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I. *Table of* $\Delta^m 0^n \div \Pi(m)$ *up to* $m = n = 20$. By A. CAYLEY, Sadlerian Professor of Pure Mathematics.

[Read October 27, 1879.]

THE differences of the powers of zero, $\Delta^m 0^n$, present themselves in the Calculus of Finite Differences, and especially in the applications of Herschel's theorem, $f(e^t) = f(1+\Delta)\, e^{t.0}$, for the expansion of the function of an exponential. A small Table up to $\Delta^{10}0^{10}$ is given in Herschel's *Examples* (Camb. 1820), and is reproduced in the treatise on Finite Differences (1843) in the *Encyclopædia Metropolitana*. But, as is known, the successive differences $\Delta 0^n$, $\Delta^2 0^n$, $\Delta^3 0^n$,... are divisible by 1, 1.2, 1.2.3,... and generally $\Delta^m 0^n$ is divisible by $1.2.3...m, = \Pi(m)$; these quotients are much smaller numbers, and it is therefore desirable to tabulate them rather than the undivided differences $\Delta^m 0^n$: it is moreover easier to calculate them. A Table of the quotients $\Delta^m 0^n \div \Pi(m)$, up to $m = n = 12$ is in fact given by Grunert, *Crelle*, t. XXV. (1843), p. 279, but without any explanation in the heading of the meaning of the tabulated numbers $C_n^k\, (= \Delta^n 0^k \div \Pi(n))$, and without using for their determination the convenient formula $C_n^{k+1} = nC_n^k + C_{n-1}^k$ given by Bjorling in a paper *Crelle*, t. XXVIII. (1844), p. 284. The formula in question, say

$$\frac{\Delta^m 0^{n+1}}{\Pi(m)} = m \cdot \frac{\Delta^m 0^n}{\Pi(m)} + \frac{\Delta^{m-1} 0^n}{\Pi(m-1)},$$

is given in the second edition (by Moulton) of Boole's *Calculus of Finite Differences*, (London, 1872), p. 28, under the form

$$\Delta^m 0^n = m\,(\Delta^{m-1} 0^{n-1} + \Delta^m 0^{n-1}).$$

It occurred to me that it would be desirable to extend the table of the quotients $\Delta^m 0^n \div \Pi(m)$, up to $m = n = 20$. The calculation is effected very readily by means of the foregoing theorem, which is used in the following form; viz. any column of the table, for instance the fifth, being

A	; then following column is	A
B		$2B + A$
C		$3C + B$
D		$4D + C$
E		$5E + D$
		$+E$;

and then we obtain a good verification by taking the sum of the terms in the new column, and comparing it with the value as calculated from the formula,

$$\text{Sum} = 2A + 3B + 4C + 5D + 6E:$$

observe that in the two calculations we take successive multiples such as $4D$ and $5D$ of each term of the preceding column, and that the verification is thus a safeguard against any error of multiplication or addition.

Table, No. 1, of $\Delta^m 0^m \div \Pi(m)$.

Ind. Δ	0^1	0^2	0^3	0^4	0^5	0^6	0^7	0^8	0^9	0^{10}	0^{11}	0^{12}	0^{13}	0^{14}
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	3	7	15	31	63	127	255	511	1 023	2 047	4 095	8 191
3			1	6	25	90	301	966	3 025	9 330	28 501	86 526	261 625	788 970
4				1	10	65	350	1 701	7 770	34 105	145 750	611 501	2 532 530	10 391 745
5					1	15	140	1 050	6 951	42 525	246 730	1 379 400	7 508 501	40 075 035
6						1	21	266	2 646	22 827	179 487	1 323 652	9 321 312	63 436 373
7							1	28	462	5 880	63 987	627 396	5 715 424	49 329 280
8								1	36	750	11 880	159 027	1 899 612	20 912 320
9									1	45	1 155	22 275	359 502	5 135 130
10										1	55	1 705	39 325	752 752
11											1	66	2 431	66 066
12												1	78	3 367
13													1	91
14														1
15														
16														
17														
18														
19														
20														

Ind. Δ	0^{15}	0^{16}	0^{17}	0^{18}	0^{19}	0^{20}	
1	1	1	1	1	1	1	1
2	16 383	32 767	65 535	131 071	262 143	524 287	2
3	2 375 101	7 141 686	21 457 825	64 439 010	193 448 101	580 606 446	3
4	42 355 950	171 798 901	694 337 290	2 798 806 985	11 259 666 950	45 232 115 901	4
5	210 766 920	1 096 190 550	5 652 751 651	28 958 095 545	147 589 284 710	749 206 090 500	5
6	420 693 273	2 734 926 558	17 505 749 898	110 687 251 039	693 081 601 779	4 306 078 895 384	6
7	408 741 333	3 281 882 604	25 708 104 786	197 462 483 400	1 492 924 634 839	11 143 554 045 652	7
8	216 627 840	2 141 764 053	20 415 995 028	189 036 065 010	1 709 751 003 480	15 170 932 662 679	8
9	67 128 490	820 784 250	9 528 822 303	106 175 395 755	1 144 614 626 805	12 011 282 644 725	9
10	12 662 650	193 754 990	2 758 334 150	37 112 163 803	477 297 033 785	5 917 584 964 655	10
11	1 479 478	28 936 908	512 060 978	8 391 004 908	129 413 217 791	1 900 842 429 486	11
12	106 470	2 757 118	62 022 324	1 256 328 866	23 466 951 300	411 016 633 391	12
13	4 550	165 620	4 910 178	125 854 638	2 892 439 160	61 068 660 380	13
14	105	6 020	249 900	8 408 778	243 577 530	6 302 524 580	14
15	1	120	7 820	367 200	13 916 778	452 329 200	15
16		1	136	9 996	527 136	22 350 954	16
17			1	153	12 597	741 285	17
18				1	171	15 675	18
19					1	190	19
20						1	20

Writing down the sloping lines as columns thus:

1 (0)	2 (2)	3 (4)	4 (6)	5 (8)	6 (10)	7 (12)	8 etc. (14) etc.
1							
1	1						
1	3	1					
1	6	7	1				
1	10	25	15	1			
1	15	65	90	31	1		
1	21	140	350	301	63	1	
1	28	266	1 050	1 701	966	127	
1	36	462	2 646	6 951	7 770	3 025	
1	45	750	5 880	22 827	42 525	34 105	
1	55	1 155	11 880	63 987	179 487	246 730	
1	66	1 705	22 275	159 027	627 396	1 323 652	
1	78	2 431	39 325	359 502	1 899 612	5 715 424	
1	91	3 367	66 066	752 752	5 135 130	20 912 320	
1	105	4 550	106 470	1 479 478	12 662 650	67 128 490	
1	120	6 020	165 620	2 757 118	28 936 908	193 754 990	
1	136	7 820	249 900	4 910 178	62 022 324	512 060 978	
1	153	9 996	367 200	8 408 778	125 854 638	1 256 328 866	
1	171	12 597	527 136	13 916 778	243 577 530	2 892 439 160	
1	190	15 675	741 285	22 350 954	452 329 200	6 302 524 580	
20	19	18	17	16	15	14	13 etc.

it appears by inspection that in the second column the second differences are constant, in the third column the fourth differences, in the fourth column the sixth differences, and so on, are constant; and we thence deduce the law of the numbers in the successive columns: viz. this can be done up to column 7, in which we have 14 numbers for taking the 12-th differences: but in column 8 we have only 13 numbers, and therefore cannot find the 14-th differences. The differences are given in the following

Table, No. 2 (*explanation infrà*).

Ind. Δ	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1
1		2	6	14	30	62	126
2		1	12	61	240	841	2 772
3			10	124	890	5 060	25 410
4			3	131	1 830	16 990	127 953
5				70	2 226	35 216	401 436
6				15	1 600	47 062	836 976
7					630	40 796	1 196 532
8					105	21 225	1 182 195
9						10 930	795 718
10						945	349 020
11							90 090
12							10 395

We have by means of this Table, the general expressions of $\Delta^r 0^r$, $\Delta^{r-1} 0^r$, $\Delta^{r-2} 0^r \ldots$ up to $\Delta^{r-6} 0^r$, viz. the formulæ are

$$\Delta^r 0^r \div \Pi(r) \qquad = 1,$$

$$\Delta^{r-1} 0^r \div \Pi(r-1) = 1 + 2\binom{r-2}{1}^1 + 1\binom{r-2}{2}^2,$$

$$\Delta^{r-2} 0^r \div \Pi(r-2) = 1 + 6\binom{r-3}{1}^1 + 12\binom{r-3}{2}^2 + 10\binom{r-3}{3}^3 + 3\binom{r-3}{4}^4,$$

&c., &c.

where the numerical coefficients are the numbers in the successive columns of the table; and where for shortness $\binom{r-m}{k}^k$ is written to denote the binomial coefficient $\frac{[r-m]^k}{[k]^k}$. For instance, $r = 10$, we have

$$\Delta^8 0^{10} \div \Pi(8) = 1 + 6 \,.\, 7 + 12 \,.\, 21 + 10 \,.\, 35 + 3 \,.\, 35, = 750,$$

agreeing with the principal Table. It will be observed that in the successive columns of the Table the last terms are 1, 1, 1.3, 1.3.5, 1.3.5.7, 1.3.5.7.9, and 1.3.5.7.9.11. This is itself a good verification: I further verified the last column by calculating from it the value of $\Delta^{14} 0^{20} \div \Pi(14), = 6\,302\,524\,580$ as above. The Table shows that we have $\Delta^{r-m} 0^r \div \Pi(r-m)$ given as an algebraical rational and integral function of r, of the degree $2m$. But the terms from the top of a column, $\Delta 0^r = 1$, $\Delta^2 0^r \div 1.2 = 2^{r-1} - 1$, &c., are not algebraical functions of r.

22 *October*, 1879.

II. *On the Schwarzian Derivative, and the Polyhedral Functions.* By A. CAYLEY, Sadlerian Professor of Pure Mathematics.

[Read March 8, 1880.]

THE quotient s of any two solutions of a linear partial differential equation of the second order, $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$, is determined by a differential equation of the third order

$$\frac{\frac{d^3s}{dx^3}}{\frac{ds}{dx}} - \frac{3}{2}\left(\frac{\frac{d^2s}{dx^2}}{\frac{ds}{dx}}\right)^2 = -\frac{1}{2}\left(p^2 + 2\frac{dp}{dx} - 4q\right),$$

where the function on the left hand is what I call the Schwarzian Derivative; or say this derivative is

$$\{s, x\}, = \frac{s'''}{s'} - \frac{3}{2}\left(\frac{s''}{s'}\right)^2,$$

where the accents denote differentiations in regard to the second variable x of the symbol.

Writing in general $(\text{a, b, c} \therefore \between X, Y, Z)^2$ to denote a quadric function

$$(\text{a, b, c}, \tfrac{1}{2}(\text{a}-\text{b}-\text{c}), \tfrac{1}{2}(-\text{a}+\text{b}-\text{c}), \tfrac{1}{2}(-\text{a}-\text{b}+\text{c}) \between X, Y, Z)^2,$$

then, if the equation of the second order be that of the hypergeometric series, generalised by a homographic transformation upon the variable x, the resulting differential equation of the third order is of the form

$$\{s, x\} = (\text{a, b, c} \therefore)\left(\frac{1}{x-a}, \frac{1}{x-b}, \frac{1}{x-c}\right)^2;$$

and, presenting themselves in connection with the algebraically integrable cases of this equation, we have rational and integral functions of s, derived from the polygon, the double pyramid, and the five regular solids, and which are called Polyhedral Functions.

The Schwarzian Derivative occurs implicitly in Jacobi's differential equation of the third order for the modulus in the transformation of an elliptic function (*Fund. Nova*, 1829, p. 79) and in Kummer's fundamental equation for the transformation of a hypergeometric series (Kummer, 1836: see list of Memoirs): but it was first explicitly considered and brought into notice in the two Memoirs of Schwarz, 1869 and 1873; the latter of these (relating to the algebraic integration of the differential equation for the hypergeometric series) is the fundamental Memoir upon the subject, but the theory is in some material points completed in the Memoirs by Klein and Brioschi.

The following list of Memoirs relating as well to the Polyhedral Functions as to the Schwarzian Derivative is arranged nearly in chronological order.

Kummer, Ueber die hypergeometrische Reihe $1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \ldots$. Crelle, t. XV. (1836), pp. 39—83 and 127—172.

Schwarz, Ueber einige Abbildungsaufgaben. Crelle-Borchardt, t. LXX. (1869), pp. 105—120.

——— Ueber diejenigen Fälle in welchen die *Gauss*ische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt. Do. t. LXXV. (1873), pp. 292—335.

Cayley, Notes on Polyhedra. Quart. Math. Jour. t. VII. (1866), pp. 304—316.

——— On the Regular Solids. Do. t. XV. (1877), pp. 127—131.

Fuchs, Ueber diejenigen Differentialgleichungen zweiter Ordnung welche algebraische Integralen besitzen, und eine Anwendung der Invariantentheorie. Crelle-Borchardt, t. 81 (1875), pp. 97—142.

Klein, Ueber binäre Formen mit linearen Transformationen in sich selbst. Math. Ann. t. IX. (1875), pp. 183—209.

Brioschi, Extrait d'une lettre à M. Klein. Math. Ann. t. XI. (1877), pp. 111—114.

Klein, Ueber lineare Differential-Gleichungen. Math. Ann. t. XI. (1877), pp. 115—118.

Brioschi, La théorie des formes dans l'intégration des équations différentielles linéaires du second ordre. Math. Ann. t. XI. (1877), pp. 401—411.

Gordan, Ueber endliche Gruppen linearer Transformationen einer Veränderlichen. Math. Ann. t. XII. (1877), pp. 23—46.

——— Binäre Formen mit verschwindenden Covarianten. Math. Ann. t. XII. (1877), pp. 147—166.

Klein, Ueber lineare Differentialgleichungen. Math. Ann. t. XII. (1877), pp. 167—179.

——— Weitere Untersuchungen über das Icosaeder. Math. Ann. t. XII. (1877), pp. 503—560.

Cayley, On the Correspondence of Homographies and Rotations. Math. Ann. t. XV. (1879), pp. 238—240.

——— On the finite Groups of linear transformations of a Variable. Math. Ann. t. XVI. (1880), pp. 260—263, and pp. 439—440.

I propose in the present Memoir to consider the whole theory: and in particular to give some additional developments in regard to the Polyhedral Functions.

I remark that Schwarz starts with the foregoing differential equation of the third order

$$\{s, x\} = (\mathrm{a}, \mathrm{b}, \mathrm{c} \;\therefore)\left(\frac{1}{x-a}, \frac{1}{x-b}, \frac{1}{x-c}\right)^2,$$

and he shows (by very refined reasoning founded on the theory of conformable figures, which will be in part reproduced) that this equation is in fact algebraically integrable for 16 different sets of values of the coefficients a, b, c. It may I think be taken to be part of his theory, although not very clearly brought out by him, that these integrals are some of them of the form, $x =$ rational function of s; others of the form, rational function of $x =$ rational function of s; the rational functions of s being in fact the same in these last as in the first set of solutions, and being quotients of Polyhedral functions.

But as regards the second set of cases, the solution of these (introducing for convenience a new variable z in place of s) may be made to depend upon the solution in the form, $x =$ rational function of z, of an equation of a somewhat similar form, but involving two quadric functions of x and z respectively, viz. the equation

$$\{x, z\} + \left(\frac{dx}{dz}\right)^2 (\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore)\left(\frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2 = (\mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1\ \therefore)\left(\frac{1}{z-a_1},\ \frac{1}{z-b_1},\ \frac{1}{z-c_1}\right)^2;$$

and we have the theorem that the solution of this equation depends upon the determination of P, Q, R rational and integral functions of z (containing each of them multiple factors) which are such that $P+Q+R=0$: (using accents to denote differentiation in regard to z, this implies $P'+Q'+R'=0$, and consequently $QR'-Q'R=RP'-R'P=PQ'-P'Q$): and are further such that the equal functions $QR'-Q'R$, $RP'-R'P$, $PQ'-P'Q$ contain only factors which are factors of P, Q or R.

In fact, writing $f,\ g,\ h = b-c,\ c-a,\ a-b$, the required relation between x, z is then expressed in the symmetrical form $f(x-a) : g(x-b) : h(x-c) = P : Q : R$.

The last mentioned differential equation is considered by Klein and Brioschi: the solutions in 13 cases, or such of them as had not been given by Schwarz, were obtained by Brioschi, and those of the remaining 3 cases (subject to a correction in one of them) were afterwards obtained by Klein.

The first part of the present Memoir relates, say to the foregoing equation

$$\{s, x\} = (\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore)\left(\frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2,$$

although the other form in $\{x, z\}$ may equally well be regarded as the fundamental form: and

We consider in the theory:

A. The Derivative $\{s, x\}$, meaning as above explained.

B. Quadric functions of any three or more inverts $\dfrac{1}{x-l}$.

C. Rational and integral functions P, Q, R having a sum $=0$, and which are such that $QR'-Q'R$, $=RP'-R'P$, $=PQ'-P'Q$, contains only the factors of P, Q, R.

D. The differential equation of the third order.

E. The Schwarzian theory in regard to conformable figures and the corresponding values of the imaginary variables s and x.

F. Connection with the differential equation for the hypergeometric series.

The Second part of the Memoir relates to the Polyhedral Functions.

The paragraphs of the whole Memoir are numbered consecutively.

PART I.

The Derivative $\{s, x\}$, Article Nos. 1 to 7.

1. If $p = \frac{s''}{s'} = \frac{d}{dx}\left(\log \frac{ds}{dx}\right)$, then $\{s, x\} = \frac{dp}{dx} - \frac{1}{2}p^2$.

2. The derivative $\{s, x\}$ may be transformed in regard to either or both of the variables.

Suppose first that s is a function of the new variable S, (hence also S is a function of x): using subscript numbers to denote differentiations in regard to S, and the accents as before for differentiations in regard to x, we have

$$s' = S' s_1,$$

whence, differentiating the logarithms,

$$\frac{s''}{s'} = S' \frac{s_2}{s_1} + \frac{S''}{S'}$$

and again differentiating

$$\frac{s'''}{s'} - \left(\frac{s''}{s'}\right)^2 = S'^2 \left[\frac{s_3}{s_1} - \left(\frac{s_2}{s_1}\right)^2\right] + S'' \frac{s_2}{s_1} + \frac{S'''}{S'} - \left(\frac{S''}{S'}\right)^2.$$

But $$-\frac{1}{2}\left(\frac{s''}{s'}\right)^2 = S'^2 \left[-\frac{1}{2}\left(\frac{s_2}{s_1}\right)^2\right] - S'' \frac{s_2}{s_1} \quad - \frac{1}{2}\left(\frac{S''}{S'}\right)^2,$$

and consequently

$$\frac{s'''}{s'} - \frac{3}{2}\left(\frac{s''}{s'}\right)^2 = S'^2 \left[\frac{s_3}{s_1} - \frac{3}{2}\left(\frac{s_2}{s_1}\right)^2\right] \quad + \frac{S'''}{S'} - \frac{3}{2}\left(\frac{S''}{S'}\right)^2;$$

that is $$\{s, x\} = \left(\frac{dS}{dx}\right)^2 \{s, S\} + \{S, x\},$$

the required formula.

In a very similar manner, taking x a function of X, it is shown that

$$\{s, x\} = \left(\frac{dX}{dx}\right)^2 (\{s, X\} - \{x, X\}).$$

3. If in this formula we write S for s, and substitute the resulting value of $\{S, x\}$ in the former formula, we have

$$\{s, x\} = \left(\frac{dS}{dx}\right)^2 \{s, S\} - \left(\frac{dX}{dx}\right)^2 \{x, X\} + \left(\frac{dX}{dx}\right)^2 \{S, X\},$$

which is the formula for the change of both variables, and it in fact includes the other two: viz. writing $X = x$, or $S = s$, and observing that $\{s, s\} = \{x, x\} = 0$, we have the other two formulæ.

4. By putting in the first formula $X = s$, we obtain

$$\{s, x\} = -\left(\frac{ds}{dx}\right)^2 \{x, s\},$$

a formula for the interchange of the variables.

5. Writing $S=\dfrac{as+b}{cs+d}$, and using for a moment the accents to denote differentiation in regard to s, we have

$$S'=\frac{ad-bc}{(cs+d)^2},\qquad \frac{S''}{S'}=\frac{-2c}{cs+d},$$

and thence

$$\frac{S'''}{S'}-\left(\frac{S''}{S'}\right)^2=\frac{2c^2}{(cs+d)^2},$$

$$-\tfrac{1}{2}\left(\frac{S''}{S'}\right)^2=\frac{-2c^2}{(cs+d)^2}.$$

Consequently $\{S,\ s\}=0$ (whence also $\{s,\ S\}=0$).

Hence in the first formula $\{S,\ x\}=\{s,\ x\}$, that is

$$\left\{\frac{as+b}{cs+d},\ x\right\}=\{s,\ x\};$$

viz. we may in the derivative $\{s,\ x\}$ write for s any homographic function $(as+b)\div(cs+d)$ of s.

6. Again if $X=\dfrac{\alpha x+\beta}{\gamma x+\delta}$, then from the second formula

$$\{s,\ x\}=\frac{(\alpha\delta-\beta\gamma)^2}{(\gamma x+\delta)^4}\{s,\ X\};$$

that is

$$\left\{s,\ \frac{\alpha x+\beta}{\gamma x+\delta}\right\}=\frac{(\gamma x+\delta)^4}{(\alpha\delta-\beta\gamma)^2}\{s,\ x\};$$

and here, changing s into $(as+b)\div(cs+d)$, we have finally

$$\left\{\frac{as+b}{cs+d},\ \frac{\alpha x+\beta}{\gamma x+\delta}\right\}=\frac{(\gamma x+\delta)^4}{(\alpha\delta-\beta\gamma)^2}\{s,\ x\},$$

which is the formula for the homographic transformation of the two variables s, x.

7. Let s be a given function of x, the equation $\{S,\ x\}=\{s,\ x\}$ is a differential equation of the third order in S, and by what precedes, its general integral is $S=\dfrac{as+b}{cs+d}$.

The direct process is as follows: we have a first integral $\dfrac{S''}{S'}=\dfrac{s''}{s'}-\dfrac{2cs'}{cs+d}$; a second integral $\log S'=\log s'-2\log(cs+d)+\text{Const.}$, that is $S'=\dfrac{cAs'}{(cs+d)^2}$; and thence a final integral $S=B-\dfrac{A}{cs+d}$, which is equivalent to the foregoing value of S.

The Quadric Function of three or more Inverts. Art. Nos. 8 to 15.

8. We consider a quadric function of any number of inverts $\frac{1}{x-\alpha}, \frac{1}{x-\beta}, \ldots$ all of them different: it is assumed that the constant term is $=0$, and also that the sum of the coefficients of the linear terms is $=0$. We have therefore square terms $\frac{a}{(x-\alpha)^2}$, product terms $\frac{h}{x-\alpha \, . \, x-\beta}$, and linear terms $\frac{A}{x-\alpha}$, where the sum of the coefficients A is $=0$. Any product term $\frac{h}{x-\alpha \, . \, x-\beta}$ is expressible in the form of a difference $\frac{h}{\alpha-\beta}\frac{1}{x-\alpha} - \frac{h}{\alpha-\beta}\frac{1}{x-\beta}$ of two linear terms, and (the coefficients of these being equal) after it is thus expressed the sum of the coefficients of the linear terms is still $=0$. The function is thus always in expressible in the form

$$\frac{a}{(x-\alpha)^2} + \frac{b}{(x-\beta)^2} + \ldots + \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \ldots$$

where the sum $A+B+\ldots$ is $=0$: this may be called the reduced form.

9. Observe that any particular invert $\frac{1}{x-\alpha}$ may disappear altogether from the reduced form: this will be the case if $a=0$ $\left(\text{that is if the original form contains no term in } \frac{1}{(x-\alpha)^2}\right)$, and if also $A=0$. An invert thus disappearing from the reduced form is said to be non-essential: and the inverts which do not disappear are said to be essential. The original form contains in appearance the non-essential inverts, but it is really a quadric function of the essential inverts only.

10. Imagine the original function expressed as a rational fraction, the denominator being the product $(x-\alpha)^2(x-\beta)^2(x-\gamma)^2\ldots$ of the squared factors corresponding to all the inverts (non-essential as well as essential): the numerator will be in general of a degree less by 2 than that of the denominator, but the coefficients of any one or more of the higher powers of x may vanish, and the numerator will then be of a lower degree. But this numerator will for any non-essential invert $\frac{1}{x-\gamma}$ contain the factor $(x-\gamma)^2$, or, dividing the numerator and denominator each by this factor, the difference of the degrees of the numerator and denominator will remain unaltered; that is the difference will have the same value whether we do or do not attend to the non-essential inverts; or say it will have the same value for the original form and for the reduced form.

11. It is to be remarked that the linear terms $\frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma} \ldots$, where $A+B+C+\ldots=0$, can be (and that in a variety of ways) expressed as a sum of differences $\frac{1}{x-\alpha} - \frac{1}{x-\beta}$, that is as a sum of product-terms $\frac{1}{x-\alpha \, . \, x-\beta}$. Hence the quadric

function can be (and that in a variety of ways) expressed as a homogeneous function $\left(\mathrm{a}, \ldots \between \frac{1}{x-\alpha}, \frac{1}{x-\beta}, \ldots\right)^2$; we must have in the form all the essential inverts, and we need have these only. Supposing that this is so, and that the number of the essential inverts is $=n$, then the number of constants is $=\frac{1}{2}n(n+1)$, whereas the number of constants in the reduced form is only $=2n-1$: hence the coefficients are not determinate; or, what is the same thing, we may have different quadric functions having each of them the same reduced function; these quadric functions, as having the same reduced function, can only differ by multiples of the evanescent expressions

$$\frac{\beta-\gamma}{x-\beta\,.\,x-\gamma}+\frac{\gamma-\alpha}{x-\gamma\,.\,x-\alpha}+\frac{\alpha-\beta}{x-\alpha\,.\,x-\beta}, \text{ \&c.}$$

In particular if the number of essential inverts is $=3$, then the quadric function is of the form $\left(\mathrm{a, b, c, f, g, h} \between \frac{1}{x-\alpha}, \frac{1}{x-\beta}, \frac{1}{x-\gamma}\right)^2$, which contains one superfluous constant, and equivalent functions differ only by a multiple of

$$\frac{\beta-\gamma}{x-\beta\,.\,x-\gamma}+\frac{\gamma-\alpha}{x-\gamma\,.\,x-\alpha}+\frac{\alpha-\beta}{x-\alpha\,.\,x-\beta}.$$

12. A quadric function such that the degree of the numerator is less by 4 than that of the denominator is said to be "curtate."

The conditions in order that the function $\left(\mathrm{a, b, c, f, g, h} \between \frac{1}{x-\alpha}, \frac{1}{x-\beta}, \frac{1}{x-\gamma}\right)^2$ may be curtate are easily found to be

$$\mathrm{a+b+c+2f+2g+2h}=0,$$

$$\alpha\,(\mathrm{a+h+g})+\beta\,(\mathrm{h+b+f})+\gamma\,(\mathrm{g+f+c})=0\,;$$

and by reason of the superfluous constant we are at liberty to assume a third condition: the three conditions may be taken to be $\mathrm{a+h+g}$, $\mathrm{h+b+f}$, $\mathrm{g+f+c}$ each $=0$; and this being so the values of f, g, h are $=\frac{1}{2}(\mathrm{a-b-c})$, $\frac{1}{2}(\mathrm{-a+b-c})$, $\frac{1}{2}(\mathrm{-a-b+c})$ respectively. Hence the form is

$$\left(\mathrm{a, b, c}, \tfrac{1}{2}(\mathrm{a-b-c}), \tfrac{1}{2}(\mathrm{-a+b-c}), \tfrac{1}{2}(\mathrm{-a-b+c}) \between \frac{1}{x-\alpha}, \frac{1}{x-\beta}, \frac{1}{x-\gamma}\right)^2,$$

which, as already mentioned, we denote by

$$\left(\mathrm{a, b, c}\ \therefore \between \frac{1}{x-\alpha}, \frac{1}{x-\beta}\ \ \frac{1}{x-\gamma}\right)^2.$$

We have thus the theorem that a curtate function of any number of inverts, but with only the three essential inverts $\frac{1}{x-\alpha}, \frac{1}{x-\beta}, \frac{1}{x-\gamma}$, is always expressible in the foregoing form

$$\left(\mathrm{a, b, c}\ \therefore \between \frac{1}{x-\alpha}, \frac{1}{x-\beta}, \frac{1}{x-\gamma}\right)^2.$$

13. It may be remarked that the function $(\mathrm{a, b, c} \,\therefore\!\!\between X, Y, Z)^2$ is a function of the differences of the variables X, Y, Z; and similarly in the case of four variables a function $(\mathrm{a, b, c, d, f, g, h, l, m, n} \between X, Y, Z, W)^2$ for which $\mathrm{a+h+g+l}$, $\mathrm{h+b+f+m}$, $\mathrm{g+f+c+n}$, $\mathrm{l+m+n+d}$ are each $=0$ is a function of the differences of the variables X, Y, Z, W: and so in general. Any such function is said to be "diaphoric:" and it is easy to see that, taking for the variables any inverts whatever, a diaphoric function is always curtate.

14. The function

$$\left\{-\frac{\mathrm{a}}{(x-\alpha)^2}-\frac{\mathrm{b}}{(x-\beta)^2}-\frac{\mathrm{c}}{(x-\gamma)^2}\ldots\right\}^2,$$

$$-\tfrac{1}{2}\left\{\quad\frac{\mathrm{a}}{x-\alpha}+\frac{\mathrm{b}}{x-\beta}+\frac{\mathrm{c}}{x-\gamma}\ldots\right\}^2,$$

where the coefficients a, b, c... satisfy the relation $\mathrm{a+b+c}+\ldots=-2$, is diaphoric, and therefore curtate. In fact forming the sum, coeff. $\dfrac{1}{(x-\alpha)^2}+\tfrac{1}{2}$ coeff. $\dfrac{1}{x-\alpha\,.\,x-\beta}+\ldots$, this is $-\mathrm{a}-\tfrac{1}{2}\mathrm{a}^2-\tfrac{1}{2}\mathrm{ab}-\tfrac{1}{2}\mathrm{ac}\ldots, =-\tfrac{1}{2}\mathrm{a}\,(2+\mathrm{a+b+c}\ldots)$ which is $=0$; and similarly the other conditions are satisfied.

15. The function

$$\left(\mathrm{a, b, c} \,\therefore\!\!\between \frac{a}{x-\alpha}+\frac{a_1}{x-\alpha_1}+\ldots,\ \frac{b}{x-\beta}+\frac{b_1}{x-\beta_1}+\ldots,\ \frac{c}{x-\gamma}+\frac{c_1}{x-\gamma_1}+\ldots\right)^2,$$

regarded as a function of the inverts $\dfrac{1}{x-\alpha}, \dfrac{1}{x-\alpha_1}, \ldots \dfrac{1}{x-\beta}, \ldots$ where

$$a+a_1+\ldots=b+b_1+\ldots=c+c_1+\ldots, =k \text{ suppose},$$

is diaphoric, and therefore curtate. In fact the condition in regard to $\dfrac{1}{x-\alpha}$ is

$$\mathrm{a}\,(a^2+aa_1+aa_2+\ldots)+\tfrac{1}{2}(-\mathrm{a}+\mathrm{b}-\mathrm{c})\,(ab+ab_1+\ldots)+\tfrac{1}{2}(-\mathrm{a}-\mathrm{b}+\mathrm{c})\,(ac+ac_1+\ldots)=0;$$

that is

$$ak\,\{\mathrm{a}+\tfrac{1}{2}(-\mathrm{a}+\mathrm{b}-\mathrm{c})+\tfrac{1}{2}(-\mathrm{a}-\mathrm{b}+\mathrm{c})\}=0,$$

which is satisfied. And similarly the other conditions are satisfied.

The functions P, Q, R. Article, Nos. 16 to 20.

16. We consider P, Q, R, rational and integral fractions of z, such that $P+Q+R=0$: hence, using the accent to denote differentiation in regard to z, we have also $P'+Q'+R'=0$; and therefore $QR'-Q'R=RP'-R'P=PQ'-P'Q, =\Theta$ suppose: and we require to find P, Q, R, such that the function Θ contains only the factors of P, Q, R.

17. It is to be observed that, effecting upon a solution P, Q, R any linear substitution $(\alpha z+\beta)\div(\gamma z+\delta)$, and omitting the common denominator, we have a solution; but this is regarded as identical with the original solution. The three functions, if not originally of the same order, can thus be made to be of the same order; or by taking account of the root $z=\infty$, we may in the original case regard them as being of the same order, and it is convenient so to regard them: say they are taken to be of the same order δ. And there is clearly no loss of generality in taking the three functions to be prime to each other; for any common factor of two of them would divide the third, and might therefore be struck out.

18. We may therefore write

$$P=F\Pi(z-l)^p, \qquad Q=G\Pi(z-m)^q, \qquad R=H\Pi(z-n)^r,$$

where $(z-l)^p$ is taken to denote the distinct simple or multiple factors of P, and the like as regards Q and R; the factors $z-l$, $z-m$, $z-n$ are thus all of them different. And we have $\delta=\Sigma p, =\Sigma q, =\Sigma r$.

19. It is at once seen that Θ is of the degree $2\delta-2$, and moreover that it contains the factors $\Pi(z-l)^{p-1}$, $\Pi(z-m)^{q-1}$, $\Pi(z-n)^{r-1}$; hence it contains the factor

$$\Pi(z-l)^{p-1}(z-m)^{q-1}(z-n)^{r-1}.$$

Suppose the number of distinct indices p is $=\sigma_1$, that of distinct indices q is σ_2, and that of distinct indices r is $=\sigma_3$; then the degree of the factor is $=3\delta-\sigma_1-\sigma_2-\sigma_3$; and if this be $=2\delta-2$, then Θ can have no other variable factor: viz. if the numbers σ_1, σ_2, σ_3 of the distinct indices p, q, r respectively are such that $\sigma_1+\sigma_2+\sigma_3=\delta+2$ (a relation which is henceforth taken to be satisfied), then we have

$$\Theta=K\Pi(z-l)^{p-1}(z-m)^{q-1}(z-n)^{r-1}.$$

As already in effect remarked the conclusion extends to the case where P, Q, R are not of the same degree; the equation $P+Q+R=0$ here implies that two functions, say P, Q, are of the same degree, and the third function R of an inferior degree; but, this being so, we have only to regard R as containing the factor $\left(1-\frac{z}{\infty}\right)^t$ of the degree t proper for raising its degree up to that of P or Q.

20. Solutions are given in the following PQR-Table: in which, where required, the proper factor $\left(1-\frac{z}{\infty}\right)^t$ has been added; the first column headed Ref. No. (Reference Number) will be explained further on. The Annex to the same Table will also be explained.

THE PQR-TABLE.

Ref. No.	$P=$	$Q=$	$R=$	$O=$
1	z^n	$-1\left(1-\frac{z}{\infty}\right)^n$	$-z^n+1$	$nz^{n-1}\left(1-\frac{z}{\infty}\right)^{n-1}$
2	$4z^n\left(1-\frac{z}{\infty}\right)^n$	$-(z^n+1)^2$	$(z^n-1)^2$	$-4nz^{n-1}(z^n+1)(z^n-1)\left(1-\frac{z}{\infty}\right)^{n-1}$
3	$(z^4+2\sqrt{-3}z^2+1)^3$	$-12\sqrt{-3}z^2(z^4-1)^2\left(1-\frac{z}{\infty}\right)^2$	$-(z^4-2\sqrt{-3}z^2+1)^3$	$24\sqrt{-3}(z^4+2\sqrt{-3}z^2..)^2$ $(z^4-2\sqrt{-3}z^2..)z(z^4-1)\left(1-\frac{z}{\infty}\right)$
4	$(z^8+14z^4+1)^3$	$-(z^{12}-33z^8-33z^4+1)^2$	$-108(z^5-z)^4\left(1-\frac{z}{\infty}\right)^4$	$576(z^8+..)^2(z^{12}-..)(z^5-z)^3\left(1-\frac{z}{\infty}\right)^3$
5	$(z^{20}-228z^{15}+494z^{10}+228z^5+1)^3$	$-(z^{30}-522z^{25}-10005z^{20}$ $+0z^{15}-10005z^{10}+522z^5+1)^2$	$-1728(z^{11}+11z^6-z)^5\left(1-\frac{z}{\infty}\right)^5$	$-8640(z^{20}-..)^2(z^{11}+..)^4\left(1-\frac{z}{\infty}\right)^4$
III, V, VII, VIII	$4z\left(1-\frac{z}{\infty}\right)$	$-(z+1)^2$	$(z-1)^2$	$-4(z+1)(z-1)$
IX	$(z-4)^3$	$-(z-1)(z+8)^2$	$27z^2\left(1-\frac{z}{\infty}\right)$	$27(z-4)^2(z+8)z$
X	$z(z+8)^3$	$-(z^2-20z-8)^2$	$-64(z-1)^3\left(1-\frac{z}{\infty}\right)$	$-64(z+8)^2(z^2-20z..)(z-1)$
XI	$4(z^2-z+1)^3$	$-(2z^3-3z^2-3z+2)^2$	$-27z^2(z-1)^2\left(1-\frac{z}{\infty}\right)^2$	$-108(z^2-z..)^2(2z^3-3z^2..)$ $z(z-1)\left(1-\frac{z}{\infty}\right)$
XII	$z^3(z+5)^2(z+8)$	$-(z^3+9z^2+12z-8)^2$	$-64(3z-1)\left(1-\frac{z}{\infty}\right)^5$	$-960z^3(z+5)(z^3+9z^2..)\left(1-\frac{z}{\infty}\right)^4$
XIII	$(z^2+14z+1)^3$	$-(z^3-33z^2-33z+1)^2$	$-108z(z-1)^4$	$-108(z^2+14z..)^2(z^3-33z^2..)(z-1)^3$
XIV	$(64z+189)(64z^2+133z+49)^3$	$-z(4096z^3+18816z^2+25725z+12005)^2$	$-27.7^7.(z+1)^2\left(1-\frac{z}{\infty}\right)^5$	$-135.7^7.(64z^2+..)^2(4096z^3+..)$ $(z+1)\left(1-\frac{z}{\infty}\right)^4$
XV	$-(5z-27)(125z^3-25z^2-265z-243)^3$	$-(-3125z^5+9375z^4+18750z^3$ $+8750z^2+30750z+19683)^2$	$+1382400000z^3(z+1)^2\left(1-\frac{z}{\infty}\right)^5$	$-1382400000(125z^3-..)^2$ $(-3125z^5+..)z^2(z+1)\left(1-\frac{z}{\infty}\right)^4$

In the second half of the table the functions P, Q, R, except in the lines XII, XIV and XV, were calculated by Brioschi: those for the lines XII and XIV were calculated by Klein, but as regards line XV there would seem to have been some error at the beginning of the calculation, and the values found by him are erroneous.

ANNEX TO THE *PQR*-TABLE.

Ref. No.	(a, b, c)*	Calculation of (ap), (bq), (cr).				Ref. No.	
1	$\frac{1}{2}(1-n^{-2}),\ \frac{1}{2}(1-n^{-2}),\ 0$	$an=0$	$bn=0$	$c1=0$	$\frac{1}{2}(1-n^{-2}),\ \frac{1}{2}(1-n^{-2}),\ 0$		Polygon
2	$\frac{1}{2}(1-n^{-2}),\ \frac{3}{8},\ \frac{3}{8}$	$an=0$	$b2=0$	$c2=0$	$\frac{1}{2}(1-n^{-2}),\ \frac{3}{8},\ \frac{3}{8}$	I	Double Pyramid
3	$\frac{4}{9},\ \frac{3}{8},\ \frac{4}{9}$	$a3=0$	$b2=0$	$c3=0$	$\frac{4}{9},\ \frac{3}{8},\ \frac{4}{9}$	II	Tetrahedron
4	$\frac{4}{9},\ \frac{3}{8},\ \frac{15}{32}$	$a3=0$	$b2=0$	$c4=0$	$\frac{4}{9},\ \frac{3}{8},\ \frac{15}{32}$	IV	Cube and Octahedron
5	$\frac{4}{9},\ \frac{3}{8},\ \frac{12}{25}$	$a3=0$	$b2=0$	$c5=0$	$\frac{4}{9},\ \frac{3}{8},\ \frac{12}{25}$	VI	Dodecahedron and Icosahedron
III	$\frac{4}{9},\ \frac{3}{8},\ \frac{4}{9}$	$a1=\frac{4}{9},\ a1=\frac{4}{9}$	$b2=0$	$c2=\frac{5}{18}$	$\left(\frac{4}{9},\ \frac{4}{9},\ \frac{5}{18}\ \therefore\ \chi\ \frac{1}{z},\ \frac{1}{z-\infty},\ \frac{1}{z-1}\right)^2$		Tetrahedron
V	$\frac{15}{32},\ \frac{3}{8},\ \frac{4}{9}$	$a1=\frac{15}{32},\ a1=\frac{15}{32}$	$b2=0$	$c2=\frac{5}{18}$	$\left(\frac{15}{32},\ \frac{15}{32},\ \frac{5}{18}\ \therefore\ \chi\ \text{,,}\ \ \text{,,}\ \ \text{,,}\right)^2$		Cube and Octahedron
VII	$\frac{4}{9},\ \frac{3}{8},\ \frac{12}{25}$	$a1=\frac{4}{9},\ a1=\frac{4}{9}$	$b2=0$	$c2=\frac{21}{50}$	$\left(\frac{4}{9},\ \frac{4}{9},\ \frac{21}{50}\ \therefore\ \chi\ \text{,,}\ \ \text{,,}\ \ \text{,,}\right)^2$		Dodecahedron and Icosahedron
VIII	$\frac{12}{25},\ \frac{3}{8},\ \frac{4}{9}$	$a1=\frac{12}{25},\ a1=\frac{12}{25}$	$b2=0$	$c2=\frac{5}{18}$	$\left(\frac{12}{25},\ \frac{12}{25},\ \frac{5}{18}\ \therefore\ \chi\ \text{,,}\ \ \text{,,}\ \ \text{,,}\right)^2$		,, ,,
IX	$\frac{4}{9},\ \frac{3}{8},\ \frac{12}{25}$	$a3=0$	$b1=\frac{3}{8},\ b2=0$	$c1=\frac{12}{25},\ c2=\frac{21}{50}$	$\left(\frac{3}{8},\ \frac{12}{25},\ \frac{21}{50}\ \therefore\ \chi\ \frac{1}{z-1},\ \frac{1}{z-\infty},\ \frac{1}{z}\right)^2$		,, ,,
X	,, ,, ,,	$a1=\frac{4}{9},\ a3=0$	$b2=0$	$c1=\frac{12}{25},\ c3=\frac{8}{25}$	$\left(\frac{4}{9},\ \frac{12}{25},\ \frac{8}{25}\ \therefore\ \chi\ \frac{1}{z},\ \frac{1}{z-\infty},\ \frac{1}{z-1}\right)^2$		,, ,,
XI	,, ,, ,,	$a3=0$	$b2=0$	$c2=\frac{21}{50},\ c2=\frac{21}{50},\ c2=\frac{21}{50}$	$\left(\frac{21}{50},\ \frac{21}{50},\ \frac{21}{50}\ \therefore\ \chi\ \frac{1}{z},\ \frac{1}{z-\infty},\ \frac{1}{z-1}\right)^2$		,, ,,
XII	,, ,, ,,	$a1=\frac{4}{9},\ a2=\frac{5}{18},\ a3=0$	$b2=0$	$c1=\frac{12}{25},\ c5=0$	$\left(\frac{4}{9},\ \frac{5}{18},\ \frac{12}{25}\ \therefore\ \chi\ \frac{1}{z+8},\ \frac{1}{z+5},\ \frac{1}{z-\frac{1}{3}}\right)^2$		,, ,,
XIII	,, ,, ,,	$a3=0$	$b2=0$	$c1=\frac{12}{25},\ c1=\frac{12}{25},\ c4=\frac{9}{50}$	$\left(\frac{12}{25},\ \frac{12}{25},\ \frac{9}{50}\ \therefore\ \chi\ \frac{1}{z},\ \frac{1}{z-\infty},\ \frac{1}{z-1}\right)^2$		,, ,,
XIV	,, ,, ,,	$a1=\frac{4}{9},\ a3=0$	$b1=\frac{3}{8},\ b2=0$	$c2=\frac{21}{50},\ c5=0$	$\left(\frac{4}{9},\ \frac{3}{8},\ \frac{21}{50}\ \therefore\ \chi\ \frac{1}{z+\frac{189}{64}},\ \frac{1}{z},\ \frac{1}{z+1}\right)^2$		,, ,,
XV	,, ,, ,,	$a1=\frac{4}{9},\ a3=0$	$b2=0$	$c2=\frac{21}{50},\ c3=\frac{8}{25},\ c5=0$	$\left(\frac{4}{9},\ \frac{21}{50},\ \frac{8}{25}\ \therefore\ \chi\ \frac{1}{z-\frac{27}{5}},\ \frac{1}{z+1},\ \frac{1}{z}\right)^2$		,, ,,

* Observe as regards the (a, b, c) column, that the line III agrees with 3; line V with 4 $\left(\text{only there is a transposition of } \frac{4}{9} \text{ and } \frac{15}{32}\right)$; lines VII and VIII agree each of them with 5 $\left(\text{only as regards VIII there is a transposition of } \frac{4}{9} \text{ and } \frac{12}{25}\right)$: the remaining lines IX to XV agree each of them with 5. Observe also as regards the Annex generally, the Roman numbering of the lines 2, 3, 4, 5: the lines after the first have thus the Roman numbers I to XV, corresponding to the Roman numbers used by Schwarz.

The Differential Equations $\{x, z\}$ *and* $\{s, x\}$. Art. Nos. 21 to 45.

21. In reference to what follows, it is convenient to put $P = XP_0$, $P' = X_1P_0$, where P_0 is written for $\Pi (z-l)^{p-1}$, the G.C.M. of P and P'; and X is consequently $= F$ into the product $\Pi (z-l)$ of the several factors taken each with the index unity; and so for Q and R: viz. we write

$$P,\ Q,\ R = XP_0,\ YQ_0,\ ZR_0,$$
$$P',\ Q',\ R' = X_1P_0,\ Y_1Q_0,\ Z_1R_0,$$

and the foregoing value of Θ then is

$$\Theta = KP_0Q_0R_0.$$

We come now to the investigation of the leading theorem. Take a, b, c arbitrary, $f, g, h = b-c,\ c-a,\ a-b$; P, Q, R functions of z as above; and write

$$f(x-a) : g(x-b) : h(x-c) = P : Q : R,$$

equations consistent with each other, and which determine x as a rational function of z. Using as before the accent to denote differentiation in regard to z, and taking the coefficients (a, b, c) arbitrary, it is required to find the value of

$$\{x, z\} + x'^2 \left(\text{a, b, c} \ \therefore\!\Big)\!\left(\frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2.$$

22. Calculation of the first term $\{x, z\}$.

We have $x =$ a function $\left(\alpha \dfrac{P}{R} + \beta\right) \div \left(\gamma \dfrac{P}{R} + \delta\right)$, and thence $\{x, z\} = \left\{\dfrac{P}{R},\ z\right\}$, $= \{\xi, z\}$ for a moment; then

$$\xi' = \left(\frac{P}{R}\right)' = \frac{RP' - R'P}{R^2},\ = \frac{P_0Q_0R_0}{Z^2R_0^{\,2}},\ = \frac{P_0Q_0}{Z^2R_0}.$$

Substituting the values

$$P_0 = \Pi (z-l)^{p-1},\ \ Q_0 = \Pi (z-m)^{q-1},\ \ R_0 = \Pi (z-n)^{r-1},\ \ Z = \Pi (z-n),$$

we have

$$\frac{\xi''}{\xi'} = \Sigma \frac{p-1}{z-l} + \Sigma \frac{q-1}{z-m} - \Sigma \frac{r+1}{z-n},$$

and thence

$$\{x, z\} = \left\{-\Sigma \frac{p-1}{(z-l)^2} - \Sigma \frac{q-1}{(z-m)^2} + \Sigma \frac{r+1}{(z-n)^2}\right\}$$
$$-\tfrac{1}{2}\left\{\Sigma \frac{p-l}{z-l} + \Sigma \frac{q-1}{z-m} - \Sigma \frac{r+1}{z-n}\right\}^2;$$

or say

$$= \left(-\frac{p-1}{(z-l)^2} - \frac{p_1-1}{(z-l_1)^2} \cdots - \frac{q-1}{(z-m)^2} - \frac{q_1-1}{(z-m_1)^2} \cdots + \frac{r+1}{(z-n)^2} + \frac{r_1+1}{(z-n_1)^2} \cdots\right)$$
$$-\tfrac{1}{2}\left(\frac{p-1}{z-l} + \frac{p_1-1}{z-l_1} \cdots + \frac{q-1}{z-m} + \frac{q_1-1}{z-m_1} \cdots - \frac{r+1}{z-n} - \frac{r_1+1}{z-n_1} \cdots\right)^2,$$

where it is to be observed that

$$\Sigma(p-1)+\Sigma(q-1)-\Sigma(r+1),\ =\delta-\sigma_1+\delta-\sigma_2-(\delta+\sigma_3),\ =\delta-\sigma_1-\sigma_2-\sigma_3,\ =-2;$$

consequently the function is diaphoric, and therefore curtate.

It is to be remarked that the function, although presenting itself in a form unsymmetric in regard to the factors of P and Q, and of R, is really symmetric as regards the three sets of factors; this is obvious *à priori*, and it will be presently verified.

23. Calculation of the second term

$$x'^2\left(\text{a, b, c} \,\therefore\!\right)\!\!\left(\frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2.$$

We have

$$f(x-a),\quad g(x-b),\quad h(x-c)=\Omega P,\ \Omega Q,\ \Omega R,$$

where Ω is a determinate function of z. Hence

$$\frac{x'}{x-a},\ \frac{x'}{x-b},\ \frac{x'}{x-c}=\frac{P'}{P}+\frac{\Omega'}{\Omega},\ \frac{Q'}{Q}+\frac{\Omega'}{\Omega},\ \frac{R'}{R}+\frac{\Omega'}{\Omega},$$

and then substituting these values, by reason that the function is diaphoric, the terms in $\frac{\Omega'}{\Omega}$ disappear, and we have

$$x'^2\left(\text{a, b, c} \,\therefore\!\right)\!\!\left(\frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2$$

$$=\left(\text{a, b, c} \,\therefore\!\right)\!\!\left(\frac{P'}{P},\ \frac{Q'}{Q},\ \frac{R'}{R}\right)^2,$$

which is

$$=\left(\text{a, b, c} \,\therefore\!\right)\!\!\left(\Sigma\frac{p}{z-l},\ \Sigma\frac{q}{z-m},\ \Sigma\frac{r}{z-n}\right)^2.$$

We have $\Sigma p=\Sigma q=\Sigma r,\ =\delta$: and hence by what precedes, this function considered as a function of the inverts $\frac{1}{z-l}$, &c., is diaphoric, and therefore curtate.

24. We have therefore

$$\{x, z\}+x'^2\left(\text{a, b, c} \,\therefore\!\right)\!\!\left(\frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2=$$

$$\left(-\Sigma\frac{p-1}{(z-l)^2}-\Sigma\frac{q-1}{(z-m)^2}+\Sigma\frac{r+1}{(z-n)^2}\right)$$

$$-\tfrac{1}{2}\left(\Sigma\frac{p-1}{z-l}+\Sigma\frac{q-1}{z-m}-\Sigma\frac{r+1}{z-n}\right)^2$$

$$+\left(\text{a, b, c} \,\therefore\!\right)\!\!\left(\Sigma\frac{p}{z-m},\ \Sigma\frac{q}{z-m},\ \Sigma\frac{r}{z-n}\right)^2,$$

where the whole function on the right hand is curtate.

25. We have to bring the function on the right hand into the reduced form

$$\frac{a}{(z-\alpha)^2}+\ldots+\frac{A}{z-\alpha}+\ldots$$

for the purpose of getting rid of the non-essential inverts (if any).

We write

$$\Sigma\,\frac{p-1}{(z-l)^2}=\frac{p-1}{(z-l)^2}+\frac{p_1-1}{(z-l_1)^2}+\ldots$$

$$=\frac{p-1}{(z-l)^2}+\Sigma'\,\frac{p_1-1}{(z-l_1)^2},$$

viz. $z-l$ here denotes any particular factor, and $z-l_1$ represents any other factor of the same set; and so in other like cases.

26. The whole coefficient of $\dfrac{1}{(z-l)^2}$ is

$$-(p-1)-\tfrac{1}{2}(p-1)^2+\mathrm{a}p^2,\quad =\tfrac{1}{2}(1-p^2)+\mathrm{a}p^2\,;$$

an expression which, regarded as a function of a and p, is represented by $(\mathrm{a}p)$: the parentheses are used only to avoid ambiguity, and are omitted when p is a number, thus $\mathrm{a}1=\mathrm{a}$, $\mathrm{a}2=-\frac{3}{8}+4\mathrm{a}$, and so in other cases.

27. The whole term in $\dfrac{1}{z-l}$ comes from

$$-\frac{p-1}{z-l}\left(\Sigma'\,\frac{p_1-1}{z-l_1}+\Sigma\,\frac{q-1}{z-m}-\Sigma\,\frac{r+1}{z-n}\right)$$

$$+\frac{p}{z-l}\left(2\mathrm{a}\Sigma'\,\frac{p_1}{z-l}+(-\mathrm{a}-\mathrm{b}+\mathrm{c})\,\Sigma\,\frac{q}{z-m}+(-\mathrm{a}+\mathrm{b}-\mathrm{c})\,\Sigma\,\frac{r}{z-n}\right),$$

viz. each term such as $\dfrac{1}{z-l\,.\,z-l_1}$ is to be replaced by $\dfrac{1}{l-l_1}\left(\dfrac{1}{z-l}-\dfrac{1}{z-l_1}\right)$, giving rise to the term $\dfrac{1}{l-l_1}\dfrac{1}{z-l}$, or contributing the term $\dfrac{1}{l-l_1}$ to the coefficient of $\dfrac{1}{z-l}$. The whole coefficient thus is

$$=-(p-1)\left(\Sigma'\,\frac{p_1-1}{l-l_1}+\Sigma\,\frac{q-1}{l-m}-\Sigma\,\frac{r+1}{l-n}\right)$$

$$+2\mathrm{a}p\Sigma'\,\frac{p_1}{l-l_1}+p\,(-\mathrm{a}-\mathrm{b}+\mathrm{c})\,\Sigma\,\frac{q}{l-m}+p\,(-\mathrm{a}+\mathrm{b}-\mathrm{c})\,\Sigma\,\frac{r}{l-n}.$$

28. Suppose first that $z-l$ is a multiple factor of P, viz. a factor with an index p greater than 1: then for $z=l$ we have $Q+R=0$, $Q'+R'=0$, and thence $\dfrac{Q'}{Q}=\dfrac{R'}{R}$, that is $\Sigma\,\dfrac{q}{l-m}=\Sigma\,\dfrac{r}{l-n}$. We have therefore

$$p\,(-\mathrm{a}-\mathrm{b}+\mathrm{c})\,\Sigma\frac{q}{l-m}+p\,(-\mathrm{a}+\mathrm{b}-\mathrm{c})\,\Sigma\frac{r}{l-n}$$

$$=-\,\mathrm{a}p\left(\Sigma\frac{q}{l-m}+\Sigma\frac{r}{l-m}\right);$$

and moreover in the top line the terms $\Sigma\frac{q}{l-m}$ and $-\Sigma\frac{r}{l-n}$ destroy each other. The whole coefficient of $\frac{1}{z-l}$ when $(z-l)$ is a multiple factor of P, thus is

$$=-\,(p-1)\left(\Sigma'\frac{p_1-1}{l-l_1}-\Sigma\frac{1}{l-m}-\Sigma\frac{1}{l-n}\right)$$

$$+\,2\mathrm{a}p\Sigma'\frac{p_1}{l-l_1}-\mathrm{a}p\left(\Sigma\frac{q}{l-m}+\Sigma\frac{r}{l-n}\right),$$

a form which is now symmetrical in regard to the inverts $\frac{1}{l-m}$ and $\frac{1}{l-n}$.

29. The value just obtained is

$$=(1-p^2+2\mathrm{a}p^2)\left(\Sigma\frac{\frac{1}{2}q-1}{l-m}+\Sigma\frac{\frac{1}{2}r-1}{l-m}-\Sigma'\frac{1}{l-l_1}\right),$$

viz. comparing the two forms and reducing, they will be identical if only

$$(1-p+2\mathrm{a}p)\left\{\Sigma'\frac{p+p_1}{l-l_1}-\Sigma\frac{\frac{1}{2}(1+p)\,q-p}{l-m}-\Sigma\frac{\frac{1}{2}(1+p)\,r-p}{l-n}\right\}=0,$$

and it can be shown that the function inside the $\{\}$ is in fact $=0$.

30. We have as before $\Sigma\frac{q}{l-m}=\Sigma\frac{r}{l-m}$, or writing each of these quantities $=\Phi$, the equation to be verified is

$$\Sigma'\frac{p+p_1}{l-l_1}=(p+1)\,\Phi-p\Sigma\frac{1}{l-m}-p\Sigma\frac{1}{l-n}.$$

We have
$$\frac{P'}{P}=\frac{p}{z-l}+\Sigma'\frac{p_1}{z-l_1},\ =\frac{X_1}{X},$$

that is
$$\Sigma'\frac{p_1}{l-l_1}=\left[\frac{X_1}{X}-\frac{p}{z-l}\right]\text{ for }z=l,$$

$$=\left[\frac{X_1(z-l)-pX}{X(z-l)}\right].$$

The first derived function of the numerator is $X_1'(z-l)+X_1-pX'$, which for $z=l$ is X_1-pX', which is $=0$; and for the denominator it is $X'(z-l)+X$, which is also $=0$; passing to the second derived functions we find

$$\Sigma'\frac{p_1}{z-l_1}=\frac{2X_1'-pX''}{2X'},\ =\frac{X_1'-\frac{1}{2}pX''}{X'}.$$

From the equation
$$\frac{X'}{X}=\frac{1}{z-l}+\Sigma'\frac{1}{z-l}$$
we find in like manner,
$$\Sigma'\frac{1}{l-l_1}=\frac{\frac{1}{2}X''}{X'},$$
and we thence obtain (z being always $=l$)
$$\Sigma'\frac{p+p_1}{z-l_1}=\frac{X_1'}{X'},$$
so that the equation to be verified becomes
$$\frac{X_1'}{X_1}=(p+1)\,\Phi-p\Sigma\frac{1}{l-m}-p\Sigma\frac{1}{l-n}.$$

31. But from the equation $\Theta, =PQ'-P'Q, =KP_0Q_0R_0$, we find $XY_1-X_1Y=KR_0$, and then, differentiating, $XY_1'+X'Y_1-X_1'Y-X_1Y'=KR_0'$: writing in these equations $z=l$, they become
$$-X_1Y=KR_0,$$
$$X'Y_1-X_1'Y-X_1Y'=KR_0,$$
and dividing the second by the first
$$-\frac{X'}{X_1}\cdot\frac{Y_1}{Y}+\frac{X_1'}{X_1}+\frac{Y'}{Y}=\frac{R_0'}{R_0};$$
or recollecting that $X_1=pX'$, and $\frac{Y_1}{Y}=\frac{Q'}{Q}$, we have
$$\frac{X_1'}{X'}=p\left(\frac{R_0'}{R_0}-\frac{Y'}{Y}\right)+\frac{Q'}{Q},$$
that is
$$\frac{X_1'}{X'}=p\left(\Sigma\frac{r-1}{l-n}-\Sigma\frac{1}{l-m}\right)+\Sigma\frac{q}{l-m},$$
$$=(p+1)\,\Phi-p\Sigma\frac{1}{l-m}-p\Sigma\frac{1}{l-n},$$
the required relation.

32. The result is that $z-l$ being a multiple factor of P the coefficient of the term $\frac{1}{z-l}$ is
$$=(1-p^2+2\mathrm{a}p^2)\left\{\Sigma\frac{\frac{1}{2}q-1}{l-m}+\Sigma\frac{\frac{1}{2}r-1}{l-n}-\Sigma'\frac{1}{l-l_1}\right\},$$
$$=2\,(\mathrm{a}p)\left[\tfrac{1}{2}\left(\frac{Q'}{Q}+\frac{R'}{R}\right)-\frac{Y'}{Y}-\frac{Z'}{Z}-\tfrac{1}{2}\frac{X''}{X'}\right].$$

33. In the case where $z-l$ is a simple factor of P we have $p=1$, and the coefficient is
$$=2\mathrm{a}\Sigma'\frac{p_1}{l-l_1}+(-\mathrm{a}-\mathrm{b}+\mathrm{c})\,\Sigma\frac{q}{l-m}+(-\mathrm{a}+\mathrm{b}-\mathrm{c})\,\Sigma\frac{r}{l-m},$$
$$=\mathrm{a}\left(2\Sigma'\frac{p_1}{l-l_1}-\Sigma\frac{q}{l-m}-\Sigma\frac{r}{l-n}\right)-(\mathrm{b}-\mathrm{c})\left(\Sigma\frac{q}{l-m}-\Sigma\frac{r}{l-n}\right).$$

34. Of course the formulæ for the coefficients of $\dfrac{1}{(z-l)^2}$ and $\dfrac{1}{z-l}$ give at once by a mere change of letters those for the coefficients of $\dfrac{1}{(z-m)^2}$, $\dfrac{1}{z-m}$, and $\dfrac{1}{(z-n)^2}$, $\dfrac{1}{z-n}$; and the function in question,

$$\{x, z\} + x'^2\left(\text{a, b, c} \;\therefore\!\!\left\langle \frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2\right.,$$

is now obtained in the required form,

$$= \frac{(\text{a}p)}{(z-l)^2}\cdots + \frac{(\text{b}q)}{(z-m)^2}\cdots + \frac{(\text{c}r)}{(z-n)^2}\cdots + \frac{A}{z-l}\cdots + \frac{B}{z-m}\cdots + \frac{C}{z-n}\cdots$$

where $(\text{a}p)$ denotes $\frac{1}{2}(1-p^2)+\text{a}p^2$, and the like for $(\text{b}q)$ and $(\text{c}r)$; and where $z-l$ being a multiple factor of P, the coefficient A contains the factor $(\text{a}p)$; and similarly for B and C.

35. Suppose that the coefficients a, b, c are no one of them $=0$; we have a1, $=$a, which does not vanish; that is, $z-l$ being a simple factor of P, the expression contains $\dfrac{1}{(z-l)^2}$, or the invert $\dfrac{1}{z-l}$ is essential: and similarly $z-m$ being a simple factor of Q, or $z-n$ a simple factor of R, the inverts $\dfrac{1}{z-m}$ and $\dfrac{1}{z-n}$ are essential. But for $z-l$ a multiple factor of P, the coefficient $(\text{a}p)$ of the term $\dfrac{1}{(z-l)^2}$ may vanish, viz. this will be the case if $\text{a}=\frac{1}{2}\left(1-\dfrac{1}{p^2}\right)$; and when this is so the coefficient A of the corresponding term $\dfrac{1}{z-l}$ also vanishes; that is $\dfrac{1}{z-l}$ is a non-essential invert. And similarly for any multiple factor $z-m$ of Q or $z-n$ of R, the invert $\dfrac{1}{z-m}$ or $\dfrac{1}{z-n}$ may be non-essential.

36. If P, Q, R contain each of them only multiple factors of the same index, say of the indices p, q, r for the three functions respectively, viz. if the functions are $F(\Pi(z-l))^p$, $G(\Pi(z-m))^q$, $H(\Pi(z-n))^r$, the result contains only the six terms written down: and then if a, b, c are $=\frac{1}{2}\left(1-\dfrac{1}{p^2}\right)$, $\frac{1}{2}\left(1-\dfrac{1}{q^2}\right)$, $\frac{1}{2}\left(1-\dfrac{1}{r^2}\right)$ respectively the result is $=0$: viz. we then have

$$\{x, z\} + x'^2\left(\text{a, b, c} \;\therefore\!\!\left\langle \frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right) = 0,\right.$$

or we in fact have for the values in question of (a, b, c) a solution

$$f(x-a) : g(x-b) : h(x-c) = P : Q : R$$

of this differential equation of the third order.

37. The reasoning applies directly to lines 2, 3, 4, 5 of the *PQR*-Table: and with a slight variation to line 1; viz. here the factors of $R\,(=-1+z^n)$ are all simple factors, but in virtue of $\text{c}=0$ and $\text{a}=\text{b}$, the corresponding inverts disappear, and, the other inverts also disappearing, the value of the function is $=0$. Hence lines 1, 2, 3, 4, 5 of the *PQR*-Table give each of them a result $=0$, for the values of (a, b, c) appearing by the table itself, and shown explicitly in the corresponding line of the Annex.

Thus line 3 shows that the function x determined by

$$f(x-a) : g(x-b) : h(x-c) = (z^4+2\sqrt{-3}\,z^2+1)^3 : -12\sqrt{-3}\,(z^5-z)^2 : -(z^4-2\sqrt{-3}\,z^2+1)^3$$

satisfies

$$\{x,\ z\} + x'^2\left(\frac{4}{9},\ \frac{3}{8},\ \frac{4}{9}\ \therefore\!\!\bigg\rangle\ \frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2 = 0,$$

and so for any other of the five lines.

38. The indices of the factors of P, Q, R may be such that for proper values of the coefficients (a, b, c) there are in all only three essential inverts, say $\frac{1}{z-a_1}$, $\frac{1}{z-b_1}$, $\frac{1}{z-c_1}$, belonging to the three functions P, Q, R respectively, or it may be two, or three, of them to the same function. When this is so, the function of these inverts is by what precedes a curtate function, and it is consequently a function

$$\left(\text{a}_1,\ \text{b}_1,\ \text{c}_1\ \therefore\!\!\bigg\rangle\ \frac{1}{z-a_1},\ \frac{1}{z-b_1},\ \frac{1}{z-c_1}\right)^2,$$

where a_1, b_1, c_1 are the values of the three which do not vanish in the series of expressions (ap), (bq), (cr).

The remaining lines (III, V, VII, VIII) and IX to XV of the *PQR*-Table give such values of P, Q, R, the values of (a, b, c), and the calculation of the values of $(\text{a}_1,\ \text{b}_1,\ \text{c}_1)$ being shown by the corresponding lines of the Annex. And we have thus values of x determined by the equations

$$f(x-a) : g(x-b) : h(x-c) = P : Q : R,$$

and giving

$$\{x,\ z\} + x'^2\left(\text{a},\ \text{b},\ \text{c}\ \therefore\!\!\bigg\rangle\ \frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2 = \left(\text{a}_1,\ \text{b}_1,\ \text{c}_1\ \therefore\!\!\bigg\rangle\ \frac{1}{z-a_1},\ \frac{1}{z-b_1},\ \frac{1}{z-c_1}\right)^2.$$

39. For instance, from line IX we have

$$f(x-a) : g(x-b : h(x-c) = (z-4)^3 : -(z-1)(z+8)^2 : 27z^2\left(1-\frac{z}{\infty}\right),$$

the values of (a, b, c) are $\frac{4}{9}$, $\frac{3}{8}$, $\frac{12}{25}$; and since P, Q, R contain factors with the exponents 3; 1, 2; and 1, 2 respectively, the coefficients which present themselves on the right hand are

$$\text{a}3;\ \text{b}1,\ \text{b}2;\ \text{c}1,\ \text{c}2,$$

which are $=0;\ \frac{3}{8},\ 0;\ \frac{12}{25},\ \frac{21}{50}$ respectively.

Hence writing $\text{a}_1, \text{b}_1, \text{c}_1 = \frac{3}{8}, \frac{12}{25}, \frac{21}{50}$ the corresponding inverts are $\frac{1}{z-1}, \frac{1}{z-\infty}, \frac{1}{z}$, and the result is

$$\{x, z\} + x'^2\left(\frac{4}{9}, \frac{3}{8}, \frac{12}{25} \therefore \right)\!\!\left(\frac{1}{x-a}, \frac{1}{x-b}, \frac{1}{x-c}\right)^2 = \left(\frac{3}{8}, \frac{12}{25}, \frac{21}{50} \therefore \right)\!\!\left(\frac{1}{z-1}, \frac{1}{z-\infty}, \frac{1}{z}\right)^2.$$

40. It is hardly necessary to remark that an expression

$$\left(\text{a}_1, \text{b}_1, \text{c}_1 \therefore \right)\!\!\left(\frac{1}{z-a_1}, \frac{1}{z-b_1}, \frac{1}{z-\infty}\right)^2$$

in fact denotes $\frac{\text{a}_1}{(z-a_1)^2} + \frac{\text{b}_1}{(z-b_1)^2} + \frac{-\text{a}_1 - \text{b}_1 + \text{c}_1}{(z-a_1)(z-b_1)}$.

The particular form of the z inverts is immaterial; we could by a general linear transformation upon the z make them to be $\frac{1}{z-a_1}, \frac{1}{z-b_1}, \frac{1}{z-c_1}$ with the (a_1, b_1, c_1) arbitrary; or we can give to the a_1, b_1, c_1 any particular values we please: there would be a propriety in making the inverts to be in every case (as in the foregoing example) $\frac{1}{z}, \frac{1}{z-\infty}, \frac{1}{z-1}$; but the numerical work would be troublesome, and it is not worth while to effect it.

41. The conclusion is that lines (III, V, VII, VIII) and IX to XV of the *PQR*-Table, give for determinate values of (a, b, c) and ($\text{a}_1, \text{b}_1, \text{c}_1$) solutions

$$f(x-a) : g(x-b) : h(x-c) = P : Q : R$$

of the equation

$$\{x, z\} + x'^2\left(\text{a}, \text{b}, \text{c} \therefore \right)\!\!\left(\frac{1}{x-a}, \frac{1}{x-b}, \frac{1}{x-c}\right)^2 = \left(\text{a}_1, \text{b}_1, \text{c}_1 \therefore \right)\!\!\left(\frac{1}{z-a_1}, \frac{1}{z-b_1}, \frac{1}{z-z_1}\right)^2,$$

where a, b, c, a_1, b_1, c_1 are or can be made arbitrary, but without any real gain of generality herein. This is the Differential Equation $\{x, z\}$.

42. Recurring to the results from the Arabic lines of the *PQR*-Table, but for convenience writing s instead of z, we have

$$f(x-a) : g(x-b) : h(x-c) = P : Q : R$$

(where P, Q, R are now functions of s), a solution of

$$\{x, s\} + \left(\frac{dx}{ds}\right)^2\left(\text{a}, \text{b}, \text{c} \therefore \right)\!\!\left(\frac{1}{x-a}, \frac{1}{x-b}, \frac{1}{x-c}\right)^2 = 0.$$

But we have

$$\{s,\ x\} = -\left(\frac{ds}{dx}\right)^2 \{x,\ s\},$$

and the foregoing is therefore a solution of

$$\{s,\ x\} = \left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\!\!\big)\!\!\big(\frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2,$$

a differential equation of the third order; this is the Differential Equation $\{s,\ x\}$.

43. From the Roman lines if we assume

$$f(x-a) : g(x-b) : h(x-c) = \mathfrak{P} : \mathfrak{Q} : \mathfrak{R}$$

(where $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$ are functions of z, not the same functions that P, Q, R are of s, since they belong to a different line of the Table): we have as before

$$\{x,\ z\} + \left(\frac{dx}{dz}\right)^2 \left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\!\!\big)\!\!\big(\frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2 = \left(\mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1\ \therefore\!\!\big)\!\!\big(\frac{1}{z-a_1},\ \frac{1}{z-b_1},\ \frac{1}{z-c_1}\right)^2.$$

44. We may combine any such result with a properly selected result of the preceding system, the two results being such that (a, b, c) have the same values in each of them. (See as to this the foot-note referring to the Annex to the PQR-Table.) The last equation then becomes

$$\{x,\ z\} + \left(\frac{dx}{dz}\right)^2 \{s,\ x\} = \left(\mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1\ \therefore\!\!\big)\!\!\big(\frac{1}{z-a_1},\ \frac{1}{z-b_1},\ \frac{1}{z-c_1}\right)^2,$$

or since $\{x,\ z\} + \left(\frac{dx}{dz}\right)^2 \{s,\ x\} = \{s,\ z\}$, this is

$$\{s,\ z\} = \left(\mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1\ \therefore\!\!\big)\!\!\big(\frac{1}{z-a_1},\ \frac{1}{z-b_1},\ \frac{1}{z-c_1}\right)^2,$$

the corresponding relation between s, z being of course obtained by the elimination of x from the two sets of equations

$$f(x-a) : g(x-b) : h(x-c) = P : Q : R, \quad \text{and } f(x-a) : g(x-b) : h(x-c) = \mathfrak{P} : \mathfrak{Q} : \mathfrak{R};$$

viz. the required relation is

$$P : Q : R = \mathfrak{P} : \mathfrak{Q} : \mathfrak{R}$$

(where P, Q, R are functions of s; $\mathfrak{P} : \mathfrak{Q} : \mathfrak{R}$ functions of z; and in virtue of

$$P + Q + R = 0, \quad \mathfrak{P} + \mathfrak{Q} + \mathfrak{R} = 0$$

the relations are equivalent to a single equation between z and s). And writing finally x in place of z, that is now considering $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$ as functions of x, we have

$$\mathfrak{P} : \mathfrak{Q} : \mathfrak{R} = P : Q : R$$

as a solution of

$$\{s,\ x\} = \left(\mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1\ \therefore\!\!\big)\!\!\big(\frac{1}{x-a_1},\ \frac{1}{x-b_1},\ \frac{1}{x-c_1}\right)^2,$$

a differential equation of the third order of the foregoing form $\{s, x\}$ = given function of x, but with different values of the coefficients, (a_1, b_1, c_1) instead of (a, b, c).

45. It thus appears that there are in all 16 sets of values of (a, b, c), for which the equation is solved, viz. the 16 sets of values are shown in the right-hand column of the Annex. For greater clearness I exhibit the integral equations as follows:

	Functions of x.		Functions of s.		
1	$f(x-a) : g(x-b) : h(x-c)$	=	$P : Q : R$	(1)	Polygon
I	„	=	„	(2)	Double Pyramid
II	„	=	„	(3)	Tetrahedron
III	$4x : -(x+1)^2 : (x-1)^2$	=	„	(3)	„
IV	$f(x-a) : g(x-b) : h(x-c)$	=	„	(4)	Cube and Octahedron
V	$(x-1)^2 : -(x+1)^2 : 4x$	=	„	(4)	„
VI	$f(x-a) : g(x-b) : h(x-c)$	=	„	(5)	Dodecahedron and Icosahedron
VII	$4x : -(x+1)^2 : (x-1)^2$	=	„	(5)	„
VIII	$(x-1)^2 : -(x+1)^2 : 4x$	=	„	(5)	„
IX	$P : Q : R$ (IX)	=	„	(5)	„
X	„ (X)	=	„	(5)	„
XI	„ (XI)	=	„	(5)	„
XII	„ (XII)	=	„	(5)	„
XIII	„ (XIII)	=	„	(5)	„
XIV	„ (XIV)	=	„	(5)	„
XV	„ (XV)	=	„	(5)	„

The values of the P, Q, R as functions of x, or of s, are taken out of the PQR-Table: only in the lines III, V, VII, VIII, where P, Q, R are given as

$$=4z, \quad -(z+1)^2, \quad (z-1)^2,$$

and where, as regards V and VIII, there is a transposition of P and R, I have inserted the actual values of the x-functions. (See as to this the foot-note referring to the Annex.)

The Schwarzian Theory. Article, Nos. 46 to 62.

46. Considering the foregoing equation

$$\{s, x\} = \left(a_1, b_1, c_1 \therefore \right)\left(\frac{1}{x-a_1}, \frac{1}{x-b_1}, \frac{1}{x-c_1}\right)^2$$

as a particular case of the equation $\{s, x\}$ = Rational function of x, $= R(x)$ suppose, then we have in 1, I, II, IV, VI solutions of the form x = Rational function of s.

Consider in general a solution of this form, $x = F(s)$ a rational function of s: s is then an irrational function of x, and if s_1, s_2 are any two of its values $\{s_1, x\} = R(x)$, $\{s_2, x\} = R(x)$; that is $\{s_2, x\} = \{s_1, x\}$, and therefore (ante, No. 7) $s_2 = \dfrac{as_1 + b}{cs_1 + d}$. And then $x = F(s_2) = F\left(\dfrac{as_1 + b}{cs_1 + d}\right)$, $= F(s_1)$: viz. $F(s)$ is a rational function of s transformable into itself by the transformation s into $\dfrac{as + b}{cs + d}$: and it is moreover clear that between any two roots s whatever of the equation $x = F(s)$ there exists a homographic relation of the form in question. It is moreover clear that these homographic transformations form a group; and consequently that $F(s)$ is a rational function of s transformable into itself by the several homographic transformations of a group of such transformations: viz. taking x to be a rational function of x, it is *only* in the case $x = F(s)$, a function of the form in question, that $\{s, x\}$ can be equal to a rational function of x.

47. We may in any equation between x and s consider these as imaginary variables $p + qi$ and $u + vi$ respectively; considering then (p, q) and (u, v) as rectangular co-ordinates of points in different planes, we have a first plane the locus of the points x, and a second plane the locus of the points s: there is between the two planes a correspondence which is in fact the correspondence of conformable figures: to the infinitesimal element dx drawn from a point x of the first figure corresponds an infinitesimal element ds drawn from the corresponding point s of the second figure, and which elements are in general connected by an equation of the form $ds = (a + bi)\,dx$ (a and b functions of x or s); and this signifies that to obtain the pencil of infinitesimal elements or radii ds proceeding in different directions from the point s, we alter in a determinate ratio the absolute lengths of the infinitesimal elements or radii proceeding from the corresponding point x, and rotate the pencil through a determinate angle: this ratio and angle of rotation, or say, the Auxesis and Streblosis, being of course variable from point to point. Or, what comes to the same thing, if dx and d_1x be consecutive elements of the path of the point x, and ds, d_1s the corresponding consecutive elements of the path of the point s, then the ratio of the lengths of the elements dx, d_1x is equal to that of the lengths of the elements ds, d_1s; and the mutual inclination of the first pair of elements is equal to that of the second pair of elements. In particular if at any point the path of x is a curved line without abrupt change of direction, then at the corresponding point the path of s is a curved line without abrupt change of direction. In what precedes we have the relation at ordinary points, but there may be critical corresponding points (x, s), the relation at a critical point between the corresponding elements dx, ds being of the form $ds = (a + bi)(dx)^\lambda$, ($\lambda$ a positive integer or fraction): here the angle between two elements ds is $= \lambda$ times that between the two elements dx; or, if the path of the point x through the critical point is without abrupt change of direction (say if the angle between the two consecutive elements is the flat angle π) then the angle between the two consecutive elements ds is $= \lambda\pi$: viz. there is in the path of the point s an abrupt change of direction.

48. I consider the foregoing equation $\{s, x\} = R(x)$, where $R(x)$ is a rational function, and is now taken to be a real function of x: we may assume $s' = \rho'\theta' e^{i\theta}$, where the accents denote differentiation in regard to x, and where ρ', θ, and therefore also θ', are real functions of x. We have

$$\frac{s''}{s'} = \frac{\rho''}{\rho'} + \frac{\theta''}{\theta'} + i\theta',$$

and thence

$$\frac{s'''}{s'} - \left(\frac{s''}{s'}\right)^2 = \frac{\rho'''}{\rho'} - \left(\frac{\rho''}{\rho'}\right)^2 + \frac{\theta'''}{\theta'} - \left(\frac{\theta''}{\theta'}\right)^2 \qquad + i\theta''$$

$$-\tfrac{1}{2}\left(\frac{s''}{s'}\right)^2 = \quad -\tfrac{1}{2}\left(\frac{\rho''}{\rho'}\right)^2 \quad -\tfrac{1}{2}\left(\frac{\theta''}{\theta'}\right)^2 + \tfrac{1}{2}\theta'^2 - \frac{\rho''\theta''}{\rho'\theta'} - i\theta'' - i\frac{\rho''\theta'}{\rho'},$$

and thence

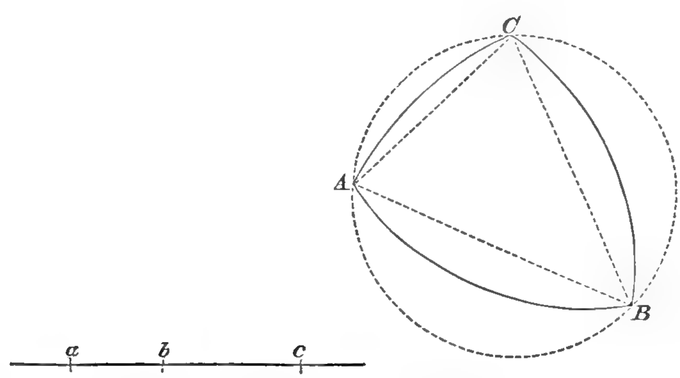
$$\{s, x\} = \{\rho, x\} + \{\theta, x\} + \tfrac{1}{2}\theta'^2 - \frac{\rho''\theta''}{\rho'\theta'} - i\frac{\rho''\theta'}{\rho'}.$$

Putting this $= R(x)$, and assuming that x is real, we have

$$\{\rho, x\} + \{\theta, x\} + \tfrac{1}{2}\theta'^2 - \frac{\rho''\theta''}{\rho'\theta'} = R(x); \quad 0 = i\frac{\rho''\theta'}{\rho'}.$$

The last equation gives $\rho''\theta' = 0$, that is $\theta' = 0$, which gives $s' = 0$, and may be disregarded; or else $\rho'' = 0$, therefore ρ', a real constant, $= \gamma$ suppose, and $\{\rho, x\} = 0$: hence for the solution of the equation $\{s, x\} = R(x)$, we have $s' = \gamma\theta' e^{i\theta}$, θ a real quantity determined by $\{\theta, x\} + \tfrac{1}{2}\theta'^2 = R(x)$: and then integrating the equation for s' we have $s = \alpha + \beta i + \gamma e^{i\theta}$, α, β, γ real constants.

49. The conclusion is that if $\{s, x\} = R(x)$, a real function of x, and if x be real, that is if the point x move along a right line (say the x-line) then $s = \alpha + \beta i + \gamma e^{i\theta}$ (θ, and the constants α, β, γ, being real), that is the point s moves in a circle, coordinates of the centre α, β, and radius $= \gamma$.



50. Suppose a, b, c are any real values of x representing points a, b, c on the x-line; and A, B, C any given imaginary values of s representing points A, B, C in the s-plane:

since $\{s, x\} = R(x)$ is a differential equation of the third order, the integral contains three arbitrary constants, and we may imagine these so determined that to the values $x = a, b, c$ shall correspond the values $s = A, B, C$ respectively.

If there is not on the x-line any critical point, as the point x moves continuously along this line the point s will move continuously along a circle, which (inasmuch as a, b, c and A, B, C are corresponding points) must be the circle through the three points A, B, C*.

51. If however the points a, b, c are critical points, such that the element ds at the corresponding points A, B, C are equal to multiples of $(dx)^\lambda$, $(dx)^\mu$, $(dx)^\nu$ respectively, then to the flat angles π at a, b, c correspond in the path of s the angles $\lambda\pi, \mu\pi, \nu\pi$ at the points A, B, C respectively: and (assuming that a, b, c are the only critical points on the x-line) the path of s is made up of the three circular arcs CA, AB, BC meeting at angles $\lambda\pi, \mu\pi, \nu\pi$ respectively. The arcs are completely determined by these conditions; for supposing the arc BC to make with the chord BC, at the points B and C, the angles f, f, and similarly the arcs CA and AB to make with the corresponding chords the angles g, g and h, h, then the conditions give $\lambda\pi, \mu\pi, \nu\pi = \angle A + g + h, \ \angle B + h + f, \ \angle C + f + g$, where the angles referred to are those of the rectilinear triangle ABC: we have thus the values of f, g, h; and the arc BC is the arc on the chord BC meeting it at angles f, f: and the like as regards the arcs CA and AB respectively.

52. The foregoing equation

$$\{s, x\} = \left(\mathrm{a}, \mathrm{b}, \mathrm{c} \ \therefore\right)\!\left(\frac{1}{x-a}, \ \frac{1}{x-b}, \ \frac{1}{x-c}\right)^2,$$

where a, b, c have the values $\tfrac{1}{2}(1-\lambda^2)$, $\tfrac{1}{2}(1-\mu^2)$, $\tfrac{1}{2}(1-\nu^2)$ (λ, μ, ν being real and positive), has $x = a, b, c$ for critical points of the kind in question: in fact, writing $x - a = h$, the equation is of the form

$$\{s, h\} = \frac{\tfrac{1}{2}(1-\lambda^2)}{h^2} + \frac{a_0}{h} + a_1 + a_2 h + \dots$$

which is satisfied by

$$\frac{d}{dh} \log \frac{ds}{dh} = -\frac{1+\lambda}{h} + b_0 + b_1 h + b_2 h^2 + \dots$$

and we thence obtain an integral of the form

$$s = kh^{-\lambda}(1 + k_1 h + k_2 h^2 + \dots), \ = k\phi \text{ for shortness.}$$

This is a particular integral, but we have from it the general integral

$$s = \frac{\alpha + \beta k\phi}{\gamma + \delta k\phi},$$

* Since there is no critical point on the x-line there can be no abrupt change of direction in the path of s, that is the path of s cannot consist of circular arcs meeting at an angle: but it is in the text further assumed that the path of s cannot consist of different arcs of circle, the one continuing the other without any abrupt change of direction.

and if A be the value of s corresponding to $h=0$, then $\beta=\delta A$, and we find

$$s=\frac{\alpha+A\delta k\phi}{\gamma+\delta k\phi},\ =\left(A+\frac{\alpha}{\delta k\phi}\right)\left(1+\frac{\gamma}{\delta k\phi}\right),\ =A+\frac{\alpha-\gamma A}{\delta k}\frac{1}{\phi}+\ldots$$

viz. reducing $\frac{1}{\phi}$ to its principal term h^λ, and then writing ds, dx for $s-A$, and $h\,(=x-a)$ respectively, we have $ds=K(dx)^\lambda$, or $x=a$ is a critical point with the exponent λ; and similarly $x=b$ and $x=c$ are critical points with the exponents μ and ν respectively.

53. Hence in the equation

$$\{s,\ x\}=\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\!\left)\!\!\left(\frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2,\right.$$

as the point x, passing successively through a, b, c describes the x-line, the point s passing successively through A, B, C describes the sides AB, BC, CA of the curvilinear triangle ABC. To points x indefinitely near the x-line correspond points s indefinitely near the boundary AB, BC, CA of the triangle, viz. to points x indefinitely near to and on one side, suppose the upper side, of the x-line, correspond the points s indefinitely near to and within the boundary of the triangle: and in like manner to whole series of the points x on the same upper side of the x-line, correspond the whole series of points s inside the triangle.

54. We have attended so far only to one of the points s which correspond to a given point x, but considering the set of points s which correspond to the same point s, we have in the s-plane entire circles forming by their intersections curvilinear triangles ABC, ABC', &c.; we have thus two systems, say ABC, &c., and ABC', &c., of triangles, such that to a point x on the upper side of the x-line correspond points s, one of them within each of the triangles ABC, &c., and to a point x on the lower side of the x-line correspond points s, one of them within each of the triangles ABC', &c.; and so consequently that to the two half-planes on opposite sides of the x-line correspond the two sets of triangles ABC, &c., and ABC', &c., respectively.

55. In order that the relation s and x may be an algebraical one it is necessary that the two sets of triangles should completely cover, once or a finite number of times, the whole of the s-plane: and this implies that the angles $\lambda\pi$, $\mu\pi$, $\nu\pi$ have certain determinate values; and, in fact, that dividing the surface of a sphere into triangles, each with these angles, the curvilinear triangles ABC, ABC', &c., are the stereographic projections of these triangles. It was by such considerations as these that Schwarz, in the Memoir of 1875, p. 323, obtained the series of values I to XV of λ, μ, ν, giving for a, b, c, $=\tfrac{1}{2}(1-\lambda^2)$, $\tfrac{1}{2}(1-\mu^2)$, $\tfrac{1}{2}(1-\nu^2)$, the series of values mentioned in the Annex of the PQR-Table: and thus showed *à priori* that the equation

$$\{s,\ x\}=\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\!\left)\!\!\left(\frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2\right.$$

is algebraically integrable for these values of a, b, c; and only for these values, or for values reducible to them.

56. As an instance take the double pyramid form: the integral equation is

$$f(x-a) : g(x-b) : h(x-c) = 4s^n : -(s^n-1)^2 : (s^n+1)^2,$$

or say

$$\frac{(c-a)(x-b)}{(a-b)(x-c)} = -\frac{(s^n-1)^2}{(s^n+1)^2};$$

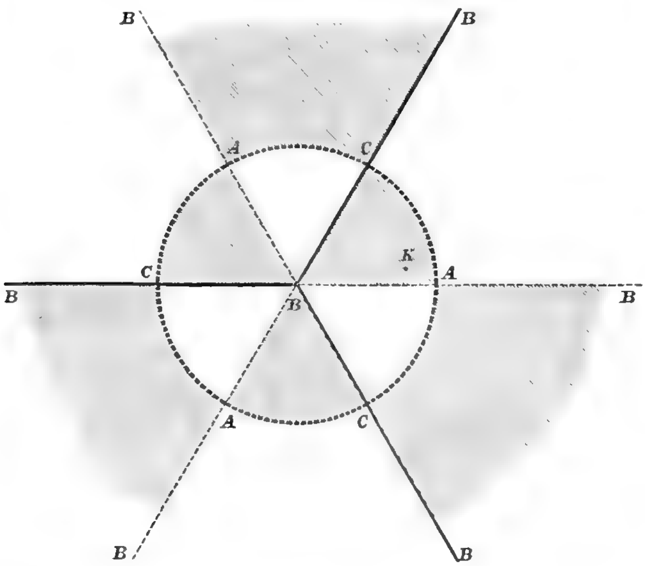
or if for greater simplicity we assume $a, b, c = 1, 0, \infty$, this is $x = \frac{(s^n-1)^2}{(s^n+1)}$, or say $-(s^n-1) = \sqrt{x}(s^n+1)$, that is, $s^n = \frac{1 \mp \sqrt{x}}{1 \pm \sqrt{x}}$, a solution of the differential equation

$$\{s, x\} = \left(\frac{3}{8}, \tfrac{1}{2}(1-n^{-2}), \tfrac{1}{2}(1-n^{-2}) \therefore \right)\left(\frac{1}{x}, \frac{1}{x-1}, \frac{1}{x-\infty}\right)^2.$$

In particular if $n=3$, we have $x = \left(\frac{s^3-1}{s^3+1}\right)$ or $s^3 = \frac{1 \mp \sqrt{x}}{1 \pm \sqrt{x}}$ an integral of

$$\{s, x\} = \left(\frac{3}{8}, \frac{4}{9}, \frac{4}{9} \therefore \right)\left(\frac{1}{x}, \frac{1}{x-1}, \frac{1}{x-\infty}\right)^2.$$

57. We have here the spherical surface divided by the equator and three meridians into twelve triangles, each with the angles $\frac{1}{2}\pi$, $\frac{1}{2}\pi$, $\frac{1}{3}\pi$: and then projecting from the



South pole on the plane of the equator we have the annexed figure of the s-plane, divided into 12 curvilinear triangles, each with these same angles 90°, 90°, 60°, and which are by the shading divided into two systems, each of 6 triangles. The figure of the x-plane is by the x-line divided into two half-planes, one shaded, the other unshaded; and we have on the line the point c at ∞, a at the origin, and b at the distance unity.

58. Take x real, then if x is positive and less than 1, s^3 is real and positive, and we have for s the infinite half-lines at the inclinations 0°, 120°, 240°, while if x is positive and greater than 1, s^3 is real and negative, and we have the infinite half-lines at the inclinations 60°, 180°, 300°. If x is real and negative, then s^3 is of the form $\frac{1-ki}{1+ki}$, $=\cos\theta + i\sin\theta$; whence s is of the same form, or the locus of the point s is a circle radius unity. Writing $s^3 = \frac{1-\sqrt{x}}{1+\sqrt{x}}$, and supposing that the point x moves along the x-line from b through a to c at $-\infty$, and then from c at $+\infty$ to b, the point s describes the sides BA, AC, CB of the shaded triangle marked K.

59. Suppose that the point x is at k, in the shaded half-plane at an indefinitely small distance from a; say we have $x=-2\kappa^2 i$, (κ small), then taking for $\sqrt{x}$ the value $\kappa(1-i)$ we have $s^3 = \frac{1-\kappa(1-i)}{1+\kappa(1-i)}$, $=1-2\kappa(1-i)$ nearly, and hence a value of s is $=1-\frac{2}{3}\kappa+\frac{2}{3}\kappa i$, which belongs to a point K near A, and within the shaded triangle: we have thus, in respect of this value of s, the shaded half of the x-plane corresponding to this shaded triangle: to the same value $x=-2\kappa^2 i$ correspond in all six values of s, giving six points K each lying near a point A within one of the shaded triangles; and hence the shaded half-plane corresponds to the six shaded triangles, and the unshaded half-plane corresponds to the six unshaded triangles.

60. Suppose the equation is

$$\{s,\ x\} = \left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore \right)\!\!\left(\frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2,$$

that is

$$= \frac{-(b-c)(c-a)(a-b)}{x-a\,.\,x-b\,.\,x-c}\left(\frac{\mathrm{a}}{b-c\,.\,x-a}+\frac{\mathrm{b}}{c-a\,.\,x-b}+\frac{\mathrm{c}}{a-b\,.\,x-c}\right),$$

where a, b, c are real, but a, b, c are imaginary. It is to be shown that if the path of x is the circle passing through the points a, b, c, then the path of s is a circle passing through the corresponding three points.

61. We may find α, β, γ, θ_0, θ_1, θ_2 such that a, b, c are $=\alpha+\beta i+\gamma e^{\theta_0 i}$, $\alpha+\beta i+\gamma e^{\theta_1 i}$, $\alpha+\beta i+\gamma e^{\theta_2 i}$ (this is in fact finding α and β the coordinates of the centre, and γ the radius of the circle through the three points a, b, c): we then have $x=\alpha+\beta i+\gamma e^{\theta i}$, θ a variable parameter, the equation which expresses that the point x is situate on the circle in question.

We have $x-a=\gamma(e^{\theta i}-e^{\theta_0 i})$, $=\gamma e^{\frac{1}{2}(\theta+\theta_0)}\{e^{\frac{1}{2}(\theta-\theta_0)i}-e^{-\frac{1}{2}(\theta-\theta_0)i}\}$; the second factor is $i\sin\frac{1}{2}(\theta-\theta_0)$, $=iP$ suppose, or the equation is $x-a=iP\gamma\,.\,e^{\frac{1}{2}(\theta+\theta_0)i}$, say

$$x-a=iP\gamma\,.\,\mathrm{expi}\,\tfrac{1}{2}(\theta+\theta_0).$$

Similarly $x-b=iQ\gamma\ \mathrm{expi}\ \frac{1}{2}(\theta+\theta_1)$, and $x-c=iR\gamma\,.\,\mathrm{expi}\ \frac{1}{2}(\theta+\theta_2)$; where P, Q, R denote $\sin\frac{1}{2}(\theta-\theta_0)$, $\sin\frac{1}{2}(\theta-\theta_1)$, $\sin\frac{1}{2}(\theta-\theta_2)$ respectively: in like manner $b-c$, $c-a$, $a-b$, $=iF\gamma\ \mathrm{expi}\ \frac{1}{2}(\theta_1+\theta_2)$, $iG\gamma\ \mathrm{expi}\ \frac{1}{2}(\theta_2+\theta_0)$, $iH\gamma\ \mathrm{expi}\ \frac{1}{2}(\theta_0+\theta_1)$, where F, G, H denote $\sin\frac{1}{2}(\theta_1-\theta_2)$, $\sin\frac{1}{2}(\theta_2-\theta_0)$, $\sin\frac{1}{2}(\theta_0-\theta_1)$ respectively.

We have

$$-\frac{b-c\,.\,c-a\,.\,a-b}{x-a\,.\,x-b\,.\,x-c}=\frac{-FGH}{PQR}\text{ expi }\tfrac{1}{2}(\theta_0+\theta_1+\theta_2-3\theta),$$

$$\frac{1}{b-c\,.\,x-a}=\frac{-1}{\gamma^2PF}\text{ expi }-\tfrac{1}{2}(\theta_0+\theta_1+\theta_2+\theta),$$

with the like values for $\frac{1}{c-a\,.\,x-b}$ and $\frac{1}{a-b\,.\,x-c}$. Hence the right-hand side of the equation is

$$=\frac{FGH}{PQR}\left(\frac{\mathrm{a}}{PF}+\frac{\mathrm{b}}{QG}+\frac{\mathrm{c}}{RH}\right)\text{ expi }(-2\theta).$$

62. Considering now the left-hand side of the equation, we have

$$\{s,\ x\}=\frac{1}{\left(\frac{dx}{d\theta}\right)^2}(\{s,\ \theta\})-\{x,\ \theta\}),$$

or substituting for x its value $=\alpha+\beta i+\gamma e^{\theta i}$, this becomes

$$\{s,\ x\}=-\frac{1}{\gamma^2}e^{-2\theta i}(\{s,\ \theta\}-\tfrac{1}{2}),$$

that is

$$=-\frac{1}{\gamma^2}(\{s,\ \theta\}-\tfrac{1}{2})\text{ expi }(-2\theta).$$

Assume $s=L+Mi+Ne^{\Theta i}$, L, M, and N constants; then using the accent to denote differentiation in regard to θ, we find without difficulty $\{s,\ \theta\}=\{\Theta,\ \theta\}+\tfrac{1}{2}\Theta'^2$, and the value of $\{s,\ x\}$ becomes

$$=-\frac{1}{\gamma^2}\left(\{\Theta,\ \theta\}+\tfrac{1}{2}\Theta'^2-\tfrac{1}{2}\right)\text{ expi }(-2\theta).$$

Hence, substituting the values of the two sides of the equation, the imaginary factor expi (-2θ) divides out, and the equation becomes

$$\{\Theta,\ \theta\}+\tfrac{1}{2}\Theta'^2-\tfrac{1}{2}=-\frac{FGH}{PQR}\left(\frac{\mathrm{a}}{PF}+\frac{\mathrm{b}}{QG}+\frac{\mathrm{c}}{RH}\right),$$

an equation in which everything is real, and which thus determines Θ as a real function of θ: and we have therefore the theorem in question.

Connection with the differential equation for the hypergeometric series. Art. Nos. 63 to 68.

63. Take p, q given functions of x, and y a function of x determined by the equation

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0;$$

again P, Q given functions of z, and v a function of z determined by the equation

$$\frac{d^2v}{dz^2}+P\frac{dv}{dz}+Qv=0,$$

and assume

$$y=wv.$$

Substituting this value of y in the first equation, we obtain for v an equation of the second order (the coefficients of which contain w), and we may make this identical with the second equation; viz. comparing the coefficients of the two equations, we thus have two equations each containing w; and by eliminating w we obtain a differential equation of the third order between z and x. This is in fact the basis of Kummer's theory for the transformation of a hypergeometric series: the equation between z, x will be found presently in a different manner.

64. But if with Schwarz, instead of making the equation obtained for v as above identical with the given equation for v, we merely assume that the two equations are consistent, then there is nothing to determine the value of z, which may be regarded as an arbitrary function of x; y and v are then functions of x, and w denotes the quotient $y\div v$ of these two functions, and as such satisfies an equation the form of which will depend on the assumed relation between z and x. In particular if P and Q denote the same functions of z that p and q are of x; and if we assume $z=x$, P, Q will become $=p$, q respectively: the given equation in v will be

$$\frac{d^2v}{dx^2}+p\frac{dv}{dx}+qv=0\,;$$

and w will thus denote the quotient of any two solutions of the equation

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0\,;$$

viz. writing $X=p^2+2\dfrac{dp}{dx}-4q$, then by what precedes, the equation for w will be

$$\{w,\ x\}=-\tfrac{1}{2}X.$$

65. Returning now to Kummer's problem, and considering y, v as solutions of the two differential equations respectively, w is a function independent of the particular solutions denoted by these letters: we have $y=wv$, and taking any other two solutions $y_1=wv_1$, and therefore $\dfrac{y}{y_1}=\dfrac{v}{v_1}$; calling each of these equal quantities s we have s denoting the quotient of two solutions of the equation in y, and also the quotient of two solutions of the equation in v; whence writing as before $X=p^2+2\dfrac{dp}{dx}-4q$, and similarly $Z=P^2+2\dfrac{dP}{dz}-4Q$, we have

$$\{s,\ x\}=-\tfrac{1}{2}X,\quad \{s,\ z\}=-\tfrac{1}{2}Z,$$

and since in general

$$\{s,\ x\}=\left(\frac{dz}{dx}\right)^2\{s,\ z\}+\{z,\ x\},$$

we obtain

$$\{z,\ x\} = -\tfrac{1}{2}X + \tfrac{1}{2}Z\left(\frac{dz}{dx}\right)^2,$$

as the required equation for the determination of z as a function of x. The process does not give the value of w, but this can be found without difficulty, viz.

$$w^2 = Ce^{\int Pdz - \int pdx} \div \frac{dz}{dx}.$$

If z, x are regarded each of them as a function of the new independent variable θ, then the equation is

$$\{z,\ \theta\} - \tfrac{1}{2}\left(\frac{dz}{d\theta}\right)^2 Z = \{x,\ \theta\} - \tfrac{1}{2}\left(\frac{dx}{d\theta}\right)^2 X.$$

66. Jacobi's differential equation of the third order for the transformed modulus λ, *Fund. Nova*, p. 78 is

$$3(k''^2\lambda'^2 - \lambda''^2k'^2) - 2k'\lambda'(k'\lambda''' - \lambda'k''') + k'^2\lambda'^2\left\{\left(\frac{1+k^2}{k-k^3}\right)^2 k'^2 - \left(\frac{1+\lambda^2}{\lambda-\lambda^3}\right)^2\lambda'^2\right\} = 0,$$

where the accents denote differentiations in regard to an independent variable θ: viz. dividing by $2k'^2\lambda'^2$ this becomes

$$\{k,\ \theta\} + \tfrac{1}{2}k'^2\left(\frac{1+k^2}{k-k^3}\right)^2 = \{\lambda,\ \theta\} + \tfrac{1}{2}\lambda'^2\left(\frac{1+\lambda^2}{\lambda-\lambda^3}\right)^3,$$

which is thus a particular case of Kummer's equation, k, λ corresponding to x, z respectively, and the values of X, Z being

$$X = -\left(\frac{1+k^2}{k-k^3}\right)^2,\quad Z = -\left(\frac{1+\lambda^2}{\lambda-\lambda^3}\right)^3.$$

67. In the case of the hypergeometric series, the two differential equations of the second order are

$$\frac{d^2y}{dx^2} + \frac{\gamma-(\alpha+\beta+1)x}{x\,.\,1-x}\,\frac{dy}{dx} - \frac{\alpha\beta y}{x\,.\,1-x} = 0,$$

$$\frac{d^2v}{dz^2} + \frac{\gamma'-(\alpha'+\beta'+1)z}{z\,.\,1-z}\,\frac{dv}{dz} - \frac{\alpha\beta v}{z\,.\,1-z} = 0.$$

Hence $$p = \frac{\gamma(x+(1-x))-(\alpha+\beta+1)x}{x\,.\,1-x} = \frac{\gamma}{x} + \frac{\gamma-\alpha-\beta-1}{1-x},\quad q = \frac{-\alpha\beta}{x\,.\,1-x},$$

and hence

$$p^2 + 2\frac{dp}{dx} - 4q = \frac{\gamma^2-2\gamma}{x^2} + \frac{(\gamma-\alpha-\beta-1)^2+2(\gamma-\alpha-\beta-1)}{(1-x)^2} + \frac{4\alpha\beta+2\gamma(\gamma-\alpha-\beta-1)}{x\,.\,1-x},$$

viz. writing

$$\lambda^2 = (1-\gamma)^2,\qquad \mathrm{a} = \tfrac{1}{2}(1-\lambda^2),$$

$$\mu^2 = (\alpha-\beta)^2,\qquad \mathrm{b} = \tfrac{1}{2}(1-\mu^2),$$

$$\nu^2 = (\gamma-\alpha-\beta)^2,\quad \mathrm{c} = \tfrac{1}{2}(1-\nu^2),$$

and putting in the formula $x-1, =-(1-x)$, we have

$$-\tfrac{1}{2}\left(p^2+2\frac{dp}{dx}-4q\right)=\frac{\frac{1}{2}(1-\lambda^2)}{x^2}+\frac{\frac{1}{2}(1-\nu^2)}{(x-1)^2}+\frac{\frac{1}{2}(\lambda^2-\mu^2+\nu^2-1)}{x\,.\,x-1},$$

$$=\frac{\mathrm{a}}{x^2}+\frac{\mathrm{c}}{(x-1)^2}+\frac{-\mathrm{a}+\mathrm{b}-\mathrm{c}}{x\,.\,x-1},$$

$$=\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\!\!\bigg)\!\!\bigg(\frac{1}{x},\ \frac{1}{x-\infty},\ \frac{1}{x-1}\right)^2,$$

with a like formula for $\frac{1}{2}\left(P^2+2\frac{dP}{dx}-4Q\right)$. We then have

$$y=wv,$$

$$w^2=Cx^{-\gamma}(1-x)^{\gamma-\alpha-\beta-1}z^{\gamma'}(1-z)^{-\gamma'+\alpha'+\beta'+1}\frac{dx}{dz},$$

and the differential equation of the third order for the determination of z is

$$\{z,\ x\}+\left(\mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1\ \therefore\!\!\bigg)\!\!\bigg(\frac{1}{z},\ \frac{1}{z-\infty},\ \frac{1}{z-1}\right)^2\left(\frac{dz}{dx}\right)^2-\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\!\!\bigg)\!\!\bigg(\frac{1}{x},\ \frac{1}{x-\infty},\ \frac{1}{x-1}\right)^2=0,$$

where a_1, b_1, c_1 are the same functions of α', β', γ' which a, b, c are of α, β, γ. This is in effect Kummer's equation for the transformation of the hypergeometric series.

68. And in like manner the Schwarzian equation for the determination of s, the quotient of two solutions is

$$\{s,\ x\}=\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\!\!\bigg)\!\!\bigg(\frac{1}{x},\ \frac{1}{x-\infty},\ \frac{1}{x-1}\right)^2.$$

PART II. THE POLYHEDRAL FUNCTIONS.

Origin and Properties. Art. Nos. 69 to 80.

69. The functions in lines 1, ... 5 of the *PQR*-Table are connected with the geometrical forms:

{1. Polygon or
{2. Double Pyramid*.
3. Tetrahedron.
4. Octahedron and Cube.
5. Dodecahedron and Icosahedron,

(these figures being regarded as situate on a spherical surface,) and with the stereographic projections of these figures.

Consider a spherical surface and upon it any number of points: take at pleasure any point as South Pole, this determines the plane of the equator; and the stereo-

* Prof. Klein regards 1 as belonging to the polygon and 2 to the double pyramid: it seems to me that the fundamental figure, to which 1 and 2 each of them belong, is the polygon.

graphic projection of any point is the intersection with the plane of the equator of the line joining the point with the South Pole.

To fix the ideas take the radius of the sphere as unity: let the axes of x and y be drawn in the plane of the equator in longitudes 0° and 90° respectively, and the axis of z upwards through the North Pole: the position of a point on the sphere is determined by means of its N. P. D. θ and longitude f: moreover we take X, Y, Z for the coordinates of the point on the surface, and x, y for those of its projection; and we then have

$$X,\ Y,\ Z = \sin\theta\cos f,\ \ \sin\theta\sin f,\ \ \cos\theta;$$

$$x = \frac{X}{1+Z} = \tan\tfrac{1}{2}\theta\cos f,\quad y = \frac{Y}{1+Z} = \tan\tfrac{1}{2}\theta\sin f,$$

and conversely,

$$X,\ Y,\ Z = 2x,\ 2y,\ 1-x^2-y^2,\ \div(1+x^2+y^2).$$

We represent the point (X, Y, Z) on the spherical surface by means of the magnitude $x+iy\ (=\tan\tfrac{1}{2}\theta(\cos f + i\sin f))$, or say by the linear factor, $s-(x+iy)$: and similarly any system of points on the surface by means of the system of magnitudes $x+iy$, or say by the function $\Pi\{s-(x+iy)\}$, denoting in this manner the product of the linear factors which correspond to the different points respectively.

70. It will presently appear that if (considering a different stereographic projection, that is, a different position of the South Pole) we take x', y' as the coordinates of the new projection of the point, then $x'+iy'$ is a homographic function

$$a(x+iy)+b \div \{c(x+iy)+d\}$$

of $x+iy$: and consequently that the functions of s which belong to different projections are linear transformations one of the other: but at present we consider a single projection.

It may be proper to remark that the figures in question are spherical figures having summits which are points on the spherical surface, edges (or sides) which are arcs of great circle joining two summits, and faces, which are portions of the spherical surface: the mid-points of the sides, and the centres of the faces are of course points on the spherical surface.

71. (1), (2). Considering a regular polygon formed by n summits on the equator, the longitude of one of them being 0°, then the stereographic projections correspond with the points themselves, and the values of $x+iy$ are

$$1,\ \cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n},\ \ldots\ \cos\frac{(n-1)\,2\pi}{n}+i\sin\frac{(n-1)\,2\pi}{n}.$$

The corresponding function of s is s^n-1.

The values of $x+iy$ for the mid-points of the sides are

$$\cos\frac{\pi}{n}+i\sin\frac{\pi}{n},\ \ \cos\frac{3\pi}{n}+i\sin\frac{3\pi}{n},\ \ldots\ \cos\frac{(2n-1)\,\pi}{n}+i\sin\frac{(2n-1)\,\pi}{n},$$

and the corresponding function is s^n+1.

The North and South Poles, which form with the n points a double pyramid of $n+2$ summits, correspond to the values $s=0$ and $s=\infty$. We have thus

$$s\left(1-\frac{s}{\infty}\right)(s^n-1)$$

as the function corresponding to the double pyramid.

72. (3) Considering for a moment the tetrahedron as a figure with rectilinear edges, this is so placed that two opposite edges are horizontal, and that the vertical planes passing through the centre and these two edges respectively are inclined at angles $\pm 45^\circ$ to the meridian: viz. the upper edge has the longitudes 135°, 315° and the lower edge the longitudes 45°, 225°. We thus explain the position of the spherical figure.

Corresponding to the summits we have the function $s^4-2i\sqrt{3}\,s^2+1$.

In fact the equation $s^4-2i\sqrt{3}\,s^2+1=0$ gives $s^2=i(\sqrt{3}\pm 2)$, and hence the values of s are the four values of $x+iy$ shown in the annexed table for the values of X, Y, Z, and $x+iy$ for the summits of the tetrahedron,

long.	X	Y	Z	$x+iy$
45°	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$\frac{1+i}{\sqrt{3}-1}$
135°	$-$	$+$	$+$	$\frac{-1+i}{\sqrt{3}+1}$
225°	$-$	$-$	$-$	$\frac{-1-i}{\sqrt{3}-1}$
315°	$+$	$-$	$+$	$\frac{1+i}{\sqrt{3}+1}$.

Corresponding to the centres of the faces, or summits of the opposite tetrahedron we have the function $s^4+2i\sqrt{3}\,s^2+1$.

Corresponding to the mid-points of the sides we have the function $s\left(1-\frac{s}{\infty}\right)(s^4-1)$; viz. the points in question are the North Pole $s=0$, the South Pole $s=\infty$, and the four points $s=\pm 1$, $s=\pm i$ on the equator at longitudes 0°, 90°, 180°, 270° respectively.

73. (4) The octahedron is placed with two of its summits as poles, and the other four summits in the equator at longitudes 0, 90°, 180°, 270° respectively: the values of s are as in the last case 0, ∞, ± 1, $\pm i$, and the function is $s\left(1-\frac{s}{\infty}\right)(s^4-1)$.

The function for the centres of the faces, or summits of the cube is s^8+14s^4+1.

The function for the mid-points of the sides of the octahedron or of the cube is

$$s^{12}-33s^8-33s^4+1.$$

74. (5) The Icosahedron is placed with 2 of its summits for poles; five summits lying in a small circle above the plane of the equator at longitudes 0°, 72°, 144°, 288°, and the remaining 5 summits in the corresponding small circle below the equator at longitudes 36°, 108°, 180°, 252° and 324°.

The function for the summits of the Icosahedron is $s\left(1-\frac{s}{\infty}\right)(s^{10}+11s^5-1)$.

The function for the centres of the faces of the Icosahedron, or summits of the Dodecahedron is $s^{20}-228s^{15}+494s^{10}+228s^5-1$.

The function for the mid-points of the sides of the Icosahedron or the Dodecahedron is

$$s^{30}-522s^{25}+10005s^{20}+0\,s^{15}-10005s^{10}+522s^5+1.$$

I give for the present these results without demonstration.

75. Writing $\frac{x}{y}$ for s so as to obtain homogeneous functions $(*\between x,\ y)^n$—it will be recollected that the x, y of these functions have nothing to do with the x, y of the foregoing values $x+iy$—the forms which have thus presented themselves may be denoted as follows:

$$\begin{aligned}
(3)\quad f3 &= (1,\ -2i\sqrt{3},\ 1\between x^2,\ y^2)^2,\\
h3 &= (1,\ +2i\sqrt{3},\ 1\between x^2,\ y^2)^2,\\
t3 &= xy\,(x^4-y^4),\\
(4)\quad f4 &= xy\,(x^4-y^4),\\
h4 &= (1,\ 14,\ 1\between x^4,\ y^4)^2,\\
t4 &= (1,\ -33,\ -33,\ 1\between x^4,\ y^4)^3,\\
(5)\quad f5 &= xy\,(1,\ 11,\ -1\between x^5,\ y^5)^2,\\
h5 &= (1,\ -228,\ +494,\ +228,\ -1\between x^5,\ y^5)^4,\\
t5 &= (1,\ -522,\ 10005,\ 0,\ -10005,\ 522,\ 1\between x^5,\ y^5)^6,
\end{aligned}$$

where observe that $f4$ is the same function as $t3$. In each set of functions f, h, t, we have h and t covariants of f, viz. disregarding numerical factors,

h is the Hessian, or derivative $(f,\ f)$, and t is the derivative $(f,\ h)$.

76. Since $f4$ is the same function as $t3$, we have of course $f4$, $h4$ and $t4$ themselves covariants of $f3$: but it is convenient to separate the two systems.

77. It is to be observed that $f3$ is a quartic function having its quadrinvariant $(I)=0$; but independently of this, that is quà quartic function, it has only the covariants $h3$ and $t3$ the (Hessian and the cubicovariant respectively), viz. every other covariant is a rational and integral function of $f3$, $h3$ and $t3$. In particular $h4$ and $t4$ are rational and integral functions of $f3$, $h3$ and $t3$; but inasmuch as $f3$ and $h3$ are not covariants of $f4$ this is not a property of $h4$ and $t4$ considered as covariants of $f4$, and the relation in question need not be attended to.

78. It has just been stated that $f3$ quà quartic function has (in the sense explained) only the covariants $h3$ and $t3$: $f4$ quà *special* sextic function and $f5$ quà *special* dodecadic function have the like property, viz. $f4$ has only the covariants $h4$ and $t4$; $f5$ only the covariants $h5$ and $t5$. Hence $f3$, $f4$, $f5$ are "Prime-forms" in the sense defined

in the paper Fuchs, 1875, viz. a Prime-form has no covariant of a lower order than itself, and also no covariant of a higher order which is a power of a form of a lower order.

79. The same functions have also the property that they are functions transformable into themselves by means of a group of linear transformations, and in this point of view they were considered in the nearly contemporaneous paper Klein, 1875; it is in this paper shown that the functions so transformable into themselves must be Polyhedral functions as above, the linear transformations in fact corresponding to the rotations whereby the spherical polyhedron can be brought into coincidence with its own original position. This theory will be presently given.

80. It is to be observed that if U, V are functions $(*\between x, y)^n$ of the same order n, then using the accent to denote differentiation in regard to x, $UV' - U'V$ and (U, V) differ only by a numerical factor: and further that writing as before $s = \frac{x}{y}$, and in the expression $UV' - U'V$ regarding U, V as functions $(*\between s, 1)^n$ and the accent as denoting differentiation in regard to s, we have $UV' - U'V$ and (U, V) differing by a numerical factor only. We have in the PQR-table, lines 3, 4, 5, P, Q, R equal to given numerical multiples of h^β, t^γ, f^α, the indices α, β, γ being such as to make these to be functions of the same degree: hence neglecting numerical multipliers $PQ' - P'Q$ is equal to a function (h^β, t^γ), which is $= h^{\beta-1}t^{\gamma-1}(h, t)$: and the theorem that $PQ' - P'Q$, $= QR' - Q'R$, $= RP' - R'P$ contains only factors of P, Q, R is in fact the theorem that (h, t), (h, f), and (t, f) are each of them equal to a term or product of f, h, t: which is a result included in the theorem that f has only the covariants h and t. And by this last theorem we know already how from R, assumed to be known, we can derive P and Q: viz. R is a power of f; and we thence have $h = (f, f)$ and $t = (h, f)$, equations giving the functions h and t, upon which P and Q depend.

Covariantive Formulæ. Art. Nos. 81 to 84.

81. The various covariantive formulæ will be given with their proper numerical coefficients.

Tetrahedron function. f, h, t stand for the before-mentioned values, $f3$, $h3$, $t3$ $(P, Q, R = h^3, -12i\sqrt{3}\,.\,t^2, -f^3)$.

For $f3$. $(a, b, c, d, e) = 1, 0, \frac{-i}{\sqrt{3}}, 0, 1$.

$\tfrac{1}{2}(f, f)^2 = -96i\sqrt{3}\,.\,h$, $\quad \tfrac{1}{2}(h, h)^2 = 96i\sqrt{3}\,.\,f$, $\quad \tfrac{1}{2}(t, t)^2 = -25fh$,

$(f, h) = 32i\sqrt{3}\,.\,t$, $\quad (f, f)^4 = 576I = 0$, $\quad (f, h)^4 = 1152J = 1152\,.\,\frac{-4i}{\sqrt{3}}$,

$(f, t) = 4\,.\,h^2$,

$(h, t) = 4\,.\,f^2$,

$h^3 - f^3 - 12i\sqrt{3}\,t^2 = 0$,

$fh = (1, 14, 1\between x^4, y^4)^2 \;(= f4)$.

It is convenient to remark that t^2, f^3, h^3 being of the same order we have

$$t^2(f^3,\ h^3)+f^3(h^3,\ t^2)+h^3(t^2,\ f^3)=0,$$

that is $$t^2.3.3f^2h^2(f,\ h)+f^3.3.2\,h^2t(h,\ t)+h^3.2.3\,tf^2(t,\ f)=0,$$

an equation, which substituting for $(f,\ h)$, $(h,\ t)$, $(t,\ f)$ their values, reduces itself to the before-mentioned relation $h^3-f^3-12i\sqrt{3}\,t^2=0$; and we have thus a verification of the values of $(f,\ h)$, $(h,\ t)$ and $(t,\ f)$. The like remark applies to the other two cases, which follow.

82. Hexahedron function. $f,\ h,\ t$ stand for the before-mentioned values $f4,\ h4,\ t4$ $(P,\ Q,\ R=h^3,\ -t^2,\ -108f^4)$.

For $f4$. $(a,\ b,\ c,\ d,\ e,\ f,\ g)=(0,\ \tfrac{1}{4},\ 0,\ 0,\ 0,\ -\tfrac{1}{4},\ 0)$.

$$\tfrac{1}{2}(f,\ f)^2=-25h,\qquad \tfrac{1}{2}(f,\ f)^4=0,\qquad \tfrac{1}{2}(f,\ f)^6=(720)^2.\tfrac{3}{8},$$

$$(f,\ h)=-8t,\qquad \tfrac{1}{2}(h,\ h)^2=3.2^6.7^2.f^2,$$

$$(f,\ t)=-12h^2,\qquad \tfrac{1}{2}(t,\ t)^2=2^4.3^3.11^2.f^2h,$$

$$(h,\ t)=-1728f^3,$$

$$h^3-t^2-108f^4=0.$$

83. Dodecahedron function. $f,\ h,\ t$ stand for the before-mentioned values $f5,\ h5,\ t5$ $(P,\ Q,\ R=h^3,\ -t^2,\ -1728f^5)$.

For $f5$. $(a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ i,\ j,\ k,\ l,\ m)=(0,\ \tfrac{1}{12},\ 0,\ 0,\ 0,\ 0,\ \tfrac{1}{84},\ 0,\ 0,\ 0,\ 0,\ -\tfrac{1}{12},\ 0)$.

$$\tfrac{1}{2}(f,\ f)^2=-121h,\qquad \tfrac{1}{2}(f,\ f)^4=0,\qquad \tfrac{1}{2}(f,\ f)^6=\tfrac{1}{2}(924)^2(720)^2.\tfrac{-5}{84}f^*,$$

$$\tfrac{1}{2}(f,\ f)^8=\ \ 0,\qquad \tfrac{1}{2}(f,f)^{10}=0,\qquad \tfrac{1}{2}(f,\ f)^{12}=\tfrac{1}{2}(924)^2(720)^4.\tfrac{25}{84}{}^*,$$

$$(f,\ h)=-20t,\qquad \tfrac{1}{2}(h,\ h)^2=173280f^3,$$

$$(f,\ t)=-30h^2,\qquad \tfrac{1}{2}(t,\ t)^2=9082800f^3h,$$

$$(h,\ t)=-86400f^5,$$

$$h^3-t^2-1728f^5=0.$$

84. We have

$$t=(x^{10}+y^{10})(1,\ 522,\ -10006,\ -522,\ 1\between x^5,\ y^5)^4.$$

Write $$\xi=(x^2+y^2).(1,\ 2,\ 6,\ -2,\ 1\between x,\ y)^4,$$

then $$t=\xi(1,\ -10,\ 45\between \xi^2,\ f).$$

Or putting $$p=\frac{\xi}{\sqrt{f}},\quad =\frac{(x^2+y^2)(1,\ 2,\ 6,\ -2,\ 1\between x,\ y)^4}{\sqrt{xy(x^{10}+11x^5y^5-y^{10})}},$$

that is, $\xi=p\sqrt{f}$, then $$p^5-10p^3+45p=\frac{t}{\sqrt{f^5}}.\quad \text{(Klein.)}$$

* The numerical coefficients $-\frac{5}{84}$ and $\frac{25}{84}$ are Klein's B and A: the latter of them the ordinary quadrinvariant of a dodecadic function; the former is an invariant linear as regards the coefficients of f, and existing only for the special form f in question: viz. writing for a moment

$$f=\lambda(x^{11}y+11x^6y^6-xy^{11}),$$

then $(f,\ f)^6$ contains the factor λ^2, and (f containing the factor λ) the form is

$$\tfrac{1}{2}(f,f)^6=\tfrac{1}{2}(924)^2(720)^2.-\tfrac{5}{84}\lambda.f,$$

which is linear as regards λ. We have also

$$\tfrac{1}{2}(f,f)^{12}=\tfrac{1}{2}(924)^2(720)^4.\tfrac{25}{84}\lambda^2:$$

say $A=\frac{25}{84}\lambda^2$, $B=-\frac{5}{84}\lambda$; or $84B^2=A$. Of course in the case of a general dodecadic function f, we have $(f,\ f)^6$, an irreducible covariant, not breaking up into factors.

Investigation of the forms f5 and h5. Art. Nos. 85 and 86.

85. Writing for shortness* $k = \tan\alpha = \frac{\sqrt{5}-1}{2}$, and $g = \cos 36^\circ + i\sin 36^\circ$, then the values of $x+iy$ corresponding to the summits of the Icosahedron are

$$\begin{array}{lllll} 0, \\ k, & kg^2, & kg^4, & kg^6, & kg^8, \\ k^{-1}g, & k^{-1}g^3, & k^{-1}g^5, & k^{-1}g^7, & k^{-1}g^9, \\ \infty; \end{array}$$

and the function $f5$ is thus

$$= s\left(1-\frac{s}{\infty}\right)(s^5-k^5)(s-k^{-5}),$$

where the product of the last two factors is $s^{10}+(k^{-5}-k^5)s^5-1$. We have

$$k^{-5} = \tfrac{1}{32}(80\sqrt{5}+176), \quad = \tfrac{1}{2}(5\sqrt{5}+11),$$
$$k^5 = \tfrac{1}{32}(80\sqrt{5}-176), \quad = \tfrac{1}{2}(5\sqrt{5}-11),$$

and consequently $k^{-5}-k^5 = 11$; or the function is $s\left(1-\frac{s}{\infty}\right)(s^{10}+10s^5-1)$.

86. Similarly writing for shortness* $l = \tan\frac{1}{2}\gamma$, $l' = \tan\frac{1}{2}\gamma'$, where

$$\cos^2\gamma = \frac{5+2\sqrt{5}}{15}, \quad \sin^2\gamma = \frac{10-2\sqrt{5}}{15}; \text{ and therefore } \frac{\cos\gamma}{\sin\gamma} = \frac{3+\sqrt{5}}{4};$$
$$\cos^2\gamma' = \frac{5-2\sqrt{5}}{15}, \quad \sin^2\gamma' = \frac{10+2\sqrt{5}}{15}; \quad \frac{\cos\gamma'}{\sin\gamma'} = \frac{3-\sqrt{5}}{4};$$

and $g = \cos 36^\circ + i\sin 36^\circ$ as before, then the values of $x+iy$ for the summits of the dodecahedron are

$$\begin{array}{lllll} lg, & lg^3, & lg^5, & lg^7, & lg^9, \\ l'g, & l'g^3, & l'g^5, & l'g^7, & l'g^9, \\ l'^{-1}, & l'^{-1}g^2, & l'^{-1}g^4, & l'^{-1}g^6, & l'^{-1}g^8, \\ l^{-1}, & l^{-1}g^2, & l^{-1}g^4, & l^{-1}g^6, & l^{-1}g^8, \end{array}$$

and the function $h5$ is therefore

$$= s^{10}+s^5(l^5-l^{-5})+1 \;.\; s^{10}+s^5(l'^5-l'^{-5})-1.$$

We have $$l^{-5}-l^5 = \frac{(1+\cos\gamma)^5-(1-\cos\gamma)^5}{\sin^5\gamma} = \frac{2\cos\gamma}{\sin^5\gamma}(5+10\cos^2\gamma+\cos^4\gamma)$$

$$= \frac{2\cos\gamma}{\sin^5\gamma}\cdot\frac{384+64\sqrt{5}}{45} = \frac{128}{45}\,\frac{\cos\gamma}{\sin^5\gamma}(6+\sqrt{5}), \quad = 114+50\sqrt{5}.$$

(viz. this last identity depends on $\frac{32}{45}(3+\sqrt{5})(6+\sqrt{5}) = (114+50\sqrt{5})\sin^4\gamma$, that is

$$160(3+\sqrt{5})(6+\sqrt{5}) = (114+50\sqrt{5})(120-40\sqrt{5}),$$

or $$2(3+\sqrt{5})(6+\sqrt{5}) = (57+25\sqrt{5})(3-\sqrt{5}),$$

or finally $$(7+3\sqrt{5})(6+\sqrt{5}) = 57+25\sqrt{5}, \text{ which is right}).$$

* α is the α, γ is the γ, and γ' the $\alpha-\beta$ of the Table, No. 99.

Similarly $$l'^{-5} - l'^{5} = 114 - 50\sqrt{5},$$
and observing that the sum and product of $114 + 50\sqrt{5}$, $114 - 50\sqrt{5}$ are $= 228$ and 496 respectively, the required function of s is

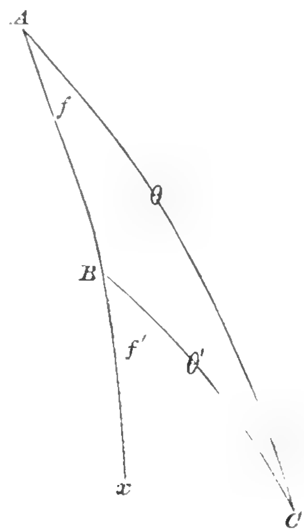
$$(s^{10} - 1)^2 - 228(s^{15} - s^5) + 496s^{10},$$
$$= s^{20} - 228s^{15} + 494s^{10} + 228s^5 + 1,$$

which is the required value of $h5$.

Invariantive property of the Stereographic Projection. Art. Nos. 87 to 93.

87. The before-mentioned theorem that the functions derived from two different stereographic projections of the same point are linear transformations one of the other, may be thus stated:

Considering on the surface of a sphere, two fixed points A and B; and determining the position of a point C first in regard to A by its distance θ and azimuth f, and



next in regard to B by its distance θ' and azimuth f', the azimuths from the great circle ABx which joins the two points A and B, then we have

$$\tan\tfrac{1}{2}\theta\,(\cos f + i\sin f),\ \text{ and }\ \tan\tfrac{1}{2}\theta'\,(\cos f' + i\sin f'),$$

homographic functions one of the other: calling them s, s', and putting the distance $AB = c$, the relation between them in fact is

$$s' = \frac{s - \tan\frac{1}{2}c}{1 + s\tan\frac{1}{2}c},$$

or, what is the same thing, $\tan\frac{1}{2}c\,(1 + ss') = s - s'$;

or observing that $ss' = \tan\frac{1}{2}\theta\tan\frac{1}{2}\theta'\,\{\cos(f+f') + i\sin(f+f')\}$, we have the two equations

$$\tan\tfrac{1}{2}c\,\{1 + \tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\theta'\cos(f+f')\} = \tan\tfrac{1}{2}\theta\cos f - \tan\tfrac{1}{2}\theta'\cos f',$$
$$\tan\tfrac{1}{2}c\,\{\quad \tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\theta'\sin(f+f')\} = \tan\tfrac{1}{2}\theta\sin f - \tan\tfrac{1}{2}\theta'\sin f'.$$

88. If we denote the angles of the spherical triangle by C, A, B and the opposite sides by c (as before), a, b, then $\theta,\ \theta' = b,\ a$; $f,\ f' = A,\ \pi - B$, whence

$$s,\ s' = \tan \tfrac{1}{2}b\,(\cos A + i \sin A), \quad -\tan \tfrac{1}{2}a\,(\cos B - i \sin B):$$

or we have between the sides a, b, c and angles A, B of a spherical triangle the relations

$$\tan \tfrac{1}{2}c\,\{1 - \tan \tfrac{1}{2}a \tan \tfrac{1}{2}b \cos (A - B)\} = \tan \tfrac{1}{2}b \cos A + \tan \tfrac{1}{2}a \cos B,$$
$$\tan \tfrac{1}{2}c\,\{\quad - \tan \tfrac{1}{2}a \tan \tfrac{1}{2}b \sin (A - B)\} = \tan \tfrac{1}{2}b \sin A - \tan \tfrac{1}{2}a \sin B;$$

equations which may be verified by means of the ordinary formulæ of Spherical Trigonometry.

89. But it is interesting to give the proof with rectangular coordinates.

Taking (X, Y, Z), (X_1, Y_1, Z_1) for the coordinates, referred to two different sets of axes, of a point on the spherical surface: also x, y, x_1, y_1 for the coordinates of the corresponding stereographic projections, we have

$$(X_1,\ Y_1,\ Z_1) = \left(\begin{array}{ccc} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array}\right)(X,\ Y,\ Z),$$

$$X : Y : Z : 1 = 2x : 2y : 1 - x^2 - y^2 : 1 + x^2 + y^2,$$
$$X_1 : Y_1 : Z_1 : 1 = 2x_1 : 2y_1 : 1 - x_1^2 - y_1^2 : 1 + x_1^2 + y_1^2,$$

and thence

$$\begin{aligned} x_1 : y_1 : 1 = &\ 2\alpha x + 2\beta y + \gamma\,(1 - x^2 - y^2) \\ : &\ 2\alpha' x + 2\beta' y + \gamma'\,(1 - x^2 - y^2) \\ : &\ 1 + x^2 + y^2 + 2\alpha'' x + 2\beta'' y + \gamma''\,(1 - x^2 - y^2). \end{aligned}$$

90. Introducing z, z_1 for homogeneity, or writing $\frac{x}{z}$, $\frac{y}{z}$ and $\frac{x_1}{z_1}$, $\frac{y_1}{z_1}$ in place of x, y and x_1, y_1, respectively, we have

$$\begin{aligned} x_1 &= 2\alpha x + 2\beta y + \gamma\,(z^2 - x^2 - y^2), &&= (-\gamma,\ -\gamma,\ \gamma,\ \beta,\ \alpha,\ 0)(x,\ y,\ z)^2, \\ y_1 &= 2\alpha' x + 2\beta' y + \gamma'\,(z^2 - x^2 - y^2), &&= (-\gamma',\ -\gamma',\ \gamma',\ \beta',\ \alpha',\ 0)(\quad ,, \quad)^2, \\ z_1 &= z^2 + x^2 + y^2 + 2\alpha'' x + 2\beta'' y + \gamma''\,(z^2 - x^2 - y^2), &&= (1 - \gamma'',\ 1 - \gamma'',\ 1 + \gamma'',\ \beta'',\ \alpha'',\ 0)(\quad ,, \quad)^2, \end{aligned}$$

and thence without difficulty

$$z_1 = \frac{1}{1 + \gamma''}\{(1 + \gamma'')z + (\alpha'' + i\beta'')(x - iy)\}\{(1 + \gamma'')z + (\ \alpha'' - i\beta'')(x + iy)\},$$

$$x_1 + iy_1 = \frac{1}{\gamma + \gamma'^2}\{(1 + \gamma'')z + (\alpha'' + i\beta'')(x - iy)\}\{(1 - \gamma'')z + (-\alpha'' + i\beta'')(x + iy)\},$$

$$x_1 - iy_1 = \frac{1}{\gamma - \gamma'^2}\{(1 - \gamma'')z - (\alpha'' + i\beta'')(x - iy)\}\{(1 + \gamma'')z + (\ \alpha'' - i\beta'')(x + iy)\},$$

viz. the form is $z_1 : x_1 + iy_1 : x_1 - iy_1 = MN : NL : LM$ (L, M, N linear functions of z, $x + iy$, $x - iy$): showing that the relation between two stereographic projections of the same spherical figure is in fact that of a quadric transformation, the fundamental points in each figure being an arbitrary point and the two circular points at infinity: or, what is the same thing, to any line in the one figure there corresponds a circle in the other figure, which is the "circular relation" of Möbius.

91. The actual values are

$$\frac{x_1+iy_1}{z_1}=\frac{1+\gamma''}{\gamma+\gamma' i}\cdot\frac{(1-\gamma'')z-(\alpha''-i\beta'')(x+iy)}{(1+\gamma'')z+(\alpha''-i\beta'')(x+iy)},$$

$$\frac{x_1-iy_1}{z_1}=\frac{1+\gamma''}{\gamma-\gamma' i}\cdot\frac{(1-\gamma'')z-(\alpha''+i\beta'')(x-iy)}{(1+\gamma'')z+(\alpha''+i\beta'')(x-iy)},$$

viz. attending only to the former of these, we have $\frac{x_1+iy_1}{z_1}$ a homographic function of $\frac{x+iy}{z}$, which is the before-mentioned theorem.

92. Supposing that the transformation from (X, Y, Z) to (X_1, Y_1, Z_1) is made by a rotation the coordinates of which are λ, μ, ν (that is, if f, g, h are the inclinations of the resultant axis to the axes of x, y, z respectively, and θ the angle of rotation, putting $\lambda, \mu, \nu=\tan\frac{1}{2}\theta\cos f,\ \tan\frac{1}{2}\theta\cos g,\ \tan\frac{1}{2}\theta\cos h$), then the coefficients of transformation are

$$\begin{array}{c|ccc|l}(\alpha, \beta, \gamma) = (& 1+\lambda^2-\mu^2-\nu^2, & 2(\lambda\mu+\nu), & 2(\lambda\nu-\mu) &)\div(1+\lambda^2+\mu^2+\nu^2),\\ \alpha', \beta', \gamma' & 2(\mu\lambda-\nu), & 1-\lambda^2+\mu^2-\nu^2, & 2(\mu\nu+\lambda) & \\ \alpha'', \beta'', \gamma'' & 2(\nu\lambda+\mu), & 2(\mu\nu-\lambda), & 1-\lambda^2-\mu^2+\nu^2 & \end{array}$$

and substituting these values, the formulæ become after an easy reduction

$$\frac{x_1+iy_1}{z_1}=\frac{-(\nu+i)(x+iy)+(\lambda+i\mu)z}{(\lambda-i\mu)(x+iy)+(\nu-i)z},$$

$$\frac{x_1-iy_1}{z_1}=\frac{-(\nu-i)(x-iy)+(\lambda-i\mu)z}{(\lambda+i\mu)(x-iy)+(\nu+i)z};$$

attending to the former of these, and writing for greater simplicity $\frac{x_1+iy_1}{z_1}$, $\frac{x+iy}{z}$ $=s_1$, s respectively, we have

$$s_1=\frac{-(\nu+i)s+(\lambda+i\mu)}{(\lambda-i\mu)s+(\nu-i)},$$

or writing this

$$s_1=\frac{As+B}{Cs+D},$$

then $$A:B:C:D=-\nu-i:\lambda+i\mu:\lambda-i\mu:\nu-i.$$

93. I call to mind that the condition in order that the homographic transformation $s_1=(As+B)\div(Cs+D)$ may be periodic of the order n is

$$(A+D)^2-4(AD-BC)\cos^2\frac{2m\pi}{n}=0,$$

in particular $n=2$, it is $A+D=0$: $n=3$, it is $A^2+AD+D^2+BC=0$: $n=4$, it is $A^2+D^2+2BC=0$: and $n=5$, it is $(A+D)^2-\frac{1}{2}(3\pm\sqrt{5})(AD-BC)=0$.

Groups of homographic transformations. Art. Nos. 94 and 95.

94. The formulæ just obtained serve to connect the theory of the rotations of a polyhedron with that of the homographic transformations s into $As+B\div(Cs+D)$: and,

corresponding to the rotations which leave the polyhedron unaltered, we have groups of homographic transformations. We have thus, corresponding to the cases of the tetrahedron, the cube and the octahedron, and the dodecahedron and icosahedron respectively, groups of 12, of 24, and of 60 homographic transformations s into $As+B \div (Cs+D)$. The group of 60 and the group of 24 include each of them as part of itself the group of 12: it is further to be remarked that the group of 12 may be regarded as that of the positive substitutions upon four letters *abcd*, the group of 24 as that of all the substitutions upon the four letters, and the group of 60 as that of the positive substitutions upon five letters *abcde*.

95. I call to mind that a group of functional symbols 1, α, β,... can always be expressed in the equivalent form 1, $\vartheta\alpha\vartheta^{-1}$, $\vartheta\beta\vartheta^{-1}$,... where ϑ is any functional symbol whatever: clearly, α, β,... being homographic transformations, then, ϑ being any homographic transformation whatever, the new symbols $\vartheta\alpha\vartheta^{-1}$, $\vartheta\beta\vartheta^{-1}$... will also be homographic transformations; and thus the group of homographic transformations can be expressed in various equivalent forms: these correspond to the different positions of the polyhedron in regard to the axes of coordinates: and there are in fact three cases which it is proper to consider, viz. attending for the moment to the dodecahedron, we may have the axis of z passing through the midpoint of a side, through the centre of a face, or through a summit; that is, in the language presently explained, the cases are 1°, Pole at a point Θ; 2°, Pole at a point A; 3°, Pole at a point B.

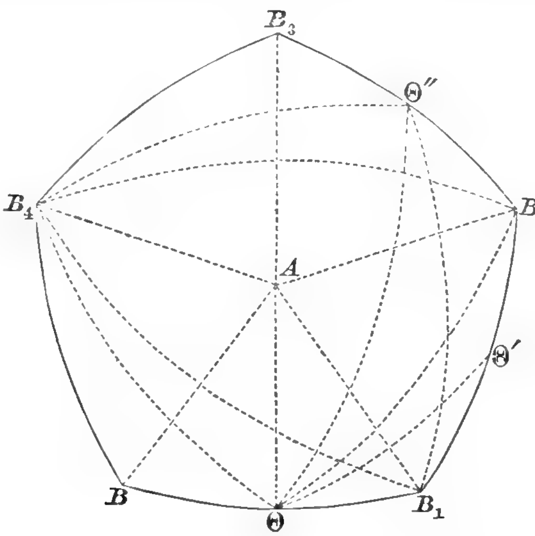
The regular Polyhedra. Art. Nos. 96 to 103.

96. We require a theory of the regular Polyhedra considered as systems of points on a sphere. I refer to my two papers (1866) and (1877). In the latter paper, I remark that considering the five regular figures drawn in proper relation to each other on the same spherical surface, the only points which have to be considered are 12 points A, 20 points B, 30 points Θ, and 60 points Φ. Describing these by reference to the dodecahedron, the points A are the centres of the faces, the points B are the summits, the points Θ are the midpoints of the sides, and the points Φ are the midpoints of the diagonals of the faces. Or describing them by reference to the icosahedron, the points A are the summits, the points B are the centres of the faces, the points Θ are the midpoints of the sides (viz. each point Θ is the common midpoint of a side of the dodecahedron and a side of the icosahedron, which there intersect at right angles), and the points Φ are points lying by three's on the faces of the icosahedron, each point Φ of the face being given as the intersection of a perpendicular $A\Theta$ of the face by a line BB joining the centres of two adjacent faces and which intersects $A\Theta$ at right angles.

97. The points Φ are comparatively unimportant, and it is proper in the first instance to attend only to the 12 points A, the 20 points B, and the 30 points Θ: these form 6 pairs of opposite points A, 10 pairs of opposite points B, and 15 pairs of opposite points Θ. Considering the diameters through each pair of opposite points Θ, we have thus a system of 15 axes, which in fact form 5 sets each of 3 rectangular axes: attending to any one of such sets, the diametral plane at right angles to one of the three axes contains of

course the other two axes: it contains also two axes each through a pair of opposite points A, and two axes each through a pair of opposite points B. If instead of the plane we consider its intersection with the sphere, we have thus on the sphere 15 circles each containing 4 points Θ, 4 points A and 4 points B. The fifteen circles intersect by fives in the pairs of opposite points A, by three's in the pairs of opposite points B, and by two's in the pairs of opposite points Θ; the mutual inclinations of successive circles at the points A, B, Θ being $= 36^\circ$, 60° and 90° respectively. The whole number $15.14, = 210$ of the intersections of the circles two and two together is thus made up of the 12 points A each counting 10 times, the 20 points B each counting 3 times, and the 30 points Θ each counting once; $210 = 120 + 60 + 30$.

98. The angular magnitudes which present themselves are all obtained from the dodecahedral pentagon, shown in the annexed figure, and in which the angle subtended by a side at the centre is $= 72^\circ$, and the angle between two adjacent sides is $= 120^\circ$.



We write $A\Theta = \alpha$, $B\Theta = \beta$, $AB = \gamma$, $B_1B_4 = x$, $\angle\, B_1B_4B = \theta$, $\Theta B_4 = g$, $\angle\, \Theta B_4 B = \phi$.

From the triangle $A\Theta B$, the angles of which are 36°, 90°, 60° and the opposite sides β, γ, α, we find the values of α, β, γ, and these are such that $\alpha + \beta + \gamma = 1$.

From the triangle B_4BB_1 where the sides B_4B, BB_1, and the included angle are 2β, 2β, 120°, we have the opposite side x, and the other two angles each $= \theta$.

From the triangle $B_4B\Theta$, where the sides B_4B, $B\Theta$, and the included angle are 2β, β, 120°, we find the opposite side g, the angle $BB_4\Theta, = \phi$, and the angle $B_4\Theta B, = 45^\circ$.

Hence each of the angles $B_4\Theta B$, $B_2\Theta B$, being $= 45^\circ$, the angle $B_4\Theta B$ is $= 90^\circ$: in this triangle the hypothenuse B_2B_4 is $= x$, and each of the other two sides is $= g$: whence we have $\cos x = \cos^2 g$ as is in fact the case, and moreover the values give $x + 2g = 180^\circ$. Also each of the other angles is found to be $= 60^\circ$; that is we have $\angle\, B_2B\Theta = 60^\circ$, or the whole angle at B_4 being $= 120^\circ$, the sum of the remaining angles $B_3B_4B_2$ and $BB_4\Theta$ is $= 60^\circ$: that is $\theta + \phi = 60^\circ$.

From the triangle $\Theta\beta_1\Theta'$ where the two sides and the included angle are β, β, 120°, we find $\Theta\Theta' = 36^\circ$.

And from the triangle $\Theta B_4\Theta''$, where the two sides and the included angle are g, g and $(120^\circ - 2\phi =)\ 2\theta$, we find $\Theta\Theta'' = 60^\circ$.

