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M.DCCC.XXXV.

CONTENTS OF THE FIFTH VOLUME.

PART I.

PART II.

PART III.

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M.DCCC.XXXIII.

I. *Mathematical Investigations concerning the Laws of the Equilibrium of Fluids analagous to the Electric Fluid, with other similar Researches.*

BY GEORGE GREEN, Esq. *Communicated by Sir Edward Ffrench Bromhead, Bart.* M.A. F.R.S.L. and E.

[Read *Nov.* 12, 1832.]

AMONGST the various subjects which have at different times occupied the attention of Mathematicians, there are probably few more interesting in themselves, or which offer greater difficulties in their investigation, than those in which it is required to determine mathematically the laws of the equilibrium or motion of a system composed of an infinite number of free particles all acting upon each other mutually, and according to some given law. When we conceive, moreover, the law of the mutual action of the particles to be such that the forces which emanate from them may become insensible at sensible distances, the researches to which the consideration of these forces lead will be greatly simplified by the limitation thus introduced, and may be regarded as forming a class distinct from the rest. Indeed they then for the most part terminate in the resolution of equations between the values of certain functions at any point taken at will in the interior of the system, and the values of the partial differentials of these functions at the same point. When on the contrary the forces in question continue sensible at every finite distance, the researches dependent upon them become far more complicated, and often require all the resources of the modern analysis for their successful prosecution. It would be easy so to exhibit the theories of the equilibrium and motion of ordinary fluids, as to offer instances of researches appertaining to the former class, whilst the mathematical investigations to which the theories of Electricity and Magnetism have given rise may be considered as interesting examples of such as belong to the latter class.

It is not my chief design in this paper to determine mathematically the density of the electric fluid in bodies under given circumstances, having elsewhere* given some general methods by which this may be effected, and applied these methods to a variety of cases not before submitted to calculation. My present object will be to determine the laws of the equilibrium of an hypothetical fluid analagous to the electric fluid, but of which the law of the repulsion of the particles, instead of being inversely as the square of the distance, shall be inversely as any power n of the distance; and I shall have more particularly in view the determination of the density of this fluid in the interior of conducting spheres when in equilibrium, and acted upon by any exterior bodies whatever, though since the general method by which this is effected will be equally applicable to circular plates and ellipsoids. I shall present a sketch of these applications also.

It is well known that in enquiries of a nature similar to the one about to engage our attention, it is always advantageous to avoid the direct consideration of the various forces acting upon any particle p of the fluid in the system, by introducing a particular function V of the co-ordinates of this particle, from the differentials of which the values of all these forces may be immediately deduced†. We have, therefore, in the present paper endeavoured, in the first place, to find the value of V, where the density of the fluid in the interior of a sphere is given by means of a very simple consideration, which in a great measure obviates the difficulties usually attendant on researches of this kind, have been able to determine the value V, where ρ, the density of the fluid in any element dv of the sphere's volume, is equal to the product of two factors, one of which is a very simple function containing an arbitrary exponent β, and the remaining one f is equal to any rational

* Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.

† This function in the present case will be obtained by taking the sum of all the molecules of a fluid acting upon p, divided by the $(n-1)^{\text{th}}$ power of their respective distances from p; and indeed the function which Laplace has represented by V in the third book of the *Mecanique Celeste,* is only a particular value of our more general one produced by writing 2 in the place of the general exponent n.

and entire function whatever of the rectangular co-ordinates of the element dv, and afterwards by a proper determination of the exponent β, have reduced the resulting quantity V to a rational and entire function of the rectangular co-ordinates of the particle p, of the same degree as the function f. This being done, it is easy to perceive that the resolution of the inverse problem may readily be effected, because the coefficients of the required factor f will then be determined from the given coefficients of the rational and entire function V, by means of linear algebraic equations.

The method alluded to in what precedes, and which is exposed in the two first articles of the following paper, will enable us to assign generally the value of the induced density ρ for any ellipsoid, whatever its axes may be, provided the inducing forces are given explicitly in functions of the co-ordinates of p; but when by supposing these axes equal we reduce the ellipsoid to a sphere, it is natural to expect that as the form of the solid has become more simple, a corresponding degree of simplicity will be introduced into the results; and accordingly, as will be seen in the fourth and fifth articles, the complete solutions both of the direct and inverse problems, considered under their most general point of view, are such that the required quantities are there always expressed by simple and explicit functions of the known ones, independent of the resolution of any equations whatever.

The first five articles of the present paper being entirely analytical, serve to exhibit the relations which exist between the density ρ of our hypothetical fluid, and its dependent function V; but in the following ones our principal object has been to point out some particular applications of these general relations.

In the seventh article, for example, the law of the density of our fluid when in equilibrium in the interior of a conductory sphere, has been investigated, and the analytical value of ρ there found admits of the following simple enunciation.

The density ρ of free fluid at any point p within a conducting sphere A, of which O is the centre, is always proportional to the $(n-4)^{\text{th}}$ power of the radius of the circle formed by the intersection of a plane perpendicular to the ray Op with the surface of the sphere itself, provided

n is greater than 2. When on the contrary n is less than 2, this law requires a certain modification; the nature of which has been fully investigated in the article just named, and the one immediately following.

It has before been remarked, that the generality of our analysis will enable us to assign the density of the free fluid which would be induced in a sphere by the action of exterior forces, supposing these forces are given explicitly in functions of the rectangular co-ordinates of the point of space to which they belong. But, as in the particular case in which our formulæ admit of an application to natural phenomena, the forces in question arise from electric fluid diffused in the inducing bodies, we have in the ninth article considered more especially the case of a conducting sphere acted upon by the fluid contained in any exterior bodies whatever, and have ultimately been able to exhibit the value of the induced density under a very simple form, whatever the given density of the fluid in these bodies may be.

The tenth and last article contains an application of the general method to circular planes, from which results, analagous to those formed for spheres in some of the preceding ones are deduced; and towards the latter part, a very simple formula is given, which serves to express the value of the density of the free fluid in an infinitely thin plate, supposing it acted upon by other fluid, distributed according to any given law in its own plane. Now it is clear, that if to the general exponent n we assign the particular value 2, all our results will become applicable to electrical phenomena. In this way the density of the electric fluid on an infinitely thin circular plate, when under the influence of any electrified bodies whatever, situated in its own plane, will become known. The analytical expression which serves to represent the value of this density, is remarkable for its simplicity; and by suppressing the term due to the exterior bodies, immediately gives the density of the electric fluid on a circular conducting plate, when quite free from all extraneous action. Fortunately, the manner in which the electric fluid distributes itself in the latter case, has long since been determined experimentally by Coulomb. We have thus had the advantage of comparing our theoretical results with those of a very accurate observer, and the differences between them are not greater than may be supposed due to the unavoidable errors of experiment, and to that which would necessarily be produced by employing plates of a finite thickness, whilst the theory supposes this thickness infinitely small. Moreover, the errors are all of the same kind with regard to sign, as would arise from the latter cause.

1. If we conceive a fluid analogous to the electric fluid, but of which the law of the repulsion of the particles instead of being inversely as the square of the distance is inversely as some power n of the distance, and suppose ρ to represent the density of this fluid, so that dv being an element of the volume of a body A through which it is diffused, ρdv may represent the quantity contained in this element, and if afterwards we write g for the distance between dv and any particle p under consideration, and these form the quantity

$$V = \int \frac{\rho dv}{g^{n-1}};$$

the integral extending over the whole volume of A, it is well known that the force with which a particle p of this fluid situate in any point of space is impelled in the direction of any line q and tending to increase this line will always be represented by

$$(1) \ldots\ldots\ldots\ldots \frac{1}{1-n}\left(\frac{dV}{dq}\right);$$

V being regarded as a function of three rectangular co-ordinates of p, one of which co-ordinates coincides with the line q, and $\left(\frac{dV}{dq}\right)$ being the partial differential of V, relative to this last co-ordinate.

In order now to make known the principal artifices on which the success of our general method for determining the function V mainly depends, it will be convenient to begin with a very simple example.

Let us therefore suppose that the body A is a sphere, whose centre is at the origin O of the co-ordinates, the radius being 1; and ρ is such a function of x', y', z', that where we substitute for x', y', z' their values in polar co-ordinates

$$x' = r' \cos\theta', \quad y' = r' \sin\theta' \cos\varpi', \quad z' = r' \sin\theta' \sin\varpi',$$

it shall reduce itself to the form

$$\rho = (1 - r'^2)^\beta . f(r'^2);$$

f being the characteristic of any rational and entire function whatever: which is in fact equivalent to supposing

$$\rho = (1 - x'^2 - y'^2 - z'^2)^\beta . f(x'^2 + y'^2 + z'^2).$$

Now, when as in the present case, ρ can be expanded in a series of the entire powers of the quantities x', y', z', and of the various products of these powers, the function V will always admit of a similar expansion in the entire powers and products of the quantities x, y, z, provided the point p continues within the body A*, and as moreover V evidently depends on the distance $Op = r$ and is independent of θ and ϖ, the two other polar co-ordinates of p, it is easy to see that the quantity V when we substitute for x, y, z these values

$$x = r\cos\theta, \quad y = r\sin\theta\cos\varpi, \quad z = r\sin\theta\sin\varpi$$

will become a function of r, only containing none but the even powers of this variable.

But since we have

$$dv = r'^2 dr'\, d\theta'\, d\varpi' \sin\theta', \quad \text{and } \rho = (1 - r'^2)^\beta . f(r'^2),$$

the value of V becomes

$$V = \int \frac{\rho\, dv}{g^{n-1}} = \int r'^2 dr'\, d\theta'\, d\varpi' \sin\theta' (1 - r'^2)^\beta f(r'^2) \cdot g^{1-n};$$

the integrals being taken from $\varpi' = 0$ to $\varpi' = 2\pi$, from $\theta' = 0$ to $\theta' = \pi$, and from $r' = 0$ to $r' = 1$.

* The truth of this assertion will become tolerably clear, if we recollect that V may be regarded as the sum of every element ρdv of the body's mass divided by the $(n-1)^{\text{th}}$ power of the distance of each element from the point p, supposing the density of the body A to be expressed by ρ, a continuous function of x', y', z'. For then the quantity V is represented by a continuous function, so long as p remains within A; but there is in general a violation of the law of continuity whenever the point p passes from the interior to the exterior space. This truth, however, as enunciated in the text, is demonstrable, but since the present paper is a long one, I have suppressed the demonstrations to save room.

Now V may be considered as composed of two parts, one V' due to the sphere B whose centre is at the origin O, and surface passes through the point p, and another V'' due to the shell S exterior to B. In order to obtain the first part, we must expand the quantity g^{1-n} in an ascending series of the powers of $\frac{r'}{r}$. In this way we get

$$g^{1-n} = [r^2 - 2rr' \{\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\varpi' - \varpi)\} + r'^2]^{\frac{1-n}{2}}$$

$$= r^{1-n} . \left(Q_0 + Q_1 \frac{r'}{r} + Q_2 \frac{r'^2}{r^2} + Q_3 \frac{r'^3}{r^3} + \&c.\right).$$

If then we substitute this series for g^{1-n} in the value of V', and after having expanded the quantity $(1-r'^2)^\beta$, we effect the integrations relative to r', θ', and ϖ', we shall have a result of the form

$$V' = r^{4-n} \{A + Br^2 + Cr^4 + \&c.\}$$

seeing that in obtaining the part of V before represented by V', the integral relative to r' ought to be taken from $r' = 0$ to $r' = r$ only.

To obtain the value of V'', we must expand the quantity g^{1-n} in an ascending series of the powers of $\frac{r}{r'}$, and we shall thus have

$$g^{1-n} = (r^2 - 2rr' [\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\varpi - \varpi')] + r'^2)^{\frac{1-n}{2}}$$

$$= r'^{1-n} . \{Q_0 + Q_1 \frac{r}{r'} + Q_2 \frac{r^2}{r'^2} + Q_3 \frac{r^3}{r'^3} + \&c.\};$$

the coefficients Q_0, Q_1, Q_2, &c. being the same as before.

The expansion here given being substituted in V'', there will arise a series of the form

$$V'' = T_0 + T_1 + T_2 + T_3 + \&c.$$

of which the general term T_s is

$$T_s = \int d\theta' d\varpi' \sin\theta' \, Q_s \int r'^2 dr' \frac{r^s}{r'^{s+n-1}} (1 - r'^2)^\beta . f(r'^2);$$

the integrals being taken from $r' = r$ to $r' = 1$, from $\theta' = 0$ to $\theta' = \pi$, and from $\varpi' = 0$ to $\varpi' = 2\pi$. This will be evident by recollecting that the triple integral by which the value of V'' is expressed, is the same as the one before given for V, except that the integration relative to r', instead of extending from $r'=0$ to $r'=1$, ought only to extend from $r'=r$ to $r=1$.

But the general term in the function $f(r'^2)$ being represented by $A_t r'^{2t}$, the part of T_s dependent on this term will evidently be

$$(2)\ldots\ldots\ldots\ldots A_t r^s \int d\theta' d\varpi' \sin\theta' . Q_s \int r'^{2t+3-s-n} dr' (1-r'^2)^\beta;$$

the limits of the integrals being the same as before.

We thus see that the value of T_s and consequently of V'' would immediately be obtained, provided we had the value of the general integral

$$\int_r^1 r'^b dr' (1-r'^2)^\beta,$$

which being expanded and integrated becomes

$$\left.\begin{array}{l} \dfrac{1}{b+1} - \dfrac{\beta}{1}\cdot\dfrac{1}{b+3} + \dfrac{\beta(\beta-1)}{1.2}\cdot\dfrac{1}{b+5} - \&c. \\ -\dfrac{r^{b+1}}{b+1} + \dfrac{\beta}{1}\cdot\dfrac{r^{b+3}}{b+3} - \dfrac{\beta(\beta-1)}{1.2}\cdot\dfrac{r^{b+5}}{b+5} + \&c. \end{array}\right\}$$

but since the first line of this expression is the well known expansion of

$$\left(\frac{p}{q}\right) \text{ or } \frac{\Gamma\left(\frac{p}{n}\right)\Gamma\left(\frac{q}{n}\right)}{n\Gamma\left(\frac{p+q}{n}\right)},$$

when $n=2.p=b+1$ and $q=2(\beta+1)$ we have ultimately,

$$(3)\ldots\ldots \int_r^1 r'^b dr'(1-r'^2)^\beta = \frac{\Gamma\left(\frac{b+1}{2}\right)\Gamma(\beta+1)}{2\Gamma\left(\frac{b+3}{2}+\beta\right)} - 1\times\frac{r^{b+1}}{b+1} + \frac{\beta}{1}\times\frac{r^{b+3}}{b+3} - \&c.$$

By means of the result here obtained, we shall readily find the value of the expression (2) which will evidently contain one term multiplied by r^s and an infinite number of others, in all of which the quantity r is affected with the exponent n. But as in the case under consideration, n may represent any number whatever, fractionary or irrational, it is clear that none of the terms last mentioned can enter into V, seeing that it ought to contain the even powers of r only, thence the terms of this kind entering into V'', must necessarily be destroyed by corresponding ones in V'. By rejecting them, therefore, the formula (2) will become

$$(2')\ldots\ldots\ldots\frac{\Gamma\left(t+2-\frac{s+n}{2}\right)\Gamma(\beta+1)}{2\Gamma\left(t+\beta+3-\frac{s+n}{2}\right)}A_t r^s.\int d\theta' d\varpi' \sin\theta' Q_s.$$

But as V ought to contain the even powers of r only, those terms in which the exponent s is an odd number, will vanish of themselves after all the integrations have been effected, and consequently the only terms which can appear in V, are of the form

$$(4)\ldots\ldots\ldots\frac{\Gamma\left(t+2-s'-\frac{n}{2}\right)\Gamma(\beta+1)}{2\Gamma\left(t+\beta+3-s'-\frac{n}{2}\right)}A_t r^{2s'}\int d\theta' d\varpi' \sin\theta' Q_{2s'};$$

where, since s is an even number, we have written $2s'$ in the place of s, and as $Q_{2s'}$ is always a rational and entire function of $\cos\theta'$, $\sin\theta'$ $\cos\varpi'$, and $\sin\theta'\sin\varpi'$, the remaining integrations may immediately be effected.

Having thus the part of $T'_{2s'}$ due to any term $A_t r'^{2t}$ of the function $f(r'^2)$ we have immediately the value of $T'_{2s'}$ and consequently of V'', since

$$V''=U'+T_0'+T_2'+T_4'+T_6'+\&c.;$$

U' representing the sum of all the terms in V'' which have been rejected on account of their form, and T_0' T_1' T_2' the value of T_0 T_1 T_2, &c. obtained by employing the truncated formula (2) in the place of the complete one (2).

$$\text{But}\quad V=V'+V''=V'+U'+T_0'+T_2'+T_4'+T_6'+\&c.$$

or by transposition,

$$V-T_0'-T_2'-T_4'-T_6'-\&c.=V'+U',$$

and as in this equation, the function on the left side contains none but the even powers of the indeterminate quantity r, whilst that on the right does not contain any of the even powers of r, it is clear that each of its sides ought to be equated separately to zero. In this way the left side gives

$$(5)\ldots\ldots\ldots V = T_0' + T_2' + T_4' + T_6' + \&c.$$

Hitherto the value of the exponent β has remained quite arbitrary, but the known properties of the function Γ will enable us so to determine β, that the series just given shall contain a finite number of terms only. We shall thus greatly simplify the value of V, and reduce it in fact to a rational and entire function of r^2.

For this purpose, we may remark that

$$\Gamma(0) = \infty, \quad \Gamma(-1) = \infty, \quad \Gamma(-2) = \infty, \textit{ in infinitum.}$$

If therefore we make $-\frac{n}{2} + \beta =$ any whole number positive or negative, the denominator of the function (4) will become infinite, and consequently the function itself will vanish when s' is so great that $-\frac{n}{2} + \beta + t + 3 - s'$ is equal to zero or any negative number, and as the value of t never exceeds a certain number, seeing that $f(r'^2)$ is a rational and entire function, it is clear that the series (4) will terminate of itself, and V become a rational and entire function of r^2.

(2) The method that has been employed in the preceding article where the function by which the density is expressed is of the particular form

$$\rho = (1 - r'^2)^\beta . f(r'^2)$$

may by means of a very slight modification, be applied to the far more general value

$$\rho = (1 - r'^2)^\beta f(x', y', z') = (1 - x'^2 - y'^2 - z'^2)^\beta f(x', y', z')$$

where f is the characteristic of any rational and entire function whatever: and the same value of β which reduces V to a rational and entire function of r^2 in the first case, reduces it in the second to a similar function of x, y, z and the rectangular co-ordinates of p.

To prove this, we may remark that the corresponding value V will become

$$V = \int r'^2 dr' d\theta' d\varpi' \sin\theta' (1 - r'^2)^\beta f(x', y', z') g^{1-n};$$

the integral being conceived to comprehend the whole volume of the sphere.

Let now the function f be divided into two parts, so that

$$f(x', y', z') = f_1(x', y', z') + f_2(x', y', z');$$

f_1 containing all the terms of the function f, in which the sum of the exponents of x', y', z' is an odd number; and f_2 the remaining terms, or those where the same sum is an even number. In this way we get

$$V = V_1 + V_2;$$

the functions V_1 and V_2 corresponding to f_1 and f_2, being

$$V_1 = \int r'^2 dr' d\theta' d\varpi' \sin\theta' (1 - r'^2)^\beta f_1(x', y', z') g^{1-n},$$

$$V_2 = \int r'^2 dr' d\theta' d\varpi' \sin\theta' (1 - r'^2)^\beta f_2(x', y', z') g^{1-n}.$$

We will in the first place endeavour to determine the value V_1; and for this purpose, by writing for x', y', z' their values before given in r', θ', ϖ', we get

$$f_1(x', y', z') = r'\psi(r'^2);$$

the coefficients of the various powers of r'^2 in $\psi(r'^2)$ being evidently rational and entire functions of $\cos\theta'$, $\sin\theta' \cos\varpi'$, and $\sin\theta' \sin\varpi'$. Thus

$$V_1 = \int r'^2 dr' d\theta' d\varpi' \sin\theta' (1 - r'^2)^\beta r'\psi(r'^2) g^{1-n};$$

this integral, like the foregoing, comprehending the whole volume of the sphere.

Now as the density corresponding to the function V_1 is

$$\rho_1 = (1 - x'^2 - y'^2 - z'^2)^\beta f_1(x', y', z'),$$

it is clear that it may be expanded in an ascending series of the entire powers of x', y', z', and the various products of these powers consequently, as was before remarked (Art. 1.), V_1 admits of an analagous expansion in entire powers and products of x, y, z. Moreover, as the density ρ_1

retains the same numerical value, and merely changes its sign when we pass from the element dv to a point diametrically opposite, where the co-ordinates x', y', z' are replaced by $-x'$, $-y'$, $-z'$: it is easy to see that the function V_1, depending upon ρ_1, possesses a similar property, and merely changes its sign when x, y, z, the co-ordinates of p, are changed into $-x$, $-y$, $-z$. Hence the nature of the function V_1 is such that it can contain none but the odd powers of r, when we substitute for the rectangular co-ordinates x, y, z, their values in the polar co-ordinates r, θ, ϖ.

Having premised these remarks, let us now suppose V_1 is divided into two parts, one V_1' due to the sphere B which passes through the particle p, and the other V_1'' due to the exterior shell S. Then it is evident by proceeding, as in the case where $\rho = (1-r'^2)^\beta f(r^2)$, that V_1' will be of the form

$$V_1' = r^{5-n}\{A + Br^2 + Cr^4 + \&c.\};$$

the coefficients A, B, C, &c. being quantities independent of the variable r.

In like manner we have also

$$V_1'' = \int r'^2 dr' d\theta' d\varpi' \sin\theta' (1-r'^2)^\beta . r' \psi(r'^2) g^{1-n};$$

the integrals being taken from $r'=r$ to $r=1$, from $\theta'=0$ to $\theta'=\pi$, and from $\varpi'=0$ to $\varpi'=2\pi$.

By substituting now the second expansion of g^{1-n} before used (Art. 1.), the last expression will become

$$V_1'' = T_0 + T_1 + T_2 + T_3 + \&c.$$

of which series the general term is

$$T_s = \int d\theta' d\varpi' \sin\theta' Q_s \int r'^{4-n} dr' (1-r'^2)^\beta \frac{r^s}{r'^s} \psi(r'^2).$$

Moreover, the general term of the function $\psi(r'^2)$ being represented by $A_t r'^{2t}$, the portion of T_s due to this term, will be

$$(a) \ldots\ldots\ldots r^s \int d\theta' d\varpi' \sin\theta' Q_s A_t \int r'^{4-n+2t-s} dr' (1-r'^2)^\beta;$$

the limits of the integrals being the same as before.

If now we effect the integrations relative to r' by means of the formula (3), Art. 1, and reject as before those powers of the variable r, in which it is affected, with the exponent n, since these ought not enter into the function V_1, the last formula will become

$$(a') \ldots\ldots\ldots \frac{\Gamma\left(\frac{5-n+2t-s}{2}\right)\Gamma(\beta+1)}{2\,\Gamma\left(\frac{7+2\beta-n+2t-s}{2}\right)} r^s \int d\theta' d\varpi' \sin\theta' Q_s A_t,$$

and as V_1 ought to contain none but the odd powers of r, we may make $s = 2s'+1$, and disregard all those terms in which s is an even number, since they will necessarily vanish after all the operations have been effected. Thus the only remaining terms will be of the form

$$\frac{\Gamma\left(\frac{4-n+2t-2s'}{2}\right)\Gamma(\beta+1)}{2.\Gamma\left(\frac{6+2\beta-n+2t-2s'}{2}\right)} r^{2s'+1} \int d\theta' d\varpi' \sin\theta' Q_{2s'+1} A_t;$$

where, as A_t and $Q_{2s'+1}$ are both rational and entire functions of $\cos\theta'$, $\sin\theta' \cos\varpi'$, $\sin\theta' \sin\varpi'$, the remaining integrations from $\theta'=0$ to $\theta'=\pi$, and $\varpi'=0$ to $\varpi'=2\pi$, may easily be effected in the ordinary way.

If now we follow the process employed in the preceding article, and suppose T_0', T_1', T_2', &c. are what T_0, T_1, T_2, &c. become when we use the truncated formula (a') instead of the complete one (a), we shall readily get

$$V_1 = T_1' + T_3' + T_5' + T_7' + \&c.$$

In like manner, from the value of V_2 before given, we get

$$V_2'' = \int r'^2 dr' d\theta' d\varpi' \sin\theta' (1-r'^2)^\beta \phi(r'^2) g^{1-n};$$

the integrals being taken from $r'=r$ to $r=1$, from $\theta'=0$ to $\theta'=\pi$, and from $\varpi=0$ to $\varpi=2\pi$.

Expanding now g^{1-n} as before, we have

$$V_2'' = U_0 + U_1 + U_2 + U_3 + \&c.$$

where

$$U_s = \int d\theta' d\varpi' \sin\varpi' Q_s \int_r^1 r'^{3-n} dr' (1-r'^2)^\beta \frac{r^s}{r'^s} \phi(r'^2),$$

and the part of U_s due to the general term $B_t r'^{2t}$ in $\phi(r'^2)$, will be

$$(b) \ldots\ldots\ldots \; r^s \int d\theta' d\varpi' \sin\theta' Q_s B_t \int_r^1 r'^{3-n+2t-s} dr' (1-r'^2)^\beta;$$

which, by employing the formula (3^r) Art. 1., and rejecting the inadmissible terms, gives for truncated formula

$$(b') \ldots\ldots \; \frac{\Gamma\left(\frac{4-n+2t-s}{2}\right)\Gamma(\beta+1)}{2\Gamma\left(\frac{6-n+2\beta+2t-s}{2}\right)} \, r^s \int d\theta' d\varpi' \sin\theta' Q_s B_t.$$

By continuing to follow exactly the same process as was before employed in finding the value of V_1, we shall see that s must always be an even number, say $2s'$; and thus the expression immediately preceding will become

$$\frac{\Gamma\left(\frac{4-n+2t-2s'}{2}\right)\Gamma(\beta+1)}{2\Gamma\left(\frac{6-n+2\beta+2t-2s'}{2}\right)} \, r^{2s'} \int d\theta' d\varpi' \sin\theta' Q_{2s'} B_t.$$

Moreover, the value of V_2 will be

$$V_2 = U_0' + U_2' + U_4' + U_6' + \&c.;$$

U_0', U_1', U_2', U_3', &c. being what U_0, U_1, U_2, &c. become when we use the formula (b') instead of the complete one (b).

The value of V answering to the density

$$\rho = \rho_1 + \rho_2 = (1-r'^2)^\beta . f(x', y', z'),$$

by adding together the two parts into which it was originally divided, therefore, becomes

$$V = V_1 + V_2 = T_1' + T_3' + T_5' + T_7' + \&c.$$
$$+ U_0' + U_2' + U_4' + U_6' + \&c.$$

When β is taken arbitrarily, the two series entering into V extend in infinitum, but by supposing as before, Art. 1.,

$$\frac{-n}{2} + \beta = \omega;$$

ω representing any whole number, positive or negative, it is clear from the form of the quantities entering into $T_{2s'+1}$ and $U_{2s'}$, and from the known properties of the function Γ, that both these series will terminate of themselves, and the value of V be expressed in a finite form; which, by what has preceded, must necessarily reduce itself to a rational and entire function of the rectangular co-ordinates x, y, z. It seems needless, after what has before been advanced, (Art. 1.) to offer any proof of this: we will, therefore, only remark that if γ represents the degree of the function $f(x', y', z')$, the highest degree to which V can ascend will be

$$\gamma + 2\omega + 4.$$

In what immediately precedes, ω may represent any whole number whatever, positive or negative; but if we make $\omega = -2$, and consequently, $\beta = \frac{n-4}{2}$, the degree of the function V is the same as that of the factor

$$f(x', y', z'),$$

comprised in ρ. This factor then being supposed the most general of its kind, contains as many arbitrary constant quantities as there are terms in the resulting function V. If, therefore, the form of the rational and entire function V be taken at will, the arbitrary quantities contained in $f(x', y', z')$ will in case $\omega = -2$ always enable us to assign the corresponding value of ρ, and the resulting value of $f(x', y', z')$ will be a rational and entire function of the same degree as V. Therefore, in the case now under consideration, we shall not only be able to determine the value of V when ρ is given, but shall also have the means of solving the inverse problem, or of determining ρ when V is given; and this determination will depend upon the resolution of a certain number of algebraical equations, all of the first degree.

3. The object of the preceding sketch has not been to point out the most convenient way of finding the value of the function V, but merely to make known the spirit of the method; and to show on what its success depends. Moreover, when presented in this simple form, it has the advantage of being, with a very slight modification, as applicable to any ellipsoid whatever as to the sphere itself. But when

spheres only are to be considered, the resulting formulæ, as we shall afterwards show, will be much more simple if we expand the density ρ in a series of functions similar to those used by Laplace (*Mec. Cel.* Liv. iii.): it will however be advantageous previously to demonstrate a general property of functions of this kind, which will not only serve to simplify the determination of V, but also admit of various other applications of $d\sigma$.

Suppose, therefore, $Y^{(i)}$ is a function of θ and ϖ, of the form considered by Laplace (*Mec. Cel.* Liv. iii.), r, θ, ϖ being the polar co-ordinates referred to the axes X, Y, Z, fixed in space, so that

$$x = r\cos\theta,\quad y = r\sin\theta\cos\varpi,\quad z = r\sin\theta\sin\varpi;$$

then, if we conceive three other fixed axes X_1, Y_1, Z_1, having the same origin but different directions, $Y^{(i)}$ will become a function of θ_1 and ϖ_1, and may therefore be expanded in a series of the form

$$(6)\ldots\ldots\ldots Y^{(r)} = Y_1^{(0)} + Y_1^{(1)} + Y_1^{(2)} + Y_1^{(3)} + \&c.$$

Suppose now we take any other point p' and mark its various co-ordinates with an accent, in order to distinguish them from those of p; then, if we designate the distance pp' by (p, p'), we shall have

$$\frac{1}{(p, p')} = \{r^2 - 2rr'[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varpi - \varpi')] + r'^2\}^{-\frac{1}{2}}$$

$$= \frac{1}{r}\left(Q^{(0)} + Q^{(1)}\cdot\frac{r'}{r} + Q^{(2)}\frac{r'^2}{r^2} + Q^{(3)}\frac{r'^3}{r^3} + \&c.\right),$$

as has been shewn by Laplace in the third book of the *Mec. Cel.*, where the nature of the different functions here employed is completely explained.

In like manner, if the same quantity is expressed in the polar co-ordinates belonging to the new system of axes X_1, Y_1, Z_1, we have, since the quantities r and r' are evidently the same for both systems,

$$\frac{1}{(p, p')} = \frac{1}{r}\left(Q_1^{(0)} + Q_1^{(1)}\frac{r'}{r} + Q_1^{(2)}\frac{r'^2}{r^2} + Q_1^{(3)}\frac{r'^3}{r^3} + \&c.\right);$$

and it is also evident from the form of the radical quantity of which the series just given are expansions, that whatever number i may represent, $Q_1^{(i)}$ will be immediately deduced from $Q^{(i)}$ by changing θ, ϖ, θ', ϖ' into θ_1, ϖ_1, θ_1', ϖ_1'. But since the quantity $\frac{r'}{r}$ is indeterminate, and may be taken at will, we get, by equating the two values of $\frac{1}{(p, p')}$ and comparing the like powers of the indeterminate quantity $\frac{r'}{r}$,

$$Q^{(i)} = Q_1^{(i)}.$$

If now we multiply the equation (6) by the element of a spherical surface whose radius is unity, and then by $Q^{(h)} = Q_1^{(h)}$, we shall have, by integrating and extending the integration over the whole of this spherical surface,

$$\int d\mu\, d\varpi\, Q^{(h)} Y^{(i)} = \int d\mu_1\, d\varpi_1\, Q_1^{(h)} \{ Y_1^{(0)} + Y_1^{(1)} + Y_1^{(2)} + \text{\&c.} \}.$$

Which equation, by the known properties of the functions $Q^{(h)}$ and $Y^{(i)}$, reduces itself to

$$0 = \int d\mu_1\, d\varpi_1\, Q_1^{(h)} Y_1^{(h)},$$

when h and i represent different whole numbers. But by means of a formula given by Laplace (*Mec. Cel.* Liv. iii. No. 17.) we may immediately effect the integration here indicated, and there will thus result

$$0 = \frac{4\pi}{2h+1} Y_1'^{(h)};$$

$Y_1'^{(h)}$ being what $Y_1^{(h)}$ becomes by changing θ_1, ϖ_1 into θ_1', ϖ_1', and as the values of these last co-ordinates, which belong to p', may be taken arbitrarily like the first, we shall have generally $Y_1^{(h)}$, except when $h = i$. Hence, the expansion (6) reduces itself to a single term, and becomes

$$Y^{(i)} = Y_1^{(i)}.$$

We thus see that the function $Y^{(i)}$ continues of the same form even when referred to any other system of axes X_1, Y_1, Z_1, having the same origin O with the first.

This being established, let us conceive a spherical surface whose center is at the origin O of the co-ordinates and radius r', covered with fluid, of which the density $\rho = Y'^{(i)}$; then, if $d\sigma'$ represent any element of this surface, and we afterwards form the quantity

$$V = \int \rho d\sigma' \psi (g^2) = \int Y'^{(i)} d\sigma' \psi (g^2);$$

the integral extending over the whole spherical surface, g being the distance p, $d\sigma'$ and ψ the characteristic of any function whatever. I say, the resulting value of V will be of the form

$$V = Y^{(i)} R;$$

R being a function of r, the distance Op only and $Y^{(i)}$ what $Y'^{(i)}$ becomes by changing θ', ϖ', the polar co-ordinates, into θ, ϖ, the like co-ordinates of the point p.

To justify this assertion, let there be taken three new axes X_1, Y_1, Z_1, so that the point p may be upon the axis X_1; then, the new polar co-ordinates of $d\sigma'$ may be written r', θ', ϖ', those of p being r, 0, ϖ, and consequently, the distance will become

$$g = \sqrt{(r'^2 - 2rr' \cos \theta_1' + r^2)};$$

and as $d\sigma' = r'^2 d\theta_1' d\varpi_1' \sin \theta_1'$, we immediately obtain

$$V = \int Y'^{(2)} r'^2 d\theta_1 d\varpi_1 \sin \theta_1 \psi (r^2 - 2rr' \cos \theta_1' + r'^2)$$
$$= r'^2 \int_0^\pi d\theta_1' \sin \theta_1' \psi (r^2 - 2rr' \cos \theta_1' + r'^2) \int_0^{2\pi} d\varpi_1' Y'^{(i)}.$$

Let us here consider more particularly the nature of the integral

$$\int_0^{2\pi} d\varpi_1' Y'^{(i)}.$$

In the preceding part of the present article, it has been shown that the value of $Y'^{(i)}$, when expressed in the new co-ordinates, will be of the form $Y_1'^{(i)}$; but all functions of this form (Vide *Mec. Cel.* Liv. iii.) may be expanded in a finite series containing $2i+1$ terms, of which the first is independent of the angle ϖ_1', and each of the others has for a factor a sine or cosine of some entire multiple of this same angle. Hence, the integration relative to ϖ_1' will cause all the last mentioned terms to vanish, and we shall only have to attend to the first here. But this term is known to be of the form

$$k \left(\mu_1'^i - \frac{i.i-1}{2.2i-1} \mu_1'^{i-2} + \frac{i.i-1.i-2.i-3}{2.4.2i-1.2i-3} m_1'^{i-4} - \&c. \right),$$

and consequently, there will result

$$\int_0^{2\pi} d\varpi_1' \, Y'^{(i)} = 2\pi k \left(\mu_1'^{i} - \frac{i.i-1}{2.2i-1}\mu_1'^{i-2} + \frac{i.i-1.i-2.i-3}{2.4.2i-1.2i-3}\mu_1'^{i-4} - \&c.\right);$$

where $\mu_1' = \cos\theta_1'$ and k is a quantity independent of θ_1' and ϖ_1', but which may contain the co-ordinates θ, ϖ, that serve to define the position of the axis X_1 passing through the point p.

It now only remains to find the value of the quantity k, which may be done by making $\theta_1' = 0$, for then the line r coincides with the axis X_1, and $Y^{(i)}$ during the integration remains constantly equal to $Y^{(i)}$, the value of the density at this axis. Thus we have

$$2\pi Y^{(i)} = 2\pi k \left(1 - \frac{i.i-1}{2.2i-1} + \frac{i.i-1.i-2.i-3}{2.4.2i-1.2i-3} - \&c.\right):$$

or, by summing the series within the parenthesis, and supplying the common factor 2π,

$$Y^{(i)} = \frac{1.2.3.........i}{1.3.5.........2i-1} k,$$

and, by substituting the value of k, drawn from this equation in the value of the required integral given above, we ultimately obtain

$$\int_0^{2\pi} d\varpi_1' \, Y'^{(i)} = 2\pi Y^{(i)} \frac{1.3.5......2i-1}{1.2.3......i} \left(\mu_1'^{i} - \frac{i.i-1}{2.2i-1}\mu_1'^{i-2} + \&c.\right).$$

If now, for abridgement, we make

$$(i) = \mu_1'^{i} - \frac{i.i-1}{2.2i-1}\mu_1'^{i-2} + \frac{i.i-1.i-2.i-3}{2.4.2i-1.2i-3}\mu_1'^{i-4} - \&c.$$

we shall obtain, by substituting the value of the integral just found in that of V before given,

$$V = Y^{(i)} . 2\pi r'^2 \frac{1.3.5......2i-1}{1.2.3......i} \int_{-1}^{1} d\mu_1' \, (i) \, \psi(r^2 - 2rr'\mu_1' + r'^2);$$

which proves the truth of our assertion.

From what has been advanced in the preceding article, it is likewise very easy to see that if the density of the fluid within a sphere of any radius be every where represented by

$$\rho = Y'^{(i)} \phi(r);$$

ϕ being the characteristic of any function whatever; and we afterwards form the quantity

$$V = \int \rho dv \, \psi (g^2),$$

where dv represents an element of the sphere's volume, and g the distance between dv and any particle p under consideration, the resulting value of V will always be of the form

$$V = Y^{(i)} . R;$$

$Y^{(i)}$ being what $Y'^{(i)}$ becomes by changing $\theta_{,}$ ϖ', the polar co-ordinates of the element dv into θ, ϖ, the co-ordinates of the point p; and R being a function of r, the remaining co-ordinate of p, only.

4. Having thus demonstrated a very general property of functions of the form $Y^{(i)}$, let us now endeavour to determine the value of V for a sphere whose radius is unity, and containing fluid of which the density is every where represented by

$$\rho = (1 - x'^2 - y'^2 - z'^2)^\beta f(x', y', z');$$

x', y', z' being the rectangular co-ordinates of dv, an element of the sphere's volume, and f, the characteristic of any rational and entire function whatever.

For this purpose we will substitute in the place of the co-ordinates x', y', z' their values

$$x' = r' \cos \theta' : \quad y' = r' \sin \theta' \cos \varpi' : \quad z' = r' \sin \theta' \sin \varpi';$$

and afterwards expand the function $f(x', y', z')$ by Laplace's simple method, (*Mec. Cel.* Liv. iii. No. 16.). Thus,

$$(7) \ldots\ldots\ldots f(x', y', z') = f'^{(0)} + f'^{(1)} + f'^{(2)} + \&c. \ldots\ldots + f'^{(s)};$$

s being the degree of the function $f(x', y', z')$.

It is likewise easy to perceive that any term $f^{(i)}$ of this expansion may be again developed thus,

$$f'^{(i)} = f_0'^{(i)} r'^i + f_1'^{(i)} r'^{i+2} + f_2'^{(i)} r'^{i+4} + \&c.;$$

and as every coefficient of the last developement is of the form $U^{(i)}$, (*Mec. Cel.* Liv. iii.), it is easy to see that the general term $f'^{(i)} r'^{i+2t}$ may always be reduced to a rational and entire function of the original co-ordinates x', y', z'.

If now we can obtain the part of V due to the term

$$f_t'^{(i)} \cdot r'^{i+2t},$$

we shall immediately have the value of V by summing all the parts corresponding to the various values of which i and t are susceptible. But from what has before been proved (Art. 3.), the part of V now under consideration must necessarily be of the form $Y^{(i)}$; representing, therefore, this part by $V_t^{(i)}$, we shall readily get

$$V_t^{(i)} = \int_0^1 r'^{i+2t+2} dr' (1 - r'^2)^\beta \int d\varpi' d\theta' \sin\theta' f_t'^{(i)} g^{1-n}.$$

Moreover from what has been shown in the same article, it is easy to see that we have generally

$$\int Y'^{(i)} d\varpi' d\theta' \sin\theta' \psi(g^2) = 2\pi\, Y^{(i)} \frac{1.3.5.\ldots.2i-1}{1.2.3.\ldots i} \int_{-1}^{1} d\mu_1' (i)\, \psi(r^2 - 2rr'\mu_1' + r'^2);$$

ψ being the characteristic of any function whatever, and $Y^{(i)}$ what $Y'^{(i)}$ becomes by substituting θ, ϖ the polar co-ordinates of p in the place of θ', ϖ', the analogous co-ordinates of the element dv. If therefore in the expression immediately preceding, we make

$$Y'^{(i)} = f_t'^{(i)} \text{ and } \psi(g^2) = g^{n-1} = (g^2)^{\frac{1-n}{2}},$$

and substitute the value of the integral thus obtained for its equal in $V_t^{(i)}$ there will arise

$$(8)\quad V_t^{(i)} = 2\pi f_t^{(i)} \frac{1.3.5\ldots2i-1}{1.2.3\ldots i} \int_0^1 r'^{i+2t+2} dr' (1-r'^2)^\beta \int_{-1}^{1} d\mu'_1 (i).(r^2 - 2rr'\mu'_1 + r^2)^{\frac{1-n}{2}};$$

where $f_t^{(i)}$ is deduced from $f_t'^{(i)}$ by changing θ', ϖ' into θ, ϖ, and (i), for abridgement, is written in the place of the function

$$\mu_1'^i - \frac{i.i-1}{2.2i-1}\mu_1'^{i-2} + \frac{i.i-1.i-2.i-3}{2.4.2i-1.2i-3}\mu_1'^{i-4} - \&c.$$

As the integral relative to μ'_1 which enters into the expression on the right side of the equation (8) is a definite one, and depends therefore on the two extreme values of μ'_1 only, it is evident that in the determination of this integral, it is altogether useless to retain the accents by which μ'_1 is affected. But by omitting these superfluous accents, we shall have to calculate the value of the quantity

$$\int_{-1}^{1} d\mu\,(i)\,.\,(r^2 - 2rr'\mu + r'^2)^{\frac{1-n}{2}};$$

where

$$(i) = \mu^i - \frac{i.i-1}{2.2i-1}\mu^{i-2} + \frac{i.i-1.i-2.i-3}{2.4.2i-1.2i-3}\mu^{i-4} - \&c.$$

The method which first presents itself for determining the value of the integral in question, is to expand the quantity $(r^2 - 2rr'\mu + r'^2)^{\frac{1-n}{2}}$ by means of the Binomial Theorem, to replace the various powers of μ by their values in functions similar to (i) and afterwards to effect the integrations by the formulæ contained in the third Book of the *Mec. Cel.* For this purpose we have the general equation

$$(9)\ldots\ldots\ldots\mu^i = (i) + \frac{i.i-1}{2.2i-1}(i-2) + \frac{i.i-1.i-2.i-3}{2.4.2i-3.2i-5}(i-4)$$
$$+\frac{i.i-1.i-2.i-3.i-4.i-5}{2.4.6.2i-5.2i-7.2i-9}(i-6) + \&c.$$

To remove all doubt of the correctness of this equation, we may multiply each of its sides by μ, and reduce the products on the right by means of the relation

$$\mu\,(i) = (i+1) + \frac{i^2}{2i-1\,.\,2i+1}(i-1),$$

which it is very easy to prove exists between functions of the form (i). In this way it will be seen that if the equation (9) holds good for any power μ^i it will do so likewise for the following power μ^{i+1}, and as it is evidently correct when $i=1$, it is therefore necessarily so, whatever whole number i may represent.

Now by means of the Binomial Theorem, we obtain when $r \angle r'$

$$r'^{n-1}.(r^2 - 2rr'\mu + r'^2)^{\frac{1-n}{2}} = \left(1 - 2\mu\frac{r}{r'} + \frac{r^2}{r'^2}\right)^{\frac{1-n}{2}}$$

$$= \Sigma_0^\infty \frac{n-1\,.\,n+1\,.\,n+3\ldots\ldots n+2s-3}{2\;.\;4\;.\;6\;\ldots\ldots\;2s}\left(2\mu\frac{r}{r'} - \frac{r^2}{r'^2}\right)^s;$$

If now we conceive the quantity $\left(2\mu\frac{r}{r'} - \frac{r^2}{r'^2}\right)^s$ to be expanded by the same theorem, it is easy to perceive that the term having $\left(\frac{r}{r'}\right)^{i+2t'}$ for factor is

$$\frac{n-1.n+1.n+3......n+2i+4t'-3}{2\;.\;4\;.\;6\;......\;2i+4t'}\,2^{i+2t'}\mu^{i+2t'}\left(\frac{r}{r'}\right)^{i+2t'}$$

$$-\frac{n-1.n+1......n+2i+4t'-5}{2\;.\;4\;......\;2i+4t'-2}\,.\,2^{i+2t'-2}\mu^{i+2t'-2}\left(\frac{r}{r'}\right)^{i+2t-2}\frac{r^2}{r'^2}\,.\,\frac{i+2t-1}{1}$$

$$+\frac{n-1.n+1......n+2i+4t'-7}{2\;.\;4\;......\;2i+4t'-4}\,(2\mu)^{i+2t'-4}\left(\frac{r}{r'}\right)^{i+2t'-4}\frac{r^4}{r'^4}\,.\,\frac{i+2t-2.i+2t'-3}{1\;.\;2}$$

$$-\ \&c.............................\&c...........................\&c................$$

and therefore the coefficient of $\left(\frac{r}{r'}\right)^{i+2t'}$ in the expansion of the function

$$\left(1-2\mu\frac{r}{r'}+\frac{r^2}{r'^2}\right)^{\frac{1-n}{2}},$$

will be expressed by

$$\Sigma\,\frac{n-1.n+1......n+2i+4t'-2s-3}{2\;.\;4\;......\;2i+4t'-2s}\,(2\mu)^{i+2t-2s}.(-1)^s$$

$$.\,\frac{i+2t'-s.i+2t'-s-1......i+2t'-2s+1}{1\;.\;2\;......\;s}.$$

Hence the portion of this coefficient containing the function (i), when the various powers of μ have been replaced by their values in functions of this kind agreeably to the preceding observation will be found, by means of the equation (9), to be

$$(i)\,\Sigma\,\frac{n-1\,.\,n+1......n+2i+4t'-2s-3}{2\;.\;4\;......\;2i+4t'-2s}$$

$$\times\,\frac{i+2t'-2s\,.\,i+2t'-2s-1......i+1}{2\,.\,4......2t'-2s\times 2i+2t'-2s+1\,.\,2i+2t'-2s-1...2i+3}$$

$$......2^{i+2t'-2s}(-1)^s\times\frac{i+2t'-s\,.\,i+2t'-s-1.........i+2t'-2s+1}{1\;.\;2\;.\;3\;.........\;s}$$

$$=(i)\,\Sigma\,\frac{n-1\,.\,n+1\,.\,n+3......n+2i+4t'-2s-3}{2\;.\;4\;.\;6\;......\;2i+4t'-2s}\,.\,2^{i+2t'-2s}(-1)^s$$

$$\ldots\ldots \times \frac{i+1 \,.\, i+2 \,.\, i+3 \,.\, i+4 \ldots\ldots\ldots i+2t'-s}{1 \,.\, 2 \,.\, 3 \ldots\ldots s \times 2 \,.\, 4 \,.\, 6 \ldots\ldots 2t'-2s \times 2i+2t'-2s+1 \ldots\ldots 2i+3}$$

$$= 2^i \,.\, (i) \,.\, \Sigma \frac{(-1)^s \,.\, n-1 \,.\, n+1 \,.\, n+3 \ldots\ldots\ldots n+2i+4t'-2s-3}{2 \,.\, 4 \ldots\ldots 2i \times 2 \,.\, 4 \ldots\ldots 2s \times 2 \,.\, 4 \ldots\ldots 2t'-2s \times 2i+2t'-2s+1 \ldots 2i+3}$$

$$= \frac{3 \,.\, 5 \,.\, 7 \ldots\ldots 2i+1}{1 \,.\, 2 \,.\, 3 \ldots\ldots \; i} (i) \times \frac{n-1 \,.\, n+1 \,.\, n+3 \ldots\ldots n+2i+2t'-3}{3 \;\;.\;\; 5 \;\;.\;\; 7 \;\;\ldots\ldots\;\; 2i+2t'+1}$$

$$\ldots\ldots \times \Sigma \,.\, (-1)^s \frac{n+2i+2t'-1 \ldots\ldots n+2i+4t'-2s-3}{2 \,.\, 4 \,.\, 6 \;\;\ldots\ldots\;\; 2t'-2s}$$

$$\times \frac{2i+2t'-2s+3 \ldots\ldots 2i+2t'+1}{2 \,.\, 4 \,.\, 6 \;\;\ldots\ldots\;\; 2s},$$

where all the finite integrals may evidently be extended from $s=0$ to $s=\infty$, and it is clear that the last of these integrals is equal to the coefficient of $x^{t'}$ in the product

$$\{1 + \frac{n+2i+2t'-1}{2} x + \frac{n+2i+2t'-1 \,.\, n+2i+2t'+1}{2 \,.\, 4} x^2 + \&c. \textit{ in inf.}\}$$

$$\times \{1 - \frac{2i+2t'+1}{2} x + \frac{2i+2t'+1 \,.\, 2i+2t'-1}{2 \,.\, 4} x^2 - \&c. \textit{ in inf.}\}$$

If now we write in the place of the series their known values, the preceding product will become

$$(1-x)^{-\frac{n+2i+2t'-1}{2}} \times (1-x)^{\frac{2i+2t'+1}{2}} = (1-x)^{\frac{2-n}{2}},$$

and consequently the value of the required coefficient of $x^{t'}$ is

$$\frac{n-2 \,.\, n \,.\, n+2 \ldots\ldots\ldots n+2t'-4}{2 \;.\; 4 \;.\; 6 \;\;\ldots\ldots\ldots\;\; 2t'}.$$

This quantity being substituted in the place of the last of the finite integrals gives for the value of that portion of the coefficient of

$$\left(\frac{r}{r'}\right)^{i+2t'} \text{ in } \left(1 - 2\mu \frac{r}{r'} + \frac{r^2}{r'^2}\right)^{\frac{1-n}{2}},$$

which contains the function (i) the expression

$$\frac{3 \,.\, 5 \,.\, 7 \ldots\ldots 2i+1}{1 \,.\, 2 \,.\, 3 \ldots\ldots \; i} \times \frac{n-1 \,.\, n+1 \ldots\ldots n+2i+2t'-3}{3 \;.\; 5 \;\;\ldots\ldots\;\; 2i+2t'+1} \times \frac{n-2 \,.\, n \ldots\ldots n+2t'-4}{2 \;.\; 4 \ldots\ldots \;\; 2t'} (i).$$

By multiplying the last expression by $\left(\frac{r}{r'}\right)^{i+2t'}$, and taking the sum of all the resulting values which arise when we make successively

$$t' = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \&\text{c. } \textit{in infinitum},$$

we shall obtain the value of the term $Y^{(i)}$ contained in the expression

$$\left(1 - 2\mu\frac{r'}{r'} + \frac{r^2}{r'^2}\right)^{\frac{1-n}{2}} = Y^{(0)} + Y^{(1)} + Y^{(2)} + Y^{(3)} + \&\text{c}.$$

Hence,

$$Y^{(i)} = \frac{3.5\ldots\ldots 2i+1}{1.2\ldots\ldots\ i}\,(i)\,\Sigma\,\frac{n-1.n+1\ldots\ldots n+2i+2t'-3}{3\ .\ 5\ \ldots\ldots\ 2i+2t'+1}$$

$$\times\frac{n-2.n\ldots\ldots n+2t'-4}{2.4\ldots\ldots\ 2t'}\left(\frac{r}{r'}\right)^{i+2t'};$$

the finite integral extending from $t' = 0$ to $t' = \infty$.

But by the known properties of functions of this kind, we have by substituting for $Y^{(i)}$ its value

$$\int_{-1}^{1} d\mu\,(i)\left(1 - 2\mu\frac{r}{r'} + \frac{r^2}{r'^2}\right)^{\frac{1-n}{2}} = \int_{-1}^{1} d\mu\,(i)\,.\,Y^{(i)}$$

$$= \frac{3.5.7\ldots\ldots 2i+1}{1.2.3\ldots\ldots\ i}\int d\mu\,(i)^2 \times \Sigma\,\frac{n-1.n+1\ldots\ldots n+2i+2t'-3}{3\ .\ 5\ \ldots\ldots\ 2i+2t'+1}$$

$$\times\frac{n-2.n\ldots\ldots n+2t'-4}{2.4\ldots\ldots\ 2t'}\left(\frac{r}{r'}\right)^{i+2t'}$$

$$= 2\,\frac{1.2.3\ldots\ldots\ i}{1.3.5\ldots\ldots 2i-1}\,\Sigma\,\frac{n-1.n+1\ldots\ldots n+2i+2t'-3}{3\ .\ 5\ \ldots\ldots\ 2i+2t'+2}$$

$$\times\frac{n-2.n\ldots\ldots n+2t'-4}{2.4\ldots\ldots\ 2t'}\left(\frac{r}{r'}\right)^{i+2t'},$$

since by what Laplace has shown (*Mec. Cel.* Liv. iii. No. 17.)

$$\int_{-1}^{1} d\mu\,(i)^2 = \frac{2}{2i+1}\left(\frac{1.2.3\ldots\ldots\ i}{1.3.5\ldots\ldots 2i-1}\right)^2.$$

If now we restore to μ the accents with which it was originally affected, and multiply the resulting quantity by r'^{n-1}, we shall have when $r<r'$

$$(10)\quad \int_{-1}^{1} d\mu'_1 (i) \left(r^2 - 2rr'\mu'_1 + r'^2\right)^{\frac{1-n}{2}} = r'^{n-1} \int_{-1}^{1} d\mu'_1 (i) \left(1 - 2\mu'_1 \frac{r}{r'} + \frac{r^2}{r'^2}\right)^{\frac{1-n}{2}}$$

$$= 2 \cdot r'^{1-n} \cdot \frac{1 \cdot 2 \cdot 3 \ldots\ldots i}{1 \cdot 3 \cdot 5 \ldots\ldots 2i-1} \Sigma \frac{n-1 \cdot n+1 \ldots\ldots n+2i+2t'-3}{3 \;.\; 5 \;\ldots\ldots\; 2i+2t'+1}$$

$$\times \frac{n-2 \cdot n \ldots\ldots n+2t'-4}{2 \cdot 4 \ldots\ldots 2t'} \left(\frac{r}{r'}\right)^{i+2t'},$$

and in order to deduce the value of the same integral when $r' \angle r$, we shall only have to change r into r', and reciprocally, in the formula just given.

We may now readily obtain the value of $V_t^{(i)}$ by means of the formula (8). For the density corresponding thereto being

$$'f_t^{(i)} r^{i+2t} (1 - r'^2)^\beta,$$

it follows from what has been observed in the former part of the present article, that $f_t'^{(i)} r^{i+2t}$ may always be reduced to a rational and entire function of x', y', z' the rectangular co-ordinates of the element dv, and therefore the density in question will admit of being expanded in a series of the entire powers of x', y', z' and the various products of these powers. Hence (Art. 1.) $V_t^{(i)}$ admits of a similar expansion in entire powers, &c. of x, y, z the rectangular co-ordinates of the point p, and by following the methods before exposed Art. 1 and 2, we readily get

$$V_t''^{(i)} = 4\pi f^{(i)} \cdot \int_r^1 r'^{i+2t'+3-n} dr' (1 - r'^2)^\beta \cdot \Sigma \frac{n-1 \cdot n+1 \ldots\ldots n+2i+2t'-3}{3 \;.\; 5 \;\ldots\ldots\; 2i+2t'+1}$$

$$\times \frac{n-2 \cdot n \cdot n+2 \ldots\ldots n+2t'-4}{2 \;.\; 4 \;.\; 6 \;\ldots\ldots\; 2t'} \left(\frac{r}{r'}\right)^{i+2t'};$$

and thence we have ultimately,

$$(11)\quad V_t^{(i)} = 2\pi f_t^{(i)} \Sigma \frac{n-1 \cdot n+1 \ldots\ldots n+2i+2t'-3}{3 \;.\; 5 \;\ldots\ldots\; 2i+2t'+1} \times \frac{n-2 \cdot n \ldots\ldots n+2t'-4}{2 \cdot 4 \ldots\ldots 2t'}$$

$$r^{i+2t} \cdot \frac{\Gamma\left(\frac{2t-2t'+4-n}{2}\right)\Gamma(\beta+1)}{\Gamma\left(\frac{2t-2t'+2\beta+6-n}{2}\right)} = 2\pi f_t^{(i)} \cdot \frac{\Gamma(\beta+1)\,\Gamma\left(\frac{4-n}{2}\right)}{\Gamma\left(\frac{6+2\beta-n}{2}\right)} r^i \ldots\ldots$$

$$\ldots\ldots \Sigma r^{2t'} \frac{4-n.6-n\ldots\ldots \quad 2t-2t'+2-n}{6+2\beta-n\ldots\ldots 2t-2t'+2\beta+4-n} \times \frac{n-2.n\ldots\ldots n+2t'-4}{2.4\ldots\ldots \quad 2t'}$$

$$\times \frac{n-1.n+1\ldots\ldots n+2i+2t'-3}{3\ .\ 5\ \ldots\ldots\ 2i+2t'+1};$$

the finite integrals being taken from $t'=0$ to $t'=\infty$ and Γ being the characteristic of the well known function Gamma, which is introduced when we effect the integrations relative to r' by means of the formula (3), Art. 1.

Having thus $V_t^{(i)}$ or the part of V corresponding to the term $f_t'^{(i)}$, in $f(x', y', z')$ we immediately deduce the complete value of V by giving to i and t the various values of which these numbers are susceptible, and taking the sum of all the parts corresponding to the different terms in the expansion of the function $f(x', y', z')$.

Athough in the present Article we have hitherto supposed f to be the characteristic of a rational and entire function, the same process will evidently be applicable, provided $f(x', y', z')$ can be expanded in an infinite series of the entire powers of x', y', z' and the various products of these powers. In the latter case we have as before, the development

$$f(x', y', z') = f'^{(0)} + f'^{(1)} + f'^{(2)} + f'^{(3)} + \&c.$$

of which any term, as for example $f'^{(i)}$ may be farther expanded as follows,

$$f'^{(i)} = f_0'^{(i)} r'^{i} + f_1'^{(i)} r'^{i+2} + f_2'^{(i)} r'^{i+4} + \&c.$$

and as we have already determined $V_t^{(i)}$ or the part of V corresponding to $f_t'^{(i)} r'^{i+2t'}$, we immediately deduce as before the required value of V, the only difference is, that the numbers i and t, instead of being as in the former case confined within certain limits, may here become indefinitely great.

In the foregoing expression (11) β may be taken at will, but if we assign to it such a value that $\frac{2\beta - n}{2}$ may be a whole number, the series contained therein will terminate of itself, and consequently the value of $V_t^{(i)}$ will be exhibited in a finite form, capable by what has been shown at the beginning of the present Article of being converted into a rational and entire function of x, y, z, the rectangular co-ordinates of p. It is moreover evident, that the complete value of V being composed of a finite number of terms of the form $V_t^{(i)}$ will possess the same property, provided the function $f(x', y', z')$ is rational and entire, which agrees with what has been already proved in the second Article, by a very different method.

(5) We have before remarked, (Art. 2.) that in the particular case where $\beta = \frac{n-4}{2}$, the arbitrary constants contained in $f(x', y', z')$ are just sufficient in number to enable us to determine this function, so as to make the resulting value of V equal to any given rational and entire function of x, y, z, the rectangular co-ordinates of p, and have proved that the corresponding functions V and f will be of the same degree. But when this degree is considerable, the method there proposed becomes impracticable, seeing that it requires the resolution of a system of

$$\frac{s+1 \,.\, s+2 \,.\, s+3}{1 \,.\, 2 \,.\, 3}$$

linear equations containing as many unknown quantities; s being the degree of the functions in question. But by the aid of what has been shown in the preceding Article, it will be very easy to determine for this particular value of β the function $f(x', y', z')$ and consequently the density ρ when V is given, and we shall thus be able to exhibit the complete solution of the inverse problem by means of very simple formulæ.

For this purpose, let us suppose agreeably to the preceding remarks, that ρ the density of the fluid in the element dv is of the form

$$\rho = (1 - r'^2)^{\frac{n-4}{2}} f(x', y', z');$$

f being the characteristic of a rational and entire function of the same degree as V, and which we will here endeavour so to determine, that the value of V thence resulting, may be equal to any given rational and entire function of x, y, z of the degree s.

Then by Laplace's simple method (*Mec. Cel.* Liv. iii. No. 16.) we may always expand V in a series of the form

$$V = V^{(0)} + V^{(1)} + V^{(2)} + \text{\&c.} \ldots\ldots + V^{(s)}.$$

In like manner as has before been remarked, we shall have the analogous expansion

$$f(x', y', z') = f'^{(0)} + f'^{(1)} + f'^{(2)} + f'^{(3)} + \text{\&c.} \ldots\ldots + f'^{(s)},$$

of which any term $f'^{(i)}$ for example, may be farther developed as follows,

$$f'^{(i)} = f_0'^{(i)} r'^i + f_1'^{(i)} r'^{i+2} + f_2'^{(i)} r'^{i+4} + \text{\&c.} = r'^i \left(f'^{(i)} + f_1'^{(i)} r'^2 + f'^{(i)} r'^4 + \text{\&c.}\right)$$

$f_0'^{(i)}$, $f_1'^{(i)}$, $f_2'^{(i)}$, &c. being quantities independent of r' and all of the form $Y'^{(i)}$ (*Mec. Cel.* Liv. iii.) Moreover $V_t^{(i)}$ the part of V due to the general term $f_t'^{(i)} r'^{i+2t}$ of the last series, will be obtained by writing $\frac{n-4}{2}$ for β in the equation (11), and afterwards substituting for

$$\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{4-n}{2}\right) \text{ its value } \frac{\pi}{\sin\left(\frac{n-2}{2}\pi\right)}.$$

In this way we get

$$V_t^{(i)} = \frac{2\pi^2 f_t^{(i)} r^i}{\sin\left(\frac{n-2}{2}\pi\right)} \Sigma r^{2t'} \frac{4-n \,.\, 6-n \ldots\ldots 2t-2t'+2-n}{2 \,.\, 4 \ldots\ldots 2t-2t'}$$

$$\times \frac{n-2 \,.\, n \ldots\ldots n+2t'-4}{2 \,.\, 4 \ldots\ldots 2t'} \times \frac{n-1 \,.\, n+1 \ldots\ldots n+2i+2t'-3}{3 \,.\, 5 \ldots\ldots 2i+2t'+1};$$

$f_t^{(i)}$ being what $f_t'^{(i)}$ becomes by changing θ', ϖ' into θ, ϖ, and the finite integral being taken from $t'=0$ to $t'=\infty$.

Let us now for a moment assume

$$\phi(t') = \frac{n-2 \,.\, n \ldots\ldots n+2t'-4}{2 \,.\, 4 \ldots\ldots 2t'} \times \frac{n-1 \,.\, n+1 \ldots\ldots n+2i+2t'-3}{3 \,.\, 5 \ldots\ldots 2i+2t'+1},$$

then the expression immediately preceding may be written

$$V_t^{(i)} = \frac{2\pi^2 . f_t^{(i)} . r^i}{\sin\left(\frac{n-2}{2}\pi\right)} \Sigma \frac{4-n.6-n......2t-2t'+2-n}{2\;.\;4\;......\;2t-2t'} \phi(t') . r^{2t'},$$

and by giving to t the various values 0, 1, 2, 3, &c. of which it is susceptible, and taking the sum of all the resulting values of $V_t^{(i)}$ the quantity thus obtained will be equal to $V^{(i)}$ or that part of V which is of the form $Y^{(i)}$. Thus we get

$$V^{(i)} = \frac{2\pi^2 . r^i}{\sin\left(\frac{n-2}{2}\pi\right)} \times$$

$$..........\phi(0) . f_0^{(i)}$$

$$+ \frac{4-n}{2} \phi(0) f_1^{(i)} + \phi(1) f_1^{(i)} . r^2$$

$$+ \frac{4-n.6-n}{2\;.\;4} . \phi(0) . f_2^{(i)} + \frac{4-n}{2} \phi(1) f_2^{(i)} . r^2 + \phi(2) f_2^{(i)} . r^4$$

$$+ \frac{4-n.6-n.8-n}{2\;.\;4\;.\;6} \phi(0) f_3^{(i)} + \frac{4-n.6-n}{2\;.\;4} \phi(1) f_3^{(i)} . r^2 + \frac{4-n}{2} \phi(2) f_3^{(i)} . r^4 + \phi(3) f_3^{(i)} . r^6$$

+ &c...............................&c..............................&c..................

since all the terms in the preceding value of $V_t^{(i)}$ in which $t' > t$ vanish of themselves in consequence of the factor

$$\frac{4-n.6-n......2t-2t'+2-n}{2\;.\;4\;......\;2t-2t'} = \frac{\Gamma\left(\frac{2t-2t'+4-n}{2}\right)}{\Gamma(t-t'+1)\,\Gamma\left(\frac{4-n}{2}\right)} = 0 \text{ (when } t' > t\text{)}.$$

But $V^{(i)}$ as deduced from the given value of V may be expanded in a series of the form

$$V^{(i)} = r^i . \{V_0^{(i)} + V_1^{(i)} r^2 + V_2^{(i)} . r^4 + V_3^{(i)} r^6 + \&c.\}$$

and if in order to simplify the remaining operations, we make generally

$$V_t^{(i)} = \frac{2\pi^2}{\sin\left(\frac{n-2}{2}\pi\right)} \times \frac{n-2.n......n+2t-4}{2.4......\;2t} \times \frac{n-1.n+1......n+2i+2t-3}{3\;.\;5\;......\;2i+2t+1} U_t^{(i)}$$

$$= \frac{2\pi^2}{\sin\left(\frac{n-2}{2}\pi\right)} \times \phi(t) \,.\, U_t^{(i)},$$

the equation immediately preceding will become

$$V^{(i)} = \frac{2\pi^2 . r^i}{\sin\left(\frac{n-2}{2}\pi\right)} \times \{\phi(0) \,.\, U_0^{(i)} + \phi(1) \,.\, U_1^{(i)} r^2 + \phi(2)\, U_2^{(i)} . r^4 + \&c.\}$$

which compared with the foregoing value of $V^{(i)}$, will give by suppressing the factor $\frac{2\pi^2 . r^i}{\sin\left(\frac{n-2}{2}\pi\right)}$, common to both, and equating separately the coefficients of the different powers of the indeterminate quantity r the following system of equations

$$U_0^{(i)} = f_0^{(i)} + \frac{4-n}{2} f_1^{(i)} + \frac{4-n \,.\, 6-n}{2 \,.\, 4} f_2^{(i)} + \frac{4-n \,.\, 6-n \,.\, 8-n}{2 \,.\, 4 \,.\, 6} f_3^{(i)} +$$

$$U_1^{(i)} = f_1^{(i)} + \frac{4-n}{2} f_2^{(i)} + \frac{4-n \,.\, 6-n}{2 \,.\, 4} f_3^{(i)} + \&c.$$

$$U_2^{(i)} = f_2^{(i)} + \frac{4-n}{2} f_3^{(i)} + \frac{4-n \,.\, 6-n}{2 \,.\, 4} f_4^{(i)} + \&c.$$

$$\&c = \ldots\&c \ldots\ldots\ldots\ldots \&c \ldots\ldots\ldots\ldots \&c.$$

for determining the unknown functions $f_0^{(i)}$, $f_1^{(i)}$, $f_2^{(i)}$, &c. by means of the known ones $U_0^{(i)}$, $U_1^{(i)}$, $U_2^{(i)}$, &c. In fact the last equation of the system gives $U_s^{(i)} = f_s^{(i)}$, and then by ascending successively from the bottom to the top equation, we shall get the values of $f_s^{(i)}$, $f_{s-1}^{(i)}$, $f_{s-2}^{(i)}$, &c. with very little trouble. It will however be simpler still to remark, that the general type of all our equations is

$$U_u^{(i)} = (1-\epsilon)^{\frac{n-4}{2}} f_u^{(i)},$$

where the symbols of operation have been separated from those of quantity and ϵ employed in its usual acceptation, so that

$$\epsilon f_u^{(i)} = f_{u+1}^{(i)}, \quad \epsilon^2 f_u^{(i)} = \epsilon f_{u+1}^{(i)} = f_{u+2}^{(i)}, \ \&c.$$

But it is evident we may satisfy the last equation by making

$$f_u^{(i)} = (1-\epsilon)^{\frac{4-n}{2}} U_u^{(i)}.$$

Expanding now and replacing $\epsilon U_u^{(i)}$; $\epsilon^2 U_u^{(i)}$, &c. by these values $U_{u+1}^{(i)}$, $U_{u+2}^{(i)}$, &c. we get

$$f_u^{(i)} = U_u^{(i)} + \frac{n-4}{2} U_{u+1}^{(i)} + \frac{n-4 \,.\, n-2}{2 \,.\, 4} U_{u+2}^{(i)} + \frac{n-4 \,.\, n-2 \,.\, n}{2 \,.\, 4 \,.\, 6} U_{u+3}^{(i)} + \&\text{c.},$$

from which we may immediately deduce $f_u'^{(i)}$ and thence successively

$$f'^{(i)} = r'^i \left(f_0'^{(i)} + f_1'^{(i)} r'^2 + f_2'^{(i)} r'^4 + f_3'^{(i)} r'^6 + \&\text{c.}\right)$$

$$f(x', y', z') = f'^{(0)} + f'^{(1)} + f'^{(2)} + \&\text{c.}\ldots\ldots + f'^{(s)},$$

$$\text{and } \rho = (1 - x'^2 - y'^2 - z'^2)^{\frac{n-4}{2}} . f(x', y', z').$$

Application of the general Methods exposed in the preceding Articles to Spherical conducting Bodies.

(6) In order to explain the phenomena which electrified bodies present, Philosophers have found it advantageous either to adopt the hypothesis of two fluids, the vitreous and resinous of Dufay for example, or to suppose with Æpinus and others, that the particles of matter when deprived of their natural quantity of electric fluid, possess a mutual repulsive force. It is easy to perceive that the mathematical laws of equilibrium deducible from these two hypotheses, ought not to differ when the quantity of fluid or fluids (according to the hypothesis we choose to adopt) which bodies in their natural state are supposed to contain, is so great, that a complete decomposition shall never be effected by any forces to which they may be exposed, but that in every part of them a farther decomposition shall always be possible by the application of still greater forces. In fact the mathematical theory of electricity merely consists in determining ρ* the analytical value of

* It may not be improper to remark that ρ is always supposed to represent the density of the free fluid, or that which manifests itself by its repulsive force; and therefore, when the hypothesis of two fluids is employed, the measure of the excess of the quantity of either fluid

the fluid's density, so that the whole of the electrical actions exerted upon any point p, situated at will in the interior of the conducting bodies may exactly destroy each other, and consequently p have no tendency to move in any direction. For the electric fluid itself, the exponent n is equal to 2, and the resulting value of ρ is always such as not to require that a complete decomposition should take place in the body under consideration, but there are certain values of n for which the resulting values of ρ will render $\int \rho dv$ greater than any assignable quantity; for some portions of the body it is therefore evident that how great soever the quantity of the fluid or fluids may be, which in a natural state this body is supposed to possess, it will then become impossible strictly to realize the analytical value of ρ, and therefore some modification at least will be rendered necessary, by the limit fixed to the quantity of fluid or fluids originally contained in the body, and as Dufay's hypothesis appears the more natural of the two, we shall keep this principally in view, when in what follows it may become requisite to introduce either.

7. The foregoing general observations being premised, we will proceed in the present article to determine mathematically the law of the density ρ, when the equilibrium has established itself in the interior of a conducting sphere A, supposing it free from the actions of exterior bodies, and that the particles of fluid contained therein repel each other with forces which vary inversely as the n^{th} power of the distance. For this purpose it may be remarked, that the formula (1), Art. 1, immediately gives the values of the forces acting on any particle p, in virtue of the repulsion exerted by the whole of the fluid contained in A. In this way we get

$$\frac{1}{1-n}\cdot\frac{dV}{dx} = \text{the force directed parallel to the axis } X,$$

$$\frac{1}{1-n}\cdot\frac{dV}{dy} = \text{the force directed parallel to the axis } Y,$$

fluid which we choose to consider as positive over that of the fluid of opposite name in any element dv of the volume of the body is expressed by ρdv, whereas on the other hypothesis ρdv serves to measure the excess of the quantity of fluid in the element dv over what it would possess in a natural state.

$$\frac{1}{1-n}\cdot\frac{dV}{dz} = \text{the force directed parallel to the axis } Z.$$

But since, in consequence of the equilibrium, each of these forces is equal to zero, we shall have

$$0 = \frac{dV}{dx}dx + \frac{dV}{dy}dy + \frac{dV}{dz}dz = dV;$$

and therefore, by integration,

$$V = \text{const.}$$

Having thus the value of V at the point p, whose co-ordinates are x, y, z, we immediately deduce, by the method explained in the fifth article,

$$\rho = \frac{\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2} V.(1-r'^2)^{\frac{n-4}{2}};$$

seeing that in the present case the general expansion of V there given reduces itself to

$$V = V^{(0)}.$$

If moreover Q serve to designate the total quantity of free fluid in the sphere, we shall have, by substituting for

$$\sin\left(\frac{n-2}{2}\pi\right) \text{ its value } \frac{\pi}{\Gamma\left(\frac{n-2}{4}\right)\Gamma\left(\frac{4-n}{2}\right)},$$

$$Q = \int\rho dv = \frac{\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2} V\int_0^1 4\pi r'^2 dr'(1-r'^2)^{\frac{n-4}{2}} = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{4-n}{2}\right)} V.$$

See Legendre. *Exer. de Cal. Int. Quatrième Partie.*

In the preceding values, as in the article cited, the radius of the sphere is taken for the unit of space; but the same formulæ may readily be adapted to any other unit by writing $\frac{r'}{a}$ in the place of r', and recollecting that the quantities ρ, V, and Q, are of the dimensions 0, $4-n$, and 3 respectively, with regard to space; a being the number which represents the radius of the sphere when we employ the new unit. In this way we obtain

$$\rho = \frac{\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2} V(a^2 - r'^2)^{\frac{n-4}{2}}, \text{ and } Q = \frac{\Gamma\left(\frac{3}{2}\right) a^{n-1}}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{4-n}{2}\right)} . V.$$

Hence, when Q, the quantity of redundant fluid originally introduced into the sphere is given, the values of V and of the density ρ are likewise given. In fact, by writing in the preceding equation for

$$\Gamma\left(\frac{3}{2}\right), \text{ and } \sin\left(\frac{n-2}{2}\pi\right),$$

their values, we thence immediately deduce

$$(12) \ldots\ldots\ldots \rho = \frac{1}{\pi\sqrt{\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)} a^{1-n} Q (a^2 - r'^2)^{\frac{n-4}{2}},$$

$$\text{and } V = \frac{2\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{4-n}{2}\right)}{\sqrt{\pi}} a^{1-n} . Q.$$

The foregoing formulæ present no difficulties where $n > 2$, but when $n < 2$, the value of ρ, if extended to the surface of the sphere A, would require an infinite quantity of fluid of one name to have been originally introduced into its interior, and therefore, agreeably to a preceding observation, could not be strictly realized. In order then to determine the modification which in this case ought to be introduced, let us in the first place make $n > 2$, and conceive an inner sphere B, whose radius is $a - \delta a$, in which the density of the fluid is still defined by the first of the equations (12); then, supposing afterwards the rest of the fluid in the exterior shell to be considered on A's surface, the portion so condensed, if we neglect quantities of the order δa, compared with those retained, will be

$$\frac{2^{\frac{n}{2}}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} a^{\frac{2-n}{2}} Q . \delta a^{\frac{n-2}{2}},$$

and since, in the transfer of the fluid to A's surface, its particles move over spaces of the order δa only, the alteration which will thence be produced in V will evidently be of the order

$$\delta a^{\frac{n-2}{2}} \times \delta a = \delta a^{\frac{n}{2}},$$

and consequently the value of V will become

$$V = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{4-n}{2}\right) a^{1-n} Q + k \,.\, \delta a^{\frac{n}{2}};$$

k being a quantity which remains finite when δa vanishes.

In establishing the preceding results, n has been supposed greater than 2, but ρ the density of the fluid within B and the quantity of it condensed on A's surface being still determined by the same formulæ, the foregoing value of V ought to hold good in virtue of the generality of analysis whatever n may be, and therefore when n is a positive quantity and δa is exceedingly small, we shall have without sensible errors

$$V = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{4-n}{2}\right) a^{1-n} . Q.$$

Conceiving now P' to represent the density of the fluid condensed on A's surface, $4\pi a^2 P'$ will be the total quantity thereon contained, which being equated to the value before given, there results

$$4\pi a^2 P' = \frac{2^{\frac{n}{2}}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} a^{\frac{2-n}{2}} Q \delta a^{\frac{n-2}{2}},$$

and hence we immediately deduce

$$P' = \frac{2^{\frac{n-4}{2}}}{\pi\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} a^{-\frac{2+n}{2}} . Q . \delta a^{\frac{n-2}{2}}.$$

Moreover as Q represents the total quantity of redundant fluid in the entire sphere A, the quantity contained in B is

$$Q - \frac{2^{\frac{n}{2}}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} a^{\frac{2-n}{2}} . Q . \delta a^{\frac{n-2}{2}} .$$

If now when n is supposed less than 2, we adopt an hypothesis similar to Dufay's, and conceive that the quantities of fluid of opposite denominations in the interior of A are exceedingly great when this body is in a natural state, then after having introduced the quantity Q of redundant fluid, we may always by means of the expression just given, determine the value of δa, so that the whole of the fluid of contrary name to Q, may be contained in the inner sphere B, the density in every part of it being determined by the first of the equations (12). If afterwards the whole of the fluid of the same name as Q is condensed upon A's surface, the value of V in the interior of B as before determined will evidently be constant, provided we neglect indefinitely small quantities of the order $\delta a^{\frac{n}{2}}$. Hence all the fluid contained in B will be in equilibrium, and as the shell included between the two concentric spheres, A and B is entirely void of fluid, it follows that the whole system must be in equilibrium.

From what has preceded, we see that the first of the formulæ (12) which served to give the density ρ within the sphere A when n is greater than 2, is still sensibly correct when n represents any positive quantity less than 2, provided we do not extend it to the immediate vicinity of A's surface. But as the foregoing solution is only approximative, and supposes the quantities of the two fluids which originally neutralized each other to be exceedingly great, we shall in the following article endeavour to exhibit a rigorous solution of the problem, in case $n < 2$, which will be totally independent of this supposition.

8. Let us here in the first place conceive a spherical surface whose radius is a, covered with fluid of the uniform density P', and suppose it is required to determine the value of the density ρ in the interior of a concentric conducting sphere, the radius of which is taken for the unit of space, so that the fluid therein contained, may be in equilibrium in virtue of the joint action of that contained in the sphere itself, and on the exterior spherical surface.

If now V' represents the value of V due to the exterior surface, it is clear from what Laplace has shown, (*Mec. Cel.* Liv. ii. No. 12.) that

$$V' = \int \frac{d\sigma P'}{g'^{1-n}} = \frac{2\pi a P'}{(3-n)r}\{(a+r)^{3-n} - (a-r)^{3-n}\};$$

$d\sigma$ being an element of this surface, and g' being the distance of this element from the point p to which V' is supposed to belong.

If afterwards we conceive that the function V is due to the fluid within the sphere itself, it is easy to prove as in the last article, that in consequence of the equilibrium we must have

$$V' + V = \text{const.}$$

But V' and consequently V is of the form $Y^{(0)}$, therefore by employing the method before explained, (Art. 4.) we get

$$f(x', y', z') = f'^{(0)} = f_0^{(0)} + f_1^{(0)}.r'^2 + f_2^{(0)}.r'^4 + \&\text{c.} = B_0 + B_1 r'^2 + B_2 r'^4 + \&\text{c.};$$

where, as in the present case, $f_0'^{(0)}$, $f_1'^{(0)}$, $f_2'^{(0)}$, &c. are all constant quantities, they have for the sake of simplicity been replaced by

$$B_0,\ B_1,\ B_2,\ \&\text{c.}$$

Hitherto the exponent β has remained quite arbitrary, but by making $\beta = \frac{n-2}{2}$, the formula (11) Art. 4. will become when $i=0$,

$$V_t^{(0)} = 2\pi B_t \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{4-n}{2}\right)}{\Gamma(2)} \Sigma r^{2t'} \frac{4-n.6-n\ldots\ldots 2t-2t'+2-n}{4\ .\ 6\ \ldots\ldots\ 2t-2t'+2}$$
$$\times \frac{n-2.n-1\ldots\ldots n+2t'-3}{2.3.4\ \ldots\ldots\ 2t'+1}$$
$$= \frac{(n-2)\pi^2 B_t}{\sin\left(\frac{n-2}{2}\pi\right)} \Sigma . r^{2t'} . \frac{4-n.6-n\ldots\ldots 2t-2t'+2-n}{4\ .\ 6\ \ldots\ldots\ 2t-2t'+2} \times \frac{n-2.n-1\ldots\ldots n+2t'-3}{2\ .\ 3\ \ldots\ldots\ 2t'+1}.$$

Giving now to t the successive values 0, 1, 2, 3, &c. and taking the sum of the functions thence resulting, there arises

$$V = V^{(0)} = V_0^{(0)} + V_1^{(0)} + V_2^{(0)} + V_3^{(0)} + \&\text{c.} = S.V_t^{(0)}$$
$$= \frac{(n-2)\pi^2}{\sin\left(\frac{n-2}{2}\pi\right)} SB_t \Sigma r^{2t'} \frac{4-n.6-n\ldots\ldots 2t-2t'+2-n}{4.6.8\ \ldots\ldots\ 2t-2t'+2} \times \frac{n-2.n-1\ldots\ldots n+2t'-3}{2\ .\ 3\ \ldots\ldots\ 2t'+1},$$

where the sign S is referred to the variable t and Σ to t'.

Again, by substituting for V and V' their values in the equation $V' + V = \text{const.}$ and expanding the function V' we obtain

$$\text{const.} = 4\pi P' a^{3-n} . \Sigma \frac{r^{2t'}}{a^{2t'}} . \frac{n-2 . n-1 . n ...... n+2t'-3}{2 \ . \ 3 \ . 4 ...... \ 2t'+1}$$

$$+ \frac{(n-2)\pi^2}{\sin\left(\frac{n-2}{2}\pi\right)} S \Sigma B_t r^{2t'} \frac{4-n . 6-n ...... 2t-2t'+2-n}{4 \ . \ 6 \ ...... 2t-2t'+2} \times \frac{n-2 . n-1 ...... n+2t'+3}{2 . 3 . 4 ...... 2t'+1},$$

which by equating separately the coefficients of the various powers of the indeterminate quantity r, and reducing, gives generally

$$\frac{2P' a^{3-n-2t'} . \sin\left(\frac{n-2}{2}\pi\right)}{\pi} = S \frac{2-n . 4-n ...... 2s-n}{2 \ . \ 4 \ ...... \ 2s} B_{t'+s-1}.$$

Then by assigning to t' its successive values 1, 2, 3, &c. there results for the determination of the quantities B_0, B_1, B_2, &c. the following system of equations,

$$\frac{2P'}{\pi} a^{1-n} . \sin\left(\frac{n-2}{2}\pi\right) = B_0 + \frac{2-n}{2} B_1 + \frac{2-n . 4-n}{2 \ . \ 4} B_2 + \&c.$$

$$\frac{2P'}{\pi} a^{1-n} . \sin\left(\frac{n-2}{2}\pi\right) . a^{-2} = B_1 + \frac{2-n}{2} B_2 + \frac{2-n . 4-n}{2 \ . \ 4} B_3 + \&c.$$

$$\frac{2P'}{\pi} a^{1-n} . \sin\left(\frac{n-2}{2}\pi\right) . a^{-4} = B_2 + \frac{2-n}{2} B_3 + \frac{2-n . 4-n}{2 \ . \ 4} B_4 + \&c.$$

&c.................&c..................&c..................&c.........

But it is evident from the form of these equations, that we may satisfy the whole system by making

$$B_1 = B_0 . a^{-2}, \quad B_2 = B_1 . a^{-2}, \quad B_3 = B_2 . a^{-2}, \quad B_4 = B_3 . a^{-2}, \ \&c.$$

provided we determine B_0 by

$$\frac{2P'}{\pi} a^{1-n} \sin\left(\frac{n-2}{2}\pi\right) = B_0 \left(1 + \frac{2-n}{2} a^{-2} + \frac{2-n . 4-n}{2 \ . \ 4} a^{-4} + \&c.\right)$$

$$= B_0 (1 - a^{-2})^{\frac{n-2}{2}}.$$

Hence as in the present case, $\beta = \frac{n-2}{2}$, we immediately deduce the successive values

$$B_0 = \frac{2P'}{\pi a} \sin\left(\frac{n-2}{2}\pi\right) . (a^2 - 1)^{\frac{2-n}{2}},$$

$$f(x', y', z') = f'^{(0)} = B_0 + B_1 r'^2 + B_2 r'^2 + \&c. = B_0 \left(1 - \frac{r'^2}{a^2}\right)^{-1},$$

$$\text{and } \rho = (1 - r'^2)^{\frac{n-2}{2}} . f(x', y', z') = \frac{2P'a}{\pi} \sin\left(\frac{n-2}{2}\pi\right) . (a^2 - 1)^{\frac{2-n}{2}} \ldots$$

$$\ldots\ldots (a^2 - r'^2)^{-1} (1 - r'^2)^{\frac{n-2}{2}}.$$

In the value of ρ just exhibited, the radius of the sphere is taken as the unit of space, but the same formula may easily be adapted to any other unit by writing $\frac{a}{b}$ and $\frac{r'}{b}$ in the place of a and r' respectively, and recollecting at the same time that in consequence of the equation

$$\text{const.} = V + V' = \int \frac{dv . \rho}{g} + \int \frac{d\sigma P'}{g'},$$

before given, $\frac{\rho}{P'}$, is a quantity of the dimension -1 with regard to space: b being the number which represents the radius of the sphere when we employ the new unit. Hence we obtain for a sphere whose radius is bg, acted upon by an exterior concentric spherical surface of which the radius is a,

$$(\beta) \ldots\ldots \rho = \frac{2P'a . \sin\left(\frac{n-2}{2}\pi\right)}{\pi} a^2 - b^2)^{\frac{2-n}{2}} (a^2 - r'^2)^{-1} (b^2 - r'^2)^{\frac{n-2}{2}};$$

P' being the density of the fluid on the exterior surface.

If now we conceive a conducting sphere A whose radius is a, and determine P' so that all the fluid of one kind, viz. that which is redundant in this sphere, may be condensed on its surface, and afterwards find b the radius of the interior sphere B from the condition that it shall just contain all the fluid of the opposite kind, it is evident that each of the fluids will be in equilibrium within A, and therefore the problem originally proposed is thus accurately solved. The reason for supposing all the fluid of one name to be completely abstracted from B, is that our formulæ may represent the state of *permanent* equilibrium, for the tendency of the forces acting within the void shell included between the surfaces A and B, is to abstract continually the fluid of the same name as that on A's surface from the sphere B.

To prove the truth of what has just been asserted, we will begin with determining the repulsion exerted by the inner sphere itself, on any point p exterior to it, and situate at the distance r from its centre O. But by what Laplace has shown (*Mec. Cel.* Liv. ii. No. 12.) the repulsion on an exterior point p, arising from a spherical shell of which the radius is r', thickness dr' and center is at O will be measured by

$$\frac{2\pi r' dr' \rho}{1-n\,.\,3-n}\cdot\frac{d}{dr}\cdot\frac{(r+r')^{3-n}-(r-r')^{3-n}}{r},$$

the general term of which when expanded in an ascending series of the powers of $\frac{r'}{r}$ is,

$$+4\pi\,.\,\frac{-2+n\times n\,.\,n+1\,.\,n+2\ldots\ldots n+2s-3\times n+2s-1}{2\;.\;3\;.\;4\;.\;5\;\ldots\ldots\qquad 2s+1}\,r^{n-2s}\,.\,r'^{2s+2}\,.\,\rho dr',$$

and the part of the required repulsion due thereto will, by substituting for ρ its value before found, become

$$+\frac{8P'}{a}\sin\left(\frac{n-2}{2}\pi\right).\,(a^2-b^2)^{\frac{2-n}{2}}\,\frac{-2+n\times n\,.\,n+1\ldots\ldots n+2s-3\times n+1s-1}{2\,.\,3\;.\;4\;\ldots\ldots\qquad 2s+1}\,r^{-n-2s}$$

$$\times\int_0^b\left(1-\frac{r'^2}{a^2}\right)^{-1}(b^2-r'^2)^{\frac{n-2}{2}}\,r'^{2s+2}dr'.$$

It now remains to find the value of the definite integral herein contained. But when $\left(1-\frac{r'^2}{a^2}\right)^{-1}$ is expanded, and the integrations are effected by known formulæ, we obtain

$$(14)\quad \int_0^b\left(1-\frac{r'^2}{a^2}\right)^{-1}(b^2-r'^2)^{\frac{n-2}{2}}\,r'^{2s+2}dr'=\int_0^b\Sigma_0^\infty\frac{r'^{2t}}{a^{2t}}(b^2-r'^2)^{\frac{n-2}{2}}.\,r'^{2s+2}dr'$$

$$=\tfrac{1}{2}b^{2s+1+n}\,\Sigma_0^\infty\frac{b^{2t}}{a^{2t}}\times\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(s+t+\frac{3}{2}\right)}{\Gamma\left(s+t+\frac{3}{2}+n\right)}=\tfrac{1}{2}b^{2s+1+n}.\,\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(s+\frac{3}{2}\right)}{\Gamma\left(s+\frac{n}{2}+\frac{3}{2}\right)}$$

$$\left\{1+\frac{2s+3}{2s+3+n}\,\frac{b^2}{a^2}+\frac{2s+3\,.\,2s+5}{2s+3+n\,.\,2s+5+n}\,\frac{b^4}{a^4}+\&c.\right\}$$

$$=\tfrac{1}{2}b^{2s+1+n}.\,\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(s+\frac{3}{2}\right)}{\Gamma\left(s+\frac{n}{2}+\frac{3}{2}\right)}\times\frac{(2s+1+n)(1-x^2)^{\frac{n-2}{2}}}{x^{2s+1+n}}\int_0\frac{x^{2s+n}dx}{(1-x^2)^{\frac{n}{2}}}$$

$$= \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{n+1}{2}\right)} \cdot \frac{1\,.\,3\,.\,5\,\ldots\ldots 2s+1\; b^{2s+1+n}}{1+n\,.\,3+n\,.\,5+n\ldots\ldots\, 2s-1+n} \times \frac{(1-x^2)^{\frac{n-2}{2}}}{x^{2s+1+n}} \int_0 \frac{x^{2s+n}\,dx}{(1-x^2)^{\frac{n}{2}}}:$$

where after the integrations have been effected, x ought to be made equal to $\dfrac{b}{a}$.

The value of the integral last found being substituted in the expression immediately preceding, and the finite integral taken relative to s from $s=0$ to $s=\infty$ gives for the repulsion of the inner sphere,

$$-\frac{4\pi P'b}{a} \cdot \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}(a^2-b^2)^{\frac{2-n}{2}}$$

$$\times\, \Sigma_0^\infty \frac{n-2\,.\,n\,.\,n+2\ldots\ldots n+2s-4}{2\,.\,4\,.\,6\;\ldots\ldots\;2s}\left(\frac{b}{r}\right)^{2s+n}\frac{(1-x^2)^{\frac{n-2}{2}}}{x^{2s+1+n}}\int_0\frac{x^{2s+n}\,dx}{(1-x^2)^{\frac{n}{2}}}$$

$$= \frac{-4\pi\sqrt{\pi}\,P'a^2r^{-n}}{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}\,\Sigma_0^\infty\,\frac{n-2\,.\,n\,.\,n+2\ldots\ldots n+2s-4}{2\,.\,4\,.\,6\;\ldots\ldots\;2s}\left(\frac{a}{r}\right)^{2s}\int_0 dx\,x^{2s+n}\,.\,(1-x^2)^{\frac{-n}{2}};$$

$$\text{since } \Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi},\ \sin\left(\frac{n-2}{2}\pi\right)=\frac{-\pi}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)},$$

and as was before observed, $x=\dfrac{b}{a}$.

But we have evidently by means of the binomial theorem,

$$\left(1-\frac{a^2x^2}{r^2}\right)^{\frac{2-n}{2}} = \Sigma_0^\infty\,\frac{n-2\,.\,n\,.\,n+2\ldots\ldots n+2s-4}{2\,.\,4\,.\,6\;\ldots\ldots\;2s}\left(\frac{ax}{r}\right)^{2s};$$

and therefore the preceding quantity becomes

$$(15)\ldots\ldots\ldots\ldots -\frac{4\pi\sqrt{\pi}\,.\,P'a^2.r^{-n}}{\Gamma\left(\frac{1-n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}\int_0^{\frac{b}{a}} dx\,x^n\left(1-\frac{a^2x^2}{r^2}\right)^{\frac{2-n}{2}}(1-x^2)^{\frac{-n}{2}}.$$

If now we make $x = \frac{rx'}{a}$, the same quantity may be written

$$(16) \ldots\ldots\ldots - \frac{4\pi\sqrt{\pi}P'a^{1-n}r}{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)} \int_0^{\frac{b}{r}} x'^n dx' (1-x'^2)^{\frac{2-n}{2}} \left(1 - \frac{r^2x'^2}{a^2}\right)^{\frac{-n}{2}}.$$

Having thus the value of the repulsion due to the inner sphere B on an exterior point p, it remains to determine that due to the fluid on A's surface. But this last is represented by

$$(17) \ldots\ldots\ldots \frac{2\pi a P'}{1-n.3-n} \frac{d.}{dr} \frac{(a+r)^{3-n} - (a-r)^{3-n}}{r}.$$

(*Mec. Cel.* Liv. ii. No. 12.) Now by expanding this function there results

$$4\pi P'a^{1-n}r.\frac{2-n}{3}.\left\{1 + \frac{n.n+1}{4.5} 2\frac{r^2}{a^2} + \frac{n.n+1.n+2.n+3}{4.5.6.7}.3\frac{r^4}{a^4} + \&\text{c.}\right\}$$

$$= 4\pi P'a^{1-n}r.\frac{2-n}{3}.\Sigma_0^\infty \frac{n.n+1.n+2 \ldots\ldots\ldots n+2s-1}{4\,.\,5\,.\,6 \ldots\ldots\ldots 2s+3}(s+1)\frac{r^{2s}}{a^{2s}}.$$

The last of these expressions may readily be exhibited under a finite form, by remarking that

$$\int_0^1 x^n dx (1-x^2)^{\frac{2-n}{2}} \left(1 - \frac{r^2x^2}{a^2}\right)^{\frac{-n}{2}} = \int_0^1 x^n dx (1-x^2)^{\frac{2-n}{n}} \Sigma \frac{n.n+2 \ldots n+2s-2}{2.4.6 \ldots 2s}.\frac{r^{2s}x^{2s}}{a^{2s}}$$

$$= \Sigma_0^\infty \frac{n.n+2.n+4 \ldots\ldots\ldots n+2s-2}{2.\,4\,.\,6 \ldots\ldots\ldots 2s}.\frac{r^{2s}}{a^{2s}}.\frac{\Gamma\left(\frac{2s+n+1}{2}\right)\Gamma\left(\frac{4-n}{2}\right)}{2\Gamma\left(\frac{2s+5}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{2-n}{2}\right)\Gamma\left(\frac{1+n}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}.\frac{2-n}{3}.\Sigma_0^\infty \frac{n.n+1.n+2 \ldots\ldots\ldots n+2s-1}{4.\,5\,.\,6 \ldots\ldots\ldots 2s+3}(s+1)\frac{r^{2s}}{a^{2s}}.$$

Hence, since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, the value of the repulsion arising from A's surface becomes

$$\frac{4\pi\sqrt{\pi}.P'a^{1-n}.r}{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)} \int_0^1 x^n dx (1-x^2)^{\frac{2-n}{2}} \left(1 - \frac{r^2x^2}{a^2}\right)^{\frac{-n}{2}}.$$

Now by adding the repulsion due to the inner sphere which is given by the formula (16), we obtain, (since it is evidently indifferent what variable enters into a definite integral, provided each of its limits remain unchanged)

$$\frac{4\pi\sqrt{\pi}P'a^{1-n}r}{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}.\left\{\int_0^1 x^n dx(1-x^2)^{\frac{2-n}{2}}\left(1-\frac{r^2x^2}{a^2}\right)^{\frac{-n}{2}}-\int_0^{\frac{b}{r}} x^n dx(1-x^2)^{\frac{2-n}{2}}.\left(1-\frac{r^2x^2}{a^2}\right)^{\frac{-n}{2}}\right\}$$

$$=\frac{4\pi\sqrt{\pi}.P'a^{1-n}r}{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}\int_{\frac{b}{r}}^1 x^n dx(1-x^2)^{\frac{2-n}{2}}.\left(1-\frac{r^2x^2}{a^2}\right)^{\frac{-n}{2}},$$

for the value of the total repulsion upon a particle p of positive fluid situate within the sphere A and exterior to B. We thus see that when P' is positive the particle p is always impelled by a force which is equal to zero at B's surface, and which continually increases as p recedes farther from it. Hence, if any particle of positive fluid is separated ever so little from B's surface, it has no tendency to return there, but on the contrary, it is continually impelled therefrom by a regularly increasing force; and consequently, as was before observed, the equilibrium can not be permanent until all the positive fluid has been gradually abstracted from B and carried to the surface of A, where it is retained by the non-conducting medium with which the sphere A is conceived to be surrounded.

Let now q represent the total quantity of fluid in the inner sphere, then the repulsion exerted on p by this will evidently be

$$qr^{-n},$$

when r is supposed infinite. Making therefore r infinite in the expression (15), and equating the value thus obtained to the one just given, there arises

$$q=\frac{-4\pi\sqrt{\pi}.P'a^2}{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}\int_0^{\frac{b}{a}} dx.x^n(1-x^2)^{\frac{-n}{2}}.$$

When the equilibrium has become permanent, q is equal to the total quantity of that kind of fluid, which we choose to consider negative, originally introduced into the sphere A; and if now q_1 represent the total quantity of fluid of opposite name contained within A, we shall have, for the determination of the two unknown quantities P' and b, the equations

$$q_1 = 4\pi a^2 . P',$$

$$\text{and } \frac{q}{q_1} = \frac{-\sqrt{\pi}}{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)} \int_0^{\frac{b}{a}} dx\, x^n (1-x^2)^{\frac{-n}{2}},$$

and hence we are enabled to assign accurately the manner in which the two fluids will distribute themselves in the interior of A; q and q_1, the quantities of the fluids of opposite names originally introduced into A being supposed given.

9. In the two foregoing articles we have determined the manner in which our hypothetical fluids will distribute themselves in the interior of a conducting sphere A when in equilibrium and free from all exterior actions, but the method employed in the former is equally applicable when the sphere is under the influence of any exterior forces. In fact, if we conceive them all resolved into three X, Y, Z, in the direction of the co-ordinates x, y, z of a point p, and then make, as in Art. 1,

$$V = \int \frac{\rho\, dv}{g^{n-1}}.$$

we shall have, in consequence of the equilibrium,

$$0 = \frac{1}{1-n}\frac{dV}{dx} + X, \quad 0 = \frac{1}{1-n}\frac{dV}{dy} + Y, \quad 0 = \frac{1}{1-n}\frac{dV}{dz} + Z,$$

which, multiplied by dx, dy and dz respectively, and integrated, give

$$\text{const.} = \frac{1}{1-n} V + \int (X dx + Y dy + Z dz);$$

where $X dx + Y dy + Z dz$ is always an exact differential.

We thus see that when X, Y, Z are given rational and entire functions V will be so likewise, and we may thence deduce (Art. 5.)

$$\rho = (1 - x'^2 - y'^2 - z'^2)^{\frac{n-4}{2}} . f(x', y', z'),$$

where f is the characteristic of a rational and entire function of the same degree as V.

The preceding method is directly applicable when the forces X, Y, Z are given explicitly in functions of x, y, z. But instead of these forces, we may conceive the density of the fluid in the exterior bodies as given, and thence determine the state which its action will induce in the conducting sphere A. For example, we may in the first place suppose the radius of A to be taken as the unit of space, and an exterior concentric spherical surface, of which the radius is a, to be covered with fluid of the density $U''^{(i)}$: $U''^{(i)}$ being a function of the two polar co-ordinates θ'' and ϖ'' of any element of the spherical surface of the same kind as those considered by Laplace (*Mec. Cel.* Liv. iii.). Then it is easy to perceive by what has been proved in the article last cited, that the value of the induced density will be of the form

$$\rho = U'^{(i)} r'^{i} (1 - r'^2)^{\frac{n-4}{2}} . f(r'^2);$$

r', θ', ϖ' being the polar co-ordinates of the element dv, and $U'^{(i)}$ what $U''^{(i)}$ becomes by changing θ'', ϖ'' into θ', ϖ'.

Still continuing to follow the methods before explained, (Art. 4. and 5.) we get in the present case

$$f(x', y', z') = U'^{(i)} r'^{i} f(r'^2) = f^{(i)},$$

and by expanding $f(r'^2)$, we have

$$f(r'^2) = B_0 + B_1 r'^2 + B_2 r'^4 + B_3 r'^6 + \&c.$$

Hence, $f_t^{(i)} = B_t U'^{(i)}$, and

$$V_t^{(i)} = \frac{2\pi^2 U^{(i)} r^i}{\sin\left(\frac{n-2}{2}\pi\right)} B_t \Sigma_0^\infty r^{2t'} \frac{4-n.6-n.....2t-2t'+2-n}{2\,.\,4\,.\,6\,.....\,2t-2t'} \times \frac{n-2.n.....n+2t'-4}{2.4.....\,2t'}$$
$$\times \frac{n-1.n+1.....n+2i+2t'-3}{3\,.\,5\,.....\,2i+2t'+1}.$$

Then, by giving to t all the values 1, 2, 3, &c. of which it is susceptible, and taking the sum of all the resulting quantities, we shall have, since in the present case V reduces itself to the single term $V^{(i)}$,

$$V = \frac{2\pi^2 U^{(i)} r^i}{\sin\left(\frac{n-2}{2}\pi\right)} S B_t \Sigma r^{2t'} . \frac{4-n.6-n.....2t-2t'+2-n}{2\,.\,4\,.....\,2t-2t'} \times \frac{n-2.n.....n+2t'-4}{2.4.....\,2t'}$$
$$\times \frac{n-1.n+1.....n+2i+2t'-3}{3\,.\,5\,.....\,2i+2t'+1};$$

the sign S belonging to the unaccented letter t.

If now V' represents the function analagous to V and due to the fluid on the spherical surface, we shall obtain by what has been proved (Art. 3.)

$$V' = U^{(i)}.\,2\pi a^2 \frac{1.3.5.....2i-1}{1.2.3.....\;\;i} \int_{-1}^{1} d\mu\,(i)\,(r^2 - 2ar\mu + a^2)^{\frac{1-n}{2}};$$

(i) representing the same function as in the article just cited.

Moreover, it is evident from the equation (10) Art. 4, that

$$\int_{-1}^{1} d\mu\,(i)\,(r^2 - 2ar\mu + a^2)^{\frac{1-n}{2}} = 2a^{1-n} \frac{1.2.3.....\;\;i}{1.3\;.....2i-1} \Sigma \frac{n-1.n+1.....n+2i+2t'-3}{3\;.\;5\;.....\;\;2i+2t'+1}$$

$$\times \frac{n-2.n.....n+2t'-4}{2.4.....\;\;2t'} \left(\frac{r}{a}\right)^{i+2t'};$$

and consequently,

$$(19)......V' = U^{(i)}.\,4\pi a^{3-n}.\,\Sigma \frac{n-1.n+1.....n+2i+2t'-3}{3\;.\;5\;.....\;\;2i+2t'+1}$$

$$\times \frac{n-2.n.....n+2t'-4}{2.4.....\;\;2t'} \left(\frac{r}{a}\right)^{i+2t};$$

the finite integrals extending from $t'=0$ to $t'=\infty$.

Substituting now for V and V' their values in the equation of equilibrium,

$$(20)\;.........\;\text{const.} = V' + V,$$

we immediately obtain

$$\text{const.} = U^{(i)}.\,4\pi a^{3-n}.\,\Sigma \frac{n-1.n+1.....n+2i+2t'-3}{3\;.\;5\;.....\;\;2i+2t'+1}$$

$$\times \frac{n-2.n.....n+2t'-4}{2\;.4.....\;\;2t'} \left(\frac{r}{a}\right)^{i+2t'}$$

$$+ \frac{2\pi^2}{\sin\left(\dfrac{n-2}{2}\pi\right)} U^{(i)} SB_i \Sigma r^{i+2t'}.\frac{n-1.n+1.....n+2i+2t'-3}{3\;.\;5\;.....\;\;2i+2t'-1}$$

$$\times \frac{n-2.n.....n+2t'-4}{2\;.4.....\;\;2t'} \times \frac{4-n.6-n.....2t-2t'+2-n}{2\;.\;4\;.....2t-2t'},$$

the constant on the left side of this equation being equal to zero, except when $i = 0$.

By equating separately the coefficients of the various powers of the indeterminate quantity r, we get the following system of equations:

$$-\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi}a^{3-n-1} = B_0 + B_1\frac{4-n}{2} + B_2\frac{4-n.6-n}{2\ .\ 4} + \&c.$$

$$-\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi}a^{3-n-i-2} = B_1 + B_2\frac{4-n}{2} + B_3\frac{4-n.6-n}{2\ .\ 4} + \&c.$$

$$-\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi}a^{3-n-i-4} = B_2 + B_3\frac{4-n}{2} + B_4\frac{4-n.6-n}{2\ .\ 4} + \&c.$$

&c.....................&c................&c.....................

But it is evident from the form of these equations, that if we make generally $B_{t+1} = a^{-2}B_t$, they will all be satisfied provided the first is, and as by this means the first equation becomes

$$-\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi}a^{3-n-i} = B_0\left(1+\frac{4-n}{2}a^{-2}+\frac{4-n.6-n}{2\ .\ 4}a^{-4}+\&c.\right)$$

$$= B_0(1-a^{-2})^{\frac{n-4}{2}} = B_0 a^{4-n}(a^2-1)^{\frac{n-4}{2}},$$

there arises

$$B_0 = -\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi}a^{-i-1}(a^2-1)^{\frac{4-n}{2}},\quad B_1 = B_0.a^{-2},\quad B_2 = B_0.a^{-4},\ \&c.$$

Hence

$$f(r'^2) = B_0 + B_1 r'^2 + B_2 r'^4 + \&c. = B_0\left(1+\frac{r'^2}{a^2}+\frac{r'^4}{a^4}+\&c.\right)$$

$$= B_0\left(1-\frac{r'^2}{a^2}\right)^{-1} = B_0 a^2(a^2-r'^2)^{-1} = -\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi}a^{i+1}(a^2-1)^{\frac{4-n}{2}}(a^2-r'^2)^{-1},$$

and the required value of ρ becomes

$$(21)\ldots\ldots\ \rho = U'^{(i)}r'^i(1-r'^2)^{\frac{n-4}{2}}f(r'^2)$$

$$= -\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi}(a^2-1)^{\frac{4-n}{2}}a\,U'^{(i)}\left(\frac{r'}{a}\right)^i(a^2-r'^2)^{-1}(1-r'^2)^{\frac{n-4}{2}}.$$

But whatever the density P on the inducing spherical surface may be, we can always expand it in a series of the form

$$P = U''^{(0)} + U''^{(1)} + U''^{(2)} + U''^{(3)} + \text{\&c. } in\ inf.$$

and the corresponding value of ρ by what precedes will be

$$\rho = -\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi} a(a^2-1)^{\frac{4-n}{2}} . (a^2-r'^2)^{-1}(1-r'^2)^{\frac{n-4}{2}} \ldots\ldots$$

$$\ldots\ldots \times \left\{U'^{(0)} + U'^{(1)}\frac{r'}{a} + U'^{(2)}\frac{r'^2}{a^2} + U'^{(3)}\frac{r'^3}{a^3} + \text{\&c. } in\ inf.\right\};$$

$U'^{(0)}$, $U'^{(1)}$, $U'^{(2)}$, &c. being what $U''^{(0)}$, $U''^{(1)}$, $U''^{(2)}$, &c. become by changing θ'', ϖ'' into θ', ϖ', the polar co-ordinates of the element dv. But, since we have generally

$$\int d\theta'' d\varpi'' \sin\theta'' PQ^{(i)} = \int d\theta'' d\varpi'' \sin\theta'' U''^{(i)} Q^{(i)} = \frac{4\pi}{2i+1} U^{(i)},$$

(*Mec. Cel.* Liv. iii.) the preceding expression becomes

$$\rho = \frac{-\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2} a(a^2-1)^{\frac{4-n}{2}} (a^2-r'^2)^{-1}(1-r'^2)^{\frac{n-4}{2}} \int d\theta'' d\varpi'' \sin\theta'' \ldots\ldots$$

$$\ldots\ldots \Sigma_0^\infty (2i+1) PQ^{(i)} \frac{r'^i}{a^i};$$

the integrals being taken from $\theta''=0$ to $\theta''=\pi$, and from ϖ'' to $\varpi''=2\pi$.

