by the abscissa. Fig. 1, which represents the trajectories described when the mass of the bridge is neglected, is here given for the sake of comparison with fig. 2. The numbers in fig. 1, refer to the values of β. The equilibrium curve represented in this figure is the true equilibrium trajectory expressed by equation (65), whereas the equilibrium curve represented in fig. 2 is the approximate equilibrium trajectory expressed by equation (64). In fig. 1, the body is represented as flying off near the second extremity of the bridge, which is in fact the case. The numerous small oscillations which would take place if the body were held down to the bridge could not be properly represented in the figure without using a much larger scale. The reader is however requested to bear in mind the existence of these oscillations, as indicated by the analysis, because, if the ratio of M to M' altered continuously from ∞ to 0, they would probably pass continuously into the oscillations which are so conspicuous in the case of the larger values of q in fig. 2. Thus the consideration of these insignificant oscillations which, strictly speaking, belong to fig. 1, aids us in mentally filling up the gap which corresponds to the cases in which the ratio of M to M' is neither very small nor very large.

As everything depends on the value of q, in the approximate investigation in which the inertia of the bridge is taken into account, it will be proper to consider further the meaning of this constant. In the first place it is to be observed that although M appears in equation (61), q is really independent of the mass of the travelling body. For, when M alone varies, β varies inversely as S, and S varies directly as M, so that q remains constant. To get rid of the apparent dependance of q on M, let S_1 be the central statical deflection produced by a mass equal to that of the bridge, and at the same time restore the general unit of length. If x continue to denote the ratio of the abscissa of the body to the length of the bridge, q will be numerical, and therefore, to restore the general unit of length, it will be sufficient to take the general expression (5) for β. Let moreover τ be the time the body takes to travel over the bridge, so that $2c = V\tau$; then we get

$$q^2 = \frac{63 g \tau^2}{31 S_1} \quad \text{.....................} (68)$$

If we suppose τ expressed in seconds, and S_1 in inches, we must put $g = 32.2 \times 12 = 386$, nearly, and we get,

$$q = \frac{28\,\tau}{\sqrt{S_1}} \quad \text{.....................} (69)$$

Conceive the mass M removed; suppose the bridge depressed through a small space, and then left to itself. The equation of motion will be got from (59) by putting $M = 0$, where M is not divided by S, and replacing $\frac{M}{S}$, by $\frac{M'}{S_1}$, and $V\frac{d}{dx}$ by $\frac{d}{dt}$. We thus get

$$\frac{d^2\mu}{dt^2} + \frac{63 g}{31 S_1}\mu = 0;$$

and therefore, if P be the period of the motion, or twice the time of oscillation from rest to rest,

$$P = 2\pi\sqrt{\frac{31 S_1}{63 g}}; \quad q = 2\pi\frac{\tau}{P} \quad \text{.....................} (70)$$

Hence the numbers 1, 2, 3, &c., written at the head of Table III. and against the curves of figs. 2 and 3, represent the number of quarter periods of oscillation of the bridge which elapse during the passage of the body over it. This consideration will materially assist us in understanding the nature of the motion. It should be remarked too that q is increased by diminishing either the velocity of the body or the inertia of the bridge.

In the trajectory 1, fig. 2, the ordinates are small because the body passed over before there was time to produce much deflection in the bridge, at least except towards the end of the body's course, where even a large deflection of the bridge would produce only a small deflection of the body. The corresponding deflection curve, (curve 1, fig. 3,) shews that the bridge was depressed, and that its deflection was rapidly increasing, when the body left it. When the body is made to move with velocities successively one half and one third of the former velocity, more time is allowed for deflecting the bridge, and the trajectories marked 2, 3, are described, in which the ordinates are far larger than in that marked 1. The deflections too, as appears from fig. 3, are much larger than before, or at least much larger than any deflection which was produced in the first case while the body remained on the bridge. It appears from Table III, or from fig. 3, that the greatest deflection occurs in the case of the third curve, nearly, and that it exceeds the central statical deflection by about three-fourths of the whole. When the velocity is considerably diminished, the bridge has time to make several oscillations while the body is going over it. These oscillations may be easily observed in fig. 3, and their effect on the form of the trajectory, which may indeed be readily understood from fig. 3, will be seen on referring to fig. 2.