99. We thus arrive at the following Table:

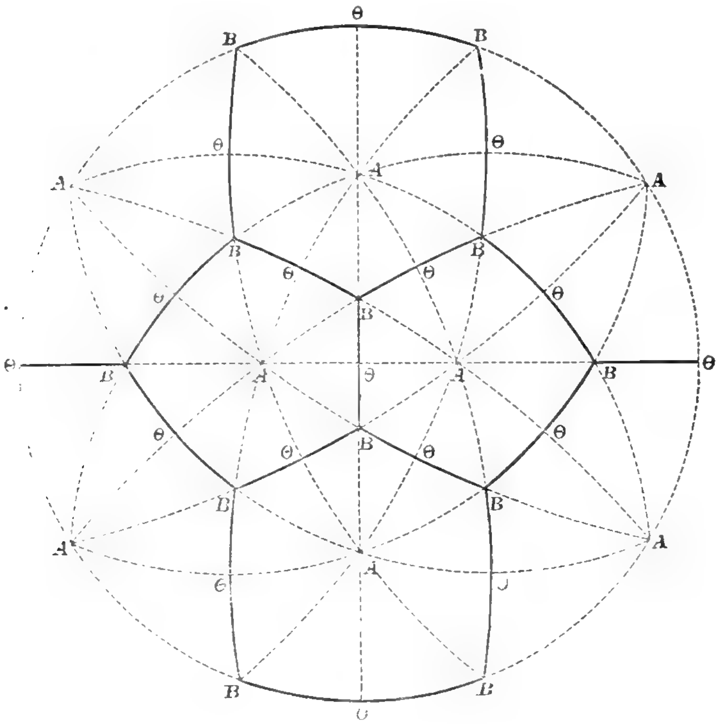
			sin	cos
$A\Theta$	α	$31^0\ 43'$	$\sqrt{\frac{5-\sqrt{5}}{10}}$	$\sqrt{\frac{5+\sqrt{5}}{10}}$
$B\Theta$	β	$20^0\ 55'$	$\frac{\sqrt{5}-1}{2\sqrt{3}}$	$\frac{\sqrt{5}+1}{2\sqrt{3}}$
AB	γ	$37^0\ 22'$	$\sqrt{\frac{10-2\sqrt{5}}{15}}$	$\sqrt{\frac{5+2\sqrt{5}}{15}}$
(BB)	x	$70^0\ 32'$	$\frac{2\sqrt{2}}{3}$	$\frac{1}{3}$
$(B\Theta)$	g	54 44	$\frac{\sqrt{2}}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$
BBB	θ	$37^0\ 46'$	$\frac{\sqrt{3}}{2\sqrt{2}}$	$\frac{\sqrt{5}}{2\sqrt{2}}$
$B\Theta B$	ϕ	22 14	$\frac{\sqrt{3}\,(\sqrt{5}-1)}{4\sqrt{2}}$	$\frac{\sqrt{5}+3}{4\sqrt{2}}$
	2α	63 26	$\frac{2}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$
	2β	41 50	$\frac{2}{3}$	$\frac{\sqrt{5}}{3}$
	2γ	74 44	$\frac{2\,(\sqrt{5}+1)}{3\sqrt{5}}$	$\frac{4-\sqrt{5}}{3\sqrt{5}}$
	$\alpha-\beta$		$\sqrt{\frac{5-2\sqrt{5}}{15}}$	$\sqrt{\frac{10+2\sqrt{5}}{15}}$
		18^0	$\frac{\sqrt{5}-1}{4}$	$\sqrt{\frac{5+\sqrt{5}}{8}}$
$\Theta\Theta$		36^0	$\sqrt{\frac{5-\sqrt{5}}{8}}$	$\frac{\sqrt{5}+1}{4}$

Where as above

$$\begin{aligned} \alpha+\beta+\gamma &= 90^\circ, \\ x + 2g \quad &= 180^\circ, \\ \theta + \phi \quad &= 60. \end{aligned}$$

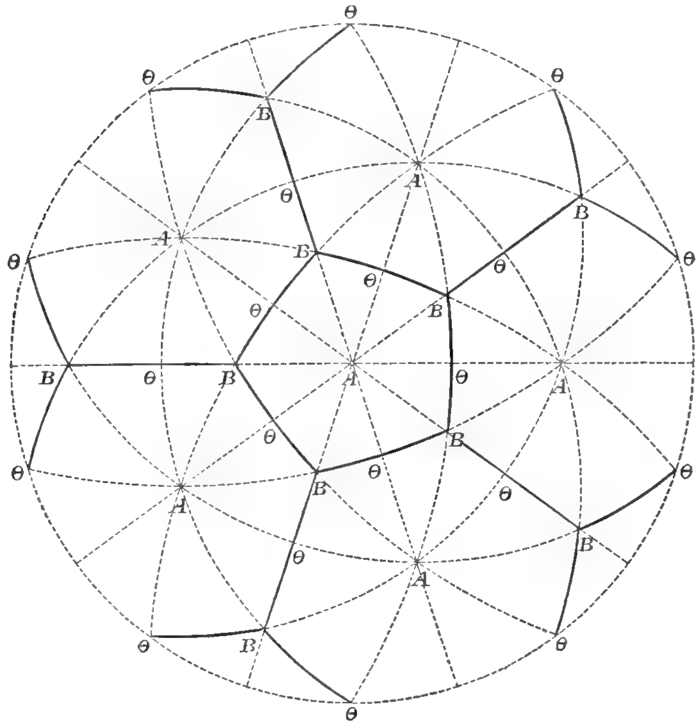
100. We now construct three figures of the points A, B, Θ; viz. these are stereographic projections, each showing the Northern hemisphere projected on the plane of the equator by lines drawn to the South Pole: hence for any pair of opposite points not on the equator only the point in the Northern hemisphere is shown: but for a pair of opposite points on the equator the two points are each of them shown. In fig. 1 the North Pole is taken to be a point Θ; in fig. 2 it is a point A; and in fig. 3 it is a point B. The Position of any point on the sphere is determined by its N.P.D. and its longitude (measured from an arbitrary origin, say from the point E of the centre left-handedly): and in the three figures the positions are as follows.

101. Fig. 1. Pole at Θ.



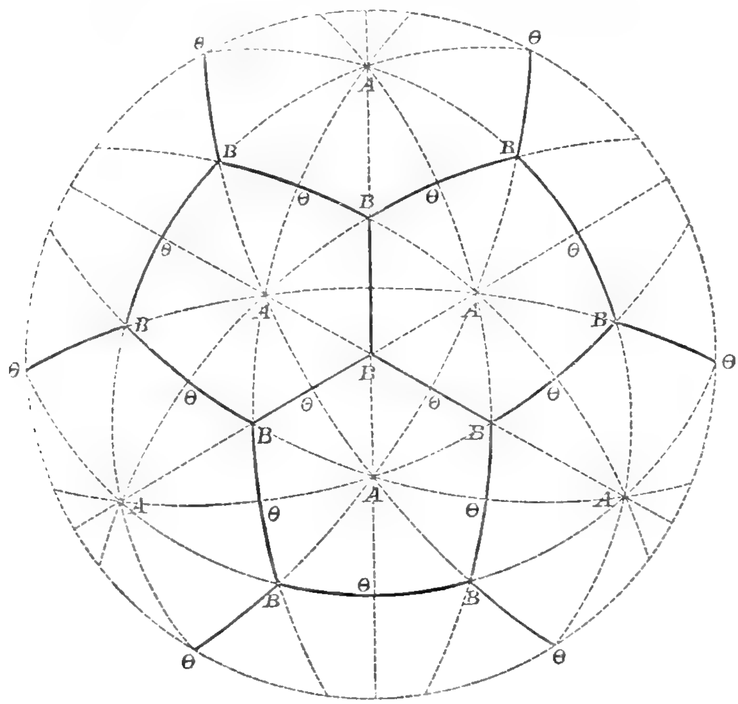
	N.P.D.'s	Longitudes.
2A	$a=$ 31° 43′	0°, 180°
2A	$90°-a=$ 58 17	90, 270
4A	90	(0, 180) $\pm a=$ 31° 43′
2A	$90°+a=$ 121 43	90, 270
2A	$180°-a=$ 148 17	0, 180
2B	$\beta=$ 20° 55′	90°, 270°
4B	$g=$ 54 44	45, 135, 225°, 315°
2B	$90°-\beta=$ 69 5	0, 180
4B	90	(90, 270) $\pm\beta=$ 20° 55′
2B	$90°+\beta=$ 110 55	0, 180
4B	$180-g=$ 125 16	45, 135, 225, 315
2B	$180°-\beta=$ 159 5	90, 270
1Θ	0°	—
4Θ	36	(90°, 270°) $\pm a=$ 31° 43′
4Θ	60	(0, 180) $\pm\beta=$ 20 55
4Θ	72	(90, 270) $\pm a=$ 31 43
4Θ	90	0, 90, 180°, 270°
4Θ	108	(90, 270) $\pm a=$ 31 43
4Θ	120	(0, 180) $\pm\beta=$ 20 55
4Θ	144	(90, 270) $\pm a=$ 31 43
1Θ	180	—

102. Fig. 2, Pole at A.



	N.P.D.'s	Longitudes.
A	0	–
$5A$	$2a=63^0\ 26'$	$0^0\ 72^0\ 144^0\ 216^0\ 288^0$
$5A$	$180^0-2a=116\ 34$	36 108 180 252 324
A	180	–
$5B$	$\gamma=\ 37\ 22$	36 108 180 252 324
$5B$	$90^0-a+\beta=\ 79\ 12$	36 108 180 252 324
$5B$	$90\ +a-\beta=100\ 48$	0 72 144 216 288
$5B$	$180\ \ -\gamma=142\ 38$	0 72 144 216 288
5Θ	$a=\ 31\ 43$	0 72 144 216 288
5Θ	$90^0-a=\ 58\ 17$	36 108 180 252 324
10Θ	90	$(36\ 108\ 180\ 252\ 324)\pm18^0$
5Θ	$90\ +a=121\ 43$	0 72 144 216 288
5Θ	$180\ -a=144\ 17$	36 108 180 252 324

103. Fig. 3, Pole at B.



	N.P.D.'s	Longitudes.
$3A$	$\gamma = 37^0\ 22'$	$30^0\ 150^0\ 270^0$
$3A$	$90^0 - a + \beta = 79\ 12$	90 210 330
$3A$	$90 + a - \beta = 100\ 48$	30 150 270
$3A$	$180 - \gamma = 142\ 38$	90 210 330
B	0	–
$3B$	$2\beta = 41\ 50$	90 210 330
$6B$	$x = 70\ 32$	$(30\ 150\ 270) \pm \vartheta = 37^0\ 46'$
$6B$	$180^0 - x = 109\ 28$	$(90\ 210\ 330) \pm \vartheta = 37\ 46$
$3B$	$180 - 2\beta = 138\ 10$	30 150 270
B	180	–
3Θ	$\beta = 20\ 55$	90 210 330
6Θ	$g = 54\ 44$	$(90\ 210\ 330) \pm \phi = 22^0\ 14'$
3Θ	$90^0 - \beta = 69\ 5$	30 150 270
6Θ	90	$0\ 60\ 120\ 180^0\ 240^0\ 300^0$
3Θ	$90 + \beta = 110\ 55$	90 210 330
6Θ	$180 - g = 125\ 16$	$(30\ 150\ 270) \pm \phi = 22^0\ 14'$
3Θ	$180 - \beta = 159\ 5$	30 150 270

The groups of homographic transformations, resumed. Art. Nos. 104 to 117.

104. The axes of rotation for the dodecahedron and the icosahedron are 15 axes each through a pair of opposite points Θ, 6 axes each through a pair of opposite points A, and 10 axes each through a pair of opposite points B; or say 15 Θ-axes, 10 B-axes and 6 A-axes: the corresponding angles of rotation are 180°, 72° and 120°; so that (excluding in each case the original position or that of a rotation 0) we have in respect of each Θ-axis 1 position, in respect of each A-axis 4 positions, and in respect of each B-axis 2 positions; in all, including the original position, $1 + 15 + (6 \times 4) + (10 \times 2), = 60$ positions, that is a group of 60 rotations.

To find in any one of the three forms the group of homographic transformations, we can in each case obtain from the foregoing tables the values $\cos f$, $\cos g$, $\cos h$ of the cosine-inclination of an axis of rotation to the axes of coordinates, and thence calculate the values of

$$\lambda,\ \mu,\ \nu = \tan \tfrac{1}{2}\vartheta \cos f, \qquad \tan \tfrac{1}{2}\vartheta \cos g, \qquad \tan \tfrac{1}{2}\vartheta \cos h,$$

and thence the values of

$$A,\ B,\ C,\ D = -\nu - i, \quad \lambda + i\mu, \quad \lambda - i\mu, \quad \nu - i,$$

viz.: in the case of a Θ-axis ϑ is $= 180^\circ$, (so that here $\tan \frac{1}{2}\vartheta = \infty$, or the values of A, B, C, D are $= -\nu,\ \lambda + i\mu,\ \lambda - i\mu,\ \nu$, that is $-\cos h,\ \cos f + i \cos g,\ \cos f - i \cos g,\ \cos h$); in the case of a B-axis the values are $\vartheta = 120^\circ, 240^\circ$, and therefore $\tan \frac{1}{2}\vartheta = \pm\sqrt{3}$; and in the case of an A-axis, they are $\vartheta = 72^\circ, 144^\circ, 216^\circ, 288^\circ$, and therefore

$$\tan \tfrac{1}{2}\vartheta = \pm \frac{\sqrt{10 + 2\sqrt{5}}}{\sqrt{5} - 1}, \quad \pm \frac{\sqrt{10 - 2\sqrt{5}}}{\sqrt{5} + 1}.$$

105. The Θ-form was first given in my paper of 1879, but in obtaining it I used results given in the paper of 1877. As regards the identification with the substitution-symbols, since there is nothing to distinguish *inter se* the letters a, b, c, d, e, any transformation A, B, C, D of the fifth order might have been taken for $abcde$, but No. 37 of the group having been taken for this substitution $abcde$, I do not recall in what manner I found that consistently herewith the transformation No. 2 $(-1, 0, 0, 1$, that is s into $-s)$ of the second order could be taken for $ab \,.\, cd$. But there is no sub-group of an order divisible by 5; and hence, these two transformations being identified with the two substitutions, the other transformations correspond each of them to a determinate substitution.

106. Homographic Transformations. The group of 60: Pole at Θ.

	$(Ax$	$+B)$ $\div$	$(Cx$	$+D)$	
1	1	0	0	1	1
2	-1	0	0	1	$ab.cd$
3	0	1	1	0	$ac.bd$
4	0	1	-1	0	$ad.bc$
5	2	$-3+\sqrt{5}+i(\ \ 1-\sqrt{5})$	$-3+\sqrt{5}+i(-1+\sqrt{5})$	-2	$bc.dc$
6	2	$-3+\sqrt{5}+i(-1+\sqrt{5})$	$-3+\sqrt{5}+i(\ \ 1-\sqrt{5})$	-2	$ae.bc$

7	2	$3-\sqrt{5}+i(-1+\sqrt{5})$	$3-\sqrt{5}+i(\ \ 1-\sqrt{5})$	-2	*ad.ce*
8	2	$3-\sqrt{5}+i(\ \ 1-\sqrt{5})$	$3-\sqrt{5}+i(-1+\sqrt{5})$	-2	*ad.be*
9	2	$-1-\sqrt{5}+i(\ \ 1-\sqrt{5})$	$-1-\sqrt{5}+i(-1+\sqrt{5})$	-2	*ae.cd*
10	2	$-1-\sqrt{5}+i(-1+\sqrt{5})$	$-1-\sqrt{5}+i(\ \ 1-\sqrt{5})$	-2	*ab.de*
11	2	$1+\sqrt{5}+i(-1+\sqrt{5})$	$1+\sqrt{5}+i(\ \ 1-\sqrt{5})$	-2	*be.cd*
12	2	$1+\sqrt{5}+i(\ \ 1-\sqrt{5})$	$1+\sqrt{5}+i(-1+\sqrt{5})$	-2	*ab.ce*
13	2	$-1-\sqrt{5}+i(-3-\sqrt{5})$	$-1-\sqrt{5}+i(\ \ 3+\sqrt{5})$	-2	*ac.be*
14	2	$-1-\sqrt{5}+i(\ \ 3+\sqrt{5})$	$-1-\sqrt{5}+i(-3-\sqrt{5})$	-2	*bd.ce*
15	2	$1+\sqrt{5}+i(\ \ 3+\sqrt{5})$	$1+\sqrt{5}+i(-3-\sqrt{5})$	-2	*ae.bd*
16	2	$1+\sqrt{5}+i(-3-\sqrt{5})$	$1+\sqrt{5}+i(\ \ 3+\sqrt{5})$	-2	*ac.de*
17	$-i$	i	1	1	*abc*
18	-1	i	1	i	*acb*
19	1	$-i$	1	i	*adc*
20	$-i$	$-i$	1	-1	*acd*
21	i	i	1	-1	*adb*
22	1	i	1	$-i$	*abd*
23	-1	$-i$	1	$-i$	*bcd*
24	i	$-i$	1	1	*bdc*
25	$-1-\sqrt{5}+i(\ \ 3+\sqrt{5})$	2	-2	$-1-\sqrt{5}+i(-3-\sqrt{5})$	*aec*
26	$1+\sqrt{5}+i(\ \ 3+\sqrt{5})$	2	-2	$1+\sqrt{5}+i(-3-\sqrt{5})$	*ace*
27	$1+\sqrt{5}+i(-3-\sqrt{5})$	2	-2	$1+\sqrt{5}+i(\ \ 3+\sqrt{5})$	*bed*
28	$-1-\sqrt{5}+i(-3-\sqrt{5})$	2	-2	$-1-\sqrt{5}+i(\ \ 3+\sqrt{5})$	*bde*
29	$-3+\sqrt{5}+i(\ \ 1-\sqrt{5})$	2	2	$3-\sqrt{5}+i(\ \ 1-\sqrt{5})$	*bec*
30	$-3+\sqrt{5}+i(-1+\sqrt{5})$	2	2	$3-\sqrt{5}+i(-1+\sqrt{5})$	*bce*
31	$3-\sqrt{5}+i(-1+\sqrt{5})$	2	2	$-3+\sqrt{5}+i(-1+\sqrt{5})$	*aed*
32	$3-\sqrt{5}+i(\ \ 1-\sqrt{5})$	2	2	$-3+\sqrt{5}+i(\ \ 1-\sqrt{5})$	*ade*
33	2	$-1-\sqrt{5}+i(-1+\sqrt{5})$	$1+\sqrt{5}+i(-1+\sqrt{5})$	2	*cde*
34	2	$1+\sqrt{5}+i(\ \ 1-\sqrt{5})$	$-1-\sqrt{5}+i(\ \ 1-\sqrt{5})$	2	*ced*
35	2	$-1-\sqrt{5}+i(\ \ 1-\sqrt{5})$	$1+\sqrt{5}+i(\ \ 1-\sqrt{5})$	2	*aeb*
36	2	$1+\sqrt{5}+i(-1+\sqrt{5})$	$-1-\sqrt{5}+i(-1+\sqrt{5})$	2	*abe*
37	$-1-\sqrt{5}+i(-3-\sqrt{5})$	2	2	$1+\sqrt{5}+i(-3-\sqrt{5})$	*abcde*
38	$-1-\sqrt{5}+i(\ \ 1-\sqrt{5})$	2	2	$1+\sqrt{5}+i(\ \ 1-\sqrt{5})$	*acebd*
39	$-1-\sqrt{5}+i(-1+\sqrt{5})$	2	2	$1+\sqrt{5}+i(-1+\sqrt{5})$	*adbec*
40	$-1-\sqrt{5}+i(\ \ 3+\sqrt{5})$	2	2	$1+\sqrt{5}+i(\ \ 3+\sqrt{5})$	*aedcb*
41	$1+\sqrt{5}+i(\ \ 3+\sqrt{5})$	2	2	$-1-\sqrt{5}+i(\ \ 3+\sqrt{5})$	*adceb*
42	$1+\sqrt{5}+i(-1+\sqrt{5})$	2	2	$-1-\sqrt{5}+i(-1+\sqrt{5})$	*acbde*
43	$1+\sqrt{5}+i(\ \ 1-\sqrt{5})$	2	2	$-1-\sqrt{5}+i(\ \ 1-\sqrt{5})$	*aedbc*
44	$1+\sqrt{5}+i(-3-\sqrt{5})$	2	2	$-1-\sqrt{5}+i(-3-\sqrt{5})$	*abecd*
45	$-1-\sqrt{5}+i(-1+\sqrt{5})$	2	-2	$-1-\sqrt{5}+i(\ \ 1-\sqrt{5})$	*acbed*
46	$-3+\sqrt{5}+i(-1+\sqrt{5})$	2	-2	$-3+\sqrt{5}+i(\ \ 1-\sqrt{5})$	*abdce*
47	$3-\sqrt{5}+i(-1+\sqrt{5})$	2	-2	$3-\sqrt{5}+i(\ \ 1-\sqrt{5})$	*aecdb*
48	$1+\sqrt{5}+i(-1+\sqrt{5})$	2	-2	$1+\sqrt{5}+i(\ \ 1-\sqrt{5})$	*adebc*
49	$1+\sqrt{5}+i(\ \ 1-\sqrt{5})$	2	-2	$1+\sqrt{5}+i(-1+\sqrt{5})$	*aecbd*
50	$3-\sqrt{5}+i(\ \ 1-\sqrt{5})$	2	-2	$3-\sqrt{5}+i(-1+\sqrt{5})$	*acdeb*
51	$-3+\sqrt{5}+i(\ \ 1-\sqrt{5})$	2	-2	$-3+\sqrt{5}+i(-1+\sqrt{5})$	*abedc*
52	$-1-\sqrt{5}+i(\ \ 1-\sqrt{5})$	2	-2	$-1-\sqrt{5}+i(-1+\sqrt{5})$	*adbce*
53	2	$-3+\sqrt{5}+i(-1+\sqrt{5})$	$3-\sqrt{5}+i(-1+\sqrt{5})$	2	*aebdc*
54	2	$-1-\sqrt{5}+i(\ \ 3+\sqrt{5})$	$1+\sqrt{5}+i(\ \ 3+\sqrt{5})$	2	*abced*
55	2	$1+\sqrt{5}+i(-3-\sqrt{5})$	$-1-\sqrt{5}+i(-3-\sqrt{5})$	2	*adecb*
56	2	$3-\sqrt{5}+i(\ \ 1-\sqrt{5})$	$-3+\sqrt{5}+i(\ \ 1-\sqrt{5})$	2	*acdbe*
57	2	$-3+\sqrt{5}+i(\ \ 1-\sqrt{5})$	$3-\sqrt{5}+i(\ \ 1-\sqrt{5})$	2	*abdec*
58	2	$-1-\sqrt{5}+i(-3-\sqrt{5})$	$1+\sqrt{5}+i(-3-\sqrt{5})$	2	*adcbe*
59	2	$1+\sqrt{5}+i(\ \ 3+\sqrt{5})$	$-1-\sqrt{5}+i(\ \ 3+\sqrt{5})$	2	*aebcd*
60	2	$3-\sqrt{5}+i(-1+\sqrt{5})$	$-3+\sqrt{5}+i(-1+\sqrt{5})$	2	*acedb*

107. Taking out of the foregoing group of 60 a group of 12 contained in it, viz. that corresponding to the positive substitutions of the four letters *abcd*, it is easy to see that there is a transformation $(i, 0, 0, 1)$, that is, s into is, which can be taken for the substitution *adbc*, and to complete thence the group of 24. And we have thus the following Table.

Groups of 12 and 24. Pole at Θ.

	$(Ax$	$+B)$	$\div\ (Cx$	$+D)$	
1	1	0	0	1	1
2	−1	0	0	1	*ab.cd*
3	0	1	1	0	*ac.bd*
4	0	1	−1	0	*ad.bc*
5	$-i$	i	1	1	*abc*
6	−1	i	1	i	*acb*
7	1	$-i$	1	i	*adc*
8	$-i$	$-i$	1	−1	*acd*
9	i	i	1	−1	*adb*
10	1	i	1	$-i$	*abd*
11	−1	$-i$	1	$-i$	*bcd*
12	i	$-i$	1	1	*bdc*
13	i	0	0	1	*adbc*
14	$-i$	0	0	1	*acbd*
15	0	i	1	0	*cd*
16	0	i	−1	0	*ab*
17	1	−1	1	1	*acdb*
18	$-i$	−1	1	i	*bd*
19	i	1	1	i	*abcd*
20	1	1	1	−1	*bc*
21	−1	−1	1	−1	*abdc*
22	i	−1	1	$-i$	*ac*
23	$-i$	1	1	$-i$	*adcb*
24	−1	1	1	1	*ad*

108. The group of 60 was obtained in the A-form by Gordan, in his paper. The passage from the Θ-form to the A-form is made as follows: let X, Y, Z be the coordinates of a point when the axes are as in the Θ-form, X_1, Y_1, Z_1 the coordinates of the same point when the axes are as in the A-form: we may write

$$X,\ Y,\ Z = \mathrm{b}X_1 - \mathrm{a}Z_1 : Y_1 : \mathrm{a}X_1 + \mathrm{b}Z, \quad \text{where a, b} = \sqrt{\frac{5-\sqrt{5}}{10}},\ \sqrt{\frac{5+\sqrt{5}}{10}};$$

then if the equations of an axis of rotation referred to the first set of coordinates are $X : Y : Z = L : M : N$, those of the same axis referred to the second set of coordinates are $\mathrm{b}X_1 + \mathrm{a}Z_1 : Y_1 : -\mathrm{a}X_1 + cZ_1 = L : M : N$; or taking these to be $X_1 : Y_1 : Z_1 = L_1 : M_1 : N_1$, we may write $L_1, M_1, N_1 = \mathrm{b}L + \mathrm{a}N,\ M,\ -\mathrm{a}L + \mathrm{b}N$: these values are such that $L_1^2 + M_1^2 + N_1^2 = L^2 + M^2 + N^2$, and hence λ, μ, ν and λ_1, μ_1, ν_1 being the rotations, we may write

$L, M, N = \vartheta\lambda, \vartheta\mu, \vartheta\nu$; $L_1, M_1, N_1 = \vartheta\lambda_1, \vartheta\mu_1, \vartheta\nu_1$; where ϑ has the same value in each set of equations. From the equations

$$A : B : C : D = -\nu - i : \lambda + i\mu : \lambda - i\mu : \nu - i,$$

we have

$$\begin{aligned} B + C : B - C : D - A : D + A &= \lambda : i\mu : \nu : -i \\ &= L : iM : N : -i\vartheta, \end{aligned}$$

and similarly

$$B_1 + C_1 : B_1 - C_1 : D_1 - A_1 : D_1 + A_1 = L_1 : iM_1 : N_1 : -i\vartheta.$$

Hence we may write

$$\begin{aligned} B_1 + C_1 &= \mathrm{b}(B + C) + \mathrm{a}(D - A), \\ B_1 - C_1 &= B - C, \\ D_1 - A_1 &= -\mathrm{a}(B + C) + \mathrm{b}(D - A), \\ D_1 + A_1 &= D + A. \end{aligned}$$

Or say,

$$\begin{aligned} A_1 &= \mathrm{a}(B + C) - \mathrm{b}(D - A) + (D + A), \\ B_1 &= \mathrm{b}(B + C) + \mathrm{a}(D - A) + (B - C), \\ C_1 &= \mathrm{b}(B + C) + \mathrm{a}(D - A) - (B - C), \\ D_1 &= -\mathrm{a}(B + C) + \mathrm{b}(D - A) + (D + A), \end{aligned}$$

which are the values for a transformation (A_1, B_1, C_1, D_1) in the A-form: of course as only the ratios are material, the values may be multiplied by any common factor.

109. The results are exhibited in terms of ϵ an imaginary fifth root of unity: taking $\epsilon = \cos 72^\circ + i \sin 72^\circ$, we have

$$\epsilon, \epsilon^4 = \frac{\sqrt{5} - 1}{4} \pm i\sqrt{\frac{5 + \sqrt{5}}{8}},$$

$$\epsilon^2, \epsilon^3 = -\frac{\sqrt{5} + 1}{4} \pm i\sqrt{\frac{5 - \sqrt{5}}{8}};$$

where the upper signs belong to ϵ, ϵ^2 and the lower to ϵ^4, ϵ^3. It may be remarked that

$$\frac{1}{\mathrm{a}} = \sqrt{\frac{5 + \sqrt{5}}{2}}, \quad \frac{1}{\mathrm{b}} = \sqrt{\frac{5 - \sqrt{5}}{2}}, \quad \frac{\mathrm{b}}{\mathrm{a}} = \frac{\sqrt{5} + 1}{2}, \quad \frac{\mathrm{a}}{\mathrm{b}} = \frac{\sqrt{5} - 1}{2}.$$

For instance, we have in the Θ-group $(A, B, C, D) = (-1, 0, 0, 1)$; $ab.cd$: and thence in the A-group $A_1, B_1, C_1, D_1 = (-2\mathrm{b}, 2\mathrm{a}, 2\mathrm{a}, 2\mathrm{b})$; $ab.cd$: or say this is $\left(-1, \frac{\mathrm{a}}{\mathrm{b}}, \frac{\mathrm{a}}{\mathrm{b}}, 1\right)$, $= (-1, \epsilon + \epsilon^4, \epsilon + \epsilon^4, 1)$; which is in the Table given as $(-\epsilon^3, \epsilon^2 + \epsilon^4, \epsilon^2 + \epsilon^4, \epsilon^3)$; $ab.cd$.

By effecting the passage to the A-group in this manner we of course obtain the proper substitution corresponding to each transformation: but I found it easier starting from two transformations and the corresponding substitutions, to obtain thence by successive compositions the entire group.

110. Homographic Transformations. The group of 60. Pole at A.

	Θ No.	$(As$	$+B)$	$\div(Cs$	$+D)$	
1	1	1			1	1
2	4	0	-1	1	0	*ad . bc*
3	13	0	$-\epsilon^4$	1	0	*ac . be*
4	9	0	$-\epsilon^3$	1	0	*ae . cd*
5	10	0	$-\epsilon^2$	1	0	*ab . de*
6	14	0	$-\epsilon$	1	0	*bd . ce*
7	6	$\epsilon+\epsilon^2$	ϵ^4	1	$-(\epsilon+\epsilon^3)$	*ae . bc*
8	5	$\epsilon+\epsilon^3$	1	ϵ^4	$-(\epsilon+\epsilon^3)$	*bc . de*
9	16	$\epsilon+\epsilon^3$	ϵ	ϵ^3	$-(\epsilon+\epsilon^3)$	*ac . de*
10	3	$\epsilon+\epsilon^3$	ϵ^2	ϵ^2	$-(\epsilon+\epsilon^3)$	*ac . bd*
11	15	$\epsilon+\epsilon^3$	ϵ^3	ϵ	$-(\epsilon+\epsilon^3)$	*ae . bd*
12	12	-1	$\epsilon+\epsilon^3$	$\epsilon^2+\epsilon^4$	1	*ab . ce*
13	11	$-\epsilon$	ϵ^3+1	$\epsilon^2+\epsilon^4$	ϵ	*be . cd*
14	7	$-\epsilon^2$	$1+\epsilon^2$	$\epsilon^2+\epsilon^4$	ϵ^2	*ad . ce*
15	2	$-\epsilon^3$	$\epsilon^2+\epsilon^4$	$\epsilon^2+\epsilon^4$	ϵ^3	*ab . cd*
16	8	$-\epsilon^4$	$\epsilon^4+\epsilon$	$\epsilon^2+\epsilon^4$	ϵ^4	*ad . be*
17	21	ϵ^3+1	ϵ	1	$-(\epsilon+\epsilon^3)$	*adb*
18	35	ϵ^3+1	ϵ^2	ϵ^4	$-(\epsilon+\epsilon^3)$	*aeb*
19	30	ϵ^3+1	ϵ^3	ϵ^3	$-(\epsilon+\epsilon^3)$	*bce*
20	34	ϵ^3+1	ϵ^4	ϵ^2	$-(\epsilon+\epsilon^3)$	*ced*
21	19	ϵ^3+1	1	ϵ	$-(\epsilon+\epsilon^3)$	*adc*
22	33	$\epsilon+\epsilon^4$	ϵ^2	1	$-(\epsilon+\epsilon^3)$	*cde*
23	20	$\epsilon+\epsilon^4$	ϵ^3	ϵ^4	$-(\epsilon+\epsilon^3)$	*acd*
24	22	$\epsilon+\epsilon^4$	ϵ^4	ϵ^3	$-(\epsilon+\epsilon^3)$	*abd*
25	36	$\epsilon+\epsilon^4$	1	ϵ^2	$-(\epsilon+\epsilon^3)$	*abe*
26	29	$\epsilon+\epsilon^4$	ϵ	ϵ	$-(\epsilon+\epsilon^3)$	*bec*
27	31	$-\epsilon$	$\epsilon^2+\epsilon^4$	$\epsilon^2+\epsilon^4$	1	*aed*
28	17	$-\epsilon^2$	$\epsilon^4+\epsilon$	$\epsilon^2+\epsilon^4$	ϵ	*abc*
29	27	$-\epsilon^3$	$\epsilon+\epsilon^3$	$\epsilon^2+\epsilon^4$	ϵ^2	*bed*
30	25	$-\epsilon^4$	ϵ^3+1	$\epsilon^2+\epsilon^4$	ϵ^3	*aec*
31	23	-1	$1+\epsilon^2$	$\epsilon^2+\epsilon^4$	ϵ^4	*bcd*
32	24	$-\epsilon^4$	$1+\epsilon^2$	$\epsilon^2+\epsilon^4$	1	*bdc*
33	32	-1	$\epsilon^2+\epsilon^4$	$\epsilon^2+\epsilon^4$	ϵ	*ade*
34	18	$-\epsilon$	$\epsilon^4+\epsilon$	$\epsilon^2+\epsilon^4$	ϵ^2	*acb*
35	28	$-\epsilon^2$	$\epsilon+\epsilon^3$	$\epsilon^2+\epsilon^4$	ϵ^3	*bde*
36	26	$-\epsilon^3$	ϵ^3+1	$\epsilon^2+\epsilon^4$	ϵ^4	*ace*
37	44	ϵ	0	0	1	*abecd*
38	43	ϵ^2	0	0	1	*aedbc*
39	42	ϵ^3	0	0	1	*acbde*
40	41	ϵ^4	0	0	1	*adceb*
41	38	$\epsilon^2+\epsilon^4$	1	1	$-(\epsilon+\epsilon^3)$	*acebd*
42	46	$\epsilon^2+\epsilon^4$	ϵ	ϵ^4	$-(\epsilon+\epsilon^3)$	*abdce*
43	58	$\epsilon^2+\epsilon^4$	ϵ^2	ϵ^3	$-(\epsilon+\epsilon^3)$	*adcbe*
44	55	$\epsilon^2+\epsilon^4$	ϵ^3	ϵ^2	$-(\epsilon+\epsilon^3)$	*adecb*
45	50	$\epsilon^2+\epsilon^4$	ϵ^4	ϵ	$-(\epsilon+\epsilon^3)$	*acdeb*
46	51	$1+\epsilon^2$	ϵ^3	1	$-(\epsilon+\epsilon^3)$	*abcde*
47	39	$1+\epsilon^2$	ϵ^4	ϵ^4	$-(\epsilon+\epsilon^3)$	*adbec*
48	47	$1+\epsilon^2$	1	ϵ^3	$-(\epsilon+\epsilon^3)$	*aecdb*
49	59	$1+\epsilon^2$	ϵ	ϵ^2	$-(\epsilon+\epsilon^3)$	*aebcd*
50	54	$1+\epsilon^2$	ϵ^2	ϵ	$-(\epsilon+\epsilon^3)$	*abced*

51	56	$-\epsilon^2$	ϵ^3+1	$\epsilon^2+\epsilon^4$	1	*acdbe*
52	49	$-\epsilon^3$	$1+\epsilon^2$	$\epsilon^2+\epsilon^4$	ϵ	*aecbd*
53	37	$-\epsilon^4$	$\epsilon^2+\epsilon^4$	$\epsilon^2+\epsilon^4$	ϵ^2	*abcde*
54	45	-1	$\epsilon^4+\epsilon$	$\epsilon^2+\epsilon^4$	ϵ^3	*acbed*
55	57	$-\epsilon$	$\epsilon+\epsilon^3$	$\epsilon^2+\epsilon^4$	ϵ^4	*abdec*
56	48	$-\epsilon^3$	$\epsilon^4+\epsilon$	$\epsilon^2+\epsilon^4$	1	*adebc*
57	60	$-\epsilon^4$	$\epsilon+\epsilon^3$	$\epsilon^2+\epsilon^4$	ϵ	*acedb*
58	53	-1	ϵ^3+1	$\epsilon^2+\epsilon^4$	ϵ^2	*aebdc*
59	52	$-\epsilon$	$1+\epsilon^2$	$\epsilon^2+\epsilon^4$	ϵ^3	*adbce*
60	40	$-\epsilon^2$	$\epsilon^2+\epsilon^4$	$\epsilon^2+\epsilon^4$	ϵ^4	*aedcb*

111. Selecting the transformations which correspond to the positive substitutions *abcd*, and completing the group of 24 we have

Homographic Transactions. The groups of 12 and 24. Pole at A.

	$(As$	$+B)$	$\div(Cs$	$+D)$	
1	1	0	0	1	1
2	0	-1	1	0	*ad . bc*
3	$\epsilon+\epsilon^3$	ϵ^2	ϵ^2	$-(\epsilon+\epsilon^3)$	*ac . bd*
4	$-\epsilon^3$	$\epsilon^2+\epsilon^4$	$\epsilon^2+\epsilon^4$	ϵ^3	*ab . cd*
5	$-\epsilon^2$	$\epsilon+\epsilon^4$	$\epsilon^2+\epsilon^4$	ϵ	*abc*
6	$-\epsilon$	$\epsilon+\epsilon^4$	$\epsilon^2+\epsilon^4$	ϵ^2	*acb*
7	$\epsilon+\epsilon^4$	ϵ^3	ϵ^4	$-(\epsilon+\epsilon^3)$	*acd*
8	ϵ^3+1	1	ϵ	$-(\epsilon+\epsilon^3)$	*adc*
9	$\epsilon+\epsilon^4$	ϵ^4	ϵ^3	$-(\epsilon+\epsilon^3)$	*abd*
10	ϵ^3+1	ϵ	1	$-(\epsilon+\epsilon^3)$	*adb*
11	-1	$1+\epsilon^2$	$\epsilon^2+\epsilon^4$	ϵ^4	*bcd*
12	$-\epsilon^4$	$1+\epsilon^2$	$\epsilon^2+\epsilon^4$	1	*bdc*
13	1	$1+2\epsilon^4$	$1+2\epsilon$	-1	*ab*
14	$-\epsilon^2+\epsilon^3$	$1+\epsilon+3\epsilon^4$	$-1-3\epsilon-{}^4$	$\epsilon^2-\epsilon^3$	*cd*
15	$\epsilon^2-\epsilon^4$	$3+\epsilon+\epsilon^3$	$-1-3\epsilon-\epsilon^3$	$-\epsilon^2+\epsilon^4$	*ac*
16	$-1+\epsilon^2$	$-1-\epsilon^2+2\epsilon^4$	$1+\epsilon^2-2\epsilon^3$	$1-\epsilon^2$	*bd*
17	$2+\epsilon^3+2\epsilon^4$	$-2-2\epsilon^2-\epsilon^3$	$2\epsilon+\epsilon^3+2\epsilon^4$	$2\epsilon+2\epsilon^2+\epsilon^3$	*ad*
18	$2+2\epsilon^2+\epsilon^3$	$2+\epsilon^3+2\epsilon^4$	$-2\epsilon-2\epsilon^2-\epsilon^3$	$2\epsilon+\epsilon^3+2\epsilon^4$	*bc*
19	$-2+\epsilon+\epsilon^3$	$-\epsilon+\epsilon^3$	$-\epsilon+\epsilon^3$	$\epsilon+\epsilon^3-2\epsilon^4$	*abcd*
20	1	-1	1	1	*abdc*
21	1	1	-1	1	*acdb*
22	$1+\epsilon+3\epsilon^4$	$\epsilon^2-\epsilon^3$	$\epsilon^2-\epsilon^3$	$1+3\epsilon+\epsilon^4$	*acbd*
23	$1+2\epsilon^4$	-1	-1	$-1-2\epsilon$	*adbc*
24	$3+\epsilon+\epsilon^3$	$-\epsilon^2+\epsilon^4$	$-\epsilon^2+\epsilon^4$	$1+3\epsilon+\epsilon^3$	*adcb*

As an example of the calculation we have $(A, B, C, D) = (0, i, -1, 0)$; *ab*. Hence

$$A_1, B_1, C_1, D_1 = \Big(\mathrm{a}(i-1),\ \mathrm{b}(i-1)+i+1,\ \mathrm{b}(i-1)-(i+1),\ -\mathrm{a}(i+1)\Big), = \Big(1,\ \frac{\mathrm{b}-i}{\mathrm{a}},\ \frac{\mathrm{b}+i}{\mathrm{a}},\ -1\Big).$$

The second and third coefficients are $\frac{\sqrt{5}+1}{2} - i\sqrt{\frac{5+\sqrt{5}}{2}}$, $\frac{\sqrt{5}+1}{2} + i\sqrt{\frac{5+\sqrt{5}}{2}}$, which in virtue of the values of ϵ and ϵ^4 are $=1+2\epsilon^4$ and $1+2\epsilon$ respectively: or the result is as above $(1,\ 1+2\epsilon^4,\ 1+2\epsilon,\ -1)$.

112. In like manner for the passage from the Θ-form to the B-form, if X, Y, Z be the coordinates of a point on the spherical surface in regard to the Θ-axes, X_2, Y_2, Z_2 those of the same point in regard to the B-axes, we may write

$$X : Y : Z = X_2 : \mathrm{b}Y_2 + \mathrm{a}Z_2 : -\mathrm{a}Y_2 + \mathrm{b}Z_2, \text{ where a, b} = \frac{\sqrt{5}-1}{2\sqrt{3}}, \ \frac{\sqrt{5}+1}{2\sqrt{3}}.$$

Hence $X : Y : Z = L : M : N$, being the equations of an axis of rotation in the first set of coordinates, those of the same axis in the second set of coordinates will be $X_2 : \mathrm{b}Y_2 : \mathrm{a}Z_2 : -\mathrm{a}Y_2 + \mathrm{b}Z_2 = L : M : N$, or calling these $X_2 : Y_2 : Z_2 = L_2 : M_2 : N_2$, we have $L_2, M_2, N_2 = L : \mathrm{b}M - \mathrm{a}N : \mathrm{a}M + \mathrm{b}N$: these values are such that $L_2^2 + M_2^2 + N_2^2 = L^2 + M^2 + N^2$, or $\lambda, \mu, \nu, \lambda_2, \mu_2, \nu_2$ being the rotations, we have $L, M, N = \vartheta\lambda, \vartheta\mu, \vartheta\nu$; $L_2, M_2, N_2 = \vartheta\lambda_2, \vartheta\mu_2, \vartheta\nu_2$, where ϑ has the same value in the two sets of equations. We have thus

$$B + C : B - C : D - A : D + A = L : 2M : N : -i\vartheta,$$
$$B_2 + C_2 : B_2 - C_2 : D_2 - A_2 : D_2 + A_2 = L_2 : 2M_2 : N_2 : -i\vartheta,$$

and hence

$$\begin{aligned} B_2 + C_2 &= B + C, \\ B_2 - C_2 &= \mathrm{b}(B - C) - \mathrm{a}i(D - A), \\ D_2 - A_2 &= -\mathrm{a}i(B - C) + \mathrm{b}(D - A), \\ D_2 + A_2 &= D + A; \end{aligned}$$

and thence

$$\begin{aligned} A_2 &= \mathrm{a}i(B - C) - \mathrm{b}(D - A) + (D + A), \\ B_2 &= \mathrm{b}(B - C) - \mathrm{a}i(D - A) + (B + C), \\ C_2 &= -\mathrm{b}(B - C) + \mathrm{a}i(D - A) + (B + C), \\ D_2 &= -\mathrm{a}i(B - C) + \mathrm{b}(D - A) + (D + A). \end{aligned}$$

113. As an example of the transformation take

$$(A, B, C, D) = \left(2, -3 + \sqrt{5} + i(1 - \sqrt{5}), -3 + \sqrt{5} + i(-1 + \sqrt{5}), -2\right) \quad [bc \,.\, de]:$$

then $B - C$, $B + C$, $D - A$, $D + A = i(1 - \sqrt{5})$, $-3 + \sqrt{5}$, -2, 0; and thence

$$\begin{aligned} A_2 &= \frac{1}{2\sqrt{3}}(6 - 2\sqrt{5}) + \frac{1}{2\sqrt{3}}(2 + 2\sqrt{5}), \\ B_2 &= \frac{1}{2\sqrt{3}}(-4i) + \frac{1}{2\sqrt{3}}\left(2i(1 + \sqrt{5}) + (-3 + \sqrt{5})\right), \\ C_2 &= \frac{1}{2\sqrt{3}}(4i) + \frac{1}{2\sqrt{3}}\left(2i(1 - \sqrt{5}) + (-3 + \sqrt{5})\right), \\ D_2 &= \frac{1}{2\sqrt{3}}(-6 + 2\sqrt{5}) + \frac{1}{2\sqrt{3}}(-2 - 2\sqrt{5}), \end{aligned}$$

viz. multiplying by $2\sqrt{3}$, these are

$$8, \quad i(-6 + 2\sqrt{5}) + 2\sqrt{3}(-3 + \sqrt{5}), \quad i(6 - 2\sqrt{5}) + 2\sqrt{3}(-3 + \sqrt{5}), \quad -8,$$

that is, $$8, \quad (-6 + 2\sqrt{5})(i + \sqrt{3}), \quad (-6 + 2\sqrt{5})(-i + \sqrt{3}), \quad -8,$$

or since $2 + \sqrt{3} = -2i\omega$ and $-2 + \sqrt{3} = 2i\omega^2$, dividing by 4 these are

$$2, \quad i(3 - \sqrt{5})\,\omega, \quad i(-3 + \sqrt{5})\,\omega^2, \quad -2,$$

as in the table.

114. Homographic Transformations. The group of 60. Pole at B.

$$\omega = \tfrac{1}{2}(-1 + i\sqrt{3}).$$

	$(As$	$+B)$	$\div(Cs$	$+D)$	
1	1	0	0	1	1
2	0	1	1	0	$ac\,.\,bd$
3	0	ω	1	0	$ae\,.\,bd$
4	0	ω^2	1	0	$bd\,.\,ce$
5	2	$i(\ \ 3-\sqrt{5})$	$i(\ -3+\sqrt{5})$	-2	$ab\,.\,cd$
6	2	$i(-3-\sqrt{5})$	$i(\ \ 3+\sqrt{5})$	-2	$ad\,.\,bc$
7	2	$i(\ \ 3-\sqrt{5})\,\omega$	$i(\ -3+\sqrt{5})\,\omega^2$	-2	$bc\,.\,de$
8	2	$i(-3-\sqrt{5})\,\omega$	$i(\ \ 3+\sqrt{5})\,\omega^2$	-2	$be\,.\,cd$
9	2	$i(\ \ 3-\sqrt{5})\,\omega^2$	$i(\ -3+\sqrt{5})\,\omega$	-2	$ad\,.\,be$
10	2	$i(-3-\sqrt{5})\,\omega^2$	$i(\ \ 3+\sqrt{5})\,\omega$	-2	$ab\,.\,de$
11	2	$(-\sqrt{3}-i\sqrt{5})\,\omega$	$(-\sqrt{3}+i\sqrt{5})\,\omega^2$	-2	$ab\,.\,ce$
12	2	$-\sqrt{3}-i\sqrt{5}$	$-\sqrt{3}+i\sqrt{5}$	-2	$ac\,.\,be$
13	2	$(-\sqrt{3}-i\sqrt{5})\,\omega^2$	$(-\sqrt{3}+i\sqrt{5})\,\omega$	-2	$ae\,.\,bc$
14	2	$\sqrt{3}-i\sqrt{5}$	$\sqrt{3}+i\sqrt{5}$	-2	$ac\,.\,de$
15	2	$(\sqrt{3}-i\sqrt{5})\,\omega$	$(\sqrt{3}+i\sqrt{5})\,\omega^2$	-2	$ad\,.\,ce$
16	2	$(\sqrt{3}-i\sqrt{5})\,\omega^2$	$(\sqrt{3}+i\sqrt{5})\,\omega$	-2	$ae\,.\,cd$
17	ω	0	0	1	ace
18	ω^2	0	0	1	aec
19	$\sqrt{3}-i\sqrt{5}$	2	-2	$\sqrt{3}+i\sqrt{5}$	bed
20	$-\sqrt{3}-i\sqrt{5}$	2	-2	$-\sqrt{3}+i\sqrt{5}$	bde
21	$-\sqrt{3}-i\sqrt{5}$	$2\omega^2$	-2ω	$-\sqrt{3}+i\sqrt{5}$	bdc
22	$\sqrt{3}-i\sqrt{5}$	$2\omega^2$	-2ω	$\sqrt{3}+i\sqrt{5}$	bcd
23	$-\sqrt{3}-i\sqrt{5}$	2ω	$-2\omega^2$	$-\sqrt{3}+i\sqrt{5}$	abd
24	$\sqrt{3}-i\sqrt{5}$	2ω	$-2\omega^2$	$\sqrt{3}-i\sqrt{5}$	adb
25	$2\omega^2$	$-\sqrt{3}-i\sqrt{5}$	$-\sqrt{3}+i\sqrt{5}$	-2ω	abc
26	2ω	$-\sqrt{3}-i\sqrt{5}$	$-\sqrt{3}+i\sqrt{5}$	$-2\omega^2$	acb
27	$2\omega^2$	$-\sqrt{3}-i\sqrt{5}$	$(-\sqrt{3}+i\sqrt{5})\,\omega^2$	-2	abe
28	2	$-\sqrt{3}-i\sqrt{5}$	$(-\sqrt{3}+i\sqrt{5})\,\omega^2$	$-2\omega^2$	aeb
29	2ω	$\sqrt{3}-i\sqrt{5}$	$\sqrt{3}+i\sqrt{5}$	$-2\omega^2$	acd
30	$2\omega^2$	$\sqrt{3}-i\sqrt{5}$	$\sqrt{3}+i\sqrt{5}$	-2ω	adc
31	$2\omega^2$	$\sqrt{3}-i\sqrt{5}$	$(\ \ \sqrt{3}+i\sqrt{5})\,\omega^2$	-2	ade
32	2	$\sqrt{3}-i\sqrt{5}$	$(\ \ \sqrt{3}+i\sqrt{5})\,\omega^2$	$-2\omega^2$	aed
33	2	$-\sqrt{3}-i\sqrt{5}$	$(-\sqrt{3}+i\sqrt{5})\,\omega$	-2ω	bce
34	2ω	$-\sqrt{3}-i\sqrt{5}$	$(-\sqrt{3}+i\sqrt{5})\,\omega$	-2	bec
35	2ω	$\sqrt{3}-i\sqrt{5}$	$(\ \ \sqrt{3}+i\sqrt{5})\,\omega$	-2	cde
36	2	$\sqrt{3}-i\sqrt{5}$	$(\ \ \sqrt{3}+i\sqrt{5})\,\omega$	-2ω	ced
37	2	$i(\ \ 3-\sqrt{5})\,\omega^2$	$i(-3+\sqrt{5})$	$-2\omega^2$	$adceb$
38	$-\sqrt{3}-i\sqrt{5}$	$+2\omega^2$	-2	$(-\sqrt{3}+i\sqrt{5})\,\omega^2$	$acbde$
39	$\sqrt{3}-i\sqrt{5}$	2	-2ω	$(\ \ \sqrt{3}+i\sqrt{5})\,\omega$	$aedbc$
40	2	$i(\ \ 3-\sqrt{5})$	$i(-3+\sqrt{5})\,\omega$	-2ω	$abecd$
41	2	$i(\ \ 3-\sqrt{5})\,\omega$	$i(-3+\sqrt{5})$	-2ω	$aedcb$
42	$-\sqrt{3}-i\sqrt{5}$	2ω	-2	$(-\sqrt{3}+i\sqrt{5})\,\omega$	$adbec$
43	$\sqrt{3}-i\sqrt{5}$	2	$-2\omega^2$	$(\ \ \sqrt{3}+i\sqrt{5})\,\omega^2$	$acebd$
44	2	$i(\ \ 3-\sqrt{5})$	$i(-3+\sqrt{5})\,\omega^2$	$-2\omega^2$	$abcde$
45	2	$i(\ \ 3-\sqrt{5})\,\omega^2$	$i(-3+\sqrt{5})\,\omega^2$	-2ω	$adebc$
46	$\sqrt{3}-i\sqrt{5}$	$2\omega^2$	$-2\omega^2$	$(\ \ \sqrt{3}+i\sqrt{5})\,\omega$	$aecdb$
47	$-\sqrt{3}-i\sqrt{5}$	2ω	-2ω	$(-\sqrt{3}+i\sqrt{5})\,\omega^2$	$abdce$
48	2	$i(\ \ 3-\sqrt{5})\,\omega$	$i(-3+\sqrt{5})\,\omega$	$-2\omega^2$	$acbed$
49	2	$i(-3-\sqrt{5})\,\omega$	$i(\ \ 3+\sqrt{5})\,\omega$	$-2\omega^2$	$acdeb$

50	$\sqrt{3}-i\sqrt{5}$	2ω	-2ω	$(\sqrt{3}+i\sqrt{5})\,\omega^2$	*adbce*
51	$-\sqrt{3}-i\sqrt{5}$	$2\omega^2$	$-2\omega^2$	$(-\sqrt{3}-i\sqrt{5})\,\omega$	*aecbd*
52	2	$i(-3-\sqrt{5})\,\omega^2$	$i(\ \ 3+\sqrt{5})\,\omega^2$	-2ω	*abedc*
53	2	$i(-3-\sqrt{5})\,\omega$	$i(\ \ 3+\sqrt{5})$	-2ω	*aebcd*
54	$-\sqrt{3}-i\sqrt{5}$	2ω	-2	$(-\sqrt{3}+i\sqrt{5})\,\omega$	*abdec*
55	$\sqrt{3}-i\sqrt{5}$	2	$-2\omega^2$	$(\ \ \sqrt{3}+i\sqrt{5})\,\omega^2$	*acedb*
56	2	$i(-3-\sqrt{5})$	$i(\ \ 3+\sqrt{5})\,\omega^2$	$-2\omega^2$	*adcbe*
57	2	$i(-3-\sqrt{5})$	$i(\ \ 3+\sqrt{5})\,\omega$	-2ω	*adecb*
58	$-\sqrt{3}-i\sqrt{5}$	2	-2ω	$(-\sqrt{3}+i\sqrt{5})\,\omega$	*aebdc*
59	$\sqrt{3}-i\sqrt{5}$	$2\omega^2$	-2	$(\ \ \sqrt{3}+i\sqrt{5})\,\omega^2$	*acdbe*
60	2	$i(-3-\sqrt{5})\,\omega^2$	$i(\ \ 3+\sqrt{5})$	$-2\omega^2$	

115. We hence derive

Homographic Transformations. The groups of 12 and 24. Pole at B.

	$(As$	$+B)$	$\div(Cs$	$+D)$	
1	1	0	0	1	1
2	2	$i(\ \ 3-\sqrt{5})$	$i(-3+\sqrt{5})$	-2	*ab . cd*
3	0	1	1	0	*ac . bd*
4	2	$i(-3-\sqrt{5})$	$i(\ \ 3+\sqrt{5})$	-2	*ad . bc*
5	$2\omega^2$	$-\sqrt{3}-i\sqrt{5}$	$-\sqrt{3}+i\sqrt{5}$	-2ω	*abc*
6	2ω	$-\sqrt{3}-i\sqrt{5}$	$\sqrt{3}+i\sqrt{5}$	$-2\omega^2$	*acb*
7	$-\sqrt{3}-i\sqrt{5}$	2ω	$-2\omega^2$	$-\sqrt{3}+i\sqrt{5}$	*abd*
8	$\sqrt{3}-i\sqrt{5}$	2ω	$-2\omega^2$	$\sqrt{3}+i\sqrt{5}$	*adb*
9	2ω	$\sqrt{3}-i\sqrt{5}$	$\sqrt{3}+i\sqrt{5}$	$-2\omega^2$	*acd*
10	$2\omega^2$	$\sqrt{3}-i\sqrt{5}$	$\sqrt{3}+i\sqrt{5}$	-2ω	*adc*
11	$\sqrt{3}-i\sqrt{5}$	$2\omega^2$	-2ω	$\sqrt{3}+i\sqrt{5}$	*bcd*
12	$-\sqrt{3}-i\sqrt{5}$	$2\omega^2$	-2ω	$-\sqrt{3}+i\sqrt{5}$	*bdc*
13	2	$\sqrt{3}(\ \ 1+\sqrt{5})+\ (-3-\sqrt{5})$	$\sqrt{3}(\ \ 1+\sqrt{5})+i(\ \ 3+\sqrt{5})$	-2	*ab*
14	2	$\sqrt{3}(-1-\sqrt{5})+\ (-3-\sqrt{5})$	$\sqrt{3}(-1-\sqrt{5})+i(\ \ 3+\sqrt{5})$	-2	*cd*
15	$\sqrt{5}$	$-i$	i	$-\sqrt{5}$	*ac*
16	1	$i\sqrt{5}$	$-i\sqrt{5}$	-1	*bd*
17	2	$\sqrt{3}(-1+\sqrt{5})+i(\ \ 3-\sqrt{5})$	$\sqrt{3}(-1+\sqrt{5})+i(-3+\sqrt{5})$	-2	*ad*
18	2	$\sqrt{3}(\ \ 1-\sqrt{5})+i(\ \ 3-\sqrt{5})$	$\sqrt{3}(\ \ 1-\sqrt{5})+i(-3+\sqrt{5})$	-2	*bc*
19	1	i	i	1	*abcd*
20	1	$-i$	$-i$	1	*adcb*
21	$\sqrt{3}(\ \ 1-\sqrt{5})+i(\ \ 3+\sqrt{5})$	2	-2	$\sqrt{3}(\ \ 1-\sqrt{5})+i(-3+\sqrt{5})$	*abdc*
22	$\sqrt{3}(\ \ 1+\sqrt{5})+i(-3+\sqrt{5})$	2	-2	$\sqrt{3}(\ \ 1+\sqrt{5})+i(\ \ 3+\sqrt{5})$	*acbd*
23	$\sqrt{3}(-1+\sqrt{5})+i(\ \ 3-\sqrt{5})$	2	-2	$\sqrt{3}(-1+\sqrt{5})+i(-3+\sqrt{5})$	*acdb*
24	$\sqrt{3}(-1-\sqrt{5})+i(-3-\sqrt{5})$	2	-2	$\sqrt{3}(-1-\sqrt{5})+i(\ \ 3+\sqrt{5})$	*adbc*

116. I give also the group of 12, (*abce*), slightly modifying the form: viz. I write first $\sqrt{3}+i\sqrt{5}=2\sqrt{2k}$, and therefore $\sqrt{3}-i\sqrt{5}=2\sqrt{2}\,.\,\frac{1}{k}$: then for x I write λx, and divide the A and B by λ: the A and B then contain $\frac{k}{\lambda}$, and the C and D contain $\frac{\lambda}{k}$, and assuming $\frac{k}{\lambda}=i$, we have $\frac{\lambda}{k}=-i$. For instance in the transformation corresponding to *abc*, the $Ax+B$ and $Cx+D$, $=2\omega^2x-(\sqrt{3}+i\sqrt{5})$ and $(-\sqrt{3}+2\sqrt{5})\,x-2\omega$ become first

$2\omega^2 x - 2\sqrt{2}k$, and $-2\sqrt{2}\,\frac{1}{k}\,x - 2\omega$, and then (omitting also the factor 2) $\omega^2 x - \sqrt{2}\,\frac{k}{\lambda}$ and $-\sqrt{2}\,\frac{\lambda}{k}\,x - \omega$, viz. when $\frac{k}{\lambda} = i$, they are $\omega^2 x - i\sqrt{2}$ and $x \,.\, i\sqrt{2} - \omega$; that is the values of A, B, C, D are ω^2, $-i\sqrt{2}$, $i\sqrt{2}$, $-\omega$. The group is

Group of 12. Pole at B.

1	0	0	1	1
ω	0	0	1	ace
ω^2	0	0	1	aec
1	$-i\omega\sqrt{2}$	$i\omega\sqrt{2}$	$-\omega^2$	abc
1	$-i\omega^2\sqrt{2}$	$i\omega^2\sqrt{2}$	$-\omega$	acb
1	$-i-\omega\sqrt{2}$	$i\sqrt{2}$	$-\omega$	abe
1	$-i\sqrt{2}$	$i\omega^2\sqrt{2}$	$-\omega^2$	aeb
1	$-i\omega^2\sqrt{2}$	$i\sqrt{2}$	$-\omega^2$	bec
1	$-i\sqrt{2}$	$i\omega\sqrt{2}$	$-\omega$	bce
1	$-i\omega\sqrt{2}$	$i\omega^2\sqrt{2}$	-1	$ab\,.\,ce$
1	$-i\omega^2\sqrt{2}$	$i\omega\sqrt{2}$	-1	$ae\,.\,bc$
1	$-i\sqrt{2}$	$i\sqrt{2}$	-1	$ac\,.\,be$

117. From the Table of the Groups of 12 and 24, Θ-form, it appears that the group of 12 is

$$x,\ \frac{1}{x},\ -x,\ -\frac{1}{x},\ \frac{i(x-1)}{x+1},\ \frac{-i(x-1)}{x+1},\ \frac{i(x+1)}{x-1},\ \frac{-i(x+1)}{x-1},\ \frac{x+i}{x-i},\ \frac{x-i}{x+i},\ \frac{-(x+i)}{x-2},\ \frac{-(x-i)}{x+i};$$

and if we proceed to form the product of the twelve factors $s-x$, $s-\frac{1}{x}$, $s+x$, &c., we have first the three products

$$s^2 - x^2 \,.\, s^2 - \frac{1}{x^2};\quad s^2 + \left(\frac{x-1}{x+1}\right)^2 \,.\, s^2 + \left(\frac{x+1}{x-1}\right)^2;\quad s^2 - \left(\frac{x+i}{x-i}\right)^2 \,.\, s^2 - \left(\frac{x-i}{x+i}\right)^2$$

$$= s^4 + \alpha s^2 + 1;\qquad s^4 + \beta s^2 + 1;\qquad s^4 + \gamma s^2 + 1;$$

if for shortness,

$$\alpha,\ \beta,\ \gamma = -\left(x^2 + \frac{1}{x^2}\right),\quad 2\,\frac{x^4 + 6x^2 + 1}{(x^2-1)^2},\quad -2\,\frac{x^4 - 6x^2 + 1}{(x^2+1)^2}.$$

The product of the three quartic functions is

$$= (s^4+1)^3 + (s^4+1)^2 s^2 (\alpha+\beta+\gamma) + (s^4+1)\, s^4 (\beta\gamma + \gamma\alpha + \alpha\beta) + s^6 \,.\, \alpha^2\beta^2\gamma^2;$$

and we have

$$\beta + \gamma = \frac{32x^2(x^4+1)}{(x^4-1)^2},\qquad \alpha + \beta + \gamma = \frac{-(x^{12} - 33s^8 - 33s^4 + 1)}{x^2(x^4-1)^2},$$

$$\beta\gamma = \frac{-4(x^8 - 34x^4 + 1)}{(x^4-1)^2},\quad \alpha(\beta+\gamma) = \frac{-32x^2(x^4+1)^2}{x^2(x^4-1)^2},\quad \beta\gamma + \gamma\alpha + \alpha\beta, = \frac{-36x^2(x^4-1)^2}{x^2(x^4-1)^2},\ = -36,$$

$$\alpha\beta\gamma = \frac{4(x^{12} - 33s^8 - 33s^4 + 1)}{x^2(x^4-1)^2}.$$

Hence the product is found to be

$$= (s^{12} - 33s^8 - 33s^8 + 1) - s^2 (s^4 - 1)^2 \cdot \frac{x^{12} - 33x^8 - 33x^4 + 1}{x^2 (x^4 - 1)^2},$$

which is

$$= s^2 (s^4 - 1)^2 \left\{ \frac{s^{12} - 33s^8 - 33s^4 + 1}{s^2 (s^4 - 1)^2} - \frac{x^{12} - 33x^8 - 33x^4 + 1}{x^2 (x^4 - 1)^2} \right\}.$$

We thus verify that the twelve transformations x into x, into $\frac{1}{x}$ &c., give each of them a transformation of the function

$$\frac{x^{12} - 33x^8 - 33x^4 + 1}{x^2 (x^4 - 1)^2}$$

into itself.

The system of 15 *circles.* Art. Nos. 118 to 127.

118. It has been already remarked that we can from the coefficients (A, B, C, D) of the homographic transformation pass back to the position of the axis of rotation: viz., we have

$$A : B : C : D = -\nu - i : \lambda + i\mu : \lambda - i\mu : \nu - i,$$

and thence

$$\lambda : \mu : \nu : 1 = \quad B + C : -i(B - C) \quad : D - A \quad : i(D + A),$$

that is

$$\lambda,\ \mu,\ \nu = -i(B + C),\ -(B - C),\ -i(D - A);\ \div (D + A).$$

The equations of the axis thus are

$$\frac{x}{B + C} = \frac{iy}{B - C} = \frac{z}{D - A},$$

and the equations of the central plane at right angles to the axis are

$$-(B + C)\,x + i(B - C)\,y + (A - D)\,z = 0.$$

119. In particular we may find the equations of the 15 planes at right angles to the Θ-axes: these are in fact the before-mentioned 15 planes, intersecting the sphere in great circles the projections of which are the circles in the three figures respectively. Taking the equation of the plane to be $Lx + My + Nz = 0$, it is at once seen that the equation of the projecting cone (vertex at the South pole) is

$$N(x^2 + y^2 + z^2 - 1) - 2(z + 1)(Lx + My + Nz) = 0,$$

and hence, writing $z = 0$, we find

$$N(x^2 + y^2 - 1) - 2(Lx + My) = 0$$

for the equation of the circle in the plane figure. We have thus the equations of a system of 15 circles related to each other in the manner before referred to.

120. Taking the Θ-form, the equations of the 15 planes are at once found: and we thence obtain the equations of the 15 circles: viz. writing for shortness

$$\Omega = x^2 + y^2 - 1,$$

the equations are

$$\begin{array}{rll}
z = 0, & (ab\,.\,cd) & \Omega = 0, \\
x = 0, & (ac\,.\,bd) & x = 0, \\
y = 0, & (ad\,.\,bc) & y = 0, \\[6pt]
(3-\sqrt{5})\,x + (\ \ 1-\sqrt{5})\,y + 2z = 0, & (ae\,.\,bc) & \Omega - [(\ \ 3-\sqrt{5})\,x + (\ \ 1-\sqrt{5})\,y] = 0, \\
(-1-\sqrt{5})\,x + (-1+\sqrt{5})\,y + 2z = 0, & (ab\,.\,ce) & \Omega - [(-1-\sqrt{5})\,x + (-1+\sqrt{5})\,y] = 0, \\
(1+\sqrt{5})\,x + (\ \ 3+\sqrt{5})\,y + 2z = 0, & (ac\,.\,be) & \Omega - [(\ \ 1+\sqrt{5})\,x + (\ \ 3+\sqrt{5})\,y] = 0; \\[6pt]
(-3+\sqrt{5})\,x + (-1+\sqrt{5})\,y + 2z = 0, & (ad\,.\,be) & \text{and similarly for the other circles.} \\
(1+\sqrt{5})\,x + (\ \ 1-\sqrt{5})\,y + 2z = 0, & (ab\,.\,de) & \\
(-1-\sqrt{5})\,x + (-3-\sqrt{5})\,y + 2z = 0, & (ae\,.\,bd) & \\[6pt]
(-3+\sqrt{5})\,x + (\ \ 1-\sqrt{5})\,y + 2z = 0, & (ad\,.\,ce) & \\
(1+\sqrt{5})\,x + (-1+\sqrt{5})\,y + 2z = 0, & (ae\,.\,cd) & \\
(-1-\sqrt{5})\,x + (\ \ 3+\sqrt{5})\,y + 2z = 0, & (ac\,.\,de) & \\[6pt]
(3-\sqrt{5})\,x + (-1+\sqrt{5})\,y + 2z = 0, & (bc\,.\,de) & \\
(-1-\sqrt{5})\,x + (\ \ 1-\sqrt{5})\,y + 2z = 0, & (be\,.\,cd) & \\
(1+\sqrt{5})\,x + (-3-\sqrt{5})\,y + 2z = 0, & (bd\,.\,ce). &
\end{array}$$

121. Observe that the arrangement is in sets of 3 planes, or circles, intersecting at right angles. One of the circles is the circle $\Omega, = x^2 + y^2 - 1, = 0$ corresponding to the equator, and two of them are the right lines $x = 0$ and $y = 0$. The equations of the remaining 12 circles may be written in the somewhat different form

$$\begin{aligned}
&\Omega + (\sqrt{5}-1)[y - \tfrac{1}{2}(\sqrt{5}-1)\,x] = 0, \\
&\Omega - (\sqrt{5}-1)[y - \tfrac{1}{2}(\sqrt{5}+3)\,x] = 0, \\
&\Omega - (\sqrt{5}+3)[y + \tfrac{1}{2}(\sqrt{5}-1)\,x] = 0, \\[4pt]
&\Omega - (\sqrt{5}-1)[y - \tfrac{1}{2}(\sqrt{5}-1)\,x] = 0, \\
&\Omega + (\sqrt{5}-1)[y - \tfrac{1}{2}(\sqrt{5}+3)\,x] = 0, \\
&\Omega + (\sqrt{5}+3)[y + \tfrac{1}{2}(\sqrt{5}-1)\,x] = 0, \\[4pt]
&\Omega + (\sqrt{5}-1)[y + \tfrac{1}{2}(\sqrt{5}-1)\,x] = 0, \\
&\Omega - (\sqrt{5}-1)[y + \tfrac{1}{2}(\sqrt{5}+3)\,x] = 0, \\
&\Omega - (\sqrt{5}+3)[y - \tfrac{1}{2}(\sqrt{5}-1)\,x] = 0, \\[4pt]
&\Omega - (\sqrt{5}-1)[y + \tfrac{1}{2}(\sqrt{5}-1)\,x] = 0, \\
&\Omega + (\sqrt{5}-1)[y + \tfrac{1}{2}(\sqrt{5}+3)\,x] = 0, \\
&\Omega + (\sqrt{5}+3)[y - \tfrac{1}{2}(\sqrt{5}-1)\,x] = 0,
\end{aligned}$$

and it hence appears that 4 and 4 circles have with $\Omega = 0$ the common chords $y + \tfrac{1}{2}(\sqrt{5}-1)\,x = 0$, $y - \tfrac{1}{2}(\sqrt{5}-1)\,x = 0$ respectively: and that 2 and 2 circles have with $\Omega = 0$ the common chords $y + \tfrac{1}{2}(\sqrt{5}+3)\,x = 0$, $y - \tfrac{1}{2}(\sqrt{5}+3)\,x = 0$ respectively.

122. The equations of the 12 circles are in fact

$$\Omega \pm (\surd 5 - 1)[y \pm \tfrac{1}{2}(\surd 5 - 1)\,x] = 0, \qquad \Omega \pm (\surd 5 + 3)[y \pm \tfrac{1}{2}(\surd 5 - 1)\,x] = 0,$$
$$\Omega \pm (\surd 5 - 1)[y \pm \tfrac{1}{2}(\surd 5 + 3)\,x] = 0:$$

hence the radii are $= \surd 5 - 1$, 2 and $\surd 5 + 1$ respectively.

The construction of the 12 circles is as follows: starting with a circle radius 1,

Lay down the diameters $y \pm \frac{1}{2}(\surd 5 - 1)\,x = 0$ (*AA* in the figure), and through the extremities of each describe 2 pairs of circles with the radii $\surd 5 - 1$, $\surd 5 + 1$ respectively.

Lay down the diameters $y \pm \frac{1}{2}(\surd 5 + 3)\,x = 0$ (*BB* in the figure), and through the extremities of each describe a pair of circles with the radius 2.

123. For the *A*-form, the equations of the fifteen planes are at once found to be

$$\begin{array}{rlrl}
 & y & = 0, & ad\,.\,bc \\
-x & + & (\epsilon + \epsilon^4)\,z = 0, & ac\,.\,bd \\
(\epsilon + \epsilon^4)\,x & + & z = 0, & ab\,.\,cd \\
\hline
(\epsilon^2 - \epsilon^3)\,x & -i\,(\epsilon^2 + \epsilon^3)\,y & = 0, & ac\,.\,be \\
-(\epsilon^2 + \epsilon^3)\,x & +i\,(\epsilon^2 - \epsilon^3)\,y + 2\,(\epsilon + \epsilon^4)\,z & = 0, & ae\,.\,bc \\
-x + i\,(\epsilon^2 + \epsilon^4 - \epsilon - \epsilon^3)\,y + & & 2z = 0, & ab\,.\,ce \\
\hline
(\epsilon - \epsilon^4)\,x & -i\,(\epsilon + \epsilon^4)\,y & = 0, & ab\,.\,de \\
-(\epsilon + \epsilon^4)\,x & +i\,(\epsilon - \epsilon^4)\,y + 2\,(\epsilon + \epsilon^4)\,z & = 0, & ae\,.\,bd \\
+(\epsilon^2 + \epsilon^3 + 2)\,x & -i\,(\epsilon^2 - \epsilon^3)\,y + & 2z = 0, & ad\,.\,be \\
\hline
(\epsilon - \epsilon^4)\,x & +i\,(\epsilon + \epsilon^4)\,y & = 0, & ae\,.\,cd \\
-(\epsilon + \epsilon^4)\,x & -i\,(\epsilon - \epsilon^4)\,y + 2\,(\epsilon + \epsilon^4)\,z & = 0, & ac\,.\,de \\
(\epsilon^2 + \epsilon^3 + 2)\,x & +i\,(\epsilon^2 - \epsilon^3)\,y + & 2z = 0, & ad\,.\,ce \\
\hline
(\epsilon^2 - \epsilon^3)\,x & +i\,(\epsilon^2 + \epsilon^3)\,y & = 0, & bd\,.\,ce \\
-(\epsilon^2 + \epsilon^3)\,x & -i\,(\epsilon^2 - \epsilon^3)\,y + 2\,(\epsilon + \epsilon^4)\,z & = 0, & bc\,.\,de \\
-x - i\,(\epsilon^2 + \epsilon^4 - \epsilon - \epsilon^3)\,y + & & 2z = 0, & be\,.\,cd.
\end{array}$$

where as before the three planes of each set intersect at right angles.

124. Passing to the circles, the first plane of each set gives a right line, and we have thus five of the circles reducing themselves to right lines inclined to the axis of x at angles 0°, 36°, 72°, 108° and 114° respectively.

The remaining 10 circles form 5 pairs, the circles of a pair having different radii, but the two radii being the same for each pair, and so that for the several pairs the common chords with the circle $\Omega = 0$, are the diameters inclined to the axis of x at the angles 18°, 54°, 90°, 126° and 162° respectively. Considering the two circles for which the inclination is 90°, these arise from the planes $-x + (\epsilon + \epsilon^4)\,z = 0$, $(\epsilon + \epsilon^4)\,x + z = 0$ respectively. The equations of the circles thus are $(\epsilon + \epsilon^4)\,\Omega + 2x = 0$, $\Omega - 2\,(\epsilon + \epsilon^4)\,x = 0$, or, recollecting that $2\,(\epsilon + \epsilon^4) = \surd 5 - 1$ and therefore $\dfrac{2}{\epsilon + \epsilon^2} = \surd 5 + 1$, the equations are $x^2 + y^2 - (\surd 5 - 1)\,x - 1 = 0$, $x^2 + y^2 + (\surd 5 + 1)\,x = 0$; hence for the first circle the x-coordinate of the centre is $\frac{1}{2}(\sqrt{5}-1)$ and radius is $=\frac{1}{2}\sqrt{(10-2\sqrt{5})}$; for the second circle the x-coordinate of the centre is $=\frac{1}{2}(\sqrt{5}+1)$, and radius $=\frac{1}{2}\sqrt{(10+2\sqrt{5})}$. We have thus the construction of these two circles, and consequently the construction of all the 12 circles.

125. For the B-form after some easy reductions, and attending to the relation $\omega-\omega^2=i\sqrt{3}$, the equations of the 15 planes become

x		$=0,$	$ac\,.\,bd$
	$(-3+\sqrt{5})\,y+$	$2z=0,$	$ab\,.\,cd$
	$(3+\sqrt{5})\,y+$	$2z=0,$	$ad\,.\,bc$
$\sqrt{3}x+$	$\sqrt{5}\,y+$	$2z=0,$	$ac\,.\,be$
$-(1+\sqrt{5})\sqrt{3}x+$	$(3-\sqrt{5})\,y+$	$4z=0,$	$ab\,.\,ce$
$(-1+\sqrt{5})\sqrt{3}x+$	$(-3-\sqrt{5})\,y+$	$4z=0,$	$ae\,.\,bc$
$x+$	$\sqrt{3}y$	$=0,$	$ae\,.\,bd$
$-\sqrt{3}x+$	$y+(3+\sqrt{5})\,z$	$=0,$	$ad\,.\,be$
$\sqrt{3}x-$	$y+(3-\sqrt{5})\,z$	$=0,$	$ab\,.\,de$
$-\sqrt{3}x+$	$\sqrt{5}\,y+$	$2z=0,$	$ac\,.\,de$
$(1-\sqrt{5})\sqrt{3}x+$	$(-3-\sqrt{5})\,y+$	$4z=0,$	$ad\,.\,ce$
$(1+\sqrt{5})\sqrt{3}x+$	$(3-\sqrt{5})\,y+$	$4z=0,$	$ae\,.\,cd$
$x-$	$\sqrt{3}y$	$=0,$	$bd\,.\,ce$
$\sqrt{3}x+$	$y+(3+\sqrt{5})\,z$	$=0,$	$bc\,.\,de$
$-\sqrt{3}x-$	$y+(3-\sqrt{5})\,z$	$=0,$	$be\,.\,cd.$

126. Of the 15 circles 3 are the lines $x-y\sqrt{3}=0$, $x=0$, $x+y\sqrt{3}=0$, viz. these are lines at inclinations 30°, 90°, 150° to the axis of x. The equations of the remaining 12 circles are

$$\begin{aligned}\Omega+\ (3-\sqrt{5})\,y&=0,\\ \Omega-\ (3+\sqrt{5})\,y&=0,\\ (3+\sqrt{5})\,\Omega-2\,(y-x\sqrt{3})&=0,\\ (3-\sqrt{5})\,\Omega+2\,(y-x\sqrt{3})&=0,\\ (3+\sqrt{5})\,\Omega-2\,(y+x\sqrt{3})&=0,\\ (3-\sqrt{5})\,\Omega+2\,(y+x\sqrt{3})&=0,\end{aligned}$$

viz. these are pairs of circles having for their common chords with $\Omega=0$ the diameters at inclinations 0, 60°, 120° respectively. And lastly we have the circles

$$\begin{array}{l|l} 2\Omega-[(-1+\sqrt{5})\sqrt{3}x-(3+\sqrt{5})\,y]=0, & 2\Omega+[(-1+\sqrt{5})\sqrt{3}x+(3+\sqrt{5})\,y]=0,\\ \Omega-[\quad-\sqrt{3}x+\quad\sqrt{5}y]=0, & \Omega-[\quad\sqrt{3}x+\quad\sqrt{5}y]=0,\\ 2\Omega+[(\ 1+\sqrt{5})\sqrt{3}x-(3-\sqrt{5})\,y]=0, & 2\Omega-[(\ 1+\sqrt{5})\sqrt{3}x+(3-\sqrt{5})\,y]=0.\end{array}$$

127. The first three of these have for common chords with $\Omega=0$, the diameters whose equations are

$$(-1+\sqrt{5})\sqrt{3}x-(3+\sqrt{5})\,y=0,\quad -\sqrt{3}x+\sqrt{5}y=0,\quad (1+\sqrt{5})\sqrt{3}x-(3-\sqrt{5})\,y=0:$$

viz. these equations are $y = (-2+\sqrt{5})\,x\sqrt{3}$, $y = \dfrac{\sqrt{3}}{\sqrt{5}}\,x$, $y = (2+\sqrt{5})\,x\sqrt{3}$. If, as in a foregoing table $\theta = 37^\circ\ 46'$, $\sin = \dfrac{\sqrt{3}}{2\sqrt{2}}$, $\cos = \dfrac{\sqrt{5}}{2\sqrt{2}}$; and therefore $\tan\theta = \dfrac{\sqrt{3}}{\sqrt{5}}$, then the inclinations of these diameters to the axis of x are respectively $60^\circ - \theta$, θ and $120^\circ - \theta$, or say $30^\circ - (\theta - 30^\circ)$, $30^\circ + (\theta - 30^\circ)$ and $90^\circ - (\theta - 30^\circ)$, where $\theta - 30^\circ = 7^\circ\ 46'$, i.e. the inclinations are $30^\circ \pm 7^\circ\ 46'$ and $90^\circ - 7^\circ\ 46'$. And for the other three circles the common chords are the diameters at the same inclinations taken negatively. The geometrical construction of the fifteen circles for the B-case in question is thus not so simple as in the Θ- and A-cases.

The Regular Polyhedra as Solid figures. Art. Nos. 128 to 134.

128. I annex some results relating to the polyhedra considered as solid figures bounded by plane faces; or say results relating to the regular solids: s is in each case taken for the length of the edge of the solid.

	Tetrahedron.	Cube.	Octahedron.	Dodecahedron.	Icosahedron.
Edge	s	s	s	s	s
Rad. of circum. sphere, R	$s\,\dfrac{\sqrt{3}}{2\sqrt{2}}$	$s\,.\,\frac{1}{2}\sqrt{3}$	$s\,\dfrac{1}{\sqrt{2}}$	$s\,\dfrac{\sqrt{3}\,(\sqrt{5}+1)}{4}$	$s\sqrt{\dfrac{5+\sqrt{5}}{8}}$
Rad. of inters. sphere, ρ	$s\,\dfrac{1}{2\sqrt{2}}$	$s\,\dfrac{1}{\sqrt{2}}$	$s\,.\,\frac{1}{2}$	$s\,\dfrac{3+\sqrt{5}}{4}$	$s\,\dfrac{1+\sqrt{5}}{4}$
Rad. of inscribed sph., r	$s\,\dfrac{1}{2\sqrt{2}\sqrt{3}}$	$s\,.\,\frac{1}{2}$	$s\,\dfrac{1}{\sqrt{2}\sqrt{3}}$	$s\sqrt{\dfrac{25+11\sqrt{5}}{40}}$	$s\,\dfrac{3+\sqrt{5}}{4\sqrt{3}}$
Rad. of circle circum. to face, R'	$s\,.\,\dfrac{1}{\sqrt{3}}$	$s\,\dfrac{1}{\sqrt{2}}$	$s\,\dfrac{1}{\sqrt{3}}$	$s\sqrt{\dfrac{5+\sqrt{5}}{10}}$	$s\,\dfrac{1}{\sqrt{3}}$
Rad. of circle inscribed to face, r'	$s\,.\,\dfrac{1}{2\sqrt{3}}$	$s\,.\,\frac{1}{2}$	$s\,\dfrac{1}{2\sqrt{3}}$	$s\sqrt{\dfrac{5+2\sqrt{5}}{20}}$	$s\,\dfrac{1}{2\sqrt{3}}$
Incl. of adjacent faces	$\cos^{-1}\frac{1}{3} = 70^\circ\ 28'$	90°	$\cos^{-1} - \frac{1}{3} = 109^\circ\ 32'$		
Incl. of edge to adjacent face	$\cos^{-1}\dfrac{1}{\sqrt{3}} = 54^\circ\ 46'$	90°	$\cos^{-1} - \dfrac{1}{\sqrt{3}} = 125^\circ\ 44'$		

But we require further data in the cases of the dodecahedron and the icosahedron respectively.

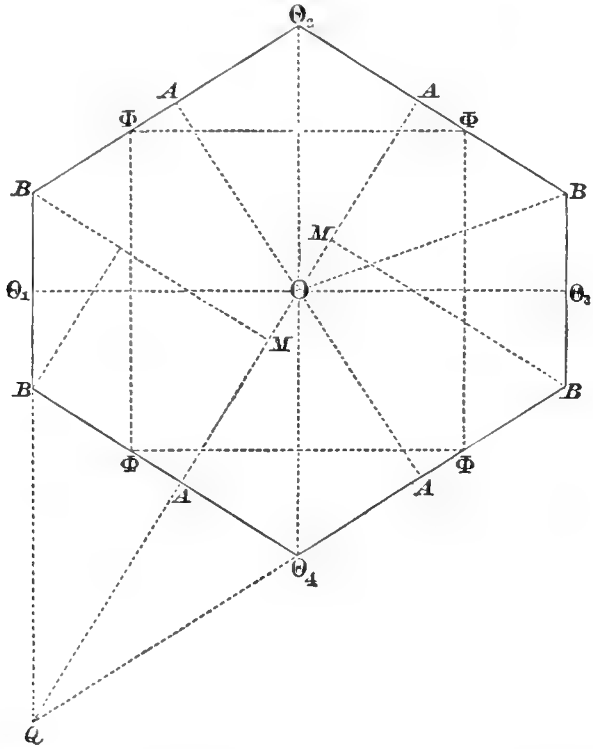
129. For the dodecahedron, taking as before the edge to be $= s$, then in the pentagonal face

$$\begin{aligned} \text{diagonal, } g \text{ is } &= s\,.\,\tfrac{1}{2}(\sqrt{5}+1), \\ \text{altitude, } k \quad ,, \ &= s\,.\,\tfrac{1}{2}\sqrt{(5+2\sqrt{5})}, \\ \text{segments of do. } e \quad ,, \ &= s\,.\,\tfrac{1}{4}\sqrt{(10-2\sqrt{5})}, \\ f \quad ,, \ &= s\,.\,\tfrac{1}{4}\sqrt{(10+2\sqrt{5})}, \end{aligned}$$

where $$k = e + f = R' + r'.$$

130. The section through a pair of opposite edges is a hexagon, as shown in the figure, viz. this is constructed by taking the four equal distances $O\Theta$, $= \rho$, $= s\,.\,\frac{1}{4}(3+\sqrt{5})$, meeting at right angles in O; then drawing the double ordinates BB, each $= s$, through Θ_1

and Θ_3 respectively, and joining their extremities with Θ_2 and Θ_4: the sides $\Theta_2 B$ and $\Theta_4 B$ are then each $=k$, $=s\,.\,\frac{1}{2}\sqrt{(5+2\sqrt{5})}$; and inserting upon them the points A, Φ from the figure of the pentagon, we have several geometrical relations; viz. the line AA cuts the



parallel sides $B\Theta_2$, $B\Theta_4$ at right angles, and when produced passes through the intersection of $B\Theta_1$ and $B\Theta_4$: we have OA, OB, $O\Theta = r$, R, ρ respectively: the four points Φ form a square, the side of which is g, $= s\,.\,\frac{1}{2}(\sqrt{5}+1)$.

131. We find also

$$AM = s\sqrt{\frac{5+\sqrt{5}}{10}},$$

$$AQ = s\sqrt{\frac{5+2\sqrt{5}}{5}},$$

$$QM = s\sqrt{\frac{25+11\sqrt{5}}{5}}, \quad = r\,.\,2\sqrt{2},$$

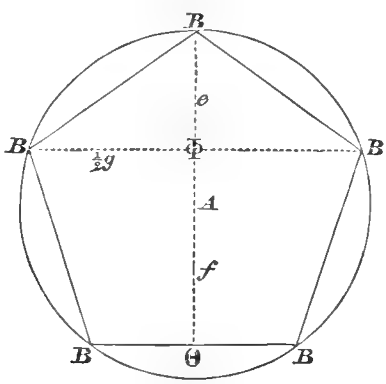
$$OQ = s\sqrt{\frac{25+11\sqrt{5}}{8}}, \quad = r\,.\,\sqrt{5},$$

$$OM = s\sqrt{\frac{5-\sqrt{5}}{40}},$$

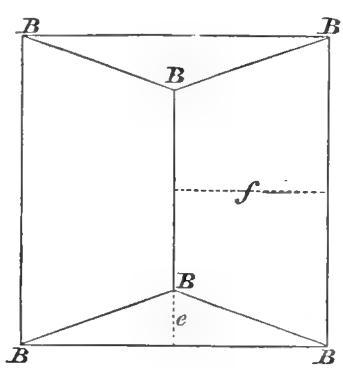
$$MB = s\sqrt{\frac{2(5+2\sqrt{5})}{B}}.$$

It may be remarked that in the figure $B\Theta_2$, $B\Theta_4$ are the projections of pentagonal faces, at right angles to the plane of the paper, having their centres at the points A, A, and the perpendicular distance between them $= AA$: the points Q, Q (only one of them shown in the figure) determine the directions of the $5+5$ sides which abut on these pentagonal faces respectively; and the $5+5$ points B which are the other extremities of these sides respectively form two pentagons, centres M, M in the planes MB and MB respectively: the remaining 10 sides of the dodecahedron are the skew decagon obtained by joining in order these 10 points B. We have thus the means of making the perspective delineation of the dodecahedron.

132. The dodecahedron is built up from the cube, by placing on each face a figure of two triangular and two quadrangular faces, the orthogonal projection of which

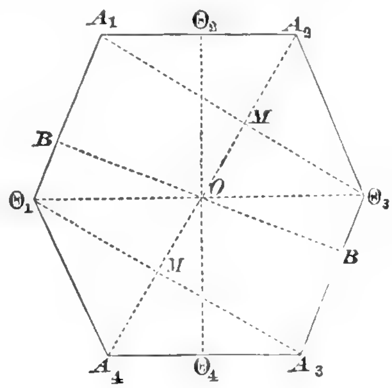


on the face of the cube is as in the figure: the side of the square is g, $= s \cdot \frac{1}{2}(\sqrt{5}+1)$: the slope-breadths of the triangular faces are e, $= s \cdot \frac{1}{4}\sqrt{(10-2\sqrt{5})}$, and those of the quadrangular faces are f, $= s \cdot \frac{1}{4}\sqrt{(10+2\sqrt{5})}$; the lines represented by the other lines of the figure are in actual length each $= s$. We have thus a section which is an isosceles triangle, base $= g$, other sides each $= f$; and the square of the altitude is thus $= f^2 - \frac{1}{4}g^2$



$= \frac{1}{4}s^2$, or altitude $= \frac{1}{2}s$; viz. the altitude of the ridge-line BB, above the face of the cube is $= \frac{1}{2}s$, the half-side of the dodecahedron: we have in this result the most simple means of forming the perspective delineation of the dodecahedron.

133. For the icosahedron the section through two opposite edges is a hexagon, as shown in the figure: to construct it we take the four distances $O\Theta$ each $=\rho$ $= s.\frac{1}{4}(1+\sqrt{5})$ meeting at right angles; and then the distances $A\Theta_2$, $A\Theta_4$ each $=\frac{1}{2}s$; and complete the hexagon. This gives the sides $A\Theta_1$, $A\Theta_3$ each $= s.\frac{1}{2}\sqrt{3}$, the altitude of the triangular face, side $= s$; and then taking $\Theta_1 B$ one-third of this, $= s\,\dfrac{1}{2\sqrt{3}}$, we have OB at right angles to $A\Theta_1$, and OA, OB, $O\Theta = R, r, \rho$ respectively.



Moreover, joining $A_1\Theta_2$, and OA_2 we have these lines cutting at right angles in a point M: we find

$$A_1\Theta_3 = s.\tfrac{1}{2}\sqrt{(5+2\sqrt{5})},$$

$$M\Theta_3 = s\sqrt{\frac{5+2\sqrt{5}}{20}},$$

$$A_1M = s\sqrt{\frac{5+\sqrt{5}}{10}},$$

$$OM = s\sqrt{\frac{5+\sqrt{5}}{40}}, \quad = \tfrac{1}{2}A_1M,$$

$$A_2M = s\sqrt{\frac{5-\sqrt{5}}{10}}.$$

134. It may be remarked that $A_1\Theta_3$, $A_3\Theta_1$ are the projections of two pentagons in planes perpendicular to that of the paper, their centres being M, M: producing OM, OM to the points A_2, A_4 respectively, we have a pentagonal pyramid, summit A_2, standing on the first pentagon, and an opposite pyramid, summit A_4, standing on the other pentagon: the $5+5$ triangular faces of the two pyramids are 10 of the faces of the icosahedron, and the remaining ten faces are the triangles each having for its base a side of the one pentagon, and for its vertex a summit of the other pentagon, viz. the sides are the sides of the skew decagon obtained by joining in order the angular points of the two pentagons. We have thus a convenient method of forming the perspective delineation of the icosahedron.

III. *On the Application of Quaternions and Grassmann's Ausdehnungslehre to different kinds of Uniform Space.* By HOMERSHAM COX, B.A., Fellow of Trinity College, Cambridge.

[Read *February* 20, 1882.]

INTRODUCTION.

THE object of this paper is, following Grassmann, to establish a pure algebraical calculus, the laws of which will coincide with those of actual geometry. Ordinary algebra may be considered a calculus of one dimensional space, either of lengths measured along a line or of time, or of any other quantity capable of only one kind of variation. It might even arise as a calculus of discrete objects for starting with the series of natural numbers, fractional, negative and imaginary quantities would arise as the results of inverse operations, although no interpretation could be found for them. A geometrical calculus of two or more dimensions must have more than one independent unit. It must include the calculus of one dimension, and therefore all algebraical quantities; even imaginary quantities, for they will arise as an indication that lines do not intersect. Whatever symbols then a proper geometrical calculus may contain, the algebraic $\sqrt{-1}$ will always exist apart and in addition to them. Starting with different independent units, the straight line, the plane, &c. may all have purely analytical definitions given of them, and this has been done by Grassmann. I have endeavoured to combine with the ideas of Grassmann Prof. Cayley's theory of distance and the applications of it made by Dr Klein. It is shewn that there are three different ways in which distance may be introduced, and consequently three different kinds of uniform geometry. These are, ordinary geometry, spherical geometry, and the non-Euclidean geometry of Lobatschewsky and Bolyai. Besides the works referred to, I have constantly used Prof. Tait's "Introduction to Quaternions" and Hamilton's "Elements." In this first part chiefly Quaternion methods are employed; in the second part those of the Ausdehnungslehre.

ADDITION OF POINTS.

Suppose A and B to be the distinct independent quantities, so that neither can be derived from the other by multiplication with an algebraical quantity.

We will call A and B points.

The expressions $2A$, $3A$, $4A$, &c., which have no distinct meaning except in combination with other symbols, will be called multiples of the point A.

Every expression of the form $pA+qB$, where p and q are numbers, is defined to be some multiple of a point.

$pA+qB$ and $rA+sB$ will be considered different points if $\frac{s}{r}$ is not equal to $\frac{q}{p}$, but if $\frac{s}{r}=\frac{q}{p}$ then they are multiples of the same point.

Hence all the points included in the series $pA+qB$ (neglecting the number of times they may happen to be multiplied), form a singly infinite series, each point depending on the value of $\frac{q}{p}$. This series of points will be defined to be a line.

If a point be considered to have no dimensions, a line will be a manifold of one dimension.

It follows from the original assumption that any equation of the form $pA+qB=rA+sB$ involves the equations $p=r$, $q=s$, for otherwise we should have $B=\frac{p-r}{s-q}A$, and this was supposed not to be possible.

We take as the *definition* of the addition of points and their multiples the equation

$$(pA+qB)+(rA+sB)=(p+r)A+(q+s)B.$$

It follows at once that addition is commutative and associative.

Similar definitions can be given for subtraction and multiplication by an algebraical quantity.

Any point $xA+yB$ can be derived from any two points $pA+qB$, $rA+sB$ on the same line by means of the equation

$$xA+yB=x'(pA+qB)+y'(rA+sB),$$

which gives

$$x=px'+ry', \qquad y=qx'+sy'.$$

There is then nothing peculiar about the points A, B; nothing to distinguish them from any other points on the same line. It follows that without violating analogy we may call the series of points $pA+qB$ a *straight line.* In ordinary Euclidean geometry the series of points might lie either on a straight line or a circle. Both would be included in the definition; for, taken in themselves without reference to outside points there is nothing to distinguish them, except that the one is infinite and the other finite; and this the definition says nothing about.

Let C be a new point not lying on the line joining A and B, or, in other words, not expressible in the form $pA+qB$.

Then all the points included in the expression $pA+qB+rC$, leaving out of consideration the number of times each point is taken, form a doubly infinite number. They will be said to constitute a plane.

If $$X = lA + mB + nC, \qquad Y = l'A + m'B + n'C,$$
be any point in the plane, then

$$Z = pX + qY = (lp + l'q)A + (mp + m'q)B + (np + n'q)C$$

will also be a point in the plane; or the plane containing any two points contains also the straight line joining them. This theorem justifies the use of the word plane.

The condition for three points lying in a straight line may be written symmetrically

$$\lambda X + \mu Y + \nu Z = 0,$$

where λ, μ, ν are any numbers.

Suppose $P = xA + yB + zC$, then x, y, z or their ratios may be called the co-ordinates of the point P. If $Q = x_1A + y_1B + z_1C$, $R = x_2A + y_2B + z_2C$ be two other points; then applying the condition that the three points P, Q, R should lie on a straight line, and equating the coefficients of A, B, C to zero, we have

$$\lambda x + \mu x_1 + \nu x_2 = 0$$
$$\lambda y + \mu y_1 + \nu y_2 = 0$$
$$\lambda z + \mu z_1 + \nu z_2 = 0.$$

Therefore

$$\begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix} = 0.$$

If x, y, z be considered variable this is the equation to a straight line and may be written in the form $lx + my + nz = 0$ where l, m, n are constant.

Now suppose $P = \lambda_1 A + \mu_1 B$, $Q = \lambda_2 A + \mu_2 B$, then the ratio $\frac{\mu_1}{\lambda_1} : \frac{\mu_2}{\lambda_2}$ or $\frac{\mu_1\lambda_2}{\mu_2\lambda_1}$ will be called the anharmonic ratio of the points $ABPQ$ taken in that order. Thus we have a definition of anharmonic ratio independent of the idea of distance*.

Also the ratio is not altered by taking in place of the points A, P, B, Q any multiples of them; for putting ρA for A and σB for B, we have

$$P = \frac{\lambda_1}{\rho}\rho A + \frac{\mu_1}{\sigma}\sigma B, \qquad Q = \frac{\lambda_2}{\rho}\rho A + \frac{\mu_2}{\sigma}\sigma B,$$

and this gives for the new anharmonic ratio

$$\frac{\rho}{\sigma}\frac{\mu_1}{\lambda_1} : \frac{\rho}{\sigma}\frac{\mu_2}{\lambda_2} = \frac{\mu_1\lambda_2}{\lambda_1\mu_2}, \text{ as before.}$$

So that in determining the anharmonic ratio of four points we may use any multiples of the points instead of the points themselves.

* Felix Klein, "Ueber die Nicht-Euclidische Geometrie," *Mathematische Annalen*, Vol. IV. p. 624, says "Die Doppel-Verhältnissè dürfen dabei natürlich nicht als Strecken-Verhältnisse definirt werden, da diess die Kenntniss einer Massbestimmung voraussetzen würde."

Now let O be any new point, join OA, OP, OB, OQ, and let a new line cut these lines in A', P', B', Q'; put

$$A' = \rho O + A, \quad B' = \sigma O + B;$$

then $\quad P'$ (the intersection of OP and $A'B'$) $= \lambda_1 A' + \mu_1 B' = (\lambda_1 \rho + \mu_1 \sigma) O + P$,

$$Q' = \lambda_2 A' + \mu_2 B' = (\lambda_2 \rho + \mu_2 \sigma) O + Q;$$

therefore the anharmonic ratio of A', P', B', Q' is $\frac{\mu_1 \lambda_2}{\lambda_1 \mu_2}$, the same as that of A, P, B, Q. This ratio may then be called the ratio of the pencil of lines OA, OP, OB, OQ, since it is the same for all lines meeting that pencil.

If $P = x_1 A + y_1 B$, $Q = x_2 A + y_2 B$, $R = x_3 A + y_3 B$, $S = x_4 A + y_4 B$ be four points on the line AB, and we wish to find the anharmonic ratio of the points P, Q, R, S, we must solve the equations

$$x_2 = \lambda_1 x_1 + \mu_1 x_3, \qquad y_2 = \lambda_1 y_1 + \mu_1 y_3,$$

$$x_4 = \lambda_2 x_1 + \mu_2 x_3, \qquad y_4 = \lambda_2 y_1 + \mu_2 y_3,$$

whence

$$\frac{\mu_1}{\lambda_1} \frac{\lambda_2}{\mu_2} = \frac{(x_1 y_2 - x_2 y_1)(x_3 y_4 - x_4 y_3)}{(x_2 y_3 - x_3 y_2)(x_4 y_1 - x_1 y_4)}.$$

The quantities $x_1 y_2 - x_2 y_1$ which occur in this expression are invariants, for if

$$P = x_1' A + y_1' B, \qquad Q = x_2' A + y_2' B,$$

where

$$A' = pA + qB, \qquad B' = rA + sB;$$

then

$$x_1 y_2 - x_2 y_1 = (ps - qr)(x_1' y_2' - x_2' y_1').$$

To find the anharmonic ratio of four straight lines $l_1 x + m_1 y + n_1 z = 0$, &c., intersecting in a point, we may take the points $n_1 B - m_1 C$, $n_2 B - m_2 C$ where they cut the line BC. This gives for the ratio

$$\frac{(m_1 n_2 - m_2 n_1)(m_3 n_4 - m_4 n_3)}{(m_2 n_3 - m_3 n_2)(m_4 n_1 - m_1 n_4)}.$$

We should have also the expressions

$$\frac{(n_1 l_2 - n_2 l_1)(n_3 l_4 - n_4 l_3)}{(n_2 l_3 - n_3 l_2)(n_4 l_1 - n_1 l_4)} \text{ and } \frac{(l_1 m_2 - l_2 m_1)(l_3 m_4 - l_4 m_3)}{(l_2 m_3 - l_3 m_2)(l_4 m_1 + l_1 m_4)}.$$

These are all equal, since, if x, y, z be the point through which the lines pass,

$$\frac{x}{m_1 n_2 - m_2 n_1} = \frac{y}{n_1 l_2 - n_2 l_1} = \frac{z}{l_1 m_2 - l_2 m_1},$$

$$\frac{x}{m_2 n_3 - m_3 n_2} = \frac{y}{n_2 l_3 - n_3 l_2} = \frac{z}{l_2 m_3 - l_3 m_2}.$$

We may find now the anharmonic ratio of the four lines joining the point x, y, z to (x_1, y_1, z_1) (x_2, y_2, z_2) (x_3, y_3, z_3) (x_4, y_4, z_4); l_1, m_1, n_1, l_2, m_2, n_2 are now the minors of

$$\begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix}.$$

Therefore
$$m_1 n_2 - m_2 n_1 = \begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix} x,$$

and the required ratio is
$$\frac{\begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix} \begin{vmatrix} x, & y, & z \\ x_3, & y_3, & z_3 \\ x_4, & y_4, & z_4 \end{vmatrix}}{\begin{vmatrix} x, & y, & z \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} \begin{vmatrix} x, & y, & z \\ x_4, & y_4, & z_4 \\ x_1, & y_1, & z_1 \end{vmatrix}}.$$

If we put this equal to a constant k, the locus of the point x, y, z becomes
$$\begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix} \begin{vmatrix} x, & y, & z \\ x_3, & y_3, & z_3 \\ x_4, & y_4, & z_4 \end{vmatrix} = k \begin{vmatrix} x, & y, & z \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} \begin{vmatrix} x, & y, & z \\ x_4, & y_4, & z_4 \\ x_1, & y_1, & z_1 \end{vmatrix},$$
an equation of the second degree.

It is equivalent to the most general equation of the second degree, for it involves besides the four points through which the curve passes another constant k, making five in all.

By means of this theorem or directly from the equation can be proved the other descriptive theorems about curves of the second degree, such as Pascal's Theorem, harmonic properties of poles and polars, self-conjugate triangles, &c.

It may be shewn in the usual way (by taking a point $X+\lambda x$, $Y+\lambda y$, $Z+\lambda z$ on the line joining (X, Y, Z) and (x, y, z)) that
$$X\frac{dF}{dx} + Y\frac{dF}{dy} + Z\frac{dF}{dz} = 0$$
is the tangent at any point x, y, z of the curve $F(x, y, z) = 0$.

Similarly, as usual, are proved all descriptive or projective theorems concerning higher plane curves, such as Plücker's equations, the Hessian, the nine points of inflexion of a cubic lying three by three on straight lines, &c.

Sir W. Hamilton's theory of nets can be easily derived from the addition of points. If $O = aA + bB + cC$ be a fourth point in the plane of ABC, then the intersections of OA, BC; OB, CA; OC, AB give three new points
$$D_1 = bB + cC, \qquad D_2 = cC + aA, \qquad D_3 = aA + bB,$$
which are called points of the first construction.

D_2D_3, D_3D_1, D_1D_2 intersect DC, CA, AB, OA, OB, OC in six points
$$E_{23} = bB - cC, \qquad E_{31} = cC - aA, \qquad E_{12} = aA - bB,$$
$$E_{01} = O + aA \qquad E_{02} = O + bB \qquad E_{03} = O + cC$$
$$= 2aA + bB + cC, \qquad = 2bB + cC + aA, \qquad = 2cC + aA + bB.$$

They lie by threes on four new straight lines $(E_{23}E_{31}E_{12})$, $(E_{02}E_{03}E_{23})$, &c. The ratios $E_{23}BD_1C$, $E_{23}D_2E_{01}D_3$, &c. are harmonic.

The intersections of the line $E_{01}B$ with CA, D_1D_2, D_1D_3 give three new points,

$$E_{011} = 2aA + cC, \quad E_{012} = 2aA - bB + cC, \quad \text{and} \quad E_{013} = 2aA + 3bB + cC\,;$$

$BE_{01}E_{011}E_{012}$ form an harmonic range.

In this way an indefinite number of points can be found, including every point of the form $xaA + ybB + zcC$, where x, y, z are whole numbers, and an indefinite number of lines including every line the coefficients of whose equation are multiples of a, b, c.

The quantities x, y, z are called by Hamilton the anharmonic co-ordinates of a point. If P be the point $xaA + ybB + zcC$, and Q_1, Q_2, Q_3 be the points where AP, BC; BP, CA; CP, AB intersect; then

$\frac{z}{y}$ is the anharmonic ratio of the points BD_1CQ_1,

$\frac{x}{z}$ .... CD_2AQ_2,

$\frac{y}{x}$ .... AD_3BQ_3.

Let us now take a fourth point D not connected with the other points by a linear relation. Then we have a space or manifold of three dimensions every point of which can be represented by $xA + yB + zC + wD$.

A single equation will represent a plane, two equations a straight line.

A plane can be determined by four homogeneous co-ordinates, a straight line by six homogeneous co-ordinates connected by a homogeneous relation.

In this way we may proceed till we have n points A_1, A_2, ... A_n unconnected. Every point $x_1A_1 + x_2A_2 ... + x_nA_n$ will belong to a space of $(n-1)$ dimensions, which may be called an n point space. (See H. D'Ovidio, *Mathematische Annalen*, Vol. XI.)

A point can be determined by the ratios of its n co-ordinates, that is by $(n-1)$ quantities. A straight line will be determined by two points on it, but as each of these two points may be anywhere on the line we shall have determined two more quantities than is necessary to fix the straight line. Therefore a straight line requires $2(n-1) - 2 = 2(n-2)$ quantities to fix it.

A plane can be determined by three points, but in determining each of these the two quantities necessary to fix its position in its plane will have been determined in excess of what is needed; therefore a plane requires $3(n-1) - 3.2 = 3(n-3)$ quantities to fix it.

In general an r point space requires $r(n-1) - r(r-1) = r(n-r)$ quantities to determine it.

Or we may begin with the $(n-1)$ point space which is represented by the general equation

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = 0.$$

This is determined by $(n-1)$ quantities the ratios of $a_1, a_2, \dots a_n$. An $(n-2)$ point space is determined by two equations,

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = 0,$$
$$b_1x_1 + b_2x_2 + \dots + b_nx_n = 0,$$

but as x_1 may be eliminated from the first equation and x_2 from the second, these may be put in the form

$$a_2x_2 + \dots + a_nx_n = 0,$$
$$b_1x_1 + b_3x_3 + \dots + b_nx_n = 0,$$

so that the space required $2(n-2)$ points to determine it.

And in this way the previous result may be confirmed.

The number of quantities requisite to determine an r point space is identical with that requisite to determine an $(n-r)$ point space.

If a p point space and a q point space have an r point space in common, together they contain $p+q-r$ independent points. Now this number cannot be greater than n. Therefore if $p+q$ be greater than n the spaces must have at least $p+q-n$ points in common, and in general will have just that number.

If, however, $p+q-r$ be less than n, the number of points required to determine

the p point space in the $p+q-r$ is $p(p+q-r-p)$,

the q point space in the $p+q-r$ is $q(p+q-r-q)$,

and $p+q-r$ point space in the n point space $(p+q-r)(n+r-p-q)$.

The difference between the sum of these numbers and the number of quantities required in general to determine a p point space and a q point space in an n point space will give the number of conditions that these two spaces have an r point space for intersection.

It is

$$p(n-p) + q(n-q) - p(p+q-r-p) - q(p+q-r-q) - (p+q-r)(n+r-p-q)$$
$$= r(n+r-p-q).$$

The n points $A_1, A_2 \dots A_n$ will constitute what may be called an n-hedron.

It will contain n points, and n, $(n-1)$ point spaces;

$\frac{n(n-1)}{2}$ lines, and $\frac{n(n-1)}{2}$, $(n-2)$ point spaces;

$\frac{n(n-1)(n-2)}{2.3}$ planes, and $\frac{n(n-1)(n-2)}{2.3}$, $(n-3)$ point spaces.

If we take another point $O = a_1A_1 + a_2A_2 + \ldots + a_nA_n$ we shall be able to construct a geometrical net.

The intersections of OA_1 and $(A_2A_3 \ldots A_n)$, &c. give n points $B_1 = a_2A_2 + \ldots + a_nA_n$.

$$B_1B_2 \text{ intersects } A_1A_2 \text{ in } a_1A_1 - a_2A_2 = C_{12},$$

$$\text{and } OA_1 \text{ intersects } B_2B_3 \ldots B_n \text{ in } B_2 + B_3 + \ldots B_n = (n-2)\,O + a_1A_1 = C_{\cdot 1}.$$

We then get $\frac{n(n+1)}{2}$ new points which lie on straight lines in threes.

In this way every point of the form $x_1a_1A_1 + x_2a_2A_2 + \ldots + x_nO_nA_n$ can be constructed, where $x_1x_2 \ldots x_n$ are whole numbers.

The equation to any $(n-1)$ space will be

$$F(x_1x_2 \ldots x_n) = 0,$$

and the equation to the tangent space derived from the $(n-1)$ points near $x_1x_2 \ldots x_n$ will be

$$X_1\frac{dF}{dx_1} + X_2\frac{dF}{dx_2} + \ldots + X_n\frac{dF}{dx_n} = 0.$$

In all that precedes, though the words points and lines are used, it is not necessary to suppose actual points and lines to be meant. For instance, we might suppose A and B to be two liquids capable of mixing in any proportions. Then $pA + qB$ will be a mixture of the two, and $rA + sB$ will be the same or a different mixture according as $\frac{s}{r}$ is equal or different from $\frac{q}{p}$, since the quantity is not considered. $-pA + B$ would in this case have no meaning if the liquids were altogether different; but we might suppose each of them to be mixture, and then some of one could be supposed contained in some of the other up to certain limits. This is an instance in which quantities of the form $pA + qB + rC$ would not always have a real meaning; so that the theorems which have been mentioned are to be regarded as purely analytical, and we cannot say in each case without knowing more about the special circumstances of each manifold whether they will have a real interpretation or not.

Again, suppose A, B denote two conics in the same plane, and therefore intersecting in four (ordinary) points. $pA + qB$ may represent any conic passing through those four points, and all these conics are equivalent to what has been called a straight line.

All the conics in a plane form a manifold of 5 dimensions, and if $s_1s_2 \ldots s_6$ denote six of them, any other may be represented by $x_1s_1 + x_2s_2 + \ldots + x_6s_6$.

We have so far substantially followed Grassmann. We now proceed to see how and under what restrictions the idea of distance can be determined.

Determination of Distance.

If A and B be two points each taken once only, it is natural to *define* $A+B$ to be some multiple of a point midway between them or bisecting them. Also all points of the form $pA+qB$, where p and q are positive, will be said to lie between A and B. This is in accordance with ordinary language, for "something between the two" is said where there is no reference to position in space.

Now, if q be greater than p, $pA+qB$ can be written $p(A+B)+(q-p)B$. The point $pA+qB$ then lies between the middle point of AB and B. It may be said to be nearer to B than to A, or the distance AC may be said to be greater than the distance CB, if $C=pA+qB$. AC is therefore less, equal or greater than CB, according as $\frac{q}{p}$ is less, equal or greater than 1. Supposing the coefficient of A to be the negative, then, if we put $-pA+qB=C$, we shall have $B=\frac{1}{q}C+\frac{p}{q}A$, so that B lies between A and C, and C may be said to lie in AB produced. Similarly $pA-qB$ will lie in BA produced.

We make these assumptions, 1st, that if C lie between A and B,

$$\text{distance } AC + \text{distance } CB = \text{distance } AB.$$

2nd. If A, B, C be single points (not multiples of point) and A', B', C' other single points, also $A+B=\lambda C$, $A'+B'=\lambda C'$, then the distance between A and B is equal to the distance between A' and B'.

The distance AB is thus some function of λ, or we may say inversely λ is some function of the (unknown) distance. Put $AC=CB=x$, and $\lambda=\phi(x)$. Take a point A_1 between C and A and a point A_2 in CA produced, and let $A_1A=AA_2=y$.

Similarly take points B_1, B_2 so that $B_1B=BB_2=y$; then

$$A_1C=CB_1=x-y,$$

$$A_2C=CB_2=x+y.$$

Therefore
$$A_1+B_1=\phi(x-y)\,C, \qquad A_2+B_2=\phi(x+y)\,C,$$

$$A_1+B_1+A_2+B_2=\{\phi(x-y)+\phi(x+y)\}\,C.$$

But
$$A_1+A_2=\phi(y)A, \qquad B_1+B_2=\phi(y)B, \qquad \theta+B=\phi(x)C;$$

$$\therefore\ A_1+A_2+B_1+B_2=\phi(x)\,\phi(y)\,C.$$

Hence
$$\phi(x+y)+\phi(x-y)=\phi(x)\,\phi(y).$$

This is Poisson's functional equation.

Its solutions are

$$\phi(x) = 2\sinh\frac{x}{k},$$

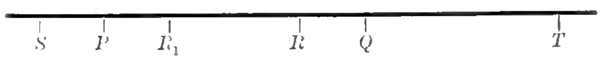
$$\phi(x) = 2,$$

$$\phi(x) = 2\sin\frac{x}{k},$$

where k is some constant.

We may next determine in an equation $pP + qQ = rR$ the connection between the quantities p, q, r and the distances PR, RQ, PQ.

Take a point R_1 between Q and P, such that $PR_1 = RQ$; also points S and T in QP and PQ produced, such that $SP = PR_1$, $R_1Q = QT$.



Also suppose $PR = m\theta$, $RQ = n\theta$, where m and n are whole numbers. Divide SP, PR_1, R_1R, RQ, QT into equal parts θ. Consider for the moment the first case only, and take each point of division $\sinh\frac{\theta}{2k}$ times, excepting R_1, which must be taken $2\sinh\frac{\theta}{2k}$ times. The points of division from S to R_1, each taken $\sinh\frac{\theta}{2k}$ times, are equivalent to a point at P taken $\sinh\frac{\theta}{2k}\left(1 + 2\cosh\frac{\theta}{k} + 2\cosh\frac{2\theta}{k} + \ldots + 2\cosh\frac{n\theta}{k}\right) = \sinh\frac{2n+1}{2}\frac{\theta}{k}$ times.

The points of division from R_1 to T, taken $\sinh\frac{\theta}{2k}$ times, are equivalent to a point at Q taken $\sinh\frac{\theta}{2k}\left(1 + 2\cosh\frac{\theta}{k} + \ldots + 2\cosh\frac{m\theta}{k}\right) = \sinh\frac{2m+1}{2}\frac{\theta}{k}$ times.

Again, the points of division from S to T, taken $\sinh\frac{\theta}{2k}$ times, are equivalent to a point at R taken $\sinh\frac{\theta}{2k}\left(1 + 2\cosh\frac{\theta}{k} + \ldots + 2\cosh\frac{m+n}{k}\theta\right) = \sinh\frac{2(m+n)+1}{2}\frac{\theta}{k}$ times.

We have then the equation

$$\sinh\frac{2n+1}{2}\frac{\theta}{k}\,.\,P + \sinh\frac{2m+1}{2}\frac{\theta}{k}\,.\,Q = \sinh\frac{\theta}{2k}\,.\,R_1 + \sinh\frac{2(m+n)+1}{2k}\theta\,.\,R.$$

Let $\qquad PR = \alpha, \qquad RQ = \beta, \qquad PQ = \gamma, \qquad \alpha + \beta = \gamma.$

Suppose m, n to increase indefinitely and θ to diminish indefinitely. Then in the limit

$$\sinh\frac{\beta}{k}\,.\,P + \sinh\frac{\alpha}{k}\,Q = \sinh\frac{\gamma}{k}\,R.$$

Or, if $pP + qQ = rR$,

$$\frac{p}{\sinh\frac{\beta}{k}} = \frac{q}{\sinh\frac{\alpha}{k}} = \frac{r}{\sinh\frac{\gamma}{k}}.$$

These equations may be replaced by

$$q^2 = p^2 + q^2 + 2pq\cosh\frac{\gamma}{k},$$

$$p\sinh\frac{\alpha}{k} = q\sinh\frac{\beta}{k}.$$

In the other two cases the equations are respectively

$$r^2 = p + q,$$

$$p\alpha = q\beta;$$

and

$$r^2 = p^2 + q^2 + 2pq\cos\frac{\gamma}{k},$$

$$p\sin\frac{\alpha}{k} = q\sin\frac{\beta}{k}.$$

We may also determine the connection between the distances of points and the quantities p, q, r by a method almost identical with that used by Klein in his article "Ueber die sogenannte nicht Euclidische Geometrie."

Suppose, as before, A, B, and C are unit points, and $A + B = \lambda C$; also suppose that by a transformation A and C become altered into C and B respectively, and any point $P_0 = x_0 A + y_0 C$ into $P_1 = x_0 C + y_0 B$. Such a transformation will be called a translation, and it will be assumed that the distance between P_0 and P_1 is the same as that between A and C or C and B.

We have $$P_1 = x_0 C + y_0 B = x_0 C + y_0(\lambda C - A) = x_1 A + y_1 C,$$

where

$$x_1 = -y_0,$$

$$y_1 = x_0 + \lambda y_0.$$

Repeating the transformation we get a point $P_2 = x_2 A + y_2 B$,

where

$$x_2 = -y_1,$$

$$y_2 = x_1 + \lambda y_1;$$

so that

$$x_2 - \lambda x_1 + x_0 = 0,$$

$$y_2 - \lambda y_1 + y_0 = 0.$$

In this way we get a point $P_n = x_n A + y_n C$ with the equations

$$x_n - \lambda x_{n-1} + x_{n-2} = 0,$$

$$y_n - \lambda y_{n-1} + y_{n-2} = 0.$$

If z, $\frac{1}{z}$ be the roots of the equation $z^2 - \lambda z + 1 = 0$,

we shall have
$$x_n = Az^n + Bz^{-n},$$
and
$$y_n = -x_{n+1} = -(Az^{n+1} + Bz^{-n-1}).$$

Now if we put
$$p(x_0 A + y_0 C) + q(x_n A + y_n C) = r(x_r A + y_r C),$$
then
$$\frac{p}{x_r y_n - x_n y_r} = \frac{q}{x_0 y_r - x_r y_0} = \frac{r}{x_0 y_n - y_0 x_n},$$
or
$$\frac{p}{x_r x_{n+1} - x_n x_{r+1}} = \frac{q}{x_0 x_{r+1} - x_r x_1} = \frac{r}{x_0 x_{n+1} - x_1 x_n}.$$

Now
$$x_r x_{n+1} - x_n x_{r+1} = (Az^r + Bz^{-r})\{Az^{n+1} + Bz^{-(n+1)}\} - (Az^n + Bz^{-n})\{Az^{-(r+1)} + Bz^{-(r+1)}\}$$
$$= AB(z - z^{-1})\{z^{n-r} - z^{-(n-r)}\};$$
so that
$$\frac{p}{z^{n-r} - z^{-(n-r)}} = \frac{q}{z^r - z^{-r}} = \frac{r}{z^n - z^{-n}}.$$

We will suppose as before the distance between P_0 and P_r to be α, that between P_r and P_n to be β, and that between P_0, P_n to be γ.

Let the distance between AC be θ, then $\alpha = r\theta$, $\beta = (n-r)\theta$, $\gamma = n\theta$.

There are three cases, according as the roots of $z^2 - \lambda z + 1 = 0$ are real, equal or imaginary, that is, according as $\lambda > = < 2$.

In the first we may put $\lambda = 2\cosh\frac{\theta}{k}$, $z = \rho^{\frac{\theta}{k}}$, $z^{-1} = \rho^{-\frac{\theta}{k}}$ where k is some constant; then
$$z^r - z^{-r} = \rho^{\frac{r\theta}{k}} - \rho^{-\frac{r\theta}{k}} = 2\sinh\frac{\alpha}{k}.$$

Hence
$$\frac{p}{\sinh\frac{\beta}{k}} = \frac{q}{\sinh\frac{\alpha}{k}} = \frac{r}{\sinh\frac{\gamma}{k}}.$$

In the third we may put $\lambda = 2\cos\theta$, $z = \rho^{\theta\iota}$, $z^{-1} = \rho^{-\theta\iota}$,
$$z^r - z^{-r} = \rho^{\frac{r\theta\iota}{k}} - \rho^{-\frac{r\theta\iota}{k}} = 2\iota\sin\frac{\alpha}{k};$$
so that
$$\frac{p}{\sin\frac{\beta}{k}} = \frac{q}{\sin\frac{\alpha}{k}} = \frac{r}{\sin\frac{\gamma}{k}}.$$

Lastly, when $z = z^{-1} = 1$,
$$z^r - z^{-r} = (z - z^{-1})(r^{r-1} + z^{r-3} + \ldots + z^{-(r-1)}) = r(z^r - z^{-1})$$
$$\text{in the limit} = \frac{\alpha}{\gamma}(z - z^{-1});$$
and therefore
$$\frac{p}{\beta} = \frac{q}{\alpha} = \frac{r}{\gamma}.$$

These equations are the same as those found before.

The identity of this method with Klein's is seen by considering the points that will remain unaltered by a translation.

We must put
$$x_1 A + \gamma_1 C = z\,(x_0 A + \gamma_0 C),$$
or
$$z x_0 + y_0 = 0,$$
$$-x_0 + (z - \lambda)\, y_0 = 0;$$
so that
$$z^2 - \lambda z + 1 = 0,$$
the same equation for determining z as before.

In the first case then there are two real points that remain unaltered by a translation. These are $A - \rho^{-\frac{\theta}{k}}\,C$ and $C - \rho^{-\frac{\theta}{k}} A$.

If H be the latter point then H is in AC produced, and its position is given by
$$\frac{\sinh CH}{\sinh AH} = \rho^{-\frac{\theta}{k}},$$
or
$$\frac{\rho^{\frac{CH}{k}} - \rho^{-\frac{CH}{k}}}{\rho^{AH} - \rho^{-AH}} = \rho^{-\frac{\theta}{k}} = \rho^{\frac{CH - AH}{k}};$$

but this is the case when CH and AH are infinite, so that H is a point at infinity. This clearly ought to be the case, since the distance of all points at a finite distance is diminished.

The number of times H is taken is $\sqrt{1 + \rho^{-\frac{2\theta}{k}} - 2\rho^{-\frac{\theta}{k}} \cosh\frac{\theta}{k}} = 0$.

In the third case there are two imaginary null points at an infinite distance.

In the second or intermediate case there is one null point $C - A$ at an infinite distance.

The connection between the quantities p, q, r in the equation $pP + qQ = rR$ and the distances PR, RQ is then determined. Whether these relations be considered to give the distances in terms $\frac{p}{r}$, $\frac{q}{r}$, or, on the other hand, to give $\frac{p}{r}$, $\frac{q}{r}$ in terms of the distances, will depend on which we are considered to know originally. In the case of addition of points on a plane or on a sphere, it is the distance that may be supposed to be immediately known. But supposing pP, qQ represent portions of different fluids that mix without condensation and p, q be the volumes of the fluids, then $r = p + q$, the second of the three possible laws of combination. We can *define* then the distance PR to be $\frac{q}{p+q}$ multiplied by some constant, and the distance RQ to be $\frac{p}{p+q}$ multiplied by the same constant. This kind of manifold might be represented by a finite straight line, the two extreme points of which would be the two unmixed fluids By taking three fluids and mixing them in all proportions a manifold would be obtained which might be represented by the portion of a plane enclosed within a triangle.

If again, according to Young's theory, all colours be made up of three fundamental colours, we shall have a manifold bounded by a triangle. The quantities p, q, r would naturally be taken to mean the intensities of the different colours. But the idea of distance cannot always be introduced. For we found that r, p, q are not independent but are connected by the relation

$$r^2 = p^2 + q^2 + 2Cpq,$$

where C is some quantity independent of p and q, and constant for the same two points.

That the relation must be of this form can be proved directly.

For assume it to be $r^n = \phi(p, q)$ where $\phi(p, q)$ is a homogeneous function of degree n is p and q, and involves besides only the distance between P and Q.

Put $$P = x_1 A + y_1 B, \qquad Q = x_2 A + y_2 B,$$

and let $\psi(x, y)$ be a function corresponding to ϕ for A and B;

then $$\psi(x_1 y_1) = 1, \qquad \psi(x_2 y_2) = 1;$$

and if $$rR = pP + qQ = (px_1 + qx_2) A + (py_1 + qy_2) B,$$

then $$r^n = \psi(px_1 + qx_2, py_1 + qy_2)$$

$$= p^n + np^{n-1} q \left(x_1 \frac{d\psi}{dx_2} + y_1 \frac{d\psi}{dy_2}\right) + \ldots + q^n.$$

The quantities $x_1 \frac{d\phi}{dx_2} + y_1 \frac{d\phi}{dy_2}$ must be independent of $x_1 y_1$, $x_2 y_2$ except in so far as they involve one single quantity, the distance. This can only be the case if n is 2, for otherwise we should get a new relation between $x_1 y_1$, $x_2 y_2$.

Therefore $$r^2 = p^2 + q^2 + 2cpq.$$

If the relation between r, p, q is not of this form, no meaning can be attached to the distance. The theorems then that have been proved before, about intersection of lines, anharmonics, ratios, conics, &c. are so general that they apply not only to the cases distinguished but even to manifolds where all measurement of distance is impossible.

If $P = \lambda_1 A + \mu_1 B$, $Q = \lambda_2 A + \mu_2 B$ then $\frac{\mu_1 \lambda_2}{\mu_2 \lambda_1}$ was defined to be the anharmonic ratios of $APBQ$. With the first form of measurement of distance this becomes

$$\frac{\sinh \frac{AP}{k} \sinh \frac{QB}{k}}{\sinh \frac{PB}{k} \sinh \frac{AQ}{k}}.$$

We will assume that lengths along all lines in a manifold of two dimensions (or three-point manifold) are measured in the same way. We will consider only the first case, and put the constant $k = 1$.

The expression for any point was $P = xA + yB + zC$.

AP intersects BC in a point D which is a multiple of $yB + zC$,

therefore $y \sinh BD = z \sinh DC$;

PB intersects CA in E where $z \sinh CE = x \sinh EA$,

CP intersects AB in F where $x \sinh AF = y \sinh FB$.

It follows that
$$\frac{\sinh BD \,.\, \sinh CE \,.\, \sinh AF}{\sinh DC \,.\, \sinh EA \,.\, \sinh FB} = 1.$$

This is the necessary and sufficient condition that AD, BE, CF meet in a point, but regard must be paid to the signs.

In particular the lines drawn from the opposite angles of a triangle to bisect the sides meet in a point which is a multiple of $A + B + C$.

A straight line whose equation is $lx + my + nz = 0$ will cut BC in a point D for which $my = -nz$, or $\frac{\sinh BD}{m} = \frac{\sinh DC}{-n}$.

Similarly BC cuts CA in E where $\frac{\sinh CE}{n} = \frac{\sinh EA}{-l}$,

AB in F where $\frac{\sinh AF}{l} = \frac{\sinh FB}{-m}$.

Therefore
$$\frac{\sinh BD \,.\, \sinh CE \,.\, \sinh AF}{\sinh DC \,.\, \sinh EA \,.\, \sinh FB} = -1,$$

and this is the necessary and sufficient condition that D, E, F lie on a straight line.

Similarly other theorems, such as Carnot's theorem, can be adapted.

Multiplication of Points.

Taking any point P and raising it to successive powers P, P^2, P^3, P^4 ... we must either have an infinite series or else we must come to some power which can be expressed numerically in terms of the preceding powers. That is, there must be some relation of the form $P^n + aP^{n-1} + \ldots = 0$. We will take the latter hypothesis and assume as the simplest relation

$$P^2 = \alpha P + \beta.$$

We will assume also that multiplication is distributive, or that

$$A(B + C) = A \,.\, B + A \,.\, C.$$

Indeed, any operation which did not satisfy this latter condition could not properly be spoken of as a multiplication.

Now square the equation $$pP + qQ = rR.$$

We have $$p^2(\alpha P+\beta) + q^2(\alpha Q+\beta) + pq(P.Q+Q.P) = r^2(\alpha R+\beta),$$

or $$pq(P.Q+Q.P) = (r^2-p^2-q^2)\beta + \alpha\{(2p-p^2)P+(rq-q^2)Q\}.$$

Now if the multiplication be uniform the relation between $P.Q$, $Q.P$, P and Q must be independent of the quantities p and q. But this can only be the case if $\alpha = 0$.

Supposing then θ the distance PQ we shall have

$$P^2 = Q^2 = R^2 = \beta,$$

and $$P.Q+Q.P = 2\beta\cosh\theta,$$

or $$P.Q+Q.P = 2\beta,$$

or $$P.Q+Q.P = 2\beta\cos\theta,$$

according as the first, second, or third law of addition holds.

This then is the most general law of multiplication consistent with giving P^2 a real meaning.

It has been seen that the two conditions $P^2 = Q^2 = \beta$, $P.Q+Q.P = 2\beta\cosh\theta$ are sufficient to ensure that $R^2 = \beta$ where R is any other point, but it has to be proved that they are sufficient to ensure that $P'.Q'+Q'.P' = 2\beta\cosh\phi$, where P', Q' are two other points, the distance between which is ϕ. Let $PP' = x$, $PQ' = y$, then $\phi = y - x$.

We are supposing, of course, that the first law of addition holds.

Then $$P' = \frac{\sinh(\theta-x)P+\sinh xQ}{\sinh\theta}, \quad Q' = \frac{\sinh(\theta-y)P+\sinh yQ}{\sinh\theta},$$

and $$P'Q' = \frac{1}{\sinh^2\theta}\{\sinh(\theta-x)\sinh(\theta-y)P^2+\sinh(\theta-x)\sinh y.PQ + \sinh x\sinh(\theta-y)\,QP+\sinh x\sinh yQ^2\};$$

hence $$P'Q'+Q'P' = \frac{2\beta}{\sinh\theta}\{\sinh(\theta-x)\sinh(\theta-y)+\sinh(\theta-x)\sinh y\cosh\theta + \sinh x\sinh(\theta-y)\cosh\theta+\sinh x\sinh y\}$$

$$= \frac{2\beta}{\sinh^2\theta}[\sinh(\theta-x)\{\sinh(\theta-y)+\sinh y\cosh\theta\} + \sinh\{x\sinh y+\sinh(\theta-y)\cosh\theta\}]$$

$$= \frac{2\beta}{\sinh\theta}\{\sinh(\theta-x)\cosh y+\sinh x\cosh(\theta-y)\}$$

$$= 2\beta(\cosh x\cosh y - \sinh x\sinh y) = 2\beta\cosh\phi.$$

The laws $$P^2 = Q^2 = \beta, \qquad PQ+QP = 2\beta\cosh\theta,$$

are thus proved to be true for all points if they are assumed for any two; and the same result can be obtained in the other two cases.

At this point several distinct assumptions can be made.

1st. We can put $\beta = 0$, $\quad P^2 = Q^2 = 0$, $\quad PQ = -QP$.

This applies to all three addition laws. With the first we shall have

$$P'.Q' = \frac{1}{\sinh^2\theta}\{\sinh(\theta - x)\sinh y - \sinh x \sinh(\theta - y)\} P.Q$$

$$= \frac{P.Q}{\sinh\theta}(\sinh y \cosh x - \sinh x \cosh y).$$

Therefore $$\frac{P'.Q'}{\sinh\phi} = \frac{P.Q}{\sinh\theta};$$

or, for any two points on the same straight line the product $P.Q$ is proportional to the hyperbolic sine of the distance between P and Q.

We can put $P.Q = \sinh\theta \times$ some constant peculiar to the line.

In the second case $P.Q = \theta \times$ some constant peculiar to the line.

In the third $P.Q = \sin\theta \times$ some constant peculiar to the line.

The second and third cases are what Grassmann calls the outer (äussere) multiplication of points and strokes (strecken). He has not considered the first case.

2nd. We can put $P^2 = Q^2 = 1, \quad PQ = QP = \cosh\theta;$

in the 2nd case $P^2 = Q^2 = 1, \quad PQ = QP = 1;$

„ 3rd case $P^2 = Q^2 = 1, \quad PQ = QP = \cos\theta.$

This is in the third case what Grassmann called the inner (innere) multiplication of strokes. He dismisses the inner multiplication of points as useless.

3rd. We can combine these two forms of multiplication and obtain the following general form,

$$P^2 = Q^2 = \beta, \qquad PQ = \beta\cosh\theta + \gamma\sinh\theta, \qquad QP = \beta\cosh\theta - \gamma\sinh\theta,$$

where β and γ are constant for the special line under consideration. β must indeed be constant for all lines since it is the square of a point.

4th. We obtain a special form of multiplication by assuming—1st, the associative principle; 2nd, that β is a mere number.

Then multiplying PQ and QP we have

$$\beta^2 = \beta^2\cosh^2\theta - \gamma\sinh^2\theta;$$

therefore $$\beta^2 = \gamma^2.$$

We will introduce a quantity ι peculiar to the line under consideration such that $\iota^2 = 1$. Then we may put $\gamma = \beta\iota$, and therefore

$$PQ = \beta(\cosh\theta \pm \iota\sinh\theta), \qquad QP = \beta(\cosh\theta \mp \iota\sinh\theta).$$

Dividing the second equation by β or P^2 we have

$$QP^{-1} = \cosh\theta \mp \iota\sinh\theta.$$

Now QP^{-1} may be considered to be the operation of transferring P to Q. If then the direction PQ be considered positive we will put

$$QP^{-1} = \cosh\theta + \iota \sinh\theta,$$

and therefore $\quad PQ = \beta^2 (\cosh\theta - \iota\sinh\theta), \qquad QP = \beta^2 (\cosh\theta + \sinh\theta).$

In the second case $\quad P^2 = Q^2 = \beta, \qquad PQ = \beta + \gamma\theta, \qquad QP = \beta - \gamma\theta;$

$$\therefore\ \beta^2 = \beta^2 - \gamma^2\theta^2.$$

Hence $\gamma^2 = 0$, and we may put $\gamma = \pm\beta\iota$ with the condition $\iota^2 = 0$.

With the same convention as before,

$$PQ = \beta^2(1 - \iota\theta), \qquad QP = \beta^2(1 + \iota\theta), \qquad QP^{-1} = 1 + \iota\theta.$$

In the third case $\quad P^2 = Q^2 = \beta, \quad PQ = \beta + \gamma\sin\theta, \quad QP = \beta\cos\theta - \gamma\sin\theta.$

$$\beta^2 = \beta^2\cos^2\theta - \gamma^2\sin^2\theta,$$

$$\beta^2 = -\gamma^2.$$

We may put $\gamma = \pm\beta\iota$ where $\iota^2 = -1$, and with the same convention

$$PQ = \beta(\cos\theta - \iota\sin\theta), \qquad QP = \beta(\cos\theta + \iota\sin\theta), \qquad \text{and } QP^{-1} = \cos\theta + \iota\sin\theta.$$

Comparing the three cases we have,

in the first $\quad QP^{-1} = \cosh\theta + \iota\sinh\theta, \qquad \iota^2 = 1,$

second $\quad QP^{-1} = 1 + \iota\theta, \qquad \iota^2 = 0,$

third $\quad QP^{-1} = \cos\theta + \iota\sin\theta, \qquad \iota^2 = -1.$

In all three cases $QP^{-1} = e^{\iota\theta}$ ($\iota^2 = 1, 0, -1$ respectively).

This result might have been arrived at directly by assuming—1st, that QP^{-1} is the operation of transferring P to Q and is the same for any two points at the same distance on the same line; 2nd, that multiplication is associative.

If then we put $\qquad QP^{-1} = f(\theta), \qquad RQ^{-1} = f(\phi),$

where θ is the distance between P and Q, ϕ between Q and R.

Then $\qquad f(\theta)f(\phi) = RQ^{-1}.QP^{-1} = RP^{-1} = f(\theta + \phi).$

This functional equation gives $f(\theta) = e^{\nu\theta}$, where ν is some constant connected with the special line.

We need not in this method assume the addition formulæ, but can deduce it. Assuming only that if R be the middle point of P and Q, then $P + Q =$ some multiple of R, we have since $P = Re^{-\nu\theta}$, $Q = Re^{\nu\theta}$, if $\theta =$ distance PR, $e^{\nu\theta} + e^{-\nu\theta} =$ some number.

Since this is true for all values of θ it follows that ν^2 is a number.

The imaginary or complex numbers of algebra may be excluded, since we are dealing with real points, and they must be reserved for cases of non-intersection of curves, &c.

It follows that either $\quad \nu^2$ is positive $= \frac{1}{k^2}$ where k is a number;

ν^2 is null $= 0$,

ν^2 is negative $= -\frac{1}{k^2}$ where k is a number.

In all three cases we may put $\nu = \frac{\iota}{k}$, $(\iota^2 = 1, 0 \text{ or } -1)$.

Then, in the first case,

$$P + Q = (\rho^{\frac{\iota\theta}{k}} + \rho^{-\frac{\iota\theta}{k}}) R = 2 \cosh \frac{\theta}{k} . R.$$

More generally, if $pP + qQ = rR$, and α be as before the distance from P to R, β the distance from R to Q, γ the distance from P to Q,

$$p\rho^{-\frac{\iota\alpha}{k}} + q\rho^{\frac{\iota\beta}{k}} = r.$$

This gives the equations

$$p \cosh \frac{\alpha}{k} + q \cosh \frac{\beta}{k} = r,$$

$$p \sinh \frac{\alpha}{k} = q \sinh \frac{\beta}{k}.$$

Squaring and adding these we have, since

$$\cosh \frac{\alpha}{k} \cosh \frac{\beta}{k} - \sinh \frac{\alpha}{k} \sinh \frac{\beta}{k} = \cosh \frac{\gamma}{k},$$

$$r^2 = p^2 + q^2 + 2pq \cosh \frac{\gamma}{k},$$

and these are the same equations as those obtained before.

In the same way the suitable formulae for the other two cases can be found.

Thus, omitting the previous section, the conception of distance might have been derived from the division of points. This naturally raises the question, Why should distance be connected with QP^{-1} rather than with $Q - P$? Or, in other words, why should QP^{-1} be supposed to be the same for two points on the same line at the same distance, when $Q - P$ is not supposed to be the same?

We may try the result of the latter supposition. If, then, again, we suppose the distance between P and Q to be θ, and that between Q and R to be ϕ, and put

$$Q - P = f(\theta), \qquad R - Q = f(\phi),$$

then

$$f(\theta) + f(\phi) = f(\theta + \phi).$$

Therefore $f(\theta) = \nu\theta$, where ν is some constant depending on the particular straight line.

Hence we have with the notation used a little while before, if $pP + qQ = rR$,

$$p(R - \nu\alpha) + q(R + \nu\beta) = r;$$

$$\therefore\ p + q = r,$$

$$p\alpha = q\beta.$$

This is the second of the three systems before found, and is that of points in ordinary plane space.

We may say then,—There are three uniform systems in which distance depends on division; there is only one uniform system in which distance depends on subtraction, and it is the special or intermediate case of the three former systems.

If in the third system we put $P^2 = Q^2 = \beta = -1$,

we have
$$PQ = -\cos\theta + \iota\sin\theta.$$

This is the Quaternion multiplication. It can be, as Grassmann has shewn in the *Mathematische Annalen* (Die Ort der Quaternionen in der Ausdehnunglehre), derived from the general form $PQ = \beta\cos\theta + \gamma\sin\theta$, by introducing the associative principle.

The corresponding forms for the other two systems will be
$$PQ = -\cosh\theta + \iota\sinh\theta,$$
and
$$PQ = -1 + \iota\theta.$$

It remains only to prove the distributive principle. That is, to prove if $pP + qP = rR$ and O be any point on the same line,
$$pO.P + qO.Q = rO.R.$$

We will take the first system and the most general form of multiplication; that is, if distance $OR = \sigma$, $PR = \theta$, $RQ = \phi$, $PQ = \chi$,
then we will put $O.P = \beta\cosh(\sigma - \theta) + \gamma\sinh(\sigma - \theta)$, and so on.

It has to be shewn then that
$$p\cosh(\sigma - \theta) + q\cosh(\sigma + \phi) = r\cosh\sigma,$$
$$p\sinh(\sigma - \theta) + q\sinh(\sigma + \phi) = r\sinh\sigma.$$

But these equations follow at once from the equations
$$p\cosh\theta + q\cosh\phi = r,$$
and
$$p\sinh\theta = q\sinh\phi.$$

DETERMINATION OF ANGLES.

If we consider the multiplication of points not merely in a straight line but in a plane every point of which is a multiple of $xA + yB + zC$, we shall have a number of different quantities ι connected with the different lines that can be drawn in the plane.

Now all the lines (l, m, n) that can be drawn through a point x, y, z satisfy the equation
$$lx + my + nz = 0,$$
and therefore they are singly infinite in number.

We will now use Hamilton's multiplication and notation.

Thus, if $PQ = -\cosh\theta + \iota\sinh\theta$, we will put $SPQ = -\cosh\theta$, and call it the scalar part of PQ, and $VPQ = \iota\sinh\theta$, and call it the vector part of PQ. Then $VPQ = -VQP$, so that Hamilton's vector multiplication is for two points identical with Grassmann's outer multiplication. What immediately follows will apply to all three kinds of geometry.

If ι_1, ι_2 be the quantities corresponding to the lines joining

$$xA + yB + zC \text{ to } x_1A + y_1B + z_1C \text{ and } x_2A + y_2B + z_2C \text{ respectively,}$$

then
$$\begin{aligned}\iota_1 &= \text{some multiple of } V(xA + yB + zC)(x_1A + y_1B + z_1C)\\ &= \text{some multiple of } (yz_1 - zy_1)\,VBC + (zx_1 - xz_1)\,(VCA + (xy_1 - yz_1)\,VAB\\ &= \text{some multiple of } l_1VBC + m_1VCA + n_1VAB,\end{aligned}$$

if $l_1m_1n_1$ be the coefficients of the equation to the line.

Similarly $\iota_2 = \text{some multiple of } l_2VBC + m_2VCA + n_2VAB.$

And if ι_3 be another line passing through the point (we may say line instead of quantity connected with line) $l_3m_3n_3$ coefficients of its equation,

$$\iota_3 = \text{some mult. of } l_3VBC + m_3VCA + n_3VAB.$$

Now $(l_1m_1n_1)$, $(l_2m_2n_2)$, $(l_3m_3n_3)$ are connected by the equation

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_2, & m_3, & n_3 \end{vmatrix} = 0,$$

and therefore we may put $l_3 = \lambda l_1 + \mu l_2$, $m_3 = \lambda m_1 + \mu m_2$, $n_3 = \lambda n_1 + \mu n_2$. It follows that we may put $r\iota_3 = p\iota_1 + q\iota_2$, so that all the lines passing through a point are lineally connected and form a system similar to that of points on a line.

We will assume then that we may introduce the conception of distance for straight lines, and we will call the quantity corresponding to the distance between two points the angle between two straight lines.

Consider the point $O(x, y, z)$ to be within the triangle ABC, so that x, y, z are all positive. Then the equation $lx + my + nz = 0$ requires that one of the quantities l, m, n should be negative if l, m, n be the coefficients of the equation to a line passing through O. This line will cut the sides in points given by $my + nz = 0$, $x = 0$; $nz + lx = 0$, $y = 0$; $lx + ny = 0$, $z = 0$. That is, in points $nB - mC$, $lC - nA$, $mA - lB$.

If l_1 be negative, the second and third points must lie between C and A and A and B respectively. So that every line drawn out from the point O in either direction cuts one of the sides of the triangle.