In order to find the value of the finite integral entering into the preceding formula, let R represent the distance between the two elements $d\sigma$, dv; then by expanding $\frac{a}{R}$ in an ascending series of the powers of $\frac{r'}{a}$ we shall obtain

$$\frac{a}{R} = \frac{a}{\sqrt{a^2 - 2ar'[\cos\theta'\cos\theta'' + \sin\theta'\sin\theta''\cos(\varpi'-\varpi'') + r'^2}} = \Sigma_0^\infty Q^{(i)} . \frac{r'^i}{a^i},$$

Mec. Cel. Liv. iii.). Hence we immediately deduce

$$\frac{a\sqrt{r'}}{R} = \Sigma_0^\infty Q^{(i)} \frac{r'^{i+\frac{1}{2}}}{a^i}, \text{ and } 2\sqrt{r'}\frac{d.}{dr'}\frac{a\sqrt{r'}}{R} = \Sigma_0^\infty (2i+1) Q^{(i)} \frac{r'^i}{a^i}.$$

If now we substitute this in the value of ρ before given, and afterwards write $\dfrac{ds}{a^2}$ and $\dfrac{a^2-r'^2}{2R^3}$ in the place of their equivalents,

$$d\theta'' d\varpi'' \sin\theta'', \text{ and } \sqrt{r'}\frac{d.}{dr'}\frac{\sqrt{r'}}{R},$$

we shall obtain

$$\rho = -\frac{\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2}(a^2-1)^{\frac{4-n}{2}}(1-r'^2)^{\frac{n-4}{2}}\int\frac{d\sigma P}{R^3};$$

the integral relative to $d\sigma$ being extended over the whole spherical surface.

Lastly, if ρ_1 represents the density of the reducing fluid disseminated over the space exterior to A, it is clear that we shall get the corresponding value of ρ by changing P into $\rho_1 da$ in the preceding expression, and then integrating the whole relative to a. Thus,

$$\rho = -\frac{\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2}(1-r'^2)^{\frac{n-4}{2}}\int(1-a^2)^{\frac{4-n}{2}}\int\frac{d\sigma da\rho_1}{R^3}.$$

But $d\sigma da = dv_1$; dv_1 being an element of the volume of the exterior space, and therefore we ultimately get

$$(22)\ldots\ldots\rho = -\frac{\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2}(1-r'^2)^{\frac{n-4}{2}}.\int\rho_1 dv_1\frac{(a^2-1)^{\frac{4-n}{2}}}{R^3};$$

where the last integral is supposed to extend over all the space exterior to the sphere and R, to represent the distance between the two elements dv and dv_1.

It is easy to perceive from what has before been shown (Art. 7.), that we may add to any of the preceding values of ρ, a term of the form

$$h(1-r'^2)^{\frac{n-4}{2}};$$

h being an arbitrary constant quantity: for it is clear from the article just cited, that the only alteration which such an addition could produce would be to change the value of the constant on the left side of the general equation of equilibrium; and as this constant is arbitrary, it is evident that the equilibrium will not be at all affected by the change in question. Moreover, it may be observed, that in general the additive term is necessary to enable us to assign the proper value of ρ, when Q, the quantity of redundant fluid originally introduced into the sphere, is given.

In the foregoing expressions the radius of the sphere has been taken as the unit of space, but it is very easy thence to deduce formulæ adapted to any other unit, by recollecting that $\frac{\rho}{\rho_1}$, $\frac{\rho}{P}$, $\frac{\rho}{U'^{(i)}}$ and $\frac{V_1}{U^{(i)}}$, are quantities of the dimensions 0, -1, -1 and $3-n$ respectively with regard to space: for if b represents the sphere's radius, when we employ any other unit we shall only have to write, $\frac{r}{b}$, $\frac{r'}{b}$, $\frac{R}{b}$, $\frac{dv_1}{b^3}$ and $\frac{a}{b}$ in the place of r, r', R, dv_1 and a, and afterwards to multiply the resulting expressions by such powers of b, as will reduce each of them to their proper dimensions.

If we here take the formula (22) of the present article as an example, there will result,

$$(23)\ldots\ldots\ldots \rho = -\frac{\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2}(b^2-r'^2)^{\frac{n-4}{2}} \int \rho_1 dv_1 \frac{(a^2-b^2)^{\frac{4-n}{2}}}{R^3},$$

for the value of the density which would be induced in a sphere A, whose radius is b, by the action of any exterior bodies whatever.

When $n > 2$, the value of ρ or of the density of the free fluid here given offers no difficulties, but if $n < 2$, we shall not be able strictly to realize it, for reasons before assigned (Art. 6. and 7.) If however n is positive, and we adopt the hypothesis of two fluids, supposing that the quantities of each contained by bodies in a natural state are exceedingly great, we shall easily perceive by proceeding as in the last of the articles here cited, that the density given by the formula (23) will be sensibly correct except in the immediate vicinity of A's surface, provided we extend it to the surface of a sphere whose radius is $b-\delta b$ only, and afterwards conceive the exterior shell entirely deprived of fluid: the surface of the conducting sphere itself having such a quantity condensed upon it, that its density may every where be represented by

$$P' = -\frac{\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2} \times \frac{b^{\frac{n-4}{2}}(2\delta b)^{\frac{n-2}{2}}}{n-2} \int \rho_1 dv_1 \frac{(a^2-b^2)^{\frac{4-n}{2}}}{R^3}.$$

Application of the general Methods to circular conducting Planes, &c.

10. Methods in every way similar to those which have been used for a sphere, are equally applicable to a circular plane as we shall immediately proceed to show, by endeavouring in the first place to determine the value of V when the density of the fluid on such a plane is of the form

$$\rho = (1 - r'^2)^\beta . f(x', y'):$$

f being the characteristic of a rational and entire function of the degree s; x', y' the rectangular co-ordinates of any element $d\sigma$ of the plane's surface, and r', θ' the corresponding polar co-ordinates.

Then we shall readily obtain the formula

$$V = \int \frac{\rho d\sigma}{g^{n-1}} = \iint \frac{r dr' d\theta' (1 - r'^2)^\beta . f(x', y')}{(r^2 - 2rr' \cos(\theta - \theta') + r'^2)^{\frac{n-1}{2}}};$$

where r, θ are the polar co-ordinates of p, and the integrals are to be taken from $\theta' = 0$ to $\theta' = 2\pi$, and from $r' = 0$ to $r' = 1$; the radius of the circular plane being for greater simplicity considered as the unit of distance.

Since the function $f(x', y')$ is rational and entire of the degree s, we may always reduce it to the form

$$(24) \ldots\ldots\ldots f(x', y') = A^{(0)} + A^{(1)} \cos\theta' + A^{(2)} \cos 2\theta' + A^{(3)} \cos 3\theta' + \\ + B^{(1)} \sin\theta' + B^{(2)} \sin 2\theta' + B^{(3)} \sin 3\theta' +$$

the coefficients $A^{(0)}$, $A^{(1)}$, $A^{(2)}$, &c. $B^{(1)}$, $B^{(2)}$, $B^{(3)}$, &c. being functions of r' only of a degree not exceeding s, and such that

$$A^{(0)} = a_0^{(0)} + a_1^{(0)} r'^2 + a_2^{(0)} r'^4 + \text{\&c.}; \quad A^{(1)} = (a_0^{(1)} + a_1^{(1)} r^2 + a_2^{(1)} r'^4 +) r';$$
$$B^{(1)} = (b_0^{(1)} + b_1^{(1)} r'^2 + b_2^{(1)} r'^4 + \text{\&c.}) r'; \quad B^{(2)} = (b_0^{(2)} + b_1^{(2)} r'^2 + \text{\&c.}) r'^2.$$

We will now consider more particularly the part of V due to any of the terms in f as $A^{(i)}\cos i\theta'$ for example. The value of this part will evidently be

$$\iint \frac{r'dr'd\theta'(1-r'^2)^\beta A^{(i)}\cos i\theta'}{(r^2-2rr'\cos(\theta-\theta')+r'^2)^{\frac{n-1}{2}}};$$

the limits of the integrals being the same as before. But if we make $\theta'=\theta+\phi$, there will result $d\theta'=d\phi$, and $\cos i\theta'=\cos i\theta\cos i\phi-\sin i\theta\sin i\phi$, and hence the double integral here given by observing that the term multiplied $\sin i\phi$ vanishes when the integration relative to ϕ is effected, becomes

$$\cos i\theta \textstyle\int_0^1 A^{(i)}r'dr'(1-r'^2)^\beta \displaystyle\int_0^{2\pi}\frac{d\phi\cos i\phi}{(r^2-2rr'\cos\phi+r'^2)^{\frac{n-1}{2}}};$$

If now we write $V_t^{(i)}$ for that portion of V which is due to the term $a_t^{(i)}.r'^{i+2t}$ in the coefficient $A^{(i)}$ we shall have

$$V_t^{(i)}=a_t^{(i)}.\cos i\theta \textstyle\int_0^1 r'^{i+2t+1}dr'(1-r'^2)^\beta \displaystyle\int_0^{2\pi}\frac{d\phi\cos i\phi}{(r^2-2rr'\cos\phi+r'^2)^{\frac{n-1}{2}}}.$$

But by well known methods we readily get

$$\int_0^{2\pi}\frac{d\phi\cos i\phi}{(r^2-2rr'\cos\phi+r'^2)^{\frac{n-1}{2}}}$$

$$=2\pi r^i.r'^{1-n-i}\Sigma_0^\infty\frac{r^{2t'}}{r'^{2t'}}.\frac{n-1.n+1\ldots\ldots n+2t'-3}{2\;.\;4\;\ldots\ldots\;2t'}\times\frac{n-1.n+1\ldots\ldots n+2i+2t'-3}{2\;.\;4\;\ldots\ldots\;2i+2t},$$

when $r'>r$, and when $r'<r$, the same expression will still be correct, provided we change r into r' and reciprocally.

This value being substituted in that of $V_t^{(i)}$ we shall readily have by following the processes before explained, (Art. 1. and 2.)

$$V_t^{(i)}=2\pi a_t^{(i)}r^i\cos i\theta\,\Sigma_0^\infty r^{2t'}\frac{n-1.n+1\ldots\ldots n+2t'-3}{2\;.\;4\;\ldots\ldots\;2t'}$$

$$\times\frac{n-1.n+1\ldots\ldots n+2i+2t'-3}{2\;.\;4\;\ldots\ldots\;2i+2t'}\times\frac{\Gamma(\beta+1)\Gamma\left(\frac{3+2t-2t'-n}{2}\right)}{2\Gamma\left(\frac{2\beta+5+2t-2t'}{2}\right)}$$

$$= \pi a_t^{(i)} r^i \cos i\theta \,.\, \frac{\Gamma(\beta+1)\,\Gamma\left(\frac{3-n}{2}\right)}{\Gamma\left(\frac{2\beta+5-n}{2}\right)} \,.$$

$$\Sigma_0^\infty\, r^{2t'} \frac{n-1\,.\,n+1\ldots\ldots n+2t'-3}{2\;.\;4\;\ldots\ldots\;2t'} \times \frac{n-1\,.\,n+1\ldots\ldots n+2i+2t'-3}{2\;.\;4\;\ldots\ldots\;2i+2t'}$$

$$\times \frac{3-n\,.\,5-n\ldots\ldots\;1+2t-2t'-n}{2\beta+5-n\ldots\ldots 2\beta+3+2t+2t'-n};$$

the sign of integration Σ belonging to the variable t'.

Having thus the part of V due to the term $a_t^{(i)} \cos i\theta'$ in the expansion of $f(x', y')$ it is clear that we may thence deduce the part due to the analogous term $b_t^{(i)} \sin i\theta'$ by simply changing $a_t^{(i)} \cos i\theta$ into $b_t^{(i)} \sin i\theta$, and consequently we shall have the total value of V itself, by taking the sum of the various parts due to all the different terms which enter into the complete expansion of $f(x', y')$.

If now we make $\beta = \frac{n-3}{2}$ and recollect that

$$\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{3-n}{2}\right) = \frac{\pi}{\sin\left(\frac{n-1}{2}\pi\right)},$$

the foregoing expression will undergo simplifications analogous to those before noticed (Art. 5.) Thus we shall obtain

$$V_t^{(i)} = \frac{\pi^2 a_t^{(i)}}{\sin\left(\frac{n-1}{2}\pi\right)}\, r^i \cos i\theta \,.\, \Sigma r^{2t'} .\, \frac{n-1\,.\,n+1\ldots\ldots n+2t'-3}{2\;.\;4\;\ldots\ldots\;2t'}$$

$$\times \frac{n-1\,.\,n+1\ldots\ldots n+2i+2t'-3}{2\;.\;4\;\ldots\ldots\;2i+2t'} \times \frac{3-n\,.\,5-n\ldots\ldots 1+2t-2t'-n}{2\;.\;4\;\ldots\ldots\;2t-2t'},$$

or by writing for abridgment

$$\phi(i, t') = \frac{n-1\,.\,n+1\ldots\ldots n+2t'-3}{2\;.\;4\;\ldots\ldots\;2t'} \times \frac{n-1\,.\,n+1\ldots\ldots n+2i+2t'-3}{2\;.\;4\;\ldots\ldots\;2i+2t'},$$

there will result this particular value of β

$$V_t^{(i)} = \frac{\pi^2 a_t^{(i)}}{\sin\left(\frac{n-1}{2}\pi\right)}\, r^i \cos i\theta \,.\, \Sigma r^{2t'} .\, \frac{3-n\,.\,5-n\ldots\ldots 1+2t-2t'-n}{2\,.\,4\,.\,6\;\ldots\ldots\;2t-2t'} \,.\, \phi(i;t'),$$

and afterwards by making

$$V^{(i)} = V_0^{(i)} + V_1^{(i)} + V_2^{(i)} + V_3^{(i)} + V_4^{(i)} + \&c.$$

we shall have

$$V^{(i)} = \frac{\pi^2}{\sin\left(\frac{n-1}{2}\pi\right)} r^i \cos i\theta \text{ into } \times \ldots\ldots$$

$$\begin{aligned}
&a_0^{(i)}.\,1.\,\phi(i;\,0)\\
&+a_1^{(i)}.\,\frac{3-n}{2}.\,\phi(i;\,0) + a_1^{(i)}.\,1.\,\phi(i;\,1).\,r^2\\
&+a_2^{(i)}.\,\frac{3-n\,.\,5-n}{2\,.\,4}.\,\phi(i;\,0) + a_2^{(i)}.\,\frac{3-n}{2}.\,\phi(i;\,1).\,r^2 + a_2^{(i)}\,\phi(i;\,2).\,r^4\\
&+a_3^{(i)}.\,\frac{3-n.5-n.7-n}{2\,.\,4\,.\,6}.\,\phi(i;\,0) + a_3^{(i)}\,\frac{3-n.5-n}{2\,.\,4}.\,\phi(i;\,1).\,r^2\\
&\qquad + a_3^{(i)}.\,\frac{3-n}{2}.\,\phi(i;\,2).\,r^4 + a_3^{(i)}.\,1.\,\phi(i;\,3).\,r^6\\
&+\&c.\ldots\ldots\ldots\ldots + \&c.\ldots\ldots\ldots\ldots + \&c.\ldots\ldots\ldots\ldots + \&c.\ldots\ldots\ldots\ldots
\end{aligned}$$

Conceiving in the next place that V is a given rational and entire function of x, y, the rectangular co-ordinates of p, we shall have since $x = r\cos\theta$, $y = r\sin\theta$.

$$(25)\ \ldots\ldots V = C^{(0)} + C^{(1)}\cos\theta + C^{(2)}\cos 2\theta + C^{(3)}\cos 3\theta + \&c.$$
$$+ E^{(1)}\sin\theta + E^{(2)}\sin 2\theta + E^{(3)}\sin 3\theta + \&c.$$

of which expansion any coefficient as $C^{(i)}$ for example, may be still farther developed in the form

$$C^{(i)} = \frac{\pi^2 . r^i}{\sin\left(\frac{n-1}{2}\pi\right)} \{c_0^{(i)}.\,\phi(i;\,0) + c_1^{(i)}.\,\phi(i;\,1).\,r^2 + c_2^{(i)}.\,\phi(i;\,2).\,r^4 + \&c.\}.$$

Now it is clear that the term $C^{(i)}\cos i\theta$ in the developement (25) corresponds to that part of V which we have designated by $V^{(i)}$, and hence by equating these two forms of the same quantity, we get

$$V^{(i)} = C^{(i)}\cos i\theta,$$

which by substituting for $V^{(i)}$ and $C^{(i)}$ their values before exhibited, and comparing like powers of the indeterminate quantity r gives

$$c_0^{(i)} = 1 \,.\, a_0^{(i)} + \frac{3-n}{2} a_1^{(i)} + \frac{3-n \,.\, 5-n}{2 \,.\, 4} a_2^{(i)} + \frac{3-n \,.\, 5-n \,.\, 7-n}{2 \,.\, 4 \,.\, 6} a_3^{(i)} + \&c.$$

$$c_1^{(i)} = 1 \,.\, a_1^{(i)} + \frac{3-n}{2} a_2^{(i)} + \frac{3-n \,.\, 5-n}{2 \,.\, 4} a_3^{(i)} + \&c.$$

$$c_2^{(i)} = 1 \,.\, a_2^{(i)} + \frac{3-n}{2} a_3^{(i)} + \&c.$$

&c. = ...&c..........&c.......

of which system the general type is

$$c_u^{(i)} = (1-\epsilon)^{\frac{n-3}{2}} \,.\, a_u^{(i)};$$

the symbols of operation being here separated from those of quantity, and ϵ being used in its ordinary acceptation with reference to the lower index u, so that we shall have generally

$$\epsilon^m \,.\, a_u^{(i)} = a_{u+m}^{(i)}.$$

The general equation between $a_u^{(i)}$ and $c_u^{(i)}$ being resolved, evidently gives by expanding the binomial and writing in the place of $\epsilon c_u^{(i)}$, $\epsilon^2 c_u^{(i)}$, $\epsilon^3 c_u^{(i)}$, &c. their values $c_{u+1}^{(i)}$, $c_{u+2}^{(i)}$, $c_{u+3}^{(i)}$, &c.

$$(26)\ldots\ldots\ldots\ldots a_u^{(i)} = (1-\epsilon)^{\frac{3-n}{2}} c_u^{(i)} = c_u^{(i)} + \frac{n-3}{2} c_{u+1}^{(i)} + \frac{n-3 \,.\, n-1}{2 \,.\, 4}$$

$$c_{u+2}^{(i)} + \frac{n-3 \,.\, n-1 \,.\, n+1}{2 \,.\, 4 \,.\, 6} c_{u+3}^{(i)} + \&c.$$

Having thus the value of $a_u^{(i)}$ we thence immediately deduce the value of $A^{(i)}$ and this quantity being known, the first line of the expansion (25) evidently becomes known.

In like manner when we suppose that the quantity $E^{(i)}$ is expanded in a series of the form

$$E^{(i)} = \frac{\pi^2 r^i}{\sin\left(\frac{n-1}{2}\pi\right)} \{e_0^{(i)} \,.\, \phi(i;\, 0) + e_1^{(i)} \phi(i;\, 1) \,.\, r^2 + e_2^{(i)} \phi(i;\, 2) \,.\, r^4 + \&c.\}$$

we shall readily deduce

$$b_u^{(i)} = (1-\epsilon)^{\frac{3-n}{2}} e_u^{(i)} = e_u^{(i)} + \frac{n-3}{2} e_{u+1}^{(i)} + \frac{n-3 \,.\, n-1}{2 \,.\, 4} e_{u+2}^{(i)} + \&c.,$$

and $b_u^{(i)}$ being thus given, $B^{(i)}$ and consequently the second line of the expansion (25) are also given.

From what has preceded, it is clear that when V is given equal to any rational and entire function whatever of x and y, the value of $f(x', y')$ entering into the expression

$$\rho = (1 - r'^2)^{\frac{n-3}{2}} \,.\, f(x', y'),$$

will immediately be determined by means of the most simple formulæ.

The preceding results being quite independent of the degree s of the function $f(x', y')$ will be equally applicable when s is infinite, or wherever this function can be expanded in a series of the entire powers of x', y', and the various products of these powers.

We will now endeavour to determine the manner in which one fluid will distribute itself on the circular conducting plane A when acted upon by fluid distributed in any way in its own plane.

For this purpose, let us in the first place conceive a quantity q of fluid concentrated in a point P, where $r=a$ and $\theta=0$, to act upon a conducting plate whose radius is unity. Then the value of V due to this fluid will evidently be

$$\frac{q}{(a^2 - 2ar\cos\theta + r^2)^{\frac{n-1}{2}}} = V',$$

and consequently the equation of equilibrium analogous to the one marked (20) Art. 10., will be

$$(27) \ldots\ldots\ldots\ldots\ldots \text{const.} = \frac{q}{(a^2 - 2ar\cos\theta + r^2)^{\frac{n-1}{2}}} + V;$$

V being due to the fluid on the conducting plate only.

If now we expand the value of V deduced from this equation, and then compare it with the formulæ (25) of the present article, we shall have generally $E^{(i)}=0$, and

$$C^{(i)}=-2qa^{-n}\frac{r^i}{a^i}.\{\phi(i;\,0)+\phi(i;\,1)\frac{r^2}{a^2}+\phi(i;\,2)\frac{r^4}{a^4}+\phi(i;\,3)\frac{r^6}{a^6}+\&\text{c.}\},$$

except when $i=0$, in which case we must take only half the quantity furnished by this expression in order to have the correct value of $C^{(0)}$. Hence whatever u may be,

$$e_u^{(i)}=0,\qquad \text{and } c_u^{(i)}=-\frac{2\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2}qa^{1-n-i-2u};$$

the particular value $i=0$ being excepted, for in this case we have agreeably to the preceding remark

$$c_u^{(0)}=-\frac{\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2}q\,.\,a^{1-n-2u},$$

and then the only remaining exception is that due to the constant quantity on the left side of the equation (27). But it will be more simple to avoid considering this last exception here, and to afterwards add to the final result the term which arises from the constant quantity thus neglected.

The equation (26) of the present article gives by substituting for $c_u^{(i)}$ its value just found.

$$a_u^{(i)}=-\frac{2\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2}qa^{1-n-i-2u}.\{1+\frac{n-3}{2}.a^{-2}$$

$$+\frac{n-3\,.\,n-1}{2\;.\;4}.a^{-4}+\frac{n-3\,.\,n-1\,.\,n-1}{2\;.\;4\;.\;6}.a^{-6}+\&\text{c.}\}$$

$$=-\frac{2\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2}qa^{1-n-i-2u}(1-a^{-2})^{\frac{3-n}{2}}$$

$$=-\frac{2\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2}qa^{-2-i-2u}.(a^2-1)^{\frac{3-n}{2}},$$

and consequently,

$$A^{(i)} = \{a_0^{(i)} + a_1^{(i)} r'^2 + a_2^{(i)} r'^4 + \&c.\} r'^i$$

$$= -\frac{2\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2} q a^{-2-i} (a^2-1)^{\frac{3-n}{2}} r'^i . \{1 + \frac{r'^2}{a^2} + \frac{r'^4}{a^2} + \&c.\}$$

$$= -\frac{2\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2} q a^{-2-i} (a^2-1)^{\frac{3-n}{2}} r'^i \left(1 - \frac{r'^2}{a^2}\right)^{-1}$$

$$= -\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi^2} q (a^2-1)^{\frac{3-n}{2}} (a^2 - r'^2)^{-1} . \frac{r'^i}{a^i};$$

the particular value $A^{(0)}$ being one half only of what would result from making $i = 0$ in this general formulæ.

But $e_u^{(i)} = 0$ evidently gives $E^{(i)} = 0$, and therefore the expansion of $f(x', y')$ before given becomes

$$f(x', y') = A^{(0)} + A^{(1)} \cos\theta' + A^{(2)} \cos 2\theta' + A^{(3)} \cos 3\theta' + \&c.$$

$$= -\frac{2\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2} q (a^2-1)^{\frac{3-n}{2}} (a^2 - r'^2)^{-1} . \{\tfrac{1}{2} + \frac{r'}{a}\cos\theta' + \frac{r'^2}{a^2}\cos 2\theta' + \&c.\}$$

or by summing the series included between the braces,

$$f(x', y') = -\frac{\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2} q \frac{(a^2-1)^{\frac{3-n}{2}}}{a^2 - 2ar'\cos\theta' + r'^2}$$

$$= -\frac{\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2} q \frac{(a^2-1)^{\frac{3-n}{2}}}{R^2};$$

R being the distance between P, the point in which the quantity of fluid q is concentrated, and that to which the density ρ is supposed to belong.

Having thus the value of $f(x', y')$ we thence deduce

$$\rho = (1 - r'^2)^{\frac{n-3}{2}} f(x', y') = -\frac{\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2} (1 - r'^2)^{\frac{n-3}{2}} q \frac{(a^2-1)^{\frac{3-n}{2}}}{R^2}.$$

The value of ρ here given being expressed in quantities perfectly independent of the situation of the axis from which the angle θ' is measured, is evidently applicable when the point P is not situated upon this axis, and in order to have the complete value of ρ, it will now only be requisite to add the term due to the arbitrary constant quantity on the left side of the equation (26), and as it is clear from what has preceded, that the term in question is of the form

$$\text{const.} \times (1-r'^2)^{\frac{n-3}{2}},$$

we shall therefore have generally, wherever P may be placed,

$$\rho = (1-r'^2)^{\frac{n-3}{2}} . \left\{ \text{const.} - \frac{\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2} q . \frac{(a^2-1)^{\frac{3-n}{2}}}{R^2} \right\}.$$

The transition from this particular case to the more general one, originally proposed is almost immediate: for if ρ represents the density of the inducing fluid on any element $d\sigma_1$ of the plane coinciding with that of the plate, $\rho_1 d\sigma_1$ will be the quantity of fluid contained in this element, and the density induced thereby will be had from the last formula, by changing q into $\rho_1 d\sigma_1$. If then we integrate the expression thus obtained, and extend the integral over all the fluid acting on the plate, we shall have for the required value of ρ

$$\rho = (1-r'^2)^{\frac{n-3}{2}} . \left\{ \text{const.} - \frac{\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2} \int \rho_1 d\sigma_1 \frac{(a^2-1)^{\frac{3-n}{2}}}{R^2} \right\};$$

R being the distance of the element $d\sigma_1$ from the point to which ρ belongs, and a the distance between $d\sigma_1$ and the center of the conducting plate.

Hitherto the radius of the circular plate has been taken as the unit of distance, but if we employ any other unit, and suppose that b is the measure of the same radius, in this case we shall only have to write $\frac{a}{b}$, $\frac{r'}{b}$, $\frac{d\sigma_1}{b^2}$ and $\frac{R}{b}$ in the place of a, r', $d\sigma_1$ and R respectively, recollecting that $\frac{\rho}{\rho_1}$ is a quantity of the dimension 0 with regard to space, by so doing the resulting value of ρ is

$$(28)\ldots\ldots\ldots\rho=(b^2-r'^2)^{\frac{n-3}{2}}.\left\{\text{const.}-\frac{\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2}\int\rho_1 d\sigma_1\frac{(a^2-b^2)^{\frac{3-n}{2}}}{R^2}\right\}.$$

By supposing $n=2$, the preceding investigation will be applicable to the electric fluid, and the value of the density induced upon an infinitely thin conducting plate by the action of a quantity of this fluid, distributed in any way at will in the plane of the plate itself will be immediately given. In fact, when $n=2$, the foregoing value of ρ becomes

$$\rho=\frac{1}{\sqrt{b^2-r'^2}}\left\{\text{const.}-\frac{1}{\pi^2}\int\rho_1 d\sigma_1\frac{\sqrt{a^2-b^2}}{R^2}\right\}.$$

If we suppose the plate free from all extraneous action, we shall simply have to make $\rho_1=0$ in the preceding formula; and thus

$$(29)\ldots\ldots\ldots\rho=\frac{\text{const.}}{\sqrt{b^2-r'^2}}.$$

Biot (*Traité de Physique*, Tom. II. p. 277.), has related the results of some experiments made by Coulomb on the distribution of the electric fluid when in equilibrium upon a plate of copper 10 inches in diameter, but of which the thickness is not specified. If we conceive this thickness to be very small compared with the diameter of the plate, which was undoubtedly the case, the formula just found ought to be applicable to it, provided we except those parts of the plate which are in the immediate vicinity of its exterior edge. As the comparison of any results mathematically deduced from the received theory of electricity with those of the experiments of so accurate an observer as Coulomb must always be interesting, we will here give a table of the values of the density at different points on the surface of the plate, calculated by means of the formula (29), together with the corresponding values found from experiment.

Distances from the Plate's edge.	Observed densities.	Calculated densities.
5 in.......	1,	1,
4	1,001	1,020
3	1,005	1,090
2	1,17	1,250
1	1,52	1,667
,5........	2,07	2,294
0	2,90	infinite.

We thus see that the differences between the calculated and observed densities are trifling; and moreover, that the observed are all something smaller than the calculated ones, which it is evident ought to be the case, since the latter have been determined by considering the thickness of the plate as infinitely small, and consequently they will be somewhat greater than when this thickness is a finite quantity, as it necessarily was in Coulomb's experiments.

It has already been remarked that the method given in the second article is applicable to any ellipsoid whatever, whose axes are a, b, c. In fact, if we suppose that x, y, z are the co-ordinates of a point p within it, and x', y', z' those of any element dv of its volume, and afterwards make

$$x = a.\cos\theta, \quad y = b.\sin\theta \cos\varpi, \quad z = c.\sin\theta \sin\varpi,$$

$$x' = a.\cos\theta', \quad y' = b.\sin\theta' \cos\varpi', \quad z' = c.\sin\theta' \sin\varpi',$$

we shall readily obtain by substitution,

$$V = abc \int \rho . r'^2 dr' d\theta' d\varpi' \sin\theta' . (\lambda r^2 - 2\mu r r' + \nu r'^2)^{\frac{1-n}{2}};$$

the limits of the integrals being the same as before (Art. 2.), and

$$\lambda = a^2 \cos\theta^2 + b^2 \sin\theta^2 \cos\varpi^2 + c^2 \sin\theta^2 \sin\varpi^2,$$

$$\mu = a^2 \cos\theta \cos\theta' + b^2 \sin\theta \sin\theta' \cos\varpi \cos\varpi' + c^2 \sin\theta \sin\theta' \sin\varpi \sin\varpi',$$

$$\nu = a^2 \cos\theta'^2 + b^2 \sin\theta'^2 \cos\varpi'^2 + c^2 \sin\theta'^2 \sin\varpi'^2.$$

Under the present form it is clear the determination of V can offer no difficulties after what has been shown (Art. 2.). I shall not therefore insist upon it here more particularly, as it is my intention in a future paper to give a general and purely analytical method of finding the value of V, whether p is situated within the ellipsoid or not. I shall therefore only observe, that for the particular value

$$(30)\ldots\ldots\ldots \rho = k\left(1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2} - \frac{z'^2}{c^2}\right)^{\frac{n-4}{2}} = k\,(1 - r'^2)^{\frac{n-4}{2}},$$

the series $U_0' + U_2' + U_4' + \&c.$ (Art. 2.) will reduce itself to the single term U_0', and we shall ultimately get

$$V = \frac{\pi k a b c}{2 \sin\left(\frac{n-2}{2}\pi\right)} \int_0^{\pi} d\theta' \sin\theta' \int_0^{2\pi} d\varpi' \left(a^2 \cos\theta'^2 + b^2 \sin\theta'^2 \cos\varpi'^2 + c^2 \sin\theta'^2 \sin\varpi'^2\right)^{\frac{1-n}{2}},$$

which is evidently a constant quantity. Hence it follows that the expression (30) gives the value of ρ when the fluid is in equilibrium within the ellipsoid, and free from all extraneous action. Moreover, this value is subject, when $n < 2$, to modifications similar to those of the analagous value for the sphere (Art. 7.).

G. GREEN.

II. *On Elimination between an Indefinite Number of Unknown Quantities.* *By the Rev.* R. MURPHY, M. A. *Fellow of Caius College, and of the Cambridge Philosophical Society.*

[Read *Nov.* 26, 1832.]

SECTION I.

INTRODUCTION.

FOURIER, in his treatise, '*Theorie de la Chaleur,*'* has given an example of the determination of an indefinite number of unknown quantities, subject to the same number of conditions. If n be the number of those quantities, in order to discover their law by this method, it will be necessary to eliminate successively the first $(m-1)$ and the last $(n-m)$ unknown quantities, thus determining the m^{th}, by a final equation containing that quantity only.

This process is obviously too laborious, and the results too complicated, to be practically useful, in most cases.

The same objection applies to the elegant method of Laplace, which makes the determination of one of the unknown quantities, depend on the discovery of all the $(n-1)$ arbitrary multipliers introduced in the process. It has besides the disadvantage of not seizing, in many cases, the facilities offered by the peculiar forms of the proposed equations.

* Vid. Fourier, p. 169 to 174.

In the physical investigations, which conduct to an indefinite number of equations, it is of great importance to discover the law of those quantities, corresponding to the law by which the given equations are connected. The method which I here propose for this object is founded on the two following principles.

First, if we make the right-hand member of the x^{th} equation disappear by transposition, the left-hand member is then a function of x, which vanishes when x is any number of the series $1, 2, 3, \ldots\ldots n$; and therefore it must be of the form

$$P.(x-1)(x-2)(x-3)\ldots\ldots(x-n).$$

Secondly, if an identity exist between two formulæ which are partly integer, partly proper algebraic fractions (of which the numerators are of lower dimensions than the denominators) the integer and fractional parts are separately equal.

To demonstrate this principle, let

$$N+\frac{P}{Q}=N'+\frac{P'}{Q'},$$

represent such an identity, where each symbol denotes an entire function of x, and the dimensions of P, P' are respectively lower than those of Q, Q'; then we have

$$(N-N')\,QQ'=P'Q-PQ'.$$

If therefore $N-N'$ be not identically nothing, we shall have the entire function, represented by the left-hand member, identical with one of lower dimensions; but this is impossible, because in integer formulæ we may equate like powers of x, hence we must have $N=N'$ and, therefore also,

$$\frac{P}{Q}=\frac{P'}{Q'}.$$

By means of this principle, we shall be able to expand a given entire function P, in terms of other given functions, whenever such an expansion is possible.

SECTION II.

Application of the First Principle.

THE first principle alone is sufficient, in a great number of instances, to resolve the proposed equations; we shall illustrate its application by selecting three distinct classes of equations to be resolved.

First, when the terms which compose the general or x^{th} equation are proper fractions.

EXAMPLE:

To find the values of the n unknown quantities z_1, z_2, z_3,......z_n subject to the n equations following,

$$\frac{z_1}{2}+\frac{z_2}{3}+\frac{z_3}{4}+\ldots\ldots\ldots\ldots\frac{z_n}{n+1}=-1,$$

$$\frac{z_1}{3}+\frac{z_2}{4}+\frac{z_3}{5}+\ldots\ldots\ldots\ldots\frac{z_n}{n+2}=-\frac{1}{2},$$

$$\frac{z_1}{4}+\frac{z_2}{5}+\frac{z_3}{6}+\ldots\ldots\ldots\ldots\frac{z_n}{n+3}=-\frac{1}{3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{z_1}{n+1}+\frac{z_2}{n+2}+\frac{z_3}{n+3}+\ldots\ldots\ldots\ldots+\frac{z_n}{2n}=-\frac{1}{n}.$$

The general, or x^{th} equation, when its right-hand member is transposed, becomes

$$\frac{1}{x}+\frac{z_1}{x+1}+\frac{z_2}{x+2}+\ldots\ldots\ldots\ldots+\frac{z_n}{x+n}=0.$$

Suppose these fractions are actually added, and let $\frac{N}{D}$ represent the sum; where $D=x\,(x+1)\,(x+2)\ldots\ldots(x+n)$ and N is some function of x of n dimensions.

Hence we have $\frac{N}{D}=0$, and therefore $N=0$, provided x is any number of the series 1, 2, 3,....n and consequently N (which is of n dimensions) has a factor $(x-1)\,(x-2)\ldots\ldots(x-n)$; and can therefore admit of no other factor, but a constant c.

Hence we have in general,

$$(a)\ldots\ldots\frac{1}{x}+\frac{z_1}{x+1}+\frac{z_2}{x+2}+\ldots\ldots\frac{z_n}{x+n}=\frac{c\,.\,(x-1)\,(x-2)\ldots(x-n)}{x\,(x+1)\,(x+2)\,(x+3)\ldots(x+n)}.$$

Multiply this equation by x, and then put $x=0$, hence $c=(-1)^n$.

Multiply the same by $x+1$, and then put $x=-1$;

$$\text{hence } z_1=-\frac{n}{1}\cdot\frac{n+1}{1}.$$

Similarly, multiply by $x+2$, and put $x=-2$,

$$\therefore\ z_2=\frac{n\,.\,(n-1)}{1\,.\,2}\cdot\frac{(n+1)\,(n+2)}{1\,.\,2},$$

and generally, if we multiply equation (a) by $x+m$, and then put $x=-m$, we get

$$z_m=(-1)^m\cdot\frac{n\,.\,(n-1)\,(n-2)\ldots\ldots(n-m+1)}{1\,.\,2\,.\,3\,\ldots\ldots\,m}\cdot\frac{(n+1)(n+2)\ldots\ldots(n+m)}{1\,.\,2\,\ldots\ldots\,m}.$$

It is clear from this example, that if the general or x^{th} equation were

$$\frac{1}{a+bx}+\frac{z_1}{a'+b'x}+\frac{z_2}{a''+b''x}+\ldots\ldots\frac{z_n}{a^{(n)}+b^{(n)}x}=0,$$

we should find the sum of the fractions composing the left-hand member to be

$$\frac{c\,.\,(x-1)\,(x-2)\quad\ldots\ldots\quad(x-n)}{(a+bx)\,(a'+b'x)\,(a''+b''x)\ldots\ldots(a^{(n)}+b^{(n)}x)},$$

then multiplying by $a+bx$ and putting $x=-\frac{a}{b}$, we should find c,

multiplying by $a'+b'x$ and putting $x=-\frac{a'}{b'}$, we should find z_1,

..........................&c......................&c....................&c......

In the example above taken, we have supposed that the number of equations and unknown quantities were the same, but if we supposed that following the same law as in that example, the number of equations were $n+m$, then the numerator N which was shown to be of n dimensions, ought to vanish when x is any number of the series $1, 2, 3 \ldots\ldots n+m$; that is, the equation $N=0$ has more roots than it has dimensions, which is impossible; it is therefore equally impossible to satisfy all the given equations.

On the other hand, if the number of the given equations was only $n-m$, then n would by the preceeding reasoning have a factor

$$(x-1)(x-2)\ldots\ldots(x-n+m),$$

and since it is of n dimensions, it must have another factor of m dimensions, as $c(x-a_1)(x-a_2)\ldots\ldots(x-a_m)$.

$$\text{Hence } \frac{1}{x}+\frac{z_1}{x+1}+\frac{z_2}{x+2}+\ldots\ldots\frac{z_n}{x+n}$$

$$=\frac{c(x-1)(x-2)\ldots\ldots(x-n+m)(x-a_1)(x-a_2)\ldots\ldots(x-a_m)}{x.(x+1)(x+2)\ldots\ldots(x+n)};$$

following now the same steps as before, we find

$$1=\frac{c(-1)^n.a_1a_2\ldots\ldots a_m}{n.(n-1)\ldots\ldots(n-m+1)}, \quad \therefore c=(-1)^n.\frac{n.(n-1)\ldots\ldots(n-m+1)}{a_1.a_2 \ \ldots\ldots \ a_m},$$

$$z_1=\frac{c(-1)^n.(1+a_1)(1+a_2)\ldots\ldots(1+a_m)}{(n-1)(n-2)\ldots\ldots(n-m+2)}=\frac{(1+a_1)(1+a_2)\ldots\ldots(1+a_m)}{a_1 \ . \ a_2 \ \ldots\ldots \ a_m}.\frac{n}{1}.\frac{n-m+1}{1}.$$

$$\text{Similarly, } z_2=-\frac{(2+a_1)(2+a_2)\ldots\ldots(2+a_m)}{a_1 \ . \ a_2 \ \ldots\ldots \ a_m}.\frac{n.(n+1)}{1.2}.\frac{(n-m+1)(n-m+2)}{1 \ . \ 2}.$$

The quantities a_1, $a_2 \ldots\ldots a_m$ are evidently arbitrary, and each of the required quantities z_1 z_2, &c. z_{n-m} are here determined in such a manner, as to contain the m arbitrary constants. This is therefore the most complete solution of the problem.

Another useful observation may be made in this place; if the function which represents the x^{th} equation were discontinuous, i. e. if any of the equations, for instance the second, were

$$\frac{z_1}{\frac{3}{2}} + \frac{z_2}{\frac{5}{2}} + \frac{z_3}{\frac{7}{2}} + \ldots\ldots = -2,$$

and consequently an exception to the general law expressed by the x^{th} equation, we should have then $N=0$ when $x = 1, 3, 4\ldots\ldots n$, also when $x = \frac{1}{2}$, but not when $x = 2$, hence in this case,

$$N = c\,.\,(x-\tfrac{1}{2})\,(x-1)\,(x-3)\,(x-4)\ldots\ldots(x-n);$$

after this the remainder of the process would be the same as before.

We have been thus particular about the preceding example, as being well calculated to shew the spirit and advantages of the present method.

The next class of equations, which may be solved by the first principle alone, consists of those in which the terms composing the x^{th} equation contain common factors; for if we then assign to x such values as may successively cause such factors to vanish, the unknown quantities will be determined.

EXAMPLE:

To find the values of z_1, z_2, $z_3\ldots\ldots z_n$ subject to the n equations following; viz.

$$\begin{gathered}
z_1 + 1\,.\,2\,.\,z_2 + 1\,.\,2\,.\,3\,.\,z_3 + \ldots\ldots + 1\,.\,2\,.\,3\ldots\ldots\ldots n\,z_n = -1,\\
2\,z_1 + 2\,.\,3\,.\,z_2 + 2\,.\,3\,.\,4\,.\,z_3 + \ldots\ldots + 2\,.\,3\,.\,4\ldots(n+1)\,z_n = -1,\\
3\,z_1 + 3\,.\,4\,.\,z_2 + 3\,.\,4\,.\,5\,.\,z_3 + \ldots\ldots + 3\,.\,4\,.\,5\ldots(n+2)\,z_n = -1,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
n z_1 + n(n+1) z_2 + n\,(n+1)(n+2)\,z_3 + \ldots\ldots + n\,(n+1)\,(n+2)\ldots 2n\,z_n = -1.
\end{gathered}$$

If we transpose the right-hand member of the above equations, the x^{th} or general equation becomes

$$1 + x z_1 + x\,(x+1)\,z_2 + x\,(x+1)\,(x+2)\,.\,z_3 + \ldots\ldots + x(x+1)(x+2)\ldots 2x\,.\,z_n = 0.$$

This equation is evidently of n dimensions with respect to x, and its roots by the first principle are 1, 2, 3......n; the left-hand member must therefore be identical with the product

$$c(x-1)(x-2)(x-3)......(x-n),$$

whatever value may be assigned to x.

Put therefore $x=0$. Hence $c=\dfrac{(-1)^n}{1.2.3......n}$,

$$x=-1..........z_1=-n,$$

$$x=-2..........z_2=\frac{n.(n-1)}{1^2.2^2},$$

$$x=-3..........z_3=-\frac{n(n-1)(n-2)}{1^2.2^2.3^2},$$

&c......................&c..................

and generally, $z_m=\dfrac{n(n-1)(n-2)......(n-m+1)}{(1.2\;.\;3\;......\;m)^2}.(-1)^m.$

We may verify this result by observing, that if we substitute this quantity for z_m in the general or x^{th} equation, then its left-hand member becomes

$$1-nx+\frac{n.(n-1)}{1.2}.\frac{x.(x+1)}{1.2}-\frac{n.(n-1)(n-2)}{1.2\;.\;3}.\frac{x.(x+1)(x+2)}{1.2\;.\;3}+\&c.$$

This quantity is evidently the part which does not contain h in the product,

$$\left\{1+xh+\frac{x(x+1)}{1.2}.h^2+\frac{x(x+1)(x+2)}{1.2\;.\;3}.h^3+\&c.\right\}\times\left\{1-\frac{n}{h}+\frac{n(n-1)}{1.2}.\frac{1}{h^2}-\&c.\right\},$$

or in
$$(1-h)^{-x}.\left(1-\frac{1}{h}\right)^n,$$

it is therefore the coefficient of h^n in the expansion of

$$(-1)^n(1-h)^{n-x}.$$

But this coefficient is manifestly 0 when x is any positive integer, which evidently agrees with the proposed conditions.

Another class of equations which may easily be resolved by the first principle, occurs when the x^{th} equation is of n dimensions, and arranged according to the powers of some function of x; it is then merely necessary to expand

$$c \,.\, x(x-1)(x-2)\,\ldots\ldots\ldots\,(x-n)$$

according to the powers of that function; and equate the coefficients of like powers in both cases.

EXAMPLE:

$$\begin{aligned}
z_1 + z_2 + z_3 + \ldots\ldots\ldots + z_n &= -1,\\
2z_1 + 2^2 z_2 + 2^3 z_3 + \ldots\ldots\ldots + 2^n z_n &= -1,\\
3z_1 + 3^2 z_2 + 3^3 z_3 + \ldots\ldots\ldots + 3^n z_n &= -1,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
n z_1 + n^2 z_2 + n^3 z_3 + \ldots\ldots\ldots + n^n z_n &= -1,
\end{aligned}$$

to find $z_1, z_2 \ldots\ldots z_n$.

The general or x^{th} equation in this case, is

$$1 + x z_1 + x^2 z_2 + \ldots\ldots + x^n z_n = 0,$$

the roots of which equation are $x = 1, 2, 3 \ldots\ldots n$.

Hence, the left-hand member is identical with the product

$$c \,.\, (x-1)(x-2)(x-3) \ldots\ldots\ldots (x-n),$$

$$\text{or} \quad c(-1)^n \{S_n - x S_{n-1} + x^2 S_{n-2} - \ldots\ldots (-1)^n \,.\, x^n\},$$

where S_m denotes the sum of the quantities $1, 2, 3 \ldots\ldots n$ when taken in products m and m together.

Hence, by equating, we get

$$\begin{aligned}
c(-1)^n \,.\, S_n &= 1; \qquad \therefore\ c = \frac{(-1)^n}{1\,.\,2\,.\,3 \ldots\ldots n};\\
-\,c(-1)^n \,.\, S_{n-1} &= z_1; \qquad \therefore\ z_1 = -S_{-1};\\
c(-1)^n \,.\, S_{n-2} &= z_2; \qquad \therefore\ z_2 = -S_{-2};
\end{aligned}$$

and generally $\qquad z_m = S_m,$

where S_{-m} denotes the sum of the reciprocals of the quantities of which S_m represents the sum.

SECTION III.

Application of the Second Principle.

To expand a given function of x as P, in terms of other given functions

$$Q_0,\ Q_1,\ Q_2 \ldots\ldots Q_n,$$

all being supposed of n dimensions in x.

$$\text{Let } P = a_0 Q_0 + a_1 Q_1 + a_2 Q_2 + \ldots\ldots + a_n Q_n,$$

where a_0, a_1, $a_2 \ldots\ldots a_n$ are constants to be determined.

Divide all the functions by Q_0, and let the corresponding quotients be respectively

$$P',\ Q'_0,\ Q'_1,\ Q'_2 \ldots\ldots Q'_n,$$

and the remainders

$$p',\ q'_0,\ q'_1,\ q'_2 \ldots\ldots q'_n.$$

Then by attending to the second principle, we have

$$P' = a_0 Q'_0 + a_1 Q'_1 + a_2 Q'_2 + \ldots\ldots + a_n Q'_n,$$
$$p' = a_0 q'_0 + a_1 q'_1 + a_2 q'_2 + \ldots\ldots + a_n q'_n,$$

when we obviously have $Q'_0 = 1$ and $q'_0 = 0$.

Dividing the *last* equation by q'_1 and using a similar notation, we get in like manner

$$P'' = a_1 Q''_1 + a_2 Q''_2 + \ldots\ldots a_n Q''_n,$$
$$p'' = a_1 q''_1 + a_2 q''_2 + \ldots\ldots a_n q''_n,$$

where $Q''_1 = 1$ and $q''_1 = 0$.

Divide the equation last obtained by q''_2, and we obtain

$$P''' = a_2 Q'''_2 + \ldots\ldots + a_n Q'''_n,$$
$$p''' = a_2 q'''_2 + \ldots\ldots + a_n q'''_n,$$

in the latter of which equations the first term $= 0$ and in the former it equals unity.

The systems of the first equations thus obtained may be written in an inverse order thus,

$$\begin{aligned}
P''^{(n+1)} &= a_n Q_n''^{(n+1)} = a_n,\\
P''^{(n)} &= a_{n-1} + a_n Q_n''^{(n)},\\
P''^{(n-1)} &= a_{n-2} + a_{n-1} Q''^{(n-1)}_{n-1} + a_n Q_n''^{(n-1)},\\
\&c. &= \dots\dots\dots\dots \&c. \dots\dots\dots\dots\dots\dots
\end{aligned}$$

whence a_n, a_{n-1}, a_{n-2}, &c. are successively known.

We have supposed all the functions to be of n dimensions, for these necessarily comprise all of lower degrees.

EXAMPLE:

To expand *unity* in terms of the functions

$$x^n,\ (x+h)^n,\ (x+2h)^n, \dots\dots (x+nh)^n.$$

Put $1 = a_0 x^n + a_1 (x+h)^n + a_2 (x+2h)^n + \dots\dots + a_n (x+nh)^n$; dividing by x^n, we get

$$\begin{aligned}
0 &= a_0 + a_1 + a_2 + \dots\dots + a_n,\\
1 &= \quad a_1 q'_1 + a_2 q'_2 + \dots\dots + a_n q'_n,
\end{aligned}$$

where we have $q'_m = h \,.\, \{n x^{n-1} . m + \dfrac{n(n-1)}{1.2} h m^2 x^{n-2} + \dots\dots\}$.

Divide now by q'_1 and we obtain

$$\begin{aligned}
0 &= a_1 + 2a_2 + \dots\dots + n a_n,\\
1 &= \quad a_2 q''_2 + \dots\dots + a_n q''_n,
\end{aligned}$$

where in general $q''_m = h^2 \left\{\dfrac{n(n-1)}{1.2} x^{n-2} (m^2 - m) + \&c.\right\}$.

This process is easily continued, and we obtain successively the equations

$$\begin{aligned}
0 &= 1.2 a_2 + 2.3 a_3 + \dots\dots (n-1).n a_n,\\
0 &= \quad 1.2.3 a_3 + \dots\dots (n-2)(n-1) n a_n,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

and lastly, $\dfrac{1}{h^n} = \quad 1.2.3 \dots\dots\dots n a_n.$

From these equations taken in the inverse order, we get

$$a_n = \frac{1}{1 \,.\, 2 \,.\, 3 \ldots\ldots n h^n},$$

$$a_{n-1} = -\, n\, a_n,$$

$$a_{n-2} = \frac{n\,(n-1)}{1 \,.\, 2} \,.\, a_n,$$

$$\&c. = \ldots\ldots\ldots \&c. \ldots\ldots$$

Hence the required expansion is

$$1 = \frac{(-1)^n}{1 \,.\, 2 \,.\, 3 \ldots\ldots n h^n} \,.\, \{x^n - n\,(x+h)^n + \frac{n\,(n-1)}{1 \,.\, 2}\,(x+2h)^n - \&c.\}\,.$$

To apply this principle to equations, we may observe that when the general or x^{th} equation is cleared of fractions and its right-hand member transposed, it is of the form

$$-P + z_1 X_1 + z_2 X_2 + \ldots\ldots + z_n X_n = 0,$$

where z_1, $z_2 \ldots\ldots z_n$ are the unknown quantities, and P, X_1, $X_2 \ldots.$ known functions of x.

The left-hand member must, by the reasoning of the preceding Section, be divisible by $(x-1)\,(x-2)\ldots\ldots(x-n)$.

Let X_1, X_2, &c. when divided by this quantity leave the remainders Q'_1, Q'_2, &c. and P, the remainder P', hence

$$P' = z_1 Q'_1 + z_2 Q'_2 + \ldots\ldots + z_n Q_n,$$

where all the functions are necessarily of less than n dimensions, the application of the process above described, would then determine the quantities z_1, $z_2, \ldots\ldots z_n$.

R. MURPHY.

CAIUS COLLEGE,
March 5, 1833.

III. *On the General Equation of Surfaces of the Second Degree.*

By Augustus De Morgan, *of Trinity College.*

[Read *Nov.* 12, 1832.]

The present investigations are a continuation of those upon lines of the second degree, published in Vol. IV. Part I. of these Transactions. I have omitted various algebraical developments, as unnecessary, and tending to swell this communication to a length more than proportional to its importance.

As the theory of the reduction of oblique to rectangular co-ordinates is a very necessary part of what follows, I proceed first to give the equations which will be required under this head. Let x, y, z, be oblique, and x', y', z' rectangular co-ordinates to the same point, with a common origin. Let the angles made by the first system be

$$\widehat{yz} = \xi, \quad \widehat{zx} = \eta, \quad \widehat{xy} = \zeta,$$

and let the rectangular and oblique co-ordinates be so related that

$$\cos \widehat{xx'} = \alpha, \quad \cos \widehat{yx'} = \beta, \quad \cos \widehat{xy'} = \alpha', \text{ \&c.};$$

whence the following equations:

$$\begin{aligned} x' &= \alpha x + \beta y + \gamma z, \\ y' &= \alpha' x + \beta' y + \gamma' z \;\ldots\ldots\ldots\ldots (1), \\ z' &= \alpha'' x + \beta'' y + \gamma'' z; \end{aligned}$$

$$\begin{aligned} 1 &= \alpha^2 + \alpha'^2 + \alpha''^2, & \cos\xi &= \beta\gamma + \beta'\gamma' + \beta''\gamma'', \\ 1 &= \beta^2 + \beta'^2 + \beta''^2, & \cos\eta &= \gamma\alpha + \gamma'\alpha' + \gamma''\alpha'' \ldots\ldots (2), \\ 1 &= \gamma^2 + \gamma'^2 + \gamma''^2, & \cos\zeta &= \alpha\beta + \alpha'\beta' + \alpha''\beta''. \end{aligned}$$

Make the following abbreviations, to which, for facility of reference, are annexed those which will afterwards appear in treating the general equation of the surface,

$$ax^2+by^2+cz^2+2\bar{a}yz+2\bar{b}zx+2\bar{c}xy+2\bar{\bar{a}}x+2\bar{\bar{b}}y+2\bar{\bar{c}}z+f=0 \text{......(3)},$$

the co-ordinates of the center of which call X, Y, and Z. Throughout this paper, all *subscript* indices indicate the dimension of the quantity signified, in terms of the coefficients of (3):

$$\begin{array}{lll} p\ =\beta'\gamma''-\beta''\gamma', & q\ =\gamma'\alpha''-\gamma''\alpha', & r\ =\alpha'\beta''-\alpha''\beta', \\ p'\ =\beta''\gamma-\beta\gamma'', & q'\ =\gamma''\alpha-\gamma\alpha'', & r'\ =\alpha''\beta-\alpha\beta'' \text{.......(4)}, \\ p''=\beta\gamma'-\beta'\gamma, & q''=\gamma\alpha'-\gamma'\alpha, & r''=\alpha\beta'-\alpha'\beta. \end{array}$$

$$\begin{array}{lll} a_{\prime\prime}=bc-\bar{a}^2, & a_{\prime}=b+c-2\bar{a}\cos\xi, & a_0=\sin^2\xi, \\ b_{\prime\prime}=ca-\bar{b}^2, & b_{\prime}=c+a-2\bar{b}\cos\eta, & b_0=\sin^2\eta \text{........(5)}, \\ c_{\prime\prime}=ab-\bar{c}^2, & c_{\prime}=a+b-2\bar{c}\cos\zeta, & c_0=\sin^2\zeta. \end{array}$$

$$\begin{array}{lll} l_{\prime\prime}\ =\bar{b}\bar{c}-a\bar{a}, & l_{\prime}\ =\bar{b}\cos\zeta+\bar{c}\cos\eta-\bar{a}-a\cos\xi, & l_0\ =\cos\eta\cos\zeta-\cos\xi, \\ m_{\prime\prime}=\bar{c}\bar{a}-b\bar{b}, & m_{\prime}=\bar{c}\cos\xi+\bar{a}\cos\zeta-\bar{b}-b\cos\eta, & m_0=\cos\zeta\cos\xi-\cos\eta \text{...(6)}, \\ n_{\prime\prime}\ =\bar{a}\bar{b}-c\bar{c}, & n_{\prime}\ =\bar{a}\cos\eta+\bar{b}\cos\xi-\bar{c}-c\cos\zeta, & n_0\ =\cos\xi\cos\eta-\cos\zeta. \end{array}$$

$$V_0=1+2\cos\xi\cos\eta\cos\zeta-\cos^2\xi-\cos^2\eta-\cos^2\zeta \begin{cases} =b_0c_0-l_0^2 \\ =c_0a_0-m_0^2 \\ =a_0b_0-n_0^2 \end{cases} \begin{cases} =m_0n_0-a_0\ l_0\div\cos\xi \\ =n_0\ l_0-b_0m_0\div\cos\eta \\ =l_0m_0-c_0\ n_0\div\cos\zeta \end{cases} \text{...(7)},$$

$$V_1=aa_0+bb_0+cc_0+2\bar{a}l_0+2\bar{b}m_0+2\bar{c}n_0 \text{ ....(8)},$$

$$V_2=a_{\prime\prime}+b_{\prime\prime}+c_{\prime\prime}+2l_{\prime\prime}\cos\xi+2m_{\prime\prime}\cos\eta+2n_{\prime\prime}\cos\zeta \text{ ....(9)},$$

$$V_3=abc+2\bar{a}\bar{b}\bar{c}-a\bar{a}^2-b\bar{b}^2-c\bar{c}^2 \begin{cases} =b_{\prime\prime}c_{\prime\prime}-l_{\prime\prime}^2\ \div a \\ =c_{\prime\prime}a_{\prime\prime}-m_{\prime\prime}^2\div b \\ =a_{\prime\prime}b_{\prime\prime}-n_{\prime\prime}^2\ \div c \end{cases} \begin{cases} =m_{\prime\prime}n_{\prime\prime}-a_{\prime\prime}l_{\prime\prime}\ \div\bar{a} \\ =n_{\prime\prime}l_{\prime\prime}\ -b_{\prime\prime}m_{\prime\prime}\div\bar{b} \\ =l_{\prime\prime}m_{\prime\prime}\ -c_{\prime\prime}n_{\prime\prime}\ \div\bar{c} \end{cases} \text{...(10)},$$

$$V_4=a_{\prime\prime}\bar{\bar{a}}^2+b_{\prime\prime}\bar{\bar{b}}^2+c_{\prime\prime}\bar{\bar{c}}^2+2l_{\prime\prime}\bar{\bar{b}}\bar{\bar{c}}+2m_{\prime\prime}\bar{\bar{c}}\bar{\bar{a}}+2n_{\prime\prime}\bar{\bar{a}}\bar{\bar{b}} \text{ ....(11)},$$

$$W=-\frac{V_4}{V_3}+f=\bar{\bar{a}}X+\bar{\bar{b}}Y+\bar{\bar{c}}Z+f \text{ ....(12)}.$$

From (4), we find by inspection that the following six quantities are severally equal:

$$\begin{array}{ll} p\alpha + q\beta + r\gamma, & p\alpha + p'\alpha' + p''\alpha'', \\ p'\alpha' + q'\beta' + r'\gamma', & q\beta + q'\beta' + q''\beta'' \ldots\ldots\ldots\ldots(13), \\ p''\alpha'' + q''\beta'' + r''\gamma'', & r\gamma + r'\gamma' + r''\gamma'', \end{array}$$

and moreover, that any symmetrical interchanges of accents in the first three, or of letters in the second, give results severally equal to nothing. Such are $p\alpha'+q\beta'+r\gamma'$, $p\beta+p'\beta'+p''\beta''$, &c. Let the common value of the first six be T. We have then

$$\begin{aligned} p\alpha + q\beta + r\gamma &= T, \\ p\alpha' + q\beta' + r\gamma' &= 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14), \\ p\alpha'' + q\beta'' + r\gamma'' &= 0. \end{aligned}$$

From which, by obvious multiplications and additions, looking at equations (2), we have

$$\begin{aligned} p + q\cos\zeta + r\cos\eta &= T\alpha, \\ p\cos\zeta + q + r\cos\xi &= T\beta \ldots\ldots\ldots\ldots\ldots (15), \\ p\cos\eta + q\cos\xi + r &= T\gamma. \end{aligned}$$

From either of which sets we deduce

$$p^2+q^2+r^2+2qr\cos\xi+2rp\cos\eta+2pq\cos\zeta = T^2 \ldots\ldots\ldots(16),$$

and similar relations may be deduced between p', q', r', and p'', q'', r''; T being the same throughout.

Again, form the several quantities

$$a_0,\ \ l_0,\ \ \text{\&c.}\ \ \text{or}\ \ 1-\cos^2\xi,\ \ \cos\eta\cos\zeta-\cos\xi,\ \ \text{\&c.}$$

from the second set of equations in (2), and make the results homogeneous and symmetrical from the first set; for example, write for a_0 and l_0

$$[\beta^2+\beta'^2+\beta''^2][\gamma^2+\gamma'^2+\gamma''^2]-(\beta\gamma+\beta'\gamma'+\beta''\gamma'')^2$$

$$(\gamma\alpha+\gamma'\alpha'+\gamma''\alpha'')(\alpha\beta+\alpha'\beta'+\alpha''\beta'')-[\alpha^2+\alpha'^2+\alpha''^2](\beta\gamma+\beta'\gamma'+\beta''\gamma''),$$

in which the factors equal to unity, and introduced for symmetry, have the brackets [ ]. Develope these expressions, from which we obtain the following equations:

$$a_0=p^2+p'^2+p''^2, \qquad l_0=qr+q'r'+q''r'',$$
$$b_0=q^2+q'^2+q''^2, \qquad m_0=rp+r'p'+r''p'' \dots\dots\dots (17),$$
$$c_0=r^2+r'^2+r''^2, \qquad n_0=pq+p'q'+p''q''.$$

These, added together, the three last having been respectively multiplied by $2\cos\xi$, $2\cos\eta$, $2\cos\zeta$, give from (16)

$$a_0+b_0+c_0+2l_0\cos\xi+2m_0\cos\eta+2n_0\cos\zeta=3T^2.$$

The first side of which, developed from (5) and (6) gives $3V_0$, whence

$$T=\sqrt{V_0} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (18).$$

If the process by which (17) was obtained from (2) be repeated upon (17), that is, if $a_0b_0-l_0^2$, $m_0n_0-a_0l_0$, &c. be formed, we shall have equations of a similar form, substituting instead of p, p' &c. such functions of them, as they themselves are of α, β, &c., the first sides of the equations being from (7), V_0 in the first three, and $V_0\cos\xi$, $V_0\cos\eta$, $V_0\cos\zeta$, in the last three. These equations are such as would arise from substituting in (2),

$$\frac{q'r''-q''r'}{\sqrt{V_0}} \text{ instead of } \alpha \quad \frac{q''r-qr''}{\sqrt{V_0}} \text{ and } \frac{qr'-rq'}{\sqrt{V_0}} \text{ for } \alpha' \text{ and } \alpha'', \text{ \&c.} \dots (19),$$

which are therefore the values of α, α', &c. in terms of p, q, &c.

From (1), by means of (14) and (18), can be deduced the following:

$$\sqrt{V_0}\,x=px'+p'y'+p''z',$$
$$\sqrt{V_0}\,y=qx'+q'y'+q''z' \dots\dots\dots\dots\dots\dots\dots\dots\dots (20),$$
$$\sqrt{V_0}\,z=rx'+r'y'+r''z',$$

and the equations of the axis of x', referred to the oblique axes x, y, and z, are any two of the three,

$$qx-py=0, \qquad ry-qz=0, \qquad pz-rx=0 \dots\dots\dots\dots\dots (21).$$

The equations of the center, central line, or central plane, as the case may be, of the surface expressed by (3) are

$$\begin{aligned} &aX+\bar{c}Y+\bar{b}Z+\bar{\bar{a}}=0, \\ &\bar{c}X+bY+\bar{a}Z+\bar{\bar{b}}=0 \quad \text{.................................(22)}, \\ &\bar{b}X+\bar{a}Y+cZ+\bar{\bar{c}}=0, \end{aligned}$$

and in the two following sets of quantities, it will be found that the sum of the products made by taking a term from each in the *same* horizontal line is $=V_3$; while if the terms be taken from *different* horizontal lines, it will be $=0$.

$$\begin{matrix} a & \bar{c} & \bar{b}, \\ \bar{c} & b & \bar{a}, \\ \bar{b} & \bar{a} & c, \end{matrix} \qquad\qquad \begin{matrix} a_{\prime\prime} & n_{\prime\prime} & m_{\prime\prime}, \\ n_{\prime\prime} & b_{\prime\prime} & l_{\prime\prime}, \\ m_{\prime\prime} & l_{\prime\prime} & c_{\prime\prime}. \end{matrix}$$

Thus

$$aa_{\prime\prime}+\bar{c}n_{\prime\prime}+\bar{b}m_{\prime\prime}=V_3, \qquad an_{\prime\prime}+\bar{c}b_{\prime\prime}+\bar{b}l_{\prime\prime}=0, \text{ \&c.}$$

Hence, if the three equations in (22) be independent of one another, the co-ordinates of the center are

$$X=-\frac{a_{\prime\prime}\bar{\bar{a}}+n_{\prime\prime}\bar{\bar{b}}+m_{\prime\prime}\bar{\bar{c}}}{V_3}, \quad Y=-\frac{n_{\prime\prime}\bar{\bar{a}}+b_{\prime\prime}\bar{\bar{b}}+l_{\prime\prime}\bar{\bar{c}}}{V_3}, \quad Z=-\frac{m_{\prime\prime}\bar{\bar{a}}+l_{\prime\prime}\bar{\bar{b}}+c_{\prime\prime}\bar{\bar{c}}}{V_3} \quad (23)$$

The equation of the surface, referred to this center, and to axes parallel to the primitive axes, becomes, calling $\phi(x, y, z)$ the first side of equation (3),

$$ax^2+by^2+cz^2+2\bar{a}yz+2\bar{b}zx+2\bar{c}xy+\phi(X, Y, Z)=0 \quad \text{..................(24)},$$

and by multiplying the three equations in (22) by X, Y, and Z respectively, and adding, we get

$$\phi(X, Y, Z)=\bar{\bar{a}}X+\bar{\bar{b}}Y+\bar{\bar{c}}Z+f=W \quad \text{....................(25)}.$$

When only two of the equations (22) are independent, there is a central line. The conditions of this case are, that the numerators and denominators in (23) must be severally equal to nothing; but if $V_3=0$, the equations in (10) shew that it is sufficient that one of the numerators should be equal to nothing; or that the conditions may be stated thus,

$$V_3=0, \qquad \sqrt{a_{\prime\prime}}\,\overline{\overline{a}}+\sqrt{b_{\prime\prime}}\,\overline{\overline{b}}+\sqrt{c_{\prime\prime}}\,\overline{\overline{c}}=0\text{...........}(26).$$

When $V_3=0$, V_4 is a perfect square, (10) and (11), its root being the second expression in (26). Hence W appears in the form $\frac{0}{0}$. From two of equations (22), substitute in (25) values of any two co-ordinates of the center in terms of the third; it will be found that the coefficient of the third disappears under the conditions in (26), and that the resulting value of W, which we denote by W', may be expressed in either of the following ways:

$$W'=-\frac{b\overline{\overline{a}}^2-2\bar{c}\,\overline{\overline{a}}\,\overline{\overline{b}}+a\overline{\overline{b}}^2}{ab-\bar{c}^2}+f=-\frac{c\overline{\overline{b}}^2-2\bar{a}\,\overline{\overline{c}}\,\overline{\overline{b}}+b\overline{\overline{c}}^2}{bc-\bar{a}^2}+f$$

$$=-\frac{a\overline{\overline{c}}^2-2\bar{b}\,\overline{\overline{a}}\,\overline{\overline{c}}+c\overline{\overline{a}}^2}{ac-\bar{b}^2}+f\text{..................}(27).$$

When no two of the equations (22) are independent, there is a central plane. The conditions of this case are, as appears from the equations, that $a_{\prime\prime}$, $b_{\prime\prime}$, $c_{\prime\prime}$, $l_{\prime\prime}$, $m_{\prime\prime}$, $n_{\prime\prime}$, must be severally $=0$; of which however it is sufficient that any three should exist. We have moreover

$$\overline{\overline{a}}:\overline{\overline{b}}:\overline{\overline{c}}::a:\bar{c}:\bar{b}\text{....................}(28).$$

From all which it appears that W' is now in the form $\frac{0}{0}$. From one of the equations (22) substitute in (25) the value of one of the co-ordinates in terms of the other two; the coefficients of the last two will disappear, as before, and the different forms of the value of W', which we call W'', will be

$$W''=-\frac{\overline{\overline{a}}^2}{a}+f=-\frac{\overline{\overline{b}}^2}{b}+f=-\frac{\overline{\overline{c}}^2}{c}+f\text{..............}(29).$$

By substituting W' or W'', when necessary, for W or $\phi(X, Y, Z)$ in (24) the equation of the surface will be obtained, referred to *any* point in its central line or plane.

Let the equation of the surface, referred to the principal axes, be

$$Ax'^2 + A'y'^2 + A''z'^2 + W = 0 \text{.............................} (30),$$

which must be identical with (24) when the values of x', y', z', found in (1) are substituted. We must then have

$$\left.\begin{aligned}
a &= A\alpha^2 + A'\alpha'^2 + A''\alpha''^2, \\
b &= A\beta^2 + A'\beta'^2 + A''\beta''^2, \\
c &= A\gamma^2 + A'\gamma'^2 + A''\gamma''^2, \\
\bar{a} &= A\beta\gamma + A'\beta'\gamma' + A''\beta''\gamma'', \\
\bar{b} &= A\gamma\alpha + A'\gamma'\alpha' + A''\gamma''\alpha'', \\
\bar{c} &= A\alpha\beta + A'\alpha'\beta' + A''\alpha''\beta'',
\end{aligned}\right\} \text{..........................} (31),$$

which equations are reduced to those in (2) by substituting unity for A, A', A'', a, b, and c; and $\cos\xi$, $\cos\eta$, and $\cos\zeta$ for $\bar{a}$, $\bar{b}$, and $\bar{c}$. Thus, whatever equation is deduced from these, we immediately find another, containing α, β, &c. in the same way, by the last mentioned substitution. Multiplying the first of these by p, the last by q, and the last but one by r; and adding, we obtain by the use of (14),

$$\left.\begin{aligned}
pa + q\bar{c} + r\bar{b} &= A\alpha\sqrt{V_0} \\
p + q\cos\zeta + r\cos\eta &= \alpha\sqrt{V_0}
\end{aligned}\right\} \text{........................} (32),$$

from which, and similar processes, we obtain

$$\begin{aligned}
p(A-a) + q(A\cos\zeta - \bar{c}) + r(A\cos\eta - \bar{b}) &= 0, \\
p(A\cos\zeta - \bar{c}) + q(A-b) + r(A\cos\xi - \bar{a}) &= 0 \text{......} (33), \\
p(A\cos\eta - \bar{b}) + q(A\cos\xi - \bar{a}) + r(A-c) &= 0;
\end{aligned}$$

which agree in form with (22), if $\bar{\bar{a}}$, $\bar{\bar{b}}$, and $\bar{\bar{c}}$ be struck out, and $A-a$ substituted for a, $A\cos\xi - \bar{a}$ for $\bar{a}$, &c. But $V_3 = 0$ is the result of (22), with the last terms erased; that is, if in V_3 the substitutions just mentioned be made for a, $\bar{a}$, &c. the result developed and equated to zero will give the equation for determining A, A', and A''. That equation is

$$V_0A^3 - V_1A^2 + V_2A - V_3 = 0 \text{....................} (34).$$

We also find from (33), for substitution in (21),

$$\frac{1}{p}:\frac{1}{q}:\frac{1}{r}::l_{\prime\prime}-l_{\prime}A+l_0A^2:m_{\prime\prime}-m_{\prime}A+m_0A^2:n_{\prime\prime}-n_{\prime}A+n_0A^2\ldots\ldots(35).$$

The equation (34) must have all its roots possible. For from (31) it appears that A' and A'' cannot be of the forms $\lambda+\mu\sqrt{-1}$ and $\lambda-\mu\sqrt{-1}$, unless α' and α'', β' and β'', γ' and γ'' are of the same form; from which, since

$$(\kappa+\lambda\sqrt{-1})(\theta-\phi\sqrt{-1})-(\kappa-\lambda\sqrt{-1})(\theta+\phi\sqrt{-1})$$

is of the form $\kappa\sqrt{-1}$, it will follow that p, q, and r (4) must be of this form: which is inconsistent with (32), if we suppose V_0 positive; since it may be seen from (31), and will presently appear otherwise, that a is possible when A is possible.

We might find equations of the third degree to determine p, q, &c. but it will be more convenient to express them in terms of A, &c., supposed to be found from (34). To do this, let $a_{\prime\prime}$, $a_{\prime}$, $l_{\prime\prime}$, $l_{\prime}$, &c. (5) and (6), be found in terms of A, a, &c. by substituting the values of a, b, $\bar{a}$, $\bar{b}$, &c. from (31). The results, after reduction, are

$$\begin{aligned}
a_{\prime\prime}&=A'A''p^2+A''Ap'^2+AA'p''^2, & a_{\prime}&=(A'+A'')p^2+(A''+A)p'^2+(A+A')p''^2,\\
b_{\prime\prime}&=A'A''q^2+A''Aq'^2+AA'q''^2, & b_{\prime}&=(A'+A'')q^2+(A''+A)q'^2+(A+A')q''^2,\\
c_{\prime\prime}&=A'A''r^2+A''Ar'^2+AA'r''^2, & c_{\prime}&=(A'+A'')r^2+(A''+A)r'^2+(A+A')r''^2,\\
&&&\qquad\qquad\qquad\qquad\ldots\ldots(36),\\
l_{\prime\prime}&=A'A''qr+A''Aq'r'+AA'q''r'', & l_{\prime}&=(A'+A'')qr+(A''+A)q'r'+(A+A')q''r'',\\
m_{\prime\prime}&=A'A''rp+A''Ar'p'+AA'r''p'', & m_{\prime}&=(A'+A'')rp+(A''+A)r'p'+(A+A')r''p'',\\
n_{\prime\prime}&=A'A''pq+A''Ap'q'+AA'p''q'', & n_{\prime}&=(A'+A'')pq+(A''+A)p'q'+(A+A')p''q'',
\end{aligned}$$

which equations, with those marked (17), give the following values of p^2, qr, &c.

$$\begin{aligned}
p^2&=\frac{a_{\prime\prime}-a_{\prime}A+a_0A^2}{(A-A')(A-A'')}, & qr&=\frac{l_{\prime\prime}-l_{\prime}A+l_0A^2}{(A-A')(A-A'')},\\
q^2&=\frac{b_{\prime\prime}-b_{\prime}A+b_0A^2}{(A-A')(A-A'')}, & rp&=\frac{m_{\prime\prime}-m_{\prime}A+m_0A^2}{(A-A')(A-A'')}\ldots\ldots\ldots(37).\\
r^2&=\frac{c_{\prime\prime}-c_{\prime}A+c_0A^2}{(A-A')(A-A'')}, & pq&=\frac{n_{\prime\prime}-n_{\prime}A+n_0A^2}{(A-A')(A-A'')}.
\end{aligned}$$

In which equations the letter p, q, A, &c. may be accented throughout singly or doubly, striking off three accents from any A which thus obtains three or more.

By squaring the equations (15), writing V_0 for T'^2, substituting the values just obtained for p^2, qr, &c. and then multiplying the same equations together two and two, and making

$$L_2 = b_{\prime\prime}c_0 + b_0c_{\prime\prime} - 2l_{\prime\prime}l_0, \qquad \overline{L}_2 = m_{\prime\prime}n_0 + m_0n_{\prime\prime} - a_0l_{\prime\prime} - a_{\prime\prime}l_0,$$

$$M_2 = c_{\prime\prime}a_0 + c_0a_{\prime\prime} - 2m_{\prime\prime}m_0, \qquad \overline{M}_2 = n_{\prime\prime}l_0 + n_0l_{\prime\prime} - b_0m_{\prime\prime} - b_{\prime\prime}m_0,$$

$$N_2 = a_{\prime\prime}b_0 + a_0b_{\prime\prime} - 2n_{\prime\prime}n_0, \qquad \overline{N}_2 = l_{\prime\prime}m_0 + l_0m_{\prime\prime} - c_0n_{\prime\prime} - c_{\prime\prime}n_0,$$

we get

$$\alpha^2 = \frac{V_2 - L_2 - (V_1 - aV_0)A + V_0A^2}{V_0(A - A')(A - A'')}, \quad \beta^2 = \frac{V_2 - M_2 - (V_1 - bV_0)A + V_0A^2}{V_0(A - A')(A - A'')} \cdots\cdots$$

$$\cdots\cdots\gamma^2 = \frac{V_2 - N_2 - (V_1 - cV_0)A + V_0A^2}{V_0(A - A')(A - A'')} \cdots\cdots\cdots (38),$$

$$\beta\gamma = \frac{V_2 \cos\xi - \overline{L}_2 - (V_1 \cos\xi - \bar{a}V_0)A + V_0 \cos\xi A^2}{V_0(A - A')(A - A'')},$$

$$\gamma\alpha = \frac{V_2 \cos\eta - \overline{M}_2 - (V_1 \cos\eta - \bar{b}V_0)A + V_0 \cos\eta A^2}{V_0(A - A')(A - A'')},$$

$$\alpha\beta = \frac{V_2 \cos\zeta - \overline{N}_2 - (V_1 \cos\zeta - \bar{c}V_0)A + V_0 \cos\zeta A^2}{V_0(A - A')(A - A'')};$$

in which the letters may be singly or doubly accented as before, and from which the determination of the position of the principal diameters is made to depend directly upon the solution of (34).

Let the surface whose equation is (3) be referred to another origin and other axes, and let the quantities corresponding to those already given or deduced, which belong to the new origin or axes, be denoted by the same letters and accents enclosed in brackets [ ]. Thus the angles made by the new axes are $[\xi]$, $[\eta]$, and $[\zeta]$; the coefficients of the new equation are $[a]$, $[\bar{a}]$, &c.; the functions of these coefficients already noticed are $[a_{\prime\prime}]$, $[l_{\prime\prime}]$, $[V_1]$, &c. Since the principal diameters of the surface are the same, from whatever equation they are derived, that is, since $-\dfrac{W}{A} = -\dfrac{[W]}{[A]}$, &c. the roots of (34) bear to those of [34] the proportion of W to $[W]$; whence, λ being an indeterminate quantity, since one coefficient in (3) is indeterminate,

$$[W] = \lambda W, \qquad \left[\frac{V_1}{V_0}\right] = \lambda \frac{V_1}{V_0},$$
$$\left[\frac{V_2}{V_0}\right] = \lambda^2 \frac{V_2}{V_0}, \qquad \left[\frac{V_3}{V_0}\right] = \lambda^3 \frac{V_3}{V_0}. \qquad \ldots\ldots\ldots\ldots(39),$$

These equations* correspond to the general relations (6), (7), and (9), given in my former paper, and from them may be deduced the properties of systems of conjugate diameters, and the remarkable property of the reciprocal squares of three semi-diameters at right angles to one another.

Let X', Y', and Z', be the co-ordinates of the second origin referred to the first, so that if the co-ordinates be changed, $[f]$ and $\phi(X', Y', Z')$ will be corresponding terms of two equations, the terms of which should be respectively proportional. Assume λ, the indeterminate quantity above-mentioned, so that

$$[f] = \lambda\phi(X', Y', Z') \ldots\ldots\ldots\ldots\ldots\ldots(40),$$

and multiply together the first and last of (39), recollecting that

$$W = -\frac{V_4}{V_3} + f, \qquad [W] = -\left[\frac{V_4}{V_3}\right] + [f],$$

* These relations have been given by M. CAUCHY, for the case of rectangular co-ordinates, in his "*Leçons sur les applications du Calcul Infinitésimal à la Géométrie*," Vol. I. p. 244. The equation (34) of this paper, in as general a form, has also been given, since this was written, by Mr LUBBOCK, in the *Philosophical Magazine*.

and we obtain

$$\left[\frac{V_4}{V_0}\right] - \lambda\phi(X', Y', Z')\left[\frac{V_3}{V_0}\right] = \lambda^4\left(\frac{V_4}{V_0} - f\frac{V_3}{V_0}\right).$$

Substitute from the last of (39) for $\left[\frac{V_3}{V_0}\right]$, and develope $\phi(X', Y', Z')$, removing the term which contains it to the left hand side; which gives

$$\left[\frac{V_4}{V_0}\right] = \lambda^4\frac{V_4 + V_3(aX'^2 + bY'^2 + cZ'^2 + \text{\&c. \&c.})}{V_0} \quad\text{.........(41)},$$

answering to (8) in my former paper.

We shall afterwards proceed to some applications of these general formulæ, and now enquire into the several varieties of the equation (3), and the criteria for distinguishing between them. The following table, immediately to be explained, gives a synoptical view of the various cases.

When the Equations of the Center denote	W positive. negative.	W changes its sign.	W negative. positive.
A point	Impossible.	($W=0$) Point. ($W=\infty$) Elliptic Paraboloid.	Ellipsoid.
	Single Hyperboloid.	($W=0$) Cone. ($W=\infty$) Hyperbolic Paraboloid.	Double Hyperboloid.
A Right Line. W' substituted for W.	Impossible.	($W'=0$) Right line. ($W'=\infty$) Parabolic Cylinder.	Elliptic Cylinder.
	Hyperbolic Cylinder.	($W'=0$) Intersecting Planes. ($W'=\infty$) Parabolic Cylinder.	Hyperbolic Cylinder.
A Plane. W'' substituted for W'.	Impossible.	$\left(\begin{matrix}W''=0\\ W''=\infty\end{matrix}\right)$ Plane.	Parallel Planes.

Taking the first line of this table, and the signs of W, V_3, V_2, and V_1, (on which, as will presently be shewn, the variety of the equation depends,) being such as to denote that the equation is impossible, a change of sign in W *only* will indicate the ellipsoid, the elliptic cylinder, or parallel planes, according as the centre is a point, a line, or a plane. When the sign changes, if W be then $= 0$, the variety of the equation belongs to a point, a right line, or a plane; while if W be infinite, we have an elliptic paraboloid, a parabolic cylinder, or a plane. In using W, we mean its real value, W' or W'', when the primitive form of W becomes $\frac{0}{0}$.

The following table, from which the preceding may be deduced, and which I proceed to establish, gives the signs of W, &c., and also of V_3, &c., for the different cases. When p alone, or p and n occur on the same line, p may signify either sign, provided n stands for the other. Also when a sign is enclosed in brackets, it is a necessary consequence of what precedes it, and not an independent assumption. The numbers over the headings are references to the equations.

The last part of the table, including all the varieties under $W = \frac{0}{0}$, forms a similar synoptical table for the *curves* of the second degree. The following are the values of W', W'', V_2 and V_1, expressed in the notation of my former paper, the equation of the curve being

$$ay^2 + bxy + cx^2 + dy + ex + f = 0;$$

and the angle made by the axes being θ,

$$W' = \frac{cd^2 + ae^2 - bde}{b^2 - 4ac} + f,$$

$$W'' = -\frac{d^2 - 4af}{4d} = -\frac{e^2 - 4cf}{4c},$$

$$V_2 = -(b^2 - 4ac),$$

$$V_1 = a + c - b\cos\theta.$$

(12) W	(10) V_3	(27) W'	(9) V_2	(29) W''	(8) V_1	Variety of the Surface.	Remarks.
p	p		$+$		p	Impossible.	$a_{\prime\prime}$, $b_{\prime\prime}$, and $c_{\prime\prime}$ are positive, and V_3 has the common sign of a, b, and c.
	n		$+$		n	Ellipsoid.	
p	p		$+$ $-$		n	Single Hyperboloid.	Either V_2 or V_1 may be $=0$, but not both.
	n		$+$ $-$		p	Double Hyperboloid.	
∞	(0)		$+$			Elliptic Paraboloid.	In this, and the next V_2 has the same sign as $a_{\prime\prime}$, $b_{\prime\prime}$, and $c_{\prime\prime}$. V_1 may be $=0$.
			$-$			Hyperbolic Paraboloid.	
0	p		$+$		p	Point.	Either V_1 or V_2 may $=0$.
			$+$ $-$		n	Cone.	
$\frac{0}{0}$	(0)	p	$+$		p	Impossible.	In this, and all which succeed, $a_{\prime\prime}$ may be substituted for V_2.
					n	Elliptic Cylinder.	
			$-$			Hyperbolic Cylinder.	
		∞	(0)			Parabolic Cylinder.	Or $a_{\prime\prime}=0$, $b_{\prime\prime}=0$, $c_{\prime\prime}=0$.
		0	$+$			Straight line.	
			$-$			Intersecting Planes.	Intersecting at right angles when $V_1=0$.
		$\frac{0}{0}$	(0)	p	p	Impossible.	a may be substituted for V_1 in these cases.
					n	Parallel Planes.	
				0 ∞		Plane.	

First, with regard to the coefficients V_0, V_1, V_2, V_3 in equation (34) it appears from spherical trigonometry, that V_0 is always positive when ξ, η, and ζ are the sides of a spherical triangle; while from the possibility of the roots, as well as from the quantities themselves, we infer that if V_3 is finite, V_1 and V_2 can never vanish at the same time, while if $V_1=0$, and $V_3=0$, V_2 must be negative.

If we suppose W finite, and the order of signs in (34) to be $+-+-$ or $++++$, in which case all its roots are of one sign; that is, if V_2 be positive, and V_1 and V_3 of the same sign, the equation (30) shews that the surface is impossible or an ellipsoid, according as W and V_3 have the same or different signs. From (36) it appears that in this case, $a_{\prime\prime}$, $b_{\prime\prime}$, and $c_{\prime\prime}$ must be positive, whence a, b, and c have the same sign; which conditions, together with that of V_3 having the same sign as a, are equivalent to those given in the Table for the impossible case or the ellipsoid. If we examine independently into the conditions under which the aggregate of the first six terms of (24) always has the same sign, we shall find them to be that $a_{\prime\prime}$, $b_{\prime\prime}$, and $c_{\prime\prime}$ must be positive, and V_3 must have the common sign of a, b, and c. And it is evident that the first three terms of (30) are the first six terms of (24) in a different form. It may be worth noticing, that these conditions are equivalent to supposing $\frac{\bar{a}}{\sqrt{bc}}$, $\frac{\bar{b}}{\sqrt{ca}}$, $\frac{\bar{c}}{\sqrt{ab}}$ to be the cosines of the sides of a spherical triangle. When any other order of signs except the two already noticed, is found in (34), we shall have one positive root only, or one negative root only, according as V_3 is positive or negative; that is to say, one possible axis, or a double hyperboloid, when V_3 and W have contrary signs; and one impossible axis or a single hyperboloid, when they have the same signs.

When $W=0$, V_3 being finite, equation (30) represents a point, or a cone; the first when all the roots of (34) have the same sign, the second in any other case. When $V_3=0$, V_4 being finite, or W infinite, the center is at an infinite distance, and the equation belongs to an elliptic or hyperbolic paraboloid, according as V_2 is positive or negative. Since when $V_3=0$, $a_{\prime\prime}$, $b_{\prime\prime}$, and $c_{\prime\prime}$ have the same sign, (10), which is also the sign of V_2, $a_{\prime\prime}$ may be substituted for V_2. In this case, (10) and (9), V_2 has the form

$$P+Q+R+2\sqrt{QR}\cos\xi+2\sqrt{RP}\cos\eta+2\sqrt{PQ}\cos\zeta,$$

which, when P, Q, and R have the same sign, is always of that sign; and therefore can only be $=0$ when P, Q, and R are severally $=0$.

When $V_3=0$, and $V_4=0$, in which case W appears in the form $\frac{0}{0}$, and its real value is W' (27), the simplest criteria of which are expressed in (26) the equations (30) and (34) assume the forms

$$Ax^2+A'y'^2+W'=0\ldots\ldots\ldots\ldots\ldots\ldots(42),$$

$$V_0A^2-V_1A+V_2=0\ldots\ldots\ldots\ldots\ldots\ldots(43),$$

the first of which, if V_2 be positive, and V_1 and W' of the same sign, is impossible, and belongs to an elliptic cylinder if V_2 be positive, and V_1 and W' of different signs. As before, we may substitute $a_{\prime\prime}$ for V_2. If V_2 or $a_{\prime\prime}$ be negative, (42) belongs to an hyperbolic cylinder: and if $V_2=0$, in which case $a_{\prime\prime}=0$, $b_{\prime\prime}=0$, and $c_{\prime\prime}=0$ and W' is infinite, we have a parabolic cylinder. It appears therefore, that any surface of the second order, which has three parabolic sections, not having a common line of intersection, is a parabolic cylinder. The central line of this surface is at an infinite distance. When $W'=0$ and V_2 is positive, equation (42), considered as of *two* dimensions, represents only the origin, and therefore belongs to a straight line, the axis of z'. When V_2 is negative, W' being $=0$, (42) is the equation of two planes intersecting at an angle whose tangent is

$$\frac{2\sqrt{-AA'}}{A+A'}, \quad \text{or} \quad \frac{2\sqrt{-V_2V_0}}{V_1}.$$

When the equations of the center belong to a plane, and W' as well as W appears in the form $\frac{0}{0}$, the real value of W' is W'', given in (29) and the simplest conditions are, as in (28),

$$a_{\prime\prime}=b_{\prime\prime}=c_{\prime\prime}=0,$$

$$\overline{\overline{a}}:\overline{\overline{b}}:\overline{\overline{c}}::\overline{a}:\overline{c}:\overline{b}.$$

The equations (42) and (43) take the forms

$$Ax'^2 + W'' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(44),$$

$$V_0 A - V_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(45).$$

The first of which is impossible if W'' and V_1 have the same sign, that is, if W'' and a have the same sign; for when $a_{\prime\prime} = b_{\prime\prime} = c_{\prime\prime} = 0$, V_1 takes the same form with respect to a, b, and c which V_2 took with respect to $a_{\prime\prime}$, $b_{\prime\prime}$, and $c_{\prime\prime}$ in the last case. When a and W'' have different signs (44) belongs to two parallel planes, which coincide in one where $W''=0$. That is (29) the surface is impossible, two parallel planes, or one plane, according as $af-\bar{\bar{a}}^2$ is positive, negative, or nothing. When W' becomes infinite, or $a=0$, in which case b, c, $\bar{a}$, $\bar{b}$, and $\bar{c}$ are severally $=0$, the proposed equation (3) is in fact of the first degree.

Though oblique co-ordinates have hitherto been used, yet they might have been dispensed with so far as the criteria of distinction between the different classes of surfaces are concerned. It would take some space, and complicated algebraical operations, to prove this in all the individual cases, but the following general consideration is equally conclusive. So long as we only consider those distinctions which are implied in calling the surface bounded or unbounded, of one sheet or of two sheets, &c. in which no *numerical* relations of lines, &c. appear, it is evident that any equation will preserve the same character, however the axes on which its results are measured are inclined to one another. That is, when the sign of a quantity is alleged to be a criterion of distinction, it cannot stand as such, if by any alteration of ξ, η, or ζ, consistent with V_0 remaining positive, the sign of that quantity can be changed. Again, if the signs of two out of the three, $\bar{a}$, $\bar{b}$, and $\bar{c}$ be changed, as well as that of the third letter in $\bar{\bar{a}}$, $\bar{\bar{b}}$, and $\bar{\bar{c}}$, (those of $\bar{a}$, $\bar{b}$, and $\bar{\bar{c}}$, for example) it is evident that the surface remains the same in form and magnitude, those parts which were below one of the co-ordinate planes being transferred above it, and *vice versâ*. That is, it is impossible that any aggregate of terms of an odd degree, with respect to $\bar{a}$, $\bar{b}$, and $\bar{\bar{c}}$, $\bar{b}$, $\bar{c}$, and $\bar{\bar{a}}$, or $\bar{c}$, $\bar{a}$, and $\bar{\bar{b}}$, can affect the sign of any of the criteria. If we look at V_1, V_2, V_3, and V_4, we find that those terms, and those terms only, which are multiplied by cosines of ξ, &c., are of the first or third degree, with respect to any of the three sets just mentioned.

The case is very much altered when we consider any numerical relation, however simple. For example, I give the condition which expresses a surface of revolution, or a surface two of whose axes are equal. If A and A' belong to the equal axes, a, a', &c. become indeterminate; hence the numerators of the six equations (38), will, when equated to zero, have a common root. Eliminate $V_2 - V_1 A + V_0 A^2$ from the values of a^2 and $\beta\gamma$, &c. in (38), which gives

$$AV_0 = \frac{L_2 \cos\xi - \overline{L_2}}{a \cos\xi - \overline{a}} = \frac{M_2 \cos\eta - \overline{D}}{b \cos\eta - \overline{b}} = \frac{N_2 \cos\zeta - \overline{N_2}}{c \cos\zeta - \overline{c}} \quad \text{.........(46)},$$

which does not admit of any material simplification. There are evidently other ways of obtaining corresponding conditions from (38). I have chosen this because the corresponding formulæ have been given in the case of rectangular co-ordinates. In this case,

$$\cos\xi = \cos\eta = \cos\zeta = 0, \quad \text{and } \overline{L_2} = -\, l_{,,}, \text{ \&c.}$$

whence,

$$\frac{l_{,,}}{\overline{a}} = \frac{m_{,,}}{\overline{b}} = \frac{n_{,,}}{\overline{c}}.$$

(See Mr Hamilton's *Analytical Geometry*, p. 323.)

To apply the formulæ (39) and (41), let there be two planes whose equations, separately considered, are

$$\left.\begin{array}{l} \lambda x + \mu y + \nu z + 1 = 0 \\ \lambda' x + \mu' y + \nu' z + 1 = 0 \end{array}\right\} \text{...............} (47),$$

but which together must be one of the varieties of equation (3). Let new and rectangular axes be taken, the intersection of the planes being that of x. Their equation will then be

$$[c]\, z^2 + 2\,[\overline{a}]\, yz = 0,$$

where $-2\left[\frac{\bar{a}}{\bar{c}}\right]$ is the tangent of their angle of inclination. We have then (39),

$$[V_0]=1, \qquad \left[\frac{V_1}{V_0}\right]=[c]=\text{indet. quant.}\times\frac{V_0}{V_1}, \qquad \left[\frac{V_2}{V_0}\right]=-[\bar{a}]^2=(\text{indet. quant.})^2\times\frac{V_2}{V_1},$$

$$\text{which gives}\quad -2\left[\frac{\bar{a}}{\bar{c}}\right]=\frac{2\sqrt{-V_2V_0}}{V_1},$$

as we have already found. By multiplying together the equations (47), making the result identical with (3), and substituting in V_2 and V_1 the values of a, $2\bar{a}$, &c. thus obtained, we find for the tangent of the angle made by the planes

$$\sqrt{V_0}\,\frac{\sqrt{(\mu\nu'-\mu'\nu)^2+(\nu\lambda'-\nu'\lambda)^2+(\lambda\mu'-\lambda'\mu)^2+2(\nu\lambda'-\nu'\lambda)(\lambda\mu'-\lambda'\mu)\cos\xi+2(\lambda\mu'-\lambda'\mu)(\mu\nu'-\mu'\nu)\cos\eta+2(\mu\nu'-\mu'\nu)(\nu\lambda'-\nu'\lambda)\cos\zeta}}{\lambda\lambda'\sin^2\xi+\mu\mu'\sin^2\eta+\nu\nu'\sin^2\zeta+(\mu\nu'+\mu'\nu)(\cos\eta\cos\zeta-\cos\xi)+(\nu\lambda'+\nu'\lambda)(\cos\zeta\cos\xi-\cos\eta)+(\lambda\mu'+\lambda'\mu)(\cos\xi\cos\eta-\cos\zeta)}\ldots(48).$$

If we refer one of the paraboloids to its vertex, and axes parallel to the principal axes, the form of its equation is

$$[a]\,x^2+[b]\,y^2+2[\bar{\bar{c}}]\,z=0,$$

$$[V_0]=0, \qquad [V_1]=[a]+[b], \qquad V_2=[a][b], \qquad [V_3]=0, \qquad [V_4]=-[a][b][\bar{\bar{c}}]^2,$$

$$\text{whence}\quad (V_1+\sqrt{V_1^2-4V_2V_0})\,x^2+(V_1-\sqrt{V_1^2-4V_2V_0})\,y^2+2V_0\frac{\sqrt{a_{\prime\prime}}\,\bar{\bar{a}}+\sqrt{b_{\prime\prime}}\,\bar{\bar{b}}+\sqrt{c_{\prime\prime}}\,\bar{\bar{c}}}{\sqrt{V_2}}\,z=0\ldots\ldots(49).$$

AUGUSTUS DE MORGAN.

IV. *On a Monstrosity of the Common Mignionette.* By Rev. J. S. Henslow, M.A. *Professor of Botany in the University of Cambridge, and Secretary to the Cambridge Philosophical Society.*

[Read *May* 21, 1832.]

Having met with a very interesting monstrosity of the common Mignionette (***Reseda odorata,***) in the course of last summer (1831), I made several drawings of the peculiarities which it exhibited. I beg to present the Society with a selection from these, as I think they may both serve to throw considerable light upon the true structure of the flowers of this genus, which is at present a matter of dispute among our most eminent Botanists, and also tend to illustrate the manner in which the reproductive organs of plants generally, may be considered as resulting from a modification of the leaf.

It is well known to every Botanist, that Professor Lindley has proposed a new and highly ingenious theory, in which he considers the flowers of a Reseda to be compounded of an aggregate of florets, very analogous to the inflorescence of a Euphorbia. Mr Brown, on the other hand, maintains the ordinary opinion of each flower being simple, and possessed of calyx, corolla, stamens, and pistil. I shall not here enter upon any examination of the arguments by which these gentlemen have supported their respective views, but will refer those who are desirous of seeing them to the "Introduction to the Natural System of Botany, by Prof. Lindley," and to the "Appendix to Major Denham's Narrative, by Mr Brown." My present object will be little more than to describe the several appearances figured in plates 1 and 2.

Fig. 1. is one of the slightest deviations that was noticed from the ordinary state of the flower. It consists in an elongation of the pistil (*a*), and a general enlargement of its parts, indicating a tendency in them to pass into leaves. This is accompanied by a slight diminution in the size of the central disk. The number of the sepals was either six or seven.

Fig. 2. is a portion of the ovarium of the same flower opened, in which three of the ovules are somewhat distorted.

Fig. 3. Here the three valves of the ovarium have assumed a distinctly foliaceous character (*a*); the same has happened to some of the stamens (*b*), and to the petals (*c*); but the sepals are unaltered. The central disk has entirely disappeared.

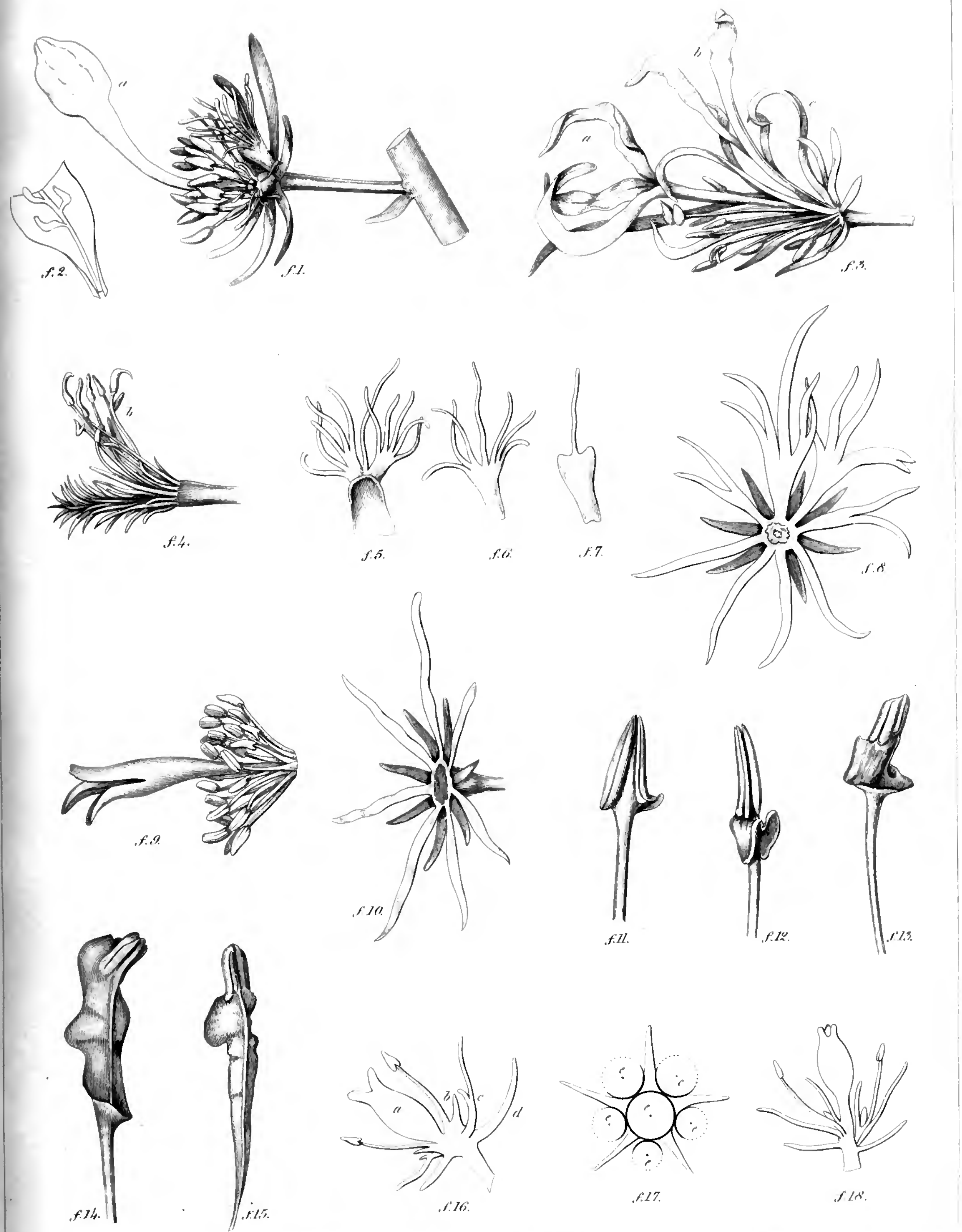
Fig. 4. This is a still closer approximation to the ordinary state of a proliferous flower bud, when developed. Those parts which would have formed the pistil, if the flower had been completed, are no longer distinguishable, and only a few of the stamens are to be seen, disguised in the form of foliaceous filaments crowned by distorted anthers (*b*).

Fig. 5. A slight deviation in one of the petals from the usual character. The fleshy unguis is somewhat diminished, and the fimbriæ are becoming green and leaf-like. These are aggregated into three distinct bundles, the middle one being composed of a single strap, and the two outer ones of five straps each, blended together at the base.

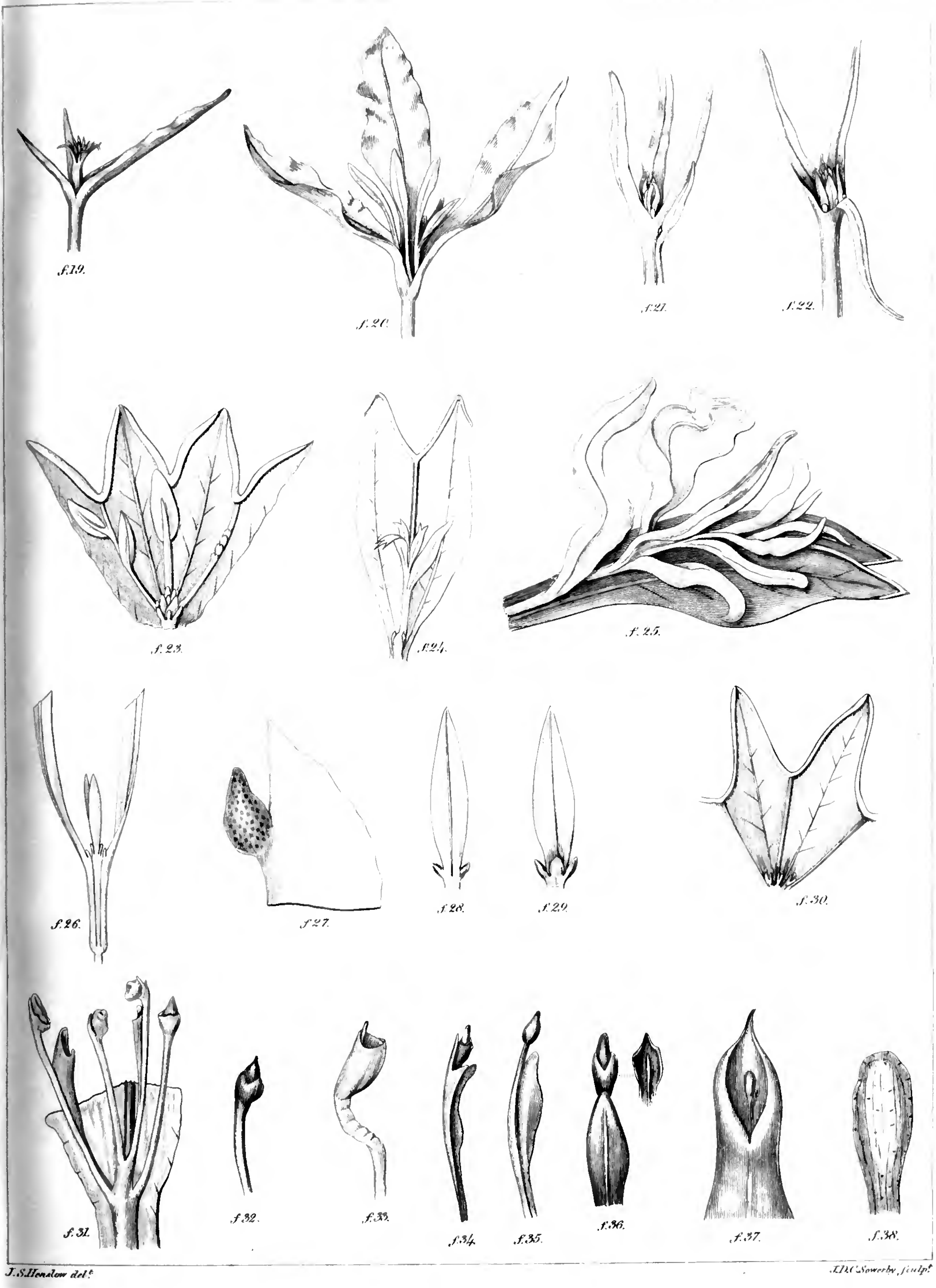
Fig. 6. The line of demarcation between the unguis and the fimbriæ has completely disappeared, and the number of the latter is considerably reduced. The whole is more green and leaf-like than fig. 5.

Fig. 7. The fimbriæ reduced to a single strap; the position of the lateral bundles being indicated by slight projections only. Other instances occurred in which the petal appeared as a single undivided uniform green strap.

Fig. 8. The two exterior whorls of a flower, consisting of seven regularly formed sepals, and eight petals. The latter deviate more or less from the forms represented in fig. 6 and 7. The whole of a green tint, and leaf-like.



J. S. Henslow delt. J. D. C. Sowerby sculpt.



J. S. Henslow del.^t J. D. C. Sowerby sculp.^t

Figs. 9, 10. These are parts of one and the same flower dissected to shew the several whorls more distinctly. The whole has assumed a regular appearance, and is composed of seven sepals, alternating with seven green strap-shaped petals, which are succeeded by about twenty stamens without any fleshy disk; the pistil is somewhat metamorphosed. This is perhaps the most remarkable deviation that was noticed from the ordinary state of the flower, and as several examples of it occurred, it is not likely that there is any error in this account of it. It appears to lead us in a very decided manner to the plan on which the flowers of the genus may be considered to be constructed, and to shew us that they are really simple and not compound.

Fig. 11 to 15, represent the appearances assumed by some of the stamens, indicating various degrees of deviation from the perfect state towards a foliaceous structure.

There were other circumstances, besides the appearances in figs. 9. and 10, which may lead us to conclude the structure of the flowers of the genus to be simple and not compound. A compound flower arises from the development of several buds in the axillæ of certain foliaceous appendages more or less degenerated from the character of leaves, and consequently these buds and the florets which they develop are always seated nearer to the axis than the foliaceous appendages themselves. If we suppose a raceme of the mignionette to degenerate into the condition of a compound flower, we must allow for the abortion of the stem on which the several flowers are seated, so that these may become condensed into a capitulum, each floret of which will be accompanied by a bractea, more or less developed, at its base. Let us compare this supposition with the diagrams represented in figs. 16, 17, 18.

Fig. 16. is an imaginary section of the flower in its ordinary state, (*a*) the pistil, (*b*) the stamens on the fleshy disk, (*c*) the petals, (*d*) the sepals alternating with them.

Fig. 17. represents the position of the several buds (*e*) which compose the florets of the flower on the supposition of its being compound. Here it will be noticed that these buds alternate with the

sepals instead of being placed in their axils where we might rather expect to find them.

Fig. 18. represents a fact which was observed in the present case, where some of the latent buds in the axils of the altered petals were partially developed. This development might perhaps be considered as indicating the construction of a compound flower, and those buds which in ordinary cases compose the outer and abortive florets, it might be said, are here manifesting themselves. But the axes of these buds lie nearer to the axis of the whole flower than the petals in whose axils they are developed; whereas it appears by fig. 17, that they ought to be further from it, since the centres of the five outer circles marked (*e*) would represent the axes of the several buds, whose partial development must be supposed to be on the side next the axis, if we allow any weight to the analogy between the position of the abortive stamens on the supposed calyx, and the fertile stamens on the central disk.

These figures are all that I have thought it necessary to give for the purpose of illustrating the structure of the flower; but as there were several interesting appearances noticed upon dissecting the pistil, I have selected some of them for the second plate, as they may possibly serve to throw some light upon the relationship which the several parts of the ovarium bear to the leaf, and to support the theory of their being all of them merely modifications of that important organ.

Fig. 19. is a pistil in which the three ovules have become foliaceous, and the central, or terminal bud of the flower-stalk is developing in the proliferous form represented in fig. 4.

Fig. 20. The central bud is not developing; but the three axillary buds in the bases of the transformed valves of the pistil are here assuming the form of branches on which one or two pair of leaves are expanded.