When q is large, as is the case in practice, it will be sufficient in equation (66) to retain only the term which is divided by the first power of q. With this simplification we get

$$\frac{D_1}{S} = \frac{25}{16}\frac{\mu_1}{S} = -\frac{25}{8q}\sin qx\,; \qquad (71)$$

so that the central deflection is liable to be alternately increased and decreased by the fraction $\frac{25}{8q}$ of the central statical deflection. By means of the expressions (61), (69), we get

$$\frac{25}{8q} = .55\sqrt{\frac{M'}{M\beta}} = .112\,\frac{\sqrt{S_1}}{\tau} \qquad (72)$$

It is to be remembered that in the latter of these expressions the units of space and time are an inch and a second respectively. Since the difference between the pressure on the bridge and weight of the body is neglected in the investigation in which the inertia of the bridge is considered, it is evident that the result will be sensibly the same whether the bridge in its natural position be straight, or be slightly raised towards the centre, or, as it is technically termed, *cambered*. The increase of deflection in the case first investigated would be diminished by a camber.

In this paper the problem has been worked out, or worked out approximately, only in the two extreme cases in which the mass of the travelling body is infinitely great and infinitely small respectively, compared with the mass of the bridge. The causes of the increase of deflection in these two extreme cases are quite distinct. In the former case, the increase of deflection depends entirely on the difference between the pressure on the bridge and the weight of the body, and may be regarded as depending on the centrifugal force. In the latter, the effect depends on the manner in which the force, regarded as a function of the time, is applied to the bridge. In practical cases the masses of the body and of the bridge are generally comparable with each other, and the two effects are mixed up in the actual result. Nevertheless, if we find that each effect, taken separately, is insensible, or so small as to be of no practical importance, we may conclude without much fear of error that the actual effect is insignificant. Now we have seen that if we take only the most important terms, the increase of deflection is measured by the fractions $\frac{1}{\beta}$ and $\frac{25}{8q}$ of S. It is only when these fractions are both small that we are at liberty to neglect all but the most important terms, but in practical cases they are actually small. The magnitude of these fractions will enable us to judge of the amount of the actual effect.

To take a numerical example lying within practical limits, let the span of a given bridge be 44 feet, and suppose a weight equal to $\frac{4}{3}$ of the weight of the bridge to cause a deflection of $\frac{1}{5}$ inch. These are nearly the circumstances of the Ewell bridge, mentioned in the report of the commissioners. In this case, $S_1 = \frac{3}{4} \times .2 = .15$; and if the velocity be 44 feet in a second, or 30 miles an hour, we have $\tau = 1$, and therefore from the second of the formulæ (72),

$$\frac{25}{8q} = .0434, \quad q = 72.1 = 45.9 \times \frac{\pi}{4}.$$

The travelling load being supposed to produce a deflection of .2 inch, we have $\beta = 127$, $\frac{1}{\beta} = .0079$. Hence in this case the deflection due to the inertia of the bridge is between 5 and 6 times as great as that obtained by considering the bridge as infinitely light, but in neither case is the deflection important. With a velocity of 60 miles an hour the increase of deflection $.0434S$ would be doubled.

In the case of one of the long tubes of the Britannia bridge β must be extremely large; but on account of the enormous mass of the tube it might be feared that the effect of the inertia of the tube itself would be of importance. To make a supposition every way disadvantageous, regard the tube as unconnected with the rest of the structure, and suppose the weight of the whole train collected at one point. The clear span of one of the great tubes is 460 feet, and the weight of the tube 1400 tons. When the platform on which the tube had been built was removed, the centre sank 10 inches, which was very nearly what had been calculated, so that the bottom became very nearly straight, since, in anticipation of the deflection which would be produced by the weight of the tube itself, it had been originally built curved upwards. Since a uniformly distributed weight produces the same deflection as $\frac{5}{8}$ths of the same weight placed at the centre, we have in this case $S_1 = \frac{8}{5} \times 10 = 16$; and supposing the train to be going at the rate of 30 miles an hour, we have $\tau = \frac{460}{44} = 10.5$, nearly. Hence in this case $\frac{25}{8q} = .043$, or $\frac{1}{23}$ nearly, so that the increase of deflection due to the inertia of the bridge is unimportant.

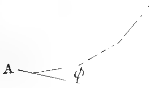
In conclusion, it will be proper to state that this "Addition" has been written on two or three different occasions, as the reader will probably have perceived. It was not until a few days after the reading of the paper itself that I perceived that the equation (16) was integrable in finite terms, and consequently that the variables were separable in (4). I was led to try whether this might not be the case in consequence of a remarkable numerical coincidence. This circumstance occasioned the complete remodelling of the paper after the first six articles. I had previously obtained for the calculation of z for values of x approaching 1, in which case the series (9) becomes inconvenient, series proceeding according to ascending powers of $1 - x$, and involving two arbitrary constants. The determination of these constants, which at first appeared to require the numerical calculation of five series, had been made to depend on that of three only, which were ultimately geometric series with a ratio equal to $\frac{1}{2}$.

The fact of the integrability of equation (4) in the form given in art. 7, to which I had myself been led from the circumstance above mentioned, has since been communicated to me by Mr. Cooper, Fellow of St John's College, through Mr. Adams, and by Professors Malmsten and A. F. Svanberg of Upsala through Professor Thomson; and I take this opportunity of thanking these mathematicians for the communication.