If then we start with the line OA, and draw lines from O to successive points of AB, then to successive points of BC, then to successive points of CA; the line OA will have returned to its old position. It follows that the equations connecting p, q, r with the angles must contain only periodic functions of the angles, and therefore the third kind of addition laws must hold. We have then, if α be the angle between ι_1 and ι_3, β between ι_3 and ι_2, and γ between ι_1 and ι_2, while $r\iota_3 = p\iota_1 + q\iota_2$,

$$r_1^2 = p^2 + q^2 + 2pq \cos \gamma,$$

$$p \sin \alpha = \sin \beta.$$

The angle a line has turned through when it has come back to its old position is said to be four right angles. The uniformity supposed to exist in the plane requires us to consider all right angles equal.

Multiplication of Lines meeting at a Point.

This will follow the same laws as multiplication of points on a line in the third kind of geometry.

We will consider for the present only the quaternion form of multiplication. Then, if ρ, σ be two lines making an angle θ,

$$\rho\sigma = \sigma^2 (\cos \theta - I \sin \theta),$$

where I is a quantity whose square is -1.

Suppose O to be the point where the lines meet; then we may identify the quantity I with the point O, and write $\rho\sigma = \sigma^2 (\cos \theta - O \sin \theta)$.

In the first kind of geometry $\sigma^2 = 1$,

$$\text{therefore } \rho\sigma = \cos \theta - O \sin \theta.$$

Therefore $\quad S\rho\sigma = \cos \theta, \qquad V\rho\sigma = - O \sin \theta, \qquad TV\rho\sigma = \sin \theta.$

In Grassmann's notation $\quad \rho|\sigma = \cos \theta, \qquad \rho\sigma = O \sin \theta.$

In the second kind $\sigma^2 = 0$,

$$\text{therefore } \rho\sigma = 0, \qquad S\rho\sigma = 0, \qquad V\rho\sigma = 0.$$

In the third kind $\sigma^2 = -1$,

$$\text{therefore } \rho\sigma = - \cos \theta + O \sin \theta,$$

$$S\rho\sigma = - \cos \theta, \qquad V\rho\sigma = O \sin \theta, \qquad TV\rho\sigma = \sin \theta.$$

In Grassmann's notation $\quad \rho|\sigma = \cos \theta, \qquad \rho\sigma = O \sin \theta.$

The inner multiplication of Grassmann corresponds (neglecting sign) to the scalar multiplication of quaternions; and the outer multiplication to the vector multiplication of quaternions.

If ρ and σ be lines at right angles, we have in the first case

$$\rho\sigma = - O, \qquad \sigma\rho = O.$$

Therefore $O = \frac{\sigma}{\rho}$, or O may be interpreted as the operation of turning a line through a right angle in the positive direction. Multiplying by σ and by ρ we have also the equations

$$O\rho = \sigma, \qquad \rho O = -\sigma,$$
$$O\sigma = -\rho, \qquad \sigma O = \rho.$$

If A be another point on the line σ and p be the distance OA, then

$$\frac{A}{O} = \cosh p + \sigma \sinh p,$$

or

$$A = O \cosh p + \rho \sinh p.$$

Let B be a third point at a distance s from O, then we have

$$B = O \cosh s + \rho \sinh s.$$

This gives

$$A \sinh s - B \sinh p = O \sinh (s - p) = O \sin \delta, \text{ if } \delta = s - p,$$

an equation equivalent to those already found, and also

$$A \cosh s - B \cosh p = -\rho \sin \delta.$$

The first equation assigns a real point for $\lambda A - \mu B$ for all values of $\frac{\mu}{\lambda}$ from 0 to $e^{-\delta}$. The point gradually moves from A to infinity along the line BA produced. When $\frac{\mu}{\lambda}$ is greater than $e^{-\delta}$ the first equation ceases to give a value for $\lambda A - \mu B$,

since

$$\frac{\sinh p}{\sinh s} = \frac{e^p - e^{-p}}{e^s - e^{-s}} < e^{-\delta} \text{ when } p \text{ and } s \text{ are positive.}$$

But the second equation gives a value for $\lambda A - \mu B$, since

$$\frac{\cosh p}{\cosh s} = \frac{e^p + e^{-p}}{e^s + e^{-s}} > e^{-\delta}.$$

As then $\frac{\mu}{\lambda}$ increases from $e^{-\delta}$ to $\frac{1}{\cosh \delta}$, $\lambda A - \mu B$ is a line perpendicular to AB, and this line moves from infinity in BA produced up to A. When the line passes A, p becomes negative, but the sign of $\frac{\mu}{\lambda}$ is unchanged. When $\frac{\mu}{\lambda} = 1$, then

$$\cosh \frac{\delta}{2} (A - B) = -\rho \sin \delta, \quad \text{or } A - B = -2\rho \sinh \frac{\delta}{2},$$

where ρ is a line drawn through the middle point of AB at right angles to it and on the positive side of rotation. As $\frac{\mu}{\lambda}$ becomes greater than 1, we can consider the expression $\mu B - \lambda A$ and the ratio $\frac{\lambda}{\mu}$, and everything will occur as before, but in inverse order.

The line will be at an infinite distance in AB produced when $\frac{\lambda}{\mu} = e^{-\delta}$ or $\frac{\mu}{\lambda} = e^{\delta}$. The expression will then represent a point which will move up to B and reach it when $\frac{\lambda}{\mu} = 0$ or $\frac{\mu}{\lambda} = \infty$.

Therefore the expression $\lambda A - \mu B$, as $\frac{\mu}{\lambda}$ varies from 0 to ∞, represents first a point which moves from A to infinity in BA produced; then a line at right angles to AB which moves from infinity in BA produced to infinity in AB produced; then a point which moves from infinity in AB produced to B.

Since
$$\sinh^2 p + \sinh^2 s - 2 \sinh p \sinh s \cosh \delta = \sinh^2 \delta,$$
$$2 \cosh p \cosh s - \cosh^2 p - \cosh^2 s = \sinh^2 \delta,$$

the magnitude of $\lambda A - \mu B$ when it represents a point is $\sqrt{\lambda^2 + \mu^2 - 2\lambda\mu \cosh \delta}$, and when it represents a straight line $\sqrt{2\lambda\mu \cosh \delta - \lambda^2 - \mu^2}$, so that it can never represent a real point and straight line at the same time.

Returning to the equation
$$A = O \cosh p + \rho \sinh p,$$

and multiplying by σ we have, if we call ρ' the perpendicular to OA at A,
$$\rho' = \rho \cosh p + O \sinh p$$
$$= (\cosh p + \sigma \sinh p)\, \rho,$$

so that $\frac{\rho'}{\rho} = \cosh p + \sigma \sinh p$, as ought to be the case, since $\rho'\rho^{-1}$ represents the operation of transferring a line along σ in the positive direction.

Also
$$S\rho'\rho = S\rho\rho' = \cosh p,$$
$$V\rho'\rho = - V\rho\rho' = \sigma \sinh p.$$

Now if the lines ρ, ρ' were to intersect at an angle θ we should have, by what has gone before, $S\rho\rho' = \cos\theta$, but it is impossible that $\cos\theta = \cosh p$, since $\cos\theta$ is always less than 1 and $\cosh p$ always greater than 1. It follows that *two lines at right angles to the same line can never intersect.*

If the equation $A = O \cosh p + \rho \sinh p$ be multiplied by ρ, we find
$$SA\rho = S\rho A = \sinh p,$$
$$VA\rho = - V\rho A = \sigma \cosh p.$$

In words,—The scalar of the product of a point and a line is the hyperbolic sine of the perpendicular from the point on the line, and the magnitude of the vector part is the hyperbolic cosine of the perpendicular; supposing always the point to be on the positive side of the straight line.

If we take a point C at a distance ϕ along ρ, then

$$C = (\cosh\phi + \rho\sinh\phi)\,O$$
$$= O\cosh\phi - \sigma\sinh\phi.$$

If p' be the distance of C from ρ', then, since C is on the negative side of ρ',

$$-\sinh p' = SC\rho' = S(O\cosh\phi - \sigma\sinh\phi)(O\sinh p + \rho\sinh p)$$
$$= -\cosh\phi\sinh p.$$

Therefore $\sinh p'$ is greater than $\sinh p$ and p' greater than p, or the distance of C from ρ' is greater than the distance from O.

Hence lines drawn at right angles to the same line diverge from one another.

In the third kind of geometry, if ρ and σ be at right angles, we shall have equations

$$\rho\sigma = O, \qquad \sigma\rho = -O,$$
$$O\rho = \sigma, \qquad \rho O = -\sigma,$$
$$\sigma O = \rho, \qquad O\sigma = -\rho\,;$$

and also, if the letters have the same meaning as before,

$$A\sin s - B\sin p = O\sin\delta,$$
$$A\cos s - B\cos p = -\rho\sin\delta.$$

But if O' be a point at a distance $\frac{\pi}{2}$ from O in the positive direction

$$A\sin(s - \tfrac{1}{2}\pi) - B\sin(p - \tfrac{1}{2}\pi) = O\sin\delta.$$

It follows that $O' = \rho$.

Now σ may be any line perpendicular to ρ, therefore all these lines pass through a point O' at a distance $\frac{1}{2}\pi$ from ρ.

This may be confirmed by the other equations analogous to those obtained above. Thus $\sin p' = \cos\phi\sin p$ shews that p' is less than p and the perpendiculars converge, and also that $p' = 0$ when $\phi = \frac{\pi}{2}$ whatever p may be; so that all perpendiculars to the same straight line pass also through the same point. The point and line are called pole and polar, and considered as symbolic quantities may be identified.

Now, if ρ, ρ' be the perpendiculars and θ their angle of intersection, then

$$S\rho\rho' = \cos p \quad\text{and}\quad S\rho\rho' = \cos\theta\,;$$

so that $\theta = p$, if the signs be properly chosen.

That is to say, the distance between two lines along their common perpendicular is equal to the angle at which they intersect.

If at points distant $\frac{1}{2}\pi$, perpendiculars be raised to the line joining them, these perpendiculars will intersect at right angles, and we shall have a triangle all of whose sides and angles are equal to $\frac{1}{2}\pi$.

If i, j, k be the points or sides of this triangle they may be identified (except in sign) with O, ρ, σ, and it follows that

$$jk = i, \quad kj = -i, \quad ki = j, \quad ik = -j, \quad ij = k, \quad ji = -k.$$

These are the symbols of Quaternions.

In the intermediate kind of geometry O, ρ, σ may be taken to satisfy the following laws

$$\rho\sigma = 0, \; \sigma\rho = 0, \; O\rho = \sigma, \; \rho O = -\sigma, \; O\sigma = -\rho, \; \sigma O = \rho.$$

The equation $$\frac{A}{O} = 1 + \sigma p$$

gives $$A = O + \rho p.$$

We have also $$B - A = \rho\delta,$$
but if ρ' be the perpendicular at A $$B - A = \rho'\delta.$$

Therefore all lines drawn at right angles to the same line are to be considered identical.

If C be a point distant ϕ from O, and on the line drawn through O perpendicular to σ, and D be a point at the same distance on the line through A, then

$$C = O - \sigma\phi,$$
$$D = A - \sigma\phi.$$

Therefore $$D - C = A - O = \rho p.$$

But if ρ' be the imaginary quantity perpendicular to the line CD, and p' be the length CD, then $D - C = \rho' p'$.

Therefore $$\rho' p' = \rho p, \; \rho' = \rho, \; p' = p.$$

It follows that lines perpendicular to the same line are everywhere equidistant. Also CD must be perpendicular to OC and AD, since ρ' coincides with ρ, the quantity belonging to OC and AD.

The figure $OCDA$ has then its angles, right angles, and its opposite sides equal. It is said to be a rectangle and the lines perpendicular to the same line are said to be parallel.

Since in general $xA + yO$ represents $x + y$ times a point P such that $xAP = yPO$, $A - O$ will represent a small point at an infinite distance. But $D - C = A - O$ so that parallel lines may be said to intersect at infinity.

General Expression for Ratios of Lines and Points.

We obtained for the first kind of geometry in the preceding section three points O, A, C expressed literally in terms of O, ρ, σ. But by definition any point in the plane can be expressed lineally in terms of three points O, A, C. Therefore any point in the plane can be expressed lineally in terms of O, ρ, σ. So also can any line, since any line is the vector part of the product of two points. To find these expressions let P be the point, a the distance OP, and α the angle OP makes with the line ρ, then the imaginary quantity corresponding to the line OP will be $\rho\cos\alpha + \sigma\sin\alpha$,

and $$PO^{-1} = \cosh a + (\rho\cos\alpha + \sigma\sin\alpha)\sinh\alpha.$$

Therefore $$P = O\cosh a - \sigma\sinh a\cos\alpha + \rho\sinh a\sin\alpha.$$

If then $x\rho + y\sigma + zO$ be identified with any multiple rP of a point P we must have $$x = r\sinh a\sin\alpha,\quad y = -r\sinh a\cos\alpha,\quad z = r\cosh a.$$

These equations give

$$r^2 = z^2 - x^2 - y^2,\quad \tan\alpha = -\frac{x}{y},\quad \tanh a = \frac{\sqrt{x^2+y^2}}{z}.$$

It is always possible then to find values of r, a, α in terms of z, x, y if $z^2 - x^2 - y^2$ be positive.

Next take a line drawn through P perpendicular to OP and let λ be the corresponding imaginary quantity. The imaginary quantity for a line at O perpendicular to OP is $$\sigma\cos\alpha - \rho\sin\alpha.$$

Therefore
$$\begin{aligned}\lambda &= \{\cosh a + (\rho\cos\alpha + \sigma\sin\alpha)\sinh a\}(\sigma\cos\alpha - \rho\sin\alpha)\\ &= -O\sinh a + \rho\cosh a\cos\alpha - \rho\cosh a\sin\alpha.\end{aligned}$$

If then $x\rho + y\sigma + zO$ be identified with $r\lambda$,

$$x = -r\cosh a\sin\alpha,\quad y = r\cosh a\cos\alpha,\quad z = -r\sinh a.$$

So that $$r^2 = x^2 + y^2 - z^2,\quad \tan\alpha = -\frac{x}{y},\quad \tanh a = \frac{z}{\sqrt{(x^2+y^2)}}.$$

It is always possible to find values of r, a, α in terms of x, y, z, if $x^2 + y^2 - z^2$ be positive.

It follows then $x\rho + y\sigma + zO$ is a point on a line according as $z^2 >< x^2 + y^2$.

If $x\rho + y\sigma + zO$ be a point then $w + x\rho + y\sigma + zO = w + rP$ may be identified with some multiple of the ratio of two lines meeting at the point P. Let θ be the angle at which they meet, s the multiple of their ratio. Then $s(\cos\theta + P\sin\theta) = w + rP$,

$$s\cos\theta = w,\quad s\sin\theta = r,\quad s^2 = w^2 + r^2,\quad \tan\theta = \frac{z}{w}.$$

Substituting the value of r^2

$$s^2 = w^2 + z^2 - x^2 - y^2.$$

If $x\rho+y\sigma+zO$ be a line then we may identify $w+x\rho+y\sigma+zO$ either with the ratio of two points on the line or else with the ratio of a point to a line. In the first case we must put

$$w+x\rho+y\sigma+zO=w+r\lambda=s(\cosh\theta+\lambda\sinh\theta).$$

Therefore
$$w=s\cosh\theta,\quad r=s\sinh\theta,$$

$$\tanh\theta=\frac{r}{w},\quad s^2=w^2-r^2=w^2+z^2-x^2-y^2.$$

The condition that this substitution may be possible

$$\text{is } w^2>r^2 \text{ or } w^2+z^2-x^2-y^2>0.$$

In the second case we must put

$$w+r\lambda=s(\sinh\theta+\lambda\cosh\theta)$$

$$\tanh\theta=\frac{w}{r},\quad s^2=r^2-w^2=x^2+y^2-w^2-z^2.$$

This will be possible if $r^2>w^2$ or $x^2+y^2-w^2-z^2>0$.

Thus a meaning can always be found for $w+x\rho+y\sigma+zO$.

If we form the product of two such expressions

$$\begin{aligned}(w+x\rho+y\sigma+zO)&(w'+x'\rho+y'\sigma+z'O)\\ &=ww'+xx'+yy'-zz'\\ &+(wx'+w'x+yz'-y'z)\rho\\ &+(wy'+w'y+zx'-z'x)\sigma\\ &+(wz'+w'z+yx'-y'x)O,\end{aligned}$$

then since the product of the magnitudes of two ratios is the magnitude of their product,

$$\begin{aligned}(w^2+z^2-x^2-y^2)&(w'^2+z'^2-x'^2-y'^2)\\ &=(ww'+xx'+yy'-zz')^2+(wz'+w'z+yx'-y'x)^2\\ &-(wx'+w'x+yz'-y'z)^2-(wy'+w'y+zx'-z'x)^2,\end{aligned}$$

a formula analogous to Euler's for the product of two sums of four squares.

Since the multiplication of the quantities O, ρ, σ is associative and all others can be formed lineally from them, multiplication will be always associative.

All the terms of Quaternions such as conjugate, tensor, versor, can be employed.

In the third kind of geometry every point can be expressed in terms of the three points i, j, k.

If we put $rP = xi + yj + zk$ and call α, β, γ the distances of P from the points i, j, k, then multiplying by i, and taking scalars

$$r\cos\alpha = -rSPi = x,$$
$$r\cos\beta = \quad y,$$
$$r\cos\gamma = \quad z.$$

Also squaring the equation $\quad rP = xi + yj + zk,$

$$r^2 = x^2 + y^2 + z^2,$$

and therefore $\quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$

If $w + xi + yj + zk$ be the ratio of two points at a distance θ, we may put

$$w + xi + yj + zk = w + rP = s(\cos\theta + P\sin\theta),$$

where P is the pole of the line on which the points are.

Then $\quad s^2 = w^2 + r^2 = w^2 + x^2 + y^2 + z^2,$

$$\tan\theta = \frac{r}{w}.$$

We arrive at Euler's formula in the way shewn above, as is pointed out in Hamilton's *Elements of Quaternions*.

Formulae in Coordinates.

If OX, OY be any lines at right angles, P any point in their plane, then the position of P may be determined by the three quantities

$$x = \sinh OP \cos POX, \qquad y = \sinh OP \sin POX, \qquad z = \cosh OP.$$

These quantities are not independent, but are connected by the equation

$$z^2 - x^2 - y^2 = 1.$$

If ρ, σ be the imaginary quantities corresponding to the lines OX, OY, then, by the preceding section,

$$P = zO - x\sigma + y\rho.$$

Now if PL, PM be the perpendiculars from P on OX, OY,

$$\sinh PL = SP\rho = y, \qquad \sinh PM = -SP\sigma = x.$$

Since $\sinh PL = \sinh OP \sin POX$ we have the theorem,—In a right-angled triangle the hyperbolic sine of the perpendicular divided by the hyperbolic sign of the hypothenuse is equal to the sine of the angle.

If OQ be the perpendicular from the origin on any line, then the position of the line may be determined by the three quantities

$$l = \sinh OQ, \qquad m = \cosh OQ \cos QOX, \qquad n = \cosh OQ \sin QOX.$$

They are connected by the relation

$$l^2 + m^2 - n^2 = 1.$$

If λ be the imaginary quantity corresponding to the line

$$\lambda = -nO + l\sigma - y\rho.$$

Supposing PR to be the perpendicular from P on the line λ,

$$\sinh PR = -SP\lambda = lx + my - nz.$$

If P be on the line, $PR = 0$ and therefore its equation is

$$lx + my - nz = 0.$$

The distance between two points $P = zO - x\sigma - y\rho$, $P' = z'O - x'\sigma + y'\rho$ is given by

$$\cosh\theta = -SPP' = zz' - xx' - yy'.$$

The angle or shortest distance between two lines is given by

$$\cos\theta = S\lambda\lambda' = ll' + mm' - nn',$$

or

$$\cosh\theta = ll' + mm' - nn'.$$

These are all the formulae that are required for questions concerning distances and angles in this kind of geometry. As an example the following problem may be taken: To find the locus of two lines meeting at right angles and passing through two fixed points.

If $(\sinh\phi,\ o\cosh\phi)\,(-\sinh\phi,\ o\cosh\phi)$ be the points, so that they are at equal distances ϕ on either side of O, $(x,\ y,\ z)$ the points whose locus is required; then, if $(l,\ m,\ n)$, $(l',\ m',\ n')$ be the lines

$$\frac{l}{y\cosh\phi} = \frac{m}{z\sinh\phi - x\cosh\phi} = \frac{n}{y\sinh\phi},$$

$$\frac{l'}{y\cosh\phi} = \frac{m'}{-z\sinh\phi - x\cosh\phi} = \frac{-n'}{y\sinh\phi};$$

and, since $ll' + mm' - nn' = 0$,

$$y^2\cosh^2\phi + (x\cosh\phi - z\sinh\phi)(x\cosh\phi + z\sinh\phi) + y\sinh^2\phi = 0,$$

$$x^2\cosh^2\phi + y^2\cosh 2\phi - z^2\sinh^2\phi = 0.$$

The formulae for the third kind of geometry will be identical with those of spherical geometry referred to ordinary rectangular co-ordinates.

In the second or intermediate kind, if P be any point we can put

$$P = O + xi + yj,$$

where, if PL be the perpendicular on OX, then $OL = x$, $PL = y$.

It follows that $\qquad L - O = xi, \qquad P - M = yj.$

If r be the length of OP then $\frac{P-O}{r}$ represents a unit line perpendicular to PO; so that if OP makes an angle θ with OX, then

$$\frac{P-O}{r} = i\cos\theta + j\sin\theta.$$

Therefore $$x = r\cos\theta, \qquad y = r\sin\theta,$$
and hence $$x^2 + y^2 = r^2,$$
relations connecting the angles and sides of the right-angled triangle POL.

If θ' be the angle OPL, then in the same way

$$x = r\sin\theta', \qquad y = r\cos\theta',$$
so that $$\cos\theta = \sin\theta', \qquad \sin\theta = \cos\theta'.$$

The angles θ, θ' therefore together make a right angle, and it follows that every straight line cuts parallel straight lines, so that the alternate angles are equal.

The distance between two points is found in the usual way (which assumes nothing not already proved) to be $(x-x')^2 + (y-y')^2$.

This is not at all a convenient system for deriving properties of the ordinary plane, but it is the one which offers the closest analogy with Quaternions for the sphere.

Relations between the Sides and Angles of a Triangle.

Let A, B, C be the points of a triangle in the first kind of geometry;
α, β, γ the corresponding angles;
ρ, σ, τ the imaginary quantities belonging to the opposite sides;
a, b, c the length of these sides.

Then
$$CB^{-1} = \cosh a + \rho\sinh a, \qquad BC^{-1} = \cosh a - \rho\sinh a,$$
$$AC^{-1} = \cosh b + \sigma\sinh b, \qquad CA^{-1} = \cosh b - \sigma\sinh b,$$
$$BA^{-1} = \cosh c + \tau\sinh c, \qquad AB^{-1} = \cosh c - \tau\sinh c.$$

Therefore $$\cosh a - \rho\sinh a = (\cosh c + \tau\sinh c)(\cosh b + \sigma\sinh b).$$

Take the scalar parts: then, since $S\tau\sigma = \cos(\pi - \alpha) = -\cos\alpha$,
$$\cosh a = \cosh b\cosh c - \sinh b\sinh c\cos\alpha.$$

Take the vector parts:
$$-\rho\sinh a = \tau\sinh c\cosh b + \sigma\sinh b\cosh c + A\sinh b\sinh c\sin\alpha,$$
since $$V\tau\sigma = A\sin(\pi-\alpha) = A\sin\alpha.$$

Multiply by A and take scalars,

$$SA\rho \, . \sinh a = \sinh b \sinh c \sin \alpha.$$

Now, if p be the perpendicular from A on BC, $SA\rho = \sinh p$.

Also, $$SA\rho \sinh a = SAVBC = SABC, \text{ since } S(ASBC) = 0$$
$$= S\rho A \sinh a = SVBC \, . A = SBCA = SBVCA$$
$$= SB\sigma = SC\tau.$$

Hence the quantity $SABC$ is unaltered by the interchange in cyclic order of A, B, C, and we have the equations

$$\sinh b \sinh c \sin \alpha = \sinh c \sinh a \sin \beta = \sinh a \sinh b \sin \gamma$$
$$= \sinh a \sinh p = \sinh b \sinh q = \sinh c \sinh r$$
$$= \sqrt{\sinh^2 b \sinh^2 c - (\cosh b \cosh c - \cosh a)^2}$$
$$= \sqrt{1 + 2 \cosh a \cosh b \cosh c - \cosh^2 a - \cosh^2 b - \cosh^2 c},$$

by the preceding equation.

We have thus three independent equations between a, b, c, α, β, γ, and can determine any three in terms of the others. The equation giving α in terms of β, γa can however be obtained directly in the same form as that giving a in terms of b, c, α.

For $$\tau\sigma^{-1} = -\cos\alpha + A \sin\alpha, \qquad \sigma\tau^{-1} = -\cos\alpha - A\sin\alpha,$$
$$\rho\tau^{-1} = -\cos\beta + B \sin\beta, \qquad \tau\rho^{-1} = -\cos\beta - B\sin\beta,$$
$$\sigma\rho^{-1} = -\cos\gamma + C\sin\gamma, \qquad \rho\sigma^{-1} = -\cos\gamma - C\sin\gamma.$$

Therefore $$-\cos\alpha - A\sin\alpha = (-\cos\gamma + C\sin\gamma)(-\cos\beta + B\sin\beta).$$

Taking scalars $$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cosh a.$$

From this last equation it is seen that

$$\cos(\pi - \alpha) < \cos(\beta + \gamma),$$
$$\pi - \alpha > \beta + \gamma,$$
$$\pi > \alpha + \beta + \gamma;$$

or, the sum of the angles of a triangle is always less than two right angles.

Suppose C to be a right angle, then

$$\cosh c = \cosh a \cosh b,$$
$$\cos\alpha = \sin\beta \cosh a,$$
$$\cot\alpha = \frac{\sin\beta}{\sin\alpha}\cosh a = \sinh b \cosh a.$$

Now let B move off to infinity while A is unaltered, then AB will be said to be parallel to CB, and $\cot\alpha = \sinh b$.

This gives the angle the line drawn from any point parallel to a given line makes with the perpendicular on that line. This angle is less than a right angle, and continually decreases as the length of the perpendicular increases. From the symmetry of any line it is obvious that two parallels can be drawn from any point, one on either side of the perpendicular and making equal angles with it. An infinite number of lines can be drawn not meeting the given line. The equations found are sufficient to determine all properties of this kind of geometry. The trigonometrical relations of the other two kinds will be those of the sphere and the ordinary plane.

THE DIFFERENT KINDS OF UNIFORM SPACE.

We have by a purely analytical method arrived at three different kinds of space relations, and as no assumptions have been made except those necessary to ensure uniformity, these are the only possible uniform relations. We may now bring together the chief distinctive properties.

In the first case.

The angles of a triangle are together always less than two right angles.

Lines perpendicular to the same line diverge from one another.

From any point outside a line, two lines can be drawn meeting that line at infinity and an infinite number of lines not meeting it.

In the second case.

The angles of a triangle are together always equal to two right angles.

Lines perpendicular to the same line are always equidistant.

From any point outside a straight line only one line can be drawn to meet that line at infinity, and every other line will meet it at a finite distance.

In the third case.

The angles of a triangle are together always greater than two right angles.

Lines perpendicular to the same line approach one another.

From any point outside a straight line no lines can be drawn to meet that line at infinity. All lines meet at a finite distance.

The first kind of geometry is the imaginary or non-Euclidean geometry of Gauss, Lobatschewsky and Bolyai.

The second is the geometry of the ordinary plane.

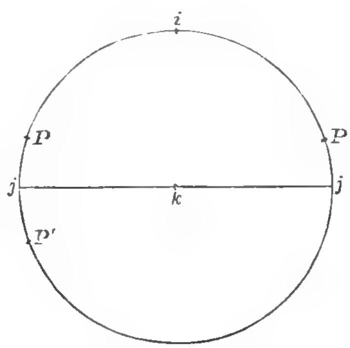
The third kind may be divided into two sub-cases according as we treat A and $-A$ as distinct or identical points.

In the first sub-case since if the equation $lx + my + nz = 0$ is satisfied by (x, y, z), it is also satisfied by $(-x, -y, -z)$; all the lines drawn through a given point pass also through another point which may be called the opposite point. The distance from i to j is $\frac{1}{2}\pi$, from j to $-i$ $\frac{1}{2}\pi$, from $-i$ to $-j$ $\frac{1}{2}\pi$, from $-j$ to i $\frac{1}{2}\pi$; so that the whole length of a line returning into itself is 2π. Every line, for instance that joining j, k, divides the space symmetrically, and it is impossible to pass from a point in one half to a point in the other without cutting the line. This is the geometry of the sphere.

The other sub-case gives a geometry which has been considered by Simon Newcomb (*Crelle*, Vol. 83, for the year 1877, p. 293); Killing (*Crelle*, Vol. 86, for the year 1879, p. 72); and Frankland "On the simplest continuous Manifoldness of two dimensions" in *Nature*, 1878.

The distance from i to j is $\frac{1}{2}\pi$, and from j to $-i$ is $\frac{1}{2}\pi$, and therefore as $-i$ coincides with i the length of all straight lines is π. As in the sphere, all the points distant $\frac{1}{2}\pi$ from i lie on a straight line.

The surface may be represented by the figure if it be remembered that opposite



points on the circle are identical. If a man were to start from j and walk along the line jk till he returned to j, he would then be in the position $P'j$, that is to say upside down, and he would have to complete the line twice to return to his original position. For this reason, the line jk may be considered a double line, and just as in the sphere every line has two poles through which the same lines pass, so we may say in this geometry every point has two polar lines on which the same points lie. The properties are therefore reciprocal to those of the sphere, as is pointed out by Killing. Any two points P, P' for instance can be joined without cutting jk, so that the straight line does not divide the plane.

This kind of surface could not like the sphere exist in a non-Euclidean or Euclidean space of three dimensions.

It is obvious that the ordinary trigonometry can be obtained from the imaginary or spherical trigonometry by putting $(1, a)$ for $(\cosh a, \sinh a)$, or $(\cos a, \sin a)$ respectively.

Or else reintroducing the constant k we have only to put in the equations

$$\frac{\sin\frac{a}{k}}{\sin\alpha}=\frac{\sin\frac{b}{k}}{\sin\beta}=\frac{\sin\frac{c}{k}}{\sin\gamma},$$

$k=\infty$ after having multiplied by k to obtain the corresponding plane equations. Similarly those of imaginary trigonometry can be obtained by making k imaginary. Thus the three kinds of relations are those of a sphere of imaginary, infinite and real radius respectively. This could have been seen originally from the fact that the other two addition equations can be derived from the third by making k imaginary or infinite.

Only linear manifolds have been considered, but a non-linear manifold such as that of the points $p^2A+pqB+q^2C$ will obviously lie in a linear manifold $xA+yB+zC$ of higher dimensions.

The assumptions that must be made to identify our actual space with the second of the three kinds here considered seem to be the following. They are arranged in the order in which they have been successively introduced.

1. There exists a continuous line determined in a single way by any two points on it and capable of being drawn between any two points.

2. There exists a continuous surface determined in a single way by any three points on it.

3. If the line be called a straight line, then a straight line can be moved along itself. To determine the motion it is sufficient to know the new position of any one point, then that of any other will be known.

4. Keeping a point fixed a straight line can be moved in only one way while remaining in the same plane (if the surface be called a plane), so as to coincide with any other line in that plane passing through the point.

5. The line equidistant from a straight line, is itself a straight line.

6. Space is of three dimensions.

The first two assumptions are equivalent to the definitions that a line consists of all the points $pA+qB$ and a plane of all the points $pA+qB+rC$. The third is requisite to enable us to measure distance. It is verified every time a distance is measured in the usual way. In fact the ordinary measure of distance coincides, as Klein points out, with the definition given in a former section. It must be observed that the lines spoken of are actual physical lines, yard measures, stiff wires, &c. We find the properties nearly true of these actual lines, and that as far as our knowledge goes they can be approximated to, indefinitely.

Geometry of course merely expresses in abstract language a special class of relations between existing bodies. Moreover, as Riemann observes, our observation is limited just as much on the side of the very small as on the side of the very great. In the same way as the conception of rays of light is found to give a very accurate explanation of shadows, images, &c., but a closer observation shews a different class of optical phenomena;

so it might be that if our microscopes were powerful enough, we should discover different geometrical relations among the particles of matter.

The fourth axiom shews how to measure angles and compare distances on different straight lines, for the preceding axiom only shews how to compare them on the same straight line. The fifth is one of the many forms in which an axiom distinguishing our space from the other two kinds of uniform space may be stated. The sixth requires no comment.

It is difficult in such an enumeration to be sure of having included all the axioms and of not having combined two distinct propositions in one. But at least it is clear that the number of axioms must be limited. Grassmann's results refute the view that geometry contains an indefinite number of distinct synthetic propositions. When once the axioms have been assumed, all other geometrical truths can be derived by the mere rules of calculation.

IMAGINARY GEOMETRY OF THREE DIMENSIONS.

We proceed next to construct a calculus analogous to Quaternions for the imaginary geometry of Bolyai, when it is in three dimensions.

All the points equidistant from a given point form a closed uniform surface identical in all its properties with the ordinary sphere.

Draw from the point O three lines at right angles, and let i, j, k be the corresponding imaginary quantities. Let also I denote the operation of turning the line j round O till it coincides with k, and J, K the operations of turning k round to i and i to j. If we consider the points where i, j, k cut the sphere, then these operations will move the points along three arcs, distances equal to right angles. I, J, K then must be identical in their properties with the quantities before found for the spherical calculus.

Therefore $\quad JK = I, \quad KI = J, \quad IJ = K, \quad I^2 = -1, \quad J^2 = -1, \quad K^2 = -1.$

But since I turns j to k and therefore k to $-j$,

$$Ij = k, \quad Ik = -j,$$

and therefore since $\quad j^2 = k^2 = 1, \quad kj = I, \quad jk = -I, \quad jI = -k, \quad kI = j.$

Similarly $\quad Jk = i, \quad kJ = -i, \quad Ji = -k, \quad iJ = k, \quad ik = J, \quad ki = -J,$

$$Ki = j, \quad iK = -j, \quad Kj = -i, \quad jK = i, \quad ji = K, \quad ij = -K.$$

The equations may be written thus

$$\begin{aligned} I &= JK = kj, & i &= jK = Jk, \\ J &= KI = ik, & j &= kI = Ki, \\ K &= IJ = ji, & k &= iJ = Ij, \\ I^2 = J^2 &= K^2 = -1, & i^2 = j^2 &= k^2 = 1\,; \end{aligned}$$

remembering always that the products change sign when the order of the factors is inverted.

Multiplying the equations $i = jK = Jk$ by I and assuming the associative principle

$$Ii = kK = Kk.$$

Therefore the quantities Ii, Jj, Kk, iI, jJ, kK must be all equal.

The equation $I = -jk$ shews that $iI = -ijk$.

Put $ijk = v$, $iI = -v$, then $i^2 = 1$, $I^2 = -1$,

$$vI = i, \quad vi = -I, \quad v^2 = -1,$$

and since $Ii = -v$, $Iv = i$, $iv = -I$.

The quantity v is therefore commutative with each of the six quantities I, J, K, i, j, k.

A unit line through the origin will be $li + mj + nk$ if (l, m, n) be its direction-cosines. A rotation through a right angle about this line will be $lI + mJ + nK$. The product $(li + mj + nk)(lI + mJ + nK)$ is equal to v as it ought to be.

If r times the line $li + mj + nk$ be transferred through a distance θ along a line at right angles whose direction-cosines are (l', m', n') we shall obtain the line

$$r\{\cosh\theta + (l'i + m'j + n'k)\sinh\theta\}(li + mj + nk)$$
$$= rl\cosh\theta i + rm\cosh\theta j + rn\cosh\theta k$$
$$+ r(mn' - m'n)\sinh\theta I + r(nl' - n'l)\sinh\theta J + r(lm' - l'm)\sinh\theta K.$$

If we write this $Xi + Yj + Zk + LI + MJ + NK$,

then $$XL + YM + ZN = 0,$$

and $X^2 + Y^2 + Z^2 - L^2 - M^2 - N^2 = r^2$ a positive quantity,

since $(mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2 = (l^2 + m^2 + n^2)(l'^2 + m'^2 + n'^2) - (ll' + mm' + nn') = 1.$

If these two conditions hold $Xi + Yj + Zk + LI + MJ + NK$ can always be identified with a translation along a certain line, for there will be five independent equations to determine five quantities fixing the line. We shall have, in fact,

$$r = \sqrt{X^2 + Y^2 + Z^2 - L^2 - M^2 - N^2}, \quad \tanh\theta = \sqrt{\frac{L^2 + M^2 + N^2}{X^2 + Y^2 + Z^2}},$$

$$\rho = \frac{X}{\sqrt{(X^2 + Y^2 + Z^2)}}, \qquad l' = \frac{ZM - YN}{\sqrt{(X^2 + Y^2 + Z^2)}\sqrt{(L^2 + M^2 + N^2)}}.$$

When r becomes 0 and θ infinite while X, Y, Z, L, M, N still keep finite values,

$$X^2 + Y^2 + Z^2 - L^2 - M^2 - N^2 = 0,$$

and $Xi + Yj + Zk + LI + MJ + NK$ represents a null translation along a line at an infinite distance.

If r times the rotation $lI + mJ + nK$ be transferred through a distance θ along the line whose direction-cosines are (l', m', n') where $ll' + mm' + nn' = 0$, we shall obtain the rotation

$$r\{\cosh\theta + (l'i + m'j + n'k)\sinh\theta\}\{lI + mJ + nK\}$$
$$= rl\cosh\theta I + rm\cosh\theta J + rn\cosh\theta K$$
$$+ r(m'n - mn')\sinh\theta i + r(n'l - nl')\sinh\theta j + r(l'm - l'm)\sinh\theta k.$$

We may put this equal to

$$-Li - Mj - Nk + XI + YJ + ZK,$$

and if this be multiplied by v, it gives the corresponding line as it ought to do. It follows that, if

$$XL + YM + ZN = 0,$$

then $$Xi + Yj + Zk + LI + MJ + NK$$

represents a line or rotation according as

$$X^2 + Y^2 + Z^2 - L^2 - M^2 - N^2 > \text{ or } < 0.$$

Take a unit rotation, or as it will perhaps be better to call it, couple

$$+\lambda i + \mu j + \nu k + \rho I + \sigma J + \tau K.$$

This may be written $(\rho + \lambda\sqrt{-1})I + (\sigma + \mu\sqrt{-1})J + (\tau + \nu\sqrt{-1})K$, for the quantity v used above has all the properties of the algebraic $\sqrt{-1}$, and may be identified with it.

The two conditions $$\rho^2 + \sigma^2 + \tau^2 - \lambda^2 - \mu^2 - \nu^2 = 1$$

and $$\rho\lambda + \sigma\mu + \tau\nu = 0$$

are equivalent to $$(\rho + \lambda\sqrt{-1})^2 + (\sigma + \mu\sqrt{-1})^2 + (\tau + \nu\sqrt{-1})^2 = 1.$$

It follows that a unit couple in imaginary geometry is identical with a unit bivector in the sense Hamilton gives that word in his Quaternions. A force along the same line will be identical with a bivector of magnitude or tensor $\sqrt{-1}$. Hence a force F along a line and couple G about that line may be identified with a bivector of magnitude $G + F\sqrt{-1}$. Now any expression

$$Xi + Yj + Zk + LI + MJ + NK = (L + X\sqrt{-1})I + (M + Y\sqrt{-1})J + (N + Z\sqrt{-1})K$$

may be equated to

$$(G + F\sqrt{-1})\{(\rho + \lambda\sqrt{-1})I + (\sigma + \mu\sqrt{-1})J + (\tau + \nu\sqrt{-1})K\}.$$

For we have only to solve the equations

$$L + X\sqrt{-1} = (G + F\sqrt{-1})(\rho + \lambda\sqrt{-1}),$$
$$M + Y\sqrt{-1} = (G + F\sqrt{-1})(\sigma + \mu\sqrt{-1}),$$
$$N + Z\sqrt{-1} = (G + F\sqrt{-1})(\tau + \nu\sqrt{-1}),$$

and this is analytically the same problem as in ordinary space finding the length and direction-cosines of a line whose rectangular co-ordinates are given. Squaring and adding the three equations, we have

$$(G + F\sqrt{-1})^2 = (L + X\sqrt{-1})^2 + (M + Y\sqrt{-1})^2 + (N + Z\sqrt{-1})^2;$$

and this determines F, G in a single way if we agree that the sign of F shall always be positive. Moreover $Xi + Yj + Zk + LI + MJ + NK$ can represent the sum of any number of forces and couples.

Therefore we have, as in ordinary geometry, the proposition that any number of forces and couples are equivalent to a force along a certain line and a couple about that line. We may call this line the Central Axis, X, Y, Z may be called the components of the resultant force at the point O, and L, M, N the components of the resultant couple. Then it is found from the above equations, that

$$X^2 + Y^2 + Z^2 = F^2(\rho^2 + \sigma^2 + \tau^2) + G^2(\lambda^2 + \mu^2 + \nu^2).$$

Now the method by which the expression for a line was obtained shews that, if θ be the distance of line $(\rho\sigma\tau, \lambda\mu\nu)$ from the origin,

$$\cosh^2\theta = \rho^2 + \sigma^2 + \tau^2, \qquad \sinh^2\theta = \lambda^2 + \mu^2 + \nu^2.$$

Therefore $$X^2 + Y^2 + Z^2 = F^2\cosh^2\theta + G^2\sinh^2\theta.$$

This shews, first, that $X^2 + Y^2 + Z^2$ is always greater than F^2, and therefore the resultant force is least at any point on the central axis; secondly, that $X^2 + Y^2 + Z^2$ for given values of F and G depends only on θ, and therefore the resultant force is the same on all points of cylinders described about the central axis.

The square of the resultant couple at O is

$$L^2 + M^2 + N^2 = F^2\sinh^2\theta + G^2\cosh^2\theta,$$

and the same theorems are true with respect to it.

Let α, β be two unit lines and let θ be their shortest distance, ϕ the angle the planes passing through either and that shortest distance make with one another. Then the operation of transferring α to β is equivalent to turning α through an angle ϕ about the shortest distance, and moving it through a distance θ along the shortest distance.

If δ be a unit line along the shortest distance, D unit rotation about the shortest distance, then we may put

$$\begin{aligned}\beta\alpha^{-1} &= (\cosh\theta + \delta\sinh\theta)(\cos\phi + D\sin\phi)\\ &= (\cos\theta\sqrt{-1} + D\sin\theta\sqrt{-1})(\cos\phi + D\sin\phi)\\ &= \cos(\phi + \theta\sqrt{-1}) + D\sin(\phi + \theta\sqrt{-1}).\end{aligned}$$

This is a biversor with angle $\phi + \theta\sqrt{-1}$.

If we identify

$$\begin{aligned}&P + QI + RJ + SK + p\sqrt{-1} + qi + rj + sk\\ &\quad = (P + p\sqrt{-1}) + (Q + q\sqrt{-1})I + (R + r\sqrt{-1})J + (S + s\sqrt{-1})K,\end{aligned}$$

with n times the ratio of two lines, we must have

$$n\cos(\phi - \theta\sqrt{-1}) = P + p\sqrt{-1},$$

and $$n^2\sin^2(\phi - \theta\sqrt{-1}) = (Q + q\sqrt{-1})^2 + (R + r\sqrt{-1})^2 + (S + s\sqrt{-1})^2.$$

Therefore $$n^2 = (P + p\sqrt{-1})^2 + (Q + q\sqrt{-1})^2 + (R + r\sqrt{-1})^2 + (S + s\sqrt{-1})^2,$$

giving $$n^2 = P^2 + Q^2 + R^2 + S^2 - p^2 - q^2 - r^2 - s^2,$$

$$Pp + Qq + Rr + Ss = 0.$$

Only then, if this last condition holds, can the general expression be identified with the ratio of two forces or of two couples or of a couple to a force. Otherwise it will be the ratio of a force and a couple to a couple; that is, some multiple of the ratio of a screw to a couple.

If β, α be two rotations, we still have

$$\beta\alpha^{-1} = \cos(\phi + \theta\sqrt{-1}) + D\sin(\phi + \theta\sqrt{-1});$$

and, as with real quaternions,

$$S\alpha\beta = -\cos(\phi + \theta\sqrt{-1}).$$

But if $(\rho, \sigma, \tau, \lambda, \mu, \nu)$ $(\rho', \sigma', \tau', \lambda', \mu', \nu')$ be the co-ordinates of the lines α, β

$$S\alpha\beta = -(\rho + \lambda\sqrt{-1})(\rho' + \lambda'\sqrt{-1}) - (\sigma + \mu\sqrt{-1})(\sigma' + \mu'\sqrt{-1}) - (\tau + \nu\sqrt{-1})(\tau' + \nu'\sqrt{-1}).$$

Equating the real and imaginary parts in the two expressions $S\alpha\beta$,

$$\cosh\theta\cos\phi = \rho\rho' + \sigma\sigma' + \tau\tau' - \lambda\lambda' - \mu\mu' - \nu\nu',$$

$$\sinh\theta\sin\phi = \rho\lambda' + \sigma\mu' + \tau\nu' + \rho'\lambda + \sigma'\mu + \tau'\nu.$$

More generally, if the system of forces (X, Y, Z, L, M, N) reduce to a force F and a couple G, while (X', Y', Z', L', M', N') reduce to a force F' and a couple G', and ϕ, θ have the same meaning as before, and are referred to the central axes

$$(G + F\sqrt{-1})(G' + F'\sqrt{-1})\cos(\phi + \theta\sqrt{-1})$$
$$= (L + X\sqrt{-1})(L' + X'\sqrt{-1}) + (M + Y\sqrt{-1})(M' + Y'\sqrt{-1}) + (N + Z\sqrt{-1})(N' + Z'\sqrt{-1}).$$

Therefore $$(FF' - GG')\cosh\theta\cos\phi - (FG' + F'G)\sinh\theta\sin\phi$$
$$= XX' + YY' + ZZ' - LL' - MM' - NN',$$

$$(FG' + F'G)\cosh\theta\cos\phi + (FF' - GG')\sinh\theta\sin\phi$$
$$= XL' + YM' + ZN' + LX' + MY' + NZ'.$$

These are the simplest expressions for the two invariants of a system of forces. The second is identical with that called by Prof. Ball, in his "Theory of Screws," the virtual coefficient of two screws.

If O be the origin and l, m, n the direction-cosines of the line OP, θ the length of OP, then $li + my + nk$ is the imaginary quantity corresponding to the line OP; we have

$$PO^{-1} = \cosh\theta + (li + mj + nk)\sinh\theta$$
$$= w + xi + yj + zk$$

we may say, where $$w^2 - x^2 - y^2 - z^2 = 1,$$

(w, x, y, z) may be taken for homogeneous co-ordinates of the point P.

Also,
$$OP^{-1} = \cosh\theta - (li + mj + nk)\sinh\theta$$
$$= w - xi - yj - zk.$$

If (w', x', y', z') be the co-ordinates of any other point Q,
$$QO^{-1} = w' + x'i + y'j + z'k.$$

Therefore
$$QP^{-1} = (w' + x'i + y'j + z'k)(w - xi - yj - zk)$$
$$= ww' - xx' - yy' - zz'$$
$$+ (wx' - w'x)i + (wy' - w'y)j + (wz' - w'z)k$$
$$- (yz' - y'z)I - (zx' - zx)J - (xy' - x'y)K.$$

But if $(\rho, \sigma, \tau, \lambda, \mu, \nu)$ be the co-ordinates of the line PQ, ψ its length,
$$QP^{-1} = \cosh\psi + (\rho i + \sigma j + \tau k - \lambda I - \lambda J - \nu K)\sinh\psi.$$

Therefore
$$\cosh\psi = ww' - xx' - yy' - zz,$$
$$\rho = \frac{wx' + w'x}{\sinh\psi}, \qquad \lambda = \frac{yz' - y'z}{\sinh\psi},$$
$$\sinh^2\psi = (wx' - w'x)^2 + (wy' - w'y)^2 + (wz' - w'z)^2$$
$$- (yz' - y'z)^2 - (zx' - z'x)^2 - (xy' - x'y)^2.$$

We may use these formulae to find the equation to the cylindroid.

Suppose $(X, 0, 0, L, 0, 0)$ $(0, Y, 0, 0, M, 0)$ to be two screws about axes at right angles to each other, then the cylindroid is the locus of the axes of screws that are obtained by adding different multiples of these two screws together. (X, O, O, L, O, O) will represent any multiple of a screw about the axis of x, but the ratio $\frac{L}{X}$ must be taken constant, and we will put it equal to p_a in accordance with Prof. Ball's notation. Also, $\frac{M}{Y} = p_\beta$.

If F, G, be the force and couple of the resultant screw, $(\rho, \sigma, \tau, \lambda, \mu, \nu)$ its co-ordinates,
$$L + X\sqrt{-1} = (G + F\sqrt{-1})(\rho + \lambda\sqrt{-1}),$$
$$M + Y\sqrt{-1} = (G + F\sqrt{-1})(\sigma + \mu\sqrt{-1}),$$
$$0 = (G + F\sqrt{-1})(\tau + \nu\sqrt{-1}).$$

The last equation shews that $\tau = 0$, $\nu = 0$, or the axis meets the axis of Z at right angles.

Multiplying the first by $\rho - \lambda\sqrt{-1}$,
$$(L + X\sqrt{-1})(\rho - \lambda\sqrt{-1}) = (G + F\sqrt{-1})(\rho^2 + \lambda^2).$$

Therefore
$$\frac{G}{F} = \frac{X\lambda + L\rho}{X\rho - L\lambda} = \frac{\lambda + p_a\rho}{\rho - p_a\lambda}.$$

Similarly
$$\frac{G}{F} = \frac{Y\mu + M\sigma}{Y\sigma - M\mu} = \frac{\mu + p_\beta\sigma}{\sigma - p_\beta\mu}.$$

Now, if w, x, y, z, be any point on the axis of the screw, w', o, o, z' the point where it meets the axis,

$$\rho = -w'x, \qquad \sigma = -w'y, \qquad \lambda = yz', \qquad \mu = -xz'.$$

But $\tau = wz' - w'z = 0$, so that $\frac{w}{w'} = \frac{z}{z'}$, and as only the ratios of $\rho, \sigma, \lambda, \mu$ are required, we may write

$$\rho = wx, \qquad \sigma = wy, \qquad \lambda = -yz, \qquad \mu = xz.$$

Hence

$$\frac{-yz + p_\alpha wx}{wx + p_\alpha yz} = \frac{xz + p_\beta wy}{wy - p_\beta xz}.$$

That is

$$(p_\alpha - p_\beta)(w^2 - z^2)\, xy = (1 + p_\alpha p_\beta)(x^2 + y^2)\, wz.$$

This is the equation to the cylindroid and it is a surface of the fourth degree as has already been shewn by Lindemann.

SPHERICAL GEOMETRY OF THREE DIMENSIONS.

In this geometry we shall have three imaginary quantities, i, j, k represent translations along lines, such that $i^2 = j^2 = k^2 = -1$, and three quantities I, J, K representing rotations about the lines, such that

$$I^2 = J^2 = K^2 = -1, \qquad JK = I, \qquad KI = J, \qquad IJ = K.$$

Also $$i = Jk = -kJ, \qquad j = Ki = -iK, \qquad k = Ij = -jI.$$

We write the equations thus

$$i = Jk = jK, \qquad I = JK = jk,$$
$$j = Ki = kI, \qquad J = KI = ki,$$
$$k = Ij = iJ, \qquad K = IJ = ij.$$

It follows that $$Ii = Jj = Kk = iI = jJ = kK = -\omega \text{ say,}$$

and $$\omega^2 = Ii \,.\, iI = 1.$$

Hence $$\omega I = i, \qquad \omega J = j, \qquad \omega K = k.$$

The general vector expression is

$$(X + L\omega)\, I + (Y + M\omega)\, J + (N + K\omega)\, K.$$

This is a different sort of bivector, and has been considered by Prof. Clifford in the *Proceedings of the London Mathematical Society* and the *American Journal of Mathematics*.

Results exactly similar to those of the last section can be obtained by using this imaginary quantity ω, which is commutative with I, J, K in the place of $\sqrt{-1}$. It must be noticed that a rotation about a line is always equivalent to a translation along another real line which may be called the conjugate line. In fact, the rotation I moves all points in the plane j, k along circles with O for centre. One of these circles, namely, that of radius $\frac{\pi}{2}$, is a straight line, so that this straight line moves along itself, and the distance of all points from it remains unchanged.

The equation to the cylindroid is

$$(p_\alpha - p_\beta)(w^2 + z^2)\, xy = (1 - p_\alpha p_\beta)(x^2 + y^2)\, wz.$$

Ordinary Geometry of Three Dimensions.

In this case $\quad i^2 = j^2 = k^2 = 0, \qquad I^2 = J^2 = K^2 = -1.$

We may take a quantity ω such that $\omega^2 = 0$, and put

$$\omega I = i, \qquad \omega J = j, \qquad \omega K = k.$$

As has been shewn already for two dimensions, all parallel translations will have to be considered equal when they are of equal magnitude. A rotation about a line through the origin whose direction-cosines are (l, m, n) will be expressed by $lI + mJ + nK$. If this be transferred to a point x, y, z along a perpendicular line we have the rotation

$$\begin{aligned}(1 + x\omega I + y\omega J + z\omega K)&(lI + mJ + nK)\\ &= lI + mJ + nK + \omega\{(yn - zm) I + (zl - xn) J + (xl - ym) K\}.\end{aligned}$$

In general the quantity

$$(X + L\omega) I + (Y + M\omega) J + (Z + N\omega) K$$

will represent a rotation F and a translation G, or we may say a rotation of magnitude $F + Gw$.

Putting $\rho = l$, $\sigma = m$, $\tau = n$, $\lambda = yn - zm$, $\mu = zl - xn$, $\nu = xl - ym$ where (l, m, n) are the direction-cosines of the rotations and x, y, z a point on its line of application, we may call $\rho + \lambda\omega$, $\sigma + \mu\omega$, $\tau + \nu\omega$ the direction-cosines of the line considered in position as well as magnitude.

Since the two equations

$$\rho^2 + \sigma^2 + \tau^2 = 1, \quad \rho\lambda + \sigma\mu + \tau\nu = 0$$

are equivalent to the equation

$$(\rho + \lambda\omega)^2 + (\sigma + \mu\omega)^2 + (\tau + \nu\omega)^2 = 1.$$

To find then the magnitudes of F and G and the lines about which they act, we have to solve the equations

$$\begin{aligned} X + L\omega &= (F + G\omega)(\rho + \lambda\omega),\\ Y + M\omega &= (F + G\omega)(\sigma + \mu\omega),\\ Z + N\omega &= (F + G\omega)(\tau + \nu\omega).\end{aligned}$$

We see then that in all three kinds of geometry the properties of lines in space can be derived from those of lines meeting at a point if we write $X+L\omega$, &c., for the co-ordinates of the extremity of the line, $\rho+\lambda\omega$, &c., for its direction-cosines, $F+G\omega$ for its length, where in the first kind of geometry $\omega^2=-1$, in the second $\omega^2=0$, and in the third $\omega^2=1$.

In spherical geometry if $(\rho, \sigma, \tau, \lambda, \mu, \nu)$ are the co-ordinates of a unit force along a line, $(\lambda, \mu, \nu, \rho, \sigma, \tau)$ will be the co-ordinates of a unit couple about that line. But it is easy to see that these are the co-ordinates of a line conjugate to the original line with respect to the sphere at infinity $w^2+x^2+y^2+z^2=0$.

Hence decomposing a system of forces into a force along a line and a couple about that line is a particular case of decomposing the system into forces along conjugate lines with respect to a given surface of the second degree. This is also true in imaginary geometry and ordinary geometry, though in the one case the conjugate line becomes imaginary and in the other passes off to infinity. Now in all three kinds of geometry the equation to any surface of the second degree may be written

$$w^2+x^2+y^2+z^2=0.$$

We shall have then the following theorems:

Any system of forces may be decomposed into forces along conjugate lines with respect to a given surface of the second degree.

If two systems of forces be compounded in different proportions, and the resultant systems be decomposed into pairs of conjugate lines, the locus of these conjugate lines is a surface of the fourth degree.

When the given surface degenerates into the imaginary circle at infinity then the surface of the fourth degree degenerates into a surface of the third degree, the ordinary cylindroid.

SPACES OF HIGHER DIMENSIONS.

The units of an imaginary space of three dimensions were derived from three independent units i, j, k.

They could be considered as the commutative product of two systems

$$1,\ i,\ j,\ ij,$$

$$1,\ ijk;$$

which may be written more shortly

$$1,\ i,\ j,\ K,$$

$$1,\ \nu,$$

where 1, i, j, K form an imaginary plane system and $\nu^2=-1$.

The units of the imaginary space of four dimensions are got by multiplying these units by a new unit l or instead by $\mu = ijl$.

$\mu^2 = -1$ and μ is commutative with i, j and therefore with k but not with ν. In fact

$$\mu\nu = ijlijk = lijijk = -lk = kl = -\nu\mu.$$

Hence ν, μ, $\nu\mu$ form a quaternion system.

The system of four dimensions is therefore the commutative product of the two systems

$$1,\ i,\ j,\ k,$$

$$1,\ \nu,\ \mu,\ \nu\mu,$$

the one an imaginary system of two dimensions, and the other a spherical system of two dimensions.

The system of five dimensions is obtained by multiplying this by a new unit m or else by $\omega = ijklm$,

$$\omega^2 = ijklmijklm = ijklmmijkl = ijklijkl = -ijkllijk = -ijkijk = 1,$$

w is commutative with each of the units i, j, k, l and therefore with their products.

Hence the system of five dimensions is the commutative product of

$$1,\ i,\ j,\ k,$$

$$1,\ \nu,\ \mu,\ \nu\mu,$$

$$1,\ w.$$

The system of six dimensions is obtained by multiplying the system of five by a new unit n or else by $\pi = ijkln$.

$\pi^2 = 1$ and π is commutative with each of the units formed out of i, j, k, l but not with ω. In fact

$$w\pi = ijklmijkln = mijklijkln = mn = -nm = -\pi w,$$

so that

$$(w\pi)^2 = -w\pi\pi w = -1.$$

Hence w, π, $w\pi$ form an imaginary plane system, and the system of six dimensions is the commutative product of the three systems

$$1,\ i,\ j,\ k,$$

$$1,\ \nu,\ \mu,\ \nu\mu,$$

$$1,\ w,\ \pi,\ w\pi,$$

or of two imaginary systems of two dimensions and one spherical system of two dimensions.

The system of seven dimensions is formed by multiplying this by a new unit o or by

$$\nu' = ijklmno,$$

$$\nu' = ijklmnoijklmno = jklmnoi^2jklmno = jklmnoklmno = -klmnojklmno = -1,$$

and ν' is commutative with all the former units i, j, k, l, m, n and their products.

Hence the system of seven dimensions is the product of the systems

$$1,\ i,\ j,\ k,$$
$$1,\ \nu,\ \mu,\ \nu\mu,$$
$$1,\ w,\ \pi,\ w\pi,$$
$$1,\ \nu'.$$

If p be another unit and $\mu' = ijklmnp$, the system of eight dimensions will be the product of

$$1,\ i,\ j,\ k,$$
$$1,\ \nu,\ \mu,\ \nu\mu,$$
$$1,\ w,\ \pi,\ w\pi,$$
$$1,\ \nu',\ \mu',\ \nu'\mu',$$

that is of two imaginary systems and two quaternion systems.

We arrive then at the following results:

The system of $4m$ dimensions is the product of m spherical systems and m imaginary system of two dimensions.

The system of $4m+1$ dimensions is the product of the system of $4m$ dimensions by the system $1,\ \omega$, where $\omega^2 = 1$.

The system of $4m+2$ dimensions is the product of m spherical systems and $m+1$ imaginary systems.

The system of $4m+3$ dimensions is the product of the system of $4m+2$ dimensions by the system $1,\ \nu$ where $\nu^2 = -1$.

The only difference in the spherical systems is that the squares of the fundamental units are -1 instead of 1. It is easily seen that the following laws will hold:

The system of $4m$ dimensions is the product m spherical and m imaginary systems.

The system of $4m+1$ dimensions is the product of the system of $4m$ dimensions by $1,\ \nu$ where $\nu^2 = -1$.

The system of $4m+2$ dimensions is the product of $m+1$ spherical systems and m imaginary systems.

The system of $4m+3$ dimensions is the product of the system of $4m+2$ dimensions by $1,\ \omega$ where $\omega^2 = 1$.

The most general quantity in a system of the n^{th} degree will contain 2^n terms.

Grassmann's Ausdehnungslehre. The Outer Multiplication.

We return to the point we left off at in the first section of the former part, and put aside for the time all considerations respecting distance. We have to determine a multiplication which will be independent of these considerations. If we assume that the square of a point is always the same, we have

$$(pA+qB)^2 = A^2 = B^2 \text{ for all values of } p \text{ and } q.$$

Hence $$(p^2+q^2-1)A^2 + pq\,(A\,.\,B + B\,.\,A) = 0.$$

This can only be the case if $A^2=0$, $B^2=0$, and $A\,.\,B+B\,.\,A = 0$.

That this multiplication is the only one which can include all three laws of distance is seen by recalling what were shewn in the former part to be the most general laws of uniform multiplication.

They were in the three cases

$$A\,.\,B+B\,.\,A = 2\cosh\theta\,.\,A^2,$$
$$A\,.\,B+B\,.\,A = 2\theta^2,$$
$$A\,.\,B+B\,.\,A = 2\cos\theta\,.\,A^2,$$

and these can only be collected in one by putting

$$A\,.\,B+B\,.\,A = 0, \qquad A^2=0.$$

This multiplication, called by Grassmann "the outer multiplication," is therefore the proper one for treating descriptive theorems, as it involves no ideas of distance*.

If there be three points A, B, C, we must put

$$A^2=B^2=C^2=0, \qquad BC=-CB, \qquad CA=-AC, \qquad AB=-BA.$$

Assuming the associative principle to determine the products of three factors, we have

$$ABC=-ACB=-BAC=BCA=-CBA=CAB,$$

or the product is the same when the cyclical order is unchanged.

If $$P=xA+yB+zC, \quad Q=x'A+y'B+z'C, \quad R=x''A+y''B+z''C$$

be any three other points in the plane of A, B, C, then, by the distributive law of multiplication,

$$P^2=Q^2=R^2=0, \quad QR=-RQ, \quad RP=-PR, \quad PQ=-QP,$$

and $$PQR=-QPR=\ldots=\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix} ABC.$$

If P lie on the line QR, then $P=\lambda Q+\mu R$,

and therefore $$PQR=\lambda Q^2R-\mu R^2Q=0.$$

* *Die Ausdehnungslehre*, Ed. of 1862, pp. 6—30. "Die verschiedenen Arten des Produktbildung."

Hence the equation to the line QR is $\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix} = 0.$

The product of any two points may be called a line, and every line in the plane may be expressed as the sum of three lines, BC, CA, AB.

For $$QR = (y'z'' - y''z')\,BC + (z'x'' - z''x')\,CA + (x'y'' - y'x'')\,AB.$$

Conversely, every expression of the form $lBC + mCA + nAB$ may be considered a line, for we may write $\frac{1}{l}(lB - mA)(lC - nA)$.

By multiplying two new points on the same line we only obtain a different multiple of the line, since
$$(\lambda Q + \mu R)(\lambda' Q + \mu' R) = (\lambda\mu' - \lambda'\mu)\,QR.$$

The sum of any number of lines in a plane is itself a line in the plane, for it must always be of the form $lBC + mBA + nAB$.

If there be four independent points, all the points derived from them form a space of three dimensions, any point of which is represented by $P = xA + yB + zC + wD$.

If four points P, Q, R, S be in the same plane, $PQRS = 0$, and hence the equation to the plane is
$$\begin{vmatrix} x, & y, & z, & w \\ x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \end{vmatrix} = 0.$$

The product of any three points may be called a plane, since only the degree of multiplicity is altered by taking different points in the same plane. The general expression for a plane will be
$$lBCD + mCDA + nDAB + rABC.$$

Conversely, every expression of this form is the product of three points; for
$$lBCD + mCDA = CD\,(lB + mA), \qquad nDAB = \frac{n}{l}\,DA\,(lB + mA),$$

since $A^2 = 0$, $$lBCD + mCDA + nDAB = \frac{1}{l}\,D\,(nA - lC)(lB + mA),$$
$$rABC = \frac{r}{l^2}\,A\,(nA - lC)(lB + mB).$$

Hence $$lBCD + mCDA + nDAB + rABC = \frac{1}{l^2}(lD - rA)(nA - lC)(lB + mA).$$

It follows that the sum of any number of planes in a four point space is itself a plane.

Any line or product of two points in a four-point space may be represented by

$$F = XAB + YAC + ZAD + LCD + MDB + NBC,$$

where X, Y, Z, L, M, N are numbers. But such an expression need not represent a line; for it involves (disregarding multiples of the same expression) five constants, and a straight line only involves four.

The condition that it should be a line is found by putting $F^2 = 0$ to be

$$LX + MY + NZ = 0.$$

In general, let $e_1 e_2 e_3 e_4 \ldots e_n$ be n unconnected points, then the expression for an $(n-1)$-point space is

$$p_1 e_2 e_3 \ldots e_n + p_2 e_1 e_3 \ldots e_n + p_3 e_1 e_2 e_4 \ldots e_n + \ldots;$$

and, conversely, such an expression will always be the product of $(n-1)$ points. For

$$p_1 e_2 \ldots e_n + p_2 e_1 e_3 \ldots e_n = (p_1 e_2 + p_2 e_1)\, e_3 e_4 \ldots e_n,$$

$$p_3 e_1 e_2 e_4 \ldots e_n = \frac{p_3}{p} (p_1 e_2 + p_2 e_1)\, e_2 e_4 \ldots e_n,$$

$$p_1 e_2 \ldots e_n + p_2 e_1 e_3 \ldots e_n + p_3 e_1 e_2 e_4 \ldots e_n = \frac{1}{p_2} (p_1 e_2 + p_2 e_1)(p_2 e_3 + p_3 e_2)\, e_4 \ldots e_n,$$

and so term after term may be joined together.

The general expression of the form

$$a_{12\ldots r} e_1 e_2 \ldots e_r + a_{2 \ldots r+1} e_2 e_3 \ldots e_{r+1} + \ldots$$

involves (disregarding multiplicity) $\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r} - 1$ constants, and a r-point space in an n-point space requires $r(n-r)$ constants to determine it. Therefore

$$\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r} - r(n-r) - 1$$

conditions are necessary that the two may coincide. For instance, in a 5-point space, if

$$F = a_{12} e_1 e_2 + a_{23} e_2 e_3 + \ldots$$

is equivalent to a single line, we must have $\frac{5 \cdot 4}{1 \cdot 2} - 2 \times 3 - 1 = 3$ conditions satisfied. In fact, putting $F^2 = 0$ we obtain the five equations,

$$\begin{aligned} a_{23}a_{45} + a_{24}a_{53} + a_{25}a_{34} &= 0, \qquad (a_{23} = -a_{32}) \\ a_{34}a_{51} + a_{35}a_{14} + a_{31}a_{45} &= 0, \\ a_{45}a_{12} + a_{41}a_{26} + a_{42}a_{51} &= 0, \\ a_{51}a_{23} + a_{52}a_{31} + a_{53}a_{12} &= 0, \\ a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23} &= 0, \end{aligned}$$

of which it is easily seen only three are independent.

APPLICATIONS TO SYSTEMS OF FORCES AND LINEAR COMPLEXES.

Although the general expression for the sum of any number of lines in space of three dimensions

$$F = Xe_0e_1 + Ye_0e_2 + Ze_0e_3 + Le_2e_3 + Me_3e_1 + Ne_1e_2$$

cannot be reduced to a line, it can be reduced to the sum of two lines in an infinite number of ways.

For if l be any line and λ a number the equation

$$(F - \lambda l)^2 = 0, \text{ or } F^2 - \lambda Fl = 0, \text{ since } l^2 = 0,$$

always gives a value of λ for which $F - \lambda l$ is a straight line.

If μm be this straight line (where μ is a number) then

$$F = \lambda l + \mu m.$$

Thus we can in general take any straight line and find another corresponding to it, such that F will be the sum of the two. The exception is when $Fl = 0$, for then the equation gives no finite value of λ. All the lines satisfying the equation $Fl = 0$ form what is called by Plücker a linear complex of the first degree, and it is easy to see that this equation is equivalent to the most general linear equation between the co-ordinates of a straight line.

If $F^2 = 0$ and F be consequently a straight line, the equation $Fl = 0$ represents all the lines meeting this line; for the product of two intersecting lines is 0, since it is equal to the product of four points in the same plane.

Decomposing F into two lines so that $F = \lambda l' + \mu m'$, the equation Fl will be satisfied if $l'l = 0$, $m'l = 0$; and, conversely, if $Fl = 0$ and $l'l = 0$, then also $m'l = 0$, so that the complex consists of all the lines that can be drawn intersecting any pair of corresponding lines. It follows that the lines of the complex which can be drawn through any point lie in a plane, and this is seen directly by putting $l = xy$ where x, y are two points, then keeping x fixed the equation $Fxy = 0$ is satisfied by all the points y which lie on the plane Fx. To every point there exists therefore a polar plane, and the polar planes of all the points lying on a straight line pass through another straight line, since

$$F(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 Fx_1 + \lambda_2 Fx_2,$$

and $\lambda_1 x_1 + \lambda_2 x_2$ is any point on the line joining x_1, x_2, while $\lambda Fx_1 + \lambda_2 Fx_2$ is a plane passing through the planes Fx_1, Fx_2.

If F_1, F_2, F_3, F_4, F_5, F_6 be any six given sums of lines not connected by any linear relation, then the quantities e_0e_1, e_1e_2 ... may be expressed in terms of F_1, F_2, &c.; and therefore any system of line F may be expressed in the form

$$F = \lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 + \lambda_4 F_4 + \lambda_5 F_5 + \lambda_6 F_6,$$

and this only in one way. All the quantities of F may then be said to form a 6-point space or space of five dimensions. If the co-ordinates of F satisfy any linear relation, then F may be derived from five independent line-systems satisfying that relation; just as all the points in a linear n-point space satisfying an equation of the first degree belong to a $(n-1)$-point space. The most general relation of the first degree between the coefficients of F may be expressed by the equation $F'F=0$ where F' is another line-system.

If $$F = Xe_0e_1 + \dots + Le_2e_3 + \dots, \qquad F' = X'e_0e_1 + \dots + L'e_2e_3 + \dots,$$

then $$FF' = F'F = XL' + YM' + ZN' + LX' + MY' + NZ',$$

since $$e_0e_1e_2e_3 = e_2e_3e_0e_1, \qquad e_0e_1e_0e_2 = 0.$$

When $F'F=0$ either line-system may be said to be reciprocal to the other or the line-systems may be said to be co-reciprocal.

Thus all the line-systems reciprocal to a given line-system can be derived from five systems reciprocal to the given system.

Similarly, all the systems reciprocal to two, three, or four systems can be derived from four, three or two reciprocal systems, and there always exists one system of lines reciprocal to five given systems. Hence we may choose F_1, $F_2 \dots F_6$, so that each is reciprocal to the rest. The lines of a complex $Fl=0$ belong to the screws (we may call the quantities F screws for shortness) reciprocal to F.

The lines common to two complexes $F_1 l=0$, $F_2 l=0$ are said to form a congruence. If F_1, F_2 are real, the congruence will always contain real lines. For, taking any real line l not belonging to it, F_1 may be put equal to $\lambda l + \mu l'$ and F_2 to $\lambda' l + \mu' l''$, so that a line intersecting the line l, l', l'' will belong to the congruence. But an infinite number of real lines can be drawn to intersect three real lines.

From this may be proved that if F_1, $F_2 \dots F_6$ be co-reciprocal, three of the quantities $F_1^2 F_2^2 \dots F_6^2$ must be positive and three negative. For the lines of the congruence $F_1 l=0$, $F_2 l=0$ can be expressed in the form

$$\lambda_3 F_3 + \lambda_4 F_4 + \lambda_5 F_5 + \lambda_6 F_6,$$

since they are a particular case of screws conjugate to two given screws.

Squaring this last expression, and remembering that $F_3F_4=0$, &c.,

$$\lambda_3^2 F_3^2 + \lambda_4^2 F_4^2 + \lambda_5^2 F_5^2 + \lambda_6^2 F_6^2 = 0.$$

But this last equation can only be satisfied by real values of λ_3, &c. if one at least of the quantities F_3^2, F_4^2, &c. is different in sign from the others, and so with any other four. In the language of mechanics, If six screws be co-reciprocal three must be right-handed and three left-handed. It is understood all along, of course, that the word "real" is used in an algebraical sense, and means "expressed by co-ordinate not involving the algebraic $\sqrt{-1}$.

The lines belonging to two complexes $F_1 l=0$, $F_2 l=0$ belong also to any complex of the system $(\lambda_1 F_1+\lambda_2 F_2)\, l=0$.

Now $\lambda_1 F_1+\lambda_2 F_2$ is a straight line when

$$(\lambda_1 F_1+\lambda_2 F_2)^2=0, \text{ or } \lambda_1^2 F_1^2+2\lambda_1\lambda_2 F_1 F_2+\lambda_2^2 F_2^2=0.$$

This equation gives in general two real or imaginary values of $\frac{\lambda_2}{\lambda_1}$, and thus in general a congruence consists of all the lines intersecting two given lines.

The lines belonging to three complexes $F_1 l=0$, $F_2 l=0$, $F_3 l=0$ are singly infinite in number. They consist in fact of all the lines that can be made by combining three screws of the reciprocal system F_4, F_5, F_6. Now if $\lambda_4 F_4+\lambda_5 F_5+\lambda_6 F_6$ be a straight line

$$\lambda_4^2 F_4^2+\lambda_5^2 F_5^2+\lambda_6^2 F_6^2+2\lambda_5\lambda_6 F_5 F_6+2\lambda_6\lambda_4 F_6 F_4+2\lambda_4\lambda_5 F_4 F_5=0;$$

and this equation leaves one of the ratios $\frac{\lambda_5}{\lambda_4}$, $\frac{\lambda_6}{\lambda_4}$ arbitrary.

All these lines intersect the lines contained in the expression $\lambda_1 F_1+\lambda_2 F_2+\lambda_3 F_3$.

These latter lines therefore lie on the surface formed by the former lines, and are a second set of generators.

The lines belonging to three complexes, although not the same, generate the same surface as the lines belonging to three reciprocal complexes. Through any point on the surface can be drawn one generator of each set. In fact the lines that can be drawn through x are the intersection of the planes $F_1 x=0$, $F_2 x=0$, $F_3 x=0$, and these must intersect in a straight line if x lie on the surface.

There are two lines common to four complexes $F_1 l=0$, $F_2 l=0$, $F_3 l=0$, $F_4 l=0$. They are in fact the two lines included in the expression $\lambda_5 F_5+\lambda_6 F_6$, where F_5, F_6 are reciprocal screws to $F_1 F_2 F_3 F_4$.

The ratio $\frac{\lambda_6}{\lambda_5}$ is obtained from the equation $\lambda_5^2 F_5^2+2\lambda_5\lambda_6 F_5 F_6+\lambda_6^2 F_6^2=0$. In particular if F_4 be a straight line, it is seen that there are two lines on the surface consisting of the lines common to $F_1 l=0$, $F_2 l=0$, $F_3 l=0$ which also intersect a given line; or in other words that any straight line cuts that surface in two points and it is therefore a surface of the second degree. If $l=\lambda_5 F_5+\lambda_6 F_6$ gives only one line, that is to say, if the equation $\lambda_5^2 F_5^2+2\lambda_5\lambda_6 F_5 F_6+\lambda_6^2 F_6^2=0$ has equal roots, we must have

$$\lambda_5^2 F_5^2+\lambda_6 F_5 F_6=0, \ \lambda_5 F_5 F_6+\lambda_6 F_6^2=0,$$

or $F_5 l=0$, $F_6 l=0$. Hence l can be derived from the screws $F_1 F_2 F_3 F_4$, and we may put

$$\lambda_1 F_1+\lambda_2 F_2+\lambda_3 F_3+\lambda_4 F_4=l.$$

Substituting in the equations

$$F_1 l=0, \ F_2 l=0, \ F_3 l=0, \ F_4 l=0$$

we have $$\lambda_1 F_1^2+\lambda_2 F_1 F_2+\lambda_3 F_1 F_3+\lambda_4 F_1 F_4=0,$$

with three others. Hence

$$\begin{vmatrix} F_1^2, & F_1F_2, & F_1F_3, & F_1F_4 \\ F_1F_2, & F_2^2, & F_2F_3, & F_2F_4 \\ F_1F_3, & F_2F_3, & F_3^2, & F_3F_4 \\ F_1F_4, & F_2F_4, & F_3F_4, & F_4^2 \end{vmatrix} = 0$$

is the condition that four complexes should have only one line in common. In particular the condition that the four lines l_1, l_2, l_3, l_4, should only be intersected by one line, reduces to

$$\sqrt{l_1l_2}\sqrt{l_3l_4} \pm \sqrt{l_1l_3}\sqrt{l_2l_4} \pm \sqrt{l_1l_4}\sqrt{l_2l_3} = 0.$$

If the five complexes $F_1l=0$, $F_2l=0$, $F_3l=0$, $F_4l=0$, $F_5l=0$ have a line in common, then since l is reciprocal to itself, it must belong to the screws derived from the five F_1, F_2, F_3, F_4, F_5.

Hence $$\lambda_1F_1 + \lambda_2F_2 + \lambda_3F_3 + \lambda_4F_4 + \lambda_5F_5 = l,$$

and since $F_1l=0$ $$\lambda_1F_1^2 + \lambda_2F_1F_2 + \lambda_3F_1F_3 + \lambda_4F_1F_4 + \lambda_5F_1F_5 = 0,$$

with four others. Eliminating $\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5$,

$$\begin{vmatrix} F_1^2, & F_1F_2, & F_1F_3, & F_1F_4, & F_1F_5 \\ F_1F_2, & F_2^2, & F_2F_3, & F_2F_4, & F_2F_5 \\ F_1F_3, & F_2F_3, & F_3^2, & F_3F_4, & F_3F_5 \\ F_1F_4, & F_2F_4, & F_3F_4, & F_4^2, & F_4F_5 \\ F_1F_5, & F_2F_5, & F_3F_5, & F_4F_5, & F_5^2 \end{vmatrix} = 0.$$

Lastly the condition that there should be a linear relation among the screws $F_1F_2F_3F_4F_5F_6$ is found by putting

$$\lambda_1F_1 + \lambda_2F_2 + \lambda_3F_3 + \lambda_4F_4 + \lambda_5F_5 + \lambda_6F_6 = 0,$$

and multiplying by F_1, &c. to be

$$\begin{vmatrix} F_1^2, & F_1F_2, & F_1F_3, & F_1F_4, & F_1F_5, & F_1F_6 \\ F_1F_2, & F_2^2, & F_2F_3, & F_2F_4, & F_2F_5, & F_2F_6 \\ F_1F_3, & F_2F_3, & F_3^2, & F_3F_4, & F_3F_5, & F_3F_6 \\ F_1F_4, & F_2F_4, & F_3F_4, & F_4^2, & F_4F_5, & F_4F_6 \\ F_1F_5, & F_2F_5, & F_3F_5, & F_4F_5, & F_5^2, & F_5F_6 \\ F_1F_6, & F_2F_6, & F_3F_6, & F_4F_6, & F_5F_6, & F_6^2 \end{vmatrix} = 0.$$

If F_1^2, F_1F_2, &c. be interpreted in ordinary geometry these are identical with the equations Prof. Ball has given in his *Theory of Screws*, in generalisation of those of Profs. Cayley and Sylvester*.

* Sturm "Sulle Forze in Equilibrie" in the *Annali di Matematica*.

THE REGRESSIVE MULTIPLICATION.

If there be only three units e_1, e_2, e_3, or in other words if only points in a plane are being considered, we may put $e_1e_2e_3 = 1$, since no higher products can occur. With this supposition the product of any three points will be a number. It may be noticed that there is a reciprocity between the line and the points, in so far that just as the sum of any two points is a point on the same line, so the sum of any two lines is a line through the same point. Again any point can be expressed in the form $x = x_1e_1 + x_2e_2 + x_3e_3$, and so if we put $E_1 = e_2e_3$, $E_2 = e_3e_1$, $E_3 = e_1e_2$, any line can be expressed in the form

$$X = X_1E_1 + X_2E_2 + X_3E_3.$$

We may carry this reciprocity further, and introduce a multiplication of line exactly corresponding to the multiplication of points.

We may put $E_2E_3 = e_1 = -E_3E_2$, $E_3E_1 = e_2 = -E_1E_3$, $E_1E_2 = e_3 = -E_2E_1$,

$$E_1^2 = 0,\ E_2^2 = 0,\ E_3^2 = 0.$$

It must be observed that the associative principle cannot hold when the two kinds of multiplication are combined.

For $e_3e_1 \,.\, e_1e_2 = e_1$ and $e_3e_1^2e_2 = 0$, since $e_1^2 = 0$.

The product of two lines $X = X_1E_1 + X_2E_2 + X_3E_3$, $Y = Y_1E_1 + Y_2E_2 + Y_3E_3$ is defined by

$$XY = (X_1E_1 + X_2E_2 + X_3E_3)(Y_1E_1 + Y_2E_2 + Y_3E_3)$$
$$= (X_2Y_3 - X_3Y_2)E_2E_3 + (X_3Y_1 - X_1Y_3)E_3E_1 + (X_1Y_2 - X_2Y_1)E_1E_2,$$

and this definition involves the distributive principle,

or
$$X(Y+Z) = XY + XZ.$$

Now the equation $-E_3E_2 = e_1$ may be written $e_1e_2 \,.\, e_1e_3 = e_1 = (e_1e_2e_3)\,e_1$, and we will shew that if x, y, z be any points

$$xy \,.\, xz = (xyz)\,x,$$

or that just as the product of any two points is some multiple of the line joining them, so the product of any two lines is some multiple of their point of intersection.

For if $\quad x = x_1e_1 + x_2e_2 + x_3e_3$, $y = y_1e_1 + y_2e_2 + y_3e_3$, $z = z_1e_1 + z_2e_2 + z_3e_3$,

$$yz = X_1E_1 + X_2E_2 + X_3E_3,\ zx = Y_1E_1 + Y_2E_2 + Y_3E_3,\ xy = Z_1E_1 + Z_2E_2 + Z_3E_3,$$

then X_1, Y_1, &c. are the minors of

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix},$$

and $$zx\ .\ xy=(Y_2Z_3-Y_3Z_2)\,e_1+(Y_3Z_1-Y_1Z_3)\,e_2+(Y_1Z_2-Y_2Z_1)\,e_3$$

$$=\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}(x_1e_1+x_2e_2+x_3e_3)$$

$$=(xyz)\,x.$$

Just as the multiplication of not more than three points is associative, so the multiplication of not more than three lines is associative.

For $$yz\,(zx\ .\ xy)=xyz\ .\ yzx=(xyz)^2,$$

since xyz is a number

$$(yz\ .\ zx)\,xy=(xyz\ .\ z)\,xy=(xyz)^2.$$

The multiplication of points to make lines is called by Grassmann progressive; the multiplication of lines to make points, regressive. These results may be applied to find the equation of any locus generated by linear constructions.

For example, the three sides of a triangle pass through fixed points and two of the angles lie on fixed straight lines; to find the locus of the third angle. If x be the point whose locus is required, a, b, c the fixed points, A, B the fixed straight lines, then xa is the line joining x and a, and therefore one of the sides of the triangle, xaA is the point where xa intersects A and one of the angles, $xaAb$ another side, $xaAbB$ the next angle, $xaAbBc$ the third side, and as this must pass through x,

$$xaAbBcx=0,$$

and this is therefore the equation to the locus, since the product of three points in a straight line must vanish.

This curve is cut by a straight line in two points since if $\lambda x+\mu y$ be a point on the line joining x, y and also a point on the locus

$$(\lambda x+\mu y)\,aAbBc\,(\lambda x+\mu y)=0,$$

or $$\lambda^2\{xaAbBcx\}+\lambda\mu\{xaAbBcy+yaAbBcx\}+\mu^2\{yaAbBcy\}=0.$$

The coefficients of λ^2, $\lambda\mu$, μ^2 are numbers, and this equation gives two values of $\frac{\mu}{\lambda}$ determining the points of intersection. It is clear that in general the number of times x appears in an equation will represent the number of times the curve can be cut by a straight line, and therefore its degree in the usual sense. The equation in y,

$$xaAbBcy+yaAbBcx=0,$$

will represent the tangent to the curve in x, and from the symmetry of this equation in x and y follows the theory of poles and polars*.

* This example and the proof of Pascal's theorem are given in the *Ausdehnungslehre* of 1862, p. 195, or in that of 1864, p. 226. The generation of a cubic is given in Crelle's *Journal.*

The condition that the opposite sides of the hexagon formed by x, a, b, c, d, e should intersect in points lying on a straight line is

$$(xa \cdot cd)(ab \cdot dc)(bc \cdot ex) = 0,$$

and the equation shews that the locus of x is a curve of the second degree.

Moreover the curve passes through the points a, b, c, d, e.

For it is obvious that it passes through a and e since $a^2 = 0$, $e^2 = 0$, and using the sign $\equiv$ to mean, is congruent to, is a multiple of

$$(ba \cdot cd)(ab \cdot de) \equiv ab, \quad (bc \cdot eb) \equiv b,$$

and

$$ab \cdot b = 0,$$

$$ca \cdot cd \equiv c, \quad bc \cdot ec \equiv c,$$

$$c(ab \cdot de)c = 0,$$

$$da \cdot cd \equiv d, \quad (ab \cdot de)(bc \cdot ed) \equiv ed,$$

$$d \cdot ed = 0.$$

Therefore it passes through b, c, d also.

As a curve of the second degree is determined by five points, this is the most general form of its equation and Pascal's theorem is thus proved. Again, from a variable point, lines are drawn through fixed points to meet fixed straight lines and the points of intersection lie on a straight line. Find the locus of the variable point.

If a, b, c be the fixed points, A, B, C the fixed straight lines, its locus is

$$(xa \cdot A)(xb \cdot B)(xc \cdot C) = 0,$$

and is therefore a curve of the third degree.

It obviously passes through the points a, b, c. It also passes through BC, CA, AB; since $BCbB \equiv BC$, $BCcC \equiv BC$, and lastly through the points $bc \cdot A$, $ca \cdot B$, $ab \cdot c$.

For if x be any point on the line bc, $xb \equiv bc$, $xc \equiv bc$,

$$(xb \cdot B)(xc \cdot C) \equiv (bc \cdot B)(bc \cdot C) \equiv bc,$$

and

$$xa \cdot A \cdot bc = 0.$$

xa, A, bc must pass through a point or x must be on the intersection of A and bc. The cubic therefore passes through nine points, but these nine points are not arbitrarily situated. However as a, b, c, A, B, C involve twelve constants a cubic can be generated by this construction in an infinite number of ways.

All these results are included by Grassmann in the following general theorem:

The locus of any point determined by linear constructions leading to the condition that three points should lie on a straight line or three lines pass through a point, is an algebraical curve whose degree can be found by mere counting. The degree is equal to the number of times the variable point is introduced in the construction.

If a, $\lambda a+\mu b$, b, $\lambda' a+\mu' b$ are four points on a straight line their anharmonic ratio was defined to be $\frac{\mu}{\lambda}\frac{\lambda'}{\mu'}$. Similarly the anharmonic ratio of four lines A, $\lambda A+\mu B$, B, $\lambda' A+\mu' B$ is defined to be $\frac{\mu}{\lambda}\frac{\lambda'}{\mu'}$.

If o be any other point the lines joining it to a, $\lambda a+\mu b$, b, $\lambda' a+\mu' b$ are oa, $\lambda oa+\mu ob$, ob, $\lambda' oa+\mu' ob$, and therefore the anharmonic ratio of any range of four points is the same as that of the pencil of lines joining these points to any given point.

If we put $c=\lambda a+\mu b$, $d=\lambda' a+\mu b$, then the anharmonic ratio of the points a, c, b, d is $\frac{ac}{cb}\cdot\frac{db}{ad}$ where it must be remembered that $\frac{ac}{cb}$, $\frac{db}{ad}$ are numbers, and that ac, db are not to be multiplied together. If a, b, c, d be any four points in a plane, we may put $a+b+c+d=0$, since the proper multiples can be included in the symbols for the points.

Let ab, cd intersect in e, ac, bd in f, and ad, bc in g, then since

$$(a+c)(b+d)=-(a+b)^2=0,$$

$$ab+cd=-(ad+bc)=\text{some multiple of line } eg \text{ since it passes through } e \text{ and } g;$$

again

$$(a+d)(b+c)=0,$$

$$ab-cd=-(ac+db)=\text{some multiple of } ef.$$

Hence since the lines ea, ef, eb, eg are multiples of ab, $ab-cd$, cd, $ab+cd$, they form an harmonic pencil.

The anharmonic ratio of the four lines A, B, C, D will be

$$\frac{AB}{BC}\cdot\frac{DC}{AD},$$

but if these be the lines joining x to four points a, b, c, d

$$AB=xa\,.\,xb=(xab)\,x,\quad BC=(xb\,.\,xc)=(xbc)\,x,$$

and the ratio becomes

$$\frac{xab}{xbc}\cdot\frac{xdc}{xad}.$$

If this be constant

$$(xab)(xdc)=k\,(xbc)(xac),$$

a curve of the second degree passing through a, b, c, d.

This proof is really identical with that given in the first section of the former part but Grassmann's notation enables it to be written more concisely.

It must be noticed that though the product ab corresponds to $V\alpha\beta$ in Quaternions, the product abc corresponds to $S\alpha\beta\gamma$.

The theorem corresponding to $ab \cdot ac = (abc)\, a$ is

$$V \cdot V\alpha\beta V\alpha\gamma = -\alpha S\alpha\beta\gamma,$$

but in Quaternions α, β, γ could, if taken to be points, only be points on a sphere. In Grassmann's system if a, b, c were points on an ordinary plane, abc is double the area of the triangle they enclose. The Quaternion expression is $V\beta\gamma + V\gamma\alpha + V\alpha\beta$, so that there is no correspondence except for points on a sphere. The proof of Pascal's theorem, given in Tait's *Quaternions*, applies directly to spherical conics, or the cones joining them to the centre of the sphere, and thus indirectly to plane conics. If ρ, α, β, γ, δ, ϵ, be vectors $\quad SVV\rho\alpha V\gamma\delta,\ VV\alpha\beta V\delta\epsilon,\ VV\beta\gamma V\epsilon\rho = 0$

is the equation the expression of that theorem leads to, and it is identical with

$$(xa \cdot cd)\,(ab \cdot de)\,(bc \cdot ex) = 0.$$

Grassmann's method however proves the theorem, independently of all metrical assumptions.

We will consider now the regressive multiplication generally in an n-point space. So long as the number of points multiplied together is not greater than n and so long as the points are comprised in a space of lower dimensions, the laws of the progressive multiplication will hold. Thus the product of three points in a straight line, or of four points in a plane will vanish if the space be of higher dimensions than a three-point space. It is seen then, that though when we are considering only points in a plane, the product of two lines is their point of intersection, yet when we are considering points in space the product of two intersecting lines vanishes. This can give rise to no confusion any more than the fact that the reciprocal of a point is different in the two cases.

If $e_1e_2 \ldots e_n$ be n points then since no product can have a higher term than $e_1e_2 \ldots e_n$ we may put $e_1e_2 \ldots e_n = 1$.

If follows that the product of any other n points unconnected by a linear relation is some number different from 0.

If E, F, G be different products formed from e_1, e_2, ... e_n, such that EFG contains all the points $e_1e_2 \ldots e_n$ without repetition and is therefore ± 1, we will say that

$$EF \cdot EG = (EFG)\, E.$$

For example $\quad e_1e_2 \cdot e_1e_3 \ldots e_n = (e_1e_2 \ldots e_n)\, e_1 = e_1,$

$$e_1e_2 \ldots e_re_{r+1} \ldots e_s \cdot e_1e_2 \ldots e_re_{s+1} \ldots e_n = (e_1e_2 \ldots e_n)\, e_1e_2 \ldots e_r = e_1e_2 \ldots e_r.$$

This assumption, with the distributive principle, will form the definition of the regressive multiplication.

*We will now prove that if E', F', G' be products of any points such that $E'F'G'$ is a number $\quad E'F' \cdot E'G' = (E'F'G')\, E'.$

For this purpose let $E = e_1e_2 \ldots e_r$, $F = e_{r+1} \ldots e_s$, $G = e_{s+1} \ldots e_n$, and change successively the points e_1, $e_2 \ldots e_n$ into a_1, $a_2 \ldots a_n$.

* This proof is given in nearly the same form in *Ausdehnungslehre* of 1862, p. 68.

Let $a_1 = x_1e_1 + x_2e_2 + \dots x_ne_n$ then

$$E' = a_1e_2 \dots e_r = x_1E + x_{r+1}e_{r+1}e_2 \dots e_r + x_{r+2}e_{r+2} \dots e_r + \dots + x_ne_ne_2 \dots e_r,$$

since
$$e_2e_2 \dots e_r = 0, \quad e_3e_2 \dots e_r = 0, \text{ \&c.}$$

$$E'F = x_1EF + x_{s+1}e_{s+1}e_2 \dots e_s + \dots + x_ne_ne_2 \dots e_s,$$

$$E'G = x_1EG + x_{r+1}e_{r+1}e_2 \dots e_re_{s+1} \dots e_n + \dots + x_se_se_2 \dots e_re_{s+1} \dots e_n.$$

Now
$$EF \,.\, EG = (EFG)\, E$$

$$\begin{aligned} EF \,.\, e_{r+1}e_2 \dots e_re_{s+1} \dots e_n &= e_1e_2 \dots e_re_{r+1} \dots e_s \,.\, e_{r+1}e_2 \dots e_re_{s+1} \dots e_n \\ &= -e_{r+1}e_2 \dots e_re_1e_{r+2} \dots e_s \,.\, e_{r+1}e_2 \dots e_re_{s+1} \dots e_n \\ &= -(e_{r+1}e_2 \dots e_re_1e_{r+2} \dots e_n)\, e_{r+1}e_2 \dots e_r \\ &= (e_1e_2 \dots e_n)\, e_{r+1}e_2 \dots e_r. \end{aligned}$$

Similarly
$$EF \,.\, e_se_2 \dots e_re_{s+1} \dots e_n = (e_1e_2 \dots e_n)\, e_se_2 \dots e_r.$$

Also
$$\begin{aligned} e_{s+1}e_2 \dots e_s \,.\, EG &= e_{s+1}e_2 \dots e_s \,.\, e_1e_2 \dots e_re_{s+1} \dots e_n \\ &= -e_{s+1}e_2 \dots e_re_{r+1} \dots e_s \,.\, e_{s+1}e_2 \dots e_re_1e_{s+2} \dots e_n \\ &= -(e_{s+1}e_2 \dots e_se_1e_{s+2} \dots e_n)\, e_{s+1}e_2 \dots e_r \\ &= (e_1e_2 \dots e_n)\, e_{s+1}e_2 \dots e_r, \end{aligned}$$

and $e_{s+1}e_2 \dots e_s \,.\, e_{r+1}e_2 \dots e_re_{s+1} \dots e_n = 0$, since one of the points e is wanting.

Hence collecting all the terms

$$E'F \,.\, E'G = x_1(e_1e_2 \dots e_n)\{x_1E_1 + x_{r+1}e_{r+1}e_2 \dots e_r + \dots + x_ne_ne_2 \dots e_r\} = (E'FG)\, E',$$

since
$$x_1e_1e_2 \dots e_n = (x_1e_1 + x_2e_2 + \dots)\, a_2 \dots e_n = a_1e_2 \dots e_n = E'FG.$$

Likewise a point in F for instance e_{r+1} may be changed into a_{r+1}.

If
$$a_{r+1} = x_1e_1 + x_2e_2 + \dots x_{r+1}e_{r+1} + \dots$$

then
$$\begin{aligned} EF' &= e_1e_2 \dots e_r(x_1e_1 + \dots x_{r+1}e_{r+1} \dots x_ne_n)\, e_{r+2} \dots e_s \\ &= x_{r+1}EF + x_{s+1}e_1e_2 \dots e_re_{s+1}e_{r+2} \dots e_s + \dots + x_ne_1e_2 \dots e_re_ne_{r+2} \dots e_s, \end{aligned}$$

but
$$e_1e_2 \dots e_re_{s+1}e_{r+2} \dots e_s \,.\, e_1e_2 \dots e_re_{s+1} \dots e_n = 0, \text{ since } e_{r+1} \text{ is wanting.}$$

Therefore
$$EF' \,.\, EG = x_{r+1}EF \,.\, EG = x_{r+1}(EFG)\, E = (EF'G)\, E;$$

and the same reasoning will apply to a point in G.

Thus point by point all the points in E, F, G may be changed, and we shall have generally $E'F' \,.\, E'G' = (E'F'G')\, E'$, or in words, the product of two spaces is the space common to both, provided always the two spaces cannot be included in any space lower than the n-point space.

Let $\quad E_1 = e_2e_3 \dots e_n, \quad E_2 = -e_1e_3 \dots e_n, \quad E_3 = e_1e_2e_4 \dots e_n, \quad E_n = \pm e_1e_2 \dots e_{n-1},$
where the signs of $E_1, E_2 \dots$ are taken so that

$$e_1E_1 = 1, \quad e_2E_2 = 1, \quad e_3E_3 = 1, \text{ \&c.}$$

Then $$E_1E_2 = -e_3e_4 \dots e_ne_2 . e_3e_4 \dots e_ne_1 = (e_3e_4 \dots e_ne_1e_2)\, e_3e_4 \dots e_n$$
$$= (e_1e_2 \dots e_n)\, e_3e_4 \dots e_n = e_3e_4 \dots e_n,$$
$$E_1E_2 . E_3 = (e_3e_4 \dots e_n)\, e_1e_2e_4 \dots e_n = \pm\, e_4 \dots e_ne_3 . e_4 \dots e_ne_1e_2 = \pm\, (e_4 \dots e_ne_3e_1e_2)\, e_4 \dots e_n$$
$$= (e_1e_2 \dots e_n)\, e_4e_5 \dots e_n = e_4e_5 \dots e_n,$$
and $$E_2E_3 = -\, e_1e_3e_4 \dots e_n . e_1e_2e_4 \dots e_n$$
$$= -\, e_1e_4 \dots e_ne_3 . e_1e_4 \dots e_ne_2$$
$$= -\, (e_1e_4 \dots e_ne_3e_2)\, e_1e_4 \dots e_n$$
$$= (e_1e_2e_3e_4 \dots e_n)\, e_1e_4 \dots e_n = e_1e_4 \dots e_n,$$
$$E_1 . E_2E_3 = e_2e_3e_4 \dots e_n . e_1e_4 \dots e_n = \pm\, e_4 \dots e_ne_2e_3 . e_4 \dots e_ne_1$$
$$= \pm\, (e_4 \dots e_ne_2e_3e_1)\, e_4 \dots e_n = (e_1e_2 \dots e_n)\, e_4 \dots e_n$$
$$= e_4 \dots e_n = E_1E_2 . E_3.$$

Thus the multiplication of the quantities E_1, E_2, E_3 is associative. Proceeding in the same way $E_1E_2 \dots E_r = e_{r+1} \dots e_n$.

We can arrange the quantities $e_1e_2 \dots$ so that $e_re_se_t \dots = 1$, where e_re_s are any two of them; then in the same way

$$E_rE_s = e_t \dots = \text{product of all other units so arranged that } e_re_s . E_rE_s = 1$$

And for any three of the quantities $E_rE_sE_t \dots$

$$E_r . E_sE_t = E_rE_s . E_t = \text{product of all the units not containing } e_re_se_t, \text{ so arranged that}$$
$$e_re_se_t . E_rE_sE_t = 1.$$

It follows that any product of three terms is associative, so long as the multiplications involved are either all progressive or all regressive. This was assumed originally for the progressive multiplication. For the regressive let A, B, C be the products of $n-r$, $n-s$, $n-t$ points, then they can be expressed as the product of r, s, t quantities E. None of the quantities E can be equal for they correspond to all the e's which are absent from A, B, C and if the same e were absent from two of these quantities, either AB or ABC would be enclosed in a space of lower than n dimensions and would vanish. Again AB consists of the product of the points which occur in A and B, therefore the degree of AB is $n-r+n-s-n=n-r-s$. Similarly the degree of ABC is $n-r-s-t$, and hence $r+s+t$ is less than n. Therefore there are less than n quantities E and their product is therefore associative. A product in general including both progressive and regressive multiplications will not be associative.

Grassmann writes the quantity E in the form $|\,e$, and calls it the complement (Ergänzung) of e_1. In general if A be any product of the units e_1, $e_2 \dots$, $|\,A$ is the product of the remainining units so arranged that $A\,|\,A = 1$.

If $B = |\,A$, then $AB = 1$ and therefore $BA = \pm 1$, and $|\,B = \pm A$.

BA can only be -1 when B and A both contain an odd number of points, and therefore n is even.

For example in a two-point space, if $e_1e_2 = 1$, $e_2e_1 = -1$, $e_2 = | e$, $e_1 = -| e_2$.

If $A = e_1e_2 \ldots e_r$ then it has been shewn that $| A = E_1E_2 \ldots E_r$, let also

$$B = e_{r+1} \ldots e_s, \qquad | B = E_{r+1} \ldots E_s,$$

then

$$| A | B = E_1E_2 \ldots E_rE_{r+1} \ldots E_s = | e_1e_2 \ldots e_s = | AB.$$

This is for the case when the product AB is progressive and does not include all the factors, but the same result is shewn by Grassmann to be true in the other cases. (*Ausdehnungslehre*, p. 64.)

We may apply this general theory to multiplication in space of three dimensions or four-point space. We will write down for comparison the definitions of progressive multiplication along with the definitions of regressive multiplication.

The product of two points is the line joining them.

The product of two planes is their line of intersection.

The product of three points is the plane containing them.

The product of three planes is their point of intersection.

The product of four points is a number.

The product of four planes is a number.

The product of a line and point is a plane.

The product of a line and plane is a point.

The product of three points on the same straight line is zero.

The product of three planes passing through the same straight line is zero.

The product of four points on the same plane is zero.

The product of four planes passing through the same point is zero.

The product of two intersecting straight lines is zero.

These results may be used as in the case of plane multiplication for proving descriptive theorems.

Thus the equation to the surface generated by a line which meets three given lines A, B, C is $xABCx = 0$.

For xA is the plane containing x and A, xAB the point where it meets B, so that the line joining x to xAB meets both A and B: and $xABCx = 0$ expresses that xAB, C, x lie in the same plane so that the line joining x to xAB meets C. Hence x lies on a line meeting A, B, and C. It might seem at first as if we could write the equation to the surface in the form $xA \,.\, xB \,.\, xC = 0$, as this expresses that the three planes have a line in common, and that therefore the surface must be of the third degree.

But $xA \, . \, xB \, . \, xC$ is always some multiple of the point $x = mx$ say, where m is a number of the second degree in x.

Therefore the equation $xA \, . \, xB \, . \, xC = 0$ reduces to $m = 0$ again an equation of the second degree in x. In fact in every equation employed the left-hand side must, if x were unrestricted, be a number. If it were a point or a plane, equating it to 0, would give four equations and in general only determine special points.

To ensure the left-hand side being a number, we have only to add up the number of points that are altogether multipled together and see that it is divisible by four. Thus in $xABCx$, x and x each give one point, A, B, C two points each; so that the whole number of points is eight and this is divisible by four.

Pascal's theorem may be stated thus: If from a variable point lines be drawn through two fixed points to meet two fixed straight lines and the line joining the points of intersection passes through a fixed point, then the locus of the variable point is a conic.

In this form the corresponding theorem in space is

If from a variable point lines be drawn through two fixed points to meet two fixed planes and the line joining the points of intersection, intersects a fixed straight line then the locus of the variable point is a surface of the second degree.

In fact if a, b be the fixed points, A, B the planes, L the straight line, its equation is

$$(xaA)\,(xbB)\,L = 0.$$

This surface, as is easily seen from the equation, passes through the points a, b, and the points LA, LB and it contains the line AB.

Since passing through two fixed points involves two conditions and containing a given generator involves three more, and this, together with the four constants in L, makes nine, enough to determine the surface; we may say:—If through any two fixed points on a surface of the second degree and through a variable point on the surface lines be drawn to meet any two planes passing through a generator the line joining the points of intersection will always meet a certain fixed line.

The equation $(xaA)\,(xbB)\,(xcC)\,(xdD) = 0$ gives a generation of a quartic surface analogous to that for cubic curves, but it does not involve enough constants to generate any quartic surface.

THE INNER MULTIPLICATION.

If $$a = a_1 e_1 + a_2 e_2 + a_3 e_3 + \ldots + a_n e_n,$$

where $a_1 a_2 \ldots$ are numbers and $e_1 e_2 \ldots e_n$ a system of points such that $e_1 e_2 \ldots e_n = 1$, then

$$| \, a \text{ is defined to be } \alpha_1 \, | \, e_1 + \alpha_2 \, | \, e_2 + \alpha_3 \, | \, e_3 + \ldots + \alpha_n \, | \, e_n.$$

It is clear that $$| \, (b + c) = | \, b + | \, c.$$

Now $$e_1 \mid e_1 = 1, \quad e_2 \mid e_2 = 1, \text{ \&c.}$$
$$e_1 \mid e_2 = e_1 . e_1 e_3 \dots e_n = 0, \quad e_1 \mid e_3 = 0, \dots ;$$
therefore $$a \mid a = \alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2.$$

It will be assumed that when a is a simple point and not a multiple of a point $a \mid a = 1$ just as $e_1 \mid e_1 = 1$, &c.

This assumption limits the spaces treated of to those in which distance is possible.

A system of points $e_1, e_2 \dots e_n$ such that $e_2 \mid e_r = 1$, $e_r \mid e_s = 0$ is called by Grassmann a normal system.

If $f_1, f_2 \cdots f_n$ form another normal system, and if
$$f_1 = \alpha_{11} e_1 + \alpha_{12} e_2 + \dots + \alpha_{1n} e_n,$$
$$f_2 = \alpha_{21} e_1 + \alpha_{22} e_2 + \dots + \alpha_{2n} e_n,$$
&c., &c.

then since $$f_1 \mid f_1 = 1, \quad \alpha_{11}^2 + \alpha_{12}^2 + \dots + \alpha_{1n}^2 = 1,$$
and since $$f_1 \mid f_2 = 0, \quad \alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22} + \dots + \alpha_{1n}\alpha_{2n} = 0,$$
&c., &c.

From these equations
$$\begin{vmatrix} \alpha_{11} & \alpha_{12} \cdots \alpha_{1n} \\ \alpha_{21} & \alpha_{22} \cdots \alpha_{2n} \\ \cdots\cdots & \cdots\cdots \\ \alpha_{n1} & \alpha_{n2} \cdots \alpha_{nn} \end{vmatrix} = \pm 1,$$
and we will take it equal to $+1$.

Then $$f_1 f_2 \cdots f_n = 1.$$

Now with reference to the system $f_1, f_2, \cdots f_n$,
$$\mid f_1 = f_2 f_3 \cdots f_n .$$

But $$f_2 f_3 \cdots f_n = \begin{vmatrix} \alpha_{22} \cdots \alpha_{2n} \\ \alpha_{n1} \cdots \alpha_{nn} \end{vmatrix} e_2 e_3 \cdots e_n + \cdots$$
$$= \alpha_{11} \mid e_1 + \alpha_{12} \mid e_2 + \dots + \alpha_{1n} \mid e_n,$$
and this is the meaning of $\mid f_1$ referred to the system $e_1 e_2 \dots e_n$. Similarly the meanings of $\mid f_2$, &c. are identical.

Now if $$x = x_1 f_1 + x_2 f_2 + \dots + x_n f_n,$$
then $$\mid x = x_1 \mid f_1 + x_2 \mid f_2 + \dots + x_n \mid f_n ;$$
and this is the same whether x be referred to the system $e_1 e_2 \dots e_n$ or to $f_1 f_2 \cdots f_n$. Hence $\mid x$ does not depend on any special set of points, but is the same for any normal system.

As any point can be expressed in terms of $e_1, e_2 \dots e_n$, so any quantity of the rth order can be expressed in terms of the products $e_1e_2 \dots e_r$, $e_2e_3 \dots e_{r+1}r \dots$. If $A_1, A_2 \dots A_m$ be these products and

$$A = \alpha_1 A_1 + \alpha_2 A_2 + \dots + \alpha_m A_m,$$

then $\qquad | A$ is defined to be $\alpha_1 \mid A_1 + \alpha_2 \mid A_2 + \dots + \alpha_m \mid A_m$.

Let $\qquad | B = \beta_1 \mid A_1 + \beta_2 \mid A_2 + \dots + \beta_m \mid A_m$.

Then

$$\begin{aligned} | A \mid B &= (\alpha_1\beta_2 - \alpha_2\beta_1) \mid A_1 \mid A_2 + \dots\dots \\ &= (\alpha_1\beta_2 - \alpha_2\beta_1) \mid A_1A_2 + \dots\dots \\ &= \mid AB, \end{aligned}$$

since it was shewn in the last section that $| A_1 \mid A_2 = \mid A_1A_2$ where A_1, A_2 are products of the original units.

And this is also true when A, B are not of the same order.

It follows that $\qquad | A \mid B \mid C$ &c. $= \mid ABC$, &c.

and in particular $\qquad | f_1f_2 \dots f_r = | f_1 \mid f_2 \dots \mid f_r$;

and therefore $| f_1f_2 \dots f_r$ is independent of the particular normal system to which it is referred since $| f_1, | f_2 \dots | f_r$ are independent. Since any quantity A of the rth order can be expressed in terms of the quantities $f_1f_2 \dots f_r$, $f_2f_3 \dots f_{r+1}$, &c. it follows as before that $| A$ or the complement of A is independent of the special normal system to which it is referred.

Since

$$| (B + C) = \mid B + \mid C,$$

$$A \mid (B + C) = A \mid B + A \mid C,$$

and therefore the quantity $A \mid B$ may be considered to result from a new kind of multiplication between A and B.

If $A_1, A_2, A_3 \dots A_m$ be products of the rth order formed from the original units $e_1e_2 \dots e_n$ then by the definition of $| A_1$, $| A_2$, &c.

$$A_1 \mid A_1 = 1, \quad A_2 \mid A_2 = 1, \text{ \&c.,}$$

and if $\qquad A_1 = e_1e_2 \dots e_r, \quad A_2 = e_2e_3 \dots e_{r+1}$,

so that $\qquad | A_2 = \pm e_1e_{r+2} \dots e_n$,

then $\qquad A_1 \mid A_2 = \pm e_1e_2 \dots e_re_1e_{r+2} \dots e_n = 0$,

since e_1 is repeated twice and e_{r+1} is missing; and similarly with all the other products.

If then A and B be of the same order, and

$$A = \alpha_1A_1 + \alpha_2A_2 + \dots + \alpha_mA_m,$$

$$B = \beta_1A_1 + \beta_2A_2 + \dots + \beta_mA_m,$$

$$A \mid B = \alpha_1\beta_1 + \alpha_2\beta_2 + \dots + \alpha_m\beta_m = B \mid A,$$

so that the inner multiplication of two quantities of the same order is commutative and the product is a number.

Take two quantities $e_1e_2 \ldots e_re_{r+1} \ldots e_s$, $e_1e_2 \ldots e_r$ which are not of the same order, then

$$e_1e_2 \ldots e_re_{r+1} \ldots e_s \mid e_1e_2 \ldots e_r$$
$$= e_1e_2 \ldots e_re_{r+1} \ldots e_s \,.\, e_{r+1} \ldots e_s \ldots e_n$$
$$= \pm\, e_{r+1} \ldots e_se_1e_2 \ldots e_r \,.\, e_{r+1} \ldots e_s \ldots e_n$$
$$= \pm\, (e_{r+1} \ldots e_se_1e_2 \ldots e_2e_{s+1} \ldots e_n)\, e_{r+1} \ldots e_s$$
$$= (e_1e_2 \ldots e_n)\, e_{r+1} \ldots e_s = e_{r+1} \ldots e_s,$$

so that $E \mid F$ when E contains F is equal to the product of the remaining limits so arranged always that

$$F\{E \mid F\} = E.$$

Again,
$$e_1e_2 \ldots e_r \mid e_1e_2 \ldots e_s = e_1e_2 \ldots e_re_{s+1} \ldots e_n$$
$$= \text{product of units except } e_{r+1} \ldots e_s$$
$$= (-1)^{r(s-r)} \mid e_{r+1} \ldots e_s$$
$$= (-1)^{r(s-1)} \mid e_{r+1} \ldots e_s \,.$$

That is to say if E be of higher order than F and s, r be their orders

$$F \mid E = (-1)^{r(s-1)} \mid (E \mid F).$$

From the distributive principle it follows that this result must be true for any quantities A, B of orders s, r.

If both E and F contain quantities which do not occur in the other, then $E \mid F = 0$ for $\mid F$ will contain quantities which occur in E since F does not contain all the quantities in E, hence in $E \mid F$ some points will occur twice over; also $E \mid F$ will not contain all the points, for since E does not contain all the points in F, there will be some points neither in E nor in $\mid F$.

If, in the equation $F\{E \mid F\} = E$, for F we substitute any quantity B of the same order where $B = \beta_1F_1 + \beta_2F_2 + \ldots$ and F_1, F_2 are only made up of points contained in E, then since
$$F_1\{E \mid F_1\} = E, \qquad F_2\{E \mid F_2\} = E,$$
and it is easily seen that
$$F_1\{E \mid F_2\} = 0, \qquad F_2\{E \mid F_1\} = 0,$$
we have
$$B\{E \mid B\} = \{\beta_1F_1 + \beta_2F_2 + \ldots\}\{\beta_1E \mid F + \beta_2E \mid F_2 + \ldots\}$$
$$= (\beta_1^2 + \beta_2^2 + \ldots)\, E = \{B \mid B\}\, E.$$

Also with the notation before used for F and E,

$$F \mid E = e_1e_2 \ldots e_re_{s+1} \ldots e_n,$$
$$E\{F \mid E\} = e_1e_2 \ldots e_r \ldots e_s \,.\, e_1e_2 \ldots e_re_{s+1} \ldots e_n,$$
$$= e_1e_2 \ldots e_r = F,$$

if F be entirely contained in the points of E. But if not, and $F = F' + F''$, then
$$F'' \mid E = 0, \text{ so that } F \mid E = F' \mid E,$$
and
$$E\{F \mid E\} = E\{F' \mid E\} = F'.$$

Hence when E is a product of normal points and F any quantity of lower degree, $E\{F \mid E\}$ expresses the part of F which is contained in E. If E were some multiple of a product instead of the product itself, we should have $E\{F \mid E\} = \{E \mid E\} F'$.

For example, if $x = pe_1 + qe_2 + re_3$, where p, q, r are numbers,

then $$x \mid e_2e_3 = (qe_2 + re_3) \mid e_2e_3 = qe_2e_1 + re_3e_1,$$

and $$e_2e_3\{x \mid e_2e_3\} = qe_2 + re_3.$$

The quantity F' may be called the projection of F on E. Also $x \mid e_2e_3 = xe_1$ passes through the point x and is normal to e_2e_3, since $xe_1 \mid e_2e_3 = xe_1e_1 = 0$. It may therefore be called the perpendiculars from x on e_2e_3; or the inner product of a point into a line represents the perpendicular from the point on the line. Again $e_2e_3 \mid x$ is the point where the complement of x or its polar intersects e_2e_3.

If $a_1a_2 \dots a_n$ be any points and x a point belonging to the system

$$a_1a_2 \dots a_n \mid x = (a_1 \mid x)(a_2a_3 \dots a_n) - (a_2 \mid x)\, a_1a_3 \dots a_n + (a_3 \mid x)(a_1a_2a_4 \dots a_n) - \&c.$$

For let $e_1e_2 \dots e_n$ be a normal system including $a_1a_2 \dots a_n$ and therefore x.

Let $$a_1 = \alpha_{11}e_1 + \alpha_{12}e_2 + \dots \alpha_{1n}e_n,$$

$$\&c. \qquad \&c.$$

$$x = x_1e_1 + x_2e_2 \dots + x_ne_n;$$

and let A_{11}, A_{12} be the minors of $D = \begin{vmatrix} \alpha_{11}, & \alpha_{12}, & \dots & \alpha_{1n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1}, & \alpha_{n2}, & \dots & \alpha_{nn} \end{vmatrix}$.

Then $$a_2a_3 \dots a_n = A_{11} \mid e_1 + A_{12} \mid e_2 + \dots A_{1n} \mid e_n,$$

$$a_1 \mid x = \alpha_{11}x_1 + \alpha_{12}x_2 + \dots$$

Thus the coefficient of $x_1 \mid e_1$ on the right-hand side is $\alpha_{11}A_{11} + \alpha_{21}A_{21} + \dots = D$, and of $x_1 \mid e_2$ is $\alpha_{11}A_{12} + \alpha_{21}A_{22} + \alpha_{31}A_{32} + \dots = 0$.

Hence the right-hand side become $D\{x_1 \mid e_1 + x_2 \mid e_2 + \dots\} = a_1a_2 \dots a_n \mid x$,

since $\quad a_1a_2 \dots a_n = D$, and $\mid x = x_1 \mid e_1 + x_2 \mid e_2 + \dots$

As particular cases $$ab \mid c = (a \mid c)\, b - (b \mid c)\, a,$$

$$abc \mid d = (a \mid d)\, bc + (b \mid d)\, ca + (c \mid d)\, ab,$$

$$abcd \mid e = (a \mid e)\, bcd - (b \mid e)\, cda + (c \mid e)\, dab - (d \mid e)\, abc^*.$$

From this first equation and two similar

$$bc \mid a + ca \mid b + ab \mid c = 0.$$

Hence, as $$a \mid bc = -\{bc \mid a\}, \qquad a \mid bc + b \mid ca + c \mid ab = 0.$$

This equation expresses that the perpendiculars from the angles of a triangle on the opposite sides meet in a point.

* *Ausdehnungslehre*, pp. 131, 134.

Again, $$bcd \mid a - cda \mid b + dab \mid c - abc \mid d = 0,$$

and since $$a \mid bcd = \mid \{bcd \mid a\},$$

$$a \mid bcd - b \mid cda + c \mid dab - d \mid abc = 0.$$

That is to say, the four perpendiculars from the angles of a tetrahedron on the opposite faces are lines such that forces along them can be in equilibrium and they are therefore generators of a surface of the second degree.

The locus of the third of three points forming a normal system, in a three point space, is a conic when the other two points lie on fixed straight lines. For if x be the variable point, A, B the fixed straight lines, then $A \mid x$, $B \mid x$, are the other two points and since these are normal

$$(A \mid x) \mid (B \mid x) = 0,$$

an equation of the second degree*. This is MacCullagh's theorem.

If $a, b, c \ldots\ a', b', c', \ldots$ be two sets of points of equal number,

then $$abc\ldots \mid a'b'c'\ldots = \begin{vmatrix} a \mid a', & a \mid b', & a \mid c'\ldots \\ b \mid a', & b \mid b', & b \mid c'\ldots \\ c \mid a', & c \mid b', & c \mid c'\ldots \end{vmatrix}$$

For we may write

$$a = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n,$$
$$b = \beta_1 e_1 + \beta_2 e_2 + \ldots + \beta_n e_n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a' = \alpha_1' e_1 + \alpha_2' e_2 + \alpha_n' e_n + \ldots + \alpha_{2n}' e_{2n}.$$

Then $$abc\ldots = \begin{vmatrix} \alpha_1, & \alpha_2 & \ldots & \alpha_n \\ \beta_1, & \beta_2 & \ldots & \beta_n \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} e_1 e_2 \ldots e_n,$$

$$a'b'c' = \begin{vmatrix} \alpha_1' & \alpha_2' & \ldots & \alpha_n' \\ \beta_1' & \beta_2' & \ldots & \beta_n' \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} e_1 e_2 \ldots e_n + \ldots$$

Hence $$abc\ldots \mid a'b'c\ldots = \begin{vmatrix} \alpha_1 \alpha_2 & \ldots & \alpha_n \\ \beta_1 \beta_2 & \ldots & \beta_n \\ \ldots & \ldots & \ldots \end{vmatrix} \begin{vmatrix} \alpha_1' \alpha_2' & \ldots & \alpha_n' \\ \beta_1' \beta_2' & \ldots & \beta_n' \\ \ldots & \ldots & \ldots \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_1 \alpha_1' + \alpha_2 \alpha_2' + \ldots, & \alpha_1 \beta_1' + \alpha_2 \beta_2' + \ldots \\ \beta_1 \alpha_1' + \beta_2 \alpha_2' + \ldots, & \beta_1 \beta_1' + \beta_2 \beta_2' + \ldots \\ \ldots & \ldots \end{vmatrix}$$

$$= \begin{vmatrix} a \mid a', & a \mid b'\ldots \\ b \mid a', & b \mid b'\ldots \\ \ldots & \ldots \end{vmatrix}.$$

In particular $$ab \mid a'b' = (a \mid a')(b \mid b') - (a \mid b')(b \mid a'),$$

and therefore $$ab \mid cd + ac \mid db + ad \mid bc = 0.$$

* Tait's *Quaternions*, p. 146.

MEASURE OF DISTANCE.

If a, b be two simple points, we will put $a \mid b = \cos\theta$, and call θ the distance between a and b.

Since
$$ab \mid ab = (a \mid a)(b \mid b) - (a \mid b)^2 = 1 - \cos^2\theta = \sin^2\theta,$$
we may put $ab = \sin\theta e_1 e_2$, where $e_1 e_2$ are two normal points on the same line. The distance between two normal points is always $\frac{\pi}{2}$, since $e_1 \mid e_2 = 0 = \cos\frac{\pi}{2}$.

Again if A, B be two intersecting lines of length $\frac{\pi}{2}$, we will put $A \mid B = \cos\phi$, and call ϕ the angle between A and B.

If o be the point of intersection of A, B; a, b, the points distant $\frac{\pi}{2}$ from o, so that $oa = A$, $ob = B$, then
$$\cos\phi = oa \mid ob = (o \mid o)(a \mid b) - (o \mid a)(o \mid b) = a \mid b;$$
and thus ϕ is equal to the distance between a and b, or the angle between two lines is equal to the distance between the points where they cut the polar of their point of intersection.

Again if a, b be the poles of A and B in a three point space
$$A \mid B = \mid a \mid \mid b = \mid a . b = b \mid a,$$
or the angle between two lines is the distance between their poles. Since
$$AB \mid AB = (A \mid A)(B \mid B) - (A \mid B)(B \mid A) = 1 - \cos^2\phi = \sin^2\phi;$$
it follows that $AB = \sin\phi . o$, where o is their point of intersection. If a, b, c be points of a triangle whose sides are α, β, γ and angles θ, ϕ, χ, then
$$ab \mid ac = \sin\beta \sin\gamma \cos\theta,$$
but
$$ab \mid ac = (b \mid c) - (a \mid b)(a \mid c) = \cos\alpha - \cos\beta\cos\gamma,$$
so that
$$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos\theta^*.$$

Again
$$ab . ac = \sin\beta\sin\gamma\sin\theta a,$$
so that
$$abc = \sin\beta\sin\gamma\sin\theta = \sin\gamma\sin\alpha\sin\phi = \sin\alpha\sin\beta\sin\chi,$$
and hence
$$\frac{\sin\theta}{\sin\alpha} = \frac{\sin\phi}{\sin\beta} = \frac{\sin\chi}{\sin\gamma}.$$

The locus of a point such that the product of the cosines of its distances from two fixed points is constant, is
$$(x \mid a)(x \mid b) = \text{const.} = cx \mid x,$$

* When $\theta = \pi$, $\alpha = \beta + \gamma$, and this justifies the use of the term distance.

and it is therefore a curve of the second degree. It is equivalent too to the most general equation, since the two points and the constant distance make up five constants.

We may also say that in any conic the product of the sines of the perpendiculars on two straight lines is constant, the straight lines being the polars of the former points, and then in the usual way it may be shewn that the portions of any line cut off between the curve and these lines are equal.

The locus of the point such that lines drawn from it to two fixed points are at right angles is given by

$$xa \mid xb = 0, \quad \text{or} \quad (x \mid x)(a \mid b) - (x \mid a)(x \mid b) = 0,$$

and is a particular case of the former curve.

If a, b, be any two points distant $\frac{\pi}{2}$ from c and d in a space containing more than three independent points, then

$$(\alpha a + \beta b) \mid (\gamma c + \delta d) = 0,$$

and therefore every point in the line ab is distant $\frac{\pi}{2}$ from every point in the line cd.

Also $$ca \mid cd = (c \mid c)(a \mid d) - (c \mid a)(c \mid d) = 0,$$

or ca is at right angles to cd.

Again, $$ca \mid cb = a \mid b = da \mid db,$$

or the angle between two lines ca and cb is equal that between two lines da and db. This angle may be defined to be the angle between the planes acd, bcd, since it is the angle between lines drawn in the respective planes at right angles to the line of intersection from any point in that line.

If θ be this angle, and ϕ be equal to the distance cd, then

$$abcd \mid abcd = \begin{vmatrix} 1, & a \mid b, & 0, & c \\ a \mid b, & 1, & 0, & c \\ 0, & c, & 1, & c \mid d \\ 0, & c, & c \mid d, & 1 \end{vmatrix} = \sin^2\theta \sin^2\phi,$$

so that in a four-point space, where $abcd$ is a number,

$$abcd = \pm \sin\theta \sin\phi.$$

But this is the product of the lines ac, bd with its sign changed. Hence the product of two lines of length $\frac{\pi}{2}$ is equal to the product of the sines of the perpendicular distances between them.

If ac, bd, instead of being of lengths $\frac{\pi}{2}$, be of lengths ψ, χ, then

$$acbd = \pm \sin\psi \sin\chi \sin\theta \sin\phi,$$

or the product of two lines is equal to the product of the sines of their lengths multiplied by the sines of their perpendicular distances.

Making again ac, bd of lengths $\frac{\pi}{2}$,

$$ac \mid bd = \begin{vmatrix} a \mid b, & 0 \\ 0, & c \mid d \end{vmatrix} = \cos\theta\cos\phi,$$

and thus the inner product of two lines of length $\frac{\pi}{2}$ is the product of the cosines of their perpendicular distances.

If L, L' be two lines the equation $LL' = 0$ expresses that they meet, and when also $L \mid L' = 0$, they must meet at right angles.

If F be any sum of lines it can always be expressed as the sum of two conjugate lines; that is to say, we can put $F = L + \alpha \mid L$, where α is a number called the pitch of F.

For $$\mid F = \mid L + \alpha L.$$

Hence $$F - \alpha \mid F = (1 - \alpha^2) L;$$

and therefore $$(F - \alpha \mid F)^2 = 0,$$

or $$F^2 (1 + \alpha^2) - 2\alpha F \mid F = 0.$$

This gives a quadratic equation for determining α the roots of which are α, $\frac{1}{\alpha}$. The meaning of this is that if $L + \alpha \mid L$ be one solution, then $\alpha \mid L + \frac{1}{\alpha} \mid (\alpha \mid L)$ is an identical solution.

Either L or $\mid L$ may be called the axis of F.

Suppose we have two screws $e_1e_2 + \alpha e_3e_4$, $e_1e_3 + \beta e_4e_2$, where $e_1e_2e_3e_4$ form a normal system, so that $e_3e_4 = \mid e_1e_2$, $e_4e_2 = \mid e_1e_3$, and wish to find the cylindroid or the locus of the axis of a screw compounded of these.

Put $$L + \gamma \mid L = \lambda (e_1e_2 + \alpha e_3e_4) + \mu (e_1e_3 + \beta e_4e_2),$$

then $$e_1e_2L + \gamma e_1e_2 \mid L = \lambda\alpha,$$

$$e_1e_2 \mid L + \gamma e_1e_2L = \lambda,$$

since $$L \mid e_1e_2 = e_1e_2 \mid L \text{ and } \mid e_1e_2 \mid L = \mid Le_1e_2 = Le_1e_2.$$

Hence $$e_1e_2L - \alpha e_1e_2 \mid L = \gamma (\alpha e_1e_2L - e_1e_2 \mid L);$$

similarly $$e_1e_3L - \beta e_1e_3 \mid L = \gamma (\beta e_1e_3L - e_1e_3 \mid L).$$

Eliminating γ,

$$(e_1e_2L \,.\, e_1e_3L - e_1e_2 \mid L \,.\, e_1e_3 \mid L)(\alpha - \beta) = (\alpha\beta + 1)(e_1e_2L \,.\, e_1e_3 \mid L - e_1e_3 \mid L \,.\, e_1e_2L).$$

Now, since $Le_1e_4 = 0$, $Le_2e_3 = 0$, L meets the lines e_1e_4, e_2e_3, and we may put $L = \sigma xe_1 + \tau xe_4$ where x is any point on L,

$$Le_2e_3 = 0 \text{ gives } -\sigma x \mid e_4 + \tau x \mid e_1 = 0,$$

and the previous equation gives

$$(\tau^2 x \mid e_3 . x \mid e_2 + \sigma^2 x \mid e_2 . x \mid e_3)(\alpha - \beta) = (\alpha\beta + 1)\{(x \mid e_3)^2 + (x \mid e_2)^2\}\sigma\tau.$$

And therefore

$$(\alpha - \beta)(x \mid e_2)(x \mid e_3)\{(x \mid e_1)^2 + (x \mid e_4)^2\} = (\alpha\beta + 1)(x \mid e_1)(x \mid e_4)\{(x \mid e_2)^2 + (x \mid e_3)^2\}.$$

The product of two screws $L + \gamma \mid L$, $M + \delta \mid M$ is

$$LM(1 + \gamma\delta) + (\gamma + \delta) L \mid M.$$

If M meets L at right angles, or if it meets it, and has an opposite pitch, the product vanishes. Any three screws on the cylindroid, F_1, F_2, F_3, are connected by a linear relation, and therefore, if the product of a screw into any two of them vanishes, the product of the same screw into the third will vanish. Now from a point of the cylindroid draw a line M perpendicular to the line passing through that point. It will also meet at right angles the polar line, and it will meet two other lines of the cylindroid in addition.

We may suppose a screw with pitch equal and opposite to one of these last two lines and with axis M. Then $M + \delta \mid M$ is reciprocal to two screws of the cylindroid, and therefore to all.

Hence, if $L + \gamma \mid L$ be the last line it meets since $LM = 0$,

$$(\gamma + \delta) L \mid M = 0 \text{ and } \gamma = -\delta,$$

that is, a line perpendicular to a line of the cylindroid meets it again in two lines corresponding to screws of equal pitch*.

If x be any point, $x + \frac{dx}{dt}dt$ its consecutive position,

then $$x\frac{dx}{dt}dt \text{ is a small portion of its path,}$$

and $$x\frac{dx}{dt} \text{ its velocity;}$$

$$\frac{d}{dt}\left(x\frac{dx}{dt}\right) = x\frac{d^2x}{dt^2} \text{ is the acceleration.}$$

If F be the force acting, m the mass,

$$mx\frac{d^2x}{dt^2} = F,$$

* Ball, *Theory of Screws*.

and for a system of particles, if F be the system of forces acting

$$\Sigma mx \frac{d^2x}{dt^2} = F,$$

and this includes the six equations required to determine the motion.

For a particle $F = xl$, since F must pass through x,

$$\therefore\ mx \frac{d^2x}{dt^2} = xl.$$

Multiply by $|x\frac{dx}{dt}$, then

$$x\frac{d^2x}{dt^2} \mid x\frac{dx}{dt} = (x \mid x)\frac{d^2x}{dt^2}\Big|\frac{dx}{dt} - x\Big|\frac{dx}{dt}\,.\,x\Big|\frac{d^2x}{dt^2} = \frac{dx}{dt}\Big|\frac{d^2x}{dt^2},$$

since $$x\Big|\frac{dx}{dt} = 0, \text{ and } x \mid x = 1,$$

and therefore $$m\frac{dx}{dt}\Big|\frac{d^2x}{dt^2} = l\frac{dx}{dt},$$

$$\tfrac{1}{2}m\frac{dx}{dt}\Big|\frac{dx}{dt} = \int l \mid dx.$$

If l depends only on x, this is the equation of *vis viva*.

For a mass $$\tfrac{1}{2}\Sigma m\frac{dx}{dt}\,.\,\frac{dx}{dt} = \int(l \mid dx_1 + l_2 \mid dx_2 + \ldots).$$

Suppose l_1', l_2' the parts of l_1, l_2 which arise from the mutual actions of x_1 and x_2. Then, if $\alpha x_1 x_2$ be this mutual action where α does not depend on the time,

$$l_1' = \alpha x_2, \qquad l_2' = \alpha x,$$

$$l_1' \; dx_1 + l_2' \; dx_2 = \alpha\,(x_2\,.\,dx_1 + x_1 \mid dx_2) = 0,$$

so that the mutual action disappears.

Considering only the external forces,

$$\tfrac{1}{2}\Sigma m\frac{dx}{dt}\;\frac{dx}{dt} = \Sigma \int l dx,$$

and this is the equation of *vis viva*, which is therefore true for any space allowing the measure of distance.

The motion of a particle under the action of a central force is given by

$$mx\frac{d^2x}{dt^2} = Pxa, \text{ if } P \text{ be a number.}$$

Hence $$max\,\frac{d^2x}{dt^2} = 0,$$

$$max\,\frac{dx}{dt} = \text{const.}$$

If v be velocity, p perpendicular on tangent,

$$mv \sin p = h, \text{ a constant.}$$

Imaginary and Flat Geometry.

Instead of putting $a\,|\,b = \cos\theta$, we may put $a\,|\,b = \cos\frac{\theta}{k}$, where k is any number; and we shall have three distinct cases according as k is real, infinite or imaginary.

If k be imaginary, we may take it equal to the algebraic $\sqrt{-1}$, and put

$$a\,|\,b = \cosh\theta;$$

then $$ab\,|\,ab = 1 - \cosh^2\theta = -\sinh^2\theta.$$

Take a line E of length γ on ab such that $\sinh\gamma = 1$;

then $$ab = \sinh\theta\,.\,E, \text{ and } E\,|\,E = -1.$$

For any two lines E_1, E_2 of length γ we may put

$$E_1\,|\,E_2 = -\cos\phi,$$

where ϕ is the angle between them. Then

$$E_1E_2 = \sin\phi\,.\,a, \text{ if } a \text{ is the point of intersection.}$$

For lines ab, ac of any lengths θ_1, θ_2,

$$ab\,.\,ac = \sinh\theta_1 \sinh\theta_2 \sin\phi a,$$

and therefore $$abc = \sinh\theta_1 \sinh\theta_2 \sin\phi.$$

The points $|\,a$, $|\,b$ are imaginary.

In ordinary geometry, making k infinite, $a\,|\,b = 1$, and therefore this quantity does not give a measure of distance.

But if $$a = \beta b + \gamma c,$$

the equation $$a\,|\,a = \beta a\,|\,b + \gamma a\,|\,c$$

gives $$1 = \beta + \gamma,$$

and hence $$\beta(a - b) = \gamma(c - a).$$

For points on the same line therefore $a - b$, $c - a$ are always in a numerical ratio, and we may take $a - b$ proportional to the distance between a and b.

Since $$ab = a(b-a), \qquad ac = a(c-a),$$

ab, ac are also proportional to these distances.

The quantity $b-a$ is called by Grassmann a stroke (streike), and the quantity ab a line (Linientheil).

A stroke may also be considered a point at an infinite distance, for since $\beta b + \gamma c$ represents a point dividing bc internally, so that the distance ab is to ac in the ratio γ to β; $c-b$ will represent outside bc, whose distance from c is equal to its distance from b, and therefore a point at an infinite distance.

Hence strokes are equal when they lie on equal and parallel lines, and a stroke involves magnitude and direction but not position. Since the inner multiplication for points leads to no results, Grassmann discards it and introduces an independent inner multiplication for strokes. If a, b be strokes of unit length making an angle θ, $a_1 | b_1 = \cos\theta$, so that $a_1 | a_1 = 1$, $b_1 | b_1 = 1$. Therefore $a_1 b_1 | a_1 b_1 = \sin^2\theta$; and, if $a_1 b_1$ be of any length, and U be the product of two unit strokes in the same plane at right angles,

$$a_1 b_1 = \alpha\beta \sin\theta U,$$

so that the outer product of two strokes is always proportional to the magnitude of the parallelogram they involve. This product may be called a parallelogram, and it is the same for all parallel planes, since strokes are unaltered by being moved parallel to themselves.

A parallelogram thus depends only on its magnitude and the direction of the normal to the plane in which it lies.

A line involves position as well as direction, and if cd, ab be equal and parallel lines,

$$cd - ab = c(d-c) - a(b-a) = (c-a)(b-a),$$

(since $d - c = b - a$); = parallelogram of which cd, ab are opposite sides.

The product of three points

$$abc = a(b-a)(c-a) = b(b-a)(c-a) = c(b-a)(c-a)$$

= product of any point in the plane into a parallelogram double the area of the triangle enclosed by the points.

This quantity is called by Grassmann a plane.

The product of three strokes is a parallelepiped with the strokes for edge.

The difference of two planes is a parallelepiped with the planes for opposite faces.

If a_1, b_1, c_1 be three strokes forming the sides of a triangle so that $c_1 = b_1 - a_1$, then

$$c_1 | c_1 = a_1 | a_1 + b_1 | b_1 - 2a_1 | b_1;$$

or, if a, b, c be the sides in magnitude, α, β the angles,

$$c^2 = a^2 + b^2 - 2ab \cos c.$$

Calculations with strokes will be identical with calculations with vectors as far as only the product of two quantities is concerned. The chief differences between Grassmann's system and Quaternions are that Grassmann takes into account the outer multiplication of points, the regressive multiplication, and the position of lines as well as their direction*.

* A great portion of the "Ausdehnungslehre" is also devoted to the algebraical multiplication of different quantities which has not been mentioned here; also expressions analogous to Hamilton's linear and vector functions, and called by Grassmann fractions, are considered.

IV. *Table of the Descending Exponential Function to Twelve or Fourteen Places of Decimals.* By F. W. Newman, Emeritus Professor of University College, London.

[Read *December* 4, 1876.]

My object in constructing these tables was to facilitate the calculation of those Anti-cyclic functions which become prominent in Elliptic Integrals.

Mode of Formation. This must be explained, in order to give confidence in their use. First, a table was constructed of e^{-x} (to 16 decimals) with x an integer, by a method which afforded a systematic verification of the results. Put generally

$$M \pm N = e^{-x \pm h} = e^{-x}\left\{1 \pm \frac{h}{1} + \frac{h^2}{1.2} \pm \frac{h^3}{1.2.3} + \&c.\right\}$$

in which we may make $h=1$, or $h=\cdot 1$, or $h=\cdot 01$, or $h=\cdot 001$ or any convenient small fraction. At starting, take $h=1$. To find e^{-x-1} when we know e^{-x+1} and e^{-x}, calculate the separate terms of

$$e^{-x}\left\{1 - \frac{1}{1} + \frac{1}{1.2} - \frac{1}{1.2.3} + \frac{1}{1.2.3.4} - \&c.\right\},$$

each term being derived by a simple division. Add in separate sums the odd terms $(=M)$ and the even terms $(=N)$. If the work is done correctly, it ought to yield $M+N=e^{-x+1}$ a quantity already known; and if (in working with 18 decimals) the last equation is found to have 16 decimals correct, it is almost certain that our M and N are attained accurately for those 16. Any small error in the two last decimals (for some *must* exist) will be less in $M-N$ than in $M+N$. We may then trust the result $e^{-x-1}=M-N$. Thus from e^{-1} we calculate e^{-2}, by making $x=1$, and the test of accuracy is $M+N=e^0=1$. For the next step we make $x=2$, and seek for e^{-3} from e^{-2}, and our new test is $M+N=e^{-1}$. This being fulfilled, we attain $e^{-4}=M-N$; and so on continually.

After this first skeleton table was finished in 37 entries, from e^{-1} to e^{-37} (since e^{-37} does not yield any digits to 16 decimals), I proceeded to make $h=\cdot 1$, so as to calculate e^{-1}, e^{-2}, e^{-3} ... up to e^{-37}, with the same method of verification. But I found it convenient first to halve the intervals, by making $h=\frac{1}{2}$, then from e^{-x} I deduced simultaneous $e^{-x-\frac{1}{2}}$ and $e^{-x+\frac{1}{2}}$. For instance, having deduced $e^{-2\cdot 5}$ and $e^{-3\cdot 5}$ from e^{-3}, I proceeded to deduce $e^{-3\cdot 5}$ and $e^{-4\cdot 5}$ from e^{-4}, and compared the two values thus obtained for $e^{-3\cdot 5}$. If they agreed to 16 figures, I trusted my work. When disagreement was found, I searched out the error.

After thus halving the intervals, I made $h=\cdot 1$, and had a new check after every five steps, so that any error was sure to be discovered.

Since I worked with 18 decimals, in order to get 16 always accurate, the first idea was to correct the 16th by the two which follow, and drop the two last. But this leaves it uncertain whether the figure presented as 16th is too small or too large; and if further deductions are needed (as in interpolation) we no longer have 16 figures accurate. For this reason I have left the complete 18, which can be dealt with as pleases him who uses the table.

Problem. Given $e^{-2\cdot3}$ and $e^{-2\cdot4}$ to find $e^{-2\cdot5}$, true and verified.

We shall divide $e^{-2\cdot4}$ by 10, 20, 30,... successively. Put M for sum of odd rows, and N for sum of even rows. Then $M+N$ ought to make $e^{-2\cdot3}$ accurately, if we are to trust $M-N=e^{-2\cdot5}$.

The error in $M+N$ is naturally greater than in $M-N$; hence $M+N$ varying from $e^{-2\cdot3}$ (as given) by only 4 in the 18th place, we have just confidence that $M-N$ gives $e^{-2\cdot5}$ correct to the 16th place.

·0907 1795 3289 4124 98 $= e^{-2\cdot4}$ } trustworthy for 16 places.
·1002 5884 3722 8037 29 $= e^{-2\cdot3}$ }

10)	907 1795 3289 4125 00
20)	− 90 7179 5328 9412 50
30)	4 5358 9766 4470 62
40)	− 1511 9658 8815 68
50)	37 7991 4720 39
60)	− 7559 8294 40
70)	125 9971 57
80)	− 1 7999 59
90)	224 99
100)	− 2 49
	2

$M=$ 911 7192 1173 3512 59

$N=$ 90 8692 2549 4524 66

$M+N=$ ·1002 5884 3722 8037 25 $=e^{-2\cdot3}$, which is sufficiently correct.

$M-N=$ 820 8499 8623 8987 93 $=e^{-2\cdot5}$, which was sought.

A third step was, to make $h=\cdot01$, and limit the new table to 12 decimals. This is compact enough, and occupies barely 26 pages of a full sized quarto copy book. In fact $e^{-27\cdot7}$ yields only zero to the first 12 decimals. But this table remains in MS., being superseded by a fourth (also carried to 12 decimals), in which h, the increment of x, is only ·001. The entries in this great table are all made by interpolation into the preceding; and the interpolation is in all cases conducted by the same perfectly accurate formula, the series for e^h. Thus error can scarcely exist without detection, except errors of copying out and errors of printing.

It has been my good fortune to find in Mr J. W. L. Glaisher, a mathematician who has given his time and valuable superintendence to the differencing of this large table; whereby he has exterminated errors of copying or printing, not at all numerous on the whole, yet enough to have hurtfully infused suspicion. I feel much indebted for his zealous co-operation, so critically important.

From not having retained the 13th decimal in my third table, I have had occasional hesitation as to the accuracy of the 12th in my large table founded upon it. Though I believe it could only affect a unit in the last place, it sometimes gave me much trouble of recalculation, until I reached about $x = 4{\cdot}3$, after which I fell back on my second table, and worked from it with 14 decimals, checking myself from the third table after every ten entries, and by my second after every hundred.

Mode of using the tables. When x does not exceed 15·349, and the decimal part contains only tenths, hundredths and thousandths, the value of e^{-x} will be found in its own place, somewhere in the first 77 pages of the table (pp. 151—227).