Fig. 21. 22. unite the appearances in fig. 19 and 20, with the addition of a glandular body seated between the leaves at their

junction. This apparently originates in the union of the two glandular stipules seated at the base of the leaves of this genus, and which may also be seen to accompany the scale-like leaves on the central bud within.

Figs. 23. to 25. Interior views of metamorphosed pistils, in which the ovules are seen transformed to leaves, and the glandular stipules are all that remain of the leaves which should compose the central bud, their limbs having entirely disappeared.

Fig. 26. The appearance of these stipules on a leaf-bud, developing under ordinary circumstances.

Fig. 27. One of them more highly magnified.

Figs. 28. 29. Their appearance on the small scale-like leaves of the central buds in fig. 21, 22.

Fig. 30. Similar to fig. 23, but without any appearance of the transformed ovules; the glandular stipules are seen in the bottom of the ovarium.

These glandular bodies assume a very prominent character in the anatomy of the metamorphosed pistils, and I was for some time puzzled to account for them, thinking that they might represent an altered condition of the ovules. I believe however that I have rightly considered them as the only representatives of the various leaves which would have made their appearance on the branch if the bud had developed in the ordinary way. They do not appear to diminish in size though the limb of the leaf has disappeared.

Fig. 31. Four pedicillated semitransformed ovules, seated on a placenta of a pistil metamorphosed similarly to that in fig. 9.

Figs. 32. to 35. Other appearances of a similar kind, all representing various approaches of the ovules to a foliaceous character. The little theca-shaped appendages are hollow, with a perforation at their apex, representing the foramen.

Fig. 36. One of these dissected, exhibiting a free clavate cellular body within, resembling the columella in the theca of a moss, and here probably representing the nucleus of the ovule.

Fig. 37. In this case the theca-shaped body was partially open exposing the included nucleus.

Fig. 38. This nucleus more highly magnified.

These appearances surely indicate a development of the investing coats of the nucleus into leaves; but how far these developments might be extended, and whether the nucleus itself is capable of being further separated into a series of investing coats does not appear from these specimens.

J. S. HENSLOW.

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V. *On the Calculation of Newton's Experiments on Diffraction.* By GEORGE BIDDELL AIRY, M.A. *late Fellow of Trinity College, and Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge.*

[Read *May* 7, 1833.]

SINCE the publication of Fresnel's experiments on Diffraction, it has been usual to employ as the source of light, in all experiments of this class, the image of the Sun formed by a lens of short focal length. On the undulatory theory, the effect of light thus produced is precisely the same as if the minute image of the Sun were the real origin of the light diverging with equal intensity through a solid angle whose diameter is many degrees. The spherical or chromatic aberration of the lens produces no sensible effect in any of the common experiments, in all which the angle, made by rays which afterwards interfere, is small. In calculating experiments thus conducted we proceed therefore with full confidence that no consideration is left out of sight, the omission of which could cause sensible error.

Newton's experiments however were conducted in a different way. His origin of light was a hole, from $\frac{1}{42}$ to $\frac{1}{4}$ of an inch in diameter, through which the Sun's light was made to pass. The effect of this light, on the undulatory theory, is *not* the same as if the bright hole were the origin of light. It becomes then a matter of some interest to examine mathematically what is the effect produced by transmitting the sun-beams directly through a hole of sensible size; and whether this effect, in practice, will differ much from the effect produced by forming an image of the Sun with a lens of short focal length.

The integrals which occur in this investigation are of such a kind that their values cannot be exhibited even in tables of numbers (except of course in any particular case, when by very tedious summation numerical results might be obtained). The only thing which can be attempted is, to shew that the integrals are precisely the same as those that occur in a very different instance where Fresnel's method of experimenting is adopted. Even thus far however I have not succeeded except in one case, namely, where the hole is a rectangular parallelogram of any length, and where the diffracting aperture is also a rectangular parallelogram in a similar position; including in this general case the particular instance in which one or both parallelograms have no boundary on one side.

To consider, in the first place, a case similar to Newton's. A plane wave is supposed to enter an external parallelogram and then to pass through a slit with sides parallel to those of the parallelogram; and the intensity of the light which falls upon a screen at a given distance is to be found. First, it is to be observed, that in estimating the *comparative* intensity of light in a direction parallel to one side of the parallelograms (suppose for instance the shorter) there is no necessity to take into account the length of the parallelograms in the other direction; as it will easily be seen, upon attempting an integration, that the intensity of light is expressed by the product of two quantities, of which one depends only on the lengths of the parallelograms and the position of the point of the screen in one dimension, and the other depends only on the breadth of the parallelograms and the position of the point of the screen in the other dimension. The intensity of light along a given line parallel to one side of the parallelogram will therefore, so far as it depends on the other side, be affected only with a constant multiplier. Neglecting therefore the lengths (by which term I designate that dimension of the parallelograms which is perpendicular to the line on which the comparative brightness is to be ascertained), suppose a normal to the front of the wave to be drawn, and suppose the limits of the breadth of the external aperture measured from this line to be α, β, (the distance of any point of the aperture being v), and suppose the limits of the breadth of the slit to be γ, δ, (the distance of any point of the slit being w): and suppose the distance of the point on the screen, whose illumination we wish to ascertain, to be x. Let the distance of the external aperture from the slit be a, and the distance of the slit from the screen b. Suppose the front of the wave where it enters the external aperture to be divided into a great number of small parts δv; and suppose each of these to be the origin of a small wave which diverges from it as a center. The distance from the point v of the aperture to the point w of the slit is

$$\sqrt{\{a^2+(v-w)^2\}}=a+\frac{1}{2a}(v-w)^2;$$

and the disturbance produced at w by the small wave spreading from the space δv at v will therefore be proportional to

$$\delta v\,.\sin.\frac{2\pi}{\lambda}\{\mathrm{v}t-A-a-\frac{1}{2a}(v-w)^2\}.$$

Integrating this with respect to v, the coefficient of $\sin\frac{2\pi}{\lambda}(\mathrm{v}t-A-a)$ will be

$$\textstyle\int_v \cos\frac{\pi}{a\lambda}(v-w)^2,$$

and the coefficient of $\cos\frac{2\pi}{\lambda}(\mathrm{v}t-A-a)$ will be

$$-\textstyle\int_v \sin\frac{\pi}{a\lambda}(v-w)^2.$$

The first of these integrals $=\int_v \cos\frac{\pi}{2}\left(v\sqrt{\frac{2}{a\lambda}}-w\sqrt{\frac{2}{a\lambda}}\right)^2$:

and putting $\phi(z)$ for $\int_z \cos\left(\frac{\pi}{2}z^2\right)$, this integral between the limits $v=\alpha$, $v=\beta$, will be proportional to

$$\phi\left(\beta\sqrt{\frac{2}{a\lambda}}-w\sqrt{\frac{2}{a\lambda}}\right)-\phi\left(\alpha\sqrt{\frac{2}{a\lambda}}-w\sqrt{\frac{2}{a\lambda}}\right).$$

Similarly putting $\psi(z)$ for $\int_z \sin\left(\frac{\pi}{2}z^2\right)$, the integral $-\int_v \sin\frac{\pi}{a\lambda}(v-w)^2$ between the same limits will be proportional to

$$-\psi\left(\beta\sqrt{\frac{2}{a\lambda}}-w\sqrt{\frac{2}{a\lambda}}\right)+\psi\left(\alpha\sqrt{\frac{2}{a\lambda}}-w\sqrt{\frac{2}{a\lambda}}\right).$$

The whole displacement at the point w will therefore be

$$\sin\frac{2\pi}{\lambda}(\mathrm{v}t-A-a)\times\left\{\phi\left(\beta\sqrt{\frac{2}{a\lambda}}-w\sqrt{\frac{2}{a\lambda}}\right)-\phi\left(\alpha\sqrt{\frac{2}{a\lambda}}-w\sqrt{\frac{2}{a\lambda}}\right)\right\}$$

$$+\cos\frac{2\pi}{\lambda}(\mathrm{v}t-A-a)\times\left\{-\psi\left(\beta\sqrt{\frac{2}{a\lambda}}-w\sqrt{\frac{2}{a\lambda}}\right)+\psi\left(\alpha\sqrt{\frac{2}{a\lambda}}-w\sqrt{\frac{2}{a\lambda}}\right)\right\}.$$

Suppose now this displacement to be the origin of a small wave which diverges from it as a center. The distance of the point w of the slit from the point x of the screen is

$$\sqrt{\{b^2+(w-x)^2\}}=b+\frac{1}{2b}(w-x)^2,$$

and this distance must be added to $A+a$ in the expressions

$$\sin\frac{2\pi}{\lambda}(\mathrm{v}t-A-a) \text{ and } \cos\frac{2\pi}{\lambda}(\mathrm{v}t-A-a),$$

in order to find an expression proportional to the displacement produced by it on the screen at the point x. The expression must also be multiplied by δw, the breadth of the small space from which the wave proceeds. Thus we find for the whole displacement at the point x of the screen,

$$\sin\frac{2\pi}{\lambda}(\mathrm{v}t-A-a-b)\times\int_w\left\{\begin{array}{l}\cos\frac{\pi}{b\lambda}(w-x)^2\times\left\{\phi\left(\beta\sqrt{\frac{2}{a\lambda}}-w\sqrt{\frac{2}{a\lambda}}\right)-\phi\left(\alpha\sqrt{\frac{2}{a\lambda}}-w\sqrt{\frac{2}{a\lambda}}\right)\right\}\\+\sin\frac{\pi}{b\lambda}(w-x)^2\times\left\{-\psi\left(\beta\sqrt{\frac{2}{a\lambda}}-w\sqrt{\frac{2}{a\lambda}}\right)+\psi\left(\alpha\sqrt{\frac{2}{a\lambda}}-w\sqrt{\frac{2}{a\lambda}}\right)\right\}\end{array}\right\}$$

$$+\cos\frac{2\pi}{\lambda}(\mathrm{v}t-A-a-b)\times\int_w\left\{\begin{array}{l}\sin\frac{\pi}{b\lambda}(w-x)^2\times\left\{-\phi\left(\beta\sqrt{\frac{2}{a\lambda}}-w\sqrt{\frac{2}{a\lambda}}\right)+\phi\left(\alpha\sqrt{\frac{2}{a\lambda}}-w\sqrt{\frac{2}{a\lambda}}\right)\right\}\\+\cos\frac{\pi}{b\lambda}(w-x)^2\times\left\{-\psi\left(\beta\sqrt{\frac{2}{a\lambda}}-w\sqrt{\frac{2}{a\lambda}}\right)+\psi\left(\alpha\sqrt{\frac{2}{a\lambda}}-w\sqrt{\frac{2}{a\lambda}}\right)\right\}\end{array}\right\}$$

where the integrals are to be taken between the limits $w=\gamma$, $w=\delta$. The brightness at the point x of the screen will then be proportional to the sum of the squares of the coefficients of $\sin\frac{2\pi}{\lambda}(\mathrm{v}t-A-a-b)$ and $\cos\frac{2\pi}{\lambda}(\mathrm{v}t-A-a-b)$.

To consider in the second place a case in which the illumination is produced in Fresnel's method. Let the distance from the origin of light to the aperture be a', and from the aperture to the screen b'. Let a line be drawn from the origin of light perpendicular to the screen, and let the limits of the aperture measured from this line, in the same direction as the breadths of the parallelograms in Newton's case, be ϵ and ζ (the general letter for the distance of any point in this direction being p), and let the limits in the direction perpendicular to this be $\eta+np$, $\theta+np$, where n is constant. (It is readily seen that this implies the figure to be rhomboidal, with two sides parallel to the length of the parallelograms in Newton's case.) Let q be the general letter for distance in this second direction: also let x' and y' be the distances, in the directions of p and q, of a point on the screen from the same line. The distance from the origin of light to the point p, q, in the aperture is

$$\sqrt{(a'^2+p^2+q^2)}=a'+\frac{p^2}{2a'}+\frac{q^2}{2a'}:$$

and the displacement there will therefore be proportional to

$$\sin\frac{2\pi}{\lambda}\left(\mathrm{v}t-A'-a'-\frac{p^2}{2a'}-\frac{q^2}{2a'}\right).$$

The distance from the point p, q, in the aperture to the point x', y', on the screen, is

$$\sqrt{\{b'^2+(p-x')^2+(q-y')^2\}}=b'+\frac{(p-x')^2}{2b'}+\frac{(q-y')^2}{2b'}:$$

and this must be added to

$$A+a'+\frac{p^2}{2a'}+\frac{q^2}{2a'},$$

in the expression for the displacement, in order to find the displacement produced at the point x', y', of the screen by the wave diverging from the point p, q, of the aperture. For the effect of the wave spreading from the small rectangle whose sides are δp, δq, we must multiply by δp, δq. Hence we find that the quantity to be integrated is

$$\int_p \cdot \int_q \cdot \sin \frac{2\pi}{\lambda}\left(vt - A' - a' - b' - \frac{p^2}{2a'} - \frac{q^2}{2a'} - \frac{(p-x')^2}{2b'} - \frac{(q-y')^2}{2b'}\right);$$

where, after integrating with respect to q, the limits of q must be expressed in terms of p before the next integration.

Putting $A' + a' + b' + \dfrac{x'^2 + y'^2}{2(a'+b')} = B'$, this expression becomes

$$\int_p \cdot \int_q \cdot \sin \frac{2\pi}{\lambda}\left\{vt - B' - \frac{a'+b'}{2a'b'}\left(p - \frac{a'x'}{a'+b'}\right)^2 - \frac{a'+b'}{2a'b'}\left(q - \frac{a'y'}{a'+b'}\right)^2\right\}.$$

The first integral is

$$\sin \frac{2\pi}{\lambda}\left\{vt - B' - \frac{a'+b'}{2a'b'}\left(p - \frac{a'x'}{a'+b'}\right)^2\right\} \int_q \cos\left\{\frac{\pi}{2}\cdot\frac{2(a'+b')}{a'b'\lambda}\left(q - \frac{a'y'}{a'+b'}\right)^2\right\}$$

$$-\cos \frac{2\pi}{\lambda}\left\{vt - B' - \frac{a'+b'}{2a'b'}\left(p - \frac{a'x'}{a'+b'}\right)^2\right\} \int_q \sin\left\{\frac{\pi}{2}\cdot\frac{2(a'+b')}{a'b'\lambda}\left(q - \frac{a'y'}{a'+b'}\right)^2\right\}$$

and $\displaystyle\int_q \cos\left\{\frac{\pi}{2}\cdot\frac{2(a'+b')}{a'b'\lambda}\left(q - \frac{a'y'}{a'+b'}\right)^2\right\} = \int_q \cos\frac{\pi}{2}\left(q\sqrt{\frac{2(a'+b')}{a'b'\lambda}} - y'\sqrt{\frac{2a'}{b'(a'+b')\lambda}}\right)^2,$

which between the limits $q = \eta + np$, $q = \theta + np$, is proportional to

$$\phi\left\{\theta\sqrt{\frac{2(a'+b')}{a'b'\lambda}} - y'\sqrt{\frac{2a'}{b'(a'+b')\lambda}} + p \cdot n\sqrt{\frac{2(a'+b')}{a'b'\lambda}}\right\}$$

$$-\phi\left\{\eta\sqrt{\frac{2(a'+b')}{a'b'\lambda}} - y'\sqrt{\frac{2a'}{b'(a'+b')\lambda}} + p \cdot n\sqrt{\frac{2(a'+b')}{a'b'\lambda}}\right\}.$$

The quantity proportional to $\displaystyle\int_q \sin\left\{\frac{\pi}{2}\cdot\frac{2(a'+b')}{a'b'\lambda}\left(q - \frac{a'y'}{a'+b'}\right)^2\right\}$ will be expressed in the same manner, putting ψ in the place of ϕ.

The whole displacement of ether at the point x', y', will therefore be found to be

$$\sin\frac{2\pi}{\lambda}(\mathrm{v}t-B')\times\int_p\left\{\begin{aligned}&\cos\frac{\pi}{\lambda}\cdot\frac{a'+b'}{a'b'}\left(p-\frac{a'x'}{a'+b'}\right)^2\times\Bigg[\ \phi\left\{\theta\sqrt{\frac{2(a'+b')}{a'b'\lambda}}-y'\sqrt{\frac{2a'}{b'(a'+b')\lambda}}\right.\\&\left.+p.n\sqrt{\frac{2(a'+b')}{a'b'\lambda}}\right\}-\phi\left\{\eta\sqrt{\frac{2(a'+b')}{a'b'\lambda}}-y'\sqrt{\frac{2a'}{b'(a'+b')\lambda}}+p.n\sqrt{\frac{2(a'+b')}{a'b'\lambda}}\right\}\Bigg]\\&+\sin\frac{\pi}{\lambda}\cdot\frac{a'+b'}{a'b'}\left(p-\frac{a'x'}{a'+b'}\right)^2\times\Bigg[-\psi\left\{\theta\sqrt{\frac{2(a'+b')}{a'b'\lambda}}-y'\sqrt{\frac{2a'}{b'(a'+b')\lambda}}\right.\\&\left.+p.n\sqrt{\frac{2(a'+b')}{a'b'\lambda}}\right\}+\psi\left\{\eta\sqrt{\frac{2(a'+b')}{a'b'\lambda}}-y'\sqrt{\frac{2a'}{b'(a'+b')\lambda}}+p.n\sqrt{\frac{2(a'+b')}{a'b'\lambda}}\right\}\Bigg]\end{aligned}\right\}$$

$$+\cos\frac{2\pi}{\lambda}(\mathrm{v}t-B')\times\int_p\left\{\begin{aligned}&\sin\frac{\pi}{\lambda}\cdot\frac{a'+b'}{a'b'}\left(p-\frac{a'x'}{a'+b'}\right)^2\times\Bigg[-\phi\left\{\theta\sqrt{\frac{2(a'+b')}{a'b'\lambda}}-y'\sqrt{\frac{2a'}{b'(a'+b')\lambda}}\right.\\&\left.+p.n\sqrt{\frac{2(a'+b')}{a'b'\lambda}}\right\}+\phi\left\{\eta\sqrt{\frac{2(a'+b')}{a'b'\lambda}}-y'\sqrt{\frac{2a'}{b'(a'+b')\lambda}}+p.n\sqrt{\frac{2(a'+b')}{a'b'\lambda}}\right\}\Bigg]\\&+\cos\frac{\pi}{\lambda}\cdot\frac{a'+b'}{a'b'}\left(p-\frac{a'x'}{a'+b'}\right)^2\times\Bigg[-\psi\left\{\theta\sqrt{\frac{2(a'+b')}{a'b'\lambda}}-y'\sqrt{\frac{2a'}{b'(a'+b')\lambda}}\right.\\&\left.+p.n\sqrt{\frac{2(a'+b')}{a'b'\lambda}}\right\}+\psi\left\{\eta\sqrt{\frac{2(a'+b')}{a'b'\lambda}}-y'\sqrt{\frac{2a'}{b'(a'+b')\lambda}}+p.n\sqrt{\frac{2(a'+b')}{a'b'\lambda}}\right\}\Bigg]\end{aligned}\right\},$$

where the integrals are to be taken between the limits $p=\epsilon$, $p=\zeta$. The brightness at the point x', y', of the screen will then be proportional to the sum of the squares of the coefficients of $\sin\frac{2\pi}{\lambda}(\mathrm{v}t-B')$ and $\cos\frac{2\pi}{\lambda}(\mathrm{v}t-B')$.

We have now to shew that, for a constant value of y', and a variable value of x', these expressions may be made similar to those obtained in the first case. For this purpose it will be necessary, first, to make the coefficients of the expressions under the integral sign equal: secondly, to make the limits of integration the same.

The first consideration gives us $\frac{\pi}{b\lambda} = \frac{\pi}{\lambda} \cdot \frac{a'+b'}{a'b'}$: $x = \frac{a'x'}{a'+b'}$;

$$\beta\sqrt{\frac{2}{a\lambda}} = \theta\sqrt{\frac{2(a'+b')}{a'b'\lambda}} - y'\sqrt{\frac{2a'}{b'(a'+b')\lambda}};$$

$$\alpha\sqrt{\frac{2}{a\lambda}} = \eta\sqrt{\frac{2(a'+b')}{a'b'\lambda}} - y'\sqrt{\frac{2a'}{b'(a'+b')\lambda}}; \quad \sqrt{\frac{2}{a\lambda}} = n\sqrt{\frac{2(a'+b')}{a'b'\lambda}}:$$

and the second consideration gives $\gamma=\epsilon$; $\delta=\zeta$; whence $\delta-\gamma=\zeta-\epsilon$. The first set of equations, reduced, are

$$\frac{1}{b} = \frac{1}{a'} + \frac{1}{b'}; \quad x = \frac{b}{b'}x'; \quad \beta\sqrt{\frac{1}{a}} = \theta\sqrt{\frac{1}{b}} - y'\sqrt{\frac{b}{b'^2}};$$

$$\alpha\sqrt{\frac{1}{a}} = \eta\sqrt{\frac{1}{b}} - y'\sqrt{\frac{b}{b'^2}}; \text{ whence } (\beta-\alpha)\sqrt{\frac{b}{a}} = \theta-\eta; \text{ and } n = -\sqrt{\frac{b}{a}}.$$

The purport of these equations, in common language, may be stated thus:

If in Newton's method light pass through a rectangular hole whose horizontal breadth is $\beta-\alpha$, and through a slit whose horizontal breadth is $\delta-\gamma$, at the distance a from the former, and fall finally on a screen at the distance b from the slit:

And if in Fresnel's method light pass through a rhomboidal hole, with two vertical sides, at the distance a' from the Sun's image; and fall on a screen or eyepiece at the distance b' from the hole, so that $\frac{1}{a'} + \frac{1}{b'} = \frac{1}{b}$:

And if the length of the vertical sides of the rhomboid be $\sqrt{\frac{b}{a}} \times$ the horizontal breadth of the external hole in the first case (or $\beta-\alpha$); and the horizontal breadth of the rhomboid be equal to the horizontal breadth of the slit in the first case (or $\delta-\gamma$); and the tangent of the angle made by the sides of the rhomboid be $\sqrt{\frac{a}{b}}$, (the acute angle of the rhomboid being on the side where x is negative and y positive).

Then the proportion of the intensities of light along the horizontal line in the first case will be the same as the proportion of the intensities of light along a horizontal line in the second case: the distance $x' = x \times \frac{b'}{b}$ in the second case corresponding to the distance x in the first case.

If in the first case the center of the hole is opposite to the center of the slit, the horizontal line in the second case must be drawn over the middle of the illumination on the screen. But if in the first case the center of the hole is not opposite to the center of the slit, but deviates in the direction which makes x positive, then the horizontal line in the second case must not be drawn over the middle of the illumination, but on that side on which y' is negative. In general, or when one side of either aperture in the first case is wanting, the equations

$$\gamma = \epsilon, \quad \delta = \zeta, \quad \beta \sqrt{\frac{1}{a}} = \theta \sqrt{\frac{1}{b}} - y' \sqrt{\frac{b}{b'^2}}, \quad \alpha \sqrt{\frac{1}{a}} = \eta \sqrt{\frac{1}{b}} - y' \sqrt{\frac{b}{b'^2}},$$

may be used.

When the inequality of the sides of the rhomboid is considerable, the form of the illumination is not very different from the illumination when the hole is parallelogrammic. The coloured bars will be a little inclined, so that those which for a parallelogram would be perpendicular to its longest sides, will approach towards the direction perpendicular to the longer diagonal of the rhomboid. Besides these, there is a faint brush of light projecting from each part which corresponds to an obtuse angle, and nearly in the direction of a line bisecting that angle produced. These general notions will assist the reader in judging what ought, theoretically, to be expected in the different circumstances of Newton's experiments.

In Newton's experiments the external hole was in fact circular. What would be the effect of this form it is impossible (theoretically) to say: but judging from the insignificance of the effect produced by a rectangular hole, I am inclined to think that, when the apertures are centrally opposite, the same investigation will apply well to it.

I may now without impropriety mention the circumstances which induced me to make this investigation.

In Newton's Optics, Book III. Observation 6, Newton describes in very striking language the effect of narrowing a slit on which the sun-light fell after having passed through a hole a quarter of an inch in diameter. He states that when the breadth of the slit was about $\frac{1}{400}$th of an inch, the illumination on the screen was interrupted by a black shadow in the middle. It is certain, theoretically and practically, that if the experiment had been made in Fresnel's method the center would be the brightest part. It seemed therefore worth while to ascertain, by the best kind of investigation that such an unmanageable case admits of, whether the size of the external hole could account for the dark shadow. From consideration of the form of the illumination in the second case above, it appears certain that it could not. The only resource (which the dullness of the weather at that time denied me) was to repeat the experiment. This I have now done three separate times in the presence of as many different persons: I have used both parallelogrammic and circular holes of different sizes (the largest circular hole being $\frac{1}{4}$ inch in diameter) and have sometimes diminished the aperture to as little as $\frac{1}{1000}$ inch (by estimation). The distances have been 30 inches each, which appear to have been the distances in Newton's experiments. In every instance the center has been bright. I can account for this inaccuracy in Newton's observation only by supposing that his eye was in such a state as not to recover from the sudden impression which is produced by rapidly diminishing the central light on the screen (which makes it for an instant appear black), and by referring to his candid avowal in the Advertisement, that "the third book and the last proposition of the second were put together out of scattered papers," and that "The subject of the third book I have also left imperfect, "not having tried all the experiments which I intended when I was "about these matters, nor repeated some of those which I did try "until I had satisfied myself about all their circumstances." I may add that Newton's measures of the distances at which the first dark bar was formed are so irreconcileable with those of his admirer Biot that, referring to the avowal above-cited, I think no reliance ought to be placed on the accuracy of his observations of diffraction.

Since writing the above, I find that Biot has repeated the experiment with the same result which I have obtained (*Traité de Physique*, Tom. IV. p. 749). He has not commented on or even mentioned Newton's observation.

G. B. AIRY.

OBSERVATORY,
May 6, 1833.

VI. *Second Memoir on the Inverse Method of Definite Integrals.*
By the Rev. R. MURPHY, M.A. *Fellow of Caius College, and of the Cambridge Philosophical Society.*

[Read *Nov.* 11, 1833.]

INTRODUCTION.

THE object of my former Memoir on the present subject, published in the Fourth Volume of the Society's Transactions, was to investigate the principles by which we might revert from a function outside the sign of definite integration, to the function under that sign, whenever the latter belonged to any of those classes usually received in analysis. In that case the function outside the sign of integration possessed the characteristic property of converging to zero when a variable quantity x was made to increase indefinitely; in the present Memoir I have endeavoured to complete this theory, by the research of the forms and properties of the functions under the sign of integration, when the characteristic above mentioned is not possessed by the function resulting from integration: and as the subject increased in difficulty, those methods of analysis which possessed greater simplicity and uniformity have been most adhered to, in the following investigations.

The fourth Section is devoted to the research of the nature and properties of the function under the sign of integration, when the integral always vanishes between the limits (0 and 1) of the independent variable which have been uniformly adopted in this as in the first Memoir. The class of functions thus investigated possess the remarkable property of vanishing an indefinitely great number of times in a finite extent; such functions correspond to an extended and curious class of phænomena in nature, when any principles of action which have been observed, under peculiar circumstances cease to produce the observed effects, as when equal charges of opposite electricities are communicated to a body, or when a body electrised by influence is removed from the vicinity of the influencing system; or lastly, as when heat in its thermometric effects disappears in the chemical changes which bodies undergo.

The properties of this class of functions are of great use and importance in analysis, as they conduct directly to the theory of reciprocal functions. This term I have here employed to denote such functions, two of which being multiplied together the integral of the product vanishes, except in one particular case. That function which is in this sense reciprocal to another, is also in general different in its nature. There are however many functions which are reciprocal to functions of their own nature, and to this class belong the only two species of reciprocal functions hitherto introduced into analysis; namely, the sines or cosines of the multiples of an angle, the integral of the product of which always vanishes (when taken between proper limits) except in the particular case of equimultiples; and secondly, such functions as satisfy the well-known partial differential equation in the third book of the *Mecanique Celeste;* where the integral of the product also vanishes except in the particular case where the functions are of the same order. It is this exception which renders reciprocal functions particularly useful, as is evident from the application of the trigonometrical functions in the theory of heat, and of Laplace's functions in investigations relative to the distribution of electricity. In the same Section I have shewn generally the means of discovering all species of reciprocal functions, and given several examples: as an instance of one of the most simple species possessing properties very analogous to those of Laplace's functions, but giving a simpler integral in the case where that integral does not vanish, it is proved in the succeeding Section that if T_n be the coefficient of h^n in $\frac{t^{\frac{h}{1-h}}}{1-h}$, then when n and m are unequal $\int_t T_n T_m = 0$, but when $n = m$, $\int_t T_n T_m = 1$.

The theory of reciprocal functions is applied in the fifth Section to the complete solution of the question, which was the object of this and the preceding Memoir, namely, to revert from any function whatever to that under the sign of definite integration, those reciprocal functions being employed which are most convenient in each particular instance.

The last application in this Memoir of the theory of reciprocal functions, is to the development of given functions of x in descending powers or other forms which vanish when x is infinitely great; the results of which may be applied to the valuation of functions of very great numbers, and to a great variety of physical problems. These series have also the peculiarity, generally, to terminate for the functions of integer numbers.

SECTION IV.

Inverse Method for Definite Integrals which vanish; and Theory of Reciprocal Functions.

1. When the equation $\int_t f(t) . t^x = \phi(x)$ is supposed to be restricted to particular values of x, then whatever may be the form of $\phi(x)$, $f(t)$ may always be determined; the values to which x is restricted we shall suppose to be the natural numbers 0, 1, 2, 3......$(n-1)$, and the method here pursued will also apply if the values of n should be different from those mentioned.

2. * First, let $\int_t f(t) . t^x = 0$, the limits of t being always 0 and 1, and let us seek for $f(t)$ *a rational function of t of the lowest possible dimensions, which shall satisfy this equation when x is any integer from* 0 *to* $n-1$ *inclusive.*

Any value of $f(t)$ which answers the proposed conditions may be divided by the absolute term, and the quotient, it is evident, will equally fulfil those conditions; we may therefore take the first or absolute term in $f(t)$ to be unity, and as the conditions to be satisfied are n in number, we must have n coefficients in $f(t)$, which will hence be a rational function of the form

$$1 + A_1 t + A_2 t^2 + \ldots\ldots + A_n t^n;$$

and therefore $$\phi(x) = \frac{1}{x+1} + \frac{A_1}{x+2} + \frac{A_2}{x+3} + \ldots\ldots + \frac{A_n}{x+n+1},$$

$$\text{or } 0 = \frac{P}{Q} \text{ by actual addition,}$$

putting Q for $(x+1)(x+2)\ldots\ldots(x+n+1)$, and P representing a function of x of n dimensions.

Hence $P=0$, provided x be any number of the series 0, 1, 2....$(n-1)$; these are therefore all the roots of that equation, P being of n dimensions; hence we must have

$$P = c . x . (x-1)(x-2)\ldots\ldots(x-n+1);$$

c representing a constant quantity.

* I have resolved this question in a different manner in the "Treatise on Electricity."

We have thus

$$\frac{1}{x+1}+\frac{A_1}{x+2}+\frac{A_2}{x+3}+\ldots\ldots\frac{A_n}{x+n+1}=\frac{c\,.\,x\,.\,(x-1)\;\ldots.(x-n+1)}{(x+1)\,.\,(x+2)\ldots.(x+n+1)}.$$

Multiply by $x+1$, and then put $x=-1$; hence $c=(-1)^n$,

.... by $x+2$, ... $x=-2$; ......$A_1=-\frac{n}{1}.\frac{n+1}{1}$,

.... by $x+3$, ... $x=-3$; ......$A_2=-\frac{n.(n-1)}{1\,.\,2}.\frac{(n+1)(n+2)}{1\,.\,2}$;

&c. .... &c.

$$\text{hence }\; f(t)=1-\frac{n}{1}.\frac{n+1}{1}.t+\frac{n\,.\,(n-1)}{1\,.\,2}.\frac{(n+1).(n+2)}{1\,.\,2}.t^2-\&c.$$

$$=\frac{d^n}{dt^n}\frac{\left\{t^n.\left(1-nt+\dfrac{n\,.\,(n-1)}{1\,.\,2}.t^2-\&c.\right)\right\}}{1\,.\,2\,.\,3\ldots.n}$$

$$=\frac{d^n\,.\,(tt')^n}{1\,.\,2\,.\,3\ldots n\,.\,dt^n},\ \text{putting } t'=1-t.^*$$

3. Denoting by P_n the value of $f(t)$ which has been investigated in the preceding article, it possesses the remarkable property: *that* $\int_t P_m P_n=0$, *except when* $m=n$, *and then*

$$\int_t P_m P_n=\frac{1}{2n+1};$$

the limits being always 0 *and* 1.

For when m and n are unequal, one of them as n is the greater, P_m contains then only powers of t inferior to n, the integral of each of which vanishes by the nature of P_n.

When $m=n$, the last term of P_n, namely

$$\frac{(n+1)\,(n+2)\ldots.2\,n}{1\;\;.\;\;2\;\;\ldots.\,n}\,(-t)^n,$$

is the only term of which, when multiplied by P_n, the integral does

* This value of $f(t)$ has been shewn in the "Treatise on Electricity" to be the coefficient of h^n in $\{1-2h.(1-2t)+h^2\}^{-\frac{1}{2}}$.

not vanish; and since in general

$$\int_t P_n t^x = (-1)^n \cdot \frac{x \cdot (x-1) \quad \ldots (x-n+1)}{(x+1)(x+2) \ldots (x+n+1)},$$

it is evident that in this case $\int_t P_n^2 = \dfrac{1}{2n+1}$.

4. To illustrate the observation in Art. 1, with respect to the generality of this method, let it now be required, *to find a rational function of t, as $f(t)$, of the lowest possible dimensions, to satisfy the equation $\int_t f(t) \cdot t^x = 0$, when x is any number of the series*

$$p, \; p+1, \; p+2, \ldots\ldots\ldots p+n-1.$$

Putting as before $f(t) = 1 + A_1 t + A_2 t^2 + \ldots\ldots + A_n t^n$, we have

$$\int_t f(t) \cdot t^x = \frac{1}{x+1} + \frac{A_1}{x+2} + \frac{A_2}{x+3} + \ldots\ldots \frac{A_n}{x+n+1},$$

the sum of all which fractions must by the reasoning of Art. 2, be

$$\frac{c \cdot (x-p)(x-p-1) \ldots (x-p-n+1)}{(x+1)(x+2)(x+3) \ldots (x+p+x)};$$

and determining c, A_1, A_2, &c. in the same manner as in the Article referred to, we have

$$c = (-1)^n \cdot \frac{1 \cdot 2 \cdot 3 \ldots n}{(p+1) \cdot (p+2) \ldots (p+n)},$$

$$A_1 = -\frac{n}{1} \cdot \frac{n+p+1}{p+1},$$

$$A_2 = \frac{n \cdot (n-1)}{1 \cdot 2} \cdot \frac{(n+p+1) \cdot (n+p+2)}{(p+1) \cdot (p+2)},$$

&c.............................&c.

and therefore

$$f(t) = 1 - \frac{n}{1} \cdot \frac{n+p+1}{p+1} \cdot t + \frac{n \cdot (n-1)}{1 \cdot 2} \cdot \frac{(n+p+1) \cdot (n+p+2)}{(p+1) \cdot (p+2)} \cdot t^2 + \&c.$$

$$= \frac{t^{-p}}{(p+1) \cdot (p+2) \ldots (p+n)} \cdot \frac{d^n}{dt^n} \cdot \left\{ t^{n+p} \cdot \left(1 - \frac{n}{1} \cdot t + \frac{n \cdot (n-1)}{1 \cdot 2} \cdot t^2 - \&c. \right\};$$

or, putting $1-t=t'$, we obtain

$$f(t)=\frac{t^{-p}}{(p+1).(p+2)\ldots.(p+n)}\cdot\frac{d^n.(t^n t'^n t^p)}{dt^n}.$$

5. From this result it follows that if we put

$$f(t)=\frac{d^n.(t^n t'^n.a_p t^p)}{dt^n},$$

then shall $\int_t f(t).t^x=0$, provided x is any number of the series

$$0,\ 1,\ 2\ldots\ldots(n-1);$$

a_p representing any constant quantity.

Now $a_p t^p$ may be taken for the general term of an arbitrary function V; hence the most general function which satisfies the equation $\int_t f(t).t^x=0$, is expressed by

$$f(t)=\frac{d^n.(t^n t'^n V)}{dt^n}.$$

In fact we have (supposing the integrals to commence from $t=0$,)

$$\int_t f(t).t^x = t^x f_1(t) - xt^{x-1}f_2(t) + x.(x-1).f_3(t),\ \&c.$$

representing by $f_n(t)$ the n^{th} successive integral of $f(t)$, and putting for x 0, 1, 2....$(n-1)$ successively, it follows that

$$f_1(t)=0,\ \ f_2(t)=0\ldots\ldots f_n(t)=0,\ \text{when}\ t=1;$$

that is, $f_n(t)$ and its n differential coefficients vanish when $t=0$ and when $t=1$; therefore $f_n(t)$ contains a factor of the form $t^n.(1-t)^n$, and consequently the most general form of $f(t)$ is

$$\frac{d^n(t^n t'^n V)}{dt^n}.$$

6. Hence we deduce the following general property: "*If $f(t)$ be any function which satisfies the equation $\int_t f(t).t^x=0$, x being any integer from* 0 *to* $(n-1)$ *inclusive, then the equation* $f(t)=0$ *will always have n real roots lying between* 0 *and* 1."

For the equation $t^n.t'^n V=0$ has n roots $t=0$ and n roots $t=1$; and therefore $f(t)$ which is the n^{th} derived equation must have n roots between 0 and 1.

Hence, if we suppose the equation $\int_t f(t) \,.\, t^x = 0$ to hold true for an indefinite number of entire values of x, the equation $f(t) = 0$ will also have an indefinitely great number of roots all lying between 0 and 1, and the curve, of which the ordinate is $f(t)$, and the abscissa t, would intersect that portion of the axis of x, of which the length is unity measured from the origin in an indefinitely great number of points; thus we have a property characteristic of this class of functions.*

7. We have supposed $f(t)$ to consist of terms involving the powers of t, but as we may proceed in like manner for any other assumed form, we take the following as an example, because it leads to some remarkable results.

To find a rational function of h. l. (t) *as* $f(\text{h. l. } t)$ *of the lowest possible dimensions, which may satisfy the equation* $\int_t f(\text{h. l. } t) \,.\, t^x = 0$, *$x$ being any integer from* 0 *to* $n-1$ *inclusive.*

Put $f(\text{h. l. } t) = 1 + A_1 \text{ h. l. } t + A_2 (\text{h. l. } t)^2 + \ldots\ldots + A_n (\text{h. l. } t)^n$,

and observing that $\int_t \{\text{h. l. } (t)\}^m . t^x = (-1)^m . \dfrac{1 . 2 . 3 \ldots . m}{(x+1)^{m+1}}$,

we get $\int_t f(\text{h. l. } t) \,.\, t^x = \dfrac{1}{x+1} - \dfrac{A_1}{(x+1)^2} + \dfrac{A_2 . 1 . 2}{(x+1)^3} - \ldots\ldots \pm \dfrac{1 . 2 \ldots\ldots n A_n}{(x+1)^{n+1}}$,

and actually adding the fractions in the right-hand member of this equation, the numerator which is a function of n dimensions, ought to vanish when x is any number of the series 0, 1, 2...$(n-1)$; that is,

$$(x+1)^n - A_1 (x+1)^{n-1} + 1 . 2 A_2 (x+1)^{n-2} - 1 . 2 . 3 A_3 (x+1)^{n-3} \ldots\ldots$$
$$= C . x . (x-1)(x-2) \ldots\ldots (x-n+1).$$

Let S_1 represent the sum of the natural numbers 1, 2, 3....$(n-1)$, n,
S_2 the sum of their products two by two,
S_3 the sum of their products three by three, &c.

* Vide Art. (4) in my first Memoir on the Inverse Method of Definite Integrals.

Then by the theory of equations, the right-hand member of this equation is equivalent to

$$c\,\{(x+1)^n - S_1 (x+1)^{n-1} + S_2 (x+1)^{n-2} - S_3 (x+1)^{n-3}, \text{ \&c.}\}$$

whence $c=1$, $A_1=S_1$, $A_2=\dfrac{S_2}{1.2}$, $A_3=\dfrac{S_3}{1.2.3}$, &c. hence the required function is

$$1+S_1 \,.\, \text{h. l. } t+\frac{S_2}{1.2}\,.\,(\text{h. l. } t)^2+\frac{S_3}{1.2.3}\,.\,(\text{h. l. } t)^3+\ldots .+\frac{S_n}{1.2\ldots .n}\,.\,(\text{h. l. } t)^n.$$

8. It has been proved, that the function thus obtained (which we shall denote by L_n) in common with all others which possess the property that $\int_t f(t)\,.\,t^x=0$, when x is any integer from 0 to $n-1$ inclusive, is of the form

$$\frac{d^n.\,(t^n t'^n V)}{dt^n};$$

to verify this in the present case, we must sum the preceding series which is represented by L_n.

First, by the nature of multiplication, we have

$$h^n + S_1 h^{n-1} + S_2 h^{n-2} + \ldots\ldots + S_n = (h+1)\,(h+2)\ldots\ldots(h+n),$$

and the development of an exponential gives

$$1 + h \text{ h. l. } (t) + \frac{h^2\,(\text{h. l. } t)^2}{1.2} + \ldots\ldots + \frac{h^n\,(\text{h. l. } t)^n}{1.2.3\ldots .n} + \text{\&c.} = t^h,$$

the coefficient of h^n in the product of both the latter series is identical with that by which L_n is expressed.

But since that product $=t^h\,(h+1)\,(h+2)\ldots\ldots(h+n)$

$$=\frac{d^n\,(t^{h+n})}{dt^n}$$

$$=\frac{d^n}{dt^n}\,\{t^n\,(1+h \text{ h. l. } t+\frac{h^2\,(\text{h. l. } t)^2}{1.2}+\text{\&c.})\},$$

it follows that the coefficient of h^n is also expressed by

$$\frac{d^n\,\{t^n\,(\text{h. l. } t)^n\}}{1.2.3\ldots\ldots n\,dt^n};$$

this quantity is therefore the sum of the series which we proposed to find.

Now the equation h. l. $(t) = 0$ is satisfied by $t = 1$; hence h. l. t is of the form $t'.Q$, {where $Q = -\left(1 + \frac{t'}{2} + \frac{t'^2}{3} + \&c.\right)$}, and therefore if we put $\frac{Q^n}{1.2....n} = V$, we get the value of L_n to be

$$\frac{d^n.(t^n t'^n V)}{dt^n},$$

which was the formula we had required to verify.

We may also observe that since in the equation $L_n = 0$, t must have n values lying between 0 and 1, therefore h. l. (t), according to the powers of which L_n is arranged, must have n real negative roots, which we see confirmed by the positive signs of all the terms which compose L_n.

9. If we form the equation

$$u\,(1 - h \text{ h. l. } u) = t,$$

we have by Lagrange's theorem

$$u = t + ht \text{ h. l. } (t) + \frac{h^2}{1.2}\cdot\frac{d\,(t \text{ h. l. } t)^2}{dt} + \frac{h^3}{1.2.3}\cdot\frac{d^2\,(t \text{ h. l. } t)^3}{dt^2} + \&c.$$

from whence it appears that L_n is the coefficient of h^n in the value of $\frac{du}{dt}$. Similarly if in Article (12) we form the equation

$$u\,\{1 - h.(1 - u)\} = t,$$

$$\text{we have } P_n = \text{the coefficient of } h^n \text{ in } \frac{du}{dt}.$$

10. If Q_n *be the coefficient of* h^n *in* $\frac{du}{dt}$, *supposing* u *to be determined by the equation* $u\,(1 - hU) = t$, U *being a function of* u *which vanishes when* $u = 1$, *and* T *the same function of* t, *then shall*

$$\frac{\int_t Q_n t^x}{\int_t T^n t^x} = \frac{x.(x-1)(x-2)......(x-n+1)}{1.2.3......n}.(-1)^n.$$

For if we put $u=0$ in the equation $u(1-hU)=t$ we get $t=0$, and putting $u=1$ we have by supposition $U=0$ and therefore $t=1$, hence the limits of u are the same as the limits of t.

$$\text{But } \textstyle\int_t Q_n t^x = \text{the coefficient of } h^n \text{ in } \displaystyle\int_t \frac{du}{dt}\cdot t^x,$$

$$\text{and } \int_t \frac{du}{dt}\cdot t^x = \textstyle\int_u t^x = \int_u u^x (1-hU)^x$$

$$= \textstyle\int_t t^x (1-hT)^x,$$

expanding the part under the sign of integration, and taking the coefficient of h^n we obtain

$$\textstyle\int_t Q_n t^x = \dfrac{x.(x-1)(x-2)\ldots\ldots(x-n+1)}{1.2.3\ldots\ldots n}.(-1)^n \int_t T^n.t^x.$$

11. *If U be a rational and entire function of u which vanishes when $u=1$, and if Q_n be the term independent of u in the product $U^n.\left(1-\frac{t}{u}\right)^{-(n+1)}$, then shall Q_n be itself a rational and entire function of t possessing the property of $\int_t Q_n t^x = 0$, x being any integer from 0 to $n-1$ inclusive.*

For it has been proved in my former Memoir on the Resolution of Equations*, that the root of the rational equation $\phi(x)=0$ is the coefficient of $\frac{1}{x}$ in $-\text{h.l.}\frac{\phi x}{x}$, hence the value of u in the equation $u(1-hU)=t$, is the coefficient of $\frac{1}{u}$ in $-\text{h.l.}\left\{\left(1-\frac{t}{u}\right)-hU\right\}$, and differentiating, it follows that the value of $\frac{du}{dt}$ is the term independant of u in

$$\frac{1}{\left(1-\frac{t}{u}\right)-hU},$$

* Camb. Trans. Vol. IV. p. 131.

because the u under the logarithmic sign is the same as if we had placed there, a or any arbitrary symbol, and is therefore treated as a constant in the differentiation; hence the coefficient of h^n in $\frac{du}{dt}$ is the term independant of u in

$$\frac{U^n}{\left(1-\frac{t}{u}\right)^{n+1}},$$

that is, its value is Q_n, and therefore by the preceding Article $\int_t Q_n t^x$ vanishes between the limits of x, 0 and $n-1$, its general value being

$$\frac{x.(x-1)(x-2)\ldots\ldots(x-n+1)}{1.2.3\ldots\ldots n}(-1)^n \int_t T^n t^x,$$

T being the same function of t that U is of u.

By this theorem, every possible variety of rational and entire functions which possess the above-mentioned property may be found, as in the following

EXAMPLE:

To find a rational function of t, in which the powers of the variable are in arithmetical progression, such that $\int_t Q_n t^x = 0$ *when x is any number of the series* 0, 1, 2......$(n-1)$.

In this instance put $U = 1 - u^m$, m being any positive integer.

Hence $Q_n =$ term independent of u in

$$(1-u^m)^n \cdot \left(1-\frac{t}{u}\right)^{-(n+1)}$$

$$=1-\frac{n}{1}\cdot\frac{(n+1)(n+2)\ldots(n+m)}{1.2\ldots m}.t^m+\frac{n.(n-1)}{1.2}\cdot\frac{(n+1)(n+2)\ldots(n+2m)}{1.2\ldots 2m}.t^{2m}-\&c.,$$

in which if we take in particular $m=1$, we get the value of P_n before found in Art. (2).

This formula for Q_n may be written in another form by which it will comprise the case where m is a fraction, thus

$$Q_n=1-\frac{n}{1}\cdot\frac{(m+1)(m+2)\ldots(m+n)}{1.2\ldots n}.t^m+\frac{n(n+1)}{1.2}\cdot\frac{(2m+1)(2m+2)\ldots(2m+n)}{1.2\ldots n}.t^{2m}-\&c.$$

and it is, moreover, evident that either of those values are identical with

$$\frac{d^n . \{t^n (1-t^m)^n\}}{1 . 2 \ldots n dt^n},$$

which is included in the general form given in Art. 5. viz.

$$\frac{d^n . (t^n t'^n V)}{dt^n}.$$

12. *To find a rational and entire function of t^p of n dimensions, which if multiplied by a rational and entire function of t^q of less than n dimensions, the integral of the product may vanish between the limits $t=0$ and $t=1$.*

Let the required function be represented by $(p, q)_n$, so that

$$(p, q)_n = 1 + A_1 t^p + A_2 t^{2p} + \ldots\ldots A_n t^{np},$$

and by the proposed conditions we must have

$$\textstyle\int_t (p, q)_n t^{mq} = 0,$$

m being any integer from 0 to $n-1$ inclusive, put $t^q = T$, the limits of T are the same as those of t.

$$\text{Hence } \textstyle\int_T (p, q)_n T^{\frac{1}{q}-1} . T^m = 0.$$

Now $(p, q)_n T^{\frac{1}{q}-1}$, is a function of T of which the indices are in arithmetical progression, $\frac{p}{q}$ being the common difference, and $T^{\frac{1}{q}-1}$ the first term; and as the nature of the question affords n independant equations for the determination of the n coefficients $A_1, A_2 \ldots A_n$, it follows that there is only one function of the kind, which will satisfy the proposed conditions, and by Art. 5, it is evident that the function

$$\frac{\frac{d^n}{dT^n} . \{T^{n+\frac{1}{q}-1} (1 - T^{\frac{p}{q}})^n\}}{\left(n+\frac{1}{q}-1\right)\left(n+\frac{1}{q}-2\right) \ldots \frac{1}{q}},$$

answers those conditions, and is manifestly of the required form, it follows that if we divide this function by $T^{\frac{1}{q}-1}$, and then substitute t^q for T, we shall obtain the value of $(p, q)_n$; we have thus,

$$(p, q)_n = \frac{T^{-\frac{1}{q}+1}}{\frac{1}{q}\left(\frac{1}{q}+1\right)\dots\left(\frac{1}{q}+n-1\right)} \cdot \frac{d^n}{dT^n}\left\{T^{n+\frac{1}{q}-1} - nT^{n+\frac{p+1}{q}-1}\right.$$

$$\left. + \frac{n.(n-1)}{1.2} . T^{n+\frac{2p+1}{q}-1} - \&c.\right\}$$

$$= 1 - \frac{(p+1)(p+1+q)(p+1+2q)\dots\{p+1+(n-1).q\}}{1\,(1+q)\,(1+2q)\dots\{1+(n-1).q\}} \cdot \frac{n}{1} \cdot t^p$$

$$+ \frac{(2p+1)\,(2p+1+q)\dots\{2p+1+(n-1).q\}}{1.(1+q)\dots\{1+(n-1).q\}} \cdot \frac{n.(n-1)}{1.2}\, t^{2p} - \&c.$$

COR. $(p, p)_n = 1 - \frac{n}{1} \cdot \frac{np+1}{1} . t^p + \frac{n.(n-1)}{1.2} \cdot \frac{\{(n+1).p+1\}\,(np+1)}{1.(1+p)}\, t^{2p} - \&c.$

13. The functions $(p, q)_n$ and $(q, p)_n$ may be termed reciprocal functions, and possess the remarkable property, that if n and n' are any different integers, then shall

$$\textstyle\int_t (p, q)_n . (q, p)_{n'} = 0.$$

For if $n > n'$ then $(q, p)_{n'}$ is a rational and entire function of t^q of less than n dimensions, and therefore by the preceding Article the integral of the product must vanish; again if $n' > n$, then $(p, q)_n$ is a function of t^p of less than n' dimensions, and therefore when multiplied by $(q, p)_{n'}$ the integral ought to vanish.

To determine the value of the same integral when $n = n'$, it is evident by the nature of the function $(p, q)_n$ that we need only attend to the last term in the expansion of $(q, p)_n$, namely

$$(-1)^n . t^{nq} . \frac{(nq+1)(nq+1+p)\dots\{nq+1+(n-1).p\}}{1.(1+p)\dots\{1+(n-1).p\}}.$$

Now if we put for $(p, q)_n$ the series assumed in Art. (12) and multiplying then by t^x, integrate from $t=0$ to $t=1$, we have

$$\int_t (p, q)_n t^x = \frac{1}{x+1} + \frac{A_1}{x+p+1} + \frac{A_2}{x+2p+1} + \ldots + \frac{A_n}{x+np+1},$$

and actually adding these fractions, the denominator of the sum is

$$(x+1)(x+p+1)(x+2p+1)\ldots\ldots(x+np+1);$$

and since the numerator is of n dimensions in x, and vanishes when

$$x = 0,\ q,\ 2q\ldots.(n-1).q,$$

it follows that the sum is of the form

$$\frac{c.x.(x-q)(x-2q)\ldots.\{x-(n-1).q\}}{(x+1).(x+p+1)\ldots.(x+np+1)}.$$

Multiply by $x+1$ and then put $x=-1$; hence

$$1 = \frac{c.(-1)^n.1.(q+1)(2q+1)\ldots.\{(n-1).q+1\}}{p\,.\,2p\,.\,3p\ldots.np},$$

whence deducing the value of c, and substituting in the above integral, we obtain

$$\int_t (p, q)_n . t^x = (-p)^n . \frac{1.2.3\ldots.n}{1.(q+1)(2q+1)\ldots.\{(n-1).q+1\}}$$
$$\times \frac{x.(x-q)(x-2q)\ldots.\{x-(n-1).q\}}{(x+1)(x+p+1)\ldots.(x+np+1)},$$

hence $$\int_t (p, q)_n . t^{nq} = (-p)^n . \frac{1.2.3\ldots.n}{1(q+1)\ldots\{(n-1).q+1\}}$$
$$\times \frac{nq(nq-q)(nq-2q)\ldots.\{nq-(n-1).q\}}{(nq+1)(nq+p+1)\ldots.(nq+np+1)},$$

from whence we obtain finally

$$\int_t (p, q)_n (q, p)_n = \frac{p^n}{n(p+q)+1} . \frac{1.2.3\ldots.n}{1(q+1)\ldots.\{(n-1).q+1\}}$$
$$\times \frac{nq(nq-q)(nq-2q)\ldots.\{nq-(n-1).q\}}{1(p+1)(2p+1)\ldots.\{(n-1).p+1\}}.$$

that is, it

$$= \frac{(pq)^n}{n(p+q)+1} \cdot \frac{1^2 . 2^2 . 3^2 \ldots\ldots n^2}{1.(p+1)(q+1)(2p+1)(2q+1)\ldots\{(n-1).p+1\}\{(n-1).q+1\}}.$$

COR. $\int_t (p, p)_n^2 = \frac{p^{2n}}{2n+1} \cdot \left\{\frac{1 . 2 . 3 \ldots n}{1.(p+1).(2p+1)\ldots\{(n-1).p+1\}}\right\}^2.$

14. *To find the reciprocal function to that denoted by* L_n *in Art.* 8, *namely,* $\frac{d^n\{t^n(\text{h. l. } t)^n\}}{1.2\ldots n \quad dt^n}$.

L_n consists of the powers of h. l. t, and possesses the property of $\int_t L_n t^x = 0$ when $x < n$; suppose now that we investigate a rational function λ_n which shall possess the property $\int_t \lambda_n (\text{h. l. } t)^x = 0$ when $x < n$; then it is evident that $\int_t \lambda_n L_{n'} = 0$ when n and n' are unequal; and therefore they are reciprocal functions.

Put $\lambda_n = 1 + A_1 t + A_2 t^2 + \ldots . A_n t^n$,

$$\int_t \lambda_n (\text{h. l. } t)^x = (-1)^x . 1.2.3 \ldots . x \left\{\frac{1}{1^{x+1}} + \frac{A_1}{2^{x+1}} + \frac{A_2}{3^{x+1}} + \ldots . \frac{A_n}{(n+1)^{x+1}}\right\}.$$

Put $A_1 = 2^{n+1} B_1, \quad A_2 = 3^{n+1} B_2 \ldots\ldots A_n = (n+1)^{n+1} . B_n$;

hence we must have when $x < n$,

$$1^{n-x} + 2^{n-x} B_1 + 3^{n-x} B_2 + \ldots\ldots (n+1)^{n-x} B_n = 0.$$

Now the left-hand member of the equation is the same as

$$-\frac{1}{n+1} \frac{d^{n-x}}{dt^{n-x}} \{1 - (n+1)\epsilon^t - (n+1) B_1 \epsilon^{2t} - \ldots . - (n+1) B_n \epsilon^{(n+1).t}\},$$

putting $t = 0$ after the differentiations.

Hence the differential coefficients from the 1st to the n^{th} inclusive of the function between the brackets vanishes when $t = 0$; that function of ϵ^t ought therefore to contain no power of t inferior to the $(n+1)^{\text{th}}$, and conversely, a function of ϵ^t which does not contain such a power of t, will fulfil the required conditions.

Now this is the case with $(1-\epsilon^t)^{n+1}$, which is also when expanded of the same form as the part between the brackets; hence equating like terms, we have

$$(n+1).B_1 = -\frac{(n+1).n}{1.2} \quad (n+1).B_2 = \frac{(n+1).n(n-1)}{1.2.3} \ldots (n+1)B_{n+1} = (-1)^n.$$

Hence $A_1 = -\frac{n}{1}.2^n, \quad A_2 = \frac{n.(n-1)}{1.2}.3^n \ldots.. A_{n+1} = (-1)^n.(n+1)^n;$

and therefore $\lambda_n = 1 - \frac{n}{1}.2^n t + \frac{n.(n-1)}{1.2}.3^n t^2 - \ldots\ldots(-1)^n.(n+1)^n.t^n.$

Cor. 1. When n and n' are unequal, then $\int_t L_n \lambda_{n'} = 0$.

But when $n' = n$, we need only take the last term of L_n, namely, $(\text{h. l. } t)^n$; hence

$$\int_t \lambda_n L_n = \int_t (\text{h. l. } t)^n \left\{1 - \frac{n}{1}.2^n t + \frac{n.(n-1)}{1.2}.3^n t^2 - \&c.\right\}$$

$$= (-1)^n.1.2.3\ldots.n \left\{1 - \frac{n}{1}.\frac{1}{2} + \frac{n.(n-1)}{1.2}.\frac{1}{3} - \&c.\right\}$$

$$= \frac{(-1)^n.1.2.3\ldots.n}{n+1}.$$

Cor. 2. $\int_t \lambda_n (\text{h. l. } t)^x$

$$= (-1)^x.1.2.3\ldots.x \left\{1^{n-x-1} - n\,2^{n-x-1} + \frac{n.(n-1)}{1.2}.3^{n-x-1} - \&c.\right\}.$$

$$= (-1)^{n-x}.1.2.3\ldots.x\,\Delta^n.(h^{n-x-1}),$$

h being put $= 1$ after the operation of taking the n^{th} finite difference on the supposition that the increment of h is unity; from whence it is easy to deduce

$$\int_t \lambda_n t^x = (-1)^n \Delta^n.\frac{h^n}{h+x}.$$

Cor. 3. All the roots of the equation $\lambda_n = 0$ are real, and lie between 0 and 1.

For if we put $\text{h. l. } (t) = u$, and $\lambda_n \epsilon^u = U$,

then $\int_t \lambda_n (\text{h. l. } t)^x = \int_u U u^x = u^x \int_u U - x u^{x-1} \int_u^2 U + \frac{x.(x-1)}{1.2} \int_u^3 U$, &c.

and putting $x=0, 1, 2$, &c. successively, it follows that $\int_u^n U$ and its $(n-1)$ successive differential coefficient vanish when $u=0$ and $u=-\infty$. Hence $U=0$ has n real negative roots; and therefore $\lambda_n=0$ has n real positive and fractional roots.

15. In general let U_n, V_n be any functions of the variable t and the integer n, and let $A_1 \ldots A_n$, $a_1 \ldots a_n$ represent constant quantities; or depending on n only.

$$\text{Put} \quad T_n = U_0 + A_1 U_1 + A_2 U_2 + \ldots + A_n U_n,$$
$$\text{and} \quad T_n' = V_0 + a_1 V_1 + a_2 V_2 + \ldots + a_n V_n.$$

Then the n equations

$$\int_t T_n V_0 = 0, \quad \int_t T_n V_1 = 0, \quad \int_t T_n V_2 = 0 \ldots\ldots \int_t T_n V_{n-1} = 0,$$

will serve to determine the constants $A_1, A_2 \ldots A_n$.

In like manner let the corresponding constants $a_1, a_2 \ldots a_n$ be determined from the n equations

$$\int_t T_n' U_0 = 0, \quad \int_t T_n' U_1 = 0, \quad \int_t T_n' U_2 = 0 \ldots \int_t T_n' U_{n-1} = 0,$$

the functions T_n and T_n' which are thus determined, are reciprocal functions, and possess the general property $\int_t T_n T_{n'}' = 0$, except when $n=n'$, and then

$$\int_t T_n T_n' = a_n \int_t T_n V_n = A_n \int_t T_n' U_n;$$

this is the general principle of reciprocal functions.

COR. Let $f(t)$ be any function of t represented by the series

$$f(t) = c_0 T_0 + c_1 T_1 + c_2 T_2 \ldots \text{\&c.}$$

where c_0, c_1, c_2, &c. are constant coefficients to be determined, then multiply by T_0', T_1', T_2', &c. and integrate the successive products, and we get

$$c_0 \int_t T_0 T_0' = \int_t f(t)\, T_0',$$
$$c_1 \int_t T_1 T_1' = \int_t f(t) \,.\, T_1',$$
$$c_2 \int_t T_2 T_2' = \int_t f(t) \,.\, T_2',$$
$$\text{\&c.} \ldots\ldots\ldots\ldots \text{\&c.}$$

by means of which equations the required coefficients are given.

16. Let a_n, b_n, c_n, &c. be any functions of t, the reciprocal functions to which for simple integration are a_n', b_n', c_n', &c.

Let α_n, &c. be any function of another variable T, and let α_n', &c. represent the corresponding reciprocal function.

$$\text{Put } S_n = a_n\alpha_0 + b_n\alpha_1 + c_n\alpha_2 + \ldots\ldots$$
$$\text{and } S_n' = a_n'\alpha_0' + b_n'\alpha_1' + c_n'\alpha_2' + \ldots\ldots$$

then S_n, S_n' are general forms for reciprocal functions with respect to the double integration relative both to t and T.

For if we put m for n in the latter series, and multiply the series for S_n and S_m' together, the integral of the products of any two terms which do not hold the same place in either series when taken relative to T must vanish, since α_n, α_n' are reciprocal functions.

$$\text{Hence } \int_T S_n S_m' = a_n a_m' \int_T \alpha_0\alpha_0' + b_n b_m' \int_T \alpha_1\alpha_1' + c_n c_m' \int_T \alpha_2\alpha_2' + \ldots\ldots$$

Integrate now with respect to t, observing that when m and n are unequal, then

$$\int_t a_n a_m' = 0, \quad \int_t b_n b_m' = 0, \quad \int_t c_n c_m' = 0, \text{ \&c.}$$

Hence $\int_t \int_T S_n S_m' = 0$, when m is not equal to n,

$$\text{and } \int_t \int_T S_n S_n' = \int_t \int_T \{a_n a_n' \alpha_0\alpha_0' + b_n b_n' \alpha_1\alpha_1' + c_n c_n' \alpha_2\alpha_2' + \ldots\}.$$

Cor. 1. In the same manner reciprocal functions of any number of independent variables may be formed.

Cor. 2. The equation $S_n = 0$ has n real roots or values of t lying between 0 and 1, whatever value be assigned to T, when a_n, b_n, c_n, &c. are functions possessing the property $\int_t a_n t^x = 0$, &c. x being any integer from 0 to $n-1$ inclusive; for then a_n must be of the form $\dfrac{d^n(t^n t'^n V)}{dt^n}$, by Art. 5, and similarly

$$b_n = \frac{d^n.(t^n t'^n V')}{dt^n}, \quad c_n = \frac{d^n.(t^n t'^n V'')}{dt^n};$$

and therefore

$$S_n = \frac{d^n(t^n t'^n U)}{dt^n} \quad \text{where } U = \alpha_0 V + \alpha_1 V' + \alpha_2 V'' + \text{\&c.}$$

Hence $S_n = 0$ must have n real roots between 0 and 1. (Art. 6.)

17. If it is necessary that the terms which compose the reciprocal functions S_n, S_n' should follow a simple law, it will be most convenient to get first two reciprocal functions of t, as R_n, R_n', which may contain an arbitrary constant r, and to put for a_n, b_n, c_n &c. the values acquired by R_n when $r=0$, 1, 2, &c.; and similarly for a_n', b_n', c_n', &c. the corresponding values of R'_n.

EXAMPLE:

Thus, put $R_n = (tt')^{\frac{r}{2}-\alpha}\dfrac{d^r P_n}{dt^r}$, and $R_n' = (tt')^{\frac{r}{2}+\alpha}\dfrac{d^r P_n}{dt^r}$, P_n being the function so denominated in Art. 3, namely, $\dfrac{d^n (tt')^n}{1.2\ldots n dt^n}$; then, integrating by parts, we have

$$\int_t R_n R_m' = (tt')^r \frac{d^r P_m}{dt^r} \cdot \frac{d^{r-1} P_n}{dt^{r-1}} - \int_t \frac{d^{r-1} P_n}{dt^{r-1}} \cdot \frac{d}{dt}\left(t^r t'^r \frac{d^r P_m}{dt^r}\right),$$

the part outside the sign of integration vanishes between the limits of t, and repeating the same operation any number of times, the part outside the sign of integration is evidently of the form

$$\frac{d^{r-k} P_n}{dt^{r-k}} \cdot \frac{d^{k-1}}{dt^{k-1}}\left(t^r t'^r \frac{d^r P_m}{dt^r}\right),$$

the latter differential coefficient will vanish between limits when k is any number from 0 to r inclusive, because it will always contain the factor $(tt')^{r-k+1}$; also when n and m are unequal we may suppose n to be the greater, and since $t^r t'^r \dfrac{d^r P_m}{dt^r}$ is of $m+r$ dimensions, it follows that if $k > n+r$, then $k-1 > m+r$; and consequently the latter differential coefficient will be identically zero.

The only instance in which the factor $\dfrac{d^{k-1}}{dt^{k-1}}\left(t^r t'^r \dfrac{d^r P_m}{dt^r}\right)$ does not vanish between limits is, therefore, where k lies between $r+1$ and $r+n$ inclusive, but then the first factor is changed to $\int_t^{k-r} P_n$; and since $k-r$ is now some number from 1 to n inclusive, this factor vanishes between limits (vid. Art. 5.), and therefore the part outside the sign of integration vanishes in all cases, and we thus obtain

$$\int_t R_m R_n' = (-1)^k \int_t \frac{d^{r-k} P_n}{dt^{r-k}} \cdot \frac{d^k}{dt^k}\left(t^r t'^r \frac{d^r P_m}{dt^r}\right).$$

Put now $k=r$, the first factor under the sign of integration becomes simply P_n, and the second factor is then of m dimensions; and therefore, by the nature of P_n, the integral vanishes; and therefore, when $n>m$, $\int_t R_n R_m'=0$: and the same reasoning applies when $m>n$, only substituting R_m' instead of R_m throughout the process, hence R_n and R_m' are reciprocal functions.

When $m=n$, then in the general expression

$$\int_t R_n R_n' = (-1)^r \int_t P_n \frac{d^r}{dt^r}\left(t^r t'^r \frac{d^r P_n}{dt^r}\right);$$

we need only take the term involving the highest power of t in

$$\frac{d^r}{dt^r}\left(t^r t'^r \frac{d^r P_n}{dt^r}\right),$$

namely,

$$(-1)^{n+r}\frac{(n+1).(n+2)\dots(2n)}{1\;.\;2\;\dots\;n}\,\frac{d^r}{dt^r}\left(t^{2r}\frac{d^r.t^n}{dt^r}\right)$$

$$=(-1)^{n+r}\frac{(n+1).(n+2)\dots 2n}{1.2\dots(n-r+1)}.(n+r).(n+r-1)\dots(n+1).\,t^n,$$

and observing that $\int_t P_n t^n = (-1)^n . \dfrac{1.2\dots.n}{(n+1).(n+2)\dots(2n+1)}$;

it follows that $\int_t R_n R_n' = \dfrac{1}{2n+1}.(n+r)(n+r-1)(n+r-2)\dots(n-r)$.

The reciprocal functions a_n, a_n' may be obtained by putting $r=0$ in R_n and R_n'; similarly, if we put $r=1$, we get b_n, b_n', &c., and thence we obtain the reciprocal functions relative to double integration, namely,

$$S_n = a_0(tt')^{-\alpha}.P_n + a_1(tt')^{\frac{1}{2}-\alpha}\frac{dP_n}{dt} + a_2(tt')^{1-\alpha}\frac{d^2P_n}{dt^2} + a_3(tt')^{\frac{3}{2}-\alpha}\frac{d^3P_n}{dt^3}, \text{ \&c.}$$

$$S_n' = a_0'(tt')^{\alpha}\;P_n + a_1'(tt')^{\frac{1}{2}+\alpha}\frac{dP_n}{dt} + a_2'(tt')^{1+\alpha}\frac{d^2P_n}{dt^2} + a_3'(tt')^{\frac{3}{2}-\alpha}\frac{d^3P_n}{dt^3}, \text{ \&c.}$$

In the same manner if we vary the constant a while r remains constant, we obtain the reciprocal functions

$$\sigma_n = (tt')^{\frac{r}{2}}\frac{d^rP_n}{dt^r}\{a_0(tt')^{-\alpha} + a_1(tt')^{-\beta} + a_2(tt')^{-\gamma} + \text{\&c.}\}$$

$$\sigma_n' = (tt')^{\frac{r}{2}}.\frac{d^rP_n}{dt^r}\{a_0'(tt')^{\alpha} + a_1'(tt')^{\beta} + a_2'(tt')^{\gamma} + \dots\}.$$

COR. 1. The simplest form for a_n is the sine or cosine of the n^{th} multiple of an arc of which the limits are 0 and $2n\pi$, as

$$A_n \sin(2n\pi T) + B_n \cos(2n\pi T),$$

where A_n, B_n are arbitrary constants, then we have (putting for simplicity $a = 0$),

$$S_n = A_0 P_n + (A_1 \sin 2\pi\tau + B_1 \cos 2\pi\tau)\frac{dP_n}{dt}$$
$$+ (A_2 \sin 4\pi\tau + B_2 \cos 4\pi\tau)\frac{d^2 P_n}{dt^2},$$

this is the most general form for all the reciprocal functions which occur in the *Mécanique Céleste.* (Vid. Prop. XI. Treatise on Electricity.)

COR. 2. If T_n, T_n' are arbitrary functions of t, which do not become infinite when $t=0$ or 1, then, putting

$$R_n = (tt')^{\frac{r}{2}}\, T_r \frac{d^r P_n}{dt^r}, \text{ and } R_n' = (tt')^{\frac{r}{2}}\, T_r' \cdot \frac{d^r P_n}{dt^r},$$

the same reasoning as that used in the preceding example will show that R_n, R_n' are reciprocal functions, and thus we get for S_n, S_n' the very general forms

$$S_n = a_0 T_0 P_n + a_1 T_1 \frac{dP_n}{dt}(tt')^{\frac{1}{2}} + a_2 T_2 \frac{d^2 P_n}{dt^2}(tt') + a_3 T_3 \frac{d^3 P_n}{dt^3}(tt')^{\frac{3}{2}} + \&c.$$

$$S_n' = a_0' T_0' P_n + a_1' T_1' \frac{dP_n}{dt}(tt')^{\frac{1}{2}} + a_2' T_2' \frac{d^2 P_n}{dt^2}(tt') + a_3' T_3' \frac{d^3 P_n}{dt^3}(tt')^{\frac{3}{2}} + \&c.$$

COR. 3. If $f(t, \tau)$ is any function of the variables t, τ, which is expanded under the form

$$f(t, \tau) = a_0 S_0 + a_1 S_1 + a_2 S_2 + \ldots\ldots$$

then, to determine the coefficients a_0, a_1, a_2, &c., multiply successively by S_0', S_1', S_2'.... and integrate from $t=0$ to $t=1$, and from $\tau=0$ to $\tau=1$: we thus get

$$a_0 \int_t \int_\tau S_0 S_0' = \int_t \int_\tau S_0' f(t, \tau),$$
$$a_1 \int_t \int_\tau S_1 S_1' = \int_t \int_\tau S_1' f(t, \tau),$$
$$a_2 \int_t \int_\tau S_2 S_2' = \int_t \int_\tau S_2' f(t, \tau);$$

from whence the required coefficients are known.

SECTION V.

Inverse Method for Definite Integrals which are expressed in positive powers of x, or under any form.

18. Let $\phi(x)$ represent any function of x, such that $\int_t f(t) . t^x = \phi(x)$ when x is any integer from 0 to $n-1$ inclusive, then excluding the case of $\phi(x) = 0$, which has been considered in the preceding Section, it is evident that by putting

$$f(t) = A_0 + A_1 t + A_2 t^2 + \dots\dots + A_{n-1} t^{n-1},$$

the conditions of the question give n equations, which suffice to determine the coefficients A_0, A_1, $A_2, \dots\dots A_{n-1}$; if we represent the particular value of $f(t)$ thus deduced by T_{n-1}, and seek its most general value, we have

$$\int_t f(t) . t^x = \phi(x),$$
$$\int_t T_{n-1} . t^x = \phi(x),$$
$$\therefore \int_t \{f(t) - T_{n-1}\} . t^x = 0.$$

Hence by the preceding Section, the most general value of $f(t) - T_{n-1}$ is

$$\frac{d^n (t^n t'^n V)}{dt^n},$$

and therefore the most general value of $f(t)$ is found by adding this appendage to its prime value T_{n-1}.

19. When $\phi(x)$ is a rational and entire function of x, of m dimensions, we have by the proposed conditions

$$\phi(x) = \frac{A_0}{x+1} + \frac{A_1}{x+2} + \frac{A_2}{x+3} + \dots\dots \frac{A_{n-1}}{x+n},$$

and actually adding the terms which compose the right-hand member of this equation, the common denominator is

$$(x+1)(x+2)\dots\dots(x+n),$$

and the numerator will be a function of $n-1$ dimensions, represented by v_n, so that

$$\phi(x) = \frac{v_n}{(x+1)(x+2)\ldots\ldots(x+n)},$$

when x is any integer from 0 to $(n-1)$ inclusive; and if we multiply by $x+1$ and put $x=-1$, and again by $x+2$ and put $x=-2$, &c. as in the preceding Section, we get

$$A_0 = \frac{v_{-1}}{1.2.3\ldots.(n-1)},$$

$$A_1 = -\frac{n-1}{1} \cdot \frac{v_{-2}}{1.2.3\ldots.(n-1)},$$

$$A_2 = \frac{(n-1)(n-2)}{1.2} \cdot \frac{v_{-3}}{1.2.3\ldots.(n-1)},$$

&c. = ...&c.

Now the equation

$$\phi(x).(x+1)(x+2)\ldots\ldots(x+n) - v_n = 0,$$

is of $m+n$ dimensions, and is by hypothesis satisfied, when

$$x = 0,\ 1,\ 2,\ldots\ldots(n-1);$$

therefore if u_x represent some function of x of m dimensions, we must have the identity

$$\phi(x).(x+1)(x+2)\ldots\ldots(x+n) - v_n = u_x.x.(x-1)(x-2)\ldots\ldots(x-n+1),$$

hence if we divide

$$\phi(x)(x+1)(x+2)\ldots\ldots(x+n) \text{ by } x(x-1)(x-2)\ldots\ldots(x-n+1),$$

and retain only the part of the quotient which is an entire function of x, u_x will be completely determined.

Put now $-1, -2, \ldots -n$ successively for x in the preceding identity, and we get

$$v_{-1} = (-1)^{n+1}.1.2.3\ldots.n.u_{-1},$$

$$v_{-2} = (-1)^{n+1} . 1 . 2 . 3 .... n . \frac{n+1}{1} . u_{-2},$$

$$v_{-3} = (-1)^{n+1} . 1 . 2 . 3 .... n . \frac{(n+1)(n+2)}{1 . 2} . u_{-3},$$

$$\&c. = ...\&c..........$$

from whence the values of A_0, A_1, A_2, &c. are known, and being substituted, give

$$T_{n-1} = (-1)^{n-1}\left\{nu_{-1} - \frac{n.(n+1)}{1} . \frac{(n-1)}{1} . u_{-2}t \right.$$

$$\left. + \frac{n.(n+1)(n+2)}{1 . 2} . \frac{(n-1)(n-2)}{1 . 2} . u_{-3}t^2 - \&c.\right\}.$$

EXAMPLE:

Let $\phi(x) = 1$, then $u_x = 1$, and therefore

$$T_{n-1} = (-1)^{n-1}\left\{n - \frac{n.(n+1)}{1} . \frac{(n-1)}{1} . t \right.$$

$$\left. + \frac{n.(n+1)(n+2)}{1 . 2} . \frac{(n-1)(n-2)}{1 . 2} . t^2 - \&c.\right\}.$$

20. The function T_{n-1} possesses a property analogous to the characteristic property of those in the former Section, that is, *the equation* $T_{n-1} = 0$ *admits of* $n - m - 1$ *roots between* 0 *and* 1, and consequently vanishes an indefinitely great number of times between the limits $t = 0$ and $t = 1$ when n is taken indefinitely great.

For since $T_{n-1} = (-1)^{n-1}\left\{nu_{-1} - \frac{n(n+1)}{1} . \frac{n-1}{1} . u_{-2}t\right.$

$$\left. + \frac{n(n+1)(n+2)}{1 . 2} . \frac{(n-1)(n-2)}{1 . 2} . u_{-3}t^2, \&c.\right\}$$

$$= \frac{(-1)^{n-1}}{1 . 2 . 3 ....(n-1)} . \frac{d^n}{dt^n}\left\{t^n\left(u_{-1} - \frac{n-1}{1} . u_{-2} . t + \frac{(n-1).(n-2)}{1 . 2} . u_{-3}t^2 - \&c.\right)\right\}$$

$$= \frac{1}{1 . 2 . 3 ....(n-1)} . \frac{d^n}{dt^n}\{t^n \Delta^{n-1}u_{-x}t^{x-1}\},$$

the operation Δ being performed on the supposition that the finite increment of x is unity, and x being put $=1$ after the operation Δ^{n-1} has been performed.

Put $t=1-t'$, and therefore,

$$\Delta^{n-1}(u_{-x}t^{x-1}) = \Delta^{n-1}u_{-x} - t'\,\Delta^{n-1}u_{-x}(x-1) + \frac{t'^2}{1\,.\,2}\,\Delta^{n-1}u_{-x}(x-1)(x-2) - \&c.$$

and since u_{-x} is of m dimensions, the first term of this series which does not vanish is

$$\pm\frac{t'^{n-m-1}}{1\,.\,2\ldots.(n-m-1)}\,.\,\Delta^{n-1}u_{-x}(x-1)(x-2)\ldots.(x-n+m+1),$$

and therefore the whole expansion is of the form

$$t'^{n-m-1}V\,.\,1\,.\,2\,.\,3\ldots\ldots(n-1),$$

which being substituted gives

$$T_{n-1} = \frac{d^n\{t^n t'^{n-m-1}V\}}{dt^n},$$

and since the equation $t^n t'^{n-m-1}V=0$ has at least $2n-m-1$ real roots, viz. n of them $=0$, and $n-m-1$ of them $=1$, it follows that the n^{th} derived equation $T_n=0$ has $n-m-1$ real roots lying between 0 and 1.

COR. Since $T_{n-1} = \dfrac{1}{1\,.\,2\,.\,3\ldots\ldots(n-1)}\,.\,\dfrac{d^n}{dt^n}\{t^n\Delta^{n-1}u_{-x}t^{x-1}\}$,

if we actually differentiate we get

$$T_{n-1} = \frac{1}{1\,.\,2\,.\,3\ldots.(n-1)}\,.\,\Delta^{n-1}\{x\,.\,(x+1)\ldots.(x+n-1)\,u_{-x}t^{x-1}\}.$$

21. Let now $\phi(x)$ be any function whatever, and let it be required, in general, to find $f(t)$, so that $\int_t f(t)\,.\,t^x = \phi(x)$, provided x be any integer from 0 to $n-1$ inclusive.

It has been shewn in Art. 18, that a function T_{n-1} of $n-1$ dimensions may always be found to satisfy the imposed conditions, and for the most general value of $f(t)$ we shall then have

$$f(t) = T_{n-1} + \frac{d^n(t^n t'^n V)}{dt^n}.$$

Now T_{n-1} contains only n constants, being of $n-1$ dimensions, and therefore if we denote by P_n the same quantity as in the preceding Section, namely the coefficient of h^n in

$$\{1 - 2h(1-2t) + h^2\}^{-\frac{1}{2}},$$

we may put

$$T_{n-1} = a_0 P_0 + a_1 P_1 + a_2 P_2 + \ldots\ldots + a_{n-1} P_{n-1},$$

the right-hand member being of the same dimensions with the left, and containing the same number of constants.

Now by the properties of P_n we have $\int_t P_m P_n = 0$, when m and n are unequal, and

$$\int_t P_n P_n = \frac{1}{2n+1}.$$

$$\text{Hence we have } \int_t P_0 T_{n-1} = a_0,$$

$$\int_t P_1 T_{n-1} = \frac{a_1}{3},$$

$$\int_t P_2 T_{n-1} = \frac{a_2}{5}.$$

But by the conditions of the question,

$$\int_t T_{n-1} \,.\, t^x = \phi(x),$$

x being any integer less than n.

Hence

$$\int_t P_0 T_{n-1} = \int_t T_{n-1} = \phi(0) = \phi(h) \text{ when } h \text{ is put } = 0,$$

$$\int_t P_1 T_{n-1} = \int_t T_{n-1}(1-2t) = \phi(0) - 2\phi(1) = -\Delta \frac{(h+1)}{1} \,.\, \phi(h),$$

$$\int_t P_2 T_{n-1} = \int_t T_{n-1}(1-6t+6t^2) = \phi(0) - 6\phi(1) + 6\phi(2) = \Delta^2 \,.\, \frac{(h+1)(h+2)}{1\,.\,2} \phi(h),$$

and generally

$$\textstyle\int_t P_m T_{n-1} = \int_t T_{n-1} \left\{1 - \frac{m}{1} \cdot \frac{m+1}{1} \cdot t + \frac{m \cdot (m-1)}{1 \cdot 2} \cdot \frac{(m+1)(m+2)}{1 \cdot 2} \cdot t^2 - \&c.\right\}$$

$$= \phi(0) - \frac{m}{1} \cdot \frac{m+1}{1} \cdot \phi(1) + \frac{m \cdot (m-1)}{1 \cdot 2} \cdot \frac{(m+1) \cdot (m+2)}{1 \cdot 2} \cdot \phi(2) - \&c.$$

$$= (-1)^m \Delta^m \cdot \frac{(h+1)(h+2) \ldots\ldots (h+m)}{1 \cdot 2 \ldots\ldots m} \cdot \phi(h). \quad \text{When } h \text{ is put } = 0.$$

and by comparing the former integrals with the latter, the values of a_0, a_1, a_2, &c. are known, and being substituted give

$$T_{n-1} = P_0 \cdot \phi(h) - 3P_1 \Delta \frac{h+1}{1} \cdot \phi(h) + 5P_2 \Delta^2 \cdot \frac{(h+1) \cdot (h+2)}{1 \cdot 2} \cdot \phi(h)$$

$$- 7P_3 \Delta^3 \frac{(h+1) \cdot (h+2) \cdot (h+3)}{1 \cdot 2 \cdot 3} \cdot \phi h + \&c.$$

h being put $=0$, after the operations are performed.

It should be observed here that the terms of this expansion are perfectly independant of n, which only fixes the number of the terms; hence this series may be continued to any number of terms, and we shall always have $\int_t T_{n-1} t^x = \phi(x)$ provided x is any integer less than that number, and consequently if the series be continued *ad infinitum*, the equation will be true for all integer and positive values of x.

COR. Multiply both sides by t^x and integrate from $t=0$ to $t=1$,

$$\text{hence } \phi(x) = \frac{1}{x+1} \cdot \phi(h) + 3 \cdot \frac{x}{(x+1)(x+2)} \Delta \frac{h+1}{1} \cdot \phi h$$

$$+ 5 \cdot \frac{x \cdot (x-1)}{(x+1)(x+2)(x+3)} \Delta^2 \frac{(h+1) \cdot (h+2)}{1 \cdot 2} \phi h + \&c.$$

when h is put $=0$.

This series may be used, not only for the integer and positive values of x, but for any values which will not render it divergent. (Vid. First Memoir, 'On the Inverse method of Definite Integrals,' Art. 2.)

22. When $\phi(x)$ is given we may obtain $f(t)$ in an infinite variety of forms by means of the theory of reciprocal functions given in the preceding Section. For instance, if we denote by S_m the sum of the products of the natural numbers 1, 2, 3......n when taken m and m together, and put

$$L_n = 1 + S_1 . \text{h. l. } t + \frac{S_2}{1.2} . (\text{h. l. } t)^2 + \frac{S_3}{1.2.3} . (\text{h. l. } t)^3 + ... + \frac{S_n}{1.2....n} . (\text{h. l. } t)^n$$

$$= \frac{d^n \{t^n (\text{h. l. } t)^n\}}{1.2....n dt^n}. \quad (\text{Art. 8. Section IV.})$$

$$\text{and } \lambda_n = 1 - \frac{n}{1} . 2^n t + \frac{n.(n-1)}{1.2} . 3^n t^2 - ...... \pm (n+1)^n t^n$$

$$= (-1)^n \Delta^n \{(h+1)^n t^h\}, \text{ when } h \text{ is put } = 0,$$

then L_n and λ_n are reciprocal functions. (Sect. IV. Art. 14.)

$$\text{Put therefore } f(t) = a_0 L_0 + a_1 L_1 + a_2 L_2 + a_3 L_3 + \&\text{c.}$$

and observing that

$$\textstyle\int_t \lambda_n L_n = (-1)^n . \frac{1.2.3....n}{n+1},$$

$$\text{we have } a_n = (-1)^n . \frac{n+1}{1.2.3....n} . \textstyle\int_t f(t) . \lambda_n.$$

But $\int_t f(t) . \lambda_n = \int_t f(t) . (-1)^n . \Delta^n . (h+1)^n . t^h$. When h is put $=0$,

$$= (-1)^n \Delta^n (h+1)^n \textstyle\int_t f(t) . t^h$$

$$= (-1)^n . \Delta^n . (h+1)^n . \phi(h), \quad \text{since } \textstyle\int_t f(t) . t^x = \phi(x).$$

$$\text{Hence } a_n = \frac{n+1}{1.2.3....n} . \Delta^n . (h+1)^n . \phi(h),$$

and therefore

$$f(t) = L_0 \phi(h) + 2 L_1 \frac{\Delta.(h+1).\phi h}{1} + 3 L_2 \frac{\Delta^2 (h+1)^2 \phi h}{1.2} + 5 L_3 . \frac{\Delta^3.(h+1)^3 \phi h}{1.2.3}, \&\text{c.}$$

which series when convergent will satisfy the equation $\int_t f(t) \cdot t^x = \phi(x)$ for all values of x; but even if not convergent, it will satisfy that equation for all the integer values of x from 0 to $n-1$ inclusive, provided it be continued for at least n terms.

If we multiply by t^x and integrate as before, we get

$$\phi(x) = \frac{1}{x+1} \cdot \phi(h) + \frac{2 \cdot x}{(x+1)^2} \cdot \Delta(h+1) \cdot \phi h + \frac{3 \cdot x(x-1)}{(x+1)^3} \cdot \frac{\Delta^2 (h+1)^2 \phi h}{1 \cdot 2} + \&c.$$

which series when convergent may be used for any value of x, but only positive and integer values when divergent.

23. In Art. 21. when $\int_t f(t) \cdot t^x = \phi(x)$ a given function of x, we have found $f(t)$ in a series expressed by functions of t of the same nature as P_n, now P_n is only a particular value of the general function $(p, q)_n$ investigated in the former Section, Art. 12., namely, when $p = q = 1$; we shall now express $f(t)$ according to this more general class of functions, that is, under the form

$$f(t) = a_0 (p, q)_0 + a_1 (p, q)_1 + a_2 (p, q)_2 + \&c.$$

Now in Art. 12. above referred to, we have found

$$(p, q)_m = 1 - \frac{(p+1)(p+1+q) \dots \{p+1+(m-1) \cdot q\}}{1(1+q) \dots \{1+(m-1) \cdot q\}} \frac{m}{1} \cdot t^p$$

$$+ \frac{(2p+1)(2p+1+q) \dots \{2p+1+(m-1) \cdot q\}}{1 \cdot (1+q) \dots \{1+(m-1) \cdot q\}} \cdot \frac{m \cdot (m-1)}{1 \cdot 2} \cdot t^{2p} - \&c.$$

To simplify this expression, put

$$H_{p,q} = \frac{(ph+1)(ph+1+q) \dots \{ph+1+(m-1) \cdot q\}}{1(1+q) \dots \{1+(m-1) \cdot q\}}.$$

Let ψ express the *operation* of changing h into $h+1$ (Vid. former Memoir, Note B. 2.), ψ^2 the repetition of this operation a second time, &c.; the preceding series will then become

$$(p, q)_m = H_{p,q} t^{ph} - \frac{m}{1} \cdot \psi H_{p,q} t^{ph} + \frac{m \cdot (m-1)}{1 \cdot 2} \psi^2 H_{p,q} t^{ph}$$

$$- \frac{m \cdot (m-1) \cdot (m-2)}{1 \cdot 2 \cdot 3} \cdot \psi^3 H_{p,q} t^{ph} + \&c.$$

on the supposition that we put $h=0$ after the operations above indicated, are performed.

Separate in this expression the symbols of operation and of quantity, and we shall obtain the equation

$$(p, q)_m = (1 - \psi)^m \cdot H_{p,q} t^{ph}.$$

But $\psi - 1$ or $\psi - \psi^0$ indicates that we must subtract the original value of $H_{p,q}$, from the value it receives when $h+1$ is put for h, that is, it is the same as performing the operation Δ of finite differences; this consideration transforms the preceding equation, to this

$$(p, q)_m = (-1)^m \Delta^m \cdot H_{p,q} t^{ph}, \text{ when } h \text{ is put } = 0.$$

In like manner if we put

$$H_{q,p} = \frac{(qh+1)(qh+1+p) \ldots\ldots \{qh+1+(m-1) \cdot p\}}{1 (1+p) \ldots\ldots \{1 + (m-1) \cdot p\}},$$

we have $(q, p)_m = (-1)^m \Delta^m H_{q,p} t^{qh}$, when $h=0$.

Now observing that by the nature of reciprocal functions we have

$$\textstyle\int_t (p, q)_m (q, p)_n = 0, \text{ except when } m = n,$$

and by Art. 13., $\int_t (p, q)_m (q, p)_m$

$$= \frac{(p, q)^m}{1+m(p+q)} \cdot \frac{1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \ldots\ldots m \cdot m}{1 \cdot 1 \cdot (1+p)(1+q)(1+2p)(1+2q) \ldots \{1+(m-1) \cdot p\}\{1+(m-1) \cdot q\}},$$

then since $f(t) = a_0 (p, q)_0 + a_1 (p, q)_1 + a_2 (p, q)_2 + \&c.$

we have $\int_t f(t) \cdot (q, p)_m$

$$= a_m \cdot \frac{(pq)^m}{1+m(p+q)} \cdot \frac{1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \ldots\ldots m \cdot m}{1 \cdot 1 (1+p)(1+q) \ldots\ldots \{1+(m-1) \cdot p\}\{1+(m-1) \cdot q\}}.$$

But if we put for $(q, p)_m$ the value above found, and observe that the operations Δ and $\int_t$ are with respect to different variables h and t, and therefore their order is transmutable, we have also,

$$\int_t f(t) \cdot (q, p)_m = (-1)^m \Delta^m H_{q,p} \cdot \int_t f(t) \cdot t^{qh}$$

$$= (-1)^m \Delta^m H_{q,p} \phi(qh), \text{ by hypothesis.}$$

Comparing this value of the integral with that already found, we get

$$a_m = (-1)^m \frac{1+m(p+q)}{(pq)^m} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1+p}{2} \cdot \frac{1+q}{2} \cdot \frac{1+2p}{3} \cdot \frac{1+2q}{3} \cdots$$

$$\cdots \frac{1+(m-1) \cdot p}{m} \cdot \frac{1+(m-1) \cdot q}{m}$$

$$\times \Delta^m H_{q,p} \phi(qh), \text{ when } h = 0,$$

from whence we have finally

$$f(t) = (p, q)_0 \phi(qh) - (p, q)_1 \cdot \frac{1+p+q}{pq} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \Delta H'_{q,p} \phi(qh)$$

$$+ (p, q)_2 \cdot \frac{1+2(p+q)}{(pq)^2} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1+p}{2} \cdot \frac{1+q}{2} \cdot \Delta^2 H''_{q,p} \phi(qh)$$

$$- (p, q)_0 \cdot \frac{1+3(p+q)}{(pq)^3} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1+p}{2} \cdot \frac{1+q}{2} \cdot \frac{1+2p}{3} \cdot \frac{1+2q}{3} \cdot \Delta^3 H'''_{q,p} \phi(qh)$$

+ &c............................&c.

h being put $= 0$, after the operation, and H', H'', H''', &c. being the values of $H_{p,q}$ when $m = 1, 2, 3$, &c. successively.

COR. 1. Multiply by t^x, and then integrate from $t=0$ to $t=1$; for $\int_t f(t) \cdot t^x$ put its value $\phi(x)$, and for $\int_t (p, q)_m t^x$ its value

$$(-1)^m p^m \cdot \frac{1 \cdot 2 \cdot 3 \ldots m}{1(1+q)(1+2q)\ldots\{1+(m-1\} \cdot q)} \cdot \frac{x(x-q)\ldots\{x-(m-1) \cdot q\}}{(x+1)(x+p+1)\ldots(x+mp+1)},$$

by Art. 12; and lastly, put for $H_{q,p}$ its value

$$\frac{(qh+1)(qh+1+p)\ldots.\{qh+1+(m-1).p\}}{1.(1+p)\ldots\{1+(m-1).p\}};$$

we thus obtain

$$\phi(x)=\frac{1}{x+1}.\phi(qh)+\frac{x}{(x+1)(x+p+1)}\,\frac{1+p+q}{q}\Delta(qh+1)\,\phi(qh)$$

$$+\frac{x(x-q)}{(x+1)(x+p+1)(x+2p+1)}\,\frac{1+2(p+q)}{2q^2}\Delta^2(qh+1)(qh+p+1)\,\phi(qh)$$

$$+\frac{x(x-q)(x-2q)}{(x+1)(x+p+1)(x+2p+1)(2+3p+1)}$$

$$\times\frac{1+3(p+q)}{1.2.3q^3}\Delta^3(qh+1)(qh+p+1)(qh+2p+1)\,\phi(qh)$$

$+$ &c. when h is put $=0$,

and where p and q are perfectly arbitrary.

COR. 2. Put $p=q=0$, and make

$$A_n=\phi(0)+n\phi'(0)+\frac{n.(n-1)}{1.2}.\frac{\phi''(0)}{1.2}+\&\text{c.}$$

where $\phi(0)$, $\phi'(0)$, $\phi''(0)$, and the values of $\phi(x)$ and its successive differential coefficients when $x=0$, and the above expansion will become

$$\phi(x)=A_0.\frac{1}{x+1}+A_1.\frac{x}{(x+1)^2}+A_2.\frac{x^2}{(x+1)^3}+\&\text{c.}$$

If, moreover, we put

$$T_n=1+\frac{n}{1}.\text{h. l. }t+\frac{n.(n-1)}{1.2}.\frac{(\text{h. l. }t)^2}{1.2}+\&\text{c.}$$

which is the same as A_n when we put t^x for $\phi(x)$, then it is easily seen by the principles of the first Memoir, that $\int_t T_n t^x=\frac{x^n}{(x+1)^{n+1}}$, and since we have also $\int_t f(t).t^x=\phi(x)$, it follows that

$$f(t)=A_0T_0+A_1T_1+A_2T_2+\&\text{c.}$$

24. The functions which have been all along designated by $(p, q)_n$ and $(q, p)_n$, have been already shewn to be reciprocal one to the other; putting $p=q$, the resulting function $(p, p)_n$ must be reciprocal to itself; that is, $\int_t (p, p)_n (p, p)_m = 0$ when m and n are unequal positive integers; when $p=1$ the function $(p, p)_n$ is then identical with that denoted by P_n, which has been before shewn to be reciprocal to itself; again, the function T_n or

$$1 + \frac{n}{1}.\text{h. l. } t + \frac{n.(n-1)}{1.2}.\frac{(\text{h. l. } t)^2}{1.2} + \frac{n.(n-1).(n-2)}{1.2.3}.\frac{(\text{h. l. } t)^3}{1.2.3} + \&\text{c.}$$

is reciprocal to itself, for if we multiply by $(\text{h. l. } t)^m$, and integrate, we get

$$\int_t T_n (\text{h. l. } t)^m = 1.2.3...m(-1)^m \left\{1 - \frac{n}{1}.\frac{m+1}{1} \div \frac{n.(n-1)}{1.2}\ \frac{(m+1).(m+2)}{1.2} - \&\text{c.}\right\}.$$

The expression between the brackets is the term independent of h in the product $(1+h)^n \left(1+\frac{1}{h}\right)^{-(m+1)}$, or the coefficient of $h^{-(m+1)}$ in $(1+h)^{n-m-1}$; it is therefore zero when $n>m$, but when $n=m$ its value is $(-1)^m$, and when $n<m$, its value is

$$(-1)^m.\frac{(m+1-n)(m+2-n)...m}{1.2...n}.$$

Hence $\int_t T_n T_m = 0$, when m and n are unequal, and

$$\int_t T_n T_n = \int_t T_n.\frac{(\text{h. l. } t)^n}{1.2...n} = 1.$$

25. Put $\text{h. l.}(t) = \tau$, and substituting in T_n, we have

$$1.2...n\, T_n \epsilon^\tau = \epsilon^\tau \left\{1.2...n + n.2.3...n\tau + \frac{n.(n-1)}{1.2}.3.4...n\tau^2 + \&\text{c.}\right\}$$

$$= \epsilon^\tau \left\{\frac{d^n \tau^n}{d\tau^n} + n.\frac{d^{n-1}\tau^n}{d\tau^{n-1}} + \frac{n.(n-1)}{1.2}.\frac{d^{n-2}\tau^n}{d\tau^{n-2}} + \&\text{c.}\right\}$$

$$= \frac{d^n(\epsilon^\tau \tau^n)}{d\tau^n}.$$

Hence $$T_n = \frac{\epsilon^{-\tau} d^n (\epsilon^\tau \tau^n)}{1.2...n\, d\tau^n}.$$

From this formulæ it appears that the equation $T_n = 0$ has n real values of τ all negative; and therefore n corresponding values of t, which are all included between 0 and 1.

Moreover, if we form the equation

$$u = \tau + hu, \qquad \text{or } u = \frac{\tau}{1-h},$$

it follows by the theorem of Lagrange, that T_n is the coefficient of h^n in $\epsilon^{-\tau}.\dfrac{d\epsilon^u}{d\tau}$, that is, in $\dfrac{\epsilon^{\frac{h\tau}{1-h}}}{1-h}$, and putting t for ϵ^τ, T_n is clearly the coefficient of h^n in the expansion of the function $\dfrac{t^{\frac{h}{1-h}}}{1-h}$.

Conversely, we may now prove that the coefficient of h^n in the expansion of $\dfrac{t^{\frac{h}{1-h}}}{1-h}$ is a reciprocal function; for when $h=0$, this function is reduced to unity, we may therefore put generally

$$\frac{t^{\frac{h}{1-h}}}{1-h} = T_0 + T_1 h + T_2 h^2 + \text{\&c. where } T_0 = 1.$$

Let h' represent any other arbitrary quantity, and we have

$$\frac{t^{\frac{h'}{1-h'}}}{1-h'} = T_0 + T_1 h' + T_2 h'^2 + \text{\&c.}$$

Multiply both series term by term and integrate, the result in the left-hand members is

$$\frac{1}{(1-h)(1-h')} \int_t t^{\frac{h}{1-h} + \frac{h'}{1-h'}} = \frac{1}{1-hh'},$$

which expanded becomes $1 + hh' + h^2h'^2 + \text{\&c.}$; which being identical with the integral of the product of the right-hand members, will necessarily require that the integrals of those terms which are not in corresponding places in both series must vanish, and the integrals of the products of the corresponding coefficients to be unity, which are the same properties that have been demonstrated in Art. 24.

COR. Put $\dfrac{h}{1-h} = x$, and the series

$$t^{\frac{h}{1-h}} = (1-h)\left\{T_0 + T_1 h + T_2 h^2 + \&c.\right\} \text{ becomes}$$

$$t^x = \frac{T_0}{x+1} + T_1 \cdot \frac{x}{(x+1)^2} + T_2 \cdot \frac{x^2}{(x+1)^3} + \&c.$$

The principles which have been used in this Section to obtain expansions such as the preceding by means of reciprocal functions relative to simple integration, will apply with equal simplicity to reciprocal functions relative to any number of integrations.

R. MURPHY.

CAIUS COLLEGE,
Dec. 18, 1833.

VII. *On the Nature of the Truth of the Laws of Motion.* *By the Rev.* W. Whewell, M.A. *Fellow and Tutor of Trinity College.*

[Read *Feb.* 17, 1834.]

1. The long continuance of the disputes and oppositions of opinion which have occurred among theoretical writers concerning the elementary principles of Mechanics, may have made such discussions appear to some persons wearisome and unprofitable. I might, however, not unreasonably plead this very circumstance as an apology for offering a new view of the subject; since the extent to which these discussions have already gone shews that some men at least take a great interest in them; and it may be stated, I think, without fear of contradiction, that these controversies have not terminated in the general and undisputed establishment of any one of the antagonist opinions.

The question to which my remarks at present refer is this: "What is the kind and degree of cogency of the best proofs of the laws of motion, or of the fundamental principles of mechanics, exprest in any other way?" Are these laws, philosophically considered, *necessary*, and capable of demonstration by means of self-evident axioms, like the truths of geometry; or are they *empirical*, and only known to be true by trial and observation, like such general rules as we obtain in natural history?

It certainly appears, at first sight, very difficult to answer the arguments for either side of this alternative. On the one hand it is said, the laws of motion cannot be necessarily true, for if they were so, the denial of them would involve a contradiction. But this it does not, for we can readily conceive them to be other than they are. We can conceive that a body in motion should have a natural tendency to move slower and slower. And we know that, historically speaking, men did at first suppose the laws of motion to be different from what they are now proved to be. This would have been impossible if the negation of these laws had involved a contradiction of self-evident principles, and consequently had been not only false but inconceivable. These laws, therefore, cannot be necessary; and can be duly established in no other way than by a reference to experience.

On the other hand, those who deduce their mechanical principles without any express reference to experiment, may urge, on their side, that, by the confession even of their adversaries, the laws of motion are proved to be true beyond the limits of experience;—that they are assumed to be true of any new kind of motion when first detected, as well as of those already examined;—and that it is inexplicable how such truths should be established empirically. They may add that the consequences of these laws are allowed to hold with the most complete and absolute universality; for instance, the proposition that "the quantity of motion in the world in a given direction cannot be either increased or diminished," is conceived to be rigorously exact; and to have a degree and kind of certainty beyond and above all mere facts of experience; what other kind of truth than necessary truth this can be, it is difficult to say. And if the conclusions be necessarily true, the principles must be so too.

This apparent contradiction therefore, that a law should be necessarily true and yet the contrary of it conceivable, is what I have now to endeavour to explain; and this I must do by pointing out what appear to me the true grounds of the laws of motion.

2. The science of Mechanics is concerned about motions as determined by their causes, namely, forces; the nature and extent of the truth of the first principles of this science must therefore depend upon the way in which we can and do reason concerning *causes*. In what manner we obtain the conception of cause, is a question for the metaphysician, and has been the subject of much discussion. But the general principle which governs our mode of viewing occurrences with reference to this conception, so far as our present subject is concerned, does not appear to be disturbed by any of the arguments which have been

adduced in this controversy. This principle I shall state in the form of an axiom, as follows.

AXIOM I. *Every change is produced by a cause.*

It will probably be allowed that this axiom expresses a universal and constant conviction of the human mind; and that in looking at a series of occurrences, whether for theoretical or practical purposes, we inevitably and unconsciously assume the truth of this axiom. If a body at rest moves, or a body in motion stops, or turns to the right or the left, we cannot conceive otherwise than that there is some cause for this change. And so far as we can found our mechanical principles on this axiom, they will rest upon as broad and deep a basis as any truths which can come within the circle of our knowledge.

I shall not attempt to analyse this axiom further. Different persons may, according to their different views of such subjects, call it a law of our nature that we should think thus, or a part of the constitution of the human mind, or a result of our power of seeing the true relations of things. Such variety of opinion or expression would not affect the fundamental and universal character of the conviction which the axiom expresses; and would therefore not interfere with our future reasonings.

3. There is another axiom connected with this, which is also a governing and universal principle in all our reasoning concerning causes. It may be thus stated.

AXIOM II. *Causes are measured by their effects.*

Every effect, that is, every change in external objects, implies a cause, as we have already said: and the existence of the cause is known only by the effects it produces. Hence the intensity or magnitude of the cause cannot be known in any other manner than by these effects: and, therefore, when we have to assign a measure of the cause, we must take it from the effects produced.

In what manner the effects are to be taken into account, so as to measure the cause for any particular purpose, will have to be further considered; but the axiom, as now stated, is absolutely and universally true, and is acted upon in all parts of our knowledge in which causes are measured.

4. But something further is requisite. We not only consider that all changes of motion in a body have a cause, but that this cause may reside in other bodies. Bodies are conceived to act upon one another, and thus to influence each other's motions, as when one billiard ball strikes another. But when this happens, it is also supposed that the body struck influences the motion of the striking body. This is included in our notion of body or matter. If one ball could strike and affect the motions of any number of others without having its own motion in any degree affected, the struck balls would be considered, not as bodies, but as mere shapes or appearances. Some reciprocal influence, some resistance, in short some *reaction*, is necessarily involved in our conception of action among bodies. All mechanical action upon matter implies a corresponding reaction; and we might describe matter as that which resists or reacts when acted on by force. Not only must there be a reaction in such cases, but this reaction is defined and determined by the action which produces it, and is of the same kind as the action itself. The action which one body exerts upon another is a blow, or a pressure; but it cannot press or strike without receiving a pressure or a blow in return. And the reciprocal pressure or blow depends upon the direct, and is determined altogether and solely by that. But this action being mutual, and of the same kind on each body, the effect on each body will be determined by the effect on the other, according to the same rule; each effect in turn being considered as action and the other as reaction. But this cannot be otherwise than by the equality and opposite direction of the action and reaction. And since this reasoning applies in all cases in which bodies influence each others motions, we have the following axiom which is universally true, and is a fundamental principle with regard to all mechanical relations.

AXIOM III. *Action is always accompanied by an equal and opposite Reaction.*

5. I now proceed to shew in what manner the Laws of Motion depend upon these three axioms.

Bodies move in lines straight or curved, they move more or less rapidly, and their motions are variously affected by other bodies. This succession of occurrences suggests the conceptions of certain properties or attributes of the motions of bodies, as their direction and velocity, by means of which the laws of such occurrences may be exprest. And these properties or attributes are conceived as belonging to the body at each *point* of its motion, and as changing from one point to another. Thus the body, at each point of its path, moves in a certain direction, and with a certain velocity.

These properties, direction and velocity for instance, are subject to the rule stated in the first axiom: they cannot change without some cause; and when any changes in the motions of a body are seen to depend on its position relative to another body or to any part of space, such other body, or such other part of space, is said to exert a *force* upon the moving body. Also the force exerted upon the moving body is considered to be of a certain value at each point of the body's motion; and though it may change from one point to another, its changes must depend upon the position of the points only, and not upon the velocity and direction of the moving body. For the force which acts upon the body is conceived as a property of the bodies, or points, or lines, or surfaces among which the moving body is placed; the force at all points therefore depends upon the position with regard to the bodies and spaces of which the force is a property; but remains the same, whatever be the circumstances of the body moved. The circumstances of the body moved cannot be a cause which shall change the force acting at any point of space, although they may alter the *effect* which that force produces upon the body. Thus, gravity is the same force at the same point of space, whether it have to act upon a body at rest or in motion; although it still remains to be seen whether it will produce the same effect in the two cases.

6. This being established, we can now see of what nature the laws of motion must be, and can state in a few words the proofs

of them. We shall have a law of motion corresponding to each of the above three axioms; the first law will assert that when no force acts, the properties of the motion will be constant; the second law will assert that when a force acts, its quantity is measured by the effect produced; the third law will assert that, when one body acts upon another, there will be a reaction, equal and opposite to the action. And so far as the laws are announced in this form, they will be of absolute and universal truth, and independent of any particular experiment or observation whatever.

But though these laws of motion are necessarily and infallibly true, they are, in the form in which we have stated them, entirely useless and inapplicable. It is impossible to deduce from them any definite and positive conclusions, without some additional knowledge or assumption. This will be clear by stating, as we can now do in a very small compass, the proofs of the laws of motion in the form in which they are employed in mechanical reasonings.

7. First, of the first Law;—that *a body not acted upon by any force will go on in a straight line with an invariable velocity.*

The body will go on in a straight line: for, at any point of its motion, it has a certain direction, which direction will, by Axiom I, continue unchanged, except some cause make it deviate to one side or other of its former position. But any cause which should make the direction deviate towards any part of space would be a force, and the body is not acted upon by any force. Therefore, the direction cannot change, and the body will go on in the same straight line from the first.

The body will move with an invariable velocity. For the velocity at any point will, by Axiom I, continue unchanged, except some cause make it increase or decrease. And since, by supposition, the body is not acted upon by any force, there can be no such cause depending upon position, that is, upon relations of *space;* for any cause of change of motion which has a reference to space is force.

Therefore there can be no cause of change of motion, except there be one depending upon *time,* such, for instance, as would exist if bodies had a natural tendency to move slower and slower, according to a rate depending on the time elapsed.

But if such cause existed, its effects ought to be considered separately; and it would still be requisite to assume the permanence of the same velocity, as the first law of motion; and to obtain, in addition to this, the laws of the retardation depending on the time.

Whether there is any such cause of retardation in the actual motions of bodies, can be known only by a reference to experience; and by such reference it appears that there is no such cause of the diminution of velocity depending on time alone; and therefore that the first law of motion may, in all cases in which bodies are exempt from the action of external forces, be applied without any addition or correction depending upon the time elapsed.