G. G. STOKES.

PEMBROKE COLLEGE,
Oct. 22, 1849.

Fig. 1.



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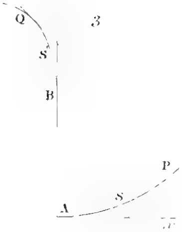

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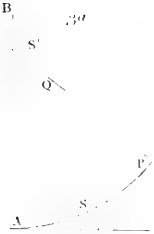

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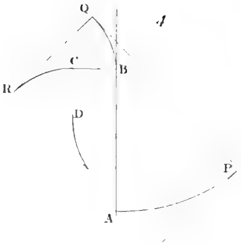

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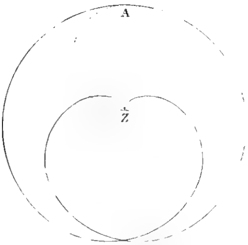
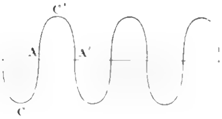


$(s = l \sin \frac{\varphi}{3})$

10

$(\varphi = \sin s)$

11



$\varphi = \frac{\pi}{2} \sin s$

15

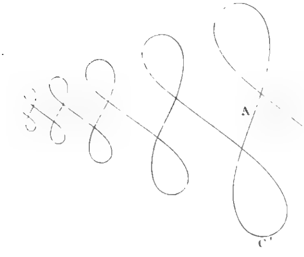
$\varphi = \frac{\pi}{2} \frac{m + \cos s}{(1 + m \cos s)^2} \quad (m = \frac{1}{2})$

16

$\varphi = \frac{\pi}{2} \frac{m + \cos s}{(1 + m \cos s)^2} \quad (m = \frac{5}{8})$

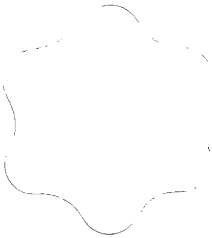
20

$\varphi = \frac{2\pi}{3} \frac{m + \cos (s^2 \cdot 1)}{(1 + m \cos (s^2 \cdot 1))^2}$