But on p. 228 a Second Part of the table begins, carried to 14 decimals, and from $x = 15{\cdot}350$ to $x = 17{\cdot}298$ the values of x which end with an odd digit, are omitted. But we get the intermediate values of e^{-x}, true to 12 decimals, by taking an arithmetic mean. Thus to find $e^{-15{\cdot}611}$, we have $e^{-15{\cdot}610} = 1662\ 1229$; $e^{-15{\cdot}612} = 1658\ 8020$ (each to 14 decimals), sum $= 3320\ 9249$, half sum $= 1660\ 4624$. Therefore $e^{-15{\cdot}611}$, true to 12 decimals, is 166046 (*six* zeros must be prefixed). *Proof* of this rule. Given consecutive entries $A = e^{-a+h}$, $B = e^{-a-h}$, to find the intermediate $U = e^{-a}$ which is not in (this part of) the table. Here $h = {\cdot}001$, A and B begin with 6 zeros, therefore each is less than 10^{-6}; so is h^2. Now

$$A = e^{-a} . e^{h} = U\left(1 + \frac{h}{1} + \frac{h^2}{1 . 2}\right), \qquad B = e^{-a} . e^{-h} = U\left(1 - \frac{h}{1} + \frac{h^2}{1 . 2}\right),$$

since Uh^3 is less than 10^{-6-9}, or does not affect the 15th decimal; therefore

$$\tfrac{1}{2}(A + B) = U(1 + \tfrac{1}{2}h^2), \text{ whence } U = \tfrac{1}{2}(A + B)(1 + \tfrac{1}{2}h^2)^{-1} = \tfrac{1}{2}(A + B)(1 - \tfrac{1}{2}h^2 + \tfrac{1}{4}h + \&c.).$$

But h^2 being $< 10^{-6}$, $\tfrac{1}{2}(A+B).\tfrac{1}{2}h^2$ is less than $\tfrac{1}{2}10^{-12}$. Thus, true to 12 decimals, $U = \tfrac{1}{2}(A+B)$, as was asserted. To obtain accuracy to 14 decimals, we must take account of the small factor $\tfrac{1}{2}h^2$. For this, divide the half sum by $2 . 10^6$ and add the quotient to the half sum.

After $x = 17{\cdot}298$, x increases by ·005 at each step, as 17·300, 17·305, 17·310, 17·315... and the new question rises how to find an intermediate x. Its last figure may be, first, either 9 or 1, on the two sides of a zero; next, 4 or 6, on the two sides of a 5, giving to our x the form $-a \pm h$; where e^{-a} is in the table, and $h = {\cdot}001$. Or again the x may end in 2 or 3, else in 7 or 8, so as to have the form $-a \pm 2h$. Thus either by assuming $y = x \pm {\cdot}001$, else by assuming $u = x \pm {\cdot}002$, y or u will be in the series of the table (ending either in 0 or in 5), and e^{-y} or e^{-u} will be known to us by the table. Either then $x = y \mp h$, and $e^{-x} = e^{-y}(1 \mp h)$, else $x = u \mp 2h$, $e^{-x} = e^{-u}(1 \mp 2h)$. Each will be true to 16 decimals.

Hence the RULE. If x is a unit greater than y, subtract from e^{-y} its thousandth part, to find e^{-x}; or add, if x is less than y. But if x be two units greater than y, subtract from e^{-y} its five hundredth part to find e^{-x}, or when x is less than y, add the five hundredth part.

Examples. Given in the table $e^{-y} = 104\ 6740$, $e^{-y-{\cdot}005} = 104\ 1519$, where $y = 18{\cdot}375$. It is required to fill up for the intervals between 0 and 5. Divide by 1000, and we get 1046 and 1041. Subtract the former from e^{-y} and add the latter to $e^{-y-{\cdot}005}$, then $e^{-18{\cdot}376} = 104\ 5694$; $e^{-18{\cdot}379} = 104\ 2560$; else, divide by 500, which yields 2093 and 2083. Subtract the former from e^{-y}, and add the latter to $e^{-y-{\cdot}005}$, then $e^{-18{\cdot}377} = 104\ 4647$, $e^{-18{\cdot}378} = 104\ 3602$. In these four new results, only the last (14th) figure is in error.

Table of e^{-x} to eighteen decimal places (sixteen exact).

x	e^{-x}	x	e^{-x}	x	e^{-x}
·1	9048 3741 80359 59545	5·1	60 9674 65655 15637	10·1	4107 95552 25302
·2	8187 3075 30779 81848	5·2	55 1656 44207 60774	10·2	3717 03186 84128
·3	7408 1822 06817 17871	5·3	49 9159 39069 10218	10·3	3363 30951 85721
·4	6703 2004 60356 39307	5·4	45 1658 09426 12670	10·4	3043 24830 08403
·5	6065 3065 97126 33423	5·5	40 8677 14384 64068	10·5	2753 64493 49746
·6	5488 1163 60940 26441	5·6	36 9786 37164 82931	10·6	2491 60097 31501
·7	4965 8530 37914 09523	5·7	33 4596 54574 71272	10·7	2254 49379 13206
·8	4493 2896 41172 21599	5·8	·30 2755 47453 75813	10·8	2039 95034 11166
·9	4065 6965 97405 99120	5·9	27 3944 48187 68370	10·9	1845 82339 95777
1·0	3678 7944 11714 42321	6·0	24 7875 21766 66358	11·0	1670 17007 90246
1·1	3328 7108 36980 79553	6·1	22 4286 77194 85802	11·1	1511 23238 19857
1·2	3011 9421 19122 02096	6·2	20 2943 06362 95735	11·2	1367 41960 65685
1·3	2725 3179 30340 12603	6·3	18 3630 47770 28910	11·3	1237 29242 61791
1·4	2465 9696 39416 06475	6·4	16 6155 72731 73937	11·4	1119 54848 42595
1·5	2231 3016 01484 29829	6·5	15 0343 91929 77572	11·5	1013 00935 98631
1·6	2018 9651 79946 55407	6·6	13 6036 80375 47893	11·6	916 60877 36245
1·7	1826 8352 40527 34648	6·7	12 3091 19026 73481	11·7	829 38191 60755
1·8	1652 9888 82215 86535	6·8	11 1377 51478 44802	11·8	750 45579 15075
1·9	1495 6861 92226 35054	6·9	10 0778 54290 48510	11·9	679 04048 07381
2·0	1353 3528 32366 12691	7·0	9 1188 19655 54515	12·0	614 42123 53327
2·1	1224 5642 82529 81909	7·1	8 2510 49232 65905	12·1	555 95132 41665
2·2	1108 0315 83623 33881	7·2	7 4658 58083 76681	12·2	503 04556 07114
2·3	1002 5884 37228 03731	7·3	6 7553 87751 93846	12·3	455 17444 63084
2·4	907 1795 32894 12500	7·4	6 1125 27611 29574	12·4	411 85887 07538
2·5	820 8499 86238 98791	7·5	5 5308 43701 47832	12·5	372 66531 72085
2·6	742 7357 82143 33876	7·6	5 0045 14334 40611	12·6	337 20152 34153
2·7	672 0551 27397 49761	7·7	4 5282 71828 86790	12·7	305 11255 58050
2·8	608 1006 26252 17961	7·8	4 0973 49789 79781	12·8	276 07725 72053
2·9	550 2322 00564 07225	7·9	3 7074 35404 59080	12·9	249 80503 25884
3·0	497 8706 83678 63943	8·0	3 3546 26279 02501	13·0	226 03294 06997
3·1	450 4920 23935 57806	8·1	3 0353 91380 78857	13·1	204 52306 24491
3·2	407 6220 39783 66216	8·2	2 7465 35699 72135	13·2	185 06011 97553
3·3	368 8316 74012 40006	8·3	2 4851 68271 07947	13·3	167 44932 09446
3·4	333 7326 99603 26081	8·4	2 2486 73241 78844	13·4	151 51441 12156
3·5	301 0738 34223 18502	8·5	2 0346 83690 10644	13·5	137 09590 86393
3·6	273 2372 24472 92561	8·6	1 8410 57936 67577	13·6	124 04950 79965
3·7	247 2352 64703 39390	8·7	1 6658 58109 87632	13·7	112 24463 65241
3·8	223 7077 18561 65595	8·8	1 5073 30750 95474	13·8	101 56314 71020
3·9	202 4191 14458 04390	8·9	1 3638 89264 82008	13·9	91 89813 57913
4·0	183 1563 88887 34179	9·0	1 2340 98040 86675	14·0	83 15287 19119
4·1	165 7267 54017 61246	9·1	1 1166 58084 90111	14·1	75 23982 99227
4·2	149 9557 68204 77705	9·2	1 0103 94018 37091	14·2	68 07981 34408
4·3	135 6855 90122 00932	9·3	9142 42314 78171	14·3	61 60116 26191
4·4	122 7733 99030 68440	9·4	8272 40655 56631	14·4	55 73903 69323
4·5	111 0899 65382 42306	9·5	7485 18298 87702	14·5	50 43476 62588
4·6	100 5183 57446 33583	9·6	6772 87364 90855	14·6	45 63526 36810
4·7	90 9527 71016 95819	9·7	6128 34950 53224	14·7	41 29249 41607
4·8	82 2974 70490 20030	9·8	5545 15994 32180	14·8	37 36299 38007
4·9	74 4658 30709 24342	9·9	5017 46820 56176	14·9	33 80743 48400
5·0	67 3794 69990 85467	10·0	4539 99297 62485	15·0	30 59023 20519

Table of e^{-x} to eighteen decimal places (sixteen exact).

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
15·1	27 67918 65864	20·1	18650 08918	25·1	125 66332	30·1	84671	35·1	571
15·2	25 04516 37241	20·2	16875 29854	25·2	113 70489	30·2	76612	35·2	517
15·3	22 66180 12790	20·3	15269 40156	25·3	102 88446	30·3	69323	35·3	467
15·4	20 50524 57575	20·4	13816 32570	25·4	93 09369	30·4	62725	35·4	423
15·5	18 55391 36271	20·5	12501 52863	25·5	84 23462	30·5	56757	35·5	383
15·6	16 78827 53003	20·6	11311 85098	25·6	76 21864	30·6	51356	35·6	346
15·7	15 19065 96759	20·7	10235 38612	25·7	68 96548	30·7	46469	35·7	313
15·8	13 74507 72802	20·8	9261 36038	25·8	62 40260	30·8	42047	35·8	283
15·9	12 43706 02371	20·9	8380 02554	25·9	56 46419	30·9	38044	35·9	256
16·0	11 25351 74726	21·0	7582 56070	26·0	51 09089	31·0	34424	36·0	232
16·1	10 18260 36937	21·1	6860 98471	26·1	46 22985	31·1	31149	36·1	210
16·2	9 21360 08336	21·2	6208 07569	26·2	41 82968	31·2	28184	36·2	190
16·3	8 33681 07883	21·3	5617 29937	26·3	37 84905	31·3	25502	36·3	172
16·4	7 54345 83479	21·4	5082 74242	26·4	34 27424	31·4	23075	36·4	156
16·5	6 82560 33757	21·5	4599 05558	26·5	30 98820	31·5	20878	36·5	141
16·6	6 17606 13351	21·6	4161 39757	26·6	28 03927	31·6	18891	36·6	128
16·7	5 58833 13920	21·7	3765 38823	26·7	25 37102	31·7	17094	36·7	116
16·8	5 05653 13478	21·8	3407 06418	26·8	22 95663	31·8	15466	36·8	105
16·9	4 57533 87708	21·9	3082 83916	26·9	20 72200	31·9	13994	36·9	95
17·0	4 13993 77202	22·0	2789 46822	27·0	18 79528	32·0	12662	37·0	86
17·1	3 74597 05575	22·1	2524 01519	27·1	17 00667	32·1	11457		
17·2	3 38949 43271	22·2	2283 82340	27·2	15 38828	32·2	10367		
17·3	3 06694 12954	22·3	2066 48887	27·3	13 92387	32·3	9381		
17·4	2 77508 32429	22·4	1869 83647	27·4	12 59884	32·4	8487		
17·5	2 51099 91571	22·5	1691 89802	27·5	11 39991	32·5	7680		
17·6	2 27204 59942	22·6	1530 89264	27·6	10 31506	32·6	6949		
17·7	2 05583 22310	22·7	1385 20895	27·7	9 33346	32·7	6288		
17·8	1 86019 39278	22·8	1253 38887	27·8	8 44526	32·8	5689		
17·9	1 68317 30706	22·9	1134 11313	27·9	7 64157	32·9	5149		
18·0	1 52299 79752	23·0	1026 18800	28·0	6 91435	33·0	4658		
18·1	1 37806 55555	23·1	928 53333	28·1	6 25638	33·1	4215		
18·2	1 24692 52791	23·2	840 17171	28·2	5 66101	33·2	3812		
18·3	1 12826 46525	23·3	760 21882	28·3	5 12231	33·3	3450		
18·4	1 02089 60750	23·4	687 87436	28·4	4 63485	33·4	3122		
18·5	92374 49702	23·5	622 41450	28·5	4 19376	33·5	2825		
18·6	83583 90136	23·6	563 18394	28·6	3 79466	33·6	2556		
18·7	75629 84148	23·7	509 58993	28·7	3 43356	33·7	2313		
18·8	68432 71049	23·8	461 09586	28·8	3 10681	33·8	2093		
18·9	61920 47706	23·9	417 21690	28·9	2 81116	33·9	1894		
19·0	56027 96459	24·0	377 51347	29·0	2 54364	34·0	1715		
19·1	50696 19869	24·1	341 58831	29·1	2 30158	34·1	1552		
19·2	45871 81754	24·2	309 08189	29·2	2 08255	34·2	1404		
19·3	41506 53683	24·3	279 66885	29·3	1 88442	34·3	1270		
19·4	37556 66761	24·4	253 05484	29·4	1 70511	34·4	1150		
19·5	33982 67815	24·5	228 97350	29·5	1 54280	34·5	1040		
19·6	30748 79877	24·6	207 18380	29·6	1 39598	34·6	941		
19·7	27822 66367	24·7	187 46766	29·7	1 26313	34·7	852		
19·8	25174 98715	24·8	169 62776	29·8	1 14293	34·8	771		
19·9	22779 27037	24·9	153 48556	29·9	1 03418	34·9	698		
20·0	20611 53619	25·0	138 87944	30·0	93576	35·0	631		

TABLE OF e^{-x}.

PART I. From $x=0$ to $x=15{\cdot}349$ at intervals of ·001 to twelve decimal places; pp. 151—227.

PART II. From $x=15{\cdot}350$ to $x=17{\cdot}298$ at intervals of ·002 and from $x=17{\cdot}300$ to $x=27{\cdot}635$ at intervals of ·005, to fourteen decimal places; pp. 228—241.

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
000	1·0000 0000 0000	·050	·9512 2942 4501	·100	·9048 3741 8036	·150	·8607 0797 6425
·001	·9990 0049 9833	·051	02 7867 0533	·101	39 3303 2886	·151	·8598 4769 8659
·002	80 0199 8667	·052	·9493 2886 6843	·102	30 2955 1669	·152	89 8828 0741
·003	70 0449 5504	·053	83 8001 2482	·103	21 2697 3481	·153	81 2972 1811
·004	60 0798 9344	·054	74 3210 6502	·104	12 2529 7421	·154	72 7202 1011
·005	50 1247 9193	·055	64 8514 7953	·105	03 2452 2586	·155	64 1517 7483
·006	40 1796 4054	·056	55 3913 5890	·106	·8994 2464 8076	·156	55 5919 0371
·007	30 2444 2933	·057	45 9406 9366	·107	85 2567 2990	·157	47 0405 8817
·008	20 3191 4837	·058	36 4994 7437	·108	76 2759 6430	·158	38 4978 1968
·009	10 4037 8773	·059	27 0676 9157	·109	67 3041 7498	·159	29 9635 8969
·010	00 4983 3749	·060	17 6453 3584	·110	58 3413 5296	·160	21 4378 8966
·011	·9890 6027 8774	·061	08 2323 9777	·111	49 3874 8928	·161	12 9207 1107
·012	80 7171 2861	·062	·9398 8288 6792	·112	40 4425 7500	·162	04 4120 4540
·013	70 8413 5018	·063	89 4347 3690	·113	31 5066 0115	·163	·8495 9118 8414
·014	60 9754 4261	·064	80 0499 9531	·114	22 5795 5882	·164	87 4202 1880
·015	51 1193 9603	·065	70 6746 3378	·115	13 6614 3906	·165	78 9370 4088
·016	41 2732 0054	·066	61 3086 4292	·116	04 7522 3297	·166	70 4623 4189
·017	31 4368 4635	·067	51 9520 1337	·117	·8895 8519 3163	·167	61 9961 1337
·018	21 6103 2358	·068	42 6047 3578	·118	86 9605 2615	·168	53 5383 4685
·019	11 7936 2244	·069	33 2668 0079	·119	78 0780 0763	·169	45 0890 3386
·020	01 9867 3307	·070	23 9381 9906	·120	69 2043 6717	·170	36 6481 6596
·021	·9792 1896 4570	·071	14 6189 2128	·121	60 3395 9593	·171	28 2157 3471
·022	82 4023 5051	·072	05 3089 5812	·122	51 4836 8503	·172	19 7917 3168
·023	72 6248 3773	·073	·9296 0083 0026	·123	42 6366 2561	·173	11 3761 4844
·024	62 8570 9758	·074	86 7169 3842	·124	33 7984 0883	·174	02 9689 7658
·025	53 0991 2028	·075	77 4348 6329	·125	24 9690 2585	·175	·8394 5702 0769
·026	43 3508 9608	·076	68 1620 6560	·126	16 1484 6784	·176	86 1798 3336
·027	33 6124 1524	·077	58 8985 3607	·127	07 3367 2598	·177	77 7978 4522
·028	23 8836 6801	·078	49 6442 6544	·128	·8798 5337 9145	·178	69 4242 3488
·029	14 1646 4466	·079	40 3992 4446	·129	89 7396 5546	·179	61 0589 9396
·030	04 4553 3548	·080	31 1634 6388	·130	80 9543 0921	·180	52 7021 1411
·031	·9694 7557 3075	·081	21 9369 1446	·131	72 1777 4392	·181	44 3535 8695
·032	85 0658 2079	·082	12 7195 8697	·132	63 4099 5080	·182	36 0134 0415
·033	75 3855 9588	·083	03 5114 7221	·133	54 6509 2108	·183	27 6815 5737
·034	65 7150 4637	·084	·9194 3125 6096	·134	45 9006 4603	·184	19 3580 3827
·035	56 0541 6257	·085	85 1228 4402	·135	37 1591 1688	·185	11 0428 3852
·036	46 4029 3483	·086	75 9423 1221	·136	28 4263 2488	·186	02 7359 4982
·037	36 7613 5349	·087	66 7709 5634	·137	19 7022 6131	·187	·8294 4373 6385
·038	27 1294 0891	·088	57 6087 6724	·138	10 9869 1745	·188	86 1470 7232
·039	17 5070 9146	·089	48 4557 3575	·139	02 2802 8458	·189	77 8650 6694
·040	07 8943 9152	·090	39 3118 5272	·140	·8693 5823 5399	·190	69 5913 3943
·041	·9598 2912 9947	·091	30 1771 0900	·141	84 8931 1699	·191	61 3258 8151
·042	88 6978 0572	·092	21 0514 9546	·142	76 2125 6487	·192	53 0686 8491
·043	79 1139 0067	·093	11 9350 0297	·143	67 5406 8896	·193	44 8197 4139
·044	69 5395 7473	·094	02 8276 2242	·144	58 8774 8061	·194	36 5790 4268
·045	59 9748 1833	·095	·9093 7293 4469	·145	50 2229 3111	·195	28 3465 8056
·046	50 4196 2191	·096	84 6401 6069	·146	41 5770 3185	·196	20 1223 4678
·047	40 8739 7590	·097	75 5600 6134	·147	32 9397 7417	·197	11 9063 3312
·048	31 3378 7077	·098	66 4890 3754	·148	24 3111 4942	·198	03 6985 3137
·049	21 8112 9699	·099	57 4270 8024	·149	15 6911 4899	·199	·8195 4989 3332

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
·200	·8187 3075 3078	·250	·7788 0078 3071	·300	·7408 1822 0682	·350	·7046 8808 9719
·201	79 1243 1553	·251	80 2237 1559	·301	00 7777 2747	·351	39 8375 3856
·202	70 9492 7942	·252	72 4473 8069	·302	·7393 3806 4890	·352	32 8012 1977
·203	62 7824 1425	·253	64 6788 1823	·303	85 9909 6371	·353	25 7719 3378
·204	54 6237 1187	·254	56 9180 2046	·304	78 6086 6451	·354	18 7496 7356
·205	46 4731 6411	·255	49 1649 7961	·305	71 2337 4392	·355	11 7344 3209
·206	38 3307 6282	·256	41 4196 8792	·306	63 8661 9456	·356	04 7262 0236
·207	30 1964 9987	·257	33 6821 3765	·307	56 5060 0907	·357	·6997 7249 7735
·208	22 0703 6711	·258	25 9523 2106	·308	49 1531 8009	·358	90 7307 5007
·209	13 9523 5643	·259	18 2302 3043	·309	41 8077 0026	·359	83 7435 1352
·210	05 8424 5970	·260	10 5158 5803	·310	34 4695 6224	·360	76 7632 6071
·211	·8097 7406 6881	·261	02 8091 9614	·311	27 1387 5869	·361	69 7899 8467
·212	89 6469 7567	·262	·7695 1102 3707	·312	19 8152 8228	·362	62 8236 7842
·213	81 5613 7217	·263	87 4189 7310	·313	12 4991 2569	·363	55 8643 3499
·214	73 4838 5023	·264	79 7353 9656	·314	05 1902 8159	·364	48 9119 4743
·215	65 4144 0177	·265	72 0594 9975	·315	·7297 8887 4269	·365	41 9665 0878
·216	57 3530 1873	·266	64 3912 7501	·316	90 5945 0167	·366	35 0280 1209
·217	49 2996 9305	·267	56 7307 1465	·317	83 3075 5126	·367	28 0964 5044
·218	41 2544 1666	·268	49 0778 1103	·318	76 0278 8415	·368	21 1718 1688
·219	33 2171 8153	·269	41 4325 5648	·319	68 7554 9306	·369	14 2541 0450
·220	25 1879 7962	·270	33 7949 4337	·320	61 4903 7074	·370	07 3433 0637
·221	17 1668 0290	·271	26 1649 6405	·321	54 2325 0990	·371	00 4394 1558
·222	09 1536 4334	·272	18 5426 1090	·322	46 9819 0330	·372	·6893 5424 2524
·223	01 1484 9294	·273	10 9278 7629	·323	39 7385 4368	·373	86 6523 2843
·224	·7993 1513 4369	·274	03 3207 5261	·324	32 5024 2380	·374	79 7691 1828
·225	85 1621 8759	·275	·7595 7212 3225	·325	25 2735 3642	·375	72 8927 8790
·226	77 1810 1665	·276	88 1293 0761	·326	18 0518 7432	·376	66 0233 3041
·227	69 2078 2290	·277	80 5449 7111	·327	10 8374 3027	·377	59 1607 3895
·228	61 2425 9835	·278	72 9682 1514	·328	03 6301 9705	·378	52 3050 0665
·229	53 2853 3505	·279	65 3990 3215	·329	·7196 4301 6747	·379	45 4561 2667
·230	45 3360 2503	·280	57 8374 1456	·330	89 2373 3432	·380	38 6140 9212
·231	37 3946 6035	·281	50 2833 5481	·331	82 0516 9040	·381	31 7788 9620
·232	29 4612 3307	·282	42 7368 4534	·332	74 8732 2854	·382	24 9505 3205
·233	21 5357 3524	·283	35 1978 7861	·333	67 7019 4156	·383	18 1289 9286
·234	13 6181 5895	·284	27 6664 4707	·334	60 5378 2227	·384	11 3142 7179
·235	05 7084 9628	·285	20 1425 4319	·335	53 3808 6352	·385	04 5063 6204
·236	·7897 8067 3932	·286	12 6261 5946	·336	46 2310 5816	·386	·6797 7052 5680
·237	89 9128 8018	·287	05 1172 8837	·337	39 0883 9903	·387	90 9109 4926
·238	82 0269 1094	·288	·7497 6159 2239	·338	31 9528 7898	·388	84 1234 3263
·239	74 1488 2373	·289	90 1220 5402	·339	24 8244 9089	·389	77 3427 0013
·240	66 2786 1066	·290	82 6356 7578	·340	17 7032 2763	·390	70 5687 4498
·241	58 4162 6388	·291	75 1567 8018	·341	10 5890 8206	·391	63 8015 6039
·242	50 5617 7551	·292	67 6853 5974	·342	03 4820 4709	·392	57 0411 3960
·243	42 7151 3771	·293	60 2214 0697	·343	·7096 3821 1560	·393	50 2874 7586
·244	34 8763 4262	·294	52 7649 1443	·344	89 2892 8049	·394	43 5405 6240
·245	27 0453 8241	·295	45 3158 7466	·345	82 2035 3468	·395	36 8003 9249
·246	19 2222 4925	·296	37 8742 8020	·346	75 1248 7107	·396	30 0669 5937
·247	11 4069 3530	·297	30 4401 2362	·347	68 0532 8258	·397	23 3402 5633
·248	03 5994 3277	·298	23 0133 9748	·348	60 9887 6215	·398	16 6202 7662
·249	·7795 7997 3384	·299	15 5940 9435	·349	53 9313 0270	·399	09 9070 1353

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
·400	·6703 2004 6036	·450	·6376 2815 1622	·500	·6065 3065 9713	·550	·5769 4981 0380
·401	·6696 5006 1038	·451	69 9084 2178	·501	59 2443 2218	·551	63 7314 8948
·402	89 8074 5690	·452	63 5416 9725	·502	53 1881 0647	·552	57 9706 3890
·403	83 1209 9323	·453	57 1813 3626	·503	47 1379 4395	·553	52 2155 4629
·404	76 4412 1269	·454	50 8273 3246	·504	41 0938 2856	·554	46 4662 0589
·405	69 7681 0858	·455	44 4796 7948	·505	35 0557 5427	·555	40 7226 1196
·406	63 1016 7425	·456	38 1383 7098	·506	29 0237 1504	·556	34 9847 5876
·407	56 4419 0301	·457	31 8034 0063	·507	22 9977 0483	·557	29 2526 4053
·408	49 7887 8822	·458	25 4747 6207	·508	16 9777 1762	·558	23 5262 5157
·409	43 1423 2322	·459	19 1524 4899	·509	10 9637 4739	·559	17 8055 8612
·410	36 5025 0136	·460	12 8364 5507	·510	04 9557 8812	·560	12 0906 3849
·411	29 8693 1600	·461	06 5267 7398	·511	·5998 9538 3381	·561	06 3814 0294
·412	23 2427 6052	·462	00 2233 9942	·512	92 9578 7845	·562	00 6778 7378
·413	16 6228 2827	·463	·6293 9263 2508	·513	86 9679 1605	·563	·5694 9800 4529
·414	10 0095 1265	·464	87 6355 4467	·514	80 9839 4062	·564	89 2879 1178
·415	03 4028 0705	·465	81 3510 5190	·515	75 0059 4618	·565	83 6014 6757
·416	·6596 8027 0484	·466	75 0728 4048	·516	69 0339 2674	·566	77 9207 0695
·417	90 2091 9944	·467	68 8009 0413	·517	63 0678 7634	·567	72 2456 2426
·418	83 6222 8425	·468	62 5352 3658	·518	57 1077 8900	·568	66 5762 1381
·419	77 0419 5268	·469	56 2758 3157	·519	51 1536 5877	·569	60 9124 6994
·420	70 4681 9815	·470	50 0226 8283	·520	45 2054 7970	·570	55 2543 8699
·421	63 9010 1409	·471	43 7757 8412	·521	39 2632 4583	·571	49 6019 5928
·422	57 3403 9393	·472	37 5351 2918	·522	33 3269 5123	·572	43 9551 8118
·423	50 7863 3112	·473	31 3007 1178	·523	27 3965 8995	·573	38 3140 4704
·424	44 2388 1909	·474	25 0725 2568	·524	21 4721 5607	·574	32 6785 5121
·425	37 6978 5130	·475	18 8505 6465	·525	15 5536 4366	·575	27 0486 8806
·426	31 1634 2120	·476	12 6348 2247	·526	09 6410 4681	·576	21 4244 5196
·427	24 6355 2228	·477	06 4252 9293	·527	03 7343 5960	·577	15 8058 3728
·428	18 1141 4798	·478	00 2219 6982	·528	·5897 8335 7612	·578	10 1928 3841
·429	11 5992 9181	·479	·6194 0248 4693	·529	91 9386 9048	·579	04 5854 4974
·430	05 0909 4723	·480	87 8339 1806	·530	86 0496 9678	·580	·5598 9836 6565
·431	·6498 5891 0774	·481	81 6491 7703	·531	80 1665 8912	·581	93 3874 8054
·432	92 0937 6685	·482	75 4706 1764	·532	74 2893 6164	·582	87 7968 8882
·433	85 6049 1804	·483	69 2982 3373	·533	68 4180 0844	·583	82 2118 8490
·434	79 1225 5485	·484	63 1320 1912	·534	62 5525 2366	·584	76 6324 6319
·435	72 6466 7078	·485	56 9719 6764	·535	56 6929 0144	·585	71 0586 1811
·436	66 1772 5935	·486	50 8180 7313	·536	50 8391 3591	·586	65 4903 4410
·437	59 7143 1410	·487	44 6703 2944	·537	44 9912 2122	·587	59 9276 3557
·438	53 2578 2857	·488	38 5287 3042	·538	39 1491 5152	·588	54 3704 8697
·439	46 8077 9629	·489	32 3932 6993	·539	33 3129 2097	·589	48 8188 9274
·440	40 3642 1083	·490	26 2639 4184	·540	27 4825 2374	·590	43 2728 4734
·441	33 9270 6573	·491	20 1407 4001	·541	21 6579 5399	·591	37 7323 4521
·442	27 4963 5455	·492	14 0236 5832	·542	15 8392 0589	·592	32 1973 8081
·443	21 0720 7087	·493	07 9126 9065	·543	10 0262 7363	·593	26 6679 4858
·444	14 6542 0826	·494	01 8078 3090	·544	04 2191 5140	·594	21 1440 4306
·445	08 2427 6032	·495	·6095 7090 7296	·545	·5798 4178 3339	·595	15 6256 5867
·446	01 8377 2061	·496	89 6164 1073	·546	92 6223 1380	·596	10 1127 8990
·447	·6395 4390 8274	·497	83 5298 3811	·547	86 8325 8683	·597	04 6054 3125
·448	89 0468 4032	·498	77 4493 4902	·548	81 0486 4670	·598	·5499 1035 7721
·449	82 6609 8694	·499	71 3749 3739	·549	75 2704 8761	·599	93 6072 2226

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
·600	·5488 1163 6094	·650	·5220 4577 6761	·700	·4965 8530 3791	·750	·4723 6655 2741
·601	82 6309 8772	·651	15 2399 1920	·701	60 8896 6697	·751	18 9442 2293
·602	77 1510 9714	·652	10 0272 8603	·702	55 9312 5692	·752	14 2276 3739
·603	71 6706 8370	·653	04 8198 6289	·703	50 9778 0281	·753	09 5157 6608
·604	66 2077 4194	·654	·5199 6176 4457	·704	46 0292 9967	·754	04 8086 0429
·605	60 7442 6639	·655	94 4206 2587	·705	41 0857 4256	·755	00 1061 4730
·606	55 2862 5159	·656	89 2288 0159	·706	36 1471 2653	·756	·4695 4083 9043
·607	49 8336 9207	·657	84 0421 6654	·707	31 2134 4666	·757	90 7153 2896
·608	44 3865 8238	·658	78 8607 1553	·708	26 2846 9800	·758	86 0269 5820
·609	38 9449 1709	·659	73 6844 4338	·709	21 3608 7562	·759	81 3432 7348
·610	33 5086 9074	·660	68 5133 4492	·710	16 4419 7461	·760	76 6642 7010
·611	28 0778 9790	·661	63 3474 1497	·711	11 5279 9004	·761	71 9899 4338
·612	22 6525 3314	·662	58 1866 4837	·712	06 6189 1699	·762	67 3202 8865
·613	17 2325 9103	·663	53 0310 3995	·713	01 7147 5057	·763	62 6553 0125
·614	11 8180 6615	·664	47 8805 8457	·714	·4896 8154 8586	·764	57 9949 7650
·615	06 4089 5309	·665	42 7352 7707	·715	91 9211 1796	·765	53 3393 0974
·616	01 0052 4644	·666	37 5951 1230	·716	87 0316 4199	·766	48 6882 9633
·617	·5395 6069 4080	·667	32 4600 8513	·717	82 1470 5305	·767	44 0419 3160
·618	90 2140 3076	·668	27 3301 9042	·718	77 2673 4626	·768	39 4002 1092
·619	84 8265 1094	·669	22 2054 2304	·719	72 3925 1673	·769	34 7631 2963
·620	79 4443 7595	·670	17 0857 7787	·720	67 5225 5960	·770	30 1306 8311
·621	74 0676 2040	·671	11 9712 4978	·721	62 6574 6999	·771	25 5028 6672
·622	68 6962 3892	·672	06 8618 3367	·722	57 7972 4304	·772	20 8796 7584
·623	63 3302 2613	·673	01 7575 2442	·723	52 9418 7388	·773	16 2611 0583
·624	57 9695 7668	·674	·5096 6583 1692	·724	48 0913 5767	·774	11 6471 5208
·625	52 6142 8519	·675	91 5642 0608	·725	43 2456 8955	·775	07 0378 0999
·626	47 2643 4632	·676	86 4751 8681	·726	38 4048 6468	·776	02 4330 7493
·627	41 9197 5471	·677	81 3912 5401	·727	33 5688 7821	·777	·4597 8329 4230
·628	36 5805 0503	·678	76 3124 0261	·728	28 7377 2531	·778	93 2374 0751
·629	31 2465 9193	·679	71 2386 2752	·729	23 9114 0115	·779	88 6464 6595
·630	25 9180 1007	·680	66 1699 2366	·730	19 0899 0090	·780	84 0601 1305
·631	20 5947 5413	·681	61 1062 8598	·731	14 2732 1974	·781	79 4783 4420
·632	15 2768 1879	·682	56 0477 0940	·732	09 4613 5285	·782	74 9011 5483
·633	09 9641 9872	·683	50 9941 8887	·733	04 6542 9543	·783	70 3285 4037
·634	04 6568 8862	·684	45 9457 1934	·734	·4799 8520 4267	·784	65 7604 9623
·635	·5299 3548 8318	·685	40 9022 9575	·735	95 0545 8975	·785	61 1970 1785
·636	94 0581 7709	·686	35 8639 1307	·736	90 2619 3189	·786	56 6381 0067
·637	88 7667 6506	·687	30 8305 6625	·737	85 4740 6429	·787	52 0837 4013
·638	83 4806 4179	·688	25 8022 5026	·738	80 6909 8216	·788	47 5339 3168
·639	78 1998 0201	·689	20 7789 6007	·739	75 9126 8073	·789	42 9886 7075
·640	72 9242 4043	·690	15 7606 9066	·740	71 1391 5521	·790	38 4479 5282
·641	67 6539 5178	·691	10 7474 3701	·741	66 3704 0083	·791	33 9117 7334
·642	62 3889 3077	·692	05 7391 9411	·742	61 6064 1282	·792	29 3801 2776
·643	57 1291 7216	·693	00 7359 5696	·743	56 8471 8642	·793	24 8530 1157
·644	51 8746 7068	·694	·4995 7377 2053	·744	52 0927 1686	·794	20 3304 2023
·645	46 6254 2107	·695	90 7444 7985	·745	47 3429 9940	·795	15 8123 4922
·646	41 3814 1809	·696	85 7562 2991	·746	42 5980 2928	·796	11 2987 9403
·647	36 1426 5649	·697	80 7729 6573	·747	37 8578 0176	·797	06 7897 5013
·648	30 9091 3103	·698	75 7946 8232	·748	33 1223 1209	·798	02 2852 1302
·649	25 6808 3648	·699	70 8213 7471	·749	28 3915 5556	·799	·4497 7851 7820

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
·800	·4493 2896 4117	·850	·4274 1493 1949	·900	·4065 6965 9740	·950	·3867 4102 3454
·801	88 7985 9742	·851	69 8773 0653	·901	61 6329 3298	·951	63 5447 5737
·802	84 3120 4248	·852	65 6095 6345	·902	57 5733 3018	·952	59 6831 4374
·803	79 8299 7184	·853	61 3460 8598	·903	53 5177 8496	·953	55 8253 8979
·804	75 3523 8104	·854	57 0868 6986	·904	49 4662 9326	·954	51 9714 9167
·805	70 8792 6559	·855	52 8319 1082	·905	45 4188 5103	·955	48 1214 4552
·806	66 4106 2102	·856	48 5812 0462	·906	41 3754 5421	·956	44 2752 4749
·807	61 9464 4286	·857	44 3347 4700	·907	37 3360 9877	·957	40 4328 9374
·808	57 4867 2665	·858	40 0925 3371	·908	33 3007 8067	·958	36 5943 8043
·809	53 0314 6792	·859	35 8545 6052	·909	29 2694 9587	·959	32 7597 0371
·810	48 5806 6223	·860	31 6208 2318	·910	25 2422 4034	·960	28 9288 5975
·811	44 1343 0512	·861	27 3913 1746	·911	21 2190 1005	·961	25 1018 4472
·812	39 6923 9214	·862	23 1660 3914	·912	17 1998 0098	·962	21 2786 5479
·813	35 2549 1885	·863	18 9449 8398	·913	13 1846 0911	·963	17 4592 8613
·814	30 8218 8082	·864	14 7281 4776	·914	09 1734 3042	·964	13 6437 3494
·815	26 3932 7361	·865	10 5155 2628	·915	05 1662 6091	·965	09 8319 9739
·816	21 9690 9280	·866	06 3071 1530	·916	01 1630 9656	·966	06 0240 6967
·817	17 5493 3395	·867	02 1029 1064	·917	·3997 1639 3338	·967	02 2199 4798
·818	13 1339 9266	·868	·4197 9029 0808	·918	93 1687 6736	·968	·3798 4196 2851
·819	08 7230 6450	·869	93 7071 0343	·919	89 1775 9451	·969	94 6231 0746
·820	04 3165 4506	·870	89 5154 9248	·920	85 1904 1084	·970	90 8303 8103
·821	·4399 9144 2994	·871	85 3280 7105	·921	81 2072 1236	·971	87 0414 4543
·822	95 5167 1473	·872	81 1448 3494	·922	77 2279 9509	·972	83 2562 9688
·823	91 1233 9505	·873	76 9657 7998	·923	73 2527 5505	·973	79 4749 3158
·824	86 7344 6648	·874	72 7909 0199	·924	69 2814 8826	·974	75 6973 4575
·825	82 3499 2465	·875	68 6201 9679	·925	65 3141 9075	·975	71 9235 3563
·826	77 9697 6517	·876	64 4536 6021	·926	61 3508 5856	·976	68 1534 9743
·827	73 5939 8366	·877	60 2912 8808	·927	57 3914 8771	·977	64 3872 2738
·828	69 2225 7574	·878	56 1370 7624	·928	53 4360 7426	·978	60 6247 2172
·829	64 8555 3705	·879	51 9790 2054	·929	49 4846 1425	·979	56 8659 7668
·830	60 4928 6321	·880	47 8291 1682	·930	45 5371 0372	·980	53 1109 8851
·831	56 1345 4987	·881	43 6833 6093	·931	41 5935 3873	·981	49 3597 5345
·832	51 7805 9266	·882	39 5417 4872	·932	37 6539 1533	·982	45 6122 6775
·833	47 4309 8723	·883	35 4042 7605	·933	33 7182 2958	·983	41 8685 2767
·834	43 0857 2924	·884	31 2709 3878	·934	29 7864 7756	·984	38 1285 2945
·835	38 7448 1433	·885	27 1417 3279	·935	25 8586 5532	·985	34 3922 6936
·836	34 4082 3816	·886	23 0166 5394	·936	21 9347 5894	·986	30 6597 4367
·837	30 0759 9641	·887	18 8956 9811	·937	18 0147 8449	·987	26 9309 4863
·838	25 7480 8472	·888	14 7788 6117	·938	14 0987 2806	·988	23 2058 8053
·839	21 4244 9879	·889	10 6661 3901	·939	10 1865 8573	·989	19 4845 3563
·840	17 1052 3429	·890	06 5575 2752	·940	06 2783 5358	·990	15 7669 1022
·841	12 7902 8689	·891	02 4530 2259	·941	02 3740 2772	·991	12 0530 0057
·842	08 4796 5228	·892	·4098 3526 2011	·942	·3898 4736 0423	·992	08 3428 0298
·843	04 1733 2615	·893	94 2563 1598	·943	94 5770 7921	·993	04 6363 1373
·844	·4299 8713 0419	·894	90 1641 0611	·944	90 6844 4877	·994	00 9335 2912
·845	95 5735 8211	·895	86 0759 8640	·945	86 7957 0902	·995	·3697 2344 4544
·846	91 2801 5560	·896	81 9919 5277	·946	82 9108 5606	·996	93 5390 5899
·847	86 9910 2036	·897	77 9120 0114	·947	79 0298 8601	·997	89 8473 6609
·848	82 7061 7212	·898	73 8361 2741	·948	75 1527 9499	·998	86 1593 6303
·849	78 4256 0659	·899	69 7643 2752	·949	71 2795 7913	·999	82 4750 4613

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
1·000	·3678 7944 1171	1·050	·3499 3774 9111	1·100	·3328 7108 3698	1·150	·3166 3676 9379
1·001	5 1174 5608	1·051	5 8798 6272	1·101	5 3837 8995	1·151	3 2029 0875
1·002	1 4441 7557	1·052	2 3857 3022	1·102	2 0600 6830	1·152	0 0412 8691
1·003	·3667 7745 6651	1·053	·3488 8950 9010	1·103	·3318 7396 6870	1·153	·3156 8828 2512
1·004	4 1086 2522	1·054	5 4079 3887	1·104	5 4225 8785	1·154	3 7275 2021
1·005	0 4463 4804	1·055	1 9242 7306	1·105	2 1088 2242	1·155	0 5753 6903
1·006	·3656 7877 3130	1·056	·3478 4440 8917	1·106	·3308 7983 6910	1·156	·3147 4263 6842
1·007	3 1327 7136	1·057	4 9673 8372	1·107	5 4912 2458	1·157	4 2805 1525
1·008	·3649 4814 6454	1·058	1 4941 5324	1·108	2 1873 8555	1·158	1 1378 0635
1·009	5 8338 0721	1·059	·3468 0243 9426	1·109	·3298 8868 4871	1·159	·3137 9982 3859
1·010	2 1897 9571	1·060	4 5581 0330	1·110	5 5896 1075	1·160	4 8618 0883
1·011	·3638 5494 2640	1·061	1 0952 7690	1·111	2 2956 6839	1·161	1 7285 1393
1·012	4 9126 9565	1·062	·3457 6359 1159	1·112	·3289 0050 1832	1·162	·3128 5983 5075
1·013	1 2795 9980	1·063	4 1800 0392	1·113	5 7176 5725	1·163	5 4713 1618
1·014	·3627 6501 3524	1·064	0 7275 5043	1·114	2 4335 8191	1·164	2 3474 0708
1·015	4 0242 9832	1·065	·3447 2785 4767	1·115	·3279 1527 8900	1·165	·3119 2266 2033
1·016	0 4020 8543	1·066	3 8329 9219	1·116	5 8752 7524	1·166	6 1089 5280
1·017	·3616 7834 9295	1·067	0 3908 8054	1·117	2 6010 3735	1·167	2 9944 0138
1·018	3 1685 1724	1·068	·3436 9522 0928	1·118	·3269 3300 7207	1·168	·3109 8829 6296
1·019	·3609 5571 5471	1·069	3 5169 7497	1·119	6 0623 7612	1·169	6 7746 3442
1·020	5 9494 0173	1·070	0 0851 7419	1·120	2 7979 4623	1·170	3 6694 1266
1·021	2 3452 5470	1·071	·3426 6568 0348	1·121	·3259 5367 7914	1·171	0 5672 9456
1·022	·3598 7447 1002	1·072	3 2318 5944	1·122	6 2788 7158	1·172	·3097 4682 7703
1·023	5 1477 6408	1·073	·3419 8103 3862	1·123	3 0242 2031	1·173	4 3723 5697
1·024	1 5544 1329	1·074	6 3922 3762	1·124	·3249 7728 2206	1·174	1 2795 3129
1·025	·3587 9646 5406	1·075	·3412 9775 5301	1·125	6 5246 7358	1·175	·3088 1897 9688
1·026	4 3784 8279	1·076	·3409 5662 8137	1·126	3 2797 7163	1·176	5 1031 5066
1·027	0 7958 9590	1·077	6 1584 1931	1·127	0 0381 1296	1·177	2 0195 8955
1·028	·3577 2168 8980	1·078	2 7539 6340	1·128	·3236 7996 9433	1·178	·3078 9391 1046
1·029	3 6414 6093	1·079	·3399 3529 1025	1·129	3 5645 1249	1·179	5 8617 1030
1·030	0 0696 0569	1·080	5 9552 5645	1·130	0 3325 6422	1·180	2 7873 8601
1·031	·3566 5013 2052	1·081	2 5609 9860	1·131	·3227 1038 4628	1·181	·3069 7161 3451
1·032	2 9366 0186	1·082	·3389 1701 3332	1·132	3 8783 5545	1·182	6 6479 5272
1·033	·3559 3754 4613	1·083	5 7826 5721	1·133	0 6560 8850	1·183	3 5828 3758
1·034	5 8178 4978	1·084	2 3985 6688	1·134	·3217 4370 4220	1·184	0 5207 8602
1·035	2 2638 0925	1·085	·3379 0178 5895	1·135	4 2212 1334	1·185	·3057 4617 9499
1·036	·3548 7133 2098	1·086	5 6405 3004	1·136	1 0085 9870	1·186	4 4058 6141
1·037	5 1663 8142	1·087	2 2665 7676	1·137	·3207 7991 9507	1·187	1 3529 8225
1·038	1 6229 8704	1·088	·3368 8959 9576	1·138	4 5929 9924	1·188	·3048 3031 5443
1·039	·3538 0831 3427	1·089	5 5287 8365	1·139	1 3900 0801	1·189	5 2563 7492
1·040	4 5468 1959	1·090	2 1649 3707	1·140	·3198 1902 1816	1·190	2 2126 4067
1·041	1 0140 3945	1·091	·3358 8044 5266	1·141	4 9936 2650	1·191	·3039 1719 4863
1·042	·3527 4847 9033	1·092	5 4473 2705	1·142	1 8002 2984	1·192	6 1342 9576
1·043	3 9590 6870	1·093	2 0935 5688	1·143	·3188 6100 2498	1·193	3 0996 7902
1·044	0 4368 7102	1·094	·3348 7431 3881	1·144	5 4230 0873	1·194	0 0680 9539
1·045	·3516 9181 9378	1·095	5 3960 6949	1·145	2 2391 7790	1·195	·3027 0395 4182
1·046	3 4030 3346	1·096	2 0523 4556	1·146	·3179 0585 2931	1·196	4 0140 1530
1·047	·3509 8913 8654	1·097	·3338 7119 6368	1·147	5 8810 5978	1·197	0 9915 1278
1·048	6 3832 4952	1·098	5 3749 2052	1·148	2 7067 6613	1·198	·3017 9720 3126
1·049	2 8786 1887	1·099	2 0412 1273	1·149	·3169 5356 4519	1·199	4 9555 6771

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
1·200	·3011 9421 1912	1·250	·2865 0479 6860	1·300	·2725 3179 3034	1·350	·2592 4026 0646
1·201	·3008 9316 8247	1·251	2 1843 5268	1·301	2 5939 7462	1·351	·2589 8114 9962
1·202	5 9242 5475	1·252	·2859 3235 9894	1·302	·2719 8727 4149	1·352	7 2229 8260
1·203	2 9198 3296	1·253	6 4657 0453	1·303	7 1542 2823	1·353	4 6370 5279
1·204	·2999 9184 1409	1·254	3 6106 6658	1·304	4 4384 3212	1·354	2 0537 0763
1·205	6 9199 9513	1·255	0 7584 8224	1·305	1 7253 5046	1·355	·2579 4729 4452
1·206	3 9245 7310	1·256	·2847 9091 4867	1·306	·2709 0149 8052	1·356	6 8947 6088
1·207	0 9321 4499	1·257	5 0626 6300	1·307	6 3073 1959	1·357	4 3191 5414
1·208	·2987 9427 0781	1·258	2 2190 2239	1·308	3 6023 6498	1·358	1 7461 2171
1·209	4 9562 5858	1·259	·2839 3782 2401	1·309	0 9001 1396	1·359	·2569 1756 6104
1·210	1 9727 9430	1·260	6 5402 6500	1·310	·2698 2005 6385	1·360	6 6077 6953
1·211	·2978 9923 1199	1·261	3 7051 4253	1·311	5 5037 1194	1·361	4 0424 4464
1·212	6 0148 0868	1·262	0 8728 5377	1·312	2 8095 5553	1·362	1 4796 8379
1·213	3 0402 8138	1·263	·2828 0433 9588	1·313	0 1180 9193	1·363	·2558 9194 8442
1·214	0 0687 2713	1·264	5 2167 6603	1·314	·2687 4293 1845	1·364	6 3618 4397
1·215	·2967 1001 4294	1·265	2 3929 6140	1·315	4 7432 3239	1·365	3 8067 5988
1·216	4 1345 2585	1·266	·2819 5719 7917	1·316	2 0598 3109	1·366	1 2542 2960
1·217	1 1718 7290	1·267	6 7538 1651	1·317	·2679 3791 1184	1·367	·2548 7042 5057
1·218	·2958 2121 8112	1·268	3 9384 7060	1·318	6 7010 7197	1·368	6 1568 2025
1·219	5 2554 4755	1·269	1 1259 3863	1·319	4 0257 0880	1·369	3 6119 3608
1·220	2 3016 6924	1·270	·2808 3162 1778	1·320	1 3530 1966	1·370	1 0695 9553
1·221	·2949 3508 4323	1·271	5 5093 0525	1·321	·2668 6830 0187	1·371	·2538 5297 9604
1·222	6 4029 6657	1·272	2 7051 9823	1·322	6 0156 5277	1·372	5 9925 3509
1·223	3 4580 3631	1·273	·2799 9038 9392	1·323	3 3509 6968	1·373	3 4578 1013
1·224	0 5160 4951	1·274	7 1053 8951	1·324	0 6889 4994	1·374	0 9256 1862
1·225	·2937 5770 0323	1·275	4 3096 8221	1·325	·2658 0295 9089	1·375	·2528 3959 5805
1·226	4 6408 9453	1·276	1 5167 6922	1·326	5 3728 8987	1·376	5 8688 2587
1·227	1 7077 2047	1·277	·2788 7266 4774	1·327	2 7188 4422	1·377	3 3442 1955
1·228	·2928 7774 7811	1·278	5 9393 1500	1·328	0 0674 5130	1·378	0 8221 3659
1·229	5 8501 6453	1·279	3 1547 6819	1·329	·2647 4187 0844	1·379	·2518 3025 7444
1·230	2 9257 7681	1·280	0 3730 0453	1·330	4 7726 1300	1·380	5 7855 3060
1·231	0 0043 1201	1·281	·2777 5940 2125	1·331	2 1291 6233	1·381	3 2710 0254
1·232	·2917 0857 6721	1·282	4 8178 1557	1·332	·2639 4883 5379	1·382	0 7589 8776
1·233	4 1701 3950	1·283	2 0443 8470	1·333	6 8501 8474	1·383	·2508 2494 8373
1·234	1 2574 2596	1·284	·2769 2737 2587	1·334	4 2146 5255	1·384	5 7424 8795
1·235	·2908 3476 2368	1·285	6 5058 3632	1·335	1 5817 5456	1·385	3 2379 9792
1·236	5 4407 2975	1·286	3 7407 1328	1·336	·2628 9514 8816	1·386	0 7360 1112
1·237	2 5367 4125	1·287	0 9783 5398	1·337	6 3238 5071	1·387	·2498 2365 2506
1·238	·2899 6356 5529	1·288	·2758 2187 5565	1·338	3 6988 3958	1·388	5 7395 3724
1·239	6 7374 6897	1·289	5 4619 1555	1·339	1 0764 5215	1·389	3 2450 4516
1·240	3 8421 7939	1·290	2 7078 3090	1·340	·2618 4566 8580	1·390	0 7530 4632
1·241	0 9497 8365	1·291	·2749 9564 9896	1·341	5 8395 3791	1·391	·2488 2635 3823
1·242	·2888 0602 7886	1·292	7 2079 1698	1·342	3 2250 0586	1·392	5 7765 1841
1·243	5 1736 6213	1·293	4 4620 8221	1·343	0 6130 8703	1·393	3 2919 8437
1·244	2 2899 3057	1·294	1 7189 9190	1·344	·2608 0037 7881	1·394	0 8099 3362
1·245	·2879 4090 8131	1·295	·2738 9786 4331	1·345	5 3970 7860	1·395	·2478 3303 6368
1·246	6 5311 1145	1·296	6 2410 3370	1·346	2 7929 8379	1·396	5 8532 7207
1·247	3 6560 1813	1·297	3 5061 6033	1·347	0 1914 9176	1·397	3 3786 5631
1·248	0 7837 9846	1·298	0 7740 2047	1·348	·2597 5925 9993	1·398	0 9065 1393
1·249	·2867 9144 4957	1·299	·2728 0446 1138	1·349	4 9963 0570	1·399	·2468 4368 4246

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
1·400	·2465 9696 3942	1·450	·2345 7028 8094	1·500	·2231 3016 0148	1·550	·2122 4797 3827
1·401	3 5049 0235	1·451	3 3583 5052	1·501	·2229 0714 1516	1·551	0 3583 1942
1·402	1 0426 2879	1·452	1 0161 6346	1·502	6 8434 5791	1·552	·2118 2390 2093
1·403	·2458 5828 1628	1·453	·2338 6763 1741	1·503	4 6177 2750	1·553	6 1218 4067
1·404	6 1254 6234	1·454	6 3388 1004	1·504	2 3942 2171	1·554	4 0067 7654
1·405	3 6705 6453	1·455	4 0036 3901	1·505	0 1729 3832	1·555	1 8938 2641
1·406	1 2181 2039	1·456	1 6708 0199	1·506	·2217 9538 7510	1·556	·2109 7829 8818
1·407	·2448 7681 2747	1·457	·2329 3402 9663	1·507	5 7370 2983	1·557	7 6742 5973
1·408	6 3205 8332	1·458	7 0121 2062	1·508	3 5224 0030	1·558	5 5676 3896
1·409	3 8754 8549	1·459	4 6862 7161	1·509	1 3099 8429	1·559	3 4631 2375
1·410	1 4328 3153	1·460	2 3627 4730	1·510	·2209 0997 7959	1·560	1 3607 1201
1·411	·2438 9926 1901	1·461	0 0415 4535	1·511	6 8917 8399	1·561	·2099 2604 0163
1·412	6 5548 4548	1·462	·2317 7226 6343	1·512	4 6859 9529	1·562	7 1621 9051
1·413	4 1195 0850	1·463	5 4060 9925	1·513	2 4824 1127	1·563	5 0660 7655
1·414	1 6866 0565	1·464	3 0918 5046	1·514	0 2810 2973	1·564	2 9720 5765
1·415	·2429 2561 3448	1·465	0 7799 1477	1·515	·2198 0818 4847	1·565	0 8801 3173
1·416	6 8280 9257	1·466	·2308 4702 8986	1·516	5 8848 6530	1·566	·2088 7902 9669
1·417	4 4024 7749	1·467	6 1629 7342	1·517	3 6900 7801	1·567	6 7025 5044
1·418	1 9792 8681	1·468	3 8579 6315	1·518	1 4974 8441	1·568	4 6168 9090
1·419	·2419 5585 1811	1·469	1 5552 5673	1·519	·2189 3070 8231	1·569	2 5333 1597
1·420	7 1401 6897	1·470	·2299 2548 5187	1·520	7 1188 6952	1·570	0 4518 2357
1·421	4 7242 3697	1·471	6 9567 4626	1·521	4 9328 4384	1·571	·2078 3724 1163
1·422	2 3107 1969	1·472	4 6609 3761	1·522	2 7490 0310	1·572	6 2950 7805
1·423	·2409 8996 1472	1·473	2 3674 2362	1·523	0 5673 4511	1·573	4 2198 2078
1·424	7 4909 1966	1·474	0 0762 0200	1·524	·2178 3878 6768	1·574	2 1466 3772
1·425	5 0846 3208	1·475	·2287 7872 7045	1·525	6 2105 6865	1·575	0 0755 2681
1·426	2 6807 4959	1·476	5 5006 2670	1·526	4 0354 4582	1·576	·2068 0064 8598
1·427	0 2792 6978	1·477	3 2162 6844	1·527	1 8624 9703	1·577	5 9395 1315
1·428	·2397 8801 9025	1·478	0 9341 9340	1·528	·2169 6917 2010	1·578	3 8746 0626
1·429	5 4835 0860	1·479	·2278 6543 9929	1·529	7 5231 1287	1·579	1 8117 6325
1·430	3 0892 2244	1·480	6 3768 8384	1·530	5 3566 7316	1·580	·2059 7509 8205
1·431	0 6973 2936	1·481	4 1016 4476	1·531	3 1923 9880	1·581	7 6922 6060
1·432	·2388 3078 2698	1·482	1 8286 7979	1·532	1 0302 8764	1·582	5 6355 9684
1·433	5 9207 1291	1·483	·2269 5579 8665	1·533	·2158 8703 3751	1·583	3 5809 8872
1·434	3 5359 8476	1·484	7 2895 6306	1·534	6 7125 4625	1·584	1 5284 3418
1·435	1 1536 4014	1·485	5 0234 0676	1·535	4 5569 1170	1·585	·2049 4779 3117
1·436	·2378 7736 7668	1·486	2 7595 1549	1·536	2 4034 3171	1·586	7 4294 7764
1·437	6 3960 9200	1·487	0 4978 8698	1·537	0 2521 0412	1·587	5 3830 7153
1·438	4 0208 8371	1·488	·2258 2385 1896	1·538	·2148 1029 2678	1·588	3 3387 1081
1·439	1 6480 4944	1·489	5 9814 0919	1·539	5 9558 9755	1·589	1 2963 9343
1·440	·2369 2775 8682	1·490	3 7265 5539	1·540	3 8110 1427	1·590	·2039 2561 1734
1·441	6 9094 9347	1·491	1 4739 5532	1·541	1 6682 7481	1·591	7 2178 8051
1·442	4 5437 6704	1·492	·2249 2236 0673	1·542	·2139 5276 7701	1·592	5 1816 8090
1·443	2 1804 0515	1·493	6 9755 0736	1·543	7 3892 1874	1·593	3 1475 1647
1·444	·2359 8194 0545	1·494	4 7296 5497	1·544	5 2528 9786	1·594	1 1153 8519
1·445	7 4607 6556	1·495	2 4860 4730	1·545	3 1187 1223	1·595	·2029 0852 8503
1·446	5 1044 8313	1·496	0 2446 8212	1·546	0 9866 5972	1·596	7 0572 1395
1·447	2 7505 5581	1·497	·2238 0055 5719	1·547	·2128 8567 3820	1·597	5 0311 6993
1·448	0 3989 8123	1·498	5 7686 7026	1·548	6 7289 4554	1·598	3 0071 5094
1·449	·2348 0497 5706	1·499	3 5340 1911	1·549	4 6032 7960	1·599	0 9851 5495

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
1·600	·2018 9651 7995	1·650	·1920 4990 8621	1·700	·1826 8352 4053	1·750	·1737 7394 3450
1·601	6 9472 2392	1·651	·1918 5795 4705	1·701	5 0093 1840	1·751	6 0025 6365
1·602	4 9312 8483	1·652	6 6619 2647	1·702	3 1852 2128	1·752	4 2674 2879
1·603	2 9173 6067	1·653	4 7462 2256	1·703	1 3629 4735	1·753	2 5340 2821
1·604	0 9054 4944	1·654	2 8324 3339	1·704	·1819 5424 9478	1·754	0 8023 6016
1·605	·2008 8955 4911	1·655	0 9205 5705	1·705	7 7238 6175	1·755	·1729 0724 2291
1·606	6 8876 5767	1·656	·1909 0105 9164	1·706	5 9070 4645	1·756	7 3442 1474
1·607	4 8817 7312	1·657	7 1025 3523	1·707	4 0920 4706	1·757	5 6177 3391
1·608	2 8778 9345	1·658	5 1963 8593	1·708	2 2788 6175	1·758	3 8929 7870
1·609	0 8760 1666	1·659	3 2921 4183	1·709	0 4674 8872	1·759	2 1699 4738
1·610	·1998 8761 4075	1·660	1 3898 0101	1·710	·1808 6579 2617	1·760	0 4486 3823
1·611	6 8782 6371	1·661	·1899 4893 6159	1·711	6 8501 7227	1·761	·1718 7290 4953
1·612	4 8823 8356	1·662	7 5908 2166	1·712	5 0442 2522	1·762	7 0111 7956
1·613	2 8884 9828	1·663	5 6941 7932	1·713	3 2400 8322	1·763	5 2950 2660
1·614	0 8966 0590	1·664	3 7994 3267	1·714	1 4377 4446	1·764	3 5805 8893
1·615	·1988 9067 0441	1·665	1 9065 7982	1·715	·1799 6372 0713	1·765	1 8678 6485
1·616	6 9187 9182	1·666	0 0156 1888	1·716	7 8384 6944	1·766	0 1568 5263
1·617	4 9328 6616	1·667	·1888 1265 4795	1·717	6 0415 2959	1·767	·1708 4475 5057
1·618	2 9489 2543	1·668	6 2393 6516	1·718	4 2463 8578	1·768	6 7399 5696
1·619	0 9669 6765	1·669	4 3540 6860	1·719	2 4530 3622	1·769	5 0340 7009
1·620	·1978 9869 9083	1·670	2 4706 5639	1·720	0 6614 7911	1·770	3 3298 8825
1·621	7 0089 9301	1·671	0 5891 2666	1·721	·1788 8717 1267	1·771	1 6274 0975
1·622	5 0329 7219	1·672	·1878 7094 7751	1·722	7 0837 3509	1·772	·1699 9266 3287
1·623	3 0589 2640	1·673	6 8317 0708	1·723	5 2975 4460	1·773	8 2275 5591
1·624	1 0868 5368	1·674	4 9558 1347	1·724	3 5131 3941	1·774	6 5301 7719
1·625	·1969 1167 5204	1·675	3 0817 9482	1·725	1 7305 1773	1·775	4 8344 9499
1·626	7 1486 1952	1·676	1 2096 4925	1·726	·1779 9496 7778	1·776	3 1405 0763
1·627	5 1824 5414	1·677	·1869 3393 7490	1·727	8 1706 1778	1·777	1 4482 1341
1·628	3 2182 5395	1·678	7 4709 6988	1·728	6 3933 3595	1·778	·1689 7576 1064
1·629	1 2560 1698	1·679	5 6044 3233	1·729	4 6178 3052	1·779	8 0686 9763
1·630	·1959 2957 4127	1·680	3 7397 6039	1·730	2 8440 9970	1·780	6 3814 7269
1·631	7 3374 2485	1·681	1 8769 5219	1·731	1 0721 4173	1·781	4 6959 3413
1·632	5 3810 6576	1·682	0 0160 0587	1·732	·1769 3019 5483	1·782	3 0120 8026
1·633	3 4266 6206	1·683	·1858 1569 1956	1·733	7 5335 3723	1·783	1 3299 0940
1·634	1 4742 1179	1·684	6 2996 9141	1·734	5 7668 8716	1·784	·1679 6494 1988
1·635	·1949 5237 1299	1·685	4 4443 1956	1·735	4 0020 0286	1·785	7 9706 1000
1·636	7 5751 6371	1·686	2 5908 0215	1·736	2 2388 8257	1·786	6 2934 7810
1·637	5 6285 6201	1·687	0 7391 3734	1·737	0 4775 2451	1·787	4 6180 2249
1·638	3 6839 0594	1·688	·1848 8893 2326	1·738	·1758 7179 2693	1·788	2 9442 4150
1·639	1 7411 9355	1·689	7 0413 5807	1·739	6 9600 8807	1·789	1 2721 3345
1·640	·1939 8004 2291	1·690	5 1952 3993	1·740	5 2040 0617	1·790	·1669 6016 9667
1·641	7 8615 9206	1·691	3 3509 6698	1·741	3 4496 7947	1·791	7 9329 2949
1·642	5 9246 9908	1·692	1 5085 3738	1·742	1 6971 0623	1·792	6 2658 3025
1·643	3 9897 4202	1·693	·1839 6679 4929	1·743	·1749 9462 8468	1·793	4 6003 9728
1·644	2 0567 1895	1·694	7 8292 0087	1·744	8 1972 1307	1·794	2 9366 2890
1·645	0 1256 2794	1·695	5 9922 9028	1·745	6 4498 8967	1·795	1 2745 2346
1·646	·1928 1964 6705	1·696	4 1572 1568	1·746	4 7043 1271	1·796	·1659 6140 7930
1·647	6 2692 3436	1·697	2 3239 7524	1·747	2 9604 8046	1·797	7 9552 9475
1·648	4 3439 2794	1·698	0 4925 6712	1·748	1 2183 9117	1·798	6 2981 6816
1·649	2 4205 4586	1·699	·1828 6629 8949	1·749	·1739 4780 4310	1·799	4 6426 9788

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
1·800	·1652 9888 8221	1·850	·1572 3716 6314	1·900	·1495 6861 9223	1·950	·1422 7407 1586
1·801	1 3367 1955	1·851	0 8000 7740	1·901	4 1912 5363	1·951	1 3186 8628
1·802	·1649 6862 0822	1·852	·1569 2300 6246	1·902	2 6978 0922	1·952	·1419 8980 7802
1·803	8 0373 4659	1·853	7 6616 1675	1·903	1 2058 5751	1·953	8 4788 8965
1·804	6 3901 3298	1·854	6 0947 3871	1·904	·1489 7153 9701	1·954	7 0611 1977
1·805	4 7445 6577	1·855	4 5294 2675	1·905	8 2264 2622	1·955	5 6447 6694
1·806	3 1006 4331	1·856	2 9656 7933	1·906	6 7389 4366	1·956	4 2298 2976
1·807	1 4583 6394	1·857	1 4034 9487	1·907	5 2529 4784	1·957	2 8163 0681
1·808	·1639 8177 2603	1·858	·1559 8428 7182	1·908	3 7684 3727	1·958	1 4041 9668
1·809	8 1787 2794	1·859	8 2838 0861	1·909	2 2854 1047	1·959	·1409 9934 9795
1·810	6 5413 6803	1·860	6 7263 0368	1·910	0 8038 6595	1·960	8 5842 0921
1·811	4 9056 4466	1·861	5 1703 5549	1·911	·1479 3238 0224	1·961	7 1763 2906
1·812	3 2715 5620	1·862	3 6159 6246	1·912	7 8452 1786	1·962	5 7698 5609
1·813	1 6391 0101	1·863	2 0631 2304	1·913	6 3681 1132	1·963	4 3647 8888
1·814	0 0082 7745	1·864	0 5118 3569	1·914	4 8924 8114	1·964	2 9611 2604
1·815	·1628 3790 8390	1·865	·1548 9620 9885	1·915	3 4183 2586	1·965	1 5588 6616
1·816	6 7515 1874	1·866	7 4139 1097	1·916	1 9456 4400	1·966	0 1580 0784
1·817	5 1255 8032	1·867	5 8672 7051	1·917	0 4744 3408	1·967	·1398 7585 4968
1·818	3 5012 6704	1·868	4 3221 7592	1·918	·1469 0046 9464	1·968	7 3604 9027
1·819	1 8785 7725	1·869	2 7786 2565	1·919	7 5364 2420	1·969	5 9638 2823
1·820	0 2575 0934	1·870	1 2366 1815	1·920	6 0696 2130	1·970	4 5685 6215
1·821	·1618 6380 6169	1·871	·1539 6961 5190	1·921	4 6042 8447	1·971	3 1746 9064
1·822	7 0202 3268	1·872	8 1572 2534	1·922	3 1404 1225	1·972	1 7822 1230
1·823	5 4040 2069	1·873	6 6198 3693	1·923	1 6780 0316	1·973	0 3911 2575
1·824	3 7894 2410	1·874	5 0839 8515	1·924	0 2170 5575	1·974	·1389 0014 2959
1·825	2 1764 4130	1·875	3 5496 6845	1·925	·1458 7575 6856	1·975	7 6131 2243
1·826	0 5650 7068	1·876	2 0168 8530	1·926	7 2995 4013	1·976	6 2262 0288
1·827	·1608 9553 1062	1·877	0 4856 3417	1·927	5 8429 6900	1·977	4 8406 6956
1·828	7 3471 5952	1·878	·1528 9559 1352	1·928	4 3878 5371	1·978	3 4565 2108
1·829	5 7406 1577	1·879	7 4277 2183	1·929	2 9341 9281	1·979	2 0737 5606
1·830	4 1356 7775	1·880	5 9010 5757	1·930	1 4819 8484	1·980	0 6923 7311
1·831	2 5323 4388	1·881	4 3759 1921	1·931	0 0312 2835	1·981	·1379 3123 7085
1·832	0 9306 1253	1·882	2 8523 0522	1·932	·1448 5819 2190	1·982	7 9337 4791
1·833	·1599 3304 8212	1·883	1 3302 1409	1·933	7 1340 6403	1·983	6 5565 0290
1·834	7 7319 5103	1·884	·1519 8096 4429	1·934	5 6876 5329	1·984	5 1806 3444
1·835	6 1350 1768	1·885	8 2905 9429	1·935	4 2426 8824	1·985	3 8061 4117
1·836	4 5396 8046	1·886	6 7730 6259	1·936	2 7991 6743	1·986	2 4330 2170
1·837	2 9459 3779	1·887	5 2570 4766	1·937	1 3570 8942	1·987	1 0612 7467
1·838	1 3537 8806	1·888	3 7425 4799	1·938	·1439 9164 5277	1·988	·1369 6908 9870
1·839	·1589 7632 2968	1·889	2 2295 6206	1·939	8 4772 5604	1·989	8 3218 9241
1·840	8 1742 6107	1·890	0 7180 8836	1·940	7 0394 9778	1·990	6 9542 5445
1·841	6 5868 8063	1·891	·1509 2081 2538	1·941	5 6031 7656	1·991	5 5879 8345
1·842	5 0010 8678	1·892	7 6996 7161	1·942	4 1682 9095	1·992	4 2230 7803
1·843	3 4168 7793	1·893	6 1927 2554	1·943	2 7348 3950	1·993	2 8595 3684
1·844	1 8342 5250	1·894	4 6872 8566	1·944	1 3028 2079	1·994	1 4973 5850
1·845	0 2532 0890	1·895	3 1833 5046	1·945	·1429 8722 3338	1·995	0 1365 4167
1·846	·1578 6737 4555	1·896	1 6809 1845	1·946	8 4430 7584	1·996	·1358 7770 8497
1·847	7 0958 6088	1·897	0 1799 8813	1·947	7 0153 4675	1·997	7 4189 8705
1·848	5 5195 5330	1·898	·1498 6805 5798	1·948	5 5890 4467	1·998	6 0622 4654
1·849	3 9448 2125	1·899	7 1826 2651	1·949	4 1641 6819	1·999	4 7068 6210

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
2·000	·1353 3528 3237	2·050	·1287 3490 3588	2·100	·1224 5642 8253	2·150	·1164 8415 7773
2·001	2 0001 5599	2·051	6 0623 3030	2·101	3 3403 3032	2·151	3 6773 1838
2·002	0 6488 3161	2·052	4 7769 1079	2·102	2 1176 0146	2·152	2 5142 2271
2·003	·1349 2988 5788	2·053	3 4927 7605	2·103	0 8960 9471	2·153	1 3522 8955
2·004	7 9502 3344	2·054	2 2099 2481	2·104	·1219 6758 0886	2·154	0 1915 1774
2·005	6 6029 5696	2·055	0 9283 5578	2·105	8 4567 4269	2·155	·1159 0319 0613
2·006	5 2570 2708	2·056	·1279 6480 6767	2·106	7 2388 9497	2·156	7 8734 5354
2·007	3 9124 4246	2·057	8 3690 5921	2·107	6 0222 6449	2·157	6 7161 5883
2·008	2 5692 0175	2·058	7 0913 2913	2·108	4 8068 5004	2·158	5 5600 2084
2·009	1 2273 0361	2·059	5 8148 7613	2·109	3 5926 5039	2·159	4 4050 3841
2·010	·1339 8867 4669	2·060	4 5396 9895	2·110	2 3796 6433	2·160	3 2512 1038
2·011	8 5475 2967	2·061	3 2657 9631	2·111	1 1678 9066	2·161	2 0985 3560
2·012	7 2096 5119	2·062	1 9931 6693	2·112	·1209 9573 2815	2·162	0 9470 1292
2·013	5 8731 0992	2·063	0 7218 0955	2·113	8 7479 7560	2·163	·1149 7966 4119
2·014	4 5379 0452	2·064	·1269 4517 2289	2·114	7 5398 3180	2·164	8 6474 1926
2·015	3 2040 3366	2·065	8 1829 0568	2·115	6 3328 9553	2·165	7 4993 4597
2·016	1 8714 9601	2·066	6 9153 5665	2·116	5 1271 6560	2·166	6 3524 2018
2·017	0 5402 9023	2·067	5 6490 7454	2·117	3 9226 4080	2·167	5 2066 4074
2·018	·1329 2104 1499	2·068	4 3840 5808	2·118	2 7193 1992	2·168	4 0620 0652
2·019	7 8818 6895	2·069	3 1203 0601	2·119	1 5172 0176	2·169	2 9185 1635
2·020	6 5546 5080	2·070	1 8578 1705	2·120	0 3162 8511	2·170	1 7761 6910
2·021	5 2287 5921	2·071	0 5965 8995	2·121	·1199 1165 6879	2·171	0 6349 6363
2·022	3 9041 9285	2·072	·1259 3366 2345	2·122	7 9180 5158	2·172	·1139 4948 9879
2·023	2 5809 5039	2·073	8 0779 1629	2·123	6 7207 3229	2·173	8 3559 7345
2·024	1 2590 3050	2·074	6 8204 6720	2·124	5 5246 0972	2·174	7 2181 8647
2·025	·1319 9384 3188	2·075	5 5642 7493	2·125	4 3296 8267	2·175	6 0815 3670
2·026	8 6191 5320	2·076	4 3093 3823	2·126	3 1359 4995	2·176	4 9460 2301
2·027	7 3011 9313	2·077	3 0556 5584	2·127	1 9434 1037	2·177	3 8116 4428
2·028	5 9845 5037	2·078	1 8032 2650	2·128	0 7520 6273	2·178	2 6783 9935
2·029	4 6692 2360	2·079	0 5520 4897	2·129	·1189 5619 0585	2·179	1 5462 8710
2·030	3 3552 1149	2·080	·1249 3021 2199	2·130	8 3729 3852	2·180	0 4153 0640
2·031	2 0425 1273	2·081	8 0534 4431	2·131	7 1851 5957	2·181	·1129 2854 5611
2·032	0 7311 2602	2·082	6 8060 1468	2·132	5 9985 6781	2·182	8 1567 3511
2·033	·1309 4210 5005	2·083	5 5598 3186	2·133	4 8131 6204	2·183	7 0291 4226
2·034	8 1122 8349	2·084	4 3148 9460	2·134	3 6289 4109	2·184	5 9026 7645
2·035	6 8048 2504	2·085	3 0712 0166	2·135	2 4459 0376	2·185	4 7773 3654
2·036	5 4986 7340	2·086	1 8287 5178	2·136	1 2640 4889	2·186	3 6531 2140
2·037	4 1938 2726	2·087	0 5875 4374	2·137	0 0833 7527	2·187	2 5300 2992
2·038	2 8902 8531	2·088	·1239 3475 7628	2·138	·1178 9038 8174	2·188	1 4080 6097
2·039	1 5880 4625	2·089	8 1088 4817	2·139	7 7255 6712	2·189	0 2872 1343
2·040	0 2871 0878	2·090	6 8713 5817	2·140	6 5484 3022	2·190	·1119 1674 8617
2·041	·1298 9874 7160	2·091	5 6351 0504	2·141	5 3724 6987	2·191	8 0488 7808
2·042	7 6891 3341	2·092	4 4000 8755	2·142	4 1976 8489	2·192	6 9313 8804
2·043	6 3920 9290	2·093	3 1663 0446	2·143	3 0240 7411	2·193	5 8150 1493
2·044	5 0963 4879	2·094	1 9337 5453	2·144	1 8516 3635	2·194	4 6997 5764
2·045	3 8018 9977	2·095	0 7024 3654	2·145	0 6803 7044	2·195	3 5856 1505
2·046	2 5087 4456	2·096	·1229 4723 4925	2·146	·1169 5102 7521	2·196	2 4725 8604
2·047	1 2168 8185	2·097	8 2434 9143	2·147	8 3413 4950	2·197	1 3606 6951
2·048	·1289 9263 1036	2·098	7 0158 6186	2·148	7 1735 9213	2·198	0 2498 6433
2·049	8 6370 2880	2·099	5 7894 5930	2·149	6 0070 0193	2·199	·1109 1401 6941

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
2·200	·1108 0315 8362	2·250	·1053 9922 4562	2·300	·1002 5884 3723	2·350	953 6916 2215
2·201	6 9241 0587	2·251	2 9387 8020	2·301	1 5863 4992	2·351	2 7384 0721
2·202	5 8177 3504	2·252	1 8863 6771	2·302	0 5852 6419	2·352	1 7861 4502
2·203	4 7124 7003	2·253	0 8350 0711	2·303	999 5851 7906	2·353	0 8348 3461
2·204	3 6083 0973	2·254	·1049 7846 9734	2·304	8 5860 9350	2·354	949 8844 7503
2·205	2 5052 5304	2·255	8 7354 3736	2·305	7 5880 0654	2·355	8 9350 6534
2·206	1 4032 9886	2·256	7 6872 2612	2·306	6 5909 1716	2·356	7 9866 0458
2·207	0 3024 4608	2·257	6 6400 6256	2·307	5 5948 2437	2·357	7 0390 9181
2·208	·1099 2026 9360	2·258	5 5939 4565	2·308	4 5997 2718	2·358	6 0925 2609
2·209	8 1040 4032	2·259	4 5488 7432	2·309	3 6056 2458	2·359	5 1469 0645
2·210	7 0064 8515	2·260	3 5048 4755	2·310	2 6125 1560	2·360	4 2022 3196
2·211	5 9100 2698	2·261	2 4618 6428	2·311	1 6203 9922	2·361	3 2585 0167
2·212	4 8146 6473	2·262	1 4199 2347	2·312	0 6292 7447	2·362	2 3157 1464
2·213	3 7203 9729	2·263	0 3790 2409	2·313	989 6391 4034	2·363	1 3738 6993
2·214	2 6272 2357	2·264	·1039 3391 6508	2·314	8 6499 9586	2·364	0 4329 6659
2·215	1 5351 4248	2·265	8 3003 4541	2·315	7 6618 4002	2·365	939 4930 0368
2·216	0 4441 5292	2·266	7 2625 6404	2·316	6 6746 7185	2·366	8 5539 8027
2·217	·1089 3542 5381	2·267	6 2258 1994	2·317	5 6884 9035	2·367	7 6158 9541
2·218	8 2654 4405	2·268	5 1901 1206	2·318	4 7032 9454	2·368	6 6787 4816
2·219	7 1777 2256	2·269	4 1554 3937	2·319	3 7190 8343	2·369	5 7425 3760
2·220	6 0910 8825	2·270	3 1218 0083	2·320	2 7358 5604	2·370	4 8072 6278
2·221	5 0055 4003	2·271	2 0891 9542	2·321	1 7536 1139	2·371	3 8729 2276
2·222	3 9210 7681	2·272	1 0576 2210	2·322	0 7723 4849	2·372	2 9395 1662
2·223	2 8376 9751	2·273	0 0270 7983	2·323	979 7920 6636	2·373	2 0070 4342
2·224	1 7554 0105	2·274	·1028 9975 6759	2·324	8 8127 6403	2·374	1 0755 0222
2·225	0 6741 8635	2·275	7 9690 8435	2·325	7 8344 4051	2·375	0 1448 9210
2·226	·1079 5940 5232	2·276	6 9416 2908	2·326	6 8570 9482	2·376	929 2152 1212
2·227	8 5149 9789	2·277	5 9152 0075	2·327	5 8807 2599	2·377	8 2864 6136
2·228	7 4370 2197	2·278	4 8897 9834	2·328	4 9053 3304	2·378	7 3586 3889
2·229	6 3601 2348	2·279	3 8654 2082	2·329	3 9309 1500	2·379	6 4317 4378
2·230	5 2843 0136	2·280	2 8420 6716	2·330	2 9574 7089	2·380	5 5057 7510
2·231	4 2095 5452	2·281	1 8197 3634	2·331	1 9849 9974	2·381	4 5807 3192
2·232	3 1358 8189	2·282	0 7984 2734	2·332	1 0135 0057	2·382	3 6566 1332
2·233	2 0632 8240	2·283	·1019 7781 3914	2·333	0 0429 7241	2·383	2 7334 1838
2·234	0 9917 5497	2·284	8 7588 7072	2·334	969 0734 1430	2·384	1 8111 4618
2·235	·1069 9212 9853	2·285	7 7406 2106	2·335	8 1048 2526	2·385	0 8897 9579
2·236	8 8519 1201	2·286	6 7233 8914	2·336	7 1372 0432	2·386	919 9693 6628
2·237	7 7835 9435	2·287	5 7071 7394	2·337	6 1705 5053	2·387	9 0498 5675
2·238	6 7163 4447	2·288	4 6919 7445	2·338	5 2048 6290	2·388	8 1312 6626
2·239	5 6501 6130	2·289	3 6777 8966	2·339	4 2401 4048	2·389	7 2135 9391
2·240	4 5850 4379	2·290	2 6646 1854	2·340	3 2763 8230	2·390	6 2968 3877
2·241	3 5209 9086	2·291	1 6524 6008	2·341	2 3135 8740	2·391	5 3809 9993
2·242	2 4580 0145	2·292	0 6413 1328	2·342	1 3517 5480	2·392	4 4660 7646
2·243	1 3960 7450	2·293	·1009 6311 7712	2·343	0 3908 8356	2·393	3 5520 6747
2·244	0 3352 0895	2·294	8 6220 5059	2·344	959 4309 7272	2·394	2 6389 7203
2·245	·1059 2754 0373	2·295	7 6139 3268	2·345	8 4720 2130	2·395	1 7267 8922
2·246	8 2166 5779	2·296	6 6068 2239	2·346	7 5140 2835	2·396	0 8155 1814
2·247	7 1589 7006	2·297	5 6007 1870	2·347	6 5569 9292	2·397	909 9051 5788
2·248	6 1023 3950	2·298	4 5956 2062	2·348	5 6009 1405	2·398	8 9957 0752
2·249	5 0467 6503	2·299	3 5915 2713	2·349	4 6457 9078	2·399	8 0871 6616

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
2·400	907 1795 3289	2·450	862 9358 6499	2·500	820 8499 8624	2·550	780 8166 6001
2·401	6 2728 0679	2·451	2 0733 6045	2·501	0 0295 4654	2·551	0 0362 3363
2·402	5 3669 8697	2·452	1 2117 1798	2·502	819 2099 2687	2·552	779 2565 8728
2·403	4 4620 7252	2·453	0 3509 3673	2·503	8 3911 2641	2·553	8 4777 2019
2·404	3 5580 6253	2·454	859 4910 1582	2·504	7 5731 4435	2·554	7 6996 3158
2·405	2 6549 5609	2·455	8 6319 5441	2·505	6 7559 7985	2·555	6 9223 2067
2·406	1 7527 5231	2·456	7 7737 5163	2·506	5 9396 3212	2·556	6 1457 8669
2·407	0 8514 5029	2·457	6 9164 0662	2·507	5 1241 0032	2·557	5 3700 2884
2·408	899 9510 4911	2·458	6 0599 1853	2·508	4 3093 8364	2·558	4 5950 4637
2·409	9 0515 4789	2·459	5 2042 8650	2·509	3 4954 8128	2·559	3 8208 3849
2·410	8 1529 4572	2·460	4 3495 0967	2·510	2 6823 9241	2·560	3 0474 0443
2·411	7 2552 4170	2·461	3 4955 8719	2·511	1 8701 1622	2·561	2 2747 4342
2·412	6 3584 3494	2·462	2 6425 1821	2·512	1 0586 5191	2·562	1 5028 5469
2·413	5 4625 2453	2·463	1 7903 0187	2·513	0 2479 9865	2·563	0 7317 3746
2·414	4 5675 0959	2·464	0 9389 3732	2·514	809 4381 5564	2·564	769 9613 9096
2·415	3 6733 8921	2·465	0 0884 2371	2·515	8 6291 2207	2·565	9 1918 1442
2·416	2 7801 6251	2·466	849 2387 6019	2·516	7 8208 9713	2·566	8 4230 0707
2·417	1 8878 2859	2·467	8 3899 4591	2·517	7 0134 8001	2·567	7 6549 6815
2·418	0 9963 8656	2·468	7 5419 8002	2·518	6 2068 6990	2·568	6 8876 9688
2·419	0 1058 3552	2·469	6 6948 6167	2·519	5 4010 6600	2·569	6 1211 9250
2·420	889 2161 7459	2·470	5 8485 9001	2·520	4 5960 6750	2·570	5 3554 5424
2·421	8 3274 0288	2·471	5 0031 6420	2·521	3 7918 7360	2·571	4 5904 8134
2·422	7 4395 1949	2·472	4 1585 8340	2·522	2 9884 8349	2·572	3 8262 7303
2·423	6 5525 2354	2·473	3 3148 4676	2·523	2 1858 9636	2·573	3 0628 2854
2·424	5 6664 1415	2·474	2 4719 5343	2·524	1 3841 1142	2·574	2 3001 4712
2·425	4 7811 9042	2·475	1 6299 0257	2·525	0 5831 2787	2·575	1 5382 2799
2·426	3 8968 5147	2·476	0 7886 9334	2·526	799 7829 4490	2·576	0 7770 7040
2·427	3 0133 9642	2·477	839 9483 2490	2·527	8 9835 6171	2·577	0 0166 7360
2·428	2 1308 2438	2·478	9 1087 9641	2·528	8 1849 7751	2·578	759 2570 3680
2·429	1 2491 3448	2·479	8 2701 0703	2·529	7 3871 9149	2·579	8 4981 5927
2·430	0 3683 2582	2·480	7 4322 5592	2·530	6 5902 0286	2·580	7 7400 4023
2·431	879 4883 9753	2·481	6 5952 4224	2·531	5 7940 1082	2·581	6 9826 7894
2·432	8 6093 4873	2·482	5 7590 6516	2·532	4 9986 1457	2·582	6 2260 7462
2·433	7 7311 7854	2·483	4 9237 2383	2·533	4 2040 1333	2·583	5 4702 2653
2·434	6 8538 8608	2·484	4 0892 1743	2·534	3 4102 0628	2·584	4 7151 3392
2·435	5 9774 7048	2·485	3 2555 4512	2·535	2 6171 9265	2·585	3 9607 9601
2·436	5 1019 3085	2·486	2 4227 0606	2·536	1 8249 7163	2·586	3 2072 1207
2·437	4 2272 6632	2·487	1 5906 9943	2·537	1 0335 4244	2·587	2 4543 8134
2·438	3 3534 7603	2·488	0 7595 2439	2·538	0 2429 0429	2·588	1 7023 0306
2·439	2 4805 5908	2·489	829 9291 8011	2·539	789 4530 5637	2·589	0 9509 7648
2·440	1 6085 1462	2·490	9 0996 6575	2·540	8 6639 9791	2·590	0 2004 0085
2·441	0 7373 4176	2·491	8 2709 8049	2·541	7 8757 2811	2·591	749 4505 7543
2·442	869 8670 3964	2·492	7 4431 2351	2·542	7 0882 4619	2·592	8 7014 9945
2·443	8 9976 0739	2·493	6 6160 9397	2·543	6 3015 5136	2·593	7 9531 7218
2·444	8 1290 4414	2·494	5 7898 9105	2·544	5 5156 4282	2·594	7 2055 9286
2·445	7 2613 4902	2·495	4 9645 1392	2·545	4 7305 1981	2·595	6 4587 6074
2·446	6 3945 2115	2·496	4 1399 6175	2·546	3 9461 8152	2·596	5 7126 7509
2·447	5 5285 5969	2·497	3 3162 3372	2·547	3 1626 2718	2·597	4 9673 3514
2·448	4 6634 6375	2·498	2 4933 2901	2·548	2 3798 5601	2·598	4 2227 4017
2·449	3 7992 3247	2·499	1 6712 4679	2·549	1 5978 6721	2·599	3 4788 8942

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
2·600	742 7357 8214	2·650	706 5121 3060	2·700	672 0551 2740	2·750	639 2786 1207
2·601	1 9934 1760	2·651	5 8059 7161	2·701	1 3834 0818	2·751	8 6396 5299
2·602	1 2517 9506	2·652	5 1005 1843	2·702	0 7123 6035	2·752	8 0013 3255
2·603	0 5109 1376	2·653	4 3957 7034	2·703	0 0419 8324	2·753	7 3636 5011
2·604	739 7707 7298	2·654	3 6917 2665	2·704	669 3722 7616	2·754	6 7266 0504
2·605	9 0313 7197	2·655	2 9883 8665	2·705	8 7032 3846	2·755	6 0901 9669
2·606	8 2927 0999	2·656	2 2857 4964	2·706	8 0348 6947	2·756	5 4544 2443
2·607	7 5547 8631	2·657	1 5838 1492	2·707	7 3671 6850	2·757	4 8192 8763
2·608	6 8176 0017	2·658	0 8825 8178	2·708	6 7001 3491	2·758	4 1847 8564
2·609	6 0811 5086	2·659	0 1820 4952	2·709	6 0387 6801	2·759	3 5509 1784
2·610	5 3454 3763	2·660	699 4822 1745	2·710	5 3680 6715	2·760	2 9176 8360
2·611	4 6104 5974	2·661	8 7830 8485	2·711	4 7030 3166	2·761	2 2850 8227
2·612	3 8762 1647	2·662	8 0846 5105	2·712	4 0386 6086	2·762	1 6531 1323
2·613	3 1427 0707	2·663	7 3869 1532	2·713	3 3749 5411	2·763	1 0217 7583
2·614	2 4099 3081	2·664	6 6898 7698	2·714	2 7119 1074	2·764	0 3910 6946
2·615	1 6778 8696	2·665	5 9935 3533	2·715	2 0495 3007	2·765	629 7609 9348
2·616	0 9465 7479	2·666	5 2978 8968	2·716	1 3878 1146	2·766	9 1315 4727
2·617	0 2159 9356	2·667	4 6029 3932	2·717	0 7267 5423	2·767	8 5027 3018
2·618	729 4861 4256	2·668	3 9086 8357	2·718	0 0663 5773	2·768	7 8745 4159
2·619	8 7570 2104	2·669	3 2151 2172	2·719	659 4066 2130	2·769	7 2469 8089
2·620	8 0286 2827	2·670	2 5222 5309	2·720	8 7475 4426	2·770	6 6200 4742
2·621	7 3009 6354	2·671	1 8300 7698	2·721	8 0891 2598	2·771	5 9937 4058
2·622	6 5740 2611	2·672	1 1385 9271	2·722	7 4313 6579	2·772	5 3680 5973
2·623	5 8478 1525	2·673	0 4477 9957	2·723	6 7742 6303	2·773	4 7430 0425
2·624	5 1223 3023	2·674	689 7576 9688	2·724	6 1178 1705	2·774	4 1185 7352
2·625	4 3975 7034	2·675	9 0682 8394	2·725	5 4620 2718	2·775	3 4947 6690
2·626	3 6735 3485	2·676	8 3795 6008	2·726	4 8068 9278	2·776	2 8715 8377
2·627	2 9502 2303	2·677	7 6915 2459	2·727	4 1524 1318	2·777	2 2490 2352
2·628	2 2276 3417	2·678	7 0041 7680	2·728	3 4985 8773	2·778	1 6270 8552
2·629	1 5057 6752	2·679	6 3175 1601	2·729	2 8454 1578	2·779	1 0057 6914
2·630	0 7846 2239	2·680	5 6315 4154	2·730	2 1928 9668	2·780	0 3850 7377
2·631	0 0641 9804	2·681	4 9462 5270	2·731	1 5410 2977	2·781	619 7649 9879
2·632	719 3444 9375	2·682	4 2616 4881	2·732	0 8898 1441	2·782	9 1455 4357
2·633	8 6255 0881	2·683	3 5777 2918	2·733	0 2392 4993	2·783	8 5267 0750
2·634	7 9072 4250	2·684	2 8944 9312	2·734	649 5893 3568	2·784	7 9084 8995
2·635	7 1896 9409	2·685	2 2119 3996	2·735	8 9400 7104	2·785	7 2908 9031
2·636	6 4728 6286	2·686	1 5300 6901	2·736	8 2914 5533	2·786	6 6739 0796
2·637	5 7567 4812	2·687	0 8488 7959	2·737	7 6434 8792	2·787	6 0575 4229
2·638	5 0413 4913	2·688	0 1683 7103	2·738	6 9961 6814	2·788	5 4417 9267
2·639	4 3266 6518	2·689	679 4885 4263	2·739	6 3494 9536	2·789	4 8266 5850
2·640	3 6126 9556	2·690	8 8093 9372	2·740	5 7034 6893	2·790	4 2121 3915
2·641	2 8994 3955	2·691	8 1309 2361	2·741	5 0580 8820	2·791	3 5982 3402
2·642	2 1868 9644	2·692	7 4531 3164	2·742	4 4133 5254	2·792	2 9849 4248
2·643	1 4750 6552	2·693	6 7760 1712	2·743	3 7692 6129	2·793	2 3722 6392
2·644	0 7639 4607	2·694	6 0995 7938	2·744	3 1258 1380	2·794	1 7601 9774
2·645	0 0535 3739	2·695	5 4238 1774	2·745	2 4830 0944	2·795	1 1487 4332
2·646	709 3438 3876	2·696	4 7487 3152	2·746	1 8408 4757	2·796	0 5379 0005
2·647	8 6348 4948	2·697	4 0743 2005	2·747	1 1993 2753	2·797	609 9276 6732
2·648	7 9265 6883	2·698	3 4005 8266	2·748	0 5584 4870	2·798	9 3180 4451
2·649	7 2189 9611	2·699	2 7275 1866	2·749	639 9182 1042	2·799	8 7090 3103

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
2·800	608 1006 2625	2·850	578 4432 0875	2·900	550 2322 0056	2·950	523 3970 5948
2·801	7 4928 2957	2·851	7 8650 5467	2·901	549 6822 4338	2·951	2 8739 2403
2·802	6 8856 4038	2·852	7 2874 7845	2·902	9 1328 3589	2·952	2 3513 1146
2·803	6 2790 5809	2·853	6 7104 7952	2·903	8 5839 7753	2·953	1 8292 2124
2·804	5 6730 8207	2·854	6 1340 5730	2·904	8 0356 6775	2·954	1 3076 5284
2·805	5 0677 1172	2·855	5 5582 1121	2·905	7 4879 0601	2·955	0 7866 0575
2·806	4 4629 4644	2·856	4 9829 4068	2·906	6 9406 9176	2·956	0 2660 7945
2·807	3 8587 8563	2·857	4 4082 4513	2·907	6 3940 2445	2·957	519 7460 7342
2·808	3 2552 2867	2·858	3 8341 2399	2·908	5 8479 0353	2·958	9 2265 8713
2·809	2 6522 7497	2·859	3 2605 7669	2·909	5 3023 2846	2·959	8 7076 2007
2·810	2 0499 2392	2·860	2 6876 0265	2·910	4 7572 9869	2·960	8 1891 7172
2·811	1 4481 7492	2·861	2 1152 0130	2·911	4 2128 1368	2·961	7 6712 4156
2·812	0 8470 2737	2·862	1 5433 7206	2·912	3 6688 7288	2·962	7 1538 2907
2·813	0 2464 8067	2·863	0 9721 1436	2·913	3 1254 7575	2·963	6 6369 3373
2·814	599 6465 3421	2·864	0 4014 2764	2·914	2 5826 2175	2·964	6 1205 5502
2·815	9 0471 8740	2·865	569 8313 1132	2·915	2 0403 1033	2·965	5 6046 9244
2·816	8 4484 3964	2·866	9 2617 6483	2·916	1 4985 4095	2·966	5 0893 4547
2·817	7 8502 9032	2·867	8 6927 8760	3·917	0 9573 1307	2·967	4 5745 1358
2·818	7 2527 3886	2·868	8 1243 7906	2·918	0 4166 2614	2·968	4 0601 9627
2·819	6 6557 8464	2·869	7 5565 3865	2·919	539 8764 7963	2·969	3 5463 9302
2·820	6 0594 2709	2·870	6 9892 6580	2·920	9 3368 7300	2·970	3 0331 0331
2·821	5 4636 6559	2·871	6 4225 5993	2·921	8 7978 0571	2·971	2 5203 2664
2·822	4 8684 9956	2·872	5 8564 2049	2·922	8 2592 7721	2·972	2 0080 6249
2·823	4 2739 2839	2·873	5 2908 4690	2·923	7 7212 8697	2·973	1 4963 1035
2·824	3 6799 5150	2·874	4 7258 3861	2·924	7 1838 3446	2·974	0 9850 6970
2·825	3 0865 6829	2·875	4 1613 9504	2·925	6 6469 1912	2·975	0 4743 4004
2·826	2 4937 7817	2·876	3 5975 1563	2·926	6 1105 4044	2·976	509 9641 2085
2·827	1 9015 8054	2·877	3 0341 9982	2·927	5 5746 9786	2·977	9 4544 1163
2·828	1 3099 7481	2·878	2 4714 4704	2·928	5 0393 9086	2·978	8 9452 1186
2·829	0 7189 6039	2·879	1 9092 5674	2·929	4 5046 1890	2·979	8 4365 2104
2·830	0 1285 3669	2·880	1 3476 2834	2·930	3 9703 8145	2·980	7 9283 3865
2·831	589 5387 0312	2·881	0 7865 6129	2·931	3 4366 7797	2·981	7 4206 6419
2·832	8 9494 5909	2·882	0 2260 5503	2·932	2 9035 0792	2·982	6 9134 9715
2·833	8 3608 0401	2·883	559 6661 0900	2·933	2 3708 7077	2·983	6 4068 3703
2·834	7 7727 3728	2·884	9 1067 2263	2·934	1 8387 6600	2·984	5 9006 8331
2·835	7 1852 5833	2·885	8 5478 9536	2·935	1 3071 9306	2·985	5 3950 3549
2·836	6 5983 6657	2·886	7 9896 2665	2·936	0 7761 5143	2·986	4 8898 9307
2·837	6 0120 6141	2·887	7 4319 1593	2·937	0 2456 4058	2·987	4 3852 5554
2·838	5 4263 4225	2·888	6 8747 6263	2·938	529 7156 5998	2·988	3 8811 2240
2·839	4 8412 0853	2·889	6 3181 6621	2·939	9 1862 0909	2·989	3 3774 9313
2·840	4 2566 5964	2·890	5 7621 2611	2·940	8 6572 8738	2·990	2 8743 6724
2·841	3 6726 9502	2·891	5 2066 4178	2·941	8 1288 9434	2·991	2 3717 4422
2·842	3 0893 1406	2·892	4 6517 1265	2·942	7 6010 2942	2·992	1 8696 2358
2·843	2 5065 1619	2·893	4 0973 3817	2·943	7 0736 9210	2·993	1 3680 0481
2·844	1 9243 0083	2·894	3 5435 1779	2·944	6 5468 8186	2·994	0 8668 8741
2·845	1 3426 6740	2·895	2 9902 5095	2·945	6 0205 9816	2·995	0 3662 7087
2·846	0 7616 1531	2·896	2 4375 3710	2·946	5 4948 4048	2·996	499 8661 5470
2·847	0 1811 4398	2·897	1 8853 7568	2·947	4 9696 0830	2·997	9 3665 3839
2·848	579 6012 5283	2·898	1 3337 6616	2·948	4 4449 0110	2·998	8 8674 2146
2·849	9 0219 4128	2·899	0 7827 0797	2·949	3 9207 1833	2·999	8 3688 0339

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
3·000	497 8706 8368	3·050	473 5892 4391	3·100	450 4920 2393	3·150	428 5212 6867
3·001	7 3730 6185	3·051	3 1158 9138	3·101	0 0417 5708	3·151	8 0929 6159
3·002	6 8759 3739	3·052	2 6430 1197	3·102	449 5919 4027	3·152	7 6650 8260
3·003	6 3793 0981	3·053	2 1706 0520	3·103	9 1425 7305	3·153	7 2376 3128
3·004	5 8831 7860	3·054	1 6986 7060	3·104	8 6936 5497	3·154	6 8106 0720
3·005	5 3875 4328	3·055	1 2272 0770	3·105	8 2451 8559	3·155	6 3840 0992
3·006	4 8924 0335	3·056	0 7562 1603	3·106	7 7971 6445	3·156	5 9578 3903
3·007	4 3977 5831	3·057	0 2856 9511	3·107	7 3495 9111	3·157	5 5320 9410
3·008	3 9036 0767	3·058	469 8156 4448	3·108	6 9024 6512	3·158	5 1067 7470
3·009	3 4099 5093	3·059	9 3460 6367	3·109	6 4557 8603	3·159	4 6818 8041
3·010	2 9167 8760	3·060	8 8769 5220	3·110	6 0095 5340	3·160	4 2574 1080
3·011	2 4241 1719	3·061	8 4083 0961	3·111	5 5637 6678	3·161	3 8333 6545
3·012	1 9319 3920	3·062	7 9401 3542	3·112	5 1184 2572	3·162	3 4097 4393
3·013	1 4402 5315	3·063	7 4724 2918	3·113	4 6735 2978	3·163	2 9865 4582
3·014	0 9490 5853	3·064	7 0051 9041	3·114	4 2290 7851	3·164	2 5637 7069
3·015	0 4583 5487	3·065	6 5384 1864	3·115	3 7850 7147	3·165	2 1414 1813
3·016	489 9681 4166	3·066	6 0721 1342	3·116	3 3415 0822	3·166	1 7194 8772
3·017	9 4784 1842	3·067	5 6062 7426	3·117	2 8983 8831	3·167	1 2979 7902
3·018	8 9891 8466	3·068	5 1409 0071	3·118	2 4557 1130	3·168	0 8768 9162
3·019	8 5004 3989	3·069	4 6759 9230	3·119	2 0134 7674	3·169	0 4562 2510
3·020	8 0121 8362	3·070	4 2115 4857	3·120	1 5716 8420	3·170	0 0359 7903
3·021	7 5244 1536	3·071	3 7475 6905	3·121	1 1303 3322	3·171	419 6161 5300
3·022	7 0371 3463	3·072	3 2840 5328	3·122	0 6894 2338	3·172	9 1967 4658
3·023	6 5503 4093	3·073	2 8210 0079	3·123	0 2489 5423	3·173	8 7777 5937
3·024	6 0640 3378	3·074	2 3584 1112	3·124	439 8089 2533	3·174	8 3591 9093
3·025	5 5782 1270	3·075	1 8962 8381	3·125	9 3693 3623	3·175	7 9410 4084
3·026	5 0928 7720	3·076	1 4346 1840	3·126	8 9301 8651	3·176	7 5233 0870
3·027	4 6080 2679	3·077	0 9734 1442	3·127	8 4914 7571	3·177	7 1059 9409
3·028	4 1236 6098	3·078	0 5126 7142	3·128	8 0532 0341	3·178	6 6890 9657
3·029	3 6397 7930	3·079	0 0523 8893	3·129	7 6153 6916	3·179	6 2726 1575
3·030	3 1563 8126	3·080	459 5925 6649	3·130	7 1779 7253	3·180	5 8565 5121
3·031	2 6734 6638	3·081	9 1332 0364	3·131	6 7410 1307	3·181	5 4409 0251
3·032	2 1910 3417	3·082	8 6742 9993	3·132	6 3044 9035	3·182	5 0256 6926
3·033	1 7090 8415	3·083	8 2158 5489	3·133	5 8684 0394	3·183	4 6108 5104
3·034	1 2276 1584	3·084	7 7578 6807	3·134	5 4327 5340	3·184	4 1964 4742
3·035	0 7466 2875	3·085	7 3003 3900	3·135	4 9975 3829	3·185	3 7824 5801
3·036	0 2661 2242	3·086	6 8432 6724	3·136	4 5627 5818	3·186	3 3688 8238
3·037	479 7860 9635	3·087	6 3866 5231	3·137	4 1284 1263	3·187	2 9557 2011
3·038	9 3065 5007	3·088	5 9304 9378	3·138	3 6945 0121	3·188	2 5429 7079
3·039	8 8274 8309	3·089	5 4747 9118	3·139	3 2610 2348	3·189	2 1306 3402
3·040	8 3488 9494	3·090	5 0195 4405	3·140	2 8279 7902	3·190	1 7187 0939
3·041	7 8707 8514	3·091	4 5647 5194	3·141	2 3953 6738	3·191	1 3071 9647
3·042	7 3931 5321	3·092	4 1104 1439	3·142	1 9631 8814	3·192	0 8960 9486
3·043	6 9159 9867	3·093	3 6565 3096	3·143	1 5314 4086	3·193	0 4854 0414
3·044	6 4393 2105	3·094	3 2031 0118	3·144	1 1001 2511	3·194	0 0751 2391
3·045	5 9631 1987	3·095	2 7501 2460	3·145	0 6692 4047	3·195	409 6652 5376
3·046	5 4873 9465	3·096	2 2976 0078	3·146	0 2387 8649	3·196	9 2557 9327
3·047	5 0121 4493	3·097	1 8455 2925	3·147	429 8087 6275	3·197	8 8467 4204
3·048	4 5373 7021	3·098	1 3939 0957	3·148	9 3791 6882	3·198	8 4380 9965
3·049	4 0630 7003	3·099	0 9427 4128	3·149	8 9500 0427	3·199	8 0298 6570

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
3·200	407 6220 3978	3·250	387 7420 7832	3·300	368 8316 7401	3·350	350 8435 4101
3·201	7 2146 2149	3·251	7 3545 3005	3·301	8 4630 2669	3·351	0 4928 7284
3·202	6 8076 1041	3·252	6 9673 6913	3·302	8 0947 4783	3·352	0 1425 5515
3·203	6 4010 0613	3·253	6 5805 9518	3·303	7 7268 3707	3·353	349 7925 8761
3·204	5 9948 0826	3·254	6 1942 0781	3·304	7 3592 9404	3·354	9 4429 6986
3·205	5 5890 1638	3·255	5 8082 0663	3·305	6 9921 1836	3·355	9 0937 0155
3·206	5 1836 3009	3·256	5 4225 9127	3·306	6 6253 0968	3·356	8 7447 8234
3·207	4 7786 4899	3·257	5 0373 6132	3·307	6 2588 6762	3·357	8 3962 1187
3·208	4 3740 7266	3·258	4 6525 1642	3·308	5 8927 9182	3·358	8 0479 8980
3·209	3 9699 0071	3·259	4 2680 5616	3·309	5 5270 8192	3·359	7 7001 1578
3·210	3 5661 3272	3·260	3 8839 8017	3·310	5 1617 3754	3·360	7 3525 8945
3·211	3 1627 6831	3·261	3 5002 8807	3·311	4 7967 5832	3·361	7 0054 1048
3·212	2 7598 0705	3·262	3 1169 7947	3·312	4 4321 4390	3·362	6 6585 7851
3·213	2 3572 4856	3·263	2 7340 5398	3·313	4 0678 9391	3·363	6 3120 9320
3·214	1 9550 9242	3·264	2 3515 1123	3·314	3 7040 0799	3·364	5 9659 5421
3·215	1 5533 3824	3·265	1 9693 5083	3·315	3 3404 8577	3·365	5 6201 6118
3·216	1 1519 8561	3·266	1 5875 7240	3·316	2 9773 2690	3·366	5 2747 1377
3·217	0 7510 3413	3·267	1 2061 7556	3·317	2 6145 3100	3·367	4 9296 1164
3·218	0 3504 8341	3·268	0 8251 5993	3·318	2 2520 9771	3·368	4 5848 5444
3·219	399 9503 3304	3·269	0 4445 2512	3·319	1 8900 2668	3·369	4 2404 4182
3·220	9 5505 8261	3·270	0 0642 7075	3·320	1 5283 1754	3·370	3 8963 7343
3·221	9 1512 3174	3·271	379 6843 9645	3·321	1 1669 6993	3·371	3 5526 4895
3·222	8 7522 8001	3·272	9 3049 0183	3·322	0 8059 8348	3·372	3 2092 6802
3·223	8 3537 2704	3·273	8 9257 8652	3·323	0 4453 5784	3·373	2 8662 3030
3·224	7 9555 7242	3·274	8 5470 5013	3·324	0 0850 9264	3·374	2 5235 3545
3·225	7 5578 1576	3·275	8 1686 9229	3·325	359 7251 8753	3·375	2 1811 8312
3·226	7 1604 5666	3·276	7 7907 1262	3·326	9 3656 4215	3·376	1 8391 7297
3·227	6 7634 9472	3·277	7 4131 1074	3·327	9 0064 5613	3·377	1 4975 0465
3·228	6 3669 2954	3·278	7 0358 8627	3·328	8 6476 2912	3·378	1 1561 7783
3·229	5 9707 6073	3·279	6 6590 3884	3·329	8 2891 6075	3·379	0 8151 9218
3·230	5 5749 8788	3·280	6 2825 6807	3·330	7 9310 5067	3·380	0 4745 4734
3·231	5 1796 1062	3·281	5 9064 7358	3·331	7 5732 9853	3·381	0 1342 4298
3·232	4 7846 2853	3·282	5 5307 5500	3·332	7 2159 0396	3·382	339 7942 7875
3·233	4 3900 4123	3·283	5 1554 1195	3·333	6 8588 6660	3·383	9 4546 5431
3·234	3 9958 4832	3·284	4 7804 4405	3·334	6 5021 8610	3·384	9 1153 6933
3·235	3 6020 4940	3·285	4 4058 5093	3·335	6 1458 6211	3·385	8 7764 2346
3·236	3 2086 4409	3·286	4 0316 3222	3·336	5 7898 9426	3·386	8 4378 1636
3·237	2 8156 3198	3·287	3 6577 8754	3·337	5 4342 8220	3·387	8 0995 4771
3·238	2 4230 1269	3·288	3 2843 1652	3·338	5 0790 2558	3·388	7 7616 1716
3·239	2 0307 8583	3·289	2 9112 1879	3·339	4 7241 2403	3·389	7 4240 2436
3·240	1 6389 5099	3·290	2 5384 9396	3·340	4 3695 7721	3·390	7 0867 6899
3·241	1 2475 0779	3·291	2 1661 4168	3·341	4 0153 8476	3·391	6 7498 5071
3·242	0 8564 5584	3·292	1 7941 6155	3·342	3 6615 4633	3·392	6 4132 6918
3·243	0 4657 9475	3·293	1 4225 5323	3·343	3 3080 6156	3·393	6 0770 2406
3·244	0 0755 2412	3·294	1 0513 1633	3·344	2 9549 3009	3·394	5 7411 1502
3·245	389 6856 4357	3·295	0 6804 5047	3·345	2 6021 5158	3·395	5 4055 4171
3·246	9 2961 5271	3·296	0 3099 5530	3·346	2 2497 2567	3·396	5 0703 0382
3·247	8 9070 5114	3·297	369 9398 3044	3·347	1 8976 5201	3·397	4 7354 0100
3·248	8 5183 3847	3·298	9 5700 7552	3·348	1 5459 3025	3·398	4 4008 3291
3·249	8 1300 1433	3·299	9 2006 9017	3·349	1 1945 6003	3·399	4 0665 9922

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
3·400	333 7326 9960	3·450	317 4563 6378	3·500	301 9738 3422	3·550	287 2463 9654
3·401	3 3001 3371	3·451	7 1390 6609	3·501	1 6720 1133	3·551	6 9592 9372
3·402	3 0659 0122	3·452	6 8220 8554	3·502	1 3704 9010	3·552	6 6724 7785
3·403	2 7330 0180	3·453	6 5054 2182	3·503	1 0692 7025	3·553	6 3859 4866
3·404	2 4004 3510	3·454	6 1890 7459	3·504	0 7683 5146	3·554	6 0997 0586
3·405	2 0682 0081	3·455	5 8730 4356	3·505	0 4677 3344	3·555	5 8137 4916
3·406	1 7362 9859	3·456	5 5573 2840	3·506	0 1674 1589	3·556	5 5280 7827
3·407	1 4047 2811	3·457	5 2419 2880	3·507	299 8673 9851	3·557	5 2426 9291
3·408	1 0734 8903	3·458	4 9268 4444	3·508	9 5676 8100	3·558	4 9575 9279
3·409	0 7425 8102	3·459	4 6120 7500	3·509	9 2682 6305	3·559	4 6727 7762
3·410	0 4120 0376	3·460	4 2976 2018	3·510	8 9691 4437	3·560	4 3882 4714
3·411	0 0817 5690	3·461	3 9834 7966	3·511	8 6703 2466	3·561	4 1040 0104
3·412	329 7518 4013	3·462	3 6696 5312	3·512	8 3718 0362	3·562	3 8200 3904
3·413	9 4222 5311	3·463	3 3561 4025	3·513	8 0735 8095	3·563	3 5363 6086
3·414	9 0929 9552	3·464	3 0429 4073	3·514	7 7756 5636	3·564	3 2529 6622
3·415	8 7640 6701	3·465	2 7300 5426	3·515	7 4780 2954	3·565	2 9698 5484
3·416	8 4354 6727	3·466	2 4174 8052	3·516	7 1807 0020	3·566	2 6870 2642
3·417	8 1071 9597	3·467	2 1052 1920	3·517	6 8836 6804	3·567	2 4044 8069
3·418	7 7792 5277	3·468	1 7932 6998	3·518	6 5869 3277	3·568	2 1222 1737
3·419	7 4516 3735	3·469	1 4816 3256	3·519	6 2904 9408	3·569	1 8402 3616
3·420	7 1243 4939	3·470	1 1703 0661	3·520	5 9943 5168	3·570	1 5585 3680
3·421	6 7973 8854	3·471	0 8592 9184	3·521	5 6985 0528	3·571	1 2771 1899
3·422	6 4707 5450	3·472	0 5485 8792	3·522	5 4029 5457	3·572	0 9959 8247
3·423	6 1444 4693	3·473	0 2381 9456	3·523	5 1076 9927	3·573	0 7151 2694
3·424	5 8184 6550	3·474	309 9281 1143	3·524	4 8127 3908	3·574	0 4345 5212
3·425	325 4928 0989	3·475	9 6183 3823	3·525	294 5180 7369	3·575	0 1542 5774
3·426	5 1674 7977	3·476	9 3088 7465	3·526	4 2237 0283	3·576	279 8742 4351
3·427	4 8424 7482	3·477	8 9997 2038	3·527	3 9296 2619	3·577	9 5945 0916
3·428	4 5177 9471	3·478	8 6908 7511	3·528	3 6358 4348	3·578	9 3150 5440
3·429	4 1934 3912	3·479	8 3823 3853	3·529	3 3423 5440	3·579	9 0358 7896
3·430	3 8694 0773	3·480	8 0741 1033	3·530	3 0491 5867	3·580	8 7569 8255
3·431	3 5457 0020	3·481	7 7661 9020	3·531	2 7562 5599	3·581	8 4783 6490
3·432	3 2223 1622	3·482	7 4585 7785	3·532	2 4636 4607	3·582	8 2000 2573
3·433	2 8992 5546	3·483	7 1512 7295	3·533	2 1713 2860	3·583	7 9219 6476
3·434	2 5765 1760	3·484	6 8442 7520	3·534	1 8793 0331	3·584	7 6441 8171
3·435	2 2541 0232	3·485	6 5375 8429	3·535	1 5875 6990	3·585	7 3666 7630
3·436	1 9320 0929	3·486	6 2311 9993	3·536	1 2961 2807	3·586	7 0894 4826
3·437	1 6102 3819	3·487	5 9251 2179	3·537	1 0049 7754	3·587	6 8124 9731
3·438	1 2887 8871	3·488	5 6193 4958	3·538	0 7141 1802	3·588	6 5358 2317
3·439	0 9676 6051	3·489	5 3138 8299	3·539	0 4235 4921	3·589	6 2594 2557
3·440	0 6468 5328	3·490	5 0087 2171	3·540	0 1332 7082	3·590	5 9833 0423
3·441	0 3263 6670	3·491	4 7038 6544	3·541	289 8432 8257	3·591	5 7074 5887
3·442	0 0062 0044	3·492	4 3993 1388	3·542	9 5535 8416	3·592	5 4318 8922
3·443	319 6863 5419	3·493	4 0950 6671	3·543	9 2641 7530	3·593	5 1565 9500
3·444	9 3668 2762	3·494	3 7911 2364	3·544	8 9750 5571	3·594	4 8815 7594
3·445	9 0476 2043	3·495	3 4874 8436	3·545	8 6862 2509	3·595	4 6068 3176
3·446	8 7287 3228	3·496	3 1841 4857	3·546	8 3976 8316	3·596	4 3323 6218
3·447	8 4101 6286	3·497	2 8811 1597	3·547	8 1094 2963	3·597	4 0581 6694
3·448	8 0919 1185	3·498	2 5783 8624	3·548	7 8214 6421	3·598	3 7842 4576
3·449	7 7739 7893	3·499	2 2759 5909	3·549	7 5337 8661	3·599	3 5105 9836

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
3·600	273 2372 2447	3·650	259 9112 8779	3·700	247 2352 6470	3·750	235 1774 5856
3·601	2 9641 2382	3·651	9 6515 0642	3·701	6 9881 5301	3·751	4 9423 9865
3·602	2 6912 9613	3·652	9 3919 8469	3·702	6 7412 8831	3·752	4 7075 7368
3·603	2 4187 4114	3·653	9 1327 2236	3·703	6 4946 7036	3·753	4 4729 8342
3·604	2 1464 5856	3·654	8 8737 1916	3·704	6 2482 9889	3·754	4 2386 2764
3·605	1 8744 4813	3·655	8 6149 7483	3·705	6 0021 7368	3·755	4 0045 0609
3·606	1 6027 0957	3·656	8 3564 8912	3·706	5 7562 9446	3·756	3 7706 1855
3·607	1 3312 4262	3·657	8 0982 6177	3·707	5 5106 6101	3·757	3 5369 6478
3·608	1 0600 4700	3·658	7 8402 9252	3·708	5 2652 7306	3·758	3 3035 4454
3·609	0 7891 2244	3·659	7 5825 8110	3·709	5 0201 3038	3·759	3 0703 5761
3·610	0 5184 6866	3·660	7 3251 2726	3·710	4 7752 3272	3·760	2 8374 0375
3·611	0 2480 8541	3·661	7 0679 3076	3·711	4 5305 7983	3·761	2 6046 8273
3·612	269 9779 7240	3·662	6 8109 9132	3·712	4 2861 7147	3·762	2 3721 9431
3·613	9 7081 2937	3·663	6 5543 0869	3·713	4 0420 0741	3·763	2 1399 3826
3·614	9 4385 5605	3·664	6 2978 8261	3·714	3 7980 8738	3·764	1 9079 1435
3·615	9 1692 5217	3·665	6 0417 1284	3·715	3 5544 1115	3·765	1 6761 2235
3·616	8 9002 1746	3·666	5 7857 9910	3·716	3 3109 7848	3·766	1 4445 6203
3·617	8 6314 5165	3·667	5 5301 4115	3·717	3 0677 8912	3·767	1 2132 3315
3·618	8 3629 5447	3·668	5 2747 3873	3·718	2 8248 4282	3·768	0 9821 3549
3·619	8 0947 2565	3·669	5 0195 9159	3·719	2 5821 3935	3·769	0 7512 6881
3·620	7 8267 6493	3·670	4 7646 9947	3·720	2 3396 7846	3·770	0 5206 3287
3·621	7 5590 7203	3·671	4 5100 6211	3·721	2 0974 5991	3·771	0 2902 2746
3·622	7 2916 4669	3·672	4 2556 7926	3·722	1 8554 8346	3·772	0 0600 5234
3·623	7 0244 8865	3·673	4 0015 5066	3·723	1 6137 4887	3·773	229 8301 0728
3·624	6 7575 9763	3·674	3 7476 7607	3·724	1 3722 5588	3·774	9 6003 9205
3·625	266 4909 7336	3·675	253 4940 5523	3·725	1 1310 0427	3·775	9 3709 0642
3·626	6 2246 1559	3·676	3 2406 8787	3·726	0 8899 9380	3·776	9 1416 5016
3·627	5 9585 2405	3·677	2 9875 7377	3·727	0 6492 2421	3·777	8 9126 2305
3·628	5 6926 9846	3·678	2 7347 1264	3·728	0 4086 9527	3·778	8 6838 2484
3·629	5 4271 3856	3·679	2 4821 0425	3·729	0 1684 0673	3·779	8 4552 5532
3·630	5 1618 4409	3·680	2 2297 4835	3·730	239 9283 5837	3·780	8 2269 1426
3·631	4 8968 1479	3·681	1 9776 4467	3·731	9 6885 4994	3·781	7 9988 0142
3·632	4 6320 5037	3·682	1 7257 9298	3·732	9 4489 8119	3·782	7 7709 1658
3·633	4 3675 5060	3·683	1 4741 9300	3·733	9 2096 5190	3·783	7 5432 5951
3·634	4 1033 1519	3·684	1 2228 4451	3·734	8 9705 6181	3·784	7 3158 2998
3·635	3 8393 4388	3·685	0 9717 4723	3·735	8 7317 1069	3·785	7 0886 2777
3·636	3 5756 3641	3·686	0 7209 0093	3·736	8 4930 9831	3·786	6 8616 5265
3·637	3 3121 9252	3·687	0 4703 0534	3·737	8 2547 2441	3·787	6 6349 0439
3·638	3 0490 1194	3·688	0 2199 6023	3·738	8 0165 8878	3·788	6 4083 8276
3·639	2 7860 9441	3·689	249 9698 6534	3·739	7 7786 9116	3·789	6 1820 8755
3·640	2 5234 3966	3·690	9 7200 2042	3·740	7 5410 3131	3·790	5 9560 1851
3·641	2 2610 4744	3·691	9 4704 2522	3·741	7 3036 0901	3·791	5 7301 7544
3·642	1 9989 1748	3·692	9 2210 7949	3·742	7 0664 2402	3·792	5 5045 5809
3·643	1 7370 4952	3·693	8 9719 8298	3·743	6 8294 7609	3·793	5 2791 6624
3·644	1 4754 4329	3·694	8 7231 3544	3·744	6 5927 6499	3·794	5 0539 9968
3·645	1 2140 9854	3·695	8 4745 3662	3·745	6 3562 9048	3·795	4 8290 5817
3·646	0 9530 1501	3·696	8 2261 8628	3·746	6 1200 5233	3·796	4 6043 4149
3·647	0 6921 9243	3·697	7 9780 8417	3·747	5 8840 5030	3·797	4 3798 4941
3·648	0 4316 3054	3·698	7 7302 3003	3·748	5 6482 8415	3·798	4 1555 8171
3·649	0 1713 2908	3·699	7 4826 2363	3·749	5 4127 5365	3·799	3 9315 3817

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
3·800	223 7077 1856	3·850	212 7973 6438	3·900	202 4191 1446	3·950	192 5470 1775
3·801	3 4841 2266	3·851	2 5846 7338	3·901	2 2167 9652	3·951	2 3545 6698
3·802	3 2607 5024	3·852	2 3721 9496	3·902	2 0146 8080	3·952	2 1623 0855
3·803	3 0376 0109	3·853	2 1599 2892	3·903	1 8127 6709	3·953	1 9702 4229
3·804	2 8146 7497	3·854	1 9478 7504	3·904	1 6110 5519	3·954	1 7783 6801
3·805	2 5919 7166	3·855	1 7360 3310	3·905	1 4095 4491	3·955	1 5866 8549
3·806	2 3694 9095	3·856	1 5244 0290	3·906	1 2082 3604	3·956	1 3951 9457
3·807	2 1472 3261	3·857	1 3129 8422	3·907	1 0071 2837	3·957	1 2038 9504
3·808	1 9251 9641	3·858	1 1017 7686	3·908	0 8062 2172	3·958	1 0127 8672
3·809	1 7033 8214	3·859	0 8907 8060	3·909	0 6055 1586	3·959	0 8218 6940
3·810	1 4817 8957	3·860	0 6799 9523	3·910	0 4050 1062	3·960	0 6311 4291
3·811	1 2604 1849	3·861	0 4694 2054	3·911	0 2047 0578	3·961	0 4406 0705
3·812	1 0392 6866	3·862	0 2590 5632	3·912	0 0046 0114	3·962	0 2502 6164
3·813	0 8183 3987	3·863	0 0489 0236	3·913	199 8046 9651	3·963	0 0601 0647
3·814	0 5976 3190	3·864	209 8389 5845	3·914	9 6049 9168	3·964	189 8701 4136
3·815	0 3771 4453	3·865	9 6292 2437	3·915	9 4054 8645	3·965	9 6803 6612
3·816	0 1568 7754	3·866	9 4196 9993	3·916	9 2061 8064	3·966	9 4907 8057
3·817	219 9368 3070	3·867	9 2103 8490	3·917	9 0070 7403	3·967	9 3013 8450
3·818	9 7170 0381	3·868	9 0012 7909	3·918	8 8081 6642	3·968	9 1121 7774
3·819	9 4973 9662	3·869	8 7923 8227	3·919	8 6094 5763	3·969	8 9231 6008
3·820	9 2780 0894	3·870	8 5836 9425	3·920	8 4109 4744	3·970	8 7343 3135
3·821	9 0588 4053	3·871	8 3752 1481	3·921	8 2126 3567	3·971	8 5456 9136
3·822	8 8398 9119	3·872	8 1669 4375	3·922	8 0145 2211	3·972	8 3572 3991
3·823	8 6211 6068	3·873	7 9588 8085	3·923	7 8166 0655	3·973	8 1689 7682
3·824	8 4026 4879	3·874	7 7510 2592	3·924	7 6188 8882	3·974	7 9809 0189
3·825	218 1843 5531	3·875	207 5433 7873	3·925	197 4213 6871	3·975	187 7930 1495
3·826	7 9662 8001	3·876	7 3359 3909	3·926	7 2240 4602	3·976	7 6053 1580
3·827	7 7484 2268	3·877	7 1287 0679	3·927	7 0269 2055	3·977	7 4178 0425
3·828	7 5307 8309	3·878	6 9216 8161	3·928	6 8299 9211	3·978	7 2304 8013
3·829	7 3133 6104	3·879	6 7148 6336	3·929	6 6332 6050	3·979	7 0433 4323
3·830	7 0961 5630	3·880	6 5082 5182	3·930	6 4367 2553	3·980	6 8563 9338
3·831	6 8791 6865	3·881	6 3018 4678	3·931	6 2403 8699	3·981	6 6696 3038
3·832	6 6623 9789	3·882	6 0956 4805	3·932	6 0442 4469	3·982	6 4830 5405
3·833	6 4458 4378	3·883	5 8896 5542	3·933	5 8482 9843	3·983	6 2966 6421
3·834	6 2295 0613	3·884	5 6838 6868	3·934	5 6525 4803	3·984	6 1104 6066
3·835	6 0133 8470	3·885	5 4782 8762	3·935	5 4569 9327	3·985	5 9244 4323
3·836	5 7974 7929	3·886	5 2729 1203	3·936	5 2616 3398	3·986	5 7386 1171
3·837	5 5817 8967	3·887	5 0677 4172	3·937	5 0664 6994	3·987	5 5529 6594
3·838	5 3663 1563	3·888	4 8627 7648	3·938	4 8715 0097	3·988	5 3675 0572
3·839	5 1510 5696	3·889	4 6580 1610	3·939	4 6767 2687	3·989	5 1822 3087
3·840	4 9360 1345	3·890	4 4534 6038	3·940	4 4821 4745	3·990	4 9971 4120
3·841	4 7211 8486	3·891	4 2491 0911	3·941	4 2877 6251	3·991	4 8122 3653
3·842	4 5065 7100	3·892	4 0449 6209	3·942	4 0935 7186	3·992	4 6275 1667
3·843	4 2921 7165	3·893	3 8410 1912	3·943	3 8995 7530	3·993	4 4429 8143
3·844	4 0779 8659	3·894	3 6372 7999	3·944	3 7057 7265	3·994	4 2586 3064
3·845	3 8640 1561	3·895	3 4337 4449	3·945	3 5121 6369	3·995	4 0744 6411
3·846	3 6502 5849	3·896	3 2304 1243	3·946	3 3187 4826	3·996	3 8904 8165
3·847	3 4367 1502	3·897	3 0272 8360	3·947	3 1255 2613	3·997	3 7066 8308
3·848	3 2233 8499	3·898	2 8243 5780	3·948	2 9324 9714	3·998	3 5230 6822
3·849	3 0102 6818	3·899	2 6216 3482	3·949	2 7396 6108	3·999	3 3396 3689

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
4·000	183 1563 8889	4·050	174 2237 4639	4·100	165 7267 5402	4·150	157 6441 6485
4·001	2 9733 2405	4·051	4 0496 0973	4·101	5 5611 1010	4·151	7 4865 9948
4·002	2 7904 4218	4·052	3 8756 4712	4·102	5 3956 3174	4·152	7 3291 9160
4·003	2 6077 4310	4·053	3 7018 5838	4·103	5 2303 1878	4·153	7 1719 4104
4·004	2 4252 2663	4·054	3 5282 4334	4·104	5 0651 7105	4·154	7 0148 4766
4·005	2 2428 9259	4·055	3 3548 0183	4·105	4 9001 8838	4·155	6 8579 1130
4·006	2 0607 4079	4·056	3 1815 3368	4·106	4 7353 7062	4·156	6 7011 3179
4·007	1 8787 7105	4·057	3 0084 3871	4·107	4 5707 1759	4·157	6 5445 0898
4·008	1 6969 8319	4·058	2 8355 1674	4·108	4 4062 2913	4·158	6 3880 4272
4·009	1 5153 7702	4·059	2 6627 6762	4·109	4 2419 0507	4·159	6 2317 3284
4·010	1 3339 5237	4·060	2 4901 9115	4·110	4 0777 4526	4·160	6 0755 7920
4·011	1 1527 0905	4·061	2 3177 8718	4·111	3 9137 4953	4·161	5 9195 8163
4·012	0 9716 4690	4·062	2 1455 5552	4·112	3 7499 1771	4·162	5 7637 3998
4·013	0 7907 6570	4·063	1 9734 9601	4·113	3 5862 4964	4·163	5 6080 5410
4·014	0 6100 6530	4·064	1 8016 0847	4·114	3 4227 4515	4·164	5 4525 2382
4·015	0 4295 4551	4·065	1 6298 9273	4·115	3 2594 0409	4·165	5 2971 4900
4·016	0 2492 0615	4·066	1 4583 4863	4·116	3 0962 2629	4·166	5 1419 2947
4·017	0 0690 4704	4·067	1 2869 7598	4·117	2 9332 1159	4·167	4 9868 6509
4·018	179 8890 6799	4·068	1 1157 7462	4·118	2 7703 5981	4·168	4 8319 5569
4·019	9 7092 6884	4·069	0 9447 4437	4·119	2 6076 7081	4·169	4 6772 0113
4·020	9 5296 4939	4·070	0 7738 8507	4·120	2 4451 4442	4·170	4 5226 0124
4·021	9 3502 0948	4·071	0 6031 9655	4·121	2 2827 8047	4·171	4 3681 5587
4·022	9 1709 4891	4·072	0 4326 7862	4·122	2 1205 7880	4·172	4 2138 6488
4·023	8 9918 6752	4·073	0 2623 3113	4·123	1 9585 3926	4·173	4 0597 2809
4·024	8 8129 6512	4·074	0 0921 5390	4·124	1 7966 6167	4·174	3 9057 4537
4·025	178 6342 4153	4·075	169 9221 4677	4·125	161 6349 4588	4·175	153 7519 1655
4·026	8 4556 9658	4·076	9 7523 0956	4·126	1 4733 9173	4·176	3 5982 4148
4·027	8 2773 3008	4·077	9 5826 4209	4·127	1 3119 9904	4·177	3 4447 2002
4·028	8 0991 4186	4·078	9 4131 4421	4·128	1 1507 6767	4·178	3 2913 5199
4·029	7 9211 3174	4·079	9 2438 1575	4·129	0 9896 9745	4·179	3 1381 3726
4·030	7 7432 9954	4·080	9 0746 5653	4·130	0 8287 8823	4·180	2 9850 7567
4·031	7 5656 4508	4·081	8 9056 6638	4·131	0 6680 3982	4·181	2 8321 6706
4·032	7 3881 6819	4·082	8 7368 4513	4·132	0 5074 5209	4·182	2 6794 1128
4·033	7 2108 6868	4·083	8 5681 9263	4·133	0 3470 2487	4·183	2 5268 0818
4·034	7 0337 4639	4·084	8 3997 0869	4·134	0 1867 5799	4·184	2 3743 5761
4·035	6 8568 0113	4·085	8 2313 9315	4·135	0 0266 5130	4·185	2 2220 5942
4·036	6 6800 3273	4·086	8 0632 4585	4·136	159 8667 0463	4·186	2 0699 1344
4·037	6 5034 4101	4·087	7 8952 6661	4·137	9 7069 1784	4·187	1 9179 1954
4·038	6 3270 2579	4·088	7 7274 5526	4·138	9 5472 9074	4·188	1 7660 7755
4·039	6 1507 8690	4·089	7 5598 1164	4·139	9 3878 2320	4·189	1 6143 8733
4·040	5 9747 2416	4·090	7 3923 3558	4·140	9 2285 1504	4·190	1 4628 4873
4·041	5 7988 3739	4·091	7 2250 2691	4·141	9 0693 6612	4·191	1 3114 6159
4·042	5 6231 2642	4·092	7 0578 8547	4·142	8 9103 7626	4·192	1 1602 2576
4·043	5 4475 9108	4·093	6 8909 1109	4·143	8 7515 4531	4·193	1 0091 4109
4·044	5 2722 3118	4·094	6 7241 0360	4·144	8 5928 7312	4·194	0 8582 0742
4·045	5 0970 4656	4·095	6 5574 6283	4·145	8 4343 5951	4·195	0 7074 2462
4·046	4 9220 3703	4·096	6 3909 8862	4·146	8 2760 0434	4·196	0 5567 9252
4·047	4 7472 0243	4·097	6 2246 8080	4·147	8 1178 0745	4·197	0 4063 1098
4·048	4 5725 4257	4·098	6 0585 3920	4·148	7 9597 6868	4·198	0 2559 7985
4·049	4 3980 5728	4·099	5 8925 6366	4·149	7 8018 8786	4·199	0 1057 9898

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
4·200	149 9557 6820	4·250	142 6423 3909	4·300	135 6855 9012	4·350	129 0681 2580
4·201	9 8058 8739	4·251	2 4997 6805	4·301	5 5499 7235	4·351	8 9391 2219
4·202	9 6561 5638	4·252	2 3573 3951	4·302	5 4144 9013	4·352	8 8102 4752
4·203	9 5065 7502	4·253	2 2150 5332	4·303	5 2791 4333	4·353	8 6815 0165
4·204	9 3571 4318	4·254	2 0729 0935	4·304	5 1439 3180	4·354	8 5528 8447
4·205	9 2078 6069	4·255	1 9309 0745	4·305	5 0088 5542	4·355	8 4243 9584
4·206	9 0587 2741	4·256	1 7890 4749	4·306	4 8739 1404	4·356	8 2960 3564
4·207	8 9097 4319	4·257	1 6473 2931	4·307	4 7391 0754	4·357	8 1678 0373
4·208	8 7609 0787	4·258	1 5057 5278	4·308	4 6044 3578	4·358	8 0396 9999
4·209	8 6122 2132	4·259	1 3643 1776	4·309	4 4698 9863	4·359	7 9117 2428
4·210	8 4636 8338	4·260	1 2230 2410	4·310	4 3354 9594	4·360	7 7838 7649
4·211	8 3152 9390	4·261	1 0818 7166	4·311	4 2012 2759	4·361	7 6561 5649
4·212	8 1670 5274	4·262	0 9408 6031	4·312	4 0670 9344	4·362	7 5285 6414
4·213	8 0189 5975	4·263	0 7999 8990	4·313	3 9330 9336	4·363	7 4010 9932
4·214	7 8710 1477	4·264	0 6592 6028	4·314	3 7992 2721	4·364	7 2737 6190
4·215	7 7232 1767	4·265	0 5186 7133	4·315	3 6654 9486	4·365	7 1465 5175
4·216	7 5755 6829	4·266	0 3782 2289	4·316	3 5318 9618	4·366	7 0194 6875
4·217	7 4280 6648	4·267	0 2379 1484	4·317	3 3984 3102	4·367	6 8925 1277
4·218	7 2807 1211	4·268	0 0977 4702	4·318	3 2650 9927	4·368	6 7656 8368
4·219	7 1335 0501	4·269	139 9577 1930	4·319	3 1319 0078	4·369	6 6389 8136
4·220	6 9864 4505	4·270	9 8178 3153	4·320	2 9988 3542	4·370	6 5124 0568
4·221	6 8395 3207	4·271	9 6780 8359	4·321	2 8659 0307	4·371	6 3859 5651
4·222	6 6927 6593	4·272	9 5384 7532	4·322	2 7331 0357	4·372	6 2596 3372
4·223	6 5461 4649	4·273	9 3990 0659	4·323	2 6004 3681	4·373	6 1334 3720
4·224	6 3996 7359	4·274	9 2596 7726	4·324	2 4679 0265	4·374	6 0073 6681
4·225	146 2533 4709	4·275	139 1204 8719	4·325	132 3355 0096	4·375	125 8814 2242
4·226	6 1071 6685	4·276	8 9814 3624	4·326	2 2032 3160	4·376	5 7556 0392
4·227	5 9611 3271	4·277	8 8425 2427	4·327	2 0710 9446	4·377	5 6299 1117
4·228	5 8152 4454	4·278	8 7037 5114	4·328	1 9390 8938	4·378	5 5043 4406
4·229	5 6695 0217	4·279	8 5651 1672	4·329	1 8072 1623	4·379	5 3789 0244
4·230	5 5239 0548	4·280	8 4266 2086	4·330	1 6754 7490	4·380	5 2535 8621
4·231	5 3784 5432	4·281	8 2882 6343	4·331	1 5438 6524	4·381	5 1283 9523
4·232	5 2331 4853	4·282	8 1500 4429	4·332	1 4123 8713	4·382	5 0033 2938
4·233	5 0879 8797	4·283	8 0119 6330	4·333	1 2810 4042	4·383	4 8783 8853
4·234	4 9429 7250	4·284	7 8740 2032	4·334	1 1498 2500	4·384	4 7535 7256
4·235	4 7981 0198	4·285	7 7362 1521	4·335	1 0187 4073	4·385	4 6288 8134
4·236	4 6533 7625	4·286	7 5985 4784	4·336	0 8877 8748	4·386	4 5043 1475
4·237	4 5087 9518	4·287	7 4610 1807	4·337	0 7569 6511	4·387	4 3798 7267
4·238	4 3643 5861	4·288	7 3236 2576	4·338	0 6262 7350	4·388	4 2555 5497
4·239	4 2200 6641	4·289	7 1863 7077	4·339	0 4957 1252	4·389	4 1313 6152
4·240	4 0759 1843	4·290	7 0492 5297	4·340	0 3652 8203	4·390	4 0072 9220
4·241	3 9319 1453	4·291	6 9122 7222	4·341	0 2349 8191	4·391	3 8833 4689
4·242	3 7880 5455	4·292	6 7754 2838	4·342	0 1048 1203	4·392	3 7595 2547
4·243	3 6443 3837	4·293	6 6387 2132	4·343	129 9747 7224	4·393	3 6358 2780
4·244	3 5007 6583	4·294	6 5021 5089	4·344	9 8448 6244	4·394	3 5122 5377
4·245	3 3573 3679	4·295	6 3657 1697	4·345	9 7150 8248	4·395	3 3888 0325
4·246	3 2140 5111	4·296	6 2294 1941	4·346	9 5854 3223	4·396	3 2654 7612
4·247	3 0709 0864	4·297	6 0932 5809	4·347	9 4559 1157	4·397	3 1422 7226
4·248	2 9279 0924	4·298	5 9572 3285	4·348	9 3265 2036	4·398	3 0191 9154
4·249	2 7850 5277	4·299	5 8213 4358	4·349	9 1972 5849	4·399	2 8962 3384

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
4·400	122 7733 9903	4·450	116 7856 6970	4·500	111 0899 6538	4·550	105 6720 4384
4·401	2 6506 8700	4·451	6 6689 4241	4·501	0 9789 3094	4·551	5 5664 2461
4·402	2 5280 9761	4·452	6 5523 3178	4·502	0 8680 0748	4·552	5 4609 1095
4·403	2 4056 3076	4·453	6 4358 3770	4·503	0 7571 9489	4·553	5 3555 0275
4·404	2 2832 8631	4·454	6 3194 6006	4·504	0 6464 9306	4·554	5 2501 9991
4·405	2 1610 6415	4·455	6 2031 9874	4·505	0 5359 0187	4·555	5 1450 0232
4·406	2 0389 6414	4·456	6 0870 5363	4·506	0 4254 2122	4·556	5 0399 0987
4·407	1 9169 8618	4·457	5 9710 2460	4·507	0 3150 5099	4·557	4 9349 2247
4·408	1 7951 3013	4·458	5 8551 1154	4·508	0 2047 9108	4·558	4 8300 3999
4·409	1 6733 9588	4·459	5 7393 1434	4·509	0 0946 4137	4·559	4 7252 6235
4·410	1 5517 8330	4·460	5 6236 3287	4·510	109 9846 0176	4·560	4 6205 8943
4·411	1 4302 9227	4·461	5 5080 6703	4·511	9 8746 7213	4·561	4 5160 2114
4·412	1 3089 2267	4·462	5 3926 1670	4·512	9 7648 5238	4·562	4 4115 5736
4·413	1 1876 7438	4·463	5 2772 8176	4·513	9 6551 4239	4·563	4 3071 9799
4·414	1 0665 4728	4·464	5 1620 6210	4·514	9 5455 4205	4·564	4 2029 4293
4·415	0 9455 4125	4·465	5 0469 5760	4·515	9 4360 5127	4·565	4 0987 9207
4·416	0 8246 5616	4·466	4 9319 6815	4·516	9 3266 6992	4·566	3 9947 4531
4·417	0 7038 9190	4·467	4 8170 9362	4·517	9 2173 9789	4·567	3 8908 0254
4·418	0 5832 4834	4·468	4 7023 3392	4·518	9 1082 3508	4·568	3 7869 6367
4·419	0 4627 2536	4·469	4 5876 8892	4·519	8 9991 8138	4·569	3 6832 2858
4·420	0 3423 2285	4·470	4 4731 5850	4·520	8 8902 3668	4·570	3 5795 9718
4·421	0 2220 4067	4·471	4 3587 4256	4·521	8 7814 0087	4·571	3 4760 6935
4·422	0 1018 7872	4·472	4 2444 4098	4·522	8 6726 7385	4·572	3 3726 4500
4·423	119 9818 3688	4·473	4 1302 5364	4·523	8 5640 5549	4·573	3 2693 2403
4·424	9 8619 1501	4·474	4 0161 8044	4·524	8 4555 4570	4·574	3 1661 0632
4·425	9 7421 1301	4·475	113 9022 2124	4·525	108 3471 4436	4·575	103 0629 9178
4·426	9 6224 3075	4·476	3 7883 7595	4·526	8 2388 5137	4·576	2 9599 8030
4·427	9 5028 6811	4·477	3 6746 4445	4·527	8 1306 6662	4·577	2 8570 7178
4·428	9 3834 2497	4·478	3 5610 2663	4·528	8 0225 9001	4·578	2 7542 6612
4·429	9 2641 0122	4·479	3 4475 2236	4·529	7 9146 2141	4·579	2 6515 6322
4·430	9 1448 9673	4·480	3 3341 3155	4·530	7 8067 6073	4·580	2 5489 6296
4·431	9 0258 1138	4·481	3 2208 5406	4·531	7 6990 0785	4·581	2 4464 6526
4·432	8 9068 4506	4·482	3 1076 8980	4·532	7 5913 6267	4·582	2 3440 7000
4·433	8 7879 9765	4·483	2 9946 3864	4·533	7 4838 2509	4·583	2 2417 7708
4·434	8 6692 6903	4·484	2 8817 0048	4·534	7 3763 9499	4·584	2 1395 8641
4·435	8 5506 5907	4·485	2 7688 7521	4·535	7 2690 7226	4·585	2 0374 9788
4·436	8 4321 6767	4·486	2 6561 6270	4·536	7 1618 5681	4·586	1 9355 1138
4·437	8 3137 9470	4·487	2 5435 6284	4·537	7 0547 4851	4·587	1 8336 2682
4·438	8 1955 4004	4·488	2 4310 7553	4·538	6 9477 4728	4·588	1 7318 4409
4·439	8 0774 0358	4·489	2 3187 0065	4·539	6 8408 5298	4·589	1 6301 6310
4·440	7 9593 8520	4·490	2 2064 3809	4·540	6 7340 6553	4·590	1 5285 8373
4·441	7 8414 8477	4·491	2 0942 8774	4·541	6 6273 8482	4·591	1 4271 0590
4·442	7 7237 0219	4·492	1 9822 4948	4·542	6 5208 1073	4·592	1 3257 2949
4·443	7 6060 3733	4·493	1 8703 2320	4·543	6 4143 4316	4·593	1 2244 5440
4·444	7 4884 9007	4·494	1 7585 0880	4·544	6 3079 8201	4·594	1 1232 8054
4·445	7 3710 6031	4·495	1 6468 0615	4·545	6 2017 2716	4·595	1 0222 0781
4·446	7 2537 4791	4·496	1 5352 1515	4·546	6 0955 7852	4·596	0 9212 3610
4·447	7 1365 5277	4·497	1 4237 3568	4·547	5 9895 3597	4·597	0 8203 6530
4·448	7 0194 7477	4·498	1 3123 6764	4·548	5 8835 9941	4·598	0 7195 9533
4·449	6 9025 1379	4·499	1 2011 1091	4·549	5 7777 6873	4·599	0 6189 2608