It is not here necessary to explain at any length in what manner we obtain from experience the knowledge of the truth just stated, that there is not in the mere lapse of time any cause of the retardation of moving bodies. The proposition is established by shewing that in all the cases in which such a cause appears to exist, the cause of retardation resides in surrounding bodies and not in time alone, and is therefore an external force. And as this can be shewn in every instance, there remains only the negation of all ground for the assumption of such a cause of retardation. We therefore reject it altogether.

Thus it appears that in proving the first law of motion, we obtain from our conception of cause the conviction that velocity will be uniform except some cause produce a change in it; but that we are compelled to have recourse to experience in order to learn that time alone is not a cause of change of velocity.

8. I now proceed to the second Law:—that *when a force acts upon a body in motion, the effect is the same as that which the same force produces upon a body at rest.*

This law requires some explanation. How is the effect produced upon a moving body to be measured, so that we may compare it with the effect upon a body at rest? The answer to this is, that we here take for the measure of the effect of the force, that motion which must be *compounded* with the motion existing before the change, in order to produce the motion which exists after the change: the rules for the composition of motion being established on independent grounds by the aid of definition alone. Thus if gravity act upon a body which is falling vertically, the effect of gravity upon the body is measured by the velocity *added* to that which the body already has: if gravity act upon a body which is moving horizontally, its effect is measured by the distance to which the body falls below the horizontal line.

The effect of the force which we consider in the second Law of motion, is its effect upon velocity only: and it is proper to mark this restriction by an appropriate term: we shall call this the *accelerative effect* of force; and the cause, as measured by this effect, may be termed the *accelerative quantity* of the force.*

A law of motion which necessarily results from our second Axiom is, that the accelerative quantity of a force is measured by the accelerative effect. But whether the accelerative effect depends upon the velocity and direction of the moving body, cannot be known independently of experience. It is very conceivable, for instance, that the force of gravity being every where the same, shall yet produce, upon falling bodies, a smaller accelerative effect in proportion to the velocity which they already have in a downward direction. Indeed if gravity resembled in its operation the effect of any other mode of mechanical agency, the result would be so. If a body moved downwards in

* The accelerative quantity of a force (the *quantitas acceleratrix vis cujusvis* of Newton) is often called *the accelerating force;* and we may thus have to speak of *the accelerating force of a certain force,* which is at any rate an awkward phraseology. It would perhaps have been fortunate if Newton, or some other writer of authority, at the time when the principles of mechanics were first clearly developed, had invented an abstract term for this quantity: it might for instance have been called *accelerativity.* And the second law of motion would then have been, that the *accelerativity* of the same force is the same, whatever be the motion of the body acted on.

consequence of the action of a hand pushing it with a constant effort, or of a spring, or of a stream of fluid rushing in the same direction, the accelerative effect of such agents would be smaller and smaller as the velocity of the body propelled was larger and larger. We can learn from experience alone that the effects of the action of gravity do not follow the same rule.

We assert that the accelerative quantity of the *same force* of gravity is the same whatever be the motion of the body acted on. It may be asked how we know that the force of gravity *is* the same in cases so compared; for instance, when it acts on a body at rest and in motion? The answer to this question we have given already. By the very process of considering gravity as a force, we consider it as an attribute of something independent of the body acted on. The amount of the force may depend upon place, and even time, for any thing we know *à priori;* but we do not find that the weight of bodies depends on these circumstances, and therefore, having no evidence of a difference in the force of gravity, we suppose it the same at different times and places. And as to the rest, since the force is a force which acts on the body, it is considered as the same force, whatever be the circumstances of the passive body, although the *effects* may vary with these circumstances. If the effects are liable to such change, this change must be considered separately, and its laws investigated; but it cannot be allowed to unsettle our assumption of the permanence of the force itself. It is precisely this assumption of a constant cause, which gives us a fixed term, as a means of estimating and expressing by what conditions the effects are regulated.

It appears by observation and experiment, that the accelerative quantity of the same force is not affected by the velocity or direction of the body acted on: for instance, a body falling vertically receives, in any second of time, an accession of velocity as great as that which it received in the first second, notwithstanding the velocity with which it is already moving. The proof of this and similar assertions from experiment produced, historically speaking, the establishment of the second law of motion in the sense in which we now assert it. And here, as in the case of the first law, we may observe that an important portion of the process of proof consisted in shewing that in those cases in which the *accelerative effect* of a force appeared to be changed by the circumstances of the motion of the body acted on, the change was, in fact, due to other external forces; so that all evidence of a cause of change residing in those circumstances was entirely negatived; and thus the law, that the accelerative effect of the same force is the same, appeared to be absolutely and rigorously true.

9. When the motions of bodies are not affected merely by forces like gravity, which are only perceived by their effects, but are acted upon by other bodies, the case requires other considerations.

It is in such cases that we originally form the conception of force; we ourselves pull and push, thrust and throw bodies, with a view, it may be, either to put them in motion, or to prevent their moving, or to alter their figure. Such operations, and the terms by which they are described, are all included in the term *force*, and in other terms of cognate import. And in using this term, we necessarily assume and imply the co-existence of these various effects of force which we have observed universally to accompany each other. Thus the same kind of force which is the cause of motion, may also be the cause of a body having a form different from its natural form; when we draw a bow, the same kind of pull is needed to move the string, and to hold it steady when the bow is bent. And a weight might be hung to the string, so as to produce either the one or the other of these effects. By an infinite multiplicity of experiments of this kind, we become imbued with the conviction that the same pressure may be the cause of tension and of motion. Also as the cause can be known by its effects only, each of these effects may be taken as its measure; and therefore, so long as one of them is the same, since the cause is the same, the other must be the same also. That is, so long as the pressure or force which shews itself in tension is the same, the motion which it would produce must, under the same circumstances, be the same also. This general fact is not a result of any particular observations, but of the general observation or suggestion arising unavoidably from universal experience, that both tension and motion may be referred to force as their cause, and have no other cause.

We come therefore to this principle with regard to the actions of bodies upon each other, that so long as the tension or pressure is the same, the force, as shewn by its effect in producing motion, must also be the same.

10. This force or action of bodies upon one another, is that which is meant in the Third Axiom, and we now proceed to consider the application of this axiom in mechanics.

Pressures or forces such as I have spoken of, may be employed in producing tension only, and not motion; in this case, each force prevents the motion which would be produced by the others, and the forces are said to balance each other, or to be in equilibrium. The science which treats of such cases is called Statics, and it depends entirely upon the above third axiom, applied to pressures producing rest. It follows from that axiom, that pressures, which acting in opposite directions thus destroy each other's effects, must be equal, each measuring the other. Thus if a man supports a stone in his hand, the force or effort exerted by the man upwards is equal to the weight or force of the stone downwards. And if a second stone, just equal to the first, were supported at the same time in the same hand, the force or effort must be twice as great; for the two stones may be considered as one body of twice the magnitude, and of twice the weight; and therefore the effort which supports it must also be twice as great. And thus we see in what manner statical forces are to be measured in virtue of this third axiom; and no further principle is requisite to enable us to establish the whole doctrine of statics.

11. The third axiom, when applied to the actions of bodies in motion, gives rise to the third law of motion, which we must now consider. Here, as in the cases of the other axioms, we must inquire how we are to measure the quantities to which the axiom applies. What is the measure of the *action* which takes place when a body is put in motion by pressure or force? In order to answer this question, we

must consider what circumstances make it requisite that the force should be greater or less. If we have to lift a stone, the force which we exert must be greater when the stone is greater: again, we must exert a greater force to lift it quickly than slowly. It is clear, therefore, that that property of a force with which we are here concerned, and which we may call the *motive quantity* of the force,* increases both when the velocity communicated, and when the mass moved, increase, and depends upon both these quantities, though we have not yet shewn what is the law of this dependence.

The condition that a quantity P shall increase when each of two others V and M does so, may be satisfied in many ways: for instance, by supposing P proportional to the sum $M+V$ (all the quantities being expressed in numbers), or to the product, MV, or to MV^2, or in many other ways.

When, however, the quantities V and M are altogether heterogeneous, as when one is velocity, and the other weight, the first of the above suppositions, that P varies as $M+V$, is inadmissible. For the law of variation of the formula $M+V$ depends upon the relation of the units by which M and V respectively are measured; and as these units are arbitrary in each case, the result is, in like manner, arbitrary, and therefore cannot express a law of nature.

12. The supposition that the motive quantity of a force varies as $M+V$, where M is the mass moved and V the velocity, being thus inadmissible, we have to select upon due grounds, among the other formulæ MV, MV^2, M^2V, &c.

And in the first place I observe that the formula must be proportional to M simply (excluding M^2 &c.) for both the forces which

* The motive quantity of a force (*vis cujusvis quantitas motrix* of Newton) is sometimes called *moving force;* we are thus led to speak of the moving force of a force, as we have already observed concerning accelerating force. Hence, as in that case, we might employ a single term, as *motivity*, to denote this property of force; and might thus speak of it and of its measures without the awkwardness which arises from the usual phrase.

produce motion and the masses in which motion is produced are capable of addition by juxtaposition, and it is easily seen by observation that such addition does not modify the motion of each mass. If a certain pressure upon one brick (as its own weight) cause it to fall with a certain velocity, an equal pressure on another equal brick will cause it also to fall with the same velocity; and these two bricks being placed in contact, may be considered as one mass, which a double force will cause to fall with still the same velocity. And thus all bodies, whatever be their magnitude, will fall with the same velocity by the action of gravity. Those who deny this (as the Aristotelians did) must maintain, that by establishing between two bodies such a contact as makes them one body, we modify the motion which a certain pressure will produce in them. And when we find experimentally (as we do find) that large bodies and small ones fall with the same velocity, excluding the effects of extraneous forces, this result shews that there is not, in the union of small bodies into a larger one, any cause which affects the motion produced in the bodies.

It appears, therefore, that the motive quantity of force which puts a body in motion is, *cæteris paribus*, proportional to the mass of the body; so that for a double mass a double force is requisite, in order that the velocity produced may be the same. Mass considered with reference to this rule, is called *Inertia*.

13. The measure of mass which is used in expressing a law of motion, must be obtained in some way independent of motion, otherwise the law will have no meaning. Therefore, mass measured in order to be considered as *Inertia* must be measured by the statical effects of bodies, for instance, by comparison of weights. Thus two masses are equal which each balance the same weight in the same manner; and a mass is double of one of them which produces the same effect as the two. And we find, by universal observations, that the weight of a mass is not affected by the figure or the arrangement of parts, so long as the matter continues the same. Hence it appears that the mass of bodies must be compared by comparing their weights, and Inertia is proportional to weight at the same place.

Since all bodies, small or large, light or heavy, fall downwards with equal velocities, when we remove or abstract the effect of extraneous circumstances, the motive quantity of the force of gravity on equal bodies is as their masses; or as their weight, by what has just been said.

14. For the measure of the motive quantity of force, or of the action and reaction of bodies in motion, we have, therefore, now to chuse among such expressions as MV, and MV^2. And our choice must be regulated by finding what is the measure which will enable us to assert, in all cases of action between bodies in motion, that action and reaction are equal and opposite.

Now the fact is, that either of the above measures may be taken, and each has been taken by a large body of mathematicians. The former however (MV) has obtained the designation which naturally falls to the lot of such a measure; and is called *momentum*, or sometimes simply *quantity of motion:* the latter quantity (MV^2) is called *vis viva* or *living force*.

I have said that either of these measures may be taken: the former must be the measure of action, if we are to measure it by the effect produced *in a given time;* the latter is the measure if we take the *whole* effect produced. In either way the third law of motion would be true.

Thus if a ball B, lying on a smooth table, be drawn along by a weight A hanging by a thread over the edge of the table, the motion of B is produced by the action of A, and on the other hand the motion of A is diminished by the reaction of B; and the equality of action and reaction here consists in this, that the momentum (MV) which B acquires in any time is equal to that which A loses: that is, so much is taken from the momentum which A would have had, if it had fallen freely *in the same time;* so that A falls more slowly by just so much.

But if the weight A fall through a given space from rest, as 1 foot, and then cease to act, the equality of action and reaction consists in this, that the *vis viva* which B acquires on the whole, is equal to the *vis viva* which A loses; that is, the *vis viva* of A thus acting on B is smaller by so much than it would have been, if A had fallen freely *through the same space.*

15. In fact, these two propositions are necessarily connected, and one of them may be deduced from the other. The former way of stating the third law of motion appears, however, to be the simplest mode of treating the subject, and we may put the third law of motion in this form.

In the direct mutual action of bodies, the momentum gained and lost in any time are equal.

This law depends upon experiment, and is perhaps best proved by some of its consequences. It follows from the law so stated, that the motive quantity of a force is proportional to the momentum generated in a given time; since the motive quantity of force is to be equivalent to that action and reaction which is understood in the third law of motion. Now, if the pressure arising from the weight of a body P produce motion in a mass Q, since the momentum gained by Q and that lost by P in any time are equal, the momentum of the whole at any time will be the same as if P's weight had been employed in moving P alone. Therefore, the velocity of the mass Q will be less, in the same proportion in which the mass or inertia is greater: and thus the accelerating quantity of the force is inversely proportioned to the mass moved. This rule enables us to find the accelerative quantity of the force in various cases, as for instance, when bodies oscillate, or when a smaller weight moves a large mass; and we can hence calculate the circumstances of the motion, which are found to agree with the consequences of the above law.

16. But the argument may be reduced to a simpler form. Our object is to shew that, for an equal mass, the velocity produced by a force acting for a given time is as the pressure which produces the motion; for instance, that a double pressure will produce a double velocity. Now a double pressure may be considered as the union of two equal pressures, and if these two act *successively*, the first will communicate to the body a certain velocity, and the second will com-

municate an additional velocity, equal to the first, by the second law of motion; so that the whole velocity thus communicated will be the double of the first. Therefore, if the velocity communicated be not also the double of the first when the two pressures act *together*, the difference must arise from this, that the effect of one force is modified by the simultaneous action of the other. And when we find by experience (as we do find) that there is no such difference, but that the velocity communicated in a given time is as the pressure which communicates it, this result shews that there is nothing in the circumstance of a body being already acted on by one pressure, which modifies the effect of an additional pressure acting along with the first.

17. I have above asserted the law, of the *direct* action of bodies only. But it is also true when the action is indirect, as when by turning a winch we move a wheel, the main mass of which is farther from the axis than the handle of the winch. In this case the pressure we exert acts at a mechanical disadvantage on the main mass of the wheel, and we may ask whether this circumstance introduces any new law of motion. And to this we may reply, that we can *conceive* pressure to produce different effects in moving bodies, according as it is exerted directly or by the intervention of machines; but that we *find* no reason to believe that such a difference exists. The relations of the pressures in different parts of a machine are determined by considering the machine at rest. But if we suppose it to be put in motion by such pressures, we see no reason to expect that these pressures should have a different relation to the motions produced from what they would have done if they were direct pressures. And as we find in experiment a negation of all evidence of such a difference, we reject the supposition altogether. We assert, therefore, the third law of motion to be true, whatever be the mechanism by the intervention of which action and reaction are opposed to each other.

From this consideration it is easy to deduce the following rule, which is known by the designation of D'Alembert's principle, and may be considered as a fourth law of motion.

When any forces produce motion in any connected system of matter, the motive quantities of force gained and lost by the different parts must balance each other according to the connexion of the system.

By the motive quantity of force *gained* by any body, is here meant the quantity by which that motive force which the body's motion implies (according to the measures already established) exceeds the quantity of motive force which acts immediately upon the body. It is the excess of the *effective* above the *impressed* force, and of course arises from the force transmitted from the other bodies of the system in consequence of the connexion of the parts. The motive quantity of force *lost* is in like manner the excess of the impressed above the effective force. And these two excesses, in different parts of the system, must balance each other according to the mechanical advantage or disadvantage at which they act for each part.

This completes our system of mechanical principles, and authorizes us to extend to bodies of any size and form the rules which the second law of motion gives for the motion of bodies considered as points. And by thus enabling us to trace what the motions of bodies will be according to the rule asserted in the third law of motion, (namely, that the motive quantity of forces is as the momentum produced in a given time,) it leads us to verify that supposition by experiments in which bodies oscillate or revolve or move in any regular and measurable manner, as has been done by Atwood, Smeaton, and many others.

18. We have thus a complete view of the nature and extent of the fundamental principles of mechanics; and we now see the reason why the laws of motion are so many and no more, in what way they are independent of experience, and in what way they depend upon experiment. The form, and even the language of these laws is of necessity what it is; but the interpretation and application of them is not possible without reference to fact. We may imagine many rules according to which bodies might move (for many sets of rules, different from the existing ones, are, so far as we can see, possible) and we should still have to assert—that velocity could not change without

a cause,—that change of action is proportional to the force which produces it,—and that action and reaction are equal and opposite. The truth of these assertions is involved in those notions of causation and matter, which the very attempt to know any thing concerning the relations of matter and motion presupposes. But, according to the facts which we might find, in such imaginary cases as I have spoken of, we should settle in a different way—what is a cause of change of velocity,—what is the measure of the force which changes motion,—and what is the measure of action between bodies. The law is necessary, if there is to be a law; the meaning of its terms is decided by what we find, and is therefore regulated by our special experience.

19. It may further illustrate this matter to point out that this view is confirmed by the history of mathematics. The laws of motion were assented to as soon as propounded; but were yet each in its turn the subject of strenuous controversy. The terms of the law, the form, which is necessarily true, were recognised and undisputed; but the meaning of the terms, the substance of the law, was loudly contested; and though men often tried to decide the disputed points by pure reasoning, it was easily seen that this could not suffice; and that since it was a case where experience *could* decide, experience *must* be the proper test: since the matter came within her jurisdiction, her authority was single and supreme.

Thus with regard to the first law of motion, Aristotle allowed that *natural* motions continue unchanged, though he asserted the motions of terrestrial bodies to be *constrained* motions, and therefore, liable to diminution. Whether this was the cause of their diminution was a question of fact, which was, by examination of facts, decided against Aristotle. In like manner, in the first case of the second law of motion which came under consideration, both Galileo and his opponent agree that falling bodies are *uniformly* accelerated; that is, that the force of gravity accelerates a body uniformly whatever be the velocity it has already; but the question arises, what is uniform acceleration? It so happened in this case, that the first conjecture of Galileo, afterwards defended by Casræus, (that the velocity was proportional to the space from the beginning of the motion) was not only contradictory to fact, but involved a self-contradiction; and was, therefore, easily disposed of. But this accident did not supersede the necessity of Galileo and his pupils verifying their assertion by reference to experiment, since there were many suppositions which were different from theirs, and still possible, though that of Casræus was not.

The mistake of Aristotle and his followers, in maintaining that large bodies fall more quickly than small ones, in exact proportion to their weight, arose from perceiving half of the third law of motion, that the velocity increases with the force which produces it; and from overlooking the remaining half, that a greater force is required for the same velocity, according as the mass is larger. The ancients never attained to any conception of the force which moves and the body which is moved, as distinct elements to be considered when we enquire into the subject of motion, and therefore could not even propose to themselves in a clear manner the questions which the third law of motion answered.

But, when, in more modern times, this distinction was brought into view, the progress of opinion in this case was nearly the same as with regard to the other laws.

It was allowed at once, and by all, that action and reaction are equal; but the controversy concerning the sense in which this law is to be interpreted, was one of the longest and fiercest in the history of mathematics, and the din of the war has hardly yet died away. The disputes concerning the measure of the force of bodies in motion, or the *vis viva,* were in fact a dispute which of two measures of action that I have mentioned above should be taken; the effect in a given time, or the whole effect: in the one case the momentum (MV) in the other the *vis viva,* (MV^2) was the proper measure.

20. It may be observed that the word *momentum,* which one party appropriated to their views, was employed to designate the motive quantity of force, or the action of bodies in motion, before it was

determined what the true measure of such action was. Thus Galileo, in his "Discorso intorno alle cose che stanno in su l'Acqua," says, that momentum "is the force, efficacy, or virtue with which the motion moves and the body moved resists; depending not on weight only, but on the velocity, inclination, and any other cause of such virtue."

The adoption of the phrase *vis viva* is another instance of the extent to which men are tenacious of those terms which carry along with their use a reference to the fundamental laws of our thought on such matters. The party which used this phrase maintained that the mass multiplied into the square of the velocity was the proper measure of the force of bodies in motion; but finding the term *moving force* appropriated by their opponents, they still took the same term *force*, with the peculiar distinction of its being *living* force, in opposition to *dead* force or pressure, which they allowed to be rightly measured by the momentum generated in a given time. The same tendency to adopt, in a limited and technical sense, the words of most general and fundamental use in the subject, has led some writers (Newton for instance,) to employ the term *motion* or *quantity of motion* as synonymous with momentum, or the product of the numbers which express the mass and the velocity. And this use being established, the quantities of motion gained and lost are always equal and opposite; and, therefore the quantity which exists in any given direction cannot be increased or diminished by any mutual action of bodies. Thus we are led to the assertion which has already been noticed, that the quantity of motion in the world is always the same. And we now see how far the necessary truth of this proposition can be asserted. The proposition is necessarily true according to our notions of material causation; but the measure of "quantity of motion," which is a condition of its truth, is inevitably obtained from experience.

21. It is not surprising that there should have been a good deal of confusion and difference of opinion on these matters: for it appears that there is, in the intellectual constitution and faculties of man, a source of self-delusion in such reasonings. The actual rules of the motion and mutual action of bodies are, and must be, obtained from

observation of the external world: but there is a constant wish and propensity to express these rules in such terms as shall make them appear self-evident, because identical with the universal and necessary rules of causation. And this propensity is essential to the progress of our knowledge; and in the success of this effort consists, in a great measure, the advance of the science to its highest point of simplicity and generality.

22. The nature of the truth which belongs to the laws of motion will perhaps appear still more clearly, if we state, in the following tabular form, the analysis of each law into the part which is necessary, and the part which is empirical.

	Necessary.	*Empirical.*
First Law.	Velocity does not change without a cause.	The time for which a body has already been in motion is not a cause of change of velocity.
Second Law.	The accelerating quantity of a force is measured by the acceleration produced.	The velocity and direction of the motion which a body already possesses are not, either of them, causes which change the acceleration produced.
Third Law.	Reaction is equal and opposite to action.	The connexion of the parts of a body, or of a system of bodies, and the action to which the body or system is already subject, are not, either of them, causes which change the effects of any additional action.

Of course, it will be understood that, when we assert that the connexion of the parts of a system does not change the effect of any action upon it, we mean that this connexion does not introduce any *new* cause of change, but leaves the effect to be determined by the previously established rules of equilibrium and motion. The connexion will modify the application of such rules; but it introduces no additional rule: and the same observation applies to all the above stated empirical propositions.

This being understood, it will be observed that the part of each law which is here stated as empirical, consists, in each case, of a negation of the supposition that the condition of the moving body with respect to motion and action, is a cause of any change in the circumstances of its motion; and from this it follows that these circumstances are determined entirely by the forces extraneous to the body itself.

23. This mode of considering the question shews us in what manner the laws of motion may be said to be proved by their simplicity, which is sometimes urged as a proof. They undoubtedly have this distinction of the greatest possible simplicity, for they consist in the negation of all causes of change, except those which are essential to our conception of such causation. We may conceive the motions of bodies, and the effect of forces upon them, to be regulated by the lapse of time, by the motion which the bodies have, by the forces previously acting; but though we may imagine this as possible, we do not find that it is so in reality. If it were, we should have to consider the effect of these conditions of the body acted on, and to combine this effect with that of the acting forces; and thus the motion would be determined by more numerous conditions and more complex rules than those which are found to be the laws of nature. The laws which, in reality, govern motion are the fewest and simplest possible, because all are excluded, except those which the very nature of laws of motion necessarily implies. The prerogative of simplicity is possessed by the actual laws of the universe, in the highest perfection which is imaginable or possible. Instead of having to take into account all the circumstances of the moving bodies, we find that we have only to reject all these circumstances. Instead of having to combine empirical with necessary laws, we learn empirically that the necessary laws are entirely sufficient.

24. Since all that we learn from experience is, that she has nothing to teach us concerning the laws of motion, it is very natural that some persons should imagine that experience is not necessary to their proof. And accordingly many writers have undertaken to establish all the fundamental principles of mechanics by reasoning alone.

This has been done in two ways:—sometimes by attending only to the necessary part of each law (as the parts are stated in the last paragraph but one) and by overlooking the necessity of the empirical supplement and limitation to it;—at other times by asserting the part which I have stated as empirical to be self-evident, no less than the other part. The former way of proceeding may be found in many English writers on the subject; the latter appears to direct the reasonings of many eminent French mathematicians. Some (as Laplace) have allowed the empirical nature of two out of the three laws; others, as M. Poisson, have considered the first as alone empirical; and others, as D'Alembert, have assumed the self-evidence of all the three independently of any reference whatever to observation.

25. The parts of the laws which I have stated as empirical, appear to me to be clearly of a different nature, as to the cogency of their truth, from the parts which are necessary; and this difference is, I think, established by the fact that these propositions were denied, contested, and modified, before they were finally established. If these truths could not be denied without a self-contradiction, it is difficult to understand how they could be (as they were) long and obstinately controverted by mathematicians and others fully sensible to the cogency of necessary truth.

I will not however go so far as to assert that there may not be some point of view in which that which I have called the empirical part of these laws, (which, as we have seen, contains negatives only,) may be properly said to be self-evident. But however this may be, I think it can hardly be denied that there is a difference of a fundamental kind in the nature of these truths,—which we can, in our imagination at least, contradict and replace by others, and which, historically speaking, have been established by experiment;—and those other truths, which have been assented to from the first, and by all, and which we cannot deny without a contradiction in terms, or reject without putting an end to all use of our reason on this subject.

26. On the other hand, if any one should be disposed to maintain that, inasmuch as the laws are interpreted by the aid of experience

only, they must be considered as entirely empirical laws, I should not assert this to be placing the science of mechanics on a wrong basis. But at the same time I would observe, that the form of these laws is not empirical, and would be the same if the results of experience should differ from the actual results. The laws may be considered as a formula derived from *à priori* reasonings, where experience assigns the value of the terms which enter into the formula.

Finally, it may be observed, that if any one can convince himself that matter is either necessarily and by its own nature determined to move slower and slower, or necessarily and by its own nature determined to move uniformly, he must adopt the latter opinion, not only of the truth, but of the necessity of the truth of the first law of motion, since the former branch of the alternative is certainly false: and similar assertions may be made with regard to the other laws of motion.

27. This enquiry into the nature of the laws of motion, will, I hope, possess some interest for those who attach any importance to the logic and philosophy of science. The discussion may be said to be rather metaphysical than mechanical; but the views which I have endeavoured to present, appear to explain the occurrence and result of the principal controversies which the history of this science exhibits; and, if they are well founded, ought to govern the way in which the principles of the science are treated of, whether the treatise be intended for the mathematical student or the philosopher.

VIII. *Researches in the Theory of the Motion of Fluids. By the Rev.* JAMES CHALLIS, *late Fellow of Trinity College, Cambridge, and Fellow of the Cambridge Philosophical Society.*

[Read *March* 3, 1834.]

1. THE subjects treated of in this communication are of a miscellaneous character, referring to several points of the theory of fluid motion, respecting which the author conceived he had something new to advance. In illustration of the principles he has attempted to establish, solutions are given of two problems of considerable interest:—the resistance to the motion of a ball-pendulum; and, the resistance to the motion of a body partly immersed in water and drawn along at the surface in the horizontal direction. The principal object in the solution of the latter problem is to account for the rising of the body in the vertical direction on increasing the velocity of draught, which in some recent experiments on canal navigation has been observed to take place. In the course of the paper I have had occasion to refer several times to a previous communication* to this Society respecting fluid motion, for the purpose of giving to the views there advanced some corrections and confirmations which have been suggested by more mature consideration. For the sake of distinctness the subjects of the present essay are divided into sections.

* Camb. Phil. Trans. Vol. III. Part III.

SECTION I.

On the Integral of the Equation $\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0.$

2. This equation is applicable to all problems respecting the motion of incompressible fluids, which require for their solutions the consideration of motion in one plane only. Mathematicians have obtained integrals of it suited to the particular questions they were discussing; for instance, in solving the problem of waves propagated in a canal of uniform width, M. Poisson has given a value of ϕ, which, while it satisfies the equation in question, is exclusively applicable to that problem. But it is well known that by the common method of finding the integrals of linear partial differential equations of the second order between three variables, a value of ϕ may be found *prior* to any consideration of the circumstances under which the fluid was put in motion. Therefore any inferences respecting the nature of the motion, which may be drawn from this integral, must be equally applicable to all problems of this class. To obtain such inferences is the object of the following reasoning.

3. The integral I speak of is,

$$\phi=F(x+y\sqrt{-1})+f(x-y\sqrt{-1}).$$

To ascertain its *general* signification, I propose to determine the forms of the functions F and f, independently of any hypothesis respecting the mode in which the fluid was put in motion. The quantity ϕ is subject to the condition $(d\phi)=udx+vdy$, where u and v are the velocities at the point xy in the directions of the axes of x and y respectively. Hence $\frac{d\phi}{dx}=u$, $\frac{d\phi}{dy}=v$, and

$$u=F'(x+y\sqrt{-1})+f'(x-y\sqrt{-1}),$$

$$v=\sqrt{-1}\,F'(x+y\sqrt{-1})-\sqrt{-1}f'(x-y\sqrt{-1}).$$

First, it may be observed that u and v are not both possible for *any* values of x and y, unless the functions F' and f' be the same. Again, as the form of F' we are seeking for is to be independent of all that is arbitrary, it will remain the same whatever direction we arbitrarily assign to the axes of co-ordinates. Let therefore the axis of y pass through the point to which the velocities u, v, belong. Then

$$y=0, \quad u=2F'(x), \quad v=0.$$

If now the axes be supposed to take any other position, the origin remaining the same, u will be equal to $\dfrac{2x}{\sqrt{x^2+y^2}} F'(\sqrt{x^2+y^2})$. Hence

$$F'(x+y\sqrt{-1})+F'(x-y\sqrt{-1})=\frac{2x}{\sqrt{x^2+y^2}}\cdot F'(\sqrt{x^2+y^2}),$$

a functional equation for determining the form of F'. Let

$$x+y\sqrt{-1}=m, \text{ and } x-y\sqrt{-1}=n;$$

then

$$2x=m+n, \text{ and } \sqrt{x^2+y^2}=\sqrt{mn}.$$

Therefore,

$$F'(m)+F'(n)=\frac{m+n}{\sqrt{mn}}F'(\sqrt{mn})=\frac{m}{\sqrt{mn}}F'(\sqrt{mn})+\frac{n}{\sqrt{mn}}F'(\sqrt{mn}).$$

It is easily seen that if $F'(\sqrt{mn})=\dfrac{C}{\sqrt{mn}}$, the equation is satisfied. Hence

$$\frac{d\phi}{dx}=\frac{C}{x+y\sqrt{-1}}+\frac{C}{x-y\sqrt{-1}}=\frac{2Cx}{x^2+y^2}, \text{ and } \frac{d\phi}{dy}=\frac{2Cy}{x^2+y^2};$$

and consequently the velocity at xy, or $\sqrt{u^2+v^2}=\dfrac{2C}{\sqrt{x^2+y^2}}$.

These results shew that the velocity is directed to or from the origin of co-ordinates, and varies inversely as the distance from it. But we must observe that this limitation as to the point to which the velocity is directed, is owing to the particular forms, $x+y\sqrt{-1}$, $x-y\sqrt{-1}$, of the quantities which the function F' involves. For the equation $\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0$, is also satisfied by the following,

$$\phi = F\{(\alpha + x\cos\theta - y\sin\theta) + (\beta + x\sin\theta + y\cos\theta)\sqrt{-1}\}$$
$$+ f\{(\alpha + x\cos\theta - y\sin\theta) - (\beta + x\sin\theta + y\cos\theta)\sqrt{-1}\};$$

and this analytical circumstance has its interpretation in reference to the motion of the fluid. By supposing the function f to be the same as F, and giving to F' the same form as before, we shall find,

$$\frac{d\phi}{dx} = \frac{2C(x + \alpha\cos\theta + \beta\sin\theta)}{(\alpha + x\cos\theta - y\sin\theta)^2 + (\beta + x\sin\theta + y\cos\theta)^2}$$

$$\frac{d\phi}{dy} = \frac{2C(y + \beta\cos\theta - \alpha\sin\theta)}{(\alpha + x\cos\theta - y\sin\theta)^2 + (\beta + x\sin\theta + y\cos\theta)^2}$$

$$\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 = \frac{4C^2}{(\alpha + x\cos\theta - y\sin\theta)^2 + (\beta + x\sin\theta + y\cos\theta)^2}.$$

Or, if $\alpha\cos\theta + \beta\sin\theta = -a$, and $\beta\cos\theta - \alpha\sin\theta = -b$,

$$\frac{d\phi}{dx} \text{ or } u = \frac{2C(x-a)}{(x-a)^2 + (y-b)^2},$$

$$\frac{d\phi}{dy} \text{ or } v = \frac{2C(y-b)}{(x-a)^2 + (y-b)^2},$$

$$\sqrt{u^2+v^2} = \frac{2C}{\sqrt{(x-a)^2+(y-b)^2}}.$$

This shews that the velocity is directed to the point whose co-ordinates are a, b, and varies inversely as the distance from it. And as we have arrived at this result without considering any circumstances under which the fluid was caused to move, the inference to be drawn is, that such is the general character of the motion. Nothing forbids our considering C, a, and b, functions depending on the time and the given conditions of motion in any proposed problem. Also if at a given instant, a line commencing at any point, be drawn continually in the direction of the motions of the particles through which it passes, C, a, and b, may be supposed to vary in any manner along this line. The foregoing reasoning only proves that in passing at a given instant from one point to another indefinitely near along the line, these quantities may be considered constant.

4. The nature of the integral we have been discussing will perhaps be understood by comparing it to the general integral of a common differential equation, which has a particular solution. The latter, we know, is that which gives the answer to a proposed problem, and the general integral is used (though not necessarily) to obtain this solution. So, I conceive, the integral above is useful for arriving at the particular functions of x, y, and t, which give the velocity and direction of the velocity at any point and instant in any proposed question. The integral of $\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0$, which M.M. Poisson and Cauchy have obtained for the solution of the problem of waves, may be called the particular solution of the equation, for that particular problem; and I think it probable that the same might have been obtained by employing what I would call the general integral, though I am not prepared to state exactly the process.

5. The following considerations are added in confirmation of the foregoing reasoning. In whatever manner the fluid is put in motion, we may conceive a line, commencing at any point, to be continually drawn in a direction perpendicular to the directions of the motions at a given instant of the particles through which it passes. This line may be of any arbitrary and irregular shape, not defined by a single equation between x and y. But it must be composed of parts either finite or indefinitely small, which obey the law of continuity. Consequently the motion, being at all the points of the line in the directions of the normals, must tend to or from the centres of curvature, and vary, in at least elementary portions of the fluid, inversely as the distances from those centres. An unlimited number of such lines may be drawn through the whole extent of the fluid mass in motion.

6. If we put $\phi = \phi_1 + \phi_2 + \phi_3 + \&c.$ we shall have

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = \left(\frac{d^2\phi_1}{dx^2} + \frac{d^2\phi_1}{dy^2}\right) + \left(\frac{d^2\phi_2}{dx^2} + \frac{d^2\phi_2}{dy^2}\right) + \left(\frac{d^2\phi_3}{dx^2} + \frac{d^2\phi_3}{dy^2}\right) + \&c. = 0.$$

Hence if there be any number of functions which severally satisfy the given equation, the sum of these will satisfy it. But from what has been proved above, if

$$\frac{d\phi_1}{dx} = \frac{C_1(x-\alpha_1)}{(x-\alpha_1)^2 + (y-\beta_1)^2}, \quad \frac{d\phi_1}{dy} = \frac{C_1(y-\beta_1)}{(x-\alpha_1)^2 + (y-\beta_1)^2},$$

$$\frac{d\phi_2}{dx} = \frac{C_2(x-\alpha_2)}{(x-\alpha_2)^2 + (y-\beta_2)^2}, \quad \frac{d\phi_2}{dy} = \frac{C_2(y-\beta_2)}{(x-\alpha_2)^2 + (y-\beta_2)^2},$$

$$\&c. = \&c. \qquad \&c. = \&c.$$

ϕ_1, ϕ_2, ϕ_3, &c. will severally satisfy it; therefore $\phi_1 + \phi_2 + \phi_3 + \&c.$ will also. And we have,

$$u = \frac{d\phi_1}{dx} + \frac{d\phi_2}{dx} + \frac{d\phi_3}{dx} + \&c.$$

$$v = \frac{d\phi_1}{dy} + \frac{d\phi_2}{dy} + \frac{d\phi_3}{dy} + \&c.$$

These equations prove that the velocity at any point may be the resultant of several velocities produced by different causes; and that any given cause will have the same effect in producing velocity at a given point, whether or not other causes may be operating to produce velocities at the same point.

7. We may here also determine the manner in which the motion of the fluid is affected, when the rectilinear transmission of an impulse tending from any centre is interrupted by a plane surface. For suppose two impulses tending from two centres to be of equal magnitude and in every respect alike. Then if the straight line joining these centres be bisected at right angles by a plane, there will be no motion of the particles contiguous to the plane in a direction perpendicular to it, because the resultant of the velocities from the two causes must lie wholly in

the plane. Hence as the *division of fluids** may be effected without the application of force, nothing will be altered if we suppose the plane to become rigid and to intercept the communication of the fluid on one side with that on the other. The motion on each side will then be reflected, and the angle of incidence will be equal to the angle of reflection.

8. I propose now to adduce an application of the proposition above demonstrated (Art. 3.) respecting the general law of fluid motion, which may serve to shew its utility. Suppose water in a cylindrical vessel (for instance, a glass tumbler,) to be caused to revolve with considerable rapidity about the axis of the cylinder. There is no practical difficulty in making the fluid revolve so that every particle shall describe approximately a horizontal circle about the axis. Then, the fluid being left to itself after the disturbance, each particle may be considered to move as it does, by reason of a centripetal force tending to the axis in a horizontal plane. This force must be owing to the action of the cylindrical surface on the fluid particles in contact with it, deflecting them continually from a rectilinear course. If V be the velocity of the particles in contact with the surface, and a the radius of the cylinder, the force tending to the axis is $\frac{V^2}{a}$, the effect of friction being neglected. The deflections which this force is continually producing in the directions of radii, are transmitted through the fluid, and as they tend to a centre, will vary, according to the proposition above proved, inversely as the distance from the centre.† Hence the centripetal force at the distance r is $\frac{V^2}{a} \times \frac{a}{r}$, or $\frac{V^2}{r}$. This shews that at any distance r the velocity is still V. Experience seems to confirm this result. For if light substances be strewed on the surface of the water, those nearer the centre always perform their revolutions

* The introduction of this consideration here is merely reverting to a principle which Professor Airy (very properly, I think,) has proposed to make the basis of the mathematical treatment of fluids. Without referring to a principle of this nature, I do not see that problems of reflection can be satisfactorily solved.

† The total motion is compounded of these deflections and rectilinear motions along tangents to the circles, which by Art. 6. may be considered separately.

in less time than those more remote. This is particularly observable in two of the floating particles which are near each other, and at nearly equal distances from the centre. That which is less distant *overtakes* the other, as it ought to do, supposing it to describe a less circle with equal velocity. At the centre a kind of eddy is formed, the more observable as the motion at every point of the surface is more nearly in concentric circles. When the revolving motion takes place in a conical tunnel from which the water is issuing, the appearance at the axis is very remarkable, a hollow space like a sack, being formed a considerable way down the axis. What has been here said may serve to explain in some measure the manner in which *eddies* in any case are produced.

SECTION II.

On the Integration of the Equation $\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0.$

9. M. Poisson has expressed the general integral of this equation by means of definite integrals; (*Memoires de l'Academie des Sciences*, Ann. 1818), and this, I believe, admits of a discussion similar to that applied above (Art. 3.) to the integral of $\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0$. But perhaps the following reasoning, analogous to what was indicated in Art. 5., may be considered sufficient. In whatever manner the fluid is put in motion, we may conceive a surface to be described, which shall be every where perpendicular to the directions of the motions at a given instant of the particles through which it passes. This surface may be of an arbitrary and irregular shape, not necessarily defined by a single equation between x, y, and z. But it must be composed of parts either finite or indefinitely small, which are continuous, and consequently have radii of curvature subject to the same conditions as those of regular curve surfaces. Hence the normals to all the points of any element of the surface will pass through two *focal lines*, situated at the centres and in the planes of greatest and least curvature, and cutting the

directions of the normals at right angles. The motion, being in the normals, will be directed to the focal lines. If we describe another surface indefinitely near the first, and cutting in like manner the directions of the motion at right angles, all the points of any fluid element intercepted between two opposite elements of the surfaces, will at a given instant ultimately have their motion directed to the same focal lines: but this cannot be said *in general* of more than an elementary portion. If we suppose the form of the superficial element to be a rectangle, the normals through all the points of its sides, will inclose a wedge-shaped mass, the transverse section of which at any point, it is easy to shew, will vary as the product of the distances of that point from the focal lines. Hence the velocity in passing at a given instant from the first to the second of the surfaces above-mentioned will vary inversely as this product. Let therefore r and $r+l$ be the distances of the point whose velocity is V, from the focal lines to which the motion is directed. Then $V = \dfrac{C}{r(r+l)}$, in which expression C, l, and the positions of the focal lines are constant at a given instant, when r varies through a space which may either be finite or indefinitely small. Let α, β, γ, be the co-ordinates of the middle of that focal line which is distant by r from the point in question. The velocity (u) in x will then be $V.\dfrac{x-\alpha}{r}$; the velocity (v) in y, $V.\dfrac{y-\beta}{r}$; and the velocity (w) in z, $V.\dfrac{z-\gamma}{r}$. Hence

$$udx + vdy + wdz = V\left(\frac{x-\alpha}{r}dx + \frac{y-\beta}{r}dy + \frac{z-\gamma}{r}dz\right).$$

Now since $r^2 = (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2$, if we make r vary with x, y, and z, while α, β, γ, remain constant according to what has just been said, we shall have $rdr = (x-\alpha)\,dx + (y-\beta)\,dy + (z-\gamma)\,dz$. Hence $udx + vdy + wdz = Vdr$; and as V is a function of r and t, the right side of the equation is a complete differential of a function of x, y, z, and t, with respect to the three first variables, t being constant. Therefore also the left side is the same. Let the function be ϕ.

Then

$$\frac{d\phi}{dr} = V, \quad \frac{d\phi}{dx} = u, \quad \frac{d\phi}{dy} = v, \quad \frac{d\phi}{dz} = w.$$

We proceed to shew next that the equation

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0, \text{ or } \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0,$$

is satisfied by the kind of motion we have been describing.

10. Let P (Fig. 1.) be the point whose motion we are considering; Or, Nq, the focal lines to which the motion of the element at P is directed. Let PNO be the straight line which passes through P and the focal lines, cutting them in N and O. Suppose O to be the origin of a system of axes, of which ONP is the axis of x, Oy coinciding with the focal line Or the axis of y, and Oz perpendicular to the plane yOx, the axis of z. The co-ordinates of P referred to another system of rectangular axes AX, AY, AZ, are X, Y, Z: p is a point indefinitely near to P, Pp is parallel to AZ, and the co-ordinates of p are X, Y, $Z+\delta Z$: pqr is the straight line which passes through p and the focal lines cutting them in q and r. Now let the equations of Pp referred to the system Ox, Oy, Oz, be $x = az + \alpha$, $y = bz + \beta$, and the equations of pqr, $x = a'z + \alpha'$, $y = b'z + \beta'$. Then

$$\cos \angle Ppq = \frac{1 + aa' + bb'}{\sqrt{1 + a^2 + b^2}\sqrt{1 + a'^2 + b'^2}}.$$

Let $ON = l$, $NP = r$. Hence because Pp passes through P whose co-ordinates referred to the axes Ox, Oy, Oz, are $l + r$, 0, 0, it follows that $l + r = \alpha$, and $\beta = 0$. Thus the equations of Pp become $x = az + l + r$, $y = bz$. Again, because the line pqr passes through r, whose co-ordinates are $x = 0$, $z = 0$, we have $\alpha' = 0$; and because it passes through q, whose co-ordinates are $y = 0$, $x = l$, we have $l = a'z$, and $0 = b'z + \beta'$. Hence $z = \dfrac{l}{a'} = -\dfrac{\beta'}{b'}$, and consequently $\beta' = -\dfrac{lb'}{a'}$. Thus the equations of pqr become $x = a'z$, $y = b'z - \dfrac{b'l}{a'}$. Also because Pp and pqr pass through the same point p, $x = a'z = az + l + r$, and therefore $z = \dfrac{l+r}{a'-a}$. And $y = bz = b'z - \dfrac{b'l}{a'}$: therefore $z = \dfrac{b'l}{a'(b'-b)}$. Hence $\dfrac{l+r}{a'-a} = \dfrac{b'l}{a'(b'-b)}$, which gives $b' = \dfrac{a'b(l+r)}{a'r+la}$. From p draw ps perpendicular on Ox, and let $Ps = \delta$. Then $\delta = x - (r+l)$. But $x = a'z = \dfrac{a'(l+r)}{a'-a}$. Therefore $\delta = \dfrac{a(l+r)}{a'-a}$. Hence it will be found that $a' = \dfrac{a(l+r+\delta)}{\delta}$, and $b' = \dfrac{b(l+r+\delta)}{r+\delta}$. This latter quantity, if we neglect powers of δ above the first, is equal to $\dfrac{b(l+r)}{r}\left(1 + \dfrac{l\delta}{r(l+r)}\right)$. Therefore by substitution

$$\cos \angle Ppq = \frac{1 + \dfrac{a^2}{\delta}(r+l+\delta) + \dfrac{b^2(l+r)}{r}\left(1 + \dfrac{l\delta}{r(l+r)}\right)}{\sqrt{1+a^2+b^2}\sqrt{1 + \dfrac{a^2}{\delta^2}(r+l+\delta)^2 + \dfrac{b^2(r+l)^2}{r^2}\left(1 + \dfrac{l\delta}{r(l+r)}\right)^2}}$$

$$= (\text{neglecting } \delta^2, \&c.)\ \frac{a^2(r+l) + \left(1 + a^2 + b^2\dfrac{l+r}{r}\right)\delta}{a\sqrt{1+a^2+b^2}\,(r+l+\delta)}$$

$$= \frac{a}{\sqrt{1+a^2+b^2}}\left\{1 + \left(\frac{1}{a^2(l+r)} + \frac{b^2}{a^2 r}\right)\delta\right\}, \text{ neglecting as before.}$$

Here $\dfrac{a}{\sqrt{1+a^2+b^2}}$ is the cosine of the angle pPs. Hence if V = the velocity at P in Ox, and w the part resolved in the direction parallel to AZ, $w = \dfrac{Va}{\sqrt{1+a^2+b^2}}$. Let w' = the resolved portion of the velocity at p in the same direction. Now the velocity at p is ultimately the same as that at s, and is therefore equal to $V \cdot \dfrac{r(r+l)}{(r+\delta)(r+l+\delta)}$, according to the law of variation from P to s determined by the considerations with which we commenced this investigation. Neglecting powers of δ above the first, this quantity becomes $V\left(1-\frac{\delta}{r}-\frac{\delta}{r+l}\right)$. Consequently

$$w' = V\left(1-\frac{\delta}{r}-\frac{\delta}{r+h}\right)\cos\angle Ppq$$

$$=\frac{Va}{\sqrt{1+a^2+b^2}}\left(1+\frac{\delta}{a^2(l+r)}+\frac{b^2\delta}{a^2r}\right)\left(1-\frac{\delta}{l+r}-\frac{\delta}{r}\right)$$

$$=w\left(1+\frac{(1-a^2)\delta}{a^2(l+r)}+\frac{(b^2-a^2)\delta}{a^2r}\right).$$

But $\delta = \delta Z\cos pPs = \delta Z \times \frac{a}{\sqrt{1+a^2+b^2}}$. Hence

$$\frac{w'-w}{\delta Z} = V\left(\frac{1-a^2}{1+a^2+b^2}\cdot\frac{1}{l+r}+\frac{b^2-a^2}{1+a^2+b^2}\cdot\frac{1}{r}\right).$$

If now α, β, γ, be the angles which the axis AZ makes with Ox, Oy, Oz, respectively, we have

$$\cos^2\alpha = \frac{a^2}{1+a^2+b^2},\quad \cos^2\beta = \frac{b^2}{1+a^2+b^2},\quad \cos^2\gamma = \frac{1}{1+a^2+b^2}.$$

Hence passing from differences to differentials,

$$\frac{dw}{dz} = (\cos^2\gamma-\cos^2\alpha)\frac{1}{l+r}+(\cos^2\beta-\cos^2\alpha)\frac{1}{r}\;\ldots\ldots(1).$$

So if α', β', γ', be the corresponding angles for the axis of Y, and α'', β'', γ'', for that of X, v the velocity in Y, and u that in X, we shall have by a like process,

$$\frac{dv}{dY} = (\cos^2\gamma'-\cos^2\alpha')\frac{1}{l+r}+(\cos^2\beta'-\cos^2\alpha')\frac{1}{r}\;\ldots\ldots(2),$$

$$\frac{du}{dX} = (\cos^2\gamma''-\cos^2\alpha'')\frac{1}{l+r}+(\cos^2\beta''-\cos^2\alpha'')\frac{1}{r}\;\ldots\ldots(3).$$

But as a, a', a'', are the angles which Ox makes with three rectangular axes,

$$\cos^2 a + \cos^2 a' + \cos^2 a'' = 1,$$

$$\text{so}\quad \cos^2\beta + \cos^2\beta' + \cos^2\beta'' = 1,$$

$$\text{and}\quad \cos^2\gamma + \cos^2\gamma' + \cos^2\gamma'' = 1.$$

Therefore by adding the equations (1), (2), (3),

$$\frac{du}{dX} + \frac{dv}{dY} + \frac{dw}{dZ} = 0.$$

11. The general conclusion from all that precedes is, that the law of the variation of the velocity from any point to another indefinitely near in the direction of the motion, at a given instant, may be expressed by $\frac{C}{r(r+l)}$, the quantities C, r, and l, being such as we have stated in Art. 9. If $l=0$, we have as a particular case, $V = \frac{C}{r^2}$. In my former paper on the motion of fluids, I assumed, as it now appears, incorrectly, that $\frac{C}{r^2}$ represents the general law of the variation of the velocity. None, however, of the results in that paper are affected by the assumption. For instance, the expression for

$$\frac{d\phi}{dt}, \left(\text{viz. } \int \frac{dV}{dt} ds + f'(t)\right),$$

as it only requires that ϕ should be a function of r and t, will remain the same. This expression may also be briefly obtained thus. We have seen that $\frac{d\phi}{dr} = V$. Now as r is ultimately in the direction in which the velocity V takes place, if a line commencing at a given point be drawn constantly in the direction of the motion at a given instant of the points through which it passes, dr may be considered the increment of this line. Hence if we call its length s reckoned from the fixed point, $\frac{d\phi}{dr} = \frac{d\phi}{ds} = V$. Then integrating, $\phi = \int V ds + f(t)$;

and differentiating under the sign $\int$, $\frac{d\phi}{dt} = \int \frac{dV}{dt} ds + f'(t)$. Hence substituting for $\frac{d\phi}{dt}$ in the known expression for the pressure (p),

$$p = \int (Xdx + Ydy + Zdz) - \int \frac{dV}{dt} ds - \frac{V^2}{2} - f'(t).$$

If V be always the same in quantity and direction at the same point,

$$\frac{dV}{dt} = 0: \text{ so that, } p = \int (Xdx + Ydy + Zdz) - \frac{V^2}{2} - f'(t).$$

This equation may thus be considered to be strictly deduced from the general equations of fluid motion.

Considerations analogous to those applied (Arts. 6 and 7) to motion in a plane, might here be introduced to shew that the motion at any point, when due to several causes, is the resultant of the motions which would be produced by the causes acting separately; and also to determine the same law of reflection at a plane surface.

12. The following simple instance of fluid motion may serve to illustrate some points of the preceding theory. BCD (Fig. 2.) is a conical vessel with its axis vertical. A mass of fluid $DBbd$ is made to descend so that its lower surface bd is bounded by a horizontal plane to which any arbitrary velocity is given. The upper surface is also supposed to be plane and to be kept horizontal by the force of gravity. It is required to find the consequent velocity and pressure at every point of the fluid.

It is evident that the motion will be in vertical planes passing through the axis, and will be the same in all such planes. Take therefore two planes making an indefinitely small angle with each other, and let AB, AE, be their intersections with the upper surface, ab, ae, with the lower. Let $PQSR$ be an element of the upper surface, P and R being equidistant from A, as also Q and S. If now at any instant lines commencing at the four points P, Q, R, S, be continually drawn in the direction of the motion at the points through which they pass, these lines must be rectilinear, because there is no curvilinear motion at the boundaries of the fluid, and therefore no cause to impress a curvilinear motion on the parts interior. The straight lines commencing at P and R will intersect ab and ae at p and r, points equidistant from a, and those commencing at Q and S will intersect the same lines at q and s also equidistant from a. Now from the law of the variation of the velocity above found, at every point of the cuneiform element Ps, the velocity will be inversely proportional to its transverse section. Let therefore $V =$ the vertical velocity with which db is made to descend, and v the vertical velocity with which the surface DB descends. Let $AB=a$, $AQ=x$, $PQ=\lambda$, $ab=b$, $aq=x'$, $pq=\lambda'$, and the angle $BAE=\epsilon$. Then the element $PQSR=x\epsilon\lambda$, and $pqsr=x'\epsilon\lambda'$. These elements are proportional to the transverse sections at P and p; and the vertical velocities V, v, are to each other as the velocities at p and P in Pp. Hence $\frac{V}{v}=\frac{x\epsilon\lambda}{x'\epsilon\lambda'}=\frac{x\lambda}{x'\lambda'}$. But $\frac{V}{v}$ also $=\frac{a^2}{b^2}$. Hence $\frac{x\lambda}{x'\lambda'}=\frac{a^2}{b^2}$. If we take $x=a$, x' must $=b$, because the motion is along the slant surface. Therefore in this case $\frac{\lambda}{\lambda'}=\frac{a}{b}$. Suppose λ to be given, and let λ_1 be the consequent value of λ'. Then $\frac{\lambda}{\lambda_1}=\frac{a}{b}$, and $\frac{a-\lambda}{b-\lambda_1}=\frac{a}{b}$. If now x be taken $= a-\lambda$, from what has been just shewn, x' will $= b-\lambda_1$. Hence $\frac{(a-\lambda)\lambda}{(b-\lambda_1)\lambda_2}=\frac{a^2}{b^2}$, and consequently $\frac{\lambda}{\lambda_2}=\frac{a}{b}$. Therefore $\lambda_2=\lambda_1$; and so on. From this it follows that if AB and ab be divided into the same number of indefinitely small equal parts, the straight lines joining the corresponding points of division will give the directions of the motion, which is consequently every where directed to the vertex of the cone. Hence the velocity at any point W whose distance CpW from C is ρ, varies as $\frac{1}{\rho^2}$. Let $CA=h$, $Ca=k$, $\angle ACW=\theta$; then the velocity at $p=V\sec\theta$, and the velocity at $W=V\sec\theta\,.\,\frac{k^2\sec^2\theta}{\rho^2}$; this resolved in the vertical direction gives $\dfrac{Vk^2\sec^2\theta}{\rho^2}$, which $= \dfrac{Vk^2}{\rho^2\cos^2\theta}$ = velocity at Z. Hence the vertical velocity is the same at all points of any horizontal plane, and the fluid will consequently descend in parallel slices.* Let us now determine the pressure at any point on the particular supposition that V is uniform. Then if

$$w = \frac{Vk^2\sec^3\theta}{\rho^2} \text{ the velocity at } W, \ \frac{dw}{dt} = \frac{V\sec^3\theta}{\rho^2} \times 2k\frac{dk}{dt} = -\frac{2V^2k\sec^3\theta}{\rho^2}.$$

And

$$\int\frac{dw}{dt}ds = -\int\frac{dw}{dt}d\rho = \int\frac{2V^2k\sec^3\theta\, d\rho}{\rho^2} = -\frac{2V^2k\sec^3\theta}{\rho} + C.$$

Hence

$$p = C - gz + \frac{2V^2k\sec^3\theta}{\rho} - \frac{V^2k^4\sec^6\theta}{2\rho^4}.$$

And as when $z = h$, $p = 0$, and $\rho\cos\theta = h$, it follows that

$$p = g(h-z) + 2V^2\sec^2\theta\left\{\frac{k}{z} - \frac{k^4}{4z^4} - \frac{k}{h} + \frac{k^4}{4h^4}\right\}.$$

The above solution I do not consider to be of any value, except as illustrating the *process* to be followed in determining mathematically the way in which the interior of a mass of fluid is affected as to velocity and pressure, in consequence of given conditions at the boundaries. This part of the theory of fluid motion is very defective.

* I obtained this result in the number of the *Phil. Mag.* and *Annals of Philosophy* for Jan. 1831, but omitted to shew that it is entirely dependent on the arbitrary condition that the *inferior* surface of the fluid is bounded by a horizontal plane. On any other supposition the problem would be one of much greater difficulty. This omission has not without reason caused a misapprehension as to the application of the solution, on the part of Berzelius in a notice taken of it in his Annual Review. (*Jahres-Bericht,* 1833.)

SECTION III.

Application of the Principles of the foregoing Section to an instance of the Resistance of an Incompressible Fluid to a Body bounded by a Spherical Surface moving in it.

13. Let a solid sphere, partially immersed in water, being of less specific gravity than the fluid, be drawn along in a horizontal direction with a given uniform velocity; it is required to find the height of its centre above the horizontal surface of the water.

We shall suppose for the sake of simplicity, that the fluid is unlimited in extent both in the vertical and horizontal directions, and that the surface of the sphere is so smooth that it impresses no velocity on the water in contact with it in the direction of a tangent plane. Let $CDBE$ (Fig. 3.) be the sphere, O its centre, ADE the intersection of the horizontal surface of the fluid by a vertical plane through the centre of the ball; OQ a line through the centre parallel to ADE. This will be the direction of the motion of O, since the velocity is supposed to have become uniform, and ON to be constant. Let A, a fixed point in ADE, be the origin of co-ordinates, $AN = \alpha$, $NO = \gamma$, at any instant. Then the velocity (V) of $O = \frac{d\alpha}{dt}$. Draw OB vertical; let P be any point of the surface immersed; through P draw the spherical arcs PQ, PB, and let the angle $QOP = \theta$, and the angle $PQB = \omega$. The velocity impressed by the sphere on the fluid at P is $V\cos\theta$, as none is impressed in the direction of a tangent plane. This velocity is directed to the point O, because in the case of a spherical surface $l = 0$. Hence if a = the radius of the sphere, $V\cos\theta = \frac{C}{a^2}$. (Art. 11.) The velocity at every point of the line OP produced, will at a given instant be in the direction of this line, because when the fluid is of unlimited extent, there is no cause* to produce motion at any point of the line, but the impression made at P, which is transmitted instantaneously, varying at different distances from O according to the law of the inverse square. Hence if R be a point in OP produced, and $OR = r$, the velocity at $R = \frac{C}{r^2} = \frac{Va^2 \cos\theta}{r^2}$. Let ADE be the axis of x, a vertical through A the axis of z reckoned positive downwards, and a line through A perpendicular to the plane of these two the axis of y. Then if the co-ordinates of R be x, y, z, we shall have $r^2 = (x-\alpha)^2 + y^2 + (z+\gamma)^2$; and $\cos\theta = \frac{x-\alpha}{r}$. Therefore the velocity (v) at R,

$$= \frac{Va^2(x-\alpha)}{\{(x-\alpha)^2 + y^2 + (z+\gamma)^2\}^{\frac{3}{2}}}.$$

And

$$\frac{dv}{dt} = \frac{dv}{d\alpha} \cdot \frac{d\alpha}{dt}, \text{ (for } V \text{ and } \gamma \text{ are constant)},$$

$$= \frac{Va^2(3\cos^2\theta - 1)}{r^3} \cdot \frac{d\alpha}{dt}$$

$$= \frac{V^2a^2(3\cos^2\theta - 1)}{r^3}.$$

Hence

$$\int \frac{dv}{dt} dr = f(t) - \frac{V^2a^2}{2r^2}(3\cos^2\theta - 1).$$

Therefore, gravity being the only force acting on the fluid, the pressure (p) at R,

* This cannot be said of the parts of the fluid adjacent to the radii produced which pass through the circle in which the surface of the water meets the surface of the sphere, because the water outside of the conical surface formed by these radii must be put in motion by that within by reason of the difference of pressure occasioned by the motion. On account of the difficulty of estimating this effect, it is left out of consideration in our solution, which can therefore be only considered approximate.

$$= \int g\,dz - \int \frac{dv}{dt}\,dr - \frac{v^2}{2}$$

$$= gz + \frac{V^2 a^2}{2r^2}(3\cos^2\theta - 1) - \frac{V^2 a^4}{2r^4}\cos^2\theta - f(t).$$

When r is indefinitely great this equation becomes $p = gz - f(t)$; and as for this value of r the velocity $= 0$, p must $= gz$; therefore $f(t) = 0$. If now we put $r = a$, and $z = z_1$, the co-ordinate of P, we obtain the pressure (p_1) at P, $= gz_1 + \dfrac{V^2\cos 2\theta}{2}$. The portion of this resolved in the vertical direction is $p_1 \times \cos \angle POB$. But from the spherical triangle PQB, $\cos \angle POB = \cos\omega \sin\theta$. Therefore the vertical pressure is $p_1 \cos\omega\sin\theta$. The element of the surface at $P = a\,d\theta \times a\sin\theta\,d\omega$. Hence the whole vertical pressure $= \iint p_1 a^2 \sin^2\theta\cos\omega\,d\theta\,d\omega$

$$= ga^2 \iint z_1 \sin^2\theta\cos\omega\,d\theta\,d\omega + \frac{V^2 a^2}{2}\iint \sin^2\theta\cos 2\theta\cos\omega\,d\theta\,d\omega.$$

The first term is plainly the weight of water displaced, and is therefore equal to $\dfrac{\pi g}{3}(2a^3 - 3a^2\gamma + \gamma^3)$, the specific gravity of the water being 1. The integrations with respect to ω must be taken from $\omega = -\cos^{-1}\dfrac{\gamma}{a\sin\theta}$ to $+\cos^{-1}\dfrac{\gamma}{a\sin\theta}$, and the integrations with respect to θ from $\sin^{-1}\dfrac{\gamma}{a}$ to the supplement of that arc. Between these limits of ω, $\int\cos\omega\,d\omega = 2\sqrt{1 - \dfrac{\gamma^2}{a^2\sin^2\theta}}$; and between the limits of θ,

$$2\int \sin^2\theta\cos 2\theta\,d\theta\sqrt{1 - \frac{\gamma^2}{a^2\sin^2\theta}} = -\frac{\pi}{2}\left(1 - \frac{\gamma^4}{a^4}\right).$$

Therefore if $W =$ the weight of the sphere, which is the same as the whole vertical pressure, and $w =$ the weight of fluid displaced,

$$W = w - \frac{\pi V^2 a^2}{4}\left(1 - \frac{\gamma^4}{a^4}\right).$$

This result shews that the weight of fluid displaced is greater than the weight of the sphere, and consequently that the centre O is *lower* than it would be in a state of rest.

Suppose a portion of the sphere to be cut off by a horizontal section at the distance of b from the centre; and let γ become γ', the centre being still above the surface of the water. Then if we suppose the motion to be always in the direction of the radii*, and the horizontal bottom to have no effect in impressing motion, the equation for this case will be,

$$W = w - \frac{\pi V^2 a^2}{4}\left\{\left(1 - \frac{\gamma'^4}{a^4}\right) - \left(1 - \frac{b^4}{a^4}\right)\right\}$$

$$= w - \frac{\pi V^2 a^2}{4} \cdot \frac{b^4}{a^4}\left(1 - \frac{\gamma'^4}{b^4}\right).$$

The difference between W and w is here less than before on account of both the factors $\frac{b^4}{a^4}$ and $1 - \frac{\gamma'^4}{b^4}$; for $\frac{\gamma'}{b}$ is greater than $\frac{\gamma}{a}$. This seems to shew that curved bottoms tend to depress the vessel when it begins to move, and consequently to increase the resistance.

As the term $\frac{V^2 a^2}{2} \iint \sin^2\theta \cos 2\theta \cos\omega \, d\theta \, d\omega$ is positive from $\theta = \sin^{-1}\frac{\gamma}{a}$ to $\theta = 45°$, and from $\theta = 135°$ to θ = the supplement of $\sin^{-1}\frac{\gamma}{a}$, let us integrate for the portion of the surface corresponding to these limits, or what amounts to the same, take the double of the integral between the first limits, those of ω remaining the same as before. In order to abstract from the consideration of the portion of the surface not taken into account in this integration, we may suppose the portions for which we integrate to be connected by a cylindrical surface, the radius of which $= a \sin 45°$. The length of this cylindrical part may be any we please: the vertical pressure against it will be only equal to the

* This again cannot be true in the direction of the radii which pass through the lower circular boundary of the surface.

weight of fluid displaced. Also the shape of the floating body above the part immersed is of no importance to the problem. The form of the whole body may be such as is described in Fig. 4, *ABCDEF* being a half cylinder of which the axis is *GH*, and *ALC*, *FKD*, the extreme portions of the body, bounded by spherical surfaces which have their centres at *M* and *N*. Now in general $2\iint \sin^2\theta \cos 2\theta \cos \omega \, d\theta d\omega$, commencing at $\theta = \sin^{-1}\frac{\gamma}{a}$, and ending at any other value of θ, will be found to be

$$\cos\theta\left(2\sin^2\theta + 1 - \frac{\gamma^2}{a^2}\right)\sqrt{\sin^2\theta - \frac{\gamma^2}{a^2}} - \frac{1}{2}\left(1 - \frac{\gamma^4}{a^4}\right)\cos^{-1}\frac{\cos\theta}{\sqrt{1-\frac{\gamma^2}{a^2}}}.$$

And if we put $\cos\theta = \frac{1}{\sqrt{2}}$, we shall have

$$W = w + \frac{V^2 a^2}{4}\left\{\left(2 - \frac{\gamma^2}{a^2}\right)\sqrt{1 - \frac{2\gamma^2}{a^2}} - \left(1 - \frac{\gamma^4}{a^4}\right)\cos^{-1}\frac{1}{\sqrt{2 - \frac{2\gamma^2}{a^2}}}\right\}.$$

As the second term is necessarily positive, the floating body will be higher than it would be in a state of rest, and consequently the surface against which the resistance acts becomes less by an increase of velocity.

To obtain a numerical result respecting the rise of the body corresponding to a given velocity, we will suppose for the sake of simplicity of calculation that when the vessel is at rest, the centres of the spherical ends and consequently the axis of the cylindrical part, are in the plane of the horizontal surface of the water. This circumstance may be produced by *loading* the upper part of the body without altering its specific gravity. Let l = the length of the axis of the cylindrical portion. Then the area of the horizontal section of the vessel at the level of the water surface is $lD + \frac{\pi D^2}{4} - \frac{D^2}{2}$, its breadth being D. Now $W - w$ must be equal to the difference of the quantities of fluid displaced in the states of rest and motion, and is therefore equal to $\gamma g\left(lD+\frac{\pi D^2}{4}-\frac{D^2}{2}\right)$, γ being small. Therefore neglecting powers of $\frac{\gamma}{a}$ above the first,

$$\left(lD+\frac{\pi D^2}{4}-\frac{D^2}{2}\right)\gamma g=\frac{V^2 D^2}{8}\left(2-\frac{\pi}{4}\right).$$

Let $\frac{l}{D}=3$. It will then be found that $V^2=696^{\text{ft}}\times\gamma$. And if $\gamma=$ one inch, or $\frac{1}{12}$, this equation gives $V=5{\cdot}19$ miles per hour; consequently if $V=10{\cdot}4$ miles per hour, $\gamma=4$ inches.

In general, neglecting $\frac{\gamma^2}{a^2}$, &c.

$$W-w=\frac{V^2a^2}{2}\left(\sin\theta\cos\theta\,(2\sin^2\theta+1)-\frac{\theta}{2}\right),$$

$$\text{also}\quad W-w=\gamma g\left\{lD+\frac{D^2}{2\sin^2\theta}(\theta-\sin\theta\cos\theta)\right\}\ \text{nearly};$$

therefore, as $D=2a\sin\theta$, it will be found that

$$\gamma=\frac{V^2}{4g}\cdot\frac{\sin 2\theta\,(2\sin^2\theta+1)-\theta}{4m\sin^2\theta-\sin 2\theta+2\theta},\quad m \text{ being put for } \frac{l}{D}.$$

If θ be assumed equal to 15°, and $m=3$, this equation gives $V=7{\cdot}35$ miles per hour when $\gamma=4$ inches.

These results, which probably are but very rough approximations to matters of fact, may yet suffice to shew that when vessels and boats of the usual forms sail in the open sea, they may be expected to rise in some degree upon an increase of their velocity, and so much the more as they are less adapted to *cleave* the water. Our theory shews that the rise is the same for bodies of the same shape and proportions, moving with the same velocity, whatever be their absolute magnitudes; also that this effect is equally due to the pressures on the front and stern of the vessel. The theory, in fact, determines these pressures to be in every respect alike, so that if we proceeded to investigate the total pressure in the horizontal direction, we should find it to be nothing, when the motion is uniform. This may serve to shew that, if friction be left out of consideration, a front ill-adapted to cleave the water, is not unfavorable to speedy motion, if the stern be of the same shape; and that the resistance to the motion of vessels in the open sea is principally owing to the friction of the water against their surface. This cause operates to produce unequal actions on the front and stern, making the directions of the motions of the particles in contact with the surface of the former, *less* inclined to the horizon than they would be in the case of no friction, and of those in contact with the surface of the latter *more* inclined. To counteract this inequality probably the stern should be less curved than the front.

SECTION IV.

General Propositions respecting the Motion of Compressible Fluids.

14. THE considerations applied at the beginning of Section II. to incompressible fluids, are equally applicable to compressible. I shall therefore assume that in a mass of fluid in which the density varies as the pressure, the directions of the motion at all the points of any element pass at a given instant through two focal lines. Let ρ be the density at a point distant by r and $r+l$ from the focal lines, and V the velocity: ρ' and V' the same for a point indefinitely near the former. Also let the transverse section of a cuneiform element $aclk$ (Fig. 5.) which is bounded by four planes passing through the focal lines kl, mn, be at the first point $efgh$, and at the other, $abcd$. The pressure and consequently the density will be the same at all points of the section eg; as also the velocity; at least our reasoning does not apply to cases in which this condition is not fulfilled. The same may be said of the section ac and of all sections intermediate to ac and eg.

Let now the area of $eg=m$, and that of $ac=m'$. Then if the motion which exists at a given instant, be supposed to be continued uniform for the small time τ, the quantity of fluid which passes the section eg in that time, is $m\rho V\tau$, and that which passes ac is $m'\rho' V'\tau$. Hence the increment of matter between the two sections is $-(m'\rho' V'\tau - m\rho V\tau)$, whether the velocity tend from or to the focal lines, being considered *negative* in the latter case. The increment of density ($\delta\rho$) of the element in the time τ, is consequently $-\dfrac{(m'\rho' V' - m\rho V)\tau}{m(r'-r)}$. But $\dfrac{m'}{m} = \dfrac{r'(r'+l)}{r(r+l)}$.

Hence

$$-\frac{\rho' V' r'(r'+l) - \rho V r(r+l)}{r'-r} = r(r+l)\frac{\delta\rho}{\tau};$$

And passing from differences to differentials,

$$-r(r+l)\frac{d\rho}{dt} = \frac{d.V\rho r(r+l)}{dr};$$

$$\text{or} \quad -\frac{d\rho}{dt} = \rho\frac{dV}{dr} + V\frac{d\rho}{dr} + \rho V\left(\frac{1}{r} + \frac{1}{r+l}\right) \ldots\ldots (\text{A.})$$

As before $udx + vdy + wdz = V\dfrac{x-\alpha}{r}dx + V\dfrac{y-\beta}{r}dy + V\dfrac{z-\gamma}{r}dz = Vdr$, if α, β, γ, be the co-ordinates of the middle point of the focal line kl. Now as we have supposed that in passing from one point to another of the element $acge$, the change of velocity at a given instant depends only on the change of r, we may consider V a function of r and t, and Vdr a differential of a function of r and t. Then $udx + vdy + wdz = d\phi$, a complete differential of a function of x, y, and z; and $\dfrac{d\phi}{dr} = V$. But in this case we have the known equation,

$$a^2 \text{ Nap. log. } \rho = \int(Xdx + Ydy + Zdz) - \frac{d\phi}{dt} - \frac{V^2}{2} \ldots\ldots (\text{B.})$$

Therefore considering X, Y, Z, to be independent of the time,

$$\frac{a^2 d\rho}{\rho dt} = -\frac{d^2\phi}{dt^2} - V\frac{dV}{dt} = -\frac{d^2\phi}{dt^2} - \frac{d\phi}{dr}\cdot\frac{d^2\phi}{drdt}.$$

But from (A),

$$\frac{a^2 d\rho}{\rho dt} = -\frac{a^2 d\rho}{\rho dr} \cdot \frac{d\phi}{dr} - a^2 \frac{d^2\phi}{dr^2} - a^2 \frac{d\phi}{dr}\left(\frac{1}{r} + \frac{1}{r+l}\right).$$

And differentiating (B) with respect to space only,

$$\frac{a^2 . d\rho}{\rho} = X dx + Y dy + Z dz - d . \frac{d\phi}{dt} - V dV.$$

If the variation be from one point to another in the direction of the motion, $dx = \frac{x-a}{r} dr$, $dy = \frac{y-\beta}{r} dr$, $dz = \frac{z-\gamma}{r} dr$. Hence,

$$\frac{a^2 . d\rho}{\rho dr} = X . \frac{x-a}{r} + Y . \frac{y-\beta}{r} + Z . \frac{z-\gamma}{r} - \frac{d^2\phi}{drat} - \frac{d\phi}{dr} \cdot \frac{d^2\phi}{dr^2}.$$

Substituting this value of $\frac{a^2 d\rho}{\rho dr}$ in the foregoing equation, and then equating the two values of $\frac{a^2 d\rho}{\rho dt}$, we shall obtain,

$$\left(a^2 - \frac{d\phi^2}{dr^2}\right)\frac{d^2\phi}{dr^2} - 2\frac{d\phi}{dr} \cdot \frac{d^2\phi}{drat} - \frac{d^2\phi}{dt^2} + a^2\frac{d\phi}{dr}\left(\frac{1}{r} + \frac{1}{r+l}\right)$$

$$+ \frac{d\phi}{dr}\left(X\frac{x-a}{r} + Y\frac{y-\beta}{r} + Z\frac{z-\gamma}{r}\right) = 0 \ldots\ldots (\mathrm{C.})$$

This is an equation of general application. If, as a particular case, l, a, β, γ, each $= 0$, we shall have the equation I obtained in my former paper (Art. 4.) by assuming ϕ to be a function of $\sqrt{x^2+y^2+z^2}$, and t in the equation (n) of the *Mecanique Analytique* (Part II. Sect. XII. Art. 8.)

It may be proved as in Art. 11, that $\frac{d\phi}{dt} = \int \frac{dV}{dt} ds$, as for incompressible fluids, and that the equation applicable to steady motion is,

$$a^2 \text{ h. l. } \rho = f(X dx + Y dy + Z dz) - \frac{V^2}{2} + f'(t).$$

15. If r be indefinitely great in equation (C), the motion is in parallel lines, and putting $r=c+s$, $\frac{d\phi}{dr}=\frac{d\phi}{ds}$. Let $\frac{d\phi}{ds}=\omega$, and suppose no force to act; the equation for this case becomes

$$\frac{d^2\phi}{ds^2}-\frac{2\omega}{a^2-\omega^2}\cdot\frac{d^2\phi}{dsdt}+\frac{1}{a^2-\omega^2}\cdot\frac{d^2\phi}{dt^2}=0.$$

This equation combined with $a^2\,\mathrm{N.l.}\,\rho=-\frac{d\phi}{dt}-\frac{\omega^2}{2}$, gives as a particular integral, $\omega=a\,\mathrm{N.l.}\,\rho=f\{s-(a+\omega)t\}$. By varying a little the mode of integrating, I found also $\omega=a\,\mathrm{N.l.}\,\rho=f\left(\frac{as}{a+\omega}-at\right)$, (*Camb. Phil. Trans.* Vol. III. Part III. p. 399), and endeavoured to shew the way in which each integral ought to be applied. But this enquiry was unnecessary; for the integral may present itself under an unlimited number of different forms. The equations

$$\omega=a\,\mathrm{N.l.}\,\rho=f\{s-(a+\omega)t+\psi(\omega)\},\ \text{ or }\ \omega=a\,\mathrm{N.l.}\,\rho=f\left\{\frac{s-(a+\omega)t}{\chi(\omega)}\right\},$$

will equally satisfy the differential equations, being, in fact, only different forms of the first-mentioned integral. The principle according to which it now appears to me, an integral of this nature should be employed, is to apply it *immediately* only to the parts of the fluid immediately acted upon by the arbitrary disturbance, in order to determine the law according to which the initial velocity is transmitted to the contiguous parts; then to determine the law of transmission from these to the next; and so on in succession. In the present instance by making s and t vary so that ω and ρ remain the same, we shall find $a+\omega$ for $\frac{ds}{dt}$ the velocity of transmission, under whatever form the integral may appear. The second term ω of this quantity is due to the transmission of velocity through space by the motion of the particles themselves; the other a is the velocity of propagation along the particles. In this example, as the velocity and density are propagated uniformly and undiminished, it is easy to determine at any instant the velocity and density at any given point, which result from a given disturbance. In other cases in which the velocity of propagation is variable, the determination would be more difficult, but must probably be arrived at by the same principle of reasoning. Variable propagation is analogous to variable motion, as uniform propagation to uniform motion, and would seem to require integration to determine the time at which the effect of a given disturbance is felt at a given place.

16. If in the equation (C), a be an indefinitely great quantity, the terms which do not contain a^2 as a factor may be neglected in comparison of those which do, and the equation will become

$$\frac{d^2\phi}{dr^2} + \frac{d\phi}{dr}\left(\frac{1}{r} + \frac{1}{r+l}\right) = 0,$$

which by integration gives $\dfrac{d\phi}{dr} = \dfrac{C}{r(r+l)}$, the same as for incompressible fluids. This result was to be expected, because a, as is well known, is the velocity of propagation in the compressible fluid, and when this becomes infinite, the propagation is instantaneous, and the fluid therefore incompressible.

If l be indefinitely great, it will be found in the same way that $\dfrac{d\phi}{dr} = \dfrac{C}{r}$, and the motion is such as was considered Art. 3.

Let now $\dfrac{d\phi}{dr}$ be very small compared to a, and X, Y, Z, and l each $= 0$. The equation (C) reduces itself to

$$a^2\frac{d^2\phi}{dr^2} + 2a^2\frac{d\phi}{dr} - \frac{d^2\phi}{dt^2} = 0; \text{ or } a^2 . \frac{d^2 . r\phi}{dr^2} = \frac{d^2 . r\phi}{dt^2};$$

a particular integral of which is $r\phi = F(r - at)$. This gives

$$\frac{d\phi}{dr} = \frac{F'(r-at)}{r} - \frac{F(r-at)}{r^2} \ldots\ldots (1.)$$

At the same time, because $a^2 \,\text{N.l.}\,\rho = -\dfrac{d\phi}{dt}$ nearly, we shall have

$$a\,.\,\text{N.l.}\,\rho = \frac{F'(r-at)}{r} \ldots\ldots (2.)$$

The equations (1) and (2), involving but *one* arbitrary function, can apply only to a *single* disturbance, which takes place in a direction tending from a centre, as I have elsewhere shewn*. It is important to observe that when r is very small, the term of equation (1) which involves r^2 in the denominator may be much greater than that involving r. In fact, if we expand the functions, supposing r to be very small,

$$\frac{d\phi}{dr} = \frac{F'(-at)}{r} + F''(-at) + \&\text{c}.$$

$$-\frac{F(-at)}{r^2} - \frac{F'(-at)}{r} - \frac{F''(-at)}{2} - \&\text{c}.$$

$$= -\frac{F(-at)}{r^2} \text{ (nearly) } = \frac{\psi(t)}{r^2}.$$

When therefore the disturbance is made by a sphere of very small radius r, the motion is transmitted from its surface to other parts of the fluid, nearly as if the fluid were incompressible.

SECTION V.

Application of the Principles of the foregoing Section to determine the Resistance of the Air to the Motion of a Ball-Pendulum.

17. FOR the sake of simplicity, I will suppose gravity not to act. The ball being spherical and perfectly smooth, the direction of the motion of the aerial particles in contact with its surface tends at every

* *Camb. Phil. Trans.* Vol. III. p. 402.

instant from its centre. Therefore $l=0$. Also if the radius of the ball be supposed very small, the equation $\frac{d\phi}{dr}=\frac{\psi(t)}{r^2}$, obtained at the end of the preceding Article, will be *approximately* applicable to the motion of the fluid in contact with the ball. Hence the velocity which is impressed at any point of the spherical surface may be considered to be transmitted instantaneously in the direction of the radius through that point, and to decrease according to the law of the inverse square of the distance. The problem, with the limitations above made is solved in the same manner for air as for water.

Let now the origin of co-ordinates be A, (Fig. 6.), the position of the centre of the ball when it hangs at rest. Let its centre perform oscillations of very small extent in nAN, which we will consider to be rectilinear. Suppose N to be the position of the centre at the time t reckoned from a given epoch, and call AN, a. Take P any point of the surface, join NP and produce it to R, and let NPR make an angle θ with ANQ, and the plane RNQ an angle β with the plane SAQ. The velocity of the centre $=\frac{da}{dt}$; and the velocity of the air at $P=\frac{da}{dt}\cos\theta$. Hence if $NP=b$, and $NR=r$, the velocity at $R=\frac{b^2\cos\theta}{r^2}\cdot\frac{da}{dt}$. Now if AN be the axis of x, AS of z, and a line through A perpendicular to the plane SAN, the axis of y, and the co-ordinates of R be x, y, z, then $r^2=(x-a)^2+y^2+z^2$. Consequently the velocity (V) at $R=\frac{b^2\cos\theta}{(x-a)^2+y^2+z^2}\cdot\frac{da}{dt}$. Hence differentiating V with respect to the time only,

$$\frac{dV}{dt}=\frac{d^2a}{dt^2}\cdot\frac{b^2\cos\theta}{r^2}+\frac{2b^2\cos\theta(x-a)}{r^4}\cdot\frac{da^2}{dt^2}+\frac{b^2}{r^2}\cdot\frac{da}{dt}\cdot\frac{d.\cos\theta}{dt}.$$

But as $\cos\theta=\frac{x-a}{r}$, $\frac{d.\cos\theta}{dt}=-\frac{1}{r}\cdot\frac{da}{dt}+\frac{\cos^2\theta}{r}\cdot\frac{da}{dt}=-\frac{\sin^2\theta}{r}\cdot\frac{da}{dt}$.

Therefore

$$\frac{dV}{dt}=\frac{d^2a}{dt^2}\cdot\frac{b^2\cos\theta}{r^2}+\frac{2b^2\cos^2\theta}{r^3}\cdot\frac{da^2}{dt^2}-\frac{b^2\sin^2\theta}{r^3}\cdot\frac{da^2}{dt^2}.$$

Hence

$$\int \frac{dV}{dt}\,dr = -\frac{d^2a}{dt^2}\cdot\frac{b^2\cos\theta}{r} - \frac{2b^2\cos^2\theta - b^2\sin^2\theta}{2r^2}\cdot\frac{da^2}{dt^2} + f(t).$$

Substituting in equation (B),

$$a^2\,\text{N.l.}\,\rho = \frac{d^2a}{dt^2}\cdot\frac{b^2\cos\theta}{r} + \frac{b^2}{2r^2}(2\cos^2\theta - \sin^2\theta)\frac{da^2}{dt^2} - \frac{b^4\cos^2\theta}{2r^4}\cdot\frac{da^2}{dt^2} - f(t).$$

When $r =$ infinity, $\rho = 1$: therefore $f(t) = 0$. Hence when $r = b$,

$$a^2\,\text{N.l.}\,\rho = \frac{d^2a}{dt^2}\,b\cos\theta + \frac{\cos 2\theta}{2}\cdot\frac{da^2}{dt^2}.$$

Where $\rho = 1$, let $p = \Pi = a^2$. Hence when $\rho = 1 + \sigma$, $p = a^2(1+\sigma) = \Pi + a^2\sigma$. But because σ is very small, $a^2\,\text{N.l.}\,\rho = a^2\sigma$ very nearly. Therefore,

$$p = \Pi + \frac{d^2a}{dt^2}\cdot b\cos\theta + \frac{\cos 2\theta}{2}\cdot\frac{da^2}{dt^2}.$$

The total pressure resolved in the direction NA is $\iint p\,b^2\cos\theta\sin\theta\,d\theta\,d\beta$, from $\beta = 0$ to $\beta = 2\pi$, and from $\theta = 0$ to $\theta = \pi$. It will consequently be found to be equal to $\frac{4\pi b^3}{3}\cdot\frac{d^2a}{dt^2}$: and if $\Delta =$ the ratio of the specific gravity of the ball to that of air, the accelerative force produced by this pressure is $\frac{1}{\Delta}\cdot\frac{d^2a}{dt^2}$. But the accelerative force of gravity in the same direction, if $SA = h$, is $\frac{ga}{h}\left(1 - \frac{1}{\Delta}\right)$, taking account of the weight of air displaced. Hence

$$-\frac{d^2a}{dt^2} = \frac{ga}{h}\left(1 - \frac{1}{\Delta}\right) + \frac{1}{\Delta}\cdot\frac{d^2a}{dt^2},$$

$$\text{or}\quad \frac{d^2a}{dt^2} = -\frac{ga}{h}\,\frac{1 - \frac{1}{\Delta}}{1 + \frac{1}{\Delta}} = -\frac{ga}{h}\left(1 - \frac{2}{\Delta}\right)\ \text{nearly.}$$

Therefore if L be the length of the seconds pendulum in vacuum, l in air, $l = L\left(1 - \frac{2}{\Delta}\right)$.*

The correction of the length of the pendulum is thus determined to be double of what it would be if the motion of the air were not considered. It is to be observed that these calculations apply strictly only to the case of a very small ball. The experiments of M. Bessel give 1·956 for the coefficient of $\frac{1}{\Delta}$. Those of Mr Baily, which were made most nearly under the circumstances which the theory supposes, give 1·864. The effects of friction and of the suspending wire, would tend to make the coefficient rather greater than less than 2. I am therefore unable to account for the difference between the experimental and theoretical determinations, which it appears by Mr Baily's experiments, is greater as the radius of the ball is greater, excepting perhaps the confined space of the apparatus may have had some effect on the experimental results.

It would not be difficult to shew from the nature of the analytical expressions, that if the confined space in which the balls vibrate were taken into account in the theory, the same results would be obtained for two balls of different diameters, vibrating in different spaces, if the linear dimensions of the spaces were in the proportion of the diameters, their forms being alike. If this could be verified experimentally, it would shew that the difference of the values of the numerical coefficient which Mr Baily calls n, for balls of different diameters, as well as its deviation from the theoretical value 2, is very probably owing to the confined space of the vacuum apparatus. It would at any rate be desirable to ascertain by experiment whether the same ball gives the same value of n, when it oscillates in apparatus of different dimensions.

PAPWORTH ST EVERARD,
March 3, 1834.

* This result I obtained in the *London* and *Edinburgh Philosophical Magazine* (September, 1833), by assuming the principle of the conservation of *vis viva,* without employing equation (B).

IX. *Theory of Residuo-Capillary Attraction; being an Explanation of the Phenomena of Endosmose and Exosmose on Mechanical Principles.* By the *Rev.* J. POWER, M.A. *Fellow and Tutor of Trinity Hall, and late Fellow of Clare Hall, Cambridge.*

[Read *March* 17, 1834.]

1. THE curious and elegant law, according to which an interchange takes place between two fluids separated from each other by a thin membrane, one of the fluids generally (but not universally) the lighter of the two, being transmitted in greater abundance, was discovered a few years ago by Dutrochet.*

His experiments tended to show that the unknown force which operated this effect, whether measured by the fluid transmitted in a given time, or by the pressure required to stop the process, was, for the same substances, proportional to the difference of densities of the mixtures on each side of the membrane.

The vast importance of this law in animal and vegetable physiology, renders it highly desirable that its theory should be investigated on mechanical principles, and such is the object of the present enquiry.

2. The opinion which would attribute this phenomenon to the existence of electrical currents, is now pretty nearly abandoned, even by Dutrochet himself, with whom it originated, and who maintained it with great zeal, until the publication of his later researches, in which he

* *L'Agent immediat du Mouvement Vital,* (Paris, 1826), and *Nouvelles Recherches sur l'Endosmose et l'Exosmose* (Paris, 1828).

confesses himself compelled to resign it, though he does so with manifest reluctance. That electricity, artificially excited, is capable of accelerating the process, is indeed sufficiently established by the experiments of Dutrochet; but it is equally certain that this agent is by no means essential to the operation, since, in the natural process, the most delicate galvanometer gives no indication of its existence.

3. To me it appears unquestionable, that the phenomenon results from the corpuscular attractions, which the particles constituting the membrane and the fluids, exert upon each other: that electricity, by heightening or modifying these attractions, should produce a sensible effect upon the operation, is nothing more than its ordinary chemical agency would lead us to expect.

4. By corpuscular attractions are meant the forces which the ultimate atoms of different materials, whether simple or compound, exert upon each other. These forces are enormously great (though not infinite) when the particles are in immediate contact, but diminish with extreme rapidity, as the particles separate, becoming insensible at a sensible distance. The effects of corpuscular attraction are different, according as it is exerted between particle and particle, or between mass and mass. In the former case it gives rise to the phenomena of chemical affinity; and in the latter, to those of cohesion, adhesion, and capillary attraction, which may be regarded in general, as the mutual attraction of contiguous masses, being the combined effect of the corpuscular attractions of their integrant particles. It is under this point of view that La Place has considered the subject of capillary attraction, and his theory will be of the greatest use in the present investigation.

5. Although no pores can be detected in the membranous partition by the help of the most powerful microscope, yet the fact that the fluids are transmitted, is a certain proof that such pores exist. They must indeed be extremely minute, and it will be seen that it is on this very minuteness that the energy of the sustaining force depends. These pores must be regarded as communicating with the opposite fluids

at their two extremities, while the fluids meet and mix in the interior.

6. Dutrochet argues that capillary attraction cannot be the cause of endosmose, because it can only raise a fluid to a small height in a capillary tube, and is utterly incapable of drawing it beyond the limits of the tube.

In stating these objections, he perhaps does not consider that the height at which a fluid may be sustained in a capillary tube is inversely as its diameter, and consequently in a tube of so extremely small a diameter as those of which it is necessary to suppose the membrane to consist, that height might be almost indefinitely great. It is true that in the case of a *single* fluid, this effect would require for its production that the tubes themselves should be coextensive with the fluid raised; but this is no longer necessary when the two ends of the tube are immersed in *different* fluids. The reason why a homogeneous fluid cannot be drawn beyond the limits of the tube, is, that, were it to be so, the tube, acting equally at its two ends, would produce no effect whatever upon the fluid. But the circumstances are very different when the extremities communicate with different fluids. In that case the full residual effect, consisting of the difference of effects, which the same tube indefinitely extended, is capable of impressing separately upon the two fluids, might be produced by an extremely small length of tube, not exceeding a small multiple of the sphere of attraction of the particles of the tube, and there is no doubt that the thickness of the finest membrane is a considerable multiple of this magnitude. In fact, if we cut off from the ends of the tube a distance greater than the tube's sphere of sensible attraction, it is plain that the fluids which occupy the intermediate part, in whatever way they may communicate there, will suffer no effective attraction from the tube, since every elementary portion will be drawn by it equally in both directions. The only effective attractions will therefore be those exerted by an insensible portion at each extremity; we may therefore imagine these two portions to be brought together as near as we please without any diminution of effect.

7. In order to form some sort of estimate of the forces which may be expected to result from residual attractions of this kind, let us suppose the fluids to be water and alcohol, and the tube to be of glass. Now Gay Lussac found by experiments of great accuracy, that in a tube of glass whose diameter was 1.29441 millimetres, water would stand at the height of $23^{mi}.3791$, and alcohol of specific gravity 0.8196 (that of water being 1) at the height of $9^{mi}.39808$. This column of alcohol would be equivalent to $7^{mi}.7176$ of water; the difference of effects would therefore be measured by a column of water of $15^{mi}.6615$. Suppose now the diameter of the tube to be diminished a thousand times, or to become $0^{ml}.001294$, the column of water which measures the difference of effects would be $15661^{mi}.5$: or, since the French millimetre = .0393708 of an English inch, a glass tube of diameter $0^{in}.0000507$, or about the twenty-thousandth of an inch, would produce a residual effect, with water and alcohol, measured by 616.6 inches or $51^{ft}\ 4^{in}$ of water, which is equivalent to the pressure of nearly two atmospheres. When it is considered that a platina wire of one three-thousandth of an inch in diameter may be seen by the naked eye, it is probable that the magnitude we have assigned to the capillary tube is considerably greater than the diameter of the membranous pores, which evade the powers of the strongest microscope. From this example I think the conclusion may be fairly drawn, that, so far at least as the *magnitude* of the force is concerned, we need be under no apprehension but that the residual capillary forces are sufficient to account for the sustaining force of endosmose. How far they will account for the law of its variation will be seen hereafter.

8. An attempt to explain the phenomenon by the principles of capillary attraction has been already made by a distinguished mathematician, Mons. Poisson. He first abstracts from the pressure of the adjacent fluids, by supposing their altitudes above the membrane to be inversely as their densities. The fluid in the tube being now equally pressed on both sides, he supposes that that liquid, for which the tube has the stronger attraction is drawn by this attraction to the opposite end, thus filling the whole tube. The fluid within the tube, he now argues, will be urged by two forces: 1st, the attraction of the liquid

to which it belongs; 2dly, the attraction of the opposite liquid. If then the latter attraction be superior to the former, the fluid which fills the tube, he says, will be drawn in an uninterrupted stream into the opposite vessel.

Dutrochet justly objects to this theory, that it will only account for a motion in one direction, whereas the phenomenon of exosmose requires a corresponding motion in the opposite direction.

Professor Henslow, in a number of the *Foreign Quarterly*, suggests as a modification of Poisson's theory, that whilst the fluid within the tube is carried in the direction of the stronger attraction, the natural tendency of the fluids to mix, may carry the other fluid (or, perhaps, a slight infusion of it) in the opposite direction, and thus produce the exosmose.

I perfectly agree with Professor Henslow that the natural process of mixture is the cause of the exosmose, it being only necessary to suppose that the rapidity with which this process extends itself within the tube is somewhat greater than the velocity with which the whole mass of fluid which fills the tube is drawn in the opposite direction.

But the theory of Poisson is further objectionable on this account, that it makes the continuation of the process solely dependent on the action of the fluids, whereas the experiments of Dutrochet incontestably demonstrate that it depends mainly on the action of the membrane. No doubt, the effect both of the fluids upon themselves, and of the membrane upon the fluids, ought to be taken into consideration, and this will be done in the following theory.

9. If a capillary tube be divided into two parts by a plane perpendicular to its axis; the attraction of one of these parts upon a fluid which exactly fills the *other* part is $\frac{1}{2}cH$, c being the contour of the inner surface of the tube, and H a certain definite integral or constant, depending solely on the materials of which the tube and the fluid consist. The contour of the tube may be of any shape whatever, curved or polygonal. (See *Mec. Cel.* Sup. au X^e Liv. pp. 14—21.)

It is convenient to give a name to the quantity H; we will call it the *capillary affinity* between the two materials of which the tube and fluid are composed.

It is easy to see that the quantity H will remain unchanged if we conceive the tube and the fluid to exchange their materials; for, by the equality of action and reaction, the elementary attractions, of which $\frac{cH}{2}$ is the sum, will be equal in the two hypotheses. The tube may be regarded either as solid or fluid, and this fluid may be either the same as that which fills its interior or a different one.

If we conceive the density of the inner fluid to be diminished in any ratio, all the elementary attractions, and therefore H, will be diminished in the same ratio; and if, further, the density of the tube be diminished in any ratio, H will be diminished in the compound ratio.

10. Next, let u and v be the original quantities by volume of two unmixed fluids. Then, if no penetration of dimensions takes place, $u+v$ will be their volume after mixture. If we regard the fluids after mixture as coexisting, each with a diminished density, within the same volume $u+v$, calling r_1 and ρ_1 these diminished or partial densities, (r) and (ρ) the densities of the unmixed fluids, we shall have

$$\frac{r_1}{(r)} = \frac{u}{u+v}, \quad \text{and} \quad \frac{\rho_1}{(\rho)} = \frac{v}{u+v};$$

whence

$$\frac{r_1}{(r)} + \frac{\rho_1}{(\rho)} = 1.$$

Again,

$$r_1 + \rho_1 = r,$$

r being the total or ordinary density of the mixture. The two last equations serve to express r_1 and ρ_1 in terms of r, (r) and (ρ).

If then we have a second mixture of the same two original fluids, we shall have

$$\frac{r_2}{(r)} + \frac{\rho_2}{(\rho)} = 1,$$

$$\text{and} \quad r_2 + \rho_2 = \rho,$$

where r_2 and ρ_2 are the two partial densities, and ρ the total density of this second mixture. These equations serve in like manner to express r_2 and ρ_2 in terms of ρ, (r) and (ρ).

11. Let us now endeavour to express the mutual capillary affinities which exist between the two mixtures just mentioned, and a third material (as that of a membrane or tube), in terms of the densities of these mixtures and the mutual capillary affinities between this same material and the unmixed fluids.

Let the former affinities be denoted by H, K, L, M, N, namely,

H between the tube and the first mixture,
K between the tube and the second mixture,
L between the first mixture and the second,
M between the first mixture and its like,
N between the second mixture and its like;

and let the latter affinities be denoted by (H), (K), (L), (M), (N), namely,

(H) between the tube and the fluid of density (r),
(K) between the tube and the fluid of density (ρ),
(L) between the fluids of densities (r) and (ρ),
(M) between the fluid of density (r) and its like,
(N) between the fluid of density (ρ) and its like.

The attraction $\frac{1}{2}cH$ of No. (9) will be the sum of two partial attractions, namely, that of the tube upon two coexistent cylinders of the opposite fluids, whose densities are those of the original unmixed fluids diminished in the ratios $r_1 : (r)$ and $\rho_1 : (\rho)$. Hence by the latter part of that No.,

$$\tfrac{1}{2}cH = \tfrac{1}{2}c(H)\frac{r_1}{(r)} + \tfrac{1}{2}c(K)\,.\frac{\rho_1}{(\rho)};$$

whence

$$H = (H)\frac{r_1}{(r)} + (K)\frac{\rho_1}{(\rho)}.$$

By similar reasoning, superposing all the different attractions, each diminished in the ratio of the densities of the attracting and attracted materials, we shall have

$$K = (H)\frac{r_2}{(r)} + (K)\frac{\rho_2}{(\rho)},$$

$$L = (L).\frac{r_1}{(r)}\cdot\frac{\rho_2}{(\rho)} + (L)\frac{r_2}{(r)}\cdot\frac{\rho_1}{(\rho)} + (M)\frac{r_1}{(r)}\cdot\frac{r_2}{(r)} + (N).\frac{\rho_1}{(\rho)}\cdot\frac{\rho_2}{(\rho)},$$

$$\therefore M = 2(L)\frac{r_1}{(r)}\cdot\frac{\rho_1}{(\rho)} + (M)\cdot\frac{r_1^2}{(r)^2} + (N)\frac{\rho_1^2}{(\rho)^2},$$

$$N = 2(L)\frac{r_2}{(r)}\cdot\frac{\rho_2}{(\rho)} + (M)\cdot\frac{r_2^2}{(r)^2} + (N).\frac{\rho_2^2}{(\rho)^2}.$$

By combining each of the last five equations with the four equations of No. 10, and eliminating r_1, r_2, ρ_1, ρ_2, we shall obtain H, K, L, M, N, in terms of the actual densities r, ρ, the original affinities (H), (K), (L), (M), (N), and the original densities (r) and (ρ).

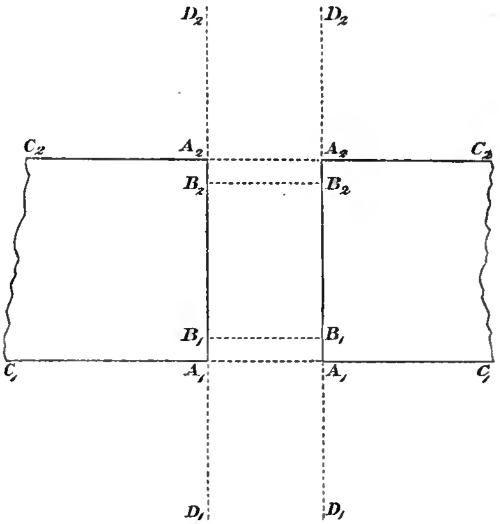
12. Let us now proceed to apply the principles of the three last numbers to explain the experiments of Dutrochet. And first let us consider those which relate to the statical force of endosmose. In these experiments the process was allowed to continue until the fluid raised, or rather the mercurial column which was hydrostatically substituted for it, attained its maximum altitude; at this moment the densities of the two liquids were experimentally determined; and instituting different experiments with different mixtures of the same substances, Dutrochet found that the maximum altitudes were proportional to the corresponding differences of densities.

The substances employed in his experiments were saccharine or gummy solutions on the one hand, and water on the other, and the water was found to be transmitted in greater abundance. Common treacle is a very convenient substance for experiments.

Let us suppose then that the lower part of the endosmometer is filled with treacle, and having a thin membrane tied over its mouth, is immersed in water; and let us suppose that the fluid is allowed to ascend until the operation ceases.

At this moment we may regard the capillary pore which traverses the membrane, as communicating at its two extremities with fluid in the same state of mixture as the fluid in the contiguous vessels, there being a gradual transition from one end to the other.

Let $C_1C_1C_2C_2$, be a portion of the membrane, $A_1A_1A_2A_2$ one of its capillary pores, with its axis at right angles to the plane of the membrane, communicating originally with the water at A_1A_1, and with the treacle at A_2A_2, but when the fluid has reached its maximum altitude, communicating with the first mixture of No. (10) at A_1A_1, and with the second mixture of that No. at A_2A_2.



Imagine the geometrical figure of the tube, (not its material) to be produced both ways to D_1 and D_2, and cut off from each end of the tube a distance A_1B_1, A_2B_2, equal to the tube's sphere of sensible attraction.

Since A_1B_1, and A_2B_2, are insensible, we may regard the fluids in $A_1A_1B_1B_1$, and $A_2A_2B_2B_2$, as in the same state of mixture as the fluids in the contiguous vessels.

Let us now estimate all the forces which tend to move the central column $D_1D_1D_2D_2$ in direction of its axis.

It is plain that, in whatever manner the fluids may communicate in the interior of the tube, the tube can produce no effect upon $B_1B_1B_2B_2$, since every elementary portion of this part of the fluid will be drawn in both directions as by an infinitely extended tube.

We may also neglect, as producing equal and opposite forces in both directions, the attraction between the tube A_1B_1, and the fluid $A_1A_1B_1B_1$; between the tube A_2B_2, and the fluid $A_2A_2B_2B_2$; between the fluid tube $C_1A_1D_1$, and $A_1A_1D_1D_1$; between $C_2A_2D_2$, and $A_2A_2D_2D_2$, between the membrane and $C_1A_1D_1$; between the membrane and $C_2A_2D_2$.

Lastly, we may neglect all the mutual actions of the particles composing the central column $D_1D_1D_2D_2$, their tendency being only to mix the opposite fluids, and not to move the column as a mass.

Of the remaining attractions we shall have at one end the attraction of the tube B_1B_2, upon $B_1B_1A_1A_1$, $(=\frac{1}{2}cH)$ + the attraction of the tube A_1B_1, upon $D_1D_1A_1A_1$, $(=\frac{1}{2}cH)$ − the attraction of the fluid tube $C_1A_1D_1$, upon $A_1A_1B_1B_1$, $(=\frac{1}{2}cM)$; constituting the capillary force $\frac{c}{2}$ $(2H-M)$. This will be opposed by a similar force $\frac{c}{2}$ $(2K-N)$ exerted at the other end of the tube. The residual sustaining force is therefore

$$\frac{c}{2}\,.\,(2H-2K-M+N).$$

It now only remains to express this force in terms of the actual densities r and ρ, and the initial constants

$$(r),\ (\rho),\ (H),\ (K),\ (L),\ (M),\ (N).$$

13. For this purpose let

$$\frac{r_1}{(r)} = s_1, \quad \text{and} \quad \frac{r_2}{(r)} = s_2\,;$$

therefore by No. (10).

$$\frac{\rho_1}{(\rho)} = 1 - s_1, \quad \text{and} \quad \frac{\rho_2}{(\rho)} = 1 - s_2\,;$$

making these substitutions in the equations of No. 11., we have

$$H = s_1\,(H) + (1 - s_1)\,(K).$$

$$K = s_2\,(H) + (1 - s_2)\,(K).$$

$$L = s_1 \cdot (1 - s_2)\,(L) + s_2 \cdot (1 - s_1)\,(L) + s_1 s_2\,(M) + (1 - s_1)\,(1 - s_2)\,(N).$$

$$M = 2s_1\,(1 - s_1)\,(L) + s_1^2\,(M) + (1 - s_1)^2\,(N).$$

$$N = 2s_2\,(1 - s_2)\,(L) + s_2^2\,(M) + (1 - s_2)^2\,(N).$$

Hence $2H - 2K - M + N = A\,(H) + B\,(K) + C\,(L) + D\,(M) + E\,(N)$,

where $A = 2 \cdot (s_1 - s_2)$.

$$B = 2 \cdot (1 - s_1) - 2 \cdot (1 - s_2)$$

$$= -2\,(s_1 - s_2).$$

$$C = -2s_1 \cdot (1 - s_1) + 2s_2\,(1 - s_2)$$

$$= -2\,(s_1 - s_2) + 2\,(s_1^2 - s_2^2).$$

$$D = -(s_1^2 - s_2^2).$$

$$E = -(1 - s_1)^2 + (1 - s_2)^2$$

$$= 2\,(s_1 - s_2) - (s_1^2 - s_2^2).$$

The residual force is therefore

$$\frac{c}{2}\,(s_1-s_2)\,\{2(H)-2(K)-2(L)+2(N)\}$$

$$+\frac{c}{2}\cdot(s_1^2-s_2^2)\,\{2(L)-(M)-(N)\}.$$

$$\text{Again,}\quad r=r_1+\rho_1=(r)\cdot\frac{r_1}{(r)}+(\rho)\cdot\frac{\rho_1}{(\rho)}$$

$$=(r)\,s_1+(\rho)\,(1-s_1)\,;$$

$$\therefore\quad s_1=\frac{(\rho)-r}{(\rho)-(r)},$$

$$\rho=r_2+\rho_2=(r)\cdot\frac{r_2}{(r)}+(\rho)\cdot\frac{\rho_2}{(\rho)}$$

$$=(r)\,s_2+(\rho)\,.\,(1-s_2)\,;$$

$$\therefore\quad s_2=\frac{(\rho)-\rho}{(\rho)-(r)};$$

$$\therefore\quad s_1-s_2=\frac{\rho-r}{(\rho)-(r)},$$

$$s_1+s_2=\frac{2(\rho)-\{\rho+r\}}{(\rho)-(r)};$$

$$\therefore\quad s_1^2-s_2^2=2(\rho)\cdot\frac{\rho-r}{\{(\rho)-(r)\}^2}-\frac{\rho^2-r^2}{\{(\rho)-(r)\}^2}.$$

The expression for the residual force is, therefore,

$$\frac{c}{2}\cdot\frac{\rho-r}{(\rho)-(r)}\,\{2(H)-2(K)-2(L)+2(N)+\frac{2(\rho)}{(\rho)-(r)}\,[2(L)-(M)-(N)]\}$$

$$-\frac{c}{2}\cdot\frac{\rho^2-r^2}{\{(\rho)-(r)\}^2}\,\{2(L)-(M)-(N)\},$$

which may be put under the form

$$cA(\rho - r) + cB(\rho^2 - r^2)^*,$$

making

$$A = \frac{1}{(\rho)-(r)} \cdot \left\{(H)-(K) + \frac{(\rho)+(r)}{(\rho)-(r)}\left[(L) - \frac{(\rho)(M)+(r)(N)}{(\rho)+(r)}\right]\right\},$$

$$\text{and } B = \frac{1}{\{(\rho)-(r)\}^2}\left\{(L) - \frac{(M)+(N)}{2}\right\}.$$

The agreement of theory with experiment, then, requires that

$$(L) - \frac{(M)+(N)}{2}$$

should be either nothing, or very small compared with

$$(H)-(K) + \frac{(\rho)+(r)}{(\rho)-(r)} \cdot \left\{(L) - \frac{(\rho)(M)+(r)(N)}{(\rho)+(r)}\right\}.$$

14. When I first began to investigate this subject, certain considerations, which it would be tedious to detail, led me to imagine that the fluids might communicate in the interior of the tube, forming a series of interlacing cylinders one within another, and I found the forces which tended to protrude the cylinders into the opposite fluids, all multiplied by $(L) - \frac{(M)+(N)}{2}$. I therefore looked upon this expression as a measure of the tendency of the fluids to mix, and this tendency being, as experience shows, very small in the case of treacle and water, as well as in the case of the gummy solutions and water, afforded an explanation why the force should be so nearly proportional to the difference of densities, as Dutrochet's

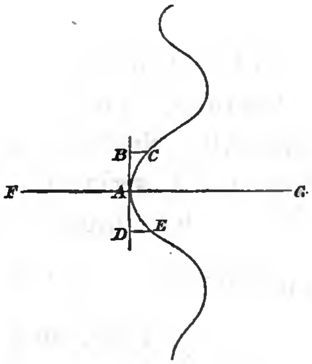
* I have elsewhere erroneously stated, that the residual force is $cA(\rho - r) + cB(\rho - r)^2$, a mistake which I am glad to have this opportunity of correcting.

experiments seemed to indicate. But the preceding theory being perfectly independent of the mode in which the fluids communicate, it is better not to have recourse to a supposition, which is in the slightest degree precarious, especially as I am now prepared to show, that, in whatever way the fluids may arrange themselves within the tube, the rapidity of the mixing process will depend upon the magnitude of $(L) - \frac{(M)+(N)}{2}$.