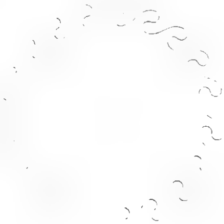
21

$\varphi = \sin \frac{s}{p} + \frac{s}{1}$



22

$\varphi = \frac{\pi}{2} \frac{m + \cos s}{(1 + m \cos s)^2} + \frac{s}{1}$



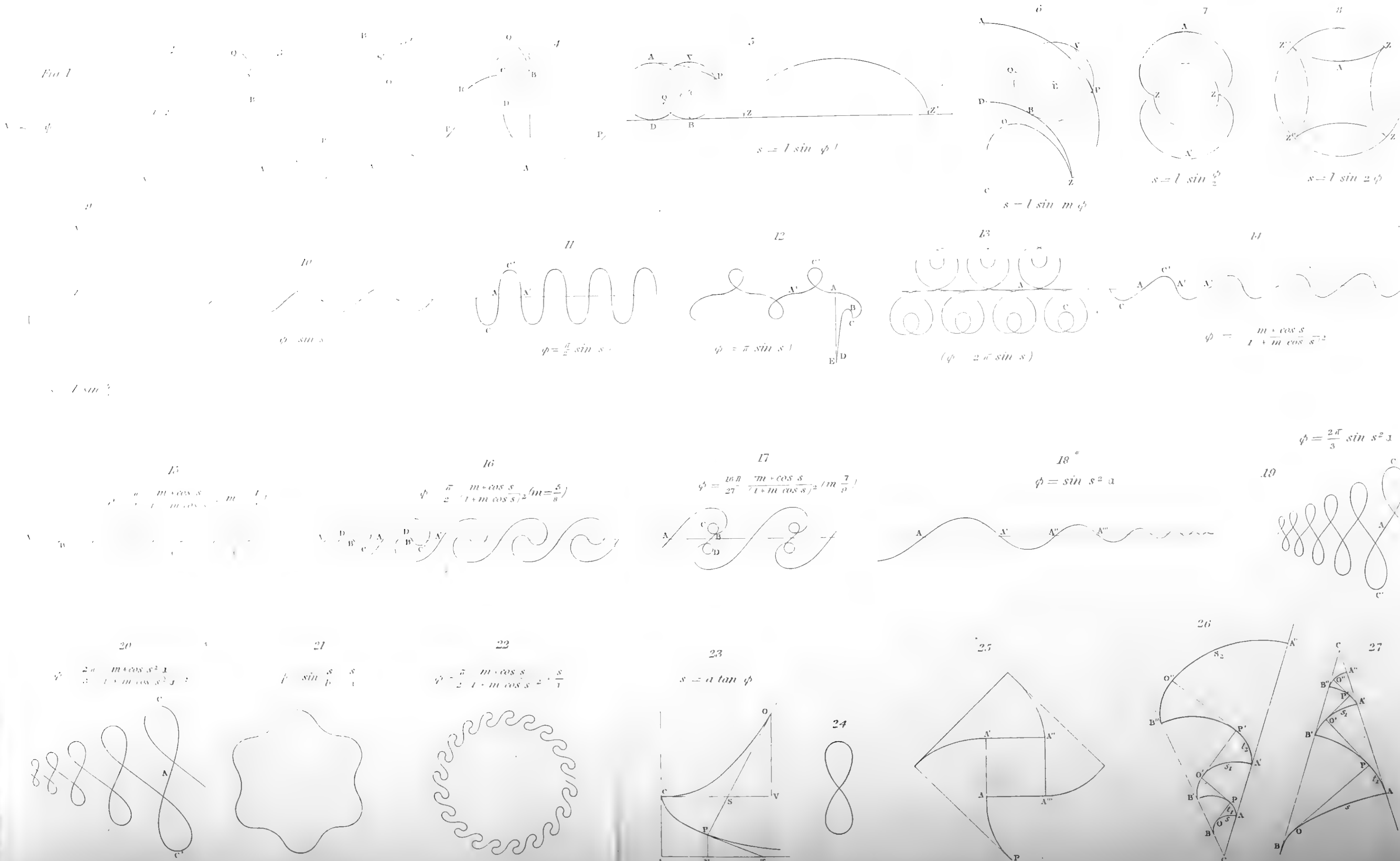


Fig. 1
5
s = l sin φ
6
s = l sin m φ
7
s = l sin φ/2
8
s = l sin 2 φ
10
11
φ = π/2 sin s
12
φ = π sin s
13
(φ = 2π sin s)
14
15
16
17
18
φ = sin s² a
φ = 2π/3 sin s² a
19
20
21
22
23
s = a tan φ
24
25
26
27

J.W. Lowry fc.

Fig. 1. Forms of the trajectory when $\frac{M}{M'}$ is very large.

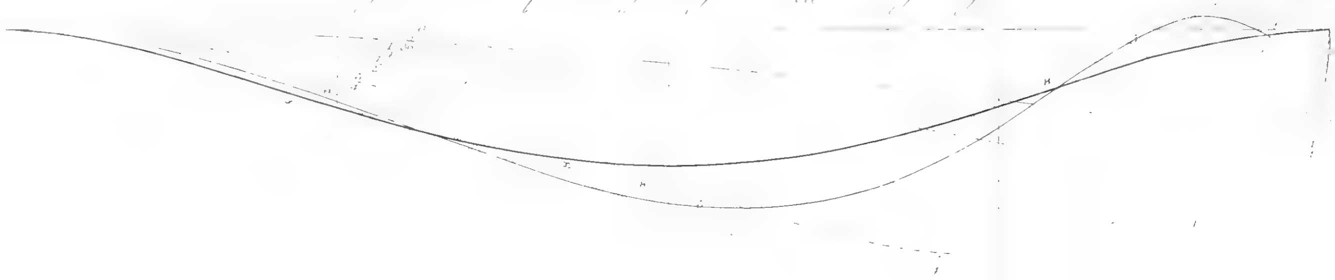
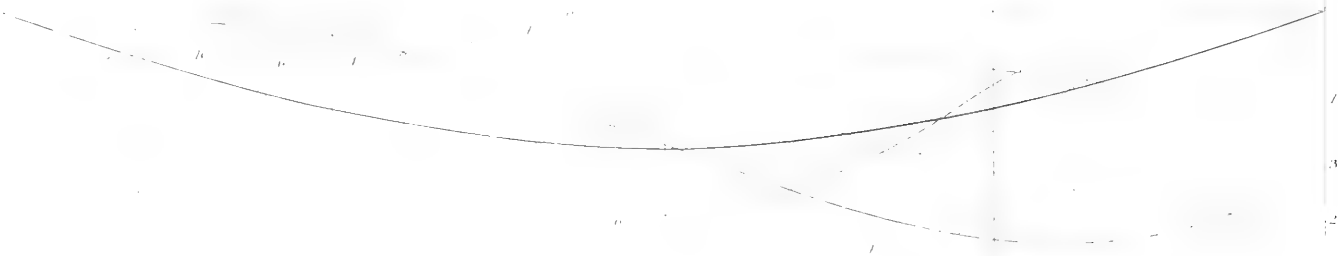


Fig. 2. Forms of the trajectory when $\frac{M}{M'}$ is very small.

Fig. 3. Corresponding curves of deflexion



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