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
4·600	100 5183 5745	4·650	95 6160 1930	4·700	90 9527 7102	4·750	86 5169 5203
4·601	0 4178 8933	4·651	5 5204 5108	4·701	0 8618 6371	4·751	6 4304 7832
4·602	0 3175 2163	4·652	5 4249 7837	4·702	0 7710 4726	4·752	6 3440 9104
4·603	0 2172 5425	4·653	5 3296 0109	4·703	0 6803 2158	4·753	6 2577 9011
4·604	0 1170 8709	4·654	5 2343 1914	4·704	0 5896 8658	4·754	6 1715 7544
4·605	0 0170 2005	4·655	5 1391 3242	4·705	0 4991 4218	4·755	6 0854 4693
4·606	99 9170 5302	4·656	5 0440 4084	4·706	0 4086 8827	4·756	5 9994 0451
4·607	9 8171 8591	4·657	4 9490 4431	4·707	0 3183 2477	4·757	5 9134 4809
4·608	9 7174 1861	4·658	4 8541 4272	4·708	0 2280 5159	4·758	5 8275 7759
4·609	9 6177 5104	4·659	4 7593 3599	4·709	0 1378 6864	4·759	5 7417 9291
4·610	9 5181 8308	4·660	4 6646 2402	4·710	0 0477 7582	4·760	5 6560 9397
4·611	9 4187 1464	4·661	4 5700 0671	4·711	89 9577 7306	4·761	5 5704 8069
4·612	9 3193 4562	4·662	4 4754 8397	4·712	9 8678 6025	4·762	5 4849 5298
4·613	9 2200 7591	4·663	4 3810 5571	4·713	9 7780 3731	4·763	5 3995 1076
4·614	9 1209 0543	4·664	4 2867 2183	4·714	9 6883 0414	4·764	5 3141 5393
4·615	9 0218 3407	4·665	4 1924 8223	4·715	9 5986 6067	4·765	5 2288 8242
4·616	8 9228 6173	4·666	4 0983 3683	4·716	9 5091 0679	4·766	5 1436 9614
4·617	8 8239 8831	4·667	4 0042 8553	4·717	9 4196 4242	4·767	5 0585 9500
4·618	8 7252 1372	4·668	3 9103 2823	4·718	9 3302 6748	4·768	4 9735 7892
4·619	8 6265 3785	4·669	3 8164 6484	4·719	9 2409 8186	4·769	4 8886 4782
4·620	8 5279 6061	4·670	3 7226 9527	4·720	9 1517 8548	4·770	4 8038 0160
4·621	8 4294 8190	4·671	3 6290 1942	4·721	9 0626 7826	4·771	4 7190 4018
4·622	8 3311 0162	4·672	3 5354 3720	4·722	8 9736 6010	4·772	4 6343 6349
4·623	8 2328 1966	4·673	3 4419 4851	4·723	8 8847 3091	4·773	4 5497 7143
4·624	8 1346 3594	4·674	3 3485 5327	4·724	8 7958 9061	4·774	4 4652 6392
4·625	98 0365 5036	4·675	93 2552 5138	4·725	88 7071 3910	4·775	84 3808 4087
4·626	7 9385 6281	4·676	3 1620 4274	4·726	8 6184 7630	4·776	4 2965 0221
4·627	7 8406 7320	4·677	3 0689 2726	4·727	8 5299 0212	4·777	4 2122 4784
4·628	7 7428 8143	4·678	2 9759 0485	4·728	8 4414 1646	4·778	4 1280 7768
4·629	7 6451 8740	4·679	2 8829 7542	4·729	8 3530 1925	4·779	4 0439 9166
4·630	7 5475 9102	4·680	2 7901 3887	4·730	8 2647 1040	4·780	3 9599 8967
4·631	7 4500 9219	4·681	2 6973 9511	4·731	8 1764 8980	4·781	3 8760 7165
4·632	7 3526 9081	4·682	2 6047 4405	4·732	8 0883 5739	4·782	3 7922 3750
4·633	7 2553 8678	4·683	2 5121 8559	4·733	8 0003 1306	4·783	3 7084 8715
4·634	7 1581 8000	4·684	2 4197 1965	4·734	7 9123 5673	4·784	3 6248 2050
4·635	7 0610 7038	4·685	2 3273 4612	4·735	7 8244 8832	4·785	3 5412 3748
4·636	6 9640 5783	4·686	2 2350 6492	4·736	7 7367 0773	4·786	3 4577 3800
4·637	6 8671 4223	4·687	2 1428 7596	4·737	7 6490 1487	4·787	3 3743 2197
4·638	6 7703 2351	4·688	2 0507 7914	4·738	7 5614 0967	4·788	3 2909 8933
4·639	6 6736 0156	4·689	1 9587 7437	4·739	7 4738 9202	4·789	3 2077 3997
4·640	6 5769 7627	4·690	1 8668 6156	4·740	7 3864 6185	4·790	3 1245 7382
4·641	6 4804 4757	4·691	1 7750 4062	4·741	7 2991 1907	4·791	3 0414 9079
4·642	6 3840 1535	4·692	1 6833 1145	4·742	7 2118 6359	4·792	2 9584 9081
4·643	6 2876 7951	4·693	1 5916 7397	4·743	7 1246 9531	4·793	2 8755 7378
4·644	6 1914 3995	4·694	1 5001 2807	4·744	7 0376 1417	4·794	2 7927 3963
4·645	6 0952 9660	4·695	1 4086 7368	4·745	6 9506 2006	4·795	2 7099 8828
4·646	5 9992 4933	4·696	1 3173 1069	4·746	6 8637 1290	4·796	2 6273 1963
4·647	5 9032 9807	4·697	1 2260 3903	4·747	6 7768 9260	4·797	2 5447 3361
4·648	5 8074 4270	4·698	1 1348 5859	4·748	6 6901 5908	4·798	2 4622 3013
4·649	5 7116 8315	4·699	1 0437 6928	4·749	6 6035 1225	4·799	2 3798 0912

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
4·800	82 2974 7049	4·850	78 2837 7549	4·900	74 4658 3071	4·950	70 8340 8929
4·801	2 2152 1415	4·851	8 2055 3084	4·901	4 3914 0210	4·951	0 7632 9061
4·802	2 1330 4003	4·852	8 1273 6440	4·902	4 3170 4788	4·952	0 6925 6268
4·803	2 0509 4805	4·853	8 0492 7609	4·903	4 2427 6798	4·953	0 6219 0546
4·804	1 9689 3811	4·854	7 9712 6583	4·904	4 1685 6232	4·954	0 5513 1885
4·805	1 8870 1014	4·855	7 8933 3353	4·905	4 0944 3083	4·955	0 4808 0280
4·806	1 8051 6406	4·856	7 8154 7913	4·906	4 0203 7343	4·956	0 4103 5722
4·807	1 7233 9979	4·857	7 7377 0255	4·907	3 9463 9006	4·957	0 3399 8206
4·808	1 6417 1724	4·858	7 6600 0370	4·908	3 8724 8063	4·958	0 2696 7723
4·809	1 5601 1633	4·859	7 5823 8251	4·909	3 7986 4507	4·959	0 1994 4268
4·810	1 4785 9698	4·860	7 5048 3891	4·910	3 7248 8331	4·960	0 1292 7833
4·811	1 3971 5910	4·861	7 4273 7281	4·911	3 6511 9528	4·961	0 0591 8410
4·812	1 3158 0263	4·862	7 3499 8414	4·912	3 5775 8090	4·962	69 9891 5993
4·813	1 2345 2747	4·863	7 2726 7282	4·913	3 5040 4009	4·963	9 9192 0576
4·814	1 1533 3355	4·864	7 1954 3877	4·914	3 4305 7279	4·964	9 8493 2150
4·815	1 0722 2078	4·865	7 1182 8191	4·915	3 3571 7892	4·965	9 7795 0709
4·816	0 9911 8908	4·866	7 0412 0218	4·916	3 2838 5841	4·966	9 7097 6240
4·817	0 9102 3837	4·867	6 9641 9948	4·917	3 2106 1118	4·967	9 6400 8754
4·818	0 8293 6858	4·868	6 8872 7375	4·918	3 1374 3716	4·968	9 5704 8226
4·819	0 7485 7961	4·869	6 8104 2491	4·919	3 0643 3628	4·969	9 5009 4655
4·820	0 6678 7139	4·870	6 7336 5288	4·920	2 9913 0847	4·970	9 4314 8035
4·821	0 5872 4384	4·871	6 6569 5758	4·921	2 9183 5364	4·971	9 3620 8357
4·822	0 5066 9688	4·872	6 5803 3894	4·922	2 8454 7174	4·972	9 2927 5616
4·823	0 4262 3042	4·873	6 5037 9688	4·923	2 7726 6267	4·973	9 2234 9804
4·824	0 3458 4439	4·874	6 4273 3132	4·924	2 6999 2639	4·974	9 1543 0914
4·825	80 2655 3870	4·875	76 3509 4219	4·925	72 6272 6280	4·975	69 0851 8939
4·826	0 1853 1328	4·876	6 2746 2941	4·926	2 5546 7184	4·976	9 0161 3874
4·827	0 1051 6805	4·877	6 1983 9290	4·927	2 4821 5343	4·977	8 9471 5709
4·828	0 0251 0292	4·878	6 1222 3260	4·928	2 4097 0750	4·978	8 8782 4440
4·829	79 9451 1782	4·879	6 0461 4841	4·929	2 3373 3399	4·979	8 8094 0058
4·830	9 8652 1266	4·880	5 9701 4028	4·930	2 2650 3281	4·980	8 7406 2557
4·831	9 7853 8737	4·881	5 8942 0811	4·931	2 1928 0390	4·981	8 6719 1931
4·832	9 7056 4186	4·882	5 8183 5183	4·932	2 1206 4718	4·982	8 6032 8171
4·833	9 6259 7605	4·883	5 7425 7138	4·933	2 0485 6258	4·983	8 5347 1272
4·834	9 5463 8988	4·884	5 6668 6667	4·934	1 9765 5003	4·984	8 4662 1226
4·835	9 4668 8325	4·885	5 5912 3762	4·935	1 9046 0946	4·985	8 3977 8027
4·836	9 3874 5609	4·886	5 5156 8417	4·936	1 8327 4079	4·986	8 3294 1668
4·837	9 3081 0831	4·887	5 4402 0623	4·937	1 7609 4395	4·987	8 2611 2142
4·838	9 2288 3984	4·888	5 3648 0373	4·938	1 6892 1888	4·988	8 1928 9442
4·839	9 1496 5060	4·889	5 2894 7659	4·939	1 6175 6549	4·989	8 1247 3561
4·840	9 0705 4052	4·890	5 2142 2475	4·940	1 5459 8372	4·990	8 0566 4492
4·841	8 9915 0950	4·891	5 1390 4812	4·941	1 4744 7350	4·991	7 9886 2229
4·842	8 9125 5747	4·892	5 0639 4663	4·942	1 4030 3475	4·992	7 9206 6765
4·843	8 8336 8436	4·893	4 9889 2020	4·943	1 3316 6741	4·993	7 8527 8094
4·844	8 7548 9007	4·894	4 9139 6876	4·944	1 2603 7139	4·994	7 7849 6207
4·845	8 6761 7455	4·895	4 8390 9224	4·945	1 1891 4664	4·995	7 7172 1099
4·846	8 5975 3770	4·896	4 7642 9055	4·946	1 1179 9308	4·996	7 6495 2763
4·847	8 5189 7945	4·897	4 6895 6363	4·947	1 0469 1063	4·997	7 5819 1191
4·848	8 4404 9971	4·898	4 6149 1140	4·948	0 9758 9923	4·998	7 5143 6378
4·849	8 3620 9842	4·899	4 5403 3379	4·949	0 9049 5881	4·999	7 4468 8316

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
5·000	67 3794 6999	5·050	64 0933 3446	5·100	60 9674 6565	5·150	57 9940 4727
5·001	7 3121 2426	5·051	4 0292 7316	5·101	0 9065 2866	5·151	7 9360 8221
5·002	7 2448 4572	5·052	3 9652 7589	5·102	0 8456 5258	5·152	7 8781 7508
5·003	7 1776 3448	5·053	3 9013 4259	5·103	0 7848 3734	5·153	7 8203 2584
5·004	7 1104 9043	5·054	3 8374 7319	5·104	0 7240 8288	5·154	7 7625 3441
5·005	7 0434 1348	5·055	3 7736 6762	5·105	0 6633 8915	5·155	7 7048 0075
5·006	6 9764 0358	5·056	3 7099 2583	5·106	0 6027 5608	5·156	7 6471 2479
5·007	6 9094 6065	5·057	3 6462 4775	5·107	0 5421 8362	5·157	7 5895 0648
5·008	6 8425 8463	5·058	3 5826 3331	5·108	0 4816 7170	5·158	7 5319 4576
5·009	6 7757 7546	5·059	3 5190 8246	5·109	0 4212 2025	5·159	7 4744 4257
5·010	6 7090 3306	5·060	3 4555 9513	5·110	0 3608 2924	5·160	7 4169 9686
5·011	6 6423 5737	5·061	3 3921 7125	5·111	0 3004 9858	5·161	7 3596 0856
5·012	6 5757 4832	5·062	3 3288 1076	5·112	0 2402 2822	5·162	7 3022 7762
5·013	6 5092 0585	5·063	3 2655 1361	5·113	0 1800 1810	5·163	7 2450 0398
5·014	6 4427 2989	5·064	3 2022 7972	5·114	0 1198 6816	5·164	7 1877 8759
5·015	6 3763 2037	5·065	3 1391 0903	5·115	0 0597 7834	5·165	7 1306 2839
5·016	6 3099 7723	5·066	3 0760 0148	5·116	59 9997 4858	5·166	7 0735 2632
5·017	6 2437 0039	5·067	3 0129 5700	5·117	9 9397 7883	5·167	7 0164 8132
5·018	6 1774 8980	5·068	2 9499 7554	5·118	9 8798 6901	5·168	6 9594 9333
5·019	6 1113 4539	5·069	2 8870 5703	5·119	9 8200 1907	5·169	6 9025 6231
5·020	6 0452 6709	5·070	2 8242 0141	5·120	9 7602 2895	5·170	6 8456 8819
5·021	5 9792 5484	5·071	2 7614 0861	5·121	9 7004 9859	5·171	6 7888 7092
5·022	5 9133 0856	5·072	2 6986 7857	5·122	9 6408 2793	5·172	6 7321 1043
5·023	5 8474 2820	5·073	2 6360 1123	5·123	9 5812 1691	5·173	6 6754 0668
5·024	5 7816 1368	5·074	2 5734 0653	5·124	9 5216 6548	5·174	6 6187 5960
5·025	65 7158 6495	5·075	62 5108 6440	5·125	59 4621 7356	5·175	56 5621 6914
5·026	5 6501 8193	5·076	2 4483 8478	5·126	9 4027 4111	5·176	6 5056 3524
5·027	5 5845 6456	5·077	2 3859 6761	5·127	9 3433 6806	5·177	6 4491 5785
5·028	5 5190 1278	5·078	2 3236 1282	5·128	9 2840 5436	5·178	6 3927 3691
5·029	5 4535 2652	5·079	2 2613 2036	5·129	9 2247 9993	5·179	6 3363 7236
5·030	5 3881 0570	5·080	2 1990 9016	5·130	9 1656 0474	5·180	6 2800 6414
5·031	5 3227 5028	5·081	2 1369 2216	5·131	9 1064 6870	5·181	6 2238 1221
5·032	5 2574 6018	5·082	2 0748 1629	5·132	9 0473 9178	5·182	6 1676 1650
5·033	5 1922 3534	5·083	2 0127 7250	5·133	8 9883 7390	5·183	6 1114 7696
5·034	5 1270 7569	5·084	1 9507 9073	5·134	8 9294 1501	5·184	6 0553 9353
5·035	5 0619 8117	5·085	1 8888 7090	5·135	8 8705 1505	5·185	5 9993 6615
5·036	4 9969 5171	5·086	1 8270 1297	5·136	8 8116 7396	5·186	5 9433 9478
5·037	4 9319 8724	5·087	1 7652 1686	5·137	8 7528 9168	5·187	5 8874 7934
5·038	4 8670 8771	5·088	1 7034 8251	5·138	8 6941 6816	5·188	5 8316 1980
5·039	4 8022 5304	5·089	1 6418 0987	5·139	8 6355 0333	5·189	5 7758 1609
5·040	4 7374 8318	5·090	1 5801 9887	5·140	8 5768 9713	5·190	5 7200 6815
5·041	4 6727 7806	5·091	1 5186 4945	5·141	8 5183 4951	5·191	5 6643 7593
5·042	4 6081 3760	5·092	1 4571 6155	5·142	8 4598 6041	5·192	5 6087 3938
5·043	4 5435 6176	5·093	1 3957 3511	5·143	8 4014 2977	5·193	5 5531 5843
5·044	4 4790 5046	5·094	1 3343 7006	5·144	8 3430 5753	5·194	5 4976 3304
5·045	4 4146 0364	5·095	1 2730 6635	5·145	8 2847 4364	5·195	5 4421 6315
5·046	4 3502 2123	5·096	1 2118 2391	5·146	8 2264 8803	5·196	5 3867 4870
5·047	4 2859 0317	5·097	1 1506 4268	5·147	8 1682 9064	5·197	5 3313 8963
5·048	4 2216 4940	5·098	1 0895 2260	5·148	8 1101 5143	5·198	5 2760 8590
5·049	4 1574 5985	5·099	1 0284 6361	5·149	8 0520 7032	5·199	5 2208 3744

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
5·200	55 1656 4421	5·250	52 4751 8399	5·300	49 9159 3907	5·350	47 4815 0999
5·201	5 1105 0614	5·251	2 4227 3503	5·301	9 8660 4808	5·351	7 4340 5222
5·202	5 0554 2318	5·252	2 3703 3850	5·302	9 8162 0696	5·352	7 3866 4187
5·203	5 0003 9527	5·253	2 3179 9434	5·303	9 7664 1565	5·353	7 3392 7892
5·204	4 9454 2237	5·254	2 2657 0250	5·304	9 7166 7411	5·354	7 2919 6330
5·205	4 8905 0441	5·255	2 2134 6292	5·305	9 6669 8228	5·355	7 2446 9497
5·206	4 8356 4134	5·256	2 1612 7555	5·306	9 6173 4013	5·356	7 1974 7389
5·207	4 7808 3311	5·257	2 1091 4035	5·307	9 5677 4759	5·357	7 1503 0001
5·208	4 7260 7966	5·258	2 0570 5726	5·308	9 5182 0461	5·358	7 1031 7328
5·209	4 6713 8093	5·259	2 0050 2622	5·309	9 4687 1116	5·359	7 0560 9365
5·210	4 6167 3688	5·260	1 9530 4719	5·310	9 4192 6718	5·360	7 0090 6107
5·211	4 5621 4744	5·261	1 9011 2011	5·311	9 3698 7261	5·361	6 9620 7551
5·212	4 5076 1256	5·262	1 8492 4493	5·312	9 3205 2741	5·362	6 9151 3691
5·213	4 4531 3219	5·263	1 7974 2160	5·313	9 2712 3154	5·363	6 8682 4522
5·214	4 3987 0628	5·264	1 7456 5007	5·314	9 2219 8494	5·364	6 8214 0040
5·215	4 3443 3476	5·265	1 6939 3028	5·315	9 1727 8755	5·365	6 7746 0240
5·216	4 2900 1759	5·266	1 6422 6219	5·316	9 1236 3934	5·366	6 7278 5118
5·217	4 2357 5471	5·267	1 5906 4574	5·317	9 0745 4026	5·367	6 6811 4669
5·218	4 1815 4606	5·268	1 5390 8088	5·318	9 0254 9025	5·368	6 6344 8887
5·219	4 1273 9160	5·269	1 4875 6756	5·319	8 9764 8926	5·369	6 5878 7769
5·220	4 0732 9126	5·270	1 4361 0573	5·320	8 9275 3725	5·370	6 5413 1310
5·221	4 0192 4500	5·271	1 3846 9533	5·321	8 8786 3417	5·371	6 4947 9505
5·222	3 9652 5276	5·272	1 3333 3632	5·322	8 8297 7997	5·372	6 4483 2350
5·223	3 9113 1448	5·273	1 2820 2864	5·323	8 7809 7459	5·373	6 4018 9839
5·224	3 8574 3011	5·274	1 2307 7225	5·324	8 7322 1800	5·374	6 3555 1968
5·225	53 8035 9960	5·275	51 1795 6708	5·325	48 6835 1014	5·375	46 3091 8733
5·226	3 7498 2289	5·276	1 1284 1310	5·326	8 6348 5096	5·376	6 2629 0129
5·227	3 6960 9993	5·277	1 0773 1024	5·327	8 5862 4042	5·377	6 2166 6152
5·228	3 6424 3067	5·278	1 0262 5846	5·328	8 5376 7847	5·378	6 1704 6796
5·229	3 5888 1505	5·279	0 9752 5770	5·329	8 4891 6505	5·379	6 1243 2056
5·230	3 5352 5303	5·280	0 9243 0793	5·330	8 4407 0012	5·380	6 0782 1930
5·231	3 4817 4453	5·281	0 8734 0907	5·331	8 3922 8363	5·381	6 0321 6411
5·232	3 4282 8952	5·282	0 8225 6109	5·332	8 3439 1554	5·382	5 9861 5496
5·233	3 3748 8793	5·283	0 7717 6393	5·333	8 2955 9579	5·383	5 9401 9179
5·234	3 3215 3972	5·284	0 7210 1755	5·334	8 2473 2433	5·384	5 8942 7456
5·235	3 2682 4484	5·285	0 6703 2188	5·335	8 1991 0112	5·385	5 8484 0322
5·236	3 2150 0322	5·286	0 6196 7688	5·336	8 1509 2611	5·386	5 8025 7773
5·237	3 1618 1481	5·287	0 5690 8251	5·337	8 1027 9925	5·387	5 7567 9805
5·238	3 1086 7957	5·288	0 5185 3870	5·338	8 0547 2050	5·388	5 7110 6412
5·239	3 0555 9744	5·289	0 4680 4542	5·339	8 0066 8980	5·389	5 6653 7591
5·240	3 0025 6836	5·290	0 4176 0260	5·340	7 9587 0710	5·390	5 6197 3336
5·241	2 9495 9228	5·291	0 3672 1019	5·341	7 9107 7237	5·391	5 5741 3643
5·242	2 8966 6915	5·292	0 3168 6816	5·342	7 8628 8554	5·392	5 5285 8507
5·243	2 8437 9892	5·293	0 2665 7644	5·343	7 8150 4658	5·393	5 4830 7924
5·244	2 7909 8154	5·294	0 2163 3499	5·344	7 7672 5543	5·394	5 4376 1889
5·245	2 7382 1694	5·295	0 1661 4375	5·345	7 7195 1205	5·395	5 3922 0399
5·246	2 6855 0509	5·296	0 1160 0268	5·346	7 6718 1639	5·396	5 3468 3447
5·247	2 6328 4592	5·297	0 0659 1173	5·347	7 6241 6840	5·397	5 3015 1030
5·248	2 5802 3938	5·298	0 0158 7084	5·348	7 5765 6804	5·398	5 2562 3144
5·249	2 5276 8542	5·299	49 9658 7997	5·349	7 5290 1525	5·399	5 2109 9783

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
5·400	45 1658 0943	5·450	42 9630 4691	5·500	40 8677 1438	5·550	38 8745 7243
5·401	5 1206 6619	5·451	2 9201 0533	5·501	0 8268 6710	5·551	8 8357 1729
5·402	5 0755 6808	5·452	2 8772 0668	5·502	0 7860 6064	5·552	8 7969 0099
5·403	5 0305 1504	5·453	2 8343 5091	5·503	0 7452 9496	5·553	8 7581 2348
5·404	4 9855 0703	5·454	2 7915 3797	5·504	0 7045 7003	5·554	8 7193 8473
5·405	4 9405 4401	5·455	2 7487 6782	5·505	0 6638 8581	5·555	8 6806 8470
5·406	4 8956 2593	5·456	2 7060 4042	5·506	0 6232 4225	5·556	8 6420 2335
5·407	4 8507 5274	5·457	2 6633 5572	5·507	0 5826 3931	5·557	8 6034 0064
5·408	4 8059 2441	5·458	2 6207 1369	5·508	0 5420 7695	5·558	8 5648 1653
5·409	4 7611 4088	5·459	2 5781 1428	5·509	0 5015 5514	5·559	8 5262 7099
5·410	4 7164 0211	5·460	2 5355 5745	5·510	0 4610 7383	5·560	8 4877 6398
5·411	4 6717 0806	5·461	2 4930 4315	5·511	0 4206 3298	5·561	8 4492 9545
5·412	4 6270 5868	5·462	2 4505 7135	5·512	0 3802 3255	5·562	8 4108 6537
5·413	4 5824 5393	5·463	2 4081 4199	5·513	0 3398 7250	5·563	8 3724 7371
5·414	4 5378 9376	5·464	2 3657 5505	5·514	0 2995 5279	5·564	8 3341 2041
5·415	4 4933 7813	5·465	2 3234 1047	5·515	0 2592 7338	5·565	8 2958 0545
5·416	4 4489 0699	5·466	2 2811 0821	5·516	0 2190 3423	5·566	8 2575 2879
5·417	4 4044 8030	5·467	2 2388 4824	5·517	0 1788 3530	5·567	8 2192 9038
5·418	4 3600 9801	5·468	2 1966 3050	5·518	0 1386 7655	5·568	8 1810 9020
5·419	4 3157 6009	5·469	2 1544 5496	5·519	0 0985 5794	5·569	8 1429 2819
5·420	4 2714 6648	5·470	2 1123 2158	5·520	0 0584 7942	5·570	8 1048 0433
5·421	4 2272 1714	5·471	2 0702 3031	5·521	0 0184 4096	5·571	8 0667 1857
5·422	4 1830 1203	5·472	2 0281 8110	5·522	39 9784 4252	5·572	8 0286 7088
5·423	4 1388 5110	5·473	1 9861 7393	5·523	9 9384 8406	5·573	7 9906 6121
5·424	4 0947 3431	5·474	1 9442 0874	5·524	9 8985 6554	5·574	7 9526 8954
5·425	44 0506 6162	5·475	41 9022 8550	5·525	39 8586 8692	5·575	37 9147 5582
5·426	4 0066 3297	5·476	1 8604 0416	5·526	9 8188 4816	5·576	7 8768 6002
5·427	3 9626 4834	5·477	1 8185 6468	5·527	9 7790 4921	5·577	7 8390 0209
5·428	3 9187 0766	5·478	1 7767 6701	5·528	9 7392 9004	5·578	7 8011 8200
5·429	3 8748 1091	5·479	1 7350 1113	5·529	9 6995 7062	5·579	7 7633 9971
5·430	3 8309 5803	5·480	1 6932 9698	5·530	9 6598 9089	5·580	7 7256 5519
5·431	3 7871 4898	5·481	1 6516 2452	5·531	9 6202 5082	5·581	7 6879 4839
5·432	3 7433 8371	5·482	1 6099 9372	5·532	9 5806 5038	5·582	7 6502 7928
5·433	3 6996 6219	5·483	1 5684 0452	5·533	9 5410 8951	5·583	7 6126 4782
5·434	3 6559 8437	5·484	1 5268 5689	5·534	9 5015 6818	5·584	7 5750 5397
5·435	3 6123 5021	5·485	1 4853 5079	5·535	9 4620 8636	5·585	7 5374 9770
5·436	3 5687 5966	5·486	1 4438 8618	5·536	9 4226 4400	5·586	7 4999 7896
5·437	3 5252 1268	5·487	1 4024 6301	5·537	9 3832 4106	5·587	7 4624 9773
5·438	3 4817 0922	5·488	1 3610 8124	5·538	9 3438 7750	5·588	7 4250 5395
5·439	3 4382 4924	5·489	1 3197 4083	5·539	9 3045 5329	5·589	7 3876 4761
5·440	3 3948 3271	5·490	1 2784 4174	5·540	9 2652 6838	5·590	7 3502 7865
5·441	3 3514 5956	5·491	1 2371 8393	5·541	9 2260 2274	5·591	7 3129 4704
5·442	3 3081 2977	5·492	1 1959 6736	5·542	9 1868 1632	5·592	7 2756 5274
5·443	3 2648 4329	5·493	1 1547 9198	5·543	9 1476 4909	5·593	7 2383 9572
5·444	3 2216 0007	5·494	1 1136 5777	5·544	9 1085 2101	5·594	7 2011 7594
5·445	3 1784 0008	5·495	1 0725 6465	5·545	9 0694 3204	5·595	7 1639 9335
5·446	3 1352 4326	5·496	1 0315 1262	5·546	9 0303 8214	5·596	7 1268 4794
5·447	3 0921 2957	5·497	0 9905 0162	5·547	8 9913 7126	5·597	7 0897 3965
5·448	3 0490 5898	5·498	0 9495 3160	5·548	8 9523 9938	5·598	7 0526 6845
5·449	3 0060 3144	5·499	0 9086 0254	5·549	8 9134 6645	5·599	7 0156 3430

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
5·600	36 9786 3716	5·650	35 1751 6775	5·700	33 4596 5457	5·750	31 8278 0796
5·601	6 9416 7701	5·651	5 1400 1016	5·701	3 4262 1164	5·751	1 7959 9606
5·602	6 9047 5380	5·652	5 1048 8772	5·702	3 3928 0214	5·752	1 7642 1596
5·603	6 8678 6749	5·653	5 0698 0038	5·703	3 3594 2603	5·753	1 7324 6762
5·604	6 8310 1805	5·654	5 0347 4810	5·704	3 3260 8328	5·754	1 7007 5102
5·605	6 7942 0544	5·655	4 9997 3087	5·705	3 2927 7385	5·755	1 6690 6611
5·606	6 7574 2963	5·656	4 9647 4863	5·706	3 2594 9772	5·756	1 6374 1287
5·607	6 7206 9057	5·657	4 9298 0136	5·707	3 2262 5484	5·757	1 6057 9127
5·608	6 6839 8823	5·658	4 8948 8902	5·708	3 1930 4520	5·758	1 5742 0128
5·609	6 6473 2258	5·659	4 8600 1157	5·709	3 1598 6874	5·759	1 5426 4286
5·610	6 6106 9358	5·660	4 8251 6898	5·710	3 1267 2545	5·760	1 5111 1598
5·611	6 5741 0118	5·661	4 7903 6122	5·711	3 0936 1528	5·761	1 4796 2062
5·612	6 5375 4536	5·662	4 7555 8825	5·712	3 0605 3821	5·762	1 4481 5673
5·613	6 5010 2608	5·663	4 7208 5003	5·713	3 0274 9419	5·763	1 4167 2429
5·614	6 4645 4330	5·664	4 6861 4654	5·714	2 9944 8321	5·764	1 3853 2327
5·615	6 4280 9698	5·665	4 6514 7773	5·715	2 9615 0522	5·765	1 3539 5364
5·616	6 3916 8709	5·666	4 6168 4357	5·716	2 9285 6019	5·766	1 3226 1536
5·617	6 3553 1359	5·667	4 5822 4403	5·717	2 8956 4808	5·767	1 2913 0840
5·618	6 3189 7645	5·668	4 5476 7907	5·718	2 8627 6888	5·768	1 2600 3273
5·619	6 2826 7563	5·669	4 5131 4866	5·719	2 8299 2254	5·769	1 2287 8832
5·620	6 2464 1109	5·670	4 4786 5276	5·720	2 7971 0902	5·770	1 1975 7514
5·621	6 2101 8279	5·671	4 4441 9134	5·721	2 7643 2831	5·771	1 1663 9316
5·622	6 1739 9071	5·672	4 4097 6437	5·722	2 7315 8036	5·772	1 1352 4234
5·623	6 1378 3480	5·673	4 3753 7180	5·723	2 6988 6514	5·773	1 1041 2266
5·624	6 1017 1503	5·674	4 3410 1361	5·724	2 6661 8261	5·774	1 0730 3409
5·625	36 0656 3136	5·675	34 3066 8976	5·725	32 6335 3276	5·775	31 0419 7659
5·626	6 0295 8375	5·676	4 2724 0022	5·726	2 6009 1554	5·776	1 0109 5013
5·627	5 9935 7218	5·677	4 2381 4495	5·727	2 5683 3092	5·777	0 9799 5468
5·628	5 9575 9660	5·678	4 2039 2392	5·728	2 5357 7887	5·778	0 9489 9021
5·629	5 9216 5697	5·679	4 1697 3709	5·729	2 5032 5935	5·779	0 9180 5668
5·630	5 8857 5327	5·680	4 1355 8443	5·730	2 4707 7234	5·780	0 8871 5408
5·631	5 8498 8546	5·681	4 1014 6591	5·731	2 4383 1779	5·781	0 8562 8237
5·632	5 8140 5349	5·682	4 0673 8149	5·732	2 4058 9569	5·782	0 8254 4151
5·633	5 7782 5734	5·683	4 0333 3114	5·733	2 3735 0599	5·783	0 7946 3147
5·634	5 7424 9696	5·684	3 9993 1482	5·734	2 3411 4867	5·784	0 7638 5223
5·635	5 7067 7233	5·685	3 9653 3250	5·735	2 3088 2368	5·785	0 7331 0376
5·636	5 6710 8341	5·686	3 9313 8414	5·736	2 2765 3101	5·786	0 7023 8602
5·637	5 6354 3015	5·687	3 8974 6972	5·737	2 2442 7061	5·787	0 6716 9898
5·638	5 5998 1253	5·688	3 8635 8919	5·738	2 2120 4246	5·788	0 6410 4261
5·639	5 5642 3052	5·689	3 8297 4253	5·739	2 1798 4651	5·789	0 6104 1688
5·640	5 5286 8406	5·690	3 7959 2969	5·740	2 1476 8275	5·790	0 5798 2176
5·641	5 4931 7314	5·691	3 7621 5066	5·741	2 1155 5114	5·791	0 5492 5723
5·642	5 4576 9770	5·692	3 7284 0538	5·742	2 0834 5164	5·792	0 5187 2324
5·643	5 4222 5773	5·693	3 6946 9383	5·743	2 0513 8422	5·793	0 4882 1977
5·644	5 3868 5318	5·694	3 6610 1598	5·744	2 0193 4886	5·794	0 4577 4679
5·645	5 3514 8401	5·695	3 6273 7179	5·745	1 9873 4552	5·795	0 4273 0427
5·646	5 3161 5020	5·696	3 5937 6123	5·746	1 9553 7416	5·796	0 3968 9217
5·647	5 2808 5170	5·697	3 5601 8426	5·747	1 9234 3476	5·797	0 3665 1047
5·648	5 2455 8848	5·698	3 5266 4085	5·748	1 8915 2728	5·798	0 3361 5914
5·649	5 2103 6051	5·699	3 4931 3097	5·749	1 8596 5169	5·799	0 3058 3814

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
5·800	30 2755 4745	5·850	28 7989 9158	5·900	27 3944 4819	5·950	26 0584 0518
5·801	0 2452 8704	5·851	8 7702 0698	5·901	7 3670 6743	5·951	6 0323 5980
5·802	0 2150 5687	5·852	8 7414 5116	5·902	7 3397 1404	5·952	6 0063 4046
5·803	0 1848 5691	5·853	8 7127 2407	5·903	7 3123 8799	5·953	5 9803 4711
5·804	0 1546 8714	5·854	8 6840 2570	5·904	7 2850 8926	5·954	5 9543 7975
5·805	0 1245 4753	5·855	8 6553 5601	5·905	7 2578 1781	5·955	5 9284 3835
5·806	0 0944 3804	5·856	8 6267 1498	5·906	7 2305 7361	5·956	5 9025 2287
5·807	0 0643 5864	5·857	8 5981 0257	5·907	7 2033 5665	5·957	5 8766 3329
5·808	0 0343 0931	5·858	8 5695 1876	5·908	7 1761 6689	5·958	5 8507 6959
5·809	0 0042 9002	5·859	8 5409 6352	5·909	7 1490 0431	5·959	5 8249 3174
5·810	29 9743 0072	5·860	8 5124 3683	5·910	7 1218 6887	5·960	5 7991 1972
5·811	9 9443 4140	5·861	8 4839 3864	5·911	7 0947 6056	5·961	5 7733 3350
5·812	9 9144 1203	5·862	8 4554 6894	5·912	7 0676 7934	5·962	5 7475 7304
5·813	9 8845 1257	5·863	8 4270 2769	5·913	7 0406 2519	5·963	5 7218 3834
5·814	9 8546 4299	5·864	8 3986 1487	5·914	7 0135 9808	5·964	5 6961 2936
5·815	9 8248 0327	5·865	8 3702 3045	5·915	6 9865 9799	5·965	5 6704 4607
5·816	9 7949 9338	5·866	8 3418 7440	5·916	6 9596 2488	5·966	5 6447 8846
5·817	9 7652 1328	5·867	8 3135 4670	5·917	6 9326 7873	5·967	5 6191 5649
5·818	9 7354 6294	5·868	8 2852 4730	5·918	6 9057 5951	5·968	5 5935 5014
5·819	9 7057 4234	5·869	8 2569 7619	5·919	6 8788 6720	5·969	5 5679 6938
5·820	9 6760 5145	5·870	8 2287 3334	5·920	6 8520 0177	5·970	5 5424 1419
5·821	9 6463 9023	5·871	8 2005 1872	5·921	6 8251 6319	5·971	5 5168 8454
5·822	9 6167 5866	5·872	8 1723 3229	5·922	6 7983 5143	5·972	5 4913 8041
5·823	9 5871 5670	5·873	8 1441 7404	5·923	6 7715 6648	5·973	5 4659 0177
5·824	9 5575 8433	5·874	8 1160 4394	5·924	6 7448 0829	5·974	5 4404 4860
5·825	29 5280 4152	5·875	28 0879 4194	5·925	26 7180 7685	5·975	25 4150 2087
5·826	9 4985 2824	5·876	8 0598 6804	5·926	6 6913 7213	5·976	5 3896 1855
5·827	9 4690 4446	5·877	8 0318 2220	5·927	6 6646 9410	5·977	5 3642 4162
5·828	9 4395 9014	5·878	8 0038 0439	5·928	6 6380 4273	5·978	5 3388 9006
5·829	9 4101 6527	5·879	7 9758 1458	5·929	6 6114 1800	5·979	5 3135 6383
5·830	9 3807 6980	5·880	7 9478 5275	5·930	6 5848 1989	5·980	5 2882 6292
5·831	9 3514 0372	5·881	7 9199 1887	5·931	6 5582 4836	5·981	5 2629 8730
5·832	9 3220 6698	5·882	7 8920 1290	5·932	6 5317 0338	5·982	5 2377 3694
5·833	9 2927 5957	5·883	7 8641 3483	5·933	6 5051 8494	5·983	5 2125 1182
5·834	9 2634 8146	5·884	7 8362 8462	5·934	6 4786 9300	5·984	5 1873 1191
5·835	9 2342 3260	5·885	7 8084 6225	5·935	6 4522 2754	5·985	5 1621 3719
5·836	9 2050 1298	5·886	7 7806 6769	5·936	6 4257 8854	5·986	5 1369 8762
5·837	9 1758 2257	5·887	7 7529 0091	5·937	6 3993 7596	5·987	5 1118 6320
5·838	9 1466 6133	5·888	7 7251 6188	5·938	6 3729 8978	5·988	5 0867 6389
5·839	9 1175 2923	5·889	7 6974 5058	5·939	6 3466 2997	5·989	5 0616 8966
5·840	9 0884 2626	5·890	7 6697 6697	5·940	6 3202 9651	5·990	5 0366 4050
5·841	9 0593 5237	5·891	7 6421 1103	5·941	6 2939 8937	5·991	5 0116 1637
5·842	9 0303 0754	5·892	7 6144 8274	5·942	6 2677 0852	5·992	4 9866 1726
5·843	9 0012 9175	5·893	7 5868 8206	5·943	6 2414 5394	5·993	4 9616 4313
5·844	8 9723 0495	5·894	7 5593 0896	5·944	6 2152 2561	5·994	4 9366 9397
5·845	8 9433 4713	5·895	7 5317 6343	5·945	6 1890 2348	5·995	4 9117 6974
5·846	8 9144 1825	5·896	7 5042 4543	5·946	6 1628 4755	5·996	4 8868 7042
5·847	8 8855 1828	5·897	7 4767 5493	5·947	6 1366 9778	5·997	4 8619 9599
5·848	8 8566 4720	5·898	7 4492 9191	5·948	6 1105 7415	5·998	4 8371 4642
5·849	8 8278 0498	5·899	7 4218 5634	5·949	6 0844 7662	5·999	4 8123 2169

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
6·000	24 7875 2177	6·050	23 5786 2006	6·100	22 4286 7719	6·150	21 3348 1770
6·001	4 7627 4663	6·051	3 5550 5322	6·101	2 4062 5973	6·151	1 3134 9355
6·002	4 7379 9626	6·052	3 5315 0995	6·102	2 3838 6467	6·152	1 2921 9070
6·003	4 7132 7063	6·053	3 5079 9020	6·103	2 3614 9199	6·153	1 2709 0916
6·004	4 6885 6971	6·054	3 4844 9396	6·104	2 3391 4168	6·154	1 2496 4888
6·005	4 6638 9349	6·055	3 4610 2121	6·105	2 3168 1370	6·155	1 2284 0985
6·006	4 6392 4192	6·056	3 4375 7191	6·106	2 2945 0804	6·156	1 2071 9205
6·007	4 6146 1499	6·057	3 4141 4605	6·107	2 2722 2468	6·157	1 1859 9546
6·008	4 5900 1268	6·058	3 3907 4361	6·108	2 2499 6358	6·158	1 1648 2005
6·009	4 5654 3496	6·059	3 3673 6456	6·109	2 2277 2474	6·159	1 1436 6581
6·010	4 5408 8180	6·060	3 3440 0887	6·110	2 2055 0813	6·160	1 1225 3272
6·011	4 5163 5319	6·061	3 3206 7653	6·111	2 1833 1372	6·161	1 1014 2074
6·012	4 4918 4909	6·062	3 2973 6751	6·112	2 1611 4149	6·162	1 0803 2987
6·013	4 4673 6948	6·063	3 2740 8179	6·113	2 1389 9143	6·163	1 0592 6007
6·014	4 4429 1434	6·064	3 2508 1934	6·114	2 1168 6350	6·164	1 0382 1134
6·015	4 4184 8364	6·065	3 2275 8014	6·115	2 0947 5769	6·165	1 0171 8364
6·016	4 3940 7737	6·066	3 2043 6417	6·116	2 0726 7398	6·166	0 9961 7697
6·017	4 3696 9548	6·067	3 1811 7141	6·117	2 0506 1234	6·167	0 9751 9128
6·018	4 3453 3797	6·068	3 1580 0182	6·118	2 0285 7275	6·168	0 9542 2658
6·019	4 3210 0480	6·069	3 1348 5540	6·119	2 0065 5518	6·169	0 9332 8282
6·020	4 2966 9595	6·070	3 1117 3211	6·120	1 9845 5963	6·170	0 9123 6000
6·021	4 2724 1140	6·071	3 0886 3192	6·121	1 9625 8606	6·171	0 8914 5809
6·022	4 2481 5112	6·072	3 0655 5483	6·122	1 9406 3445	6·172	0 8705 7708
6·023	4 2239 1509	6·073	3 0425 0081	6·123	1 9187 0478	6·173	0 8497 1694
6·024	4 1997 0328	6·074	3 0194 6982	6·124	1 8967 9703	6·174	0 8288 7764
6·025	24 1755 1567	6·075	22 9964 6186	6·125	21 8749 1118	6·175	20 8080 5917
6·026	4 1513 5224	6·076	2 9734 7689	6·126	1 8530 4720	6·176	0 7872 6151
6·027	4 1272 1296	6·077	2 9505 1490	6·127	1 8312 0508	6·177	0 7664 8464
6·028	4 1030 9781	6·078	2 9275 7586	6·128	1 8093 8479	6·178	0 7457 2854
6·029	4 0790 0676	6·079	2 9046 5974	6·129	1 7875 8630	6·179	0 7249 9318
6·030	4 0549 3979	6·080	2 8817 6653	6·130	1 7658 0961	6·180	0 7042 7855
6·031	4 0308 9687	6·081	2 8588 9620	6·131	1 7440 5468	6·181	0 6835 8462
6·032	4 0068 7798	6·082	2 8360 4873	6·132	1 7223 2149	6·182	0 6629 1137
6·033	3 9828 8310	6·083	2 8132 2409	6·133	1 7006 1003	6·183	0 6422 5879
6·034	3 9589 1221	6·084	2 7904 2227	6·134	1 6789 2026	6·184	0 6216 2684
6·035	3 9349 6527	6·085	2 7676 4324	6·135	1 6572 5218	6·185	0 6010 1553
6·036	3 9110 4227	6·086	2 7448 8698	6·136	1 6356 0575	6·186	0 5804 2481
6·037	3 8871 4318	6·087	2 7221 5346	6·137	1 6139 8096	6·187	0 5598 5467
6·038	3 8632 6798	6·088	2 6994 4266	6·138	1 5923 7778	6·188	0 5393 0509
6·039	3 8394 1664	6·089	2 6767 5457	6·139	1 5707 9620	6·189	0 5187 7605
6·040	3 8155 8914	6·090	2 6540 8915	6·140	1 5492 3618	6·190	0 4982 6753
6·041	3 7917 8545	6·091	2 6314 4638	6·141	1 5276 9772	6·191	0 4777 7951
6·042	3 7680 0556	6·092	2 6088 2625	6·142	1 5061 8078	6·192	0 4573 1197
6·043	3 7442 4943	6·093	2 5862 2872	6·143	1 4846 8535	6·193	0 4368 6488
6·044	3 7205 1705	6·094	2 5636 5378	6·144	1 4632 1140	6·194	0 4164 3823
6·045	3 6968 0839	6·095	2 5411 0141	6·145	1 4417 5892	6·195	0 3960 3200
6·046	3 6731 2342	6·096	2 5185 7157	6·146	1 4203 2788	6·196	0 3756 4616
6·047	3 6494 6213	6·097	2 4960 6426	6·147	1 3989 1825	6·197	0 3552 8070
6·048	3 6258 2449	6·098	2 4735 7944	6·148	1 3775 3003	6·198	0 3349 3559
6·049	3 6022 1048	6·099	2 4511 1709	6·149	1 3561 6319	6·199	0 3146 1082

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
6·200	20 2943 0636	6·250	19 3045 4136	6·300	18 3630 4777	6·350	17 4674 7136
6·201	0 2740 2220	6·251	9 2852 4647	6·301	8 3446 9390	6·351	7 4500 1262
6·202	0 2537 5831	6·252	9 2659 7086	6·302	8 3263 5838	6·352	7 4325 7133
6·203	0 2335 1468	6·253	9 2467 1452	6·303	8 3080 4118	6·353	7 4151 4747
6·204	0 2132 9127	6·254	9 2274 7743	6·304	8 2897 4229	6·354	7 3977 4103
6·205	0 1930 8809	6·255	9 2082 5956	6·305	8 2714 6169	6·355	7 3803 5199
6·206	0 1729 0509	6·256	9 1890 6090	6·306	8 2531 9936	6·356	7 3629 8032
6·207	0 1527 4227	6·257	9 1698 8143	6·307	8 2349 5528	6·357	7 3456 2602
6·208	0 1325 9960	6·258	9 1507 2113	6·308	8 2167 2945	6·358	7 3282 8906
6·209	0 1124 7706	6·259	9 1315 7998	6·309	8 1985 2182	6·359	7 3109 6944
6·210	0 0923 7464	6·260	9 1124 5797	6·310	8 1803 3239	6·360	7 2936 6712
6·211	0 0722 9231	6·261	9 0933 5506	6·311	8 1621 6115	6·361	7 2763 8209
6·212	0 0522 3005	6·262	9 0742 7125	6·312	8 1440 0806	6·362	7 2591 1435
6·213	0 0321 8784	6·263	9 0552 0651	6·313	8 1258 7312	6·363	7 2418 6386
6·214	0 0121 6567	6·264	9 0361 6083	6·314	8 1077 5631	6·364	7 2246 3061
6·215	19 9921 6350	6·265	9 0171 3418	6·315	8 0896 5761	6·365	7 2074 1459
6·216	9 9721 8133	6·266	8 9981 2655	6·316	8 0715 7699	6·366	7 1902 1578
6·217	9 9522 1913	6·267	8 9791 3792	6·317	8 0535 1445	6·367	7 1730 3416
6·218	9 9322 7689	6·268	8 9601 6827	6·318	8 0354 6995	6·368	7 1558 6970
6·219	9 9123 5457	6·269	8 9412 1758	6·319	8 0174 4350	6·369	7 1387 2241
6·220	9 8924 5217	6·270	8 9222 8583	6·320	7 9994 3506	6·370	7 1215 9225
6·221	9 8725 6966	6·271	8 9033 7300	6·321	7 9814 4462	6·371	7 1044 7922
6·222	9 8527 0703	6·272	8 8844 7908	6·322	7 9634 7217	6·372	7 0873 8329
6·223	9 8328 6424	6·273	8 8656 0404	6·323	7 9455 1767	6·373	7 0703 0445
6·224	9 8130 4129	6·274	8 8467 4786	6·324	7 9275 8113	6·374	7 0532 4268
6·225	19 7932 3815	6·275	18 8279 1054	6·325	17 9096 6250	6·375	17 0361 9796
6·226	9 7734 5481	6·276	8 8090 9204	6·326	7 8917 6179	6·376	7 0191 7027
6·227	9 7536 9124	6·277	8 7902 9235	6·327	7 8738 7898	6·377	7 0021 5961
6·228	9 7339 4742	6·278	8 7715 1145	6·328	7 8560 1403	6·378	6 9851 6595
6·229	9 7142 2334	6·279	8 7527 4932	6·329	7 8381 6694	6·379	6 9681 8927
6·230	9 6945 1897	6·280	8 7340 0594	6·330	7 8203 3769	6·380	6 9512 2957
6·231	9 6748 3429	6·281	8 7152 8130	6·331	7 8025 2626	6·381	6 9342 8681
6·232	9 6551 6929	6·282	8 6965 7537	6·332	7 7847 3263	6·382	6 9173 6099
6·233	9 6355 2395	6·283	8 6778 8814	6·333	7 7669 5679	6·383	6 9004 5208
6·234	9 6158 9824	6·284	8 6592 1959	6·334	7 7491 9871	6·384	6 8835 6008
6·235	9 5962 9214	6·285	8 6405 6970	6·335	7 7314 5839	6·385	6 8666 8496
6·236	9 5767 0564	6·286	8 6219 3844	6·336	7 7137 3579	6·386	6 8498 2670
6·237	9 5571 3872	6·287	8 6033 2581	6·337	7 6960 3091	6·387	6 8329 8530
6·238	9 5375 9136	6·288	8 5847 3179	6·338	7 6783 4372	6·388	6 8161 6073
6·239	9 5180 6354	6·289	8 5661 5634	6·339	7 6606 7421	6·389	6 7993 5297
6·240	9 4985 5523	6·290	8 5475 9947	6·340	7 6430 2237	6·390	6 7825 6201
6·241	9 4790 6642	6·291	8 5290 6114	6·341	7 6253 8816	6·391	6 7657 8784
6·242	9 4595 9709	6·292	8 5105 4134	6·342	7 6077 7159	6·392	6 7490 3043
6·243	9 4401 4722	6·293	8 4920 4005	6·343	7 5901 7262	6·393	6 7322 8977
6·244	9 4207 1679	6·294	8 4735 5725	6·344	7 5725 9123	6·394	6 7155 6585
6·245	9 4013 0578	6·295	8 4550 9293	6·345	7 5550 2743	6·395	6 6988 5864
6·246	9 3819 1417	6·296	8 4366 4706	6·346	7 5374 8117	6·396	6 6821 6812
6·247	9 3625 4194	6·297	8 4182 1963	6·347	7 5199 5246	6·397	6 6654 9429
6·248	9 3431 8908	6·298	8 3998 1062	6·348	7 5024 4126	6·398	6 6488 3713
6·249	9 3238 5556	6·299	8 3814 2000	6·349	7 4849 4757	6·399	6 6321 9661

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
6·400	16 6155 7273	6·450	15 8052 2169	6·500	15 0343 9193	6·550	14 3011 5598
6·401	6 5989 6546	6·451	5 7894 2437	6·501	5 0193 6505	6·551	4 2868 6197
6·402	6 5823 7479	6·452	5 7736 4283	6·502	5 0043 5319	6·552	4 2725 8225
6·403	6 5658 0071	6·453	5 7578 7707	6·503	4 9893 5634	6·553	4 2583 1681
6·404	6 5492 4319	6·454	5 7421 2707	6·504	4 9743 7448	6·554	4 2440 6561
6·405	6 5327 0222	6·455	5 7263 9282	6·505	4 9594 0759	6·555	4 2298 2867
6·406	6 5161 7778	6·456	5 7106 7428	6·506	4 9444 5566	6·556	4 2156 0595
6·407	6 4996 6986	6·457	5 6949 7146	6·507	4 9295 1867	6·557	4 2013 9745
6·408	6 4831 7843	6·458	5 6792 8434	6·508	4 9145 9661	6·558	4 1872 0315
6·409	6 4667 0349	6·459	5 6636 1289	6·509	4 8996 8947	6·559	4 1730 2304
6·410	6 4502 4502	6·460	5 6479 5710	6·510	4 8847 9723	6·560	4 1588 5710
6·411	6 4338 0300	6·461	5 6323 1697	6·511	4 8699 1987	6·561	4 1447 0532
6·412	6 4173 7741	6·462	5 6166 9246	6·512	4 8550 5738	6·562	4 1305 6769
6·413	6 4009 6824	6·463	5 6010 8358	6·513	4 8402 0975	6·563	4 1164 4418
6·414	6 3845 7547	6·464	5 5854 9029	6·514	4 8253 7696	6·564	4 1023 3479
6·415	6 3681 9908	6·465	5 5699 1259	6·515	4 8105 5899	6·565	4 0882 3951
6·416	6 3518 3906	6·466	5 5543 5046	6·516	4 7957 5584	6·566	4 0741 5831
6·417	6 3354 9540	6·467	5 5388 0389	6·517	4 7809 6748	6·567	4 0600 9119
6·418	6 3191 6807	6·468	5 5232 7285	6·518	4 7661 9390	6·568	4 0460 3812
6·419	6 3028 5706	6·469	5 5077 5734	6·519	4 7514 3508	6·569	4 0319 9910
6·420	6 2865 6235	6·470	5 4922 5733	6·520	4 7366 9102	6·570	4 0179 7412
6·421	6 2702 8393	6·471	5 4767 7282	6·521	4 7219 6170	6·571	4 0039 6315
6·422	6 2540 2177	6·472	5 4613 0378	6·522	4 7072 4709	6·572	3 9899 6619
6·423	6 2377 7588	6·473	5 4458 5020	6·523	4 6925 4720	6·573	3 9759 8322
6·424	6 2215 4622	6·474	5 4304 1207	6·524	4 6778 6200	6·574	3 9620 1422
6·425	16 2053 3278	6·475	15 4149 8937	6·525	14 6631 9147	6·575	13 9480 5918
6·426	6 1891 3555	6·476	5 3995 8209	6·526	4 6485 3561	6·576	3 9341 1810
6·427	6 1729 5450	6·477	5 3841 9020	6·527	4 6338 9439	6·577	3 9201 9094
6·428	6 1567 8963	6·478	5 3688 1370	6·528	4 6192 6781	6·578	3 9062 7771
6·429	6 1406 4092	6·479	5 3534 5257	6·529	4 6046 5585	6·579	3 8923 7838
6·430	6 1245 0834	6·480	5 3381 0679	6·530	4 5900 5850	6·580	3 8784 9295
6·431	6 1083 9190	6·481	5 3227 7635	6·531	4 5754 7573	6·581	3 8646 2139
6·432	6 0922 9156	6·482	5 3074 6123	6·532	4 5609 0754	6·582	3 8507 6370
6·433	6 0762 0731	6·483	5 2921 6142	6·533	4 5463 5391	6·583	3 8369 1986
6·434	6 0601 3914	6·484	5 2768 7691	6·534	4 5318 1483	6·584	3 8230 8986
6·435	6 0440 8702	6·485	5 2616 0767	6·535	4 5172 9028	6·585	3 8092 7368
6·436	6 0280 5096	6·486	5 2463 5369	6·536	4 5027 8024	6·586	3 7954 7130
6·437	6 0120 3092	6·487	5 2311 1495	6·537	4 4882 8471	6·587	3 7816 8273
6·438	5 9960 2689	6·488	5 2158 9145	6·538	4 4738 0367	6·588	3 7679 0793
6·439	5 9800 3886	6·489	5 2006 8317	6·539	4 4593 3710	6·589	3 7541 4691
6·440	5 9640 6681	6·490	5 1854 9008	6·540	4 4448 8499	6·590	3 7403 9964
6·441	5 9481 1072	6·491	5 1703 1218	6·541	4 4304 4732	6·591	3 7266 6610
6·442	5 9321 7058	6·492	5 1551 4945	6·542	4 4160 2409	6·592	3 7129 4630
6·443	5 9162 4637	6·493	5 1400 0188	6·543	4 4016 1527	6·593	3 6992 4021
6·444	5 9003 3808	6·494	5 1248 6944	6·544	4 3872 2085	6·594	3 6855 4781
6·445	5 8844 4569	6·495	5 1097 5213	6·545	4 3728 4083	6·595	3 6718 6911
6·446	5 8685 6918	6·496	5 0946 4993	6·546	4 3584 7517	6·596	3 6582 0407
6·447	5 8527 0855	6·497	5 0795 6283	6·547	4 3441 2387	6·597	3 6445 5269
6·448	5 8368 6376	6·498	5 0644 9080	6·548	4 3297 8692	6·598	3 6309 1496
6·449	5 8210 3481	6·499	5 0494 3384	6·549	4 3154 6429	6·599	3 6172 9086

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
6·600	13 6036 8037	6·650	12 9402 2105	6·700	12 3091 1903	6·750	11 7087 9621
6·601	3 5900 8349	6·651	2 9272 8730	6·701	2 2968 1606	6·751	1 6970 9326
6·602	3 5765 0020	6·652	2 9143 6647	6·702	2 2845 2539	6·752	1 6854 0202
6·603	3 5629 3049	6·653	2 9014 5856	6·703	2 2722 4700	6·753	1 6737 2246
6·604	3 5493 7434	6·654	2 8885 6355	6·704	2 2599 8089	6·754	1 6620 5457
6·605	3 5358 3174	6·655	2 8756 8143	6·705	2 2477 2704	6·755	1 6503 9834
6·606	3 5223 0267	6·656	2 8628 1219	6·706	2 2354 8543	6·756	1 6387 5377
6·607	3 5087 8713	6·657	2 8499 5580	6·707	2 2232 5606	6·757	1 6271 2083
6·608	3 4952 8509	6·658	2 8371 1227	6·708	2 2110 3892	6·758	1 6154 9952
6·609	3 4817 9655	6·659	2 8242 8157	6·709	2 1988 3398	6·759	1 6038 8983
6·610	3 4683 2149	6·660	2 8114 6370	6·710	2 1866 4124	6·760	1 5922 9174
6·611	3 4548 5990	6·661	2 7986 5864	6·711	2 1744 6070	6·761	1 5807 0524
6·612	3 4414 1177	6·662	2 7858 6638	6·712	2 1622 9232	6·762	1 5691 3032
6·613	3 4279 7708	6·663	2 7730 8691	6·713	2 1501 3611	6·763	1 5575 6698
6·614	3 4145 5581	6·664	2 7603 2020	6·714	2 1379 9204	6·764	1 5460 1519
6·615	3 4011 4796	6·665	2 7475 6626	6·715	2 1258 6012	6·765	1 5344 7494
6·616	3 3877 5351	6·666	2 7348 2507	6·716	2 1137 4032	6·766	1 5229 4623
6·617	3 3743 7245	6·667	2 7220 9661	6·717	2 1016 3263	6·767	1 5114 2905
6·618	3 3610 0476	6·668	2 7093 8087	6·718	2 0895 3705	6·768	1 4999 2337
6·619	3 3476 5043	6·669	2 6966 7784	6·719	2 0774 5356	6·769	1 4884 2920
6·620	3 3343 0946	6·670	2 6839 8751	6·720	2 0653 8214	6·770	1 4769 4651
6·621	3 3209 8181	6·671	2 6713 0986	6·721	2 0533 2279	6·771	1 4654 7530
6·622	3 3076 6749	6·672	2 6586 4489	6·722	2 0412 7549	6·772	1 4540 1555
6·623	3 2943 6647	6·673	2 6459 9257	6·723	2 0292 4023	6·773	1 4425 6726
6·624	3 2810 7875	6·674	2 6333 5290	6·724	2 0172 1701	6·774	1 4311 3042
6·625	13 2678 0431	6·675	12 6207 2586	6·725	12 0052 0580	6·775	11 4197 0500
6·626	3 2545 4314	6·676	2 6081 1144	6·726	1 9932 0659	6·776	1 4082 9100
6·627	3 2412 9522	6·677	2 5955 0963	6·727	1 9812 1938	6·777	1 3968 8841
6·628	3 2280 6054	6·678	2 5829 2042	6·728	1 9692 4415	6·778	1 3854 9722
6·629	3 2148 3909	6·679	2 5703 4379	6·729	1 9572 8089	6·779	1 3741 1742
6·630	3 2016 3086	6·680	2 5577 7973	6·730	1 9453 2958	6·780	1 3627 4898
6·631	3 1884 3583	6·681	2 5452 2823	6·731	1 9333 9022	6·781	1 3513 9191
6·632	3 1752 5398	6·682	2 5326 8927	6·732	1 9214 6280	6·782	1 3400 4620
6·633	3 1620 8532	6·683	2 5201 6284	6·733	1 9095 4729	6·783	1 3287 1182
6·634	3 1489 2981	6·684	2 5076 4894	6·734	1 8976 4370	6·784	1 3173 8877
6·635	3 1357 8745	6·685	2 4951 4754	6·735	1 8857 5200	6·785	1 3060 7704
6·636	3 1226 5823	6·686	2 4826 5864	6·736	1 8738 7219	6·786	1 2947 7661
6·637	3 1095 4213	6·687	2 4701 8222	6·737	1 8620 0425	6·787	1 2834 8748
6·638	3 0964 3914	6·688	2 4577 1827	6·738	1 8501 4818	6·788	1 2722 0963
6·639	3 0833 4925	6·689	2 4452 6678	6·739	1 8383 0395	6·789	1 2609 4306
6·640	3 0702 7244	6·690	2 4328 2773	6·740	1 8264 7157	6·790	1 2496 8774
6·641	3 0572 0870	6·691	2 4204 0112	6·741	1 8146 5101	6·791	1 2384 4368
6·642	3 0441 5802	6·692	2 4079 8693	6·742	1 8028 4226	6·792	1 2272 1085
6·643	3 0311 2038	6·693	2 3955 8514	6·743	1 7910 4532	6·793	1 2159 8925
6·644	3 0180 9577	6·694	2 3831 9575	6·744	1 7792 6017	6·794	1 2047 7887
6·645	3 0050 8418	6·695	2 3708 1875	6·745	1 7674 8679	6·795	1 1935 7969
6·646	2 9920 8560	6·696	2 3584 5411	6·746	1 7557 2519	6·796	1 1823 9171
6·647	2 9791 0001	6·697	2 3461 0184	6·747	1 7439 7534	6·797	1 1712 1490
6·648	2 9661 2739	6·698	2 3337 6190	6·748	1 7322 3723	6·798	1 1600 4927
6·649	2 9531 6775	6·699	2 3214 3431	6·749	1 7205 1086	6·799	1 1488 9480

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
6·800	11 1377 5148	6·850	10 5945 5693	6·900	10 0778 5429	6·950	9 5863 5154
6·801	1 1266 1929	6·851	0 5839 6767	6·901	0 0677 8147	6·951	9 5767 6998
6·802	1 1154 9824	6·852	0 5733 8899	6·902	0 0577 1872	6·952	9 5671 9799
6·803	1 1043 8829	6·853	0 5628 2089	6·903	0 0476 6603	6·953	9 5576 3558
6·804	1 0932 8945	6·854	0 5522 6334	6·904	0 0376 2339	6·954	9 5480 8272
6·805	1 0822 0171	6·855	0 5417 1636	6·905	0 0275 9078	6·955	9 5385 3941
6·806	1 0711 2505	6·856	0 5311 7991	6·906	0 0175 6820	6·956	9 5290 0564
6·807	1 0600 5946	6·857	0 5206 5399	6·907	0 0075 5564	6·957	9 5194 8140
6·808	1 0490 0493	6·858	0 5101 3860	6·908	09 9975 5309	6·958	9 5099 6667
6·809	1 0379 6144	6·859	0 4996 3371	6·909	9 9875 6053	6·959	9 5004 6146
6·810	1 0269 2900	6·860	0 4891 3933	6·910	9 9775 7796	6·960	9 4909 6575
6·811	1 0159 0758	6·861	0 4786 5543	6·911	9 9676 0537	6·961	9 4814 7952
6·812	1 0048 9718	6·862	0 4681 8201	6·912	9 9576 4275	6·962	9 4720 0278
6·813	0 9938 9778	6·863	0 4577 1906	6·913	9 9476 9008	6·963	9 4625 3552
6·814	0 9829 0938	6·864	0 4472 6657	6·914	9 9377 4737	6·964	9 4530 7771
6·815	0 9719 3196	6·865	0 4368 2453	6·915	9 9278 1459	6·965	9 4436 2936
6·816	0 9609 6551	6·866	0 4263 9292	6·916	9 9178 9173	6·966	9 4341 9045
6·817	0 9500 1003	6·867	0 4159 7174	6·917	9 9079 7880	6·967	9 4247 6097
6·818	0 9390 6549	6·868	0 4055 6097	6·918	9 8980 7577	6·968	9 4153 4092
6·819	0 9281 3189	6·869	0 3951 6061	6·919	9 8881 8264	6·969	9 4059 3029
6·820	0 9172 0922	6·870	0 3847 7064	6·920	9 8782 9940	6·970	9 3965 2906
6·821	0 9062 9747	6·871	0 3743 9107	6·921	9 8684 2604	6·971	9 3871 3723
6·822	0 8953 9662	6·872	0 3640 2186	6·922	9 8585 6255	6·972	9 3777 5478
6·823	0 8845 0667	6·873	0 3536 6302	6·923	9 8487 0891	6·973	9 3683 8171
6·824	0 8736 2761	6·874	0 3433 1453	6·924	9 8388 6513	6·974	9 3590 1801
6·825	10 8627 5941	6·875	10 3329 7639	6·925	9 8290 3118	6·975	9 3496 6367
6·826	0 8519 0208	6·876	0 3226 4857	6·926	9 8192 0706	6·976	9 3403 1868
6·827	0 8410 5561	6·877	0 3123 3109	6·927	9 8093 9276	6·977	9 3309 8303
6·828	0 8302 1997	6·878	0 3020 2391	6·928	9 7995 8827	6·978	9 3216 5671
6·829	0 8193 9516	6·879	0 2917 2703	6·929	9 7897 9358	6·979	9 3123 3972
6·830	0 8085 8118	6·880	0 2814 4045	6·930	9 7800 0868	6·980	9 3030 3203
6·831	0 7977 7800	6·881	0 2711 6415	6·931	9 7702 3356	6·981	9 2937 3365
6·832	0 7869 8562	6·882	0 2608 9812	6·932	9 7604 6821	6·982	9 2844 4456
6·833	0 7762 0402	6·883	0 2506 4235	6·933	9 7507 1262	6·983	9 2751 6476
6·834	0 7654 3321	6·884	0 2403 9683	6·934	9 7409 6679	6·984	9 2658 9423
6·835	0 7546 7315	6·885	0 2301 6155	6·935	9 7312 3069	6·985	9 2566 3297
6·836	0 7439 2386	6·886	0 2199 3651	6·936	9 7215 0432	6·986	9 2473 8096
6·837	0 7331 8530	6·887	0 2097 2168	6·937	9 7117 8768	6·987	9 2381 3820
6·838	0 7224 5748	6·888	0 1995 1706	6·938	9 7020 8075	6·988	9 2289 0467
6·839	0 7117 4038	6·889	0 1893 2264	6·939	9 6923 8351	6·989	9 2196 8039
6·840	0 7010 3400	6·890	0 1791 3841	6·940	9 6826 9597	6·990	9 2104 6532
6·841	0 6903 3831	6·891	0 1689 6436	6·941	9 6730 1812	6·991	9 2012 5946
6·842	0 6796 5332	6·892	0 1588 0048	6·942	9 6633 4993	6·992	9 1920 6280
6·843	0 6689 7900	6·893	0 1486 4676	6·943	9 6536 9141	6·993	9 1828 7533
6·844	0 6583 1536	6·894	0 1385 0318	6·944	9 6440 4255	6·994	9 1736 9704
6·845	0 6476 6237	6·895	0 1283 6975	6·945	9 6344 0332	6·995	9 1645 2793
6·846	0 6370 2002	6·896	0 1182 4644	6·946	9 6247 7374	6·996	9 1553 6798
6·847	0 6263 8833	6·897	0 1081 3325	6·947	9 6151 5377	6·997	9 1462 1719
6·848	0 6157 6725	6·898	0 0980 3017	6·948	9 6055 4343	6·998	9 1370 7555
6·849	0 6051 5679	6·899	0 0879 3719	6·949	9 5959 4268	6·999	9 1279 4304

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
7·000	9 1188 1965	7·050	8 6740 8957	7·100	8 2510 4923	7·150	7 8486 4081
7·001	9 1097 0539	7·051	8 6654 1982	7·101	8 2428 0231	7·151	7 8407 9610
7·002	9 1006 0024	7·052	8 6567 5873	7·102	8 2345 6362	7·152	7 8329 5922
7·003	9 0915 0419	7·053	8 6481 0630	7·103	8 2263 3318	7·153	7 8251 3017
7·004	9 0824 1723	7·054	8 6394 6251	7·104	8 2181 1096	7·154	7 8173 0896
7·005	9 0733 3935	7·055	8 6308 2737	7·105	8 2098 9695	7·155	7 8094 9555
7·006	9 0642 7055	7·056	8 6222 0086	7·106	8 2016 9116	7·156	7 8016 8996
7·007	9 0552 1081	7·057	8 6135 8296	7·107	8 1934 9357	7·157	7 7938 9217
7·008	9 0461 6012	7·058	8 6049 7369	7·108	8 1853 0417	7·158	7 7861 0217
7·009	9 0371 1848	7·059	8 5963 7301	7·109	8 1771 2296	7·159	7 7783 1996
7·010	9 0280 8588	7·060	8 5877 8094	7·110	8 1689 4992	7·160	7 7705 4553
7·011	9 0190 6231	7·061	8 5791 9745	7·111	8 1607 8505	7·161	7 7627 7887
7·012	9 0100 4776	7·062	8 5706 2254	7·112	8 1526 2835	7·162	7 7550 1997
7·013	9 0010 4221	7·063	8 5620 5620	7·113	8 1444 7979	7·163	7 7472 6883
7·014	8 9920 4567	7·064	8 5534 9843	7·114	8 1363 3938	7·164	7 7395 2543
7·015	8 9830 5812	7·065	8 5449 4920	7·115	8 1282 0711	7·165	7 7317 8977
7·016	8 9740 7955	7·066	8 5364 0852	7·116	8 1200 8297	7·166	7 7240 6185
7·017	8 9651 0995	7·067	8 5278 7638	7·117	8 1119 6694	7·167	7 7163 4165
7·018	8 9561 4932	7·068	8 5193 5277	7·118	8 1038 5903	7·168	7 7086 2916
7·019	8 9471 9765	7·069	8 5108 3767	7·119	8 0957 5922	7·169	7 7009 2439
7·020	8 9382 5493	7·070	8 5023 3109	7·120	8 0876 6751	7·170	7 6932 2731
7·021	8 9293 2114	7·071	8 4938 3301	7·121	8 0795 8389	7·171	7 6855 3793
7·022	8 9203 9628	7·072	8 4853 4342	7·122	8 0715 0834	7·172	7 6778 5623
7·023	8 9114 8034	7·073	8 4768 6232	7·123	8 0634 4087	7·173	7 6701 8221
7·024	8 9025 7332	7·074	8 4683 8969	7·124	8 0553 8146	7·174	7 6625 1587
7·025	8 8936 7519	7·075	8 4599 2554	7·125	8 0473 3010	7·175	7 6548 5718
7·026	8 8847 8596	7·076	8 4514 6984	7·126	8 0392 8679	7·176	7 6472 0615
7·027	8 8759 0562	7·077	8 4430 2259	7·127	8 0312 5152	7·177	7 6395 6277
7·028	8 8670 3415	7·078	8 4345 8379	7·128	8 0232 2429	7·178	7 6319 2702
7·029	8 8581 7155	7·079	8 4261 5342	7·129	8 0152 0507	7·179	7 6242 9891
7·030	8 8493 1780	7·080	8 4177 3148	7·130	8 0071 9387	7·180	7 6166 7842
7·031	8 8404 7291	7·081	8 4093 1796	7·131	7 9991 9068	7·181	7 6090 6555
7·032	8 8316 3685	7·082	8 4009 1284	7·132	7 9911 9549	7·182	7 6014 6029
7·033	8 8228 0963	7·083	8 3925 1613	7·133	7 9832 0829	7·183	7 5938 6263
7·034	8 8139 9123	7·084	8 3841 2781	7·134	7 9752 2907	7·184	7 5862 7256
7·035	8 8051 8165	7·085	8 3757 4787	7·135	7 9672 5783	7·185	7 5786 9008
7·036	8 7963 8087	7·086	8 3673 7631	7·136	7 9592 9455	7·186	7 5711 1518
7·037	8 7875 8888	7·087	8 3590 1312	7·137	7 9513 3924	7·187	7 5635 4785
7·038	8 7788 0569	7·088	8 3506 5828	7·138	7 9433 9187	7·188	7 5559 8808
7·039	8 7700 3127	7·089	8 3423 1180	7·139	7 9354 5245	7·189	7 5484 3587
7·040	8 7612 6562	7·090	8 3339 7365	7·140	7 9275 2096	7·190	7 5408 9120
7·041	8 7525 0873	7·091	8 3256 4385	7·141	7 9195 9741	7·191	7 5333 5408
7·042	8 7437 6060	7·092	8 3173 2236	7·142	7 9116 8177	7·192	7 5258 2449
7·043	8 7350 2121	7·093	8 3090 0920	7·143	7 9037 7404	7·193	7 5183 0243
7·044	8 7262 9055	7·094	8 3007 0434	7·144	7 8958 7422	7·194	7 5107 8789
7·045	8 7175 6863	7·095	8 2924 0779	7·145	7 8879 8229	7·195	7 5032 8085
7·046	8 7088 5541	7·096	8 2841 1952	7·146	7 8800 9825	7·196	7 4957 8132
7·047	8 7001 5091	7·097	8 2758 3955	7·147	7 8722 2209	7·197	7 4882 8929
7·048	8 6914 5511	7·098	8 2675 6784	7·148	7 8643 5380	7·198	7 4808 0474
7·049	8 6827 6800	7·099	8 2593 0441	7·149	7 8564 9338	7·199	7 4733 2768

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
7·200	7 4658 5808	7·250	7 1017 4389	7·300	6 7553 8775	7·350	6 4259 2360
7·201	7 4583 9596	7·251	7 0946 4569	7·301	6 7486 3574	7·351	6 4195 0089
7·202	7 4509 4129	7·252	7 0875 5459	7·302	6 7418 9048	7·352	6 4130 8460
7·203	7 4434 9407	7·253	7 0804 7058	7·303	6 7351 5196	7·353	6 4066 7472
7·204	7 4360 5430	7·254	7 0733 9365	7·304	6 7284 2017	7·354	6 4002 7125
7·205	7 4286 2196	7·255	7 0663 2379	7·305	6 7216 9511	7·355	6 3938 7417
7·206	7 4211 9705	7·256	7 0592 6100	7·306	6 7149 7678	7·356	6 3874 8350
7·207	7 4137 7956	7·257	7 0522 0527	7·307	6 7082 6516	7·357	6 3810 9921
7·208	7 4063 6949	7·258	7 0451 5659	7·308	6 7015 6025	7·358	6 3747 2130
7·209	7 3989 6682	7·259	7 0381 1495	7·309	6 6948 6204	7·359	6 3683 4976
7·210	7 3915 7155	7·260	7 0310 8036	7·310	6 6881 7052	7·360	6 3619 8459
7·211	7 3841 8368	7·261	7 0240 5279	7·311	6 6814 8569	7·361	6 3556 2579
7·212	7 3768 0318	7·262	7 0170 3225	7·312	6 6748 0755	7·362	6 3492 7334
7·213	7 3694 3007	7·263	7 0100 1872	7·313	6 6681 3607	7·363	6 3429 2724
7·214	7 3620 6432	7·264	7 0030 1221	7·314	6 6614 7127	7·364	6 3365 8748
7·215	7 3547 0594	7·265	6 9960 1270	7·315	6 6548 1313	7·365	6 3302 5406
7·216	7 3473 5491	7·266	6 9890 2018	7·316	6 6481 6164	7·366	6 3239 2697
7·217	7 3400 1123	7·267	6 9820 3465	7·317	6 6415 1680	7·367	6 3176 0621
7·218	7 3326 7488	7·268	6 9750 5611	7·318	6 6348 7861	7·368	6 3112 9176
7·219	7 3253 4587	7·269	6 9680 8454	7·319	6 6282 4705	7·369	6 3049 8362
7·220	7 3180 2419	7·270	6 9611 1994	7·320	6 6216 2211	7·370	6 2986 8179
7·221	7 3107 0982	7·271	6 9541 6230	7·321	6 6150 0380	7·371	6 2923 8626
7·222	7 3034 0277	7·272	6 9472 1161	7·322	6 6083 9210	7·372	6 2860 9702
7·223	7 2961 0301	7·273	6 9402 6787	7·323	6 6017 8701	7·373	6 2798 1406
7·224	7 2888 1056	7·274	6 9333 3107	7·324	6 5951 8853	7·374	6 2735 3739
7·225	7 2815 2539	7·275	6 9264 0121	7·325	6 5885 9663	7·375	6 2672 6698
7·226	7 2742 4751	7·276	6 9194 7827	7·326	6 5820 1133	7·376	6 2610 0285
7·227	7 2669 7689	7·277	6 9125 6225	7·327	6 5754 3261	7·377	6 2547 4498
7·228	7 2597 1355	7·278	6 9056 5314	7·328	6 5688 6046	7·378	6 2484 9336
7·229	7 2524 5746	7·279	6 8987 5094	7·329	6 5622 9489	7·379	6 2422 4799
7·230	7 2452 0863	7·280	6 8918 5564	7·330	6 5557 3587	7·380	6 2360 0886
7·231	7 2379 6704	7·281	6 8849 6723	7·331	6 5491 8341	7·381	6 2297 7597
7·232	7 2307 3270	7·282	6 8780 8570	7·332	6 5426 3750	7·382	6 2235 4931
7·233	7 2235 0558	7·283	6 8712 1105	7·333	6 5360 9813	7·383	6 2173 2887
7·234	7 2162 8568	7·284	6 8643 4328	7·334	6 5295 6530	7·384	6 2111 1465
7·235	7 2090 7300	7·285	6 8574 8236	7·335	6 5230 3900	7·385	6 2049 0664
7·236	7 2018 6753	7·286	6 8506 2831	7·336	6 5165 1922	7·386	6 1987 0483
7·237	7 1946 6927	7·287	6 8437 8110	7·337	6 5100 0596	7·387	6 1925 0922
7·238	7 1874 7819	7·288	6 8369 4074	7·338	6 5034 9921	7·388	6 1863 1981
7·239	7 1802 9431	7·289	6 8301 0722	7·339	6 4969 9896	7·389	6 1801 3658
7·240	7 1731 1760	7·290	6 8232 8053	7·340	6 4905 0521	7·390	6 1739 5953
7·241	7 1659 4807	7·291	6 8164 6066	7·341	6 4840 1795	7·391	6 1677 8866
7·242	7 1587 8570	7·292	6 8096 4760	7·342	6 4775 3717	7·392	6 1616 2396
7·243	7 1516 3049	7·293	6 8028 4136	7·343	6 4710 6287	7·393	6 1554 6541
7·244	7 1444 8244	7·294	6 7960 4192	7·344	6 4645 9504	7·394	6 1493 1302
7·245	7 1373 4153	7·295	6 7892 4927	7·345	6 4581 3368	7·395	6 1431 6678
7·246	7 1302 0775	7·296	6 7824 6342	7·346	6 4516 7877	7·396	6 1370 2669
7·247	7 1230 8111	7·297	6 7756 8434	7·347	6 4452 3032	7·397	6 1308 9273
7·248	7 1159 6159	7·298	6 7689 1205	7·348	6 4387 8831	7·398	6 1247 6490
7·249	7 1088 4918	7·299	6 7621 4652	7·349	6 4323 5274	7·399	6 1186 4320

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
7·400	6 1125 2761	7·450	5 8144 1612	7·500	5 5308 4370	7·550	5 2611 0127
7·401	6 1064 1814	7·451	5 8086 0461	7·501	5 5253 1562	7·551	5 2558 4280
7·402	6 1003 1477	7·452	5 8027 9891	7·502	5 5197 9307	7·552	5 2505 8958
7·403	6 0942 1751	7·453	5 7969 9901	7·503	5 5142 7603	7·553	5 2453 4162
7·404	6 0881 2634	7·454	5 7912 0491	7·504	5 5087 6451	7·554	5 2400 9890
7·405	6 0820 4125	7·455	5 7854 1660	7·505	5 5032 5850	7·555	5 2348 6142
7·406	6 0759 6225	7·456	5 7796 3407	7·506	5 4977 5799	7·556	5 2296 2917
7·407	6 0698 8933	7·457	5 7738 5733	7·507	5 4922 6298	7·557	5 2244 0216
7·408	6 0638 2247	7·458	5 7680 8636	7·508	5 4867 7347	7·558	5 2191 8037
7·409	6 0577 6168	7·459	5 7623 2115	7·509	5 4812 8944	7·559	5 2139 6380
7·410	6 0517 0694	7·460	5 7565 6171	7·510	5 4758 1089	7·560	5 2087 5244
7·411	6 0456 5826	7·461	5 7508 0803	7·511	5 4703 3781	7·561	5 2035 4629
7·412	6 0396 1563	7·462	5 7450 6010	7·512	5 4648 7021	7·562	5 1983 4534
7·413	6 0335 7903	7·463	5 7393 1791	7·513	5 4594 0807	7·563	5 1931 4960
7·414	6 0275 4847	7·464	5 7335 8146	7·514	5 4539 5139	7·564	5 1879 5904
7·415	6 0215 2393	7·465	5 7278 5074	7·515	5 4485 0017	7·565	5 1827 7368
7·416	6 0155 0542	7·466	5 7221 2575	7·516	5 4430 5439	7·566	5 1775 9349
7·417	6 0094 9292	7·467	5 7164 0649	7·517	5 4376 1406	7·567	5 1724 1849
7·418	6 0034 8643	7·468	5 7106 9294	7·518	5 4321 7916	7·568	5 1672 4865
7·419	5 9974 8594	7·469	5 7049 8510	7·519	5 4267 4970	7·569	5 1620 8399
7·420	5 9914 9145	7·470	5 6992 8297	7·520	5 4213 2566	7·570	5 1569 2449
7·421	5 9855 0296	7·471	5 6935 8653	7·521	5 4159 0704	7·571	5 1517 7014
7·422	5 9795 2045	7·472	5 6878 9579	7·522	5 4104 9384	7·572	5 1466 2094
7·423	5 9735 4391	7·473	5 6822 1074	7·523	5 4050 8605	7·573	5 1414 7689
7·424	5 9675 7336	7·474	5 6765 3137	7·524	5 3996 8367	7·574	5 1363 3799
7·425	5 9616 0877	7·475	5 6708 5768	7·525	5 3942 8668	7·575	5 1312 0422
7·426	5 9556 5014	7·476	5 6651 8965	7·526	5 3888 9509	7·576	5 1260 7558
7·427	5 9496 9746	7·477	5 6595 2729	7·527	5 3835 0889	7·577	5 1209 5207
7·428	5 9437 5074	7·478	5 6538 7060	7·528	5 3781 2807	7·578	5 1158 3367
7·429	5 9378 0996	7·479	5 6482 1955	7·529	5 3727 5263	7·579	5 1107 2040
7·430	5 9318 7512	7·480	5 6425 7415	7·530	5 3673 8257	7·580	5 1056 1223
7·431	5 9259 4621	7·481	5 6369 3440	7·531	5 3620 1787	7·581	5 1005 0917
7·432	5 9200 2322	7·482	5 6313 0028	7·532	5 3566 5853	7·582	5 0954 1121
7·433	5 9141 0616	7·483	5 6256 7180	7·533	5 3513 0455	7·583	5 0903 1835
7·434	5 9081 9501	7·484	5 6200 4894	7·534	5 3459 5592	7·584	5 0852 3057
7·435	5 9022 8977	7·485	5 6144 3170	7·535	5 3406 1263	7·585	5 0801 4788
7·436	5 8963 9043	7·486	5 6088 2007	7·536	5 3352 7469	7·586	5 0750 7027
7·437	5 8904 9698	7·487	5 6032 1406	7·537	5 3299 4208	7·587	5 0699 9774
7·438	5 8846 0943	7·488	5 5976 1364	7·538	5 3246 1480	7·588	5 0649 3028
7·439	5 8787 2776	7·489	5 5920 1883	7·539	5 3192 9285	7·589	5 0598 6788
7·440	5 8728 5197	7·490	5 5864 2960	7·540	5 3139 7622	7·590	5 0548 1054
7·441	5 8669 8206	7·491	5 5808 4597	7·541	5 3086 6490	7·591	5 0497 5826
7·442	5 8611 1801	7·492	5 5752 6791	7·542	5 3033 5889	7·592	5 0447 1102
7·443	5 8552 5982	7·493	5 5696 9543	7·543	5 2980 5818	7·593	5 0396 6883
7·444	5 8494 0749	7·494	5 5641 2852	7·544	5 2927 6277	7·594	5 0346 3168
7·445	5 8435 6100	7·495	5 5585 6717	7·545	5 2874 7265	7·595	5 0295 9957
7·446	5 8377 2036	7·496	5 5530 1138	7·546	5 2821 8782	7·596	5 0245 7248
7·447	5 8318 8556	7·497	5 5474 6115	7·547	5 2769 0827	7·597	5 0195 5042
7·448	5 8260 5659	7·498	5 5419 1646	7·548	5 2716 3400	7·598	5 0145 3338
7·449	5 8202 3344	7·499	5 5363 7731	7·549	5 2663 6500	7·599	5 0095 2135

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
7·600	5 0045 1433	7·650	4 7604 4129	7·700	4 5282 7183	7·750	4 3074 2541
7·601	4 9995 1232	7·651	4 7556 8323	7·701	4 5237 4582	7·751	4 3031 2013
7·602	4 9945 1531	7·652	4 7509 2992	7·702	4 5192 2433	7·752	4 2988 1916
7·603	4 9895 2329	7·653	4 7461 8137	7·703	4 5147 0737	7·753	4 2945 2249
7·604	4 9845 3626	7·654	4 7414 3756	7·704	4 5101 9492	7·754	4 2902 3012
7·605	4 9795 5421	7·655	4 7366 9849	7·705	4 5056 8698	7·755	4 2859 4203
7·606	4 9745 7715	7·656	4 7319 6416	7·706	4 5011 8354	7·756	4 2816 5823
7·607	4 9696 0506	7·657	4 7272 3456	7·707	4 4966 8461	7·757	4 2773 7871
7·608	4 9646 3794	7·658	4 7225 0969	7·708	4 4921 9017	7·758	4 2731 0347
7·609	4 9596 7578	7·659	4 7177 8954	7·709	4 4877 0023	7·759	4 2688 3251
7·610	4 9547 1858	7·660	4 7130 7411	7·710	4 4832 1477	7·760	4 2645 6581
7·611	4 9497 6634	7·661	4 7083 6339	7·711	4 4787 3380	7·761	4 2603 0337
7·612	4 9448 1905	7·662	4 7036 5738	7·712	4 4742 5730	7·762	4 2560 4520
7·613	4 9398 7670	7·663	4 6989 5607	7·713	4 4697 8528	7·763	4 2517 9128
7·614	4 9349 3929	7·664	4 6942 5946	7·714	4 4653 1773	7·764	4 2475 4162
7·615	4 9300 0682	7·665	4 6895 6755	7·715	4 4608 5464	7·765	4 2432 9620
7·616	4 9250 7928	7·666	4 6848 8033	7·716	4 4563 9602	7·766	4 2390 5502
7·617	4 9201 5666	7·667	4 6801 9779	7·717	4 4519 4185	7·767	4 2348 1809
7·618	4 9152 3896	7·668	4 6755 1993	7·718	4 4474 9213	7·768	4 2305 8538
7·619	4 9103 2618	7·669	4 6708 4675	7·719	4 4430 4686	7·769	4 2263 5691
7·620	4 9054 1831	7·670	4 6661 7824	7·720	4 4386 0604	7·770	4 2221 3267
7·621	4 9005 1534	7·671	4 6615 1439	7·721	4 4341 6965	7·771	4 2179 1265
7·622	4 8956 1728	7·672	4 6568 5521	7·722	4 4297 3770	7·772	4 2136 9684
7·623	4 8907 2411	7·673	4 6522 0068	7·723	4 4253 1017	7·773	4 2094 8525
7·624	4 8858 3583	7·674	4 6475 5080	7·724	4 4208 8708	7·774	4 2052 7787
7·625	4 8809 5243	7·675	4 6429 0557	7·725	4 4164 6840	7·775	4 2010 7469
7·626	4 8760 7392	7·676	4 6382 6499	7·726	4 4120 5414	7·776	4 1968 7572
7·627	4 8712 0028	7·677	4 6336 2904	7·727	4 4076 4429	7·777	4 1926 8094
7·628	4 8663 3152	7·678	4 6289 9773	7·728	4 4032 3885	7·778	4 1884 9036
7·629	4 8614 6762	7·679	4 6243 7105	7·729	4 3988 3781	7·779	4 1843 0396
7·630	4 8566 0858	7·680	4 6197 4899	7·730	4 3944 4117	7·780	4 1801 2175
7·631	4 8517 5440	7·681	4 6151 3155	7·731	4 3900 4893	7·781	4 1759 4371
7·632	4 8469 0507	7·682	4 6105 1872	7·732	4 3856 6107	7·782	4 1717 6986
7·633	4 8420 6059	7·683	4 6059 1051	7·733	4 3812 7760	7·783	4 1676 0017
7·634	4 8372 2095	7·684	4 6013 0690	7·734	4 3768 9851	7·784	4 1634 3466
7·635	4 8323 8614	7·685	4 5967 0789	7·735	4 3725 2380	7·785	4 1592 7330
7·636	4 8275 5617	7·686	4 5921 1348	7·736	4 3681 5347	7·786	4 1551 1611
7·637	4 8227 3103	7·687	4 5875 2366	7·737	4 3637 8750	7·787	4 1509 6307
7·638	4 8179 1071	7·688	4 5829 3843	7·738	4 3594 2589	7·788	4 1468 1418
7·639	4 8130 9521	7·689	4 5783 5778	7·739	4 3550 6864	7·789	4 1426 6944
7·640	4 8082 8452	7·690	4 5737 8172	7·740	4 3507 1575	7·790	4 1385 2884
7·641	4 8034 7864	7·691	4 5692 1022	7·741	4 3463 6721	7·791	4 1343 9238
7·642	4 7986 7756	7·692	4 5646 4329	7·742	4 3420 2302	7·792	4 1302 6005
7·643	4 7938 8128	7·693	4 5600 8093	7·743	4 3376 8316	7·793	4 1261 3186
7·644	4 7890 8980	7·694	4 5555 2313	7·744	4 3333 4765	7·794	4 1220 0779
7·645	4 7843 0310	7·695	4 5509 6988	7·745	4 3290 1647	7·795	4 1178 8784
7·646	4 7795 2119	7·696	4 5464 2119	7·746	4 3246 8961	7·796	4 1137 7201
7·647	4 7747 4406	7·697	4 5418 7704	7·747	4 3203 6709	7·797	4 1096 6030
7·648	4 7699 7170	7·698	4 5373 3743	7·748	4 3160 4888	7·798	4 1055 5269
7·649	4 7652 0411	7·699	4 5328 0236	7·749	4 3117 3499	7·799	4 1014 4919

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
7·800	4 0973 4979	7·850	3 8975 1968	7·900	3 7074 3540	7·950	3 5266 2165
7·801	4 0932 5449	7·851	3 8936 2411	7·901	3 7037 2982	7·951	3 5230 9679
7·802	4 0891 6328	7·852	3 8897 3243	7·902	3 7000 2794	7·952	3 5195 7545
7·803	4 0850 7616	7·853	3 8858 4464	7·903	3 6963 2976	7·953	3 5160 5763
7·804	4 0809 9312	7·854	3 8819 6074	7·904	3 6926 3528	7·954	3 5125 4333
7·805	4 0769 1417	7·855	3 8780 8072	7·905	3 6889 4449	7·955	3 5090 3255
7·806	4 0728 3929	7·856	3 8742 0458	7·906	3 6852 5739	7·956	3 5055 2527
7·807	4 0687 6849	7·857	3 8703 3231	7·907	3 6815 7398	7·957	3 5020 2149
7·808	4 0647 0176	7·858	3 8664 6391	7·908	3 6778 9424	7·958	3 4985 2122
7·809	4 0606 3909	7·859	3 8625 9938	7·909	3 6742 1819	7·959	3 4950 2445
7·810	4 0565 8048	7·860	3 8587 3871	7·910	3 6705 4580	7·960	3 4915 3117
7·811	4 0525 2592	7·861	3 8548 8190	7·911	3 6668 7709	7·961	3 4880 4139
7·812	4 0484 7542	7·862	3 8510 2895	7·912	3 6632 1205	7·962	3 4845 5509
7·813	4 0444 2897	7·863	3 8471 7984	7·913	3 6595 5067	7·963	3 4810 7228
7·814	4 0403 8656	7·864	3 8433 3459	7·914	3 6558 9295	7·964	3 4775 9294
7·815	4 0363 4820	7·865	3 8394 9317	7·915	3 6522 3888	7·965	3 4741 1709
7·816	4 0323 1386	7·866	3 8356 5560	7·916	3 6485 8847	7·966	3 4706 4471
7·817	4 0282 8357	7·867	3 8318 2186	7·917	3 6449 4170	7·967	3 4671 7580
7·818	4 0242 5730	7·868	3 8279 9195	7·918	3 6412 9858	7·968	3 4637 1036
7·819	4 0202 3505	7·869	3 8241 6588	7·919	3 6376 5910	7·969	3 4602 4838
7·820	4 0162 1683	7·870	3 8203 4362	7·920	3 6340 2326	7·970	3 4567 8986
7·821	4 0122 0262	7·871	3 8165 2519	7·921	3 6303 9106	7·971	3 4533 3480
7·822	4 0081 9242	7·872	3 8127 1057	7·922	3 6267 6248	7·972	3 4498 8319
7·823	4 0041 8623	7·873	3 8088 9977	7·923	3 6231 3753	7·973	3 4464 3503
7·824	4 0001 8405	7·874	3 8050 9277	7·924	3 6195 1620	7·974	3 4429 9032
7·825	3 9961 8586	7·875	3 8012 8958	7·925	3 6158 9850	7·975	3 4395 4905
7·826	3 9921 9167	7·876	3 7974 9019	7·926	3 6122 8441	7·976	3 4361 1122
7·827	3 9882 0148	7·877	3 7936 9460	7·927	3 6086 7393	7·977	3 4326 7682
7·828	3 9842 1527	7·878	3 7899 0280	7·928	3 6050 6706	7·978	3 4292 4586
7·829	3 9802 3304	7·879	3 7861 1479	7·929	3 6014 6379	7·979	3 4258 1833
7·830	3 9762 5480	7·880	3 7823 3057	7·930	3 5978 6413	7·980	3 4223 9422
7·831	3 9722 8053	7·881	3 7785 5013	7·931	3 5942 6806	7·981	3 4189 7354
7·832	3 9683 1024	7·882	3 7747 7347	7·932	3 5906 7559	7·982	3 4155 5628
7·833	3 9643 4391	7·883	3 7710 0058	7·933	3 5870 8671	7·983	3 4121 4243
7·834	3 9603 8155	7·884	3 7672 3146	7·934	3 5835 0142	7·984	3 4087 3199
7·835	3 9564 2315	7·885	3 7634 6612	7·935	3 5799 1971	7·985	3 4053 2496
7·836	3 9524 6870	7·886	3 7597 0453	7·936	3 5763 4158	7·986	3 4019 2134
7·837	3 9485 1821	7·887	3 7559 4671	7·937	3 5727 6702	7·987	3 3985 2112
7·838	3 9445 7166	7·888	3 7521 9264	7·938	3 5691 9604	7·988	3 3951 2429
7·839	3 9406 2906	7·889	3 7484 4232	7·939	3 5656 2863	7·989	3 3917 3087
7·840	3 9366 9040	7·890	3 7446 9575	7·940	3 5620 6478	7·990	3 3883 4083
7·841	3 9327 5568	7·891	3 7409 5293	7·941	3 5585 0450	7·991	3 3849 5418
7·842	3 9288 2489	7·892	3 7372 1384	7·942	3 5549 4777	7·992	3 3815 7092
7·843	3 9248 9803	7·893	3 7334 7850	7·943	3 5513 9460	7·993	3 3781 9104
7·844	3 9209 7509	7·894	3 7297 4688	7·944	3 5478 4498	7·994	3 3748 1454
7·845	3 9170 5608	7·895	3 7260 1900	7·945	3 5442 9891	7·995	3 3714 4141
7·846	3 9131 4098	7·896	3 7222 9484	7·946	3 5407 5638	7·996	3 3680 7166
7·847	3 9092 2980	7·897	3 7185 7441	7·947	3 5372 1740	7·997	3 3647 0527
7·848	3 9053 2252	7·898	3 7148 5769	7·948	3 5336 8195	7·998	3 3613 4224
7·849	3 9014 1915	7·899	3 7111 4469	7·949	3 5301 5003	7·999	3 3579 8258

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
8·000	3 3546 2628	8·050	3 1910 1922	8·100	3 0353 9138	8·150	2 8873 5360
8·001	3 3512 7333	8·051	3 1878 2980	8·101	3 0323 5751	8·151	2 8844 6769
8·002	3 3479 2373	8·052	3 1846 4356	8·102	3 0293 2666	8·152	2 8815 8466
8·003	3 3445 7748	8·053	3 1814 6051	8·103	3 0262 9885	8·153	2 8787 0451
8·004	3 3412 3457	8·054	3 1782 8064	8·104	3 0232 7407	8·154	2 8758 2725
8·005	3 3378 9501	8·055	3 1751 0395	8·105	3 0202 5230	8·155	2 8729 5286
8·006	3 3345 5878	8·056	3 1719 3043	8·106	3 0172 3356	8·156	2 8700 8134
8·007	3 3312 2589	8·057	3 1687 6009	8·107	3 0142 1783	8·157	2 8672 1270
8·008	3 3278 9633	8·058	3 1655 9291	8·108	3 0112 0512	8·158	2 8643 4692
8·009	3 3245 7010	8·059	3 1624 2890	8·109	3 0081 9542	8·159	2 8614 8400
8·010	3 3212 4719	8·060	3 1592 6805	8·110	3 0051 8873	8·160	2 8586 2395
8·011	3 3179 2760	8·061	3 1561 1036	8·111	3 0021 8505	8·161	2 8557 6675
8·012	3 3146 1133	8·062	3 1529 5583	8·112	2 9991 8436	8·162	2 8529 1241
8·013	3 3112 9838	8·063	3 1498 0445	8·113	2 9961 8668	8·163	2 8500 6093
8·014	3 3079 8874	8·064	3 1466 5622	8·114	2 9931 9199	8·164	2 8472 1229
8·015	3 3046 8240	8·065	3 1435 1114	8·115	2 9902 0029	8·165	2 8443 6650
8·016	3 3013 7937	8·066	3 1403 6920	8·116	2 9872 1158	8·166	2 8415 2356
8·017	3 2980 7964	8·067	3 1372 3040	8·117	2 9842 2587	8·167	2 8386 8345
8·018	3 2947 8321	8·068	3 1340 9474	8·118	2 9812 4313	8·168	2 8358 4619
8·019	3 2914 9007	8·069	3 1309 6221	8·119	2 9782 6338	8·169	2 8330 1176
8·020	3 2882 0023	8·070	3 1278 3281	8·120	2 9752 8660	8·170	2 8301 8016
8·021	3 2849 1367	8·071	3 1247 0654	8·121	2 9723 1280	8·171	2 8273 5140
8·022	3 2816 3040	8·072	3 1215 8340	8·122	2 9693 4198	8·172	2 8245 2546
8·023	3 2783 5041	8·073	3 1184 6337	8·123	2 9663 7412	8·173	2 8217 0235
8·024	3 2750 7370	8·074	3 1153 4647	8·124	2 9634 0923	8·174	2 8188 8205
8·025	3 2718 0026	8·075	3 1122 3268	8·125	2 9604 4730	8·175	2 8160 6458
8·026	3 2685 3010	8·076	3 1091 2200	8·126	2 9574 8833	8·176	2 8132 4992
8·027	3 2652 6320	8·077	3 1060 1443	8·127	2 9545 3232	8·177	2 8104 3808
8·028	3 2619 9957	8·078	3 1029 0997	8·128	2 9515 7927	8·178	2 8076 2905
8·029	3 2587 3920	8·079	3 0998 0861	8·129	2 9486 2916	8·179	2 8048 2282
8·030	3 2554 8209	8·080	3 0967 1035	8·130	2 9456 8201	8·180	2 8020 1940
8·031	3 2522 2823	8·081	3 0936 1519	8·131	2 9427 3780	8·181	2 7992 1878
8·032	3 2489 7763	8·082	3 0905 2312	8·132	2 9397 9653	8·182	2 7964 2096
8·033	3 2457 3028	8·083	3 0874 3414	8·133	2 9368 5820	8·183	2 7936 2594
8·034	3 2424 8617	8·084	3 0843 4825	8·134	2 9339 2281	8·184	2 7908 3371
8·035	3 2392 4530	8·085	3 0812 6545	8·135	2 9309 9036	8·185	2 7880 4427
8·036	3 2360 0768	8·086	3 0781 8572	8·136	2 9280 6083	8·186	2 7852 5762
8·037	3 2327 7329	8·087	3 0751 0907	8·137	2 9251 3423	8·187	2 7824 7375
8·038	3 2295 4213	8·088	3 0720 3550	8·138	2 9222 1056	8·188	2 7796 9267
8·039	3 2263 1420	8·089	3 0689 6500	8·139	2 9192 8981	8·189	2 7769 1437
8·040	3 2230 8950	8·090	3 0658 9757	8·140	2 9163 7198	8·190	2 7741 3884
8·041	3 2198 6802	8·091	3 0628 3321	8·141	2 9134 5707	8·191	2 7713 6609
8·042	3 2166 4976	8·092	3 0597 7190	8·142	2 9105 4507	8·192	2 7685 9611
8·043	3 2134 3472	8·093	3 0567 1366	8·143	2 9076 3598	8·193	2 7658 2890
8·044	3 2102 2289	8·094	3 0536 5848	8·144	2 9047 2979	8·194	2 7630 6445
8·045	3 2070 1427	8·095	3 0506 0634	8·145	2 9018 2652	8·195	2 7603 0277
8·046	3 2038 0886	8·096	3 0475 5726	8·146	2 8989 2614	8·196	2 7575 4384
8·047	3 2006 0666	8·097	3 0445 1123	8·147	2 8960 2866	8·197	2 7547 8768
8·048	3 1974 0765	8·098	3 0414 6824	8·148	2 8931 3408	8·198	2 7520 3427
8·049	3 1942 1184	8·099	3 0384 2829	8·149	2 8902 4239	8·199	2 7492 8361

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
8·200	2 7465 3570	8·250	2 6125 8557	8·300	2 4851 6827	8·350	2 3639 6518
8·201	2 7437 9054	8·251	2 6099 7429	8·301	2 4826 8434	8·351	2 3616 0240
8·202	2 7410 4812	8·252	2 6073 6562	8·302	2 4802 0290	8·352	2 3592 4198
8·203	2 7383 0844	8·253	2 6047 5956	8·303	2 4777 2394	8·353	2 3568 8391
8·204	2 7355 7150	8·254	2 6021 5610	8·304	2 4752 4745	8·354	2 3545 2821
8·205	2 7328 3729	8·255	2 5995 5525	8·305	2 4727 7344	8·355	2 3521 7486
8·206	2 7301 0582	8·256	2 5969 5699	8·306	2 4703 0190	8·356	2 3498 2386
8·207	2 7273 7708	8·257	2 5943 6133	8·307	2 4678 3284	8·357	2 3474 7521
8·208	2 7246 5107	8·258	2 5917 6827	8·308	2 4653 6624	8·358	2 3451 2891
8·209	2 7219 2778	8·259	2 5891 7780	8·309	2 4629 0210	8·359	2 3427 8495
8·210	2 7192 0721	8·260	2 5865 8991	8·310	2 4604 4043	8·360	2 3404 4334
8·211	2 7164 8936	8·261	2 5840 0461	8·311	2 4579 8122	8·361	2 3381 0406
8·212	2 7137 7423	8·262	2 5814 2190	8·312	2 4555 2447	8·362	2 3357 6713
8·213	2 7110 6182	8·263	2 5788 4177	8·313	2 4530 7017	8·363	2 3334 3253
8·214	2 7083 5211	8·264	2 5762 6422	8·314	2 4506 1833	8·364	2 3311 0026
8·215	2 7056 4511	8·265	2 5736 8924	8·315	2 4481 6893	8·365	2 3287 7033
8·216	2 7029 4082	8·266	2 5711 1684	8·316	2 4457 2199	8·366	2 3264 4272
8·217	2 7002 3923	8·267	2 5685 4701	8·317	2 4432 7749	8·367	2 3241 1744
8·218	2 6975 4034	8·268	2 5659 7974	8·318	2 4408 3543	8·368	2 3217 9448
8·219	2 6948 4415	8·269	2 5634 1505	8·319	2 4383 9582	8·369	2 3194 7385
8·220	2 6921 5065	8·270	2 5608 5291	8·320	2 4359 5864	8·370	2 3171 5554
8·221	2 6894 5984	8·271	2 5582 9334	8·321	2 4335 2390	8·371	2 3148 3954
8·222	2 6867 7173	8·272	2 5557 3632	8·322	2 4310 9159	8·372	2 3125 2586
8·223	2 6840 8630	8·273	2 5531 8187	8·323	2 4286 6172	8·373	2 3102 1449
8·224	2 6814 0356	8·274	2 5506 2996	8·324	2 4262 3427	8·374	2 3079 0543
8·225	2 6787 2349	8·275	2 5480 8061	8·325	2 4238 0925	8·375	2 3055 9867
8·226	2 6760 4611	8·276	2 5455 3380	8·326	2 4213 8665	8·376	2 3032 9423
8·227	2 6733 7140	8·277	2 5429 8954	8·327	2 4189 6647	8·377	2 3009 9209
8·228	2 6706 9936	8·278	2 5404 4782	8·328	2 4165 4872	8·378	2 2986 9224
8·229	2 6680 3000	8·279	2 5379 0864	8·329	2 4141 3337	8·379	2 2963 9470
8·230	2 6653 6330	8·280	2 5353 7200	8·330	2 4117 2045	8·380	2 2940 9945
8·231	2 6626 9927	8·281	2 5328 3790	8·331	2 4093 0993	8·381	2 2918 0650
8·232	2 6600 3790	8·282	2 5303 0632	8·332	2 4069 0183	8·382	2 2895 1584
8·233	2 6573 7919	8·283	2 5277 7728	8·333	2 4044 9613	8·383	2 2872 2747
8·234	2 6547 2314	8·284	2 5252 5077	8·334	2 4020 9283	8·384	2 2849 4138
8·235	2 6520 6975	8·285	2 5227 2678	8·335	2 3996 9194	8·385	2 2826 5758
8·236	2 6494 1900	8·286	2 5202 0531	8·336	2 3972 9345	8·386	2 2803 7607
8·237	2 6467 7091	8·287	2 5176 8637	8·337	2 3948 9735	8·387	2 2780 9683
8·238	2 6441 2546	8·288	2 5151 6994	8·338	2 3925 0365	8·388	2 2758 1987
8·239	2 6414 8266	8·289	2 5126 5603	8·339	2 3901 1235	8·389	2 2735 4519
8·240	2 6388 4249	8·290	2 5101 4463	8·340	2 3877 2343	8·390	2 2712 7278
8·241	2 6362 0497	8·291	2 5076 3574	8·341	2 3853 3690	8·391	2 2690 0264
8·242	2 6335 7008	8·292	2 5051 2935	8·342	2 3829 5275	8·392	2 2667 3478
8·243	2 6309 3783	8·293	2 5026 2548	8·343	2 3805 7099	8·393	2 2644 6917
8·244	2 6283 0821	8·294	2 5001 2410	8·344	2 3781 9161	8·394	2 2622 0584
8·245	2 6256 8121	8·295	2 4976 2523	8·345	2 3758 1461	8·395	2 2599 4476
8·246	2 6230 5684	8·296	2 4951 2885	8·346	2 3734 3998	8·396	2 2576 8595
8·247	2 6204 3510	8·297	2 4926 3497	8·347	2 3710 6773	8·397	2 2554 2939
8·248	2 6178 1597	8·298	2 4901 4358	8·348	2 3686 9784	8·398	2 2531 7509
8·249	2 6151 9946	8·299	2 4876 5468	8·349	2 3663 3033	8·399	2 2509 2304

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
8·400	2 2486 7324	8·450	2 1390 0415	8·500	2 0346 8369	8·550	1 9354 5099
8·401	2 2464 2569	8·451	2 1368 6622	8·501	2 0326 5002	8·551	1 9335 1651
8·402	2 2441 8039	8·452	2 1347 3042	8·502	2 0306 1839	8·552	1 9315 8396
8·403	2 2419 3733	8·453	2 1325 9676	8·503	2 0285 8879	8·553	1 9296 5334
8·404	2 2396 9651	8·454	2 1304 6523	8·504	2 0265 6121	8·554	1 9277 2465
8·405	2 2374 5794	8·455	2 1283 3583	8·505	2 0245 3566	8·555	1 9257 9789
8·406	2 2352 2160	8·456	2 1262 0855	8·506	2 0225 1214	8·556	1 9238 7306
8·407	2 2329 8749	8·457	2 1240 8341	8·507	2 0204 9064	8·557	1 9219 5015
8·408	2 2307 5562	8·458	2 1219 6039	8·508	2 0184 7116	8·558	1 9200 2916
8·409	2 2285 2598	8·459	2 1198 3949	8·509	2 0164 5369	8·559	1 9181 1009
8·410	2 2262 9857	8·460	2 1177 2071	8·510	2 0144 3825	8·560	1 9161 9294
8·411	2 2240 7338	8·461	2 1156 0404	8·511	2 0124 2482	8·561	1 9142 7770
8·412	2 2218 5042	8·462	2 1134 8950	8·512	2 0104 1340	8·562	1 9123 6438
8·413	2 2196 2968	8·463	2 1113 7706	8·513	2 0084 0399	8·563	1 9104 5297
8·414	2 2174 1116	8·464	2 1092 6674	8·514	2 0063 9659	8·564	1 9085 4347
8·415	2 2151 9486	8·465	2 1071 5853	8·515	2 0043 9120	8·565	1 9066 3588
8·416	2 2129 8077	8·466	2 1050 5243	8·516	2 0023 8781	8·566	1 9047 3020
8·417	2 2107 6890	8·467	2 1029 4843	8·517	2 0003 8642	8·567	1 9028 2642
8·418	2 2085 5923	8·468	2 1008 4653	8·518	1 9983 8703	8·568	1 9009 2455
8·419	2 2063 5178	8·469	2 0987 4673	8·519	1 9963 8965	8·569	1 8990 2457
8·420	2 2041 4653	8·470	2 0966 4903	8·520	1 9943 9425	8·570	1 8971 2650
8·421	2 2019 4348	8·471	2 0945 5343	8·521	1 9924 0086	8·571	1 8952 3032
8·422	2 1997 4264	8·472	2 0924 5993	8·522	1 9904 0945	8·572	1 8933 3604
8·423	2 1975 4400	8·473	2 0903 6851	8·523	1 9884 2004	8·573	1 8914 4365
8·424	2 1953 4755	8·474	2 0882 7919	8·524	1 9864 3261	8·574	1 8895 5315
8·425	2 1931 5330	8·475	2 0861 9195	8·525	1 9844 4717	8·575	1 8876 6454
8·426	2 1909 6124	8·476	2 0841 0680	8·526	1 9824 6372	8·576	1 8857 7782
8·427	2 1887 7138	8·477	2 0820 2374	8·527	1 9804 8224	8·577	1 8838 9298
8·428	2 1865 8370	8·478	2 0799 4276	8·528	1 9785 0275	8·578	1 8820 1003
8·429	2 1843 9821	8·479	2 0778 6385	8·529	1 9765 2524	8·579	1 8801 2896
8·430	2 1822 1490	8·480	2 0757 8703	8·530	1 9745 4970	8·580	1 8782 4977
8·431	2 1800 3378	8·481	2 0737 1228	8·531	1 9725 7614	8·581	1 8763 7246
8·432	2 1778 5483	8·482	2 0716 3960	8·532	1 9706 0455	8·582	1 8744 9703
8·433	2 1756 7807	8·483	2 0695 6900	8·533	1 9686 3493	8·583	1 8726 2347
8·434	2 1735 0348	8·484	2 0675 0046	8·534	1 9666 6728	8·584	1 8707 5178
8·435	2 1713 3106	8·485	2 0654 3400	8·535	1 9647 0159	8·585	1 8688 8196
8·436	2 1691 6081	8·486	2 0633 6960	8·536	1 9627 3787	8·586	1 8670 1402
8·437	2 1669 9274	8·487	2 0613 0726	8·537	1 9607 7612	8·587	1 8651 4793
8·438	2 1648 2683	8·488	2 0592 4698	8·538	1 9588 1632	8·588	1 8632 8372
8·439	2 1626 6308	8·489	2 0571 8876	8·539	1 9568 5848	8·589	1 8614 2137
8·440	2 1605 0150	8·490	2 0551 3260	8·540	1 9549 0260	8·590	1 8595 6088
8·441	2 1583 4208	8·491	2 0530 7850	8·541	1 9529 4868	8·591	1 8577 0224
8·442	2 1561 8482	8·492	2 0510 2644	8·542	1 9509 9670	8·592	1 8558 4547
8·443	2 1540 2971	8·493	2 0489 7644	8·543	1 9490 4668	8·593	1 8539 9055
8·444	2 1518 7676	8·494	2 0469 2849	8·544	1 9470 9861	8·594	1 8521 3749
8·445	2 1497 2595	8·495	2 0448 8258	8·545	1 9451 5248	8·595	1 8502 8628
8·446	2 1475 7730	8·496	2 0428 3872	8·546	1 9432 0830	8·596	1 8484 3692
8·447	2 1454 3080	8·497	2 0407 9691	8·547	1 9412 6607	8·597	1 8465 8940
8·448	2 1432 8644	8·498	2 0387 5713	8·548	1 9393 2577	8·598	1 8447 4374
8·449	2 1411 4423	8·499	2 0367 1939	8·549	1 9373 8741	8·599	1 8428 9991

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
8·600	1 8410 5794	8·650	1 7512 6848	8·700	1 6658 5811	8·750	1 5846 1325
8·601	1 8392 1780	8·651	1 7495 1809	8·701	1 6641 9308	8·751	1 5830 2943
8·602	1 8373 7950	8·652	1 7477 6944	8·702	1 6625 2972	8·752	1 5814 4719
8·603	1 8355 4304	8·653	1 7460 2255	8·703	1 6608 6802	8·753	1 5798 6653
8·604	1 8337 0841	8·654	1 7442 7740	8·704	1 6592 0799	8·754	1 5782 8746
8·605	1 8318 7562	8·655	1 7425 3399	8·705	1 6575 4961	8·755	1 5767 0996
8·606	1 8300 4466	8·656	1 7407 9233	8·706	1 6558 9289	8·756	1 5751 3404
8·607	1 8282 1553	8·657	1 7390 5241	8·707	1 6542 3782	8·757	1 5735 5969
8·608	1 8263 8823	8·658	1 7373 1422	8·708	1 6525 8441	8·758	1 5719 8692
8·609	1 8245 6275	8·659	1 7355 7778	8·709	1 6509 3265	8·759	1 5704 1571
8·610	1 8227 3910	8·660	1 7338 4307	8·710	1 6492 8254	8·760	1 5688 4608
8·611	1 8209 1728	8·661	1 7321 1009	8·711	1 6476 3409	8·761	1 5672 7802
8·612	1 8190 9727	8·662	1 7303 7885	8·712	1 6459 8727	8·762	1 5657 1153
8·613	1 8172 7908	8·663	1 7286 4933	8·713	1 6443 4211	8·763	1 5641 4660
8·614	1 8154 6271	8·664	1 7269 2155	8·714	1 6426 9859	8·764	1 5625 8323
8·615	1 8136 4815	8·665	1 7251 9549	8·715	1 6410 5671	8·765	1 5610 2143
8·616	1 8118 3541	8·666	1 7234 7116	8·716	1 6394 1648	8·766	1 5594 6119
8·617	1 8100 2448	8·667	1 7217 4855	8·717	1 6377 7788	8·767	1 5579 0251
8·618	1 8082 1536	8·668	1 7200 2766	8·718	1 6361 4092	8·768	1 5563 4539
8·619	1 8064 0805	8·669	1 7183 0849	8·719	1 6345 0560	8·769	1 5547 8982
8·620	1 8046 0255	8·670	1 7165 9104	8·720	1 6328 7191	8·770	1 5532 3581
8·621	1 8027 9885	8·671	1 7148 7531	8·721	1 6312 3985	8·771	1 5516 8335
8·622	1 8009 9695	8·672	1 7131 6129	8·722	1 6296 0943	8·772	1 5501 3244
8·623	1 7991 9685	8·673	1 7114 4898	8·723	1 6279 8063	8·773	1 5485 8308
8·624	1 7973 9855	8·674	1 7097 3839	8·724	1 6263 5347	8·774	1 5470 3527
8·625	1 7956 0205	8·675	1 7080 2951	8·725	1 6247 2793	8·775	1 5454 8901
8·626	1 7938 0735	8·676	1 7063 2233	8·726	1 6231 0401	8·776	1 5439 4429
8·627	1 7920 1444	8·677	1 7046 1686	8·727	1 6214 8172	8·777	1 5424 0112
8·628	1 7902 2332	8·678	1 7029 1310	8·728	1 6198 6104	8·778	1 5408 5949
8·629	1 7884 3399	8·679	1 7012 1104	8·729	1 6182 4199	8·779	1 5393 1940
8·630	1 7866 4645	8·680	1 6995 1067	8·730	1 6166 2456	8·780	1 5377 8085
8·631	1 7848 6070	8·681	1 6978 1201	8·731	1 6150 0874	8·781	1 5362 4384
8·632	1 7830 7673	8·682	1 6961 1505	8·732	1 6133 9454	8·782	1 5347 0836
8·633	1 7812 9454	8·683	1 6944 1978	8·733	1 6117 8195	8·783	1 5331 7442
8·634	1 7795 1414	8·684	1 6927 2621	8·734	1 6101 7098	8·784	1 5316 4201
8·635	1 7777 3551	8·685	1 6910 3433	8·735	1 6085 6161	8·785	1 5301 1114
8·636	1 7759 5867	8·686	1 6893 4414	8·736	1 6069 5385	8·786	1 5285 8179
8·637	1 7741 8360	8·687	1 6876 5564	8·737	1 6053 4770	8·787	1 5270 5397
8·638	1 7724 1030	8·688	1 6859 6883	8·738	1 6037 4315	8·788	1 5255 2768
8·639	1 7706 3878	8·689	1 6842 8370	8·739	1 6021 4022	8·789	1 5240 0292
8·640	1 7688 6902	8·690	1 6826 0026	8·740	1 6005 3888	8·790	1 5224 7968
8·641	1 7671 0104	8·691	1 6809 1850	8·741	1 5989 3914	8·791	1 5209 5796
8·642	1 7653 3482	8·692	1 6792 3842	8·742	1 5973 4100	8·792	1 5194 3776
8·643	1 7635 7037	8·693	1 6775 6002	8·743	1 5957 4446	8·793	1 5179 1908
8·644	1 7618 0768	8·694	1 6758 8330	8·744	1 5941 4951	8·794	1 5164 0192
8·645	1 7600 4675	8·695	1 6742 0826	8·745	1 5925 5616	8·795	1 5148 8628
8·646	1 7582 8758	8·696	1 6725 3489	8·746	1 5909 6440	8·796	1 5133 7215
8·647	1 7565 3017	8·697	1 6708 6319	8·747	1 5893 7423	8·797	1 5118 5953
8·648	1 7547 7452	8·698	1 6691 9316	8·748	1 5877 8565	8·798	1 5103 4843
8·649	1 7530 2063	8·699	1 6675 2480	8·749	1 5861 9866	8·799	1 5088 3883

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
8·800	1 5073 3075	8·850	1 4338 1736	8·900	1 3638 8926	8·950	1 2973 7160
8·801	1 5058 2417	8·851	1 4323 8426	8·901	1 3625 2606	8·951	1 2960 7488
8·802	1 5043 1910	8·852	1 4309 5259	8·902	1 3611 6421	8·952	1 2947 7945
8·803	1 5028 1553	8·853	1 4295 2236	8·903	1 3598 0373	8·953	1 2934 8532
8·804	1 5013 1347	8·854	1 4280 9355	8·904	1 3584 4460	8·954	1 2921 9248
8·805	1 4998 1291	8·855	1 4266 6617	8·905	1 3570 8684	8·955	1 2909 0093
8·806	1 4983 1384	8·856	1 4252 4021	8·906	1 3557 3043	8·956	1 2896 1068
8·807	1 4968 1628	8·857	1 4238 1569	8·907	1 3543 7538	8·957	1 2883 2171
8·808	1 4953 2021	8·858	1 4223 9258	8·908	1 3530 2168	8·958	1 2870 3403
8·809	1 4938 2564	8·859	1 4209 7090	8·909	1 3516 6933	8·959	1 2857 4764
8·810	1 4923 3256	8·860	1 4195 5064	8·910	1 3503 1834	8·960	1 2844 6254
8·811	1 4908 4097	8·861	1 4181 3180	8·911	1 3489 6870	8·961	1 2831 7872
8·812	1 4893 5088	8·862	1 4167 1437	8·912	1 3476 2040	8·962	1 2818 9618
8·813	1 4878 6227	8·863	1 4152 9837	8·913	1 3462 7345	8·963	1 2806 1492
8·814	1 4863 7515	8·864	1 4138 8378	8·914	1 3449 2785	8·964	1 2793 3495
8·815	1 4848 8952	8·865	1 4124 7060	8·915	1 3435 8360	8·965	1 2780 5625
8·816	1 4834 0537	8·866	1 4110 5884	8·916	1 3422 4069	8·966	1 2767 7884
8·817	1 4819 2271	8·867	1 4096 4848	8·917	1 3408 9912	8·967	1 2755 0270
8·818	1 4804 4153	8·868	1 4082 3954	8·918	1 3395 5889	8·968	1 2742 2783
8·819	1 4789 6182	8·869	1 4068 3200	8·919	1 3382 2000	8·969	1 2729 5424
8·820	1 4774 8360	8·870	1 4054 2588	8·920	1 3368 8245	8·970	1 2716 8192
8·821	1 4760 0686	8·871	1 4040 2115	8·921	1 3355 4623	8·971	1 2704 1087
8·822	1 4745 3159	8·872	1 4026 1783	8·922	1 3342 1135	8·972	1 2691 4110
8·823	1 4730 5779	8·873	1 4012 1592	8·923	1 3328 7781	8·973	1 2678 7259
8·824	1 4715 8547	8·874	1 3998 1540	8·924	1 3315 4560	8·974	1 2666 0535
8·825	1 4701 1462	8·875	1 3984 1629	8·925	1 3302 1472	8·975	1 2653 3938
8·826	1 4686 4524	8·876	1 3970 1857	8·926	1 3288 8517	8·976	1 2640 7467
8·827	1 4671 7733	8·877	1 3956 2225	8·927	1 3275 5695	8·977	1 2628 1123
8·828	1 4657 1089	8·878	1 3942 2732	8·928	1 3262 3005	8·978	1 2615 4905
8·829	1 4642 4591	8·879	1 3928 3379	8·929	1 3249 0449	8·979	1 2602 8813
8·830	1 4627 8239	8·880	1 3914 4165	8·930	1 3235 8024	8·980	1 2590 2847
8·831	1 4613 2034	8·881	1 3900 5091	8·931	1 3222 5733	8·981	1 2577 7008
8·832	1 4598 5975	8·882	1 3886 6155	8·932	1 3209 3573	8·982	1 2565 1293
8·833	1 4584 0062	8·883	1 3872 7359	8·933	1 3196 1545	8·983	1 2552 5705
8·834	1 4569 4295	8·884	1 3858 8700	8·934	1 3182 9650	8·984	1 2540 0242
8·835	1 4554 8674	8·885	1 3845 0181	8·935	1 3169 7886	8·985	1 2527 4904
8·836	1 4540 3198	8·886	1 3831 1800	8·936	1 3156 6254	8·986	1 2514 9692
8·837	1 4525 7867	8·887	1 3817 3557	8·937	1 3143 4754	8·987	1 2502 4605
8·838	1 4511 2682	8·888	1 3803 5453	8·938	1 3130 3384	8·988	1 2489 9643
8·839	1 4496 7642	8·889	1 3789 7486	8·939	1 3117 2147	8·989	1 2477 4806
8·840	1 4482 2747	8·890	1 3775 9658	8·940	1 3104 1040	8·990	1 2465 0093
8·841	1 4467 7996	8·891	1 3762 1967	8·941	1 3091 0065	8·991	1 2452 5505
8·842	1 4453 3391	8·892	1 3748 4414	8·942	1 3077 9220	8·992	1 2440 1042
8·843	1 4438 8929	8·893	1 3734 6998	8·943	1 3064 8506	8·993	1 2427 6703
8·844	1 4424 4613	8·894	1 3720 9720	8·944	1 3051 7923	8·994	1 2415 2489
8·845	1 4410 0440	8·895	1 3707 2579	8·945	1 3038 7470	8·995	1 2402 8398
8·846	1 4395 6412	8·896	1 3693 5575	8·946	1 3025 7148	8·996	1 2390 4432
8·847	1 4381 2527	8·897	1 3679 8708	8·947	1 3012 6956	8·997	1 2378 0589
8·848	1 4366 8787	8·898	1 3666 1977	8·948	1 2999 6894	8·998	1 2365 6871
8·849	1 4352 5190	8·899	1 3652 5384	8·949	1 2986 6962	8·999	1 2353 3276

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
9·000	1 2340 9804	9·050	1 1739 1037	9·100	1 1166 5808	9·150	1 0621 9803
9·001	1 2328 6456	9·051	1 1727 3704	9·101	1 1155 4198	9·151	1 0611 3636
9·002	1 2316 3231	9·052	1 1715 6489	9·102	1 1144 2700	9·152	1 0600 7575
9·003	1 2304 0129	9·053	1 1703 9391	9·103	1 1133 1313	9·153	1 0590 1621
9·004	1 2291 7151	9·054	1 1692 2410	9·104	1 1122 0037	9·154	1 0579 5772
9·005	1 2279 4295	9·055	1 1680 5546	9·105	1 1110 8873	9·155	1 0569 0029
9·006	1 2267 1562	9·056	1 1668 8799	9·106	1 1099 7820	9·156	1 0558 4392
9·007	1 2254 8952	9·057	1 1657 2169	9·107	1 1088 6877	9·157	1 0547 8860
9·008	1 2242 6464	9·058	1 1645 5655	9·108	1 1077 6046	9·158	1 0537 3434
9·009	1 2230 4099	9·059	1 1633 9258	9·109	1 1066 5325	9·159	1 0526 8113
9·010	1 2218 1856	9·060	1 1622 2976	9·110	1 1055 4715	9·160	1 0516 2898
9·011	1 2205 9735	9·061	1 1610 6812	9·111	1 1044 4216	9·161	1 0505 7788
9·012	1 2193 7736	9·062	1 1599 0763	9·112	1 1033 3827	9·162	1 0495 2782
9·013	1 2181 5860	9·063	1 1587 4830	9·113	1 1022 3548	9·163	1 0484 7882
9·014	1 2169 4105	9·064	1 1575 9013	9·114	1 1011 3379	9·164	1 0474 3087
9·015	1 2157 2471	9·065	1 1564 3312	9·115	1 1000 3321	9·165	1 0463 8396
9·016	1 2145 0960	9·066	1 1552 7726	9·116	1 0989 3373	9·166	1 0453 3810
9·017	1 2132 9569	9·067	1 1541 2256	9·117	1 0978 3534	9·167	1 0442 9328
9·018	1 2120 8300	9·068	1 1529 6902	9·118	1 0967 3806	9·168	1 0432 4951
9·019	1 2108 7153	9·069	1 1518 1663	9·119	1 0956 4187	9·169	1 0422 0678
9·020	1 2096 6126	9·070	1 1506 6539	9·120	1 0945 4677	9·170	1 0411 6510
9·021	1 2084 5220	9·071	1 1495 1529	9·121	1 0934 5277	9·171	1 0401 2445
9·022	1 2072 4436	9·072	1 1483 6635	9·122	1 0923 5987	9·172	1 0390 8485
9·023	1 2060 3772	9·073	1 1472 1856	9·123	1 0912 6805	9·173	1 0380 4628
9·024	1 2048 3228	9·074	1 1460 7192	9·124	1 0901 7733	9·174	1 0370 0875
9·025	1 2036 2805	9·075	1 1449 2642	9·125	1 0890 8770	9·175	1 0359 7226
9·026	1 2024 2502	9·076	1 1437 8206	9·126	1 0879 9915	9·176	1 0349 3681
9·027	1 2012 2320	9·077	1 1426 3885	9·127	1 0869 1170	9·177	1 0339 0239
9·028	1 2000 2258	9·078	1 1414 9679	9·128	1 0858 2533	9·178	1 0328 6900
9·029	1 1988 2315	9·079	1 1403 5586	9·129	1 0847 4005	9·179	1 0318 3665
9·030	1 1976 2493	9·080	1 1392 1607	9·130	1 0836 5585	9·180	1 0308 0533
9·031	1 1964 2790	9·081	1 1380 7743	9·131	1 0825 7274	9·181	1 0297 7504
9·032	1 1952 3207	9·082	1 1369 3992	9·132	1 0814 9070	9·182	1 0287 4578
9·033	1 1940 3744	9·083	1 1358 0355	9·133	1 0804 0975	9·183	1 0277 1755
9·034	1 1928 4400	9·084	1 1346 6831	9·134	1 0793 2988	9·184	1 0266 9034
9·035	1 1916 5175	9·085	1 1335 3421	9·135	1 0782 5109	9·185	1 0256 6417
9·036	1 1904 6070	9·086	1 1324 0124	9·136	1 0771 7338	9·186	1 0246 3902
9·037	1 1892 7083	9·087	1 1312 6941	9·137	1 0760 9675	9·187	1 0236 1489
9·038	1 1880 8215	9·088	1 1301 3870	9·138	1 0750 2119	9·188	1 0225 9179
9·039	1 1868 9466	9·089	1 1290 0913	9·139	1 0739 4670	9·189	1 0215 6971
9·040	1 1857 0836	9·090	1 1278 8068	9·140	1 0728 7329	9·190	1 0205 4865
9·041	1 1845 2325	9·091	1 1267 5337	9·141	1 0718 0096	9·191	1 0195 2861
9·042	1 1833 3932	9·092	1 1256 2718	9·142	1 0707 2969	9·192	1 0185 0959
9·043	1 1821 5657	9·093	1 1245 0211	9·143	1 0696 5950	9·193	1 0174 9159
9·044	1 1809 7500	9·094	1 1233 7817	9·144	1 0685 9037	9·194	1 0164 7461
9·045	1 1797 9462	9·095	1 1222 5536	9·145	1 0675 2232	9·195	1 0154 5864
9·046	1 1786 1541	9·096	1 1211 3366	9·146	1 0664 5533	9·196	1 0144 4369
9·047	1 1774 3739	9·097	1 1200 1309	9·147	1 0653 8940	9·197	1 0134 2975
9·048	1 1762 6054	9·098	1 1188 9363	9·148	1 0643 2455	9·198	1 0124 1683
9·049	1 1750 8487	9·099	1 1177 7530	9·149	1 0632 6076	9·199	1 0114 0492

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
9·200	1 0103 9402	9·250	9611 1652	9·300	9142 4231	9·350	8696 5419
9·201	1 0093 8413	9·251	9601 5588	9·301	9133 2853	9·351	8687 8497
9·202	1 0083 7526	9·252	9591 9621	9·302	9124 1566	9·352	8679 1662
9·203	1 0073 6738	9·253	9582 3749	9·303	9115 0370	9·353	8670 4914
9·204	1 0063 6051	9·254	9572 7973	9·304	9105 9265	9·354	8661 8252
9·205	1 0053 5466	9·255	9563 2293	9·305	9096 8251	9·355	8653 1677
9·206	1 0043 4980	9·256	9553 6709	9·306	9087 7328	9·356	8644 5189
9·207	1 0033 4596	9·257	9544 1220	9·307	9078 6496	9·357	8635 8787
9·208	1 0023 4311	9·258	9534 5826	9·308	9069 5755	9·358	8627 2471
9·209	1 0013 4127	9·259	9525 0528	9·309	9060 5105	9·359	8618 6242
9·210	1 0003 4043	9·260	9515 5325	9·310	9051 4545	9·360	8610 0099
9·211	0 9993 4059	9·261	9506 0217	9·311	9042 4076	9·361	8601 4042
9·212	9983 4175	9·262	9496 5205	9·312	9033 3697	9·362	8592 8071
9·213	9973 4390	9·263	9487 0287	9·313	9024 3408	9·363	8584 2185
9·214	9963 4706	9·264	9477 5464	9·314	9015 3210	9·364	8575 6386
9·215	9953 5121	9·265	9468 0736	9·315	9006 3102	9·365	8567 0673
9·216	9943 5636	9·266	9458 6103	9·316	8997 3084	9·366	8558 5045
9·217	9933 6250	9·267	9449 1564	9·317	8988 3156	9·367	8549 9502
9·218	9923 6963	9·268	9439 7119	9·318	8979 3318	9·368	8541 4046
9·219	9913 7776	9·269	9430 2769	9·319	8970 3569	9·369	8532 8674
9·220	9903 8688	9·270	9420 8514	9·320	8961 3910	9·370	8524 3388
9·221	9893 9698	9·271	9411 4352	9·321	8952 4341	9·371	8515 8188
9·222	9884 0808	9·272	9402 0285	9·322	8943 4862	9·372	8507 3072
9·223	9874 2017	9·273	9392 6312	9·323	8934 5472	9·373	8498 8041
9·224	9864 3324	9·274	9383 2432	9·324	8925 6171	9·374	8490 3096
9·225	9854 4730	9·275	9373 8647	9·325	8916 6959	9·375	8481 8235
9·226	9844 6235	9·276	9364 4955	9·326	8907 7837	9·376	8473 3459
9·227	9834 7838	9·277	9355 1357	9·327	8898 8803	9·377	8464 8768
9·228	9824 9539	9·278	9345 7852	9·328	8889 9859	9·378	8456 4162
9·229	9815 1338	9·279	9336 4441	9·329	8881 1004	9·379	8447 9640
9·230	9805 3236	9·280	9327 1123	9·330	8872 2237	9·380	8439 5202
9·231	9795 5232	9·281	9317 7899	9·331	8863 3559	9·381	8431 0849
9·232	9785 7326	9·282	9308 4768	9·332	8854 4970	9·382	8422 6581
9·233	9775 9517	9·283	9299 1729	9·333	8845 6469	9·383	8414 2396
9·234	9766 1807	9·284	9289 8784	9·334	8836 8057	9·384	8405 8296
9·235	9756 4194	9·285	9280 5932	9·335	8827 9733	9·385	8397 4280
9·236	9746 6678	9·286	9271 3172	9·336	8819 1497	9·386	8389 0347
9·237	9736 9260	9·287	9262 0505	9·337	8810 3350	9·387	8380 6499
9·238	9727 1940	9·288	9252 7931	9·338	8801 5291	9·388	8372 2734
9·239	9717 4716	9·289	9243 5450	9·339	8792 7319	9·389	8363 9053
9·240	9707 7590	9·290	9234 3060	9·340	8783 9436	9·390	8355 5456
9·241	9698 0561	9·291	9225 0763	9·341	8775 1640	9·391	8347 1942
9·242	9688 3629	9·292	9215 8559	9·342	8766 3933	9·392	8338 8512
9·243	9678 6794	9·293	9206 6446	9·343	8757 6313	9·393	8330 5165
9·244	9669 0055	9·294	9197 4426	9·344	8748 8780	9·394	8322 1902
9·245	9659 3414	9·295	9188 2497	9·345	8740 1335	9·395	8313 8722
9·246	9649 6869	9·296	9179 0661	9·346	8731 3977	9·396	8305 5624
9·247	9640 0420	9·297	9169 8916	9·347	8722 6707	9·397	8297 2610
9·248	9630 4068	9·298	9160 7263	9·348	8713 9524	9·398	8288 9679
9·249	9620 7812	9·299	9151 5701	9·349	8705 2428	9·399	8280 6831

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
9·400	8272 4065	9·450	7868 9565	9·500	7485 1830	9·550	7120 1263
9·401	8264 1383	9·451	7861 0915	9·501	7477 7015	9·551	7113 0097
9·402	8255 8783	9·452	7853 2343	9·502	7470 2276	9·552	7105 9003
9·403	8247 6265	9·453	7845 3850	9·503	7462 7611	9·553	7098 7979
9·404	8239 3830	9·454	7837 5436	9·504	7455 3020	9·554	7091 7027
9·405	8231 1477	9·455	7829 7099	9·505	7447 8505	9·555	7084 6145
9·406	8222 9207	9·456	7821 8841	9·506	7440 4063	9·556	7077 5334
9·407	8214 7019	9·457	7814 0662	9·507	7432 9697	9·557	7070 4595
9·408	8206 4913	9·458	7806 2560	9·508	7425 5404	9·558	7063 3925
9·409	8198 2889	9·459	7798 4537	9·509	7418 1186	9·559	7056 3327
9·410	8190 0947	9·460	7790 6591	9·510	7410 7042	9·560	7049 2799
9·411	8181 9087	9·461	7782 8723	9·511	7403 2972	9·561	7042 2341
9·412	8173 7309	9·462	7775 0933	9·512	7395 8976	9·562	7035 1954
9·413	8165 5613	9·463	7767 3221	9·513	7388 5054	9·563	7028 1637
9·414	8157 3998	9·464	7759 5587	9·514	7381 1206	9·564	7021 1391
9·415	8149 2465	9·465	7751 8030	9·515	7373 7431	9·565	7014 1214
9·416	8141 1013	9·466	7744 0551	9·516	7366 3731	9·566	7007 1108
9·417	8132 9643	9·467	7736 3149	9·517	7359 0104	9·567	7000 1072
9·418	8124 8354	9·468	7728 5825	9·518	7351 6551	9·568	6993 1106
9·419	8116 7146	9·469	7720 8577	9·519	7344 3071	9·569	6986 1210
9·420	8108 6019	9·470	7713 1407	9·520	7336 9664	9·570	6979 1384
9·421	8100 4974	9·471	7705 4315	9·521	7329 6331	9·571	6972 1627
9·422	8092 4009	9·472	7697 7300	9·522	7322 3072	9·572	6965 1940
9·423	8084 3126	9·473	7690 0360	9·523	7314 9885	9·573	6958 2323
9·424	8076 2323	9·474	7682 3498	9·524	7307 6772	9·574	6951 2776
9·425	8068 1601	9·475	7674 6713	9·525	7300 3732	9·575	6944 3298
9·426	8060 0960	9·476	7667 0005	9·526	7293 0764	9·576	6937 3889
9·427	8052 0399	9·477	7659 3373	9·527	7285 7870	9·577	6930 4550
9·428	8043 9919	9·478	7651 6818	9·528	7278 5049	9·578	6923 5280
9·429	8035 9519	9·479	7644 0339	9·529	7271 2300	9·579	6916 6079
9·430	8027 9200	9·480	7636 3937	9·530	7263 9624	9·580	6909 6948
9·431	8019 8961	9·481	7628 7611	9·531	7256 7021	9·581	6902 7885
9·432	8011 8802	9·482	7621 1362	9·532	7249 4490	9·582	6895 8892
9·433	8003 8723	9·483	7613 5189	9·533	7242 2032	9·583	6888 9967
9·434	7995 8724	9·484	7605 9091	9·534	7234 9646	9·584	6882 1112
9·435	7987 8806	9·485	7598 3070	9·535	7227 7332	9·585	6875 2325
9·436	7979 8967	9·486	7590 7125	9·536	7220 5091	9·586	6868 3607
9·437	7971 9208	9·487	7583 1256	9·537	7213 2922	9·587	6861 4958
9·438	7963 9528	9·488	7575 5463	9·538	7206 0825	9·588	6854 6377
9·439	7955 9929	9·489	7567 9745	9·539	7198 8800	9·589	6847 7865
9·440	7948 9408	9·490	7560 4103	9·540	7191 6848	9·590	6840 9422
9·441	7940 0968	9·491	7552 8537	9·541	7184 4967	9·591	6834 1046
9·442	7932 1606	9·492	7545 3046	9·542	7177 3158	9·592	6827 2739
9·443	7924 2325	9·493	7537 7631	9·543	7170 1420	9·593	6820 4501
9·444	7916 3122	9·494	7530 2291	9·544	7162 9755	9·594	6813 6330
9·445	7908 3998	9·495	7522 7026	9·545	7155 8161	9·595	6806 8228
9·446	7900 4954	9·496	7515 1837	9·546	7148 6638	9·596	6800 0194
9·447	7892 5988	9·497	7507 6722	9·547	7141 5188	9·597	6793 2228
9·448	7884 7102	9·498	7500 1683	9·548	7134 3808	9·598	6786 4329
9·449	7876 8294	9·499	7492 6719	9·549	7127 2500	9·599	6779 6499