15. In fact, in whatever manner the mixing process may be effected; we may at any moment imagine the fluid to be divided into an indefinite number of contiguous strata, of any arbitrary or convoluted form, the density being the same for the whole extent of any one stratum, but varying from one to another.

If the surface which separates two contiguous strata be a perfect plane, it is evident, by the equality of action and re-action, that this would be a position of momentary equilibrium, (abstracting from gravity, which I am not here considering.)

Suppose, now, that this surface becomes undulated in an arbitrary way, and take any point A upon it, and draw a tangent plane BAD, including with the surface EAC, a kind of lens $BDEC$, which, with La Place, we may call a meniscus. Draw the normal FAG; and let R_1, and R_2 be the radii of greatest and least curvature at the point A.



Now La Place has shown that the attraction of such a meniscus upon the column of fluid AF is $\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \cdot \frac{H}{2}$, where H is the capillary affinity between the material of the meniscus, and that of the fluid in the sense already defined. (See Supp. au X^e Liv. page 14—17.)

He has also shown that the attraction of the meniscus is the same whichever way it be turned.

If the meniscus instead of consisting of the left hand fluid, (as in the figure), consisted of the right hand fluid, the common boundary being the plane *BAD*, there would be equilibrium, the column *AF* being attracted by the right hand fluid, just as much as the column *AG* is by the left.

Since then the meniscus consists of the left hand fluid instead of the right, the effect of the disturbance upon the column *AF*, tending to draw it in the direction *FA*, is the attraction of the meniscus upon *AF*, regarding it as consisting of the left hand fluid, minus the attraction of the same meniscus regarding it as consisting of the right, that is

$$\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\left(\frac{M}{2}-\frac{L}{2}\right),$$

supposing the left hand fluid to be the first mixture of No. (10), or the lower fluid of No. (12).

If then we estimate the effect in the direction *AF*, it is

$$\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\cdot\left(\frac{L}{2}-\frac{M}{2}\right).$$

In the same way, the effect of the disturbance upon *AG*, in the direction *GA*, is the attraction of the meniscus, regarded as consisting of the left hand fluid, minus the attraction of the same meniscus, regarded as consisting of the right, that is

$$\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\cdot\left(\frac{L}{2}-\frac{N}{2}\right).$$

Hence the whole attraction in the direction *GF*, is

$$\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\left(L-\frac{M+N}{2}\right).$$

If we substitute for L, M, N, the expressions at the commencement of No. (13), we shall find

$$2L - M - N = A(L) + B(M) + C(N), \text{ where}$$

$$\begin{aligned} A &= 2s_1.(1-s_2) + 2s_2.(1-s_1) - 2s_1.(1-s_1) - 2s_2.(1-s_2) \\ &= 2s_1 - 2s_1s_2 + 2s_2 - 2s_1s_2 - 2s_1 + 2s_1^2 - 2s_2 + 2s_2^2 \\ &= 2s_1^2 - 4s_1s_2 + 2s_2^2 \\ &= 2(s_1 - s_2)^2. \\ B &= 2s_1s_2 - s_1^2 - s_2^2 \\ &= -(s_1 - s_2)^2. \\ C &= 2(1-s_1)(1-s_2) - (1-s_1)^2 - (1-s_2)^2 \\ &= -\{(1-s_1) - (1-s_2)\}^2 \\ &= -(s_1 - s_2)^2; \end{aligned}$$

$$\therefore\ 2L - M - N = (s_1 - s_2)^2.\{2(L) - (M) - (N)\}$$

$$= \frac{(\rho - r)^2}{\{(\rho) - (r)\}^2}.\{2(L) - (M) - (N)\}.$$

The effect of the disturbance in the direction GF, is therefore

$$\left(\frac{1}{R_1} + \frac{1}{R_2}\right).\frac{(\rho - r)^2}{\{(\rho) - (r)\}^2}.\left\{(L) - \frac{(M) + (N)}{2}\right\},$$

$$\text{or } \left(\frac{1}{R_1} + \frac{1}{R_2}\right) B\,(\rho - r)^2,$$

consequently if (L) be greater than $\frac{(M) + (N)}{2}$, or, if the capillary affinity of the opposite fluids exceed an arithmetic mean between the capillary affinities of the two fluids for fluids of their own kind, the tendency will be to depart still farther from the position of equilibrium, and the tendency is the greatest where the curvature is the greatest.

16. Hence it is easy to see that the protruding segments of each fluid will become more and more pointed at their summits of greatest curvature as they advance into the opposite fluids, thus forming interlacing spiculæ, shooting into the opposite fluids, and at the same time inosculating with each other by their lateral protrusion, and that this process cannot cease until the fluids have divided each other into segments of a magnitude comparable with that of the sphere of sensible attraction.

Beyond this limit the theory does not hold. It is very possible then, that in some cases a limit may be attained where the mixing fluids have arrived at such a state of subdivision, that the conditions for continuing the subdivision are no longer satisfied; in other cases it is possible that the subdivision may proceed until the ultimate atoms of the opposite fluids act upon each other by ones, twos, and threes, thus effecting a chemical decomposition: nature presents numerous instances of both kinds.

17. But though the mathematical theory is not strictly applicable when the subdivided segments are of less magnitude than the sphere of sensible attraction, it may be considered as an approximation to the truth considerably beyond this limit. For, the most effective part of the attraction of each segment being that exerted by the particles in immediate contact with the normal column, the diminution of the segments will only have the effect of removing the more feeble part of the attractions which the theory takes into the account. It is therefore probable that, even in cases where no chemical decomposition takes place, the subdivision of the fluids may be carried to a limit far beyond that to which the theory is strictly applicable. Besides, the processes of nature are not interrupted of a sudden; the tendency therefore to farther subdivision cannot be suddenly arrested, but in cases where it is ultimately reduced to nothing, it must be so by passing through all degrees of magnitude. This reasoning is further

confirmed by those experiments which demonstrate the almost infinite subdivision of matter by repeated dilution, experiments which are familiar to every one. This infinite subdivision is, in fact, involved in the mathematical conception upon which this theory is founded, namely, that in the state of mixture the two fluids may be regarded as coexisting within the same volume, each with a diminished density. This conception cannot of course be a rigorous representation of nature; but is sufficiently so for the application of La Place's theory, or, which comes to the same thing, for the summation of the attractions by the principles of the Integral Calculus.

18. In cases of simple mixture, unattended with a chemical change, the ultimate segments of the opposite fluids, though in an extreme state of subdivision, have a separate and independent existence, which renders it highly probable, that the volume of the mixed fluids should equal the sum of the volumes of the unmixed fluids. This supposition has been made in the preceding theory, and I find by experiment that in mixtures of treacle and water it is accurately true. The same, I believe, is true in all cases of simple mixture, where no chemical result takes place, such as the precipitation of solids, or the disengagement of heat or other volatile constituents. To liquids whose union is accompanied by such phenomena the present theory is inapplicable, not only on account of the penetration of dimensions, with which such phenomena are generally attended, but on account of the change of affinities, which the escape of some of the constituents must necessarily produce, including heat, which, regarded as a chemical constituent, is as important as any.

19. The addition of a third fluid to one of the liquids, by altering the chemical affinities, must likewise alter the capillary affinities, which are only a different modification of the same corpuscular attractions which produce the former. It is not surprising then, that Dutrochet should have discovered some substances which accelerated the process in his experiments, and others which retarded it or stopped it altogether.

Water impregnated with sulphuretted hydrogen was found not only to stop the process, but to destroy the energy of the membrane for subsequent experiments with pure water and pure saccharine solutions. No doubt the sulphuretted hydrogen had decomposed the surface of the capillary pore, leaving a coating of putrid matter, which was not possessed of such capillary properties as to supply the place of the material of the membrane. That this is the true explanation is shown by the fact, that when the membrane was for a long time steeped in water and well washed, its energy was restored: in fact, the putrid matter being washed away, the membrane presented an unvitiated surface to the fluids. Heat and electricity may be classed amongst these chemical agents, as they operate their effect precisely in the same way, namely, by changing the chemical and consequently the capillary affinities.

20. If we wish to compute the height to which the fluid will rise in the endosmometer, let ζ be the height of the supported column above the surface of the membrane, and z the height of the lower fluid above the same, ω the transverse section of the tube; the difference of the pressures of the cylindrical columns $\omega\zeta$ and ωz, having the common section ω, is $g\rho\omega\zeta - gr\omega z$: this must be counterbalanced by the sustaining force $cA(\rho - r) + cB(\rho^2 - r^2)$, which denotes a pressure on the same scale;

$$\therefore \zeta = \frac{c}{\omega} \cdot \frac{A}{g} \cdot \left(\frac{\rho - r}{\rho}\right) + \frac{c}{\omega} \cdot \frac{B}{g} \cdot \left(\frac{\rho^2 - r^2}{\rho}\right) + \frac{r}{\rho} z.$$

If a column of mercury be hydrostatically substituted for the ascending fluid, as in the experiments of Dutrochet, calling Z the altitude of the mercury, and R its density, we must have

$$Z = \frac{c}{\omega} \cdot \frac{A}{g} \cdot \left(\frac{\rho - r}{R}\right) + \frac{c}{\omega}\frac{B}{g} \cdot \left(\frac{\rho^2 - r^2}{R}\right) + \frac{r}{R} z;$$

this of course being subject to a correction when the cistern of the mercury is not on a level with the membrane.

21. If the pore be circular, let δ be its diameter, then

$$c = 2\pi \cdot \frac{\delta}{2}, \text{ and } \omega = \pi \cdot \frac{\delta^2}{4};$$

$$\therefore \frac{c}{\omega} = \frac{4}{\delta};$$

the sustaining force is therefore inversely proportional to the diameter of the pore, as in ordinary capillary attraction.

Hence we see how the membrane's delicacy of texture contributes to the intensity of the sustaining force.

22. It is now easy enough to see in what manner the process is effected. The residual force $cA(\rho - r) + cB(\rho^2 - r^2)$, which would result if the ends of the tube communicated with fluid of the densities r and ρ, being greater than the altitudinal pressure upon the section ω, would cause the fluid within the tube to move as a mass into the endosmometer, thus bringing fluid more and more diluted to the issuing orifice; this will continue until the residual force is weakened to such a degree as exactly to counterbalance the altitudinal pressure. Contemporaneously with the former motion, the mixing process will transfer the two fluids in opposite directions, the current from the endosmometer towards the water producing the exosmose, and the opposite current supplying the deficiency caused by the exosmose, and therefore not contributing to the endosmose. The diluted fluid which was carried into the endosmometer by the residual force, will gradually mix with the treacle within, whether that mixture be carried on near the orifice of the tube, or whether the diluted fluid be raised by its specific levity higher up in the endosmometer. The extremely small portion of diluted fluid which has thus been transmitted, and the viscosity of the treacle, render it most* probable that it would not be

* This probability amounts nearly to certainty when we consider that the denser fluid has no access to the *lower* part of the transmitted fluid. It is only when a lighter body is *insulated,* or partially insulated, in a denser that it rises by its specific levity.

carried up by its specific levity, but rather adhere to the membrane in the way that bubbles of air adhere to the sides of vessels containing water or mercury. But, be this as it may, the end of the tube which communicates with the endosmometer, will soon be surrounded by a stronger infusion of the treacle, which will again bring the residual force into action; thus a fresh portion of the fluid will be introduced into the endosmometer, and the same process will be repeated as before. For the sake of explanation, I have supposed the residual force to produce its effect discontinuously, but it is easy to see that the process will really be continuous, the united actions of the endosmose and exosmose always keeping the orifices of the tube surrounded by fluid in such a state of dilution that the magnitude of the residual force will be exactly sufficient to create a supply proportioned to the demand arising from the mixing process which is continually proceeding within the endosmometer. The residual force cannot be less than this, for if it were, the encroachment of the treacle upon the issuing orifice would immediately increase it; nor can it be greater, for then the accumulation of the more diluted fluid at that same orifice would immediately diminish it.

23. The quantity transmitted in a given time must depend more upon the rapidity with which the mixing process is carried on within the endosmometer than on the magnitude of the residual force. This force is certainly essential to the transmission, but its effect is no other than that of a pump which supplies the fluid from below as fast as it is wanted, and no faster, and that of a catch or valve to sustain it when it is once elevated. The moving force at the summit of any protruding spicula is by No. (14) represented by $\left(\frac{1}{R_1}+\frac{1}{R_2}\right) B(\rho-r)^2$, and is, therefore, for spiculæ of given shape, as the square of the difference of densities. It might appear then, at first sight, more probable that the quantity of the lower fluid absorbed by the fluid in the endosmometer in a given time, would be more nearly as the square of the difference of densities, than as the simple power of this difference, which is the law the experiments of Dutrochet tend to establish. But such a conclusion would be very precarious, as will appear by the following considerations.

24. Let us imagine two different experiments, all circumstances, as regards the materials, form and disposition of the apparatus, being exactly similar, but the proportions in which the substances are mixed on each side the membrane, being different in the two experiments. Let us suppose also that the mixing process takes place in both experiments after exactly the same *type*, only with different velocities, that is to say, that at certain times, t and t', $t+\tau$ and $t'+\tau'$, $t+2\tau$ and $t'+2\tau'$, &c., the protruding spiculæ from the lighter fluid exist in exactly the same state in both experiments, as regards their number, shape, size and situation.

This supposition being made, the volume of the lighter fluid absorbed by the fluid in the endosmometer in the two experiments, will be equal in the intervals τ and τ': also the summits of the spiculæ will have described the same paths in the two experiments during these same corresponding intervals. Let τ and τ' be indefinitely small, and let us equate the spaces described by the summits of any two corresponding spiculæ between the epochs t and $t+\tau$, t' and $t'+\tau'$, and also between the epochs t and $t+2\tau$, t' and $t'+2\tau'$.

Let a be the sphere of sensible attraction, and imagine a small normal column $2a$ at the vertex of each spicula, being half in one fluid and half in the other.

The two spiculæ having by the hypothesis the same shape, the moving forces upon these columns are as $(\rho-r)^2$ and $(\rho'-r')^2$, and the masses moved are as $a\rho+ar$ and $a\rho'+ar'$, that is, as $\rho+r$ and $\rho'+r'$; the accelerating forces will therefore be as $\dfrac{(\rho-r)^2}{\rho+r}$ and $\dfrac{(\rho'-r')^2}{\rho'+r'}$; let them be $k\dfrac{(\rho-r)^2}{\rho+r}$ and $k.\dfrac{(\rho'-r')^2}{\rho'+r'}$. Then if v and v' be the velocities of the two summits at times t and t', equating the corresponding spaces, we shall have

$$v\tau+\tfrac{1}{2}k\frac{(\rho-r)^2}{\rho+r}\tau^2=v'\tau'+\tfrac{1}{2}k\frac{(\rho'-r')^2}{\rho'+r'}\tau'^2,$$

and

$$2v\tau+\tfrac{1}{2}k\frac{(\rho-r)^2}{\rho+r}4\tau^2=2v'\tau'+\tfrac{1}{2}k\frac{(\rho'-r')^2}{\rho'+r'}4\tau'^2.$$

These equations are equivalent to the following:

$$v\tau = v'\tau', \text{ and } \frac{(\rho - r)^2\tau^2}{\rho + r} = \frac{(\rho' - r')^2\tau'^2}{\rho' + r'};$$

whence

$$\frac{v}{v'} = \frac{\tau'}{\tau} = \frac{\rho - r}{\rho' - r'} \cdot \sqrt{\frac{\rho' + r'}{\rho + r}}.$$

Let q be the volume of fluid absorbed in the times τ and τ', which we have seen to be the same in each experiment; and let $\mathbf{Q}$ and $\mathbf{Q}'$ be the quantities absorbed during a given time T, T not being so great but that r, ρ, r' and ρ' may be considered the same during this interval.

If then there be a law connecting the quantities absorbed in a given time with the densities, we must regard this absorption in each experiment as uniform during the time T;

$$\therefore \mathbf{Q} : q :: T : \tau,$$

$$\text{and } \mathbf{Q}' : q :: T : \tau';$$

$$\therefore \mathbf{Q}\tau = qT = \mathbf{Q}'\tau';$$

$$\therefore \frac{\mathbf{Q}}{\mathbf{Q}'} = \frac{\tau'}{\tau} = \frac{\rho - r}{\rho' - r'}\sqrt{\frac{\rho' + r'}{\rho + r}}.$$

The supposition we have made, as to the exactitude of type in the two mixing processes, is particular; but if there be a general law which is applicable to all cases, that must include the case supposed, and therefore the result of the particular case must coincide with that of the general law. If then there be such a law, it is expressed by the proportion

$$\mathbf{Q} : \mathbf{Q}' :: \frac{\rho - r}{\sqrt{\rho + r}} : \frac{\rho' - r'}{\sqrt{\rho' + r'}}.$$

This being true in different experiments, must be true in different stages of the same experiment.

Now in the same experiment ρ diminishes and r increases as the experiment proceeds, and therefore the variation of $\rho+r$ is small compared with that of $\rho-r$; the quantities absorbed will therefore be pretty nearly in the ratio of the difference of densities, as Dutrochet found them to be. Whether the proportion

$$Q : Q' :: \frac{\rho-r}{\sqrt{\rho+r}} : \frac{\rho'-r'}{\sqrt{\rho'+r'}}$$

may be a more accurate representation of nature than the law of Dutrochet, is left to the test of experiment.

25. It may perhaps be objected to the theory of No. (12), that the ordinary theory of capillary attraction supposes the dimensions of the tube to be incomparably greater than the sphere of sensible attraction, whereas the fact, that these pores are so small as to elude microscopic observation, might lead us to apprehend that their dimensions were of a size comparable with that sphere. The example which has been calculated in No. (7), does not seem to leave any cause for such an apprehension. But supposing this were the case, the only difference it would make in the theory is this: that, whereas, on the former supposition, the quantities $\frac{1}{2}cH$, $\frac{1}{2}cK$, &c., denoted the results of integrations extending from nothing to infinity, and not otherwise depending on the form of the tubes than by involving the contour c as a multiplier; on the second supposition, the limits of the integration will depend on the form of the tubes and the texture of the membrane: but *these limits being the same in the cases compared,* it is easy to see that the theory will be still true on the latter hypothesis, provided we look upon $\frac{1}{2}c(H)$, $\frac{1}{2}c(K)$, &c., as denoting certain unknown limited integrals depending not only upon the nature of the materials, but also upon the form and size of the capillary pores. The residual force will, therefore, on this hypothesis also, be of the form $a(\rho-r)+b(\rho^2-r^2)$.

26. By the application of similar reasoning to the theory of No. (15), it is not difficult to conclude that the moving forces upon the normal

columns at the summits of spiculæ of ***given shape and size*** will be as $(\rho - r)^2$, even when the dimensions of the spiculæ are indefinitely less than the sphere of sensible attraction. For, the attraction of a meniscus bounded on one side by a plane surface, upon the conterminous normal column, will in all cases be a definite integral depending on the shape and size of the meniscus, and the demonstration of La Place, by which he shows that the attraction of such a meniscus is the same whichever way it be turned, is perfectly independent of its size and the shape of its curved surface.

Let then l be the attraction of any meniscus upon the conterminous normal, the meniscus consisting of one mixture, and the normal of the other; m, the attraction of the *same* meniscus when the meniscus and column consist both of the first mixture; and n, the same thing when they consist of the second mixture. Then reasoning exactly as in No. (15), the moving force upon the column GF will be $2l - m - n$; and if (l), (m), (n), be the initial values of l, m, n, it may be shown exactly as before, that $2l - m - n = \frac{(\rho - r)^2}{\{(\rho) - (r)\}^2} \cdot \{2(l) - (m) - (n)\}$, the theory of No. (11) being equally applicable in this case. Hence, however minute the spiculæ may be, the moving force upon the central column will, for spiculæ of given shape, be as the square of the difference of densities.

This consideration applied to the theory of No. (25), gives it a generality which renders it as satisfactory as can well be desired.

J. POWER.

TRINITY HALL,
March 29, 1834.

X. *On Aerial Vibrations in Cylindrical Tubes.* By WILLIAM HOPKINS, M.A. *Mathematical Lecturer of St Peter's College, and Fellow of the Cambridge Philosophical Society.*

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THE problem which has for its object the determination of the motion of a small vibration propagated in an elastic medium along a prismatic tube of indefinite length (the motion of every particle in each section of the tube perpendicular to its axis being the same) was long since solved by Euler and Lagrange. The problem, so nearly allied to this—to determine the motion of an aerial pulsation in a tube of definite length—has not been so satisfactorily solved, the tube being either open at the extremity or stopped with a substance possessing some degree of elasticity. In addition to the difficulties of the former problem, we have in this latter one those still more formidable difficulties which exist in the determination of the circumstances of the motion at the confines of two elastic media in the closed tube, or at the extremity of the open one, where the air in the tube communicates with the circumambient air. These motions must no doubt be determinable from the nature of the media, and the causes producing and maintaining the vibrations, having nothing arbitrary in them, except what may be so in the original disturbance; but I am not aware of any progress having been made in the direct solution of these questions, which now forms one of the greatest desiderata in the application of mathematics to physical science; and in our inability to determine these motions at the extremity of the tube, either by theory or direct observation, we are driven to the necessity of *assumptions.* It is from a difference in these assumed conditions that we have the different solutions which mathematicians have given of the problem in question. The principle on which we ought to proceed in making such assumptions is obvious; they should be subjected to no restrictions, (not imposed on them by our theory), which are not necessary to draw those deductions and inferences from our mathematical results, which admit of verification by experiment, to the test of which an assumption, in any degree arbitrary, must necessarily be subjected before it can claim our confidence. The physical conditions however on which the solutions of this problem depend, (as far as it is distinct from that of the motion of a wave along a uniform tube of indefinite length), have neither been assumed on this principle, nor subjected, as far as I am aware, to this experimental test. It has been principally with the view of remedying these defects that I have prosecuted the researches, an account of which I have now the honour of laying before the Society.

1. The physical conditions assumed by Euler, and by most of those who have since written on the subject, are, that the particles of air at the extremity of a *closed* tube are always at rest; and that no condensation of the air takes place at the extremity of an *open* one. The first condition involves the supposition of the perfect rigidity of the material with which the tube is stopped. This cannot be accurately true, but probably leads to no error very appreciable to observation. The second condition assumes an equality in the densities of the external air, and of that within the tube immediately at its open extremity, during the whole time of the vibrating motion, in the same manner as if the air were at rest. This supposition carries with it but little appearance of being even very approximately true; for it is difficult to conceive how a sonorous wave could thus be produced and maintained in the surrounding air from the open extremity of the tube, and it appears perfectly irreconcileable with the fact of the sudden cessation of sound after the cause producing it has ceased. M. Poisson, struck with these objections, has assumed another physical condition as applicable to any tube, whether open or stopped, viz. that there exists at the extremity of the tube, during the whole motion, a constant relation between the velocity of the particles of the fluid at any instant, and its condensation, this relation depending on the nature of the substance

with which the fluid at the extremity of the tube is in immediate contact. This condition is manifestly less restrictive than those of Euler, since it involves no supposition of the perfect rigidity of bodies, and leaves room for a certain degree of condensation and rarefaction of the fluid at the extremity of the open tube, thus removing the difficulty above-mentioned respecting the maintaining of aerial pulsations from the open end, in the circumambient air; while it enables us also to account in some measure for the rapid cessation of sound with the cessation of the cause producing the vibratory motion of the air in the tube.

2. The two authors above-mentioned have written elaborately on this subject of the vibrations of elastic fluids in tubes. Mr Challis also in his paper published in the Transactions of this Society, (Vol. III.), has been led to the consideration of the conditions which hold at the closed or open extremity of the tube in which the air is in a state of sonorous vibration, though the determination of this point forms with him a collateral rather than a principal object. He assumes that a pulse proceeding along a cylindrical tube will be reflected from the further extremity if the tube be stopped, the intensity of the reflected pulse being equal to that of the incident one; and that if the extremity of the tube be open, it will pass into the circumambient air, sending back no reflected wave within the tube. If this were the case, it would immediately account for the apparently instantaneous cessation of sound above-mentioned; but there are other equally obvious phenomena, for which this hypothesis appears to offer no adequate solution.

3. It will be observed, that Euler has supposed either the velocity of the particles or their condensation to have, at the extremity of the tube, a constant value, independently of the time; while M. Poisson has supposed this constancy of value to belong to the quantity expressing the relation between the velocity and condensation. It does not however appear to me probable that any such conditions, independently of the time, should hold. All the above assumptions are equally arbitrary, and equally require to be put to the test of experiment. In

applying this test, I find that the deductions from the results, derived from any of the three hypotheses above-mentioned, do not sufficiently accord with the observed phenomena to be perfectly satisfactory. This discrepancy is more particularly observable in the position of the nodes or points of minimum vibration in the open tube. According to Euler's hypothesis, these nodes would be places of perfect rest; and they would be distant from the open end by an exact odd multiple of $\frac{\lambda}{4}$, where λ = length of a whole undulation. From the hypothesis of M. Poisson, their positions would be the same as in the above case, but they would become points of minimum vibration, and not of perfect rest. Mr Challis's supposition would lead to the conclusion that no nodes existed in this case, except they should be produced by some vibration of the tube itself, a cause the total inadequacy of which to produce any appreciable effect, must be immediately recognized by every one who has made experiments on this subject. The facts, as determined by experiment, are very obvious; and it appears that there are nodes, which are points of minimum vibration and not of perfect rest; that they are equidistant, but that denoting this distance by $\frac{\lambda}{2}$, the distance between the open extremity and the nearest node is considerably less than $\frac{\lambda}{4}$. I shall not in this place proceed further with the detail of experimental facts; but shall first shew how the theory of this subject may be generalized by the assumption of conditions less restrictive than those which have been made by the writers I have mentioned. In the second section, I shall describe the experiments which have suggested these assumptions; and shall conclude with some observations on the resonance of tubes, so far, more particularly, as it is allied to the investigations contained in this paper.

The form under which I shall consider the problem, is that under which it presents itself, as nearly as possible, in the experiments I have to describe.

SECTION I.

4. SUPPOSE the tube AB, (fig. I.), open at A, and stopped at B, with some substance possessing any degree of elasticity; and suppose the vibrations first produced and kept up by a rigid diaphragm, vibrating according to a given law at A, and perfectly excluding the air within the tube from any communication with the external air. We have the usual equations

$$\left.\begin{aligned} v &= f(at-x) + F(at+x) \\ as &= f(at-x) - F(at+x) \end{aligned}\right\} \ldots\ldots(\mathrm{A}),$$

v denoting the velocity of a particle at distance x from the origin, and s the condensation at the same point at the time t, and a being the velocity of propagation of an aerial pulse along the tube.

One of our conditions must necessarily be, that the velocity of the air within the tube and immediately in contact with the diaphragm, must constantly have the same velocity as the diaphragm itself, constrained to move according to a given law. Let this velocity $= \phi(at)$. Then shall we have

$$\phi(at) = f(at) + F(at) \ldots\ldots(1).$$

5. To ascertain the nature of the second condition, which must hold at B, where the motion of the wave propagated along the tube is interrupted, we must consider the effect which will be produced on the stop by the action of the air within the tube. The vibratory motion will produce alternations of condensation and rarefaction at the extremity B, which will tend to put the substance forming the stop in vibration; and if it will admit of vibrations having the same period as those of the air in the tube, this effect will be produced by the constant reiteration of the cause above-mentioned. If the substance is not susceptible of vibrations of this kind, no appreciable effect will be produced upon it.

The determination of the nature of these vibrations, or of the function expressing the velocity at any instant of the extreme section of the stop, will necessarily depend on the material of which it is made; and any solution of the problem in question, independently of this consideration, cannot be regarded as complete. Still, whatever may be the nature of the stop, we know that the period of its vibrations must be the same as for those in the tube; and it is also manifest, that each vibration of the stop must begin at a time later by an interval at least nearly $= \frac{l}{a}$, ($l =$ the length of the tube), than the corresponding vibration in the diaphragm at A, whence the original disturbance is supposed to proceed. I say that this interval is *nearly* equal $\frac{l}{a}$, because certain phenomena, of which I shall speak hereafter, seem inconsistent with its being in particular cases exactly $= \frac{l}{a}$. I shall therefore, to give the assumption all the generality possible, consider it as generally $= \frac{l}{a} +$ arbitrary quantity, to be determined in each particular case by experiment. Hence then, if ψ denote the form of the function of the time expressing the velocity of the extreme section of the stop, we shall have the velocity $= \psi\{at - (l + c)\}$, c being arbitrary. This must also be the velocity of the extreme section of the air at B, consequently we have as a second condition

$$\psi\{at - (l + c)\} = f(at - l) + F(at + l) \ldots\ldots (2).$$

We have from (1)

$$\phi(at + l) = f(at + l) + F(at + l);$$

and eliminating $F(at + l)$,

$$f(at + l) - f(at - l) = \phi(at + l) - \psi\{at - (l + c)\};$$

or, writing $at + l$ for at,

$$f(at + 2l) = f(at) - \psi(at - c) + \phi(at + 2l) \ldots\ldots (\mathrm{B}).$$

The substance forming the stop being known, so that we might regard the vibrations produced in it under given circumstances determinable, the relation between the functions ψ and f would be known, and the function f would be the only unknown one in the above functional equation, from which, any particular form being assigned to ϕ, that of f must be determined. The arbitrary quantity which will be involved in the solution of this equation, must be determined by the original value of the function f.

6. We have here supposed the tube to be stopped, but the equation (B) will still be true for the open tube, $\psi\{at-(l+c)\}$, denoting always the velocity of the extreme section at the time t.

Equation (2) gives us

$$F(at+l)=-f(at-l)+\psi\{at-(l+c)\},$$

and writing $at+x$, for $at+l$,

$$F(at+x)=-f\{at-(2l-x)\}+\psi\{at-(2l+c-x)\}.$$

Hence,

$$\left.\begin{aligned} v&=f(at-x)-f\{at-(2l-x)\}+\psi\{at-(2l+c-x)\}\\ as&=f(at-x)+f\{at-(2l-x)\}-\psi\{at-(2l+c-x)\}\end{aligned}\right\}\text{......(C)}.$$

The form of f being determined by equation B, these last equations will give the complete solution of the problem.

7. Before we proceed to consider particular cases, we will exhibit these equations (C) under another form, which will be useful in deducing some general inferences as to the nature of the motion in the tube.

Let τ denote a period of time, from the commencement of the motion at A, less than that which is necessary for the pulse to travel twice the length of the tube; consequently at will be less than $2l$.

Equation (B) gives us

$$f(a\tau+2l)=f(a\tau)-\psi(a\tau-c)+\phi(a\tau+2l),$$

and for $a\tau$, writing $a\tau - x$,

$$f\{(a\tau+2l)-x\} = f(a\tau-x) - \psi\{a\tau-(x+c)\} + \phi(a\tau+2l-x) \ldots\ldots (3).$$

Also putting $a\tau + 4l - x$, for $a\tau$,

$$\begin{aligned} f\{(a\tau+4l)-x\} &= f(a\tau+2l-x) - \psi\{a\tau+2l-(x+c)\} + \phi(a\tau+4l-x) \\ &= f(a\tau-x) - \psi\{a\tau-(x+c)\} \\ &\quad - \psi\{a\tau+2l-(x+c)\} \\ &\quad + \phi(a\tau+2l-x) \\ &\quad + \phi(a\tau+4l-x). \end{aligned}$$

And similarly, we have

$$f\left\{a\left(\tau+\frac{2nl}{a}\right)-x\right\}$$

$$= f(a\tau-x) - \begin{cases} \psi\{a\tau-(x+c)\}, \\ \psi\left\{a\left(\tau+\dfrac{2l}{a}\right)-(x+c)\right\}, \\ \psi\left\{a\left(\tau+\dfrac{4l}{a}\right)-(x+c)\right\}, \\ \qquad \&c. \\ \psi\left\{a\left[\tau+\dfrac{2(n-1).l}{a}\right]-(x+c)\right\}. \end{cases}$$

$$+ \begin{cases} \phi\left\{a\left(\tau+\dfrac{2l}{a}\right)-x\right\}, \\ \phi\left\{a\left(\tau+\dfrac{4l}{a}\right)-x\right\}, \\ \qquad \&c. \\ \phi\left\{a\left(\tau+\dfrac{2nl}{a}\right)-x\right\}. \end{cases}$$

In the same manner,

$$f\left\{a\left(\tau+\frac{2nl}{a}\right)-(2l-x)\right\}$$

$$=f\{a\tau-(2l-x)\}-\begin{cases}\psi\{a\tau-(2l+c-x)\},\\ \psi\left\{a\left(\tau+\frac{2l}{a}\right)-(2l+c-x)\right\},\\ \qquad\text{\&c.}\\ \psi\left\{a\left(\tau+\frac{2(n-1)\,l}{a}\right)-(2l+c-x)\right\}.\end{cases}$$

$$+\begin{cases}\phi\left\{a\left(\tau+\frac{2l}{a}\right)-(2l-x)\right\},\\ \phi\left\{a\left(\tau+\frac{4l}{a}\right)-(2l-x)\right\},\\ \qquad\text{\&c.}\\ \phi\left\{a\left(\tau+\frac{2nl}{a}\right)-(2l-x)\right\}.\end{cases}$$

Hence we have at the time $\left(\tau+\dfrac{2nl}{a}\right)$,

$$v=f(a\tau-x)-f\{a\tau-(2l-x)\}$$
$$-\{\psi[a\tau-(x+c)]-\psi[a\tau-(2l+c-x)]\}$$
$$-\ \text{\&c.}$$
$$-\left\{\psi\left\{a\left[\tau+\frac{2(n-1).l}{a}\right]-(x+c)\right\}-\psi\left\{a\left[\tau+\frac{2(n-1).l}{a}\right]-(2l+c-x)\right\}\right\}$$
$$+\psi\left\{a\left(\tau+\frac{2nl}{a}\right)-(2l+c-x)\right\}$$
$$+\phi\left\{a\left(\tau+\frac{2l}{a}\right)-x\right\}-\phi\left\{a\left(\tau+\frac{2l}{a}\right)-(2l-x)\right\}$$
$$+\ \text{\&c.}$$
$$+\phi\left\{a\left(\tau+\frac{2nl}{a}\right)-x\right\}-\phi\left\{a\left(\tau+\frac{2nl}{a}\right)-(2l-x)\right\},$$

or,

$$\left.\begin{aligned}
v &= f(a\tau - x) - f\{a\tau - (2l - x)\}\\
&- \Sigma_{r=1}^{r=n}\left\{\psi\left[a\left(\tau + \frac{2(r-1)l}{a}\right) - (x+c)\right] - \psi\left[a\left(\tau + \frac{2(r-1)l}{a}\right) - (2l + c - x)\right]\right\}\\
&\qquad\qquad + \psi\left\{a\left(\tau + \frac{2nl}{a}\right) - (2l + c - x)\right\}\\
&+ \Sigma_{r=1}^{r=n}\left\{\phi\left[a\left(\tau + \frac{2rl}{a}\right) - (x+c)\right] - \phi\left[a\left(\tau + \frac{2rl}{a}\right) - (2l + c - x)\right]\right\},
\end{aligned}\right\}\ldots(\mathrm{D})(1).$$

Similarly, we find

$$\left.\begin{aligned}
as &= f(a\tau - x) + f\{a\tau - (2l - x)\},\\
&- \Sigma_{r=1}^{r=n}\left\{\psi\left[a\left(\tau + \frac{2(r-1)l}{a}\right) - (x+c)\right] + \psi\left[a\left(\tau + \frac{2(r-1)l}{a}\right) - (2l + c - x)\right]\right\}\\
&\qquad\qquad - \psi\left\{a\left(\tau + \frac{2nl}{a}\right) - (2l + c - x)\right\}\\
&+ \Sigma_{r=1}^{r=n}\left\{\phi\left[a\left(\tau + \frac{2rl}{a}\right) - (x+c)\right] + \phi\left[a\left(\tau + \frac{2rl}{a}\right) - (2l + c - x)\right]\right\}.
\end{aligned}\right\}\ldots(\mathrm{D})(2).$$

8. The function $f(a\tau - x)$, in the expression for v, represents the velocity of any particle produced by the first wave, propagated along the tube from the original disturbance at A, so long as τ is less than $\frac{l}{a}$; and if this wave were reflected entirely from B, the first line of the above expression for v, would give us the velocity of any particle within the sphere of the reflected wave, the time τ not exceeding $\frac{2l}{a}$.

With our supposition as to the original disturbance, the form of f $\left(\tau \text{ less than } \frac{2l}{a}\right)$ will be immediately known from that of ϕ. The other terms in the general value of v, shew how the general waves in which we have

$$v_1 = f_1(at - x), \quad \text{and} \quad v_2 = f_1\{at - (2l - x)\},$$

are formed by the superposition of successive waves, as the time increases. If the velocity becomes by this superposition so large, that it can no longer be considered extremely small as compared with the velocity of propagation (a), our analysis will be no longer applicable; but if v never exceed a certain value, the motion will become regular, and follow the law which our investigations indicate. Let us consider in what cases we may expect these effects to be produced.

9. We have at present imposed no restrictions on the forms of the functions denoted by ϕ, f and ψ, except that their greatest values shall be small compared with a. In order however that the undulations may be *sonorous*, ϕ, and consequently f and ψ, must denote periodical functions, so that the values of $\phi(z)$, $f(z)$, and $\psi(z)$, will recur as often as z is increased by a certain quantity. We will also impose an additional limitation upon them, to which, in all practical cases they will probably be subject very nearly, as will certainly be the case in the experiments to which I shall hereafter more immediately refer. Supposing then their values to recur, when z becomes $z \pm m\lambda$, (m any whole number), we will also suppose them to recur with different signs when z becomes $z \pm m'\frac{\lambda}{2}$; ($m'$ being any odd number).

10. First suppose the greatest value of ψ, small as compared with that of f or ϕ, as must be the case in a closed tube. In the above expression for v, it will be observed that the quantity represented by z increases as we proceed from one term to the next, in a vertical line by $2l$.

Suppose then

$$2l = m' \cdot \frac{\lambda}{2}, \quad \text{or} \quad l = m'\frac{\lambda}{4}.$$

In this case it is manifest that the consecutive terms taken in the order just mentioned will destroy each other; and there will consequently be no accumulation of motion in the tube, and the vibrations will go on uniformly. Again, let

$$2l = m\lambda, \quad \text{or} \quad l = 2m \cdot \frac{\lambda}{4}.$$

In this case the values of the successive terms taken as before in the expression for v will be equal, and with the same sign. Hence, if we take x of any value, except such as would render

$$\phi(at - x) = \phi\{at - (2l - x)\},$$

$\left(\text{which value of } x \text{ is } l - m\frac{\lambda}{2}\right)$, it is manifest (since the value of ϕ is greater than that of ψ), that the motion will constantly increase for such points, and will soon become greater than is consistent with our original suppositions. Such a vibration then cannot be maintained.

11. Again suppose the functions ϕ, f, and ψ, to be continuous, and suppose

$$2l = m'\frac{\lambda}{2} + 2\lambda', \quad \text{or} \quad l = m'\frac{\lambda}{4} + \lambda',$$

λ' being any quantity less than $\frac{\lambda}{2}$; the consecutive terms of $\Sigma . \phi(z)$, will not then destroy each other, but as the number of pairs of terms increases, the sum will increase till $\phi(z + 2rl)$ becomes negative, it will then decrease, after having thus attained a maximum value. Maxima and minima values will thus occur alternately, and the same will hold for $\Sigma . \psi(z)$. If these maxima values do not render v greater than our original suppositions allow, the vibrations may be maintained.

Since these maxima values are 0, when $l = m' . \frac{\lambda}{4}$, and greatest when $l = m' . \frac{\lambda}{2}$, we conclude that they will be intermediate for intermediate values of l, following some continuous law. Hence we infer the possibility of maintaining sonorous vibrations of which the period is $\frac{\lambda}{a}$, in stopped tubes of which the length differs considerably from $m' . \frac{\lambda}{4}$, particularly if the greatest value of ψ should not be very small. If the supposition we have made respecting the continuity of the function ϕ more particularly, should not be quite true, it is not likely in those practical cases to which we can best refer, to be so far wrong as to render the above reasoning otherwise than at least approximately true.

12. Our supposition has been that the intensity of the disturbance denoted by ψ, is considerably less than that indicated by ϕ, the tube being stopped with some substance having a certain degree of elasticity; if the tube be open, it seems probable from certain phenomena, that the reverse of this supposition is true.

Assuming this to be the case, the expansion of the expression for v may be put under a more convenient form.

Let

$$\psi\{at-(2l+c-x)\} = 2f\{at-(2l-x)\} - \psi_1\{at-(2l+c'-x)\},$$

Then

$$v = f(at-x) + f\{at-(2l-x)\} - \psi_1\{at-(2l+c'-x)\} \ldots\ldots\ldots (a),$$

and equation (3) becomes

$$f(a\tau+2l-x) = -f(a\tau-x) + \psi_1\{a\tau-(x+c')\} + \phi(a\tau+2l-x) \ldots\ldots (4).$$

By proceeding exactly as in the former case, we obtain

$$\left.\begin{aligned} v &= (-1)^n\{f(a\tau-x) + f[a\tau-(2l-x)]\} \\ &+ \Sigma_{\substack{r=1\\r=n}}(-1)^{n-r}\left\{\psi_1\left[a\left(\tau+\frac{2(r-1)l}{a}\right)-(x+c')\right]+\psi_1\left[a\left(\tau+\frac{2(r-1)l}{a}\right)-(2l+c'-x)\right]\right\} \\ &\qquad - \psi_1\left\{a\left(\tau+\frac{2nl}{a}\right)-(2l+c'-x)\right\} \\ &+ \Sigma_{\substack{r=1\\r=n}}(-1)^{n-r}\left\{\phi\left[a\left(\tau+\frac{2rl}{a}\right)-x\right] + \phi\left[a\left(\tau+\frac{2rl}{a}\right)-(2l-x)\right]\right\} \end{aligned}\right\} (E)(1).$$

Similarly, we find

$$\left.\begin{aligned}as &= (-1)^n\{f(a\tau - x) - f[a\tau - (2l - x)]\}\\ &+ \Sigma_{\substack{r=1\\r=n}}(-1)^{n-r}\{\psi_1[a\left(\tau+\frac{2(r-1)l}{a}\right)-(x+c')]-\psi_1[a\left(\tau+\frac{2(r-1)l}{a}\right)-(2l+c'-x)]\}\\ &\qquad + \psi_1\{a\left(\tau+\frac{2nl}{a}\right)-(2l+c'-x)\}\\ &+ \Sigma_{\substack{r=1\\r=n}}(-1)^{n-r}\{\phi[a\left(\tau+\frac{2rl}{a}\right)-x]-\phi[a\left(\tau+\frac{2rl}{a}\right)-(2l-x)]\}.\end{aligned}\right\}\text{(E)(2)}.$$

Reasoning on the expression for v, exactly similar to that used above, will in this case show that sonorous vibrations cannot be maintained if l be too nearly equal to an *odd* multiple of $\frac{\lambda}{4}$; but that they can be continued, if l do not differ too much from an *even* multiple of $\frac{\lambda}{4}$.*

13. If we examine the expressions for as in the last article, and in Art. 7, it will appear that the condensations and rarefactions at the surface of the vibrating plate within the tube, are such as to produce forces opposing more strongly the motion of the plate as the lengths of the tubes approximate respectively to those particular lengths for which it will be impossible to maintain the vibrations in the tube; and when the lengths differ from the above by $\frac{\lambda}{4}$, these condensations and rarefactions are such as to promote the motion of the plate, instead of opposing it.

14. The expanded expression for v may be put also under another form, which it may be useful to point out for the case in which the intensity of the disturbance denoted by ψ, is considerably greater than that denoted by ϕ.

* The quantity c' in these general inferences is not taken into account. Its value however is considerable, as will be seen hereafter.

This is deduced, by assuming

$$\psi_1\{at-(x+c')\}=\beta f(at-x)+\psi'\{at-(x+c'')\},$$

or,

$$\psi\{at-(x+c)\}=(2-\beta)f(at-x)+\psi'\{at-(x+c'')\}.$$

Then the equation (a) (Art. 12) becomes

$$v=f(at-x)+(1-\beta)f\{at-(2l-x)\}-\psi'\{at-(2l+c''-x)\}\ldots\ldots(\beta).$$

We may observe, that since the vibration denoted by ψ, is produced by that denoted by f, it seems a necessary consequence that their periods must be the same. Their phases also are nearly so; and if in addition we assume that the *form* of the function expressing the one motion, does not differ very widely from that expressing the other, (however the *intensity* of the vibrations may differ) it is manifest that β may be so taken that the intensity of the vibrations denoted by the unknown function ψ' shall be small compared with that indicated by ϕ.

Equation (4) becomes

$$f(a\tau+2l-x)=-(1-\beta)f(a\tau-x)+\psi'\{a\tau-(x+c'')\}+\phi(a\tau+2l-x)\ldots\ldots(5),$$

$$=-bf(a\tau-x)+\psi'\{a\tau-(x+c'')\}+\phi(a\tau+2l-x),$$

if $1-\beta=b$.

This gives us

$$f\left\{a\left(\tau+\frac{2nl}{a}\right)-x\right\}=(-b)^n f(a\tau-x)$$

$$+\Sigma_{r=1}^{r=n}(-b)^{n-r}\psi'\left\{a\left[\tau+\frac{2(r-1).l}{a}\right]-(x+c'')\right\}$$

$$+\Sigma_{r=1}^{r=n}(-b)^{n-r}\phi\left\{a\left(\tau+\frac{2rl}{a}\right)-x\right\}.$$

And the equation (β) becomes, $\left(\text{when } t=\tau+\dfrac{2nl}{a}\right)$,

$$v=(-b)^n\{f(a\tau-x)+bf[a\tau-(2l-x)]\}$$

$$+\Sigma_{r=1}^{r=n}(-b)^{n-r}\left\{\psi'\left\{a\left[\tau+\frac{2(r-1).l}{a}\right]-(x+c'')\right\}+b\psi'\left\{a\left[\tau+\frac{2(r-1).l}{a}\right]-(2l+c''-x)\right\}\right\}$$

$$-\psi'\left\{a\left(\tau+\frac{2nl}{a}\right)-(2l+c''-x)\right\}.$$

$$+\Sigma_{r=1}^{r=n}(-b)^{n-r}\left\{\phi\left[a\left(\tau+\frac{2rl}{a}\right)-x\right]+b\phi\left[a\left(\tau+\frac{2rl}{a}\right)-(2l-x)\right]\right\}.$$

Since b is less than unity, and n soon becomes a very high number, after an extremely short time the first line in this expression may be neglected, as may also all the terms in the other lines involving high powers of b.

Whence it follows that the original disturbance (on which the form of the function f will depend), will cease in an extremely short space of time to have any effect on the form of the existing vibration, supposing the vibrations maintained by some cause distinct from that producing the original disturbance.

Also, if the cause maintaining the vibrations cease, the vibrations themselves may cease in an extremely small space of time.

The inferences we have drawn from the former developement (E) of the expression for v, may be drawn from this and perhaps with still greater facility.

15. If we suppose $\psi'(z)$ always $= 0$, the expression for v will reduce itself to the same as that given by M. Poisson. But in this case it will be observed that all the functions involving the quantity c'' disappear, which renders it impossible to account on this theory for the position of the nodes or points of minimum vibration as determined by experiment*. For the purpose of determining the positions of these points theoretically we will recur to the equations (C), the first of which is

$$v = f(at - x) - f\{at - (2l - x)\} + \psi\{at - (2l + c - x)\} \ldots\ldots(6).$$

If we neglect $\psi\{at - (2l + c - x)\}$, (or suppose the substance with which the tube is stopped perfectly rigid) we shall have $v = 0$, whenever

$$(at - x) - \{at - (2l - x)\} = 0, \quad \text{or } m\lambda,$$

(m being any whole number), or when

$$(l - x) = m \cdot \frac{\lambda}{2}.$$

* See Art. 36, Sec. II.

This condition is independent of t, and consequently at all points distant from the stopped end, any multiple of $\frac{\lambda}{2}$, the motion will be the same as at that extremity, i.e. it will always equal 0, and there will be perfect nodes at those points.

16. We may take the general case, and let

$$f\{at-(2l-x)\}-\psi\{at-(2l+c-x)\}=\chi\{at-(2l+c_1-x)\},$$

$$\text{and } \therefore v=f(at-x)-\chi\{at-(2l+c_1-x)\},$$

ψ being still small. The forms of f and ψ being known, that of χ will be determined; its period will also be the same as that of f and ψ. It expresses the velocity of each particle produced by the whole wave actually reflected from B. The nodes will in this case be points of minimum vibration, and not of perfect rest.

For the sake of clearness we will assume that $f(z)$, and $\psi(z)$, are such that

$$f(-z)=-f(z); \quad \psi(-z)=-\psi(z),$$

and therefore

$$\chi(-z)=-\chi(z),$$

that $f(z)$, and $\psi(z)$, {and therefore $\chi(z)$} admit of only one maximum value between $z=0$, and $z=\frac{\lambda}{2}$; and that the ratio which $f(z)$ bears to $\psi(z)$ is always considerable, as by hypothesis it is when those functions have their maximum values. There can be little doubt but that these assumptions are at least approximately true in all practical cases; and appear as simple as any we can make (and some must be made), in order to give distinctness to our inferences as to the positions of these points of minimum vibration.

17. For the determination of c_1 in terms of c, let the origin of t and x be so taken that $f(0)=0$, then making $at-(2l-x)=0$, we have

$$-\psi(-c)=\chi(-c_1);$$

$$\therefore \chi(c_1)=-\psi(c),$$

$$\text{or } =\psi(-c).$$

By our hypotheses, $\chi(z)$ must be always greater than $\psi(z)$; and if we suppose c and c_1 less than the least value of z, which gives to $\psi(z)$, or $\chi(z)$ its maximum value, it is manifest that from this last equation, c_1 must be considerably smaller than c, and must be affected with a different sign. Suppose $c_1 = \frac{c}{k}$, where k is considerably greater than unity. It follows then that if the phase of the vibration of the extreme section of a stopped tube be retarded by a certain quantity c, the phase of the actually reflected wave will be accelerated by a quantity $\frac{c}{k}$.

18. Giving then the proper sign to c_1, we have

$$v = f(at - x) - \chi\{at - (2l - \frac{c}{k} - x)\} \dots\dots\dots\dots\dots\dots (7),$$

and to determine the points of minimum vibration, we may observe that this expression is exactly the same, as if the wave for which

$$v_2 = \chi\{at - (2l - \frac{c}{k} - x)\},$$

were reflected immediately from a section B' whose distance from $A = l - \frac{c}{2k}$.

Suppose a rigid diaphragm at this section constrained to move exactly as the fluid does there; we may then suppose the actual stop B removed, and the points of minimum vibration will remain the same.

Now to determine them in this case, we observe that whenever

$$at - x = at - (2l - \frac{c}{k} - x) + m\lambda,$$

the value of v will be the same as when

$$at - x = at - (2l - \frac{c}{k} - x).$$

In the latter case

$$\left(l - \frac{c}{2k}\right) - x = 0,$$

and in the former

$$\left(l - \frac{c}{2k}\right) - x = m \cdot \frac{\lambda}{2},$$

$$\text{or} \quad l - x = m\frac{\lambda}{2} + \frac{c}{2k};$$

consequently, at any point in the tube whose distance from $B' = m \cdot \frac{\lambda}{2}$, the velocity will be the same as at B'. These then will be points of minimum vibration in this hypothetical case, and therefore also, from what precedes, in the actual case.

Making $c = 0$, we have $l - x = m \cdot \frac{\lambda}{2}$, which will give the positions of the nodes when there is no retardation.

Hence we have this general conclusion with respect to the stopped tube—that if there be a retardation in the phase of the vibration of the extreme section, the positions of the points of minimum vibration will all be further from the stopped end by $\frac{c}{2k}$, than if there were no such retardation, the distances between these points respectively remaining unaltered.

19. We will now consider the case of the open tube, in which we suppose $\psi(z)$ to be always considerably larger than $f(z)$. Assume, as in Art. (12),

$$\psi\{at - (2l + c - x)\} = 2f\{at - (2l - x)\} - \psi_1\{at - (2l + c' - x)\} \ldots\ldots (8),$$

$$v = f(at - x) + f\{at - (2l - x)\} - \psi_1\{at - (2l + c' - x)\}.$$

First neglecting the function ψ_1, v will $= 0$ whenever

$$f(at - x) = -f\{at - (2l - x)\};$$

i. e. whenever

$$at - x = at - (2l - x) + m' \cdot \frac{\lambda}{2} \quad (m' \text{ an odd number}),$$

$$\text{or} \quad l - x = m' \cdot \frac{\lambda}{2},$$

a condition independent of t. Consequently, at every point whose distance from the open end is an odd multiple of $\frac{\lambda}{4}$, there would be a perfect node.

20. Put

$$f\{at-(2l-x)\}-\psi_1\{at-(2l+c'-x)\}=\chi'\{at-(2l+c_2-x)\}\ldots\ldots(9).$$

Then

$$v=f(at-x)+\chi'\{at-(2l+c_2-x)\}\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

To find the relation between c_2 and c, we have from equation (8), (proceeding as in Art. 7, and with the same assumptions),

$$\psi(-c)=-\psi_1(-c'),$$

$$\text{or}\quad \psi_1(c')=-\psi(c);$$

and since $\psi(z)$ is much larger than $\psi_1(z)$, we shall have c' considerably larger than c, and affected with a different sign. We may therefore put

$$-c'=k_1c,$$

k_1 being greater than unity.

Again from equation (9),

$$-\psi_1(-c')=\chi'(-c_2),$$

$$\text{or}\quad \chi'(c_2)=-\psi_1(c').$$

If we suppose $\chi'(z)$ nearly equal to $\psi_1(z)$, (which probably is not far from the truth), we shall have

$$c_2=-c'\ \text{nearly},$$

$$=+k_1c.$$

Hence in this case if the phase of the vibration of the extreme section be retarded by a quantity c, that of the actually reflected wave will be retarded by k_1c; and it will appear by the same reasoning as in the case of the closed tube, that the distance of the points of minimum vibration from the open end will be $m'\frac{\lambda}{4}-\frac{k_1c}{2}$, ($m'$ being any odd number).

21. If ϵ and ϵ' be the distances through which the nodes are moved by a supposed given retardation of phase, the same for each, at the extremities of the open and closed tubes respectively,

$$\epsilon = -kk_1\epsilon';$$

ϵ will consequently be much larger than ϵ'.

The quantities $m'\frac{\lambda}{4} - \frac{k_1 c}{2}$ in the open tube, and $m\frac{\lambda}{4} + \frac{c}{2k}$ in the closed one, must be determined by experiment.

22. I will recapitulate the principal inferences from this theory.

I. In the tube *AB*, *open* at the extremity *B* opposite to that at which the vibrations are produced, there will be a series of nodes equidistant from each other by $\frac{\lambda}{2}$, or half a whole undulation, the distance of the nearest node from the open extremity being considerably less than $\frac{\lambda}{4}$, the whole system of nodes being thus brought nearer to the open end than the position assigned to it by the investigations of Euler or of M. Poisson. The distance of each node from the open end will be independent of the length of the tube. (Art. 20.)

II. If the tube be closed at *B*, the nodes will still be equidistant as before by $\frac{\lambda}{2}$. The distance from *B* of the node nearest that extremity will be $\frac{\lambda}{2}$, or a quantity rather greater than that, if we suppose a cause of displacement of the whole system of nodes to exist in this case of the closed tube, similar to that which exists in the open one; the displacement however being necessarily much smaller in the former than in the latter case, and in the opposite direction. (Art. 18.)

III. These nodes are not places in which the air is perfectly at rest, but points of minimum vibration. (See Art. 16.)

IV. Sonorous vibrations, whatever be their period, may be maintained in a tube of any length, except that of which the length does not approximate too nearly to something less than an *even* multiple of $\frac{\lambda}{4}$ in the *closed* tube, or to an *odd* multiple of $\frac{\lambda}{4}$ in the *open* one. (Arts. 11, 12.)

V. The intensity of the general vibrations in the tube varies with the length of the tube, being greatest for the lengths just mentioned, and least in the *closed* tube when its length is rather greater than an *odd* multiple of $\frac{\lambda}{4}$; and in the *open* one, when it is something less than an *even* multiple of $\frac{\lambda}{4}$. (Art. 10.)

VI. In these latter cases also of both tubes, the opposition afforded by the vibratory motion of the air within the tube, to the vibrating of the plate, is *least;* and *greatest* for the lengths which approximate to those mentioned in (IV.), as those with which the vibrations cannot be maintained. (Art 13.)

VII. When the cause producing the vibrations in a tube ceases, the vibrations themselves may cease, not instantaneously, but in a period of time not exceeding the small fraction of a second, supposing the tube not to exceed a few feet in length. (Art. 14.)

VIII. If we suppose the original disturbance to produce an undulation different in any respect to those produced by the cause which afterwards maintains the vibratory motion of the aerial column, this original disturbance will cease to affect the form of subsequent undulations in a period of time not exceeding the small fraction of a second, depending on the length of the tube*. (Art. 14.)

* Similar inferences to the above may be drawn equally from M. Poisson's investigations, except that the phenomena according to his solution would take place for lengths of the open tube materially different from those above-mentioned.

SECTION II.

23. I WILL now proceed to describe the experiments which have been made with a view of putting the different theories on this subject to an experimental test. Sonorous vibrations are usually excited in a tube, either by directing a stream of air across the open end, as in blowing across the embouchure of the flute; by means of a vibrating tongue, as in all reed instruments; or by placing an open end of the tube close to the surface of a vibrating body. In the two first cases it seems impossible to conceive that the same disturbance can be communicated to each part of the extreme section of the air in the tube where the original motion is produced, a condition which is always assumed to hold at least approximately in all our mathematical investigations of the subject. This irregularity of the motion will no doubt extend to some distance within the tube, and it is impossible to say how it will affect the phenomena even in those parts of the tube in which the motion may become more uniform. In the second case too in particular, a stream of air must constantly be passing through the tube, a circumstance not contemplated in our analysis of the problem. This may or may not influence materially the observed phenomena, but at all events the danger of derangement from any such cause must be avoided, if we would render our experiments decisive tests of the truth of any theory professing to account for phenomena of so delicate a nature as those which are now the objects of our investigation. The third method, however, above-mentioned, is entirely free from the latter objection, and may be made almost entirely so from the former, and is, therefore, that which I have adopted.

24. The apparatus is very simple. Figure I. represents it. A plate of common window glass is held firmly in a horizontal position by a pair of pincers at its middle point. *AB* is a *glass* tube, having a short brass tube closely sliding within it at the upper end *B*, so that the whole tube *AB* can be lengthened or shortened at pleasure. Within the tube a small* brass frame *M*, having a delicate membrane

* Fig. (2) represents this frame with the membrane *ab*, which may be *tuned*, or rendered sensitive in different degrees, to the vibrations produced by any proposed note, either by

stretched across it, is suspended by a fine wire or thread from the upper extremity of the tube, in such a manner that it can be heightened or lowered at pleasure. The other parts of the apparatus are merely such as are adapted for facility and accuracy of arrangement of the tube and plate.

25. The air in the tube is put in a state of sonorous vibration by means of the plate, which is made to vibrate by drawing the bow of a violin equably across its edge in a direction perpendicular to its plane; the vibratory motion of the air is communicated to the membrane suspended in the tube, and the degree of motion is indicated by the agitation of a small quantity of light dry sand sprinkled upon it*. Suppose the tube to be open at the upper end B, and let the membrane be drawn up near that extremity. If the sand indicate a considerable motion when the plate is vibrating, let the membrane be gradually lowered; a position will thus be found in which the sand has little or no apparent motion, thus indicating the existence of a *node*. On lowering the membrane still further, the sand will again become strongly agitated, and will then come to another place of rest, (or at least of minimum vibration), and so on till it reach the lower end of the tube. These alternations of points of rest and motion can of course only take place when the tube is sufficiently long in comparison with the length of an undulation produced by the vibrating plate, to admit of them. These nodal points are thus found to be at equal distances from each other, the distance of the upper one from the top of the tube being *less* than half that between the nodes. This is independent of the length of the tube. These results are accordant with our theory, (Art. 22, I.), from which it appears that this constant distance between two consecutive nodes must be $\frac{\lambda}{2}$.

If we call the distance of the upper node from B, $\frac{\lambda}{4} - C$, C denotes what I have termed the *displacement* of the nodes.

altering the tension by means of the small cylinder round which the end b of the membrane passes, or by moving the small *bridge* cd, and thus altering the length of the vibrating part.

* This was the method adopted by Savart in such a variety of cases, in which he wished to ascertain the intensity of sonorous vibrations in air.

26. If the membrane be rendered very sensitive by being exactly *tuned* to the note produced by the vibrating plate, it will not indicate perfect rest at the nodal points, shewing them in fact to be points of minimum vibration, which agrees with our theory, (Art. 22, III.). With such a membrane it will be difficult to determine the position of these points with accuracy, and its sensibility should be diminished, till the sand appears perfectly at rest when it is placed exactly at the node. If the membrane be rendered still less sensitive, it will appear at rest for a space on each side of the node, the position of which will in such case, be determined by observing those points immediately above, and below the node at which the motion of the sand is just sensible. The middle point between them will of course be the node.

27. Now suppose the length of the tube to be any *odd* multiple of $\frac{\lambda}{4}$, and the membrane to have such a degree of sensibility, as just to remain at rest only when placed in a node or within a very small distance of it. After it has been placed in this position, let the brass tube sliding within the upper part of the glass one be raised through a space less than $\frac{\lambda}{4}$. While the whole tube is thus lengthened, let the distance of the membrane from the upper end B remain the same; the membrane will consequently be still in a node. The plate being now put in vibration, the membrane will remain perfectly at rest, not only in this position, but also when moved to one considerably above or below the node, the new length of the tube remaining the same. This indicates a less degree of motion in the tube than in the former case, and we find that the intensity of the vibration in the *open* tube is least when its length is equal to something less than an *even* multiple of $\frac{\lambda}{4}$, or $2m \cdot \frac{\lambda}{4} - C$; and becomes greater as the length approximates to rather less than an *odd* multiple of $\frac{\lambda}{4}$, or $(2m'+1)\frac{\lambda}{4} - C$, m and m' being any whole numbers. (Art. 22. V.). This diminution of motion is also very obvious when the membrane is placed in those

parts of the tube where the motion is most sensible. In all cases, however, the distances of the nodes from B is independent of the length of the tube.

28. If we take a tube closed at B instead of the open one, we observe the same alternations of points of greatest and least vibration, and (the plate being made to vibrate in the same manner as before) at exactly the same distances from each other as in the closed tube; but the distance of the upper node from the closed extremity of the tube is now observed to be $\frac{\lambda}{2}$, the same as the distance between the nodes. Proceeding as in the former case, it is found also that the strongest vibrations are excited when the length of the tube is about equal to a multiple of $\frac{\lambda}{2}$; and the least vibrations when the length = an odd multiple of $\frac{\lambda}{4}$. I find also that in the open tube stronger vibrations exist in the nodal points than for corresponding cases of the closed tube.

29. In performing the above experiments with reference to the intensity of the vibrations in the tube, care must of course be taken to prevent the influence of any other cause than that of which I have spoken, viz. the length of the tube with respect to λ. It has been assumed that the vibration of the part of the plate immediately in contact with the mouth of the tube is in all cases the same, which requires that the tube should always be placed over exactly the same portion of the plate. This portion also should be included in one and the *same ventral segment*; for if a nodal line on the plate pass across the mouth of the tube, the vibrations transmitted from opposite sides of this line will be in exactly opposite phases, and will consequently neutralize each other in a degree depending on the ratio which the intensity of one of these undulations bears to the other. If the nodal line divides the part of the plate in contact with the mouth of the tube into two equal portions, parts of similar ventral segments, the interference will be so complete as to destroy all sensible motion in the

tube*. It is only however as regards the intensity of the vibrations that this precaution respecting the relative position of the nodal lines and mouth of the tube is important; it does not affect the *positions of the nodes.* The reason is obvious—it does not affect the value of λ.

30. Again, taking the tube open at *B*, let the extreme section at *A* be made to coincide nearly with the surface of the vibrating plate. If the plate (the bow being applied to it) vibrate freely, let the length of the tube be gradually increased or diminished. It will thus be found, that as the tube approximates to certain lengths, the plate vibrates with less facility, requiring a greater pressure of the bow, and continuing to vibrate audibly for a shorter time after its removal; and in many cases, between certain limits in the length of the tube, it becomes almost impossible to make the plate assume that state of vibration which it assumes freely for other lengths; and the vibration, if it be produced, appears to cease almost instantaneously on the removal of the bow, instead of being audible for several seconds, as it would be if the tube were removed, or were of a different length. These phenomena recur for every increase of $\frac{\lambda}{2}$ in the length of the tube; and if l be any length with which it becomes almost impossible to make the plate vibrate in the manner proposed, then will $l + \frac{\lambda}{4}$ be that length with which it vibrates with the same facility as if the tube were removed.

* It is easy by a very simple experiment to give *ocular* demonstration of the fact that the union of two intense sounds may produce perfect silence. Take a branch tube *ABA'* (Fig. 3.), and stretch over the open end *B* a fine membrane or a piece of common writing paper. Place the open extremities *A, A'* of the equal and similar branches *CA, CA'* over portions of two ventral segments of a vibratory plate in the *same phase* of vibration. A small quantity of sand strewed over the membrane at *B*, will immediately shew it to be in a state of strong vibration. Let *A, A'* be then carefully placed over similar portions of similar ventral segments of the plate, in *opposite* phases of vibration; the sand on the membrane will remain perfectly at rest, shewing that the waves propagated along *AC* and *A'C* in opposite phases so completely interfere at *C* as to produce no undulation along *CB*. In other words, no sound would in this case be transmitted along the tube to its mouth *B*.

So far these phenomena are in accordance with the results of theory, (Art. 22, VI.); but when we examine the length l just mentioned, we find it entirely at variance with them. In fact on investigating the circumstances more narrowly, we find that the value of l depends in a considerable degree on the small distance between the vibrating plate, and the extreme section A of the tube, a circumstance which nothing in our theoretical deductions has led us to anticipate. This will be seen in the results of the following experiment made with an *open* tube.

Diameter of the tube = 1.35 inches.

Value of $\frac{\lambda}{2}$ .... = 4.82 for temperature 63°.

Position of the mouth (A) of the tube (Fig. I.)	Value of the length l above mentioned.	Theoretical value of l.
As close to the plate as possible without interfering with its vibrations...........	.....12.25 inches.	* 11.46 inches.
About $\frac{1}{16}$ inch from the vibrating plate...............	12. 6	

31. This discrepancy however between the theoretical and experimental results is only apparent. It arises from the circumstance of one of the conditions assumed in our mathematical investigation, not being accurately satisfied, namely, the perfect prevention of all communication between the external air and that within the tube at the extremity next the plate. And this is easily proved experimentally, by placing the extremity of the tube as near as possible

* In this value of l I have taken account of the *displacement* of the nodes, which is .59 inches, as determined by experiment. (See Table, Art. 36.)

to the surface of the plate, without interfering with its vibrating motion, and then putting round the edge of the tube, a small quantity of fluid which by its adherence to the tube and the plate fills up the interstice between them, and prevents communication with the external air. When this precaution is taken, the lengths of the tube which correspond to the above mentioned phenomena exactly agree with theory; that is—

The vibration of the plate is unaffected by the presence of the open tube, when its length is equal to something less than an *even* multiple of $\frac{\lambda}{4}$, or $2m \cdot \frac{\lambda}{4} - C$, and of the closed one when its length is equal to an *odd* multiple of $\frac{\lambda}{4}$; but as the lengths of the tubes approximate respectively to quantities differing by $\frac{\lambda}{4}$, from the above lengths it becomes almost impossible to make the plate assume the same vibratory motion. (Art. 22, VI.)

32. It might at first appear probable that the neglect of this precaution might have some effect on the position of the nodes, as well as on the phenomena above mentioned. This however is not the case; and the reason will be obvious if we recollect that the position of the nodes depends on the *periodicity* of the vibrations, or the value of λ, which is unaffected by the communication with the external air at A; whereas the force opposing the vibration of the plate depends on the condensations and rarefactions of the air, at the surface of the plate within the tube, which will necessarily be much affected by the communication just mentioned.*

33. If we take a closed tube, a similar discrepancy or accordance in the results of theory and experiment will be found under the same circumstances as above described.

* It does not appear so easy to account for the phenomena as above described, when the influence of external air is not prevented. This, however, does not immediately belong to the object I have proposed to myself in this paper, which is, to establish as accurately as possible the identity of the results of theory and of experiment in those cases in which the conditions assumed in our mathematical investigations are experimentally satisfied.

The phenomena above mentioned, agree with those observed by Mr Willis, and described in his paper on the Vowel sounds, published in the Transactions of this Society, Vol. III. The manner however in which his experiments (having a different object from mine) were conducted, render them unfit for the verification of any of our mathematical results in this subject.

34. From what I have above stated, respecting the difficulty of making the plate vibrate with certain lengths of the tube, it is manifest how we may avail ourselves of this phenomenon, in the determination of the value of λ, corresponding to any particular mode of vibration of the plate, supposing those particular lengths of the tube can be ascertained with sufficient accuracy. Now this can be done almost as accurately as the position of a node can be determined by the vibrating membrane, and consequently the value of λ may thus be found. For if l_1 and l_2 denote two observed values of l, we shall have $\frac{\lambda}{2} = \frac{l_1 - l_2}{n}$, n being a whole number easily ascertained. (See Arts. 30, 31.)

35. Though I have had frequent occasion to speak of this *displacement* of the nodes in the open tube, from the positions assigned to them by the common theory, I have hitherto said nothing as to the experimental determination of its magnitude. The most direct way of accomplishing this, is to determine the actual positions of the nodal points by means of the vibrating membrane; but this method becomes inconvenient when the diameter of the tube is small, as, for instance, less than an inch. Those which I have used most commonly are from 1.3 in. to 1.5 in. diameter. If the tube be larger than this, it will generally be too large to admit of the extreme section of it being placed entirely upon the same ventral segment of the plate, as is always desirable, (see Art. 29.); and if much smaller it becomes necessary to make the surface of the membrane so small as to be inconvenient, in order that it may not bear too great a ratio to the area of the section of the tube, in which case the presence of the membrane might be supposed to render the vibrations in the tube materially different from what they would otherwise be.

The best method therefore of determining the positions of the nodes in tubes considerably smaller than those I have mentioned, is that by which the value of λ is determined, as described in the last Article.

Thus, suppose l to be the length of tube, with which it is found most difficult to make the plate vibrate; then (the tube being open) we shall have

$$l=(2m+1)\frac{\lambda}{4}-C,$$

where m is a whole number, which will be known when λ is determined by either of the methods pointed out above. The quantity C evidently shews how much the distance between the open extremity, and the nearest node differs from $\frac{\lambda}{4}$, or it expresses the *displacement.*

From the above equation,

$$C=(2m+1)\frac{\lambda}{4}-l,$$

and the displacement is thus determined.

36. The following table exhibits the magnitude of this displacement in a tube of given diameter, as determined experimentally for different values of $\frac{\lambda}{2}$. The positions of the nodes were in these cases carefully ascertained by means of the membrane suspended in the tube.

Diameter of the tube = 1.35.*			
Value of $\frac{\lambda}{2}$. at temp. 63°.	Computed dist. of a Node from B, (fig. 1.)	Observed dist. of the same Node.	Displacement of the Node.
2.044	...... 11.24 7.15	10.88 6.78	.36 .37 mean = .365
3.994	9.98	9.51	.47
4. 82	7.23	6.64	.59

The above values of $\frac{\lambda}{2}$ were determined by means of a membrane and a tube closed at the upper end, nearly 100 inches in length. The distance of a node from the closed end being found $= b$, we must have $n . \frac{\lambda}{2} = b$, or $\frac{\lambda}{2} = \frac{b}{n}$. Or, if b† be the observed distance, subject to an error β, and therefore $b \pm \beta$ the true distance, we have $\frac{\lambda}{2} = \frac{b}{n} \pm \frac{\beta}{n}$. The value of β will probably be less than $\frac{1}{20}$ inch, and in the determination, for example, of the first of the above values of $\frac{\lambda}{2}$, n was about 45, so that that value of $\frac{\lambda}{2}$ may probably not be subject to an error exceeding .001 inch. We may also remark, as an indication of accuracy in the numbers 10.88 and 6.78, given in the third

* The measures are all given in inches.

† In the determination of the quantity b, the temperature at the time of observation must be carefully noted, since the variation in the velocity of aerial undulations produced by a variation of temperature of even less than 1°, is sufficient to make a very sensible difference in the value of b, this value being as much as nearly 100 inches.

Since the distance of any proposed node from the upper end of the tube will be proportional to the velocity of the undulation, it is manifest that by observing the values of b, corresponding to different temperatures, we may estimate directly the effect of temperature on the velocity of sound. This method is capable of great accuracy.

column, that $10.88 - 6.78 = 4.10$ must $= 2 \cdot \frac{\lambda}{2}$, which gives us $\frac{\lambda}{2} = 2.05$, differing but .006 from the more accurate value. The error in the two numbers above mentioned, 10.88 and 6.78, does not probably exceed .01 or .02, and cannot, I conceive, exceed .04, and consequently, I think, the utmost limit to the error in the corresponding numbers in the fourth column cannot exceed .05, and is probably considerably less. The same may be concluded respecting the numbers .47, .59, in the same column.

The above results may, then, be considered sufficiently accurate to determine the fact of the magnitude of the displacement increasing with increased values of λ, though not sufficiently so to determine with certainty the law of this corresponding increase.

The displacement does not depend only on the value of λ; it depends also on the area of the mouth of the tube, as appears from the following table.

Values of $\frac{\lambda}{2}$.	Displacement.	
	Diameter of tube = 1.35.	Diameter of tube = .8.
2.044	.23	.08
3.994	.4	.1

These values of the displacement of the nodes have been obtained by the method mentioned in Art. 35, as that best applicable to small tubes. The results in the second column of this table ought to be the same as the two first in the last column of the former table; but this method is liable, I conceive, to greater error and uncertainty than the former, and to this, I doubt not, the discrepancy is due. All these latter results, however, are probably subject to an error of the same kind, and are too small both in the large and small tube. They can leave no doubt of the fact of the magnitude of the displacement being dependent on the diameter of the tube.

It is important to observe, that the values of λ determined in the large tube and the small one, from the consideration that the distance between any two nodes must equal some multiple of $\frac{\lambda}{2}$, was exactly the same, being for the first case in the table 2.05, very nearly agreeing with the accurate value 2.044. *This proves that the distance between the nodes is independent of the diameter of the tube, provided the disturbance take place uniformly throughout its extreme section.*

37. I have before remarked, that there can be nothing arbitrary or indeterminate in the vibratory motion of the air at the extremity of the open tube when the vibrations in it are excited according to some known law; and consequently, if our theoretical knowledge of the subject were complete, we should undoubtedly find in our investigations the cause of the retardation of phase, of which I have spoken, in the reflected wave of the open tube, supposing it to be the actual cause of that displacement of the whole system of nodes which I have established as an experimental fact. Our knowledge at present, however, is totally inadequate to this purpose, and therefore we can only conjecture what may be the probable cause of this retardation in the reflected wave; but at all events, our formulæ, with the modifications I have introduced into them, do become perfect representations of all those phenomena which we can distinctly determine by experiment, in the cases to which our mathematical investigations apply. The fact too, of a retardation of phase in the reflected wave may not be very difficult to conceive, or appear improbable, if we suppose the undulation proceeding from the open end of the tube to advance through a certain space before it assumes that form in diverging into free space, which it must ultimately assume when it sends back no reflected wave from any point of its path. Before it reaches this state, a partial wave may be reflected in its course from each point towards the tube; and an indefinite number of these reflected waves will form a general reflected

wave, of which the period will be the same as that of each of its component waves, but the phase of which will be retarded as compared with that of a wave reflected immediately from the extremity of the tube. This is equivalent to our supposing a certain space beyond the extremity of the tube as subject to a disturbance (acting at consecutive instants along this space) such as to produce a wave diverging in all directions, and consequently sending a portion of this general wave back along the tube.

To give generality to the investigations of the preceding section, I have considered the effect on the position of the nodes which would be produced by any retardation of the phase of the wave reflected from the *stopped* end of a tube. It appears, however, that there is not in this case any displacement of the nodes appreciable by the mode of experimenting I have described. The only reason, in fact, for supposing any retardation of phase in this case, is founded in the imperfect analogy between the cases of the open tube and the tube closed with an elastic substance. The cases are far too different, however, to admit of any thing but vague inferences from such analogy; and it is manifest that no reasoning similar to that above applied to the open tube, can be applied to the closed one. If any retardation do exist in this case, I can only conceive it to arise from a cause similar to that suggested by Mr Willis*, viz. that *time* must be necessary for the action between the elastic stop and the air to produce its effect. This, however, appears much less probable in this case than in that which suggested the idea to Mr Willis, in which the action between the air and the vibrating body (a membrane) was *lateral* instead of being *direct*, as in the present instance. I have not been able to detect any indication of such law of force in a displacement of the nodes in the closed tube, though I have examined the case with great care, conceiving that any facts bearing directly upon the nature of the mutual action of two elastic media at their common surface must necessarily be of importance.

* *Cambridge Transactions,* Vol. IV. Part III. p. 346.

The experimental deductions in the preceding part of this section are based on the evidence afforded by the exploring membrane, because it is more direct than any other evidence which the phenomena appear to admit of, and therefore better calculated to supply those decisive and positive tests for ascertaining the accuracy or fallacy of our theoretical results, which it is my object to supply. We have seen the perfect accordance of these results with the general indications of the membrane, and also with the striking and well-defined phenomenon of the impossibility of making the plate vibrate in a certain manner with tubes of certain lengths. It remains for us to consider also how far our theory agrees with the phenomena of *resonance*, in those cases in which the conditions assumed in our mathematical investigations are satisfied, viz. where the communication between the external air and that in the tube at the surface of the plate is prevented, and the disturbance extends uniformly over the whole orifice. In such cases it will appear from the following enunciation, that the intensity of the sound is proportional to that of the aerial vibrations, as indicated by the membrane, and by the difficulty or facility with which the vibrations of the plate may be maintained. (See Arts. 27, 31.)

The resonance of the open *tube is scarcely perceptible when the length of it does not differ much from something less than an even multiple of* $\frac{\lambda}{4}$, *or* $2m \cdot \frac{\lambda}{4} - C$; *but as it approximates to something less than an odd multiple of that quantity, or* $(2m'+1)\frac{\lambda}{4} - C$, *the resonance increases, and at length becomes of painful intensity, increasing till it is no longer possible to maintain the same mode of vibration of the plate.* Whether the length of the tube be gradually increased or diminished in approximating to the above-mentioned lengths, the phenomena are precisely the same.

I was the better pleased to obtain this result, inasmuch as those which I first obtained (when the precaution of preventing communication with the external air was not attended to*), as well as those of previous

* In such cases the resonance was always greatest (as in the case considered in the text) when the difficulty of making the plate vibrate was greatest. The corresponding lengths of the tube may be seen in Art. 30.

experimenters, appeared either to contradict theory, or at least to be altogether anomalous. According to our common notion on the subject, an open tube gives the strongest resonance when its length is nearly equal to an *even* multiple of $\frac{\lambda}{4}$, instead of an *odd* multiple, as above stated; and Savart* has given this as the result of his own experiments for tubes of about the same diameter as those I have usually employed†; but asserting also that the length is less as the diameter is increased, and this too whether the disturbance extend over the whole orifice of the tube or not. My results, however, are entirely at variance with this latter assertion, for I confidently conclude from them that if the disturbance extend uniformly and equably over the orifice of the tube, the phenomena will be independent of its diameter‡, with the exception of the effect it may have on the displacement of the nodes§. If, however, the disturbance extend but partially over the orifice, I see no reason to doubt the accuracy of the last-mentioned results of M. Savart; and this supposition will also account for the apparent discrepancy between his results and mine as respects the length of the open tube (of which the diameter does not much exceed an inch) producing the greatest resonance; for it is manifest that with this partial disturbance none of that condensation and rarefaction on the surface of the plate can take place, which in my experiments necessarily attends, and may be considered as causing, that powerful resonance of which I have spoken. It is easily seen, in fact, that when the length of the tube is nearly equal to an odd multiple of $\frac{\lambda}{4}$, the phase of the waves reflected from any considerable part of the orifice not occupied by the vibrating plate, will be directly opposite to that of the waves propagated by the plate itself; and that thus a great part of the vibration within the tube will be destroyed by interference.

There is no difficulty, therefore, in explaining the non-existence of resonance in this case. If the tube, however, be lengthened or shortened by about $\frac{\lambda}{4}$, (still supposing the disturbance at its mouth partial), a

* *Annales de Chimie,* Tom. XXIV. p. 56.

† See Art. 36.

‡ See Art. 36, p. 264.

§ Art. 36.

resonance will be heard, though extremely feeble as compared with that I have found in my experiments. This is, in fact, the kind of resonance which has been observed by all experimenters. It does not appear to me to admit of the same obvious explanation which the other admits of; that which is usually received being, as I conceive, in itself insufficient, when subjected to those restrictions which must be imposed upon it by the general laws which govern the communication of motion from one particle of matter to another. At present, however, it is not my object to enter on the discussion of this and of some other points relative to this part of the subject. It is sufficient for me now to have shewn that that powerful resonance which I have observed in my experiments is exactly accordant with the results of our mathematical investigations, when the conditions assumed in those investigations are fully satisfied. I hope to return to the careful examination of other cases at a future period.

I have already alluded* to a paper by Mr Willis, published in the Transactions of this Society, in which he has described some experiments bearing on this subject, and affording a general corroboration of some of the results above stated. He fixed a *reed* to a sliding tube, and observed the intensity of the sound, when the reed was made to speak, produced by different lengths of the tube, and by means of a microscope carefully adjusted, he was able to observe the excursions of the reed in its vibration, and to obtain micrometer admeasurements of them. He thus found that when the length of the tube equalled about an even multiple of $\frac{\lambda}{4}$, it gave the exact note of the reed with no perceptible resonance. As the tube was gradually lengthened, the tone was flattened, and as the length approximated to about an odd multiple of $\frac{\lambda}{4}$, the extent of the reed's excursions was diminished, its vibrations became irregular and convulsive, till at length it ceased to produce any musical tone. When the tube, however, was a little lengthened beyond this point, the reed suddenly assumed its original form of vibration, producing a note of painful intensity, similar to that which I have

* See page 260.

described in my own experiments, although the extent of excursion of the reed was in this case less than in that in which no resonance was produced.

One discrepancy is observable between this experiment and mine, inasmuch as the intensity of the sound, instead of increasing as the length of the tube approximated to the odd multiple of $\frac{\lambda}{4}$, as in my experiments, gradually decreased*. The explanation, however, of this fact, is easily found in the diminished excursion of the reed, and still more, I suspect, in the *irregularity of its vibration*, by which the undulations produced by it are probably rendered imperfectly *sonorous*†. With this explanation of this apparent discrepancy, the general results of Mr Willis's experiments afford as strong a corroboration of those which I have obtained, as the difference between our modes of experimenting will allow. The flexibility of the reed, however, and its consequent ready obedience to the vibrations of the air, as compared with the inflexible obstinacy of a glass plate, together with the *partial* disturbance produced by the reed, render it a totally unfit agent in obtaining experimental tests for our mathematical results, though it presents to us in its own motions many interesting points of enquiry.

Our theory will also perfectly account for one of the most striking phenomena observable in wind instruments, viz. the rapidity with which different states of vibration are assumed within the tube, corresponding to different effective lengths of it, as determined by the opening or closing of the finger holes. We have seen (Art. 22, VII. VIII.) that

* For a very clear and distinct account of these experiments, I must refer the reader to the excellent paper from which the above is taken. It will be observed, however, that the results mentioned in the text were not the direct objects of Mr Willis's investigations, but were such as naturally offered themselves in the course of his experiments on the production of the vowel sounds.

† I think it very possible that the *form* of the aerial vibrations may have more to do with our sense of the intensity of sound than has been generally supposed; and perhaps some cases of resonance may admit the most satisfactory explanation on this hypothesis.

according to theory, if the cause maintaining the vibratory motion in a tube be suddenly changed, (as in passing from one note to another), the effect of the former mode of disturbance on the form of the succeeding vibration will become inappreciable in an exceedingly short period of time. Now in the most rapid musical passages, the number of notes played in a second never probably exceeds ten or twelve, and these usually embrace only the higher notes of the scale, for which there must be many hundred vibrations in a second. Suppose this number, however, not greater than about two hundred; any undulation transmitted from the reed or embouchure would still be reflected about twenty times at the open end in the interval between two consecutive notes in the most rapid musical passage. Now assuming unity to represent the intensity of a wave incident at the open extremity of the instrument*, let $1-\beta$ represent that of the reflected wave, $(1-\beta)^n$ will represent (at least sufficiently approximately) its intensity after n reflections; and consequently, as we have no reason to suppose β very small as compared with unity, it is probable that after five or six reflections, the intensity of this wave will be quite inappreciable. Hence the apparently instantaneous cessation of sound after the exciting cause has ceased, and the most rapid transition from one note to another, are perfectly accordant with theory.

M. Poisson, in the Memoir referred to in the early part of this paper, has also investigated the vibratory motion of air in two tubes of different diameters united together at one extremity. I hope to examine this case also experimentally. His results must necessarily be erroneous, as far as they depend on the physical condition he has assumed to exist at the extremity of the open tube, and which I have shewn to be inconsistent with observed phenomena in the uniform tube.

* See Art. 14.

W. HOPKINS.

ST PETER'S COLLEGE,
May 20, 1833.

XI. *On the Latitude of Cambridge Observatory.* *By* George Biddell Airy, M.A. *late Fellow of Trinity College, Plumian Professor of Astronomy and Experimental Philosophy, and one of the Vice-Presidents of the Society.*

[Read *April* 14, 1834.]

The accurate determination of the latitude, with an instrument like the Mural Circle now in use at the Observatory, seems at first sight to be an easy business. In practice, however, it is not without difficulties. I do not here allude to the correction for refraction; since, though there may be a trifling uncertainty in regard to its magnitude, it is easy to leave a result subject to that uncertainty, and admitting of correction without any trouble whenever a correction of the refraction shall be established. Nor do I allude to the uncertainty in the corrections by which, from a star's apparent place on any day of observation, its mean place at a fixed epoch can be inferred; since the uncertainty about any of these is far less than the smallest quantity for which we could pretend to answer in fixing the latitude of any place; and its effects being periodical, would in a comparatively short series of observations, produce no sensible effect. The difficulties to which I allude are instrumental: they are not periodic in time, like the latter; nor do they admit of correction from posterior researches, like the former of the causes of uncertainty which I have mentioned; they are moreover such as would scarcely be suspected to exist, until their effects are discovered from the discordance of the results of observations.

The Mural Circle is an instrument which gives simply the reading of that point of the graduated limb which is opposite to an imaginary fixed index when the telescope is pointed to the object of observation.

A single observation therefore gives us no tangible result. It is necessary to have one other observation, or a series of observations, by which the reading of that point of the limb can be found which is opposite to the same index when the telescope is directed to some point of reference; then the difference between this reading and the former is the angular distance of the object observed from the point of reference. It was intended originally by the maker that this point of reference should be the celestial pole. In practice, however, it is found necessary to descend one step nearer to terrestrial things, and to adopt for the point of reference the zenith; a point which, though not marked any more than the pole by any obvious phenomena, can yet be discovered by a process which involves less of astronomical assumptions, and requires a shorter time for the complete determination.

The method of determining the zenith point from observations by reflexion at the surface of mercury, has been introduced into observatories almost entirely by the practice of the present Astronomer Royal at the Greenwich Observatory. The use of two similar circles (as at Greenwich) makes the process one of little labour, though requiring the co-operation of two observers. The same celestial objects being repeatedly observed by direct vision with both circles, the differences of the corresponding readings of the two circles are found; and any observations made with one can be referred to the other. Then when any bright star passes the meridian, one circle is employed in observing it by direct vision, and the other at the same time is employed in observing it by reflexion at the surface of mercury; the reading of the latter circle is referred to the former circle; and then the reading which is a mean between the reading for the direct observation and the referred reading for the reflected observation, is the reading that corresponds to a horizontal position of the telescope; and by adding or subtracting a quadrant, the reading which corresponds to a zenithal position of the telescope is obtained.

With a single circle this process cannot be adopted. In some instances it has been imitated by observing a star directly on one night, and observing the same star by reflexion on another night. The calculation for the zenith point then relies on our perfect acquaintance with the variations of refraction and other corrections from one night to another; and thus a cause of inaccuracy is introduced, which does not exist in the other method. In the Cambridge Observatory a different method is regularly employed (for the idea of which I am indebted to a suggestion of Mr Sheepshanks). When a star is to be observed by reflexion, the circle is set approximately for the reflected observation, and the six microscopes are read; when the star has entered the field, and before it has reached the center, it is bisected by the micrometer wire, (which in fact measures its distance from the fixed wire, and thus gives a correction to be applied to the mean of the six microscopes,) and then there is ample time to allow the circle to be turned to the position in which the star can be observed directly, shortly after it has passed the center of the field. Thus a direct and reflected observation are obtained at the same transit. This method is, in my opinion, much preferable to the second that I have mentioned, and in some respects superior to the first.

Either of the methods which applies to one circle enables us, as will shortly be seen, to examine severely into the consistency of the results obtained in different positions of the circle; and this must be considered as a most valuable property of this method of determining the zenith point, and one which places it far above the use of a collimator or any similar instrument.

I had hoped, on commencing observations with the Mural Circle at the beginning of the year 1833, to be able in a very short time to obtain a very approximate latitude. I proposed to observe some stars every night in the manner above described, as well as circumpolar stars (which might or might not be observed in the mercury): by the former I should obtain a very good zenith point; and then each observation of the latter, above and below the pole, would give me a value of the co-latitude.

But after a few nights' observations, I found that the reading for the zenith point, as determined by different stars, was not the same.

Had the discordance been wholly without regularity, this would have given me no anxiety. But the first week's observations enabled me to see with certainty that one general rule could be laid down: the reading for the zenith point as determined by northern stars was invariably greater than that found from southern stars. As the readings increase while the telescope is turned towards the south, this discordance is of the same kind as that which would be produced if the object end of the telescope dropped by its own weight.

After much anxious thought and many fruitless attempts to explain this discordance, I was obliged to give it up entirely. The method which was adopted for approximate reduction of the observations, easily admitting of future correction, was the following. When in one night, or in several nights which it appeared practicable to group together, stars had been observed by reflexion in different parts of the meridian, I took the three means of zenith points determined by stars far north, by stars far south, and by stars near the zenith, as three separate results; and then I took the mean of these three for the zenith point. For an approximate co-latitude I used 37°. 47′. 6″,83.

At the beginning of March the telescope was moved about thirty degrees on the circle; at the beginning of August it was again moved thirty degrees, and on this occasion (as it appeared that the circle was not exactly balanced) a pound of lead was attached to the eye end of the telescope; at the beginning of December it was again moved about thirty degrees. It does not appear however that the fact of the discordance has been affected, but its law seems to have been in some degree altered.

A discordance of the same kind exists, I believe, in every circle that has been properly examined. I am informed by Mr Henderson (late Cape Astronomer) that he has found it in the Cape Circle. It was recognized as existing in the Greenwich Circles: and, though the system of observing there, which I have described, does not allow us to trace the unmixed faults of either circle, yet from the discordance in the places of stars as determined by the two circles, and its variation in different points of the meridian, I am inclined to think that the defect in one circle is different from that in the other.

In vain have I endeavoured to discover the cause of this discordance. I once thought that it might be owing to the circumstance, that for the reflection-observation the circle is at rest for some minutes after the microscopes are read, and possibly it might (though clamped) have changed its position. A series of observations expressly made, showed, however, that there was no sensible change either in a few minutes or in many hours. I thought that the surface of the mercury might be sensibly curved, and that from a habit of observing in one part of the trough, an error might be produced. A set of experiments proved, however, that there was not the least sensible difference in the results found from observing at one or the other end of the trough. A flexure of the wire in the field of view would not explain it, as the discordance which that would produce is of the opposite kind. There appeared to be no reason for supposing an error in the determination of the coincidence of the micrometer wire with the fixed wire, in the value of the micrometer screw, or in the observation with the micrometer wire. The object glass, repeatedly examined by myself and once by Mr Simms, did not appear to be loose in its cell. I am driven at last to the supposition that the circle sensibly changes its figure; but I have no proof of this, nor do I see distinctly how it should produce the discordance in question. Three sets of readings of every 10° under all the microscopes, have not assisted me to discover such change. My *à priori* opinion is, that a change in figure is hardly possible. The telescope, it must be remembered, is attached at its ends to the limb of the circle: the limb is in one piece (cast in several pieces and burnt together); and the whole arrangement of parts seems admirably adapted to prevent any change. If I had to fix on an astronomical instrument which appeared less likely to change than any other, I should certainly choose the Mural Circle.

To discover experimentally the law of discordance, I proceeded as follows. The observations being reduced, and those of each star being digested under the heads of D, R, SP. D., and SP. R., I selected for the three first positions of the telescope all the unexceptionable corresponding observations D and R. (The stormy weather of December made it impracticable to observe low stars by reflexion). In each case of a double observation, the difference of the results D and R would be double the difference between the zenith point as found from that star, and the zenith point adopted in the reductions. The mean of the differences of all the corresponding results D and R, would therefore be double the mean of all the differences between the zenith points found from the particular star, and the zenith points found from all by a tolerably uniform system: and thus it might be considered as double the difference between the zenith point found without error of observation from that star, and a certain imaginary well defined point. These values for all the stars, and for each position of the telescope, were arranged in tables (for which, as well as for some other numerical values, I must refer to the Cambridge Observations, Vol. VI.)

The next step was, to connect these, approximately at least, by a law. I soon found that to attempt this by calculation was almost hopeless. Combinations of constants, sin Z.D., sin Z.D. $\cos^2$ Z.D., cos 2 Z.D., were tried in vain. I therefore adopted a graphical method similar to that used by Sir John Herschel, in the reduction of his sweeps, and described by him in the *Phil. Trans.* 1833. Taking the line of abscissæ for zenith distances, and the ordinates to represent the mean of the differences above mentioned, I made a curve to pass among the points so determined, as well as I could, giving to each point an importance depending on the number of observations. From this curve I measured off the ordinates for every 10° of zenith distances; half of this quantity I considered to be the correction to the observed zenith distance, to be applied with different signs to the direct and the reflected observation. The only respect in which theoretical consideration may be said to have assisted me is the following. Since the error in the relation between the position of the telescope and the reading of the circle, to which the discordance must be due, is periodical and never infinite, it may be expressed by sines and cosines of the Z. D. and its multiples. Now it is useless to take sines of even multiples, or cosines of odd multiples, because when $180° - Z.D.$ is substituted for $Z.D.$, the result is equal in magnitude but opposite in sign; and therefore when the two are added together, (as they are in finding the zenith point from each star), no trace of these terms would remain. Thus there may be sensible flexure in the circle which cannot be discovered from observation by reflexion. The sines of odd multiples, and the cosines of even ones, (all which may be expressed in finite series of powers of $\sin Z.D.$), will produce the same values with the same signs for $180° - Z.D.$ as for $Z.D.$, and these will affect the zenith point. Thus it appears that the terms which affect the zenith point are the same for a direct observation and for the corresponding observation by reflexion, and it is this which justifies us in applying half the discordance to each. It appears also that when $Z.D. = 90°$, the function is maximum or minimum, and hence the curve in the graphical process above described must there be parallel to the line of abscissæ.

The tables of corrections being thus formed, I now considered myself entitled to apply them to the reduced results of all the observations, whether there were corresponding observations of the opposite kind or not.

The principal steps of the succeeding process may be gathered from the subjoined table. The first column contains the name of the star, its position with regard to the pole, (the lower transit being marked by S.P.), and the method of observing it (the letters D and R being always used for direct and reflected vision). Here it is to be observed that a star above the pole and the same star below the pole are reduced as separate stars, which is necessary, because the observations have been reduced with an *assumed* co-latitude, or an assumed place of the pole, the error in which assumption can be found only by comparing the separate results for the same star above and below. The second column contains the number of observations. The third contains its mean N.P.D. for *Jan.* 1, 1833, as found from the mean of all the results in each position and mode of observation, and reduced with the assumed co-latitude $37^\circ.47'.6,''83$: those determined from the lower transits of the star have the negative sign. For refraction, Bessel's tables are used. The fourth column contains the seconds only, as corrected for the errors above described; this has been done by taking the number of observations in each position of the telescope on the circle, and finding the mean correction, supposing that to each observation the correction proper to that position was applied. The negative sign has still been retained for the lower observations. The fifth column contains the whole number of observations in each position of the star: and the sixth contains the mean N.P.D. for each position, as inferred from the combination of direct and reflected observations. The seventh contains the whole number of observations for both positions. The eighth contains the algebraic sum of the two determinations of N.P.D., as the star is above or below the pole. If the assumed co-latitude were correct, this sum would $=0$; if the assumed co-latitude be increased by x, this sum would be increased by $2x$, and therefore to make it now $=0$, x must be taken $= -\frac{1}{2} \times$ sum in 8th column. The results, as might be expected, are however different for different stars, though the difference is much smaller than I could almost have hoped; the extreme difference in the correction of latitude being $1,''3$, and this being the difference between two results from stars nearly in the same parallel (shewing that it does not arise from error in the corrections above described), and which had been not much observed. It now becomes necessary to determine how the relative importance of these results shall be estimated. It would not be right to give a value proportionate to the number of observations, because part of the discordance may be produced by errors of division and other causes which, in the observations of a single star, produce constant errors. The ninth column contains the numbers by which (from my estimation of the comparative influence of constant and variable errors) I suppose the value of each result to be estimated. The tenth contains the product of the corresponding numbers in columns 8 and 9. The sum of the numbers in column 10 being divided by the sum of those in column 9, gives $+2'',82$ for the double correction, or $+1'',41$

for the single correction, of the co-latitude; and the co-latitude thus corrected is $37^\circ . 47' . 8'',24$, or the latitude $52^\circ . 12' . 51'',76$. This result I conceive to be correct within a small fraction of a second. The number of circumpolar stars used for this determination is 10, and the whole number of observations 917.

In describing the process by which I have arrived at the above result, it has been my wish to present to the Society not only a determination possessing considerable local interest, but also an account of instrumental anomalies which are of general scientific importance. In further illustration of the latter point I will allude to the discordances in the determinations of the obliquity of the ecliptic. It is well known that most astronomers have found the obliquity smaller from observations at the winter solstice than from those at the summer solstice. Now if I had used only the latitude found from direct observations of circumpolar stars, and had applied no correction to the observations of the Sun, I should also have found two values for the obliquity discordant by about $5''$, the winter obliquity being the smaller. With the corrections above described, (and which were formed entirely from observations of stars, and before I had even examined my sun observations) the two values of the obliquity agree within $1''$. I might have altered the corrections so as to remove part of this discordance, but I prefer leaving them in the shape in which they were given by independent considerations. Indeed if I had confined myself to the January observations for the winter solstice, and omitted those of December when the correction is less certain, the discordance would wholly have disappeared. A very small alteration of the constant of refraction (such as would not alter the latitude much more than $0'',1$), or a very small alteration in the law of refraction (which would not be sensible in the latitude) would remove this difference. But I hardly venture to assume that observations of the Sun, near the winter solstice, can be relied on to this degree of accuracy.

I will only add, in conclusion, that I believe the method which I have used to be the only one of those in practice from which a good result can be obtained. Had I determined my zenith points by a floating collimator, the result of observations on Polaris and δ U. Minoris would have given the latitude more than a second wrong, and the polar distance of every southern body more than two seconds wrong: the result of observations on the Sun would have given nearly the same error in the latitude but with the opposite sign. If a circle reversible round a vertical axis had been used (as at Dublin, Palermo, &c.) its errors would (supposing the mere circle exactly as good,) have been just as great as if a collimator were employed. The method adopted above appears most valuable, not only because it gives numerical conclusions more accurate than any other, but also because it enables us to observe discordances and to suspect faults which, though they confused our results, might otherwise have wholly eluded our discovery.

G. B. AIRY.

OBSERVATORY,
March 23, 1834.

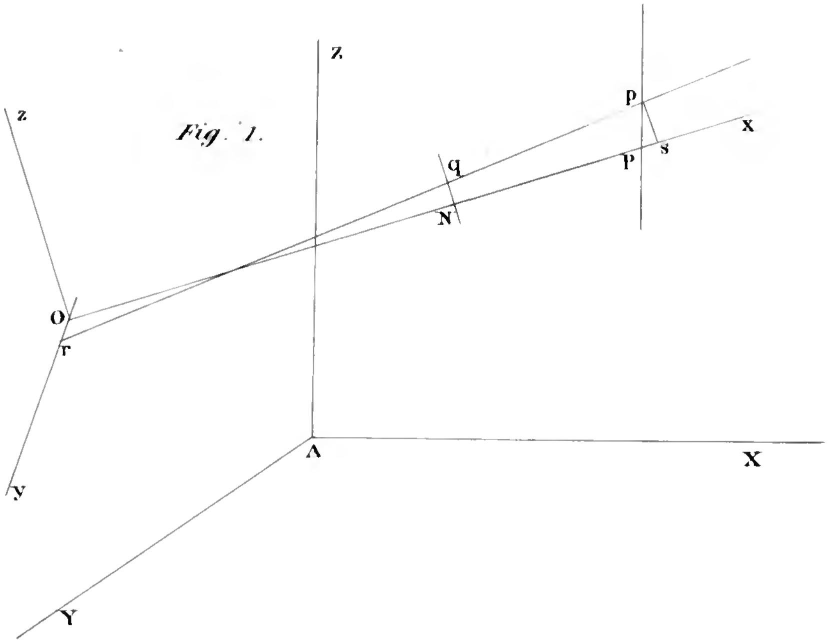


Fig. 1.

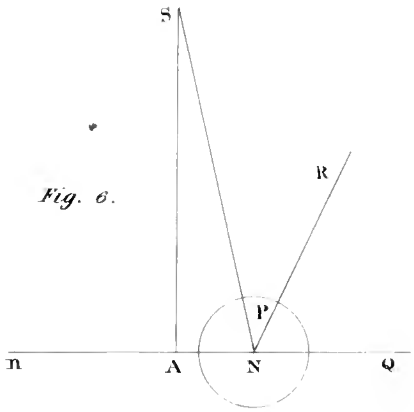


Fig. 6.

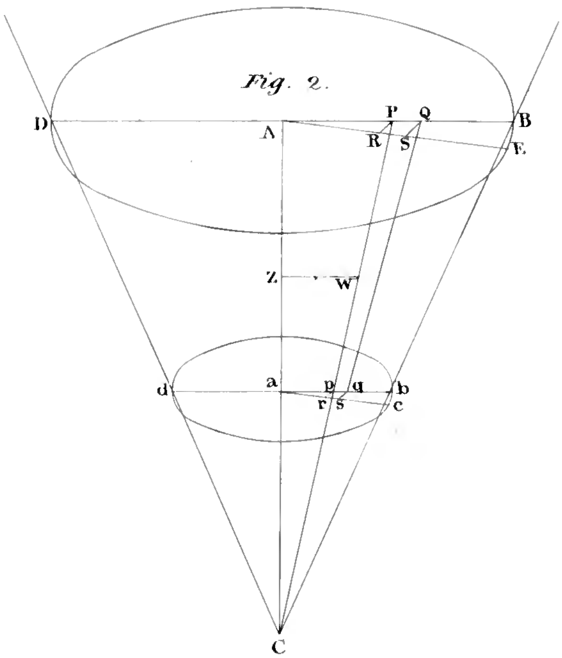


Fig. 2.

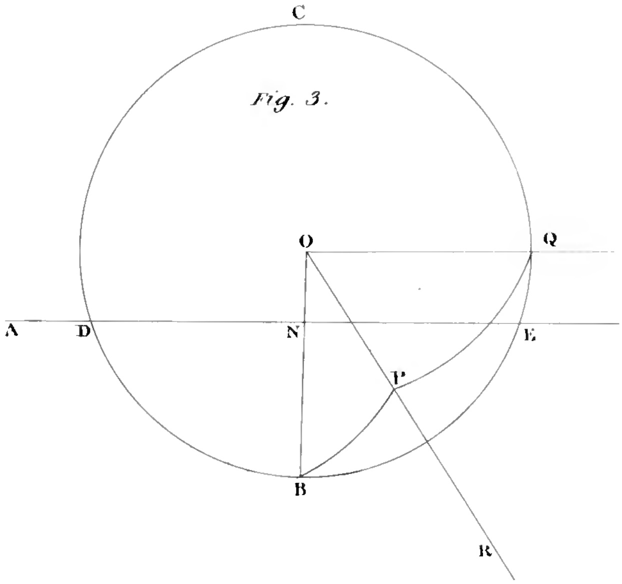


Fig. 3.

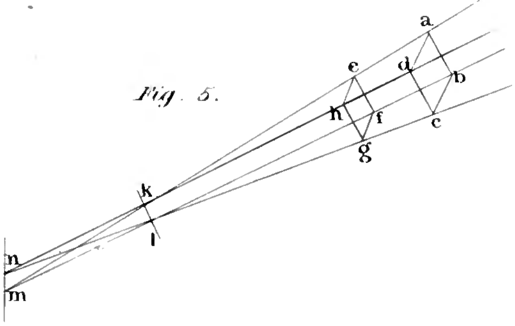


Fig. 5.

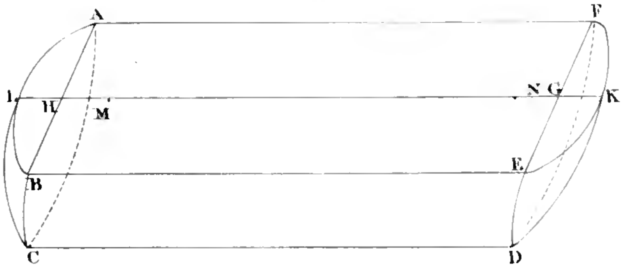
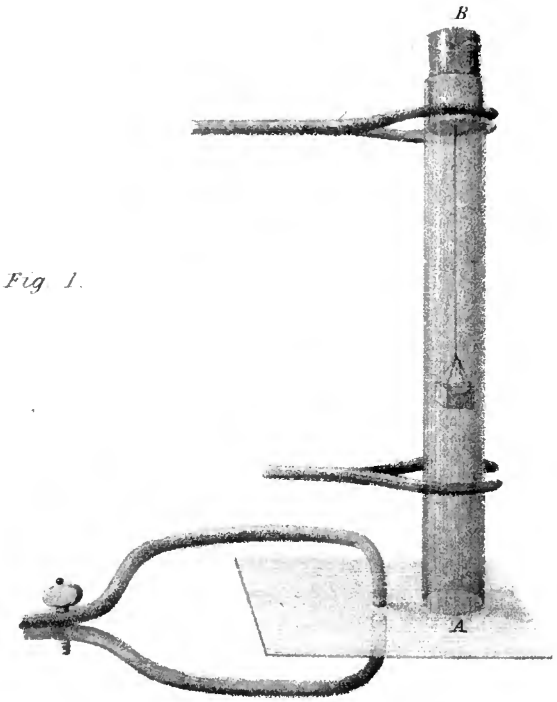


Fig. 4.

Fig. 1.



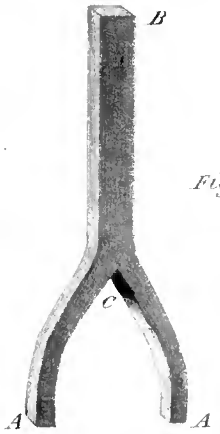


Fig. 3.

Fig. 2.

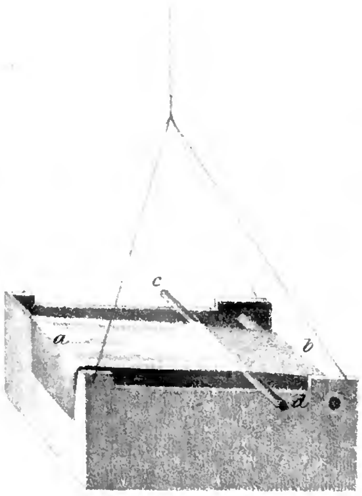


TABLE *exhibiting the* CALCULATIONS *for correcting the* LATITUDE *of* CAMBRIDGE OBSERVATORY; *the Observations having been reduced with the assumed Latitude* 52°. 12′. 53″,17.