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
9·600	6772 8736	9·650	6442 5567	9·700	6128 3495	9·750	5829 4664
9·601	6766 1042	9·651	6436 1173	9·701	6122 2242	9·751	5823 6398
9·602	6759 3414	9·652	6429 6844	9·702	6116 1050	9·752	5817 8191
9·603	6752 5855	9·653	6423 2580	9·703	6109 9920	9·753	5812 0042
9·604	6745 8363	9·654	6416 8379	9·704	6103 8851	9·754	5806 1951
9·605	6739 0938	9·655	6410 4243	9·705	6097 7842	9·755	5800 3918
9·606	6732 3581	9·656	6404 0171	9·706	6091 6895	9·756	5794 5943
9·607	6725 6291	9·657	6397 6163	9·707	6085 6008	9·757	5788 8026
9·608	6718 9068	9·658	6391 2219	9·708	6079 5183	9·758	5783 0167
9·609	6712 1913	9·659	6384 8339	9·709	6073 4418	9·759	5777 2365
9·610	6705 4824	9·660	6378 4522	9·710	6067 3714	9·760	5771 4622
9·611	6698 7803	9·661	6372 0770	9·711	6061 3071	9·761	5765 6936
9·612	6692 0849	9·662	6365 7081	9·712	6055 2488	9·762	5759 9308
9·613	6685 3961	9·663	6359 3455	9·713	6049 1966	9·763	5754 1737
9·614	6678 7141	9·664	6352 9894	9·714	6043 1504	9·764	5748 4225
9·615	6672 0387	9·665	6346 6396	9·715	6037 1103	9·765	5742 6769
9·616	6665 3700	9·666	6340 2961	9·716	6031 0762	9·766	5736 9371
9·617	6658 7079	9·667	6333 9590	9·717	6025 0481	9·767	5731 2030
9·618	6652 0526	9·668	6327 6282	9·718	6019 0261	9·768	5725 4747
9·619	6645 4038	9·669	6321 3037	9·719	6013 0101	9·769	5719 7521
9·620	6638 7618	9·670	6314 9855	9·720	6007 0000	9·770	5714 0352
9·621	6632 1263	9·671	6308 6737	9·721	6000 9960	9·771	5708 3240
9·622	6625 4975	9·672	6302 3682	9·722	5994 9981	9·772	5702 6185
9·623	6618 8753	9·673	6296 0690	9·723	5989 0060	9·773	5696 9188
9·624	6612 2597	9·674	6289 7760	9·724	5983 0200	9·774	5691 2247
9·625	6605 6508	9·675	6283 4894	9·725	5977 0400	9·775	5685 5363
9·626	6599 0484	9·676	6277 2091	9·726	5971 0660	9·776	5679 8536
9·627	6592 4527	9·677	6270 9350	9·727	5965 0979	9·777	5674 1766
9·628	6585 8635	9·678	6264 6672	9·728	5959 1358	9·778	5668 5053
9·629	6579 2810	9·679	6258 4057	9·729	5953 1796	9·779	5662 8396
9·630	6572 7050	9·680	6252 1504	9·730	5947 2294	9·780	5657 1796
9·631	6566 1356	9·681	6245 9013	9·731	5941 2851	9·781	5651 5252
9·632	6559 5727	9·682	6239 6586	9·732	5935 3468	9·782	5645 8765
9·633	6553 0164	9·683	6233 4220	9·733	5929 4144	9·783	5640 2335
9·634	6546 4667	9·684	6227 1917	9·734	5923 4880	9·784	5634 5961
9·635	6539 9235	9·685	6220 9676	9·735	5917 5675	9·785	5628 9643
9·636	6533 3868	9·686	6214 7498	9·736	5911 6529	9·786	5623 3381
9·637	6526 8567	9·687	6208 5381	9·737	5905 7442	9·787	5617 7176
9·638	6520 3331	9·688	6202 3327	9·738	5899 8414	9·788	5612 1027
9·639	6513 8160	9·689	6196 1335	9·739	5893 9445	9·789	5606 4934
9·640	6507 3055	9·690	6189 9404	9·740	5888 0535	9·790	5600 8897
9·641	6500 8014	9·691	6183 7536	9·741	5882 1684	9·791	5595 2916
9·642	6494 3039	9·692	6177 5729	9·742	5876 2891	9·792	5589 6991
9·643	6487 8128	9·693	6171 3984	9·743	5870 4158	9·793	5584 1122
9·644	6481 3282	9·694	6165 2301	9·744	5864 5483	9·794	5578 5309
9·645	6474 8501	9·695	6159 0680	9·745	5858 6867	9·795	5572 9552
9·646	6468 3785	9·696	6152 9120	9·746	5852 8309	9·796	5567 3850
9·647	6461 9134	9·697	6146 7621	9·747	5846 9810	9·797	5561 8204
9·648	6455 4547	9·698	6140 6185	9·748	5841 1370	9·798	5556 2614
9·649	6449 0025	9·699	6134 4809	9·749	5835 2987	9·799	5550 7079

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
9·800	5545 1599	9·850	5274 7193	9·900	5017 4682	9·950	4772 7634
9·801	5539 6175	9·851	5269 4472	9·901	5012 4532	9·951	4767 9930
9·802	5534 0807	9·852	5264 1804	9·902	5007 4433	9·952	4763 2274
9·803	5528 5494	9·853	5258 9188	9·903	5002 4383	9·953	4758 4666
9·804	5523 0236	9·854	5253 6625	9·904	4997 4384	9·954	4753 7105
9·805	5517 5033	9·855	5248 4115	9·905	4992 4435	9·955	4748 9591
9·806	5511 9886	9·856	5243 1657	9·906	4987 4535	9·956	4744 2126
9·807	5506 4794	9·857	5237 9252	9·907	4982 4686	9·957	4739 4707
9·808	5500 9756	9·858	5232 6899	9·908	4977 4886	9·958	4734 7336
9·809	5495 4774	9·859	5227 4598	9·909	4972 5136	9·959	4730 0012
9·810	5489 9847	9·860	5222 2350	9·910	4967 5436	9·960	4725 2736
9·811	5484 4974	9·861	5217 0154	9·911	4962 5785	9·961	4720 5507
9·812	5479 0157	9·862	5211 8010	9·912	4957 6184	9·962	4715 8325
9·813	5473 5394	9·863	5206 5918	9·913	4952 6633	9·963	4711 1190
9·814	5468 0686	9·864	5201 3878	9·914	4947 7131	9·964	4706 4103
9·815	5462 6033	9·865	5196 1890	9·915	4942 7678	9·965	4701 7062
9·816	5457 1434	9·866	5190 9954	9·916	4937 8275	9·966	4697 0068
9·817	5451 6890	9·867	5185 8070	9·917	4932 8922	9·967	4692 3122
9·818	5446 2400	9·868	5180 6238	9·918	4927 9618	9·968	4687 6222
9·819	5440 7965	9·869	5175 4457	9·919	4923 0363	9·969	4682 9369
9·820	5435 3584	9·870	5170 2729	9·920	4918 1157	9·970	4678 2563
9·821	5429 9258	9·871	5165 1052	9·921	4913 2000	9·971	4673 5804
9·822	5424 4986	9·872	5159 9427	9·922	4908 2893	9·972	4668 9092
9·823	5419 0768	9·873	5154 7853	9·923	4903 3834	9·973	4664 2426
9·824	5413 6604	9·874	5149 6331	9·924	4898 4825	9·974	4659 5807
9·825	5408 2495	9·875	5144 4860	9·925	4893 5865	9·975	4654 9234
9·826	5402 8439	9·876	5139 3441	9·926	4888 6953	9·976	4650 2708
9·827	5397 4438	9·877	5134 2073	9·927	4883 8091	9·977	4645 6229
9·828	5392 0490	9·878	5129 0757	9·928	4878 9277	9·978	4640 9796
9·829	5386 6597	9·879	5123 9492	9·929	4874 0512	9·979	4636 3409
9·830	5381 2757	9·880	5118 8278	9·930	4869 1796	9·980	4631 7069
9·831	5375 8971	9·881	5113 7115	9·931	4864 3129	9·981	4627 0775
9·832	5370 5239	9·882	5108 6004	9·932	4859 4510	9·982	4622 4528
9·833	5365 1560	9·883	5103 4943	9·933	4854 5940	9·983	4617 8326
9·834	5359 7936	9·884	5098 3934	9·934	4849 7418	9·984	4613 2171
9·835	5354 4365	9·885	5093 2975	9·935	4844 8945	9·985	4608 6062
9·836	5349 0847	9·886	5088 2068	9·936	4840 0520	9·986	4603 9999
9·837	5343 7383	9·887	5083 1211	9·937	4835 2144	9·987	4599 3982
9·838	5338 3972	9·888	5078 0405	9·938	4830 3816	9·988	4594 8011
9·839	5333 0615	9·889	5072 9650	9·939	4825 5536	9·989	4590 2086
9·840	5327 7311	9·890	5067 8946	9·940	4820 7305	9·990	4585 6207
9·841	5322 4060	9·891	5062 8292	9·941	4815 9121	9·991	4581 0373
9·842	5317 0863	9·892	5057 7689	9·942	4811 0986	9·992	4576 4586
9·843	5311 7719	9·893	5052 7137	9·943	4806 2899	9·993	4571 8844
9·844	5306 4627	9·894	5047 6635	9·944	4801 4861	9·994	4567 3148
9·845	5301 1589	9·895	5042 6184	9·945	4796 6870	9·995	4562 7498
9·846	5295 8604	9·896	5037 5783	9·946	4791 8927	9·996	4558 1893
9·847	5290 5672	9·897	5032 5432	9·947	4787 1032	9·997	4553 6334
9·848	5285 2793	9·898	5027 5132	9·948	4782 3185	9·998	4549 0820
9·849	5279 9966	9·899	5022 4882	9·949	4777 5385	9·999	4544 5352

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
10·000	4539 9930	10·050	4318 5749	10·100	4107 9555	10·150	3907 6082
10·001	4535 4552	10·051	4314 2585	10·101	4103 8496	10·151	3903 7025
10·002	4530 9221	10·052	4309 9464	10·102	4099 7478	10·152	3899 8008
10·003	4526 3934	10·053	4305 6386	10·103	4095 6501	10·153	3895 9029
10·004	4521 8693	10·054	4301 3351	10·104	4091 5565	10·154	3892 0089
10·005	4517 3497	10·055	4297 0359	10·105	4087 4670	10·155	3888 1189
10·006	4512 8346	10·056	4292 7410	10·106	4083 3816	10·156	3884 2327
10·007	4508 3240	10·057	4288 4504	10·107	4079 3002	10·157	3880 3504
10·008	4503 8179	10·058	4284 1641	10·108	4075 2230	10·158	3876 4720
10·009	4499 3163	10·059	4279 8821	10·109	4071 1498	10·159	3872 5975
10·010	4494 8193	10·060	4275 6044	10·110	4067 0807	10·160	3868 7268
10·011	4490 3267	10·061	4271 3309	10·111	4063 0156	10·161	3864 8600
10·012	4485 8386	10·062	4267 0617	10·112	4058 9546	10·162	3860 9971
10·013	4481 3550	10·063	4262 7968	10·113	4054 8977	10·163	3857 1380
10·014	4476 8759	10·064	4258 5361	10·114	4050 8448	10·164	3853 2828
10·015	4472 4013	10·065	4254 2797	10·115	4046 7960	10·165	3849 4315
10·016	4467 9311	10·066	4250 0275	10·116	4042 7512	10·166	3845 5839
10·017	4463 4654	10·067	4245 7796	10·117	4038 7105	10·167	3841 7403
10·018	4459 0042	10·068	4241 5360	10·118	4034 6738	10·168	3837 9005
10·019	4454 5474	10·069	4237 2966	10·119	4030 6412	10·169	3834 0645
10·020	4450 0951	10·070	4233 0614	10·120	4026 6125	10·170	3830 2323
10·021	4445 6472	10·071	4228 8305	10·121	4022 5879	10·171	3826 4040
10·022	4441 2038	10·072	4224 6037	10·122	4018 5674	10·172	3822 5795
10·023	4436 7648	10·073	4220 3813	10·123	4014 5508	10·173	3818 7589
10·024	4432 3303	10·074	4216 1630	10·124	4010 5383	10·174	3814 9420
10·025	4427 9001	10·075	4211 9489	10·125	4006 5297	10·175	3811 1290
10·026	4423 4745	10·076	4207 7391	10·126	4002 5252	10·176	3807 3197
10·027	4419 0532	10·077	4203 5334	10·127	3998 5247	10·177	3803 5143
10·028	4414 6363	10·078	4199 3320	10·128	3994 5282	10·178	3799 7127
10·029	4410 2239	10·079	4195 1348	10·129	3990 5356	10·179	3795 9149
10·030	4405 8159	10·080	4190 9417	10·130	3986 5471	10·180	3792 1209
10·031	4401 4123	10·081	4186 7529	10·131	3982 5625	10·181	3788 3307
10·032	4397 0131	10·082	4182 5682	10·132	3978 5820	10·182	3784 5442
10·033	4392 6183	10·083	4178 3877	10·133	3974 6054	10·183	3780 7616
10·034	4388 2278	10·084	4174 2114	10·134	3970 6327	10·184	3776 9827
10·035	4383 8418	10·085	4170 0393	10·135	3966 6641	10·185	3773 2076
10·036	4379 4601	10·086	4165 8714	10·136	3962 6994	10·186	3769 4363
10·037	4375 0829	10·087	4161 7076	10·137	3958 7387	10·187	3765 6687
10·038	4370 7100	10·088	4157 5479	10·138	3954 7819	10·188	3761 9049
10·039	4366 3415	10·089	4153 3925	10·139	3950 8291	10·189	3758 1449
10·040	4361 9773	10·090	4149 2412	10·140	3946 8803	10·190	3754 3887
10·041	4357 6175	10·091	4145 0940	10·141	3942 9354	10·191	3750 6361
10·042	4353 2621	10·092	4140 9510	10·142	3938 9944	10·192	3746 8874
10·043	4348 9110	10·093	4136 8121	10·143	3935 0574	10·193	3743 1424
10·044	4344 5642	10·094	4132 6773	10·144	3931 1243	10·194	3739 4011
10·045	4340 2218	10·095	4128 5467	10·145	3927 1951	10·195	3735 6636
10·046	4335 8838	10·096	4124 4202	10·146	3923 2699	10·196	3731 9298
10·047	4331 5501	10·097	4120 2979	10·147	3919 3486	10·197	3728 1997
10·048	4327 2207	10·098	4116 1796	10·148	3915 4312	10·198	3724 4734
10·049	4322 8956	10·099	4112 0655	10·149	3911 5177	10·199	3720 7508

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
10·200	3717 0319	10·250	3535 7501	10·300	3363 3095	10·350	3199 2790
10·201	3713 3167	10·251	3532 2161	10·301	3359 9479	10·351	3196 0813
10·202	3709 6052	10·252	3528 6856	10·302	3356 5896	10·352	3192 8868
10·203	3705 8975	10·253	3525 1587	10·303	3353 2347	10·353	3189 6955
10·204	3702 1934	10·254	3521 6353	10·304	3349 8831	10·354	3186 5074
10·205	3698 4931	10·255	3518 1154	10·305	3346 5349	10·355	3183 3225
10·206	3694 7964	10·256	3514 5991	10·306	3343 1901	10·356	3180 1408
10·207	3691 1035	10·257	3511 0863	10·307	3339 8486	10·357	3176 9622
10·208	3687 4142	10·258	3507 5769	10·308	3336 5104	10·358	3173 7868
10·209	3683 7287	10·259	3504 0711	10·309	3333 1755	10·359	3170 6146
10·210	3680 0468	10·260	3500 5688	10·310	3329 8440	10·360	3167 4456
10·211	3676 3686	10·261	3497 0700	10·311	3326 5158	10·361	3164 2798
10·212	3672 6940	10·262	3493 5746	10·312	3323 1910	10·362	3161 1171
10·213	3669 0232	10·263	3490 0828	10·313	3319 8695	10·363	3157 9575
10·214	3665 3560	10·264	3486 5944	10·314	3316 5512	10·364	3154 8011
10·215	3661 6925	10·265	3483 1096	10·315	3313 2364	10·365	3151 6479
10·216	3658 0326	10·266	3479 6282	10·316	3309 9248	10·366	3148 4978
10·217	3654 3764	10·267	3476 1504	10·317	3306 6165	10·367	3145 3509
10·218	3650 7238	10·268	3472 6759	10·318	3303 3115	10·368	3142 2071
10·219	3647 0749	10·269	3469 2050	10·319	3300 0099	10·369	3139 0665
10·220	3643 4297	10·270	3465 7375	10·320	3296 7115	10·370	3135 9290
10·221	3639 7881	10·271	3462 2735	10·321	3293 4165	10·371	3132 7946
10·222	3636 1501	10·272	3458 8130	10·322	3290 1247	10·372	3129 6634
10·223	3632 5158	10·273	3455 3559	10·323	3286 8362	10·373	3126 5353
10·224	3628 8851	10·274	3451 9023	10·324	3283 5510	10·374	3123 4103
10·225	3625 2580	10·275	3448 4521	10·325	3280 2691	10·375	3120 2885
10·226	3621 6346	10·276	3445 0054	10·326	3276 9905	10·376	3117 1698
10·227	3618 0147	10·277	3441 5621	10·327	3273 7151	10·377	3114 0542
10·228	3614 3985	10·278	3438 1222	10·328	3270 4431	10·378	3110 9417
10·229	3610 7859	10·279	3434 6858	10·329	3267 1742	10·379	3107 8323
10·230	3607 1770	10·280	3431 2529	10·330	3263 9087	10·380	3104 7260
10·231	3603 5716	10·281	3427 8233	10·331	3260 6464	10·381	3101 6228
10·232	3599 9698	10·282	3424 3972	10·332	3257 3874	10·382	3098 5227
10·233	3596 3717	10·283	3420 9745	10·333	3254 1316	10·383	3095 4258
10·234	3592 7771	10·284	3417 5553	10·334	3250 8791	10·384	3092 3319
10·235	3589 1861	10·285	3414 1394	10·335	3247 6299	10·385	3089 2411
10·236	3585 5987	10·286	3410 7270	10·336	3244 3839	10·386	3086 1534
10·237	3582 0149	10·287	3407 3180	10·337	3241 1411	10·387	3083 0688
10·238	3578 4347	10·288	3403 9124	10·338	3237 9016	10·388	3079 9873
10·239	3574 8580	10·289	3400 5101	10·339	3234 6653	10·389	3076 9088
10·240	3571 2850	10·290	3397 1113	10·340	3231 4323	10·390	3073 8334
10·241	3567 7155	10·291	3393 7159	10·341	3228 2024	10·391	3070 7611
10·242	3564 1495	10·292	3390 3239	10·342	3224 9759	10·392	3067 6919
10·243	3560 5872	10·293	3386 9353	10·343	3221 7525	10·393	3064 6258
10·244	3557 0283	10·294	3383 5500	10·344	3218 5324	10·394	3061 5627
10·245	3553 4731	10·295	3380 1682	10·345	3215 3154	10·395	3058 5026
10·246	3549 9214	10·296	3376 7897	10·346	3212 1017	10·396	3055 4457
10·247	3546 3733	10·297	3373 4146	10·347	3208 8912	10·397	3052 3917
10·248	3542 8287	10·298	3370 0429	10·348	3205 6839	10·398	3049 3409
10·249	3539 2876	10·299	3366 6745	10·349	3202 4799	10·399	3046 2931

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
10·400	3043 2483	10·450	2894 8273	10·500	2753 6449	10·550	2619 3481
10·401	3040 2066	10·451	2891 9339	10·501	2750 8927	10·551	2616 7301
10·402	3037 1679	10·452	2889 0434	10·502	2748 1431	10·552	2614 1147
10·403	3034 1322	10·453	2886 1558	10·503	2745 3964	10·553	2611 5019
10·404	3031 0996	10·454	2883 2711	10·504	2742 6523	10·554	2608 8917
10·405	3028 0700	10·455	2880 3893	10·505	2739 9111	10·555	2606 2840
10·406	3025 0435	10·456	2877 5103	10·506	2737 1725	10·556	2603 6790
10·407	3022 0199	10·457	2874 6342	10·507	2734 4367	10·557	2601 0766
10·408	3018 9994	10·458	2871 7611	10·508	2731 7036	10·558	2598 4768
10·409	3015 9819	10·459	2868 8907	10·509	2728 9733	10·559	2595 8797
10·410	3012 9675	10·460	2866 0233	10·510	2726 2457	10·560	2593 2851
10·411	3009 9560	10·461	2863 1587	10·511	2723 5208	10·561	2590 6931
10·412	3006 9476	10·462	2860 2969	10·512	2720 7987	10·562	2588 1037
10·413	3003 9421	10·463	2857 4381	10·513	2718 0792	10·563	2585 5169
10·414	3000 9397	10·464	2854 5821	10·514	2715 3625	10·564	2582 9327
10·415	2997 9402	10·465	2851 7289	10·515	2712 6485	10·565	2580 3510
10·416	2994 9438	10·466	2848 8786	10·516	2709 9372	10·566	2577 7720
10·417	2991 9503	10·467	2846 0312	10·517	2707 2286	10·567	2575 1955
10·418	2988 9599	10·468	2843 1865	10·518	2704 5227	10·568	2572 6216
10·419	2985 9724	10·469	2840 3448	10·519	2701 8196	10·569	2570 0503
10·420	2982 9879	10·470	2837 5059	10·520	2699 1191	10·570	2567 4815
10·421	2980 0064	10·471	2834 6698	10·521	2696 4213	10·571	2564 9153
10·422	2977 0279	10·472	2831 8365	10·522	2693 7263	10·572	2562 3517
10·423	2974 0524	10·473	2829 0061	10·523	2691 0339	10·573	2559 7906
10·424	2971 0798	10·474	2826 1785	10·524	2688 3442	10·574	2557 2321
10·425	2968 1102	10·475	2823 3537	10·525	2685 6572	10·575	2554 6761
10·426	2965 1436	10·476	2820 5318	10·526	2682 9729	10·576	2552 1227
10·427	2962 1799	10·477	2817 7127	10·527	2680 2912	10·577	2549 5719
10·428	2959 2192	10·478	2814 8964	10·528	2677 6123	10·578	2547 0236
10·429	2956 2615	10·479	2812 0829	10·529	2674 9360	10·579	2544 4778
10·430	2953 3067	10·480	2809 2722	10·530	2672 2624	10·580	2541 9346
10·431	2950 3549	10·481	2806 4643	10·531	2669 5915	10·581	2539 3940
10·432	2947 4060	10·482	2803 6593	10·532	2666 9233	10·582	2536 8559
10·433	2944 4601	10·483	2800 8570	10·533	2664 2577	10·583	2534 3203
10·434	2941 5171	10·484	2798 0576	10·534	2661 5947	10·584	2531 7872
10·435	2938 5770	10·485	2795 2609	10·535	2658 9345	10·585	2529 2567
10·436	2935 6399	10·486	2792 4670	10·536	2656 2769	10·586	2526 7287
10·437	2932 7058	10·487	2789 6760	10·537	2653 6219	10·587	2524 2032
10·438	2929 7745	10·488	2786 8877	10·538	2650 9696	10·588	2521 6803
10·439	2926 8462	10·489	2784 1022	10·539	2648 3200	10·589	2519 1599
10·440	2923 9208	10·490	2781 3195	10·540	2645 6730	10·590	2516 6420
10·441	2920 9984	10·491	2778 5396	10·541	2643 0287	10·591	2514 1266
10·442	2918 0788	10·492	2775 7624	10·542	2640 3870	10·592	2511 6137
10·443	2915 1622	10·493	2772 9881	10·543	2637 7479	10·593	2509 1034
10·444	2912 2484	10·494	2770 2165	10·544	2635 1115	10·594	2506 5955
10·445	2909 3377	10·495	2767 4476	10·545	2632 4777	10·595	2504 0902
10·446	2906 4298	10·496	2764 6816	10·546	2629 8465	10·596	2501 5873
10·447	2903 5248	10·497	2761 9183	10·547	2627 2180	10·597	2499 0870
10·448	2900 6227	10·498	2759 1577	10·548	2624 5921	10·598	2496 5892
10·449	2897 7236	10·499	2756 3999	10·549	2621 9688	10·599	2494 0938

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
10·600	2491 0010	10·650	2370 0841	10·700	2254 4938	10·750	2144 5408
10·601	2489 1106	10·651	2367 7153	10·701	2252 2404	10·751	2142 3974
10·602	2486 6227	10·652	2365 3487	10·702	2249 9893	10·752	2140 2560
10·603	2484 1374	10·653	2362 9846	10·703	2247 7404	10·753	2138 1168
10·604	2481 6545	10·654	2360 6227	10·704	2245 4938	10·754	2135 9798
10·605	2479 1741	10·655	2358 2633	10·705	2243 2494	10·755	2133 8449
10·606	2476 6961	10·656	2355 9062	10·706	2241 0073	10·756	2131 7121
10·607	2474 2207	10·657	2353 5515	10·707	2238 7674	10·757	2129 5815
10·608	2471 7477	10·658	2351 1991	10·708	2236 5298	10·758	2127 4529
10·609	2469 2772	10·659	2348 8491	10·709	2234 2944	10·759	2125 3265
10·610	2466 8091	10·660	2346 5014	10·710	2232 0612	10·760	2123 2023
10·611	2464 3435	10·661	2344 1561	10·711	2229 8302	10·761	2121 0801
10·612	2461 8804	10·662	2341 8131	10·712	2227 6015	10·762	2118 9601
10·613	2459 4198	10·663	2339 4725	10·713	2225 3750	10·763	2116 8422
10·614	2456 9616	10·664	2337 1342	10·714	2223 1508	10·764	2114 7264
10·615	2454 5059	10·665	2334 7982	10·715	2220 9287	10·765	2112 6128
10·616	2452 0526	10·666	2332 4646	10·716	2218 7089	10·766	2110 5012
10·617	2449 6018	10·667	2330 1333	10·717	2216 4913	10·767	2108 3918
10·618	2447 1534	10·668	2327 8043	10·718	2214 2759	10·768	2106 2844
10·619	2444 7074	10·669	2325 4777	10·719	2212 0628	10·769	2104 1792
10·620	2442 2640	10·670	2323 1533	10·720	2209 8518	10·770	2102 0761
10·621	2439 8229	10·671	2320 8314	10·721	2207 6431	10·771	2099 9750
10·622	2437 3843	10·672	2318 5117	10·722	2205 4365	10·772	2097 8761
10·623	2434 9481	10·673	2316 1943	10·723	2203 2322	10·773	2095 7793
10·624	2432 5144	10·674	2313 8793	10·724	2201 0301	10·774	2093 6846
10·625	2430 0831	10·675	2311 5666	10·725	2198 8301	10·775	2091 5919
10·626	2427 6542	10·676	2309 2562	10·726	2196 6324	10·776	2089 5014
10·627	2425 2278	10·677	2306 9481	10·727	2194 4369	10·777	2087 4129
10·628	2422 8038	10·678	2304 6423	10·728	2192 2435	10·778	2085 3265
10·629	2420 3822	10·679	2302 3388	10·729	2190 0524	10·779	2083 2423
10·630	2417 9630	10·680	2300 0376	10·730	2187 8634	10·780	2081 1601
10·631	2415 5463	10·681	2297 7387	10·731	2185 6767	10·781	2079 0799
10·632	2413 1319	10·682	2295 4421	10·732	2183 4921	10·782	2077 0019
10·633	2410 7200	10·683	2293 1478	10·733	2181 3097	10·783	2074 9259
10·634	2408 3105	10·684	2290 8558	10·734	2179 1295	10·784	2072 8521
10·635	2405 9034	10·685	2288 5661	10·735	2176 9514	10·785	2070 7802
10·636	2403 4987	10·686	2286 2787	10·736	2174 7756	10·786	2068 7105
10·637	2401 0964	10·687	2283 9935	10·737	2172 6019	10·787	2066 6428
10·638	2398 6965	10·688	2281 7107	10·738	2170 4303	10·788	2064 5772
10·639	2396 2990	10·689	2279 4301	10·739	2168 2610	10·789	2062 5137
10·640	2393 9039	10·690	2277 1518	10·740	2166 0938	10·790	2060 4522
10·641	2391 5112	10·691	2274 8758	10·741	2163 9288	10·791	2058 3928
10·642	2389 1209	10·692	2272 6021	10·742	2161 7660	10·792	2056 3354
10·643	2386 7329	10·693	2270 3306	10·743	2159 6053	10·793	2054 2801
10·644	2384 3474	10·694	2268 0614	10·744	2157 4468	10·794	2052 2268
10·645	2381 9642	10·695	2265 7945	10·745	2155 2904	10·795	2050 1756
10·646	2379 5835	10·696	2263 5298	10·746	2153 1362	10·796	2048 1265
10·647	2377 2051	10·697	2261 2674	10·747	2150 9841	10·797	2046 0794
10·648	2374 8291	10·698	2259 0073	10·748	2148 8342	10·798	2044 0343
10·649	2372 4554	10·699	2256 7494	10·749	2146 6864	10·799	2041 9913

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
10·800	2039 9503	10·850	1940 4608	10·900	1845 8234	10·950	1755 8015
10·801	2037 9114	10·851	1938 5213	10·901	1843 9785	10·951	1754 0466
10·802	2035 8745	10·852	1936 5837	10·902	1842 1354	10·952	1752 2934
10·803	2033 8397	10·853	1934 6481	10·903	1840 2942	10·953	1750 5420
10·804	2031 8068	10·854	1932 7144	10·904	1838 4548	10·954	1748 7923
10·805	2029 7760	10·855	1930 7827	10·905	1836 6173	10·955	1747 0444
10·806	2027 7473	10·856	1928 8528	10·906	1834 7816	10·956	1745 2983
10·807	2025 7205	10·857	1926 9250	10·907	1832 9477	10·957	1743 5538
10·808	2023 6958	10·858	1924 9990	10·908	1831 1157	10·958	1741 8112
10·809	2021 6732	10·859	1923 0750	10·909	1829 2855	10·959	1740 0702
10·810	2019 6525	10·860	1921 1528	10·910	1827 4571	10·960	1738 3310
10·811	2017 6338	10·861	1919 2327	10·911	1825 6306	10·961	1736 5935
10·812	2015 6172	10·862	1917 3144	10·912	1823 8059	10·962	1734 8578
10·813	2013 6026	10·863	1915 3980	10·913	1821 9830	10·963	1733 1238
10·814	2011 5900	10·864	1913 4836	10·914	1820 1619	10·964	1731 3916
10·815	2009 5794	10·865	1911 5711	10·915	1818 3427	10·965	1729 6610
10·816	2007 5709	10·866	1909 6604	10·916	1816 5252	10·966	1727 9323
10·817	2005 5643	10·867	1907 7517	10·917	1814 7096	10·967	1726 2052
10·818	2003 5597	10·868	1905 8449	10·918	1812 8958	10·968	1724 4798
10·819	2001 5572	10·869	1903 9400	10·919	1811 0838	10·969	1722 7562
10·820	1999 5566	10·870	1902 0371	10·920	1809 2736	10·970	1721 0343
10·821	1997 5581	10·871	1900 1360	10·921	1807 4653	10·971	1719 3141
10·822	1995 5615	10·872	1898 2368	10·922	1805 6587	10·972	1717 5957
10·823	1993 5669	10·873	1896 3395	10·923	1803 8539	10·973	1715 8789
10·824	1991 5744	10·874	1894 4441	10·924	1802 0510	10·974	1714 1639
10·825	1989 5838	10·875	1892 5506	10·925	1800 2498	10·975	1712 4506
10·826	1987 5952	10·876	1890 6590	10·926	1798 4505	10·976	1710 7390
10·827	1985 6086	10·877	1888 7693	10·927	1796 6529	10·977	1709 0291
10·828	1983 6240	10·878	1886 8815	10·928	1794 8572	10·978	1707 3210
10·829	1981 6413	10·879	1884 9955	10·929	1793 0632	10·979	1705 6145
10·830	1979 6607	10·880	1883 1115	10·930	1791 2711	10·980	1703 9097
10·831	1977 6820	10·881	1881 2293	10·931	1789 4807	10·981	1702 2067
10·832	1975 7053	10·882	1879 3490	10·932	1787 6921	10·982	1700 5053
10·833	1973 7306	10·883	1877 4706	10·933	1785 9053	10·983	1698 8057
10·834	1971 7579	10·884	1875 5941	10·934	1784 1203	10·984	1697 1077
10·835	1969 7871	10·885	1873 7194	10·935	1782 3371	10·985	1695 4114
10·836	1967 8183	10·886	1871 8466	10·936	1780 5556	10·986	1693 7169
10·837	1965 8514	10·887	1869 9757	10·937	1778 7760	10·987	1692 0240
10·838	1963 8866	10·888	1868 1067	10·938	1776 9981	10·988	1690 3328
10·839	1961 9237	10·889	1866 2395	10·939	1775 2220	10·989	1688 6433
10·840	1959 9627	10·890	1864 3742	10·940	1773 4476	10·990	1686 9556
10·841	1958 0037	10·891	1862 5108	10·941	1771 6751	10·991	1685 2694
10·842	1956 0467	10·892	1860 6492	10·942	1769 9043	10·992	1683 5850
10·843	1954 0917	10·893	1858 7895	10·943	1768 1353	10·993	1681 9023
10·844	1952 1385	10·894	1856 9316	10·944	1766 3680	10·994	1680 2212
10·845	1950 1874	10·895	1855 0756	10·945	1764 6025	10·995	1678 5418
10·846	1948 2382	10·896	1853 2215	10·946	1762 8388	10·996	1676 8641
10·847	1946 2909	10·897	1851 3692	10·947	1761 0768	10·997	1675 1881
10·848	1944 3456	10·898	1849 5187	10·948	1759 3166	10·998	1673 5138
10·849	1942 4022	10·899	1847 6701	10·949	1757 5582	10·999	1671 8411

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
11·000	1670 1701	11·050	1588 7149	11·100	1511 2324	11·150	1437 5287
11·001	1668 5007	11·051	1587 1270	11·101	1509 7219	11·151	1436 0919
11·002	1666 8331	11·052	1585 5407	11·102	1508 2129	11·152	1434 6565
11·003	1665 1671	11·053	1583 9559	11·103	1506 7055	11·153	1433 2226
11·004	1663 5027	11·054	1582 3727	11·104	1505 1995	11·154	1431 7901
11·005	1661 8401	11·055	1580 7912	11·105	1503 6951	11·155	1430 3590
11·006	1660 1791	11·056	1579 2112	11·106	1502 1921	11·156	1428 9293
11·007	1658 5197	11·057	1577 6327	11·107	1500 6907	11·157	1427 5011
11·008	1656 8620	11·058	1576 0559	11·108	1499 1907	11·158	1426 0743
11·009	1655 2060	11·059	1574 4806	11·109	1497 6923	11·159	1424 6490
11·010	1653 5516	11·060	1572 9069	11·110	1496 1954	11·160	1423 2250
11·011	1651 8989	11·061	1571 3348	11·111	1494 6999	11·161	1421 8025
11·012	1650 2478	11·062	1569 7643	11·112	1493 2060	11·162	1420 3814
11·013	1648 5984	11·063	1568 1953	11·113	1491 7135	11·163	1418 9618
11·014	1646 9506	11·064	1566 6279	11·114	1490 2225	11·164	1417 5435
11·015	1645 3045	11·065	1565 0620	11·115	1488 7331	11·165	1416 1267
11·016	1643 6600	11·066	1563 4977	11·116	1487 2451	11·166	1414 7113
11·017	1642 0172	11·067	1561 9350	11·117	1485 7586	11·167	1413 2973
11·018	1640 3760	11·068	1560 3739	11·118	1484 2735	11·168	1411 8847
11·019	1638 7364	11·069	1558 8143	11·119	1482 7900	11·169	1410 4735
11·020	1637 0985	11·070	1557 2562	11·120	1481 3080	11·170	1409 0637
11·021	1635 4622	11·071	1555 6998	11·121	1479 8274	11·171	1407 6554
11·022	1633 8276	11·072	1554 1448	11·122	1478 3483	11·172	1406 2484
11·023	1632 1946	11·073	1552 5915	11·123	1476 8707	11·173	1404 8429
11·024	1630 5632	11·074	1551 0397	11·124	1475 3946	11·174	1403 4387
11·025	1628 9334	11·075	1549 4894	11·125	1473 9199	11·175	1402 0360
11·026	1627 3053	11·076	1547 9407	11·126	1472 4467	11·176	1400 6347
11·027	1625 6788	11·077	1546 3935	11·127	1470 9750	11·177	1399 2347
11·028	1624 0539	11·078	1544 8479	11·128	1469 5048	11·178	1397 8362
11·029	1622 4307	11·079	1543 3038	11·129	1468 0360	11·179	1396 4390
11·030	1620 8091	11·080	1541 7613	11·130	1466 5687	11·180	1395 0433
11·031	1619 1891	11·081	1540 2203	11·131	1465 1029	11·181	1393 6490
11·032	1617 5707	11·082	1538 6808	11·132	1463 6385	11·182	1392 2560
11·033	1615 9539	11·083	1537 1429	11·133	1462 1756	11·183	1390 8644
11·034	1614 3388	11·084	1535 6066	11·134	1460 7142	11·184	1389 4743
11·035	1612 7253	11·085	1534 0717	11·135	1459 2542	11·185	1388 0855
11·036	1611 1133	11·086	1532 5384	11·136	1457 7956	11·186	1386 6981
11·037	1609 5030	11·087	1531 0066	11·137	1456 3386	11·187	1385 3121
11·038	1607 8963	11·088	1529 4764	11·138	1454 8830	11·188	1383 9275
11·039	1606 2872	11·089	1527 9477	11·139	1453 4288	11·189	1382 5442
11·040	1604 6818	11·090	1526 4205	11·140	1451 9761	11·190	1381 1624
11·041	1603 0779	11·091	1524 8949	11·141	1450 5249	11·191	1379 7819
11·042	1601 4756	11·092	1523 3707	11·142	1449 0751	11·192	1378 4028
11·043	1599 8749	11·093	1521 8481	11·143	1447 6267	11·193	1377 0251
11·044	1598 2759	11·094	1520 3270	11·144	1446 1798	11·194	1375 6488
11·045	1596 6784	11·095	1518 8075	11·145	1444 7343	11·195	1374 2738
11·046	1595 0825	11·096	1517 2894	11·146	1443 2903	11·196	1372 9002
11·047	1593 4882	11·097	1515 7729	11·147	1441 8478	11·197	1371 5280
11·048	1591 8955	11·098	1514 2579	11·148	1440 4066	11·198	1370 1572
11·049	1590 3044	11·099	1512 7444	11·149	1438 9670	11·199	1368 7877

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
11·200	1367 4196	11·250	1300 7298	11·300	1237 2924	11·350	1176 9490
11·201	1366 0529	11·251	1299 4297	11·301	1236 0557	11·351	1175 7726
11·202	1364 6875	11·252	1298 1309	11·302	1234 8203	11·352	1174 5974
11·203	1363 3235	11·253	1296 8334	11·303	1233 5861	11·353	1173 4234
11·204	1361 9608	11·254	1295 5372	11·304	1232 3531	11·354	1172 2506
11·205	1360 5996	11·255	1294 2423	11·305	1231 1214	11·355	1171 0789
11·206	1359 2396	11·256	1292 9487	11·306	1229 8909	11·356	1169 9084
11·207	1357 8811	11·257	1291 6564	11·307	1228 6616	11·357	1168 7391
11·208	1356 5239	11·258	1290 3654	11·308	1227 4336	11·358	1167 5709
11·209	1355 1680	11·259	1289 0757	11·309	1226 2067	11·359	1166 4039
11·210	1353 8135	11·260	1287 7873	11·310	1224 9812	11·360	1165 2381
11·211	1352 4604	11·261	1286 5001	11·311	1223 7568	11·361	1164 0735
11·212	1351 1086	11·262	1285 2143	11·312	1222 5336	11·362	1162 9100
11·213	1349 7582	11·263	1283 9297	11·313	1221 3117	11·363	1161 7476
11·214	1348 4091	11·264	1282 6464	11·314	1220 0910	11·364	1160 5865
11·215	1347 0614	11·265	1281 3644	11·315	1218 8715	11·365	1159 4265
11·216	1345 7150	11·266	1280 0837	11·316	1217 6533	11·366	1158 2676
11·217	1344 3699	11·267	1278 8042	11·317	1216 4362	11·367	1157 1099
11·218	1343 0262	11·268	1277 5261	11·318	1215 2204	11·368	1155 9534
11·219	1341 6839	11·269	1276 2492	11·319	1214 0058	11·369	1154 7980
11·220	1340 3429	11·270	1274 9736	11·320	1212 7924	11·370	1153 6438
11·221	1339 0032	11·271	1273 6993	11·321	1211 5802	11·371	1152 4907
11·222	1337 6649	11·272	1272 4262	11·322	1210 3692	11·372	1151 3388
11·223	1336 3279	11·273	1271 1544	11·323	1209 1595	11·373	1150 1881
11·224	1334 9922	11·274	1269 8839	11·324	1207 9509	11·374	1149 0384
11·225	1333 6579	11·275	1268 6146	11·325	1206 7436	11·375	1147 8900
11·226	1332 3249	11·276	1267 3466	11·326	1205 5374	11·376	1146 7427
11·227	1330 9932	11·277	1266 0799	11·327	1204 3325	11·377	1145 5965
11·228	1329 6629	11·278	1264 8145	11·328	1203 1288	11·378	1144 4515
11·229	1328 3339	11·279	1263 5503	11·329	1201 9262	11·379	1143 3076
11·230	1327 0062	11·280	1262 2874	11·330	1200 7249	11·380	1142 1649
11·231	1325 6799	11·281	1261 0257	11·331	1199 5248	11·381	1141 0233
11·232	1324 3549	11·282	1259 7653	11·332	1198 3259	11·382	1139 8828
11·233	1323 0312	11·283	1258 5062	11·333	1197 1281	11·383	1138 7435
11·234	1321 7088	11·284	1257 2483	11·334	1195 9316	11·384	1137 6053
11·235	1320 3878	11·285	1255 9917	11·335	1194 7363	11·385	1136 4683
11·236	1319 0680	11·286	1254 7363	11·336	1193 5421	11·386	1135 3324
11·237	1317 7496	11·287	1253 4822	11·337	1192 3492	11·387	1134 1976
11·238	1316 4325	11·288	1252 2294	11·338	1191 1574	11·388	1133 0640
11·239	1315 1168	11·289	1250 9778	11·339	1189 9669	11·389	1131 9315
11·240	1313 8023	11·290	1249 7274	11·340	1188 7775	11·390	1130 8001
11·241	1312 4892	11·291	1248 4783	11·341	1187 5893	11·391	1129 6699
11·242	1311 1773	11·292	1247 2305	11·342	1186 4023	11·392	1128 5408
11·243	1309 8668	11·293	1245 9839	11·343	1185 2165	11·393	1127 4128
11·244	1308 5576	11·294	1244 7385	11·344	1184 0319	11·394	1126 2860
11·245	1307 2497	11·295	1243 4944	11·345	1182 8484	11·395	1125 1602
11·246	1305 9431	11·296	1242 2515	11·346	1181 6662	11·396	1124 0356
11·247	1304 6378	11·297	1241 0099	11·347	1180 4851	11·397	1122 9122
11·248	1303 3338	11·298	1239 7695	11·348	1179 3052	11·398	1121 7898
11·249	1302 0311	11·299	1238 5303	11·349	1178 1265	11·399	1120 6686

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
11·400	1119 5485	11·450	1064 9475	11·500	1013 0093	11·550	963 6043
11·401	1118 4295	11·451	1063 8830	11·501	1011 9968	11·551	962 6412
11·402	1117 3116	11·452	1062 8197	11·502	1010 9854	11·552	961 6790
11·403	1116 1949	11·453	1061 7574	11·503	1009 9749	11·553	960 7178
11·404	1115 0792	11·454	1060 6962	11·504	1008 9654	11·554	959 7576
11·405	1113 9647	11·455	1059 6360	11·505	1007 9569	11·555	958 7983
11·406	1112 8513	11·456	1058 5769	11·506	1006 9495	11·556	957 8400
11·407	1111 7390	11·457	1057 5189	11·507	1005 9430	11·557	956 8826
11·408	1110 6278	11·458	1056 4619	11·508	1004 9376	11·558	955 9262
11·409	1109 5177	11·459	1055 4059	11·509	1003 9332	11·559	954 9708
11·410	1108 4088	11·460	1054 3511	11·510	1002 9297	11·560	954 0163
11·411	1107 3009	11·461	1053 2972	11·511	1001 9273	11·561	953 0627
11·412	1106 1942	11·462	1052 2445	11·512	1000 9259	11·562	952 1102
11·413	1105 0885	11·463	1051 1927	11·513	0999 9255	11·563	951 1585
11·414	1103 9840	11·464	1050 1421	11·514	998 9260	11·564	950 2078
11·415	1102 8806	11·465	1049 0925	11·515	997 9276	11·565	949 2581
11·416	1101 7782	11·466	1048 0439	11·516	996 9302	11·566	948 3093
11·417	1100 6770	11·467	1046 9964	11·517	995 9337	11·567	947 3615
11·418	1099 5769	11·468	1045 9499	11·518	994 9383	11·568	946 4146
11·419	1098 4779	11·469	1044 9045	11·519	993 9439	11·569	945 4687
11·420	1097 3799	11·470	1043 8601	11·520	992 9504	11·570	944 5237
11·421	1096 2831	11·471	1042 8168	11·521	991 9580	11·571	943 5796
11·422	1095 1874	11·472	1041 7745	11·522	990 9665	11·572	942 6365
11·423	1094 0927	11·473	1040 7332	11·523	989 9760	11·573	941 6943
11·424	1092 9992	11·474	1039 6930	11·524	988 9866	11·574	940 7531
11·425	1091 9067	11·475	1038 6538	11·525	987 9981	11·575	939 8128
11·426	1090 8154	11·476	1037 6157	11·526	987 0106	11·576	938 8735
11·427	1089 7251	11·477	1036 5786	11·527	986 0240	11·577	937 9351
11·428	1088 6359	11·478	1035 5425	11·528	985 0385	11·578	936 9976
11·429	1087 5478	11·479	1034 5075	11·529	984 0540	11·579	936 0611
11·430	1086 4608	11·480	1033 4735	11·530	983 0704	11·580	935 1255
11·431	1085 3749	11·481	1032 4406	11·531	982 0878	11·581	934 1908
11·432	1084 2901	11·482	1031 4086	11·532	981 1062	11·582	933 2571
11·433	1083 2063	11·483	1030 3777	11·533	980 1256	11·583	932 3243
11·434	1082 1237	11·484	1029 3479	11·534	979 1460	11·584	931 3925
11·435	1081 0421	11·485	1028 3190	11·535	978 1673	11·585	930 4615
11·436	1079 9616	11·486	1027 2912	11·536	977 1896	11·586	929 5315
11·437	1078 8821	11·487	1026 2645	11·537	976 2129	11·587	928 6025
11·438	1077 8038	11·488	1025 2387	11·538	975 2372	11·588	927 6743
11·439	1076 7265	11·489	1024 2140	11·539	974 2625	11·589	926 7471
11·440	1075 6504	11·490	1023 1903	11·540	973 2887	11·590	925 8208
11·441	1074 5752	11·491	1022 1676	11·541	972 3159	11·591	924 8955
11·442	1073 5012	11·492	1021 1459	11·542	971 3441	11·592	923 9710
11·443	1072 4282	11·493	1020 1253	11·543	970 3732	11·593	923 0475
11·444	1071 3563	11·494	1019 1057	11·544	969 4033	11·594	922 1250
11·445	1070 2855	11·495	1018 0871	11·545	968 4344	11·595	921 2033
11·446	1069 2158	11·496	1017 0695	11·546	967 4664	11·596	920 2825
11·447	1068 1471	11·497	1016 0529	11·547	966 4995	11·597	919 3627
11·448	1067 0795	11·498	1015 0374	11·548	965 5334	11·598	918 4438
11·449	1066 0129	11·499	1014 0229	11·549	964 5684	11·599	917 5258

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
11·600	916 6088	11·650	871 9052	11·700	829 3819	11·750	788 9325
11·601	915 6926	11·651	871 0338	11·701	828 5529	11·751	788 1439
11·602	914 7774	11·652	870 1632	11·702	827 7248	11·752	787 3562
11·603	913 8631	11·653	869 2934	11·703	826 8975	11·753	786 5692
11·604	912 9497	11·654	868 4246	11·704	826 0710	11·754	785 7830
11·605	912 0372	11·655	867 5566	11·705	825 2453	11·755	784 9977
11·606	911 1256	11·656	866 6895	11·706	824 4205	11·756	784 2131
11·607	910 2149	11·657	865 8232	11·707	823 5965	11·757	783 4292
11·608	909 3051	11·658	864 9578	11·708	822 7733	11·758	782 6462
11·609	908 3963	11·659	864 0933	11·709	821 9510	11·759	781 8639
11·610	907 4884	11·660	863 2296	11·710	821 1294	11·760	781 0825
11·611	906 5813	11·661	862 3668	11·711	820 3087	11·761	780 3018
11·612	905 6752	11·662	861 5049	11·712	819 4888	11·762	779 5219
11·613	904 7700	11·663	860 6438	11·713	818 6697	11·763	778 7427
11·614	903 8657	11·664	859 7836	11·714	817 8515	11·764	777 9644
11·615	902 9622	11·665	858 9242	11·715	817 0340	11·765	777 1868
11·616	902 0597	11·666	858 0657	11·716	816 2174	11·766	776 4100
11·617	901 1581	11·667	857 2081	11·717	815 4016	11·767	775 6340
11·618	900 2574	11·668	856 3513	11·718	814 5866	11·768	774 8587
11·619	899 3576	11·669	855 4954	11·719	813 7724	11·769	774 0843
11·620	898 4587	11·670	854 6403	11·720	812 9590	11·770	773 3106
11·621	897 5607	11·671	853 7861	11·721	812 1465	11·771	772 5376
11·622	896 6636	11·672	852 9328	11·722	811 3348	11·772	771 7655
11·623	895 7674	11·673	852 0803	11·723	810 5238	11·773	770 9941
11·624	894 8720	11·674	851 2286	11·724	809 7137	11·774	770 2235
11·625	893 9776	11·675	850 3778	11·725	808 9044	11·775	769 4537
11·626	893 0841	11·676	849 5279	11·726	808 0959	11·776	768 6846
11·627	892 1914	11·677	848 6788	11·727	807 2882	11·777	767 9163
11·628	891 2997	11·678	847 8305	11·728	806 4813	11·778	767 1488
11·629	890 4088	11·679	846 9831	11·729	805 6752	11·779	766 3820
11·630	889 5189	11·680	846 1365	11·730	804 8700	11·780	765 6160
11·631	888 6298	11·681	845 2908	11·731	804 0655	11·781	764 8508
11·632	887 7416	11·682	844 4460	11·732	803 2618	11·782	764 0863
11·633	886 8543	11·683	843 6019	11·733	802 4590	11·783	763 3226
11·634	885 9679	11·684	842 7587	11·734	801 6569	11·784	762 5596
11·635	885 0824	11·685	841 9164	11·735	800 8557	11·785	761 7975
11·636	884 1977	11·686	841 0749	11·736	800 0552	11·786	761 0361
11·637	883 3140	11·687	840 2343	11·737	799 2556	11·787	760 2754
11·638	882 4311	11·688	839 3944	11·738	798 4567	11·788	759 5155
11·639	881 5491	11·689	838 5555	11·739	797 6586	11·789	758 7564
11·640	880 6680	11·690	837 7173	11·740	796 8614	11·790	757 9980
11·641	879 7878	11·691	836 8800	11·741	796 0649	11·791	757 2404
11·642	878 9084	11·692	836 0436	11·742	795 2692	11·792	756 4835
11·643	878 0300	11·693	835 2080	11·743	794 4744	11·793	755 7274
11·644	877 1524	11·694	834 3732	11·744	793 6803	11·794	754 9721
11·645	876 2757	11·695	833 5392	11·745	792 8870	11·795	754 2175
11·646	875 3998	11·696	832 7061	11·746	792 0945	11·796	753 4636
11·647	874 5249	11·697	831 8738	11·747	791 3028	11·797	752 7105
11·648	873 6508	11·698	831 0423	11·748	790 5119	11·798	751 9582
11·649	872 7776	11·699	830 2117	11·749	789 7218	11·799	751 2066

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
11·800	750 4558	11·850	713 8556	11·900	679 0405	11·950	645 9233
11·801	749 7057	11·851	713 1421	11·901	678 3618	11·951	645 2777
11·802	748 9564	11·852	712 4294	11·902	677 6837	11·952	644 6327
11·803	748 2078	11·853	711 7173	11·903	677 0064	11·953	643 9884
11·804	747 4600	11·854	711 0059	11·904	676 3297	11·954	643 3447
11·805	746 7129	11·855	710 2953	11·905	675 6537	11·955	642 7017
11·806	745 9665	11·856	709 5853	11·906	674 9784	11·956	642 0593
11·807	745 2209	11·857	708 8761	11·907	674 3038	11·957	641 4176
11·808	744 4761	11·858	708 1676	11·908	673 6298	11·958	640 7765
11·809	743 7320	11·859	707 4598	11·909	672 9565	11·959	640 1360
11·810	742 9886	11·860	706 7526	11·910	672 2839	11·960	639 4962
11·811	742 2460	11·861	706 0463	11·911	671 6120	11·961	638 8571
11·812	741 5041	11·862	705 3406	11·912	670 9407	11·962	638 2185
11·813	740 7630	11·863	704 6356	11·913	670 2701	11·963	637 5806
11·814	740 0226	11·864	703 9313	11·914	669 6001	11·964	636 9454
11·815	739 2829	11·865	703 2277	11·915	668 9309	11·965	636 3067
11·816	738 5440	11·866	702 5248	11·916	668 2623	11·966	635 6707
11·817	737 8059	11·867	701 8227	11·917	667 5944	11·967	635 0354
11·818	737 0684	11·868	701 1212	11·918	666 9271	11·968	634 4007
11·819	736 3317	11·869	700 4204	11·919	666 2605	11·969	633 7666
11·820	735 5958	11·870	699 7203	11·920	665 5946	11·970	633 1331
11·821	734 8605	11·871	699 0210	11·921	664 9293	11·971	632 5003
11·822	734 1260	11·872	698 3223	11·922	664 2647	11·972	631 8681
11·823	733 3923	11·873	697 6243	11·923	663 6008	11·973	631 2366
11·824	732 6592	11·874	696 9271	11·924	662 9375	11·974	630 6057
11·825	731 9270	11·875	696 2305	11·925	662 2749	11·975	629 9754
11·826	731 1954	11·876	695 5346	11·926	661 6130	11·976	629 3457
11·827	730 4646	11·877	694 8394	11·927	660 9517	11·977	628 7167
11·828	729 7345	11·878	694 1449	11·928	660 2911	11·978	628 0883
11·829	729 0051	11·879	693 4511	11·929	659 6311	11·979	627 4605
11·830	728 2765	11·880	692 7580	11·930	658 9718	11·980	626 8334
11·831	727 5485	11·881	692 0656	11·931	658 3132	11·981	626 2068
11·832	726 8214	11·882	691 3739	11·932	657 6552	11·982	625 5809
11·833	726 0949	11·883	690 6829	11·933	656 9978	11·983	624 9557
11·834	725 3692	11·884	689 9925	11·934	656 3412	11·984	624 3310
11·835	724 6442	11·885	689 3029	11·935	655 6852	11·985	623 7070
11·836	723 9199	11·886	688 6139	11·936	655 0298	11·986	623 0836
11·837	723 1963	11·887	687 9256	11·937	654 3751	11·987	622 4608
11·838	722 4735	11·888	687 2380	11·938	653 7211	11·988	621 8387
11·839	721 7514	11·889	686 5512	11·939	653 0677	11·989	621 2172
11·840	721 0300	11·890	685 8649	11·940	652 4149	11·990	620 5963
11·841	720 3093	11·891	685 1794	11·941	651 7628	11·991	619 9760
11·842	719 5894	11·892	684 4946	11·942	651 1114	11·992	619 3563
11·843	718 8702	11·893	683 8104	11·943	650 4606	11·993	618 7373
11·844	718 1516	11·894	683 1270	11·944	649 8105	11·994	618 1188
11·845	717 4339	11·895	682 4442	11·945	649 1610	11·995	617 5010
11·846	716 7168	11·896	681 7621	11·946	648 5121	11·996	616 8838
11·847	716 0004	11·897	681 0807	11·947	647 8640	11·997	616 2673
11·848	715 2848	11·898	680 3999	11·948	647 2164	11·998	615 6513
11·849	714 5699	11·899	679 7199	11·949	646 5695	11·999	615 0360

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
12·000	614 4212	12·050	584 4555	12·100	555 9513	12·150	528 8372
12·001	613 8071	12·051	583 8714	12·101	555 3956	12·151	528 3087
12·002	613 1936	12·052	583 2878	12·102	554 8405	12·152	527 7806
12·003	612 5807	12·053	582 7048	12·103	554 2860	12·153	527 2531
12·004	611 9684	12·054	582 1224	12·104	553 7320	12·154	526 7261
12·005	611 3568	12·055	581 5406	12·105	553 1785	12·155	526 1997
12·006	610 7457	12·056	580 9593	12·106	552 6256	12·156	525 6737
12·007	610 1353	12·057	580 3787	12·107	552 0732	12·157	525 1483
12·008	609 5255	12·058	579 7986	12·108	551 5214	12·158	524 6234
12·009	608 9162	12·059	579 2191	12·109	550 9702	12·159	524 0991
12·010	608 3076	12·060	578 6401	12·110	550 4195	12·160	523 5752
12·011	607 6996	12·061	578 0618	12·111	549 8694	12·161	523 0519
12·012	607 0922	12·062	577 4840	12·112	549 3198	12·162	522 5291
12·013	606 4854	12·063	576 9068	12·113	548 7707	12·163	522 0069
12·014	605 8793	12·064	576 3302	12·114	548 2222	12·164	521 4851
12·015	605 2737	12·065	575 7541	12·115	547 6743	12·165	520 9639
12·016	604 6687	12·066	575 1787	12·116	547 1269	12·166	520 4432
12·017	604 0643	12·067	574 6038	12·117	546 5800	12·167	519 9230
12·018	603 4606	12·068	574 0295	12·118	546 0337	12·168	519 4033
12·019	602 8574	12·069	573 4557	12·119	545 4880	12·169	518 8842
12·020	602 2549	12·070	572 8825	12·120	544 9427	12·170	518 3656
12·021	601 6529	12·071	572 3099	12·121	544 3981	12·171	517 8475
12·022	601 0516	12·072	571 7379	12·122	543 8539	12·172	517 3299
12·023	600 4508	12·073	571 1665	12·123	543 3104	12·173	516 8128
12·024	599 8507	12·074	570 5956	12·124	542 7673	12·174	516 2962
12·025	599 2511	12·075	570 0253	12·125	542 2248	12·175	515 7802
12·026	598 6522	12·076	569 4555	12·126	541 6829	12·176	515 2647
12·027	598 0538	12·077	568 8864	12·127	541 1415	12·177	514 7497
12·028	597 4561	12·078	568 3178	12·128	540 6006	12·178	514 2352
12·029	596 8589	12·079	567 7497	12·129	540 0603	12·179	513 7212
12·030	596 2623	12·080	567 1823	12·130	539 5205	12·180	513 2077
12·031	595 6664	12·081	566 6154	12·131	538 9812	12·181	512 6948
12·032	595 0710	12·082	566 0490	12·132	538 4425	12·182	512 1824
12·033	594 4762	12·083	565 4833	12·133	537 9043	12·183	511 6704
12·034	593 8821	12·084	564 9181	12·134	537 3667	12·184	511 1590
12·035	593 2885	12·085	564 3534	12·135	536 8296	12·185	510 6481
12·036	592 6955	12·086	563 7894	12·136	536 2930	12·186	510 1377
12·037	592 1031	12·087	563 2259	12·137	535 7570	12·187	509 6278
12·038	591 5113	12·088	562 6629	12·138	535 2215	12·188	509 1185
12·039	590 9201	12·089	562 1005	12·139	534 6866	12·189	508 6096
12·040	590 3294	12·090	561 5387	12·140	534 1522	12·190	508 1012
12·041	589 7394	12·091	560 9775	12·141	533 6183	12·191	507 5934
12·042	589 1499	12·092	560 4168	12·142	533 0849	12·192	507 0861
12·043	588 5611	12·093	559 8566	12·143	532 5521	12·193	506 5792
12·044	587 9728	12·094	559 2970	12·144	532 0198	12·194	506 0729
12·045	587 3851	12·095	558 7380	12·145	531 4881	12·195	505 5671
12·046	586 7981	12·096	558 1796	12·146	530 9568	12·196	505 0618
12·047	586 2115	12·097	557 6217	12·147	530 4261	12·197	504 5570
12·048	585 6256	12·098	557 0643	12·148	529 8960	12·198	504 0526
12·049	585 0403	12·099	556 5075	12·149	529 3664	12·199	503 5488

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
12·200	503 0456	12·250	478 5117	12·300	455 1744	12·350	432 9753
12·201	502 5428	12·251	478 0335	12·301	454 7195	12·351	432 5425
12·202	502 0405	12·252	477 5557	12·302	454 2650	12·352	432 1102
12·203	501 5387	12·253	477 0783	12·303	453 8110	12·353	431 6783
12·204	501 0374	12·254	476 6015	12·304	453 3574	12·354	431 2469
12·205	500 5366	12·255	476 1251	12·305	452 9042	12·355	430 8158
12·206	500 0363	12·256	475 6493	12·306	452 4516	12·356	430 3852
12·207	499 5365	12·257	475 1738	12·307	451 9993	12·357	429 9551
12·208	499 0372	12·258	474 6989	12·308	451 5476	12·358	429 5253
12·209	498 5385	12·259	474 2244	12·309	451 0962	12·359	429 0960
12·210	498 0402	12·260	473 7505	12·310	450 6454	12·360	428 6671
12·211	497 5424	12·261	473 2769	12·311	450 1950	12·361	428 2387
12·212	497 0451	12·262	472 8039	12·312	449 7450	12·362	427 8107
12·213	496 5483	12·263	472 3313	12·313	449 2955	12·363	427 3831
12·214	496 0520	12·264	471 8592	12·314	448 8464	12·364	426 9559
12·215	495 5562	12·265	471 3876	12·315	448 3978	12·365	426 5291
12·216	495 0609	12·266	470 9165	12·316	447 9496	12·366	426 1028
12·217	494 5661	12·267	470 4458	12·317	447 5019	12·367	425 6769
12·218	494 0717	12·268	469 9756	12·318	447 0546	12·368	425 2515
12·219	493 5779	12·269	469 5058	12·319	446 6078	12·369	424 8264
12·220	493 0846	12·270	469 0366	12·320	446 1614	12·370	424 4018
12·221	492 5917	12·271	468 5678	12·321	445 7154	12·371	423 9776
12·222	492 0994	12·272	468 0994	12·322	445 2699	12·372	423 5539
12·223	491 6075	12·273	467 6316	12·323	444 8249	12·373	423 1305
12·224	491 1162	12·274	467 1642	12·324	444 3803	12·374	422 7076
12·225	490 6253	12·275	466 6972	12·325	443 9361	12·375	422 2851
12·226	490 1349	12·276	466 2308	12·326	443 4924	12·376	421 8630
12·227	489 6450	12·277	465 7648	12·327	443 0492	12·377	421 4414
12·228	489 1556	12·278	465 2992	12·328	442 6063	12·378	421 0202
12·229	488 6667	12·279	464 8342	12·329	442 1639	12·379	420 5993
12·230	488 1783	12·280	464 3696	12·330	441 7220	12·380	420 1790
12·231	487 6904	12·281	463 9054	12·331	441 2805	12·381	419 7590
12·232	487 2029	12·282	463 4418	12·332	440 8394	12·382	419 3394
12·233	486 7160	12·283	462 9786	12·333	440 3988	12·383	418 9203
12·234	486 2295	12·284	462 5158	12·334	439 9586	12·384	418 5016
12·235	485 7435	12·285	462 0535	12·335	439 5189	12·385	418 0833
12·236	485 2580	12·286	461 5917	12·336	439 0796	12·386	417 6654
12·237	484 7730	12·287	461 1303	12·337	438 6407	12·387	417 2480
12·238	484 2885	12·288	460 6694	12·338	438 2023	12·388	416 8309
12·239	483 8044	12·289	460 2090	12·339	437 7643	12·389	416 4143
12·240	483 3209	12·290	459 7490	12·340	437 3268	12·390	415 9981
12·241	482 8378	12·291	459 2895	12·341	436 8897	12·391	415 5823
12·242	482 3552	12·292	458 8304	12·342	436 4530	12·392	415 1669
12·243	481 8731	12·293	458 3718	12·343	436 0168	12·393	414 7520
12·244	481 3914	12·294	457 9137	12·344	435 5810	12·394	414 3374
12·245	480 9103	12·295	457 4560	12·345	435 1456	12·395	413 9233
12·246	480 4296	12·296	456 9988	12·346	434 7107	12·396	413 5096
12·247	479 9494	12·297	456 5420	12·347	434 2762	12·397	413 0963
12·248	479 4697	12·298	456 0857	12·348	433 8421	12·398	412 6834
12·249	478 9905	12·299	455 6298	12·349	433 4085	12·399	412 2709

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
12·400	411 8589	12·450	391 7723	12·500	372 6653	12·550	354 4902
12·401	411 4472	12·451	391 3807	12·501	372 2928	12·551	354 1359
12·402	411 0360	12·452	390 9895	12·502	371 9207	12·552	353 7819
12·403	410 6251	12·453	390 5987	12·503	371 5490	12·553	353 4283
12·404	410 2147	12·454	390 2083	12·504	371 1776	12·554	353 0751
12·405	409 8047	12·455	389 8183	12·505	370 8066	12·555	352 7222
12·406	409 3951	12·456	389 4287	12·506	370 4360	12·556	352 3696
12·407	408 9859	12·457	389 0394	12·507	370 0658	12·557	352 0174
12·408	408 5771	12·458	388 6506	12·508	369 6959	12·558	351 6656
12·409	408 1688	12·459	388 2621	12·509	369 3264	12·559	351 3141
12·410	407 7608	12·460	387 8741	12·510	368 9572	12·560	350 9630
12·411	407 3532	12·461	387 4864	12·511	368 5885	12·561	350 6122
12·412	406 9461	12·462	387 0991	12·512	368 2200	12·562	350 2617
12·413	406 5393	12·463	386 7122	12·513	367 8520	12·563	349 9117
12·414	406 1330	12·464	386 3257	12·514	367 4843	12·564	349 5619
12·415	405 7271	12·465	385 9395	12·515	367 1170	12·565	349 2125
12·416	405 3216	12·466	385 5538	12·516	366 7501	12·566	348 8635
12·417	404 9164	12·467	385 1684	12·517	366 3835	12·567	348 5148
12·418	404 5117	12·468	384 7835	12·518	366 0173	12·568	348 1665
12·419	404 1074	12·469	384 3989	12·519	365 6515	12·569	347 8185
12·420	403 7035	12·470	384 0147	12·520	365 2860	12·570	347 4708
12·421	403 3000	12·471	383 6308	12·521	364 9209	12·571	347 1235
12·422	402 8969	12·472	383 2474	12·522	364 5562	12·572	346 7766
12·423	402 4942	12·473	382 8643	12·523	364 1918	12·573	346 4300
12·424	402 0919	12·474	382 4817	12·524	363 8278	12·574	346 0837
12·425	401 6900	12·475	382 0994	12·525	363 4642	12·575	345 7378
12·426	401 2885	12·476	381 7175	12·526	363 1009	12·576	345 3922
12·427	400 8875	12·477	381 3359	12·527	362 7380	12·577	345 0470
12·428	400 4868	12·478	380 9548	12·528	362 3754	12·578	344 7022
12·429	400 0865	12·479	380 5740	12·529	362 0132	12·579	344 3576
12·430	399 6866	12·480	380 1936	12·530	361 6514	12·580	344 0134
12·431	399 2871	12·481	379 8136	12·531	361 2899	12·581	343 6696
12·432	398 8880	12·482	379 4342	12·532	360 9288	12·582	343 3261
12·433	398 4893	12·483	379 0548	12·533	360 5681	12·583	342 9829
12·434	398 0910	12·484	378 6759	12·534	360 2077	12·584	342 6401
12·435	397 6931	12·485	378 2974	12·535	359 8476	12·585	342 2977
12·436	397 2957	12·486	377 9193	12·536	359 4880	12·586	341 9555
12·437	396 8986	12·487	377 5416	12·537	359 1287	12·587	341 6138
12·438	396 5019	12·488	377 1642	12·538	358 7697	12·588	341 2723
12·439	396 1056	12·489	376 7873	12·539	358 4111	12·589	340 9312
12·440	395 7096	12·490	376 4107	12·540	358 0529	12·590	340 5904
12·441	395 3141	12·491	376 0344	12·541	357 6950	12·591	340 2500
12·442	394 9190	12·492	375 6586	12·542	357 3375	12·592	339 9099
12·443	394 5243	12·493	375 2831	12·543	356 9803	12·593	339 5702
12·444	394 1300	12·494	374 9080	12·544	356 6235	12·594	339 2308
12·445	393 7360	12·495	374 5333	12·545	356 2671	12·595	338 8917
12·446	393 3425	12·496	374 1590	12·546	355 9110	12·596	338 5530
12·447	392 9494	12·497	373 7850	12·547	355 5553	12·597	338 2146
12·448	392 5566	12·498	373 4114	12·548	355 1999	12·598	337 8766
12·449	392 1642	12·499	373 0382	12·549	354 8449	12·599	337 5389

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
12·600	337 2015	12·650	320 7560	12·700	305 1125	12·750	290 2320
12·601	336 8645	12·651	320 4354	12·701	304 8076	12·751	289 9419
12·602	336 5278	12·652	320 1151	12·702	304 5029	12·752	289 6521
12·603	336 1914	12·653	319 7952	12·703	304 1986	12·753	289 3626
12·604	335 8554	12·654	319 4755	12·704	303 8945	12·754	289 0734
12·605	335 5197	12·655	319 1562	12·705	303 5908	12·755	288 7845
12·606	335 1844	12·656	318 8372	12·706	303 2874	12·756	288 4959
12·607	334 8493	12·657	318 5185	12·707	302 9842	12·757	288 2075
12·608	334 5147	12·658	318 2002	12·708	302 6814	12·758	287 9194
12·609	334 1803	12·659	317 8821	12·709	302 3789	12·759	287 6317
12·610	333 8463	12·660	317 5644	12·710	302 0766	12·760	287 3442
12·611	333 5126	12·661	317 2470	12·711	301 7747	12·761	287 0570
12·612	333 1793	12·662	316 9299	12·712	301 4731	12·762	286 7701
12·613	332 8463	12·663	316 6132	12·713	301 1718	12·763	286 4834
12·614	332 5136	12·664	316 2967	12·714	300 8707	12·764	286 1971
12·615	332 1812	12·665	315 9806	12·715	300 5700	12·765	285 9110
12·616	331 8492	12·666	315 6647	12·716	300 2696	12·766	285 6253
12·617	331 5175	12·667	315 3492	12·717	299 9695	12·767	285 3398
12·618	331 1862	12·668	315 0340	12·718	299 6697	12·768	285 0546
12·619	330 8552	12·669	314 7192	12·719	299 3701	12·769	284 7697
12·620	330 5245	12·670	314 4046	12·720	299 0709	12·770	284 4851
12·621	330 1941	12·671	314 0904	12·721	298 7720	12·771	284 2007
12·622	329 8641	12·672	313 7764	12·722	298 4734	12·772	283 9167
12·623	329 5344	12·673	313 4628	12·723	298 1750	12·773	283 6329
12·624	329 2050	12·674	313 1495	12·724	297 8770	12·774	283 3494
12·625	328 8760	12·675	312 8365	12·725	297 5793	12·775	283 0662
12·626	328 5473	12·676	312 5238	12·726	297 2819	12·776	282 7833
12·627	328 2189	12·677	312 2115	12·727	296 9847	12·777	282 5006
12·628	327 8908	12·678	311 8994	12·728	296 6879	12·778	282 2183
12·629	327 5631	12·679	311 5877	12·729	296 3913	12·779	281 9362
12·630	327 2357	12·680	311 2762	12·730	296 0951	12·780	281 6544
12·631	326 9086	12·681	310 9651	12·731	295 7992	12·781	281 3729
12·632	326 5819	12·682	310 6543	12·732	295 5035	12·782	281 0916
12·633	326 2555	12·683	310 3438	12·733	295 2082	12·783	280 8107
12·634	325 9294	12·684	310 0336	12·734	294 9131	12·784	280 5300
12·635	325 6036	12·685	309 7237	12·735	294 6183	12·785	280 2496
12·636	325 2782	12·686	309 4142	12·736	294 3239	12·786	279 9695
12·637	324 9531	12·687	309 1049	12·737	294 0297	12·787	279 6897
12·638	324 6283	12·688	308 7960	12·738	293 7358	12·788	279 4101
12·639	324 3038	12·689	308 4873	12·739	293 4422	12·789	279 1309
12·640	323 9797	12·690	308 1790	12·740	293 1489	12·790	278 8519
12·641	323 6558	12·691	307 8710	12·741	292 8559	12·791	278 5732
12·642	323 3323	12·692	307 5632	12·742	292 5632	12·792	278 2947
12·643	323 0092	12·693	307 2558	12·743	292 2708	12·793	278 0166
12·644	322 6863	12·694	306 9487	12·744	291 9787	12·794	277 7387
12·645	322 3638	12·695	306 6419	12·745	291 6868	12·795	277 4611
12·646	322 0416	12·696	306 3354	12·746	291 3953	12·796	277 1838
12·647	321 7197	12·697	306 0293	12·747	291 1040	12·797	276 9067
12·648	321 3982	12·698	305 7234	12·748	290 8131	12·798	276 6300
12·649	321 0769	12·699	305 4178	12·749	290 5224	12·799	276 3535

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
12·800	276 0772	12·850	262 6128	12·900	249 8050	12·950	237 6219
12·801	275 8013	12·851	262 3503	12·901	249 5553	12·951	237 3844
12·802	275 5256	12·852	262 0881	12·902	249 3059	12·952	237 1471
12·803	275 2503	12·853	261 8261	12·903	249 0567	12·953	236 9101
12·804	274 9751	12·854	261 5644	12·904	248 8078	12·954	236 6733
12·805	274 7003	12·855	261 3030	12·905	248 5591	12·955	236 4367
12·806	274 4257	12·856	261 0418	12·906	248 3106	12·956	236 2004
12·807	274 1515	12·857	260 7809	12·907	248 0625	12·957	235 9643
12·808	273 8774	12·858	260 5203	12·908	247 8145	12·958	235 7285
12·809	273 6037	12·859	260 2599	12·909	247 5668	12·959	235 4929
12·810	273 3302	12·860	259 9998	12·910	247 3194	12·960	235 2575
12·811	273 0570	12·861	259 7399	12·911	247 0722	12·961	235 0224
12·812	272 7841	12·862	259 4803	12·912	246 8253	12·962	234 7875
12·813	272 5115	12·863	259 2209	12·913	246 5786	12·963	234 5528
12·814	272 2391	12·864	258 9618	12·914	246 3321	12·964	234 3184
12·815	271 9670	12·865	258 7030	12·915	246 0859	12·965	234 0842
12·816	271 6951	12·866	258 4444	12·916	245 8399	12·966	233 8502
12·817	271 4236	12·867	258 1861	12·917	245 5942	12·967	233 6165
12·818	271 1523	12·868	257 9281	12·918	245 3488	12·968	233 3830
12·819	270 8813	12·869	257 6703	12·919	245 1035	12·969	233 1497
12·820	270 6105	12·870	257 4127	12·920	244 8586	12·970	232 9167
12·821	270 3401	12·871	257 1554	12·921	244 6138	12·971	232 6839
12·822	270 0699	12·872	256 8984	12·922	244 3693	12·972	232 4513
12·823	269 7999	12·873	256 6416	12·923	244 1251	12·973	232 2190
12·824	269 5303	12·874	256 3851	12·924	243 8811	12·974	231 9869
12·825	269 2609	12·875	256 1289	12·925	243 6373	12·975	231 7550
12·826	268 9917	12·876	255 8729	12·926	243 3938	12·976	231 5233
12·827	268 7229	12·877	255 6171	12·927	243 1505	12·977	231 2919
12·828	268 4543	12·878	255 3616	12·928	242 9075	12·978	231 0608
12·829	268 1860	12·879	255 1064	12·929	242 6647	12·979	230 8298
12·830	267 9179	12·880	254 8514	12·930	242 4222	12·980	230 5991
12·831	267 6501	12·881	254 5967	12·931	242 1799	12·981	230 3686
12·832	267 3826	12·882	254 3422	12·932	241 9378	12·982	230 1384
12·833	267 1154	12·883	254 0880	12·933	241 6960	12·983	229 9083
12·834	266 8484	12·884	253 8340	12·934	241 4544	12·984	229 6785
12·835	266 5817	12·885	253 5803	12·935	241 2131	12·985	229 4490
12·836	266 3152	12·886	253 3269	12·936	240 9720	12·986	229 2196
12·837	266 0490	12·887	253 0737	12·937	240 7311	12·987	228 9905
12·838	265 7831	12·888	252 8207	12·938	240 4905	12·988	228 7617
12·839	265 5175	12·889	252 5680	12·939	240 2502	12·989	228 5330
12·840	265 2521	12·890	252 3156	12·940	240 0100	12·990	228 3046
12·841	264 9870	12·891	252 0634	12·941	239 7701	12·991	228 0764
12·842	264 7221	12·892	251 8114	12·942	239 5305	12·992	227 8484
12·843	264 4575	12·893	251 5598	12·943	239 2911	12·993	227 6207
12·844	264 1932	12·894	251 3083	12·944	239 0519	12·994	227 3932
12·845	263 9291	12·895	251 0571	12·945	238 8130	12·995	227 1659
12·846	263 6653	12·896	250 8062	12·946	238 5743	12·996	226 9389
12·847	263 4018	12·897	250 5555	12·947	238 3358	12·997	226 7120
12·848	263 1385	12·898	250 3051	12·948	238 0976	12·998	226 4854
12·849	262 8755	12·899	250 0549	12·949	237 8596	12·999	226 2591

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
13·000	226 0329	13·050	215 0092	13·100	204 5231	13·150	194 5483
13·001	225 8070	13·051	214 7943	13·101	204 3186	13·151	194 3539
13·002	225 5813	13·052	214 5796	13·102	204 1144	13·152	194 1596
13·003	225 3558	13·053	214 3651	13·103	203 9104	13·153	193 9656
13·004	225 1306	13·054	214 1508	13·104	203 7066	13·154	193 7717
13·005	224 9056	13·055	213 9368	13·105	203 5030	13·155	193 5780
13·006	224 6808	13·056	213 7230	13·106	203 2996	13·156	193 3846
13·007	224 4562	13·057	213 5094	13·107	203 0964	13·157	193 1913
13·008	224 2319	13·058	213 2960	13·108	202 8934	13·158	192 9982
13·009	224 0078	13·059	213 0828	13·109	202 6906	13·159	192 8053
13·010	223 7839	13·060	212 8698	13·110	202 4880	13·160	192 6126
13·011	223 5602	13·061	212 6570	13·111	202 2856	13·161	192 4200
13·012	223 3367	13·062	212 4445	13·112	202 0834	13·162	192 2277
13·013	223 1135	13·063	212 2321	13·113	201 8815	13·163	192 0356
13·014	222 8905	13·064	212 0200	13·114	201 6797	13·164	191 8436
13·015	222 6677	13·065	211 8081	13·115	201 4781	13·165	191 6519
13·016	222 4452	13·066	211 5964	13·116	201 2767	13·166	191 4603
13·017	222 2228	13·067	211 3849	13·117	201 0755	13·167	191 2690
13·018	222 0007	13·068	211 1736	13·118	200 8746	13·168	191 0778
13·019	221 7788	13·069	210 9626	13·119	200 6738	13·169	190 8868
13·020	221 5572	13·070	210 7517	13·120	200 4732	13·170	190 6960
13·021	221 3357	13·071	210 5411	13·121	200 2729	13·171	190 5054
13·022	221 1145	13·072	210 3306	13·122	200 0727	13·172	190 3150
13·023	220 8935	13·073	210 1204	13·123	199 8727	13·173	190 1248
13·024	220 6727	13·074	209 9104	13·124	199 6729	13·174	189 9348
13·025	220 4521	13·075	209 7006	13·125	199 4734	13·175	189 7449
13·026	220 2318	13·076	209 4910	13·126	199 2740	13·176	189 5553
13·027	220 0117	13·077	209 2816	13·127	199 0748	13·177	189 3658
13·028	219 7918	13·078	209 0724	13·128	198 8758	13·178	189 1766
13·029	219 5721	13·079	208 8634	13·129	198 6771	13·179	188 9875
13·030	219 3526	13·080	208 6547	13·130	198 4785	13·180	188 7986
13·031	219 1334	13·081	208 4461	13·131	198 2801	13·181	188 6099
13·032	218 9144	13·082	208 2378	13·132	198 0819	13·182	188 4214
13·033	218 6956	13·083	208 0296	13·133	197 8839	13·183	188 2330
13·034	218 4770	13·084	207 8217	13·134	197 6861	13·184	188 0449
13·035	218 2586	13·085	207 6140	13·135	197 4886	13·185	187 8569
13·036	218 0405	13·086	207 4065	13·136	197 2912	13·186	187 6692
13·037	217 8225	13·087	207 1992	13·137	197 0940	13·187	187 4816
13·038	217 6048	13·088	206 9921	13·138	196 8970	13·188	187 2942
13·039	217 3873	13·089	206 7852	13·139	196 7002	13·189	187 1070
13·040	217 1700	13·090	206 5785	13·140	196 5036	13·190	186 9200
13·041	216 9530	13·091	206 3721	13·141	196 3072	13·191	186 7332
13·042	216 7361	13·092	206 1658	13·142	196 1110	13·192	186 5465
13·043	216 5195	13·093	205 9597	13·143	195 9150	13·193	186 3601
13·044	216 3031	13·094	205 7539	13·144	195 7191	13·194	186 1738
13·045	216 0869	13·095	205 5482	13·145	195 5235	13·195	185 9877
13·046	215 8709	13·096	205 3428	13·146	195 3281	13·196	185 8018
13·047	215 6552	13·097	205 1375	13·147	195 1329	13·197	185 6161
13·048	215 4396	13·098	204 9325	13·148	194 9378	13·198	185 4306
13·049	215 2243	13·099	204 7277	13·149	194 7430	13·199	185 2453

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
13·200	185 0601	13·250	176 0346	13·300	167 4493	13·350	159 2827
13·201	184 8751	13·251	175 8587	13·301	167 2819	13·351	159 1235
13·202	184 6904	13·252	175 6829	13·302	167 1147	13·352	158 9645
13·203	184 5058	13·253	175 5073	13·303	166 9477	13·353	158 8056
13·204	184 3213	13·254	175 3319	13·304	166 7809	13·354	158 6469
13·205	184 1371	13·255	175 1566	13·305	166 6142	13·355	158 4883
13·206	183 9531	13·256	174 9816	13·306	166 4476	13·356	158 3299
13·207	183 7692	13·257	174 8067	13·307	166 2813	13·357	158 1716
13·208	183 5855	13·258	174 6320	13·308	166 1151	13·358	158 0135
13·209	183 4020	13·259	174 4574	13·309	165 9488	13·359	157 8556
13·210	183 2187	13·260	174 2831	13·310	165 7832	13·360	157 6978
13·211	183 0356	13·261	174 1089	13·311	165 6175	13·361	157 5402
13·212	182 8527	13·262	173 9349	13·312	165 4519	13·362	157 3827
13·213	182 6699	13·263	173 7610	13·313	165 2866	13·363	157 2254
13·214	182 4873	13·264	173 5873	13·314	165 1214	13·364	157 0682
13·215	182 3049	13·265	173 4138	13·315	164 9563	13·365	156 9113
13·216	182 1227	13·266	173 2405	13·316	164 7914	13·366	156 7544
13·217	181 9407	13·267	173 0673	13·317	164 6267	13·367	156 5977
13·218	181 7588	13·268	172 8944	13·318	164 4622	13·368	156 4412
13·219	181 5771	13·269	172 7216	13·319	164 2978	13·369	156 2849
13·220	181 3957	13·270	172 5489	13·320	164 1336	13·370	156 1287
13·221	181 2144	13·271	172 3765	13·321	163 9695	13·371	155 9726
13·222	181 0332	13·272	172 2042	13·322	163 8057	13·372	155 8167
13·223	180 8523	13·273	172 0321	13·323	163 6419	13·373	155 6610
13·224	180 6715	13·274	171 8601	13·324	163 4784	13·374	155 5054
13·225	180 4909	13·275	171 6883	13·325	163 3150	13·375	155 3500
13·226	180 3105	13·276	171 5167	13·326	163 1517	13·376	155 1947
13·227	180 1303	13·277	171 3453	13·327	162 9887	13·377	155 0396
13·228	179 9503	13·278	171 1740	13·328	162 8258	13·378	154 8846
13·229	179 7704	13·279	171 0030	13·329	162 6630	13·379	154 7298
13·230	179 5907	13·280	170 8320	13·330	162 5004	13·380	154 5752
13·231	179 4112	13·281	170 6613	13·331	162 3380	13·381	154 4207
13·232	179 2319	13·282	170 4907	13·332	162 1758	13·382	154 2663
13·233	179 0528	13·283	170 3203	13·333	162 0137	13·383	154 1122
13·234	178 8738	13·284	170 1501	13·334	161 8517	13·384	153 9581
13·235	178 6950	13·285	169 9800	13·335	161 6900	13·385	153 8042
13·236	178 5164	13·286	169 8101	13·336	161 5284	13·386	153 6505
13·237	178 3380	13·287	169 6404	13·337	161 3669	13·387	153 4969
13·238	178 1597	13·288	169 4708	13·338	161 2056	13·388	153 3435
13·239	177 9817	13·289	169 3014	13·339	161 0445	13·389	153 1903
13·240	177 8038	13·290	169 1322	13·340	160 8835	13·390	153 0371
13·241	177 6261	13·291	168 9632	13·341	160 7227	13·391	152 8842
13·242	177 4485	13·292	168 7943	13·342	160 5621	13·392	152 7314
13·243	177 2712	13·293	168 6256	13·343	160 4016	13·393	152 5788
13·244	177 0940	13·294	168 4570	13·344	160 2413	13·394	152 4262
13·245	176 9170	13·295	168 2887	13·345	160 0812	13·395	152 2739
13·246	176 7402	13·296	168 1205	13·346	159 9211	13·396	152 1217
13·247	176 5635	13·297	167 9524	13·347	159 7613	13·397	151 9696
13·248	176 3870	13·298	167 7845	13·348	159 6016	13·398	151 8177
13·249	176 2107	13·299	167 6168	13·349	159 4421	13·399	151 6660

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
13·400	151 5144	13·450	144 1250	13·500	137 0959	13·550	130 4097
13·401	151 3630	13·451	143 9809	13·501	136 9589	13·551	130 2793
13·402	151 2117	13·452	143 8370	13·502	136 8220	13·552	130 1491
13·403	151 0605	13·453	143 6932	13·503	136 6852	13·553	130 0190
13·404	150 9096	13·454	143 5496	13·504	136 5486	13·554	129 8891
13·405	150 7587	13·455	143 4061	13·505	136 4121	13·555	129 7592
13·406	150 6080	13·456	143 2628	13·506	136 2758	13·556	129 6295
13·407	150 4575	13·457	143 1196	13·507	136 1396	13·557	129 5000
13·408	150 3071	13·458	142 9766	13·508	136 0035	13·558	129 3706
13·409	150 1569	13·459	142 8337	13·509	135 8676	13·559	129 2413
13·410	150 0068	13·460	142 6909	13·510	135 7318	13·560	129 1121
13·411	149 8569	13·461	142 5483	13·511	135 5961	13·561	128 9831
13·412	149 7071	13·462	142 4058	13·512	135 4606	13·562	128 8542
13·413	149 5575	13·463	142 2635	13·513	135 3252	13·563	128 7254
13·414	149 4080	13·464	142 1213	13·514	135 1899	13·564	128 5967
13·415	149 2586	13·465	141 9792	13·515	135 0548	13·565	128 4682
13·416	149 1095	13·466	141 8373	13·516	134 9198	13·566	128 3398
13·417	148 9604	13·467	141 6955	13·517	134 7850	13·567	128 2115
13·418	148 8115	13·468	141 5539	13·518	134 6502	13·568	128 0833
13·419	148 6628	13·469	141 4124	13·519	134 5157	13·569	127 9553
13·420	148 5142	13·470	141 2711	13·520	134 3812	13·570	127 8274
13·421	148 3658	13·471	141 1299	13·521	134 2469	13·571	127 6996
13·422	148 2175	13·472	140 9888	13·522	134 1127	13·572	127 5720
13·423	148 0693	13·473	140 8479	13·523	133 9787	13·573	127 4445
13·424	147 9213	13·474	140 7071	13·524	133 8448	13·574	127 3172
13·425	147 7735	13·475	140 5665	13·525	133 7110	13·575	127 1899
13·426	147 6258	13·476	140 4260	13·526	133 5773	13·576	127 0627
13·427	147 4782	13·477	140 2856	13·527	133 4438	13·577	126 9357
13·428	147 3308	13·478	140 1454	13·528	133 3105	13·578	126 8089
13·429	147 1836	13·479	140 0054	13·529	133 1772	13·579	126 6821
13·430	147 0365	13·480	139 8654	13·530	133 0441	13·580	126 5555
13·431	146 8895	13·481	139 7256	13·531	132 9111	13·581	126 4290
13·432	146 7427	13·482	139 5860	13·532	132 7783	13·582	126 3026
13·433	146 5960	13·483	139 4465	13·533	132 6456	13·583	126 1764
13·434	146 4495	13·484	139 3071	13·534	132 5130	13·584	126 0503
13·435	146 3031	13·485	139 1678	13·535	132 3805	13·585	125 9243
13·436	146 1569	13·486	139 0288	13·536	132 2482	13·586	125 7984
13·437	146 0108	13·487	138 8898	13·537	132 1160	13·587	125 6727
13·438	145 8649	13·488	138 7510	13·538	131 9840	13·588	125 5471
13·439	145 7191	13·489	138 6123	13·539	131 8521	13·589	125 4216
13·440	145 5734	13·490	138 4737	13·540	131 7203	13·590	125 2962
13·441	145 4279	13·491	138 3353	13·541	131 5886	13·591	125 1710
13·442	145 2826	13·492	138 1971	13·542	131 4571	13·592	125 0459
13·443	145 1374	13·493	138 0589	13·543	131 3257	13·593	124 9209
13·444	144 9923	13·494	137 9209	13·544	131 1945	13·594	124 7960
13·445	144 8474	13·495	137 7831	13·545	131 0633	13·595	124 6713
13·446	144 7026	13·496	137 6454	13·546	130 9323	13·596	124 5467
13·447	144 5580	13·497	137 5078	13·547	130 8015	13·597	124 4222
13·448	144 4135	13·498	137 3704	13·548	130 6707	13·598	124 2978
13·449	144 2692	13·499	137 2331	13·549	130 5401	13·599	124 1736

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
13·600	124 0495	13·650	117 9995	13·700	112 2446	13·750	106 7704
13·601	123 9255	13·651	117 8816	13·701	112 1324	13·751	106 6637
13·602	123 8016	13·652	117 7638	13·702	112 0204	13·752	106 5571
13·603	123 6779	13·653	117 6461	13·703	111 9084	13·753	106 4506
13·604	123 5543	13·654	117 5285	13·704	111 7965	13·754	106 3442
13·605	123 4308	13·655	117 4110	13·705	111 6848	13·755	106 2379
13·606	123 3074	13·656	117 2937	13·706	111 5732	13·756	106 1317
13·607	123 1842	13·657	117 1764	13·707	111 4617	13·757	106 0256
13·608	123 0611	13·658	117 0593	13·708	111 3503	13·758	105 9196
13·609	122 9381	13·659	116 9423	13·709	111 2390	13·759	105 8138
13·610	122 8152	13·660	116 8254	13·710	111 1278	13·760	105 7080
13·611	122 6924	13·661	116 7087	13·711	111 0167	13·761	105 6024
13·612	122 5698	13·662	116 5920	13·712	110 9057	13·762	105 4968
13·613	122 4473	13·663	116 4755	13·713	110 7949	13·763	105 3914
13·614	122 3249	13·664	116 3590	13·714	110 6842	13·764	105 2860
13·615	122 2026	13·665	116 2427	13·715	110 5735	13·765	105 1808
13·616	122 0805	13·666	116 1266	13·716	110 4630	13·766	105 0757
13·617	121 9585	13·667	116 0105	13·717	110 3526	13·767	104 9706
13·618	121 8366	13·668	115 8945	13·718	110 2423	13·768	104 8657
13·619	121 7148	13·669	115 7787	13·719	110 1321	13·769	104 7609
13·620	121 5932	13·670	115 6630	13·720	110 0220	13·770	104 6562
13·621	121 4716	13·671	115 5474	13·721	109 9121	13·771	104 5516
13·622	121 3502	13·672	115 4319	13·722	109 8022	13·772	104 4471
13·623	121 2289	13·673	115 3165	13·723	109 6925	13·773	104 3427
13·624	121 1078	13·674	115 2013	13·724	109 5828	13·774	104 2384
13·625	120 9867	13·675	115 0861	13·725	109 4733	13·775	104 1342
13·626	120 8658	13·676	114 9711	13·726	109 3639	13·776	104 0301
13·627	120 7450	13·677	114 8562	13·727	109 2546	13·777	103 9262
13·628	120 6243	13·678	114 7414	13·728	109 1454	13·778	103 8223
13·629	120 5037	13·679	114 6267	13·729	109 0363	13·779	103 7185
13·630	120 3833	13·680	114 5121	13·730	108 9273	13·780	103 6148
13·631	120 2630	13·681	114 3977	13·731	108 8184	13·781	103 5113
13·632	120 1428	13·682	114 2833	13·732	108 7097	13·782	103 4078
13·633	120 0227	13·683	114 1691	13·733	108 6010	13·783	103 3045
13·634	119 9027	13·684	114 0550	13·734	108 4925	13·784	103 2012
13·635	119 7829	13·685	113 9410	13·735	108 3840	13·785	103 0981
13·636	119 6631	13·686	113 8271	13·736	108 2757	13·786	102 9950
13·637	119 5435	13·687	113 7133	13·737	108 1675	13·787	102 8921
13·638	119 4241	13·688	113 5997	13·738	108 0594	13·788	102 7892
13·639	119 3047	13·689	113 4861	13·739	107 9514	13·789	102 6865
13·640	119 1854	13·690	113 3727	13·740	107 8435	13·790	102 5839
13·641	119 0663	13·691	113 2594	13·741	107 7357	13·791	102 4813
13·642	118 9473	13·692	113 1462	13·742	107 6280	13·792	102 3789
13·643	118 8284	13·693	113 0331	13·743	107 5204	13·793	102 2766
13·644	118 7097	13·694	112 9201	13·744	107 4129	13·794	102 1743
13·645	118 5910	13·695	112 8073	13·745	107 3056	13·795	102 0723
13·646	118 4725	13·696	112 6945	13·746	107 1983	13·796	101 9702
13·647	118 3541	13·697	112 5819	13·747	107 0912	13·797	101 8683
13·648	118 2358	13·698	112 4693	13·748	106 9842	13·798	101 7665
13·649	118 1176	13·699	112 3569	13·749	106 8772	13·799	101 6648

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
13·800	101 5631	13·850	96 6098	13·900	91 8981	13·950	87 4162
13·801	101 4616	13·851	96 5133	13·901	91 8063	13·951	87 3288
13·802	101 3602	13·852	96 4168	13·902	91 7145	13·952	87 2415
13·803	101 2589	13·853	96 3204	13·903	91 6228	13·953	87 1543
13·804	101 1577	13·854	96 2242	13·904	91 5313	13·954	87 0672
13·805	101 0566	13·855	96 1280	13·905	91 4398	13·955	86 9802
13·806	100 9556	13·856	96 0319	13·906	91 3484	13·956	86 8933
13·807	100 8547	13·857	95 9359	13·907	91 2571	13·957	86 8064
13·808	100 7539	13·858	95 8400	13·908	91 1659	13·958	86 7197
13·809	100 6532	13·859	95 7443	13·909	91 0748	13·959	86 6330
13·810	100 5526	13·860	95 6486	13·910	90 9837	13·960	86 5464
13·811	100 4521	13·861	95 5530	13·911	90 8928	13·961	86 4599
13·812	100 3517	13·862	95 4575	13·912	90 8019	13·962	86 3735
13·813	100 2514	13·863	95 3620	13·913	90 7112	13·963	86 2871
13·814	100 1512	13·864	95 2667	13·914	90 6205	13·964	86 2009
13·815	100 0511	13·865	95 1715	13·915	90 5299	13·965	86 1147
13·816	099 9511	13·866	95 0764	13·916	90 4395	13·966	86 0287
13·817	99 8512	13·867	94 9814	13·917	90 3491	13·967	85 9427
13·818	99 7514	13·868	94 8864	13·918	90 2588	13·968	85 8568
13·819	99 6517	13·869	94 7916	13·919	90 1685	13·969	85 7710
13·820	99 5521	13·870	94 6968	13·920	90 0784	13·970	85 6852
13·821	99 4526	13·871	94 6022	13·921	89 9884	13·971	85 5996
13·822	99 3531	13·872	94 5076	13·922	89 8984	13·972	85 5140
13·823	99 2538	13·873	94 4132	13·923	89 8086	13·973	85 4286
13·824	99 1546	13·874	94 3188	13·924	89 7188	13·974	85 3432
13·825	99 0555	13·875	94 2245	13·925	89 6292	13·975	85 2579
13·826	98 9565	13·876	94 1304	13·926	89 5396	13·976	85 1727
13·827	98 8576	13·877	94 0363	13·927	89 4501	13·977	85 0875
13·828	98 7588	13·878	93 9423	13·928	89 3607	13·978	85 0025
13·829	98 6601	13·879	93 8484	13·929	89 2714	13·979	84 9175
13·830	98 5615	13·880	93 7546	13·930	89 1821	13·980	84 8327
13·831	98 4630	13·881	93 6609	13·931	89 0930	13·981	84 7479
13·832	98 3646	13·882	93 5673	13·932	89 0039	13·982	84 6632
13·833	98 2663	13·883	93 4737	13·933	88 9150	13·983	84 5785
13·834	98 1680	13·884	93 3803	13·934	88 8261	13·984	84 4940
13·835	98 0699	13·885	93 2870	13·935	88 7373	13·985	84 4096
13·836	97 9719	13·886	93 1937	13·936	88 6486	13·986	84 3252
13·837	97 8740	13·887	93 1006	13·937	88 5600	13·987	84 2409
13·838	97 7761	13·888	93 0075	13·938	88 4715	13·988	84 1567
13·839	97 6784	13·889	92 9146	13·939	88 3831	13·989	84 0726
13·840	97 5808	13·890	92 8217	13·940	88 2947	13·990	83 9886
13·841	97 4833	13·891	92 7289	13·941	88 2065	13·991	83 9046
13·842	97 3858	13·892	92 6362	13·942	88 1183	13·992	83 8208
13·843	97 2885	13·893	92 5437	13·943	88 0303	13·993	83 7370
13·844	97 1912	13·894	92 4512	13·944	87 9423	13·994	83 6533
13·845	97 0941	13·895	92 3587	13·945	87 8544	13·995	83 5697
13·846	96 9971	13·896	92 2664	13·946	87 7666	13·996	83 4861
13·847	96 9001	13·897	92 1742	13·947	87 6788	13·997	83 4027
13·848	96 8033	13·898	92 0821	13·948	87 5912	13·998	83 3193
13·849	96 7065	13·899	91 9901	13·949	87 5037	13·999	83 2361

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
14·000	83 1529	14·050	79 0974	14·100	75 2398	14·150	71 5703
14·001	83 0698	14·051	79 0184	14·101	75 1646	14·151	71 4988
14·002	82 9867	14·052	78 9394	14·102	75 0895	14·152	71 4273
14·003	82 9038	14·053	78 8605	14·103	75 0144	14·153	71 3559
14·004	82 8209	14·054	78 7817	14·104	74 9395	14·154	71 2846
14·005	82 7381	14·055	78 7029	14·105	74 8646	14·155	71 2134
14·006	82 6554	14·056	78 6243	14·106	74 7897	14·156	71 1422
14·007	82 5728	14·057	78 5457	14·107	74 7150	14·157	71 0711
14·008	82 4903	14·058	78 4672	14·108	74 6403	14·158	71 0001
14·009	82 4078	14·059	78 3888	14·109	74 5657	14·159	70 9291
14·010	82 3255	14·060	78 3104	14·110	74 4912	14·160	70 8582
14·011	82 2432	14·061	78 2321	14·111	74 4167	14·161	70 7873
14·012	82 1610	14·062	78 1540	14·112	74 3423	14·162	70 7166
14·013	82 0789	14·063	78 0758	14·113	74 2680	14·163	70 6459
14·014	81 9968	14·064	77 9978	14·114	74 1938	14·164	70 5753
14·015	81 9149	14·065	77 9198	14·115	74 1197	14·165	70 5048
14·016	81 8330	14·066	77 8420	14·116	74 0456	14·166	70 4343
14·017	81 7512	14·067	77 7642	14·117	73 9716	14·167	70 3639
14·018	81 6695	14·068	77 6864	14·118	73 8976	14·168	70 2936
14·019	81 5879	14·069	77 6088	14·119	73 8238	14·169	70 2233
14·020	81 5063	14·070	77 5312	14·120	73 7500	14·170	70 1531
14·021	81 4249	14·071	77 4537	14·121	73 6763	14·171	70 0830
14·022	81 3435	14·072	77 3763	14·122	73 6026	14·172	70 0130
14·023	81 2622	14·073	77 2990	14·123	73 5291	14·173	69 9430
14·024	81 1810	14·074	77 2217	14·124	73 4556	14·174	69 8731
14·025	81 0998	14·075	77 1445	14·125	73 3821	14·175	69 8033
14·026	81 0188	14·076	77 0674	14·126	73 3088	14·176	69 7335
14·027	80 9378	14·077	76 9904	14·127	73 2355	14·177	69 6638
14·028	80 8569	14·078	76 9134	14·128	73 1623	14·178	69 5942
14·029	80 7761	14·079	76 8366	14·129	73 0892	14·179	69 5246
14·030	80 6953	14·080	76 7598	14·130	73 0161	14·180	69 4551
14·031	80 6147	14·081	76 6830	14·131	72 9432	14·181	69 3857
14·032	80 5341	14·082	76 6064	14·132	72 8703	14·182	69 3163
14·033	80 4536	14·083	76 5298	14·133	72 7974	14·183	69 2471
14·034	80 3732	14·084	76 4533	14·134	72 7247	14·184	69 1778
14·035	80 2929	14·085	76 3769	14·135	72 6520	14·185	69 1087
14·036	80 2126	14·086	76 3006	14·136	72 5794	14·186	69 0396
14·037	80 1324	14·087	76 2243	14·137	72 5068	14·187	68 9706
14·038	80 0524	14·088	76 1481	14·138	72 4343	14·188	68 9017
14·039	79 9724	14·089	76 0720	14·139	72 3619	14·189	68 8328
14·040	79 8924	14·090	75 9960	14·140	72 2896	14·190	68 7640
14·041	79 8126	14·091	75 9200	14·141	72 2174	14·191	68 6953
14·042	79 7328	14·092	75 8442	14·142	72 1452	14·192	68 6266
14·043	79 6531	14·093	75 7683	14·143	72 0731	14·193	68 5580
14·044	79 5735	14·094	75 6926	14·144	72 0010	14·194	68 4895
14·045	79 4940	14·095	75 6170	14·145	71 9291	14·195	68 4211
14·046	79 4145	14·096	75 5414	14·146	71 8572	14·196	68 3527
14·047	79 3351	14·097	75 4659	14·147	71 7854	14·197	68 2843
14·048	79 2558	14·098	75 3905	14·148	71 7136	14·198	68 2161
14·049	79 1764	14·099	75 3151	14·149	71 6419	14·199	68 1479

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
14·200	68 0798	14·250	64 7595	14·300	61 6012	14·350	58 5968
14·201	68 0118	14·251	64 6948	14·301	61 5396	14·351	58 5383
14·202	67 9438	14·252	64 6301	14·302	61 4781	14·352	58 4798
14·203	67 8759	14·253	64 5655	14·303	61 4166	14·353	58 4213
14·204	67 8080	14·254	64 5010	14·304	61 3552	14·354	58 3629
14·205	67 7403	14·255	64 4365	14·305	61 2939	14·355	58 3046
14·206	67 6725	14·256	64 3721	14·306	61 2327	14·356	58 2463
14·207	67 6049	14·257	64 3078	14·307	61 1715	14·357	58 1881
14·208	67 5373	14·258	64 2435	14·308	61 1103	14·358	58 1299
14·209	67 4698	14·259	64 1793	14·309	61 0492	14·359	58 0718
14·210	67 4024	14·260	64 1151	14·310	60 9882	14·360	58 0138
14·211	67 3350	14·261	64 0511	14·311	60 9272	14·361	57 9558
14·212	67 2677	14·262	63 9870	14·312	60 8663	14·362	57 8979
14·213	67 2005	14·263	63 9231	14·313	60 8055	14·363	57 8400
14·214	67 1333	14·264	63 8592	14·314	60 7447	14·364	57 7822
14·215	67 0662	14·265	63 7954	14·315	60 6840	14·365	57 7244
14·216	66 9992	14·266	63 7316	14·316	60 6233	14·366	57 6667
14·217	66 9322	14·267	63 6679	14·317	60 5627	14·367	57 6091
14·218	66 8653	14·268	63 6043	14·318	60 5022	14·368	57 5515
14·219	66 7985	14·269	63 5407	14·319	60 4418	14·369	57 4940
14·220	66 7317	14·270	63 4772	14·320	60 3814	14·370	57 4365
14·221	66 6650	14·271	63 4137	14·321	60 3210	14·371	57 3791
14·222	66 5984	14·272	63 3504	14·322	60 2607	14·372	57 3218
14·223	66 5318	14·273	63 2870	14·323	60 2005	14·373	57 2645
14·224	66 4653	14·274	63 2238	14·324	60 1403	14·374	57 2072
14·225	66 3989	14·275	63 1606	14·325	60 0802	14·375	57 1501
14·226	66 3325	14·276	63 0975	14·326	60 0202	14·376	57 0929
14·227	66 2662	14·277	63 0344	14·327	59 9602	14·377	57 0359
14·228	66 2000	14·278	62 9714	14·328	59 9002	14·378	56 9789
14·229	66 1338	14·279	62 9085	14·329	59 8404	14·379	56 9219
14·230	66 0677	14·280	62 8456	14·330	59 7806	14·380	56 8650
14·231	66 0017	14·281	62 7828	14·331	59 7208	14·381	56 8082
14·232	65 9357	14·282	62 7200	14·332	59 6611	14·382	56 7514
14·233	65 8698	14·283	62 6573	14·333	59 6015	14·383	56 6947
14·234	65 8040	14·284	62 5947	14·334	59 5419	14·384	56 6380
14·235	65 7382	14·285	62 5321	14·335	59 4824	14·385	56 5814
14·236	65 6725	14·286	62 4696	14·336	59 4230	14·386	56 5249
14·237	65 6069	14·287	62 4072	14·337	59 3636	14·387	56 4684
14·238	65 5413	14·288	62 3448	14·338	59 3042	14·388	56 4119
14·239	65 4758	14·289	62 2825	14·339	59 2450	14·389	56 3555
14·240	65 4104	14·290	62 2203	14·340	59 1857	14·390	56 2992
14·241	65 3450	14·291	62 1581	14·341	59 1266	14·391	56 2429
14·242	65 2797	14·292	62 0959	14·342	59 0675	14·392	56 1867
14·243	65 2144	14·293	62 0339	14·343	59 0085	14·393	56 1306
14·244	65 1492	14·294	61 9719	14·344	58 9495	14·394	56 0745
14·245	65 0841	14·295	61 9099	14·345	58 8906	14·395	56 0184
14·246	65 0191	14·296	61 8480	14·346	58 8317	14·396	55 9624
14·247	64 9541	14·297	61 7862	14·347	58 7729	14·397	55 9065
14·248	64 8892	14·298	61 7245	14·348	58 7141	14·398	55 8506
14·249	64 8243	14·299	61 6628	14·349	58 6555	14·399	55 7948

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
14·400	55 7390	14·450	53 0206	14·500	50 4348	14·550	47 9750
14·401	55 6833	14·451	52 9676	14·501	50 3843	14·551	47 9271
14·402	55 6277	14·452	52 9147	14·502	50 3340	14·552	47 8792
14·403	55 5721	14·453	52 8618	14·503	50 2837	14·553	47 8313
14·404	55 5165	14·454	52 8089	14·504	50 2334	14·554	47 7835
14·405	55 4610	14·455	52 7562	14·505	50 1832	14·555	47 7358
14·406	55 4056	14·456	52 7034	14·506	50 1331	14·556	47 6880
14·407	55 3502	14·457	52 6508	14·507	50 0829	14·557	47 6404
14·408	55 2949	14·458	52 5981	14·508	50 0329	14·558	47 5928
14·409	55 2396	14·459	52 5456	14·509	49 9829	14·559	47 5452
14·410	55 1844	14·460	52 4930	14·510	49 9329	14·560	47 4977
14·411	55 1293	14·461	52 4406	14·511	49 8830	14·561	47 4502
14·412	55 0742	14·462	52 3882	14·512	49 8332	14·562	47 4028
14·413	55 0191	14·463	52 3358	14·513	49 7833	14·563	47 3554
14·414	54 9641	14·464	52 2835	14·514	49 7336	14·564	47 3081
14·415	54 9092	14·465	52 2312	14·515	49 6839	14·565	47 2608
14·416	54 8543	14·466	52 1790	14·516	49 6342	14·566	47 2135
14·417	54 7995	14·467	52 1269	14·517	49 5846	14·567	47 1663
14·418	54 7447	14·468	52 0748	14·518	49 5350	14·568	47 1192
14·419	54 6900	14·469	52 0227	14·519	49 4855	14·569	47 0721
14·420	54 6353	14·470	51 9707	14·520	49 4361	14·570	47 0251
14·421	54 5807	14·471	51 9188	14·521	49 3867	14·571	46 9781
14·422	54 5262	14·472	51 8669	14·522	49 3374	14·572	46 9311
14·423	54 4717	14·473	51 8150	14·523	49 2880	14·573	46 8842
14·424	54 4172	14·474	51 7633	14·524	49 2388	14·574	46 8373
14·425	54 3628	14·475	51 7115	14·525	49 1896	14·575	46 7905
14·426	54 3085	14·476	51 6598	14·526	49 1404	14·576	46 7438
14·427	54 2542	14·477	51 6082	14·527	49 0913	14·577	46 6970
14·428	54 2000	14·478	51 5566	14·528	49 0422	14·578	46 6504
14·429	54 1458	14·479	51 5051	14·529	48 9932	14·579	46 6037
14·430	54 0917	14·480	51 4536	14·530	48 9442	14·580	46 5572
14·431	54 0376	14·481	51 4022	14·531	48 8953	14·581	46 5106
14·432	53 9836	14·482	51 3508	14·532	48 8464	14·582	46 4641
14·433	53 9297	14·483	51 2995	14·533	48 7976	14·583	46 4177
14·434	53 8757	14·484	51 2482	14·534	48 7488	14·584	46 3713
14·435	53 8219	14·485	51 1970	14·535	48 7001	14·585	46 3250
14·436	53 7681	14·486	51 1458	14·536	48 6514	14·586	46 2787
14·437	53 7144	14·487	51 0947	14·537	48 6028	14·587	46 2324
14·438	53 6607	14·488	51 0436	14·538	48 5542	14·588	46 1862
14·439	53 6070	14·489	50 9926	14·539	48 5057	14·589	46 1400
14·440	53 5535	14·490	50 9416	14·540	48 4572	14·590	46 0939
14·441	53 4999	14·491	50 8907	14·541	48 4088	14·591	46 0478
14·442	53 4465	14·492	50 8399	14·542	48 3604	14·592	46 0018
14·443	53 3930	14·493	50 7890	14·543	48 3120	14·593	45 9558
14·444	53 3397	14·494	50 7383	14·544	48 2638	14·594	45 9099
14·445	53 2864	14.495	50 6876	14·545	48 2155	14·595	45 8640
14·446	53 2331	14·496	50 6369	14·546	48 1673	14·596	45 8182
14·447	53 1799	14·497	50 5863	14·547	48 1192	14·597	45 7724
14·448	53 1267	14·498	50 5357	14·548	48 0711	14·598	45 7266
14·449	53 0736	14·499	50 4852	14·549	48 0230	14·599	45 6809

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
14·600	45 6353	14·650	43 4096	14·700	41 2925	14·750	39 2786
14·601	45 5896	14·651	43 3662	14·701	41 2512	14·751	39 2394
14·602	45 5441	14·652	43 3229	14·702	41 2100	14·752	39 2001
14·603	45 4986	14·653	43 2796	14·703	41 1688	14·753	39 1610
14·604	45 4531	14·654	43 2363	14·704	41 1276	14·754	39 1219
14·605	45 4076	14·655	43 1931	14·705	41 0865	14·755	39 0827
14·606	45 3623	14·656	43 1499	14·706	41 0455	14·756	39 0437
14·607	45 3169	14·657	43 1068	14·707	41 0044	14·757	39 0046
14·608	45 2716	14·658	43 0637	14·708	40 9635	14·758	38 9656
14·609	45 2264	14·659	43 0207	14·709	40 9225	14·759	38 9267
14·610	45 1812	14·660	42 9777	14·710	40 8816	14·760	38 8878
14·611	45 1360	14·661	42 9347	14·711	40 8408	14·761	38 8489
14·612	45 0909	14·662	42 8918	14·712	40 7999	14·762	38 8101
14·613	45 0458	14·663	42 8489	14·713	40 7592	14·763	38 7713
14·614	45 0008	14·664	42 8061	14·714	40 7184	14·764	38 7325
14·615	44 9558	14·665	42 7633	14·715	40 6777	14·765	38 6938
14·616	44 9109	14·666	42 7206	14·716	40 6371	14·766	38 6552
14·617	44 8660	14·667	42 6779	14·717	40 5964	14·767	38 6165
14·618	44 8212	14·668	42 6352	14·718	40 5559	14·768	38 5779
14·619	44 7764	14·669	42 5926	14·719	40 5153	14·769	38 5394
14·620	44 7316	14·670	42 5500	14·720	40 4748	14·770	38 5009
14·621	44 6869	14·671	42 5075	14·721	40 4344	14·771	38 4624
14·622	44 6422	14·672	42 4650	14·722	40 3940	14·772	38 4239
14·623	44 5976	14·673	42 4226	14·723	40 3536	14·773	38 3855
14·624	44 5530	14·674	42 3802	14·724	40 3133	14·774	38 3472
14·625	44 5085	14·675	42 3378	14·725	40 2730	14·775	38 3088
14·626	44 4640	14·676	42 2955	14·726	40 2327	14·776	38 2705
14·627	44 4196	14·677	42 2532	14·727	40 1925	14·777	38 2323
14·628	44 3752	14·678	42 2110	14·728	40 1523	14·778	38 1941
14·629	44 3308	14·679	42 1688	14·729	40 1122	14·779	38 1559
14·630	44 2865	14·680	42 1266	14·730	40 0721	14·780	38 1178
14·631	44 2423	14·681	42 0845	14·731	40 0321	14·781	38 0797
14·632	44 1980	14·682	42 0425	14·732	39 9920	14·782	38 0416
14·633	44 1539	14·683	42 0004	14·733	39 9521	14·783	38 0036
14·634	44 1097	14·684	41 9585	14·734	39 9121	14·784	37 9656
14·635	44 0656	14·685	41 9165	14·735	39 8722	14·785	37 9276
14·636	44 0216	14·686	41 8746	14·736	39 8324	14·786	37 8897
14·637	43 9776	14·687	41 8328	14·737	39 7926	14·787	37 8519
14·638	43 9336	14·688	41 7910	14·738	39 7528	14·788	37 8140
14·639	43 8897	14·689	41 7492	14·739	39 7131	14·789	37 7762
14·640	43 8459	14·690	41 7075	14·740	39 6734	14·790	37 7385
14·641	43 8020	14·691	41 6658	14·741	39 6337	14·791	37 7008
14·642	43 7583	14·692	41 6241	14·742	39 5941	14·792	37 6631
14·643	43 7145	14·693	41 5825	14·743	39 5545	14·793	37 6254
14·644	43 6708	14·694	41 5410	14·744	39 5150	14·794	37 5878
14·645	43 6272	14·695	41 4995	14·745	39 4755	14·795	37 5503
14·646	43 5836	14·696	41 4580	14·746	39 4361	14·796	37 5127
14·647	43 5400	14·697	41 4166	14·747	39 3966	14·797	37 4752
14·648	43 4965	14·698	41 3752	14·748	39 3573	14·798	37 4378
14·649	43 4530	14·699	41 3338	14·749	39 3179	14·799	37 4004

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
14·800	37 3630	14·850	35 5408	14·900	33 8074	14·950	32 1586
14·801	37 3256	14·851	35 5052	14·901	33 7736	14·951	32 1265
14·802	37 2883	14·852	35 4698	14·902	33 7399	14·952	32 0944
14·803	37 2511	14·853	35 4343	14·903	33 7062	14·953	32 0623
14·804	37 2138	14·854	35 3989	14·904	33 6725	14·954	32 0302
14·805	37 1766	14·855	35 3635	14·905	33 6388	14·955	31 9982
14·806	37 1395	14·856	35 3282	14·906	33 6052	14·956	31 9662
14·807	37 1024	14·857	35 2929	14·907	33 5716	14·957	31 9343
14·808	37 0653	14·858	35 2576	14·908	33 5380	14·958	31 9024
14·809	37 0282	14·859	35 2223	14·909	33 5045	14·959	31 8705
14·810	36 9912	14·860	35 1871	14·910	33 4710	14·960	31 8386
14·811	36 9542	14·861	35 1520	14·911	33 4376	14·961	31 8068
14·812	36 9173	14·862	35 1168	14·912	33 4042	14·962	31 7750
14·813	36 8804	14·863	35 0817	14·913	33 3708	14·963	31 7433
14·814	36 8435	14·864	35 0467	14·914	33 3374	14·964	31 7115
14·815	36 8067	14·865	35 0116	14·915	33 3041	14·965	31 6798
14·816	36 7699	14·866	34 9766	14·916	33 2708	14·966	31 6482
14·817	36 7332	14·867	34 9417	14·917	33 2376	14·967	31 6165
14·818	36 6965	14·868	34 9068	14·918	33 2043	14·968	31 5849
14·819	36 6598	14·869	34 8719	14·919	33 1712	14·969	31 5534
14·820	36 6231	14·870	34 8370	14·920	33 1380	14·970	31 5218
14·821	36 5865	14·871	34 8022	14·921	33 1049	14·971	31 4903
14·822	36 5500	14·872	34 7674	14·922	33 0718	14·972	31 4589
14·823	36 5134	14·873	34 7327	14·923	33 0387	14·973	31 4274
14·824	36 4769	14·874	34 6979	14·924	33 0057	14·974	31 3960
14·825	36 4405	14·875	34 6633	14·925	32 9727	14·975	31 3646
14·826	36 4041	14·876	34 6286	14·926	32 9398	14·976	31 3333
14·827	36 3677	14·877	34 5940	14·927	32 9068	14·977	31 3020
14·828	36 3313	14·878	34 5594	14·928	32 8739	14·978	31 2707
14·829	36 2950	14·879	34 5249	14·929	32 8411	14·979	31 2394
14·830	36 2587	14·880	34 4904	14·930	32 8083	14·980	31 2082
14·831	36 2225	14·881	34 4559	14·931	32 7755	14·981	31 1770
14·832	36 1863	14·882	34 4215	14·932	32 7427	14·982	31 1458
14·833	36 1501	14·883	34 3871	14·933	32 7100	14·983	31 1147
14·834	36 1140	14·884	34 3527	14·934	32 6773	14·984	31 0836
14·835	36 0779	14·885	34 3184	14·935	32 6446	14·985	31 0525
14·836	36 0418	14·886	34 2841	14·936	32 6120	14·986	31 0215
14·837	36 0058	14·887	34 2498	14·937	32 5794	14·987	30 9905
14·838	35 9698	14·888	34 2156	14·938	32 5468	14·988	30 9595
14·839	35 9339	14·889	34 1814	14·939	32 5143	14·989	30 9286
14·840	35 8980	14·890	34 1472	14·940	32 4818	14·990	30 8977
14·841	35 8621	14·891	34 1131	14·941	32 4493	14·991	30 8668
14·842	35 8262	14·892	34 0790	14·942	32 4169	14·992	30 8359
14·843	35 7904	14·893	34 0449	14·943	32 3845	14·993	30 8051
14·844	35 7547	14·894	34 0109	14·944	32 3521	14·994	30 7743
14·845	35 7189	14·895	33 9769	14·945	32 3198	14·995	30 7436
14·846	35 6832	14·896	33 9429	14·946	32 2875	14·996	30 7128
14·847	35 6476	14·897	33 9090	14·947	32 2552	14·997	30 6821
14·848	35 6119	14·898	33 8751	14·948	32 2230	14·998	30 6515
14·849	35 5763	14·899	33 8412	14·949	32 1907	14·999	30 6208

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
15·000	30 5902	15·050	29 0983	15·100	27 6792	15·150	26 3292
15·001	30 5596	15·051	29 0692	15·101	27 6515	15·151	26 3029
15·002	30 5291	15·052	29 0402	15·102	27 6239	15·152	26 2766
15·003	30 4986	15·053	29 0112	15·103	27 5963	15·153	26 2504
15·004	30 4680	15·054	28 9822	15·104	27 5687	15·154	26 2241
15·005	30 4376	15·055	28 9532	15·105	27 5411	15·155	26 1979
15·006	30 4072	15·056	28 9243	15·106	27 5136	15·156	26 1717
15·007	30 3768	15·057	28 8954	15·107	27 4861	15·157	26 1456
15·008	30 3465	15·058	28 8665	15·108	27 4586	15·158	26 1194
15·009	30 3161	15·059	28 8376	15·109	27 4312	15·159	26 0933
15·010	30 2858	15·060	28 8088	15·110	27 4038	15·160	26 0673
15·011	30 2556	15·061	28 7800	15·111	27 3764	15·161	26 0412
15·012	30 2253	15·062	28 7512	15·112	27 3490	15·162	26 0152
15·013	30 1951	15·063	28 7225	15·113	27 3217	15·163	25 9892
15·014	30 1649	15·064	28 6938	15·114	27 2944	15·164	25 9632
15·015	30 1348	15·065	28 6651	15·115	27 2671	15·165	25 9373
15·016	30 1047	15·066	28 6365	15·116	27 2398	15·166	25 9113
15·017	30 0746	15·067	28 6078	15·117	27 2126	15·167	25 8854
15·018	30 0445	15·068	28 5792	15·118	27 1854	15·168	25 8596
15·019	30 0145	15·069	28 5507	15·119	27 1582	15·169	25 8337
15·020	29 9845	15·070	28 5221	15·120	27 1311	15·170	25 8079
15·021	29 9545	15·071	28 4936	15·121	27 1040	15·171	25 7821
15·022	29 9246	15·072	28 4652	15·122	27 0769	15·172	25 7563
15·023	29 8947	15·073	28 4367	15·123	27 0498	15·173	25 7306
15·024	29 8648	15·074	28 4083	15·124	27 0228	15·174	25 7049
15·025	29 8349	15·075	28 3799	15·125	26 9958	15·175	25 6792
15·026	29 8051	15·076	28 3515	15·126	26 9688	15·176	25 6535
15·027	29 7753	15·077	28 3232	15·127	26 9418	15·177	25 6279
15·028	29 7456	15·078	28 2949	15·128	26 9149	15·178	25 6023
15·029	29 7158	15·079	28 2666	15·129	26 8880	15·179	25 5767
15·030	29 6861	15·080	28 2383	15·130	26 8611	15·180	25 5511
15·031	29 6565	15·081	28 2101	15·131	26 8343	15·181	25 5256
15·032	29 6268	15·082	28 1819	15·132	26 8075	15·182	25 5000
15·033	29 5972	15·083	28 1538	15·133	26 7807	15·183	25 4746
15·034	29 5676	15·084	28 1256	15·134	26 7539	15·184	25 4491
15·035	29 5381	15·085	28 0975	15·135	26 7272	15·185	25 4237
15·036	29 5086	15·086	28 0694	15·136	26 7004	15·186	25 3983
15·037	29 4791	15·087	28 0414	15·137	26 6738	15·187	25 3729
15·038	29 4496	15·088	28 0133	15·138	26 6471	15·188	25 3475
15·039	29 4202	15·089	27 9853	15·139	26 6205	15·189	25 3222
15·040	29 3908	15·090	27 9574	15·140	26 5939	15·190	25 2969
15·041	29 3614	15·091	27 9294	15·141	26 5673	15·191	25 2716
15·042	29 3320	15·092	27 9015	15·142	26 5407	15·192	25 2463
15·043	29 3027	15·093	27 8736	15·143	26 5142	15·193	25 2211
15·044	29 2734	15·094	27 8458	15·144	26 4877	15·194	25 1959
15·045	29 2442	15·095	27 8179	15·145	26 4612	15·195	25 1707
15·046	29 2150	15·096	27 7901	15·146	26 4348	15·196	25 1455
15·047	29 1858	15·097	27 7624	15·147	26 4083	15·197	25 1204
15·048	29 1566	15·098	27 7346	15·148	26 3820	15·198	25 0955
15·049	29 1274	15·099	27 7069	15·149	26 3556	15·199	25 0704

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
15·200	25 0452	15·240	24 0631	15·280	23 1196	15·320	22 2131
15·201	25 0201	15·241	24 0391	15·281	23 0965	15·321	22 1909
15·202	24 9951	15·242	24 0150	15·282	23 0734	15·322	22 1687
15·203	24 9701	15·243	23 9910	15·283	23 0503	15·323	22 1465
15·204	24 9452	15·244	23 9671	15·284	23 0273	15·324	22 1244
15·205	24 9202	15·245	23 9431	15·285	23 0043	15·325	22 1023
15·206	24 8953	15·246	23 9192	15·286	22 9813	15·326	22 0802
15·207	24 8704	15·247	23 8953	15·287	22 9583	15·327	22 0581
15·208	24 8456	15·248	23 8714	15·288	22 9354	15·328	22 0361
15·209	24 8208	15·249	23 8475	15·289	22 9124	15·329	22 0140
15·210	24 7960	15·250	23 8237	15·290	22 8895	15·330	21 9920
15·211	24 7712	15·251	23 8000	15·291	22 8667	15·331	21 9701
15·212	24 7464	15·252	23 7761	15·292	22 8438	15·332	21 9481
15·213	24 7217	15·253	23 7523	15·293	22 8210	15·333	21 9262
15·214	24 6970	15·254	23 7286	15·294	22 7982	15·334	21 9042
15·215	24 6723	15·255	23 7049	15·295	22 7754	15·335	21 8823
15·216	24 6476	15·256	23 6812	15·296	22 7526	15·336	21 8605
15·217	24 6229	15·257	23 6575	15·297	22 7299	15·337	21 8386
15·218	24 5984	15·258	23 6339	15·298	22 7072	15·338	21 8168
15·219	24 5738	15·259	23 6102	15·299	22 6845	15·339	21 7950
15·220	24 5492	15·260	23 5866	15·300	22 6618	15·340	21 7732
15·221	24 5247	15·261	23 5631	15·301	22 6391	15·341	21 7514
15·222	24 5002	15·262	23 5395	15·302	22 6165	15·342	21 7297
15·223	24 4757	15·263	23 5160	15·303	22 5939	15·343	21 7080
15·224	24 4512	15·264	23 4925	15·304	22 5713	15·344	21 6863
15·225	24 4268	15·265	23 4690	15·305	22 5488	15·345	21 6646
15·226	24 4024	15·266	23 4455	15·306	22 5262	15·346	21 6430
15·227	24 3780	15·267	23 4221	15·307	22 5037	15·347	21 6213
15·228	24 3536	15·268	23 3987	15·308	22 4812	15·348	21 5997
15·229	24 3293	15·269	23 3753	15·309	22 4587	15·349	21 5781
15·230	24 3050	15·270	23 3519	15·310	22 4363		
15·231	24 2807	15·271	23 3286	15·311	22 4139		
15·232	24 2564	15·272	23 3053	15·312	22 3915		
15·233	24 2322	15·273	23 2820	15·313	22 3691		
15·234	24 2079	15·274	23 2587	15·314	22 3467		
15·235	24 1837	15·275	23 2355	15·315	22 3244		
15·236	24 1596	15·276	23 2123	15·316	22 3021		
15·237	24 1354	15·277	23 1891	15·317	22 2798		
15·238	24 1113	15·278	23 1659	15·318	22 2575		
15·239	24 0872	15·279	23 1427	15·319	22 2353		

[*Second Part. Fourteen decimal places.*]

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
15·350	2155 6572	15·450	1950 5193	15·550	1764 9028	15·650	1596 9501
15·352	2151 3502	15·452	1946 6222	15·552	1761 3765	15·652	1593 7594
15·354	2147 0518	15·454	1942 7328	15·554	1757 8573	15·654	1590 5751
15·356	2142 7620	15·456	1938 8512	15·556	1754 3451	15·656	1587 3972
15·358	2138 4808	15·458	1934 9774	15·558	1750 8399	15·658	1584 2256
15·360	2134 2081	15·460	1931 1113	15·560	1747 3416	15·660	1581 0602
15·362	2129 9439	15·462	1927 2530	15·562	1743 8504	15·662	1577 9012
15·364	2125 6882	15·464	1923 4023	15·564	1740 3661	15·664	1574 7485
15·366	2121 4411	15·466	1919 5593	15·566	1736 8890	15·666	1571 6021
15·368	2117 2025	15·468	1915 7241	15·568	1733 4187	15·668	1568 4620
15·370	2112 9723	15·470	1911 8964	15·570	1729 9553	15·670	1565 3284
15·372	2108 7507	15·472	1908 0764	15·572	1726 4988	15·672	1562 2009
15·374	2104 5374	15·474	1904 2641	15·574	1723 0493	15·674	1559 0796
15·376	2100 3325	15·476	1900 4594	15·576	1719 6066	15·676	1555 9646
15·378	2096 1360	15·478	1896 6623	15·578	1716 1709	15·678	1552 8557
15·380	2091 9479	15·480	1892 8727	15·580	1712 7420	15·680	1549 7531
15·382	2087 7082	15·482	1889 0908	15·582	1709 3201	15·682	1546 6567
15·384	2083 5969	15·484	1885 3163	15·584	1705 9049	15·684	1543 5665
15·386	2079 4338	15·486	1881 5495	15·586	1702 4965	15·686	1540 4825
15·388	2075 2791	15·488	1877 7901	15·588	1699 0949	15·688	1537 4046
15·390	2071 1327	15·490	1874 0383	15·590	1695 7000	15·690	1534 3328
15·392	2066 9943	15·492	1870 2940	15·592	1692 3120	15·692	1531 2672
15·394	2062 8645	15·494	1866 5571	15·594	1688 9306	15·694	1528 2078
15·396	2058 7430	15·496	1862 8278	15·596	1685 5560	15·696	1525 1544
15·398	2054 6297	15·498	1859 1058	15·598	1682 1883	15·698	1522 1071
15·400	2050 5246	15·500	1855 3914	15·600	1678 8275	15·700	1519 0660
15·402	2046 4276	15·502	1851 6843	15·602	1675 4732	15·702	1516 0308
15·404	2042 3389	15·504	1847 9846	15·604	1672 1256	15·704	1513 0018
15·406	2038 2583	15·506	1844 2923	15·606	1668 7847	15·706	1509 9788
15·408	2034 1858	15·508	1840 6074	15·608	1665 4505	15·708	1506 9619
15·410	2030 1215	15·510	1836 9299	15·610	1662 1229	15·710	1503 9510
15·412	2026 0653	15·512	1833 2598	15·612	1658 8020	15·712	1500 9461
15·414	2022 0173	15·514	1829 5970	15·614	1655 4877	15·714	1497 9472
15·416	2017 9773	15·516	1825 9414	15·616	1652 1800	15·716	1494 9543
15·418	2013 9454	15·518	1822 2931	15·618	1648 8790	15·718	1491 9673
15·420	2009 9215	15·520	1818 6521	15·620	1645 5846	15·720	1488 9864
15·422	2005 9056	15·522	1815 0185	15·622	1642 2967	15·722	1486 0115
15·424	2001 8978	15·524	1811 3921	15·624	1639 0154	15·724	1483 0424
15·426	1997 8980	15·526	1807 7729	15·626	1635 7407	15·726	1480 0792
15·428	1993 9062	15·528	1804 1610	15·628	1632 4725	15·728	1477 1221
15·430	1989 9224	15·530	1800 5562	15·630	1629 2108	15·730	1474 1708
15·432	1985 9466	15·532	1796 9587	15·632	1625 9556	15·732	1471 2254
15·434	1981 9786	15·534	1793 3684	15·634	1622 7069	15·734	1468 2859
15·436	1978 0186	15·536	1789 7853	15·636	1619 4647	15·736	1465 3522
15·438	1974 0666	15·538	1786 2093	15·638	1616 2291	15·738	1462 4244
15·440	1970 1224	15·540	1782 6403	15·640	1613 0000	15·740	1459 5025
15·442	1966 1861	15·542	1779 0785	15·642	1609 7771	15·742	1456 5864
15·444	1962 2576	15·544	1775 5239	15·644	1606 5607	15·744	1453 6761
15·446	1958 3370	15·546	1771 9764	15·646	1603 3507	15·746	1450 7717
15·448	1954 4243	15·548	1768 4360	15·648	1600 1472	15·748	1447 8731

[*Second Part. Fourteen decimal places.*]

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
15·750	1444 9802	15·850	1307 4722	15·950	1183 0498	16·050	1070 4677
15·752	1442 0931	15·852	1304 8599	15·952	1180 6861	16·052	1068 3289
15·754	1439 2117	15·854	1302 2528	15·954	1178 3270	16·054	1066 1944
15·756	1436 3362	15·856	1299 6509	15·956	1175 9727	16·056	1064 0641
15·758	1433 4664	15·858	1297 0542	15·958	1173 6231	16·058	1061 9381
15·760	1430 6024	15·860	1294 4626	15·960	1171 2781	16·060	1059 8164
15·762	1427 7441	15·862	1291 8763	15·962	1168 9379	16·062	1057 6989
15·764	1424 8914	15·864	1289 2951	15·964	1166 6023	16·064	1055 5855
15·766	1422 0444	15·866	1286 7191	15·966	1164 2714	16·066	1053 4765
15·768	1419 2031	15·868	1284 1483	15·968	1161 9452	16·068	1051 3716
15·770	1416 3675	15·870	1281 5825	15·970	1159 6234	16·070	1049 2710
15·772	1413 5376	15·872	1279 0219	15·972	1157 3065	16·072	1047 1746
15·774	1410 7134	15·874	1276 4665	15·974	1154 9942	16·074	1045 0823
15·776	1407 8948	15·876	1273 9161	15·976	1152 6865	16·076	1042 9943
15·778	1405 0819	15·878	1271 3708	15·978	1150 3835	16·078	1040 9103
15·780	1402 2746	15·880	1268 8306	15·980	1148 0851	16·080	1038 8306
15·782	1399 4728	15·882	1266 2955	15·982	1145 7913	16·082	1036 7550
15·784	1396 6767	15·884	1263 7654	15·984	1143 5019	16·084	1034 6836
15·786	1393 8862	15·886	1261 2404	15·986	1141 2172	16·086	1032 6164
15·788	1391 1011	15·888	1258 7205	15·988	1138 9370	16·088	1030 5532
15·790	1388 3217	15·890	1256 2055	15·990	1136 6615	16·090	1028 4941
15·792	1385 5478	15·892	1253 6956	15·992	1134 3905	16·092	1026 4391
15·794	1382 7795	15·894	1251 1908	15·994	1132 1240	16·094	1024 3883
15·796	1380 0167	15·896	1248 6909	15·996	1129 8620	16·096	1022 3417
15·798	1377 2595	15·898	1246 1960	15·998	1127 6045	16·098	1020 2990
15·800	1374 5077	15·900	1243 7060	16·000	1125 3517	16·100	1018 2604
15·802	1371 7614	15·902	1241 2211	16·002	1123 1032	16·102	1016 2259
15·804	1369 0207	15·904	1238 7411	16·004	1120 8593	16·104	1014 1955
15·806	1366 2854	15·906	1236 2661	16·006	1118 6198	16·106	1012 1691
15·808	1363 5555	15·908	1233 7961	16·008	1116 3848	16·108	1010 1468
15·810	1360 8311	15·910	1231 3309	16·010	1114 1543	16·110	1008 1285
15·812	1358 1122	15·912	1228 8707	16·012	1111 9282	16·112	1006 1143
15·814	1355 3987	15·914	1226 4155	16·014	1109 7066	16·114	1004 1041
15·816	1352 6906	15·916	1223 9651	16·016	1107 4894	16·116	1002 0979
15·818	1349 9879	15·918	1221 5196	16·018	1105 2766	16·118	1000 0957
15·820	1347 2906	15·920	1219 0790	16·020	1103 0683	16·120	998 0975
15·822	1344 5987	15·922	1216 6433	16·022	1100 8644	16·122	996 1033
15·824	1341 9122	15·924	1214 2124	16·024	1098 6647	16·124	994 1131
15·826	1339 2311	15·926	1211 7864	16·026	1096 4696	16·126	992 1268
15·828	1336 5553	15·928	1209 3653	16·028	1094 2789	16·128	990 1446
15·830	1333 8849	15·930	1206 9490	16·030	1092 0926	16·130	988 1662
15·832	1331 2198	15·932	1204 5375	16·032	1089 9106	16·132	986 1919
15·834	1328 5600	15·934	1202 1308	16·034	1087 7330	16·134	984 2215
15·836	1325 9055	15·936	1199 7289	16·036	1085 5596	16·136	982 2551
15·838	1323 2564	15·938	1197 3319	16·038	1083 3907	16·138	980 2925
15·840	1320 6125	15·940	1194 9396	16·040	1081 2261	16·140	978 3338
15·842	1317 9739	15·942	1192 5521	16·042	1079 0658	16·142	976 3791
15·844	1315 3406	15·944	1190 1694	16·044	1076 9097	16·144	974 4283
15·846	1312 7126	15·946	1187 7915	16·046	1074 7581	16·146	972 4814
15·848	1310 0898	15·948	1185 4182	16·048	1072 6107	16·148	970 5383

[*Second Part. Fourteen decimal places.*]

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
16·150	968 5992	16·250	876 4248	16·350	793 0220	16·450	717 5559
16·152	966 6639	16·252	874 6737	16·352	791 4376	16·452	716 1223
16·154	964 7325	16·254	872 9261	16·354	789 8563	16·454	714 6915
16·156	962 8050	16·256	871 1820	16·356	788 2781	16·456	713 2635
16·158	960 8813	16·258	869 4414	16·358	786 7032	16·458	711 8384
16·160	958 9615	16·260	867 7042	16·360	785 1313	16·460	710 4161
16·162	957 0455	16·262	865 9706	16·362	783 5626	16·462	708 9968
16·164	955 1333	16·264	864 2403	16·364	781 9971	16·464	707 5802
16·166	953 2249	16·266	862 5136	16·366	780 4346	16·466	706 1665
16·168	951 3204	16·268	860 7903	16·368	778 8753	16·468	704 7555
16·170	949 4197	16·270	859 0704	16·370	777 3191	16·470	703 3474
16·172	947 5228	16·272	857 3540	16·372	775 7660	16·472	701 9422
16·174	945 6296	16·274	855 6410	16·374	774 2160	16·474	700 5397
16·176	943 7403	16·276	853 9314	16·376	772 6692	16·476	699 1400
16·178	941 8546	16·278	852 2253	16·378	771 1253	16·478	697 7431
16·180	939 9728	16·280	850 5225	16·380	769 5846	16·480	696 3490
16·182	938 0947	16·282	848 8232	16·382	768 0470	16·482	694 9577
16·184	936 2204	16·284	847 1273	16·384	766 5124	16·484	693 5692
16·186	934 3498	16·286	845 4347	16·386	764 9809	16·486	692 1834
16·188	932 4830	16·288	843 7455	16·388	763 4525	16·488	690 8004
16·190	930 6199	16·290	842 0597	16·390	761 9271	16·490	689 4202
16·192	928 7605	16·292	840 3773	16·392	760 4048	16·492	688 0427
16·194	926 9049	16·294	838 6982	16·394	758 8855	16·494	686 6680
16·196	925 0529	16·296	837 0225	16·396	757 3693	16·496	685 2960
16·198	923 2046	16·298	835 3501	16·398	755 8560	16·498	683 9268
16·200	921 3601	16·300	833 6811	16·400	754 3458	16·500	682 5603
16·202	919 5192	16·302	832 0154	16·402	752 8386	16·502	681 1966
16·204	917 6820	16·304	830 3530	16·404	751 3345	16·504	679 8355
16·206	915 8485	16·306	828 6940	16·406	749 8233	16·506	678 4772
16·208	914 0186	16·308	827 0382	16·408	748 3352	16·508	677 1216
16·210	912 1924	16·310	825 3858	16·410	746 8400	16·510	675 7687
16·212	910 3698	16·312	823 7367	16·412	745 3478	16·512	674 4186
16·214	908 5509	16·314	822 0909	16·414	743 8586	16·514	673 0711
16·216	906 7356	16·316	820 4484	16·416	742 3724	16·516	671 7263
16·218	904 9240	16·318	818 8091	16·418	740 8891	16·518	670 3842
16·220	903 1159	16·320	817 1731	16·420	739 4088	16·520	669 0447
16·222	901 3115	16·322	815 5404	16·422	737 9315	16·522	667 7080
16·224	899 5107	16·324	813 9110	16·424	736 4571	16·524	666 3739
16·226	897 7135	16·326	812 2848	16·426	734 9857	16·526	665 0425
16·228	895 9199	16·328	810 6618	16·428	733 5172	16·528	663 7137
16·230	894 1298	16·330	809 0421	16·430	732 0516	16·530	662 3876
16·232	892 3433	16·332	807 4257	16·432	730 5890	16·532	661 0642
16·234	890 5604	16·334	805 8124	16·434	729 1292	16·534	659 7434
16·236	888 7811	16·336	804 2024	16·436	727 6724	16·536	658 4252
16·238	887 0053	16·338	802 5956	16·438	726 2185	16·538	657 1097
16·240	885 2330	16·340	800 9920	16·440	724 7675	16·540	655 7968
16·242	883 4644	16·342	799 3917	16·442	723 3195	16·542	654 4865
16·244	881 6992	16·344	797 7945	16·444	721 8743	16·544	653 1788
16·246	879 9376	16·346	796 2005	16·446	720 4320	16·546	651 8738
16·248	878 1795	16·348	794 6097	16·448	718 9925	16·548	650 5713

[*Second Part. Fourteen decimal places.*]

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
16·550	649 2715	16·650	587 4851	16·750	531 5785	16·850	480 9921
16·552	647 9742	16·652	586 3113	16·752	530 5164	16·852	480 0311
16·554	646 6796	16·654	585 1399	16·754	529 4564	16·854	479 0720
16·556	645 3875	16·656	583 9707	16·756	528 3986	16·856	478 1149
16·558	644 0980	16·658	582 8040	16·758	527 3429	16·858	477 1596
16·560	642 8111	16·660	581 6395	16·760	526 2892	16·860	476 2062
16·562	641 5268	16·662	580 4774	16·762	525 2377	16·862	475 2547
16·564	640 2450	16·664	579 3176	16·764	524 1883	16·864	474 3052
16·566	638 9658	16·666	578 1601	16·766	523 1410	16·866	473 3575
16·568	637 6892	16·668	577 0050	16·768	522 0957	16·868	472 4118
16·570	636 4150	16·670	575 8521	16·770	521 0526	16·870	471 4679
16·572	635 1435	16·672	574 7016	16·772	520 0115	16·872	470 5259
16·574	633 8745	16·674	573 5533	16·774	518 9726	16·874	469 5858
16·576	632 6080	16·676	572 4074	16·776	517 9356	16·876	468 6476
16·578	631 3440	16·678	571 2637	16·778	516 9008	16·878	467 7112
16·580	630 0826	16·680	570 1223	16·780	515 8680	16·880	466 7767
16·582	628 8237	16·682	568 9833	16·782	514 8373	16·882	465 8441
16·584	627 5673	16·684	567 8464	16·784	513 8087	16·884	464 9133
16·586	626 3134	16·686	566 7118	16·786	512 7821	16·886	463 9844
16·588	625 0621	16·688	565 5795	16·788	511 7575	16·888	463 0574
16·590	623 8132	16·690	564 4495	16·790	510 7350	16·890	462 1322
16·592	622 5668	16·692	563 3217	16·792	509 7146	16·892	461 2088
16·594	621 3229	16·694	562 1962	16·794	508 6962	16·894	460 2873
16·596	620 0815	16·696	561 0729	16·796	507 6798	16·896	459 3677
16·598	618 8426	16·698	559 9519	16·798	506 6654	16·898	458 4499
16·600	617 6061	16·700	558 8331	16·800	505 6531	16·900	457 5339
16·602	616 3722	16·702	557 7166	16·802	504 6428	16·902	456 6197
16·604	615 1406	16·704	556 6023	16·804	503 6346	16·904	455 7074
16·606	613 9116	16·706	555 4902	16·806	502 6283	16·906	454 7969
16·608	612 6850	16·708	554 3803	16·808	501 6241	16·908	453 8882
16·610	611 4609	16·710	553 2726	16·810	500 6218	16·910	452 9813
16·612	610 2392	16·712	552 1672	16·812	499 6215	16·912	452 0763
16·614	609 0199	16·714	551 0640	16·814	498 6233	16·914	451 1730
16·616	607 8031	16·716	549 9630	16·816	497 6271	16·916	450 2716
16·618	606 5887	16·718	548 8641	16·818	496 6328	16·918	449 3719
16·620	605 3767	16·720	547 7675	16·820	495 6405	16·920	448 4741
16·622	604 1672	16·722	546 6731	16·822	494 6502	16·922	447 5780
16·624	602 9601	16·724	545 5808	16·824	493 6619	16·924	446 6838
16·626	601 7553	16·726	544 4907	16·826	492 6756	16·926	445 7913
16·628	600 5530	16·728	543 4028	16·828	491 6912	16·928	444 9006
16·630	599 3531	16·730	542 3171	16·830	490 7088	16·930	444 0117
16·632	598 1556	16·732	541 2336	16·832	489 7284	16·932	443 1246
16·634	596 9605	16·734	540 1522	16·834	488 7499	16·934	442 2392
16·636	595 7678	16·736	539 0730	16·836	487 7734	16·936	441 3556
16·638	594 5775	16·738	537 9959	16·838	486 7988	16·938	440 4738
16·640	593 3895	16·740	536 9210	16·840	485 8262	16·940	439 5937
16·642	592 2039	16·742	535 8482	16·842	484 8555	16·942	438 7154
16·644	591 0207	16·744	534 7776	16·844	483 8868	16·944	437 8389
16·646	589 8398	16·746	533 7091	16·846	482 9200	16·946	436 9641
16·648	588 6613	16·748	532 6428	16·848	481 9551	16·948	436 0910

[*Second Part. Fourteen decimal places.*]

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
16·950	435 2197	17·050	393 8030	17·150	356 3277	17·250	322 4187
16·952	434 3501	17·052	393 0162	17·152	355 6158	17·252	321 7745
16·954	433 4823	17·054	392 2310	17·154	354 9053	17·254	321 1315
16·956	432 6162	17·056	391 4473	17·156	354 1962	17·256	320 4899
16·958	431 7518	17·058	390 6652	17·158	353 4885	17·258	319 8496
16·960	430 8892	17·060	389 8846	17·160	352 7822	17·260	319 2105
16·962	430 0283	17·062	389 1056	17·162	352 0774	17·262	318 5728
16·964	429 1691	17·064	388 3282	17·164	351 3739	17·264	317 9363
16·966	428 3116	17·066	387 5523	17·166	350 6719	17·266	317 3010
16·968	427 4558	17·068	386 7780	17·168	349 9712	17·268	316 6670
16·970	426 6018	17·070	386 0052	17·170	349 2720	17·270	316 0343
16·972	425 7494	17·072	385 2340	17·172	348 5741	17·272	315 4029
16·974	424 8988	17·074	384 4643	17·174	347 8777	17·274	314 7727
16·976	424 0498	17·076	383 6961	17·176	347 1826	17·276	314 1438
16·978	423 2026	17·078	382 9295	17·178	346 4889	17·278	313 5162
16·980	422 3570	17·080	382 1644	17·180	345 7966	17·280	312 8898
16·982	421 5132	17·082	381 4008	17·182	345 1057	17·282	312 2646
16·984	420 6710	17·084	380 6388	17·184	344 4162	17·284	311 6407
16·986	419 8305	17·086	379 8783	17·186	343 7281	17·286	311 0180
16·988	418 9917	17·088	379 1193	17·188	343 0413	17·288	310 3966
16·990	418 1545	17·090	378 3618	17·190	342 3559	17·290	309 7764
16·992	417 3191	17·092	377 6058	17·192	341 6719	17·292	309 1575
16·994	416 4852	17·094	376 8514	17·194	340 9892	17·294	308 5398
16·996	415 6531	17·096	376 0984	17·196	340 3079	17·296	307 9234
16·998	414 8226	17·098	375 3470	17·198	339 6280	17·298	307 3081
17·000	413 9938	17·100	374 5970	17·200	338 9494	17·300	306 6941
17·002	413 1666	17·102	373 8486	17·202	338 2722	17·305	305 1645
17·004	412 3411	17·104	373 1016	17·204	337 5964	17·310	303 6425
17·006	411 5173	17·106	372 3562	17·206	336 9218	17·315	302 1280
17·008	410 6951	17·108	371 6122	17·208	336 2487	17·320	300 6212
17·010	409 8745	17·110	370 8697	17·210	335 5768	17·325	299 1218
17·012	409 0556	17·112	370 1287	17·212	334 9064	17·330	297 6299
17·014	408 2383	17·114	369 3892	17·214	334 2372	17·335	296 1455
17·016	407 4226	17·116	368 6512	17·216	333 5694	17·340	294 6685
17·018	406 6086	17·118	367 9146	17·218	332 9029	17·345	293 1988
17·020	405 7961	17·120	367 1795	17·220	332 2378	17·350	291 7365
17·022	404 9854	17·122	366 4459	17·222	331 5740	17·355	290 2814
17·024	404 1762	17·124	365 7137	17·224	330 9115	17·360	288 8336
17·026	403 3687	17·126	364 9830	17·226	330 2504	17·365	287 3931
17·028	402 5628	17·128	364 2538	17·228	329 5905	17·370	285 9597
17·030	401 7584	17·130	363 5260	17·230	328 9320	17·375	284 5335
17·032	400 9557	17·132	362 7997	17·232	328 2748	17·380	283 1143
17·034	400 1546	17·134	362 0748	17·234	327 6189	17·385	281 7023
17·036	399 3551	17·136	361 3514	17·236	326 9643	17·390	280 2973
17·038	398 5572	17·138	360 6294	17·238	326 3110	17·395	278 8993
17·040	397 7608	17·140	359 9089	17·240	325 6590	17·400	277 5083
17·042	396 9661	17·142	359 1898	17·242	325 0084	17·405	276 1242
17·044	396 1730	17·144	358 4721	17·244	324 3590	17·410	274 7470
17·046	395 3814	17·146	357 7559	17·246	323 7109	17·415	273 3768
17·048	394 5914	17·148	357 0411	17·248	323 0642	17·420	272 0133

[*Second Part. Fourteen decimal places.*]

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
17·425	270 6566	17·675	210 7876	17·925	164 1615	18·175	127 8491
17·430	269 3067	17·680	209 7363	17·930	163 3428	18·180	127 2115
17·435	267 9635	17·685	208 6902	17·935	162 5281	18·185	126 5770
17·440	266 6271	17·690	207 6494	17·940	161 7175	18·190	125 9457
17·445	265 2973	17·695	206 6137	17·945	160 9109	18·195	125 3175
17·450	263 9741	17·700	205 5832	17·950	160 1084	18·200	124 6925
17·455	262 6575	17·705	204 5579	17·955	159 3098	18·205	124 0706
17·460	261 3475	17·710	203 5376	17·960	158 5153	18·210	123 4518
17·465	260 0440	17·715	202 5225	17·965	157 7247	18·215	122 8361
17·470	258 7470	17·720	201 5124	17·970	156 9380	18·220	122 2234
17·475	257 4565	17·725	200 5074	17·975	156 1553	18·225	121 6139
17·480	256 1725	17·730	199 5073	17·980	155 3765	18·230	121 0073
17·485	254 8948	17·735	198 5123	17·985	154 6015	18·235	120 4038
17·490	253 6235	17·740	197 5222	17·990	153 8304	18·240	119 8033
17·495	252 3586	17·745	196 5370	17·995	153 0632	18·245	119 2057
17·500	251 0999	17·750	195 5568	18·000	152 2998	18·250	118 6112
17·505	249 8475	17·755	194 5815	18·005	151 5402	18·255	118 0196
17·510	248 6014	17·760	193 6110	18·010	150 7844	18·260	117 4310
17·515	247 3615	17·765	192 6454	18·015	150 0323	18·265	116 8453
17·520	246 1278	17·770	191 6845	18·020	149 2841	18·270	116 2625
17·525	244 9002	17·775	190 7285	18·025	148 5395	18·275	115 6827
17·530	243 6788	17·780	189 7772	18·030	147 7987	18·280	115 1057
17·535	242 4634	17·785	188 8307	18·035	147 0615	18·285	114 5316
17·540	241 2541	17·790	187 8889	18·040	146 3280	18·290	113 9604
17·545	240 0509	17·795	186 9518	18·045	145 5982	18·295	113 3920
17·550	238 8536	17·800	186 0194	18·050	144 8720	18·300	112 8264
17·555	237 6623	17·805	185 0916	18·055	144 1495	18·305	112 2637
17·560	236 4770	17·810	184 1685	18·060	143 4305	18·310	111 7038
17·565	235 2975	17·815	183 2499	18·065	142 7152	18·315	111 1467
17·570	234 1240	17·820	182 3360	18·070	142 0034	18·320	110 5923
17·575	232 9563	17·825	181 4266	18·075	141 2951	18·325	110 0407
17·580	231 7944	17·830	180 5217	18·080	140 5904	18·330	109 4919
17·585	230 6383	17·835	179 6213	18·085	139 8892	18·335	108 9458
17·590	229 4880	17·840	178 7255	18·090	139 1915	18·340	108 4025
17·595	228 3435	17·845	177 8341	18·095	138 4973	18·345	107 8618
17·600	227 2046	17·850	176 9471	18·100	137 8065	18·350	107 3238
17·605	226 0714	17·855	176 0646	18·105	137 1192	18·355	106 7886
17·610	224 9439	17·860	175 1865	18·110	136 4353	18·360	106 2559
17·615	223 8220	17·865	174 3127	18·115	135 7549	18·365	105 7260
17·620	222 7057	17·870	173 4433	18·120	135 0778	18·370	105 1987
17·625	221 5949	17·875	172 5783	18·125	134 4041	18·375	104 6740
17·630	220 4897	17·880	171 7175	18·130	133 7338	18·380	104 1519
17·635	219 3900	17·885	170 8611	18·135	133 0667	18·385	103 6325
17·640	218 2958	17·890	170 0089	18·140	132 4031	18·390	103 1156
17·645	217 2070	17·895	169 1610	18·145	131 7427	18·395	102 6013
17·650	216 1237	17·900	168 3173	18·150	131 0856	18·400	102 0896
17·655	215 0458	17·905	167 4778	18·155	130 4318	18·405	101 5804
17·660	213 9732	17·910	166 6425	18·160	129 7813	18·410	101 0738
17·665	212 9060	17·915	165 8114	18·165	129 1340	18·415	100 5697
17·670	211 8442	17·920	164 9844	18·170	128 4900	18·420	100 0681

[*Second Part. Fourteen decimal places.*]

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
18·425	99 5690	18·675	77 5444	18·925	60 3917	19·175	47 0331	19·425	36 6294
18·430	99 0724	18·680	77 1576	18·930	60 0905	19·180	46 7985	19·430	36 4467
18·435	98 5783	18·685	76 7728	18·935	59 7908	19·185	46 5651	19·435	36 2649
18·440	98 0866	18·690	76 3899	18·940	59 4925	19·190	46 3328	19·440	36 0840
18·445	97 5974	18·695	76 0089	18·945	59 1958	19·195	46 1018	19·445	35 9041
18·450	97 1106	18·700	75 6298	18·950	58 9006	19·200	45 8718	19·450	35 7250
18·455	96 6263	18·705	75 2526	18·955	58 6068	19·205	45 6430	19·455	35 5468
18·460	96 1444	18·710	74 8773	18·960	58 3145	19·210	45 4154	19·460	35 3695
18·465	95 6648	18·715	74 5039	18·965	58 0237	19·215	45 1889	19·465	35 1931
18·470	95 1877	18·720	74 1323	18·970	57 7343	19·220	44 9635	19·470	35 0176
18·475	94 7130	18·725	73 7625	18·975	57 4463	19·225	44 7392	19·475	34 8429
18·480	94 2406	18·730	73 3946	18·980	57 1598	19·230	44 5161	19·480	34 6692
18·485	93 7705	18·735	73 0286	18·985	56 8747	19·235	44 2941	19·485	34 4963
18·490	93 3029	18·740	72 6643	18·990	56 5910	19·240	44 0732	19·490	34 3242
18·495	92 8375	18·745	72 3019	18·995	56 3088	19·245	43 8534	19·495	34 1530
18·500	92 3745	18·750	71 9413	19·000	56 0280	19·250	43 6346	19·500	33 9827
18·505	91 9138	18·755	71 5825	19·005	55 7485	19·255	43 4170	19·505	33 8132
18·510	91 4554	18·760	71 2255	19·010	55 4705	19·260	43 2004	19·510	33 6445
18·515	90 9992	18·765	70 8703	19·015	55 1938	19·265	42 9850	19·515	33 4767
18·520	90 5453	18·770	70 5168	19·020	54 9185	19·270	42 7706	19·520	33 3098
18·525	90 0938	18·775	70 1651	19·025	54 6446	19·275	42 5573	19·525	33 1436
18·530	89 6444	18·780	69 8152	19·030	54 3721	19·280	42 3450	19·530	32 9783
18·535	89 1973	18·785	69 4670	19·035	54 1009	19·285	42 1338	19·535	32 8139
18·540	88 7524	18·790	69 1205	19·040	53 8311	19·290	41 9237	19·540	32 6502
18·545	88 3098	18·795	68 7757	19·045	53 5626	19·295	41 7146	19·545	32 4873
18·550	87 8693	18·800	68 4327	19·050	53 2954	19·300	41 5065	19·550	32 3253
18·555	87 4311	18·805	68 0914	19·055	53 0296	19·305	41 2995	19·555	32 1641
18·560	86 9950	18·810	67 7518	19·060	52 7651	19·310	41 0935	19·560	32 0037
18·565	86 5611	18·815	67 4139	19·065	52 5020	19·315	40 8886	19·565	31 8441
18·570	86 1294	18·820	67 0776	19·070	52 2401	19·320	40 6846	19·570	31 6852
18·575	85 6998	18·825	66 7431	19·075	51 9796	19·325	40 4817	19·575	31 5272
18·580	85 2724	18·830	66 4102	19·080	51 7203	19·330	40 2798	19·580	31 3700
18·585	84 8471	18·835	66 0790	19·085	51 4624	19·335	40 0789	19·585	31 2135
18·590	84 4239	18·840	65 7494	19·090	51 2057	19·340	39 8790	19·590	31 0578
18·595	84 0029	18·845	65 4215	19·095	50 9503	19·345	39 6801	19·595	30 9029
18·600	83 5839	18·850	65 0952	19·100	50 6962	19·350	39 4822	19·600	30 7488
18·605	83 1670	18·855	64 7706	19·105	50 4433	19·355	39 2853	19·605	30 5954
18·610	82 7522	18·860	64 4475	19·110	50 1917	19·360	39 0894	19·610	30 4428
18·615	82 3395	18·865	64 1261	19·115	49 9414	19·365	38 8944	19·615	30 2910
18·620	81 9288	18·870	63 8062	19·120	49 6923	19·370	38 7004	19·620	30 1399
18·625	81 5202	18·875	63 4880	19·125	49 4445	19·375	38 5074	19·625	29 9896
18·630	81 1136	18·880	63 1713	19·130	49 1979	19·380	38 3154	19·630	29 8400
18·635	80 7091	18·885	62 8563	19·135	48 9525	19·385	38 1243	19·635	29 6912
18·640	80 3065	18·890	62 5428	19·140	48 7084	19·390	37 9341	19·640	29 5431
18·645	79 9060	18·895	62 2309	19·145	48 4654	19·395	37 7449	19·645	29 3958
18·650	79 5075	18·900	61 9205	19·150	48 2237	19·400	37 5567	19·650	29 2492
18·655	79 1109	18·905	61 6116	19·155	47 9832	19·405	37 3693	19·655	29 1033
18·660	78 7164	18·910	61 3044	19·160	47 7439	19·410	37 1830	19·660	28 9581
18·665	78 3238	18·915	60 9986	19·165	47 5057	19·415	36 9975	19·665	28 8137
18·670	77 9331	18·920	60 6944	19·170	47 2688	19·420	36 8130	19·670	28 6700

[*Second Part. Fourteen decimal places.*]

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
19·675	28 5270	19·925	22 2168	20·175	17 3025	20·425	13 4752	20·675	10 4945
19·680	28 3847	19·930	22 1060	20·180	17 2162	20·430	13 4080	20·680	10 4421
19·685	28 2431	19·935	21 9958	20·185	17 1303	20·435	13 3411	20·685	10 3901
19·690	28 1023	19·940	21 8861	20·190	17 0449	20·440	13 2746	20·690	10 3382
19·695	27 9621	19·945	21 7769	20·195	16 9599	20·445	13 2084	20·695	10 2867
19·700	27 8227	19·950	21 6683	20·200	16 8753	20·450	13 1425	20·700	10 2354
19·705	27 6839	19·955	21 5602	20·205	16 7911	20·455	13 0769	20·705	10 1843
19·710	27 5458	19·960	21 4527	20·210	16 7074	20·460	13 0117	20·710	10 1335
19·715	27 4084	19·965	21 3457	20·215	16 6240	20·465	12 9468	20·715	10 0830
19·720	27 2717	19·970	21 2392	20·220	16 5411	20·470	12 8822	20·720	10 0327
19·725	27 1357	19·975	21 1333	20·225	16 4586	20·475	12 8180	20·725	9 9827
19·730	27 0004	19·980	21 0279	20·230	16 3765	20·480	12 7541	20·730	9 9329
19·735	26 8657	19·985	20 9230	20·235	16 2949	20·485	12 6905	20·735	9 8833
19·740	26 7317	19·990	20 8187	20·240	16 2136	20·490	12 6272	20·740	9 8340
19·745	26 5984	19·995	20 7148	20·245	16 1327	20·495	12 5642	20·745	9 7850
19·750	26 4657	20·000	20 6115	20·250	16 0523	20·500	12 5015	20·750	9 7362
19·755	26 3337	20·005	20 5087	20·255	15 9722	20·505	12 4392	20·755	9 6876
19·760	26 2024	20·010	20 4064	20·260	15 8925	20·510	12 3771	20·760	9 6393
19·765	26 0717	20·015	20 3047	20·265	15 8133	20·515	12 3154	20·765	9 5912
19·770	25 9417	20·020	20 2034	20·270	15 7344	20·520	12 2540	20·770	9 5434
19·775	25 8123	20·025	20 1026	20·275	15 6559	20·525	12 1929	20·775	9 4958
19·780	25 6835	20·030	20 0024	20·280	15 5779	20·530	12 1320	20·780	9 4484
19·785	25 5555	20·035	19 9026	20·285	15 5002	20·535	12 0715	20·785	9 4013
19·790	25 4280	20·040	19 8033	20·290	15 4229	20·540	12 0113	20·790	9 3544
19·795	25 3012	20·045	19 7046	20·295	15 3459	20·545	11 9514	20·795	9 3078
19·800	25 1750	20·050	19 6063	20·300	15 2694	20·550	11 8918	20·800	9 2614
19·805	25 0494	20·055	19 5085	20·305	15 1932	20·555	11 8325	20·805	9 2152
19·810	24 9245	20·060	19 4112	20·310	15 1175	20·560	11 7735	20·810	9 1692
19·815	24 8002	20·065	19 3144	20·315	15 0421	20·565	11 7148	20·815	9 1235
19·820	24 6765	20·070	19 2181	20·320	14 9670	20·570	11 6563	20·820	9 0780
19·825	24 5534	20·075	19 1222	20·325	14 8924	20·575	11 5982	20·825	9 0327
19·830	24 4309	20·080	19 0268	20·330	14 8181	20·580	11 5404	20·830	8 9876
19·835	24 3091	20·085	18 9319	20·335	14 7442	20·585	11 4828	20·835	8 9428
19·840	24 1879	20·090	18 8375	20·340	14 6707	20·590	11 4255	20·840	8 8982
19·845	24 0672	20·095	18 7436	20·345	14 5975	20·595	11 3685	20·845	8 8538
19·850	23 9472	20·100	18 6501	20·350	14 5247	20·600	11 3118	20·850	8 8097
19·855	23 8277	20·105	18 5571	20·355	14 4523	20·605	11 2554	20·855	8 7657
19·860	23 7089	20·110	18 4645	20·360	14 3802	20·610	11 1993	20·860	8 7220
19·865	23 5907	20·115	18 3724	20·365	14 3085	20·615	11 1434	20·865	8 6785
19·870	23 4730	20·120	18 2808	20·370	14 2371	20·620	11 0879	20·870	8 6352
19·875	23 3559	20·125	18 1896	20·375	14 1661	20·625	11 0326	20·875	8 5922
19·880	23 2394	20·130	18 0989	20·380	14 0954	20·630	10 9775	20·880	8 5493
19·885	23 1235	20·135	18 0086	20·385	14 0251	20·635	10 9228	20·885	8 5067
19·890	23 0082	20·140	17 9188	20·390	13 9552	20·640	10 8683	20·890	8 4642
19·895	22 8934	20·145	17 8294	20·395	13 8856	20·645	10 8141	20·895	8 4220
19·900	22 7793	20·150	17 7405	20·400	13 8163	20·650	10 7602	20·900	8 3800
19·905	22 6656	20·155	17 6520	20·405	13 7474	20·655	10 7065	20·905	8 3382
19·910	22 5526	20·160	17 5640	20·410	13 6788	20·660	10 6531	20·910	8 2966
19·915	22 4401	20·165	17 4764	20·415	13 6106	20·665	10 6000	20·915	8 2553
19·920	22 3282	20·170	17 3892	20·420	13 5427	20·670	10 5471	20·920	8 2141

[*Second Part. Fourteen decimal places.*]

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
20·925	8 1731	21·175	6 3652	21·425	4 9572	21·675	3 8607	21·925	3 0067
20·930	8 1323	21·180	6 3335	21·430	4 9325	21·680	3 8414	21·930	2 9917
20·935	8 0918	21·185	6 3019	21·435	4 9079	21·685	3 8223	21·935	2 9768
20·940	8 0514	21·190	6 2705	21·440	4 8834	21·690	3 8032	21·940	2 9620
20·945	8 0113	21·195	6 2392	21·445	4 8591	21·695	3 7843	21·945	2 9472
20·950	7 9713	21·200	6 2081	21·450	4 8348	21·700	3 7654	21·950	2 9325
20·955	7 9316	21·205	6 1771	21·455	4 8107	21·705	3 7466	21·955	2 9179
20·960	7 8920	21·210	6 1463	21·460	4 7867	21·710	3 7279	21·960	2 9033
20·965	7 8526	21·215	6 1156	21·465	4 7629	21·715	3 7093	21·965	2 8888
20·970	7 8135	21·220	6 0851	21·470	4 7391	21·720	3 6908	21·970	2 8744
20·975	7 7745	21·225	6 0548	21·475	4 7155	21·725	3 6724	21·975	2 8601
20·980	7 7357	21·230	6 0246	21·480	4 6920	21·730	3 6541	21·980	2 8458
20·985	7 6971	21·235	5 9945	21·485	4 6686	21·735	3 6359	21·985	2 8316
20·990	7 6588	21·240	5 9646	21·490	4 6453	21·740	3 6177	21·990	2 8175
20·995	7 6206	21·245	5 9349	21·495	4 6221	21·745	3 5997	21·995	2 8034
21·000	7 5826	21·250	5 9053	21·500	4 5990	21·750	3 5817	22·000	2 7895
21·005	7 5447	21·255	5 8758	21·505	4 5761	21·755	3 5639	22·005	2 7756
21·010	7 5071	21·260	5 8465	21·510	4 5533	21·760	3 5461	22·010	2 7617
21·015	7 4697	21·265	5 8174	21·515	4 5306	21·765	3 5284	22·015	2 7479
21·020	7 4324	21·270	5 7884	21·520	4 5080	21·770	3 5108	22·020	2 7342
21·025	7 3953	21·275	5 7595	21·525	4 4855	21·775	3 4933	22·025	2 7206
21·030	7 3585	21·280	5 7308	21·530	4 4631	21·780	3 4759	22·030	2 7070
21·035	7 3218	21·285	5 7022	21·535	4 4409	21·785	3 4585	22·035	2 6935
21·040	7 2852	21·290	5 6737	21·540	4 4187	21·790	3 4413	22·040	2 6801
21·045	7 2489	21·295	5 6454	21·545	4 3967	21·795	3 4241	22·045	2 6667
21·050	7 2127	21·300	5 6173	21·550	4 3747	21·800	3 4071	22·050	2 6534
21·055	7 1768	21·305	5 5893	21·555	4 3529	21·805	3 3901	22·055	2 6402
21·060	7 1410	21·310	5 5614	21·560	4 3312	21·810	3 3732	22·060	2 6270
21·065	7 1054	21·315	5 5337	21·565	4 3096	21·815	3 3563	22·065	2 6139
21·070	7 0699	21·320	5 5061	21·570	4 2881	21·820	3 3396	22·070	2 6009
21·075	7 0347	21·325	5 4786	21·575	4 2667	21·825	3 3229	22·075	2 5879
21·080	6 9996	21·330	5 4513	21·580	4 2455	21·830	3 3064	22·080	2 5750
21·085	6 9647	21·335	5 4241	21·585	4 2243	21·835	3 2899	22·085	2 5622
21·090	6 9299	21·340	5 3970	21·590	4 2032	21·840	3 2735	22·090	2 5494
21·095	6 8954	21·345	5 3701	21·595	4 1823	21·845	3 2571	22·095	2 5367
21·100	6 8610	21·350	5 3433	21·600	4 1614	21·850	3 2409	22·100	2 5240
21·105	6 8268	21·355	5 3167	21·605	4 1406	21·855	3 2247	22·105	2 5114
21·110	6 7927	21·360	5 2902	21·610	4 1200	21·860	3 2086	22·110	2 4989
21·115	6 7588	21·365	5 2638	21·615	4 0994	21·865	3 1926	22·115	2 4864
21·120	6 7251	21·370	5 2376	21·620	4 0790	21·870	3 1767	22·120	2 4740
21·125	6 6916	21·375	5 2114	21·625	4 0586	21·875	3 1609	22·125	2 4617
21·130	6 6582	21·380	5 1854	21·630	4 0384	21·880	3 1451	22·130	2 4494
21·135	6 6250	21·385	5 1595	21·635	4 0183	21·885	3 1294	22·135	2 4372
21·140	6 5920	21·390	5 1338	21·640	3 9982	21·890	3 1138	22·140	2 4250
21·145	6 5591	21·395	5 1082	21·645	3 9783	21·895	3 0983	22·145	2 4129
21·150	6 5264	21·400	5 0827	21·650	3 9584	21·900	3 0828	22·150	2 4009
21·155	6 4938	21·405	5 0574	21·655	3 9387	21·905	3 0674	22·155	2 3889
21·160	6 4614	21·410	5 0322	21·660	3 9190	21·910	3 0521	22·160	2 3770
21·165	6 4292	21·415	5 0071	21·665	3 8995	21·915	3 0369	22·165	2 3652
21·170	6 3971	21·420	4 9821	21·670	3 8801	21·920	3 0218	22·170	2 3534

[*Second Part. Fourteen decimal places.*]

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
22·175	2 3416	22·425	1 8237	22·675	1 4203	22·925	1 1061	23·175	8614
22·180	2 3299	22·430	1 8146	22·680	1 4132	22·930	1 1006	23·180	8571
22·185	2 3183	22·435	1 8055	22·685	1 4061	22·935	1 0951	23·185	8529
22·190	2 3068	22·440	1 7965	22·690	1 3991	22·940	1 0896	23·190	8486
22·195	2 2953	22·445	1 7876	22·695	1 3921	22·945	1 0842	23·195	8444
22·200	2 2838	22·450	1 7786	22·700	1 3852	22·950	1 0788	23·200	8402
22·205	2 2724	22·455	1 7698	22·705	1 3783	22·955	1 0734	23·205	8360
22·210	2 2611	22·460	1 7609	22·710	1 3714	22·960	1 0681	23·210	8318
22·215	2 2498	22·465	1 7522	22·715	1 3646	22·965	1 0627	23·215	8277
22·220	2 2386	22·470	1 7434	22·720	1 3578	22·970	1 0574	23·220	8235
22·225	2 2274	22·475	1 7347	22·725	1 3510	22·975	1 0522	23·225	8194
22·230	2 2163	22·480	1 7261	22·730	1 3442	22·980	1 0469	23·230	8153
22·235	2 2053	22·485	1 7175	22·735	1 3376	22·985	1 0417	23·235	8113
22·240	2 1943	22·490	1 7089	22·740	1 3309	22·990	1 0365	23·240	8072
22·245	2 1833	22·495	1 7004	22·745	1 3243	22·995	1 0313	23·245	8032
22·250	2 1724	22·500	1 6919	22·750	1 3176	23·000	1 0262	23·250	7992
22·255	2 1616	22·505	1 6835	22·755	1 3111	23·005	1 0211	23·255	7952
22·260	2 1508	22·510	1 6751	22·760	1 3045	23·010	1 0160	23·260	7912
22·265	2 1401	22·515	1 6667	22·765	1 2980	23·015	1 0109	23·265	7873
22·270	2 1294	22·520	1 6584	22·770	1 2915	23·020	1 0059	23·270	7834
22·275	2 1188	22·525	1 6501	22·775	1 2851	23·025	1 0009	23·275	7795
22·280	2 1082	22·530	1 6419	22·780	1 2787	23·030	0 9959	23·280	7756
22·285	2 0977	22·535	1 6337	22·785	1 2723	23·035	9909	23·285	7717
22·290	2 0873	22·540	1 6256	22·790	1 2660	23·040	9860	23·290	7679
22·295	2 0768	22·545	1 6174	22·795	1 2597	23·045	9810	23·295	7640
22·300	2 0665	22·550	1 6094	22·800	1 2534	23·050	9761	23·300	7602
22·305	2 0562	22·555	1 6014	22·805	1 2471	23·055	9713	23·305	7564
22·310	2 0459	22·560	1 5934	22·810	1 2409	23·060	9664	23·310	7526
22·315	2 0357	22·565	1 5854	22·815	1 2347	23·065	9616	23·315	7489
22·320	2 0256	22·570	1 5775	22·820	1 2286	23·070	9568	23·320	7452
22·325	2 0155	22·575	1 5697	22·825	1 2224	23·075	9520	23·325	7415
22·330	2 0054	22·580	1 5618	22·830	1 2163	23·080	9473	23·330	7378
22·335	1 9954	22·585	1 5540	22·835	1 2103	23·085	9426	23·335	7340
22·340	1 9855	22·590	1 5463	22·840	1 2042	23·090	9379	23·340	7304
22·345	1 9756	22·595	1 5386	22·845	1 1982	23·095	9332	23·345	7267
22·350	1 9657	22·600	1 5309	22·850	1 1922	23·100	9285	23·350	7231
22·355	1 9559	22·605	1 5233	22·855	1 1863	23·105	9239	23·355	7195
22·360	1 9461	22·610	1 5157	22·860	1 1804	23·110	9193	23·360	7159
22·365	1 9364	22·615	1 5081	22·865	1 1745	23·115	9147	23·365	7123
22·370	1 9268	22·620	1 5006	22·870	1 1686	23·120	9102	23·370	7088
22·375	1 9172	22·625	1 4931	22·875	1 1628	23·125	9056	23·375	7053
22·380	1 9076	22·630	1 4856	22·880	1 1570	23·130	9011	23·380	7018
22·385	1 8981	22·635	1 4782	22·885	1 1512	23·135	8966	23·385	6983
22·390	1 8886	22·640	1 4709	22·890	1 1455	23·140	8921	23·390	6948
22·395	1 8792	22·645	1 4635	22·895	1 1398	23·145	8877	23·395	6913
22·400	1 8698	22·650	1 4562	22·900	1 1341	23·150	8832	23·400	6879
22·405	1 8605	22·655	1 4490	22·905	1 1284	23·155	8788	23·405	6844
22·410	1 8512	22·660	1 4417	22·910	1 1228	23·160	8745	23·410	6810
22·415	1 8420	22·665	1 4345	22·915	1 1172	23·165	8701	23·415	6776
22·420	1 8328	22·670	1 4274	22·920	1 1116	23·170	8658	23·420	6742

[*Second Part. Fourteen decimal places.*]

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
23·425	6709	23·675	5225	23·925	4069	24·175	3169	24·425	2468
23·430	6675	23·680	5199	23·930	4049	24·180	3153	24·430	2455
23·435	6642	23·685	5173	23·935	4029	24·185	3137	24·435	2443
23·440	6609	23·690	5147	23·940	4009	24·190	3122	24·440	2431
23·445	6576	23·695	5122	23·945	3989	24·195	3106	24·445	2419
23·450	6543	23·700	5096	23·950	3969	24·200	3091	24·450	2407
23·455	6510	23·705	5070	23·955	3949	24·205	3076	24·455	2395
23·460	6478	23·710	5045	23·960	3929	24·210	3061	24·460	2383
23·465	6446	23·715	5020	23·965	3910	24·215	3045	24·465	2371
23·470	6414	23·720	4995	23·970	3890	24·220	3030	24·470	2359
23·475	6382	23·725	4970	23·975	3871	24·225	3015	24·475	2348
23·480	6350	23·730	4945	23·980	3852	24·230	3000	24·480	2336
23·485	6318	23·735	4920	23·985	3833	24·235	2985	24·485	2324
23·490	6286	23·740	4895	23·990	3814	24·240	2970	24·490	2313
23·495	6255	23·745	4871	23·995	3794	24·245	2955	24·495	2301
23·500	6224	23·750	4847	24·000	3775	24·250	2940	24·500	2290
23·505	6193	23·755	4823	24·005	3756	24·255	2925	24·505	2278
23·510	6162	23·760	4799	24·010	3737	24·260	2910	24·510	2267
23·515	6131	23·765	4775	24·015	3718	24·265	2895	24·515	2256
23·520	6100	23·770	4751	24·020	3700	24·270	2881	24·520	2244
23·525	6070	23·775	4728	24·025	3682	24·275	2867	24·525	2233
23·530	6040	23·780	4704	24·030	3663	24·280	2853	24·530	2222
23·535	6010	23·785	4680	24·035	3645	24·285	2839	24·535	2211
23·540	5980	23·790	4657	24·040	3627	24·290	2825	24·540	2200
23·545	5950	23·795	4634	24·045	3609	24·295	2810	24·545	2189
23·550	5921	23·800	4611	24·050	3591	24·300	2797	24·550	2178
23·555	5891	23·805	4588	24·055	3573	24·305	2783	24·555	2167
23·560	5861	23·810	4565	24·060	3555	24·310	2769	24·560	2156
23·565	5832	23·815	4542	24·065	3536	24·315	2755	24·565	2146
23·570	5803	23·820	4519	24·070	3519	24·320	2741	24·570	2135
23·575	5774	23·825	4497	24·075	3502	24·325	2728	24·575	2124
23·580	5745	23·830	4474	24·080	3485	24·330	2714	24·580	2113
23·585	5716	23·835	4452	24·085	3467	24·335	2700	24·585	2103
23·590	5687	23·840	4430	24·090	3450	24·340	2687	24·590	2092
23·595	5660	23·845	4408	24·095	3433	24·345	2674	24·595	2082
23·600	5632	23·850	4386	24·100	3416	24·350	2660	24·600	2072
23·605	5604	23·855	4364	24·105	3398	24·355	2647	24·605	2061
23·610	5576	23·860	4342	24·110	3381	24·360	2634	24·610	2051
23·615	5548	23·865	4320	24·115	3364	24·365	2621	24·615	2041
23·620	5521	23·870	4299	24·120	3347	24·370	2608	24·620	2031
23·625	5493	23·875	4278	24·125	3331	24·375	2595	24·625	2021
23·630	5465	23·880	4256	24·130	3314	24·380	2582	24·630	2010
23·635	5438	23·885	4235	24·135	3297	24·385	2569	24·635	2000
23·640	5411	23·890	4214	24·140	3281	24·390	2556	24·640	1990
23·645	5384	23·895	4193	24·145	3265	24·395	2543	24·645	1981
23·650	5357	23·900	4172	24·150	3249	24·400	2530	24·650	1971
23·655	5330	23·905	4151	24·155	3233	24·405	2518	24·655	1961
23·660	5303	23·910	4130	24·160	3217	24·410	2505	24·660	1951
23·665	5277	23·915	4109	24·165	3201	24·415	2493	24·665	1941
23·670	5251	23·920	4089	24·170	3185	24·420	2480	24·670	1932

[*Second Part. Fourteen decimal places.*]

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
24·675	1922	24·925	1497	25·175	1165	25·425	908	25·675	707
24·680	1913	24·930	1489	25·180	1160	25·430	903	25·680	704
24·685	1903	24·935	1482	25·185	1154	25·435	899	25·685	700
24·690	1894	24·940	1475	25·190	1148	25·440	894	25·690	697
24·695	1884	24·945	1467	25·195	1142	25·445	890	25·695	693
24·700	1875	24·950	1460	25·200	1137	25·450	885	25·700	690
24·705	1865	24·955	1453	25·205	1131	25·455	881	25·705	686
24·710	1856	24·960	1445	25·210	1126	25·460	877	25·710	683
24·715	1847	24·965	1438	25·215	1120	25·465	872	25·715	679
24·720	1838	24·970	1431	25·220	1115	25·470	868	25·720	676
24·725	1828	24·975	1424	25·225	1109	25·475	864	25·725	673
24·730	1819	24·980	1417	25·230	1103	25·480	859	25·730	669
24·735	1810	24·985	1410	25·235	1098	25·485	855	25·735	666
24·740	1801	24·990	1403	25·240	1092	25·490	851	25·740	663
24·745	1792	24·995	1396	25·245	1087	25·495	847	25·745	659
24·750	1783	25·000	1389	25·250	1082	25·500	842	25·750	656
24·755	1774	25·005	1382	25·255	1076	25·505	838	25·755	653
24·760	1766	25·010	1375	25·260	1071	25·510	834	25·760	649
24·765	1757	25·015	1368	25·265	1065	25·515	830	25·765	646
24·770	1748	25·020	1361	25·270	1060	25·520	826	25·770	643
24·775	1739	25·025	1354	25·275	1055	25·525	822	25·775	640
24·780	1730	25·030	1348	25·280	1050	25·530	817	25·780	636
24·785	1722	25·035	1341	25·285	1044	25·535	813	25·785	633
24·790	1713	25·040	1334	25·290	1039	25·540	809	25·790	630
24·795	1705	25·045	1328	25·295	1034	25·545	805	25·795	627
24·800	1696	25·050	1321	25·300	1029	25·550	801	25·800	624
24·805	1688	25·055	1314	25·305	1024	25·555	797	25·805	621
24·810	1679	25·060	1308	25·310	1019	25·560	793	25·810	618
24·815	1671	25·065	1301	25·315	1013	25·565	789	25·815	615
24·820	1663	25·070	1295	25·320	1008	25·570	785	25·820	612
24·825	1654	25·075	1288	25·325	1003	25·575	781	25·825	609
24·830	1646	25·080	1282	25·330	998	25·580	777	25·830	606
24·835	1638	25·085	1276	25·335	993	25·585	774	25·835	603
24·840	1630	25·090	1269	25·340	988	25·590	770	25·840	600
24·845	1622	25·095	1263	25·345	984	25·595	766	25·845	597
24·850	1614	25·100	1257	25·350	979	25·600	762	25·850	594
24·855	1606	25·105	1250	25·355	974	25·605	758	25·855	591
24·860	1598	25·110	1244	25·360	969	25·610	755	25·860	588
24·865	1590	25·115	1238	25·365	964	25·615	751	25·865	585
24·870	1582	25·120	1232	25·370	959	25·620	747	25·870	582
24·875	1574	25·125	1226	25·375	955	25·625	743	25·875	579
24·880	1566	25·130	1219	25·380	950	25·630	740	25·880	576
24·885	1558	25·135	1213	25·385	945	25·635	736	25·885	573
24·890	1550	25·140	1207	25·390	940	25·640	732	25·890	570
24·895	1543	25·145	1201	25·395	936	25·645	729	25·895	567
24·900	1535	25·150	1195	25·400	931	25·650	725	25·900	565
24·905	1527	25·155	1189	25·405	926	25·655	721	25·905	562
24·910	1520	25·160	1183	25·410	922	25·660	718	25·910	559
24·915	1512	25·165	1177	25·415	917	25·665	714	25·915	556
24·920	1504	25·170	1171	25·420	912	25·670	711	25·920	553

[*Second Part. Fourteen decimal places.*]

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
25·925	551	26·175	429	26·425	334	26·675	260	26·925	202
25·930	548	26·180	427	26·430	332	26·680	259	26·930	201
25·935	545	26·185	425	26·435	331	26·685	257	26·935	200
25·940	543	26·190	422	26·440	329	26·690	256	26·940	199
25·945	540	26·195	420	26·445	328	26·695	255	26·945	198
25·950	537	26·200	418	26·450	326	26·700	254	26·950	197
25·955	534	26·205	416	26·455	324	26·705	252	26·955	196
25·960	532	26·210	414	26·460	323	26·710	251	26·960	195
25·965	529	26·215	412	26·465	321	26·715	250	26·965	194
25·970	526	26·220	410	26·470	319	26·720	249	26·970	193
25·975	523	26·225	408	26·475	318	26·725	247	26·975	193
25·980	521	26·230	406	26·480	316	26·730	246	26·980	192
25·985	518	26·235	404	26·485	315	26·735	245	26·985	191
25·990	516	26·240	402	26·490	313	26·740	244	26·990	190
25·995	513	26·245	400	26·495	311	26·745	242	26·995	189
26·000	511	26·250	398	26·500	310	26·750	241	27·000	188
26·005	508	26·255	396	26·505	308	26·755	240	27·005	187
26·010	506	26·260	394	26·510	307	26·760	239	27·010	186
26·015	503	26·265	392	26·515	305	26·765	238	27·015	185
26·020	501	26·270	390	26·520	304	26·770	236	27·020	184
26·025	498	26·275	388	26·525	302	26·775	235	27·025	183
26·030	496	26·280	386	26·530	301	26·780	234	27·030	182
26·035	493	26·285	384	26·535	299	26·785	233	27·035	181
26·040	491	26·290	382	26·540	298	26·790	232	27·040	180
26·045	488	26·295	380	26·545	296	26·795	231	27·045	179
26·050	486	26·300	378	26·550	295	26·800	230	27·050	178
26·055	484	26·305	377	26·555	293	26·805	228	27·055	178
26·060	481	26·310	375	26·560	292	26·810	227	27·060	177
26·065	479	26·315	373	26·565	291	26·815	226	27·065	176
26·070	476	26·320	371	26·570	289	26·820	225	27·070	175
26·075	474	26·325	369	26·575	287	26·825	224	27·075	174
26·080	472	26·330	367	26·580	286	26·830	223	27·080	173
26·085	469	26·335	366	26·585	285	26·835	222	27·085	172
26·090	467	26·340	364	26·590	283	26·840	221	27·090	172
26·095	465	26·345	362	26·595	282	26·845	219	27·095	171
26·100	462	26·350	360	26·600	280	26·850	218	27·100	170
26·105	460	26·355	358	26·605	279	26·855	217	27·105	169
26·110	458	26·360	356	26·610	278	26·860	216	27·110	168
26·115	455	26·365	355	26·615	276	26·865	215	27·115	168
26·120	453	26·370	353	26·620	275	26·870	214	27·120	167
26·125	451	26·375	351	26·625	273	26·875	213	27·125	166
26·130	449	26·380	349	26·630	272	26·880	212	27·130	165
26·135	446	26·385	348	26·635	271	26·885	211	27·135	164
26·140	444	26·390	346	26·640	269	26·890	210	27·140	163
26·145	442	26·395	344	26·645	268	26·895	209	27·145	162
26·150	440	26·400	342	26·650	267	26·900	208	27·150	162
26·155	437	26·405	341	26·655	265	26·905	207	27·155	161
26·160	435	26·410	339	26·660	264	26·910	206	27·160	160
26·165	433	26·415	337	26·665	263	26·915	204	27·165	159
26·170	431	26·420	336	26·670	261	26·920	203	27·170	158

[*Second Part. Fourteen decimal places.*]

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
27·175	158	27·275	143	27·375	129	27·475	117	27·575	106
27·180	157	27·280	142	27·380	128	27·480	116	27·580	105
27·185	156	27·285	141	27·385	128	27·485	116	27·585	105
27·190	155	27·290	140	27·390	127	27·490	115	27·590	104
27·195	155	27·295	140	27·395	127	27·495	114	27·595	104
27·200	154	27·300	139	27·400	126	27·500	114	27·600	103
27·205	153	27·305	138	27·405	125	27·505	113	27·605	103
27·210	152	27·310	138	27·410	125	27·510	113	27·610	102
27·215	152	27·315	137	27·415	124	27·515	112	27·615	102
27·220	151	27·320	136	27·420	124	27·520	112	27·620	101
27·225	150	27·325	136	27·425	123	27·525	111	27·625	101
27·230	149	27·330	135	27·430	122	27·530	111	27·630	100
27·235	148	27·335	134	27·435	122	27·535	110	27·635	99
27·240	148	27·340	134	27·440	121	27·540	110		
27·245	147	27·345	133	27·445	121	27·545	109		
27·250	146	27·350	132	27·450	120	27·550	108		
27·255	145	27·355	132	27·455	119	27·555	108		
27·260	145	27·360	131	27·460	118	27·560	107		
27·265	144	27·365	130	27·465	118	27·565	107		
27·270	143	27·370	130	27·470	117	27·570	106		

V. *Tables of the Exponential Function.* By J. W. L. Glaisher, M.A., F.R.S., Fellow of Trinity College, Cambridge.

[Read May 21, 1877.]

The present paper contains four tables, in each of which the functions tabulated are e^x, e^{-x}, $\log_{10} e^x$ and $\log_{10} e^{-x}$. The ranges of the four tables are as follows:

Table I. From $x = 0{\cdot}001$ to $x = 0{\cdot}100$ at intervals of $0{\cdot}001$.

Table II. From $x = 0{\cdot}01$ to $x = 2{\cdot}00$ at intervals of $0{\cdot}01$.

Table III. From $x = 0{\cdot}1$ to $x = 10{\cdot}0$ at intervals of $0{\cdot}1$.

Table IV. From $x = 1$ to $x = 500$ at intervals of unity.

In all the tables the first nine figures of e^x, and the first nine significant figures of e^{-x} are given. The logarithms are in all cases given to ten places of decimals.

Since $\log_{10}(e^x)$ and $\log_{10}(e^{-x})$ are equal respectively to $x \log_{10} e$ and $x \log_{10}(e^{-1})$ it is evident that the logarithmic results in the tables are merely multiples of $\log_{10} e$ and $\log_{10}(e^{-1})$. They were readily calculated in this manner and the values of e^x and e^{-x} were derived from them by means of ten-figure logarithms, the tenth figure being rejected. The last figure is therefore in general correctly given to the nearest unit, but it may be in error by a unit where the tenth figure is a 4, 5 or 6.

Mr Newman in the table which precedes this paper gives the values of e^{-x} from $x = 0{\cdot}001$ to $x = 15{\cdot}350$ to twelve places of decimals at intervals of $0{\cdot}001$, from $x = 15{\cdot}350$ to $x = 17{\cdot}300$ at intervals of $0{\cdot}002$, and thence to $x = 27{\cdot}635$ at intervals of $0{\cdot}005$ to fourteen places. The introduction contains (pp. 148, 149) a table of e^{-x} from $x = 0{\cdot}1$ to $x = 37{\cdot}0$ at intervals of $0{\cdot}1$ to eighteen places. The only other tables of exponential functions that I know of are the following:

(i) On p. 188 of the first volume of Schulze's *Sammlung logarithmischer trigonometrischer...Tafeln* (Berlin, 1778) there is a table giving the values of e^x, for $x = 1, 2, 3,...24$ to 28 or 29 figures, and for $x = 25$, 30 and 60 to 32 or 33 places.

(ii) A table of $\log_{10}(e^x)$ to seven decimal places and of e^x to seven figures from $x=0{\cdot}01$ to $x=10{\cdot}00$ at intervals of 0·01 was given in Vega's *Tabulæ logarithmico-trigonometricæ* (1797), and has been retained in the later editions of this work. Köhler's *Logarithmisch-trigonometrisches Handbuch* also contains the table of e^x.

(iii) In the eighth and ninth volumes* of *Crelle's Journal* Gudermann has given a table of $\log_{10}\sinh x$, $\log_{10}\cosh x$ and $\log_{10}\tanh x$ from $x=2$ to $x=5$ at intervals of 0·001 to nine decimal places, and from $x=5$ to $x=12$ at intervals of 0·01 to ten places. Gudermann's papers on the hyperbolic trigonometrical functions were afterwards collected together and published as a separate work under the title *Theorie der Potenzial- oder cyklisch-hyperbolischen Functionen* (Berlin, 1833), and this table occupies pp. 263—336.

The tables in the present paper are portions of some that I calculated as long ago as 1872. It was then my intention to calculate extensive tables of e^x and e^{-x} for publication in a separate form; but the scheme was not carried out and the tables were left in an incomplete state. In 1876 Mr Newman communicated the first part of his table to the Society; and I thought it would be desirable to supplement it by the tables here given. Owing to various causes Mr Newman's table has been a long time passing through the press, and of course this paper has been kept back so that the two might appear together.

The tables were verified as follows:

Tables I. and II. The values of e^x and e^{-x} were verified by differences. All the values of e^{-x} given in the two tables are included in Mr Newman's great table, and the values were compared. No error was found, but there were of course occasional differences of a unit in the last figure: the figure was in these cases changed so as to agree with Mr Newman's more extended value.

During the time that his large table was being printed Mr Newman sent me a table of e^x from $x=0$ to $x=1$ at intervals of 0·001 to twelve decimal places. The values of e^x in Table I. were compared with this table and, as in the case of e^{-x}, the last figure was changed so as to agree with it.

Table III. The values of e^{-x} were compared with Mr Newman's table and the last figure changed in cases where a discrepancy occurred as in Tables I. and II. The values of e^x were recalculated.

Table IV. The values of e^x and e^{-x} were recalculated.

The tables were also compared with Schulze's and Vega's tables (i) and (ii), described above, as far as the extent of the different tables permitted.

The columns giving e^x and e^{-x} are placed side by side, as the two functions are often required in combination as in the case of the hyperbolic sine, cosine and tangent.

* Vol. VIII. pp. 195—212, 301—316; Vol. IX. pp. 81—96, 193—208, 297—304.

Gudermann's table (iii) mentioned above begins at $x=2$, and it is for this reason that $x=2$ was taken as the limit of Table II.

With respect to the use of the tables, it may be remarked that they may be conveniently combined in interpolations: thus, for example,

$$e^{4\cdot237} = e^{4\cdot2} \times e^{0\cdot037} = 66\cdot6863310 \times 1\cdot03769302,$$

and

$$\log e^{4\cdot237} = \log (e^{4\cdot2}) + \log (e^{0\cdot037}) = 1\cdot8240368240 + 0\cdot0160688958.$$

For the sake of completeness I reproduce here Schulze's table referred to in (i).

x	e^x
1	2 . 718 281 828 459 045 235 360 287 471
2	7 . 389 056 098 930 650 227 230 427 460
3	20 . 085 536 923 187 667 740 928 529 652
4	54 . 598 150 033 144 239 078 110 261 19
5	148 . 413 159 102 576 603 421 115 580 01
6	403 . 428 793 492 735 122 608 387 180 5
7	1096 . 633 158 428 458 599 263 720 238 1
8	2980 . 957 987 041 728 274 743 592 099
9	8103 . 083 927 575 384 007 709 996 688
10	22026 . 465 794 806 716 516 957 900 641
11	59874 . 141 715 197 818 455 326 485 75
12	162754 . 791 419 003 920 808 005 204 77
13	442413 . 392 008 920 503 326 102 777 5
14	1202604 . 284 164 776 777 749 236 769 7
15	3269017 . 372 472 110 639 301 855 040
16	8886110 . 520 507 872 636 763 023 722
17	24154952 . 753 575 298 214 775 435 130
18	65659969 . 137 330 511 138 786 503 12
19	178482300 . 963 187 260 844 910 003 4
20	485165195 . 409 790 277 969 106 829 3
21	1318815734 . 483 214 697 209 998 880 2
22	3584912846 . 131 591 561 681 159 934
23	9744803446 . 248 902 600 034 632 654
24	26489122129 . 843 472 294 139 162 068

$e^{25} =$ 72004899337 . 385 872 524 161 351 466 126

$e^{30} =$ 10686474581524 . 462 146 990 468 650 741 2

$e^{60} =$ 114200738981568428366295718 . 314 472

This table was partially verified in the following manner. Since

$$1 + a + a^2 \dots + a^n = \frac{a^{n+1} - 1}{a - 1},$$

it follows that unity added to the sum of the first twelve powers of e is equal to $\frac{e^{13}-1}{e-1}$, and that unity added to the sum of the first twenty-four powers of e is equal to $\frac{e^{25}-1}{e-1}$. Retaining 23 places of decimals, we find by addition from Schulze's table

$$1+e+e^2\ldots+e^{12}=257473\,.\,706\ 979\ 533\ 059\ 990\ 318\ 032\ 45,$$

and by division, taking Schulze's value of e^{13},

$$\frac{e^{13}-1}{e-1}=257473\,.\,706\ 979\ 533\ 059\ 990\ 318\ 032\ 37,$$

which verifies the values of the first thirteen powers of e to 22 places of decimals.

Similarly by addition we find

$$1+e+e^2\ldots+e^{24}=41905174194\,.\,247\ 197\ 714\ 849\ 662\ 8,$$

and by division, taking Schulze's value of e^{25},

$$\frac{e^{25}-1}{e-1}=41905174194\,.\,247\ 197\ 714\ 849\ 663\ 0.$$

The values from e^{14} to e^{25} are thus verified to 15 places of decimals.

§ 2. In connexion with the exponential function I may here give the following values of $\frac{1}{2\,!}, \frac{1}{3\,!}, \frac{1}{4\,!}, \ldots \frac{1}{50\,!}$, which I worked out on account of their use in calculating the values of series having factorials in their denominators.

The figures enclosed in brackets denote the numbers of ciphers occurring between the decimal point and the first significant figure. From $\frac{1}{13!}$ to $\frac{1}{50!}$ the number of significant figures given is twenty-eight.

$$\frac{1}{2\,!}=0{\cdot}5, \qquad \frac{1}{6\,!}=0{\cdot}001\ 3\dot{8},$$

$$\frac{1}{3\,!}=0{\cdot}1\dot{6}, \qquad \frac{1}{7\,!}=0{\cdot}000\ 1\dot{9}8\ 412\ \dot{6},$$

$$\frac{1}{4\,!}=0{\cdot}041\dot{6}, \qquad \frac{1}{8\,!}=0{\cdot}000\ 024\ 8\dot{0}1\ 587\dot{3},$$

$$\frac{1}{5\,!}=0{\cdot}008\dot{3}, \qquad \frac{1}{9\,!}=0{\cdot}000\ 002\ 7\dot{5}5\ 731\ 922\ 398\ 589\ 065\ \dot{2},$$

$$\frac{1}{10\,!}=0{\cdot}000\ 000\ 27\dot{5}\ 573\ 192\ 239\ 858\ 906\ 5\dot{2},$$

$$\frac{1}{11\,!}=0{\cdot}000\ 000\ 02\dot{5}\ 052\ 108\ 385\ 441\ 718\ 7\dot{7},$$

$$\frac{1}{12\,!}=0{\cdot}000\ 000\ 002\ 0\dot{8}7\ 675\ 698\ 786\ 809\ 897\ 921\ 009\ 032\ 120\ 143\ 231\ 254\ 342\ 365\ 453\ 476\ 564\ \dot{5},$$

n	$\frac{1}{n!}$
13	0· (9) 160 590 438 368 216 145 993 923 771 7
14	0·(10) 114 707 455 977 297 247 138 516 979 8
15	0·(12) 764 716 373 181 981 647 590 113 198 6
16	0·(13) 477 947 733 238 738 529 743 820 749 1
17	0·(14) 281 145 725 434 552 076 319 894 558 3
18	0·(15) 156 192 069 685 862 264 622 163 643 5
19	0·(17) 822 063 524 662 432 971 695 598 123 7
20	0·(18) 411 031 762 331 216 485 847 799 061 9
21	0·(19) 195 729 410 633 912 612 308 475 743 7
22	0·(21) 889 679 139 245 057 328 674 889 744 3
23	0·(22) 386 817 017 063 068 403 771 691 193 2
24	0·(23) 161 173 757 109 611 834 904 871 330 5
25	0·(25) 644 695 028 438 447 339 619 485 321 9
26	0·(26) 247 959 626 322 479 746 007 494 354 6
27	0·(28) 918 368 986 379 554 614 842 571 683 7
28	0·(29) 327 988 923 706 983 791 015 204 172 7
29	0·(30) 113 099 628 864 477 169 315 587 645 8
30	0·(32) 376 998 762 881 590 564 385 292 152 6
31	0·(33) 121 612 504 155 351 794 962 997 468 6
32	0·(35) 380 039 075 485 474 359 259 367 089 3
33	0·(36) 115 163 356 207 719 502 805 868 814 9
34	0·(38) 338 715 753 552 116 184 723 143 573 3
35	0·(40) 967 759 295 863 189 099 208 981 638 1
36	0·(41) 268 822 026 628 663 638 669 161 566 1
37	0·(43) 726 546 017 915 307 131 538 274 503 1
38	0·(44) 191 196 320 504 028 192 510 072 237 7
39	0·(46) 490 246 975 651 354 339 769 415 994 0
40	0·(47) 122 561 743 912 838 584 942 353 998 5
41	0·(49) 298 931 082 714 240 451 078 912 191 5
42	0·(51) 711 740 673 129 143 931 140 267 122 5
43	0·(52) 165 521 086 774 219 518 869 829 563 4
44	0·(54) 376 184 288 123 226 179 249 612 644 0
45	0·(56) 835 965 084 718 280 398 332 472 542 3
46	0·(57) 181 731 540 156 147 912 680 972 291 8
47	0·(59) 386 662 851 396 059 388 682 919 769 8
48	0·(61) 805 547 607 075 123 726 422 749 520 4
49	0·(62) 164 397 470 831 657 903 351 581 534 8
50	0·(64) 328 794 941 663 315 806 703 163 069 6

These values were obtained by actual division, each result being deduced from the preceding one; that is to say, the value of $\frac{1}{n!}$ was derived from that of $\frac{1}{(n-1)!}$ by dividing it by n.

The results should be in all cases correct to the last figure, as several figures were rejected.

By addition we find

$$1+\frac{1}{2!}+\frac{1}{4!}\dots+\frac{1}{32!}=1{\cdot}543\ 080\ 634\ 815\ 243\ 778\ 477\ 905\ 620\ 757\ 061\ 682\ 6,$$

$$1+\frac{1}{3!}+\frac{1}{5!}\dots+\frac{1}{33!}=1{\cdot}175\ 201\ 193\ 643\ 801\ 456\ 882\ 381\ 850\ 595\ 600\ 815\ 2,$$

giving

$$e\ =2{\cdot}718\ 281\ 828\ 459\ 045\ 235\ 360\ 287\ 471\ 352\ 662\ 497\ 8,$$

$$e^{-1}=0{\cdot}367\ 879\ 441\ 171\ 442\ 321\ 595\ 523\ 770\ 161\ 460\ 867\ 4,$$

which are correct to the last figure.

We do not thus obtain, however, a good verification of the values of the reciprocals of the factorials even as far as $\frac{1}{33!}$; for, of those beyond $\frac{1}{12!}$ (in which only 28 significant figures are given in the table on the preceding page) the first only, $\frac{1}{13!}$, is verified to its full extent, the next, $\frac{1}{14!}$, is verified to only 27 figures and so on, the first figure alone being verified in the case of $\frac{1}{33!}$.

It seems worth while to give in detail the calculation of e^{10} and e^{-10} by means of the preceding values of the reciprocals of the factorials. The values of the different terms of the series are shown on the opposite page, and it will be seen that ten figures (3287949417) of $\frac{1}{50!}$ are thus verified. In order to complete the calculation it was necessary to find the values of $\frac{1}{51!}$ and the subsequent terms to a few places of decimals. The last term included is that involving $\frac{1}{62!}$.

The first column contains the values of the terms involving factorials of even numbers and the second column those involving factorials of uneven numbers in the series

$$1+10+\frac{10^2}{2!}+\frac{10^3}{3!}+\frac{10^4}{4!}+\frac{10^5}{5!}+\frac{10^6}{6!}+\&c.$$

Calculation of e^{10} and e^{-10} from the series.

(Even terms.)	(Uneven terms.)
1	
50	10
416·666 666 666 666 666 666 666 667	166·666 666 666 666 666 666 666 667
1388·888 888 888 888 888 888 888 889	833·333 333 333 333 333 333 333 333
2480·158 730 158 730 158 730 158 730	1984·126 984 126 984 126 984 126 984
2755·731 922 398 589 065 255 731 922	2755·731 922 398 589 065 255 731 922
2087·675 698 786 809 897 921 009 032	2505·210 838 544 171 877 505 210 839
1147·074 559 772 972 471 385 169 798	1605·904 383 682 161 459 939 237 717
477·947 733 238 738 529 743 820 749	764·716 373 181 981 647 590 113 199
156·192 069 685 862 264 622 163 643	281·145 725 434 552 076 319 894 558
41·103 176 233 121 648 584 779 906	82·206 352 466 243 297 169 559 812
8·896 791 392 450 573 286 748 897	19·572 941 063 391 261 230 847 574
1·611 737 571 096 118 349 048 713	3·868 170 170 630 684 037 716 912
247 959 626 322 479 746 007 494	644 695 028 438 447 339 619 485
32 798 892 370 698 379 101 520	91 836 898 637 955 461 484 257
3 769 987 628 815 905 643 853	11 309 962 886 447 716 931 559
380 039 075 485 474 359 259	1 216 125 041 553 517 949 630
33 871 575 355 211 618 472	115 163 356 207 719 502 806
2 688 220 266 286 636 387	9 677 592 958 631 890 992
191 196 320 504 028 193	726 546 017 915 307 132
12 256 174 391 283 858	49 024 697 565 135 434
711 740 673 129 144	2 989 310 827 142 405
37 618 428 812 323	165 521 086 774 220
1 817 315 401 561	8 359 650 847 183
80 554 760 708	386 662 851 396
3 287 949 417	16 439 747 083
123 979 993	644 695 964
4 331 935	23 392 452
140 647	787 625
4 254	24 675
120	721
3	20
11013·232 920 103 323 139 721 376 087	11013·232 874 703 39$\dot{3}$ 377 236 524 556

By adding and subtracting the sums of the two columns we find

$$e^{10} = 22026{\cdot}465\ 794\ 806\ 716\ 516\ 957\ 900\ 643,$$

$$e^{-10} = 0{\cdot}000\ 045\ 399\ 929\ 762\ 484\ 851\ 531.$$

The value of e^{10} to 24 decimal places is given by Schulze in the table reprinted on p. 245. It differs from that given by this calculation only in the last figure, the last three figures in Schulze's table being 641.

As an additional verification I have made the following calculation of e^{-10}:

Let
$$e^{-10} = \frac{4539992976 + h}{10^{14}},$$

then
$$-10 = \log_e (4539992976 + h) - 14 \log_e 10,$$

and therefore
$$\log_e (4539992976 + h) = 14 \log_e 10 - 10.$$

The object is to determine h from this equation, the value of $\log_e 4539992976$ being known, for

$$4539992976 = 1296 \times 1763 \times 1987,$$

and, taking the logarithms of 1296, 1763 and 1987 from Wolfram's table*, we find

$$\log_e 4539992976 = 22{\cdot}236\ 191\ 301\ 861\ 907\ 078\ 9.$$

Also
$$14 \log_e 10 - 10 = 22{\cdot}236\ 191\ 301\ 916\ 639\ 576\ 3.$$

Putting therefore $x = 4539992976$, we have

$$\log_e (x + h) - \log_e x = 0{\cdot}000\ 000\ 000\ 054\ 732\ 497\ 355\ 4.$$

Now
$$\log_e (x + h) - \log_e x = \frac{h}{x} - \tfrac{1}{2}\frac{h^2}{x^2} + \tfrac{1}{3}\frac{h^3}{x^3} - \&c.,$$

and therefore
$$h = x\{\log_e (x + h) - \log_e x\} + \tfrac{1}{2}\frac{h^2}{x} - \tfrac{1}{3}\frac{h^3}{x^2} + \&c.$$

By multiplication we find

$$x\{\log_e (x + h) - \log_e x\} = 0{\cdot}248\ 485\ 153\ 552\ 351\ 449\ 550.$$

Taking $\log_e (x + h) - \log_e x$ as an approximate value of $\frac{h}{x}$, we have

$$\frac{h}{x} = 0{\cdot}000\ 000\ 000\ 054\ 732\ 497\ 355,$$

and therefore
$$\tfrac{1}{2}\frac{h^2}{x} = 0{\cdot}000\ 000\ 000\ 006\ 800\ 106\ 505,$$

whence
$$h = 0{\cdot}248\ 485\ 153\ 559\ 151\ 556\ 055;$$

and, except for last-figure errors, this value should be correct as far as it extends. We thus find

$$e^{-10} = 0{\cdot}000\ 045\ 399\ 929\ 762\ 484\ 851\ 535\ 591\ 515\ 560\ 6,$$

which differs from the value found above for e^{-10} by 4 in the twenty-fourth place.

* This table gives the hyperbolic logarithms of all numbers up to 2,200 and of primes, and also of a great many composite numbers, up to 10,009 to 48 places of decimals. It was first published in Schulze's *Sammlung*, referred to on the first page of this paper, and was reprinted with the addition of six logarithms that were omitted through Wolfram's death, in Vega's *Thesaurus logarithmorum completus* (1794).

To verify the accuracy of this value of e^{-10} I found by division the reciprocal of

$$22026{\cdot}465\ 794\ 806\ 716\ 516\ 957\ 900\ 643,$$

the result being

$$0{\cdot}000\ 045\ 399\ 929\ 762\ 484\ 851\ 535\ 591\ 515\ 565,$$

which agrees to 32 places with the value of e^{-10} just found.

I have thought it worth while to give this calculation of e^{-10} at some length, as the method affords perhaps the most convenient means of calculating e^z for an isolated value of z when a considerable number of figures are required. The principle of the method is as follows. The first nine or ten figures of the value being obtained from the Tables, or calculated by logarithms independently, we seek for a number near to it which can be resolved into factors, none of which exceed 10,009, the limit of Wolfram's table. Denoting this number by x, we then obtain $\log_e x$, and it only remains to calculate h from the formula

$$h = x\{\log_e(x+h) - \log_e x\} + \tfrac{1}{2}\frac{h^2}{x} - \tfrac{1}{3}\frac{h^3}{x^2} + \&c.$$

by repeated approximation.

In calculating a table of e^z for successive integral values of z, such as Schulze's, it might be well to form the table by actual multiplication, and to verify the final value by an independent calculation in this manner.

In order to verify absolutely the accuracy of the values of the reciprocals of factorials given above, to the full extent of the 28 figures, I formed the value of $\log_e(50\,!)$ by adding up the logarithms of 2, 3,...50, and I also calculated the logarithm of the twenty-eight figure number given as the value of $\frac{1}{50\,!}$, and thence deduced the value of $\log_e(50\,!)$.

Adding together the values of the logarithms of the first 50 numbers given in Wolfram's table and retaining 28 decimals, we find

$$\log_e(50\,!) = 148{\cdot}477\ 766\ 951\ 773\ 032\ 067\ 537\ 193\ 850\ 9.$$

Now taking the value of $\frac{1}{50\,!}$ given on p. 247, we have

$$\frac{1}{50\,!} = \frac{328794941{\cdot}663\ 315\ 806\ 703\ 163\ 069\ 6}{10^{73}},$$

whence we ought to find

$$\log_e(50\,!) = 73\log_e 10 - \log_e 328794941{\cdot}663\ 315\ 806\ 703\ 163\ 069\ 6.$$

To calculate the logarithm of 328794941·663 315 806 703 163 069 6 we notice that $328794943 = 17 \times 19 \times 569 \times 1789$, whence, taking the logarithms from Wolfram's table,

$$\log_e 328794943 = 19{\cdot}610\ 944\ 840\ 857\ 706\ 621\ 131\ 134\ 384\ 5,$$

and putting $328794943 = x$, and denoting the number whose logarithm is required by $x - h$ we have

$$\log_e (x-h) = \log_e x - \frac{h}{x} - \tfrac{1}{2}\frac{h^2}{x^2} - \tfrac{1}{3}\frac{h^3}{x^3} - \&c.$$

where $$h = 1{\cdot}336\ 684\ 193\ 296\ 836\ 930\ 4.$$

Thus $$\log_e (50\,!) = 73 \log_e 10 + \frac{h}{x} + \tfrac{1}{2}\frac{h^2}{x^2} + \tfrac{1}{3}\frac{h^2}{x^3} + \&c. - \log_e x,$$

and the final steps of the calculation are as follows:

$73 \log_e 10 =$	168·088 711 788 565 334 933 313 376 191 9
	4 065 403 747 091 198 207 4
	8 263 753 813 4
	22 4
	168·088 711 792 630 738 688 668 328 235 1
$\log_e x =$	19·610 944 840 857 706 621 131 134 384 5
	148·477 766 951 773 032 067 537 193 850 6

which differs by 3 in the last place from the value of $\log_e (50\,!)$ found by addition.

The value of $\frac{1}{50\,!}$ is therefore verified, and as each value was derived from the preceding one by division, this affords a verification also of the values of the other reciprocals of factorials.

A table of $\log_{10} (x\,!)$ from $x = 1$ to $x = 1200$ to 18 places of decimals was given by C. F. Degen in his "Tabularum Enneas" (Copenhagen, 1824). This table was reprinted by De Morgan, in the Article "Probabilities" in the *Encyclopædia Metropolitana*, the number of decimal places, however, being reduced to six.

The value of e was calculated by Mr Shanks to 205 places, and published by him to this extent in Vol. VI., p. 397 (1854) of the *Proceedings of the Royal Society*. In his "Rectification of the Circle" (1853) Mr Shanks had given the value to 137 places, and this result I verified in 1871*. Mr Shanks calculated his value by means of the series

$$e = 1 + \frac{1}{1\,!} + \frac{1}{2\,!} + \frac{1}{3\,!} + \frac{1}{4\,!} + \&c.,$$

and I used the continued-fraction formula

$$\frac{e-1}{2} = \frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{10+\ldots}\,\frac{1}{4n+2+\ldots}.$$

The value thus verified is

$$e = 2{\cdot}718\ 281\ 828\ 459\ 045\ 235\ 360\ 287\ 471\ 352\ 662\ 497\ 757\ 247\ 093\ 699$$
$$959\ 574\ 966\ 967\ 627\ 724\ 076\ 630\ 353\ 547\ 594\ 571\ 382\ 178\ 525\ 166$$
$$427\ 427\ 466\ 391\ 932\ 003\ 059\ 921\ 817\ 413\ 596\ 629\ 043\ 57.$$

* *Report of the British Association for* 1871 (pp. 16—18), (Sectional Proceedings).

The value of the reciprocal of e, given by Schulze (*Sammlung*, 1778, p. 188), is

$$e^{-1} = 0{\cdot}367\ 879\ 441\ 171\ 442\ 321\ 595\ 523\ 770\ 161\ 460\ 867\ 445\ 811\ 12.$$

I may mention that I have calculated the value of e^{16} from the series, retaining nine decimal places. The sums of the terms involving even and uneven powers were found to be respectively 4443055·260 253 99 and 4443055·260 253 88, giving

$$e^{16} = 8886110{\cdot}520\ 507\ 87, \quad e^{-16} = 0{\cdot}000\ 000\ 11.$$

This value of e^{-16} is correct. The value of e^{16} given in Schulze's table (*ante*, p. 245) is 8886110·520 507 872..., which is thus directly verified to fifteen figures.

The modulus M of the common or Briggian logarithms is equal to $\log_{10} e$, and its value has been given by Professor J. C. Adams to 282 places of decimals in Vol. XXVII., p. 93 (1878) of the *Proceedings of the Royal Society*. This value is reprinted in the article "Logarithms" in the *Encyclopædia Britannica* (1882).

It may be here remarked that Gauss's posthumous memoir "De curva lemniscata" (*Werke* Vol. III., pp. 413—432) contains the values of $e^{-\pi}$, $e^{-\frac{1}{4}\pi}$, and $e^{-\frac{9}{4}\pi}$ to 50 or more places of decimals, and the value of $e^{\frac{1}{2}\pi}$ to 34 places. The values of $e^{-9\pi}$, $e^{-\frac{25}{4}\pi}$ and $e^{-\frac{49}{4}\pi}$ are also given to sixteen, twenty-four, and twelve significant figures respectively.

The values of $\sqrt[n]{e}$ for integral values of n may be readily calculated from the series by means of the values of the reciprocals given above, but they may also be very conveniently obtained from the formula expressing $\sqrt[n]{e}$ as a continued fraction, viz.

$$\frac{e^{\frac{1}{n}}-1}{2} = \frac{1}{2n-1+}\,\frac{1}{6n+}\,\frac{1}{10n+}\,\frac{1}{14n+\&c.}\,.$$

TABLE I.

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from $x=0·001$ to $x=0·100$ at intervals of 0·001.

x	$\log_{10}(e^x)$	e^x	e^{-x}	$\log_{10}(e^{-x})$
0·001	0·00043 42945	1·00 100 050	0·999 000 500	$\bar{1}$·99956 57055
0·002	0·00086 85890	1·00 200 200	0·998 001 999	$\bar{1}$·99913 14110
0·003	0·00130 28834	1·00 300 450	0·997 004 496	$\bar{1}$·99869 71166
0·004	0·00173 71779	1·00 400 801	0·996 007 989	$\bar{1}$·99826 28221
0·005	0·00217 14724	1·00 501 252	0·995 012 479	$\bar{1}$·99782 85276
0·006	0·00260 57669	1·00 601 804	0·994 017 964	$\bar{1}$·99739 42331
0·007	0·00304 00614	1·00 702 456	0·993 024 443	$\bar{1}$·99695 99386
0·008	0·00347 43559	1·00 803 209	0·992 031 915	$\bar{1}$·99652 56441
0·009	0·00390 86503	1·00 904 062	0·991 040 379	$\bar{1}$·99609 13497
0·010	0·00434 29448	1·01 005 017	0·990 049 834	$\bar{1}$·99565 70552
0·011	0·00477 72393	1·01 106 072	0·989 060 279	$\bar{1}$·99522 27607
0·012	0·00521 15338	1·01 207 229	0·988 071 713	$\bar{1}$·99478 84662
0·013	0·00564 58283	1·01 308 487	0·987 084 135	$\bar{1}$·99435 41717
0·014	0·00608 01227	1·01 409 846	0·986 097 544	$\bar{1}$·99391 98773
0·015	0·00651 44172	1·01 511 306	0·985 111 940	$\bar{1}$·99348 55828
0·016	0·00694 87117	1·01 612 869	0·984 127 320	$\bar{1}$·99305 12883
0·017	0·00738 30062	1·01 714 532	0·983 143 685	$\bar{1}$·99261 69938
0·018	0·00781 73007	1·01 816 298	0·982 161 032	$\bar{1}$·99218 26993
0·019	0·00825 15952	1·01 918 165	0·981 179 362	$\bar{1}$·99174 84048
0·020	0·00868 58896	1·02 020 134	0·980 198 673	$\bar{1}$·99131 41104
0·021	0·00912 01841	1·02 122 205	0·979 218 965	$\bar{1}$·99087 98159
0·022	0·00955 44786	1·02 224 378	0·978 240 235	$\bar{1}$·99044 55214
0·023	0·00998 87731	1·02 326 654	0·977 262 484	$\bar{1}$·99001 12269
0·024	0·01042 30676	1·02 429 032	0·976 285 710	$\bar{1}$·98957 69324
0·025	0·01085 73620	1·02 531 512	0·975 309 912	$\bar{1}$·98914 26380
0·026	0·01129 16565	1·02 634 095	0·974 335 090	$\bar{1}$·98870 83435
0·027	0·01172 59510	1·02 736 780	0·973 361 242	$\bar{1}$·98827 40490
0·028	0·01216 02455	1·02 839 568	0·972 388 367	$\bar{1}$·98783 97545
0·029	0·01259 45400	1·02 942 459	0·971 416 464	$\bar{1}$·98740 54600
0·030	0·01302 88345	1·03 045 453	0·970 445 534	$\bar{1}$·98697 11655
0·031	0·01346 31289	1·03 148 550	0·969 475 573	$\bar{1}$·98653 68711
0·032	0·01389 74234	1·03 251 751	0·968 506 582	$\bar{1}$·98610 25766
0·033	0·01433 17179	1·03 355 054	0·967 538 560	$\bar{1}$·98566 82821
0·034	0·01476 60124	1·03 458 461	0·966 571 505	$\bar{1}$·98523 39876
0·035	0·01520 03069	1·03 561 971	0·965 605 416	$\bar{1}$·98479 96931
0·036	0·01563 46013	1·03 665 585	0·964 640 293	$\bar{1}$·98436 53987
0·037	0·01606 88958	1·03 769 302	0·963 676 135	$\bar{1}$·98393 11042
0·038	0·01650 31903	1·03 873 123	0·962 712 941	$\bar{1}$·98349 68097
0·039	0·01693 74848	1·03 977 048	0·961 750 709	$\bar{1}$·98306 25152
0·040	0·01737 17793	1·04 081 077	0·960 789 439	$\bar{1}$·98262 82207
0·041	0·01780 60738	1·04 185 211	0·959 829 130	$\bar{1}$·98219 39262
0·042	0·01824 03682	1·04 289 448	0·958 869 781	$\bar{1}$·98175 96318
0·043	0·01867 46627	1·04 393 789	0·957 911 390	$\bar{1}$·98132 53373
0·044	0·01910 89572	1·04 498 235	0·956 953 957	$\bar{1}$·98089 10428
0·045	0·01954 32517	1·04 602 786	0·955 997 482	$\bar{1}$·98045 67483
0·046	0·01997 75462	1·04 707 441	0·955 041 962	$\bar{1}$·98002 24538
0·047	0·02041 18406	1·04 812 201	0·954 087 398	$\bar{1}$·97958 81594
0·048	0·02084 61351	1·04 917 066	0·953 133 787	$\bar{1}$·97915 38649
0·049	0·02128 04296	1·05 022 035	0·952 181 130	$\bar{1}$·97871 95704
0·050	0·02171 47241	1·05 127 110	0·951 229 425	$\bar{1}$·97828 52759

TABLE I. (*continued*).

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from $x=0·001$ to $x=0·100$ at intervals of 0·001.

x	$\log_{10}(e^x)$	e^x	e^{-x}	$\log_{10}(e^{-x})$
0·051	0·02214 90186	1·05 232 289	0·950 278 671	$\bar{1}$·97785 09814
0·052	0·02258 33131	1·05 337 574	0·949 328 867	$\bar{1}$·97741 66869
0·053	0·02301 76075	1·05 442 964	0·948 380 012	$\bar{1}$·97698 23925
0·054	0·02345 19020	1·05 548 460	0·947 432 107	$\bar{1}$·97654 80980
0·055	0·02388 61965	1·05 654 061	0·946 485 148	$\bar{1}$·97611 38035
0·056	0·02432 04910	1·05 759 768	0·945 539 136	$\bar{1}$·97567 95090
0·057	0·02475 47855	1·05 865 581	0·944 594 069	$\bar{1}$·97524 52145
0·058	0·02518 90800	1·05 971 500	0·943 649 947	$\bar{1}$·97481 09200
0·059	0·02562 33744	1·06 077 524	0·942 706 769	$\bar{1}$·97437 66256
0·060	0·02605 76689	1·06 183 655	0·941 764 534	$\bar{1}$·97394 23311
0·061	0·02649 19634	1·06 289 891	0·940 823 240	$\bar{1}$·97350 80366
0·062	0·02692 62579	1·06 396 234	0·939 882 887	$\bar{1}$·97307 37421
0·063	0·02736 05524	1·06 502 684	0·938 943 474	$\bar{1}$·97263 94476
0·064	0·02779 48468	1·06 609 240	0·938 005 000	$\bar{1}$·97220 51532
0·065	0·02822 91413	1·06 715 902	0·937 067 463	$\bar{1}$·97177 08587
0·066	0·02866 34358	1·06 822 672	0·936 130 864	$\bar{1}$·97133 65642
0·067	0·02909 77303	1·06 929 548	0·935 195 201	$\bar{1}$·97090 22697
0·068	0·02953 20248	1·07 036 531	0·934 260 474	$\bar{1}$·97046 79752
0·069	0·02996 63193	1·07 143 621	0·933 326 680	$\bar{1}$·97003 36807
0·070	0·03040 06137	1·07 250 818	0·932 393 820	$\bar{1}$·96959 93863
0·071	0·03083 49082	1·07 358 123	0·931 461 892	$\bar{1}$·96916 50918
0·072	0·03126 92027	1·07 465 534	0·930 530 896	$\bar{1}$·96873 07973
0·073	0·03170 34972	1·07 573 054	0·929 600 830	$\bar{1}$·96829 65028
0·074	0·03213 77917	1·07 680 681	0·928 671 694	$\bar{1}$·96786 22083
0·075	0·03257 20861	1·07 788 415	0·927 743 486	$\bar{1}$·96742 79139
0·076	0·03300 63806	1·07 896 257	0·926 816 207	$\bar{1}$·96699 36194
0·077	0·03344 06751	1·08 004 208	0·925 889 854	$\bar{1}$·96655 93249
0·078	0·03387 49696	1·08 112 266	0·924 964 427	$\bar{1}$·96612 50304
0·079	0·03430 92641	1·08 220 432	0·924 039 924	$\bar{1}$·96569 07359
0·080	0·03474 35586	1·08 328 707	0·923 116 346	$\bar{1}$·96525 64414
0·081	0·03517 78530	1·08 437 090	0·922 193 691	$\bar{1}$·96482 21470
0·082	0·03561 21475	1·08 545 581	0·921 271 959	$\bar{1}$·96438 78525
0·083	0·03604 64420	1·08 654 181	0·920 351 147	$\bar{1}$·96395 35580
0·084	0·03648 07365	1·08 762 889	0·919 431 256	$\bar{1}$·96351 92635
0·085	0·03691 50310	1·08 871 707	0·918 512 284	$\bar{1}$·96308 49690
0·086	0·03734 93254	1·08 980 633	0·917 594 231	$\bar{1}$·96265 06746
0·087	0·03778 36199	1·09 089 668	0·916 677 096	$\bar{1}$·96221 63801
0·088	0·03821 79144	1·09 198 812	0·915 760 877	$\bar{1}$·96178 20856
0·089	0·03865 22089	1·09 308 066	0·914 845 574	$\bar{1}$·96134 77911
0·090	0·03908 65034	1·09 417 428	0·913 931 185	$\bar{1}$·96091 34966
0·091	0·03952 07979	1·09 526 901	0·913 017 711	$\bar{1}$·96047 92021
0·092	0·03995 50923	1·09 636 482	0·912 105 150	$\bar{1}$·96004 49077
0·093	0·04038 93868	1·09 746 174	0·911 193 500	$\bar{1}$·95961 06131
0·094	0·04082 36813	1·09 855 975	0·910 282 762	$\bar{1}$·95917 63187
0·095	0·04125 79758	1·09 965 886	0·909 372 934	$\bar{1}$·95874 20242
0·096	0·04169 22703	1·10 075 906	0·908 464 016	$\bar{1}$·95830 77297
0·097	0·04212 65647	1·10 186 037	0·907 556 006	$\bar{1}$·95787 34353
0·098	0·04256 08592	1·10 296 279	0·906 648 904	$\bar{1}$·95743 91408
0·099	0·04299 51537	1·10 406 630	0·905 742 708	$\bar{1}$·95700 48463
0·100	0·04342 94482	1·10 517 092	0·904 837 418	$\bar{1}$·95657 05518

TABLE II.

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from $x=0·01$ to $x=2·00$ at intervals of 0·01.

x	$\log_{10}(e^x)$	e^x	e^{-x}	$\log_{10}(e^{-x})$
0·01	0·00434 29448	1·01 005 017	0·990 049 834	$\bar{1}$·99565 70552
0·02	0·00868 58896	1·02 020 134	0·980 198 673	$\bar{1}$·99131 41104
0·03	0·01302 88345	1·03 045 454	0·970 445 534	$\bar{1}$·98697 11655
0·04	0·01737 17793	1·04 081 077	0·960 789 439	$\bar{1}$·98262 82207
0·05	0·02171 47241	1·05 127 110	0·951 229 425	$\bar{1}$·97828 52759
0·06	0·02605 76689	1·06 183 655	0·941 764 534	$\bar{1}$·97394 23311
0·07	0·03040 06137	1·07 250 818	0·932 393 820	$\bar{1}$·96959 93863
0·08	0·03474 35586	1·08 328 707	0·923 116 346	$\bar{1}$·96525 64414
0·09	0·03908 65034	1·09 417 428	0·913 931 185	$\bar{1}$·96091 34966
0·10	0·04342 94482	1·10 517 092	0·904 837 418	$\bar{1}$·95657 05518
0·11	0·04777 23930	1·11 627 807	0·895 834 135	$\bar{1}$·95222 76070
0·12	0·05211 53378	1·12 749 685	0·886 920 437	$\bar{1}$·94788 46622
0·13	0·05645 82826	1·13 882 838	0·878 095 431	$\bar{1}$·94354 17174
0·14	0·06080 12275	1·15 027 380	0·869 358 235	$\bar{1}$·93919 87725
0·15	0·06514 41723	1·16 183 424	0·860 707 976	$\bar{1}$·93485 58277
0·16	0·06948 71171	1·17 351 087	0·852 143 789	$\bar{1}$·93051 28829
0·17	0·07383 00619	1·18 530 485	0·843 664 817	$\bar{1}$·92616 99381
0·18	0·07817 30067	1·19 721 736	0·835 270 211	$\bar{1}$·92182 69933
0·19	0·08251 59516	1·20 924 960	0·826 959 134	$\bar{1}$·91748 40484
0 20	0·08685 88964	1·22 140 276	0·818 730 753	$\bar{1}$·91314 11036
0·21	0·09120 18412	1·23 367 806	0·810 584 246	$\bar{1}$·90879 81588
0·22	0·09554 47860	1·24 607 673	0·802 518 798	$\bar{1}$·90445 52140
0·23	0·09988 77308	1·25 860 001	0·794 533 603	$\bar{1}$·90011 22692
0·24	0·10423 06757	1·27 124 915	0·786 627 861	$\bar{1}$·89576 93243
0·25	0·10857 36205	1·28 402 542	0·778 800 783	$\bar{1}$·89142 63795
0·26	0·11291 65653	1·29 693 009	0·771 051 586	$\bar{1}$·88708 34347
0·27	0·11725 95101	1·30 996 445	0·763 379 494	$\bar{1}$·88274 04899
0·28	0·12160 24549	1·32 312 981	0·755 783 741	$\bar{1}$·87839 75451
0·29	0·12594 53998	1·33 642 749	0·748 263 568	$\bar{1}$·87405 46002
0·30	0·13028 83446	1·34 985 881	0·740 818 221	$\bar{1}$·86971 16554
0·31	0·13463 12894	1·36 342 511	0·733 446 956	$\bar{1}$·86536 87106
0·32	0·13897 42342	1·37 712 777	0·726 149 037	$\bar{1}$·86102 57658
0·33	0·14331 71790	1·39 096 813	0·718 923 733	$\bar{1}$·85668 28210
0·34	0·14766 01238	1·40 494 759	0·711 770 323	$\bar{1}$·85233 98762
0·35	0·15200 30687	1·41 906 755	0·704 688 090	$\bar{1}$·84799 69313
0·36	0·15634 60135	1·43 332 942	0·697 676 326	$\bar{1}$·84365 39865
0·37	0·16068 89583	1·44 773 462	0·690 734 331	$\bar{1}$·83931 10417
0·38	0·16503 19031	1·46 228 459	0·683 861 409	$\bar{1}$·83496 80969
0·39	0·16937 48479	1·47 698 079	0·677 056 874	$\bar{1}$·83062 51521
0·40	0·17371 77928	1·49 182 470	0·670 320 046	$\bar{1}$·82628 22072
0·41	0·17806 07376	1·50 681 779	0·663 650 250	$\bar{1}$·82193 92624
0·42	0·18240 36824	1·52 196 156	0·657 046 820	$\bar{1}$·81759 63176
0·43	0·18674 66272	1·53 725 752	0·650 509 095	$\bar{1}$·81325 33728
0·44	0·19108 95720	1·55 270 722	0·644 036 421	$\bar{1}$·80891 04280
0·45	0·19543 25169	1·56 831 219	0·637 628 152	$\bar{1}$·80456 74831
0·46	0·19977 54617	1·58 407 399	0·631 283 646	$\bar{1}$·80022 45383
0·47	0·20411 84065	1·59 999 419	0·625 002 268	$\bar{1}$·79588 15935
0·48	0·20846 13513	1·61 607 440	0·618 783 392	$\bar{1}$·79153 86487
0·49	0·21280 42961	1·63 231 622	0·612 626 394	$\bar{1}$·78719 57039
0·50	0·21714 72410	1·64 872 127	0·606 530 660	$\bar{1}$·78285 27590

TABLE II. (*continued*).

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from $x=0·01$ to $x=2·00$ at intervals of 0·01.

x	$\log_{10}(e^x)$	e^x	e^{-x}	$\log_{10}(e^{-x})$
0·51	0·22149 01858	1·66 529 120	0·600 495 579	$\bar{1}$·77850 98142
0·52	0·22583 31306	1·68 202 765	0·594 520 548	$\bar{1}$·77416 68694
0·53	0·23017 60754	1·69 893 231	0·588 604 970	$\bar{1}$·76982 39246
0·54	0·23451 90202	1·71 600 686	0·582 748 252	$\bar{1}$·76548 09798
0·55	0·23886 19650	1·73 325 302	0·576 949 810	$\bar{1}$·76113 80350
0·56	0·24320 49099	1·75 067 250	0·571 209 064	$\bar{1}$·75679 50901
0·57	0·24754 78547	1·76 826 705	0·565 525 439	$\bar{1}$·75245 21453
0·58	0·25189 07995	1·78 603 843	0·559 898 367	$\bar{1}$·74810 92005
0·59	0·25623 37443	1·80 398 842	0·554 327 285	$\bar{1}$·74376 62557
0·60	0·26057 66891	1·82 211 880	0·548 811 636	$\bar{1}$·73942 33109
0·61	0·26491 96340	1·84 043 140	0·543 350 869	$\bar{1}$·73508 03660
0·62	0·26926 25788	1·85 892 804	0·537 944 438	$\bar{1}$·73073 74212
0·63	0·27360 55236	1·87 761 058	0·532 591 801	$\bar{1}$·72639 44764
0·64	0·27794 84684	1·89 648 088	0·527 292 424	$\bar{1}$·72205 15316
0·65	0·28229 14132	1·91 554 083	0·522 045 777	$\bar{1}$·71770 85868
0·66	0·28663 43581	1·93 479 233	0·516 851 334	$\bar{1}$·71336 56419
0·67	0·29097 73029	1·95 423 732	0·511 708 578	$\bar{1}$·70902 26971
0·68	0·29532 02477	1·97 387 773	0·506 616 992	$\bar{1}$·70467 97523
0·69	0·29966 31925	1·99 371 553	0·501 576 069	$\bar{1}$·70033 68075
0·70	0·30400 61373	2·01 375 271	0·496 585 304	$\bar{1}$·69599 38627
0·71	0·30834 90822	2·03 399 126	0·491 644 197	$\bar{1}$·69165 09178
0·72	0·31269 20270	2·05 443 321	0·486 752 256	$\bar{1}$·68730 79730
0·73	0·31703 49718	2·07 508 061	0·481 908 990	$\bar{1}$·68296 50282
0·74	0·32137 79166	2·09 593 551	0·477 113 916	$\bar{1}$·67862 20834
0·75	0·32572 08614	2·11 700 002	0·472 366 553	$\bar{1}$·67427 91386
0·76	0·33006 38062	2·13 827 622	0·467 666 427	$\bar{1}$·66993 61938
0·77	0·33440 67511	2·15 976 625	0·463 013 068	$\bar{1}$·66559 32489
0·78	0·33874 96959	2·18 147 227	0·458 406 011	$\bar{1}$·66125 03041
0·79	0·34309 26407	2·20 339 643	0·453 844 795	$\bar{1}$·65690 73593
0·80	0·34743 55855	2·22 554 093	0·449 328 964	$\bar{1}$·65256 44145
0·81	0·35177 85303	2·24 790 799	0·444 858 066	$\bar{1}$·64822 14697
0·82	0·35612 14752	2·27 049 984	0·440 431 654	$\bar{1}$·64387 85248
0·83	0·36046 44200	2·29 331 874	0·436 049 286	$\bar{1}$·63953 55800
0·84	0·36480 73648	2·31 636 698	0·431 710 523	$\bar{1}$·63519 26352
0·85	0·36915 03096	2·33 964 685	0·427 414 932	$\bar{1}$·63084 96904
0·86	0·37349 32544	2·36 316 069	0·423 162 082	$\bar{1}$·62650 67456
0·87	0·37783 61993	2·38 691 085	0·418 951 549	$\bar{1}$·62216 38007
0·88	0·38217 91441	2·41 089 971	0·414 782 912	$\bar{1}$·61782 08559
0·89	0·38652 20889	2·43 512 965	0·410 655 753	$\bar{1}$·61347 79111
0·90	0·39086 50337	2·45 960 311	0·406 569 660	$\bar{1}$·60913 49663
0·91	0·39520 79785	2·48 432 253	0·402 524 224	$\bar{1}$·60479 20215
0·92	0·39955 09234	2·50 929 039	0·398 519 041	$\bar{1}$·60044 90766
0·93	0·40389 38682	2·53 450 918	0·394 553 710	$\bar{1}$·59610 61318
0·94	0·40823 68130	2·55 998 142	0·390 627 835	$\bar{1}$·59176 31870
0·95	0·41257 97578	2·58 570 966	0·386 741 023	$\bar{1}$·58742 02422
0·96	0·41692 27026	2·61 169 647	0·382 892 886	$\bar{1}$·58307 72974
0·97	0·42126 56474	2·63 794 446	0·379 083 038	$\bar{1}$·57873 43526
0·98	0·42560 85923	2·66 445 624	0·375 311 099	$\bar{1}$·57439 14077
0·99	0·42995 15371	2·69 123 447	0·371 576 691	$\bar{1}$·57004 84629
1·00	0·43429 44819	2·71 828 183	0·367 879 441	$\bar{1}$·56570 55181

TABLE II. (*continued*).

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from $x=0{\cdot}01$ to $x=2{\cdot}00$ at intervals of 0·01.

x	$\log_{10}(e^x)$	e^x	e^{-x}	$\log_{10}(e^{-x})$
1·01	0·43863 74267	2·74 560 101	0·364 218 980	$\bar{1}$·56136 25733
1·02	0·44298 03715	2·77 319 476	0·360 594 940	$\bar{1}$·55701 96285
1·03	0·44732 33164	2·80 106 584	0·357 006 961	$\bar{1}$·55267 66836
1·04	0·45166 62612	2·82 921 701	0·353 454 682	$\bar{1}$·54833 37388
1·05	0·45600 92060	2·85 765 112	0·349 937 749	$\bar{1}$·54399 07940
1·06	0·46035 21508	2·88 637 099	0·346 455 810	$\bar{1}$·53964 78492
1·07	0·46469 50956	2·91 537 950	0·343 008 517	$\bar{1}$·53530 49044
1·08	0·46903 80405	2·94 467 955	0·339 595 526	$\bar{1}$·53096 19595
1·09	0·47338 09853	2·97 427 407	0·336 216 494	$\bar{1}$·52661 90147
1·10	0·47772 39301	3·00 416 602	0·332 871 084	$\bar{1}$·52227 60700
1·11	0·48206 68749	3·03 435 839	0·329 558 961	$\bar{1}$·51793 31251
1·12	0·48640 98197	3·06 485 420	0·326 279 795	$\bar{1}$·51359 01803
1·13	0·49075 27646	3·09 565 650	0·323 033 256	$\bar{1}$·50924 72354
1·14	0·49509 57094	3·12 676 837	0·319 819 022	$\bar{1}$·50490 42906
1·15	0·49943 86542	3·15 819 291	0·316 636 769	$\bar{1}$·50056 13458
1·16	0·50378 15990	3·18 993 328	0·313 486 181	$\bar{1}$·49621 84010
1·17	0·50812 45438	3·22 199 264	0·310 366 941	$\bar{1}$·49187 54562
1·18	0·51246 74886	3·25 437 420	0·307 278 739	$\bar{1}$·48753 25114
1·19	0·51681 04335	3·28 708 121	0·304 221 264	$\bar{1}$·48318 95665
1·20	0·52115 33783	3·32 011 692	0·301 194 212	$\bar{1}$·47884 66217
1·21	0·52549 63231	3·35 348 465	0·298 197 279	$\bar{1}$·47450 36769
1·22	0·52983 92679	3·38 718 773	0·295 230 167	$\bar{1}$·47016 07321
1·23	0·53418 22127	3·42 122 954	0·292 292 578	$\bar{1}$·46581 77873
1·24	0·53852 51576	3·45 561 347	0·289 384 218	$\bar{1}$·46147 48424
1·25	0·54286 81024	3·49 034 296	0·286 504 797	$\bar{1}$·45713 18976
1·26	0·54721 10472	3·52 542 149	0·283 654 027	$\bar{1}$·45278 89528
1·27	0·55155 39920	3·56 085 256	0·280 831 622	$\bar{1}$·44844 60080
1·28	0·55589 69368	3·59 663 973	0·278 037 301	$\bar{1}$·44410 30632
1·29	0·56023 98817	3·63 278 656	0·275 270 783	$\bar{1}$·43976 01183
1·30	0·56458 28265	3·66 929 667	0·272 531 793	$\bar{1}$·43541 71735
1·31	0·56892 57713	3·70 617 371	0·269 820 056	$\bar{1}$·43107 42287
1·32	0·57326 87161	3·74 342 138	0·267 135 302	$\bar{1}$·42673 12839
1·33	0·57761 16609	3·78 104 339	0·264 477 261	$\bar{1}$·42238 83391
1·34	0·58195 46058	3·81 904 351	0·261 845 669	$\bar{1}$·41804 53942
1·35	0·58629 75506	3·85 742 553	0·259 240 261	$\bar{1}$·41370 24494
1·36	0·59064 04954	3·89 619 330	0·256 660 777	$\bar{1}$·40935 95046
1·37	0·59498 34402	3·93 535 070	0·254 106 960	$\bar{1}$·40501 65598
1·38	0·59932 63850	3·97 490 163	0·251 578 553	$\bar{1}$·40067 36150
1·39	0·60366 93298	4·01 485 005	0·249 075 305	$\bar{1}$·39633 06701
1·40	0·60801 22747	4·05 519 997	0·246 596 964	$\bar{1}$·39198 77253
1·41	0·61235 52195	4·09 595 541	0·244 143 283	$\bar{1}$·38764 47805
1·42	0·61669 81643	4·13 712 044	0·241 714 017	$\bar{1}$·38330 18357
1·43	0·62104 11091	4·17 869 919	0·239 308 922	$\bar{1}$·37895 88909
1·44	0·62538 40539	4·22 069 582	0·236 927 759	$\bar{1}$·37461 59461
1·45	0·62972 69988	4·26 311 452	0·234 570 288	$\bar{1}$·37027 30012
1·46	0·63406 99436	4·30 595 953	0·232 236 275	$\bar{1}$·36593 00564
1·47	0·63841 28884	4·34 923 514	0·229 925 485	$\bar{1}$·36158 71116
1·48	0·64275 58332	4·39 294 568	0·227 637 688	$\bar{1}$·35724 41668
1·49	0·64709 87780	4·43 709 552	0·225 372 656	$\bar{1}$·35290 12220
1·50	0·65144 17229	4·48 168 907	0·223 130 160	$\bar{1}$·34855 82771

TABLE II. (*continued*).

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from $x=0·01$ to $x=2·00$ at intervals of 0·01.

x	$\log_{10}(e^x)$	e^x	e^{-x}	$\log_{10}(e^{-x})$
1·51	0·65578 46677	4·52 673 079	0·220 909 978	$\bar{1}$·34421 53323
1·52	0·66012 76125	4·57 222 520	0·218 711 887	$\bar{1}$·33987 23875
1·53	0·66447 05573	4·61 817 682	0·216 535 667	$\bar{1}$·33552 94427
1·54	0·66881 35021	4·66 459 027	0·214 381 101	$\bar{1}$·33118 64979
1·55	0·67315 64470	4·71 147 018	0·212 247 973	$\bar{1}$·32684 35531
1·56	0·67749 93918	4·75 882 125	0·210 136 071	$\bar{1}$·32250 06082
1·57	0·68184 23366	4·80 664 819	0·208 045 182	$\bar{1}$·31815 76634
1·58	0·68618 52814	4·85 495 581	0·205 975 098	$\bar{1}$·31381 47186
1·59	0·69052 82262	4·90 374 893	0·203 925 612	$\bar{1}$·30947 17738
1·60	0·69487 11710	4·95 303 242	0·201 896 518	$\bar{1}$·30512 88290
1·61	0·69921 41159	5·00 281 123	0·199 887 614	$\bar{1}$·30078 58841
1·62	0·70355 70607	5·05 309 032	0·197 898 699	$\bar{1}$·29644 29393
1·63	0·70790 00055	5·10 387 472	0·195 929 574	$\bar{1}$·29209 99945
1·64	0·71224 29503	5·15 516 951	0·193 980 042	$\bar{1}$·28775 70497
1·65	0·71658 58951	5·20 697 983	0·192 049 909	$\bar{1}$·28341 41049
1·66	0·72092 88400	5·25 931 084	0·190 138 980	$\bar{1}$·27907 11600
1·67	0·72527 17848	5·31 216 780	0·188 247 066	$\bar{1}$·27472 82152
1·68	0·72961 47296	5·36 555 597	0·186 373 976	$\bar{1}$·27038 52704
1·69	0·73395 76744	5·41 948 071	0·184 519 524	$\bar{1}$·26604 23256
1·70	0·73830 06192	5·47 394 739	0·182 683 524	$\bar{1}$·26169 93808
1·71	0·74264 35641	5·52 896 148	0·180 865 793	$\bar{1}$·25735 64359
1·72	0·74698 65089	5·58 452 846	0·179 066 148	$\bar{1}$·25301 34911
1·73	0·75132 94537	5·64 065 391	0·177 284 410	$\bar{1}$·24867 05463
1·74	0·75567 23985	5·69 734 342	0·175 520 401	$\bar{1}$·24432 76014
1·75	0·76001 53433	5·75 460 268	0·173 773 944	$\bar{1}$·23998 46567
1·76	0·76435 82881	5·81 243 739	0·172 044 864	$\bar{1}$·23564 17119
1·77	0·76870 12330	5·87 085 336	0·170 332 989	$\bar{1}$·23129 87670
1·78	0·77304 41778	5·92 985 642	0·168 638 147	$\bar{1}$·22695 58222
1·79	0·77738 71226	5·98 945 247	0·166 960 170	$\bar{1}$·22261 28774
1·80	0·78173 00674	6·04 964 746	0·165 298 889	$\bar{1}$·21826 99326
1·81	0·78607 30122	6·11 044 743	0·163 654 137	$\bar{1}$·21392 69878
1·82	0·79041 59571	6·17 185 845	0·162 025 751	$\bar{1}$·20958 40429
1·83	0·79475 89019	6·23 388 666	0·160 413 568	$\bar{1}$·20524 10981
1·84	0·79910 18467	6·29 653 826	0·158 817 426	$\bar{1}$·20089 81533
1·85	0·80344 47915	6·35 981 952	0·157 237 166	$\bar{1}$·19655 52085
1·86	0·80778 77363	6·42 373 677	0·155 672 630	$\bar{1}$·19221 22637
1·87	0·81213 06812	6·48 829 640	0·154 123 662	$\bar{1}$·18786 93188
1·88	0·81647 36260	6·55 350 486	0·152 590 106	$\bar{1}$·18352 63740
1·89	0·82081 65708	6·61 936 868	0·151 071 809	$\bar{1}$·17918 34292
1·90	0·82515 95156	6·68 589 444	0·149 568 619	$\bar{1}$·17484 04844
1·91	0·82950 24604	6·75 308 880	0·148 080 387	$\bar{1}$·17049 75396
1·92	0·83384 54053	6·82 095 847	0·146 606 962	$\bar{1}$·16615 45947
1·93	0·83818 83501	6·88 951 024	0·145 148 199	$\bar{1}$·16181 16499
1·94	0·84253 12949	6·95 875 097	0·143 703 950	$\bar{1}$·15746 87051
1·95	0·84687 42397	7·02 868 758	0·142 274 072	$\bar{1}$·15312 57603
1·96	0·85121 71845	7·09 932 707	0·140 858 421	$\bar{1}$·14878 28155
1·97	0·85556 01293	7·17 067 649	0·139 456 856	$\bar{1}$·14443 98707
1·98	0·85990 30742	7·24 274 299	0·138 069 237	$\bar{1}$·14009 69258
1·99	0·86424 60190	7·31 553 376	0·136 695 426	$\bar{1}$·13575 39810
2·00	0·86858 89638	7·38 905 610	0·135 335 283	$\bar{1}$·13141 10362

TABLE III.

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from $x=0.1$ to $x=10.0$ at intervals of 0.1.

x	$\log_{10}(e^x)$	e^x	e^{-x}	$\log_{10}(e^{-x})$
0.1	0.04342 94482	1.10 517 092	0.904 837 418	$\bar{1}$.95657 05518
0.2	0.08685 88964	1.22 140 276	0.818 730 753	$\bar{1}$.91314 11036
0.3	0.13028 83446	1.34 985 881	0.740 818 221	$\bar{1}$.86971 16554
0.4	0.17371 77928	1.49 182 470	0.670 320 046	$\bar{1}$.82628 22072
0.5	0.21714 72410	1.64 872 127	0.606 530 660	$\bar{1}$.78285 27590
0.6	0.26057 66891	1.82 211 880	0.548 811 636	$\bar{1}$.73942 33109
0.7	0.30400 61373	2.01 375 271	0.496 585 304	$\bar{1}$.69599 38627
0.8	0.34743 55855	2.22 554 093	0.449 328 964	$\bar{1}$.65256 44145
0.9	0.39086 50337	2.45 960 311	0.406 569 660	$\bar{1}$.60913 49663
1.0	0.43429 44819	2.71 828 183	0.367 879 441	$\bar{1}$.56570 55181
1.1	0.47772 39301	3.00 416 602	0.332 871 084	$\bar{1}$.52227 60699
1.2	0.52115 33783	3.32 011 692	0.301 194 212	$\bar{1}$.47884 66217
1.3	0.56458 28265	3.66 929 667	0.272 531 793	$\bar{1}$.43541 71735
1.4	0.60801 22747	4.05 519 997	0.246 596 964	$\bar{1}$.39198 77253
1.5	0.65144 17229	4.48 168 907	0.223 130 160	$\bar{1}$.34855 82771
1.6	0.69487 11710	4.95 303 242	0.201 896 518	$\bar{1}$.30512 88790
1.7	0.73830 06192	5.47 394 739	0.182 683 524	$\bar{1}$.26169 93808
1.8	0.78173 00674	6.04 964 746	0.165 298 888	$\bar{1}$.21826 99326
1.9	0.82515 95156	6.68 589 444	0.149 568 619	$\bar{1}$.17484 04843
2.0	0.86858 89638	7.38 905 610	0.135 335 283	$\bar{1}$.13141 10362
2.1	0.91201 84120	8.16 616 991	0.122 456 428	$\bar{1}$.08798 15880
2.2	0.95544 78602	9.02 501 350	0.110 803 158	$\bar{1}$.04455 21398
2.3	0.99887 73084	9.97 418 246	0.100 258 844	$\bar{1}$.00112 26916
2.4	1.04230 67566	11.0 231 764	(1)907 179 533	$\bar{2}$.95769 32434
2.5	1.08573 62048	12.1 824 940	(1)820 849 986	$\bar{2}$.91426 37952
2.6	1.12916 56529	13.4 637 380	(1)742 735 782	$\bar{2}$.87083 43471
2.7	1.17259 51011	14.8 797 317	(1)672 055 127	$\bar{2}$.82740 48989
2.8	1.21602 45493	16.4 446 468	(1)608 100 626	$\bar{2}$.78397 54507
2.9	1.25945 39975	18.1 741 454	(1)550 232 201	$\bar{2}$.74054 60025
3.0	1.30288 34457	20.0 855 369	(1)497 870 684	$\bar{2}$.69711 65543
3.1	1.34631 28939	22.1 979 513	(1)450 492 024	$\bar{2}$.65368 71061
3.2	1.38974 23421	24.5 325 302	(1)407 622 040	$\bar{2}$.61025 76579
3.3	1.43317 17903	27.1 126 389	(1)368 831 674	$\bar{2}$.56682 82096
3.4	1.47660 12385	29.9 641 001	(1)333 732 700	$\bar{2}$.52339 87615
3.5	1.52003 06867	33.1 154 520	(1)301 973 834	$\bar{2}$.47996 93133
3.6	1.56346 01349	36.5 982 344	(1)273 237 224	$\bar{2}$.43653 98651
3.7	1.60688 95830	40.4 473 044	(1)247 235 265	$\bar{2}$.39311 04170
3.8	1.65031 90312	44.7 011 845	(1)223 707 719	$\bar{2}$.34968 09688
3.9	1.69374 84794	49.4 024 491	(1)202 419 114	$\bar{2}$.30625 15206
4.0	1.73717 79276	54.5 981 500	(1)183 156 389	$\bar{2}$.26282 20724
4.1	1.78060 73758	60.3 402 876	(1)165 726 754	$\bar{2}$.21939 26242
4.2	1.82403 68240	66.6 863 310	(1)149 955 768	$\bar{2}$.17596 31760
4.3	1.86746 62722	73.6 997 937	(1)135 685 590	$\bar{2}$.13253 37278
4.4	1.91089 57204	81.4 508 687	(1)122 773 399	$\bar{2}$.08910 42796
4.5	1.95432 51686	90.0 171 313	(1)111 089 965	$\bar{2}$.04567 48314
4.6	1.99775 46168	99.4 843 157	(1)100 518 357	$\bar{2}$.00224 53832
4.7	2.04118 40649	109. 947 173	(2)909 527 710	$\bar{3}$.95881 59351
4.8	2.08461 35131	121. 510 418	(2)822 974 705	$\bar{3}$.91538 64869
4.9	2.12804 29613	134. 289 780	(2)744 658 307	$\bar{3}$.87195 70387
5.0	2.17147 24095	148. 413 159	(2)673 794 700	$\bar{3}$.82852 75905

The numbers in parentheses denote the numbers of ciphers between the decimal point and the first significant figure; for example, $e^{-5} = 0.00673794700$.

TABLE III. (*continued*).

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from $x=0·1$ to $x=10·0$ at intervals of 0·1.

x	$\log_{10}(e^x)$	e^x	e^{-x}	$\log_{10}(e^{-x})$
5·1	2·21490 18577	164· 021 907	(2) 609 674 657	$\bar{3}$·78509 81423
5·2	2·25833 13059	181· 272 242	(2) 551 656 442	$\bar{3}$·74166 86941
5·3	2·30176 07541	200· 336 810	(2) 499 159 391	$\bar{3}$·69823 92459
5·4	2·34519 02023	221· 406 416	(2) 451 658 094	$\bar{3}$·65480 97977
5·5	2·38861 96505	244· 691 932	(2) 408 677 144	$\bar{3}$·61138 03495
5·6	2·43204 90987	270· 426 407	(2) 369 786 372	$\bar{3}$·56795 09013
5·7	2·47547 85468	298· 867 401	(2) 334 596 546	$\bar{3}$·52452 14532
5·8	2·51890 79950	330· 299 560	(2) 302 755 475	$\bar{3}$·48109 20050
5·9	2·56233 74432	365· 037 468	(2) 273 944 482	$\bar{3}$·43766 25568
6·0	2·60576 68914	403· 428 794	(2) 247 875 218	$\bar{3}$·39423 31086
6·1	2·64919 63396	445· 857 770	(2) 224 286 772	$\bar{3}$·35080 36604
6·2	2·69262 57878	492· 749 041	(2) 202 943 064	$\bar{3}$·30737 42122
6·3	2·73605 52360	544· 571 910	(2) 183 630 478	$\bar{3}$·26394 47640
6·4	2·77948 46842	601· 845 038	(2) 166 155 727	$\bar{3}$·22051 53158
6·5	2·82291 41324	665· 141 633	(2) 150 343 919	$\bar{3}$·17708 58676
6·6	2·86634 35806	735· 095 189	(2) 136 036 804	$\bar{3}$·13365 64194
6·7	2·90977 30288	812· 405 825	(2) 123 091 190	$\bar{3}$·09022 69712
6·8	2·95320 24769	897· 847 292	(2) 111 377 515	$\bar{3}$·04679 75231
6·9	2·99663 19251	992· 274 716	(2) 100 778 543	$\bar{3}$·00336 80749
7·0	3·04006 13733	109 6·63 316	(3) 911 881 966	$\bar{4}$·95993 86267
7·1	3·08349 08215	121 1·96 708	(3) 825 104 923	$\bar{4}$·91650 91785
7·2	3·12692 02697	133 9·43 077	(3) 746 585 808	$\bar{4}$·87307 97303
7·3	3·17034 97179	148 0·29 993	(3) 675 538 775	$\bar{4}$·82965 02821
7·4	3·21377 91661	163 5·98 443	(3) 611 252 761	$\bar{4}$·78622 08339
7·5	3·25720 86143	180 8·04 242	(3) 553 084 370	$\bar{4}$·74279 13857
7·6	3·30063 80625	199 8·19 590	(3) 500 451 433	$\bar{4}$·69936 19375
7·7	3·34406 75107	220 8·34 799	(3) 452 827 183	$\bar{4}$·65593 24893
7·8	3·38749 69588	244 0·60 198	(3) 409 734 979	$\bar{4}$·61250 30412
7·9	3·43092 64070	269 7·28 233	(3) 370 743 540	$\bar{4}$·56907 35930
8·0	3·47435 58552	298 0·95 799	(3) 335 462 628	$\bar{4}$·52564 41448
8·1	3·51778 53034	329 4·46 808	(3) 303 539 138	$\bar{4}$·48221 46966
8·2	3·56121 47516	364 0·95 031	(3) 274 653 570	$\bar{4}$·43878 52484
8·3	3·60464 41998	402 3·87 239	(3) 248 516 827	$\bar{4}$·39535 58002
8·4	3·64807 36480	444 7·06 675	(3) 224 867 324	$\bar{4}$·35192 63520
8·5	3·69150 30962	491 4·76 884	(3) 203 468 369	$\bar{4}$·30849 69038
8·6	3·73493 25444	543 1·65 959	(3) 184 105 794	$\bar{4}$·26506 74556
8·7	3·77836 19926	600 2·91 222	(3) 166 585 811	$\bar{4}$·22163 80074
8·8	3·82179 14407	663 4·24 401	(3) 150 733 075	$\bar{4}$·17820 85593
8·9	3·86522 08889	733 1·97 354	(3) 136 388 926	$\bar{4}$·13477 91111
9·0	3·90865 03371	810 3·08 393	(3) 123 409 804	$\bar{4}$·09134 96629
9·1	3·95207 97853	895 5·29 270	(3) 111 665 808	$\bar{4}$·04792 02147
9·2	3·99550 92335	989 7·12 906	(3) 101 039 402	$\bar{4}$·00449 07665
9·3	4·03893 86817	109 38·0 192	(4) 914 242 315	$\bar{5}$·96106 13183
9·4	4·08236 81299	120 88·3 807	(4) 827 240 656	$\bar{5}$·91763 18701
9·5	4·12579 75781	133 59·7 268	(4) 748 518 299	$\bar{5}$·87420 24219
9·6	4·16922 70263	147 64·7 816	(4) 677 287 365	$\bar{5}$·83077 29737
9·7	4·21265 64744	163 17·6 072	(4) 612 834 951	$\bar{5}$·78734 35255
9·8	4·25608 59227	180 33·7 449	(4) 554 515 994	$\bar{5}$·74391 40774
9·9	4·29951 53708	199 30·3 704	(4) 501 746 821	$\bar{5}$·70048 46292
10·0	4·34294 48190	220 26·4 658	(4) 453 999 298	$\bar{5}$·65705 51810

The numbers in parentheses denote the numbers of ciphers between the decimal point and the first significant figure; for example, $e^{-10} = 0·0000453999298$.

TABLE IV.

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from 1 to 500 at intervals of unity.

x	$\log_{10}(e^x)$	e^x		e^{-x}	$\log_{10}(e^{-x})$
1	·43429 44819	2·71 828 183		0·367 879 441	$\bar{1}$·56570 55181
2	·86858 89638	7·38 905 610		0·135 335 283	$\bar{1}$·13141 10362
3	1·30288 34457	20·0 855 369		(1) 497 870 684	$\bar{2}$·69711 65543
4	1·73717 79276	54·5 981 500		(1) 183 156 389	$\bar{2}$·26282 20724
5	2·17147 24095	148·413 159		(2) 673 794 700	$\bar{3}$·82852 75905
6	2·60576 68914	403·428 793		(2) 247 875 218	$\bar{3}$·39423 31086
7	3·04006 13733	109 6·63 316		(3) 911 881 966	$\bar{4}$·95993 86267
8	3·47435 58552	298 0·95 799		(3) 335 462 628	$\bar{4}$·52564 41448
9	3·90865 03371	810 3·08 393		(3) 123 409 804	$\bar{4}$·09134 96629
10	4·34294 48190	220 26·4 658		(4) 453 999 298	$\bar{5}$·65705 51810
11	4·77723 93009	598 74·1 417		(4) 167 017 008	$\bar{5}$·22276 06991
12	5·21153 37828	162 754·791		(5) 614 421 235	$\bar{6}$·78846 62172
13	5·64582 82647	442 413·392		(5) 226 032 941	$\bar{6}$·35417 17353
14	6·08012 27466	120 260 4·28		(6) 831 528 719	$\bar{7}$·91987 72534
15	6·51441 72285	326 901 7·37		(6) 305 902 321	$\bar{7}$·48558 27715
16	6·94871 17105	888 611 0·52		(6) 112 535 175	$\bar{7}$·05128 82896
17	7·38300 61924	241 549 52·8		(7) 413 993 772	$\bar{8}$·61699 38076
18	7·81730 06743	656 599 69·1		(7) 152 299 797	$\bar{8}$·18269 93257
19	8·25159 51562	178 482 301·		(8) 560 279 644	$\bar{9}$·74840 48438
20	8·68588 96381	485 165 195·		(8) 206 115 362	$\bar{9}$·31411 03619
21	9·12018 41200	131 881 573	[1]	(9) 758 256 043	$\overline{10}$·87981 58800
22	9·55447 86019	358 491 285	[1]	(9) 278 946 809	$\overline{10}$·44552 13981
23	9·98877 30838	974 480 345	[1]	(9) 102 618 796	$\overline{10}$·01122 69162
24	10·42306 75657	264 891 221	[2]	(10) 377 513 454	$\overline{11}$·57693 24343
25	10.85736 20476	720 048 993	[2]	(10) 138 879 439	$\overline{11}$·14263 79524
26	11·29165 65295	195 729 609	[3]	(11) 510 908 903	$\overline{12}$·70834 34705
27	11·72595 10114	532 048 241	[3]	(11) 187 952 882	$\overline{12}$·27404 89886
28	12·16024 54933	144 625 707	[4]	(12) 691 440 011	$\overline{13}$·83975 45067
29	12·59453 99752	393 133 430	[4]	(12) 254 366 565	$\overline{13}$·40546 00248
30	13·02883 44571	106 864 746	[5]	(13) 935 762 297	$\overline{14}$·97116 55429
31	13·46312 89390	290 488 497	[5]	(13) 344 247 711	$\overline{14}$·53687 10610
32	13·89742 34209	789 629 602	[5]	(13) 126 641 656	$\overline{14}$·10257 65791
33	14·33171 79028	214 643 580	[6]	(14) 465 888 615	$\overline{15}$·66828 20972
34	14·76601 23847	583 461 743	[6]	(14) 171 390 843	$\overline{15}$·23398 76153
35	15·20030 68666	158 601 345	[7]	(15) 630 511 676	$\overline{16}$·79969 31334
36	15·63460 13485	431 123 155	[7]	(15) 231 952 283	$\overline{16}$·36539 86515
37	16·06889 58304	117 191 424	[8]	(16) 853 304 763	$\overline{17}$·93110 41696
38	16·50319 03123	318 559 318	[8]	(16) 313 913 279	$\overline{17}$·49680 96877
39	16·93748 47942	865 934 004	[8]	(16) 115 482 242	$\overline{17}$·06251 52058
40	17·37177 92761	235 385 267	[9]	(17) 424 835 426	$\overline{18}$·62822 07239
41	17·80607 37580	639 843 493	[9]	(17) 156 288 219	$\overline{18}$·19392 62420
42	18·24036 82399	173 927 494	[10]	(18) 574 952 227	$\overline{19}$·75963 17601
43	18·67466 27218	472 783 947	[10]	(18) 211 513 104	$\overline{19}$·32533 72782
44	19·10895 72037	128 516 001	[11]	(19) 778 113 224	$\overline{20}$·89104 27963
45	19·54325 16856	349 342 711	[11]	(19) 286 251 858	$\overline{20}$·45674 83144
46	19·97754 61675	949 611 942	[11]	(19) 105 306 174	$\overline{20}$·02245 38325
47	20·41184 06495	258 131 289	[12]	(20) 387 399 763	$\overline{21}$·58815 93505
48	20·84613 51314	701 673 591	[12]	(20) 142 516 408	$\overline{21}$·15386 48686
49	21·28042 96133	190 734 657	[13]	(21) 524 288 566	$\overline{22}$·71957 03867
50	21·71472 40952	518 470 553	[13]	(21) 192 874 985	$\overline{22}$·28527 59048

The numbers in square brackets denote the numbers of figures between the last figure given and the decimal point; for example, the first nine figures of e^{50} are 518470553, and there are 13 additional figures before the decimal point is reached. The numbers in parentheses denote the numbers of ciphers between the decimal point and the first significant figure; for example, in e^{-50} there are 21 ciphers between the decimal point and the figures 192874985.

TABLE IV. (*continued*).

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from 1 to 500 at intervals of unity.

x	$\log_{10}(e^x)$	e^x	e^{-x}	$\log_{10}(e^{-x})$
51	22·14901 85771	140 934 908 [14]	(22) 709 547 416	$\overline{23}$·85098 14229
52	22·58331 30590	383 100 800 [14]	(22) 261 027 907	$\overline{23}$·41668 69410
53	23·01760 75409	104 137 594 [15]	(23) 960 268 005	$\overline{24}$·98239 24591
54	23·45190 20228	283 075 330 [15]	(23) 353 262 857	$\overline{24}$·54809 79772
55	23·88619 65047	769 478 527 [15]	(23) 129 958 143	$\overline{24}$·11380 34953
56	24·32049 09866	209 165 950 [16]	(24) 478 089 288	$\overline{25}$·67950 90134
57	24·75478 54685	568 572 000 [16]	(24) 175 879 220	$\overline{25}$·24521 45315
58	25·18907 99504	154 553 894 [17]	(25) 647 023 493	$\overline{26}$·81092 00496
59	25·62337 44323	420 121 040 [17]	(25) 238 026 641	$\overline{26}$·37662 55677
60	26·05766 89142	114 200 739 [18]	(26) 875 651 076	$\overline{27}$·94233 10858
61	26·49196 33961	310 429 794 [18]	(26) 322 134 029	$\overline{27}$·50803 66039
62	26·92625 78780	843 835 667 [18]	(26) 118 506 487	$\overline{27}$·07374 21220
63	27·36055 23599	229 378 316 [19]	(27) 435 961 000	$\overline{28}$·63944 76401
64	27·79484 68418	623 514 908 [19]	(27) 160 381 089	$\overline{28}$·20515 31582
65	28·22914 13237	169 488 924 [20]	(28) 590 009 054	$\overline{29}$·77085 86763
66	28·66343 58056	460 718 663 [20]	(28) 217 052 201	$\overline{29}$·33656 41944
67	29·09773 02875	125 236 317 [21]	(29) 798 490 425	$\overline{30}$·90226 97125
68	29·53202 47694	340 427 605 [21]	(29) 293 748 211	$\overline{30}$·46797 52306
69	29·96631 92513	925 378 172 [21]	(29) 108 063 928	$\overline{30}$·03368 07487
70	30·40061 37332	251 543 867 [22]	(30) 397 544 974	$\overline{31}$·59938 62668
71	30·83490 82151	683 767 123 [22]	(30) 146 248 623	$\overline{31}$·16509 17849
72	31·26920 26970	185 867 175 [23]	(31) 538 018 616	$\overline{32}$·73079 73030
73	31·70349 71789	505 239 363 [23]	(31) 197 925 988	$\overline{32}$·29650 28211
74	32·13779 16608	137 338 298 [24]	(32) 728 129 018	$\overline{33}$·86220 83392
75	32·57208 61427	373 324 200 [24]	(32) 267 863 696	$\overline{33}$·42791 38573
76	33·00638 06246	101 480 039 [25]	(33) 985 415 469	$\overline{34}$·99361 93754
77	33·44067 51066	275 851 346 [25]	(33) 362 514 092	$\overline{34}$·55932 48934
78	33·87496 95885	749 841 700 [25]	(33) 133 361 482	$\overline{34}$·12503 04115
79	34·30926 40704	203 828 107 [26]	(34) 490 609 473	$\overline{35}$·69073 59296
80	34·74355 85523	554 062 238 [26]	(34) 180 485 139	$\overline{35}$·25644 14477
81	35·17785 30342	150 609 731 [27]	(35) 663 967 720	$\overline{36}$·82214 69658
82	35·61214 75161	409 399 696 [27]	(35) 244 260 074	$\overline{36}$·38785 24839
83	36·04644 19980	111 286 376 [28]	(36) 898 582 594	$\overline{37}$·95355 80020
84	36·48073 64799	302 507 732 [28]	(36) 330 570 063	$\overline{37}$·51926 35201
85	36·91503 09618	822 301 271 [28]	(36) 121 609 930	$\overline{37}$·08496 90382
86	37·34932 54437	223 524 660 [29]	(37) 447 377 931	$\overline{38}$·65067 45563
87	37·78361 99256	607 603 023 [29]	(37) 164 581 143	$\overline{38}$·21638 00744
88	38·21791 44075	165 163 626 [30]	(38) 605 460 189	$\overline{39}$·78208 55925
89	38·65220 88894	448 961 282 [30]	(38) 222 736 356	$\overline{39}$·34779 11106
90	39·08650 33713	122 040 329 [31]	(39) 819 401 262	$\overline{40}$·91349 66287
91	39·52079 78532	331 740 010 [31]	(39) 301 440 879	$\overline{40}$·47920 21468
92	39·95509 23351	901 762 841 [31]	(39) 110 893 902	$\overline{40}$·04490 76649
93	40·38938 68170	245 124 554 [32]	(40) 407 955 867	$\overline{41}$·61061 31830
94	40·82368 12989	666 317 622 [32]	(40) 150 078 576	$\overline{41}$·17631 87011
95	41·25797 57808	181 123 908 [33]	(41) 552 108 228	$\overline{42}$·74202 42192
96	41·69227 02627	492 345 829 [33]	(41) 203 109 266	$\overline{42}$·30772 97373
97	42·12656 47446	133 833 472 [34]	(42) 747 197 234	$\overline{43}$·87343 52554
98	42·56085 92265	363 797 095 [34]	(42) 274 878 501	$\overline{43}$·43914 07735
99	42·99515 37084	988 903 032 [34]	(42) 101 122 149	$\overline{43}$·00484 62916
100	43·42944 81903	268 811 714 [35]	(43) 372 007 598	$\overline{44}$·57055 18097

The numbers in square brackets denote the numbers of figures between the last figure given and the decimal point; for example, the first nine figures of e^{51} are 140934908, and there are 14 additional figures before the decimal point is reached. The numbers in parentheses denote the numbers of ciphers between the decimal point and the first significant figure; for example, in e^{-51} there are 22 ciphers between the decimal point and the figures 709547416.

TABLE IV. (*continued*).

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from 1 to 500 at intervals of unity.

x	$\log_{10}(e^x)$	e^x	e^{-x}	$\log_{10}(e^{-x})$
101	43·86374 26722	730 705 998 [35]	(43) 136 853 947	$\overline{44}$·13625 73278
102	44·29803 71541	198 626 484 [36]	(44) 503 457 536	$\overline{45}$·70196 28459
103	44·73233 16360	539 922 761 [36]	(44) 185 211 676	$\overline{45}$·26766 83640
104	45·16662 61179	146 766 223 [37]	(45) 681 355 682	$\overline{46}$·83337 38821
105	45·60092 05998	398 951 957 [37]	(45) 250 656 748	$\overline{46}$·39907 94002
106	46·03521 50817	108 446 386 [38]	(46) 922 114 642	$\overline{47}$·96478 49183
107	46·46950 95636	294 787 839 [38]	(46) 339 227 019	$\overline{47}$·53049 04364
108	46·90380 40456	801 316 426 [38]	(46) 124 794 646	$\overline{47}$·09619 59544
109	47·33809 85275	217 820 388 [39]	(47) 459 093 847	$\overline{48}$·66190 14725
110	47·77239 30094	592 097 203 [39]	(47) 168 891 188	$\overline{48}$·22760 69906
111	48·20668 74913	160 948 707 [40]	(48) 621 315 959	$\overline{49}$·79331 25087
112	48·64098 19732	437 503 945 [40]	(48) 228 569 368	$\overline{49}$·35901 80268
113	49·07527 64551	118 925 902 [41]	(49) 840 859 712	$\overline{50}$·92472 35449
114	49·50957 09370	323 274 119 [41]	(49) 309 335 001	$\overline{50}$·49042 90630
115	49·94386 54189	878 750 164 [41]	(49) 113 797 987	$\overline{50}$·05613 45811
116	50·37815 99008	238 869 060 [42]	(50) 418 639 400	$\overline{51}$·62184 00992
117	50·81245 43827	649 313 426 [42]	(50) 154 008 829	$\overline{51}$·18754 56173
118	51·24674 88646	176 501 689 [43]	(51) 566 566 818	$\overline{52}$·75325 11354
119	51·68104 33465	479 781 333 [43]	(51) 208 428 284	$\overline{52}$·31895 66535
120	52·11533 78284	130 418 088 [44]	(52) 766 764 807	$\overline{53}$·88466 21716
121	52·54963 23103	354 513 118 [44]	(52) 282 077 009	$\overline{53}$·45036 76897
122	52·98392 67922	963 666 567 [44]	(52) 103 770 332	$\overline{53}$·01607 32078
123	53·41822 12741	261 951 732 [45]	(53) 381 749 719	$\overline{54}$·58177 87259
124	53·85251 57560	712 058 633 [45]	(53) 140 437 873	$\overline{54}$·14748 42440
125	54·28681 02379	193 557 604 [46]	(54) 516 642 063	$\overline{55}$·71318 97621
126	54·72110 47198	526 144 118 [46]	(54) 190 061 994	$\overline{55}$·27889 52802
127	55·15539 92017	143 020 800 [47]	(55) 699 199 000	$\overline{56}$·84460 07983
128	55·58969 36836	388 770 841 [47]	(55) 257 220 937	$\overline{56}$·41030 63164
129	56·02398 81655	105 678 871 [48]	(56) 946 262 947	$\overline{57}$·97601 18345
130	56·45828 26474	287 264 955 [48]	(56) 348 110 684	$\overline{57}$·54171 73526
131	56·89257 71293	780 867 107 [48]	(56) 128 062 764	$\overline{57}$·10742 28707
132	57·32687 16112	212 261 687 [49]	(57) 471 116 580	$\overline{58}$·67312 83888
133	57·76116 60931	576 987 086 [49]	(57) 173 314 104	$\overline{58}$·23883 39069
134	58·19546 05750	156 841 351 [50]	(58) 637 586 958	$\overline{59}$·80453 94250
135	58·62975 50569	426 338 995 [50]	(58) 234 555 134	$\overline{59}$·37024 49431
136	59·06404 95388	115 890 954 [51]	(59) 862 880 116	$\overline{60}$·93595 04612
137	59·49834 40207	315 024 275 [51]	(59) 317 435 855	$\overline{60}$·50165 59793
138	59·93263 85026	856 324 762 [51]	(59) 116 778 125	$\overline{60}$·06736 14974
139	60·36693 29846	232 773 204 [52]	(60) 429 602 713	$\overline{61}$·63306 70154
140	60·80122 74665	632 743 171 [52]	(60) 158 042 006	$\overline{61}$·19877 25335
141	61·23552 19484	171 997 426 [53]	(61) 581 404 049	$\overline{62}$·76447 80516
142	61·66981 64303	467 537 479 [53]	(61) 213 886 597	$\overline{62}$·33018 35697
143	62·10411 09122	127 089 863 [54]	(62) 786 844 816	$\overline{63}$·89588 90878
144	62·53840 53941	345 466 066 [54]	(62) 289 464 031	$\overline{63}$·46159 46059
145	62·97269 98760	939 074 129 [54]	(62) 106 487 866	$\overline{63}$·02730 01240
146	63·40699 43579	255 266 814 [55]	(63) 391 746 966	$\overline{64}$·59300 56421
147	63·84128 88398	693 887 142 [55]	(63) 144 115 655	$\overline{64}$·15871 11602
148	64·27558 33217	188 618 081 [56]	(64) 530 171 867	$\overline{65}$·72441 66783
149	64·70987 78036	512 717 102 [56]	(64) 195 039 330	$\overline{65}$·29012 21964
150	65·14417 22855	139 370 958 [57]	(65) 717 509 597	$\overline{66}$·85582 77145

The numbers in square brackets denote the numbers of figures between the last figure given and the decimal point; for example, the first nine figures of e^{101} are 730705998, and there are 35 additional figures before the decimal point is reached. The numbers in parentheses denote the numbers of ciphers between the decimal point and the first significant figure; for example, in e^{-101} there are 43 ciphers between the decimal point and the figures 136853947.

TABLE IV. (*continued*).

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from 1 to 500 at intervals of unity.

x	$\log_{10}(e^x)$	e^x	e^{-x}	$\log_{10}(e^{-x})$
151	65·57846 67674	378 849 543 [57]	(65) 263 957 030	66·42153 32326
152	66·01276 12493	102 981 983 [58]	(66) 971 043 646	$\overline{67}$·98723 87507
153	66·44705 57312	279 934 052 [58]	(66) 357 226 994	$\overline{67}$·55294 42688
154	66·88135 02131	760 939 648 [58]	(66) 131 416 467	$\overline{67}$·11864 97869
155	67·31564 46950	206 844 842 [59]	(67) 483 454 164	$\overline{68}$·68435 53050
156	67·74993 91769	562 262 575 [59]	(67) 177 852 848	$\overline{68}$·25006 08231
157	68·18423 36588	152 838 814 [60]	(68) 654 284 062	$\overline{69}$·81576 63412
158	68·61852 81407	415 458 971 [60]	(68) 240 697 655	$\overline{69}$·38147 18593
159	69·05282 26226	112 933 457 [61]	(69) 885 477 188	$\overline{70}$·94717 73774
160	69·48711 71045	306 984 964 [61]	(69) 325 748 853	$\overline{70}$·51288 28955
161	69·92141 15864	834 471 649 [61]	(69) 119 836 306	$\overline{70}$·07858 84136
162	70·35570 60683	226 832 912 [62]	(70) 440 853 133	$\overline{71}$·64429 39317
163	70·79000 05502	616 595 783 [62]	(70) 162 180 804	$\overline{71}$·20999 94498
164	71·22429 50321	167 608 111 [63]	(71) 596 629 837	$\overline{72}$·77570 49679
165	71·65858 95140	455 606 083 [63]	(71) 219 487 851	$\overline{72}$·34141 04860
166	72·09288 39959	123 846 574 [64]	(72) 807 450 679	$\overline{73}$·90711 60041
167	72·52717 84778	336 649 891 [64]	(72) 297 044 505	$\overline{73}$·47282 15222
168	72·96147 29597	915 109 280 [64]	(72) 109 276 566	$\overline{73}$·03852 70403
169	73·39576 74416	248 752 493 [65]	(73) 402 006 022	$\overline{74}$·60423 25584
170	73·83006 19236	676 179 381 [65]	(73) 147 889 751	$\overline{74}$·16993 80764
171	74·26435 64055	183 804 612 [66]	(74) 544 055 988	$\overline{75}$·73564 35945
172	74·69865 08874	499 632 738 [66]	(74) 200 147 013	$\overline{75}$·30134 91126
173	75·13294 53693	135 814 259 [67]	(75) 736 299 712	$\overline{76}$·86705 46307
174	75·56723 98512	369 181 433 [67]	(75) 270 869 527	$\overline{76}$·43276 01488
175	76·00153 43331	100 353 918 [68]	(76) 996 473 301	$\overline{77}$·99846 56669
176	76·43582 88150	272 790 232 [68]	(76) 366 582 041	$\overline{77}$·56417 11850
177	76·87012 32969	741 520 730 [68]	(76) 134 857 996	$\overline{77}$·12987 67031
178	77·30441 77788	201 566 233 [69]	(77) 496 114 844	$\overline{78}$·69558 22212
179	77·73871 22607	547 913 828 [69]	(77) 182 510 451	$\overline{78}$·26128 77393
180	78·17300 67426	148 938 420 [70]	(78) 671 418 429	$\overline{79}$·82699 32574
181	78·60730 12245	404 856 601 [70]	(78) 247 001 036	$\overline{79}$·39269 87755
182	79·04159 57064	110 051 434 [71]	(79) 908 666 032	$\overline{80}$·95840 42936
183	79·47589 01883	299 150 814 [71]	(79) 334 279 552	$\overline{80}$·52410 98117
184	79·91018 46702	813 176 221 [71]	(79) 122 974 575	$\overline{80}$·08981 53298
185	80·34447 91521	221 044 214 [72]	(80) 452 398 179	$\overline{81}$·65552 08479
186	80·77877 36340	600 860 471 [72]	(80) 166 427 989	$\overline{81}$·22122 63660
187	81·21306 81159	163 330 810 [73]	(81) 612 254 357	$\overline{82}$·78693 18841
188	81·64736 25978	443 979 173 [73]	(81) 225 235 791	$\overline{82}$·35263 74022
189	82·08165 70797	120 686 052 [74]	(82) 828 596 168	$\overline{83}$·91834 29203
190	82·51595 15616	328 058 702 [74]	(82) 304 823 495	$\overline{83}$·48404 84384
191	82·95024 60435	891 756 007 [74]	(82) 112 138 297	$\overline{83}$·04975 39565
192	83·38454 05254	242 404 415 [75]	(83) 412 533 741	$\overline{84}$·61545 94746
193	83·81883 50073	658 923 516 [75]	(83) 151 762 682	$\overline{84}$·18116 49927
194	84·25312 94892	179 113 982 [76]	(84) 558 303 706	$\overline{85}$·74687 05108
195	84·68742 39711	486 882 283 [76]	(84) 205 388 455	$\overline{85}$·31257 60289
196	85·12171 84530	132 348 326 [77]	(85) 755 581 902	$\overline{86}$·87828 15470
197	85·55601 29349	359 760 050 [77]	(85) 277 963 048	$\overline{86}$·44398 70651
198	85·99030 74168	977 929 206 [77]	(85) 102 256 891	$\overline{86}$·00969 25832
199	86·42460 18987	265 828 719 [78]	(86) 376 182 078	$\overline{87}$·57539 81013
200	86·85889 63807	722 597 377 [78]	(86) 138 389 653	$\overline{87}$·14110 36194

The numbers in square brackets denote the numbers of figures between the last figure given and the decimal point; for example, the first nine figures of e^{151} are 378849543, and there are 57 additional figures before the decimal point is reached. The numbers in parentheses denote the numbers of ciphers between the decimal point and the first significant figure; for example, in e^{-151} there are 65 ciphers between the decimal point and the figures 263957030.

TABLE IV. (*continued*).

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from 1 to 500 at intervals of unity.

x	$\log_{10}(e^x)$	e^x	e^{-x}	$\log_{10}(e^{-x})$
201	87·29319 08626	196 422 332 [79]	(87) 509 107 081	$\overline{88}$·70680 91374
202	87·72748 53445	533 931 255 [79]	(87) 187 290 028	$\overline{88}$·27251 46555
203	88·16177 98264	145 137 563 [80]	(88) 689 001 510	$\overline{89}$·83822 01736
204	88·59607 43083	394 524 800 [80]	(88) 253 469 490	$\overline{89}$·40392 56917
205	89·03036 87902	107 242 960 [81]	(89) 932 462 145	$\overline{90}$·96963 12098
206	89·46466 32721	291 516 588 [81]	(89) 343 033 653	$\overline{90}$·53533 67279
207	89·89895 77540	792 424 244 [81]	(89) 126 195 029	$\overline{90}$·10104 22460
208	90·33325 22359	215 403 242 [82]	(90) 464 245 566	$\overline{91}$·66674 77641
209	90·76754 67178	585 526 719 [82]	(90) 170 786 399	$\overline{91}$·23245 32822
210	91·20184 11997	159 162 664 [83]	(91) 628 288 051	$\overline{92}$·79815 88003
211	91·63613 56816	432 648 977 [83]	(91) 231 134 257	$\overline{92}$·36386 43184
212	92·07043 01635	117 606 185 [84]	(92) 850 295 414	$\overline{93}$·92956 98365
213	92·50472 46454	319 686 757 [84]	(92) 312 806 202	$\overline{93}$·49527 53546
214	92·93901 91273	868 998 701 [84]	(92) 115 074 971	$\overline{93}$·06098 08727
215	93·37331 36092	236 218 338 [85]	(93) 423 337 159	$\overline{94}$·62668 63908
216	93·80760 80911	642 108 015 [85]	(93) 155 737 037	$\overline{94}$·19239 19089
217	94·24190 25730	174 543 055 [86]	(94) 572 924 543	$\overline{95}$·75809 74270
218	94·67619 70549	474 457 215 [86]	(94) 210 767 161	$\overline{95}$·32380 29451
219	95·11049 15368	128 970 843 [87]	(95) 775 369 053	$\overline{96}$·88950 84632
220	95·54478 60187	350 579 098 [87]	(95) 285 242 334	$\overline{96}$·45521 39813
221	95·97908 05006	952 972 790 [87]	(95) 104 934 791	$\overline{96}$·02091 94994
222	96·41337 49825	259 044 862 [88]	(96) 386 033 520	$\overline{97}$·58662 50175
223	96·84766 94644	704 156 941 [88]	(96) 142 013 796	$\overline{97}$·15233 05356
224	97·28196 39463	191 409 702 [89]	(97) 522 439 558	$\overline{98}$·71803 60537
225	97·71625 84282	520 305 514 [89]	(97) 192 194 773	$\overline{98}$·28374 15718
226	98·15055 29101	141 433 702 [90]	(98) 707 045 056	$\overline{99}$·84944 70899
227	98·58484 73920	384 456 663 [90]	(98) 260 107 340	$\overline{99}$·41515 26080
228	99·01914 18739	104 506 156 [91]	(99) 956 881 429	$\overline{100}$·98085 81261
229	99·45343 63558	284 077 185 [91]	(99) 352 017 006	$\overline{100}$·54656 36442
230	99·88773 08377	772 201 850 [91]	(99) 129 499 819	$\overline{100}$·11226 91623
231	100·32202 53197	209 906 226 [92]	(100) 476 403 211	$\overline{101}$·67797 46803
232	100·75631 98016	570 584 279 [92]	(100) 175 258 947	$\overline{101}$·24368 01984
233	101·19061 42835	155 100 888 [93]	(101) 644 741 635	$\overline{102}$·80938 57165
234	101·62490 87654	421 607 925 [93]	(101) 237 187 193	$\overline{102}$·37509 12346
235	102·05920 32473	114 604 916 [94]	(102) 872 562 919	$\overline{103}$·94079 67527
236	102·49349 77292	311 528 461 [94]	(102) 320 997 959	$\overline{103}$·50650 22708
237	102·92779 22111	846 822 154 [94]	(102) 118 088 550	$\overline{103}$·07220 77889
238	103·36208 66930	230 190 127 [95]	(103) 434 423 497	$\overline{104}$·63791 33070
239	103·79638 11749	625 721 640 [95]	(103) 159 815 473	$\overline{104}$·20361 88251
240	104·23067 56568	170 088 776 [96]	(104) 587 928 270	$\overline{105}$·76932 43432
241	104·66497 01387	462 349 230 [96]	(104) 216 286 723	$\overline{105}$·33502 98613
242	105·09926 46206	125 679 551 [97]	(105) 795 674 389	$\overline{106}$·90073 53794
243	105·53355 91025	341 632 440 [97]	(105) 292 712 250	$\overline{106}$·46644 08975
244	105·96785 35844	928 653 253 [97]	(105) 107 682 819	$\overline{106}$·03214 64156
245	106·40214 80663	252 434 126 [98]	(106) 396 142 952	$\overline{107}$·59785 19337
246	106·83644 25482	686 187 098 [98]	(106) 145 732 848	$\overline{107}$·16355 74518
247	107·27073 70301	186 524 992 [99]	(107) 536 121 186	$\overline{108}$·72926 29699
248	107·70503 15120	507 027 496 [99]	(107) 197 227 962	$\overline{108}$·29496 84880
249	108·13932 59939	137 824 363 [100]	(108) 725 561 126	$\overline{109}$·86067 40061
250	108·57362 04758	374 645 461 [100]	(108) 266 919 022	$\overline{109}$·42637 95242

The numbers in square brackets denote the numbers of figures between the last figure given and the decimal point; and the numbers in parentheses denote the numbers of ciphers between the decimal point and the first significant figure.

TABLE IV. (*continued*).