Star's Name.	No. of Obs.	Uncorrected mean N.P.D.	Corrected for Discordance.	No. of Obs.	Concluded N.P.D.	No. of Obs.	Algebraic Sum of Determin.	Weight of Result.	Product.
		° ′ ″	″		″		″		
Polaris . . . . D	113	1 . 34 . 52,22	51,38	155	51,49	324	− 3,21	5	− 16,05
Polaris . . . . R	42	51,00	51,78						
Polaris S.P. . . . . D	111	− 1 . 34 . 53,77	− 54,62	169	− 54,70				
Polaris S.P. . . . . R	58	55,70	− 54,83						
δ Ursæ Minoris . . . . . . D	43	3 . 24 . 45,21	44,44	80	44,35	142	− 3,00	3	− 9,00
δ Ursæ Minoris . . . . . . R	37	43,48	44,24						
δ Ursæ Minoris S.P. . . D	39	− 3 . 24 . 46,34	− 47,40	62	− 47,35				
δ Ursæ Minoris S.P. . . R	23	48,42	− 47,28						
β Ursæ Minoris . . . . . . D	20	15 . 9 . 41,65	41,25	37	41,78	61	− 3,79	2	− 7,58
β Ursæ Minoris . . . . . . R	17	42,02	42,40						
β Ursæ Minoris S.P. . . D	22	− 15 . 9 . 44,58	− 45,52	24	− 45,57				
β Ursæ Minoris S.P. . . R	2	47,54	− 46,18						
β Cephei . . . . D	4	20 . 10 . 15,91	15,74	4	15,74	23	− 1,55	1	− 1,55
β Cephei . . . . R	none	- - -	- -						
β Cephei S.P. . . . . D	12	− 20 . 10 . 16,97	− 17,74	19	− 17,29				
β Cephei S.P. . . . . R	7	17,24	− 16,52						
δ Draconis . . . . D	3	22 . 37 . 54,40	54,34	6	54,17	22	− 1,66	1	− 1,66
δ Draconis . . . . R	3	53,95	54,01						
δ Draconis S.P. . . . . . . D	10	− 22 . 37 . 54,99	− 56,14	16	− 55,83				
δ Draconis S.P. . . . . . . R	6	56,31	− 55,31						
α Draconis . . . . D	14	24 . 49 . 24,80	24,76	25	24,63	35	− 3,41	1	− 3,41
α Draconis . . . . R	11	24,51	24,47						
α Draconis S.P. . . . . . . D	7	− 24 . 49 . 26,90	− 28,10	10	− 28,04				
α Draconis S.P. . . . . . . R	3	28,90	− 27,90						
α Ursæ Majoris . . . . . . D	32	27 . 20 . 55,75	55,84	64	55,74	69	− 4,12	1	− 4,12
α Ursæ Majoris . . . . . . R	32	55,73	55,64						
α Ursæ Majoris S.P. . . D	5	− 27 . 20 . 59,30	− 59,86	5	− 59,86				
α Ursæ Majoris S.P. . . R	none	- - -	- -						
α Cephei . . . . D	43	28 . 7 . 11,43	11,58	78	11,54	99	− 1,59	2	− 3,18
α Cephei . . . . R	35	11,64	11,50						
α Cephei S.P. . . . . D	13	− 28 . 7 . 12,67	− 13,27	21	− 13,13				
α Cephei S.P. . . . . R	8	13,47	− 12,93						
δ Ursæ Majoris . . . . . . D	26	32 . 2 . 18,12	18,41	52	18,25	58	− 3,55	1	− 3,55
δ Ursæ Majoris . . . . . . R	26	18,40	18,09						
δ Ursæ Majoris S.P. . . D	3	− 32 . 2 . 20,38	− 21,16	6	− 21,80				
δ Ursæ Majoris S.P. . . R	3	23,23	− 22,45						
α Cassiopeiæ . . . . D	34	34 . 22 . 45,38	45,73	49	45,82	84	− 1,75	2	− 3,50
α Cassiopeiæ . . . . R	15	46,36	46,03						
α Cassiopeiæ S.P. . . . . D	26	− 34 . 22 . 47,37	− 47,93	35	− 47,57				
α Cassiopeiæ S.P. . . . . R	9	47,00	− 46,54						

TRANSACTIONS

OF THE

CAMBRIDGE

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VOL. V. PART III.

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M.DCCC.XXXV.

XII. *On the Diffraction of an Object-glass with Circular Aperture.* By GEORGE BIDDELL AIRY, A. M. *late Fellow of Trinity College, and Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge.*

[Read *Nov.* 24, 1834.]

THE investigation of the form and brightness of the rings or rays surrounding the image of a star as seen in a good telescope, when a diaphragm bounded by a rectilinear contour is placed upon the object-glass, though sometimes tedious is never difficult. The expressions which it is necessary to integrate are always sines and cosines of multiples of the independent variable, and the only trouble consists in taking properly the limits of integration. Several cases of this problem have been completely worked out, and the result, in every instance, has been entirely in accordance with observation. These experiments, I need scarcely remark, have seldom been made except by those whose immediate object was to illustrate the undulatory theory of light. There is however a case of a somewhat different kind; which in practice recurs perpetually, and which in theory requires for its complete investigation the value of a more difficult integral; I mean the usual case of an object-glass with a circular aperture. The desire of submitting to mathematical investigation every optical phænomenon of frequent occurrence has induced me to procure the computation of the numerical values of the integral that presents itself in this inquiry: and I now beg leave to lay before the Society the calculated table, with a few remarks upon its application.

Let a be the radius of the aperture of the object-glass, f the focal length, b the lateral distance of a point (in the plane which is normal to the axis of the telescope) from the focus. Then, the lens being supposed aplanatic, and a plane wave of light being supposed incident, the immediate effect of the lens is to give to this wave a spherical shape, its centre being the focus of the lens. Every small portion of the wave, as limited by the form of the object-glass, must now be supposed to be the origin of a little wave, whose intensity is proportional to the surface of that small portion; and the phases of all these little waves, at the time of leaving the spherical surface above alluded to, must be the same. If then $\delta x \times \delta y$ be the area of a very small part of the object-glass, q the distance of that part from the point defined by the distance b, the displacement of the ether at that point, caused by this small wave, will be represented by

$$\delta x \times \delta y \times \sin \frac{2\pi}{\lambda}(vt - q - A);$$

and the whole displacement caused by the small waves coming from every part of the spherical wave will be the integral of

$$\sin \frac{2\pi}{\lambda}(vt - q - A)$$

through the whole surface of the object-glass, q being expressed in terms of the co-ordinates of any point of the spherical surface.

Now let x be measured from the center of the lens in a direction parallel to b; y perpendicular to x and also perpendicular to the axis of the telescope; and z from the focus parallel to the axis of the telescope. Then

$$q = \sqrt{\{(x-b)^2 + y^2 + z\}} = \sqrt{(x^2 + y^2 + z^2 - 2bx)}$$

omitting squares and superior powers of b. But $x^2 + y^2 + z^2 = f^2$, since the wave is part of a sphere whose centre is the focus; therefore,

$$q = \sqrt{(f^2 - 2bx)} = f - \frac{b}{f}x \text{ nearly};$$

and the quantity to be integrated is

$$\sin \frac{2\pi}{\lambda}\left(vt - f - A + \frac{b}{f}x\right).$$

The first integration with regard to y is simple, as y does not enter into the expression, which is therefore to be considered as constant. Putting y_1 and y_2 for the smallest and greatest values of y corresponding to x, the first integral is

$$(y_2 - y_1) \times \sin\frac{2\pi}{\lambda}(vt - f - A + \frac{b}{f}x).$$

To this point of the investigation the expressions are general, including every form of contour of the object-glass.

We must now substitute the values of y_1 and y_2 in terms of x, before integrating with regard to x. For a circular aperture

$$y_2 - y_1 = 2\sqrt{a^2 - x^2}$$

where the sign of the radical is essentially positive. Hence the displacement of the ether at the point defined by the distance b is represented by

$$2\textstyle\int_x\sqrt{a^2 - x^2} \, . \, \sin\frac{2\pi}{\lambda}(vt - f - A + \frac{b}{f}x)$$

$$= 2\sin\frac{2\pi}{\lambda}(vt - f - A)\textstyle\int_x\sqrt{a^2 - x^2} \, . \, \cos\frac{2\pi}{\lambda} \, . \, \frac{b}{f}x$$

$$+ 2\cos\frac{2\pi}{\lambda}(vt - f - A)\textstyle\int_x\sqrt{a^2 - x^2} \, . \, \sin\frac{2\pi}{\lambda} \, . \, \frac{b}{f}x,$$

and the limits of integration are from $x = -a$ to $x = +a$. Between these limits it is evident that

$$\textstyle\int_x\sqrt{a^2 - x^2} \, . \, \sin\frac{2\pi}{\lambda} \, . \, \frac{b}{f}x = 0,$$

(as every positive value is destroyed by an equal negative value); and the displacement is therefore represented by

$$2\sin\frac{2\pi}{\lambda}(vt - f - A)\textstyle\int_x\sqrt{a^2 - x^2} \, . \, \cos\frac{2\pi}{\lambda} \, . \, \frac{b}{f}x,$$

the integral being taken between the limits $x = -a$, $x = +a$.

If we make $\frac{x}{a} = w$, $\frac{2\pi}{\lambda} \cdot \frac{ba}{f} = n$, the expression becomes

$$2a^2 . \sin \frac{2\pi}{\lambda}(vt - f - A) \int_w \sqrt{1 - w^2} . \cos nw, \text{ from } w = -1 \text{ to } w = +1,$$

$$\text{or } 4a^2 . \sin \frac{2\pi}{\lambda}(vt - f - A) \int_w \sqrt{1 - w^2} . \cos nw, \text{ from } w = 0 \text{ to } w = 1.$$

It does not appear, so far as I am aware, that the value of this integral can be exhibited in a finite form either for general or for particular values of w. The definite integral

$$\int_n \sqrt{1 - w^2} . \cos nw \text{ (from } w = 0 \text{ to } w = 1,)$$

(which will be a function of n only) being expressed by N, it may be shewn that N satisfies the linear differential equation

$$N + \frac{3}{n} . \frac{dN}{dn} + \frac{d^2N}{dn^2} = 0,$$

which may be depressed to an equation of the first order that does not appear to yield to any known methods of solution.

If we solve the equation by assuming a series proceeding by powers of n, or if we expand $\cos nw$ and integrate each term separately, we arrive (by either method) at this expression for the integral

$$\frac{\pi}{4} \times \left(1 - \frac{n^2}{2 . 4} + \frac{n^4}{2 . 4^2 . 6} - \frac{n^6}{2 . 4^2 . 6^2 . 8} + \&c.\right)$$

The table appended to this paper contains the values of the series in the bracket, for every 0,2 from $n = 0$ to $n = 12$. Each value has been calculated separately, the logarithms used in the calculation have been systematically checked, and the whole process has been carefully examined. The calculations were carried to one place further than the numbers here exhibited. I believe that they will seldom be found in error more than a unit of the last place; except perhaps in some of the last values, where the rapid divergence of the series for the first five or six terms made it difficult to calculate them accurately by logarithms.

In the use of this table n must be taken $= \frac{2\pi}{\lambda} \cdot \frac{ba}{f}$. If instead of using the linear distance b to define the point of the field at which we wish to ascertain the illumination, we use the number of seconds s, then $b = f.s.\sin 1''$, and n must be taken $= \frac{2\pi}{\lambda} as \sin 1''$. If λ be taken for mean rays $= 0{,}000022$ inch, n must be taken $= 1{,}3846 \times as$, a being expressed in inches. From this expression, and from the numbers of the table, we draw the following inferences.

1. The image of a star will not be a point but a bright circle surrounded by a series of bright rings. The angular diameters of these (or the value of s corresponding to a given value of n) will depend on nothing but the aperture of the telescope, and will be inversely as the aperture.

2. The intensity of the light being expressed (on the principles of the undulatory theory) by the square of the coefficient of

$$\sin \frac{2\pi}{\lambda}(vt - f - A),$$

and the intensity at the center of the circle being taken as the standard, it appears that the central spot has lost half its light when $n = 1{,}616$, or $s = \frac{1{,}17}{a}$; that there is total privation of light, or a black ring, when $n = 3{,}832$, or $s = \frac{2{,}76}{a}$; that the brightest part of the first bright ring corresponds to $n = 5{,}12$, or $s = \frac{3{,}70}{a}$, and that its intensity is about $\frac{1}{57}$ of that at the center; that there is a black ring when $n = 7{,}14$, or $s = \frac{5{,}16}{a}$; that the brightest part of the second bright ring corresponds to $n = 8{,}43$, or $s = \frac{6{,}09}{a}$, and that its intensity is about $\frac{1}{240}$ of that of the center: that there is a black ring when $n = 10{,}17$, or $s = \frac{7{,}32}{a}$; that the brightest part of the third bright ring corresponds to $n = 11{,}63$, or $s = \frac{8{,}40}{a}$, and that its intensity is about $\frac{1}{620}$ of that of the center.

The rapid decrease of light in the successive rings will sufficiently explain the visibility of two or three rings with a very bright star and the non-visibility of rings with a faint star. The difference of the diameters of the central spots (or spurious disks) of different stars (which has presented a difficulty to writers on Optics) is also fully explained. Thus the radius of the spurious disk of a faint star, where light of less than half the intensity of the central light makes no impression on the eye, is determined by making $n = 1{,}616$, or $s = \frac{1{,}17}{a}$: whereas the radius of the spurious disk of a bright star, where light of $\frac{1}{10}$ the intensity of the central light is sensible, is determined by making $n = 2{,}73$, or $s = \frac{1{,}97}{a}$.

The general agreement of these results with observation is very satisfactory. It is not easy to obtain measures of the rings; since when a is made small enough to render them very distinct as to form and separation, the intensity of their light (which varies as a^4) is so feeble that they will not bear sufficient illumination for the use of a micrometer. Fraunhofer however obtained measures agreeing pretty well (as to proportion of diameters, &c.) with the results above.

For verification of the numbers it would probably be best to use an elliptic aperture. By an investigation of exactly the same kind as that above it will be found that the rings will then be ellipses exactly similar to the ellipse of the aperture, but in a transverse position; that the major axes of the rings for the elliptic aperture will be the same as the diameters of the rings for a circular aperture whose diameter = minor axis of ellipse of aperture, but that the intensity will be greater in the proportion of the squares of the axes. I have not yet had an opportunity of examining this in practice.

I shall now apply the numbers of the table to the solution of the following problem. To find the diameters, &c. of the rings when a circular patch, whose diameter is half the diameter of the object-glass, is applied to its center, so as to leave an annular aperture.

The radius of the patch being $\frac{a}{2}$, it is easily seen that the displacement (using the same notation) is

$$2\sin\frac{2\pi}{\lambda}(vt-f-A)\int_x\sqrt{a^2-x^2}\,.\cos\frac{2\pi}{\lambda}\,.\frac{b}{f}x \text{ (from } x=-a \text{ to } x=+a)$$

$$-2\sin\frac{2\pi}{\lambda}(vt-f-A)\int_x\sqrt{\frac{a^2}{4}-x^2}\,.\cos\frac{2\pi}{\lambda}\,.\frac{b}{f}x\left(\text{from } x=-\frac{a}{2} \text{ to } x=+\frac{a}{2}\right).$$

Putting $\frac{x}{a}=w$, $\frac{2x}{a}=u$, this becomes

$$4a^2\,.\sin\frac{2\pi}{\lambda}(vt-f-A)\int_w\sqrt{1-w^2}\,.\cos\frac{2\pi}{\lambda}\,.\frac{ba}{f}w$$

$$-4\,.\frac{a^2}{4}\,.\sin\frac{2\pi}{\lambda}(vt-f-A)\int_u\sqrt{1-u^2}\,.\cos\frac{2\pi}{\lambda}\,.\frac{ba}{2f}u,$$

the limits of integration both for w and for u being 0 and 1. Omitting the factor $a^4\pi$, the intensity will be expressed by

$$\left\{\phi(n)-\tfrac{1}{4}\phi\left(\frac{n}{2}\right)\right\}^2,$$

where $\phi(n)$ is the number given in the table.

Upon forming the numerical values we find that the black rings correspond to values of $n=3{,}15$, $7{,}18$, $10{,}97$: and that the intensities of the bright rings (in terms of the intensity of the center) are $\frac{1}{10}$, $\frac{1}{80}$. Thus the magnitude of the central spot is diminished, and the brightness of the rings increased, by covering the central part of the object-glass.

In like manner, if the diameter of the circular patch $=a(1-p)$, the intensity of light would be proportional to $\{\phi(n)-(1-p)^2\,.\,\phi(n-pn)\}^2$.

The quantity under the bracket, if p is very small, is equal to

$$2p \cdot \phi(n) + pn\phi'(n) = \frac{p}{n} \cdot \frac{d}{dn}\{n^2\phi(n)\}.$$

In the case of a very narrow annulus therefore the diameters of the black rings will be determined by making $n^2\phi(n)$ maximum or minimum. It appears then that there ought to be only one black ring corresponding to each black ring with the full aperture, and that its diameter ought to be somewhat smaller. This conclusion does not agree with the experiments recorded by Sir J. Herschel, in the Encyc. Metrop. Article Light, page 488: but it is acknowledged there that the results are discordant with Fraunhofer's: and I am inclined therefore to attribute the phænomena observed by Sir J. Herschel to some other cause.

The investigation of cases of diffraction similar to that discussed here appears to me a matter of great interest to those who are occupied with the examination of theories of light. The assumption of transversal vibrations is not necessary here as for the explanation of the phænomena of polarization: and they therefore offer no arguments for the support of that principle. But they require absolutely the supposition of almost unlimited divergence of the waves coming not merely from a small aperture, but also from every point of a large wave: and the results to which they lead us, shew strikingly how small foundation there was for the original objection to the undulatory theory of light, viz. that if waves spread equally in all directions, there could be no such thing as darkness.

G. B. AIRY.

Observatory, Cambridge,
November 20, 1834.

TABLE of the values of $\phi(n) = \frac{4}{\pi} \int_w \sqrt{1-w^2} \,.\, \cos nw$ from $w=0$ to $w=1$.

n	$\phi(n)$	n	$\phi(n)$
0,0	+ 1,0000	6,0	− 0,0922
0,2	+ ,9950	6,2	− ,0751
0,4	+ ,9801	6,4	− ,0568
0,6	+ ,9557	6,6	− ,0379
0,8	+ ,9221	6,8	− ,0192
1,0	+ ,8801	7,0	− ,0013
1,2	+ ,8305	7,2	+ ,0151
1,4	+ ,7742	7,4	+ ,0296
1,6	+ ,7124	7,6	+ ,0419
1,8	+ ,6461	7,8	+ ,0516
2,0	+ ,5767	8,0	+ ,0587
2,2	+ ,5054	8,2	+ ,0629
2,4	+ ,4335	8,4	+ ,0645
2,6	+ ,3622	8,6	+ ,0634
2,8	+ ,2927	8,8	+ ,0600
3,0	+ ,2261	9,0	+ ,0545
3,2	+ ,1633	9,2	+ ,0473
3,4	+ ,1054	9,4	+ ,0387
3,6	+ ,0530	9,6	+ ,0291
3,8	+ ,0067	9,8	+ ,0190
4,0	− ,0330	10,0	+ ,0087
4,2	− ,0660	10,2	− ,0013
4,4	− ,0922	10,4	− ,0107
4,6	− ,1116	10,6	− ,0191
4,8	− ,1244	10,8	− ,0263
5,0	− ,1310	11,0	− ,0321
5,2	− ,1320	11,2	− ,0364
5,4	− ,1279	11,4	− ,0390
5,6	− ,1194	11,6	− ,0400
5,8	− ,1073	11,8	− ,0394
6,0	− ,0922	12,0	− ,0372

XIII. *On the Equilibrium of the Arch.* *By the Rev.* HENRY MOSELEY, B.A. *of St John's College; Professor of Natural Philosophy and Astronomy in King's College, London.*

[Read *Dec.* 9, 1833.]

1. LET a mass acted upon by forces applied to any number of points in it be *imagined* to be intersected by an infinite number of planes, dividing it into exceedingly small laminæ. Suppose the direction of the resultant of the forces acting upon one of these, having for its external face a portion of the surface of the body, to be determined. Combining this force with those acting upon the different points of the next, contiguous lamina; let their common resultant be ascertained. Proceed similarly with the next, and with each succeeding lamina.

These lines will then be the tangents to a curved line, called in the following paper the line of pressure, whose intersection with each lamina, marks the point where a single force might be applied so as to produce the same effect with *all* those impressed upon that lamina, this single force being impressed in the direction of a tangent to the curve.

If any of these *imaginary* intersecting planes be supposed to become real sections of the mass, so as to separate it into distinct parts, the conditions necessary that no one of these parts may slip or turn over on those contiguous to it, will manifestly be determined by the direction of the line of pressure in reference to the plane of the section.

In general it will be observed that forces applied to a system of variable form are, when in equilibrium, subject to the *same conditions as though its form were invariable, together with certain other conditions,* dependant upon the nature of the variation to which the form of the system is liable. In other words the conditions of the equilibrium of a system of invariable form are *necessary* to the equilibrium of a system of variable form; but they are not *sufficient.* We shall first determine the form and position of the line of pressure on the hypothesis, that the form of the system is invariable, and then consider the modification to which these are subjected by the opposite hypothesis.

2. Let there be conceived a mass, the connexion between the parts of which may be any whatever, and the nature of whose surface is determined by the equation

$$\Psi xyz = 0.$$

Let it be intersected by an *imaginary* plane whose position in reference to a given system of rectangular co-ordinates is determined by the arbitrary constants A, B, C, and whose equation is

$$z = Ax + By + C \ldots\ldots\ldots (1).$$

Let M_1, M_2, M_3 represent the sums of the forces acting upon one of the parts into which the mass is divided by the intersecting plane, resolved in directions parallel to the axes of x, y, z, respectively. Also let N_1, N_2, N_3 be the moments of these forces about the same axes. Then M_1, M_2, M_3; N_1, N_2, N_3 are *given* in terms of the arbitrary constants A, B, C—of the given forces—and of the constants involved in the given equation to the surface of the mass.

Let the position of the intersecting plane be supposed to be such, that the forces acting upon the above mentioned portion of the mass may have a single resultant, an hypothesis which involves the known condition

$$M_1N_3 + M_2N_2 + M_3N_1 = 0 \ldots\ldots\ldots (2).$$

The equations to the resultant in any given position of the intersecting plane, are

$$\left.\begin{aligned} x &= \frac{M_1}{M_3}\cdot z + \frac{N_2}{M_3} \\ y &= \frac{M_2}{M_3}\, z + \frac{N_1}{M_3} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Let the arbitrary constant C be eliminated from this equation, and from the equation to the intersecting plane by means of equation (2); and let the plane be then supposed to take up a series of positions, the law of which is fixed by its equation, and of which, each is immediately adjacent to the former.

Further, let it be supposed that the resultant of the forces upon the portion of the mass, cut off by the plane, in each of its positions, intersects with the resultant similarly taken in its immediately previous position—an hypothesis which introduces a new condition into the question and establishes a second relation between the quantities M_1, M_2, M_3; A, B, C.

That relation is determined as follows.

Since x, y, z are to be considered as the co-ordinates of a point of intersection of two consecutive resultants; we may differentiate the equations (3) with respect to the arbitrary constants A and B, considering x and y as constant. From this differentiation, the following equations are obtained:

$$\left.\begin{aligned} 0 &= z\left\{\frac{d\left(\frac{M_1}{M_3}\right)}{dA}dA + \frac{d\left(\frac{M_1}{M_3}\right)}{dB}dB\right\} + \left\{\frac{d\left(\frac{N_2}{M_3}\right)}{dA}dA + \frac{d\left(\frac{N_2}{M_3}\right)}{dB}dB\right\} \\ 0 &= z\left\{\frac{d\left(\frac{M_2}{M_3}\right)}{dA}dA + \frac{d\left(\frac{M_2}{M_3}\right)}{dB}dB\right\} + \left\{\frac{d\left(\frac{N_1}{M_3}\right)}{dA}dA + \frac{d\left(\frac{N_1}{M_3}\right)}{dB}dB\right\} \end{aligned}\right\} \dots\dots\dots(4),$$

whence, eliminating z

$$\left.\begin{aligned} &\left\{\frac{d\left(\frac{M_2}{M_3}\right)}{dA}dA + \frac{d\left(\frac{M_2}{M_3}\right)}{dB}dB\right\}\left\{\frac{d\left(\frac{N_2}{M_2}\right)}{dA}dA + \frac{d\left(\frac{N_2}{M_3}\right)}{dB}dB\right\} \\ -&\left\{\frac{d\left(\frac{M_1}{M}\right)}{dA}dA + \frac{d\left(\frac{M_1}{M}\right)}{dB}dB\right\}\left\{\frac{d\left(\frac{N_1}{M_3}\right)}{dA}dA + \frac{d\left(\frac{N_1}{M_3}\right)}{dB}dB\right\} \end{aligned}\right\} = 0\dots(5).$$

This last equation determines the relation between A and B necessary to the continual intersection of the consecutive resultants; and the elimination of these quantities between equations (3) and (4), produces two equations in x, y, z which are those to the *locus* of that intersection. That is, they are the equations to the LINE OF PRESSURE.

3. By the elimination of A, B and C between the equations (2), (3) and (5), a relation is obtained between the co-ordinates of a point in the direction of the resultant force, applicable to every position of the intersecting plane. Being in fact, the equation to that developable surface which is the *locus of the resultants*, and, which has for its *edge of regression*, the line of pressure. This surface will be properly called *the surface of pressure*.

It is evident that at that point where the line of pressure eventually cuts the surface of the mass, there must be applied a force equal to the resultant of all the other forces impressed upon the system and in the direction of a tangent to the line of pressure at that point, or there must be applied to the surface of the last lamina cut off by the intersecting plane, forces whose resultant is of that magnitude and in that direction.

4. These conditions may be expressed as follows.

Let P' be the force—or the resultant of the forces—applied to the last lamina, x_1, y_1, z_1 the co-ordinates of the intersection of the line of pressure with it, α, β, γ the inclinations of P' to the axes of x, y, z. Also let

$$\left.\begin{aligned} x &= F_1 z \\ y &= F_2 z \end{aligned}\right\} \text{.........}(6)$$

be the equations to the line of pressure.

Since the point x_1, y_1, z_1 is a point in the surface of the mass,

$$\therefore \quad \Psi x_1 y_1 z_1 = 0.$$

Also, since it is a point in the line of pressure,

$$\therefore \quad \left.\begin{aligned} x_1 &= F_1 z_1 \\ y_1 &= F_2 z_1 \end{aligned}\right\}.$$

Since the direction of P' is that of a tangent to the line of pressure,

$$\therefore \quad \tan\alpha = \frac{dF_1 z_1}{dz_1},$$

$$\tan\beta = \frac{dF_2 z_1}{dz_1}.$$

Also

$$P' = \sqrt{M_1^2 + M_2^2 + M_3^2},$$

where M_1, M_2, M_3 are supposed to be taken throughout the *whole* mass.

Thus there are six equations of condition, which together with the equation

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1,$$

determine the seven quantities P', x_1, y_1, z_1; α, β, γ in terms of the forces (other than P') which compose the system, and the constants which enter its equation. These fix the relations necessary to the equilibrium of the mass considered as one continued geometrical solid.

Before proceeding to the discussion of the additional conditions requisite to the equilibrium when the mass passes from the invariable form here supposed, to a *variable form*, it will be well to give an example of the application of the principles which have been already laid down to the actual determination of the line of pressure in a particular instance.

5. Let then $ABCD$ (fig. 1.) represent a heavy mass, bounded *at its extremities* by parallel planes AB and CD, and laterally, by the planes AC and BD inclined at any angle to one another.

Let the mass be imagined to be intersected by an infinite number of planes parallel to AB, of which one is mn, and to be supported by forces acting at p and p' at angles ϕ and ϕ' with the horizon.

It is required under these circumstances to determine the form and position of the line of pressure.

Let the line $P'G$ bisect AB and CD. Draw $P'E$ horizontal and PM vertical.

Let $P'M = A$, $CD = 2b$, $P'p = \kappa$,
$AB = 2a$, $P'G = k$, $Gp' = \kappa'$.

Inclination of $P'G$ to the horizon $= \gamma$,
.... AB .... $= \beta$.

Now,

$$\frac{BP' - Pm}{PP'} = \frac{BP' - DG}{GP'};$$

$$\therefore \frac{2a - (mn)}{A \sec \gamma} = \frac{2(a-b)}{k};$$

$$\therefore (mn) = 2a - \frac{2(a-b)}{k} \sec \gamma \,.\, A;$$

$$\therefore \text{area } (BAnm) = \tfrac{1}{2}\{(AB) + (mn)\} \,.\, (P'P) \,.\, \sin (PP'A)$$

$$= \sin(\beta + \gamma) \sec \gamma \left\{2aA - \frac{a-b}{k} \,.\, \sec \gamma \,.\, A^2\right\};$$

$$\therefore \frac{d\{\text{area}(BAnm)\}}{dA} = \sin(\beta + \gamma) \sec \gamma \left\{2a - \frac{2(a-b)}{k} \sec \gamma \,.\, A\right\}.$$

Now each element of the area has its centre of gravity in $P'G$;

$$\therefore \text{ moment of area} = 2g \sin(\beta + \gamma) \sec \gamma \textstyle\int_A \left\{aA - \frac{a-b}{k} \sec \gamma A^2\right\}$$

$$= g \sin(\beta + \gamma) \sec \gamma \left\{aA^2 - \frac{2(a-b)}{3k} \sec \gamma A^3\right\}.$$

Now,

$$N_2 = \text{moment of } p + \text{moment of area } (BAmn)$$

$$= pg\kappa \sin(\phi + \beta) + g \sin(\beta + \gamma) \sec \gamma \left\{aA^2 - \frac{2(a-b)}{3k} \sec \gamma A^3\right\}.$$

Also,

$$M_1 = pg \cos \phi, \qquad M_2 = 0,$$

$$M_3 = pg \sin \phi - g \sin(\beta + \gamma) \sec \gamma \left\{2aA - \frac{a-b}{k} \sec \gamma A^3\right\},$$

$$N_1 = 0, \qquad N_3 = 0.$$

Calling therefore x and z the co-ordinates of any point in the resultant of the forces applied to the area $(ABmn)$, we have for the equation to that resultant,

$$zM_1 - xM_3 = N_2;$$

$$\text{or, } z p g \cos\phi - p g \sin\phi + x g \sin(\beta+\gamma)\sec\gamma\left\{2aA - \frac{a-b}{k}\sec\gamma \,.\, A^2\right\}$$

$$= p g \kappa \sin(\phi+\beta) + g\sin(\beta+\gamma)\sec\gamma\left\{aA^2 - \frac{2(a-b)}{3k}\sec\gamma\,.\,A^3\right\}.$$

Differentiating which equation with regard to the arbitrary constant A, we obtain

$$A = x,$$

whence by elimination and reduction,

$$z = \tfrac{1}{3}\left(\frac{a-b}{pk}\right)\left\{\frac{\sec^2\gamma\,.\,\sin(\beta+\gamma)}{\cos\phi}\right\}.\,x^3$$

$$- \left(\frac{a}{p}\right)\left\{\frac{\sec\gamma\,.\,\sin(\beta+\gamma)}{\cos\phi}\right\}.\,x^2$$

$$+ \tan\phi\,.\,x$$

$$+ \kappa\frac{\sin(\phi+\beta)}{\cos\phi}.$$

The above is the equation to the line of pressure. It indicates a point of *contrary flexure* corresponding to

$$x = \frac{ak}{a-b}\cos\phi.$$

The curve is concave to the axis of x, between the origin and this point. It is afterwards continually convex.

A minimum value of z is determined by the equation

$$x = \left\{\frac{ka}{a-b}\right\}\left\{1 \pm \sqrt{1 - \left(\frac{p}{a}\right)^2 . \frac{\sin^2\phi\cos^4\gamma}{\sin^2(\beta+\gamma)}}\right\}.$$

It will be observed that since all the forces applied to the system may be supposed to act in the same plane, the two conditions,

First, "That in every position of the intersecting plane, the forces shall have a single resultant," and Secondly, "That the consecutive resultants shall intersect," are *necessarily satisfied.*

To simplify the question, let the planes *AC*, *BD* which bound the mass laterally be supposed to be parallel, the figure *ABCD* assuming the form of a rectangle. Fig. 6.

This hypothesis will introduce the following conditions:

$$a = b, \quad \beta = \frac{\pi}{2} - \gamma.$$

Hence, by substitution the equation to the line of pressure becomes

$$z = -\frac{a}{p} . \sec\gamma . \sec\phi . x^2$$
$$+ \tan\phi . x$$
$$+ \kappa . \frac{\cos(\gamma - \phi)}{\cos\phi},$$

which may be put under the form

$$\{x - \frac{p}{2a}\sin\phi\cos\gamma\}^2 = \frac{p}{a}\cos\gamma . \cos\phi . \{\kappa\frac{\cos(\gamma-\phi)}{\cos\phi} + \frac{p}{4a}\frac{\sin^2\phi\cos\gamma}{\cos\phi} - z\}.$$

It is manifest therefore, that the line of pressure is in this case a parabola—having its *axis vertical* and at a distance $= \frac{p}{2a}\sin\phi\cos\gamma$ from the origin—having its concavity *downwards*—its vertex at a height

$$= \kappa\frac{\cos(\gamma-\phi)}{\cos\phi} + \frac{p}{4a}\frac{\sin^2\phi\cos\gamma}{\cos\phi},$$

above the axis of x—and having for its parameter the quantity

$$\left(\frac{p}{a}\right) . \cos\phi\cos\gamma.$$

Let us now seek to determine what relation must exist between the forces impressed upon the mass which we have hitherto considered of invariable form, that the equilibrium, may continue *under* the same circumstances when its form and dimensions are made to admit of variation. And let us suppose

First. That certain of the sections, which we have *imagined*, become *real* sections of the mass, dividing it into separate and distinct parts, each of which retains the properties of a perfect solid.

Secondly. Let us suppose every point in the system to admit of displacement, subject, *within certain limits*, to the law of perfect elasticity.

The determination of the conditions of the equilibrium in these two cases, will constitute a complete theory of *construction*.

The discussion contained in the remainder of this paper will be confined to the first case.

6. Let the mass AB (fig. 2.) have for its line of pressure the line PP'. Now it is clear, that if this line cut the plane QQ' of any section of the mass in a point n' *without* the surface of the mass; the tendency of the opposite resultants of the forces acting upon the two parts AQQ' and BQQ', into which that section divides the mass, will be to cause them to *revolve* about the nearest point Q' of its intersection with the surface of the mass. And, *this tendency being wholly unopposed*, motion will ensue. And so in the mass represented (fig. 6.) the force p and with it the line of pressure pp' being given, it appears that, being cut transversely as shewn in the figure, the mass cannot be supported by any single force p' if it extend beyond $C'D'$: any such force must, to produce equilibrium, be applied at q; and being applied there, the portion $C'C'''D'''D'$ will be wholly *unsupported*. The line of pressure being continued cuts the planes of the sections $C'D'$, $C''D''$, &c., *without* the surface of the mass.

Thus then it is a condition of the equilibrium, *that the line of pressure should intersect the plane of every section of the body within its mass*.

This condition will be satisfied if this line nowhere *cut* the surface of the mass except at the points P and P'. Fig. 2. Or if the equation

$$\Psi F_1 z, \quad F_2 z, \quad \cdot z = 0,$$

found by eliminating the values of x and y between the equation to the surface and the equation to the line of pressure, involve only such possible values of z as correspond to the points P and P', where the intersecting plane touches the surface, or to points where the line of pressure touches it.

It is a further condition of the equilibrium that the line of pressure should not cut any section of the mass, at an angle with the perpendicular to that section greater than a certain given angle, dependant upon the friction of the surfaces in contact, and having for its tangent the coefficient of friction.

The resistance of surfaces is *not exerted exclusively* in the direction of the normal, according to *an hypothesis*, which was probably introduced into the theory of Statics in order to simplify the investigations of those who *originated* that science, but which there seems no reason for retaining any longer. It is exerted in an infinity of different directions included within a certain angle to the normal, or rather within the surface of a certain right cone, having the normal for its axis and the point of resistance for its vertex. *Any force*, however great, applied within this conical surface will be sustained by the resistance of the surface of the mass—and *no force* however small, without it.

Let R represent a single force on the resultant of any number of forces applied to a fixed surface, and let R' and R'' be the resolved parts of R in the directions perpendicular and parallel to the surface. Also let ρ be the inclination of R to the vertical, and f the coefficient of friction. The friction of the surfaces in contact is therefore represented by fR', and motion will, or will not, ensue according as R'' *is* greater or is *not* greater than fR'. Or, according as $\frac{R''}{R'}$ *is* greater or is not greater than f. Or, if $f = \tan\phi$, according as $\tan\rho$ is, or is not, greater than $\tan\phi$, or as ρ is greater or is not greater than ϕ.

In the remainder of this paper the angle ϕ, or $\tan^{-1} f$, will be called *the limiting angle of resistance**.

From the above then it appears, that unless the tangent to the line of pressure at the point where it cuts any section of the mass, make with the perpendicular to the plane of that section an angle, which is not greater than the limiting angle of resistance, the surfaces there in contact will slip upon one another.

This condition may be expressed analytically as follows:

$$z = Ax + By + C$$

is the equation to the plane of any section of the mass, therefore

$$x - x_{,} = -A\,(z - z_{,}), \quad y - y_{,} = -B\,(z - z_{,}),$$

are the equations to the perpendicular to that section. And the angles which that perpendicular makes with the co-ordinate axes have for their cosines

$$\frac{-A}{\sqrt{A^2 + B^2 + 1}}, \quad \frac{-B}{\sqrt{A^2 + B^2 + 1}}, \quad \frac{-1}{\sqrt{A^2 + B^2 + 1}}.$$

Also it appears from the given equations (3) to the resultant force, or tangent to the line of pressure, that this line makes angles with the co-ordinate axes which have for their cosines the quantities

$$\frac{M_1}{\sqrt{M_1^2 + M_2^2 + M_3^2}}, \quad \frac{M_2}{\sqrt{M_1^2 + M_2^2 + M_3^2}}, \quad \frac{M_3}{\sqrt{M_1^2 + M_2^2 + M_3^2}}.$$

Hence, therefore if I be the inclination of these lines to one another,

* It is here supposed that the coefficient of friction f is constant for the same surfaces, whatever be the force R' by which they are pressed together. This is usually assumed to be the law of friction. It is only however an approximation to that law. The experiments of Mr Rennie shew that f must be considered a function of R' increasing continually, but very slowly, up to the limits of abrasion.

$$\cos I = -\frac{AM_1 + BM_2 + M_3}{\{(A^2 + B^2 + 1)(M_1^2 + M_2^2 + M_3^2)\}^{\frac{1}{2}}};$$

in which expression M_1, M_2, M_3, and B, are *known functions* of A.

Now I must not *exceed* the limiting angle of resistance. Therefore $\cos I$ must not be *less than* the cosine of that angle.

On the whole then we have these two conditions necessary to the equilibrium of a mass intersected by a series of planes, under the circumstances supposed.

1. That the equation

$$\Psi F_1 z, \quad F_2 z, \quad z = 0,$$

shall involve no possible roots, except such as correspond to the extremities of the line of pressure, or *to points where* it touches the surface of the mass.

2. That the fraction

$$-\frac{AM_1 + BM_2 + M_3}{\{(A^2 + B^2 + 1)(M_1^2 + M_2^2 + M_3^2)\}^{\frac{1}{2}}},$$

shall for all values of A, corresponding to *real* sections of the mass, be not less than the cosine of that arc, whose tangent is the coefficient of friction.

The first of these conditions being satisfied, the parts of the mass cannot *turn upon* one another. The second being satisfied, they cannot *slip* upon one another.

We have supposed the whole of the forces impressed upon the system to be known excepting the force P', which has been determined in terms of the rest. The force P' may be supplied by the *resistance* of a point in a fixed surface, in which case the amount and direction of that resistance will be known.

If, however, there enter *two* or more resistances of surfaces among the forces which compose the equilibrium, since the magnitudes of these and also their directions may be any whatever, within the limits imposed by the friction of the surfaces; the problem remains, in so far as the known conditions of equilibrium are concerned, *indeterminate*, and recourse must be had for its solution to other principles.

7. Suppose the mass AB to be acted upon by any number of forces among which is the force P being the resultant of certain resistances, supplied by different points in a surface Bb, common to the intersected mass and to an immoveable obstacle BC.

Now it is clear that under these circumstances we may vary the force P', both as to its amount, direction, and point of application, without disturbing the equilibrium, provided only the form and direction of the line of pressure continue to satisfy the conditions imposed by the equilibrium of the system.

These are manifestly, that it no where *cut* the surface of the mass, except at P' and within the space Bb, and that it no where cut a section of the mass or the common surface of the mass and obstacle, at any angle with the perpendicular greater than the limiting angle of resistance.

Thus, varying the force P', we may destroy the equilibrium, either, first, by causing the line of pressure to take a direction without the limits prescribed by the resistance of the section through which it passes; or, secondly, by causing the point P to fall *without* the surface Bb, in which case *no resistance* can be opposed to the resultant force acting in that point; or, thirdly, the point P lying within the surface Bb, we may destroy the equilibrium by causing the line of pressure to cut the surface of the mass somewhere between that point and P'.

Let us suppose the limits of the variation of P' within which the first two conditions are satisfied, to be known; and varying it, within those limits, let us consider what may be its *least* and *greatest* values so as to satisfy the third condition; and where, and in what direction they must be applied.

In the first place it will be observed, that by diminishing the force P', its direction and point of application remaining the same, the line of pressure is made continually to assume more nearly that direction which it would have, if P' were entirely removed.

Provided then, that if P' *were* thus removed, the line of pressure would cut the surface, that is, provided the force P' be necessary to the equilibrium; it follows that by diminishing it, we may vary the direction and curvature of the line of pressure until we at length make it *touch* some point or other in the surface of the mass.

And this is the limit; for if the diminution be carried further, it will *cut* the surface, and the equilibrium will be destroyed. It appears then that under the circumstances supposed, when P' acting at a given point and in a given direction, is the least possible, the line of pressure *touches the surface of the mass.*

In the same manner it may be shewn, that when it is the greatest possible, the line of pressure touches the surface of the mass.

Now by varying the direction and point of application of P', as well as its amount, this contact may be made to take place in infinite variety of different points, and each such variety supplies a new value of P', producing the required contact. Among these, therefore, it remains to seek the *absolute* maximum and minimum values of that force.

To express these conditions analytically, let x_2, y_2, z_2 represent the co-ordinates of a point where the line of pressure touches the surface of the body.

Since the point x_2, y_2, z_2 is common to the line of pressure and to the surface of the body,

$$\therefore \ \Psi x_2 y_2 z_2 = 0, \quad x_2 = F z_2, \quad y_2 = F_2 z_2.$$

Also, since it touches the surface in the point $x_2 y_2 z_2$;

$$\therefore \quad \frac{dF_1 z_2}{dz_2} + \frac{\left(\dfrac{d\Psi x_2 y_2 z_2}{dz_2}\right)}{\left(\dfrac{d\Psi x_2 y_2 z_2}{dx_2}\right)} = 0,$$

$$\frac{dF_2 z_2}{dz_2} + \frac{\left(\dfrac{d\Psi x_2 y_2 z_2}{dz_2}\right)}{\left(\dfrac{d\Psi x_2 y_2 z_2}{dy_2}\right)} = 0.$$

Eliminating x_2, y_2, z_2 among these *five* equations *two* relations are established between the force P'*, the co-ordinates of its point of application, and the angles which fix its direction (see Art. 4); by elimination between which a further relation is established between six of these seven quantities, and, finally, by the equations of condition

$$\Psi x_1 y_1 z_1 = 0,$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1,$$

a relation is obtained between four of them.

Thus then we may obtain the value of P' in terms of three of the quantities x_1, y_1, z_1; α, β, γ.

Its *maximum* and *minimum* values are then at once determined by the known conditions of the maxima and minima of functions of three variables.

8. It is evident that the minimum value of P', being that which just counteracts the tendency of the mass to revolve about the point where the line of pressure touches its surface, is also precisely that force which would be exerted there by another equal and similar mass, acted upon by equal forces, under the same circumstances, but placed in a contrary position, so that its line of pressure shall have, at P', a common tangent with the line of pressure of the first mass.

* The line of pressure is *here* supposed to commence at P', and the force P' to enter among the other forces which determine its equation.

Two masses, therefore, thus placed together would remain in equilibrium, without the aid of any external force, and by reason only of their mutual pressures and the resistance of their abutments.

It is also evident that since the line of pressure is similarly situated in both, they cannot be thus placed together so that their lines of pressure may meet and have a common tangent at the point where they meet, unless both lines of pressure be perpendicular to the common surface at that point.

This condition throws two new equations into the system, and determines the value of P' in terms of a single variable.

The value of P' is not in this case that which we have called the absolute minimum or minimum minimorum, but simply the greatest or least force, which applied at a given point, in a given direction will support the system.

If however instead of a single point of contact we suppose the masses to be in contact throughout the *whole* surfaces of two planes, it is evident that the point P'* will take up for itself that position, which we have supposed to correspond with the absolute *minimum;* a condition to which the form of the line of pressure, and the position of its point of contact with the surface of the mass, will also be subjected.

Hence it appears that two masses, thus in contact *throughout* the surfaces of two planes, sustain a less aggregate of pressure, on their common *surface* of contact, than two similar masses in contact only by a single point, unless that point, and the position of the masses, be such as to correspond to the minimum minimorum.

In the preceding pages we have supposed the form of the solid to be given, together with the positions of the different sections made through it, and we have thence deduced the form of its line of pressure and the direction of that line through its mass.

* The point P' is here the point of application of the *resultant* of the resistances on the different points of either plane.

It is manifest that the converse of this operation is possible.

9. Having given the form and position of the line of pressure, and the positions of the different sections to be made through the mass, we may, for instance, enquire what form these conditions impose upon the surface which bounds it.

Or we may make the direction of the line of pressure and the form of the bounding surface subject to certain conditions not absolutely determining either.

For instance, if we suppose the form of the intrados of an *arch* to be given, and the direction of the intersecting plane to be always perpendicular to it, and if we suppose the line of pressure to intersect this plane always at the same given angle with the perpendicular to it, so that the tendency of the pressure to thrust each from its place may be the same,—we may determine what under these circumstances must be the extrados of the arch.

If this angle *equal* constantly the limiting angle of resistance, the arch is in a state bordering upon motion, each voussoir being upon the point of slipping downwards or upwards, according as the constant angle is measured above or below the perpendicular to the surface of the voussoir.

The systems of voussoirs which satisfy these two conditions are the greatest and least possible.

If the constant angle be zero, the line of pressure being everywhere perpendicular to the joints of the voussoirs, the arch would stand even if there were no friction of their surfaces.

It is then technically said to be equilibrated. It is impossible to conceive any arrangement of the parts of an arch by which its stability can be more effectually secured*.

10. The theory stated above readily explains the phenomena observed in the settlement and fall of the arch.

* The great arches of late years erected by Mr Rennie, in this country, have for the most part been so loaded as very nearly to satisfy this condition.

Thus let ABB' (fig. 3) represent an arch having the joints of its voussoirs perpendicular to the intrados as they are usually made.

Let $RQPQ'R'$ be the line of pressure, touching the intrados in the points Q and Q'. It is manifest that this curve is then perpendicular to the joints of the voussoirs at Q and Q', and inclined in respect to those above and below these points. The inclination being *downwards*, or towards the intrados, in reference to the former, and *upwards*, or from the intrados, in reference to the latter.

Hence, therefore, it appears that the tendency of the pressure is to cause all the voussoirs above the points Q and Q' to slide *downwards*, and those beneath those points, *upwards*.

And that these effects may be expected to follow the striking of the centre of the arch; the weight being then suddenly thrown upon the voussoirs, and these admitting of a certain degree of motion in the directions of the forces impressed upon them.

Now this is precisely what was observed at the bridge of Nogent, of the construction of which Perronet has left a detailed account.

Three straight lines were drawn upon the face of the arch before the striking of the centre, shewn in the figure 4, by the polygon $nmm'n'$, mm' being horizontal, and the other two mn and $m'n'$ stretching from the extremities of mm' towards the springing of the arch.

After the centre had been struck, the lines were observed to have assumed the curved forms indicated by the dotted lines MM', MN, $M'N'$, indicating, in accordance with the theory, a downward motion in all the voussoirs above Q and Q', and an upward motion in those beneath those points.

These observations have been confirmed by numerous others, and especially by those (made also by Perronet) at the Pont de Neuilly.

The sinking of the voussoirs at the crown necessarily tends to produce a separation of their joints at the intrados in the neighbourhood of that point, and thus to cause the actual contact of the key and adjacent voussoirs to take place only at their superior edges.

If therefore the settlement be considerable, we may conclude that the line of pressure touches the extrados at the crown, and for some distance on either side of it. The material of the arch may therefore be expected to yield more particularly about that point and the points Q and Q' than any other; a great proportion of the pressure being there thrown upon the *edges* of the voussoirs.

11. If by reason of such yielding, or from any other alteration in the forces impressed upon the mass, or in the circumstances of their application, the form of the line of pressure be altered, it may manifestly be expected to intersect the surface of the mass first about *those points;* the least possible alteration of form being there sufficient to produce the intersection. And this being the case, the portion of the arch above Q and Q' must separate into *two* portions, revolving at those points about the lower portions of the arch (see fig. 5) and at A, upon the extremities of one another.

Nevertheless this revolution is manifestly impossible unless the points Q and Q' yield outwards. And this can only take place by the yielding of the material at Q and Q', by the slipping back of the voussoirs there, or by the portions of the arch or its abutments beneath those points revolving outwards, in consequence of the intersection of the extrados by the extremities QR and $Q'R'$ of the line of pressure (fig. 3).

The last is in point of fact the cause which leads, in the great majority of cases, to the fall of the arch.

The extremity R of the line of pressure is made to cut the extrados of the arch, or the outer surface of the pier, by the diminution or removal of some force which acted there in opposition to the tendency of the arch to spread itself, and which kept the direction of the line of pressure within its mass,—the resistance of a mass of earth for instance, or the opposite thrust of some other arch springing from the same pier or abutment.

On the whole, then, it appears that in the commencement of its fall the arch will divide itself into six distinct portions, of which four will revolve about the points S, S', Q, Q' and A, as represented in the figure 5. Now this is what is uniformly observed to take place in the fall of the arch.

12. Gauthey, having occasion to destroy a bridge, caused one of its arches to be *insulated* from the rest; and the adhesion of the cement being sufficient to counteract the tendency of the pressure to rupture the piers, he caused them to be cut across. The whole then at once fell, the falling portion separating itself into four parts. Having constructed small arches of soft stone, and without cement he loaded them until they fell. Their fall was always observed to be attended with the same circumstances. Before the arch finally yielded the stone also was observed to chip at the intrados about the points Q and Q', round which the upper portions of it finally revolved.

Some experiments made by Professor Robinson with *chalk* models were attended with slightly different results. Having loaded them at the crown until they fell, he observed first, that the points where the material began to yield were not precisely those where the rupture finally took place.

This fact presents a remarkable confirmation of the theory expounded in this paper.

It is manifest, that according to that theory, with any variation in the least force P', which would support the semi-arch if applied at its crown, there will be a corresponding change in the position of the point Q.

Now as the load upon the crown is increased, this least force P' is manifestly increased. The result is a corresponding variation in the form of the line of pressure, tending to carry its point of contact with the intrados lower down upon the arch.

This is precisely what Professor Robinson observed. The arch began to chip at a point about half way between the crown and the point where the rupture finally took place.

The existence of the points Q and Q', about which the two upper portions of the arch have a tendency to turn, and about which the material is first observed to yield, has long been known to practical men. The French engineers have named these points the points of rupture of the arch; and the determination of their position by a *tentative method* forms an important feature in the very unsatisfactory theory which they have applied to this important branch of Statics.

13. The theory of the equilibrium of the groin and that of the dome are precisely analogous to the theory of the arch.

In the former case a mass springs from a small abutment spreading itself out symmetrically with regard to a vertical plane passing through the centre of its abutment. It is in fact nothing more than an arch, whose voussoirs vary as well in breadth as in depth. The centres of gravity of the different elementary voussoirs of this mass lie all in its plane of symmetry. Its line of pressure is therefore in that plane, and its theory is embraced in that which has been already laid down.

Four groins commonly spring from one abutment; each *opposite* pair being addossed, and each *adjacent* pair uniting their margins. They thus lend one another mutual support, partake in the properties of a dome, and form a continued covering.

The groined arch is of all arches the most stable; and could materials be found of sufficient strength to form its abutment and the parts about its springing, it might be safely built of any required degree of flatness, and spaces of enormous dimensions might readily be covered by it.

It is remarkable that modern builders, whilst they have erected the common arch on a scale of magnitude nearly approaching perhaps the limits to which it can be safely carried, have been remarkably timid in the use of the groin.

H. MOSELEY.

King's College, London,
October 9, 1833.

XIV. *Third Memoir on the Inverse Method of Definite Integrals.*

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INTRODUCTION.

In the two preceding Memoirs on the Inverse Method of Definite Integrals, the limits of integration had been fixed throughout at 0 and 1, but in the sixth Section, which is the first of the present Memoir, the integrations terminated by arbitrary limits are fully considered; and when performed with respect to any function of the independant variable, the proper methods for discovering reciprocal functions are given, and it is remarkable that the forms thus obtained for the trigonometrical functions, for Laplace's and an infinite variety of other reciprocal functions, are all similar, differing only by a constant.

In identities obtained between the n^{th} differential coefficient of a function *not containing* n, and its expanded value, we may, generally, by changing the sign of n, obtain a corresponding identity between the n^{th} successive integral and its expansion, abstracting from the appendage of integration which ought to contain n arbitrary constants; this property however extends also to certain reciprocal functions *which contain* n; and this consideration leads in the same section to the complete resolution of Laplace's equation for the reciprocal functions of one variable, which are the coefficients in the developement of the reciprocal of the distance of two points; the n^{th} coefficient when multiplied by an arbitrary constant, satisfies that equation, as is well known, but as the equation is of the second order, another function multiplied by an arbitrary constant must be also represented by the same equation, this function, which is here found, is altogether different in its form and properties from Laplace's coefficients.

The great class of reciprocal functions above alluded to possess the remarkable property, that their integrals vanish between any of their own maxima or minima values.

In this Section I have noticed some curious trigonometrical functions of which the properties are very elegant, particularly as affording simple means of representing by Definite Integrals the general differential coefficients of rational and integral functions; another application of trigonometrical functions is made, in representing the sum of the divisors of any given number, by means of a Definite Integral.

The seventh Section is on Transient Functions. The way of forming reciprocal functions by means of arbitrary coefficients, when the form of the general term was given, has been shewn in the Second Memoir on this subject. To this I have here added the method of finding the functions which shall be reciprocal to any proposed one, and applied the method to the cases where the given function is t^n, $(\log. t)^n$, and $\cos^n(t)$; the reciprocal functions which thence resulted are *transient*, that is, they have but a momentary existence between the limits of integration; that existence is however sufficient to make their integrals finite, and to endow them with remarkable properties. They are capable of representing the electrical state of a body when an electrical spark is infinitely near, and about to form a part of the system; they are also capable of representing, under continuous forms, the state of a body considered as composed of absolute mathematical centres of forces, separated mutually by infinitesimal intervals.

The eighth and last Section is on the Resolution of Equations which contain Definite Integrals; the first method for this purpose is to decompose the integrals into elements, and then determine the unknown functions by elimination. This tedious process is useful in verifying results otherwise obtained, and in giving numerical approximations in the most difficult cases. Afterwards I have considered separately, Equations to Definite Integrals; first, when they contain but one Definite Integral and one parameter; second, when they contain two or more Definite Integrals and as many parameters; third, Simultaneous Equations; fourth, Definite Integral Equations of superior orders and degrees; besides which, the nature of the appendage analogous to the arbitrary constant of integration is discussed in the same Section.

Throughout the whole of this Memoir, a considerable number of examples, illustrative of the corresponding theories, are dispersed.

SECTION VI.

Method of discovering Reciprocal Functions when the integrations are performed with respect to any function of the independant variable.

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(1) *When the limits of integration are arbitrary.*

1. The investigations of reciprocal functions contained in the Second Memoir *on the Inverse Method of Definite Integrals*, are founded on the supposition that 0 and 1 are always the limits of the independant variable, but it is often of importance to possess reciprocal functions in which the limits of integration are different from those quoted. The principle by which this is most easily accomplished, is to suppose the integrations performed relative to a function of the independant variable, which must be so chosen, that when the values 0 and 1 are assigned to the independant variable, the corresponding values or the function may be the proposed limits of integration.

2. Let Q_n, R_m be functions of a variable (ϕ), the limits of which are arbitrary, as a and b, between which limits $\int_\phi Q_n R_m$ always must vanish, except when the integers m and n are equal.

Suppose that a function of ϕ, as t, is found such that when $\phi = a$ $t = 0$, and when $\phi = b$, $t = 1$, conditions which it is always easy to satisfy.

We may now conversely regard ϕ as a function of t, and then the preceding integral becomes $\int_t Q_n R_m \dfrac{d\phi}{dt}$, the limits being now reduced to 0 and 1. Suppose that $\dfrac{d\phi}{dt}$ is separated into any two factors, λ and λ'; then since $\int_t Q_n \lambda \times R_m \lambda' = 0$, except when $m = n$, it follows that $Q_n \lambda$, $R_m \lambda'$ are mutually reciprocal, and may therefore be found in an indefinite variety of modes by the principles explained in Section IV; and dividing these functions respectively by λ, λ', and substituting in the quotients the value of t expressed in terms of ϕ, the required functions Q_n, R_m will be obtained.

If it be desired that Q_n, R_m should be functions of the same nature, differing only in the order expressed by m and n, that is self-reciprocal, put $\lambda=\lambda'=\sqrt{\left(\frac{d\phi}{dt}\right)}$, and having found any kind of self-reciprocal functions in which the limits are 0 and 1, as for example, the functions denoted by P_m, P_n in the preceding Memoirs, we then obtain

$$Q_n=P_n\sqrt{\left(\frac{dt}{d\phi}\right)}\,; \qquad R_m=P_m\sqrt{\left(\frac{dt}{d\phi}\right)}\,,$$

3. *If a function* V *can be determined so that the quantity*

$$\frac{d^n\{(tt')^nV\}}{dt^n}\cdot\frac{dt}{d\phi}$$

may be of n *dimensions in* t, (*where* $t'=1-t$ *as in the former Memoirs*), *this quantity will be a self-reciprocal function when the integrations are performed relative to* ϕ.

Denote this quantity by Q_n, and supposing m to be an integer less than n, it is necessary to show that $\int_\phi Q_m Q_n=0$, or that

$$\int_t Q_m\frac{d^n\{(tt')^nV\}}{dt^n}=0,$$

the limits of t being 0 and 1.

Now Q_m being of m dimensions in t, let its general term be represented by a_pt^p, where it is evident that p cannot exceed $n-1$, since $m<n$; the part of the preceding integral dependant on this term is

$$a_p\int_t t^p\frac{d^n\{(tt')^nV\}}{dt^n}.$$

The latter integral may by partial integration be put in the form,

$$t^p\frac{d^{n-1}\{(tt')^nV\}}{dt^{n-1}}-pt^{p-1}\frac{d^{n-2}\{(tt')^nV\}}{dt^{n-2}}+p.(p-1)\,t^{p-2}\frac{d^{n-3}\{(tt')^nV\}}{dt^{n-3}}-\&c.,$$

the last term being

$$(-1)^p.p.(p-1)\,(p-2)\ldots\ldots1.\frac{d^{n-p-1}\{(tt')^nV\}}{dt^{n-p-1}},$$

and therefore the index of differentiation never becomes negative.

The first term, and '*a fortiori*', all the succeeding terms of this series vanish between the limits $t=0$, and $t=1$, or $t'=0$, for

$$\frac{d^{n-1}\{(tt')^n V\}}{dt^{n-1}} = V\frac{d^{n-1}(tt')^n}{dt^{n-1}} + (n-1)\frac{dV}{dt}\frac{d^{n-2}(tt')^n}{dt^{n-2}}$$
$$+\frac{(n-1)(n-2)}{1.2}\frac{d^2V}{dt^2}\cdot\frac{d^{n-3}(tt')^n}{dt^{n-3}} + \&c.,$$

the first term of this latter series contains a factor tt', the second a factor $(tt')^2$, &c., and therefore the whole vanishes between limits.

The following *exception* to this theorem must however be attended to; V *must not be of the form* $(tt')^{-r}.V_1$, where r is equal to, or greater than unity, for the above reasoning will not be applicable, since then

$$\frac{d^{n-1}\{(tt')^n V\}}{dt^{n-1}} = \frac{d^{n-1}\{(tt')^{n-r}V_1\}}{dt^{n-1}},$$

which being expanded as above, will not vanish unless r be less than unity.

4. *If a function* V *can be determined so that the quantity*

$$\frac{d^n\{(tt')^n V\}}{dt^n}\cdot\frac{d\phi}{dt}$$

may be of n *dimensions in* t, *then the factor by which* $\frac{d\phi}{dt}$ *is here multiplied, will be a self-reciprocal function when the integrations are performed relative to* ϕ.

Denote this coefficient by q_n, then

$$\textstyle\int_\phi q_m q_n = \int_t q_n\cdot q_m\frac{d\phi}{dt} = \int_t\cdot\frac{d^n(tt')^n V}{dt^n}\cdot q_m\frac{d\phi}{dt},$$

and as we may suppose $m<n$, the general term of $q_m\frac{d\phi}{dt}$, as $a_p t^p$ cannot be of greater dimensions than $n-1$, and therefore the part of the whole integral dependant on this term vanishes, as has been shewn in the preceding article, hence $\int_\phi q_m q_n = 0$, when m and n are unequal.

We must except, as before, from the application of this theorem the case where V is of the form $(tt')^{-r}.V_1$, and r greater than, or equal to unity.

5. *If* ϕ *be any of the transcendants contained in the indefinite integral* $\int_t (tt')^m$, *where* m *is between* -1 *and* $+\infty$ *exclusive, and if*

$$Q_n = \frac{d^n (tt')^{n+m}}{1.2.3...n\,dt^n}.(tt')^{-m},$$

then shall Q_n *be a self-reciprocal function for integrations relative to* ϕ.

For Q_n is evidently of the form $\frac{d^n\{(tt')^n V\}}{dt^n}.\frac{dt}{d\phi}$, and V is not of the form excepted in Art. 3., since m is between -1 and $+\infty$.

Moreover, by actual differentiation we get

$$\frac{d^n.(tt')^{n+m}}{1.2.3...n\,dt^n} = at^m t'^{n+m} + bt^{m+1}t'^{n+m-1} + ct^{m+2}t'^{n+m-2} + \&c.,$$

where a, b, c, &c. are constant quantities.

Hence,

$$Q_n = at'^n + btt'^{n-1} + ct^2 t'^{n-2} + \&c.,$$

which is of n dimensions in t, and therefore all the conditions required in Art. 3. are here fulfilled; therefore Q_n is a self-reciprocal function relative to ϕ.

6. *If* ϕ *be any of the transcendants expressed by the indefinite integral* $\int_t (tt')^m$, *where* m *is between* $+1$ *and* $-\infty$ *exclusive, and if*

$$q_n = \frac{d^n (tt')^{n-m}}{1.2.3....n\,dt^n},$$

then is q_n *a self-reciprocal function relative to* ϕ.

For q_n is here of the form $\frac{d^n (tt')^n V}{dt^n}$, and V does not belong to the excepted cases, moreover

$$q_n \frac{d\phi}{dt} = \frac{d^n.(tt')^{n-m}}{1.2...n\,dt^n}.(tt')^m$$

is evidently of n dimensions in t, therefore all the conditions of Art. 4. are here satisfied.

7. For the purpose of convenience both in evaluating and using reciprocal functions, the knowledge of the functions which they generate is very useful. The generating function, for example, being the quantity denoted by q_n, Art. (6), the process for finding in this case the function generated, will sufficiently exhibit the general principle, and therefore it is now proposed to sum the series $q_0 + q_1 h + q_2 h^2 + q_3 h^3$, &c.

Substituting for q_n its value given in the preceding article, and representing the required sum by S we have

$$S = (tt')^{-m} + h\frac{d(tt')^{1-m}}{dt} + \frac{h^2}{1.2}\cdot\frac{d^2(tt')^{2-m}}{dt^2} + \frac{h^3}{1.2.3}\cdot\frac{d^3(tt')^{3-m}}{dt^3} + \text{\&c.}$$

But if we form the equation, $u = t + hu(1-u)$, and suppose $f'(u)$ to be the derived function from $f(u)$, we have generally

$$\frac{du}{dt}\cdot f'(u) = f'(t) + h\frac{d\{f'(t).tt'\}}{dt} + \frac{h^2}{1.2}\cdot\frac{d^2\{f'(t).(tt')^2\}}{dt^2}$$
$$+ \frac{h^3}{1.2.3}\cdot\frac{d^3\{f'(t).(tt')^3\}}{dt^3}\ \text{\&c.}$$

which is obtained by differentiating the value of $f(u)$ given by Lagrange's Theorem.

The preceding series coincide by supposing

$$f'(t) = (tt')^{-m} = t^{-m}(1-t)^{-m},$$

$$\text{and therefore } f'(u) = u^{-m}(1-u)^{-m} = \frac{(u-t)^{-m}}{h^{-m}}$$

by the assumed equation.

$$\text{Hence } S = \frac{(u-t)^{-m}}{h^{-m}}\cdot\frac{du}{dt}.$$

Now the actual solution of the assumed quadratic equation gives

$$u = \frac{R-1+h}{2h}, \quad \text{where } R = \{1-2h(1-2t)+h^2\}^{\frac{1}{2}},$$

$$\text{whence } u-t = \frac{R-1+h(1-2t)}{2h}, \text{ and } \frac{du}{dt} = \frac{1}{R};$$

$$\text{therefore } S = \{R - 1 + h(1-2t)\}^{-m} \cdot \frac{2^m h^{2m}}{R}.$$

Knowing thus the generated function S, we can conversely find q_n by taking the coefficient of h^n in the quantity S, and substituting for t its value in terms of ϕ.

An exactly similar process applied to the function Q_n of Art. (5), would give

$$\{R - 1 + h(1-2t)\}^m \cdot \frac{(tt')^{-m}}{2^m h^{2m} R}$$

as the function generated,

and observing that

$$R^2 - \{1 - h(1-2t)\}^2 = 4h^2 tt',$$

this quantity may be transformed to

$$\frac{2^m}{R} \{R + 1 - h(1-2t)\}^{-m},$$

so that Q_n is the coefficient of h^n in the expansion of this function.

8. From the theorems given in Arts. (5) and (6), we can determine reciprocal functions relative to ϕ, which quantity may denote any transcendant contained in the formula $\int_t (tt')^m$, from $m = -\infty$ to $m = +\infty$; circular arcs are amongst these transcendants, namely, when $m = -\frac{1}{2}$, and since both theorems are true simultaneously, when m is between -1 and $+1$, we shall get in this instance the two species of circular self-reciprocal functions, namely, the sines and cosines of the multiples of the simple arc.

I. To evaluate Q_n when $m = -\frac{1}{2}$.

For the variable with respect to which the integrations must be performed, we have

$$\phi = \int_t (tt')^{-\frac{1}{2}} = \int_t \frac{1}{\sqrt{(t-t^2)}} = \cos^{-1}(1-2t),$$

neglecting the constant which is unimportant.

By Art. (7),

$$Q_n = \text{coefficient of } h^n \text{ in } \frac{2^{-\frac{1}{2}}}{R}\{R+1-h(1-2t)\}^{\frac{1}{2}},$$

in which R represents $\{1-2h(1-2t)+h^2\}^{\frac{1}{2}}$.

Putting for t its value in terms of ϕ, we obtain

$$R = \{1-2h\cos\phi+h^2\}^{\frac{1}{2}} = (1-h\epsilon^{\phi\sqrt{-1}})^{\frac{1}{2}}.(1-h\epsilon^{-\phi\sqrt{-1}})^{\frac{1}{2}},$$

$$\text{and } 1-h(1-2t) = 1-h\cos\phi = \tfrac{1}{2}(1-h\epsilon^{\phi\sqrt{-1}}) + \tfrac{1}{2}(1-h\epsilon^{-\phi\sqrt{-1}}).$$

$$\text{Hence, } R+1-h(1-2t) = \tfrac{1}{2}\{(1-h\epsilon^{\phi\sqrt{-1}})^{\frac{1}{2}} + (1-h\epsilon^{-\phi\sqrt{-1}})^{\frac{1}{2}}\}^2;$$

$$\text{therefore, } Q_n = \text{coefficient of } h^n \text{ in } \tfrac{1}{2}.\frac{(1-h\epsilon^{\phi\sqrt{-1}})^{\frac{1}{2}} + (1-h\epsilon^{-\phi\sqrt{-1}})^{\frac{1}{2}}}{(1-h\epsilon^{\phi\sqrt{-1}})^{\frac{1}{2}} \times (1-h\epsilon^{-\phi\sqrt{-1}})^{\frac{1}{2}}}$$

$$= \tfrac{1}{2}\text{ coefficient of } h^n \text{ in } (1-h\epsilon^{\phi\sqrt{-1}})^{-\frac{1}{2}} + (1-h\epsilon^{-\phi\sqrt{-1}})^{-\frac{1}{2}}$$

$$= c.\frac{\epsilon^{n\phi\sqrt{-1}} + \epsilon^{-n\phi\sqrt{-1}}}{2} = c.\cos n\phi;$$

where $c = \dfrac{1.3.5\ldots(2n-1)}{2.4.6\ldots\ 2n}$, the limits of ϕ are 0 and π.

II. To evaluate q_n when $m = -\frac{1}{2}$.

As above, we have $\phi = \cos^{-1}(1-2t)$,

$$\text{and } q_n = \text{coefficient of } h^n \text{ in } \frac{2^{-\frac{1}{2}}}{hR}.\{R-1+h(1-2t)\}^{\frac{1}{2}}.$$

$$\text{But } R-1+h(1-2t) = \tfrac{1}{2}\left\{\frac{(1-h\epsilon^{-\phi\sqrt{-1}})^{\frac{1}{2}}}{\sqrt{-1}} - \frac{(1-h\epsilon^{+\phi\sqrt{-1}})^{\frac{1}{2}}}{\sqrt{-1}}\right\}^2;$$

$$\therefore\ q_n = \tfrac{1}{2}\text{ coefficient of } h^{n+1} \text{ in } \frac{(1-h\epsilon^{+\phi\sqrt{-1}})^{-\frac{1}{2}}}{\sqrt{-1}} - \frac{(1-h\epsilon^{-\phi\sqrt{-1}})^{-\frac{1}{2}}}{\sqrt{-1}}$$

$$= c\frac{\epsilon^{(1+n)\phi\sqrt{-1}} - \epsilon^{(1+n)\phi\sqrt{-1}}}{2\sqrt{-1}} = c\sin(1+n)\phi,$$

$$\text{where } c = \frac{1.3.5\ldots(2n+1)}{2.4.6\ldots(2n+2)}.$$

9. But whatever may be the value of m, the quantities Q_n, q_n may always be simply expressed in terms of t by the theorem of Leibnitz, viz.

$$\frac{d^n(uv)}{dt^n} = u\frac{d^n v}{dt^n} + n\frac{du}{dt}\cdot\frac{d^{n-1}v}{dt^{n-1}} + \frac{n\cdot(n-1)}{1\cdot 2}\cdot\frac{d^2u}{dt^2}\cdot\frac{d^{n-2}v}{dt^{n-2}} + \&c.$$

after applying which we may substitute for t its value in terms of ϕ.

Thus when $m = -\frac{1}{2}$

$$Q_n = \frac{d^n(tt')^{n-\frac{1}{2}}}{1\cdot 2\cdot 3\ldots\ldots n\,dt^n}\cdot(tt')^{\frac{1}{2}}$$

$$= \frac{1\cdot 3\cdot 5\ldots\ldots(2n-1)}{2\cdot 4\cdot 6\ldots\ldots 2n}\times\Big\{t'^n - \frac{n}{1}\cdot\frac{2n-1}{1}tt'^{n-1}$$

$$+\frac{n(n-1)}{1\cdot 2}\cdot\frac{(2n-1)(2n-3)}{1\cdot 3}t^2t'^{n-2} - \&c.\Big\}$$

$$= \frac{1\cdot 3\cdot 5\ldots\ldots(2n-1)}{2\cdot 4\cdot 6\ldots\ldots 2n}\times\Big\{t'^n - \frac{2n(2n-1)}{1\cdot 2}tt'^{n-1}$$

$$+\frac{2n(2n-1)(2n-2)(2n-3)}{1\cdot 2\cdot 3\cdot 4}t^2t'^{n-2} - \&c.\Big\}$$

$$= \frac{1}{2}\cdot\frac{1\cdot 3\cdot 5\ldots\ldots(2n-1)}{2\cdot 4\cdot 6\ldots\ldots 2n}\cdot\Big\{(t'^{\frac{1}{2}}+\sqrt{-1}\,t^{\frac{1}{2}})^{2n}+(t'^{\frac{1}{2}}-\sqrt{-1}\,t^{\frac{1}{2}})^{2n}\Big\},$$

and in the same way we have

$$q_n = \frac{d^n(tt')^{n+\frac{1}{2}}}{1\cdot 2\cdot 3\ldots\ldots n\,dt^n}$$

$$= \frac{3\cdot 5\cdot 7\ldots\ldots(2n+1)}{2\cdot 4\cdot 6\ldots\ldots 2n}\Big\{t^{\frac{1}{2}}t'^{n+\frac{1}{2}} - \frac{n}{1}\cdot\frac{2n+1}{3}t^{\frac{3}{2}}t'^{n-\frac{1}{2}}$$

$$+\frac{n(n-1)}{1\cdot 2}\cdot\frac{(2n+1)(2n-1)}{3\cdot 5}t^{\frac{5}{2}}t'^{n-\frac{3}{2}} - \&c.\Big\}$$

$$= \frac{1}{2\sqrt{-1}}\cdot\frac{3\cdot 5\cdot 7\ldots\ldots(2n+1)}{2\cdot 4\cdot 6\ldots\ldots(2n+2)}\Big\{(t'^{\frac{1}{2}}+\sqrt{-1}\,t^{\frac{1}{2}})^{2n+2}-(t'^{\frac{1}{2}}-\sqrt{-1}\,t^{\frac{1}{2}})^{2n+2}\Big\},$$

and passing to the variable ϕ, since $1-2t=\cos\phi$; therefore $t=\sin^2\frac{\phi}{2}$ and $t'=\cos^2\frac{\phi}{2}$, whence $t'^{\frac{1}{2}}\pm\sqrt{-1}\,t^{\frac{1}{2}}=\cos\frac{\phi}{2}\pm\sqrt{-1}\sin\frac{\phi}{2}$, by substituting which we obtain

$$Q_n=\frac{1.3.5\ldots.(2n-1)}{2.4.6\ldots.2n}.\cos n\phi,$$

$$q_n=\frac{1.3.5\ldots.(2n+1)}{2.4.6\ldots.(2n+2)}.\sin(n+1).\phi,$$

which values are the same with those in Art. 8.

The numerical coefficients in these formulæ may be rejected as having no importance in self-reciprocal functions; it is also observable that q_n contains a different multiple arc from that in Q_n, the reason of which is that Q_n, q_n are to be self-reciprocal functions for all entire values of n from 0 to $+\infty$, and then $\int_\phi q_n q_m=0$ except when $n=m$, this exception (on which the main value of reciprocal functions depends) would not hold universally true if q_n were of the form $\sin(n\phi)$, for then $q_0=0$, and therefore $\int_\phi q_0.q_0=0$ contrary to the principle of the exception, but in the form above found this irregularity does not occur.

10. From the results found in Art. 9, it follows that if we put

$$(t'^{\frac{1}{2}}+\sqrt{-1}\,t^{\frac{1}{2}})^{2n}=Q_n+\sqrt{-1}\,q_n,$$

the real functions Q_n, q_n possess a common property, viz.

$$\int_t\frac{Q_mQ_n}{\sqrt{(tt')}}=0;\quad\int_t\frac{q_mq_n}{\sqrt{(tt')}}=0;$$

except when $m=n$, which exception does not apply to the last integral when $m=n=0$.

From the same results the following identities are obtained:

$$\frac{2^n d^n(tt')^{n-\frac{1}{2}}}{1.3.5\ldots.(2n-1)\,dt^n}.(tt')^{\frac{1}{2}}=\cos\{n\cos^{-1}(1-2t)\}$$

$$\frac{(n+1)\,2^{n+1}d^n(tt')^{n+\frac{1}{2}}}{1.3.5\ldots.(2n+1)dt^n}=\sin\{(n+1)\cos^{-1}(1-2t)\}.$$

We shall now consider whether analogous formulæ hold true for negative values of n the index of differentiation.

Generally if u and v be functions of t and $\int_t^n u$ denote the n^{th} successive integral of u, then

$$\int_t^n uv = u\int_t^n v - n\frac{du}{dt}\int_t^{n+1} v + \frac{n(n+1)}{1.2}\cdot\frac{d^2u}{dt^2}\cdot\int_t^{n+2} v - \&c.$$

for if we take the n^{th} differential coefficient of each term in this series, all the terms resulting mutually destroy each other except the first term uv.

Putting $u = t'^{-n-\frac{1}{2}}$, $v = t^{-n-\frac{1}{2}}$, and rejecting the constants of integration in the latter, we have

$$\int_t^n v = \frac{(-2)^n}{1.3.5....(2n-1)} \times t^{-\frac{1}{2}},$$

$$\int_t^{n+1} v = \frac{(-2)^n}{1.3.5....(2n-1)}\cdot\frac{2}{1}\cdot t^{\frac{1}{2}}, \quad \int_t^{n+2} v = \frac{(-2)^n}{1.3.5....(2n-1)}\cdot\frac{2^2}{1.3}\, t^{\frac{3}{2}}, \&c.,$$

$$\text{also } \frac{du}{dt} = \frac{2n+1}{2}\, t'^{-n-\frac{3}{2}}, \quad \frac{d^2u}{dt^2} = \frac{(2n+1)(2n+3)}{2^2}\cdot t'^{-n-\frac{5}{2}}, \&c.$$

Hence $\int_t^n (tt')^{-n-\frac{1}{2}}$

$$= \frac{(-2)^n (tt')^{-\frac{1}{2}}}{1.3.5....(2n-1)}\left\{t'^{-n} - \frac{n}{1}\cdot\frac{2n+1}{1}\cdot t'^{-n-1}t + \frac{n(n-1)}{1.2}\cdot\frac{(2n+1)(2n+3)}{1.3}\, t'^{-n-2}t^2 - \&c.,\right\}$$

$$\text{or } \frac{1.3.5....(2n-1)}{(-2)^n}\cdot\frac{d^{-n}(tt')^{-n-\frac{1}{2}}}{dt^{-n}}\cdot(tt')^{\frac{1}{2}}$$

$$= \tfrac{1}{2}\{t'^{\frac{1}{2}} + \sqrt{-1}\, t^{\frac{1}{2}}\}^{-2n} + \tfrac{1}{2}\{t'^{\frac{1}{2}} - \sqrt{-1}\, t^{\frac{1}{2}}\}^{-2n}$$

$$= \cos\{n\cos^{-1}(1-2t)\},$$

the appendage which contains all the arbitrary constants being

$$\{A_0 + A_1 t + A_2 t^2 + \ldots A_{n-1}t^{n-1}\}\cdot(tt')^{\frac{1}{2}}.$$

Dividing the last equation by $(tt')^{\frac{1}{2}}$, and integrating with respect to t, we get

$$\frac{1.3.5\ldots(2n-1)}{(-2)^n}\cdot\frac{d^{-n-1}(tt')^{-n-\frac{1}{2}}}{dt^{-n-1}}=\frac{1}{n}\sin\{n\cos^{-1}(1-2t)\}.$$

Putting $n=m-1$, we get

$$(m-1)\cdot\frac{1.3.5\ldots(2m-3)}{(-2)^{m-1}}\cdot\frac{d^{-m}(tt')^{-m+\frac{1}{2}}}{dt^{-m}}=\sin\{(1-m)\cos^{-1}(1-2t)\},$$

thus are obtained the corresponding formulæ for negative indices.

11. The two series of reciprocal functions arising from the theorems in Arts. 5. and 6., differ essentially, only in reference to the independant variable of integration, for in Art. 5., m may be any quantity between -1, and $+\infty$, and in Art. 6. any quantity between $+1$ and $-\infty$; change in the latter theorem m into $-m$, and the limits of m will then be the same in both; for distinctness, also let θ be used instead of ϕ in the value of q_n.

$$\text{Hence,}\quad Q_n=\frac{d^n(tt')^{n+m}}{1.2.3\ldots n\,dt^n}\cdot(tt')^{-m},\quad\text{and}\quad \phi=\textstyle\int_t(tt')^m,$$

$$q_n=\frac{d^n(tt')^{n+m}}{1.2.3\ldots n\,dt^n},\qquad\text{and}\quad \theta=\textstyle\int_t(tt')^{-m}.$$

Now the reciprocal functions of Art. 5., give the equation

$$\textstyle\int_\phi Q_nQ_{n'}=0,\quad\text{or}\quad \int_\theta Q_nQ_{n'}\dfrac{d\phi}{d\theta}=0.$$

$$\text{But}\quad\frac{d\phi}{dt}=(tt')^m,\quad\text{and}\quad\frac{d\theta}{dt}=(tt')^{-m};\quad\text{therefore}\quad\frac{d\phi}{d\theta}=(tt')^{2m}.$$

$$\text{Hence,}\quad \textstyle\int_\theta Q_n(tt')^m\times Q_{n'}(tt')^m=0.$$

And since $Q_n(tt')^m=q_n$, and $Q_{n'}(tt')^m=q_{n'}$, it follows that $\int_\phi Q_nQ_{n'}$ is equivalent to $\int_\theta q_nq_{n'}$, the only difference being with respect to the variables ϕ and θ employed for integration.

If in the formulæ of Arts. 5. and 6., we assign to m all possible values between -1 and $+1$, we obtain two series of self-reciprocal functions, which when $m=0$ become identical with each other, and with the functions denominated P_n in the preceding memoirs. For every other value of m between those limits, there are two different kinds of reciprocal functions, one of which only is a rational and entire function of t, for instance when $m=-\frac{1}{2}$, we have found the functions $\cos n\phi$ and $\sin(n+1)\phi$, the former of which only is a rational function of $\cos\phi$.

12. (1.) *When* m $= -\frac{1}{4}$.

To determine ϕ in this case, make $\sin\theta = t^{\frac{1}{2}} - t'^{\frac{1}{2}}$, squaring and observing that $t+t'=1$, we get $\sin^2\theta = 1-2(tt')^{\frac{1}{2}}$, whence

$$t^{\frac{1}{2}} + t'^{\frac{1}{2}} = \sqrt{(2-\sin^2\theta)}, \quad \text{and} \quad 2^{\frac{1}{2}}(tt')^{\frac{1}{4}} = \cos\theta.$$

Differentiate the assumed equation, and we get

$$\cos\theta = \frac{t^{\frac{1}{2}} + t'^{\frac{1}{2}}}{2(tt')^{\frac{1}{2}}}\cdot\frac{dt}{d\theta}; \quad \text{therefore } \frac{1}{(tt')^{\frac{1}{4}}}\cdot\frac{dt}{d\theta} = 2\cos\theta\,.\frac{(tt')^{\frac{1}{4}}}{t^{\frac{1}{2}} + t'^{\frac{1}{2}}},$$

$$\text{or, } \frac{d\phi}{d\theta} = \frac{\cos^2\theta}{\sqrt{(1-\frac{1}{2}\sin^2\theta)}} = 2\sqrt{(1-\frac{1}{2}\sin^2\theta)} - \frac{1}{\sqrt{(1-\frac{1}{2}\sin^2\theta)}},$$

hence, $\phi = 2E(\theta) - F(\theta)$.

The extreme values of the amplitude θ of these elliptic functions being $-\frac{\pi}{2}$, and $+\frac{\pi}{2}$; the limits of ϕ are 0, and $4E_1 - 2F_1$, where E_1 and F_1 denote the complete functions when the amplitude extends from zero to a right angle.

The reciprocal functions for integrations relative to ϕ, are

$$Q_n = \frac{3.7.11\ldots(4n-1)}{4.8.12\ldots\ldots4n}$$

$$\left\{t'^n - \frac{4n(4n-1)}{3.4}t'^{n-1}t + \frac{4n(4n-1)(4n-4)(4n-5)}{3.4.7.8}t'^{n-2}t^2, \text{ \&c.}\right\},$$

$$q_n = \frac{5 \cdot 9 \cdot 13 \ldots (4n+1)}{4 \cdot 8 \cdot 12 \ldots\ldots 4n}$$

$$\times\, (tt')^{\frac{1}{4}} \left\{ t'^n - \frac{(4n+1) \cdot 4n}{4 \cdot 5} \cdot t'^{n-1} t + \frac{(4n+1) \cdot 4n \cdot (4n-3)(4n-4)}{4 \cdot 5 \cdot 8 \cdot 9} t'^{n-2} t^2, \text{ \&c.} \right\}.$$

(2.) When $m = -1$.

In this case $q_n = \dfrac{d^n (tt')^{n+1}}{1 \cdot 2 \cdot 3 \ldots n dt^n}$

$$= (n+1) tt' \left\{ t'^n - \frac{n}{1} \cdot \frac{n+1}{2} \cdot t'^{n-1} t + \frac{n \cdot (n-1)}{1 \cdot 2} \cdot \frac{(n+1)(n)}{2 \cdot 3} \cdot t'^{n-2} t^2, \text{ \&c.} \right\},$$

$$\text{and } \phi = \int_t (tt')^{-1} = \text{h. l.} \left(\frac{t}{t'}\right).$$

$$\text{Hence, } \frac{t}{t'} = \epsilon^{\phi}, \quad \frac{1}{t'} = 1 + \epsilon^{\phi};$$

therefore

$$q_n = \frac{(n+1)}{(1+\epsilon^{\phi})^{n+2}} \left\{ \epsilon^{\phi} - \frac{n}{1} \cdot \frac{n+1}{2} \cdot \epsilon^{2\phi} + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{(n+1)(n)}{2 \cdot 3} \cdot \epsilon^{3\phi} - \text{\&c.} \right\},$$

where the limits of ϕ are $-\infty$ and $+\infty$.

13. *To express the functions* Q_n *and* q_n *in terms of* t *alone.*

By Art. 6., we have

$$q_n = \frac{d^n (tt')^{n-m}}{1 \cdot 2 \cdot 3 \ldots n dt^n}$$

$$= \frac{(n-m)(n-m-1)\ldots(1-m)}{1 \cdot 2 \cdot 3 \ldots n} \cdot (tt')^{-m} \cdot \left\{ t'^n - \frac{n}{1} \cdot \frac{n-m}{1-m} t'^{n-1} t \right.$$

$$\left. + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{(n-m)(n-m-1)}{(1-m)(2-m)} t'^{n-2} t^2, \text{ \&c.} \right\}.$$

Suppose $1-t$ substituted for t' in each term between the brackets, then expanding each, the coefficient of t^r in the whole will be

$$\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}(-1)^r\left\{1+r.\frac{n-m}{1-m}+\frac{r(r-1)}{1.2}.\frac{(n-m)(n-m-1)}{(1-m)(2-m)}+\&c.\right\}$$

$$=\frac{n(n-1)\ldots(n-r+1).(-1)^r}{1.2\ldots r\times(1-m)(2-m)\ldots(r-m)}\left\{t^{n-m}\frac{d^r\,t^{r-m}}{dt^r}+r\,\frac{d.t^{n-m}}{dt}.\frac{d^{r-1}.t^{r-m}}{dt^{r-1}}\right.$$

$$\left.+\frac{r.(r-1)}{1.2}.\frac{d^2.t^{n-m}}{dt^2}.\frac{d^{r-2}.t^{r-m}}{dt^{r-2}}-\&c.\right\},$$

when t is put equal to unity after the differentiations.

But by the theorem of Leibnitz, the part within the latter brackets is equivalent to

$$\frac{d^r\,t^{r+n-2m}}{dt^r}=(n-2m+1)(n-2m+2)\ldots(n-2m+r).t^{n-2m},$$

hence, the required coefficient of

$$t^r=(-1)^r.\frac{n.(n-1)\ldots(n-r+1)}{1.2\ldots r}\times\frac{(n-2m+1)(n-2m+2)\ldots(n-2m+r)}{(1-m)(2-m)\ldots(r-m)}.$$

Hence,

$$q_n=\frac{(n-m)(n-m-1)\ldots(1-m)}{1.2.3\ldots n}(t-t^2)^{-m}\left\{1-\frac{n}{1}.\frac{n-2m+1}{1-m}.t\right.$$

$$\left.+\frac{n(n-1)}{1.2}.\frac{(n-2m+1)(n-2m+2)}{(1-m)(2-m)}t^2,\ \&c.\right\}.$$

Again, by Art. 5.,

$$Q_n=\frac{d^n(tt')^{n+m}}{1.2\ldots n\,dt^n}.(tt')^{-m}$$

$$=\frac{(n+m)(n+m-1)\ldots(1+m)}{1.2\ldots n}.\left\{t'^n-\frac{n}{1}.\frac{n+m}{1+m}.t'^{n-1}t\right.$$

$$\left.+\frac{n.(n-1)}{1.2}.\frac{(n+m)(n+m-1)}{(1+m)(2+m)}t'^{n-2}t^2,\ \&c.\right\},$$

the reduction of which to the powers of t is effected as before, putting $-m$ for m, whence

$$Q_n = \frac{(n+m)(n+m-1)\dots(1+m)}{1.2\dots n}\left\{1 - \frac{n}{1}.\frac{n+2m+1}{1+m}.t + \frac{n(n-1)}{1.2}.\frac{(n+2m+1)(n+2m+2)}{(1+m)(2+m)}.t^2 - \&c.\right\}.$$

When $m=0$,

$$q_n = Q_n = 1 - \frac{n}{1}.\frac{n+1}{1}.t + \frac{n.(n-1)}{1.2}.\frac{(n+1)(n+2)}{1.2}.t^2 - \&c.$$

which is the same as the value of P_n, Sect. II. Art. 2.

When $\left.\begin{array}{l} m=-\frac{1}{2} \\ \text{and } t=\sin^2\frac{\phi}{2}\end{array}\right\}$, $q_n = \frac{1}{2}.\frac{3.5\dots(2n+1)}{2.4\dots\dots 2n}.\sin\phi\left\{1 - \frac{(n+1)^2-1}{2.3}.2^2\sin^2\frac{\phi}{2} + \frac{\{(n+1)^2-1\}.\{(n+1)^2-2^2\}}{2.3.4.5}.2^4\sin^4\frac{\phi}{2} - \&c.\right\}$,

$$Q_n = \frac{1.3\dots(2n-1)}{2.4\dots\dots 2n}.\left\{1 - \frac{n^2}{1.2}.2^2\sin^2\frac{\phi}{2} + \frac{n^2.(n^2-1)}{1.2.3.4}.2^4\sin^4\frac{\phi}{2} - \&c.\right\}.$$

14. To express the quantities Q_n, q_n by means of a differential equation.

Suppose $f(t)$ is a function of t, subject to the condition

$$t(1-t).f''(t) + (m+1)(1-2t).f'(t) + n.(n+2m+1).f(t) = 0,$$

where $f''(t)$ denotes the second, and $f'(t)$ the first differential coefficient of $f(t)$ relatively to t; differentiating this equation, we get

$$t(1-t).f'''(t) + (m+2)(1-2t).f''(t) + (n-1)(n+2m+2).f'(t) = 0,$$

$$t(1-t).f''''(t) + (m+3)(1-2t).f'''(t) + (n-2)(n+2m+3).f''(t) = 0,$$

and generally,

$$t(1-t).f^{\prime\prime\prime(r)}(t) + (m+r-1)(1-2t).f^{\prime\prime\prime(r-1)}(t) + (n-r+2)(n+2m+r-1).f^{\prime\prime\prime(r-1)}(t) = 0.$$

Put $t = 0$ in all these equations successively, thence we have

$$(m+1) . f'(0) = - n . (n+2m+1) . f(0),$$

$$(m+2) . f''(0) = - (n-1)(n+2m+2) . f'(0),$$

$$m+3 . f'''(0) = - (n-2) . (n+2m+3) . f(0),$$

&c.

it follows from this by Maclaurin's Theorem, that the preceding equation will be satisfied, as a particular solution, by taking

$$f(t) = f(0)\left\{1 - \frac{n}{1} . \frac{n+2m+1}{1+m} . t + \frac{n.(n-1)}{1.2} . \frac{(n+2m+1)(n+2m+2)}{(1+m).(2+m)} . t^2 - \&c.\right\},$$

and $f(0)$ being arbitrary if we put it equal to

$$\frac{(1+m)(2+m)(3+m)......(n+m)}{1 \quad . \quad 2 \quad . \quad 3 \quad ...... \quad n},$$

this value of $f(t)$ will become the same as the value found for Q_n in the preceding article; hence, replacing $1-t$ by its equal t', we get

$$(tt')\frac{d^2Q_n}{dt^2} + (m+1)(1-2t) . \frac{dQ_n}{dt} + n . (n + 2m + 1) . Q_n = 0.$$

But if in the value of $f(t)$ we change the sign of m, putting

$$f(0) = \frac{(1-m)(2-m)......(n-m)}{1 \quad . \quad 2 \quad ...... \quad n},$$

then $f(t)$ becomes equivalent to $q_n(tt')^m$; and if we put this for $f(t)$ in the first supposed equation, and divide the result by $(tt')^m$, we get

$$(tt')\frac{d^2q_n}{dt^2} + (m+1)(1-2t) . \frac{dq_n}{dt} + (n+1)(n-2m) . q_n = 0.$$

(2) *Particular inferences resulting from the preceding theory.*

15. Denoting as before by ϕ the indefinite integral $\int_t (tt')^m$, and putting

$$Q_n = \frac{d^n(tt')^{n+m}}{1.2.3...n\,dt^n}(tt')^{-m} \text{ and } q_n = \frac{d^n(tt')^{n-m}}{1.2.3...n\,dt^n},$$

then assigning to m all possible values from $-\infty$ to $+\infty$, the functions Q_n, q_n will give an infinite series of reciprocal functions relative to all the transcendants contained in ϕ considered as the variable of integration; and when m is between -1 and $+1$, pairs of reciprocal functions will be obtained, except when $m=0$, when both coincide.

In this series are included the trigonometrical functions, namely, when $m=-\frac{1}{2}$; and Laplace's functions, when $m=0$.

In all the reciprocal functions thus arising, there exists one common property, namely, the definite integral always vanishes between the limits which make the functions themselves maxima and minima; this remarkable property I have had occasion in another place to notice, in the particular case of Laplace's functions.*

To prove this generally take the equations of the preceding article, viz.

$$tt'\frac{d^2Q_n}{dt^2} + (m+1)(1-2t).\frac{dQ_n}{dt} + n(n+2m+1)\,Q_n = 0,$$

$$tt'\frac{d^2q_n}{dt^2} + (m+1)(1-2t).\frac{dq_n}{dt} + (n+1)(n-2m)q_n = 0.$$

Multiply both equations by $(tt')^m$, and integrate reserving the constants under the integral sign; hence,

$$(tt')^{m+1}\frac{dQ_n}{dt} + n(n+2m+1)\int_t Q_n(tt')^m = 0,$$

$$(tt')^{m+1}\frac{dq_n}{dt} + (n+1)(n-2m)\int_t q_n(tt')^m = 0;$$

and changing the independant variable by the condition $\frac{dt}{d\phi}=(tt')^{-m}$, we have

$$(tt')^{2m+1}\frac{dQ_n}{d\phi} + n(n+2m+1)\int_\phi Q_n = 0,$$

$$(tt')^{2m+1}\frac{dq_n}{d\phi} + (n+1)(n-2m)\int_\phi q_n = 0.$$

* Electricity, *Introduction*.

But when Q_n, q_n are maxima and minima, $\frac{dQ_n}{d\phi}$ and $\frac{dq_n}{d\phi}$ respectively vanish; therefore, between the corresponding limits of ϕ, we must have $\int_\phi Q_n = 0$, $\int_\phi q_n = 0$, which general property is easily verified when

$$Q_n = a\cos n\phi \text{ and } q_n = a\sin(n+1)\phi.$$

16. *To find the complete integral of the differential equation*

$$tt' \cdot \frac{d^2u}{dt^2} + (m+1)(1-2t)\frac{du}{dt} + n(n+2m+1)u = 0,$$

where n *is integer and* m *any constant.*

The differential equation for Q_n (Art. 15.) is of the same form as the above equation, and therefore $u = cQ_n$ is a particular solution, c being an arbitrary constant.

The form of the differential equation for q_n will become the same as that of the given equation, if $-(n+1)$ be written instead of n in the former; hence, another particular solution is $c'q_{-(n+1)}$.

The complete solution is therefore

$$u = cQ_n + c'q_{-(n+1)}.$$

This solution fails first when $m = 0$, for then the functions Q_n, $q_{-(n+1)}$ in their expanded forms become both identical with Laplace's function P_n, and consequently the two constants c, c' merge into only one, viz. their sum; but if we put generally

$$c = a + \frac{b}{m} \text{ and } c' = -\frac{b}{m},$$

$$\text{then } u = aQ_n + b \cdot \frac{Q_n - q_{-(n+1)}}{m}.$$

And putting $m = 0$, the latter term becomes a vanishing fraction, and therefore,

$$u = aP_n + b\frac{d}{dm}\{Q_n - q_{-(n+1)}\} \text{ when } m = 0.$$

The term by which b is here multiplied, is the coefficient of m in $Q_n - q_{-(n+1)}$, which is easily found from the expansions in Art. 13; hence,

$$u = P_n \{a + b \log.(tt')\} + \frac{n}{1} . \frac{n+1}{1} . 2bt \left(1 - \frac{1}{n} - \frac{1}{n+1}\right)$$

$$- \frac{n(n-1)}{1 . 2} . \frac{(n+1)(n+2)}{1 . 2} 2bt^2 \left\{1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}\right\} + \&c.$$

The general solution also fails when m is an integer, for then some of the terms in the expansion of Q_n or $q_{-(n+1)}$ will become infinite, and the principle of vanishing fractions will simply enough in this case also be applicable in determining the complete solution; but if we put for Q_n, q_n their differential forms, the solution will never fail, for the failures arise from the entrance of logarithms into the result, and these will actually enter in the latter forms; changing our constants, the complete solution for all cases is

$$u = a(tt')^{-m} \frac{d^n (tt')^{n+m}}{dt^n} + b \frac{d^{-(n+1)} (tt')^{-(n+m+1)}}{dt^{-(n+1)}};$$

it is therefore necessary to shew that the functions by which the arbitrary constants are multiplied, are particular solutions.

Putting $v = (tt')^{n+m}$, then $\frac{dv}{dt} = (n+m)(1-2t)(tt')^{n+m-1}$,

and $\frac{d^2 v}{dt^2} = (n+m)(n+m-1)(1-2t)^2 (tt')^{n+m-2} - 2(n+m)(tt')^{n+m-1}$.

Hence $tt' . \frac{d^2 v}{dt^2} - (n+m-1)(1-2t)\frac{dv}{dt} + 2(n+m) . v = 0$,

and by successive differentiations the following equations arise:

$$(tt') . \frac{d^3 v}{dt^3} - (n+m-2)(1-2t) . \frac{d^2 v}{dt^2} + 2(2n+2m-1)\frac{dv}{dt} = 0,$$

$$(tt')\frac{d^4 v}{dt^4} - (n+m-3)(1-2t) . \frac{d^3 v}{dt^3} + 2(3n+3m-3)\frac{d^2 v}{dt^2} = 0,$$

$$(tt')\frac{d^5 v}{dt^5} - (n+m-4)(1-2t) . \frac{d^4 v}{dt^4} + 2(4n+4m-6) . \frac{d^3 v}{dt^3} = 0,$$

and the law of the successive formation of these equations being very simple, we have generally

$$(tt')\frac{d^{k+2}v}{dt^{k+2}} - (n+m-k-1)(1-2t)\frac{d^{k+1}v}{dt^{k+1}} + 2\left\{(k+1)(n+m) - \frac{k(k+1)}{2}\right\}.\frac{d^k v}{dt^k} = 0.$$

Put $k = n$, hence

$$(tt').\frac{d^{n+2}v}{dt^{n+2}} - (m-1)(1-2t).\frac{d^{n+1}v}{dt^{n+1}} + (n+1)(n+2m)\frac{d^n v}{dt^n} = 0.$$

Transpose $n(n+2m+1)\dfrac{d^n v}{dt^n}$, and multiply by $(tt')^{-m}$, hence

$$tt'\frac{d^2}{dt^2}\left\{(tt')^{-m}\frac{d^n v}{dt^n}\right\} + (m+1)(1-2t)\frac{d}{dt}\left\{(tt')^{-m}.\frac{d^n v}{dt^n}\right\} = -n(n+2m+1).\frac{d^n v}{dt^n},$$

from which it follows that $u = (tt')^{-m}.d^n.\dfrac{(tt')^{n+m}}{dt^n}$ satisfies the equation of Art. 16.

Again put $v' = (tt')^{-(n+m+1)}$, or $tt'\dfrac{dv'}{dt} + (n+m+1)(1-2t)v' = 0$, and by successive integrations we obtain

$$tt'.v' + (n+m)(1-2t)\int_t v' + 2(n+m)\int_t^2 v' = 0,$$

$$tt'.\int_t v' + (n+m-1)(1-2t).\int_t^2 v' + 2(2n+2m-1)\int_t^3 v' = 0,$$

and generally

$$tt'\int_t^{k-1} v' + (n+m-k+1)(1-2t)\int_t^k v' + 2\left\{k(n+m) - \frac{k.(k-1)}{2}\right\}.\int_t^{k-1} v = 0.$$

Put $k = n$, hence

$$tt'\int_t^{n-1} v' + (m+1)(1-2t).\int_t^n v' + n(n+2m+1).\int_t^{n+1} v' = 0;$$

from which it appears that $u = \int_t^{n+1}(v')$ is also a particular solution, and therefore the complete solution of the general equation is

$$u = a(tt')^{-m}\frac{d^n(tt')^{n+m}}{dt^n} + b\int_t^{n+1}(tt')^{-(n+m+1)}.$$

Laplace's equation occurs when we put $m=0$, and therefore

$$u=\frac{ad^n.(tt')^n}{dt^n}+b\int_t^{n+1}(tt')^{-(n+1)},$$

the first term alone of which is the type of Laplace's functions, the equation is therefore more general than the functions it was used to designate.

The term $\int_t^{n+1}(tt')^{-(n+m+1)}$ gives $n+1$ constants of integration which enter as coefficients of the appendage which is a rational function of n dimensions, but this must be rejected, since the constants must be determined so that the rational function of n dimensions may satisfy the given equation, and this only identifies the appendage with the other term in u, viz. $a(tt')^{-m}\dfrac{d^n.(tt')^{n+m}}{dt^n}$.

17. *To find explicitly the omitted part of the complete integral in Laplace's equation.*

The general equation of Art. 16. becomes in this instance

$$tt'\frac{d^2u}{dt^2}+(1-2t)\frac{du}{dt}+n(n+1)u=0,$$

and the complete solution is

$$u=a\frac{d^n(tt')^n}{dt^n}+b\int_t^{n+1}(tt')^{-(n+1)},$$

the first term being Laplace's function, and the second the transcendant, it is required to find explicitly.

Let α, β be any arbitrary quantities, then we have

$$\frac{d^n}{d\alpha^n}\left(\frac{1}{t-\alpha}\cdot\frac{1}{\beta-\alpha}\right)=\frac{1}{\beta-\alpha}\cdot\frac{d^n}{d\alpha^n}\left(\frac{1}{t-\alpha}\right)+n\frac{d}{d\alpha}\left(\frac{1}{\beta-\alpha}\right)\frac{d^{n-1}}{d\alpha^{n-1}}\left(\frac{1}{t-\alpha}\right)$$

$$+\frac{n(n-1)}{1.2}\frac{d^2}{d\alpha^2}\left(\frac{1}{\beta-\alpha}\right)\frac{d^{n-2}}{d\alpha^{n-2}}\left(\frac{1}{t-\alpha}\right),\ \&c.$$

or $$\frac{\left\{d^n \dfrac{1}{(t-\alpha)(\beta-\alpha)}\right\}}{1.2.3\ldots n d\alpha^n} = \frac{1}{\beta-\alpha}\cdot\frac{1}{(t-\alpha)^{n+1}} + \frac{1}{(\beta-\alpha)^2}\cdot\frac{1}{(t-\alpha)^n} + \frac{1}{(\beta-\alpha)^3}\cdot\frac{1}{(t-\alpha)^{n-1}} + \&c.,$$

hence

$$(-1)^n \cdot \frac{d^{2n}\left\{\dfrac{1}{(t-\alpha)(\beta-\alpha)}\right\}}{1^2.2^2.3^2\ldots n^2 d\alpha^n d\beta^n} = \frac{1}{(\beta-\alpha)^{n+1}}\left\{\frac{1}{(t-\alpha)^{n+1}} + \frac{n+1}{1}\cdot\frac{1}{(\beta-\alpha)}\cdot\frac{1}{(t-\alpha)^n}\right.$$

$$\left. + \frac{(n+1)(n+2)}{1.2}\cdot\frac{1}{(\beta-\alpha)^2}\cdot\frac{1}{(t-\alpha)^{n-1}} + \&c.\right\}$$

Commuting in this equation the quantities α and β, we have

$$(-1)^n \cdot \frac{d^{2n}\left\{\dfrac{1}{(t-\beta)(\alpha-\beta)}\right\}}{1^2.2^2.3^2\ldots n^2.d\alpha^n d\beta^n} = \frac{1}{(\alpha-\beta)^{n+1}}\left\{\frac{1}{(t-\beta)^{n+1}} + \frac{n+1}{1}\cdot\frac{1}{\alpha-\beta}\cdot\frac{1}{(t-\beta)^{n+1}}\right.$$

$$\left. + \frac{(n+1)(n+2)}{1.2}\cdot\frac{1}{(\alpha-\beta)^2}\cdot\frac{1}{(t-\beta)^n} + \&c.\right\}$$

If both equations be added observing that

$$\frac{1}{(t-\alpha)(\beta-\alpha)} + \frac{1}{(t-\beta)(\alpha-\beta)} = \frac{1}{(t-\alpha)(\beta-t)},$$

the sum of the left-hand members

$$= (-1)^n \cdot \frac{d^{2n}\dfrac{1}{(t-\alpha)(\beta-t)}}{1^2.2^2.3^2\ldots n^2 d\alpha^n d\beta^n}$$

$$= (-1)^n \cdot \frac{d^n \cdot \dfrac{1}{t-\alpha}}{1.2.3\ldots n d\alpha^n}\cdot\frac{d^n \cdot \dfrac{1}{\beta-t}}{1.2.3\ldots n d\beta^n}$$

$$= \frac{1}{(t-\alpha)^{n+1}}\cdot\frac{1}{(\beta-t)^{n+1}}.$$

Hence, we get the general identity,

$$\frac{1}{\{(t-\alpha)(\beta-t)\}^{n+1}} = \frac{1}{(\beta-\alpha)^{n+1}}$$

$$\cdot\left\{\begin{array}{l} \dfrac{1}{(t-\alpha)^{n+1}} + \dfrac{n+1}{1}\cdot\dfrac{1}{\beta-\alpha}\cdot\dfrac{1}{(t-\alpha)^n} + \dfrac{(n+1)(n+2)}{1\,.\,2}\cdot\dfrac{1}{(\beta-\alpha)^2}\cdot\dfrac{1}{(t-\alpha)^{n-1}}\ \&c. \\ + \dfrac{1}{(\beta-t)^{n+1}} + \dfrac{n+1}{1}\cdot\dfrac{1}{\beta-\alpha}\cdot\dfrac{1}{(\beta-t)^n} + \dfrac{(n+1)(n+2)}{1\,.\,2}\cdot\dfrac{1}{(\beta-\alpha)^2}\cdot\dfrac{1}{(\beta-t)^{n-1}}\ \&c. \end{array}\right\}.$$

Put now $\alpha = 0$, $\beta = 1$, and therefore $\beta - t = t'$, hence,

$$\frac{1}{(tt')^{n+1}} = \left\{\begin{array}{l} \dfrac{1}{t^{n+1}} + \dfrac{n+1}{1}\cdot\dfrac{1}{t^n} + \dfrac{(n+1)(n+2)}{1\,.\,2}\cdot\dfrac{1}{t^{n-1}} + \&c. \\ + \dfrac{1}{t'^{n+1}} + \dfrac{n+1}{1}\cdot\dfrac{1}{t'^n} + \dfrac{(n+1)(n+2)}{1\,.\,2}\cdot\dfrac{1}{t'^{n-1}} + \&c. \end{array}\right\}*,$$

in which identity n must be one of the natural numbers 0, 1, 2, 3, &c. and the number of terms in each series must be limited to $n+1$.

Suppose the $(n+1)^{\text{th}}$ successive integral of each term of this expansion is taken after multiplying, for convenience, by $1\,.\,2\,.\,3\ldots\ldots n$, the result will consist,

1st, of a logarithmic part, viz.

$$(-1)^n\,.\,\text{h.l.}\,(t)\left\{1 - \frac{n}{1}\cdot\frac{n+1}{1}\,.\,t + \frac{n\,.\,(n-1)}{1\,.\,2}\cdot\frac{(n+1)(n+2)}{1\,.\,2}\,t^2 - \&c.\right\}$$

$$-\ \text{h.l.}\,t'\left\{1 - \frac{n}{1}\cdot\frac{n+1}{1}\,.\,t' + \frac{n\,.\,(n-1)}{1\,.\,2}\cdot\frac{(n+1)(n+2)}{1\,.\,2}\,t'^2 - \&c.\right\},$$

where the part between brackets in the upper line is equivalent to the function P_n, and in the lower to $(-1)^n\,.\,P_n$, and therefore the whole to $(-1)^n\,.\,P_n\,.\,\text{h.l.}\,\dfrac{t}{t'}$.

* This method is applicable in every case to the decomposition of fractions, the denominators of which contain equal factors.

2d, of a rational and entire function p_n which satisfies the equation,

$$(tt') \cdot \frac{d^2 p_n}{dt^2} + (1-2t) \frac{dp_n}{dt} + n(n+1) \cdot p_n + 2(-1)^n \frac{dP_n}{dt} = 0,$$

since the term $2(-1)^n \cdot \frac{dP_n}{dt}$ is the result which arises if the logarithmic term $(-1)^n P_n \cdot \text{h.l.} \frac{t}{t'}$ be put for u in the actual equation.

3d, of an appendage containing $n+1$ arbitrary constants, which as before remarked must be rejected altogether.

Differentiating the equation for p_n above obtained, we get

$$(tt') \frac{d^3 p_n}{dt^3} + 2(1-2t) \cdot \frac{d^2 p_n}{dt^2} + (n-1)(n+2) \frac{dp_n}{dt} + 2(-1)^n \cdot \frac{d^2 P_n}{dt^2} = 0,$$

$$(tt') \frac{d^4 p_n}{dt^4} + 3(1-2t) \cdot \frac{d^3 p_n}{dt^3} + (n-2)(n+3) \frac{d^2 p_n}{dt^2} + 2(-1)^n \cdot \frac{d^3 P_n}{dt^3} = 0,$$

....

$$(n-1)(1-2t) \cdot \frac{d^{n-1} p_n}{dt^{n-1}} + 2(2n-1) \frac{d^{n-2} p_n}{dt^{n-2}} + 2(-1)^n \frac{d^{n-1} P_n}{dt^{n-2}} = 0,$$

$$2n \frac{d^{n-1} p_n}{dt^{n-1}} + 2(-1)^n \frac{d^n P_n}{dt^n} = 0,$$

when these equations terminate, since p_n is of $n-1$ dimensions.