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from 1 to 500 at intervals of unity.

x	$\log_{10}(e^x)$	e^x	e^{-x}	$\log_{10}(e^{-x})$
251	109·00791 49577	101 839 195 [101]	(109) 981 940 205	$\overline{110}$·99208 50423
252	109·44220 94396	276 827 633 [101]	(109) 361 235 614	$\overline{110}$·55779 05604
253	109·87650 39215	752 495 525 [101]	(109) 132 891 156	$\overline{110}$·12349 60785
254	110·31079 84034	204 549 491 [102]	(110) 488 879 241	$\overline{111}$·68920 15966
255	110·74509 28853	556 023 165 [102]	(110) 179 848 622	$\overline{111}$·25490 71147
256	111·17938 73672	151 142 767 [103]	(111) 661 626 106	$\overline{112}$·82061 26328
257	111·61368 18491	410 848 636 [103]	(111) 243 398 642	$\overline{112}$·38631 81509
258	112·04797 63310	111 680 238 [104]	(112) 895 413 564	$\overline{113}$·95202 36690
259	112·48227 08129	303 578 362 [104]	(112) 329 404 242	$\overline{113}$·51772 91871
260	112·91656 52948	825 211 544 [104]	(112) 121 181 048	$\overline{113}$·08343 47052
261	113·35085 97767	224 315 755 [105]	(113) 445 800 163	$\overline{114}$·64914 02233
262	113·78515 42587	609 753 439 [105]	(113) 164 000 715	$\overline{114}$·21484 57413
263	114·21944 87406	165 748 169 [106]	(114) 603 324 914	$\overline{115}$·78055 12594
264	114·65374 32225	450 550 237 [106]	(114) 221 950 832	$\overline{115}$·34625 67775
265	115·08803 77044	122 472 252 [107]	(115) 816 511 481	$\overline{116}$·91196 22956
266	115·52233 21863	332 914 098 [107]	(115) 300 377 787	$\overline{116}$·47766 78137
267	115·95662 66682	904 954 342 [107]	(115) 110 502 813	$\overline{116}$·04337 33318
268	116·39092 11501	245 992 094 [108]	(116) 406 517 129	$\overline{117}$·60907 88499
269	116·82521 56320	668 675 840 [108]	(116) 149 549 294	$\overline{117}$·17478 43680
270	117·25951 01139	181 764 939 [109]	(117) 550 161 108	$\overline{118}$·74048 98861
271	117·69380 45958	494 088 330 [109]	(117) 202 392 961	$\overline{118}$·30619 54042
272	118·12809 90777	134 307 133 [110]	(118) 744 562 094	$\overline{119}$·87190 09223
273	118·56239 35596	365 084 638 [110]	(118) 273 909 087	$\overline{119}$·43760 64404
274	118·99668 80415	992 402 938 [110]	(118) 100 765 522	$\overline{119}$·00331 19585
275	119·43098 25234	269 763 087 [111]	(119) 370 695 639	$\overline{120}$·56901 74766
276	119·86527 70053	733 292 098 [111]	(119) 136 371 304	$\overline{120}$·13472 29947
277	120·29957 14872	199 329 459 [112]	(120) 501 681 993	$\overline{121}$·70042 85128
278	120·73386 59691	541 833 645 [112]	(120) 184 558 491	$\overline{121}$·26613 40309
279	121·16816 04510	147 285 655 [113]	(121) 678 952 746	$\overline{122}$·83183 95490
280	121·60245 49329	400 363 920 [113]	(121) 249 772 757	$\overline{122}$·39754 50671
281	122·03674 94148	108 830 197 [114]	(122) 918 862 622	$\overline{123}$·96325 05852
282	122·47104 38967	295 831 147 [114]	(122) 338 030 668	$\overline{123}$·52895 61033
283	122·90533 83786	804 152 430 [114]	(122) 124 354 533	$\overline{123}$·09466 16214
284	123·33963 28605	218 591 294 [115]	(123) 457 474 762	$\overline{124}$·66036 71395
285	123·77392 73424	594 192 742 [115]	(123) 168 295 560	$\overline{124}$·22607 26576
286	124·20822 18243	161 518 333 [116]	(124) 619 124 764	$\overline{125}$·79177 81757
287	124·64251 63062	439 052 350 [116]	(124) 227 763 272	$\overline{125}$·35748 36938
288	125·07681 07881	119 346 803 [117]	(125) 837 894 253	$\overline{126}$·92318 92119
289	125·51110 52700	324 418 245 [117]	(125) 308 244 070	$\overline{126}$·48889 47300
290	125·94539 97519	881 860 219 [117]	(125) 113 396 656	$\overline{126}$·05460 02481
291	126·37969 42338	239 714 461 [118]	(126) 417 162 985	$\overline{127}$·62030 57662
292	126·81398 87157	651 611 463 [118]	(126) 153 465 686	$\overline{127}$·18601 12843
293	127·24828 31977	177 126 360 [119]	(127) 564 568 707	$\overline{128}$·75171 68023
294	127·68257 76796	481 479 366 [119]	(127) 207 693 220	$\overline{128}$·31742 23204
295	128·11687 21615	130 879 661 [120]	(128) 764 060 659	$\overline{129}$·88312 78385
296	128·55116 66434	355 767 804 [120]	(128) 281 082 208	$\overline{129}$·44883 33566
297	128·98546 11253	967 077 157 [120]	(128) 103 404 366	$\overline{129}$·01453 88747
298	129·41975 56072	262 878 826 [121]	(129) 380 463 403	$\overline{130}$·58024 43928
299	129·85405 00891	714 578 737 [121]	(129) 139 942 591	$\overline{130}$·14594 99109
300	130·28834 45710	194 242 640 [122]	(130) 514 820 022	$\overline{131}$·71165 54290

The numbers in square brackets denote the numbers of figures between the last figure given and the decimal point; and the numbers in parentheses denote the numbers of ciphers between the decimal point and the first significant figure.

TABLE IV. (*continued*).

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from 1 to 500 at intervals of unity.

x	$\log_{10}(e^x)$	e^x	e^{-x}	$\log_{10}(e^{-x})$
301	130.72263 90529	528 006 237 [122]	(130) 189 391 702	$\overline{131}$.27736 09471
302	131.15693 35348	143 526 976 [123]	(131) 696 733 135	$\overline{132}$.84306 64652
303	131.59122 80167	390 146 771 [123]	(131) 256 313 796	$\overline{132}$.40877 19833
304	132.02552 24986	106 052 888 [124]	(132) 942 925 762	$\overline{133}$.97447 75014
305	132.45981 69805	288 281 638 [124]	(132) 346 883 002	$\overline{133}$.54018 30195
306	132.89411 14624	783 630 737 [124]	(132) 127 611 125	$\overline{133}$.10588 85376
307	133.32840 59443	213 012 919 [125]	(133) 469 455 094	$\overline{134}$.67159 40557
308	133.76270 04262	579 029 148 [125]	(133) 172 702 878	$\overline{134}$.23729 95738
309	134.19699 49081	157 396 441 [126]	(134) 635 338 381	$\overline{135}$.80300 50919
310	134.63128 93900	427 847 886 [126]	(134) 233 727 929	$\overline{135}$.36871 06100
311	135.06558 38719	116 301 113 [127]	(135) 859 836 997	$\overline{136}$.93441 61281
312	135.49987 83538	316 139 203 [127]	(135) 316 316 354	$\overline{136}$.50012 16462
313	135.93417 28357	859 355 450 [127]	(135) 116 366 284	$\overline{136}$.06582 71643
314	136.36846 73176	233 597 031 [128]	(136) 428 087 634	$\overline{137}$.63153 26824
315	136.80276 17995	634 982 563 [128]	(136) 157 484 640	$\overline{137}$.19723 82005
316	137.23705 62814	172 606 156 [129]	(137) 579 353 612	$\overline{138}$.76294 37186
317	137.67135 07633	469 192 178 [129]	(137) 213 132 283	$\overline{138}$.32864 92367
318	138.10564 52452	127 539 657 [130]	(138) 784 069 851	$\overline{139}$.89435 47548
319	138.53993 97271	346 688 732 [130]	(138) 288 443 179	$\overline{139}$.46006 02729
320	138.97423 42090	942 397 682 [130]	(138) 106 112 315	$\overline{139}$.02576 57910
321	139.40852 86909	256 170 249 [131]	(139) 390 365 393	$\overline{140}$.59147 13091
322	139.84282 31728	696 342 934 [131]	(139) 143 607 403	$\overline{140}$.15717 68272
323	140.27711 76548	189 285 634 [132]	(140) 528 302 110	$\overline{141}$.72288 23452
324	140.71141 21367	514 531 700 [132]	(140) 194 351 485	$\overline{141}$.28858 78633
325	141.14570 66186	139 864 217 [133]	(141) 714 979 157	$\overline{142}$.85429 33814
326	141.58000 11005	380 190 360 [133]	(141) 263 026 133	$\overline{142}$.41999 88995
327	142.01429 55824	103 346 455 [134]	(142) 967 619 067	$\overline{143}$.98570 44176
328	142.44859 00643	280 924 790 [134]	(142) 355 967 162	$\overline{143}$.55140 99357
329	142.88288 45462	763 632 751 [134]	(142) 130 953 001	$\overline{143}$.11711 54538
330	143.31717 90281	207 576 903 [135]	(143) 481 749 167	$\overline{144}$.68282 09719
331	143.75147 35100	564 252 524 [135]	(143) 177 225 614	$\overline{144}$.24852 64900
332	144.18576 79919	153 379 738 [136]	(144) 651 976 599	$\overline{145}$.81423 20081
333	144.62006 24738	416 929 355 [136]	(144) 239 848 787	$\overline{145}$.37993 75262
334	145.05435 69557	113 333 149 [137]	(145) 882 354 377	$\overline{146}$.94564 30443
335	145.48865 14376	308 071 439 [137]	(145) 324 600 035	$\overline{146}$.51134 85624
336	145.92294 59195	837 424 995 [137]	(145) 119 413 680	$\overline{146}$.07705 40805
337	146.35724 04014	227 635 715 [138]	(146) 439 298 377	$\overline{147}$.64275 95986
338	146.79153 48833	618 778 027 [138]	(146) 161 608 841	$\overline{147}$.20846 51167
339	147.22582 93652	168 201 307 [139]	(147) 594 525 703	$\overline{148}$.77417 06348
340	147.66012 38471	457 218 555 [139]	(147) 218 713 783	$\overline{148}$.33987 61529
341	148.09441 83290	124 284 889 [140]	(148) 804 603 044	$\overline{149}$.90558 16710
342	148.52871 28109	337 841 356 [140]	(148) 295 996 918	$\overline{149}$.47128 71891
343	148.96300 72928	918 348 018 [140]	(148) 108 891 181	$\overline{149}$.03699 27072
344	149.39730 17747	249 632 873 [141]	(149) 400 588 267	$\overline{150}$.60269 82253
345	149.83159 62566	678 572 502 [141]	(149) 147 368 188	$\overline{150}$.16840 37434
346	150.26589 07385	184 455 130 [142]	(150) 542 137 266	$\overline{151}$.73410 92615
347	150.70018 52204	501 401 028 [142]	(150) 199 441 155	$\overline{151}$.29981 47796
348	151.13447 97023	136 294 931 [143]	(151) 733 703 005	$\overline{152}$.86552 02977
349	151.56877 41842	370 488 033 [143]	(151) 269 914 251	$\overline{152}$.43122 58158
350	152.00306 86661	100 709 089 [144]	(152) 992 959 040	$\overline{153}$.99693 13339

The numbers in square brackets denote the numbers of figures between tho last figure given and the decimal point; and the numbers in parentheses denote the numbers of ciphers between the decimal point and the first significant figure.

TABLE IV. (*continued*).

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from 1 to 500 at intervals of unity.

x	$\log_{10}(e^x)$	e^x	e^{-x}	$\log_{10}(e^{-x})$
351	152·43736 31480	273 755 686 [144]	(152) 365 289 217	$\overline{153}$·56263 68520
352	152·87165 76299	744 145 106 [144]	(152) 134 382 393	$\overline{153}$·12834 23701
353	153·30595 21118	202 279 612 [145]	(153) 494 365 196	$\overline{154}$·69404 78882
354	153·74024 65938	549 852 994 [145]	(153) 181 866 792	$\overline{154}$·25975 34062
355	154·17454 10757	149 465 540 [146]	(154) 669 050 538	$\overline{155}$·82545 89243
356	154·60883 55576	406 289 462 [146]	(154) 246 129 938	$\overline{155}$·39116 44424
357	155·04313 00395	110 440 926 [147]	(155) 905 461 441	$\overline{156}$·95686 99605
358	155·47742 45214	300 209 562 [147]	(155) 333 100 649	$\overline{156}$·52257 54786
359	155·91171 90033	816 054 198 [147]	(155) 122 540 881	$\overline{156}$·08828 09967
360	156·34601 34852	221 826 530 [148]	(156) 450 802 707	$\overline{157}$·65398 65148
361	156·78030 79671	602 987 025 [148]	(156) 165 841 048	$\overline{157}$·21969 20329
362	157·21460 24490	163 908 867 [149]	(157) 610 095 120	$\overline{158}$·78539 75510
363	157·64889 69309	445 550 495 [149]	(157) 224 441 452	$\overline{158}$·35110 30691
364	158·08319 14128	121 113 182 [150]	(158) 825 673 958	$\overline{159}$·91680 85872
365	158·51748 58947	329 219 761 [150]	(158) 303 748 474	$\overline{159}$·48251 41053
366	158·95178 03766	894 912 093 [150]	(158) 111 742 819	$\overline{159}$·04821 96234
367	159·38607 48585	243 262 328 [151]	(159) 411 078 858	$\overline{160}$·61392 51415
368	159·82036 93404	661 255 566 [151]	(159) 151 227 461	$\overline{160}$·17963 06596
369	160·25466 38223	179 747 899 [152]	(160) 556 334 737	$\overline{161}$·74533 61777
370	160·68895 83042	488 605 447 [152]	(160) 204 664 112	$\overline{161}$·31104 16958
371	161·12325 27861	132 816 731 [153]	(161) 752 917 192	$\overline{162}$·87674 72139
372	161·55754 72680	361 033 306 [153]	(161) 276 982 756	$\overline{162}$·44245 27320
373	161·99184 17499	981 390 275 [153]	(161) 101 896 262	$\overline{162}$·00815 82501
374	162·42613 62318	266 769 535 [154]	(162) 374 855 397	$\overline{163}$·57386 37682
375	162·86043 07137	725 154 779 [154]	(162) 137 901 594	$\overline{163}$·13956 92863
376	163·29472 51956	197 117 506 [155]	(163) 507 311 614	$\overline{164}$·70527 48044
377	163·72901 96775	535 820 935 [155]	(163) 186 629 513	$\overline{164}$·27098 03225
378	164·16331 41594	145 651 231 [156]	(164) 686 571 609	$\overline{165}$·83668 58406
379	164·59760 86413	395 921 094 [156]	(164) 252 575 580	$\overline{165}$·40239 13587
380	165·03190 31232	107 622 512 [157]	(165) 929 173 632	$\overline{166}$·96809 68768
381	165·46619 76051	292 548 318 [157]	(165) 341 823 876	$\overline{166}$·53380 23949
382	165·90049 20870	795 228 776 [157]	(165) 125 749 977	$\overline{166}$·09950 79130
383	166·33478 65689	216 165 593 [158]	(166) 462 608 311	$\overline{167}$·66521 34311
384	166·76908 10508	587 599 004 [158]	(166) 170 184 087	$\overline{167}$·23091 89492
385	167·20337 55328	159 725 970 [159]	(167) 626 072 268	$\overline{168}$·79662 44672
386	167·63767 00147	434 180 200 [159]	(167) 230 319 116	$\overline{168}$·36232 99853
387	168·07196 44966	118 022 415 [160]	(168) 847 296 678	$\overline{169}$·92803 55034
388	168·50625 89785	320 818 186 [160]	(168) 311 703 028	$\overline{169}$·49374 10215
389	168·94055 34604	872 074 245 [160]	(168) 114 669 136	$\overline{169}$·05944 65396
390	169·37484 79423	237 054 357 [161]	(169) 421 844 176	$\overline{170}$·62515 20577
391	169·80914 24242	644 380 552 [161]	(169) 155 187 800	$\overline{170}$·19085 75758
392	170·24343 69061	175 160 794 [162]	(170) 570 904 011	$\overline{171}$·75656 30939
393	170·67773 13880	476 136 404 [162]	(170) 210 023 848	$\overline{171}$·32226 86120
394	171·11202 58699	129 427 294 [163]	(171) 772 634 560	$\overline{172}$·88797 41301
395	171·54632 03518	351 819 860 [163]	(171) 284 236 370	$\overline{172}$·45367 96482
396	171·98061 48337	956 345 533 [163]	(171) 104 564 717	$\overline{172}$·01938 51663
397	172·41490 93156	259 961 668 [164]	(172) 384 672 096	$\overline{173}$·58509 06844
398	172·84920 37975	706 649 079 [164]	(172) 141 512 956	$\overline{173}$·15079 62025
399	173·28349 82794	192 087 135 [165]	(173) 520 597 071	$\overline{174}$·71650 17206
400	173·71779 27613	522 146 969 [165]	(173) 191 516 960	$\overline{174}$·28220 72387

The numbers in square brackets denote the numbers of figures between the last figure given and the decimal point; and the numbers in parentheses denote the numbers of ciphers between the decimal point and the first significant figure.

TABLE IV. (*continued*).

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from 1 to 500 at intervals of unity.

x	$\log_{10}(e^x)$	e^x	e^{-x}	$\log_{10}(e^{-x})$
401	174·15208 72432	141 934 262 [166]	(174) 704 551 521	$\overline{175}$·84791 27568
402	174·58638 17251	385 817 325 [166]	(174) 259 190 020	$\overline{175}$·41361 82749
403	175·02067 62070	104 876 022 [167]	(175) 953 506 796	$\overline{176}$·97932 37930
404	175·45497 06889	285 082 586 [167]	(175) 350 775 547	$\overline{176}$·54502 93111
405	175·88926 51708	774 934 812 [167]	(175) 129 043 112	$\overline{176}$·11073 48292
406	176·32355 96527	210 649 122 [168]	(176) 474 723 081	$\overline{177}$·67644 03473
407	176·75785 41346	572 603 680 [168]	(176) 174 640 862	$\overline{177}$·24214 58654
408	177·19214 86165	155 649 818 [169]	(177) 642 467 826	$\overline{178}$·80785 13835
409	177·62644 30984	423 100 071 [169]	(177) 236 350 705	$\overline{178}$·37355 69016
410	178·06073 75803	115 010 524 [170]	(178) 869 485 652	$\overline{179}$·93926 24197
411	178·49503 20622	312 631 016 [170]	(178) 319 865 896	$\overline{179}$·50496 79378
412	178·92932 65441	849 819 210 [170]	(178) 117 672 087	$\overline{179}$·07067 34559
413	179·36362 10260	231 004 812 [171]	(179) 432 891 416	$\overline{180}$·63637 89740
414	179·79791 55079	627 936 182 [171]	(179) 159 251 852	$\overline{180}$·20208 44921
415	180·23220 99898	170 690 751 [172]	(180) 585 854 824	$\overline{181}$·76779 00102
416	180·66650 44718	463 985 567 [172]	(180) 215 523 945	$\overline{181}$·33349 55282
417	181·10079 89537	126 124 354 [173]	(181) 792 868 285	$\overline{182}$·89920 10463
418	181·53509 34356	342 841 539 [173]	(181) 291 679 942	$\overline{182}$·46490 65644
419	181·96938 79175	931 939 925 [173]	(181) 107 303 054	$\overline{182}$·03061 20825
420	182·40368 23994	253 327 536 [174]	(182) 394 745 875	$\overline{183}$·59631 76006
421	182·83797 68813	688 615 639 [174]	(182) 145 218 892	$\overline{183}$·16202 31187
422	183·27227 13632	187 185 138 [175]	(183) 534 230 448	$\overline{184}$·72772 86368
423	183·70656 58451	508 821 958 [175]	(183) 196 532 399	$\overline{184}$·29343 41549
424	184·14086 03270	138 312 148 [176]	(184) 723 002 290	$\overline{185}$·85913 96730
425	184·57515 48089	375 971 399 [176]	(184) 265 977 679	$\overline{185}$·42484 51911
426	185·00944 92908	102 199 622 [177]	(185) 978 477 197	$\overline{186}$·99055 07092
427	185·44374 37727	277 807 376 [177]	(185) 359 961 645	$\overline{186}$·55625 62273
428	185·87803 82546	755 158 743 [177]	(185) 132 422 489	$\overline{186}$·12196 17454
429	186·31233 27365	205 273 429 [178]	(186) 487 155 111	$\overline{187}$·68766 72635
430	186·74662 72184	557 991 031 [178]	(186) 179 214 350	$\overline{187}$·25337 27816
431	187·18092 17003	151 677 688 [179]	(187) 659 292 750	$\overline{188}$·81907 82997
432	187·61521 61822	412 302 703 [179]	(187) 242 540 248	$\overline{188}$·38478 38178
433	188·04951 06641	112 075 495 [180]	(188) 892 255 710	$\overline{189}$·95048 93359
434	188·48380 51460	304 652 780 [180]	(188) 328 242 532	$\overline{189}$·51619 48540
435	188·91809 96279	828 132 117 [180]	(188) 120 753 679	$\overline{189}$·08190 03721
436	189·35239 41098	225 109 648 [181]	(189) 444 227 960	$\overline{190}$·64760 58902
437	189·78668 85917	611 911 467 [181]	(189) 163 422 334	$\overline{190}$·21331 14083
438	190·22098 30736	166 334 782 [182]	(190) 601 197 168	$\overline{191}$·77901 69264
439	190·65527 75555	452 144 816 [182]	(190) 221 168 078	$\overline{191}$·34472 24445
440	191·08957 20374	122 905 704 [183]	(191) 813 631 891	$\overline{192}$·91042 79626
441	191·52386 65193	334 092 341 [183]	(191) 299 318 445	$\overline{192}$·47613 34807
442	191·95816 10012	908 157 139 [183]	(191) 110 113 102	$\overline{192}$·04183 89988
443	192·39245 54831	246 862 705 [184]	(192) 405 083 466	$\overline{193}$·60754 45169
444	192·82674 99650	671 042 405 [184]	(192) 149 021 879	$\overline{193}$·17325 00350
445	193·26104 44469	182 408 237 [185]	(193) 548 220 856	$\overline{194}$·73895 55531
446	193·69533 89289	495 836 997 [185]	(193) 201 679 182	$\overline{194}$·30466 10711
447	194·12963 34108	134 782 470 [186]	(194) 741 936 248	$\overline{195}$·87036 65892
448	194·56392 78927	366 376 739 [186]	(194) 272 943 092	$\overline{195}$·43607 21073
449	194·99822 23746	995 915 232 [186]	(194) 100 411 052	$\overline{195}$·00177 76254
450	195·43251 68565	270 717 828 [187]	(195) 369 388 307	$\overline{196}$·56748 31435

The numbers in square brackets denote the numbers of figures between the last figure given and the decimal point; and the numbers in parentheses denote the numbers of ciphers between the decimal point and the first significant figure.

TABLE IV. (*continued*).

Values of e^x, e^{-x}, $\log_{10}(e^x)$, $\log_{10}(e^{-x})$ from 1 to 500 at intervals of unity.

x	$\log_{10}(e^x)$	e^x	e^{-x}	$\log_{10}(e^{-x})$
451	195·86681 13384	735 887 352 [187]	(195) 135 890 364	$\overline{196}$·13318 86616
452	196·30110 58203	200 034 922 [188]	(196) 499 912 711	$\overline{197}$·69889 41797
453	196·73540 03022	543 751 292 [188]	(196) 183 907 609	$\overline{197}$·26459 96978
454	197·16969 47841	147 806 926 [189]	(197) 676 558 284	$\overline{198}$·83030 52159
455	197·60398 92660	401 780 880 [189]	(197) 248 891 883	$\overline{198}$·39601 07340
456	198·03828 37479	109 215 367 [190]	(198) 915 622 070	$\overline{199}$·96171 62521
457	198·47257 82298	296 878 146 [190]	(198) 336 838 535	$\overline{199}$·52742 17702
458	198·90687 27117	806 998 471 [190]	(198) 123 915 972	$\overline{199}$·09312 72883
459	199·34116 71936	219 364 928 [191]	(199) 455 861 386	$\overline{200}$·65883 28064
460	199·77546 16755	596 295 697 [191]	(199) 167 702 032	$\overline{200}$·22453 83245
461	200·20975 61574	162 089 976 [192]	(200) 616 941 298	$\overline{201}$·79024 38426
462	200·64405 06393	440 606 236 [192]	(200) 226 960 020	$\overline{201}$·35594 93607
463	201·07834 51212	119 769 192 [193]	(201) 834 939 253	$\overline{202}$·92165 48788
464	201·51263 96031	325 566 419 [193]	(201) 307 156 986	$\overline{202}$·48736 03969
465	201·94693 40850	884 981 282 [193]	(201) 112 996 740	$\overline{202}$·05306 59150
466	202·38122 85669	240 562 854 [194]	(202) 415 691 777	$\overline{203}$·61877 14331
467	202·81552 30488	653 917 634 [194]	(202) 152 924 459	$\overline{203}$·18447 69512
468	203·24981 75307	177 753 242 [195]	(203) 562 577 643	$\overline{204}$·75018 24693
469	203·68411 20126	483 183 408 [195]	(203) 206 960 749	$\overline{204}$·31588 79874
470	204·11840 64945	131 342 868 [196]	(204) 761 366 047	$\overline{205}$·88159 35055
471	204·55270 09764	357 026 931 [196]	(204) 280 090 916	$\overline{205}$·44729 90236
472	204·98699 54583	970 499 818 [196]	(204) 103 039 690	$\overline{205}$·01300 45417
473	205·42128 99402	263 809 202 [197]	(205) 379 061 834	$\overline{206}$·57871 00598
474	205·85558 44221	717 107 760 [197]	(205) 139 449 056	$\overline{206}$·14441 55779
475	206·28987 89040	194 930 099 [198]	(206) 513 004 407	$\overline{207}$·71012 10960
476	206·72417 33859	529 874 947 [198]	(206) 188 723 775	$\overline{207}$·27582 66141
477	207·15846 78679	144 034 944 [199]	(207) 694 275 967	$\overline{208}$·84153 21321
478	207·59276 23498	391 527 571 [199]	(207) 255 409 855	$\overline{208}$·40723 76502
479	208·02705 68317	106 428 228 [200]	(208) 939 600 347	$\overline{209}$·97294 31683
480	208·46135 13136	289 301 919 [200]	(208) 345 659 650	$\overline{209}$·53864 86864
481	208·89564 57955	786 404 148 [200]	(208) 127 161 079	$\overline{209}$·10435 42045
482	209·32994 02774	213 766 811 [201]	(209) 467 799 467	$\overline{210}$·67005 97226
483	209·76423 47593	581 078 436 [201]	(209) 172 093 807	$\overline{210}$·23576 52407
484	210·19852 92412	157 953 496 [202]	(210) 633 097 734	$\overline{211}$·80147 07588
485	210·63282 37231	429 362 117 [202]	(210) 232 903 640	$\overline{211}$·36717 62769
486	211·06711 82050	116 712 724 [203]	(211) 856 804 611	$\overline{212}$·93288 17950
487	211·50141 26869	317 258 077 [203]	(211) 315 200 801	$\overline{212}$·49858 73131
488	211·93570 71688	862 396 865 [203]	(211) 115 955 895	$\overline{212}$·06429 28312
489	212·37000 16507	234 423 773 [204]	(212) 426 577 897	$\overline{213}$·62999 83493
490	212·80429 61326	637 229 881 [204]	(212) 156 929 239	$\overline{213}$·19570 38674
491	213·23859 06145	173 217 041 [205]	(213) 577 310 406	$\overline{214}$·76140 93855
492	213·67288 50964	470 852 734 [205]	(213) 212 380 629	$\overline{214}$·32711 49036
493	214·10717 95783	127 991 043 [206]	(214) 781 304 673	$\overline{215}$·89282 04217
494	214·54147 40602	347 915 727 [206]	(214) 287 425 926	$\overline{215}$·45852 59398
495	214·97576 85421	945 732 997 [206]	(214) 105 738 089	$\overline{215}$·02423 14579
496	215·41006 30240	257 076 882 [207]	(215) 388 988 692	$\overline{216}$·58993 69760
497	215·84435 75059	698 807 417 [207]	(215) 143 100 942	$\overline{216}$·15564 24941
498	216·27865 19878	189 955 550 [208]	(216) 526 438 947	$\overline{217}$·72134 80122
499	216·71294 64697	516 352 721 [208]	(216) 193 666 066	$\overline{217}$·28705 35303
500	217·14724 09516	140 359 222 [209]	(217) 712 457 641	$\overline{218}$·85275 90484

The numbers in square brackets denote the numbers of figures between the last figure given and the decimal point; and the numbers in parentheses denote the numbers of ciphers between the decimal point and the first significant figure.

POSTSCRIPT. The statement on p. 244 that the tables in this paper were compared with Schulze's and Vega's tables, as far as the extent of the different tables permitted, may perhaps convey the impression that no errors were detected by the comparison. It seems therefore desirable to state that no errors were found in Schulze's table, but that the following errors (in which the discrepancy amounts to at least 3 in the last figure) were found in Vega's *Tabulæ logarithmico-trigonometricæ:*

$x = 0{\cdot}46$,	$\log_{10} e^x$	is given as	0·1991755	instead of	0·1997755
„ 1·27,	e^x	„	3·560860	„	3·560853
„ 1·46,	„	„	4·305950	„	4·305960
„ 1·71,	„	„	5·528964	„	5·528961
„ 2·30,	„	„	9·974185	„	9·974182
„ 5·30,	„	„	200·3371	„	200·3368

There were also a great many cases in which the discrepancy was a unit in the last figure, and several in which it amounted to 2.

VI. *On Functions of more than two variables analogous to Tesseral Harmonics.*

BY M. J. M. HILL, M.A.

[Read *January* 29, 1883.]

THE objects of this paper are

(I) To develope a series of functions *, here called Normal Functions, of $(i-1)$ variables analogous to Tesseral Harmonics, and

(II) To show how to expand any function of $(i-1)$ variables in terms of them, the values of the variables being restricted within certain limits.

The result marked (A) corresponds to the conjugate property of Tesseral Harmonics, the expansion (B) to the expansion of an arbitrary function of two variables in a series of Laplace's Functions, and the expansion (C) to the expansion of an arbitrary function of two variables in a series of Tesseral Harmonics.

The following is an abstract.

Part I. 1. By means of the formulae of transformation

$$
\begin{aligned}
x_1 &= r \sin\theta_1 \sin\theta_2 \ldots\ldots\ldots\ldots \sin\theta_{i-2} \sin\theta_{i-1} \\
x_2 &= r \sin\theta_1 \sin\theta_2 \ldots\ldots\ldots\ldots \sin\theta_{i-2} \cos\theta_{i-1} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_m &= r \sin\theta_1 \sin\theta_2 \ldots\ldots\ldots\ldots \sin\theta_{i-m} \cos\theta_{i-m+1} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_{i-2} &= r \sin\theta_1 \sin\theta_2 \cos\theta_3 \\
x_{i-1} &= r \sin\theta_1 \cos\theta_2 \\
x_i &= r\cos\theta_1
\end{aligned}
$$

the equation $\frac{d^2u}{dx_1^2} + \frac{d^2u}{dx_2^2} + \ldots\ldots + \frac{d^2u}{dx_i^2} = 0$ is transformed into the equation

$$
\begin{aligned}
&\left(\frac{d^2u}{dr^2} + \frac{i-1}{r}\frac{du}{dr}\right) + \frac{1}{r^2}\left\{\frac{d^2u}{d\theta_1^2} + (i-2)\cot\theta_1 \frac{du}{d\theta_1}\right\} + \frac{1}{r^2\sin^2\theta_1}\left\{\frac{d^2u}{d\theta_2^2} + (i-3)\cot\theta_2\frac{du}{d\theta_2}\right\} + \ldots\ldots \\
&\quad + \frac{1}{r^2\sin^2\theta_1\sin^2\theta_2\ldots\ldots\sin^2\theta_{m-1}}\left\{\frac{d^2u}{d\theta_m^2} + (i-m-1)\cot\theta_m\frac{du}{d\theta_m}\right\} + \ldots\ldots \\
&\quad + \frac{1}{r^2\sin^2\theta_1\sin^2\theta_2\ldots\ldots\sin^2\theta_{i-3}}\left(\frac{d^2u}{d\theta_{i-2}^2} + \cot\theta_{i-2}\frac{du}{d\theta_{i-2}}\right) + \frac{1}{r^2\sin^2\theta_1\sin^2\theta_2\ldots\ldots\sin^2\theta_{i-2}}\left(\frac{d^2u}{d\theta_{i-1}^2}\right) = 0.
\end{aligned}
$$

* These functions were obtained by Green in his paper "On the determination of the attractions of ellipsoids of variable densities." He states in the preface that it would be easy to show that the functions of Laplace are particular cases of them. The present communication proves this statement. Articles 1 and 2 of Part I. of this paper correspond with the commencement of Article 14 of Green's paper. They have been inserted in order to make the paper complete in itself.

2. A solution $u = r^p (\sin\theta_1)^{p_1}\Theta_1 \ldots\ldots (\sin\theta_m)^{p_m}\Theta_m \ldots\ldots (\sin\theta_{i-2})^{p_{i-2}}\Theta_{i-2}\binom{\cos}{\sin}p_{i-2}\theta_{i-1}$ is found, where Θ_m is a function of θ_m only, satisfying the equation

$$\sin^2\theta_m \frac{d^2\Theta_m}{d\theta_m^2} + (2p_m + i - m - 1)\sin\theta_m \cos\theta_m \frac{d\Theta_m}{d\theta_m} + (p_{m-1} - p_m)(p_{m-1} + p_m + i - m - 1)\sin^2\theta_m \Theta_m = 0.$$

The indices $p, p_1, p_2, \ldots\ldots p_{i-2}$ are all integers such that no one is greater than any one preceding it in the series.

The function $(\sin\theta_1)^{p_1}\Theta_1 \ldots\ldots (\sin\theta_m)^{p_m}\Theta_m \ldots\ldots (\sin\theta_{i-2})^{p_{i-2}}\Theta_{i-2}\binom{\cos}{\sin}p_{i-2}\theta_{i-1}$ will be referred to in this paper as a Normal Function of the equation $\frac{d^2u}{dx_1^2} + \frac{d^2u}{dx_2^2} + \ldots\ldots + \frac{d^2u}{dx_i^2} = 0$.

3. Putting $\cos\theta_m = \mu$, $i - m - 1 = n$, and denoting the expansion of $(1 - 2\mu h + h^2)^{-\frac{n}{2}}$ in powers of h by

$$Q_0 + Q_1 h + \ldots\ldots + Q_{p_{m-1}} h^{p_{m-1}} + \ldots\ldots,$$

it is shown that $\left(\frac{d}{d\mu}\right)^{p_m} Q_{p_{m-1}}$ is a possible value of Θ_m.

4. If n be an even integer it is shown that

$$Q_{p_{m-1}} = \frac{1}{(n-2)(n-4)\ldots\ldots 4 \,.\, 2}\left(\frac{d}{d\mu}\right)^{\frac{n}{2}} \frac{\cos\left\{\left(p_{m-1} + \frac{n}{2}\right)\theta_m\right\}}{p_{m-1} + \frac{n}{2}}.$$

If n be an odd integer, and P_r the r^{th} Legendre's coefficient,

$$Q_{p_{m-1}} = \frac{1}{(n-2)(n-4)\ldots\ldots 3 \,.\, 1}\left(\frac{d}{d\mu}\right)^{\frac{n-1}{2}} P_{\left(p_{m-1} + \frac{n-1}{2}\right)}.$$

5. If $\Theta_m = \Theta_m'$ when $p_{m-1} = p'_{m-1}$, but $= \Theta_m''$ when $p_{m-1} = p''_{m-1}$; and if Θ_m', $\frac{d\Theta_m'}{d\theta_m}$, Θ_m'', $\frac{d\Theta_m''}{d\theta_m}$ are all finite when $\theta_m = 0$ or π; then

$$\int_0^\pi (\sin\theta_m)^{2p_m + n}\Theta_m'\Theta_m'' d\theta_m = 0.$$

6. It is shown that

$$\int_0^\pi (\sin\theta_m)^{2p_m+n}(\Theta_m)^2 d\theta_m = \frac{\lfloor n + p_{m-1} + p_m - 1}{\lfloor p_{m-1} - p_m} \cdot \frac{\lfloor p_{m-1}}{\lfloor n + p_{m-1} - 1} \cdot \int_0^\pi (\sin\theta_m)^n (Q_{p_{m-1}})^2 d\theta_m.$$

7. It is shown that

$$\int_0^\pi (\sin\theta_m)^{2p_m+n}(\Theta_m)^2\,d\theta_m = \frac{\lfloor n+p_{m-1}+p_m-1}{\lfloor p_{m-1}-p_m} \cdot \frac{1}{p_{m-1}+\frac{n}{2}} \cdot \frac{\pi}{2^{n-1}} \cdot \frac{1}{\{\Gamma(\frac{1}{2}n)\}^2}$$

where Γ is the symbol of the second Eulerian integral.

8. If $u_1 = (\sin\theta_1)^{p_1}\Theta_1 \ldots\ldots (\sin\theta_m)^{p_m}\Theta_m \ldots\ldots (\sin\theta_{i-2})^{p_{i-2}}\Theta_{i-2}\binom{\cos}{\sin}p_{i-2}\,\theta_{i-1}$;

$u_1' = (\sin\theta_1)^{p'_1}\Theta_1' \ldots\ldots (\sin\theta_m)^{p'_m}\Theta'_m \ldots\ldots (\sin\theta_{i-2})^{p'_{i-2}}\Theta'_{i-2}\binom{\cos}{\sin}p'_{i-2}\theta_{i-1}$;

and $dS = (\sin\theta_1)^{i-2}(\sin\theta_2)^{i-3} \ldots\ldots (\sin\theta_{i-2})\,d\theta_1 d\theta_2 \ldots\ldots d\theta_{i-2}d\theta_{i-1}$,

then

$$\int u_1 u_1'\,dS = 0 \ldots\ldots\ldots\ldots (A),$$

unless $u_1 = u_1'$ identically, the limits of integration for θ_{i-1} being 0 and 2π, but 0 and π for the other variables.

9. It is shown that

$$\int u_1^2 dS = \frac{\lfloor p_{i-3}+p_{i-2}}{\lfloor p_{i-3}-p_{i-2}} \cdot \frac{\lfloor p_{i-4}+p_{i-3}+1}{\lfloor p_{i-4}-p_{i-3}} \ldots\ldots \frac{\lfloor p_1+p_2+i-4}{\lfloor p_1-p_2} \cdot \frac{\lfloor p+p_1+i-3}{\lfloor p-p_1} \text{ multiplied by}$$

$$\frac{1}{p_{i-3}+\frac{1}{2}} \ldots\ldots \frac{1}{p_1+\frac{i-3}{2}} \cdot \frac{1}{p+\frac{i-2}{2}} \cdot \frac{\pi^{\frac{1}{2}i}}{2^{i-3}\,\Gamma(\frac{1}{2}i-1)\,.\,\lfloor i-4\,.\,\lfloor i-5 \ldots\ldots \lfloor 2\,\lfloor 1}$$

unless p_{i-2} vanish, when this value must be doubled.

10. It is shown that

$$r^p(\sin\theta_1)^{p_1}\Theta_1 \ldots\ldots (\sin\theta_m)^{p_m}\Theta_m \ldots\ldots (\sin\theta_{i-2})^{p_{i-2}}\Theta_{i-2}\binom{\cos}{\sin}p_{i-2}\,\theta_{i-1}$$

is a rational integral homogeneous function of the variables $x_1 x_2 \ldots\ldots x_i$ of degree p.

11. It is shown that the different functions of degree p of the form in the preceding article are all the independent rational integral homogeneous functions of $x_1 x_2 \ldots\ldots x_i$ of the same degree which satisfy the equation

$$\frac{d^2u}{dx_1^2} + \frac{d^2u}{dx_2^2} + \ldots\ldots + \frac{d^2u}{dx_i^2} = 0.$$

PART II. 1. Preliminary propositions.

(α) If R_p be the coefficient of h^p in the expansion of $(1-2h\cos\theta+h^2)^{-\frac{i}{2}+1}$, then R_p is greatest when $\cos\theta = 1$, i being a positive integer not less than 3.

(β) If $h < 1$, and i a positive integer not less than 2,

$$\int_0^{\pi} \frac{(1-h^2)(\sin\theta)^{i-2} d\theta}{(1-2h\cos\theta+h^2)^{\frac{i}{2}}} = \int_0^{\pi} (\sin\theta)^{i-2} d\theta.$$

2. In general any arbitrary function of the variables $\theta_1', \theta_2' \ldots \theta'_{i-1}$,

$$F(\theta_1'\theta_2' \ldots \theta'_{i-1}) = \frac{\Gamma(\frac{1}{2}i)}{2(i-2)\pi^{\frac{1}{2}i}} \int_0^{\pi} \ldots \int_0^{\pi} \int_0^{2\pi} F(\theta_1\theta_2 \ldots \theta_{i-1}) R d\theta_1 d\theta_2 \ldots d\theta_{i-3} d\theta_{i-2} d\theta_{i-1} \ldots \text{(B)},$$

where
$$R = \left\{\sum_{p=0}^{p=\infty} (2p+i-2) R_p\right\} (\sin\theta_1)^{i-2} (\sin\theta_2)^{i-3} \ldots\ldots (\sin\theta_{i-3})^2 (\sin\theta_{i-2}),$$

and where R_p is the coefficient of h^p in the expansion of $(1-2h\cos\theta+h^2)^{-\frac{i}{2}+1}$, and

$$\cos\theta = \frac{x_1x_1' + \ldots + x_ix_i'}{rr'}.$$

The relation between the variables $x_1'x_2' \ldots x_i'$ and $r'\theta_1' \ldots \theta'_{i-1}$ is the same as that between $x_1x_2 \ldots x_i$ and $r\theta_1 \ldots \theta_{i-1}$. θ'_{i-1} lies between 0 and 2π; $\theta_1' \ldots \theta'_{i-2}$ between 0 and π.

3. In general any arbitrary function of the variables $\theta_1'\theta_2' \ldots \theta'_{i-1}$

$$F(\theta_1'\theta_2' \ldots \theta'_{i-1})$$
$$= \Sigma (\sin\theta_1')^{p_1}\Theta_1' \ldots (\sin\theta_m')^{p_m}\Theta_m' \ldots (\sin\theta'_{i-2})^{p_{i-2}}\Theta'_{i-2} \{C\cos p_{i-2}\theta'_{i-1} + D\sin p_{i-2}\theta'_{i-1}\},$$

where $C =$

$$\frac{\int_0^{\pi}\ldots\int_0^{\pi}\int_0^{2\pi} (\sin\theta_1)^{p_1+i-2}\Theta_1 \ldots (\sin\theta_m)^{p_m+i-m-1}\Theta_m \ldots (\sin\theta_{i-2})^{p_{i-2}+1}\Theta_{i-2}\cos p_{i-2}\theta_{i-1} F(\theta_1\theta_2\ldots\theta_{i-1}) d\theta_1 d\theta_2 \ldots d\theta_{i-1}}{\int_0^{\pi}\ldots\int_0^{\pi}\int_0^{2\pi} (\sin\theta_1)^{2p_1+i-2}\Theta_1^2 \ldots (\sin\theta_m)^{2p_m+i-m-1}\Theta_m^2 \ldots (\sin\theta_{i-2})^{2p_{i-2}+1}\Theta^2_{i-2} (\cos p_{i-2}\theta_{i-1})^2 d\theta_1 d\theta_2 \ldots d\theta_{i-1}}$$

and $D =$

$$\frac{\int_0^{\pi}\ldots\int_0^{\pi}\int_0^{2\pi} (\sin\theta_1)^{p_1+i-2}\Theta_1 \ldots (\sin\theta_m)^{p_m+i-m-1}\Theta_m \ldots (\sin\theta_{i-2})^{p_{i-2}+1}\Theta_{i-2} (\sin p_{i-2}\theta_{i-1}) F(\theta_1\theta_2\ldots\theta_{i-1}) d\theta_1 d\theta_2 \ldots d\theta_{i-1}}{\int_0^{\pi}\ldots\int_0^{\pi}\int_0^{2\pi} (\sin\theta_1)^{2p_1+i-2}\Theta_1^2 \ldots (\sin\theta_m)^{2p_m+i-m-1}\Theta_m^2 \ldots (\sin\theta_{i-2})^{2p_{i-2}+1}\Theta^2_{i-2} (\sin p_{i-2}\theta_{i-1})^2 d\theta_1 d\theta_2 \ldots d\theta_{i-1}}$$

C.

The summation extends to all possible positive integral values of the indices $p_1 \ldots p_{i-2}$, such that in any one term no index is greater than the index p with which that term is connected nor any index greater than any one which precedes it. The limits between which $\theta_1'\theta_2' \ldots \theta'_{i-1}$ lie are the same as in the last article.

PART I.

1. The formulae of transformation for $x_1 x_2 \dots x_i$ given in the abstract are equivalent to the following:—

$$r = (x_1^2 + x_2^2 + \dots + x_i^2)^{\frac{1}{2}},$$

$$\theta_1 = \tan^{-1}\left\{\frac{(x_1^2 + x_2^2 + \dots + x_{i-1}^2)^{\frac{1}{2}}}{x_i}\right\},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\theta_m = \tan^{-1}\left\{\frac{(x_1^2 + x_2^2 + \dots + x_{i-m}^2)^{\frac{1}{2}}}{x_{i-m+1}}\right\},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\theta_{i-2} = \tan^{-1}\left\{\frac{(x_1^2 + x_2^2)^{\frac{1}{2}}}{x_3}\right\},$$

$$\theta_{i-1} = \tan^{-1}\left\{\frac{x_1}{x_2}\right\}.$$

But
$$\frac{du}{dx_1} = \frac{du}{dr}\cdot\frac{dr}{dx_1} + \frac{du}{d\theta_1}\,\frac{d\theta_1}{dx_1} + \dots + \frac{du}{d\theta_{i-1}}\,\frac{d\theta_{i-1}}{dx_1},$$

and
$$\frac{d^2u}{dx_1^2} = \frac{dr}{dx_1}\left(\frac{d^2u}{dr^2}\,\frac{dr}{dx_1} + \frac{d^2u}{drd\theta_1}\,\frac{d\theta_1}{dx_1} + \dots + \frac{d^2u}{drd\theta_{i-1}}\,\frac{d\theta_{i-1}}{dx_1}\right) + \frac{du}{dr}\cdot\frac{d^2r}{dx_1^2}$$
$$+ \frac{d\theta_1}{dx_1}\left(\frac{d^2u}{drd\theta_1}\,\frac{dr}{dx_1} + \frac{d^2u}{d\theta_1^2}\,\frac{d\theta_1}{dx_1} + \dots + \frac{d^2u}{d\theta_1 d\theta_{i-1}}\,\frac{d\theta_{i-1}}{dx_1}\right) + \frac{du}{d\theta_1}\cdot\frac{d^2\theta_1}{dx_1^2}$$
$$+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$+ \frac{d\theta_{i-1}}{dx_1}\left(\frac{d^2u}{drd\theta_{i-1}}\,\frac{dr}{dx_1} + \frac{d^2u}{d\theta_{i-1}d\theta_1}\,\frac{d\theta_1}{dx_1} + \dots + \frac{d^2u}{d\theta_{i-1}^2}\,\frac{d\theta_{i-1}}{dx_1}\right) + \frac{du}{d\theta_{i-1}}\,\frac{d^2\theta_{i-1}}{dx_1^2}.$$

Therefore
$$\frac{d^2u}{dx_1^2} + \frac{d^2u}{dx_2^2} + \dots + \frac{d^2u}{dx_i^2}$$
$$= \frac{d^2u}{dr^2}\left\{\left(\frac{dr}{dx_1}\right)^2 + \dots + \left(\frac{dr}{dx_i}\right)^2\right\} + \frac{du}{dr}\left(\frac{d^2r}{dx_1^2} + \dots + \frac{d^2r}{dx_i^2}\right)$$
$$+ 2\,\frac{d^2u}{dr\,d\theta_1}\left(\frac{d\theta_1}{dx_1}\,\frac{dr}{dx_1} + \dots + \frac{d\theta_1}{dx_i}\,\frac{dr}{dx_i}\right) + \dots + 2\,\frac{d^2u}{drd\theta_{i-1}}\left(\frac{d\theta_{i-1}}{dx_1}\,\frac{dr}{dx_1} + \dots + \frac{d\theta_{i-1}}{dx_i}\,\frac{dr}{dx_i}\right)$$
$$+ \frac{d^2u}{d\theta_1^2}\left\{\left(\frac{d\theta_1}{dx_1}\right)^2 + \dots + \left(\frac{d\theta_1}{dx_i}\right)^2\right\} + \dots + \frac{d^2u}{d\theta_{i-1}^2}\left\{\left(\frac{d\theta_{i-1}}{dx_1}\right)^2 + \dots + \left(\frac{d\theta_{i-1}}{dx_i}\right)^2\right\}$$
$$+ \dots + 2\,\frac{d^2u}{d\theta_m d\theta_n}\left(\frac{d\theta_m}{dx_1}\,\frac{d\theta_n}{dx_1} + \dots + \frac{d\theta_m}{dx_i}\,\frac{d\theta_n}{dx_i}\right) + \dots$$
$$+ \frac{du}{d\theta_1}\left(\frac{d^2\theta_1}{dx_1^2} + \dots + \frac{d^2\theta_1}{dx_i^2}\right) + \dots + \frac{du}{d\theta_{i-1}}\left(\frac{d^2\theta_{i-1}}{dx_1^2} + \dots + \frac{d^2\theta_{i-1}}{dx_i^2}\right).$$

And from the formulae of transformation it may be shown that

$$\left(\frac{dr}{dx_1}\right)^2+\ldots+\left(\frac{dr}{dx_i}\right)^2=1,$$

$$\frac{d^2r}{dx_1^2}+\ldots+\frac{d^2r}{dx_i^2}=\frac{i-1}{r},$$

$$\frac{d\theta_m}{dx_1}\frac{dr}{dx_1}+\ldots+\frac{d\theta_m}{dx_i}\frac{dr}{dx_i}=0,$$

$$\left(\frac{d\theta_m}{dx_1}\right)^2+\ldots+\left(\frac{d\theta_m}{dx_i}\right)^2=\frac{1}{r^2\sin^2\theta_1\sin^2\theta_2\ldots\sin^2\theta_{m-1}},$$

$$\frac{d\theta_m}{dx_1}\frac{d\theta_n}{dx_1}+\ldots+\frac{d\theta_m}{dx_i}\frac{d\theta_n}{dx_i}=0,$$

$$\frac{d^2\theta_m}{dx_1^2}+\ldots+\frac{d^2\theta_m}{dx_i^2}=\frac{(i-m-1)\cot\theta_m}{r^2\sin^2\theta_1\sin^2\theta_2\ldots\sin^2\theta_{m-1}}.$$

Therefore
$$\frac{d^2u}{dx_1^2}+\frac{d^2u}{dx_2^2}+\ldots+\frac{d^2u}{dx_i^2}$$

$$\left.\begin{aligned}&=\left(\frac{d^2u}{dr^2}+\frac{i-1}{r}\frac{du}{dr}\right)+\frac{1}{r^2}\left\{\frac{d^2u}{d\theta_1^2}+(i-2)\cot\theta_1\frac{du}{d\theta_1}\right\}\\&\qquad+\frac{1}{r^2\sin^2\theta_1}\left\{\frac{d^2u}{d\theta_2^2}+(i-3)\cot\theta_2\frac{du}{d\theta_2}\right\}+\ldots\\&+\frac{1}{r^2\sin^2\theta_1\ldots\sin^2\theta_{m-1}}\left\{\frac{d^2u}{d\theta_m^2}+(i-m-1)\cot\theta_m\frac{du}{d\theta_m}\right\}+\ldots\\&+\frac{1}{r^2\sin^2\theta_1\ldots\sin^2\theta_{i-3}}\left(\frac{d^2u}{d\theta_{i-2}^2}+\cot\theta_{i-2}\frac{du}{d\theta_{i-2}}\right)+\frac{1}{r^2\sin^2\theta_1\ldots\sin^2\theta_{i-2}}\left(\frac{d^2u}{d\theta_{i-1}^2}\right)=0\end{aligned}\right\}\ldots\ldots(\text{I}).$$

2. Put in the equation (I) $u=r^p.u_1$, where u_1 is a function of $\theta_1\theta_2\ldots\theta_{i-1}$ only. After dividing out by r^{p-2}; it becomes

$$\left.\begin{aligned}&p(p+i-2)u_1+\left\{\frac{d^2u_1}{d\theta_1^2}+(i-2)\cot\theta_1\frac{du_1}{d\theta_1}\right\}+\frac{1}{\sin^2\theta_1}\left\{\frac{d^2u_1}{d\theta_2^2}+(i-3)\cot\theta_2\frac{du_1}{d\theta_2}\right\}+\ldots\ldots\\&\qquad+\frac{1}{\sin^2\theta_1\ldots\sin^2\theta_{m-1}}\left\{\frac{d^2u_1}{d\theta_m^2}+(i-m-1)\cot\theta_m\frac{du_1}{d\theta_m}\right\}+\ldots\\&\qquad+\frac{1}{\sin^2\theta_1\ldots\sin^2\theta_{i-3}}\left(\frac{d^2u_1}{d\theta_{i-2}^2}+\cot\theta_{i-2}\frac{du_1}{d\theta_{i-2}}\right)+\frac{1}{\sin^2\theta_1\ldots\sin^2\theta_{i-2}}\left(\frac{d^2u_1}{d\theta_{i-1}^2}\right)=0\end{aligned}\right\}\ldots\ldots(\text{II}),$$

Now put in the equation (II) $u_1=(\sin\theta_1)^{p_1}.\Theta_1.u_2$, where Θ_1 is a function of θ_1 only, and u_2 of $\theta_2\theta_3\ldots\theta_{i-1}$ only. After dividing out by $(\sin\theta_1)^{p_1-2}\Theta_1$, it becomes

$$u_2\left\{\frac{(\sin\theta_1)^{p_1}\frac{d^2\Theta_1}{d\theta_1^2}+(2p_1+i-2)(\sin\theta_1)^{p_1-1}\cos\theta_1\frac{d\Theta_1}{d\theta_1}+(p-p_1)(p+p_1+i-2)(\sin\theta_1)^{p_1}\Theta_1}{(\sin\theta_1)^{p_1-2}\Theta_1}+p_1(p_1+i-3)\right\}$$

$$+\left\{\frac{d^2u_2}{d\theta_2^{\,2}}+(i-3)\cot\theta_2\frac{du_2}{d\theta_2}\right\}+\ldots+\frac{1}{\sin^2\theta_2\ldots\sin^2\theta_{m-1}}\left\{\frac{d^2u_2}{d\theta_m^{\,2}}+(i-m-1)\cot\theta_m\frac{du_2}{d\theta_m}\right\}+\ldots$$

$$+\frac{1}{\sin^2\theta_2\ldots\sin^2\theta_{i-3}}\left(\frac{d^2u_2}{d\theta_{i-2}^{\,2}}+\cot\theta_{i-2}\frac{du_2}{d\theta_{i-2}}\right)+\frac{1}{\sin^2\theta_2\ldots\sin^2\theta_{i-2}}\left(\frac{d^2u_2}{d\theta_{i-1}^{\,2}}\right)=0.$$

If Θ_1 be chosen so that the coefficient of u_2 is $p_1(p_1+i-3)$, then Θ_1 must satisfy the equation

$$\sin^2\theta_1\frac{d^2\Theta_1}{d\theta_1^{\,2}}+(2p_1+i-2)\sin\theta_1\cos\theta_1\frac{d\Theta_1}{d\theta_1}+(p-p_1)(p+p_1+i-2)\sin^2\theta_1\,.\,\Theta_1=0\ldots\ldots(1),$$

and u_2 satisfies

$$\left.\begin{aligned}p_1(p_1+i-3)u_2+\left\{\frac{d^2u_2}{d\theta_2^{\,2}}+(i-3)\cot\theta_2\frac{du_2}{d\theta_2}\right\}+\ldots+\frac{1}{\sin^2\theta_2\ldots\sin^2\theta_{m-1}}\left\{\frac{d^2u_2}{d\theta_m^{\,2}}+(i-m-1)\cot\theta_m\frac{du_2}{d\theta_m}\right\}+\ldots\\+\frac{1}{\sin^2\theta_2\ldots\sin^2\theta_{i-3}}\left(\frac{d^2u_2}{d\theta_{i-2}^{\,2}}+\cot\theta_{i-2}\frac{du_2}{d\theta_{i-2}}\right)+\frac{1}{\sin^2\theta_2\ldots\sin^2\theta_{i-2}}\left(\frac{d^2u_2}{d\theta_{i-1}^{\,2}}\right)=0\end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{III}).$$

The equation (III) is strictly similar to (II).

If therefore in it u_2 be put $=(\sin\theta_2)^{p_2}.\Theta_2.u_3$, where Θ_2 is a function of θ_2 only and u_3 is a function of $\theta_3\ldots\theta_{i-1}$ only, then proceeding as before but employing the quantity $p_2(p_2+i-4)$ instead of $p_1(p_1+i-3)$, Θ_2 and u_3 satisfy the equations

$$\sin^2\theta_2\frac{d^2\Theta_2}{d\theta_2^{\,2}}+(2p_2+i-3)\sin\theta_2\cos\theta_2\frac{d\Theta_2}{d\theta_2}+(p_1-p_2)(p_1+p_2+i-3)\sin^2\theta_2\,.\,\Theta_2=0\ldots(2),$$

$$\left.\begin{aligned}p_2(p_2+i-4)u_3+\left\{\frac{d^2u_3}{d\theta_3^{\,2}}+(i-4)\cot\theta_3\frac{du_3}{d\theta_3}\right\}+\ldots+\frac{1}{\sin^2\theta_3\ldots\sin^2\theta_{m-1}}\left\{\frac{d^2u_3}{d\theta_m^{\,2}}+(i-m-1)\cot\theta_m\frac{du_3}{d\theta_m}\right\}\\+\ldots\ldots+\frac{1}{\sin^2\theta_3\ldots\ldots\sin^2\theta_{i-3}}\left(\frac{d^2u_3}{d\theta_{i-2}^{\,2}}+\cot\theta_{i-2}\frac{du_3}{d\theta_{i-2}}\right)+\frac{1}{\sin^2\theta_3\ldots\ldots\sin^2\theta_{i-2}}\left(\frac{d^2u_3}{d\theta_{i-1}^{\,2}}\right)=0\end{aligned}\right\}\ldots(\text{IV}).$$

Proceeding in this way the solution of the transformed equation will be

$$u=r^p(\sin\theta_1)^{p_1}\Theta_1\ldots\ldots(\sin\theta_m)^{p_m}\Theta_m\ldots\ldots(\sin\theta_{i-2})^{p_{i-2}}\Theta_{i-2}\,.\,u_{i-1};$$

where the quantities $\Theta_3\ldots\Theta_{i-2}$ satisfy the equations (3) ... $(i-2)$, similar to the equations (1) and (2)

$$\sin^2\theta_3\frac{d^2\Theta_3}{d\theta_3^{\,2}}+(2p_3+i-4)\sin\theta_3\cos\theta_3\frac{d\Theta_3}{d\theta_3}+(p_2-p_3)(p_2+p_3+i-4)\sin^2\theta_3\,.\,\Theta_3=0\ldots\ldots\ldots\ldots(3).$$

....

$$\sin^2\theta_m\frac{d^2\Theta_m}{d\theta_m^{\,2}}+(2p_m+i-m-1)\sin\theta_m\cos\theta_m\frac{d\Theta_m}{d\theta_m}+(p_{m-1}-p_m)(p_{m-1}+p_m+i-m-1)\sin^2\theta_m.\Theta_m=0\ldots..(m).$$

....

$$\sin^2\theta_{i-2}\frac{d^2\Theta_{i-2}}{d\theta_{i-2}^{\,2}}+(2p_{i-2}+1)\sin\theta_{i-2}\cos\theta_{i-2}\frac{d\Theta_{i-2}}{d\theta_{i-2}}+(p_{i-3}-p_{i-2})(p_{i-3}+p_{i-2}+1)\sin^2\theta_{i-2}\,.\,\Theta_{i-2}=0\ldots(i-2).$$

The quantities similar to u_1, u_2, u_3, viz. $u_4 \ldots u_m \ldots u_{i-2}\, u_{i-1}$ will satisfy equations similar to the equations II., III., IV. the last two of which will be

$$p_{i-3}(p_{i-3}+1)\, u_{i-2} + \left(\frac{d^2 u_{i-2}}{d\theta_{i-2}^{\,2}} + \cot\theta_{i-2}\frac{du_{i-2}}{d\theta_{i-2}}\right) + \frac{1}{\sin^2\theta_{i-2}}\,\frac{d^2 u_{i-2}}{d\theta_{i-1}^{\,2}} = 0,$$

$$p^2_{i-2}\, u_{i-1} + \frac{d^2 u_{i-1}}{d\theta_{i-1}^{\,2}} = 0.$$

Hence $u = r^p (\sin\theta_1)^{p_1}\,\Theta_1 \ldots\ldots (\sin\theta_m)^{p_m}\,\Theta_m \ldots\ldots (\sin\theta_{i-2})^{p_{i-2}}\,\Theta_{i-2}\left(\begin{matrix}\cos\\ \sin\end{matrix}\, p_{i-2}\,\theta_{i-1}\right)$.

Any function of the form $(\sin\theta_1)^{p_1}\,\Theta_1 \ldots\ldots (\sin\theta_m)^{p_m}\,\Theta_m \ldots\ldots (\sin\theta_{i-2})^{p_{i-2}}\,\Theta_{i-2}\left(\begin{matrix}\cos\\ \sin\end{matrix}\, p_{i-2}\,\theta_{i-1}\right)$ will be referred to in this paper as a Normal Function of the equation

$$\frac{d^2u}{dx_1^{\,2}} + \frac{d^2u}{dx_2^{\,2}} + \ldots\ldots + \frac{d^2u}{dx_i^{\,2}} = 0,$$

or more briefly as a Normal Function.

3. Certain properties of the functions Θ will now be proved. The equations (1), (2) ...... $(i-2)$ which they satisfy are all of the same form, and may be typified by the equation (m).

Putting $\cos\theta_m = \mu$ and $i - m - 1 = n$ for brevity, this becomes

$$(1-\mu^2)\frac{d^2\Theta_m}{d\mu^2} - (2p_m + n + 1)\,\mu\,\frac{d\Theta_m}{d\mu} + (p_{m-1} - p_m)(p_{m-1} + p_m + n)\,\Theta_m = 0.$$

It is now possible to obtain further information of the nature of Θ_m.

It will be shown that if

$$\frac{1}{(1 - 2\mu h + h^2)^{\frac{n}{2}}} = Q_0 + Q_1 h + Q_2 h^2 + \ldots\ldots + Q_{p_{m-1}} h^{p_{m-1}} + \ldots\ldots$$

then $\left(\frac{d}{d\mu}\right)^{p_m} Q_{p_{m-1}}$ satisfies the equation in Θ_m.

i.e. it is required to show that

$$(1-\mu^2)\left(\frac{d}{d\mu}\right)^{p_m+2} Q_{p_{m-1}} - (2p_m + n + 1)\,\mu\left(\frac{d}{d\mu}\right)^{p_m+1} Q_{p_{m-1}} + (p_{m-1} - p_m)(p_{m-1} + p_m + n)\left(\frac{d}{d\mu}\right)^{p_m} Q_{p_{m-1}} = 0.$$

But this is $\left(\frac{d}{d\mu}\right)^{p_m}\left\{(1-\mu^2)\frac{d^2 Q_{p_{m-1}}}{d\mu^2} - (n+1)\,\mu\,\frac{dQ_{p_{m-1}}}{d\mu} + p_{m-1}(p_{m-1}+n)\,Q_{p_{m-1}}\right\} = 0.$

It will be sufficient therefore to show that the term in brackets vanishes. Multiplying it by $h^{p_{m-1}}$, it may be arranged thus:—

$$(1-\mu^2)\frac{d^2}{d\mu^2}(Q_{p_{m-1}} h^{p_{m-1}}) - (n+1)\,\mu\,\frac{d}{d\mu}(Q_{p_{m-1}} h^{p_{m-1}}) + h\frac{d}{dh}\left\{h\frac{d}{dh}(Q_{p_{m-1}} h^{p_{m-1}})\right\} + nh\frac{d}{dh}(Q_{p_{m-1}} h^{p_{m-1}}).$$

This will vanish if

$$\left\{(1-\mu^2)\frac{d^2}{d\mu^2}-(n+1)\mu\frac{d}{d\mu}+\left(h\frac{d}{dh}\left(h\frac{d}{dh}\right)\right)+nh\frac{d}{dh}\right\}\frac{1}{(1-2\mu h+h^2)^{\frac{n}{2}}}=0,$$

because the expression which must be demonstrated to vanish is the term containing $h^{p_{m-1}}$ on the left-hand side of the last equation. It is necessary therefore to show that

$$\left.\begin{aligned}&(1-\mu^2)\,n\,(n+2)\,h^2(1-2\mu h+h^2)^{-\frac{n}{2}-2}-n\,(n+1)\,h\mu\,(1-2\mu h+h^2)^{-\frac{n}{2}-1}\\&+\{n\,(1-2\mu h+h^2)^{-\frac{n}{2}-1}(h\mu-2h^2)+n\,(n+2)\,h^2(\mu-h)^2(1-2\mu h+h^2)^{-\frac{n}{2}-2}\}\\&+n^2h\,(\mu-h)\,(1-2\mu h+h^2)^{-\frac{n}{2}-1}\end{aligned}\right\}=0,$$

$$\text{i.e.}\quad \left.\begin{aligned}&(1-2\mu h+h^2)^{-\frac{n}{2}-2}\{n\,(n+2)\,(h^2-h^2\mu^2)+n\,(n+2)\,(h^2\mu^2-2h^3\mu+h^4)\}\\&+(1-2\mu h+h^2)^{-\frac{n}{2}-1}\{-n\,(n+1)\,h\mu+n\,(h\mu-2h^2)+n^2h\,(\mu-h)\}\end{aligned}\right\}=0,$$

$$\text{i.e.}\quad (1-2\mu h+h^2)^{-\frac{n}{2}-2}\,n\,(n+2)\,h^2(1-2\mu h+h^2)+(1-2\mu h+h^2)^{-\frac{n}{2}-1}(-2nh^2-n^2h^2)=0,$$

which is identically true.

4. The form of $Q_{p_{m-1}}$ must now be more closely examined.

Firstly, let n be an even integer.

Since $$\log(1-2\mu h+h^2)=-2\left(h\cos\theta_m+\frac{h^2}{2}\cos 2\theta_m+\frac{h^3}{3}\cos 3\theta_m+\ldots\ldots\right),$$

after differentiating both sides $\frac{n}{2}$ times with regard to μ, and comparing the coefficients with the coefficients in the equation

$$\frac{1}{(1-2\mu h+h^2)^{\frac{n}{2}}}=Q_0+Q_1h+\ldots\ldots+Q_{p_{m-1}}h^{p_{m-1}}+\ldots\ldots$$

it appears that

$$Q_{p_{m-1}}=\frac{1}{(n-2)\,(n-4)\ldots\ldots 4\,.\,2}\left(\frac{d}{d\mu}\right)^{\frac{n}{2}}\frac{\cos\left\{\left(p_{m-1}+\frac{n}{2}\right)\theta_m\right\}}{p_{m-1}+\frac{n}{2}}.$$

Secondly, let n be an odd integer.

Since $$\frac{1}{(1-2\mu h+h^2)^{\frac{1}{2}}}=P_0+P_1h+\ldots\ldots+P_rh^r+\ldots\ldots$$

where P_r is the r^{th} Legendre's coefficient, after differentiating both sides $\frac{n-1}{2}$ times with regard to μ, and dividing by $h^{\frac{n-1}{2}}$, this becomes

$$\{1 \cdot 3 \cdot 5 \ldots\ldots (n-2)\} \cdot \frac{1}{(1-2\mu h + h^2)^{\frac{n}{2}}} = \left(\frac{d}{d\mu}\right)^{\frac{n-1}{2}} \left\{P_{\frac{n-1}{2}} + \ldots\ldots + P_{\left(p_{m-1}+\frac{n-1}{2}\right)} h^{p_{m-1}} + \ldots\ldots\right\}.$$

Comparing with the series

$$\frac{1}{(1-2\mu h+h^2)^{\frac{n}{2}}} = Q_0 + Q_1 h + \ldots\ldots + Q_{p_{m-1}} h^{p_{m-1}} + \ldots\ldots$$

it appears that $$Q_{p_{m-1}} = \frac{1}{(n-2)(n-4)\ldots\ldots 5 \cdot 3 \cdot 1} \left(\frac{d}{d\mu}\right)^{\frac{n-1}{2}} P_{\left(p_{m-1}+\frac{n-1}{2}\right)}.$$

In both cases $Q_{p_{m-1}}$ is a rational integral function of μ of the degree p_{m-1} containing only even or only odd powers of μ; and therefore Θ_m is also a rational integral function of μ of the degree $p_{m-1} - p_m$. This is not the most general form of Θ_m, but it is the only one which will be considered in this paper.

5. Suppose now that if $p_{m-1} = p'_{m-1}$, $\Theta_m = \Theta_m'$ satisfies the equation (m) in article 2; but if $p_{m-1} = p''_{m-1}$, $\Theta_m = \Theta_m''$ satisfies it; it is required to determine the value of

$$\int_0^\pi (\sin\theta_m)^{2p_m+n} \Theta_m' \Theta_m'' \, d\theta_m.$$

Since the equations

$$\sin^2\theta_m \frac{d^2\Theta_m'}{d\theta_m^2} + (2p_m+n)\sin\theta_m\cos\theta_m \frac{d\Theta_m'}{d\theta_m} + (p'_{m-1}-p_m)(p'_{m-1}+p_m+n)\sin^2\theta_m \cdot \Theta_m' = 0,$$

$$\sin^2\theta_m \frac{d^2\Theta_m''}{d\theta_m^2} + (2p_m+n)\sin\theta_m\cos\theta_m \frac{d\Theta_m''}{d\theta_m} + (p''_{m-1}-p_m)(p''_{m-1}+p_m+n)\sin^2\theta_m \cdot \Theta_m'' = 0,$$

hold good; it appears that, multiplying the first of them by $(\sin\theta_m)^{2p_m+n-2}\Theta_m''$, and the second by $(\sin\theta_m)^{2p_m+n-2}\Theta_m'$ and subtracting, the result may be arranged as follows:

$$\frac{d}{d\theta_m}\left[(\sin\theta_m)^{2p_m+n}\left\{\Theta_m''\frac{d\Theta_m'}{d\theta_m} - \Theta_m'\frac{d\Theta_m''}{d\theta_m}\right\}\right] + (p'_{m-1}-p''_{m-1})(p'_{m-1}+p''_{m-1}+n)(\sin\theta_m)^{2p_m+n}\Theta_m'\Theta_m'' = 0.$$

Integrating with regard to θ_m between the limits 0 and π,

$$\left[(\sin\theta_m)^{2p_m+n}\left\{\Theta_m''\frac{d\Theta_m'}{d\theta_m} - \Theta_m'\frac{d\Theta_m''}{d\theta_m}\right\}\right]_0^\pi + (p'_{m-1}-p''_{m-1})(p'_{m-1}+p''_{m-1}+n)\int_0^\pi(\sin\theta_m)^{2p_m+n}\Theta_m'\Theta_m''\,d\theta_m = 0.$$

Now Θ_m', $\frac{d\Theta_m'}{d\theta_m}$, Θ_m'', $\frac{d\Theta_m''}{d\theta_m}$ are finite when $\theta_m = 0$ or π since $\Theta_m'\Theta_m''$ are rational integral functions of $\cos\theta_m$ by the last article; therefore the term in brackets is zero at both limits: $p'_{m-1} - p''_{m-1}$ is not zero, since p'_{m-1} and p''_{m-1} are different: $p'_{m-1} + p''_{m-1} + n$ is not zero, since $p'_{m-1}\, p''_{m-1}$ are positive integers or zero, and n is a positive integer the least value of which is 1.

Therefore the equation shows that $\int_0^\pi (\sin\theta_m)^{2p_m+n}\Theta_m'\Theta_m''\,d\theta_m = 0.$

6. The value of the integral $\int_0^\pi (\sin\theta_m)^{2p_m+n}\,\Theta_m^{\,2} d\theta_m$ can now be determined.

The working divides itself into two parts. The first part applies whether n be an even or odd integer; the second part breaks up into two cases, the first corresponding to n even, the second to n odd.

It has been shown that

$$\sin^2\theta_m \frac{d^2\Theta_m}{d\theta_m^{\,2}} + (2p_m+n)\sin\theta_m\cos\theta_m \frac{d\Theta_m}{d\theta_m} + (p_{m-1}-p_m)(p_{m-1}+p_m+n)\sin^2\theta_m \,.\, \Theta_m = 0.$$

Multiplying by $(\sin\theta_m)^{2p_m+n-2}\,\Theta_m$ and integrating between the limits 0 and π, the result may be arranged thus

$$(p_{m-1}-p_m)(p_{m-1}+p_m+n)\int_0^\pi (\sin\theta_m)^{2p_m+n}\Theta^2_m d\theta_m$$

$$= -\int_0^\pi \left\{(\sin\theta_m)^{2p_m+n}\,\Theta_m \frac{d^2\Theta_m}{d\theta^2_m} + (2p_m+n)(\sin\theta_m)^{2p_m+n-1}\cos\theta_m\,\Theta_m \frac{d\Theta_m}{d\theta_m}\right\} d\theta_m$$

$$= -\left[(\sin\theta_m)^{2p_m+n}\Theta_m \frac{d\Theta_m}{d\theta_m}\right]_0^\pi + \int_0^\pi (\sin\theta_m)^{2p_m+n}\left(\frac{d\Theta_m}{d\theta_m}\right)^2 d\theta_m;$$

$$\therefore\quad (p_{m-1}-p_m)(p_{m-1}+p_m+n)\int_0^\pi (\sin\theta_m)^{2p_m+n}\Theta^2_m d\theta_m = \int_0^\pi (\sin\theta_m)^{2p_m+n+2}\left(\frac{d\Theta_m}{-\sin\theta_m d\theta_m}\right)^2 d\theta_m.$$

But
$$\Theta_m = \left(\frac{d}{-\sin\theta_m d\theta_m}\right)^{p_m} Q_{p_{m-1}};$$

$$\therefore\quad (p_{m-1}-p_m)(p_{m-1}+p_m+n)\int_0^\pi (\sin\theta_m)^{2p_m+n}\left\{\left(\frac{d}{-\sin\theta_m d\theta_m}\right)^{p_m} Q_{p_{m-1}}\right\}^2 d\theta_m$$

$$= \int_0^\pi (\sin\theta_m)^{2p_m+n+2}\left\{\left(\frac{d}{-\sin\theta_m d\theta_m}\right)^{p_m+1} Q_{p_{m-1}}\right\}^2 d\theta_m.$$

Put
$$\Phi(p_m) = \int_0^\pi (\sin\theta_m)^{2p_m+n}\left\{\left(\frac{d}{-\sin\theta_m d\theta_m}\right)^{p_m} Q_{p_{m-1}}\right\}^2 d\theta_m.$$

Then the last equation may be written

$$(p_{m-1}-p_m)(p_{m-1}+p_m+n)\,\Phi(p_m) = \Phi(p_m+1).$$

This is an equation which is true for all positive integral values of p_m less than p_{m-1}.

Hence changing p_m successively into p_m-1, p_m-2, ... 2, 1 a series of equations is obtained, whence it can be shown that

$$\Phi(p_m) = \frac{|\underline{n+p_{m-1}+p_m-1}}{|\underline{n+p_{m-1}-1}} \cdot \frac{|\underline{p_{m-1}}}{|\underline{p_{m-1}-p_m}} \cdot \Phi(0),$$

i.e.
$$\int_0^\pi (\sin\theta_m)^{2p_m+n}\Theta_m^{\,2} d\theta_m = \frac{|\underline{n+p_{m-1}+p_m-1}}{|\underline{n+p_{m-1}-1}} \cdot \frac{|\underline{p_{m-1}}}{|\underline{p_{m-1}-p_m}} \cdot \int_0^\pi (\sin\theta_m)^n (Q_{p_{m-1}})^2 d\theta_m.$$

7. Let n be an even integer.

Then since $Q_{p_{m-1}}$ satisfies the equation satisfied by Θ_m when $p_m = 0$, it appears that

$$p_{m-1}(p_{m-1}+n)\int_0^\pi (\sin\theta_m)^n (Q_{p_{m-1}})^2\, d\theta_m = \int_0^\pi (\sin\theta_m)^{n+2}\left(\frac{dQ_{p_{m-1}}}{-\sin\theta_m d\theta_m}\right)^2 d\theta_m;$$

$$\therefore\quad p_{m-1}(p_{m-1}+n)\int_0^\pi (\sin\theta_m)^n \left[\left(\frac{d}{d\mu}\right)^{\frac{n}{2}} \cos\left\{\left(p_{m-1}+\frac{n}{2}\right)\theta_m\right\}\right]^2 d\theta_m$$

$$= \int_0^\pi (\sin\theta_m)^{n+2}\left[\left(\frac{d}{d\mu}\right)^{\frac{n}{2}+1} \cos\left\{\left(p_{m-1}+\frac{n}{2}\right)\theta_m\right\}\right]^2 d\theta_m.$$

This is an equation which is true whatever positive number p_{m-1} may be and whatever even integer n may be. Changing n into $n-2$, and p_{m-1} into $p_{m-1}+1$, the equation becomes

$$(p_{m-1}+1)(p_{m-1}+n-1)\int_0^\pi (\sin\theta_m)^{n-2}\left[\left(\frac{d}{d\mu}\right)^{\frac{n}{2}-1} \cos\left\{\left(p_{m-1}+\frac{n}{2}\right)\theta_m\right\}\right]^2 d\theta_m$$

$$= \int_0^\pi (\sin\theta_m)^n\left[\left(\frac{d}{d\mu}\right)^{\frac{n}{2}} \cos\left\{\left(p_{m-1}+\frac{n}{2}\right)\theta_m\right\}\right]^2 d\theta_m.$$

Proceeding in this way there arise finally $\Big($by putting in the last equation $n=4$, and for p_{m-1}, $p_{m-1}+\frac{n}{2}-2$; and $n=2$, and for p_{m-1}, $p_{m-1}+\frac{n}{2}-1\Big)$ the two equations

$$\left(p_{m-1}+\frac{n}{2}-1\right)\left(p_{m-1}+\frac{n}{2}+1\right)\int_0^\pi \sin^2\theta_m\left[\frac{d}{d\mu}\cos\left\{\left(p_{m-1}+\frac{n}{2}\right)\theta_m\right\}\right]^2 d\theta_m$$

$$= \int_0^\pi \sin^4\theta_m\left[\left(\frac{d}{d\mu}\right)^2 \cos\left\{\left(p_{m-1}+\frac{n}{2}\right)\theta_m\right\}\right]^2 d\theta_m,$$

$$\left(p_{m-1}+\frac{n}{2}\right)\left(p_{m-1}+\frac{n}{2}\right)\int_0^\pi \left[\cos\left\{\left(p_{m-1}+\frac{n}{2}\right)\theta_m\right\}\right]^2 d\theta_m = \int_0^\pi \sin^2\theta_m\left[\frac{d}{d\mu}\cos\left\{\left(p_{m-1}+\frac{n}{2}\right)\theta_m\right\}\right]^2 d\theta_m;$$

$$\therefore\quad \int_0^\pi (\sin\theta_m)^n\left[\left(\frac{d}{d\mu}\right)^{\frac{n}{2}} \cos\left\{\left(p_{m-1}+\frac{n}{2}\right)\theta_m\right\}\right]^2 d\theta_m$$

$$= \left(p_{m-1}+\frac{n}{2}\right)\frac{\underline{|p_{m-1}+n-1}}{\underline{|p_{m-1}}}\int_0^\pi \left[\cos\left\{\left(p_{m-1}+\frac{n}{2}\right)\theta_m\right\}\right]^2 d\theta_m$$

$$= \left(p_{m-1}+\frac{n}{2}\right)\frac{\underline{|p_{m-1}+n-1}}{\underline{|p_{m-1}}}\cdot\frac{\pi}{2};$$

$$\therefore\quad \int_0^\pi (\sin\theta_m)^n (Q_{p_{m-1}})^2\, d\theta_m = \frac{1}{\{(n-2)(n-4)\ldots\ldots 4.2\}^2}\cdot\frac{1}{p_{m-1}+\frac{n}{2}}\cdot\frac{\underline{|p_{m-1}+n-1}}{\underline{|p_{m-1}}}\cdot\frac{\pi}{2};$$

$\therefore$ if n be an even integer

$$\int_0^\pi (\sin\theta_m)^{2p_m+n}\Theta_m^2\, d\theta_m = \frac{\underline{|n+p_{m-1}+p_m-1}}{\underline{|p_{m-1}-p_m}}\cdot\frac{1}{p_{m-1}+\frac{n}{2}}\cdot\frac{1}{\{(n-2)(n-4)\ldots\ldots 4.2\}^2}\cdot\frac{\pi}{2}.$$

But if n be an odd integer

$$\int_0^{\pi}(\sin\theta_m)^{2p_m+n}\left[\left(\frac{d}{-\sin\theta_m d\theta_m}\right)^{p_m} Q_{p_{m-1}}\right]^2 d\theta_m$$

$$=\frac{\underline{|n+p_{m-1}+p_m-1}}{\underline{|n+p_{m-1}-1}}\cdot\frac{\underline{|p_{m-1}}}{\underline{|p_{m-1}-p_m}}\cdot\frac{1}{\{(n-2)(n-4)\ldots 5.3.1\}^2}\int_0^{\pi}(\sin\theta_m)^n\left[\left(\frac{d}{d\mu}\right)^{\frac{n-1}{2}} P_{\left(p_{m-1}-\frac{n-1}{2}\right)}\right]^2 d\theta$$

$$=\frac{\underline{|n+p_{m-1}+p_m-1}}{\underline{|n+p_{m-1}-1}}\cdot\frac{\underline{|p_{m-1}}}{\underline{|p_{m-1}-p_m}}\cdot\frac{1}{\{(n-2)(n-4)\ldots\ldots 5.3.1\}^2}\text{ multiplied by}$$

$$\left\{\frac{\underline{\left|\frac{n-1}{2}+p_{m-1}+\frac{n-1}{2}\right.}}{\underline{\left|\frac{n-1}{2}+p_{m-1}-\frac{n-1}{2}\right.}}\cdot\frac{1}{2\left(\frac{n-1}{2}+p_{m-1}\right)+1}\right\}$$

by a known property of Legendre's coefficients.

$\therefore$ if n be an odd integer

$$\int_0^{\pi}(\sin\theta_m)^{2p_m+n}\Theta_m^{\,2}d\theta_m=\frac{\underline{|n+p_{m-1}+p_m-1}}{\underline{|p_{m-1}-p_m}}\cdot\frac{1}{p_{m-1}+\frac{n}{2}}\cdot\frac{1}{\{(n-2)(n-4)\ldots\ldots 5.3.1\}^2}.$$

Both forms are included in

$$\int_0^{\pi}(\sin\theta_m)^{2p_m+n}\Theta_m^{\,2}d\theta_m=\frac{\underline{|n+p_{m-1}+p_m-1}}{\underline{|p_{m-1}-p_m}}\cdot\frac{1}{p_{m-1}+\frac{n}{2}}\cdot\frac{\pi}{2^{n-1}\{\Gamma(\frac{1}{2}n)\}^2},$$

where Γ is the symbol of the second Eulerian integral.

8. Suppose now that there are two solutions of the equation in u_1 of Art. 2, viz. u_1, u_1' where

$$u_1=(\sin\theta_1)^{p_1}\Theta_1\ldots\ldots(\sin\theta_{i-2})^{p_{i-2}}\Theta_{i-2}\binom{\cos}{\sin}p_{i-2}\theta_{i-1}$$

and

$$u'_1=(\sin\theta_1)^{p_1'}\Theta_1'\ldots\ldots(\sin\theta_{i-2})^{p'_{i-2}}\Theta'_{i-2}\binom{\cos}{\sin}p'_{i-2}\theta_{i-1}$$

and if dS be a differential element having the same relation to the variables $x_1\,x_2\ldots\ldots x_i$ as an element of area of a sphere has to the three variables $x_1\,x_2\,x_3$ so

that $$\int r^{i-1}dr\,dS=\int dx_1\,dx_2\ldots\ldots dx_i=\int J dr\,d\theta_1 d\theta_2\ldots\ldots d\theta_{i-1},$$

where J is the Jacobian of $x_1\,x_2\ldots\ldots x_i$ with regard to $r\theta_1\theta_2\ldots\ldots\theta_{i-1}$, so that

$$dS=\frac{1}{r^{i-1}}J d\theta_1 d\theta_2\ldots\ldots d\theta_{i-1}=(\sin\theta_1)^{i-2}d\theta_1(\sin\theta_2)^{i-3}d\theta_2\ldots\ldots(\sin\theta_{i-3})^2\,d\theta_{i-3}(\sin\theta_{i-2})\,d\theta_{i-2}d\theta_{i-1}.$$

Then $\int u_1 u'_1\,dS=0$ unless $u_1\,u'_1$ are identical; the limits of integration for θ_{i-1} being 0 and 2π, and for the rest of the variables 0 and π. $\Big\}$ ......(A).

The integral in question is the product of a number of definite integrals which can be separately evaluated. It is necessary to commence with the last of these, viz.

$$\int_0^{2\pi} \begin{Bmatrix}\cos \\ \sin\end{Bmatrix} p_{i-2}\theta_{i-1} \begin{Bmatrix}\cos \\ \sin\end{Bmatrix} p'_{i-2}\theta_{i-1}\, d\theta_{i-1};$$

and this vanishes if $p_{i-2}\, p'_{i-2}$ be different positive integers; and if $p_{i-2}\, p'_{i-2}$ be the same, the integral will still vanish unless the terms in brackets be both sines or both cosines.

It is therefore only necessary to consider the integral when $p_{i-2}p'_{i-2}$ are the same. Supposing this to be the case, but $p_{i-3}p'_{i-3}$ different; then by Art. 5,

$$\int_0^{\pi} (\sin \theta_{i-2})^{2p_{i-2}+1}\, \Theta_{i-2}\Theta'_{i-2}\, d\theta_{i-2} = 0.$$

It is therefore only necessary to consider the integral in the cases in which $p_{i-2}\,p_{i-3}$ are respectively equal to $p'_{i-2}\,p'_{i-3}$. If this be so, but p_{i-4} different from p'_{i-4}; then again by Art. 5,

$$\int_0^{\pi} (\sin \theta_{i-3})^{2p_{i-3}+2}\, \Theta_{i-3}\Theta'_{i-3}\, d\theta_{i-3} = 0.$$

And so on, it appears that the integral in (A) will vanish unless the quantities $p_{i-2}p_{i-3}\ldots p_1$ be respectively the same as $p'_{i-2}p'_{i-3}\ldots p'_1$. This corresponds to the conjugate property of Tesseral Harmonics.

9. It is now required to determine the value of $\int u_1^2\, dS$, i. e.

$$\int_0^{2\pi} \begin{Bmatrix}\cos \\ \sin\end{Bmatrix} p_{i-2}\theta_{i-1} \Bigg)^2 d\theta_{i-1} \int_0^{\pi} (\sin\theta_{i-2})^{2p_{i-2}+1}\, \Theta_{i-2}^2\, d\theta_{i-2} \ldots \int_0^{\pi} (\sin\theta_2)^{2p_2+i-3}\, \Theta_2^2\, d\theta_2 \int_0^{\pi} (\sin\theta_1)^{2p_1+i-2}\, \Theta_1^2\, d\theta_1 .$$

Supposing p_{i-2} not to vanish, this is

$$\pi).\left(\frac{|\underline{p_{i-3}+p_{i-2}}}{|\underline{p_{i-3}-p_{i-2}}} \cdot \frac{1}{p_{i-3}+\frac{1}{2}} \cdot \frac{\pi}{\{\Gamma(\frac{1}{2})\}^2}\right) \ldots\ldots \left(\frac{|\underline{p_1+p_2+i-4}}{|\underline{p_1-p_2}} \cdot \frac{1}{p_1+\dfrac{i-3}{2}} \cdot \frac{\pi}{2^{i-4}\left\{\Gamma\left(\dfrac{i-3}{2}\right)\right\}^2}\right)$$

$$\text{multiplied by } \left(\frac{|\underline{p+p_1+i-3}}{|\underline{p-p_1}} \cdot \frac{1}{p+\dfrac{i-2}{2}} \cdot \frac{\pi}{2^{i-3}\left\{\Gamma\left(\dfrac{i-2}{2}\right)\right\}^2}\right)$$

$$= \pi \cdot \frac{|\underline{p_{i-3}+p_{i-2}}}{|\underline{p_{i-3}-p_{i-2}}} \cdot \frac{|\underline{p_{i-4}+p_{i-3}+1}}{|\underline{p_{i-4}-p_{i-3}}} \ldots\ldots \frac{|\underline{p_1+p_2+i-4}}{|\underline{p_1-p_2}} \cdot \frac{|\underline{p+p_1+i-3}}{|\underline{p-p_1}} \cdot \frac{1}{p_{i-3}+\frac{1}{2}} \ldots\ldots$$

$$\frac{1}{p_1+\dfrac{i-3}{2}} \cdot \frac{1}{p+\dfrac{i-2}{2}} \cdot \frac{\pi}{\{\Gamma(\frac{1}{2})\}^2} \cdot \frac{\pi}{2\{\Gamma(1)\}^2} \cdot \frac{\pi}{2^2\{\Gamma(\frac{3}{2})\}^2} \ldots\ldots \frac{\pi}{2^{i-4}\left\{\Gamma\left(\dfrac{i-3}{2}\right)\right\}^2} \cdot \frac{\pi}{2^{i-3}\left\{\Gamma\left(\dfrac{i-2}{2}\right)\right\}^2}.$$

But $$\pi \cdot \frac{\pi}{\{\Gamma(\frac{1}{2})\}^2} \cdot \frac{\pi}{2\{\Gamma(1)\}^2} \cdot \frac{\pi}{2^2\{\Gamma(\frac{3}{2})\}^2} \cdots\cdots \frac{\pi}{2^{i-4}\left\{\Gamma\left(\frac{i-3}{2}\right)\right\}^2} \cdot \frac{\pi}{2^{i-3}\left\{\Gamma\left(\frac{i-2}{2}\right)\right\}^2}$$

$$= \frac{\pi^{i-1}}{2^{1+2+3+\ldots+(i-4)+(i-3)}} \cdot \frac{1}{\Gamma\left(\frac{i-2}{2}\right)} \cdot \frac{1}{\left\{\Gamma\left(\frac{i-2}{2}\right)\Gamma\left(\frac{i-3}{2}\right)\right\}\left\{\Gamma\left(\frac{i-3}{2}\right)\Gamma\left(\frac{i-4}{2}\right)\right\}\ldots\{\Gamma(\frac{3}{2})\Gamma(1)\}\{\Gamma(1)\Gamma(\frac{1}{2})\}\Gamma(\frac{1}{2})}$$

$$= \frac{\pi^{i-1}}{2^{1+2+3+\ldots+(i-4)+(i-3)}} \cdot \frac{1}{\Gamma\left(\frac{i-2}{2}\right)} \cdot \frac{1}{\left\{\frac{i-4}{2}\cdot\frac{i-5}{2}\cdot\frac{i-6}{2}\ldots\frac{1}{2}\sqrt{\pi}\right\}\left\{\frac{i-5}{2}\cdot\frac{i-6}{2}\ldots\frac{1}{2}\sqrt{\pi}\right\}\ldots\{\frac{1}{2}\sqrt{\pi}\}\{\sqrt{\pi}\}\sqrt{\pi}}$$

$$= \frac{\pi^{i-1}}{2^{1+2+3+\ldots+(i-4)+(i-3)}} \cdot \frac{1}{\Gamma\left(\frac{i-2}{2}\right)} \cdot \frac{2^{1+2+3+\ldots+(i-4)}}{\underline{|i-4}\,\underline{|i-5}\ldots\underline{|2}\,\underline{|1}\,.\,(\sqrt{\pi})^{i-2}} = \frac{\pi^{\frac{1}{2}i}}{2^{i-3}\,\Gamma\left(\frac{i-2}{2}\right)\underline{|i-4}\,\underline{|i-5}\ldots\underline{|2}\,\underline{|1}}\,;$$

$$\therefore \int u_1^2\,dS$$

$$= \frac{\underline{|p_{i-3}+p_{i-2}}}{\underline{|p_{i-3}-p_{i-2}}} \cdot \frac{\underline{|p_{i-4}+p_{i-3}+1}}{\underline{|p_{i-4}-p_{i-3}}} \cdots\cdots \frac{\underline{|p_1+p_2+i-4}}{\underline{|p_1-p_2}} \cdot \frac{\underline{|p+p_1+i-3}}{\underline{|p-p_1}} \text{ multiplied by}$$

$$\cdot \frac{1}{p_{i-3}+\frac{1}{2}} \cdots\cdots \frac{1}{p_1+\frac{i-3}{2}} \cdot \frac{1}{p+\frac{i-2}{2}} \cdot \frac{\pi^{\frac{1}{2}i}}{2^{i-3}\,\Gamma(\frac{1}{2}i-1)\,.\,\underline{|i-4}\,\underline{|i-5}\cdots\cdots\underline{|2}\,\underline{|1}},$$

where Γ is the symbol of the second Eulerian Integral. If $p_{i-2}=0$, this value must be doubled.

10. To show that

$$r^p(\sin\theta_1)^{p_1}\Theta_1(\sin\theta_2)^{p_2}\Theta_2\cdots\cdots(\sin\theta_{i-2})^{p_{i-2}}\Theta_{i-2}\left({\cos \atop \sin}\,p_{i-2}\theta_{i-1}\right)$$

is a rational integral homogeneous function of $x_1x_2\ldots\ldots x_i$ of degree p.

It is homogeneous and of degree p because r^p is of degree p in these variables, and all the quantities $\sin\theta$, Θ are of degree zero.

Moreover it was shown in Art. 4 that Θ_m contains $(\cos\theta_m)^{p_{m-1}-p_m}$ and lower powers of $\cos\theta_m$ whose indices are positive integers differing from $p_{m-1}-p_m$ by some even number which may be denoted by $2q_m$, so that the general term in Θ_m is a numerical multiple of $(\cos\theta_m)^{p_{m-1}-p_m-2q_m}$.

In like manner the general term in $\cos p_{i-2}\theta_{i-1}$ is a numerical multiple of $(\cos\theta_{i-1})^{p_{i-2}-2q_{i-1}}$, and that in $\sin p_{i-2}\theta_{i-1}$ is a numerical multiple of

$$(\sin\theta_{i-1})\,(\cos\theta_{i-1})^{p_{i-2}-2q_{i-1}-1}.$$

It is sufficient therefore to show that the quantities included in the form

$$r^p(\sin\theta_1)^{p_1}(\cos\theta_1)^{p-p_1-2q_1}(\sin\theta_2)^{p_2}(\cos\theta_2)^{p_1-p_2-2q_2}\cdots\cdots(\sin\theta_{i-3})^{p_{i-3}}(\cos\theta_{i-3})^{p_{i-4}-p_{i-3}-2q_{i-3}}$$

$$\text{multiplied by } (\sin\theta_{i-2})^{p_{i-2}}(\cos\theta_{i-2})^{p_{i-3}-p_{i-2}-2q_{i-2}}(\cos\theta_{i-1})^{p_{i-2}-2q_{i-1}-1}\left({\cos \atop \sin}\,\theta_{i-1}\right)$$

are rational integral functions of $x_1x_2\ldots\ldots x_i$.

Now $r^p (\cos\theta_1)^{p-p_1-2q_1} = (r\cos\theta_1)^{p-p_1-2q_1} \,.\, r^{p_1+2q_1} = (r\cos\theta_1)^{p-p_1-2q_1} \,.\, r^{2q_1} \,.\, r^{p_1}$;

$\therefore\ r^p (\sin\theta_1)^{p_1} (\cos\theta_1)^{p-p_1-2q_1} = (r\cos\theta_1)^{p-p_1-2q_1} \,.\, r^{2q_1} \,.\, (r\sin\theta_1)^{p_1}$;

$\therefore\ r^p (\sin\theta_1)^{p_1} (\cos\theta_1)^{p-p_1-2q_1} (\cos\theta_2)^{p_1-p_2-2q_2}$

$$= (r\cos\theta_1)^{p-p_1-2q_1} r^{2q_1} (r\sin\theta_1\cos\theta_2)^{p_1-p_2-2q_2} (r\sin\theta_1)^{p_2+2q_2}$$

$$= (r\cos\theta_1)^{p-p_1-2q_1} r^{2q_1} (r\sin\theta_1\cos\theta_2)^{p_1-p_2-2q_2} (r\sin\theta_1)^{2q_2} (r\sin\theta_1)^{p_2};$$

$\therefore\ r^p (\sin\theta_1)^{p_1} (\cos\theta_1)^{p-p_1-2q_1} (\sin\theta_2)^{p_2} (\cos\theta_2)^{p_1-p_2-2q_2}$

$$= (r\cos\theta_1)^{p-p_1-2q_1} r^{2q_1} (r\sin\theta_1\cos\theta_2)^{p_1-p_2-2q_2} (r\sin\theta_1)^{2q_2} (r\sin\theta_1\sin\theta_2)^{p_2}.$$

Proceeding in this manner it appears that

$$r^p (\sin\theta_1)^{p_1} (\cos\theta_1)^{p-p_1-2q_1} (\sin\theta_2)^{p_2} (\cos\theta_2)^{p_1-p_2-2q_2} \ldots\ldots$$

$$(\sin\theta_{i-3})^{p_{i-3}} (\cos\theta_{i-3})^{p_{i-4}-p_{i-3}-2q_{i-3}} (\sin\theta_{i-2})^{p_{i-2}} (\cos\theta_{i-2})^{p_{i-3}-p_{i-2}-2q_{i-2}}$$

$$= (r\cos\theta_1)^{p-p_1-2q_1} r^{2q_1} (r\sin\theta_1\cos\theta_2)^{p_1-p_2-2q_2} (r\sin\theta_1)^{2q_2} \ldots\ldots$$

$$(r\sin\theta_1\sin\theta_2 \ldots\ldots \sin\theta_{i-4}\cos\theta_{i-3})^{p_{i-4}-p_{i-3}-2q_{i-3}}$$

$$\times (r\sin\theta_1\sin\theta_2 \ldots\ldots \sin\theta_{i-4})^{2q_{i-3}} (r\sin\theta_1\sin\theta_2 \ldots\ldots \sin\theta_{i-3}\cos\theta_{i-2})^{p_{i-3}-p_{i-2}-2q_{i-2}}$$

multiplied by $(r\sin\theta_1\sin\theta_2 \ldots\ldots \sin\theta_{i-3})^{2q_{i-2}} (r\sin\theta_1\sin\theta_2 \ldots\ldots \sin\theta_{i-3}\sin\theta_{i-2})^{p_{i-2}}$.

Every term except the last is a rational integral function of the variables. It remains therefore to show that the expressions included in the form

$$(r\sin\theta_1\sin\theta_2 \ldots\ldots \sin\theta_{i-3}\sin\theta_{i-2})^{p_{i-2}} (\cos\theta_{i-1})^{p_{i-2}-2q_{i-1}-1} \left(\begin{matrix}\cos\\ \sin\end{matrix}\theta_{i-1}\right)$$

are rational integral functions of $x_1 x_2 \ldots\ldots x_i$. This form may be arranged thus:

$$(r\sin\theta_1\sin\theta_2 \ldots\ldots \sin\theta_{i-2}\cos\theta_{i-1})^{p_{i-2}-2q_{i-1}-1} (r\sin\theta_1\sin\theta_2 \ldots\ldots \sin\theta_{i-2})^{2q_{i-1}}$$

$$\text{multiplied by } \left(r\sin\theta_1\sin\theta_2 \ldots\ldots \sin\theta_{i-2}\begin{matrix}\cos\\ \sin\end{matrix}\theta_{i-1}\right)$$

and every term in this is a rational integral function of $x_1 x_2 \ldots\ldots x_i$.

The result is that

$$r^p (\sin\theta_1)^{p_1} (\cos\theta_1)^{p-p_1-2q_1} (\sin\theta_2)^{p_2} (\cos\theta_2)^{p_1-p_2-2q_2} \ldots\ldots$$

$$(\sin\theta_{i-3})^{p_{i-3}} (\cos\theta_{i-3})^{p_{i-4}-p_{i-3}-2q_{i-3}} (\sin\theta_{i-2})^{p_{i-2}} (\cos\theta_{i-2})^{p_{i-3}-p_{i-2}-2q_{i-2}} (\cos\theta_{i-1})^{p_{i-2}-2q_{i-1}-1} \left(\begin{matrix}\cos\\ \sin\end{matrix}\theta_{i-1}\right)$$

$$= (x_i)^{p-p_1-2q_1} (x_1^2 + x_2^2 + \ldots\ldots + x_i^2)^{q_1} (x_{i-1})^{p_1-p_2-2q_2} (x_1^2 + x_2^2 + \ldots\ldots + x_{i-1}^2)^{q_2} \ldots\ldots$$

$$(x_3)^{p_{i-3}-p_{i-2}-2q_{i-2}} (x_1^2 + x_2^2 + x_3^2)^{q_{i-2}} (x_2)^{p_{i-2}-2q_{i-1}-1} (x_1^2 + x_2^2)^{q_{i-1}} \left(\begin{matrix}x_2\\ x_1\end{matrix}\right).$$

11. To determine the number of different functions included in the form

$$r^p (\sin\theta_1)^{p_1} \Theta_1 (\sin\theta_2)^{p_2} \Theta_2 \ldots\ldots (\sin\theta_{i-2})^{p_{i-2}} \Theta_{i-2} \left(\begin{matrix}\cos\\ \sin\end{matrix} p_{i-2}\theta_{i-1}\right),$$

which can be obtained by giving all possible integral values to the exponents $p_1, p_2 \ldots\ldots p_{i-2}$ which are such that no one is ever greater than p, and no one is ever greater than any one which precedes it in the series.

The possible values of p_1 are

$$0,\ 1,\ 2,\ 3,\ \ldots p \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\alpha).$$

When $p_1=0$, the only possible value of p_2 is 0;

$p_1=1$, the possible values of p_2 are $0,\ 1$;

$p_1=2$, " " " " $0,\ 1,\ 2$;

$p_1=3$, " " " " $0,\ 1,\ 2,\ 3$;

and so on.

Thus the values of p_2 giving rise to different functions may be arranged thus:

$$0;\ 0,\ 1;\ 0,\ 1,\ 2;\ 0,\ 1,\ 2,\ 3;\ \ldots\ldots\ldots;\ 0,\ 1,\ 2,\ 3\ \ldots p \ldots\ldots\ldots\ldots (\beta).$$

(It should be noticed that though p_2 may have the same value twice over, it gives rise to different functions. Thus there is a function corresponding to $p_2=0$, $p_1=0$; whilst a distinct function corresponds to $p_2=0$, $p_1=1$; and so on.)

Again the values of p_3 giving rise to different functions may be arranged thus:

$$0;\ 0,\ 0,\ 1;\ 0,\ 0,\ 1,\ 0,\ 1,\ 2;\ 0,\ 0,\ 1,\ 0,\ 1,\ 2,\ 0,\ 1,\ 2,\ 3;\ \ldots\ldots\ldots;$$
$$0,\ 0,\ 1,\ 0,\ 1,\ 2,\ 0,\ 1,\ 2,\ 3,\ \ldots\ldots\ldots 0,\ 1,\ 2,\ 3 \ldots p \ldots\ldots\ldots\ldots (\gamma).$$

The law of formation is seen to be that the numbers which stand in the n^{th} place of the row (γ) include all the numbers which stand in the first n places of the row (β). The relation of the numbers in the row (β) to the numbers in the row (α) is the same as the relation of the numbers in the row (γ) to those in the row (β).

It is necessary therefore to find the number of the numbers standing in the row corresponding to p_{i-2}, the number of zeros being counted separately, because if $p_{i-2}=0$, $\sin p_{i-2}\theta_{i-1}=0$, and this does not give rise to a function. Every other value of p_{i-2} gives rise to two functions.

The number of zeros in the row corresponding to

p_1 is 1,

p_2 is $1+1+1+\ldots+1$ ($\overline{p+1}$ terms),

p_3 is $1+2+3+\ldots+(p+1)$ ($\overline{p+1}$ terms),

and so on.

The number of other figures in the row corresponding to

p_1 is $1+1+1+\ldots+1$ (p terms),

p_2 is $1+2+3+\ldots+p$ (p terms),

p_3 is $1+3+6+\ldots+\dfrac{p(p+1)}{2}$ (p terms),

and so on.

Now the series of figurate numbers of the first order is $1+1+1+\ldots$;

" " " " " second " $1+2+3+\ldots$;

" " " " " third " $1+3+6+\ldots$;

and so on, the law of formation being the same as in the above.

Therefore the number of zeros in the row corresponding to p_{i-2} is the sum of the first $(p+1)$ terms of the figurate numbers of order $i-3$; and the number of other figures in the same row is the sum of the first p terms of the figurate numbers of order $i-2$.

Therefore the number of zeros is the $(p+1)^{\text{th}}$ figurate number of order $i-2$, and the number of other figures is the p^{th} figurate number of order $i-1$.

But the n^{th} figurate number of r^{th} order is $\dfrac{\lfloor n+r-2}{\lfloor n-1 \lfloor r-1}$.

Therefore the number of functions arising from the numbers in the row corresponding to p_{i-2} is $1 . \dfrac{\lfloor p+i-3}{\lfloor p \lfloor i-3} + 2 . \dfrac{\lfloor p+i-3}{\lfloor p-1 \lfloor i-2}$.

It is required to show that this number is equal to the number of independent rational integral homogeneous functions of $x_1 x_2 \ldots x_i$ of degree p, which satisfy the equation

$$\frac{d^2u}{dx_1^2}+\frac{d^2u}{dx_2^2}+\ldots+\frac{d^2u}{dx_i^2}=0.$$

The most general form of a rational integral homogeneous function of i variables of degree p contains $\dfrac{\lfloor p+i-1}{\lfloor p \lfloor i-1}$ constants. If it satisfy

$$\frac{d^2u}{dx_1^2}+\frac{d^2u}{dx_2^2}+\ldots+\frac{d^2u}{dx_i^2}=0,$$

then since $\dfrac{d^2u}{dx_1^2}+\dfrac{d^2u}{dx_2^2}+\ldots+\dfrac{d^2u}{dx_i^2}$ is a rational integral homogeneous function of degree $p-2$, there must exist $\dfrac{\lfloor p+i-3}{\lfloor p-2 \lfloor i-1}$ linear relations amongst the arbitrary constants. In virtue of these, $\dfrac{\lfloor p+i-3}{\lfloor p-2 \lfloor i-1}$ of the constants may be expressed as linear functions of the remainder. Therefore the most general form of a rational integral homogeneous function of i variables of degree p which satisfies the equation $\dfrac{d^2u}{dx_1^2}+\dfrac{d^2u}{dx_2^2}+\ldots+\dfrac{d^2u}{dx_i^2}=0$ contains $\dfrac{\lfloor p+i-1}{\lfloor p \lfloor i-1} - \dfrac{\lfloor p+i-3}{\lfloor p-2 \lfloor i-1}$ arbitrary constants involved in a linear manner.

But $\dfrac{\lfloor p+i-3}{\lfloor p \lfloor i-3} + 2\dfrac{\lfloor p+i-3}{\lfloor p-1 \lfloor i-2} = \dfrac{\lfloor p+i-1}{\lfloor p \lfloor i-1} - \dfrac{\lfloor p+i-3}{\lfloor p-2 \lfloor i-1}$.

Therefore the number of functions of degree p of the form considered in this article, each of which is known to satisfy the equation

$$\frac{d^2u}{dx_1^2}+\frac{d^2u}{dx_2^2}+\ldots+\frac{d^2u}{dx_i^2}=0,$$

is equal to the number of the independent rational integral homogeneous functions of $x_1x_2\ldots x_i$ of degree p which satisfy it. But the functions considered in this article are all different, and are all rational integral homogeneous functions of $x_1x_2\ldots x_i$ of degree p.

Therefore the functions included in the form

$$r^p(\sin\theta_1)^{p_1}\Theta_1(\sin\theta_2)^{p_2}\Theta_2\ldots(\sin\theta_{i-2})^{p_{i-2}}\Theta_{i-2}\left(\begin{matrix}\cos\\ \sin\end{matrix}p_{i-2}\theta_{i-1}\right)$$

are all the independent rational integral homogeneous functions of $x_1x_2\ldots x_i$ of degree p which satisfy the equation

$$\frac{d^2u}{dx_1^2}+\frac{d^2u}{dx_2^2}+\ldots+\frac{d^2u}{dx_i^2}=0.$$

PART II.

The expansion of an arbitrary function of the variables $\theta_1\theta_2\ldots\theta_{i-1}$ will now be considered. The proof of this expansion for any rational integral function of the quantities

$\cos\theta_1$, $\sin\theta_1\cos\theta_2$, $\sin\theta_1\sin\theta_2\cos\theta_3$, ..., $\sin\theta_1\sin\theta_2\ldots\sin\theta_{i-2}\cos\theta_{i-1}$, $\sin\theta_1\sin\theta_2\ldots\sin\theta_{i-2}\sin\theta_{i-1}$,

and the proof that the expansion for any arbitrary function if possible is unique, both proceed in exactly the same manner as in the case of a function of two variables. They will therefore be omitted, and the general proposition only given. The proof here set forth is similar to one of those given of Laplace's expansion of a function of two variables. It is subject to similar criticisms.

1. The following preliminary propositions are necessary.

(α) To show that if R_p be the coefficient of h^p in the expansion of

$$\frac{1}{(1-2h\cos\theta+h^2)^{\frac{i}{2}-1}},$$

then R_p is greatest when $\cos\theta=1$, i being a positive integer not less than 3.

Suppose that $$\frac{1}{(1-2h\cos\theta+h^2)^{\frac{i-1}{2}-1}}=Q_0+Q_1h+Q_2h^2+\ldots+Q_ph^p+\ldots,$$

and $$\frac{1}{(1-2h\cos\theta+h^2)^{\frac{1}{2}}}=P_0+P_1h+P_2h^2+\ldots+P_ph^p+\ldots.$$

Then since $$\frac{1}{(1-2h\cos\theta+h^2)^{\frac{i}{2}}}=\frac{1}{(1-2h\cos\theta+h^2)^{\frac{i-1}{2}-1}}\cdot\frac{1}{(1-2h\cos\theta+h^2)^{\frac{1}{2}}}$$

$$\therefore (R_0+R_1h+R_2h^2+\ldots+R_ph^p+\ldots)$$
$$=(Q_0+Q_1h+Q_2h^2+\ldots+Q_ph^p+\ldots)(P_0+P_1h+P_2h^2+\ldots+P_ph^p+\ldots).$$

Therefore $$R_p=P_pQ_0+P_{p-1}Q_1+P_{p-2}Q_2+\ldots+P_0Q_p.$$

Now it is known that each of the quantities P_0, P_1, $P_2 \ldots P_p$ is greatest when $\cos\theta=1$; if therefore this property hold also for Q_0, Q_1, $Q_2 \ldots Q_p$ it will hold for R_p; i.e. if this property hold for the coefficients in the expansion of $\frac{1}{(1-2h\cos\theta+h^2)^{\frac{i-1}{2}-1}}$, then it holds for the coefficients in the expansion of $\frac{1}{(1-2h\cos\theta+h^2)^{\frac{i}{2}-1}}$; but it does hold for the coefficients in the expansion of $\frac{1}{(1-2h\cos\theta+h^2)^{\frac{3}{2}-1}}$; therefore it holds for the coefficients in the expansion of $\frac{1}{(1-2h\cos\theta+h^2)^{\frac{4}{2}-1}}$, and so on universally.

The proof holds good only when i is a positive integer not less than 3.

(β) To show that

$$\int_0^\pi \frac{(1-h^2)(\sin\theta)^{i-2}d\theta}{(1-2h\cos\theta+h^2)^{\frac{i}{2}}}=\int_0^\pi(\sin\theta)^{i-2}\,d\theta,$$

where $h<1$, and i is a positive integer not less than 2.

Let $$\frac{1}{(1-2h\cos\theta+h^2)^{\frac{i}{2}-1}}=R_0+R_1h+\ldots\ldots+R_ph^p+\ldots\ldots$$

$$\therefore (i-2)\frac{\cos\theta-h}{(1-2h\cos\theta+h^2)^{\frac{i}{2}}}=R_1+\ldots\ldots+pR_ph^{p-1}+\ldots\ldots$$

$$\therefore (i-2)\frac{1-h^2}{(1-2h\cos\theta+h^2)^{\frac{i}{2}}}=(i-2+2\,.\,0)R_0+(i-2+2\,.\,1)hR_1+\ldots+(i-2+2p)h^pR_p+\ldots$$

$$\therefore (i-2)\int_0^\pi\frac{(1-h^2)(\sin\theta)^{i-2}\,d\theta}{(1-2h\cos\theta+h^2)^{\frac{i}{2}}}$$
$$=(i-2+2\,.\,0)\int_0^\pi(\sin\theta)^{i-2}R_0d\theta+(i-2+2\,.\,1)h\int_0^\pi(\sin\theta)^{i-2}R_1d\theta$$
$$+\ldots\ldots+(i-2+2\,.\,p)h^p\int_0^\pi(\sin\theta)^{i-2}R_pd\theta+\ldots\ldots$$

But putting $n=i-2$, $p_{m-1}=p$, $p_m=0$, $m=1$, $\theta_1=\theta$, in equation (m) of Art. 2, Θ_m becomes R_p.

Therefore $$\sin^2\theta \frac{d^2R_p}{d\theta^2} + (i-2)\sin\theta\cos\theta\frac{dR_p}{d\theta} + p(p+i-2)\sin^2\theta \,.\, R_p = 0.$$

Multiplying by $(\sin\theta)^{i-4}$ and integrating between the limits 0 and π with regard to the variable θ

$$\left[(\sin\theta)^{i-2}\frac{dR_p}{d\theta}\right]_0^\pi + p(p+i-2)\int_0^\pi (\sin\theta)^{i-2}R_p d\theta = 0,$$

$$\therefore \quad \int_0^\pi (\sin\theta)^{i-2}R_p d\theta = 0 \text{ if } p \text{ be not zero.}$$

If p be zero, then since $R_0 = 1$

$$\int_0^\pi (\sin\theta)^{i-2}R_0 d\theta = \int_0^\pi (\sin\theta)^{i-2}d\theta;$$

$$\therefore \int_0^\pi \frac{(1-h^2)(\sin\theta)^{i-2}d\theta}{(1-2h\cos\theta+h^2)^{\frac{i}{2}}} = \int_0^\pi (\sin\theta)^{i-2}d\theta.$$

The restriction i not less than 2 arises from the fact that throughout the paper, it has been supposed that there are not less than two variables included in $x_1 x_2 \ldots\ldots x_i$.

2. Suppose now that

$$x_1^2 + x_2^2 + \ldots\ldots + x_i^2 = r^2, \quad x'^2_1 + x'^2_2 + \ldots\ldots + x'^2_i = r'^2,$$

$$d\sigma = r^{i-1}(\sin\theta_1)^{i-2}(\sin\theta_2)^{i-3}\ldots\ldots(\sin\theta_{i-3})^2(\sin\theta_{i-2})\,d\theta_1\,d\theta_2\ldots\ldots d\theta_{i-3}\,d\theta_{i-2}\,d\theta_{i-1}$$

$$D^2 = (x_1 - x'_1)^2 + (x_2 - x'_2)^2 + \ldots\ldots + (x_i - x'_i)^2 = r^2 - 2rr'\cos\theta + r'^2$$

$F(\theta_1\theta_2\ldots\ldots\theta_{i-1})$ any function of $\theta_1\theta_2\ldots\ldots\theta_{i-1}$ whatever,

$u = \int \frac{(r'^2 - r^2)F(\theta_1\theta_2\ldots\ldots\theta_{i-1})\,d\sigma}{D^i}$, the limits of integration for θ_{i-1} being 0 and 2π; for $\theta_1\theta_2\ldots\ldots\theta_{i-2}$ the limits being 0 and π.

Suppose that r' is infinitely nearly equal to r but greater than it, then every element of the integral u is very small except those which make D infinitely small; therefore for all elements which do not make D infinitely small, anything may be substituted for $F(\theta_1\theta_2\ldots\ldots\theta_{i-1})$ in calculating the value of the integral. But when D is infinitely small this cannot be done, for then both numerator and denominator of the expression to be integrated are very small; so that in this case $F(\theta_1\theta_2\ldots\ldots\theta_{i-1})$ which is very nearly equal to $F(\theta'_1\theta'_2\ldots\ldots\theta'_{i-1})$ cannot be replaced by anything else. It is possible then to replace $F(\theta_1\theta_2\ldots\ldots\theta_{i-1})$ by $F(\theta_1'\theta_2'\ldots\ldots\theta'_{i-1})$ in all the elements of the integral. (Since it has been assumed that $\theta_1\theta_2\ldots\ldots\theta_{i-1}$ can be put equal to $\theta'_1\theta'_2\ldots\ldots\theta'_{i-1}$ respectively, the limits between which $\theta'_1\theta'_2\ldots\ldots\theta'_{i-1}$ lie must be the same as those between which $\theta_1\theta_2\ldots\ldots\theta_{i-1}$ are contained.)

Therefore $$\int \frac{(r'^2 - r^2)F(\theta_1\theta_2\ldots\ldots\theta_{i-1})}{D^i}\,d\sigma = F(\theta'_1\theta'_2\ldots\ldots\theta'_{i-1})\int\frac{r'^2 - r^2}{D^i}\,d\sigma$$

$$= F(\theta_1'\theta_2'\ldots\ldots\theta'_{i-1})\int\frac{r'^2 - r^2}{(r'^2 - 2rr'\cos\theta + r^2)^{\frac{i}{2}}}\,r^{i-1}S'\,d\theta_1 d\theta_2\ldots\ldots d\theta_{i-3}\,d\theta_{i-2}\,d\theta_{i-1}$$

where $$S' = (\sin\theta_1)^{i-2}(\sin\theta_2)^{i-3}\ldots\ldots(\sin\theta_{i-3})^2(\sin\theta_{i-2})_i.$$

Let the variables $x_1 x_2 \ldots\ldots x_i$ be transformed by an orthogonal transformation to new variables $y_1 y_2 \ldots\ldots y_i$, and let $y_1 y_2 \ldots\ldots y_i$ be transformed into a set of variables $r\phi_1 \phi_2 \ldots\ldots \phi_{i-1}$ by a transformation similar to that given in the abstract of Art. 1 of Part I. Moreover let $\phi_1 = \theta$. Then

$$r^{i-1} (\sin\theta_1)^{i-2} (\sin\theta_2)^{i-3} \ldots\ldots (\sin\theta_{i-3})^2 (\sin\theta_{i-2})\, d\theta_1 d\theta_2 \ldots\ldots d\theta_{i-3} d\theta_{i-2} d\theta_{i-1}$$
$$= r^{i-1} (\sin\theta)^{i-2} (\sin\phi_2)^{i-3} \ldots\ldots (\sin\phi_{i-3})^2 \sin\phi_{i-2} d\theta d\phi_2 \ldots\ldots d\phi_{i-3} d\phi_{i-2} d\phi_{i-1};$$

and the limits for the new variables are the same as those for the old.

Therefore

$$\int \frac{(r'^2 - r^2) F(\theta_1 \theta_2 \ldots \theta_{i-1})}{D^i} d\sigma = F(\theta_1' \theta_2' \ldots \theta'_{i-1}) \int_0^\pi \frac{r^{i-1} (r'^2 - r^2)(\sin\theta)^{i-2} d\theta}{(r'^2 - 2rr'\cos\theta + r^2)^{\frac{i}{2}}} \text{ multiplied by}$$
$$\int_0^\pi (\sin\phi_2)^{i-3} d\phi_2 \ldots\ldots \int_0^\pi \sin\phi_{i-2} d\phi_{i-2} \int_0^{2\pi} d\phi_{i-1}.$$

Hence by Preliminary Proposition (β)

$$\int \frac{(r'^2 - r^2) F(\theta_1 \theta_2 \ldots \theta_{i-1})\, d\sigma}{D^i} = F(\theta_1' \theta_2' \ldots \theta'_{i-1}) \frac{r^{i-1}}{r'^{i-2}} \int_0^\pi (\sin\theta)^{i-2} d\theta \text{ multiplied by}$$
$$\int_0^\pi (\sin\phi_2)^{i-3} d\phi_2 \ldots\ldots \int_0^\pi \sin\phi_{i-2} d\phi_{i-2} \int_0^{2\pi} d\phi_{i-1}$$
$$= F(\theta_1' \theta_2' \ldots \theta'_{i-1}) \frac{r^{i-1}}{r'^{i-2}} \int_0^\pi \ldots \int_0^\pi \int_0^{2\pi} (\sin\theta_1)^{i-2} (\sin\theta_2)^{i-3} \ldots\ldots (\sin\theta_{i-2})\, d\theta_1 d\theta_2 \ldots\ldots d\theta_{i-2} d\theta_{i-1}$$

and therefore as r' approaches indefinitely near to r

$$\int \frac{(r'^2 - r^2) F(\theta_1 \theta_2 \ldots \theta_{i-1})}{D^i} d\sigma = rF(\theta_1' \theta_2' \ldots \theta'_{i-1}) \int_0^\pi \ldots \int_0^\pi \int_0^{2\pi} S' d\theta_1 d\theta_2 \ldots d\theta_{i-2} d\theta_{i-1}$$

where S' has the same value as on previous page.

A similar result would have been obtained if r' had been infinitely nearly equal to r but less than it.

Both results may be expressed thus:—

$$F(\theta_1' \theta_2' \ldots \theta_{i-1}') \int_0^\pi \ldots \int_0^\pi \int_0^{2\pi} (\sin\theta_1)^{i-2} (\sin\theta_2)^{i-3} \ldots\ldots (\sin\theta_{i-3})^2 (\sin\theta_{i-2})\, d\theta_1 d\theta_2 \ldots d\theta_{i-3} d\theta_{i-2} d\theta_{i-1}$$
$$= \int_0^\pi \ldots \int_0^\pi \int_0^{2\pi} F(\theta_1 \theta_2 \ldots \theta_{i-1}) \cdot \frac{r'^2 \sim r^2}{D^i} \cdot r^{i-2} S' d\theta_1 d\theta_2 \ldots d\theta_{i-2} d\theta_{i-1},$$

where it is supposed that $\frac{r'^2 \sim r^2}{D^i}$ is expanded, and r' put $= r$ after the expansion has been performed.

But
$$\frac{r'^2 - r^2}{D^i} = -\frac{1}{i-2} \left\{ 2r' \frac{d}{dr'} \left(\frac{1}{D^{i-2}} \right) + (i-2) \frac{1}{D^{i-2}} \right\}.$$

Also $$\frac{1}{D^{i-2}}=\frac{1}{(r'^2-2rr'\cos\theta+r^2)^{\frac{i}{2}-1}}=\frac{1}{r'^{i-2}}\cdot\frac{1}{\left(1-2\frac{r}{r'}\cos\theta+\left(\frac{r}{r'}\right)^2\right)^{\frac{i}{2}-1}}$$

$$=\frac{1}{r'^{i-2}}\left\{R_0+R_1\frac{r}{r'}+\ldots+R_p\left(\frac{r}{r'}\right)^p+\ldots\right\}\text{ if } r'>r.$$

Similarly $$\frac{1}{D^{i-2}}=\frac{1}{r^{i-2}}\left\{R_0+R_1\frac{r'}{r}+\ldots+R_p\left(\frac{r'}{r}\right)^p+\ldots\right\}\text{ if } r'<r,$$

where $$R_p=\frac{1}{(i-4)(i-6)\ldots4.2}\left(\frac{d}{-\sin\theta d\theta}\right)^{\frac{i-2}{2}}\frac{\cos\left\{\left(p+\frac{i-2}{2}\right)\theta\right\}}{p+\frac{i-2}{2}}\text{ if } i \text{ be even,}$$

but $$=\frac{1}{(i-4)(i-6)\ldots5.3.1}\left(\frac{d}{-\sin\theta d\theta}\right)^{\frac{i-3}{2}}P_{\left(p+\frac{i-3}{2}\right)}\text{ if } i \text{ be odd;}$$

therefore if $r'>r$,

$$\frac{r'^2-r^2}{D^i}=\frac{1}{i-2}\cdot\frac{1}{r'^{i-2}}\left\{(i-2+2.0)R_0+(i-2+2.1)\frac{r}{r'}R_1+\ldots+(i-2+2p)\left(\frac{r}{r'}\right)^pR_p+\ldots\ldots\right\},$$

but if $r'<r$,

$$\frac{r^2-r'^2}{D^i}=\frac{1}{i-2}\cdot\frac{1}{r^{i-2}}\left\{(i-2+2.0)R_0+(i-2+2.1)\frac{r'}{r}R_1+\ldots+(i-2+2p)\left(\frac{r'}{r}\right)^pR_p+\ldots\ldots\right\}.$$

The series are convergent* except when $r=r'$; and in this case the unexpanded value of u having been shown to be finite and equal to

$$F(\theta'_1\theta'_2\ldots\ldots\theta'_{i-1})\,r\int_0^\pi\ldots\int_0^\pi\int_0^{2\pi}(\sin\theta_1)^{i-2}\ldots\ldots(\sin\theta_{i-2})\,d\theta_1\ldots\ldots d\theta_{i-2}\,d\theta_{i-1},$$

it follows that the sum of either series, obtained by substituting for $\frac{r'^2\sim r^2}{D^i}$ the above values in u, approaches more and more nearly to this value as r' approaches to equality with r.

$$\therefore\ F(\theta'_1\theta'_2\ldots\ldots\theta'_{i-1})\int_0^\pi\ldots\int_0^\pi\int_0^{2\pi}(\sin\theta_1)^{i-2}(\sin\theta_2)^{i-3}\ldots\ldots(\sin\theta_{i-3})^2(\sin\theta_{i-2})\,d\theta_1d\theta_2\ldots\ldots d\theta_{i-3}d\theta_{i-2}d\theta_{i-1}$$

$$=\frac{1}{i-2}\int_0^\pi\ldots\int_0^\pi\int_0^{2\pi}F(\theta_1\theta_2\ldots\ldots\theta_{i-1})\,Rd\theta_1d\theta_2\ldots\ldots d\theta_{i-3}d\theta_{i-2}d\theta_{i-1}$$

where $$R=\left\{\sum_{p=0}^{p=\infty}(2p+i-2)R_p\right\}(\sin\theta_1)^{i-2}(\sin\theta_2)^{i-3}\ldots\ldots(\sin\theta_{i-3})^2(\sin\theta_{i-2}).$$

* Because by Preliminary Proposition (a) R_p is greatest when $\cos\theta=1$, and then the general term is

$$(i-2+2p)\left(\frac{r}{r'}\right)^p\frac{\lfloor i+p-3}{\lfloor p\ \lfloor i-3},$$

the ratio of which to the preceding term can be made less than unity by taking p sufficiently great.

This may be written

$$F(\theta'_1\theta'_2 \ldots\ldots \theta'_{i-1}) = \frac{1}{2(i-2)\pi^{\frac{1}{2}i}}\Gamma(\tfrac{1}{2}i)\int_0^\pi \cdots \int_0^\pi\int_0^{2\pi} F(\theta_1\theta_2 \ldots\ldots \theta_{i-1})\, R\, d\theta_1 d\theta_2 \ldots d\theta_{i-3} d\theta_{i-2} d\theta_{i-1} \ldots \text{(B)}$$

where R has the same value as on previous page.

The above reasoning holds good except when θ'_{i-1} is either 0 or 2π, or one of the other variables say θ'_m is 0 or π.

In the first of these exceptional cases, it may be shown that for $F(\theta_1'\theta_2' \ldots \theta'_{i-1})$ on the left-hand side of the last equation there must be substituted

$$\tfrac{1}{2}\{F(\theta_1'\theta_2' \ldots\ldots \theta'_{i-2}\, 0) + F(\theta_1'\theta_2' \ldots\ldots \theta'_{i-2}\, 2\pi)\}.$$

In the second if $\theta'_m = \frac{0}{\pi}$, for $F(\theta_1'\theta_2' \ldots\ldots \theta'_{i-1})$ must be substituted

$$\frac{\displaystyle\int_0^\pi \cdots \int_0^\pi\int_0^{2\pi} F\left(\theta_1'\theta_2' \ldots\ldots \theta'_{m-1}\,\frac{0}{\pi}\,\theta_{m+1} \ldots\ldots \theta_{i-1}\right)(\sin\theta_{m+1})^{i-m-2} \ldots\ldots (\sin\theta_{i-2})\, d\theta_{m+1} \ldots\ldots d\theta_{i-2}\, d\theta_{i-1}}{\displaystyle\int_0^\pi \cdots \int_0^\pi\int_0^{2\pi} (\sin\theta_{m+1})^{i-m-2} \ldots\ldots (\sin\theta_{i-2})\, d\theta_{m+1} \ldots\ldots d\theta_{i-2}\, d\theta_{i-1}}.$$

3. It remains to show that the general term of the expansion is a linear function of the Normal Functions considered in this paper satisfying the equation in u_1 marked (II) in Part I. Art. 2, and to determine the coefficients of the several terms.

Firstly, it satisfies the equation in u_1 for

$$\left(\frac{d^2}{dx_1'^2} + \frac{d^2}{dx_2'^2} + \ldots\ldots + \frac{d^2}{dx_i'^2}\right)\frac{1}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + \ldots\ldots + (x_i - x_i')^2]^{\frac{i-2}{2}}} = 0;$$

$$\therefore \left(\frac{d^2}{dx_1'^2} + \frac{d^2}{dx_2'^2} + \ldots\ldots + \frac{d^2}{dx_i'^2}\right)\frac{1}{(r'^2 - 2rr'\cos\theta + r^2)^{\frac{i-2}{2}}} = 0,$$

where $\cos\theta = \dfrac{x_1x_1' + x_2x_2' + \ldots\ldots + x_ix_i'}{rr'}$, and the relation between the symbols $x_1' \ldots\ldots x_i'$ and $r'\theta_1' \ldots\ldots \theta'_{i-1}$ is the same as that between $x_1 \ldots\ldots x_i$ and $r\theta_1 \ldots\ldots \theta_{i-1}$.

$$\therefore \left(\frac{d^2}{dx_1'^2} + \frac{d^2}{dx_2'^2} + \ldots\ldots + \frac{d^2}{dx_i'^2}\right)\frac{1}{r^{i-2}}\left(R_0 + R_1\frac{r'}{r} + \ldots\ldots + R_p\left(\frac{r'}{r}\right)^p + \ldots\right) = 0, \text{ assuming } r' < r.$$

This being true for all values of r

$$\left(\frac{d^2}{dx_1'^2} + \frac{d^2}{dx_2'^2} + \ldots\ldots + \frac{d^2}{dx_i'^2}\right)(R_p r'^p) = 0;$$

$$\therefore \left[\left(\frac{d^2}{dr'^2} + \frac{i-1}{r'}\frac{d}{dr'}\right) + \frac{1}{r'^2}\left\{\left(\frac{d^2}{d\theta_1'^2} + (i-2)\cot\theta_1'\frac{d}{d\theta_1'}\right) + \ldots\ldots \right.\right.$$

$$\left.\left. + \frac{1}{\sin^2\theta_1' \ldots\ldots \sin^2\theta'_{i-2}}\frac{d^2}{d\theta'^2_{i-1}}\right\}\right](R_p r'^p) = 0;$$

$$\therefore \left[p(p+i-2) + \left(\frac{d^2}{d\theta_1'^2} + (i-2)\cot\theta_1' \frac{d}{d\theta_1'}\right) + \ldots\ldots + \frac{1}{\sin^2\theta_1' \ldots\ldots \sin^2\theta'_{i-2}} \frac{d^2}{d\theta'^2_{i-1}}\right] R_p = 0;$$

$$\therefore \left[p(p+i-2) + \left(\frac{d^2}{d\theta_1'^2} + (i-2)\cot\theta_1' \frac{d}{d\theta_1'}\right) + \ldots\ldots + \frac{1}{\sin^2\theta_1' \ldots\ldots \sin^2\theta'_{i-2}} \frac{d^2}{d\theta'^2_{i-1}}\right] V = 0$$

where $V = \int_0^\pi \ldots \int_0^\pi \int_0^{2\pi} F(\theta_1\theta_2 \ldots\ldots \theta_{i-1}) \cdot R_p (\sin\theta_1)^{i-2} \ldots\ldots (\sin\theta_{i-2})\, d\theta_1 \ldots\ldots d\theta_{i-2}\, d\theta_{i-1}$;

$\therefore$ the general term in the expansion of $F(\theta_1'\theta_2' \ldots\ldots \theta'_{i-1})$ satisfies the equation in u_1.

Secondly, it is a linear function of the Normal Functions considered in this paper.

R_p is a rational integral function of $\cos\theta$ of degree p, containing only even or only odd powers of $\cos\theta$;

$\therefore R_p r'^p$ is a rational integral function of $x_1'x_2' \ldots\ldots x_i'$ of degree p satisfying the equation $\frac{d^2u}{dx_1'^2} + \frac{d^2u}{dx_2'^2} + \ldots\ldots + \frac{d^2u}{dx_i'^2} = 0$;

$\therefore$ by Art. 11 of Part I., it is a linear function of functions of the form

$$r'^p (\sin\theta_1')^{p_1} \Theta_1' \ldots\ldots (\sin\theta'_{i-2})^{p_{i-2}} \Theta'_{i-2} \left(\begin{matrix}\cos\\ \sin\end{matrix} p_{i-2}\theta'_{i-1}\right).$$

Let one of these functions be denoted by $r'^p U'$, then $R_p = \Sigma A \,.\, U'$; where A is a function of $\frac{x_1}{r}, \frac{x_2}{r}, \ldots\ldots \frac{x_i}{r}$ only, and therefore of $\theta_1\theta_2 \ldots\ldots \theta_{i-1}$ only.

Substituting this in the general term of the expansion, it becomes

$$\Sigma U' \int_0^\pi \ldots \int_0^\pi \int_0^{2\pi} \frac{2p+i-2}{i-2} \,.\, A \,.\, F(\theta_1\theta_2 \ldots\ldots \theta_{i-1})\, S' d\theta_1 d\theta_2 \ldots\ldots d\theta_{i-3}\, d\theta_{i-2}\, d\theta_{i-1}$$

$= \Sigma U'A'$, where A' is a constant.

But U' is a normal function of the form considered in this paper; $\therefore F(\theta_1'\theta_2' \ldots\ldots \theta'_{i-1})$ may be expanded in a series of functions each one of which is a linear function of the normal functions considered in this paper.

Thirdly, to determine the coefficients of the several terms.

The following method is adopted because it will lead to the development of R_p in Normal Functions.

R_p is by the foregoing argument the sum of such terms as

$$(\sin\theta_1')^{p_1} \Theta_1' \ldots\ldots (\sin\theta'_{i-2})^{p_{i-2}} \Theta'_{i-2} \left(\begin{matrix}\cos\\ \sin\end{matrix} p_{i-2}\theta'_{i-1}\right)$$

multiplied by some quantity which is independent of the variables $\theta_1'\theta_2' \ldots\ldots \theta'_{i-1}$. But since R_p is a symmetrical function of the accented and unaccented variables, this quantity must contain $(\sin\theta_1)^{p_1}\Theta_1 \ldots\ldots (\sin\theta_{i-2})^{p_{i-2}}\Theta_{i-2}\left(\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} p_{i-2}\theta_{i-1}\right)$ as a factor. Thus R_p is the sum of such functions as

$$B(\sin\theta_1')^{p_1}\Theta_1' \ldots\ldots (\sin\theta'_{i-2})^{p_{i-2}}\Theta'_{i-2}\left(\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} p_{i-2}\theta'_{i-1}\right)(\sin\theta_1)^{p_1}\Theta_1 \ldots\ldots (\sin\theta_{i-2})^{p_{i-2}}\Theta_{i-2}\left(\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} p_{i-2}\theta_{i-1}\right)$$

where B is a constant,

$$\therefore\ F(\theta_1'\theta_2' \ldots\ldots \theta'_{i-1})\int_0^\pi \ldots \int_0^\pi\int_0^{2\pi}(\sin\theta_1)^{i-2}(\sin\theta_2)^{i-3}\ldots\ldots(\sin\theta_{i-3})^2(\sin\theta_{i-2})\,d\theta_1 d\theta_2\ldots\ldots d\theta_{i-3}\,d\theta_{i-2}\,d\theta_{i-1}$$

$$= \frac{1}{i-2}\int_0^\pi \ldots \int_0^\pi\int_0^{2\pi} F(\theta_1\theta_2 \ldots\ldots \theta_{i-1})\sum_{p=0}^{p=\infty}\Big[(2p+i-2)\,R'_p\Big]\,S'd\theta_1 \ldots\ldots d\theta_{i-1}$$

where $R'_p = \Sigma B(\sin\theta_1')^{p_1}\Theta_1' \ldots (\sin\theta'_{i-2})^{p_{i-2}}\Theta'_{i-2}\left(\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} p_{i-2}\theta'_{i-1}\right)(\sin\theta_1)^{p_1}\Theta_1 \ldots (\sin\theta_{i-2})^{p_{i-2}}\Theta_{i-2}\left(\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} p_{i-2}\theta_{i-1}\right)$.

Now $\theta_1'\theta_2' \ldots\ldots \theta'_{i-1}$ are independent of $\theta_1\theta_2 \ldots\ldots \theta_{i-1}$; therefore in each term represented in the summation of the second member of the last equation, they may be taken outside the signs of integration. After this has been done, multiply both sides of this equation by

$$(\sin\theta_1')^{p_1+i-2}\Theta_1' \ldots\ldots (\sin\theta'_{i-2})^{p_{i-2}+1}\Theta'_{i-2}\left(\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} p_{i-2}\theta'_{i-1}\right),$$

and integrate with regard to θ'_{i-1} from 0 to 2π, and with regard to the other variables from 0 to π;

$$\therefore \int_0^\pi \ldots \int_0^\pi\int_0^{2\pi} F(\theta_1'\theta_2' \ldots\ldots \theta'_{i-1})(\sin\theta_1')^{p_1+i-2}\Theta_1' \ldots\ldots (\sin\theta'_{i-2})^{p_{i-2}+1}\Theta'_{i-2}\left(\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} p_{i-2}\theta'_{i-1}\right)d\theta_1' \ldots\ldots d\theta'_{i-1}$$

multiplied by $\int_0^\pi \ldots \int_0^\pi\int_0^{2\pi}(\sin\theta_1)^{i-2} \ldots\ldots (\sin\theta_{i-2})\,d\theta_1 \ldots\ldots d\theta_{i-1}$

$$= \frac{1}{i-2}\int_0^\pi \ldots \int_0^\pi\int_0^{2\pi} F(\theta_1\theta_2 \ldots \theta_{i-1})(2p+i-2)B(\sin\theta_1)^{p_1+i-2}\Theta_1 \ldots (\sin\theta_{i-2})^{p_{i-2}+1}\Theta_{i-2}\left(\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} p_{i-2}\theta_{i-1}\right)d\theta_1 \ldots d\theta_{i-1}$$

multiplied by $\int_0^\pi \ldots \int_0^\pi\int_0^{2\pi}(\sin\theta_1')^{2p_1+i-2}\Theta_1'^2 \ldots\ldots (\sin\theta'_{i-2})^{2p_{i-2}+1}\Theta'^2_{i-2}\left(\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} p_{i-2}\theta'_{i-1}\right)^2 d\theta_1' \ldots\ldots d\theta'_{i-1}$.

All the remaining terms disappear in virtue of the conjugate property of Art. 8;

and the first integral of the second member of the last equation is $B(2p+i-2)$ times the first integral of the first member;

$$\therefore\ B = \frac{i-2}{2p+i-2}\cdot\frac{\int_0^\pi \ldots \int_0^\pi\int_0^{2\pi}(\sin\theta_1)^{i-2} \ldots\ldots (\sin\theta_{i-2})\,d\theta_1 \ldots\ldots d\theta_{i-1}}{\int_0^\pi \ldots \int_0^\pi\int_0^{2\pi}(\sin\theta_1)^{2p_1+i-2}\Theta_1^2 \ldots\ldots (\sin\theta_{i-2})^{2p_{i-2}+1}\Theta^2_{i-2}\left(\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} p_{i-2}\theta_{i-1}\right)^2 d\theta_1 \ldots\ldots d\theta_{i-1}}.$$

Observing that $\int_0^\pi \ldots \int_0^\pi\int_0^{2\pi}(\sin\theta_1)^{i-2} \ldots\ldots (\sin\theta_{i-2})\,d\theta_1 \ldots\ldots d\theta_{i-1} = 2\frac{\pi^{\frac{1}{2}i}}{\Gamma(\frac{1}{2}i)}$; it follows by Part I. Art. 9, that

$B = 2\,\underline{|i-4}\,\underline{|i-5}\,\ldots\ldots\,\underline{|3}\,\underline{|2}\,(2p_{i-3}+1)\,(2p_{i-4}+2)\,\ldots\ldots\,(2p_1+i-3)$ multiplied by

$$\frac{\underline{|p_{i-3}-p_{i-2}}}{\underline{|p_{i-3}+p_{i-2}}}\cdot\frac{\underline{|p_{i-4}-p_{i-3}}}{\underline{|p_{i-4}+p_{i-3}+1}}\cdots\cdots\cdots\frac{\underline{|p-p_1}}{\underline{|p+p_1+i-3}},$$

except when $p_{i-2}=0$, when only half this value is to be taken.

This form is free from integrals, but the other form is more convenient for the present purpose.

The value of B is the same whether the term $\cos^2 p_{i-2}\theta_{i-1}$ or $\sin^2 p_{i-2}\theta_{i-1}$ be employed to calculate the integral in the denominator.

$$\therefore\ R_p = \Sigma\,\frac{i-2}{2p+i-2}\,\frac{T_p\int_0^\pi\cdots\int_0^\pi\int_0^{2\pi}(\sin\theta_1)^{i-2}\ldots(\sin\theta_{i-2})\,d\theta_1\ldots d\theta_{i-1}}{\int_0^\pi\cdots\int_0^\pi\int_0^{2\pi}(\sin\theta_1)^{2p_1+i-2}\,\Theta_1^2\ldots(\sin\theta_{i-2})^{2p_{i-2}+1}\Theta_{i-2}^2\left(\begin{matrix}\cos\\ \sin\end{matrix}\,p_{i-2}\theta_{i-1}\right)^2 d\theta_1\ldots d\theta_{i-1}}$$

where $T_p = (\sin\theta_1')^{p_1}\Theta_1'\ldots(\sin\theta'_{i-2})^{p_{i-2}}\Theta'_{i-2}\,(\sin\theta_1)^{p_1}\Theta_1\ldots(\sin\theta_{i-2})^{p_{i-2}}\Theta_{i-2}\cos p_{i-2}(\theta_{i-1}-\theta'_{i-1})$.

The summation extends to all possible positive integral values of the indices $p_1\ldots p_{i-2}$ not greater than p, and such that in any one term no index is greater than one which precedes it. If p_{i-2} be zero, the factor $\left(\begin{matrix}\cos\\ \sin\end{matrix}\,p_{i-2}\theta_{i-1}\right)^2$ is to be replaced by 1.

Substituting this value of R_p, it appears that

$$F(\theta_1'\theta_2'\ldots\theta'_{i-1}) = \Sigma\,(\sin\theta_1')^{p_1}\Theta_1'\ldots(\sin\theta'_{i-2})^{p_{i-2}}\Theta'_{i-2}\,\{C\cos p_{i-2}\theta'_{i-1}+D\sin p_{i-2}\theta'_{i-1}\}$$

where

$$C=\frac{\int_0^\pi\cdots\int_0^\pi\int_0^{2\pi}(\sin\theta_1)^{p_1+i-2}\Theta_1\ldots(\sin\theta_{i-2})^{p_{i-2}+1}\Theta_{i-2}\,(\cos p_{i-2}\theta_{i-1})\,F(\theta_1\theta_2\ldots\theta_{i-1})\,d\theta_1\ldots d\theta_{i-1}}{\int_0^\pi\cdots\int_0^\pi\int_0^{2\pi}(\sin\theta_1)^{2p_1+i-2}\Theta_1^2\ldots(\sin\theta_{i-2})^{2p_{i-2}+1}\Theta_{i-2}^2\,(\cos p_{i-2}\theta_{i-1})^2\,d\theta_1\ldots d\theta_{i-1}}$$

and

$$D=\frac{\int_0^\pi\cdots\int_0^\pi\int_0^{2\pi}(\sin\theta_1)^{p_1+i-2}\Theta_1\ldots(\sin\theta_{i-2})^{p_{i-2}+1}\Theta_{i-2}\,(\sin p_{i-2}\theta_{i-1})\,F(\theta_1\theta_2\ldots\theta_{i-1})\,d\theta_1\ldots d\theta_{i-1}}{\int_0^\pi\cdots\int_0^\pi\int_0^{2\pi}(\sin\theta_1)^{2p_1+i-2}\Theta_1^2\ldots(\sin\theta_{i-2})^{2p_{i-2}+1}\Theta_{i-2}^2\,(\sin p_{i-2}\theta_{i-1})^2\,d\theta_1\ldots d\theta_{i-1}}$$

(C).

The summation in this case extends to all possible positive integral values of the indices $p_1\ldots p_{i-2}$, such that in any one term no index is greater than the index p with which that term is connected, nor any index greater than any one which precedes it.

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