Put $t = 0$, in all these equations beginning with the last, observing that then

$$\frac{d^n P_n}{dt^n} = (-1)^n \cdot (n+1)(n+2) \ldots (2n),$$

$$\frac{d^{n-1} P_n}{dt^{n-1}} = -(-1)^n \cdot n(n+1)(n+2) \ldots (2n-1),$$

$$\frac{d^{n-2} P_n}{dt^{n-2}} = (-1)^n \cdot \frac{n(n-1)}{1 \cdot 2} \cdot (n+1)(n+2) \ldots\ldots (2n-2), \text{ \&c.}$$

$$\text{Hence } \frac{d^{n-1}p_n}{dt^n} = -\ 2 . (n+1)(n+2)\dots(2n-1),$$

$$\frac{d^{n-2}p_n}{dt^n} = (n+1)(n+2)\dots(2n-1), \ \&c.$$

and the value of p_n is the rational function

$$-\ 2\,\frac{(n+1)(n+2)\dots(2n-1)}{1 . 2\dots(n-1)}\,\{t^{n-1} + A_1 t^{n-2} + A_2 t^{n-3}\dots + A_{n-1}\},$$

in which the coefficients are successively formed from the equation

$$(n-m-1)^2 . A_m + (m+2)(2n-m-1) . A_{m+1}$$

$$+ 2(-1)^m . \frac{n(n-1)\dots(m+2)}{1 . 2\dots(n-m-1)} . \frac{n(n-1)\dots(n-m-1)}{2n(2n-1)\dots(2n-m)} = 0,$$

and the omitted part in the integral of the proposed equation is

$$b\left\{P_n \text{ h.l.} \left(\frac{t}{t'}\right) + (-1)^n . p_n\right\}.$$

18. When $m = -\frac{1}{2}$, the general equation of Art. 16. becomes

$$tt'\frac{d^2u}{dt^2} + \tfrac{1}{2}(1-2t)\frac{du}{dt} + n^2 . u = 0,$$

and putting $\phi = \cos^{-1}(1-2t)$, we have $Q_n = \cos n\phi$, $q_{-(n+1)} = \sin n\phi$, the complete solution is therefore $u = a\cos n\phi + b\sin n\phi$.

Though the trigonometrical functions were the first used in analysis as reciprocals, for the purposes of expressing functions by means of definite integrals and of expanding them, in the former instance of their application there remain a few remarkable cases which do not seem to have been noticed, with which we shall conclude this Section.

19. The two functions which possess the remarkable properties alluded to, are

$$\Theta = \epsilon^{x\cos\theta} . \cos(x\sin\theta), \text{ and } \Theta' = \epsilon^{x\cos\theta}\sin(x\sin\theta).$$

The successive differential coefficients with respect to x of the functions Θ, Θ' follow simple and elegant laws, thus

$$\frac{d\Theta}{dx} = \epsilon^{x\cos\theta}\cos\{x\sin\theta+\theta\}, \qquad \frac{d\Theta'}{dx} = \epsilon^{x\cos\theta}\sin\{x\sin\theta+\theta\},$$

$$\frac{d^2\Theta}{dx^2} = \epsilon^{x\cos\theta}\cos\{x\sin\theta+2\theta\}, \qquad \frac{d^2\Theta'}{dx^2} = \epsilon^{x\cos\theta}\sin\{x\sin\theta+2\theta\},$$

and generally

$$\frac{d^n\Theta}{dx^n} = \epsilon^{x\cos\theta}\cos\{x\sin\theta+n\theta\}, \qquad \frac{d^n\Theta'}{dx^n} = \epsilon^{x\cos\theta}\sin\{x\sin\theta+n\theta\}.$$

Again, the successive integrals relative to x, follow the same laws, omitting the arbitrary constants of integration,

$$\textstyle\int_x\Theta = \epsilon^{x\cos\theta}\cos\{x\sin\theta-\theta\}, \qquad \int_x\Theta' = \epsilon^{x\cos\theta}\sin\{x\sin\theta-\theta\},$$

$$\textstyle\int_x^2\Theta = \epsilon^{x\cos\theta}\cos\{x\sin\theta-2\theta\}, \qquad \int_x^2\Theta' = \epsilon^{x\cos\theta}\sin\{x\sin\theta-2\theta\},$$

$$\textstyle\int_x^n\Theta = \epsilon^{x\cos\theta}\cos\{x\sin\theta-n\theta\}, \qquad \int_x^n\Theta' = \epsilon^{x\cos\theta}\sin\{x\sin\theta-n\theta\},$$

for it will readily be seen by actual differentiation that

$$\Theta = \frac{d^n}{dx^n}\{\epsilon^{x\cos\theta}\cos(x\sin\theta-n\theta)\}, \qquad \Theta' = \frac{d^n}{dx^n}\{\epsilon^{x\cos\theta}\sin(x\sin\theta-n\theta)\}.$$

Again, changing the forms of the proposed functions, we get

$$\Theta = \tfrac{1}{2}\{\epsilon^{x\epsilon^{\theta\sqrt{-1}}} + \epsilon^{x\epsilon^{-\theta\sqrt{-1}}}\}, \qquad \Theta' = \frac{1}{2\sqrt{-1}}\{\epsilon^{x\epsilon^{\theta\sqrt{-1}}} - \epsilon^{x\epsilon^{-\theta\sqrt{-1}}}\},$$

whence, expanding and passing from the exponential to trigonometrical functions

$$\Theta = 1 + x\cos\theta + \frac{x^2}{1.2}.\cos 2\theta + \frac{x^3}{1.2.3}\cos 3\theta + \&c.$$

$$\Theta' = \qquad x\sin\theta + \frac{x^2}{1.2}.\sin 2\theta + \frac{x^3}{1.2.3}\sin 3\theta + \&c.$$

$$\left.\begin{aligned} \int_\theta \Theta \cos n\theta &= \frac{\pi}{2} \cdot \frac{x^n}{1.2.3\ldots. n} \\ \int_\theta \Theta' \sin n\theta &= \frac{\pi}{2} \cdot \frac{x^n}{1.2.3\ldots. n} \end{aligned}\right\}$$ the limits of θ being 0 and π, these formulæ apply for all integer values of n, except when $n=0$.

Now $\Theta \cos n\theta \pm \Theta' \sin n\theta = \epsilon^{x\cos\theta} \cos\{x \sin\theta \mp n\theta\}$.

Hence $\int_\theta \epsilon^{x\cos\theta} \cos\{x\sin\theta - n\theta\} = \pi \cdot \dfrac{x^n}{1.2.3\ldots. n}$,

$$\int_\theta \epsilon^{x\cos\theta} \cos\{x\sin\theta + n\theta\} = 0.$$

The particular case where $n=0$ is included in the first of these two equations.

20. By the results thus obtained, we are enabled to represent any rational and integer function of x in a form adapted to *general differentiation*.

By applying Maclaurin's theorem, we first have

$$\phi(x) = A_0 + A_1 . x + A_2 . \frac{x^2}{1.2} + A_3 . \frac{x^3}{1.2.3} + \&c.;$$

and passing to definite integrals by the formulæ of the last article,

$$\phi(x) = \frac{1}{\pi} \int_\theta \epsilon^{x\cos\theta} \{A_0 \cos(x\sin\theta) + A_1 \cos(x\sin\theta - \theta) \\ + A_2 \cos(x\sin\theta - 2\theta) + \&c.\}$$

also if A_{-1}, A_{-2}, A_{-3}, &c. represent arbitrary constants,

$$0 = \frac{1}{\pi} \int_\theta \epsilon^{x\cos\theta} \{A_{-1} \cos(x\sin\theta + \theta) + A_{-2} \cos(x\sin\theta + 2\theta) \\ + A_{-3} \cos(x\sin\theta + 3\theta) + \&c.\}$$

both of which integrals must be added before $\phi(x)$ can be subjected in a complete form to general differentiation.

We then obtain the n^{th} differential coefficient by adding $n\theta$ under each cosine in this sum, that is,

$$\frac{d^n\phi(x)}{dx^n} = \frac{1}{\pi}\int_\theta \epsilon^{x\cos\theta}\{A_0\cos(x\sin\theta + n\theta) + A_1\cos[x\sin\theta + (n-1)\theta] + A_2\cos[x\sin\theta + (n-2)\theta] + \&c.\}$$
$$+ \frac{1}{\pi}\int_\theta \epsilon^{x\cos\theta}\{A_{-1}\cos[x\sin\theta + (n+1)\theta] + A_{-2}\cos[x\sin\theta + (n+2)\theta] + A_{-3}\cos[x\sin\theta + (n+3)\theta] + \&c.\}.$$

I. *When* n *is a positive integer,* the whole of the second line vanishes, there will then be no arbitrary constants; also, the first n terms of the upper line disappear.

II. *When* n *is a negative integer,* the first n terms of the second line remain, and these contain n arbitrary constants.

III. *When* n *is fractional,* the whole of the second line remains, giving an infinite number of constants.

21. The theory of numbers as connected with definite integrals, affords another remarkable application of reciprocal functions.

Let n be any integer of which the divisors are n, n', n''......1; also let m be any intger, and θ an arc of which the limits are 0, π.

Then, generally,

$$1 - 2h\cos m\theta + h^2 = (1 - h\epsilon^{m\theta\sqrt{-1}})(1 - h\epsilon^{-m\theta\sqrt{-1}});$$

and hence,

$$\text{h. l.}(1 - 2h\cos m\theta + h^2) = -2\{h\cos m\theta + \tfrac{1}{2}h^2\cos 2m\theta + \tfrac{1}{3}h^3\cos 3m\theta + \&c.\}.$$

Suppose now that m is one of the numbers n, n', n''......1; this series must contain one term involving $\cos n\theta$, viz.

$$\frac{m}{n}h^{\frac{n}{m}}\cos n\theta;$$

and therefore,

$$\int_\theta \cos n\theta\ \text{h. l.}(1 - 2h\cos m\theta + h^2) = -\pi\cdot\frac{m}{n}\cdot h^{\frac{n}{m}}.$$

But when m is not a divisor of n, there will be no term in the expansion found to contain the arc $n\theta$, and therefore,

$$\int_\theta \cos n\theta\ \text{h. l.}(1 - 2h\cos m\theta + h^2) = 0.$$

Put now for m successively every integer from 1 to n inclusive, and take the sum of all the definite integrals thus resulting, hence

$$\int_\theta \cos n\theta \text{ h. l. } \{(1-2h\cos\theta+h^2)(1-2h\cos 2\theta+h^2)\ldots(1-2h\cos n\theta+h^2)\}$$

$$=-\pi\left\{\frac{n}{n}\,.\,h+\frac{n'}{n}\,.\,h^{\frac{n}{n'}}+\frac{n''}{n}\,.\,h^{\frac{n}{n''}}+\ldots\frac{1}{n}\,.\,h^n\right\}.$$

Now the quantities $\frac{n}{n}$, $\frac{n'}{n}$, $\frac{n''}{n}$, &c. are the reciprocals of all the possible divisors of n, and therefore this definite integral may also be expressed by

$$-\pi\left\{h+\frac{1}{n'}h^{n'}+\frac{1}{n''}h^{n''}+\ldots\frac{1}{n}h^n\right\}.$$

For θ in the preceding equation write 2ϕ, the limits of the latter variable will be 0 and $\frac{\pi}{2}$.

Also put $h=1$, and therefore,

$$1-2h\cos\theta+h^2=2(1-\cos 2\phi)=4\sin^2\phi,$$

$$1-2h\cos 2\theta+h^2=2(1-\cos 4\phi)=4\sin^2 2\phi,$$

&c.;

$$\therefore \text{ h. l. } \{(1-2h\cos\theta+h^2)(1-2h\cos 2\theta+h^2)\ldots(1-2h\cos n\theta+h^2)\}$$

$$=2n\text{ h. l. }(2)+2\text{ h. l. }\{\sin\phi\sin 2\phi\ldots\sin n\phi\}.$$

The integral of the constant multiplied by $\cos 2n\phi$ vanishes, and therefore

$$\int_\phi \text{h. l. } \{\sin\phi\sin 2\phi\sin 3\phi\ldots\sin n\phi\}\,.\,\cos 2n\phi=-\frac{\pi}{4}\left\{\frac{1}{n}+\frac{1}{n'}+\frac{1}{n''}+\ldots+1\right\};$$

and multiplying both sides by $-\frac{4n}{\pi}$, we get this theorem.

The sum of all the divisors of a given number n, including the number itself and unity, is expressed by the definite integral

$$-\frac{4n}{\pi}\int_\phi \text{h. l. } \{\sin\phi\sin 2\phi\sin 3\phi\ldots\sin n\phi\}\,.\,\cos 2n\phi.$$

SECTION VII.

On Transient Functions.

22. Let $\phi(h, t)$ be such that when h has a particular value assigned, the whole function vanishes whatever may be the value of t, except in one case; $\phi(h, t)$ under those circumstances, is a transient function having only a momentary existence.

Thus the function $\dfrac{(1-h)(1+h)}{\{1-2h(1-2t)+h^2\}^{\frac{3}{2}}}$, when h is put equal to unity is a transient function, because its value is zero in every case except when $t = 0$, for then it becomes $\dfrac{1+h}{(1-h)^2}$ when h is put equal to 1, that is, it acquires momentarily an infinite value.

If the value of the function had been always zero, its definite integral relative to t would also be zero; but if we actually integrate from $t = 0$ to $t = 1$ without previously assigning a particular value to h, the definite integral

$$= \frac{(1-h)(1+h)}{2h} \cdot \left\{\frac{1}{1-h} - \frac{1}{1+h}\right\} = 1,$$

thus this integral is independent of h, and therefore remains the same when $h = 1$, that is, for the transient function.

By the principles of the *Second Memoir* we can always form a self-reciprocal function in which the general term may be of any particular kind; thus if $f(t, n)$ were the type of the general term, and if we put generally,

$$F(t, n) = a_0 f(t, 0) + a_1 f(t, 1) + a_2 f(t, 2) + \dots\dots + a_n f(t, n),$$

lastly, if we determine the coefficients $a_1, a_2, \dots\dots a_n$ in terms of a_0 and n, by the n equations (arising from the definite integrals) following,

$$\textstyle\int_t F(t, n) \,.\, f(t, 0) = 0,$$

$$\int_t F(t, n) \,.\, f(t, 1) = 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\int_t F(t, n) \,.\, f(t, n-1) = 0\,;$$

then the function $F(t, n)$ will obviously be self-reciprocal.

But if $f(t, n)$ not containing arbitrary coefficients, but being absolutely given as t^n, $(\cos t)^n$, &c. is proposed as a function to which some unknown function is reciprocal, the discovery of the latter, which is effected in the next article, is of a more difficult nature than the process above mentioned; and in the particular cases quoted, as well as in many others, this required function is transient, it is therefore in this character that transient functions are here introduced.

23. *Given* f (t, n) *a function of known form with respect to the variable* t *and the integer* n, *it is required to find another function of* t *and* n, *as* ϕ (t, n), *such that the definite integral* $\int_t$f (t, n) ϕ (t, n′) *may always vanish when the integers* n *and* n′ *are unequal.*

Begin with forming a self-reciprocal function $F(t, n)$, the general term of which may be of the given form $f(t, n)$; thus

$$F(t, n) = a_0 f(t, 0) + a_1 f(t, 1) + a_2 f(t, 2) + \dots\dots + a_n f(t, n),$$

where the coefficients are determined in the manner indicated in the preceding article.

Suppose next that the required function $\phi(t, n)$ is expanded in an infinite series of which the general term is of the form $F(t, n)$, thus

$$\phi(t, n) = A_0 \,.\, F(t, 0) + A_1 F(t, 1) + \dots + A_n F(t, n) + A_{n+1} F\{t, (n+1)\}, \text{ \&c.}$$

Multiply by $f(t, 0)$, $f(t, 1)$, $f(t, 2)\dots\dots f(t, n-1)$ successively, and integrate the products between the given limits of t, observing that

$$\int_t F(t, 1) \,.\, f(t, 0) = 0, \qquad \int_t F(t, 2) \,.\, f(t, 0) = 0 \dots \int_t F(t, n) f(t, 0) = 0,$$

by the property of the functions $F(t, n)$;

and similarly,

$$\textstyle\int_t F(t,\,2)f(t,\,1)=0, \qquad \int_t F(t,\,3)f(t,\,1)=0 \ldots \int_t F(t,\,n)\,.\,f(t,\,1)=0,$$
&c. &c.,

we thus obtain the following equations;

$$\textstyle\int_t \phi(t,\,n)\,.\,f(t,\,0) = A_0 \int_t f(t,\,0)\,.\,F(t,\,0),$$
$$\textstyle\int_t \phi(t,\,n)\,.\,f(t,\,1) = A_1 \int_t f(t,\,1)\,.\,F(t,\,1),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\textstyle\int_t \phi(t,\,n)\,.\,f(t,\,n-1) = A_{n-1} \int_t f(t,\,n)\,.\,F(t,\,n-1);$$

hence the imposed condition of reciprocity requires that the first n coefficients A_0, $A_1 \ldots A_{n-1}$ in the expansion of $\phi(t,\,n)$, may be each equal to zero; and therefore,

$$\phi(t,\,n)=A_n F(t,\,n)+A_{n+1} F(t,\,n+1)+A_{n+2} F(t,\,n+2), \text{ \&c. } ad\ inf.$$

Multiply successively both sides by $f(t,\,n+1)$, $f(t,\,n+2)$, &c., and integrate; and since $n+1$, $n+2$, &c. are each $>n$, the definite integrals must vanish.

Hence,

$$\textstyle A_n \int_t F(t,\,n)\,.\,f(t,\,n+1)+A_{n+1}\int_t F(t,\,n+1)\,.\,f(t,\,n+1)=0,$$
$$\textstyle A_n \int_t F(t,\,n)\,.\,f(t,\,n+2)+A_{n+1}\int_t F(t,\,n+1)\,.\,f(t,\,n+2)$$
$$\textstyle +A_{n+2}\int_t F(t,\,n+2)\,.\,f(t,\,n+2)=0,$$
&c. &c.,

from whence the coefficients A_{n+1}, A_{n+2}, &c. are known in terms of A_n and n, and therefore the required function $\phi(t,\,n)$ is known.

24. *To find the function which is reciprocal to* t^n.

First, we must form a self-reciprocal function, of which the general term is of the form t^n; this has been already effected in Section IV., namely,

$$P_n=1-\frac{n}{1}\cdot\frac{n+1}{1}\,t+\frac{n(n-1)}{1\,.\,2}\cdot\frac{(n+1)(n+2)}{1\,.\,2}\,.\,t^2-\text{\&c.},$$

which has been also proved to be the coefficient of h^n, in the expansion of $\{1-2h(1-2t)+h^2\}^{-\frac{1}{2}}$, (Section IV. Art. 9.), and to be equal to $\frac{d^n.(tt')^n}{1.2.3...ndt^n}$, where $t'=1-t$, (Section IV. Art. 2.)

Then representing by V_n the required function which is reciprocal to t^n, we have by the preceding article

$$V_n = A_n P_n + A_{n+1} P_{n+1} + A_{n+2} P_{n+2} + \&c.,$$

where it is obvious that when n' is less than n $\int_t V_n t^{n'} = 0$, and it is only necessary that the coefficients may be so determined, that the same equation may remain true when n' is greater than n; and since one of these coefficients is arbitrary, we may put $A_n = 1$.

Now in general, we have by Section IV. Art. 2.

$$\int_t P_n t^x = (-1)^n . \frac{x(x-1)(x-2)...(x-n+1)}{(x+1)(x+2)(x+3)...(x+n+1)},$$

$$\text{hence, } \int_t V_n t^{n+x} = (-1)^n \left\{ \frac{(n+x)(n+x-1)...(x+1)}{(n+x+1)(n+x+2)...(2n+x+1)} \right.$$

$$\left. - A_{n+1} . \frac{(n+x)(n+x-1)...x}{(n+x+1)(n+x+2)...(2n+x+2)} + A_{n+2} . \frac{(n+x)(n+x-1)...(x-1)}{(n+x+1)(n+x+2)...(2n+x+3)} - \&c. \right\}.$$

Therefore, when x is any integer from 1 to ∞, we must have

$$0 = 1 - A_{n+1} . \frac{x}{2n+x+2} + A_{n+2} . \frac{x(x-1)}{(2n+x+2)(2n+x+3)}$$

$$- A_{n+3} . \frac{x(x-1)(x-2)}{(2n+x+2)(2n+x+3)(2n+x+4)} + \&c.,$$

and putting for x the successive integrals 1, 2, 3, &c.

$$0 = 1 - A_{n+1} . \frac{1}{2n+3},$$

$$0 = 1 - A_{n+1} . \frac{2}{2n+4} + A_{n+2} . \frac{2.1}{(2n+4)(2n+5)},$$

$$0 = 1 - A_{n+1} . \frac{3}{2n+5} + A_{n+2} . \frac{3.2}{(2n+5)(2n+6)} - A_{n+3} . \frac{3.2.1}{(2n+5)(2n+6)(2n+7)},$$

&c. &c.

From whence we obtain

$$A_{n+1} = 2n+3, \quad A_{n+2} = \frac{2n+2}{1.2}.(2n+5), \quad A_{n+3} = \frac{(2n+2)(2n+3)}{1.2.3}.(2n+7),$$

and to prove that this law of formation is general, we may observe that since

$$\left(1-\frac{1}{h}\right)^{2n+2x+1} = 1 - \frac{2n+2x+1}{1}.\frac{1}{h} + \ldots\ldots + \frac{(2n+2x+1)(2n+2x)\ldots(x+1)}{1.2\ldots(2n+x+1)}$$

$$\times\left(-\frac{1}{h}\right)^{2n+x+1}.\left\{1-\frac{x}{2n+x+2}.\frac{1}{h}+\frac{x.(x-1)}{(2n+x+2)(2n+x+3)}.\frac{1}{h^2}+\&c.\right\},$$

and $(1-h)^{-(2n+2)}(1+h) = 1+(2n+3).h + \dfrac{2n+2}{1.2}.(2n+5).h^2$

$$+\frac{(2n+2)(2n+3)}{1.2.3}.(2n+7).h^3 + \&c.$$

Multiply both, and take the coefficient of $\dfrac{1}{h^{2n+x+1}}$ in the products, and we get

$$\frac{(2n+2x+1)(2n+2x)\ldots(x+1)}{1.2.3\ldots(2n+x+1)}\left\{1-\frac{x}{2n+x+2}.(2n+3)\right.$$

$$\left.+\frac{x(x-1)}{(2n+x+2)(2n+x+3)}.\frac{(2n+2)(2n+5)}{1.2} - \&c.\right\}$$

$$= \text{coefficient of } \frac{1}{h^{2n+x+1}} \text{ in } (-1)^x.\frac{(1+h)(1-h)^{2x-1}}{h^{2n+2x+1}}$$

$$= (-1)^x.\text{ coefficient of } h^x \text{ in } (1+h)(1-h)^{2x-1}.$$

Now the coefficient of h^x in $(1+h)(1-h)^{2x-1}$, is evidently the sum of the coefficients of h^{x-1}, and of h^x, in the expansion of $(1-h)^{2x-1}$; that is, the sum of the coefficients of the two middle terms in a binomial raised to an odd power, and with alternate signs of + and

–, hence the quantity we are considering must be zero, and therefore

$$0 = 1 - \frac{2n+3}{1} \cdot \frac{x}{2n+x+2} + \frac{(2n+2)(2n+5)}{1.2} \cdot \frac{x(x-1)}{(2n+x+2)(2n+x+3)}$$

$$- \frac{(2n+2)(2n+3)(2n+7)}{1.2.3} \cdot \frac{x(x-1)(x-2)}{(2n+x+2)(2n+x+3)(2n+x+4)} + \&c.,$$

which shews the generality of the observed law of the coefficients A_{n+1}, A_{n+2}, &c.

Substituting now these values in the general formula for V_n, we get the required function which is reciprocal to t^n, namely,

$$V_n = P_n + (2n+3) \cdot P_{n+1} + \frac{(2n+2)(2n+5)}{1.2} \cdot P_{n+2}$$

$$+ \frac{(2n+2)(2n+3)(2n+7)}{1.2.3} \cdot P_{n+3}, \ \&c.$$

25. *The function which is reciprocal to* t^n *is transient.*

For in general

$$P_n = \frac{d^n.(tt')^n}{1.2.3\ldots n dt^n} = (-1)^n \cdot \frac{d^n.(tt')^n}{1.2.3\ldots n dt'^n},$$

and putting $1-t'$ for t, and expanding the binomial $(1-t')^n$, and lastly actually performing the differentiations indicated, we have

$$(-1)^n . P_n = 1 - \frac{n}{1} \cdot \frac{n+1}{1} \cdot t' + \frac{n(n-1)}{1.2} \cdot \frac{(n+1)(n+2)}{1.2} \cdot t'^2 - \&c.,$$

and therefore

$$(-1)^n V_n = \{1 - \frac{n}{1} \cdot \frac{n+1}{1} \cdot t' + \frac{n(n-1)}{1.2} \cdot \frac{(n+1)(n+2)}{1.2} \cdot t'^2 - \&c.\}$$

$$-(2n+3)\{1 - \frac{n+1}{1} \cdot \frac{n+2}{1} \cdot t' + \frac{(n+1).n}{1.2} \cdot \frac{(n+2)(n+3)}{1.2} \cdot t'^2 - \&c.\}$$

$$+ \frac{(2n+2)(2n+5)}{1.2}\{1 - \frac{n+2}{1} \cdot \frac{n+3}{1} \cdot t' + \frac{(n+2)(n+1)}{1.2} \cdot \frac{(n+3)(n+4)}{1.2} \cdot t'^2 - \&c.\}$$

$$- \&c. \qquad \&c.$$

The term which is independent of t' is

$$1 - (2n+3) + \frac{(2n+2)(2n+5)}{1 \cdot 2} - \&c.$$

but in general we have

$$(1-h)(1+h)^{-(2n+2)} = 1-(2n+3) \cdot h + \frac{(2n+2)(2n+5)}{1 \cdot 2} \cdot h^2 - \&c.$$

and putting $h=1$, we find that the term independent of t' is zero.

Again, multiplying the last equation by h^{n+m}, we get

$$(h^{n+m}-h^{n+m+1})(1+h)^{-(2n+2)} = h^{n+m} - (2n+3) \cdot h^{n+m+1} + \frac{(2n+2)(2n+5)}{1 \cdot 2} \cdot h^{n+m+2} - \&c.$$

Now it is easily seen that when $h=1$, we have

$$\frac{d^{2m}}{dh^{2m}}(h^{n+m}-h^{n+m+1}) = (n+m)(n+m-1)\ldots(n-m+1) - (n+m+1)(n+m)\ldots(n-m+2)$$

$$= -2m \cdot (n+m)(n+m-1)\ldots.(n-m+2),$$

$$\frac{d^{2m-1}}{dh^{2m-1}}(h^{n+m}-h^{n+m+1}) = -(2m-1)(n+m)(n+m-1)\ldots.(n-m+3),$$

&c. &c.

and therefore when h is put $=1$ after differentiation, we have

$$\frac{d^{2m}}{dh^{2m}}\{(h^{n+m}-h^{n+m+1})(1+h)^{-2n+2}\} = -2^{-(2n+2)} \cdot 2m \cdot (n+m)(n+m-1)\ldots(n-m+2) \times$$

$$\left\{1 - \tfrac{1}{2} \cdot \frac{2n+2}{1} \cdot \frac{2m-1}{n-m+2} + \frac{1}{2^2} \cdot \frac{(2n+2)(2n+3)}{1 \cdot 2} \cdot \frac{(2m-1)(2m-2)}{(n-m+2)(n-m+3)} - \&c.\right\}$$

which series consists of only $2m$ terms, and is equal to the infinite series obtained by differentiating the other side of the equation, viz.

$$n(n-1)\ldots.(n-m+1) \times (n+1)(n+2)\ldots.(n+m)$$

$$- (2n+3 \cdot (n+1) n \ldots.(n-m+2) \times (n+2)(n+3)\ldots.(n+m+1)$$

$$+ \frac{(2n+2)(2n+5)}{1 \cdot 2}(n+2)(n+1)\ldots.(n-m+3) \times (n+3)(n+4)\ldots.(n+m+2)$$

$-$ &c. *ad infinitum.*

Now it is obvious by putting $m = 1$, 2, &c. successively, that the finite series is always $=0$, and therefore the infinite series [which is the same as the coefficient of $\dfrac{(-t')^m}{1^2 . 2^2 .... m^2}$ in the expression for $(-1)^n V_n$] vanishes also, so that if V_n be arranged according to the powers of t', it is $0+0 . t' + 0t'^2 + \text{\&c.}$, nevertheless its value is in one instance infinite, namely, when $t=0$, for then $P_n = P_{n+1} = \text{\&c.} = 1$, and therefore

$$V_n = 1 + (2n+3) + \frac{(2n+2)(2n+5)}{1 . 2} + \frac{(2n+2)(2n+3)(2n+7)}{1 . 2 . 3} + \text{\&c.}$$

$$= (1+h)(1-h)^{-(2n+2)}, \text{ when } h \text{ is put } =1.$$

$$= \quad \infty .$$

And if V_n did not possess this infinite element $\int_t V_n t^n$, from $t=0$ to $t=1$ would vanish, whereas its actual value is the same as

$$\int_t P_n t^n \text{ or } (-1)^n . \frac{n . (n-1)(n-2)...1}{(n+1)(n+2)(n+3)...(2n+1)}, \text{ (Sect. IV. Arts. 3, 4.)}$$

26. *To express the transient function* V_n *in a finite form.*

Since by Art. (24.) $V_n = P_n + (2n+3) P_{n+1}$

$$+ \frac{(2n+2)(2n+5)}{1 . 2} . P_{n+2} + \frac{(2n+2)(2n+3)(2n+7)}{1 . 2 . 3} . P_{n+3}, \text{ \&c.}$$

therefore

$$1 . 2 . 3 ... (2n+1) V_n = 1 . 2 . 3 ... 2n \times (2n+1) P_n$$

$$+ 2 . 3 ... (2n+1) \times (2n+3) P_{n+1} . h + 3 . 4 ... (2n+2) \times (2n+5) P_{n+2} h^2, \text{ \&c.}$$

when h is put equal to unity.

But in general,

$$\{1 - 2h(1-2t) + h^2\}^{-\frac{1}{2}} = P_0 + P_1 h + P_2 h^2 + ... P_n h^n + P_{n+1} h^{n+1} + \text{\&c.};$$

$$\therefore \frac{d^{2n} h^n \{1 - 2h(1-2t) + h^2\}^{-\frac{1}{2}}}{dh^{2n}}$$

$$= 1 . 2 . 3 ... 2n . P_n + 2 . 3 ... (2n+1) . P_{n+1} h + 3 . 4 ... (2n+2) . P_{n+2} h^2 + \text{\&c.}$$

Multiply by $2h^{n+\frac{1}{2}}$, and differentiating once more, we get

$$2\frac{d}{dh}\left\{h^{n+\frac{1}{2}}\frac{d^{2n}h^n\{1-2h(1-2t)+h^2\}^{\frac{1}{2}}}{dh^{2n}}\right\}$$

$$=1.2.3...2n\times(2n+1)P_nh^{n-\frac{1}{2}}+2.3...(2n+1)\times(2n+3)P_{n+1}h^{n+\frac{1}{2}}+\&c.$$

Hence, $V_n=2h^{-(n-\frac{1}{2})}.\dfrac{d}{dh}\left\{h^{n+\frac{1}{2}}\dfrac{d^{2n}h^n\{1-2h(1-2t)+h^2\}^{-\frac{1}{2}}}{1.2.3...(2n+1).dh^{2n}}\right\}$,

when h is put $=1$.

Put for abridgment the radical $\{1-2h(1-2t)+h^2\}^{-\frac{1}{2}}=R$, then

$$\frac{d^{2n}.(Rh^n)}{dh^{2n}}=\frac{2n.(2n-1)...(n+1)}{1.2...n}.1.2...n\frac{d^nR}{dh^n}$$

$$+\frac{2n(2n-1)...n}{1.2...(n+1)}.2.3...nh\frac{d^{n+1}R}{dh^{n+1}}+\&c.$$

$$=2n(2n-1)...(n+1)\left\{\frac{d^nR}{dh^n}+\frac{n}{n+1}.\frac{h}{1}.\frac{d^{n+1}R}{dh^{n+1}}\right.$$

$$\left.+\frac{n(n-1)}{(n+1)(n+2)}.\frac{h^2}{1.2}.\frac{d^{n+2}R}{dh^{n+2}},\ \&c.\right\}$$

Whence $2\dfrac{d}{dh}\left\{h^{n+\frac{1}{2}}\dfrac{d^{2n}(Rh^n)}{dh^{2n}}\right\}=2n(2n-1)...(n+1)\left\{(2n+1)h^{n-\frac{1}{2}}\dfrac{d^nR}{dh^n}\right.$

$$+\frac{2n+3}{1}.\frac{n}{n+1}.h^{n+\frac{1}{2}}\frac{d^{n+1}R}{dh^{n+1}}+\frac{2n+5}{1.2}.\frac{h^{n+\frac{3}{2}}.n.(n-1)}{(n+1)(n+2)}:\frac{d^{n+2}R}{dh^{n+2}}+\&c.$$

$$\left.+2h^{n+\frac{1}{2}}\frac{d^{n+1}R}{dh^{n+1}}+\frac{2nh^{n+\frac{3}{2}}}{n+1}.\frac{d^{n+2}R}{dh^{n+2}},\ \&c.\right\}$$

Hence $1.2.3...nV_n=\dfrac{d^nR}{dh^n}+\dfrac{2n+3}{2n+1}.\dfrac{n}{n+1}.h\dfrac{d^{n+1}R}{dh^{n+1}}$

$$+\frac{2n+5}{2n+1}.\frac{n(n-1)}{(n+1)(n+2)}.\frac{h^2}{1.2}.\frac{d^{n+2}R}{dh^{n+2}}+\&c.$$

$$+\frac{2}{2n+1}h\frac{d^{n+1}R}{dh^{n+1}}+\frac{2}{2n+1}.\frac{n}{n+1}.h^2.\frac{d^{n+2}R}{dh^{n+2}}$$

$$+\frac{2}{2n+1}.\frac{n(n-1)}{(n+1)(n+2)}.\frac{h^3}{1.2}.\frac{d^{n+3}R}{dh^{n+3}}+\&c.$$

h being put $= 1$, after the differentiations; this value of $1.2\ldots n V_n$ is expressed in two finite series, each containing only $n+1$ terms.

If we actually add the terms in this formula, which contain the same powers of h, we get

$$V_n = \frac{1}{1.2\ldots n}\left\{\frac{d^n R}{dh^n} + \frac{n+2}{n+1}\cdot\frac{h}{1}\cdot\frac{d^{n+1}R}{dh^{n+1}} + \frac{(n+3).n}{(n+1)(n+2)}\cdot\frac{h^2}{1.2}\cdot\frac{d^{n+2}R}{dh^{n+2}}\right.$$

$$\left. + \frac{(n+4).n(n-1)}{(n+1)(n+2)(n+3)}\cdot\frac{h^3}{1.2.3}\cdot\frac{d^{n+3}R}{dh^{n+3}} + \&c.\right\}$$

when h is put equal to unity.

27. *Discussion of the transient function* V_0.

Put $n=0$ in the general expression for V_n in the preceding article,

$$\text{hence } V_0 = R + 2h\frac{dR}{dh}$$

$$= \{1-2h(1-2t)+h^2\}^{-\frac{1}{2}} + 2h(1-2t-h)\{1-2h(1-2t)+h^2\}^{-\frac{3}{2}}$$

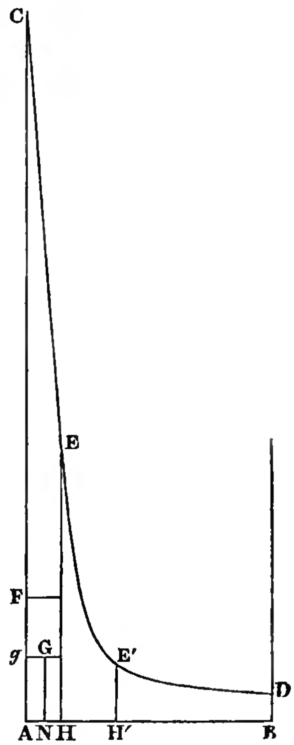
$$= \frac{(1-h)(1+h)}{\{1-2h(1-2t)+h^2\}^{\frac{3}{2}}}, \text{ when } h \text{ is put equal to unity.}$$

This function, as has been observed in Section VII. (22), is in general zero, except in the particular case when $t=0$, when its value is infinite.

If we imagine a curve of which the equation is

$$y = \frac{(1-h)(1+h)}{\{1-2h(1-2x)+h^2\}^{\frac{3}{2}}},$$

where h is less than unity but nearly equal to it, the limiting values of y as h approaches unity, will give the geometrical interpretation of the transient function V_0.



Take (Fig. 1.) $AB = 1$, $AH = 1 - h$, or $BH = h$ both along the axis of x, and make A the origin, then putting $x = 0$, we have $y = \dfrac{1+h}{(1-h)^2}$, which is very great, and tends to be infinite as h approaches unity, and is represented by AC; next putting $x = 1 - h = AH$, we get the corresponding ordinate $HE = \dfrac{1}{(1-h^{\frac{1}{2}})} \cdot \dfrac{1+h}{(1+3h)^{\frac{3}{2}}}$, which also tends to infinity; lastly, putting $x = 1$ we have $y = \dfrac{1-h}{(1+h)^{\frac{1}{2}}} = BD$, which tends to vanish in the ultimate case representing V_0.

Now varying the parameter h so as to make it approach unity, the points C and E recede indefinitely from the axis of x, and the point D approaches it indefinitely.

Yet the area $DBACE$ remains constant (for the integral between $x = 0$, and $x = 1$ of

$$\frac{(1-h)(1+h)}{\{1 - 2h(1-2x) + h^2\}^{\frac{3}{2}}}$$

relative to x is evidently unity).

And the altitude GN of the centre of gravity of this area is also constant, for

$$\tfrac{1}{2}\textstyle\int_x y^2 = \tfrac{1}{2}\displaystyle\int_x \frac{(1-h)^2(1+h)^2}{\{1-2h(1-2x)+h^3\}^3} = \frac{1}{16h}\{(1+h)^2 - (1-h)^2\} = \tfrac{1}{4},$$

and therefore is the same as that of the parallelogram HF, when $AF = \frac{1}{2}AB$, for the distance Gg from the axis of y

$$= \int_x \frac{x(1-h)(1+h)}{\{1-2h(1-2x)+h^2\}^{\frac{3}{2}}} = \frac{1-h}{2} = \frac{AH}{2}.$$

Hence G tends ultimately to the point g in the axis of y, which shews that the area $DBH'E'$ tends absolutely to vanish, $H'E'$ being an ordinate drawn near the origin at any small distance not varying with the parameter h, and since $\frac{dy}{dx}$ has the same sign in the interval from B to H', H or A, it is evident that the portion of the curve BE' tends to coincide with the axis BH', the curve therefore which represents V_0 coincides with AB, except infinitely near the origin A, when it suddenly mounts to an infinite height.

Since the general function V_n is reciprocal to t^n, it follows that $\int_t V_0 t^n = 0$, except when $n = 0$, and then the definite integral is unity; hence if $f(t)$ be any function containing only the positive and integer powers of t, the transient function V_0 possesses the remarkable property expressed by the equation $\int_t V_0 . f(t) = f(0)$.

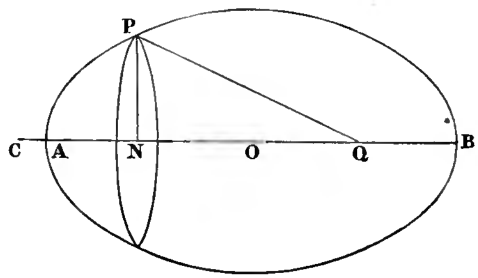


Fig. 2. Let $2a = AB$, equal the length of the axis in a solid of revolution, the surface of which is covered with an indefinitely thin stratum of fluid, let any abscissa ON measured from the centre O be put equal to $a(1-2t)$, the limits of t will evidently be O and 1.

Let the law of density or accumulation at any point P of a section perpendicular to the axis be expressed by the transient function λV_0, λ being constant, and let the total action of the fluid on any point Q in the axis be required, the law of force being capable of expansion according to the positive and integer powers of t.

Put $PN = y$, then the whole quantity E of fluid is manifestly equal to $2\lambda\pi \int_t V_0 y \frac{ds}{dt}$, s representing the arc AP.

Now it is easily seen that the value of $y\frac{ds}{dt}$ at the point A where y vanishes is $4aR$, R being the radius of curvature at that point, and by the nature of V_0 this quantity is the value of the above integral, or $E = 8\lambda\pi a R$.

Again, if we represent the distance PQ by r, and the law of force by $f(r)$ and put $AQ=k$ the initial value of r, the total action is

$$2\lambda\pi \int_t V_0 y \frac{ds}{dt} \cdot f(r) \cdot \frac{QN}{r},$$

which by the property of V_0 is equal to $8\lambda\pi a R f(k)$, or to $E \cdot f(k)$.

Let us now suppose an equal quantity of fluid, but of a contrary nature in its action, and therefore represented by $-E$ to be collected in a single point C in the axis produced to a small distance $AC=\alpha$.

The total action of the compound system on Q will then be

$$E\{f(k) - f(k+\alpha)\},$$

which tends to vanish as C approaches A.

Lastly, suppose a unit of fluid when distributed over the surface according to a law expressed by $\phi(t)$, which depends on the figure of the solid, will exert no action on any point Q in the axis; then if the law of distribution of the fluid be expressed by $\lambda V_0 + c\phi(t)$, the total action on Q including that of C, will be still $E\{f(k) - f(k+\alpha)\}$.

From which it follows that when an electrical spark $-E$ is infinitely near to the vertex of a conducting solid of revolution charged with a quantity of electricity E', the distribution of the latter under the influence of the former is expressed by the law

$$\lambda V_0 + c\phi(t) \text{ where } \lambda = \frac{E}{8\pi a R},$$

and where c is determined by the equation*

$$2\pi c \int_t \phi(t) \cdot y\frac{ds}{dt} = E' - E.$$

Having thus given the geometrical and physical interpretations of V_0, it will not be necessary to discuss the transient functions V_1, V_2, &c., of which the properties are very analogous.

* Vide First Memoir, Art. 35, the expression there obtained for a sphere being included in that obtained above, when the influencing point is infinitely near the sphere.

28. *To find the quantity to which* V_n *is the generating function.*

By Art. 26.

$$V_n = \frac{d^{2n}(Rh^n)}{1.2.3\ldots 2n\,dh^{2n}} + \frac{2h\,d^{2n+1}(Rh^n)}{1.2.3\ldots(2n+1)\,dh^{2n+1}},$$

where $R = \{1 - 2h(1-2t) + h^2\}^{-\frac{1}{2}}$, and h is ultimately equal to 1.

Forming the equation $u = h + ku^{\frac{1}{2}}$, we have

$$f(u) = f(h) + k\,.\,f'(h)\,.\,h^{\frac{1}{2}} + \frac{k^2}{1.2}\cdot\frac{d\{f'(h)\,.\,h\}}{dh} + \frac{k^3}{1.2.3}\cdot\frac{d^2\{f'(h)\,.\,h^{\frac{3}{2}}\}}{dh^2} + \&c.$$

$$\therefore f'(u)\,.\,\frac{du}{dh} = f'(h) + k\,\frac{d\{f'(h)\,.\,h^{\frac{1}{2}}\}}{dh}$$

$$+ \frac{k^2}{1.2}\cdot\frac{d^2\{f'(h)\,h\}}{dh^2} + \frac{k^3}{1.2.3}\cdot\frac{d^3\{f'(h)\,.\,h^{\frac{3}{2}}\}}{dh^3} + \&c.;$$

hence $\{1 - 2u(1-2t) + u^2\}^{-\frac{1}{2}}\,.\,k\,\dfrac{du}{dh} = kR + k^2\dfrac{d(Rh^{\frac{1}{2}})}{dh} + \dfrac{k^3}{1.2}\,\dfrac{d^2.(Rh)}{dh^2} + \&c.;$

therefore, $\dfrac{d^{2n}(Rh^n)}{1.2\ldots 2n\,dh^{2n}}$ = the coefficient of k^{2n+1} in $\dfrac{k\dfrac{du}{dh}}{\{1-2u(1-2t)+u^2\}^{\frac{1}{2}}}$.

In like manner,

$$u^{-\frac{1}{2}}\{1 - 2u(1-2t) + u^2\}^{-\frac{1}{2}}\,.\,\frac{du}{dh} = Rh^{-\frac{1}{2}} + k\,\frac{d(R)}{dh}$$

$$+ \frac{k^2}{1.2}\cdot\frac{d^2.(R\,h^{\frac{1}{2}})}{dh^2} + \frac{k^3}{1.2.3}\cdot\frac{d^3(Rh)}{dh^3}\ \&c.;$$

therefore,

$\dfrac{2h\,d^{2n+1}(Rh^n)}{1.2\ldots(2n+1)\,dh^{2n+1}}$ = the coefficient of k^{2n+1} in $\dfrac{2hu^{-\frac{1}{2}}\,.\,\dfrac{du}{dh}}{\{1-2u(1-2t)+u^2\}^{\frac{1}{2}}}$;

consequently, V_n = the coefficient of k^{2n+1} in $\dfrac{(k+2hu^{-\frac{1}{2}})\dfrac{du}{dh}}{\{1-2u(1-2t)+u^2\}^{\frac{1}{2}}}$.

Now, by the assumed equation we have

$$u^{\frac{1}{2}} - hu^{-\frac{1}{2}} = k,$$

and by differentiation $(u^{\frac{1}{2}} + hu^{-\frac{1}{2}})\dfrac{du}{dh} = 2u$;

but also $(u^{\frac{1}{2}} + hu^{-\frac{1}{2}}) = k + 2hu^{-\frac{1}{2}}$;

hence, $(k + 2hu^{-\frac{1}{2}})\dfrac{du}{dh} = 2u$;

and therefore, $V_n = 2$ the coefficient of k^{2n+1} in $u\{1-2u(1-2t)+u^2\}^{-\frac{1}{2}}$. from which it follows that if we form the two equations,

$$\left.\begin{aligned} u' &= h + (ku')^{\frac{1}{2}} \\ u'' &= h + (ku'')^{\frac{1}{2}} \end{aligned}\right\} \text{ putting } \left\{\begin{aligned} U' &= u'\{1-2u'(1-2t)+u'^2\}^{-\frac{1}{2}} \\ U'' &= u''\{1-2u''(1-2t)+u''^2\}^{+\frac{1}{2}}; \end{aligned}\right.$$

$$\text{then } \frac{U' - U''}{k^{\frac{1}{2}}} = V_0 + V_1 k + V_2 k^2 + V_3 k^3 \text{ \&c. ad inf.}$$

supposing that in the left-hand member h is finally put equal to unity. It may be observed that the quantities u', u'' are the two roots of the equation $u'^2 - (2h+k)u' + h^2 = 0$.

29. *To expand a given function* ϕ(t), *in terms of the transient function* V_n.

Let the general term of the expansion be $A_n V_n$, then by the nature of reciprocal functions we have

$$\begin{aligned} \textstyle\int_t \phi(t) \,.\, t^n &= A_n \textstyle\int_t V_n t^n, \\ &= A_n \textstyle\int_t P_n t^n, \qquad \text{(Art. 24.)} \\ &= (-1)^n \,.\, A_n \frac{n\,.\,(n-1)\ldots\ldots 1}{(n+1)(n+2)\ldots\ldots(2n+1)}; \end{aligned}$$

$$\text{or } A_n = (-1)^n \,.\, \frac{(n+1)(n+2)\ldots\ldots(2n+1)}{1\,.\,2\ldots\ldots n} \,.\, \textstyle\int_t \phi(t)\,.\,t^n.$$

$$\text{Hence, } \phi(t) = V_0 \textstyle\int_t \phi(t) - \frac{2\,.\,3}{1} V_1 \textstyle\int_t \phi(t)\, t + \frac{3\,.\,4\,.\,5}{1\,.\,2} \,.\, V_2 \textstyle\int_t \phi(t)\,.\,t^2 - \text{\&c.}$$

EXAMPLES:

$$t^n = \frac{1}{n+1} \cdot V_0 - \frac{2 \cdot 3}{1 \cdot (n+2)} \cdot V_1 + \frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot (n+3)} \cdot V_2 - \&c.$$

$$\{\text{h. l. } (t)\}^n = (-1)^n \cdot 1 \cdot 2 \cdot 3 \ldots n \left\{V_0 - \frac{2 \cdot 3}{1 \cdot 2^{n+1}} V_1 + \frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3^{n+1}} \cdot V_2 - \&c.\right\}$$

$$P_n = V_n - \frac{2n+3}{1} \cdot V_{n+1} + \frac{(2n+4)(2n+5)}{1 \cdot 2} \cdot V_{n+2} - \frac{(2n+5)(2n+6)(2n+7)}{1 \cdot 2 \cdot 3} \cdot V_{n+3} \; \&c.$$

the latter series would also result by reverting the series for V_n, in Art. 24.

30. *To find a function* U_n *which shall be reciprocal to* $(h.\, l.\, \mathrm{t})^n$.

Following the steps indicated in Art. 23, we must first form a self-reciprocal function of which the general term is a constant multiplied by $(\text{h. l. } t)^n$; this has been already effected in Sect. v, namely,

$$T_n = 1 + n \text{ h. l. } t + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{(\text{h. l. } t)^2}{1 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \cdot \frac{(\text{h. l. } t)^3}{1 \cdot 2 \cdot 3} + \&c.,$$

and then the form of the required function will be

$$U_n = T_n + a\,T_{n+1} + b\,T_{n+2} + c\,T_{n+3} + \&c.$$

Multiply by $(\text{h. l. } t)^m$, supposing $m > n$, and observing that

$$\textstyle\int_t T_n (\text{h. l. } t)^m = 1 \cdot 2 \cdot 3 \ldots m \cdot (-1)^n \cdot \frac{m \cdot (m-1)(m-2) \ldots (m-n+1)}{1 \cdot 2 \cdot 3 \ldots\ldots n} \text{ by Sect. v,}$$

$$\text{and } \textstyle\int_t U_n (\text{h. l. } t)^m = 0$$

by the nature of reciprocal functions, we get the general identity

$$0 = \frac{m(m-1)(m-2)\ldots(m-n+1)}{1 \cdot 2 \cdot 3 \ldots n} \cdot \left\{1 - a \cdot \frac{m-n}{n+1} + b \cdot \frac{(m-n)(m-n-1)}{(n+1)(n+2)} - \&c.\right\};$$

but on the same supposition that m is greater than n, we also have

$$0 = (1-1)^{m-n} = 1 - (m-n) + \frac{(m-n)(m-n-1)}{1 \cdot 2} - \&c.;$$

and by comparing the corresponding terms

$$a = \frac{n+1}{1}, \quad b = \frac{(n+1)(n+2)}{1 \cdot 2}, \quad c = \frac{(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3}, \text{ \&c.};$$

therefore,

$$U_n = T_n + \frac{n+1}{1} \cdot T_{n+1} + \frac{(n+1)(n+2)}{1 \cdot 2} \cdot T_{n+2} + \frac{(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3} \cdot T_{n+3}, \text{ \&c.}$$

31. *To express the function* $\mathrm{U_n}$ *which is reciprocal to* $(h.\ l.\ \mathrm{t})^n$ *in a finite form, and also the function which* $\mathrm{U_n}$ *generates.*

$$1 \cdot 2 \cdot 3 \ldots n\, U_n = 1 \cdot 2 \cdot 3 \ldots n\, T_n + 2 \cdot 3 \cdot 4 \ldots (n+1)\, T_{n+1} + 3 \cdot 4 \cdot 5 \ldots (n+2) \cdot T_{n+2} + \text{\&c.}$$

$$= \frac{d^n}{dh^n} \{ T_0 + T_1 h + T_2 h^2 + \ldots T_n h^n + T_{n+1} h^{n+1} + \text{\&c.} \},$$

h being put equal to unity after the differentiation.

But by Section v, we have

$$\frac{t^{\frac{h}{1-h}}}{1-h} = T_0 + T_1 h + T_2 h^2 + \text{\&c. ad inf.};$$

$$\text{therefore, } U_n = \frac{d^n \cdot \left(\dfrac{t^{\frac{h}{1-h}}}{1-h} \right)}{1 \cdot 2 \cdot 3 \ldots n\, dh^n} \text{ when } h = 1.$$

Now by Taylor's Theorem, this quantity is the coefficient of k^n in the expansion of $\dfrac{t^{\frac{h+k}{1-h-k}}}{1-h-k}$, the latter is therefore the function which U_n generates.

32. *Properties of* $\mathrm{U_n}$.

I. $\int_t U_n (\mathrm{h.\ l.}\ t)^n = \int_t T_n (\mathrm{h.\ l.}\ t)^n = 1 \cdot 2 \cdot 3 \ldots n$, by Sect. v.

II. Changing the sign of k in the quantity which U_n generates, we get

$$\frac{1}{k} t^{\frac{1}{k}-1} = U_0 - U_1 k + U_2 k^2 - U_3 k^3 + \&c.,$$

$$\text{or,} \quad t^x = t \left\{ \frac{U_0}{x} - \frac{U_1}{x^2} + \frac{U_2}{x^3} - \&c. \right\}.$$

III. Since $t^x = 1 + x \text{ h. l. } t + \frac{x^2}{1.2} . (\text{h. l. } t)^2 + \frac{x^3}{1.2.3} . (\text{h. l. } t)^3 + \&c.$

$$\therefore \int_t U_n t^x = \frac{x^n}{1.2.3 \ldots n} \int_t U_n . (\text{h. l. } t)^n$$

$$= x^n;$$

by means therefore of a single integral, x^n may be adapted to general differentiation.

As this result is remarkable, we may confirm it by the general rule in the *First Memoir.* (Vide Sect. I.) Thus,

$$\text{put } \phi(x) = \frac{1}{1-kx} = 1 + xk + x^2 k^2 + \&c.$$

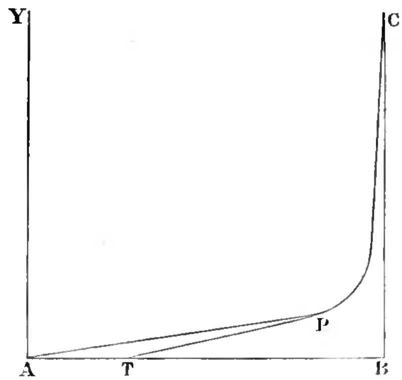
$$\text{then } f(t) = \frac{t^{\frac{k+1}{-k}}}{-k} = U_0 + U_1 k + U_2 k^2 \&c.;$$

$$\therefore \int_t U_0 t^x + k \int_t U_1 t^x + k^2 \int_t U_2 t^x + \&c. = 1 + kx + k^2 x^2 + \&c.$$

for all values of k, whence $\int_t U_n t^x = x^n$ as before.

33. *Discussion of the function* U_0.

By Art. 31. $U_0 = \frac{t^{\frac{h}{1-h}}}{1-h}$ when h is put equal to unity. Like the transient function V_0, discussed in Art. 27., the quantity U_0 is always zero for values of t between 0 and 1; but when $t = 1$ its value is infinite, and thence its integral between the limits 0 and 1 of t is finite, viz. unity.



To prove this property, conceive a curve APC, of which the abscissa measured from A along AB is taken equal to t, and the corresponding ordinate y is equal to $t^{\frac{1}{1-h}}$, and let us suppose h very nearly equal to unity, and at any point P draw a tangent PT; then since

$$y = t^{\frac{1}{1-h}}, \qquad \frac{dy}{dt} = \frac{t^{\frac{h}{1-h}}}{1-h};$$

therefore, U_0 is the limiting value of the tangent of the angle PTB.

Take $AB = 1$ and the ordinate $BC = 1$, then it is evident that A and C are constantly points of the curve when the parameter h varies so as to approach unity.

Again, for the entire area $APCB$ the expression is $\int_t t^{\frac{1}{1-h}}$, from $t=0$ to $t=1$, that is, $\dfrac{1-h}{2-h}$, which evidently tends to vanish as the parameter h approaches unity; and as no part of the area is negative, it follows that the curve APC tends ultimately to coincide with the two right lines AB, BC, and therefore when T is sensibly distant from B the tangent of the angle PTB tends to vanish, but when indefinitely near to B it tends to infinity; and therefore U_0, which ultimately represents these tangents, is zero from A to indefinitely near to B where t is unity, when its value becomes infinite.

In like manner the remaining functions U_1, U_2, &c. may be discussed with similar results.

It may be observed that for values of $t > 1$ (which however do not enter the definite integral), the values of U_0 are infinite.

34. *Expansion of given functions in terms of the functions* U_n.

The general formula for this purpose is

$$\phi(t) = U_0 \int_t \phi(t) + U_1 \int_t \phi(t) \,.\, \text{h. l.}\,(t)$$
$$+ \frac{U_2}{1\,.\,2} \int_t \phi(t) \,.\, (\text{h. l. } t)^2 + \frac{U_3}{1\,.\,2\,.\,3} \,.\, \int_t \phi(t) \,.\, (\text{h. l. } t)^3 + \&c.$$

$$\text{Thus } t^n = \frac{U_0}{n+1} - \frac{1}{(n+1)^2} \cdot U_1 + \frac{1}{(n+1)^3} \cdot U_2 - \&c.$$

$$(\text{h. l. } t)^n = (-1)^n \cdot 1 \cdot 2 \ldots n \left\{ U_0 - \frac{n+1}{1} \cdot U_1 + \frac{(n+1)(n+2)}{1 \cdot 2} \cdot U_2 - \&c. \right\}$$

$$T_n = U_n - \frac{n+1}{1} \cdot U_{n+1} + \frac{(n+1)(n+2)}{1 \cdot 2} \cdot U_{n+2} - \&c.$$

which latter series is also produced by reverting to that which expresses U_n in terms of T_n in Art. 30.

35. *To find a function reciprocal to* t^n *when the limits of* t *are* 0, *and* ∞.

Let u_n be the required function, and put $\tau = \epsilon^{-t}$,

$$\text{then } \textstyle\int_t u_n t^m = 0, \quad \text{from } t = 0 \text{ to } t = \infty;$$

$$\text{therefore } \int_\tau \frac{u_n}{\tau} (\text{h. l. } \tau)^m = 0, \quad \text{from } \tau = 0 \text{ to } \tau = 1,$$

$$\text{hence } u_n = \tau U_n = \tau \frac{d^n \cdot \dfrac{\tau^{\frac{h}{1-h}}}{1-h}}{1 \cdot 2 \ldots n dh^n} \text{ when } h = 1$$

$$= \frac{d^n \left\{ \dfrac{\epsilon^{\frac{-t}{1-h}}}{1-h} \right\}}{1 . 2 .. n dh^n}.$$

36. *To find a function* F_n *which shall be reciprocal to* $\cos^n \phi$, *the limits of* ϕ *being* $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$.

Following similar steps to those adopted in the preceding Articles we shall obtain,

$$\text{in cosines } F_n = \cos n\phi - \frac{n+2}{1} \cdot \cos (n+2)\phi$$

$$+ \frac{(n+1)(n+4)}{1 \cdot 2} \cdot \cos (n+4)\phi - \frac{(n+1)(n+2)(n+6)}{1 \cdot 2 \cdot 3} \cos (n+6)\phi, \ \&c.$$

in sines $F_n = 2 \sin \phi \{\sin (n+1)\phi - \frac{n+1}{1} . \sin (n+3)\phi$

$+ \frac{(n+1)(n+2)}{1.2} . \sin (n+5)\phi - \&c.\}$

37. *The function* $\mathrm{F_n}$ *is transient.*

Either of the preceding values of F_n give $F_n = F_n' - F_n''$, where

$$F_n' = \cos (n\phi) - \frac{n+1}{1} . \cos (n+2) . \phi + \frac{(n+1)(n+2)}{1.2} . \cos (n+4)\phi + \&c.$$

$$F_n'' = \cos (n+2)\phi - \frac{n+1}{1} . \cos (n+4) . \phi + \frac{(n+1)(n+2)}{1.2} . \cos (n+6)\phi + \&c.$$

passing from trigonometrical to exponential values,

$$2F_n' = \epsilon^{n\phi\sqrt{-1}} - \frac{n+1}{1} . \epsilon^{(n+2)\phi\sqrt{-1}} + \frac{(n+1)(n+2)}{1.2} . \epsilon^{(n+4)\phi\sqrt{-1}} - \&c.$$

$$+ \epsilon^{-n\phi\sqrt{-1}} - \frac{n+1}{1} . \epsilon^{-(n+2)\phi\sqrt{-1}} + \frac{(n+1)(n+2)}{1.2} . \epsilon^{-(n+4)\phi\sqrt{-1}} - \&c.$$

$$= \epsilon^{n\phi\sqrt{-1}}(1 + \epsilon^{2\phi\sqrt{-1}})^{-(n+1)} + \epsilon^{-n\phi\sqrt{-1}}(1 + \epsilon^{-2\phi\sqrt{-1}})^{-(n+1)}$$

$$= \epsilon^{-\phi\sqrt{-1}}(\epsilon^{\phi\sqrt{-1}} + \epsilon^{-\phi\sqrt{-1}})^{-(n+1)} + \epsilon^{\phi\sqrt{-1}}(\epsilon^{\phi\sqrt{-1}} + \epsilon^{-\phi\sqrt{-1}})^{-(n+1)}$$

$$= (\epsilon^{\phi\sqrt{-1}} + \epsilon^{-\phi\sqrt{-1}})^{-n}$$

$$= 2\cos n\phi,$$

$$2F_n'' = \epsilon^{(n+2)\phi\sqrt{-1}} - \frac{n+1}{1} . \epsilon^{(n+4)\phi\sqrt{-1}} + \frac{(n+1)(n+2)}{1.2} . \epsilon^{(n+6)\phi\sqrt{-1}} - \&c.$$

$$+ \epsilon^{-(n+2)\phi\sqrt{-1}} - \frac{n+1}{1} . \epsilon^{-(n+4)\phi\sqrt{-1}} + \frac{(n+1)(n+2)}{1.2} \epsilon^{-(n+4)\phi\sqrt{-1}} - \&c.$$

$$= \epsilon^{(n+2)\phi\sqrt{-1}}(1 + \epsilon^{2\phi\sqrt{-1}})^{-(n+1)} + \epsilon^{-(n+2)\phi\sqrt{-1}}(1 + \epsilon^{-2\phi\sqrt{-1}})^{-(n+1)}$$

$$= \epsilon^{\phi\sqrt{-1}}(\epsilon^{\phi\sqrt{-1}} + \epsilon^{-\phi\sqrt{-1}})^{-(n+1)} + \epsilon^{-\phi\sqrt{-1}}(\epsilon^{\phi\sqrt{-1}} + \epsilon^{-\phi\sqrt{-1}})^{-(n+1)}$$

$$= 2\cos n\phi,$$

hence $F_n = F_n' - F_n'' = 0.$

However, if n be even, and our limits be $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, the function becomes suddenly infinite at the limits, for the expansion of F_n is then identical with that of $(1-1)^{-(n+1)}$.

38. *To express in finite terms the transient function* F_n.

Put

$$F_n' = \cos n\phi - \frac{n+1}{1}\,.\,h\cos(n+2)\phi + \frac{(n+1)(n+2)}{1\,.\,2}\,.\,h^2\cos(n+4)\phi - \&c.$$

$$F_n'' = h\cos(n+2)\phi - \frac{n+1}{1}\,.\,h^2\cos(n+4)\,.\,\phi + \frac{(n+1)(n+2)}{1\,.\,2}\,h^3\cos(n+6)\phi - \&c.$$

Then F_n is the limit of $F_n' - F_n''$ when h approaches unity.

Put also $2\cos\phi = x + \frac{1}{x}$,

hence $2F_n'$

$$= x^n(1+hx^2)^{-(n+1)} + x^{-n}(1+hx^{-2})^{-(n+1)}$$

$$= \frac{x^{-1}}{(x^{-1}+hx)^{(n+1)}} + \frac{x}{(x+hx^{-1})^{n+1}}$$

$$= \frac{x(x^{-1}+hx)^{n+1} + x^{-1}(x+hx^{-1})^{n+1}}{\{1+h(x^2+x^{-2})+h^2\}^{n+1}}$$

$$= 2\,.\,\frac{\cos n\phi + \frac{n+1}{1}\,.\,h\cos(n-2)\phi + \frac{(n+1)\,.\,n}{1\,.\,2}\,.\,h^2\cos(n-4)\phi \ldots \frac{n+1}{1}\,.\,h^n\cos n\phi + h^{n+2}\cos(n+2)\phi}{\{1+2h\cos 2\phi + h^2\}^{n+1}},$$

the number of terms in the numerator being $n+2$.

In like manner,

$$\frac{2F_n''}{h} = \frac{x}{(x^{-1}+hx)^{n+1}} + \frac{x^{-1}}{(x+hx^{-1})^{n+1}}$$

$$= \frac{x(x+hx^{-1})^{n+1} + x^{-1}(x^{-1}+hx)^{n+1}}{\{1+h(x^2+x^{-2})+h^2\}^{n+1}}$$

$$=2.\frac{\cos(n+2)\phi+\frac{n+1}{1}.h\cos n\phi+\frac{(n+1)(n+2)}{1.2}.h^2\cos(n-2)\phi+\ldots\frac{n+1}{1}.h^n\cos(n-2)\phi+h^{n+1}\cos n\phi}{\{1+2h\cos 2\phi+h^2\}^{n+1}},$$

Hence,

$$F_n=\frac{\cos n\phi.(1-h^{n+2})+h\left\{\frac{n+1}{1}\cos(n-2)\phi-\cos(n+2)\phi\right\}+h^2\left\{\frac{(n+1).n}{1.2}\cos(n-4)\phi-\frac{n+1}{1}\cos n\phi\right\}}{\{1+2h\cos 2\phi+h^2\}^{n+1}},\&c.$$

when h is put $=1$.

Thus $F_0=\dfrac{(1-h)(1+h)}{\{1+2h\cos 2\phi+h^2\}}$,

which is evidently a transient function, as its general value for $h=1$ is zero, except ϕ is an odd multiple of $\frac{\pi}{2}$, when its value becomes infinite.

And in general F_n' and F_n'' are equal, when h is put equal to unity, and therefore F_n has a factor $1-h$ in its numerator, which causes its general vanishing state, except when $\phi=\frac{\pi}{2}$, or an odd multiple of $\frac{\pi}{2}$, when the denominator becomes $(1-h)^{2n+2}$, and as the numerator is of only $n+2$ dimensions, it is evident F_n in this case is infinite, when $h=1$.

In general $\displaystyle\int_\phi \frac{(1-h)(1+h)}{1+2h\cos 2\phi+h^2}=2\tan^{-1}.\left\{\frac{1-h}{1+h}.\tan\phi\right\}+\text{const.},$

which taken from $\phi=0$ to $\phi=\frac{\pi}{2}$ is equal to π, a quantity independent of h, a result similar to those already obtained from other transient functions.

39. When the sum of a series containing transient functions is required, the following process, with only such modifications as may simplify particular cases, will apply.

Let $S = a_0 V_0 + a_1 V_1 . z + a_2 V_2 . z^2 + \ldots + a_x V_x z^x + \&c.$

be the series proposed.

By the inverse method, put $a_x = \int_\tau f(\tau) . \tau^x$ from $\tau = 0$ to $\tau = 1$.

Then $S = \int_\tau f(\tau) \{V_0 + V_1 \tau z + V_2 \tau^2 z^2 + \&c.\}$

But $V_0 + V_1 k + V_2 k^2$, &c. is the function which V_n generates, and may be represented by $\phi(t, k)$, we have then

$$S = \int_\tau f(\tau) . \phi(t, \tau z), \text{ from } \tau = 0 \text{ to } \tau = 1.$$

SECTION VIII.

On the Resolution of Equations involving Definite Integrals.

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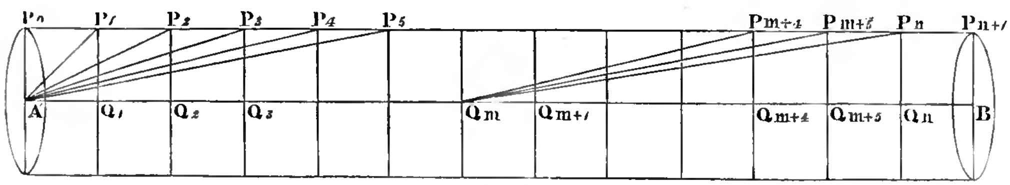
(1) *By the Decomposition of the Integrals into Elements.*

40. THE utility of the method of decomposition consists principally in the verifications it offers to results obtained by other analytical processes, the difficulty in the eliminations which it requires.



Suppose a cylindrical shell exerts no force on any point in its axis AB, the law of force tending to each particle of the shell being given, but the law of density of the shell unknown, then the application of the method of decomposition is this:

Divide the shell into $n+1$ equal portions by planes perpendicular to the axis P_1Q_1, P_2Q_2, &c.

Let the density throughout each portion be supposed uniform, and let the successive densities be ρ_0, ρ_1, $\rho_2 \ldots \rho_n$.

Let the total actions on the points of division Q_1, $Q_2 \ldots Q_n$ be equated to zero, which will give n equations, and another will be obtained by considering the mass of the shell.

From these $n+1$ equations, let ρ_0, ρ_1, ρ_2, &c. be determined in terms of n.

Finally, make n infinite.

41. *General Calculus for the Cylinder with any law of force.*

Let a_0, a_1, $a_2 \ldots a_n$ represent the total actions on the point A which would be exerted by the successive portions P_0P_1, P_1P_2 of the shell, if the density of each were unity; these quantities are given, since the law of force is supposed known.

Then $a_0\rho_0$, $a_1\rho_1$, $a_2\rho_2$, &c. represent the actual forces on A.

Again, the action of any portion as $P_{m+4}P_{m+5}$ on any point Q_m of division in the axis, will be to the action of the similarly situated portion P_4P_5 on the point A in the ratio of the corresponding densities, and in this case would be $a_4\rho_{m+4}$.

By this consideration the total actions on the points Q_1, $Q_2 \ldots Q_n$ are easily estimated, and equating each to zero, we get the following system of n equations, which serve to determine the ratios $\frac{\rho_1}{\rho_0}$, $\frac{\rho_2}{\rho_0}$, &c. viz.

$$a_0\rho_0 - a_0\rho_1 - a_1\rho_2 - a_2\rho_3 - a_3\rho_4 \ldots\ldots\ldots\ldots - a_{n-2}\rho_{n-1} - a_{n-1}\rho_n = 0,$$

$$a_1\rho_0 + a_0\rho_1 - a_0\rho_2 - a_1\rho_3 - a_2\rho_4 \ldots\ldots\ldots\ldots - a_{n-3}\rho_{n-1} - a_{n-2}\rho_n = 0,$$

$$a_2\rho_0 + a_1\rho_1 + a_0\rho_2 - a_0\rho_3 - a_1\rho_4 \ldots\ldots\ldots\ldots - a_{n-4}\rho_{n-1} - a_{n-3}\rho_n = 0,$$

$$a_3\rho_0 + a_2\rho_1 + a_1\rho_2 + a_0\rho_3 - a_0\rho_4 \ldots\ldots\ldots\ldots - a_{n-5}\rho_{n-1} - a_{n-4}\rho_n = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n-2}\rho_0 + a_{n-3}\rho_1 + a_{n-4}\rho_2 + a_{n-5}\rho_3 + a_{n-6}\rho_4 \ldots\ldots - a_2\rho_{n-1} - a_1\rho_n = 0,$$

$$a_{n-1}\rho_0 + a_{n-2}\rho_1 + a_{n-3}\rho_2 + a_{n-4}\rho_3 + a_{n-5}\rho_4 \ldots\ldots + a_1\rho_{n-1} - a_0\rho_n = 0.$$

Comparing the first equation with the n^{th}, the second with the $(n-1)^{\text{th}}$, &c. it is obvious that ρ_0 is involved in the same manner as ρ_n, ρ_1 as ρ_{n-1}, &c.

Hence, $\rho_n = \rho_0$, $\rho_{n-1} = \rho_1$, $\rho_{n-2} = \rho_2$, &c.

Form now two functions in the following manner:

a known function, $u = a_0 \sin\theta + a_1 \sin 3\theta + a_2 \sin 5\theta + \ldots a_{n-1} \sin(2n-1)\theta$,

an unknown, ... $S_n = \rho_0 \cos n\theta + \rho_1 \cos(n-2)\theta + \rho_2 \cos(n-4)\theta + \ldots\ldots$

the first series may be continued to n terms or infinity indifferently, and the last term in the second series will be $\frac{1}{2}\rho_{\frac{n}{2}}$ when n is even, and $\rho_{\frac{n-1}{2}} . \cos\theta$ when n is odd.

Suppose now that the product $2u . S_n$ is decomposed into the sines of the multiples of θ, and that all the multiples higher than the n^{th} are rejected from this product, the remaining part will evidently be,

$$-\{a_0\rho_0 - a_0\rho_1 - a_1\rho_2 \ldots\ldots\ldots\ldots - a_{n-1}\rho_0\} . \sin(n-1)\theta,$$
$$-\{a_1\rho_0 + a_0\rho_1 - a_0\rho_2 \ldots\ldots\ldots\ldots - a_{n-2}\rho_0\} . \sin(n-3)\theta,$$
$$-\{a_2\rho_0 + a_1\rho_1 + a_0\rho_2 \ldots\ldots\ldots\ldots - a_{n-3}\rho_0\} . \sin(n-5)\theta, \text{ \&c.}$$

the whole of which by the given equations is equal to zero.

Hence,

$$2S . u = A_n \sin(n+1)\theta + B_n \sin(n+3) . \theta + C_n \sin(n+5) . \theta, \text{ \&c.};$$
$$\therefore\ 4\cos\theta . S_n u = A_n \sin(n\theta) + (A_n + B_n)\sin(n+2)\theta + (B_n + C_n)\sin(n+4)\theta, \text{ \&c.}$$
$$\text{and } 2S_{n-1}u = A_{n-1}\sin(n\theta) + B_{n-1}\sin(n+2) . \theta + C_{n-1}\sin(n+4) . \theta;$$
$$\therefore\ 2\left\{2\cos\theta . S_n - \frac{A_n}{A_{n-1}} S_{n-1}\right\} . u = \left\{A_n + B_n - A_n . \frac{B_{n-1}}{A_{n-1}}\right\} . \sin(n+2) . \theta, \text{ \&c.}$$

Hence it follows that if we put $S_0 = \rho_0$, $S_1 = \rho_0 \cos\theta$,

and $u = a_0 \sin\theta + a_1 \sin 3\theta + a_2 \sin 5\theta$ &c. *ad inf.*, then,

First, Supposing S_{m-1} and S_m known, form a quantity λ_m by dividing the coefficient of $\sin(m+1)\theta$ in $2S_m u$, by the coefficient of $\sin(m\theta)$ in $2S_{m-1} . u$.

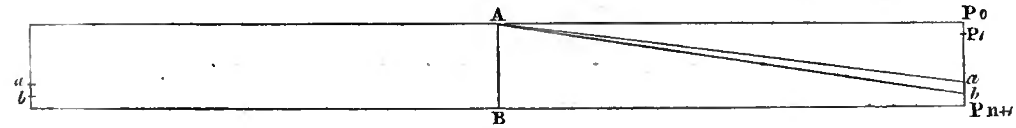
Secondly, Form a quantity S_{m+1}, by the equation

$$S_{m+1} = 2\cos\theta . S_m - \lambda_m S_{m-1},$$

by which S_2, $S_3 \ldots\ldots S_n$ may be successively formed.

Then it is obvious that the product $2S_n u$ contains no multiple of θ below the n^{th}, and therefore the coefficients in S_n must be the required quantities $\rho_0, \rho_1, \rho_2 \ldots\ldots \rho_{\frac{n-1}{2}}$ when n is odd, or $\rho_0, \rho_1, \rho_2 \ldots\ldots \frac{1}{2}\rho_{\frac{n}{2}}$ when n is even.

42. *Applications, when the law of force is the inverse square of the distance.*



(1) Let AB be the axis of a *very broad* cylindrical plate, the round side of which is covered with a fluid, attractive or repulsive, and so distributed as to exert no action on any point in the axis.

Put $AB=1$, $AP_0=a$ the radius of the base.

Let ab be one of the very small annuli into which the edge is divided, and put $aP_0=x$.

Then it is easy to prove that the action of the annulus ab on the point A is proportional to $\frac{1}{Aa}-\frac{1}{Ab}$, or ultimately to the differential coefficient of $\frac{1}{Aa}$ with respect to x, that is, to $\frac{x}{\{b^2+x^2\}^{\frac{3}{2}}}$, which quantity expanded is proportional to $x-\frac{3}{2}\frac{x^3}{b^2}+$ &c.; and as b is very great compared with x, we need only take the first term of this expansion.

In this case we may therefore put $a_0=1$, $a_1=2$, $a_2=3$, &c.,

$$\text{and therefore, } u=\sin\theta+2\sin 3\theta+3\sin 5\theta+4\sin 7\theta+\text{\&c.}$$

The calculus of S_n as indicated in the preceding article will be as follows:

$$S_0=\rho_0, \qquad S_1=\rho_0\cos\theta,$$

$$\left\{\lambda_1=\frac{\text{coefficient of }\sin 2\theta\text{ in }2S_1u}{\text{coefficient of }\sin\theta\text{ in }2S_0u}=\frac{3}{2}\right\},$$

$$S_2=2\cos\theta\,S_1-\lambda_1 S_0$$
$$=\rho_0\left\{\cos 2\theta-\tfrac{1}{2}\right\};$$

$$\left\{\lambda_2 = \frac{\text{coefficient of } \sin 3\theta \text{ in } 2S_2u}{\text{coefficient of } \sin 2\theta \text{ in } 2S_1u} = \frac{2}{3}\right\}$$

$$S_3 = 2\cos\theta S_2 - \lambda_2 S_1$$

$$= \rho_0 \{\cos 3\theta - \frac{2}{3}\cos\theta\},$$

$$\left\{\lambda_3 = \frac{\text{coefficient of } \sin 4\theta \text{ in } 2S_3u}{\text{coefficient of } \sin 3\theta \text{ in } 2S_2u} = \frac{5}{6}\right\}$$

$$S_4 = 2\cos\theta S_3 - \lambda_3 . S_2$$

$$= \rho_0 \left\{\cos 4\theta - \frac{2}{4}\cos 2\theta - \frac{1}{4}\right\},$$

$$\left\{\lambda_4 = \frac{\text{coefficient of } \sin 5\theta \text{ in } 2S_4u}{\text{coefficient of } \sin 4\theta \text{ in } 2S_3u} = \frac{9}{10}\right\}$$

$$S_5 = 2\cos\theta S_4 - \lambda_4 S_3$$

$$= \rho_0 \left\{\cos 5\theta - \frac{2}{5}\cos 3\theta - \frac{2}{5}\right\}.$$

Similarly, $$S_6 = \rho_0 \left\{\cos 6\theta - \frac{2}{6}\cos 4\theta - \frac{2}{6}\cos 2\theta - \frac{1}{6}\right\}$$

$$S_7 = \rho_0 \left\{\cos 7\theta - \frac{2}{7}\cos 5\theta - \frac{2}{7}\cos 3\theta - \frac{2}{7}\cos\theta\right\}.$$

Generally when n is an odd integer, suppose

$$\frac{S_{n-1}}{\rho_0} = \cos(n-1)\theta - \frac{2}{n-1}\{\cos(n-3)\theta + \cos(n-5)\theta + \ldots + \cos 2\theta + \tfrac{1}{2}\},$$

and $$\frac{S_n}{\rho_0} = \cos n\theta - \frac{2}{n}\{\cos(n-2)\theta + \cos(n-4)\theta + \ldots + \cos 3\theta + \cos\theta\}.$$

The coefficient of $\sin n\theta$ in $2S_{n-1}u = \dfrac{n+1}{n-1}\cdot\rho_0$,

.... of $\sin(n+1)\theta$ in $2S_n u = \dfrac{n+2}{n}\cdot\rho_0$;

therefore, $$\lambda_n = \frac{(n+2)(n-1)}{n(n+1)} = 1 - \frac{2}{n} + \frac{2}{n+1}.$$

Hence, $\dfrac{S_{n+1}}{\rho_0} = 2\cos\theta \,.\, \dfrac{S_n}{\rho_0} - \lambda_n \dfrac{S_{n-1}}{\rho_0}$

$$= \cos(n+1)\theta - \frac{2}{n+1}\{\cos(n-1)\theta + \cos(n-3)\theta + \ldots + \cos 2\theta + \tfrac{1}{2}\},$$

and by a repetition of the same process,

$$\frac{S_{n+2}}{\rho_0} = \cos(n+2)\theta - \frac{2}{n+2}\{\cos n\theta + \cos(n-2)\,.\,\theta + \ldots + \cos 3\theta + \cos\theta\}.$$

Hence the laws by which S_{n-1} and S_n are expressed are uniform, and therefore we get for the required unknown quantities,

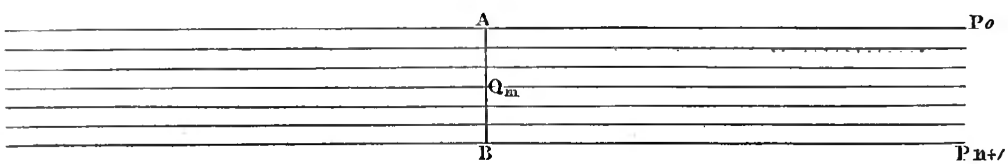
$$\rho_0 = \rho_0, \quad \rho_1 = -\frac{2}{n}\rho_0, \quad \rho_2 = -\frac{2}{n}\rho_0 \ldots\ldots \rho_{n-1} = -\frac{2}{n}\rho_0, \quad \rho_n = \rho_0.$$

The positive values may be taken for the repulsive and the negative for the attractive parts of the fluid, and if E denote the excess of the former, we have

$$E = 2\pi a\left\{\frac{\rho_0}{n} + \frac{\rho_1}{n} + \frac{\rho_2}{n} + \ldots + \frac{\rho_{n-1}}{n} + \frac{\rho_n}{n}\right\} = \frac{4\pi a \rho_0}{n}\left\{1 - \frac{n-1}{n}\right\} = \frac{4\pi a\rho_0}{n^2};$$

$\therefore\ \rho_0 = n^2 \,.\, \dfrac{E}{4\pi a}$, which gives the complete solution of the problem.

Thus the application of a process purely algebraical, conducts in this instance to a transient function, for if we suppose the final and equal densities ρ_0, ρ_n to be finite, all the intermediate values of the densities ρ_1, $\rho_2 \ldots\ldots \rho_{n-1}$ become indefinitely small when n is made infinite; yet they are not to be rejected, for if so, the total charge would be $4\pi a \dfrac{\rho_0}{n}$, whereas its actual value is only $4\pi a\, \dfrac{\rho_0}{n^2}$, an infinitesimal of the second order.



(2) Let AB be a right line perpendicular to the bounding planes, which terminate a solid composed of parallel strata of indefinite extent, but uniformly dense throughout that extent; and let the law of density of the different strata be such that there is no action on any point Q_m within.

Let the solid be decomposed into $n+1$ equal portions in which the densities are as before represented by $\rho_0, \rho_1, \rho_2 \ldots\ldots \rho_n$.

In this case the quantities $a_0, a_1, a_2 \ldots\ldots a_n$ are all equal, and putting them equal to unity, we have

$$u = \sin\theta + \sin 3\theta + \sin 5\theta + \&c.$$

$$S_0 = \rho_0, \qquad S_1 = \rho_0 \cos\theta, \qquad \lambda_1 = 1,$$

$$S_2 = 2\cos\theta \,.\, S_1 - \lambda_1 S_0 = \rho_0 \cos 2\theta, \qquad \lambda_2 = 1,$$

$$S_3 = 2\cos\theta \,.\, S_2 - \lambda_2 S_1 = \rho_0 \cos 3\theta, \qquad \lambda_3 = 1,$$

and generally, $S_n = \rho_0 \cos n\theta$, and $\lambda_n = 1$.

Hence the solution is $\rho_1 = 0,\ \rho_2 = 0 \ldots\ldots \rho_{n-1} = 0,\ \rho_n = \rho_0$.

And if E be the whole mass and A the area of the bounding planes, which is supposed very great, we have

$$E = 2A \,.\, \rho_0.$$

This result is analogous to the well-known fact, that electricity can reside only on the surfaces of bodies, and affords another instance of a transient function.

The method of decomposition may always be applied to obtain numerical approximations in cases which involve Definite Integrals; for instance, in the distribution of electricity on bodies, and in estimating the forces between bodies which are electrised.

(2) *By means of Reciprocal Functions.*

43. *Equations which contain only one definite integral.*

Let $f(t, a)$ be a function involving a variable t, and an arbitrary parameter a; $F(a)$ a function containing a only, and $\phi(t)$ a function containing t only, the first and second of these functions being given, it is required to find the third so as to satisfy the definite integral equation

$$\int_t \phi(t) \,.\, f(t, a) = F(a),$$

the limits of t being given.

Suppose $\phi(t)$ expanded according to any given class of self-reciprocal functions as P_n, that is,

$$\phi(t) = c_0 P_0 + c_1 P_1 + c_2 P_2 + c_3 P_3, \text{ \&c. } ad\ infinitum,$$

where the coefficients c_0, c_1, c_2, &c. are unknown.

Let $f(t, a)$ be expanded according to the same reciprocal functions,

$$f(t, a) = A_0 P_0 + A_1 P_1 + A_2 P_2 + A_3 P_3, \text{ \&c. } ad\ infinitum.$$

Then $\int_t P_m P_n = 0$, and $\int_t P_n P_n = \alpha_n$ a known numerical quantity dependant on n, and on the particular species of reciprocal functions which are employed.

Multiply both series and integrate between the given limits of t, and the proposed equation gives us

$$F(a) = A_0 \alpha_0 \,.\, c_0 + A_1 \alpha_1 \,.\, c_1 + A_2 \alpha_2 \,.\, c_2 + A_3 \alpha_3 \,.\, c_3, \text{ \&c. } ad\ infinitum.$$

Now A_n being a known function of a and n, we can by Art. 23. Sect. VII., find another function of a and n, as A_n' such that $\int_t A_n A_m' = 0$, when m and n are unequal integers.

Multiply the equation successively by A_0', A_1', A_2', &c. and take the definite integrals relative to a, hence

$$\int_a A_0' F(a) = c_0 \alpha_0 \int_a A_0 A_0'; \qquad \therefore\ c_0 = \frac{\int_a A_0' \,.\, F(a)}{\alpha_0 \,.\, \int_a A_0' A_0},$$

$$\int_a A_1' F(a) = c_1 \alpha_1 \int_a A_1 A_1'; \qquad \therefore\ c_1 = \frac{\int_a A_1' F(a)}{\alpha_1 \int_a A_1' A_1},$$

$$\text{and generally } c_n = \frac{\int_a \,.\, A_n' F(a)}{\alpha_n \,.\, \int_a A_n' A_n}.$$

$$\text{Hence } \phi(t) = \frac{P_0}{\alpha_0} \,.\, \frac{\int_a A_0' \,.\, F(a)}{\int_a A_0' A_0} + \frac{P_1}{\alpha_1} \,.\, \frac{\int_a A_1' F(a)}{\int_a A_1' A_1} + \frac{P_2}{\alpha_2} \,.\, \frac{\int_a \,.\, A_2' F(a)}{\int_a A_2' A_2} + \text{\&c.}$$

44. EXAMPLES.

In the following examples two things are to be observed. First, that the given functions are supposed to be continuous, and therefore the equation proposed must hold true for all values of the parameter a.

Secondly, In the final equation for determining the unknown coefficients, instead of using a reciprocal multiplier any means more simple may be occasionally employed.

Ex. 1. Given $\int_t \phi(t) . \cos(at) = 1$ to determine $\phi(t)$, the limits of t being 0 and π.

Put $\phi(t) = c_0 + c_1 \cos t + c_2 \cos(2t) + c_3 \cos(3t)$, &c. *ad infinitum*,

$$\text{and } \cos(at) = A_0 + A_1 \cos t + A_2 \cos(2t) + A_3 \cos(3t), \text{ \&c.,}$$

where to determine A_0, A_1, A_2, &c. we multiply successively by 1, $\cos t$, $\cos 2t$, &c., and integrate from $t = 0$ to $t = \pi$, whence

$$A_0 = \frac{\sin(a\pi)}{a\pi}, \qquad A_1 = -\frac{2a \sin a\pi}{\pi(a^2 - 1)}, \qquad A_2 = \frac{2a \sin a\pi}{\pi(a^2 - 2^2)},$$

$$\text{and generally } A_n = (-1)^n . \frac{2a \sin a\pi}{\pi(a^2 - n^2)} \text{ when } n > 0.$$

Multiply both series and integrate, and we get by the proposed equation,

$$1 = \sin a\pi \left\{\frac{c_0}{a} - \frac{a . c_1}{a^2 - 1} + \frac{ac_2}{a^2 - 2^2} - \frac{ac_3}{a^2 - 3^2} + \text{\&c.}\right\}$$

Put $a = 0, 1, 2, 3$, &c. successively, and we get

$$c_0 = \frac{1}{\pi}, \quad c_1 = \frac{2}{\pi}, \quad c_2 = \frac{3}{\pi}, \text{ \&c.}$$

$$\text{Hence } \pi\phi(t) = 1 + 2\cos t + 2\cos 2t + 2\cos 3t, \text{ \&c.}$$

The value of $\phi(t)$ is therefore the transient function $\frac{1}{\pi} . \frac{(1-h)(1+h)}{1 - 2h\cos t + h^2}$. (*Vide Art.* 38. *Function* F_0), when h is put equal to unity.

Ex. 2. Given $\int_t \phi(t) \,.\, \cos(at) = \cos(a\theta)$.

As before $\phi(t) = c_0 + c_1 \cos t + c_2 \cos 2t + c_3 \cos 3t + \&c.$

$$\cos at = \frac{\sin a\pi}{\pi}\left\{\frac{1}{a} - \frac{2a\cos t}{a^2-1} + \frac{2a\cos 2t}{a^2-2^2} - \frac{2a\cos 3t}{a^2-3^2} + \&c.\right\}$$

therefore $\cos a\theta = \sin(a\pi)\left\{\frac{c_0}{a} - \frac{ac_1}{a^2-1} + \frac{ac_2}{a^2-2^2} - \frac{ac_3}{a^2-3^2} + \&c.\right\}$

But also by reciprocal functions we get

$$\cos a\theta = \frac{\sin a\pi}{\pi}\left\{\frac{1}{a} - \frac{2a\cos\theta}{a^2-1} + \frac{2a\cos 2\theta}{a^2-2^2} - \frac{2a\cos 3\theta}{a^2-3^2} + \&c.\right\}$$

Hence $c_0 = \frac{1}{\pi}$, $\quad c_1 = \frac{2\cos\theta}{\pi}$, $\quad c_2 = \frac{2\cos 2\theta}{\pi}$, $\quad c_3 = \frac{2\cos 3\theta}{\pi}$, &c.

therefore $\pi\phi(t) = 1 + 2\cos\theta\cos t + 2\cos 2\theta\cos 2t + 2\cos 3\theta\cos 3t + \&c.$

$$\begin{aligned} \text{or } 2\pi\phi t &= 1 + 2\cos(\theta+t) + 2\cos 2(\theta+t) + 2\cos 3(\theta+t) + \&c. \\ &+ 1 + 2\cos(\theta-t) + 2\cos 2(\theta-t) + 2\cos 3(\theta-t) + \&c. \\ &= \frac{(1-h)(1+h)}{1-2h\cos(\theta+t)+h^2} + \frac{(1-h)(1+h)}{1-2h\cos(\theta-t)+h^2} \end{aligned}$$

when h is put equal to unity.

Ex. 3. Given $\int_t \phi(t) \,.\, \cos(at) = F(a)$.

$F(a)$ must be such (in continuous functions) as not to change when $-a$ is put for a, since $\cos(at)$ which is under the sign of integration will not then alter its value.

Proceeding as in the former examples we get

$$F(a) = \sin a\pi\left\{\frac{c_0}{a} - \frac{ac_1}{a^2-1} + \frac{ac_2}{a^2-2^2} - \frac{ac_3}{a^2-3^2} + \&c.\right\}$$

Put successively $a = 0, 1, 2, 3,$ &c. hence

$$c_0 = \frac{1}{\pi}\,.\,F(0), \qquad c_1 = \frac{2}{\pi}\,.\,F(1), \qquad c_2 = \frac{3}{\pi}\,.\,F(2), \ \&c.$$

hence $\pi\phi(t) = F(0) + 2F(1)\,.\,\cos t + 2F(2)\,.\,\cos 2t + 2F(3)\cos(3t) + \&c.$

Ex. 4. Given $\int_t \phi(t) \cdot \{f(a+t) + f(a-t)\} = F(a)$,

where the forms of the functions f and F are known, and that of ϕ required.

$$\text{Put } \phi(t) = c_0 + c_1 \cos t + c_2 \cos 2t + c_3 \cos 3t + \&\text{c.}$$

$$f(a) = \alpha_0 + \alpha_1 \cos a + \alpha_2 \cos 2a + \alpha_3 \cos 3a + \&\text{c.}$$

where α_0, α_1, α_2, &c. are known numerical quantities; hence

$$f(a+t) + f(a-t) = 2\alpha_0 + 2\alpha_1 \cos a \cos t + 2\alpha_2 \cos 2a \cos 2t + 2\alpha_3 \cos 3a \cos 3t + \&\text{c.}$$

$$\text{and } F(a) = 2\pi\alpha_0 c_0 + \pi\alpha_1 c_1 \cos a + \pi\alpha_2 c_2 \cos 2a + \pi\alpha_3 c_3 \cos 3a + \&\text{c.};$$

$$\text{therefore } c_0 = \frac{F(a)}{2\alpha_0\pi}, \quad c_1 = \frac{2}{\pi^2\alpha_1} \cdot \int_a F(a) \cdot \cos a, \quad c_2 = \frac{2}{\pi^2\alpha_2} \int_a F(a) \cos 2a, \ \&\text{c.}$$

$$\text{and } \pi\phi(t) = \frac{1}{\pi^2} \int_a F(a) \left\{ \frac{1}{2\alpha_0} + \frac{2\cos a \cos t}{\alpha_1} + \frac{2\cos 2a \cos 2t}{\alpha_2} + \&\text{c.} \right\}$$

the limits of all the integrals being 0 and π.

Ex. 5. $\displaystyle\int_t \frac{\phi(t)}{a-t} = \frac{1}{a-b}$.

In this case we shall employ the functions V_n reciprocal to t^n.

$$\text{Put } \phi(t) = c_0 V_0 + c_1 V_1 + c_2 V_2 + \&\text{c. } ad\ infinitum,$$

$$\text{and } \frac{1}{a-t} = \frac{1}{a} + \frac{t}{a^2} + \frac{t^2}{a^3} + \&\text{c. } ad\ infinitum;$$

$$\text{therefore } \frac{1}{a-b} = \frac{c_0}{a} - \frac{1}{2 \cdot 3} \cdot \frac{c_1}{a^2} + \frac{2 \cdot 1}{3 \cdot 4 \cdot 5} \cdot \frac{c_2}{a^3} - \frac{3 \cdot 2 \cdot 1}{4 \cdot 5 \cdot 6 \cdot 7} \cdot \frac{c_3}{a^4} + \&\text{c.}$$

$$= \frac{1}{a} + \frac{b}{a^2} + \frac{b^2}{a^3} + \frac{b^3}{a^4}, \ \&\text{c.}$$

$$\text{Hence } c_0 = 1, \quad c_1 = -\frac{2 \cdot 3}{1} \cdot b, \quad c_2 = \frac{3 \cdot 4 \cdot 5}{1 \cdot 2} \cdot b^2, \quad c_3 = \frac{4 \cdot 5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3} \cdot b^3, \ \&\text{c.}$$

$$\text{and } \phi(t) = V_0 - \frac{2 \cdot 3}{1} \cdot b V_1 + \frac{3 \cdot 4 \cdot 5}{1 \cdot 2} \cdot b^2 V_2 - \frac{4 \cdot 5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3} \cdot b^3 V_3, \ \&\text{c.}$$

$$= V_0 - \frac{3}{2} \cdot V_1 \cdot (4b) + \frac{3 \cdot 5}{2 \cdot 4} \cdot V_2 (4b)^2 - \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6} \cdot V_3 (4b)^3 + \&\text{c.}$$

Put $F(k) = V_0 + V_1 k + V_2 k^2 + \&c.$ *ad infinitum*,
as found in Art. 28. Sect. VII.

$$\text{Hence } F(-k\tau^2) = V_0 - V_1 k\tau^2 + V_2 k\tau^4 - \&c.$$

$$\text{therefore } \int_\tau \frac{F(-k\tau^2)}{\sqrt{1-\tau^2}} = \frac{\pi}{2}\left\{V_0 - \tfrac{1}{2} V_1 k + \frac{1.3}{2.4} . V_2 k^2 - \frac{1.3.5}{2.4.6} . k^3, \&c.\right\}$$

the limits of τ being 0 and 1;

$$\therefore \frac{2b^{\frac{1}{2}}}{\pi} \int_\tau \frac{F(-4b\tau^2)}{\sqrt{1-\tau^2}} = V_0 b^{\frac{1}{2}} - \tfrac{1}{2} . V_1 . 4b^{\frac{3}{2}} + \frac{1.3}{2.4} . V_2 4^2 b^{\frac{5}{2}} - \frac{1.3.5}{2.4.6} . 4^3 b^{\frac{7}{2}} . V_3 + \&c.$$

$$\therefore \frac{4b^{\frac{1}{2}}}{\pi} . \int_\tau \frac{d}{db} . \frac{b^{\frac{1}{2}} F(-4b\tau^2)}{\sqrt{1-\tau^2}} = V_0 - \tfrac{3}{2} V_1 (4b) + \frac{3.5.7}{2.4.6} . V_2 (4b)^2, \&c.$$

$$= \phi(t).$$

Ex. 6. $\int_t \phi(t) . f(a-t) = f(a-b)$.

Denote by $P_{t,n}$ the reciprocal function P_n when t is the variable,
by $P_{b,n}$ .... when b is the variable.

$$\text{Let } f(a-t) = A_0 P_{t,0} + A_1 P_{t,1} + A_2 P_{t,2} + A_3 P_{t,3} + \&c.$$

$$\text{and } \phi(t) = c_0 P_{t,0} + c_1 P_{t,1} + c_2 P_{t,2} + c_3 . P_{t,3} + \&c.$$

$$\therefore f(a-b) = A_0 c_0 + \tfrac{1}{3} . A_1 c_1 + \tfrac{1}{5} . A_2 c_2 + \tfrac{1}{7} . A_3 c_3 + \&c.$$

but changing t into b in the expansion of $f(a-t)$ we get

$$f(a-b) = A_0 P_{b,0} + A_1 P_{b,1} + A_2 P_{b,2} + A_3 P_{b,3} + \&c.$$

which values are identical when $c_0 = P_{b,0}$, $c_1 = 3P_{b,1}$, $c_2 = 5P_{b,2}$, &c.

$$\text{therefore } \phi(t) = P_{b,0} P_{t,0} + 3P_{b,1} P_{t,1} + 5P_{b,2} P_{t,2} + 7P_{b,3} P_{t,3} + \&c.$$

45. *On the appendage necessary to complete the Solution of a Definite-integral Equation.*

In the examples in which $f(a, t) = \cos at$ given in the last article, the function $F(a)$ is adapted to general differentiation relative to a, under the definite integral; but besides the *prime* value thus obtained, there must be an appendage to represent the same operation on zero, which contains an infinite number of constants multiplied by functions of a, which may vanish or not, and be connected or unconnected according both to the nature of the particular operation and the nature of the calculus in which it is employed; this has been already shewn by Mr Peacock*, and in Art. 20. Sect. VI. of this Memoir. The same remark applies to the value of $\phi(t)$ in the general equation

$$\int_t \phi(t) f(t, a) = F(a),$$

to complete it we must add $\psi(t)$ where $\int_t \psi(t) . f(t, a) = 0$.

To obtain $\psi(t)$ in the equation $\int_t \phi(t) . \cos(at) = F(a)$ above mentioned.

Let us suppose $\phi_1(t)$, $\phi_2(t)$, found by the method of Art. 44., to satisfy the equations

$$\int_t \phi_1(t) . \cos(at) = 1 \text{ for continuity,}$$
$$\int_t \phi_2(t) . \sin(at) = 1 \text{ for discontinuity,}$$

differentiating with respect to a, the first $2n$ times, the second $2n-1$ times, we get

$$\int_t \phi_1(t) . t^{2n} \cos(at) = 0,$$
$$\int_t \phi_2(t) . t^{2n-1} \cos(at) = 0.$$

Hence,

$$\psi(t) = \phi_2(t)\{At + Bt^3 + Ct^5, \&c.\} + \phi_1(t)\{A't^2 + B't^4 + C't^6 \&c.\},$$

where A, B, C, &c. A', B', C', &c. are absolute constants.

When transient functions appear in the appendage or even in the *prime* solution, they must not be neglected (particularly in the molecular investigations) except they are inadmissible by the nature of the particular question, for they have a physical as well as a geometrical meaning, as they are capable of expressing in continuous analytical forms, the state of bodies and their mutual actions when they are composed of absolute mathematical centres of forces, all separated mutually by infinitesimal intervals.

* Third Vol. Report of British Assoc. p. 212, &c.

Thus let the ratio of the weight to the extent of an element P of a straight rod AB be expressed by the transient function

$$\frac{(1-h)(1+h)}{1-2h\cos(2n\phi)+h^2}, \text{ when } h=1;$$

and where $AP=\phi$, and the whole length $AB=\pi$, and n is very great and integer.

Then the whole weight is finite, viz. $\int_\phi \frac{(1-h)(1+h)}{1-2h\cos 2n\phi+h^2}=1$, yet this function has only an existence when $\phi=0$, $\frac{\pi}{n}$, $\frac{2\pi}{n}$, $\frac{4\pi}{n}$ ... &c., and therefore the rod is actually composed of disjoint particles Q_1, Q_2, Q_3, &c. which are separated by equal intervals, each infinitesimals, viz. $\frac{\pi}{n}$, when n is very great, and equal to the actual number of particles; the action of such a system on another given one, may always be estimated by using the transient function in its general form, and lastly, putting h equal unity.

46. *Equations which contain two or more Definite Integrals.*

Given, $\int_t \phi(t) \,.\, f(t, a, b) + \int_t \psi(t) \,.\, F(t, a, b) = E(a, b)$,

the forms of the functions f, F, E being known, the forms of ϕ and ψ are required.

Put $f(t, a, b) = A_0P_0 + A_1P_1 + A_2P_2 + A_3P_3 +$ &c. *ad inf.*

where A_0, A_1, A_2, &c. are known functions of a and b, and P_0, P_1, &c. any self-reciprocal functions of t, such that $\int_t P_n^2 = \alpha_n$, which will be a known numerical quantity.

Similarly, $F(t, a, b) = B_0P_0 + B_1P_1 + B_2P_2 + B_3P_3 +$ &c. *ad inf.*,

where B_0, B_1, B_2, &c. are known functions of a and b.

Again, let $\phi(t) = c_0P_0 + c_1P_1 + c_2P_2 + c_3P_3$, &c. *ad inf.*

where c_0, c_1, c_2, &c. are unknown numerical quantities,

and $\psi(t) = e_0P_0 + e_1P_1 + e_2P_2 + e_3P_3$, &c. *ad inf.*, where e_0, e_1, e_2, &c. are also unknown.

The proposed equation then becomes

$$E(a, b) = \left\{ \begin{array}{l} c_0 a_0 A_0 + c_1 a_1 A_1 + c_2 a_2 A_2 + \&c. \\ + e_0 a_0 B_0 + e_1 a_1 B_1 + e_2 a_2 B_2 + \&c. \end{array} \right\}.$$

Now to the function A_n there may be found a function $\overline{A}_n$ reciprocal relative to a, and to .... B_n .... $\overline{B}_n$ .... b.

Let $\int_a \overline{A}_0 B_n = U_n$ a function of b only,

$\int_b \overline{B}_0 A_n = V_n$ .... a only.

Hence, $\int_a \overline{A}_0 E(a, b) - c_0 a_0 \int_a A_0 A_0 = e_0 a_0 U_0 + e_1 a_1 U_1 + e_2 a_2 U_2 + \&c.$ *ad inf.*

$\int_b \overline{B}_0 E(a, b) - e_0 a_0 \int_b B_0 B_0 = c_0 a_0 V_0 + c_1 a_1 V_1 + c_2 a_2 V_2 + \&c.$ *ad inf.*

Let $\overline{U}_n$ be the function of b, which is reciprocal to U_n,

$\overline{V}_n$ .... of a, ... V_n.

Hence,
$$\left\{ \begin{array}{l} \int_a \int_b (\overline{A}_0 \overline{U}_0 E(a, b) - c_0 a_0 A_0 \overline{A}_0 \overline{U}_0) = e_0 a_0 \int_b U_0 \overline{U}_0 \\ \int_a \int_b (\overline{B}_0 \overline{V}_0 E(a, b) - e_0 a_0 B_0 \overline{B}_0 \overline{V}_0) = c_0 a_0 \int_a V_0 \overline{V}_0 \end{array} \right\},$$

by which equations the constants c_0, e_0 are immediately determined.

Also,
$$\left\{ \begin{array}{l} \int_a \int_b (\overline{A}_0 \overline{U}_n E(a, b) - c_0 a_0 A_0 \overline{A}_0 \overline{U}_n) = e_n a_n \int_b U_n \overline{U}_n \\ \int_a \int_b (\overline{B}_0 \overline{V}_n E(a, b) - e_0 a_0 B_0 \overline{B}_0 \overline{V}_n) = c_n a_n \int_a V_n \overline{V}_n \end{array} \right\};$$

and since c_0, e_0 have been found, the latter equations determine generally the coefficients c_n, e_n, and therefore the required functions $\phi(t)$, $\psi(t)$ are known.

In like manner by employing reciprocal functions relative to double integration, we may solve equations containing three unknown functions, &c.

The problem of the distribution of electricity on bodies of which the surfaces are not continuous, introduces equations of this nature.

47. *Simultaneous Equations to Definite Integrals.*

$$\text{Given } \begin{Bmatrix} \int_t \phi(t) \cdot f(t, a) + \int_t \psi(t) \cdot F(t, a) = E(a) \\ \int_t \phi(t) f_1(t, a) + \int_t \psi(t) \cdot F_1(t, a) = E_1(a) \end{Bmatrix},$$

the forms of the functions f, F, E, f_1, F_1, E_1, being known, the forms of ϕ and ψ are required.

Multiply the second equation by an arbitrary quantity λ, and adding to the first, put

$$f(t, a) + \lambda f_1(t, a) = A_0 P_0 + A_1 P_1 + A_2 P_2 + \&c.$$

$$F(t, a) + \lambda F_1(t, a) = A_0 Q_0 + A_1 Q_1 + A_2 Q_2 + \&c.$$

$$\phi(t) = c_0 P_0' + c_1 P_1' + c_2 P_2' + \&c.$$

$$\psi(t) = e_0 Q_0' + e_1 Q_1 + e_2 Q_2' + \&c.$$

where $\begin{Bmatrix} P_0, & P_1, & P_2, & \&c. \\ Q_0, & Q_1, & Q_2, & \&c. \end{Bmatrix}$ are functions of t only,

A_0, A_1, A_2,} known functions of a, λ, and self-reciprocal relative to a,

P_n', Q_n' reciprocal to P_n, Q_n respectively, hence

(putting $\int_b P_n P_n' = p_n$, $\int_t Q_n Q_n' = q_n$) $\quad E(a) + \lambda E_1(a)$

$$= c_0 p_0 A_0 + c_1 p_1 A_1 + c_2 p_2 A_2 + \&c. + e_0 q_0 A_0 + e_1 q_1 A_1 + e_2 q_2 A_2, \ \&c.;$$

$$\therefore \int_a A_0 E(a) + \lambda \int_a A_0 E_1(a) = (c_0 p_0 + e_0 q_0) \int_a A_0^2,$$

$$\int_a A_1 E(a) + \lambda \int_a A_1 E_1(a) = (c_1 p_1 + e_1 q_1) \int_a A_1^2,$$

$$\&c. \qquad\qquad \&c.$$

and giving to λ any two values in each of these equations, the first will produce two equations which determine c_0, e_0, the second will similarly give c_1, e_1, &c., and thence the functions $\phi(t)$, ψt are known.

The same method is applicable to any number (n) of simultaneous equations involving n unknown functions.

48. *Definite-integral Equations of superior orders and degrees.*

Methods similar to the preceding are applicable in most cases of the former class thus:

$$\text{Given } \int_t\int_\tau \phi(t, \tau) f(t, \tau, a) = F(a),$$

the forms F and f being known to determine ϕ.

By Art. 16. Sect. IV. let a function Q_n be formed which shall be self-reciprocal, relative to double integration for t and τ.

$$\left.\begin{array}{r}\text{Put } \phi(t, \tau) = c_0 Q_0 + c_1 Q_1 + c_2 Q_2 + \&\text{c.}\\ \text{and } f(t, \tau, a) = A_0 Q_0 + A_1 Q_1 + A_2 Q_2 + \&\text{c.}\end{array}\right\} \text{ and let } \alpha_n = \int_t\int_\tau Q_n^2,$$

$$\text{hence } F(a) = \alpha_0 c_0 A_0 + \alpha_1 c_1 A_1 + \alpha_2 c_2 A_2 + \&\text{c.}$$

Let A_n' be a function of a reciprocal to A_n,

$$\text{then } \int_a A_0' F(a) = c_0 \alpha_0 \int_a A_0 A_0',$$

$$\int_a A_1' F(a) = c_1 \alpha_1 \int_a A_1 A_1,$$

$$\&\text{c.} \qquad\qquad \&\text{c.}$$

whence c_0, c_1, &c. being determined, the function $\phi(t, \tau)$ is known.

Equations of superior degrees must generally be converted into equations of superior orders to be easily solved, thus;

$$\text{Given } \int_t \phi(t) . f(t, a) \times \int_t \phi(t) . F(t, a) = \psi(a),$$

the forms f, F, and ψ being given to find the function ϕ.

Introduce another variable τ having the same limits as t, then it is evident that

$$\int_t \phi(t) . F(t, a) = \int_\tau \phi(\tau) . F(\tau, a);$$

$$\therefore \int_t\int_\tau \phi(t) . \phi(\tau) . f(t, a) . F(\tau, a) = \psi(a),$$

and since $f(t, a) . F(\tau, a)$ is a given function of t, τ and a, the unknown function $\phi(t) . \phi(\tau)$ will be determined as above, and representing it by $\phi_1(t, \tau)$, let α be a root of the equation $\phi(\tau) = 1$, then since $\phi(t) . \phi(\tau) = \phi_1(t, \tau)$, we get the required function $\phi(t) = \phi_1(t, \alpha)$, and again putting $t = \alpha$ we get $\phi_1(\alpha, \alpha) = 1$, from which equation α is known, and therefore $\phi(t) = \phi_1(t, \alpha)$ is also known.

49. In researches on the subjects of electricity, and the phænomena dependent on the molecular construction of bodies, the only data which can be furnished by experience are the total actions, and consequently

the analytical processes of calculation require the solution of definite integral equations: some of these have been resolved by Laplace and others, by means of particular artifices by which the unknown functions were subjected to differential equations; but as no general method existed for this purpose, the resolution of such equations has been extremely limited, and apparently simple physical problems, such as the distribution of electricity on surfaces, (with the exception of a very few cases) have consequently defied the powers of analysis. Besides, an abundance of facts connected with the interior arrangement of the molecules of bodies are of such a nature, that mathematics possessed but little power of reducing them to analytical forms, calculated to produce any valuable inferences; these facts are daily increasing in number, and the analyst is far behind the cultivator of Experimental Physics. The Memoirs on the Inverse Method of Definite Integrals which are now concluded, and which have been pursued when the absence of ordinary engagements permitted, originated in the belief that by proceeding gradually from the simplest classes of Definite Integrals to the more complex, the general principles of an Inverse Method would be discoverable. The formation of all possible classes of Reciprocal Functions, and the Transient Functions included amongst them, have at length furnished means for the resolution of equations to Definite Integrals. The author is however well aware that there must exist numerous imperfections in the manner in which his design is executed, but believing also that by those endeavours, however weak, some fresh powers have accrued to analysis, as an instrument of investigation, he trusts they will deserve the approbation of the Society.

R. MURPHY.

CAIUS COLLEGE,
Dec. 24, 1834.

Analytical Table *of Reference to the "Memoirs on the Inverse Method of Definite Integrals."*

FIRST MEMOIR, Vol. IV. Page 353, &c.

SECOND MEMOIR. VOL. V. Page 113, &c.

THIRD MEMOIR. VOL. V. Page 315, &c.

ERRATA.

PAGE

First Memoir. Vol. IV. 359, line 9. 16. 18. *dele* y in the sign ∞^y.
377, line 8. *for* $h < \beta$ *read* $h < 1$.
406, lowest line and third from bottom, *for* terms *read* times.
407, line 17. supply the word, equation.

Second Memoir. Vol. V. 134, line 6. *after* $\frac{dP_n}{dt}$ *supply* $(tt')^{\frac{1}{2}}$.
line 7. *after* $\frac{d^2P_n}{dt^2}$ *supply* (tt').
136, line 3, 4. 18. *for* v_n put v_x.

Third Memoir. Vol. V. 332, lowest line, *for* $f'''^{(r-1)}(t)$ *read* $f'''^{(r-2)}(t)$.
333, line 5, *for* $f(0)$ *read* $f''(0)$.
for $m+3$ *read* $(m+3)$.
337, line 8, *put* $(tt')^{-m}$ *before* $\frac{d^n v}{dt^n}$ in the last term.
345, line 16, *for* intger *read* integer.
357, line 8, *for* $(1-h^{\frac{1}{2}})$ *read* $(1-h)^{\frac{1}{2}}$.

XV. *On the Determination of the Exterior and Interior Attractions of Ellipsoids of Variable Densities.* *By* George Green, Esq., *Caius College.*

[Read *May* 6, 1833.]

The determination of the attractions of ellipsoids, even on the hypothesis of a uniform density, has, on account of the utility and difficulty of the problem, engaged the attention of the greatest mathematicians. Its solution, first attempted by Newton, has been improved by the successive labours of Maclaurin, d'Alembert, Lagrange, Legendre, Laplace, and Ivory. Before presenting a new solution of such a problem, it will naturally be expected that I should explain in some degree the nature of the method to be employed for that end, in the following paper; and this explanation will be the more requisite, because, from a fear of encroaching too much upon the Society's time, some very comprehensive analytical theorems have been in the first instance given in all their generality.

It is well known, that when the attracted point p is situated within the ellipsoid, the solution of the problem is comparatively easy, but that from a breach of the law of continuity in the values of the attractions when p passes from the interior of the ellipsoid into the exterior space, the functions by which these attractions are given in the former case will not apply to the latter. As however this violation of the law of continuity may always be avoided by simply adding a positive quantity, u^2 for instance, to that under the radical signs in the original integrals, it seemed probable that some advantage might thus be obtained, and the attractions in both cases, deduced from one common formula which would only require the auxiliary variable u to become evanescent in the final result. The principal advantage however which arises from the introduction of the new variable u, depends on the property which a certain function V* then possesses of satisfying a partial differential equation, whenever the law of the attraction is inversely as any power n of the distance. For by a proper application of this equation we may avoid all the difficulty usually presented by the integrations, and at the same time find the required attractions when the density ρ' is expressed by the product of two factors, one of which is a simple algebraic quantity, and the remaining one any rational and entire function of the rectangular co-ordinates of the element to which ρ' belongs.

The original problem being thus brought completely within the pale of analysis, is no longer confined as it were to the three dimensions of space. In fact, ρ' may represent a function of any number s, of independent variables, each of which may be marked with an accent, in order to distinguish this first system from another system of s analogous and unaccented variables, to be afterwards noticed, and V may represent the value of a multiple integral of s dimensions, of which every element is expressed by a fraction having for numerator the continued product of ρ' into the elements of all the accented variables, and for denominator a quantity containing the whole of these, with the unaccented ones also formed exactly on the model of the corresponding one in the value of V belonging to the original problem. Supposing now the auxiliary variable u is introduced, and the s integrations are effected, then will the resulting value of V be a function of u and of the s unaccented variable to be determined. But after the introduction

* This function in its original form is given by

$$V = \int \frac{\rho' \, dx' \, dy' \, dz'}{\{(x'-x)^2 + (y'-y)^2 + (z'-z)^2\}^{\frac{n-1}{2}}},$$

where $dx'dy'dz'$ represents the volume of any element of the attracting body of which ρ' is the density and x', y', z' are the rectangular co-ordinates; x, y, z being the co-ordinates of the attracted point p. But when we introduce the auxiliary variable u which is to be made equal to zero in the final result,

$$V = \int \frac{\rho' \, dx' \, dy' \, dz'}{\{(x'-x)^2 + (y'-y)^2 + (z'-z)^2 + u^2\}^{\frac{n-1}{2}}};$$

both integrals being supposed to extend over the whole volume of the attracting body.

of u, the function V has the property of satisfying a partial differential equation of the second order, and by an application of the Calculus of Variations it will be proved in the sequel that the required value of V may always be obtained by merely satisfying this equation, and certain other simple conditions when ρ' is equal to the product of two factors, one of which may be any rational and entire function of the s accented variables, the remaining one being a simple algebraic function whose form continues unchanged, whatever that of the first factor may be.

The chief object of the present paper is to resolve the problem in the more extended signification which we have endeavoured to explain in the preceding paragraph, and, as is by no means unusual, the simplicity of the conclusions corresponds with the generality of the method employed in obtaining them. For when we introduce other variables connected with the original ones by the most simple relations, the rational and entire factor in ρ' still remains rational and entire of the same degree, and may under its altered form be expanded in a series of a finite number of similar quantities, to each of which there corresponds a term in V, expressed by the product of two factors; the first being a rational and entire function of s of the new variables entering into V, and the second a function of the remaining new variable h, whose differential coefficient is an algebraic quantity. Moreover the first is immediately deducible from the corresponding part of ρ' without calculation.

The solution of the problem in its extended signification being thus completed, no difficulties can arise in applying it to particular cases. We have therefore on the present occasion given two applications only. In the first, which relates to the attractions of ellipsoids, both the interior and exterior ones are comprised in a common formula agreeably to a preceding observation, and the discontinuity before noticed falls upon one of the independent variables, in functions of which both these attractions are expressed; this variable being constantly equal to zero so long as the attracted point p remains within the ellipsoid, but becoming equal to a determinate function of the co-ordinates of p, when p is situated in the exterior space. Instead too of seeking directly the value of V, all its differentials have first been deduced, and thence the value of V obtained by integration. This slight modification has been given to our method, both because it renders the determination of V in the case considered more easy, and may likewise be usefully employed in the more general one before mentioned. The other application is remarkable both on account of the simplicity of the results to which it leads, and of their analogy with those obtained by Laplace. (Méc. Cél. Liv. III. Chap. 2.) In fact, it would be easy to shew that these last are only particular cases of the more general ones contained in the article now under notice.

The general solution of the partial differential equation of the second order, deducible from the seventh and three following articles of this paper, and in which the principal variable V is a function of $s+1$ independent variables, is capable of being applied with advantage to various interesting physico-mathematical enquiries. Indeed the law of the distribution of heat in a body of ellipsoidal figure, and that of the motion of a non-elastic fluid over a solid obstacle of similar form, may be thence almost immediately deduced; but the length of our paper entirely precludes any thing more than an allusion to these applications on the present occasion.

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1. The object of the present paper will be to exhibit certain general analytical formulæ, from which may be deduced as a very particular case the values of the attractions exerted by ellipsoids upon any exterior or interior point, supposing their densities to be represented by functions of great generality.

Let us therefore begin with considering ρ' as a function of the s independent variables

$$x_1',\quad x_2',\quad x_3' \ldots\ldots\ldots\ldots x_s',$$

and let us afterwards form the function

$$V = \int \frac{dx_1'\, dx_2'\, dx_3' \ldots\ldots\ldots dx_s' \,.\, \rho'}{\{(x_1 - x_1')^2 + (x_2 - x_2')^2 + \ldots\ldots\ldots + (x_s - x_s')^2 + u^2\}^{\frac{n-1}{2}}} \ldots\ldots\ldots (1)$$

the sign $\int$ serving to indicate s integrations relative to the variables x_1', x_2', $x_3',\ldots\ldots x_s'$, and similar to the double and triple ones employed in the solution of geometrical and mechanical problems. Then it is easy to perceive that the function V will satisfy the partial differential equation

$$0 = \frac{d^2V}{dx_1^2} + \frac{d^2V}{dx_2^2} + \ldots\ldots\ldots + \frac{d^2V}{dx_s^2} + \frac{d^2V}{du^2} + \frac{n-s}{u}\frac{dV}{du} \ldots\ldots\ldots (2)$$

seeing that in consequence of the denominator of the expression (1), every one of its elements satisfies for V to the equation (2).

To give an example of the manner in which the multiple integral is to be taken, we may conceive it to comprise all the real values both positive and negative of the variables x_1', $x_2',\ldots\ldots x_s'$, which satisfy the condition

$$\frac{x_1'^2}{a_1'^2} + \frac{x_2'^2}{a_2'^2} + \frac{x_3'^2}{a_3'^2} + \ldots\ldots\ldots + \frac{x_s'^2}{a_s'^2} < 1 \ldots\ldots\ldots (a)$$

the symbol $\angle$, as is the case also in what follows, not excluding equality.

2. In order to avoid the difficulties usually attendant on integrations like those of the formula (1), it will here be convenient to notice two or three very simple properties of the function V.

In the first place, then, it is clear that the denominator of the formula (1) may always be expanded in an ascending series of the entire powers of the increments of the variables x_1, $x_2,\ldots\ldots x_s$, u, and their various products by means of Taylor's Theorem, unless we have simultaneously

$$x_1 = x_1', \quad x_2 = x_2', \ldots\ldots\ldots x_s = x_s' \text{ and } u = 0;$$

and therefore V may always be expanded in a series of like form, unless the $s+1$ equations immediately preceding are all satisfied for one at least of the elements of V. It is thus evident that the function V possesses the property in question, except only when the two conditions

$$\frac{x_1^2}{a_1'^2} + \frac{x_2^2}{a_2'^2} + \frac{x_3^2}{a_3'^2} + \dots\dots\dots + \frac{x_s^2}{a_s'^2} \angle 1 \text{ and } u = 0\dots\dots\dots(3)$$

are satisfied simultaneously, considering as we shall in what follows the limits of the multiple integral (1) to be determined by the condition (a)*.

In like manner it is clear that when

$$\frac{x_1^2}{a_1'^2} + \frac{x_2^2}{a_2'^2} + \dots\dots\dots + \frac{x_s^2}{a_s'^2} > 1\dots\dots\dots(4),$$

the expansion of V in powers of u will contain none but the even powers of this variable.

Again, it is quite evident from the form of the function V that when any one of the $s+1$ independent variables therein contained becomes infinite, this function will vanish of itself.

3. The three foregoing properties of V combined with the equation (2) will furnish some useful results. In fact, let us consider the quantity

$$\int dx_1\, dx_2 \dots dx_s\, du\, u^{n-s} . \left\{ \left(\frac{dV}{dx_1}\right)^2 + \left(\frac{dV}{dx_2}\right)^2 + \dots\dots\dots + \left(\frac{dV}{dx_s}\right)^2 + \left(\frac{dV}{du}\right)^2 \right\} \dots\dots(5)$$

where the multiple integral comprises all the real values whether positive or negative of $x_1, x_2, \dots\dots x_s$, with all the real and positive values of u which satisfy the condition

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \dots\dots\dots + \frac{x_s^2}{a_s^2} + \frac{u^2}{h^2} \angle 1\dots\dots\dots(6)$$

* The necessity of this first property does not explicitly appear in what follows, but it must be understood in order to place the application of the method of integration by parts, in Nos. 3, 4, and 5, beyond the reach of objection. In fact, when V possesses this property, the theorems demonstrated in these Nos. are certainly correct: but they are not necessarily so for every form of the function V, as will be evident from what has been shewn in the third article of my Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.

a_1, a_2,......a_s and h being positive constant quantities; and such that we may have generally

$$a_r > a'_r.$$

In this case the multiple integral (5) will have two extreme limits, viz. one in which the conditions

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \dots\dots\dots + \frac{x_s^2}{a_s^2} + \frac{u^2}{h^2} = 1 \text{ and } u = \text{a positive quantity}\dots\dots\dots(7)$$

are satisfied; and another defined by

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \dots\dots\dots + \frac{x_s^2}{a_s^2} \angle 1 \text{ and } u = 0.$$

Moreover, for greater distinctness, we shall mark the quantities belonging to the former with two accents, and those belonging to the latter with one only.

Let us now suppose that V'' is completely given, and likewise V_1' or that portion of V' in which the condition (3) is satisfied; then if we regard V_2' or the rest of V' as quite arbitrary, and afterwards endeavour to make the quantity (5) a minimum, we shall get in the usual way, by applying the Calculus of Variations,

$$0 = -\int dx_1\, dx_2 \dots\dots dx_s\, du\, u^{n-s}\, \delta V \left\{ \Sigma_1^{s+1} \frac{d^2 V}{dx_r^2} + \frac{d^2 V}{du^2} + \frac{n-s}{u} \frac{dV}{du} \right\}$$

$$-\int dx_1\, dx_2 \dots\dots dx_s\, u'^{n-s}\, \delta V_2' \frac{dV_2'}{du} \dots\dots\dots(8)$$

seeing that $\delta V'' = 0$ and $\delta V_1' = 0$, because the quantities V'' and V_1' are supposed given.

The first line of the expression immediately preceding gives generally

$$0 = \Sigma_1^{s+1} \frac{d^2 V}{dx_r^2} + \frac{d^2 V}{du^2} + \frac{n-s}{u} \frac{dV}{du} \dots\dots\dots(2')$$

which is identical with the equation (2) No. 1, and the second line gives

$$0 = u'^{n-s} \frac{dV_2'}{du} (u' \text{ being evanescent}) \dots\dots\dots (9).$$

From the nature of the question *de minimo* just resolved, there can be little doubt but that the equations (2′) and (9) will suffice for the complete determination of V, where V'' and V_1' are both given. But as the truth of this will be of consequence in what follows, we will, before proceeding farther, give a demonstration of it; and the more willingly because it is simple and very general.

4. Now since in the expression (5) u is always positive, every one of the elements of this expression will therefore be positive; and as moreover V'' and V_1' are given, there must necessarily exist a function V_0 which will render the quantity (5) a proper minimum. But it follows, from the principles of the Calculus of Variations, that this function V_0, whatever it may be, must moreover satisfy the equations (2′) and (9). If then there exists any other function V_1 which satisfies the last-named equations, and the given values of V'' and V_1', it is easy to perceive that the function

$$V = V_0 + A(V_1 - V_0)$$

will do so likewise, whatever the value of the arbitrary constant quantity A may be. Suppose therefore that A originally equal to zero is augmented successively by the infinitely small increments δA, then the corresponding increment of V will be

$$\delta V = (V_1 - V_0)\,\delta A,$$

and the quantity (5) will remain constantly equal to its minimum value, however great A may become, seeing that by what precedes the variation of this quantity must be equal to zero whatever the variation of V may be, provided the foregoing conditions are all satisfied. If then, besides V_0 there exists another function V_1 satisfying them all, we might give to the partial differentials of V, any values however great, by augmenting the quantity A sufficiently, and thus cause the quantity (5) to exceed any finite positive one, contrary to what has just been proved. Hence no such value as V_1 exists.

We thus see that when V'' and V_1' are both given, there is one and only one way of satisfying simultaneously the partial differential equation (2), and the condition (9).

5. Again, it is clear that the condition (4) is satisfied for the whole of V_2'; and it has before been observed (No. 2.) that when V is determined by the formula (1), it may always be expanded in a series of the form

$$V = A + Bu^2 + Cu^4 + \&c.$$

Hence the right side of the equation (9) is a quantity of the order u'^{n-s+1}; and u' being evanescent, this equation will then evidently be satisfied, provided we suppose, as we shall in what follows, that

$$n - s + 1 \text{ is positive.}$$

If now we could by any means determine the values of V'' and V_1' belonging to the expression (1), the value of V would be had without integration by simply satisfying (2′) and (9), as is evident from what precedes. But by supposing all the constant quantities a_1, a_2, a_3 ......a_s and h infinite, it is clear that we shall have

$$0 = V'',$$

and then we have only to find V_1', and thence deduce the general value of V.

6. For this purpose let us consider the quantity

$$\int dx_1\, dx_2 \dots dx_s\, du\, u^{n-s} \left\{\frac{dV}{dx_1}\frac{dU}{dx_1} + \frac{dV}{dx_2}\frac{dU}{dx_2} + \dots + \frac{dV}{dx_s}\frac{dU}{dx_s} + \frac{dV}{du}\frac{dU}{du}\right\}; \dots\dots (10)$$

the limits of the multiple integral being the same as those of the expression (5), and U being a function of x_1, x_2,......x_s and u, satisfying the condition $0 = U''$ when a_1, a_2,......a_s and h are infinite.

But the method of integration by parts reduces the quantity (10) to

$$-\int dx_1\, dx_2 \dots\dots dx_s \frac{dU'}{du} u'^{n-s} \,.\, V'$$

$$-\int dx_1\, dx_2 \dots dx_s\, du\, u^{n-s}\, V \left\{\Sigma_1^{s+1} \frac{d^2 U}{dx_r^2} + \frac{d^2 U}{du^2} + \frac{n-s}{u}\frac{dU}{du}\right\} \dots\dots\dots (11)$$

since $0 = V''$; and as we have likewise $0 = U''$, the same quantity (10) may also be put under the form

$$-\int dx_1\, dx_2 \ldots\ldots dx_s \frac{dV'}{du} u'^{n-s} \,.\, U'$$

$$-\int dx_1\, dx_2 \ldots dx_s\, du\, u^{n-s} \,.\, U \left\{ \Sigma_1^{s+1} \frac{d^2 V}{dx_r^2} + \frac{d^2 V}{du^2} + \frac{n-s}{u} \frac{dV}{du} \right\} \ldots\ldots\ldots (12).$$

Supposing therefore that U like V also satisfies the equation (2′), each of the expressions (11) and (12) will be reduced to its upper line, and we shall get by equating these two forms of the same quantity:

$$\int dx_1\, dx_2 \ldots dx_s \frac{dU'}{du} u'^{n-s} V' = \int dx_1\, dx_2 \ldots dx_s \frac{dV'}{du} u'^{n-s} U';$$

the quantities bearing an accent belonging, as was before explained, to one of the extreme limits.

Because V satisfies the condition (9), the equation immediately preceding may be written

$$\int dx_1\, dx_2 \ldots dx_s \frac{dU'}{du} u'^{n-s} V' = \int dx_1\, dx_2 \ldots dx_s \frac{dV_1'}{du} u'^{n-s} U_1'.$$

If now we give to the general function U the particular value

$$U = \{(x_1 - x_1'')^2 + (x_2 - x_2'')^2 + \ldots\ldots + (x_s - x_s'')^2 + u^2\}^{\frac{1-n}{2}};$$

which is admissible, since it satisfies for V to the equation (2), and gives $U'' = 0$, the last formula will become

$$\int \frac{dx_1\, dx_2 \ldots\ldots dx_s\, u'^{n-s} \dfrac{dV_1'}{du}}{\{(x_1 - x_1'')^2 + (x_2 - x_2'')^2 + \ldots\ldots + (x_s - x_s'')^2 + u'^2\}^{\frac{n-1}{2}}}$$

$$= \int \frac{dx_1\, dx_2 \ldots\ldots dx_s \,.\, (1-n)\, u'^{n-s+1} V'}{\{(x_1 - x_1'')^2 + (x_2 - x_2'')^2 + \ldots\ldots + (x_s - x_s'')^2 + u'^2\}^{\frac{n+1}{2}}} \ldots\ldots (13);$$

in which expression u' must be regarded as an evanescent positive quantity.

In order now to effect the integrations indicated in the second member of this equation, let us make

$x_1 - x_1'' = u'\rho \cos\theta_1$; $x_2 - x_2'' = u'\rho \sin\theta_1 \cos\theta_2$; $x_3 - x_3'' = u'\rho \sin\theta_1 \sin\theta_2 \cos\theta_3$, &c. until we arrive at the two last, viz.,

$$x_{s-1} - x''_{s-1} = u'\rho \sin\theta_1 \sin\theta_2 \ldots\ldots\ldots\ldots \sin\theta_{s-2} \cos\theta_{s-1},$$
$$x_s \;-\; x_s'' = u'\rho \sin\theta_1 \sin\theta_2 \ldots\ldots\ldots\ldots \sin\theta_{s-2} \sin\theta_{s-1};$$

u' being, as before, a vanishing quantity.

Then by the ordinary formulæ for the transformation of multiple integrals we get

$$dx_1\, dx_2 \ldots\ldots\ldots dx_s = u'^s \rho^{s-1} \sin\theta_1^{s-2} \sin\theta_2^{s-3} \ldots\ldots\ldots \sin\theta'_{s-2} d\rho\, d\theta_1\, d\theta_2 \ldots d\theta_{s-1},$$

and the second number of the equation (13) by substitution will become

$$\int \frac{d\rho\, d\theta_1\, d\theta_2 \ldots\ldots d\theta_{s-1} \rho^{s-1} \sin\theta_1^{s-2} \sin\theta_2^{s-3} \ldots\ldots \sin\theta_{s-2} . (1-n) V'}{(1+\rho^2)^{\frac{n+1}{2}}} \ldots\ldots (14).$$

But since u' is evanescent, we shall have ρ infinite, whenever $x_1, x_2, \ldots x_s$ differ sensibly from $x_1'', x_2'', \ldots x_s''$; and as moreover $n-s+1$ is positive, it is easy to perceive that we may neglect all the parts of the last integral for which these differences are sensible. Hence V' may be replaced with the constant value V_0' in which we have generally

$$x_r = x_r''.$$

Again, because the integrals in (14) ought to be taken from $\theta_{s-1} = 0$ to $\theta_{s-1} = 2\pi$, and afterwards from $\theta_r = 0$ to $\theta_r = \pi$, whatever whole number less than $s-1$ may be represented by r, we easily obtain by means of the well known function Gamma:

$$\int \sin\theta_1^{s-2} \sin\theta_2^{s-3} \sin\theta_3^{s-4} \ldots\ldots\ldots \sin\theta_{s-2}\, d\theta_1\, d\theta_2 \ldots d\theta_{s-1} = \frac{2\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)};$$

and as by the aid of the same function we readily get

$$\int_0^\infty \frac{\rho^{s-1} d\rho}{(1+\rho^2)^{\frac{n+1}{2}}} = \frac{\Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{n-s+1}{2}\right)}{2\Gamma\left(\frac{n+1}{2}\right)},$$

the integral (14) will in consequence become

$$\frac{-2\pi^{\frac{s}{2}}.\Gamma\left(\frac{n-s+1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}V_0',$$

and thus the equation (13) will take the form

$$\int\frac{dx_1\,dx_2\ldots\ldots\ldots dx_s\,u'^{n-s}\,\frac{dV_1'}{du}}{\{(x_1-x_1'')^2+(x_2-x_2'')^2+\ldots.+(x_s-x_s'')^2+u'^2\}^{\frac{n-1}{2}}}=\frac{-2\pi^{\frac{s}{2}}.\Gamma\left(\frac{n-s+1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}V_0'.$$

In this equation V is supposed to be such a function of x_1, $x_2\ldots\ldots x_s$ and u, that the equation (2) and condition (9) are both satisfied. Moreover $V''=0$, and V_0' is the particular value of V for which

$$x_1=x_1'';\quad x_2=x_2'';\ldots\ldots\ldots x_s=x_s'',\ \text{and}\ u=0.$$

Let us now make, for abridgment,

$$P=u^{n-s}\,\frac{dV}{du},\ (\text{when } u=0)\ldots\ldots\ldots\ldots\ldots\ldots(b),$$

and afterwards change x into x', and x'' into x in the expression immediately preceding, there will then result

$$\int\frac{dx_1'\,dx_2'\ldots\ldots\ldots dx_s'\,P_1'}{\{(x_1'-x_1)^2+(x_2'-x_2)^2+\ldots+(x_s'-x_s)^2+u'^2\}^{\frac{n-1}{2}}}=\frac{-2\pi^{\frac{s}{2}}\Gamma\left(\frac{n-s+1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}V'\ldots(15),$$

P' being what P becomes by changing generally x_r into x_r', the unit attached to the foot of P' indicating, as before, that the multiple integral comprises only the values admitted by the condition (a), and V' being what V becomes when we make $u=0$.

The equation just given supposes u' evanescent; but if we were to replace u' with the general value u in the first member, and make a corresponding change in the second by replacing V' with the general value V, this equation would still be correct, and we should thus have

$$\int \frac{dx_1'\,dx_2'\ldots\ldots\ldots dx_s'\,P_1'}{\{(x_1'-x_1)^2+(x_2'-x_2)^2+\ldots+(x_s'-x_s)^2+u^2\}^{\frac{n-1}{2}}} = \frac{-2\pi^{\frac{s}{2}}\Gamma\left(\frac{n-s+1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} V\ldots(16).$$

For under the present form both its members evidently satisfy the equation (2), the condition (9), and give $V''=0$. Moreover, when the condition (3) is satisfied, the same members are equal in consequence of (15). Hence by what has before been proved (No. 4), they are necessarily equal in general.

By comparing the equation (16) with the formula (1), it will become evident, that whenever we can by any means obtain a value of V satisfying the foregoing conditions, we shall always be able to assign a value of ρ' which substituted in (1) shall reproduce this value of V. In fact, by omitting the unit at the foot of P', which only serves to indicate the limits of the integral, we readily see that the required value of ρ' is

$$\rho' = -\frac{\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{s}{2}}\Gamma\left(\frac{n-s+1}{2}\right)} P'\ldots\ldots\ldots\ldots\ldots\ldots\ldots(c).$$

7. The foregoing results being obtained, it will now be convenient to introduce other independent variables in the place of the original ones, such that

$$x_1 = a_1\xi_1, \quad x_2 = a_2\xi_2,\ldots\ldots\ldots x_s = a_s\xi_s, \quad u = hv,$$

a_1, a_2,……a_s being functions of h, one of the new independent variables, determined by

$$a_1^2 = a_1'^2 + h^2, \quad a_2^2 = a_2'^2 + h^2,\ldots\ldots\ldots a_s^2 = a_s'^2 + h^2,$$

and v a function of the remaining new variables, ξ_1, ξ_2, ξ_3,……ξ_s satisfying the equation

$$1 = v^2 + \xi_1^2 + \xi_2^2 + \ldots\ldots\ldots + \xi_s^2;$$

a_1', a_2', a_3',……a_s' being the same constant quantities as in the equation (a), No 1. Moreover, a_1, a_2,……a_s will take the values belonging to the extreme limit before marked with two accents, by simply assigning to h an infinite value.

The easiest way of transforming the equation (2) will be to remark, that it is the general one which presents itself when we apply the Calculus of Variations to the quantity (5), in order to render it a minimum. We have therefore in the first place

$$\left(\frac{dV}{du}\right)^2+\Sigma_1^{s+1}\left(\frac{dV}{dx_r}\right)^2=\Sigma_1^{s+1}\left(\frac{dV}{a_r d\xi_r}\right)^2+\left\{\left(\frac{dV}{dh}\right)^2-\left(\Sigma_1^{s+1}\frac{h\xi_r}{a_r^2}\frac{dV}{d\xi_r}\right)^2\right\}\left(1-\Sigma_1^{s+1}\frac{a_r'^2\xi_r^2}{a_r^2}\right)^{-1};$$

and by the ordinary formula for the transformation of multiple integrals,

$$dx_1\,dx_2\ldots\ldots\ldots dx_s\,du=\frac{a_1a_2\ldots a_s}{v}\left(1-\Sigma_1^{s+1}\frac{a_r'^2\xi_r^2}{a_r^2}\right)d\xi_1\,d\xi_2\ldots d\xi_s\,dh.$$

But since $1-\Sigma_1^{s+1}\dfrac{a_r'^2\xi_r^2}{a_r^2}=v^2+h^2\Sigma_1^{s+1}\dfrac{\xi_r^2}{a_r^2}$,

the expression (5) after substitution will become

$$\int d\xi_1\,d\xi_2\ldots\ldots\ldots d\xi_s dh\;a_1\,a_2\,a_3\ldots\ldots a_s h^{n-s}v^{n-s-1}\ldots\ldots\ldots\ldots\ldots$$

$$\left\{\left(v^2+h^2\Sigma_1^{s+1}\frac{\xi_r^2}{a_r^2}\right)\Sigma_1^{s+1}\left(\frac{dV}{a_r d\xi_r}\right)^2+\left(\frac{dV}{dh}\right)^2-h^2\left(\Sigma_1^{s+1}\frac{\xi_r}{a_r^2}\frac{dV}{d\xi_r}\right)^2\right\}.$$

Applying now the method of integration by parts to the variation of this quantity, by reduction, we get for the equivalent of (2) the equation

$$0=\frac{d^2V}{dh^2}+\left(n-\Sigma\frac{a_r'^2}{a_r^2}\right)\frac{dV}{hdh}+(1-\Sigma\xi_r^2)\,\Sigma\frac{d^2V}{a_r^2d\xi_r^2}+(s-n-1)\,\Sigma\frac{\xi_r}{a_r^2}\frac{dV}{d\xi_r},$$

$$+\,h^2\Sigma\frac{\xi_r^2}{a_r^2}\times\Sigma\frac{d^2V}{a_r^2d\xi_r^2}-h^2\,\Sigma\Sigma\frac{\xi_r\,\xi_{r'}}{a_r^2\,a_{r'}^2}\frac{d^2V}{d\xi_r d\xi_{r'}}\ldots\ldots\ldots\ldots(2'')$$

$$+\,h^2\Sigma\frac{\xi_r dV}{a_r^4d\xi_r}-h^2\Sigma\frac{1}{a_r^2}\times\Sigma\frac{\xi_r dV}{a_r^2d\xi_r};$$

where the finite integrals are all supposed taken from $r=1$ to $r=s+1$, and from $r'=1$ to $r'=s+1$.

The last equation may be put under the abridged form,

$$0=\frac{d^2V}{dh^2}+\left(n-\Sigma\frac{a_r'^2}{a_r^2}\right)\frac{dV}{hdh}+\nabla V\ldots\ldots\ldots\ldots(2'''),$$

provided we have generally

$$\text{coefficient of } \frac{d^2V}{d\xi_r^2} \text{ in } \nabla V = \frac{1}{a_r^2}\left\{1 - \xi_r^2 - \Sigma_1^{s+1}\frac{a_r'^2}{a_{r'}}\xi_{r'}^2 + \frac{a_r'^2}{a_r^2}\xi_r^2\right\},$$

$$\text{coefficient of } \frac{d^2V}{d\xi_r d\xi_{r'}} \text{ in } \nabla V = -\frac{2h^2}{a_r^2 a_{r'}^2}\xi_r\xi_{r'},$$

$$\text{coefficient of } \frac{dV}{d\xi_r} \text{ in } \nabla V = \frac{\xi_r}{a_r^2}\left\{-n + \Sigma\frac{a_{r'}'^2}{a_{r'}^2} - \frac{a_r'^2}{a_r^2}\right\}.$$

Moreover, when we employ the new variables

$$\frac{dV}{du} = -\upsilon\left(1 - \Sigma\frac{a_r'^2\xi_r^2}{a_r^2}\right)^{-1}.\left\{\Sigma\frac{h\xi_r}{a_r^2}\frac{dV}{d\xi_r} - \frac{dV}{dh}\right\},$$

and therefore the condition (9) in like manner will become

$$0 = \upsilon^{n-s+1}h^{n-s}\left(1 - \Sigma\frac{a_r'^2\xi_r^2}{a_r^2}\right)^{-1}.\left\{\Sigma\frac{h\xi_r}{a_r^2}\frac{dV}{d\xi_r} - \frac{dV}{dh}\right\}\text{.........}(9');$$

where the values of the variables $\xi_1, \xi_2, \ldots\ldots \xi_s$ must be such as satisfy the equation $\upsilon^2 = 0$, whatever h may be; and as $n - s + 1$ is positive, it is clear that this condition will always be satisfied, provided the partial differentials of V relative to the new variables are all finite.

8. Let us now try whether it is possible to satisfy the equation (2‴) by means of a function of the form

$$V = H\phi\text{.......................}(\beta);$$

H depending on the variable h only, and ϕ being a rational and entire function of $\xi_1, \xi_2, \ldots\ldots \xi_s$ of the degree γ, and quite independent of h.

By substituting this value of V in (2‴) and making

$$0 = \frac{d^2H}{dh^2} + \left(n - \Sigma\frac{a_r'^2}{a_r^2}\right)\frac{dH}{hdh} + \kappa H\text{.................}(17),$$

we readily get

$$0 = \nabla\phi - \kappa\phi\text{.............................}(18);$$

where, in virtue of (17) κ must necessarily be a function of h only; and as the required value of ϕ, if it exist, must be independent of h, we have, by making $h = 0$ in the equation immediately preceding,

$$0 = \nabla'\phi - k_0\phi\text{............................}(19);$$

k_0 being the value κ, and $\nabla'\phi$ that of $\nabla\phi$ when $h = 0$.

We shall demonstrate almost immediately that every function ϕ of the form (20), No. 9, which satisfies the equation (19), and which therefore is independent of h, will likewise satisfy the equation (18); and the corresponding value of κ obtained from the latter being substituted in the ordinary differential equation (17), we shall only have to integrate this last in order to have a proper value of V.

9. To satisfy the equation (19) let us assume

$$\phi = F(\xi_1^2, \xi_2^2, \xi_3^2, \ldots\ldots\ldots \xi_s^2)\, \xi_p, \xi_q, \text{ \&c.} \ldots\ldots\ldots\ldots\ldots (20);$$

F being the characteristic of a rational and entire function of the degree $2\gamma'$, and the most general of its kind, and ξ_p, ξ_q, &c. designating the variables in ϕ which are affected with odd exponents only; so that if their number be ν we shall have

$$\gamma = 2\gamma' + \nu,$$

the remaining variables having none but even exponents. Then it is easy to perceive, that after substitution the second member of the equation (19) will be precisely of the same form as the assumed value of ϕ, and by equating separately to zero the coefficients of the various powers and products of $\xi_1, \xi_2, \ldots\ldots \xi_s$, we shall obtain just the same number of linear algebraic equations as there are coefficients in ϕ, and consequently be enabled to determine the ratios of these coefficients together with the constant quantity k_0.

In fact, by writing the foregoing value of ϕ under the form

$$\phi = SA_{m_1, m_2, \ldots\ldots\ldots m_s}\, \xi_1^{m_1}\, \xi_2^{m_2} \ldots\ldots\ldots \xi_s^{m_s} \ldots\ldots\ldots\ldots (20');$$

and proceeding as above described, the coefficient of $\xi_1^{m_1}\, \xi_2^{m_2} \ldots\ldots \xi_s^{m_s}$ will give the general equation

$$\begin{aligned} 0 = {} & \left\{\Sigma_1^{s+1} \frac{m_r(m_r - s + n)}{a_r'^2} + k_0\right\} A_{m_1, m_2 \ldots\ldots m_s} \\ & + \Sigma\Sigma \frac{(m_r+2)(m_r+1)}{a_r'^2} A_{m_1, m_2, \ldots m_r+2 \ldots m_{r'}-2, \ldots m_s} \ldots\ldots\ldots (21) \\ & - \Sigma_1^{s+1} \frac{(m_r+2)(m_r+1)}{a_r'^2} A_{m_1, m_2, \ldots m_r+2, \ldots m_s}; \end{aligned}$$

the double finite integral comprising all the values of r and r', except those in which $r = r'$, and consequently containing when completely expanded $s(s-1)$ terms.

For the terms of the highest degree γ and of which the number is

$$\frac{\gamma'+1 \,.\, \gamma'+2 \,.\, \ldots\ldots\ldots\ldots \gamma'+s-1}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots s-1} = N,$$

the last line of the expression (21) evidently vanishes, and thus we obtain N distinct linear equations between the coefficients of the degree γ in ϕ and k_0.

Moreover, from the form of these equations it is evident that we may obtain by elimination one equation in k_0 of the degree N, of which each of the N roots will give a distinct value of the function $\phi^{(\gamma)}$, having one arbitrary constant for factor; the homogeneous function $\phi^{(\gamma)}$ being composed of all the terms of the highest degree, γ in ϕ. But the coefficients of $\phi^{(\gamma)}$ and k_0 being known, we may thence easily deduce all the remaining coefficients in ϕ, by means of the formula (21).

Now, since the N linear equations have no terms except those of which the coefficients of $\phi^{(\gamma)}$ are factors, it follows that if k_0 were taken at will, the resulting values of all these coefficients would be equal to zero. If however we obtain the values of $N-1$ of the coefficients in terms of the remaining one A from $N-1$ of the equations, by the ordinary formulæ, and substitute these in the remaining equation, we shall get a result of the form

$$K \,.\, A = 0,$$

where K is a function of k_0 of the degree N. We shall thus have only two cases to consider: First, that in which $A=0$, and consequently also all the other coefficients of $\phi^{(\gamma)}$ together with the remaining ones in ϕ, as will be evident from the formulæ (21). Hence, in this case

$$\phi = 0:$$

Secondly, that in which k_0 is one of the N roots of $0=K$, as for instance, k_0' in this case all the coefficients of ϕ will become multiples of A, and we shall have

$$\phi = A\,\phi_1:$$

ϕ_1 being a determinate function of $\xi_1, \xi_2, \ldots\ldots \xi_s$.

We thus see that when we consider functions of the form (20) only, the most general solution that the equation

$$0 = \nabla'\overline{\phi} - k_0'\overline{\phi} \ldots\ldots\ldots (19')$$

admits is ....

$$\text{or, } \overline{\phi} = 0; \qquad \text{or, } \overline{\phi} = \alpha\phi;$$

α being a quantity independent of $\xi_1, \xi_2, \ldots\ldots \xi_s$, and ϕ any function which satisfies for $\overline{\phi}$ to the equation (19′). But by affecting both sides of the equation

$$0 = \nabla'\phi - k_0'\phi$$

with the symbol ∇, we get

$$0 = \nabla \,.\, \nabla'\phi - k_0' \,.\, \nabla\phi;$$

and we shall afterwards prove the operations indicated by ∇ and ∇' to be such, that whatever ϕ may be,

$$\nabla\nabla'\phi = \nabla'\nabla\phi.$$

Hence, the last equation becomes

$$\nabla'(\nabla\phi) - k_0'\nabla\phi;$$

and as $\nabla\phi$ like ϕ is of the form (20), it follows from what has just been shewn, that

$$\text{either } \; 0 = \nabla\phi, \quad \text{or, } \nabla\phi = \alpha\phi,$$

α being a quantity independent of $\xi_1, \xi_2, \ldots\ldots \xi_s$.

The first is inadmissible, since it would give $\phi = 0$; therefore when ϕ satisfies (19′), we have

$$\nabla\phi' = \alpha\phi, \text{ i. e. } 0 = \nabla\phi - \alpha\phi.$$

But since α is independent of $\xi_1, \xi_2, \ldots\ldots \xi_s$, this last equation is evidently identical with (18), since the equation (18) merely requires that κ should be independent of $\xi_1, \xi_2, \ldots\ldots \xi_s$.

Having thus proved that every function of the form (20) which satisfies (19) will likewise satisfy (18), it will be more simple to determine the remaining coefficients of ϕ from those of $\phi^{(\gamma)}$ by means of the last equation, than to employ the formula (21) for that purpose.

Making therefore h infinite in (18), and writing $\frac{k_1}{h^2}$ in the place of κ, we get

$$0=\Sigma_1^{s+1}.(1-\xi_r^2)\frac{d^2\phi}{d\xi_r^2}-2(\Sigma\Sigma)\,\xi_r\xi_{r'}\frac{d^2\phi}{d\xi_r d\xi_{r'}}-n\Sigma_1^{s+1}\xi_r\frac{d\phi}{d\xi_r}-k_1\phi;$$

where $(\Sigma\Sigma)$ comprises the $\frac{s(s-1)}{1.2}$ combinations which can be formed of the s indices taken in pairs.

If now we substitute the value of ϕ before given (20′), and recollect that for the terms of the highest degree we have $\Sigma m_r=\gamma$, we shall readily get

$$0=(\gamma-\Sigma m_r)(\gamma+\Sigma m_r+n-1)A_{m_1,\,m_2,\ldots m_s}+(m_r+2)(m_r+1)A_{m_1,\,m_r+2,\ldots m_s}\ldots(22),$$

from which all the remaining coefficients in ϕ will readily be deduced, when those of the part $\phi^{(\gamma)}$ are known.

10. It now remains, as was before observed, to integrate the ordinary differential equation (17) No. 8. But, by the known theory of linear equations, the integration of (17) will always become more simple when we have a particular value satisfying it, and fortunately in the present case such a value may always be obtained from ϕ by simply changing ξ_r into $\frac{a_r}{\sqrt{(\Sigma a_r'^2)}}$. In fact if we represent the value thus obtained by H_0 we shall have

$$\frac{dH_0}{dh}=\Sigma_1^{s+1}\frac{d\phi}{d\xi_r}\cdot\frac{h}{a_r\sqrt{(\Sigma a_r'^2)}},$$

and by a second differentiation

$$\frac{d^2H_0}{dh^2}=\Sigma\frac{d\phi}{d\xi_r}\cdot\frac{a_r'^2}{a_r^3\sqrt{(\Sigma a_r'^2)}}+\Sigma\frac{d^2\phi}{d\xi_r^2}\cdot\frac{h^2}{a_r^2\,.\,\Sigma a_r'^2}+2(\Sigma\Sigma)\frac{d^2\phi}{d\xi_r d\xi_{r'}}\cdot\frac{h^2}{a_r a_{r'}\Sigma a_r'^2},$$

($\Sigma\Sigma$) as before comprising all the $\frac{s \,.\, s-1}{1 \,.\, 2}$ combinations of the s indices taken in pairs.

Hence, the quantity on the right side of the equation (17), when we make $H = H_0$, becomes

$$\Sigma \frac{d\phi}{d\xi_r} \cdot \frac{a_r'^2}{a_r^3 \sqrt{(\Sigma a_r'^2)}} + \Sigma \frac{d^2\phi}{d\xi_r^2} \cdot \frac{h^2}{a_r^2 \Sigma a_r'^2} + \kappa\phi$$

$$+ 2(\Sigma\Sigma) \frac{d^2\phi}{d\xi_r\, d\xi_{r'}} \cdot \frac{h^2}{a_r a_{r'} \Sigma a_r'^2} + \left(n - \Sigma \frac{a_r'^2}{a_r^2}\right) \Sigma \frac{d\phi}{d\xi_r} \cdot \frac{1}{a_r \sqrt{(\Sigma a_r'^2)}} \ldots\ldots (23).$$

But if we recollect that we have generally

$$\xi_r = \frac{a_r}{\sqrt{(\Sigma a_r'^2)}} \ldots\ldots (24),$$

it is easy to perceive that in consequence of the equation (18) the quantity (23) will vanish, and therefore the foregoing value of H_0 will always satisfy the equation (17).

Having thus a particular value of H, we immediately get the general one by assuming

$$H = H_0 \int z\, dh.$$

In fact, there thence results

$$H = KH_0 \int \frac{h^{s-n}\, dh}{H_0^2\, a_1,\, a_2,\, a_3 \ldots\ldots a_s};$$

the two arbitrary constants which the general integral ought to contain being K, and that which enters implicitly into the indefinite integral. But the condition $0 = V''$ requires that H should vanish when h is infinite, and consequently the particular value adapted to the present investigation is

$$H_0 = K \,.\, H_0 \int_\infty \frac{h^{s-n}\, dh}{H_0^2\, a_1,\, a_2 \ldots\ldots a_s}.$$

11. The values of ϕ and H being known, we may readily find the corresponding values of V and ρ'. For we have immediately

$$V = H\phi = K\phi H_0 \int_\infty \frac{h^{s-n}\,dh}{H_0^2 a_1, a_2 \ldots\ldots a_s} \ldots\ldots (26),$$

and as the function ϕ is rational and entire, and the partial differential of V relative to h is finite, it follows that all the partial differentials of V are finite; and consequently, by what precedes (No. 7.) the condition (9′) is satisfied by the foregoing value of V, as well as the equation (2) and condition $0 = V''$. Hence the equations (b) and (c) No. 6 will give, since

$$\frac{dV}{du} = -\upsilon\left(1 - \Sigma_1^{s+1}\frac{a_r'^2\xi_r^2}{a_r^2}\right)^{-1}.\left\{\Sigma_1^{s+1}\frac{h\xi_r}{a_r^2}.\frac{dV}{d\xi_r} - \frac{dV}{dh}\right\},$$

and h must be supposed equal to zero in these equations

$$\rho' = \frac{-\Gamma\left(\dfrac{n-1}{2}\right)}{2\pi^{\frac{s}{2}}\Gamma\left(\dfrac{n-s+1}{2}\right)}\upsilon^{n-s-1}.h^{n-s}\frac{dV}{dh} \ldots\ldots (\text{where } h = 0):$$

since where $h = 0$, $a_r = a_r'$; and therefore,

$$1 - \Sigma_1^{s+1}\frac{a_r'^2\xi_r^2}{a^2 r} = 1 - \Sigma_1^{s+1}\xi_r^2 = \upsilon^2.$$

If now we substitute for V its value (26), and recollect that $n-s+1$ is always positive, we get

$$\rho' = \frac{-\Gamma\left(\dfrac{n-1}{2}\right)}{2\pi^{\frac{s}{2}}\Gamma\left(\dfrac{n-s+1}{2}\right)}\upsilon^{n-s-1}\phi'\frac{K}{H_0' a_1', a_2', \ldots\ldots a_s'} \ldots\ldots (27),$$

since it is clear from the form of H_0 that this quantity may always be expanded in a series of the entire powers of h^2. In the preceding expression, (27), H_0' indicates the value of H_0 when $h = 0$, and ϕ' the corresponding value of ϕ or that which would be obtained by simply changing the unaccented letter $\xi_1, \xi_2, \ldots\ldots \xi_s$ into the accented ones $\xi_1', \xi_2', \ldots\ldots \xi_s'$ deduced from

$$(\gamma) \qquad x_1' = a_1'\xi_1'; \qquad x_2' = a_2'\xi_2'; \qquad x_s' = a_s'\xi_s'.$$

It will now be easy to obtain the value of V corresponding to

$$\rho' = \left(1 - \frac{x_1'^2}{a_1'^2} - \frac{x_2'^2}{a_2'^2} - \ldots\ldots - \frac{x_s'^2}{a_s'^2}\right)^{\frac{n-s-1}{2}} F(x_1', x_2' \ldots\ldots x_s') \ldots\ldots (28),$$

without integrating the formula (1) No 1, where F is the characteristic of any rational and entire function. In fact it is easy to see from what precedes (No. 9), that we may always expand F in a finite series of the form

$$F(x_1', x_2' \ldots\ldots x_s') = b_0\phi_0' + b_1\phi_1' + b_2\phi_2' + b_3\phi_3' + \&c.$$

after x_1', x_2', &c. have been replaced with their values (γ). Hence, we immediately get

$$\rho' = v^{n-s-1} . \{b_0\phi_0' + b_1\phi_1' + b_2\phi_2' + \&c.\} \ldots\ldots (29).$$

By comparing the formulæ (26) and (27) it is clear that any term, as $b_r\phi_r'$ for instance, of the series entering into ρ', will have for corresponding term in the required value of V, the quantity

$$-\frac{2\pi^{\frac{s}{2}}\Gamma\left(\frac{n-s+1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} H_0' a_1' a_2' \ldots\ldots\ldots a_s' . b_r\phi_r H_0 \int_\infty \frac{h^{s-n}\,dh}{H_0^2 a_1 a_2 \ldots\ldots a_s} \ldots\ldots (30):$$

H_0 being a particular value of H satisfying the equation (17), and immediately deducible from ϕ by the method before explained.

12. All that now remains, is to demonstrate that

$$\nabla'\nabla\phi = \nabla\nabla'\phi \ldots\ldots (31),$$

whatever ϕ may be. For this purpose let us here resume the value of $\Delta\phi$, as immediately deduced from the equation (2″) No. 7, viz.

$$\Delta\phi = (1 - \Sigma\xi^2)\Sigma\frac{d^2\phi}{a^2 d\xi^2} + (s-n-1)\Sigma\frac{\xi d\phi}{a^2 d\xi}$$

$$+ h^2\Sigma\frac{\xi^2}{a^2} \times \Sigma\frac{d^2\phi}{a^2 d\xi^2} - h^2\Sigma\Sigma\frac{\xi\xi'}{a^2 a'^2}\frac{d^2\phi}{d\xi d\xi'}$$

$$+ h^2\Sigma\frac{\xi d\phi}{a^4 d\xi} - h^2\Sigma\frac{1}{a_2} \times \Sigma\frac{\xi d\phi}{a^2 d\xi} \ldots\ldots (32),$$

where for simplicity the indices at the foot of the letters ξ and a have been omitted, and their accents transferred to the letters themselves. Moreover all the finite integrals are supposed taken from 1 to $s+1$.

By making $h=0$ in the last expression we immediately get $\nabla'\phi$, and if for a moment, to prevent ambiguity, we write b_r in the place of the original a'_r and omit the lower indices as before, we obtain

$$\nabla'\phi = (1-\Sigma\xi^2)\,\Sigma\frac{d^2\phi}{b''^2 d\xi''^2} + (s-n-1)\,\Sigma\frac{\xi'' d\phi}{b''^2 d\xi''} \;\ldots\ldots (33);$$

where to avoid all risk of confusion r has been changed into r'', and the double accent of this index transferred to the letters ξ and b themselves.

We will now conceive the expression (32) to be written in the abridged form

$$\nabla\phi = \nabla_1\phi + h^2\nabla_2\phi - h^2\nabla_3\phi + h^2\nabla_4\phi - h^2\nabla_5\phi,$$

the order of the terms remaining unchanged.

If then we recollect that the accents have no other office to perform than to keep the various finite integrations quite distinct, and consequently that in the final results they may be permuted in any way at will, we shall readily get

$$\nabla'\nabla_1\phi - \nabla_1\nabla'\phi =$$

$$(1-\Sigma\xi^2)\left\{4\Sigma\Sigma\left(\frac{1}{a'^2b^2}-\frac{1}{a^2b'^2}\right)\frac{\xi' d^3\phi}{d\xi^2 d\xi'}_{(1)} + 2\Sigma\frac{1}{a^2}\times\Sigma\frac{d^2\phi}{b^2 d\xi^2}_{(2)} - 2\Sigma\frac{1}{b^2}\times\Sigma\frac{d^2\phi}{a^2 d\xi^2}_{(3)}\right\}$$

$$+2(s-n-1)\left\{\Sigma\frac{\xi^2}{a^2}\times\Sigma\frac{d^2\phi}{b^2 d\xi^2}_{(4)} - \Sigma\frac{\xi^2}{b^2}\times\Sigma\frac{d^2\phi}{a^2 d\xi^2}_{(5)}\right\},$$

$$\nabla'\nabla_2\phi - \nabla_2\nabla'\phi =$$

$$(1-\Sigma\xi^2)\left\{4\Sigma\Sigma\frac{\xi'}{a^2a'^2b'^2}\cdot\frac{d^3\phi}{d\xi^2 d\xi'}_{(6)} + 2\Sigma\frac{1}{a^2b^2}\times\Sigma\frac{d^2\phi}{a^2 d\xi^2}_{(7)}\right\}$$

$$+4\Sigma\frac{\xi^2}{a^2}\times\Sigma\Sigma\frac{\xi}{a^2b'^2}\cdot\frac{d^3\phi}{d\xi d\xi'^2}_{(8)} + 2\Sigma\frac{1}{a^2}\times\Sigma\frac{\xi^2}{a_2}\times\Sigma\frac{d^2\phi}{b^2 d\xi^2}_{(9)}$$

$$+2(s-n-1)\left\{\Sigma\frac{\xi^2}{a^2b^2}\times\Sigma\frac{d^2\phi}{a^2d\xi^2}{}_{(10)}\quad-\Sigma\frac{\xi^2}{a^2}\times\Sigma\frac{1}{a^2b^2}\cdot\frac{d^2\phi}{d\xi^2}{}_{(11)}\right\}$$

$$\nabla_3\nabla'\phi-\nabla'\nabla_3\phi=$$

$$(1-\Sigma\xi^2)\left\{-4\Sigma\Sigma\frac{\xi d^3\phi}{a^2a'^2b'^2d\xi d\xi'^2}{}_{(12)}\quad-2\Sigma\frac{1}{a^4b^2}\cdot\frac{d^2\phi}{d\xi^2}{}_{(13)}\right\}$$

$$-4\Sigma\frac{\xi^2}{a^2}\times\Sigma\Sigma\frac{\xi}{a^2b'^2}\cdot\frac{d^3\phi}{d\xi d\xi'^2}{}_{(14)}\quad-2\Sigma\frac{\xi^2}{a^4}\times\Sigma\frac{d^2\phi}{b^2d\xi^2}{}_{(15)}$$

$$\nabla'\nabla_4\phi-\nabla_4\nabla'\phi=$$

$$2(1-\Sigma\xi^2)\Sigma\frac{1}{b^2a^4}\cdot\frac{d^2\phi}{d\xi^2}{}_{(16)}\quad+2\Sigma\frac{\xi^2}{a^4}\times\Sigma\frac{d^2\phi}{b^2d\xi^2}{}_{(17)}$$

$$\nabla_5\nabla'\phi-\nabla'\nabla_5\phi=$$

$$-2.(1-\Sigma\xi^2)\Sigma\frac{1}{a^2}\times\Sigma\frac{1}{a^2b^2}\cdot\frac{d^2\phi}{d\xi^2}{}_{(18)}\quad-2.\Sigma\frac{1}{a^2}\times\Sigma\frac{\xi^2}{a^2}\times\Sigma\frac{d^2\phi}{b^2d\xi^2}{}_{(19)}$$

all the finite integrals being taken from $r=1$ to $r=s+1$, and from $r'=1$ to $r'=s+1$.

In order to obtain the required value

$$\nabla'\nabla\phi-\nabla\nabla'\phi,$$

it is clear that we shall only have to add the first of the five preceding quantities to the sum of the four following ones multiplied by h^2, and to render this more easy, we have appended to each of the terms in the preceding quantities a number inclosed in a small parenthesis.

Now since the accents may be permuted at will, and we have likewise $a^2=b^2+h^2$, it is easy to see that the terms marked (1), (6) and (12) mutually destroy each other. In like manner, (2), (3), (7) and (18) mutually destroy each other; the same may evidently be said of (13) and (16), of (15) and (17), of (9) and (19), and of (8) and (14). Moreover the four quantities (4), (5), (10) and (11) will do so likewise, and consequently, we have

$$\nabla'\nabla\phi-\nabla\nabla'\phi=0.$$

Hence the truth of the equation (31) is manifest.

Application of the preceding General Theory to the Determination of the Attractions of Ellipsoids.

13. Suppose it is required to determine the attractions exerted by an ellipsoid whose semi-axes are a', b', c' whether the attracted point p is situated within the ellipsoid or not, the law of the attraction being inversely as the n'^{th} power of the distance. Then it is well known that the required attractions may always be deduced from the function

$$V = \int \frac{\rho'\, dx'\, dy'\, dz'}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{n'-1}{2}}};$$

ρ' being the density of the element $dx'\, dy'\, dz'$ of the ellipsoid, and x, y, z being the rectangular co-ordinates of p.

We may avoid the breach of the law of continuity which takes place in the value of V, when the point p passes from the interior of the ellipsoid into the exterior space, by adding the positive quantity u^2 to that inclosed in the braces, and may afterwards suppose u evanescent in the final result. Let us therefore now consider the function,

$$V = \int \frac{\rho'\, dx'\, dy'\, dz'}{\{(x-x')^2 + (y-y')^2 + (z-z')^2 + u^2\}^{\frac{n'-1}{2}}};$$

this triple integral like the preceding including all the values of x', y', z', admitted by the condition

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} + \frac{z'^2}{c'^2} \angle 1.$$

If now we suppose the density ρ' is of the form

$$\rho' = \left(1 - \frac{x'^2}{a'^2} - \frac{y'^2}{b'^2} - \frac{z'^2}{c'^2}\right)^{\frac{n'-2}{2}} f(x', y', z') \ldots\ldots\ldots\ldots\ldots\ldots (34),$$

which will simplify $f(x', y', z')$ when ρ' is constant and $n' = 2$, and then compare this value with the one immediately deducible from the general expression (28) by supposing for a moment $n' = n$, viz.

$$\rho' = \left(1 - \frac{x'^2}{a'^2} - \frac{y'^2}{b'^2} - \frac{z'^2}{c'^2}\right)^{\frac{n'-4}{2}} F(x', y', z'),$$

we see that the function f will always be two degrees higher than F. But since our formulæ become more complicated in proportion as the degree of F is higher, it will be simpler to determine the differentials of V, because for these differentials the degree of F and f is the same. Let us therefore make

$$A = \frac{1}{1-n'}\frac{dV}{dx} = \int \frac{\rho'\,(x-x')\,dx'\,dy'\,dz'}{\{(x-x')^2+(y-y')^2+(z-z')^2+u^2\}^{\frac{n'+1}{2}}},$$

then this quantity naturally divides itself into two parts, such that

$$A = xA' + A'',$$

$$\text{where } A' = +\int \frac{\rho'\,dx'\,dy'\,dz'}{\{(x-x')^2+(y-y')^2+(z-z')^2+u^2\}^{\frac{n'+1}{2}}},$$

$$\text{and } A'' = -\int \frac{x'\rho'\,dx'\,dy'\,dz'}{\{(x-x')^2+(y-y')^2+(z-z')^2+u^2\}^{\frac{n'+1}{2}}}.$$

By comparing these with the general formula (1), it is clear that $n-1=n'+1$, and consequently $n=n'+2$. In this way the expression (28) gives

$$\rho' = \left(1-\frac{x'^2}{a'^2}-\frac{y'^2}{b'^2}-\frac{z'^2}{c'^2}\right)^{\frac{n'-2}{2}} F(x', y', z'),$$

which coincides with (34) by supposing $F=f$.

The simplest case of the present theory is where $f(x', y', z')=1$, and then by No 11, we have $\phi_0'=1$ and $b_0=1$, when A' is the quantity required, and as the general series (29), No 11, then reduces itself to its first term, we immediately obtain from the formula (30), the value of A' following,

$$A' = -\frac{2\pi^{\frac{3}{2}}\Gamma\left(\frac{n'}{2}\right)}{\Gamma\left(\frac{n'+1}{2}\right)}\,a'b'c'\int_{\infty}\frac{h^{1-n}dh}{abc} \quad\text{..................(35)},$$

because in the present case $H_0=1$, $s=3$, and $n=n'+2$.

Again, the same general theory being applied to the value of A'' given above, we get

$$F(x', y', z') = -x' f(x', y', z') = -x' \text{ (when } f = 1),$$

and hence by No 11, $F(x', y', z') = -a'\xi'$. In this way the series (29) again reduces itself to a single term, in which

$$\phi_0' = \xi', \text{ and } b_0 = -a',$$

and the particular value H_0 corresponding thereto, by omitting the superfluous constant $\dfrac{1}{\sqrt{(a'^2 + b'^2 + c'^2)}}$ will be (No 10),

$$H_0 = a.$$

These substituted in the general formula (30) as before, immediately give

$$A'' = + \frac{2\pi^{\frac{3}{2}} \Gamma\left(\dfrac{n'}{2}\right)}{\Gamma\left(\dfrac{n'+1}{2}\right)} a'^3 b' c' \xi a \int_\infty \frac{h^{1-n'} dh}{a^3 bc},$$

and consequently by reduction since $a\xi = x$,

$$A = xA' + A'' = - \frac{2\pi^{\frac{3}{2}} \Gamma\left(\dfrac{n'}{2}\right)}{\Gamma\left(\dfrac{n'+1}{2}\right)} a'b'c'x \int_\infty \frac{h^{3-n'} dh}{a^3 bc} \quad \text{..............(36)}.$$

The value of A just given belongs to the density

$$\rho' = \left(1 - \frac{x'^2}{a'^2} - \frac{y'^2}{b'^2} - \frac{z'^2}{c'^2}\right)^{\frac{n'-2}{2}}.$$

Hence we immediately obtain without calculation the corresponding values

$$B = \frac{1}{1-n'} \frac{dV}{dy} = - \frac{2\pi^{\frac{3}{2}} \Gamma\left(\dfrac{n'}{2}\right)}{\Gamma\left(\dfrac{n'+1}{2}\right)} a'b'c'y \int_\infty \frac{h^{3-n'} dh}{ab^3 c},$$

$$C = \frac{1}{1-n'} \frac{dV}{dz} = - \frac{2\pi^{\frac{3}{2}} \Gamma\left(\dfrac{n'}{2}\right)}{\Gamma\left(\dfrac{n'+1}{2}\right)} a'b'c'z \int_\infty \frac{h^{3-n'} dh}{abc^3}.$$

If now we suppose moreover

$$D = \frac{1}{1-n'}\frac{dV}{du} = u\int \frac{\rho'\,dx'\,dy'\,dz'}{\{(x-x')^2+(y-y')^2+(z-z')^2+u^2\}^{\frac{n'+1}{2}}},$$

the method before explained (No 11), will immediately give

$$D = -\frac{2\pi^{\frac{3}{2}}\Gamma\left(\frac{n'}{2}\right)}{\Gamma\left(\frac{n'+1}{2}\right)}a'b'c'u\int_\infty \frac{h^{1-n'}dh}{abc},$$

and therefore if for abridgment we make

$$M = (n'-1)\frac{\pi^{\frac{3}{2}}\Gamma\left(\frac{n'}{2}\right)}{\Gamma\left(\frac{n'+1}{2}\right)}a'b'c',$$

the total differential of V may be written

$$dV = M\left\{2xdx\int_\infty \frac{h^{3-n'}dh}{a^3bc} + 2ydy\int_\infty \frac{h^{3-n'}dh}{ab^3c} + 2zdz\int_\infty \frac{h^{3-n'}dh}{abc^3} + 2udu\int_\infty \frac{h^{1-n'}dh}{abc}\right\},$$

which being integrated in the usual way by first supposing h constant, and then completing the integral with a function of h, to be afterwards determined by making every thing in V variable, we get

$$V = M\left\{x^2\int_\infty \frac{h^{3-n'}dh}{a^3bc} + y^2\int_\infty \frac{h^{3-n'}dh}{ab^3c} + z^2\int_\infty \frac{h^{3-n'}dh}{abc^3} + u^2\int_\infty \frac{h^{1-n'}dh}{abc}\right\} + k;$$

k being a quantity absolutely constant, which is equal to zero when $n' > 1$. What has just been advanced will be quite clear if we recollect that h may be regarded as a function of x, y, z and u, determined by the equation

$$1 = \frac{x^2}{a'^2+h^2} + \frac{y^2}{b'^2+h^2} + \frac{z^2}{c'^2+h^2} + \frac{u^2}{h^2} = \xi^2 + \eta^2 + \zeta^2 + v^2 \dots\dots\dots\dots (37);$$

seeing that $a^2 = a'^2 + h^2$, $b^2 = b'^2 + h^2$, and $c^2 = c'^2 + h^2$.

After what precedes, it seems needless to enter into an examination of the values of V belonging to other values of the density ρ', since it must be clear that the general method is equally applicable when

$$\rho' = \left(1 - \frac{x'^2}{a'^2} - \frac{y'^2}{b'^2} - \frac{z'^2}{c'^2}\right)^{\frac{n'-2}{2}} f(x', y', z');$$

where f is the characteristic of any rational and entire function.

The quantity A before determined when we make $u = 0$, serves to express the attraction in the direction of the co-ordinate x of an ellipsoid on any point p, situated at will either within or without it. But by making $u = 0$ in (37) we have

$$1 = \frac{x^2}{a'^2 + h^2} + \frac{y^2}{b'^2 + h^2} + \frac{z^2}{c'^2 + h^2} + \frac{o^2}{h^2} \ldots\ldots\ldots\ldots\ldots (38),$$

and it is thence easy to perceive that when p is within the ellipsoid, h must constantly remain equal to zero, and the equation (38) will always be satisfied by the indeterminate positive quantity $\frac{o^2}{o^2}$. When on the contrary p is exterior to it, h can no longer remain equal to zero, but must be such a function of x, y, z, as will satisfy the equation (38), of which the last term now evidently vanishes in consequence of the numerator o^2. Thus the forms of the quantities A, B, C, D and V all remain unchanged, and the discontinuity in each of them falls upon the quantity h.

To compare the value of A here found with that obtained by the ordinary methods, we shall simply have to make $n' = 2$ in the expression (36), recollecting that $\Gamma(1) = 1$, and $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$. In this way

$$A = -4\pi a'b'c'x \int_\infty \frac{hdh}{a^3bc} = -4\pi a'b'c'x \int_\infty \frac{da}{a^2bc}$$

$$= +4a'b'c'x \int_a^\infty \frac{da}{a^2bc} = 4\pi a'b'c' \int_a^\infty \frac{da}{a^2\sqrt{(a^2 - a'^2 + b'^2)(a^2 - a'^2 + c'^2)}}.$$

But the last quantity may easily be put under the form of a definite integral, by writing $\frac{a}{v}$ in the place of a under the sign of integration, and again inverting the limits. Thus there will result

$$A = \frac{4\pi a'b'c'}{a^3} \int_0^1 \frac{v^2dv}{\sqrt{\left(1 + \frac{b'^2 - a'^2}{a^2} v^2\right)\left(1 + \frac{c'^2 - a'^2}{a^2} v^2\right)}},$$

which agrees with the ordinary formula, since the mass of the ellipsoid is $\frac{4\pi a'b'c'}{3}$ and $a^2 = a'^2 + h^2$.

Examination of a particular Case of the General Theory exposed in the former Part of this Paper.

14. There is a particular case of the general theory first considered, which merits notice, in consequence of the simplicity of the results to which it leads. The case in question is that where we have generally whatever r may be

$$a_r' = a'.$$

Then the equation (19) which serves to determine ϕ, becomes by supposing $k_0 = k \cdot a'^2$

$$0 = (1 - \Sigma_1^{s+1}\xi_r^2)\,\Sigma_1^{s+1}\frac{d^2\phi}{d\xi_r^2} + (s - n - 1)\,\Sigma_1^{s+1}\xi_r\frac{d\phi}{d\xi_r} - k\phi \;\ldots\ldots\ldots\ldots(39).$$

If now we employ a transformation similar to that used in obtaining the formula (14), No 6, by making

$$\xi_1 = \rho\cos\theta_1,\ \xi_2 = \rho\sin\theta_1\cos\theta_2,\ \xi_3 = \rho\sin\theta_1\sin\theta_2\cos\theta_3,\ \&\text{c.}$$

and then conceive the equation (39) deduced from the condition that

$$\int d\xi_1\,d\xi_2\ldots\ldots d\xi_s\,(1 - \Sigma\xi_r^2)^{\frac{n-s+1}{2}}\left\{\Sigma_1^{s+1}\left(\frac{d\phi}{d\xi_r}\right)^2 + \frac{k\phi^2}{1 - \Sigma\xi_r^2}\right\}$$

must be a minimum (vide No 8), we shall have

$$d\xi_1\,d\xi_2\ldots\ldots d\xi_s = \rho^{s-1}\sin\theta_1^{s-2}\sin\theta_2^{s-3}\ldots\ldots\sin\theta_{s-2}\,d\rho\,d\theta_1\,d\theta_2\ldots\ldots d\theta_{s-1},$$

$$\Sigma_1^{s+1}\left(\frac{d\phi}{d\xi_r}\right)^2 = \left(\frac{d\phi}{d\rho}\right)^2 + \frac{1}{\rho^2}\,\Sigma_1^s\,\frac{\left(\frac{d\phi}{d\xi_r}\right)^2}{\sin\theta_1^2\sin\theta_2^2\ldots\ldots\ldots\sin\theta_{r-1}^2},$$

$$\text{and } 1 - \Sigma\xi_r^2 = 1 - \rho^2.$$

Proceeding now in the manner before explained, (No 8), we obtain for the equivalent of (39) by reduction

$$0 = \frac{d^2\phi}{d\rho^2} + \frac{s - 1 - n\rho^2}{\rho(1-\rho^2)}\cdot\frac{d\phi}{d\rho} + \frac{1}{\rho^2}\,\Sigma_1^s\,\frac{\frac{d^2\phi}{d\theta_r^2} + (s - r - 1)\frac{\cos\theta_r}{\sin\theta_r}\frac{d\phi}{d\theta_r}}{\sin\theta_1^2\sin\theta_2^2\ldots\ldots\sin\theta_{r-1}^2} - \frac{k}{1-\rho^2}\,\phi\ldots(40).$$

But this equation may be satisfied by a function of the form

$$\phi = P\Theta_1\Theta_2\Theta_3\ldots\ldots\ldots\Theta_{s-1};$$

P being a function of ρ only, and afterwards generally Θ_r a function of θ_r only. In fact, if we substitute this value of ϕ in (40), and then divide the result by ϕ, it is clear that it will be satisfied by the system

$$\frac{d^2\Theta_{s-1}}{\Theta_{s-1}d\theta^2_{s-1}} = \lambda_{s-1}$$

$$\frac{d^2\Theta_{s-2}}{\Theta_{s-2}\,d\theta^2_{s-2}} + 1\,.\frac{\cos\theta_{s-2}}{\sin\theta_{s-2}}\,\frac{d\Theta_{s-2}}{\Theta_{s-2}d\theta_{s-2}} + \frac{\lambda_{s-1}}{\sin\theta^2_{s-2}} = \lambda_{s-2}\,\ldots\ldots\ldots\ldots\ldots(41),$$

$$\frac{d^2\Theta_{s-3}}{\Theta_{s-3}\,d\theta^2_{s-3}} + 2\,.\frac{\cos\theta_{s-3}}{\sin\theta_{s-3}}\,\frac{d\Theta_{s-3}}{\Theta_{s-3}d\theta_{s-3}} + \frac{\lambda_{s-2}}{\sin\theta^2_{s-3}} = \lambda_{s-3}$$

&c. &c. &c. &c.

combined with the following equation,

$$\frac{d^2P}{Pd\rho^2} + \frac{s-1-n\rho^2}{\rho(1-\rho^2)}\cdot\frac{dP}{Pd\rho} + \frac{\lambda_1}{\rho^2} - \frac{k}{1-\rho^2} = 0\,\ldots\ldots\ldots\ldots\ldots\ldots(42),$$

where k, λ_1, λ_2, λ_3, &c. are constant quantities.

In order to resolve the system (41), let us here consider the general type of the equations therein contained, viz.

$$0 = \frac{d^2\Theta_{s-r}}{d\theta^2_{s-r}} + (r-1)\,\frac{\cos\theta_{s-r}}{\sin\theta_{s-r}}\cdot\frac{d\Theta_{s-r}}{d\theta_{s-r}} + \left(\frac{\lambda_{s-r+1}}{\sin\theta^2_{s-r}} - \lambda_{s-r}\right)\Theta_{s-r}.$$

Now if we reflect on the nature of the results obtained in a preceding part of this paper, it will not be difficult to see that Θ_{s-r} is of the form

$$\Theta_{s-r} = (\sin\theta_{s-r})^i p = (1-\mu^2)^{\frac{i}{2}}p;$$

where p is a rational and entire function of $\mu = \cos\theta_{s-r}$, and i a whole number.

By substituting this value in the general type and making

$$\lambda_{s-r+1} = -\,i\,(i+r-2)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(43)$$

we readily obtain

$$0 = (1-\mu^2)\,\frac{d^2p}{d\mu^2} - (2i+r)\,\mu\,\frac{dp}{d\mu} - \{\lambda_{s-r} + i\,(i+r-1)\}\,p.$$

To satisfy this equation, let us assume

$$p = \Sigma_0^\infty A_t \mu^{e-i-2t}.$$

Then by substituting in the above and equating separately the coefficients of the various powers of μ, we have in the first place from the highest

$$\lambda_{s-r} = -e(e+r-1) \ldots\ldots\ldots\ldots\ldots\ldots(44),$$

and afterwards generally

$$A_{t+1} = -\frac{e-i-2t \,.\, e-i-2t-1}{2t+2 \times 2e+r-2t-3} A_t \,.$$

But the equation (43) may evidently be made to coincide with (44), by writing $i^{(r)}$ for i, and $i^{(r+1)}$ for e, since then both will be comprised in

$$\lambda_{s-r+1} = -i^{(r)}\{i^{(r)}+r-2\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(45).$$

Hence we readily get for the general solution of the system (41),

$$\Theta_{s-r} = (1-\mu^2)^{\frac{i^{(r)}}{2}} \Big[\mu^{i^{(r+1)}-i^{(r)}} - \frac{\{i^{(r+1)}-i^{(r)}\}\{i^{(r+1)}-i^{(r)}-1\}}{2 \times 2i^{(r)}+r-3} \mu^{i^{(r+1)}-i^{(r)}-2}$$

$$+ \frac{\{i^{(r+1)}-i^{(r)}\}\{i^{(r+1)}-i^{(r)}-1\}\{i^{(r+1)}-i^{(r)}-2\}\{i^{(r+1)}-i^{(r)}-3\}}{2.4 \times \{2i^{(r)}+r-3\}\{2i^{(r)}+r-5\}} \mu^{i^{(r+1)}-i^{(r)}-4} - \text{\&c.}\Big];$$

where $\mu = \cos\theta_{s-r}$, and $i^{(r)}$ represents any positive integer whatever, provided $i^{(r)}$ is never greater than $i^{(r+1)}$.

Though we have thus the solution of every equation in the system (41), yet that of the first may be obtained under a simpler form by writing therein for λ_{s-1} its value $-i^{(2)^2}$ deduced from (45). We shall then immediately perceive that it is satisfied by

$$\Theta_{s-1} = \frac{\sin}{\cos}\left\{i^{(2)}\theta_{s-1}\right\}.$$

In consequence of the formula (45), the equation (42) becomes

$$0 = \frac{d^2P}{d\rho^2} + \frac{s-1-n\rho^2}{\rho(1-\rho^2)} \cdot \frac{dP}{d\rho} - \left\{\frac{i^{(s)}(i^{(s)}+s-2)}{\rho^2} + \frac{k}{1-\rho^2}\right\}P,$$

which is satisfied by making $k = -\lambda_1 - (i^{(s)}+2\omega)(i^{(s)}+2\omega+n-1)$, and

$$P = \rho^{i^{(s)}}\{\rho^{2\omega} - \frac{2\omega \times 2i^{(s)} + 2\omega + s - 2}{2 \times 2i^{(s)} + 4\omega + n - 3}\rho^{2\omega-2}$$

$$+ \frac{2\omega \,.\, 2\omega - 2 \times 2i^{(s)} + 2\omega + s - 2 \,.\, 2i^{(s)} + 2\omega + s - 4}{2 \,.\, 4 \times 2i^{(s)} + 4\omega + n - 3 \,.\, 2i + 4\omega + n - 5}\rho^{2\omega-4} - \&c.\}$$

where ω represents any whole positive number.

Having thus determined all the factors of ϕ, it now only remains to deduce the corresponding value of H. But H_0 the particular value satisfying the differential equation in H, will be had from ϕ by simply making therein

$$\xi_r = \frac{a_r}{\sqrt{(\Sigma a_r'^2)}} = \frac{a}{a'\sqrt{s}},$$

since in the present case we have generally $a_r' = a'$.

Hence, it is clear that the proper values of θ_1, θ_2, θ_3, &c. to be here employed are all constant, and consequently the factor

$$\Theta_1 \Theta_2 \Theta_3 \ldots\ldots\ldots\ldots\ldots\ldots \Theta_{s-1}$$

entering into ϕ is likewise constant. Neglecting therefore this factor as superfluous, we get for the particular value of H,

$$H_0 = P_{\frac{a}{a'}};$$

$$\text{since } \rho^2 = \xi_1^2 + \xi_2^2 + \ldots\ldots + \xi_s^2 = \frac{sa^2}{sa'^2} = \frac{a^2}{a'^2},$$

and $P_{\frac{a}{a'}}$ represents what P becomes when ρ is changed into $\dfrac{a}{a'}$.

Substituting this value of H_0 in the equation (25), No 10, there results since $a^2 = a'^2 + h^2$

$$H = K \,.\, P_{\frac{a}{a'}} \int_{\infty} \frac{h^{s-n}\,dh}{P_{\frac{a}{a'}}^2\,(a'^2 + h^2)^{\frac{s}{2}}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (46)$$

K being an arbitrary constant quantity.

Thus the complete value of V for the particular case considered in the present number is

$$V = P\Theta_1\Theta_2\ldots\ldots\Theta_{s-1}.KP_{\frac{a}{a'}}\int_\infty \frac{h^{s-n}dh}{P_{\frac{a}{a'}}^2(a'^2+h^2)^{\frac{s}{2}}} \ldots\ldots\ldots\ldots(47)$$

and the equation (27), No 11, will give for the corresponding value of ρ',

$$\rho' = \frac{-\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{s}{2}}\Gamma\left(\frac{n-s+1}{2}\right)}(1-\rho^2)^{\frac{n-s-1}{2}}\frac{K}{P_1 a'^s}P'\Theta_1'\Theta_2'\ldots\ldots\Theta'_{s-1}\ldots\ldots(48);$$

where P_1', Θ_1', Θ_2', &c. are the values which the functions P, Θ_1, Θ_2, &c. take when we change the unaccented variables $\xi_1, \xi_2,\ldots\ldots\xi_s$ into the corresponding accented ones $\xi_1', \xi_2',\ldots\ldots\xi_s'$, and

$$P_1 = \frac{n-s+1\,.\,n-s+3\,.\ldots\ldots\ldots\ldots.\,n-s+2\omega-1}{n+2i+2\omega-1\,.\,n+2i+2\omega+1\,.\ldots\ldots\ldots.\,n+2i+4\omega-3},$$

or the value of P when $\rho=1$; where as well as in what follows i is written in the place of $i^{(s)}$.

The differential equation which serves to determine H when we introduce a instead of h as independent variable, may in the present case be written under the form

$$0 = a^2(a^2-a'^2)\frac{d^2H}{da^2} + a^2\{na^2-(s-1).a'^2\}\frac{dH}{ada}$$
$$+\{i(i+s-2)a'^2-(i+2\omega)(i+2\omega+n-1)a^2\}H,$$

and the particular integral here required is that which vanishes when h is infinite. Moreover it is easy to prove, by expanding in series, that this particular integral is

$$H = k'a^i\Delta^\omega.a^{2r}\int_\infty a^{1-2r-s-2i}da\,(a^2-a'^2)^{\frac{s-1-n-2\omega}{2}};$$

provided we make the variable r to which Δ^ω refers, vanish after all the operations have been effected.

But the constant k' may be determined by comparing the coefficient of the highest power of a in the expansion of the last formula with the like coefficient in that of the expression (46), and thus we have

$$k' = Ka'^{i+2\omega}(-1)^\omega\frac{n+2i+2\omega-1\,.\,n+2i+2\omega+1\,.\ldots\ldots.\,n+2i+4\omega-3}{2\,.\,4\,.\,6\ldots\ldots.\,2\omega}.$$

Hence we readily get for the equivalent of (47),

$$V = P\Theta_1\Theta_2\ldots\ldots\Theta_{s-1} \times \frac{n+2i+2\omega-1\,.\,n+2i+2\omega+1\,.\ldots\ldots.\,n+2i+4\omega-3}{2\,.\,4\,.\,6\ldots\ldots.\,2\omega}$$

$$\ldots\ldots \times Ka'^{i+2\omega}(-1)^{\omega}a^{i}\Delta^{\omega}a^{2r}\int_{\infty} da\,a^{1-2r-s-2i}(a^2-a'^2)^{\frac{s-1-n-2\omega}{2}}.$$

In certain cases the value of V just obtained will be found more convenient than the foregoing one (47). Suppose for instance we represent the value of V when $h=0$, or $a=a'$ by V_0. Then we shall hence get

$$V_0 = P\Theta_1\Theta_2\ldots\ldots\Theta_{s-1} \times \frac{n+2i+2\omega-1\,.\,n+2i+2\omega+1\,.\ldots\ldots.\,n+2i+4\omega-3}{2\,.\,4\,.\,6\,.\ldots\ldots.\,2\omega}$$

$$\times Ka'^{2i+2\omega}(-1)^{\omega}\,.\,\Delta^{\omega}a'^{2r}\int_{\infty}^{a'} da\,a^{1-2r-s-2i}(a^2-a'^2)^{\frac{s-1-n-2\omega}{2}} \ldots\ldots (\delta),$$

which in consequence of the well known formula

$$\int_{\infty}^{a'} a^{-m}da\,(a^2-a'^2)^{-p} = -a'^{1-m-2p} \times \frac{\Gamma(1-p)\,\Gamma\left(\frac{m+2p-1}{2}\right)}{2\Gamma\left(\frac{1+m}{2}\right)},$$

by reduction becomes

$$V_0 = -P\Theta_1\Theta_2\ldots\ldots\Theta_{s-1} \times \frac{\Gamma\left(\frac{1+s-n}{2}\right)\Gamma\left(\frac{n+2i+4\omega-1}{2}\right)}{2\Gamma(\omega+1)\,\Gamma\left(\frac{s+2i+2\omega}{2}\right)} Ka'^{1-n} \ldots\ldots (49);$$

since in the formula (δ), r ought to be made equal to zero at the end of the process.

By conceiving the auxiliary variable u to vanish, it will become clear from what has been advanced in the preceding number, that the values of the function V *within* circular planes and spheres, are only particular cases of the more general one, (49), which answer to $s=2$ and $s=3$ respectively. We have thus by combining the expressions (48) and (49), the means of determining V_0 when the density ρ' is given, and *vice versa;* and the present method of resolving these problems seems more simple if possible than that contained in the articles (4) and (5) of my former paper.

GEORGE GREEN.

XVI. *On the Position of the Axes of Optical Elasticity in Crystals belonging to the Oblique-Prismatic System. By* W. H. MILLER, A.M. *Fellow and Tutor of St John's College, and Professor of Mineralogy.*

[Read *Dec.* 8, 1834.]

1. FRESNEL has proved that whatever be the regular arrangement of the medium which by its elasticity produces the optical properties of a crystal, there are always three directions at right angles to each other, which may be considered as axes of optical elasticity. This being understood, it is further already established, that crystals belonging to the tesseral system have three equal axes of optical elasticity; that rhombohedral and pyramidal crystals have two axes of elasticity equal to each other and perpendicular to the crystallographic axis, which therefore is the third axis of elasticity and also an optic axis; and that crystals belonging to the remaining systems have three unequal axes of elasticity, and consequently two optic axes (that is, axes of optical phenomena) making with each other angles which are bisected by the axes of greatest and least elasticity.

Sir David Brewster, who discovered the mutual dependence of the forms and optical properties of crystals, has determined the angles between the optic axes of a great number of biaxal crystals; his observations, however, do not contain any data from which the positions of the axes with respect to the faces of the crystals can be found.

2. In the right prismatic system the axes of elasticity coincide (as might have been expected) with the rectangular crystallographic axes. In the oblique prismatic system, if the three axes be XX', YY', ZZ', the crystallographic axis (YY'), which is perpendicular to the other two (XX', ZZ'), is always one of the axes of elasticity. This, in Gypsum, at the ordinary temperature of the air, and in many other crystals, is the mean axis, or it is perpendicular to the optic axes; in

Borax, Acetate of Soda, Felspar, Tartaric Acid and Gypsum, when heated to about $100^{\circ} C$, as was first observed by Mitscherlich, it is the greatest or least axis of elasticity, and is therefore in the same plane with the optic axes and makes equal angles with them.

The position of one axis of elasticity having thus an evident relation to the crystallographic form, we are naturally led to inquire if any relation can be discovered between the other two axes of elasticity and the crystallographic form. The only attempts to discover any such relation, with which I am acquainted, are those of M. Soret, (Mémoires de la Société de Physique de Gèneve, tome I.) and Professor Neumann of Königsberg (Poggendorff's Annalen, B. XXVII. S. 240). Neumann shews, that in Gypsum the axes of elasticity and also the thermal axes, or the three lines in the crystal which remain at right angles to each other at all temperatures, constitute a system of rectangular crystallographic axes. It appeared at first sight not improbable that a similar relation might be found to exist between the form and axes of elasticity of other oblique-prismatic crystals. Though my observations appear to disprove the law which has thus been suggested, they do not establish any other in the place of it. The only general fact which I have noticed is, that in many instances, though not in all, one of the two axes of elasticity which are perpendicular to YY', is also the axis of one of the principal zones of the crystal.

3. To find the angle between a normal to any face (T) of a crystal, and the apparent direction of one of the optic axes as seen in air through any parallel faces of the crystal.

Let the crystal be attached to an index, moveable on a graduated circle having its plane parallel to the axis of the polarizing instrument, or a table on which the position of the index may be marked by a line drawn along its edge with a tracing point. Let the crystal be placed in such a position, that the apparent direction of the optic axis in air and a normal to T may be parallel to the circle. Move the index till the center of the coloured rings coincides with a mark in the axis of the polarizing instrument, and observe the points in which it meets the circle. Turn the crystal half round in the plane of T, taking care not to alter the inclination of T to the index, (this may be effected by moving the crystal, the index being fixed, till the image of some well defined object seen by reflexion in T appears in the same direction after the crystal is turned as it did before.) If the index be now turned till the center of the coloured rings coincides with the mark, the angle it has described between the observations will be manifestly equal to twice the angle between the apparent direction of the optic axis in air and a normal to T. The angle between the optic axis in air and a normal to any other known face of the crystal being found in the same manner, the direction of the optic axis in air will be completely determined.

4. To find the optic axes, their apparent directions in air being known.

Let QR, $Q'R'$ (Fig. 1.) be tangents to the circular and elliptic sections of a wave diverging from O made by a plane through the optic axes, and therefore OQ, OQ', perpendiculars to QR, will be the optic axes; OP the direction in which the optic axis OQ is seen in air; OS a perpendicular to the faces through which it is seen.

The vibrations in that part of the wave which has a circular section are perpendicular to the plane QOQ', consequently a ray polarized in the plane QOQ' is refracted in that plane according to the law of sines. Let μ be the ratio of the sine of incidence to the sine of refraction for such a ray out of air into the crystal, D the minimum division of the ray when refracted in the plane QOQ' through the prism formed by two natural or artificial planes meeting at an angle I in a line perpendicular to QOQ'. Then $\mu \sin \frac{1}{2} I = \sin \frac{1}{2}(D + I)$, and $\mu \sin QOS = \sin POS$. Whence the direction of QO is known. $Q'O$ being found in the same manner, the axes of elasticity $O\xi$, $O\zeta$, which bisect the angles qOQ, QOQ', are also known.

5. The diagram which accompanies the description of each crystal, is the representation of a sphere, to the surface of which the faces of the crystal are referred by means of perpendiculars drawn from the center of the sphere. The point in which the perpendicular to any

face meets the surface of the sphere, will be called the pole of that face. The measurements express the angles between the perpendiculars to the faces, or the supplements to the angles between the faces themselves. This method of representing crystalline forms appears to have been first employed by Neumann, in his Beiträge zur Krystallonomie, and afterwards by Grassmann and Uhde. It has the advantage of exhibiting all the faces of a crystal without confusion in one figure, each zone being distinguished by a great circle drawn through the poles of the faces composing it, and also of allowing all the requisite calculations to be performed by spherical trigonometry applied to the equations

$$\frac{a}{h}\cos PX = \frac{b}{k}\cos PY = \frac{c}{l}\cos PZ,$$

or to formulæ deduced therefrom, X, Y, Z being the points in which radii parallel to the axes of the crystal meet the surface of the sphere, and P the pole of the face (h; k; l), which is parallel to the plane

$$h\frac{x}{a} + k\frac{y}{b} + l\frac{z}{c} = 0.$$

$\alpha\alpha'$, $\beta\beta'$, $\xi\xi'$, $\zeta\zeta'$ will be used to denote the extremities of diameters drawn parallel to the optic axes, and the two axes of elasticity which are perpendicular to YY'. In Figs. 5, 6, 7, 8 the faces are denoted by the same letters as in the treatises of Mohs and Naumann. The inclinations of the faces of crystal (1) and (2) are deduced from a mean of the best measurements of thirty or forty crystals, and are probably within 1′ of the truth.

The chemical notation and atomic weights are those employed by Dr Turner, in the fifth edition of his Elements of Chemistry.

EXAMINATION OF VARIOUS CRYSTALS ACCORDING TO THE METHODS ABOVE EXPLAINED.

(1). *Sulphate of Oxide of Iron and Ammonia.* According to Mitscherlich (Jahresbericht 13), the composition of this salt, which belongs to an extensive plesiomorphous group, is expresssed by the formula $H^3N\ddot{S} + Fe\ddot{S} + 7\dot{H}$. Fig. 2. represents the poles of its faces. Their symbols are $A(1;\ 0;\ 0)$, $C(0;\ 0;\ 1)$, $H(0;\ 1;\ 1;)$ $M(1;\ 1;\ 0)$, $P(1;\ 1;\ 1)$, $Q(-1;\ 1;\ 1)$, $T(2;\ 0;\ 1)$.

AT	42°,14′	CYQ	28°,48′	HH'	129°,18′	MQ	42°,23′
TC	64 ,34	QYA'	44 ,54	CH	25 ,21	QC	34 ,20
CA'	73 ,12	MM'	109 ,36	QQ'	140 ,55	TP	35 ,14
AYP	68 ,12½	AM	35 ,12	CP	44 ,45	MT	52 ,46.
PYC	38 ,35½	PP'	130 ,37	PM	58 ,32		

When yellow light is refracted through the faces TC' in the plane ACA', the minimum deviation of a ray polarized in the plane ACA', is 41°, 26′. The apparent direction of the optic axis aa' in air, when seen through the faces TT'', makes an angle of 7°,10′ with TT''; and the optic axes appear to be inclined to each other at an angle of 79° when the crystal is immersed in oil, of which the index of refraction is 1,47. From these data we find $T\alpha = 4°,47'$, $T\beta = 71°,2'$, $T\xi = 33°,8'$, $A\xi = 9°,6'$.

Tan $T\xi$ is nearly equal to $4 \tan A\xi$. The value of $A\xi$ deduced from the equations $\tan T\xi = 4 \tan A\xi$, $T\xi + A\xi = 42°, 14'$ is 9°, 13′½. This would make $C\xi = 82°,25'\frac{1}{2}$. Now, $46 \tan 9°, 13'\frac{1}{2} = \tan 82°,22'\frac{1}{2}$; therefore, if we refer the faces T, A, C, to the rectangular axes $\xi\xi'$, YY', $\zeta\zeta'$, neglecting the difference of 3′ in the value of $C'\xi$, their simplest symbols will be $(1;\ 0;\ 1)$, $(4;\ 0;\ -1)$, $(2;\ 0;\ -23)$. The magnitude of the last index renders the hypothesis that $\xi\xi'$, $\zeta\zeta'$ are crystallographic axes highly improbable.

(2). The composition of *Tartrate of Ammonia* is expressed, according to Dulk, (Jahrbuch für Chemie und Physik, 1831. B. 1.) by the formula $H^3N\overline{T} + 2\dot{H}$. The poles of its faces are represented in Fig. 3.

$A(1;\ 0;\ 0)$, $C(0;\ 0;\ 1)$, $H(0;\ 1;\ 1)$, $K(1;\ 0;\ 1)$, $L(-1;\ 0;\ 1)$, $M(1;\ 1;\ 0)$, $P(1;\ 1;\ 1)$, $Q(-1;\ 1;\ 1)$.

Cleavage parallel to the face A.

AK	$52^0,31'$	AM	$55^0,2'$	QQ'	$97^0,27'$	QA'	$60^0,54'\frac{1}{2}$
KC	$39\ ,53$	HH'	$81\ ,46$	QL	$41\ ,16\frac{1}{2}$	CP	$55\ ,34$
CL	38	CH	$49\ ,7$	AP	$63\ ,22$	PM	$35\ ,48\frac{1}{2}$
LA'	$49\ ,36$	PP'	$94\ ,55$	PH	$28\ ,12$	MQ'	$34\ ,53$
MM'	$69\ ,56$	PK	$42\ ,32$	HQ	$27\ ,31\frac{1}{2}$	QC	$53\ ,44\frac{1}{2}$.

$D=25^0,17'$, the light being refracted through CK. The apparent angle in air between the optic axes aa' and AA', is $4^0,55'$. In oil, the index of refraction of which is 1,741, the apparent angle between the optic axes $=42^0,20'$. This gives $A\alpha=3^0,7'$, $A\beta=35^0,54'$, $A\xi=16^0,24'$, $L'\xi=33^0,12$.

In this case the positions of some of the faces A, K, C, L must be altered half a degree before they can be referred to the rectangular axes $\xi\xi'$, YY', $\zeta\zeta'$ with tolerably simple indices.

(3). A solution of *Benzoic acid* in alcohol, when suffered to evaporate, affords *crystals* of which the faces C, K, I (Fig. 4) alone are bright. $CK=69^0,25'$, $CI=97^0,20'$ nearly. $D=64^0,45'$, refraction taking place through the faces CK. The apparent direction of aa' in air when seen through CC' makes with CC' an angle of $4^0, 30'$. When immersed in oil of which the index of refraction is 1,471, the apparent angle between the optic axes is 75^0. Hence $C\alpha=2^0,47'$, $C\beta=59^0,50'$, $C\zeta=28^0,31'$, $K\zeta=40^0,54'$.

$$\tan K\xi,\ \tan I\xi,\ \tan C\xi \text{ are nearly as the numbers } 3,\ 1,\ 5.$$

The equation $\frac{1}{3}\tan K\xi=\tan I\xi=\frac{1}{5}\tan C\xi$ is satisfied by making $C\xi=27^0,56'\frac{1}{2}$, $IC=97^0,17'$. Hence the faces C, I, K may be referred to the rectangular axes $\xi\xi'$, YY', $\zeta\zeta'$ without greatly altering the observed angles, and their symbols will be $(-1;\ 0;\ 5)$, $(1;\ 0;\ 1)$, $(1;\ 0;\ 3)$ respectively.

(4). In *Felspar* (Fig. 5.) the optic axes lie in the plane of the most perfect cleavage, and make with a normal to *M*, angles of about 57^0 or 58^0, ($58^0\frac{1}{2}$ according to Sir David Brewster) which increase when the crystal is heated. Hence, $\xi\xi'$ is the axis of the zone *PM*.

(5). The optic axes of *Pyroxene* (Fig. 6.) seen in air through a slice cut perpendicular to *MM* are in the plane *Pr*, and make angles of 16^0 with the axis of the zone *MM*. Hence, $\zeta\zeta'$ is the axis of the zone *MM'*. α, β approach ζ when the crystal is heated. At ordinary temperatures $\alpha\beta$ is probably about $19^0\frac{1}{2}$. The best measurements of Pyroxene shew that *Pr*, *tr* are nearly but not exactly equal, and therefore, that its faces cannot be referred to $\xi\xi'$, YY', $\zeta\zeta'$ as crystallographic axes. In all the crystals of Pyroxene which I have examined, the rings surrounding $\alpha\alpha'$ are brighter than the rings surrounding $\beta\beta'$.

(6). The form of *Borax* (Fig. 7.) closely resembles that of Pyroxene; its optic axes however are very differently situated. It was observed by Sir John Herschel and also by Professor Nörrenberg, that the optic axes for different colours do not lie in the same plane. This being the case, we cannot expect to find any simple connexion between the form and the directions of the axes of elasticity.

The mean directions of the axes seen in air through the faces *TT'* make angles of $29^0\frac{1}{2}$, with a normal to the faces *TT'*, and a perpendicular to them makes an angle of 55^0 with *MM'*. The rings surrounding $\alpha\alpha'$, $\beta\beta'$ are indistinct on the sides towards *M'P* and *MP'* respectively, the extremities α, β of the axes being next to the eye of the observer. This shews that the positions of $\xi\xi'$, $\zeta\zeta'$ vary slightly with the colour of the light employed.

(7). In *Chromate of Oxide of Lead*, as I have been informed by Professor Nörrenberg of Tübingen, the axis of the zone $M\dot{M}$ (see the figure in Phillips or Naumann) bisects the angle between the optic axes, and is therefore one of the axes of elasticity. The other two axes of elasticity are, without doubt, the lines which bisect the angles formed by normals to *MM'*.

(8). In *Epidote*, (Fig. 8.) the optic axis aa' seen in air through the faces r, r', makes with rr' an angle of $8^{\circ}, 50'$, $\beta\beta'$ seen in air through the faces M, M', makes with MM' an angle of $31^{\circ},50'$. The determination of μ is rendered difficult by the complete absorption of the light polarized in the plane MT. Assuming $\mu=1{,}7$, which is probably near the truth, we get $ra=5^{\circ},11'$, $M\beta=18^{\circ},5'$. According to Mohs $Tr=51^{\circ},41'$, $TM=64^{\circ},30$; therefore, $Ta=46^{\circ},30'$, $T\beta=46^{\circ},31'$. Hence $\zeta\zeta'$ is the axis of the zone PT. The near approximation of the values of Ta, $T\beta$ to equality must be considered accidental, as the positions of the optic axes are usually uncertain to the amount of some minutes.

The question whether any proposed lines are crystallographic axes must be decided, as has already been intimated, by the simplicity and symmetry of the numerical relations which the expression of the faces requires with reference to these axes. This according to the old Hauyian views of the structure of crystals, is equivalent to saying that the primitive form must be such that the other forms can be derived from it by simple laws of decrement. Now, we find that by assuming the axes of elasticity to be crystallographical axes, we have in the crystal (1) a face $(2;\ 0;\ -23)$, which though not very probable is not impossible, and in (5) a face $(-1;\ 0;\ 5)$; in (2) the observed and computed positions of some of the faces differ half a degree.

In (6), the optical properties are not symmetrical.

In (4), (5), (7), (8) one of the axes of elasticity $\xi\xi'$ or $\zeta\zeta'$ is the axis of a zone.

W. H. MILLER.

ST JOHN'S COLLEGE,
Dec. 8, 1834.

Fig. I.

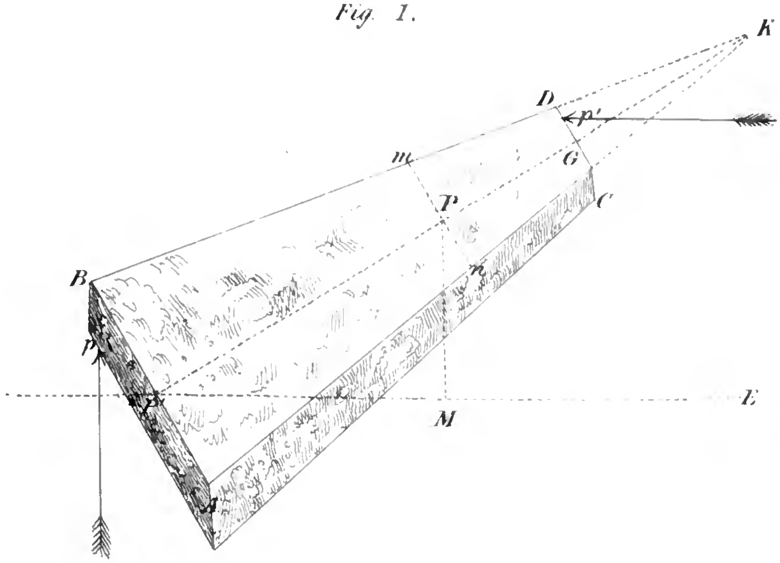
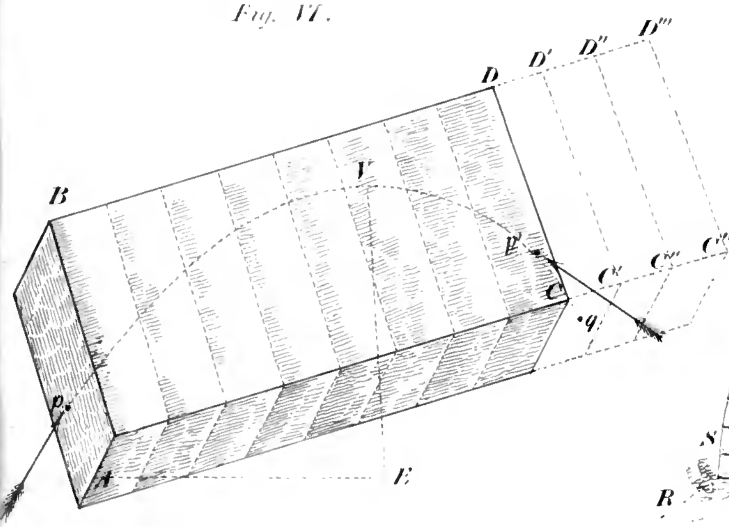


Fig. II.

Fig. VI.



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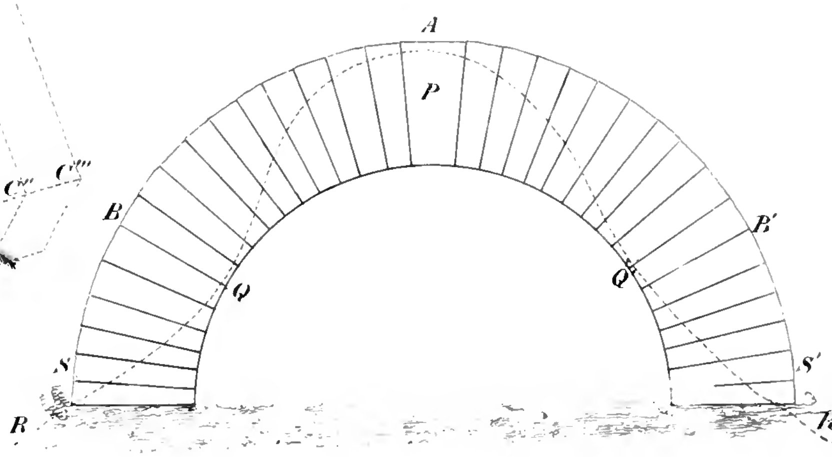

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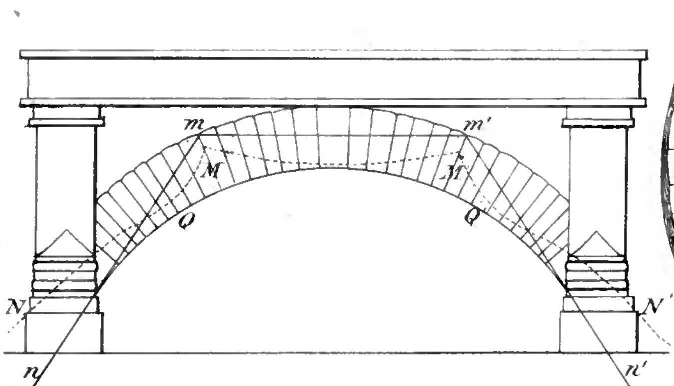

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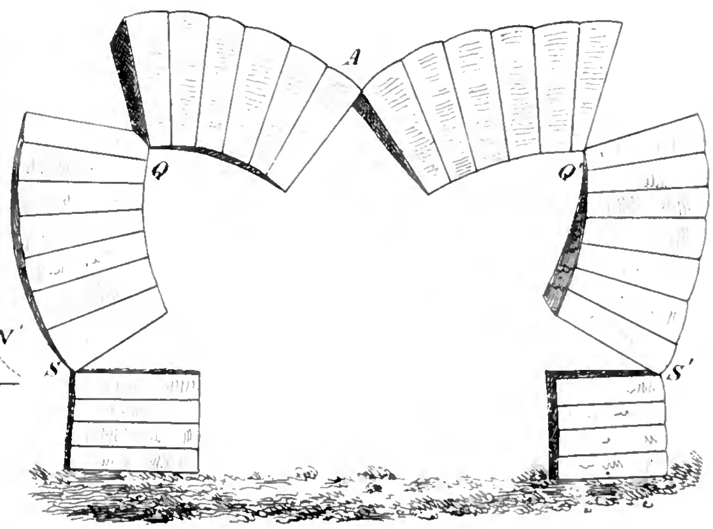
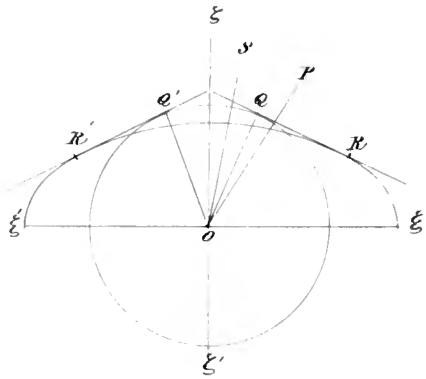


Fig. I.



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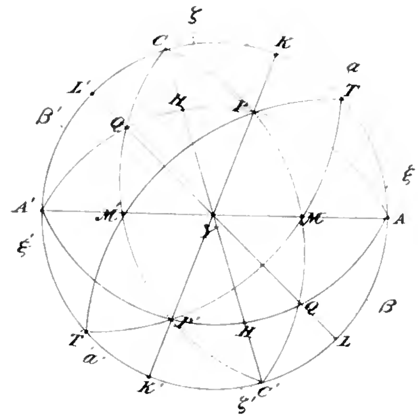

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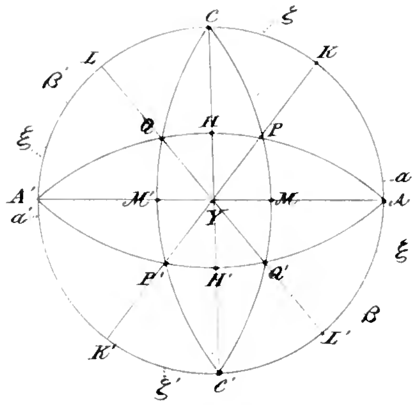

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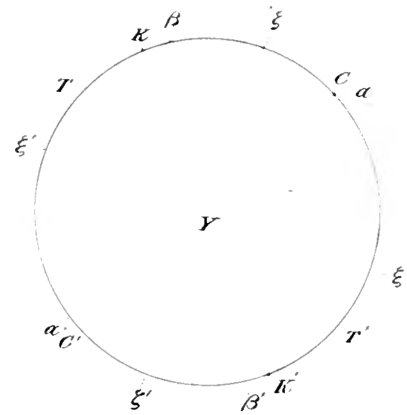

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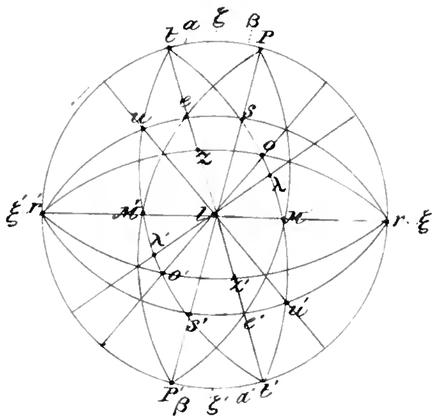

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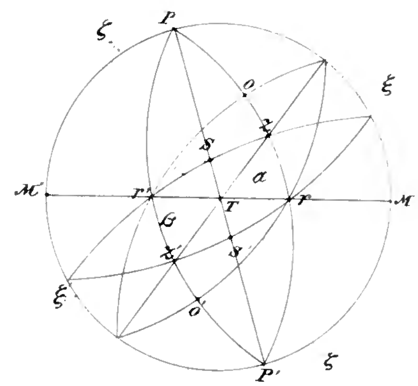

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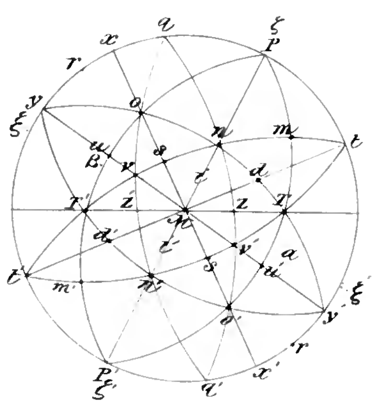

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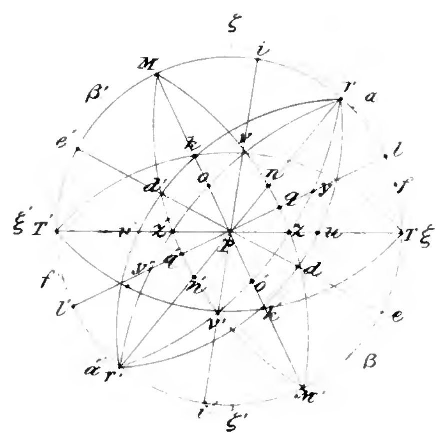

Metcalfe, lith. Cambridge.

93