TRANSACTIONS

OF THE

CAMBRIDGE

PHILOSOPHICAL SOCIETY.

ESTABLISHED NOVEMBER 15, 1819.

VOLUME THE SIXTH.

CAMBRIDGE:
PRINTED AT THE PITT PRESS;
AND SOLD BY
JOHN WILLIAM PARKER, WEST STRAND, LONDON;
J. & J. J. DEIGHTON; AND T. STEVENSON, CAMBRIDGE.

M.DCCC.XXXVIII.

CONTENTS OF THE SIXTH VOLUME.

Part I.

Part II.

PART III.

ADVERTISEMENT.

THE Society as a body is not to be considered responsible for any facts and opinions advanced in the several Papers, which must rest entirely on the credit of their respective Authors.

THE Society takes this opportunity of expressing its grateful acknowledgments to the Syndics of the University Press, for their liberality in taking upon themselves the expense of printing this Part of its TRANSACTIONS.

TRANSACTIONS

OF THE

CAMBRIDGE

PHILOSOPHICAL SOCIETY.

VOLUME VI. PART I.

CAMBRIDGE:
PRINTED BY JOHN SMITH, PRINTER TO THE UNIVERSITY;
AND SOLD BY
JOHN WILLIAM PARKER, 445 WEST STRAND, LONDON;
J. & J. J. DEIGHTON; AND T. STEVENSON,
CAMBRIDGE.

M.DCCC.XXXVI.

I. *Researches in Physical Geology. By* WILLIAM HOPKINS, M.A., *Fellow of the Cambridge Philosophical Society, and of the Geological Society, and Mathematical Lecturer of St Peter's College, Cambridge.*

[Read *May* 4, 1835.]

INTRODUCTION.

NOTWITHSTANDING the appearances of irregularity and confusion in the formation of the crust of our globe which are presented to the eye in the contemplation of its external features, geologists have been able in numerous instances to detect, in the arrangement and position of its stratified masses, distinct approximations to geometrical laws. In the phenomena of anticlinal lines, faults, fissures, mineral veins, &c., such laws are easily recognized; and though, when we consider how large a portion of the surface of the earth remains geologically unexplored, it may appear premature to assert that these are perfectly *general laws*, yet, founding our reasoning on our knowledge, and not on our ignorance, and feeling that confidence which we are entitled to feel in the universality of the laws and operations of nature, we shall, I conceive, be justified, if not in the absolute conclusion, at least in the presumption, that the laws already observed in phenomena such as those above mentioned will be found, by the wider extension and increased accuracy of geological research, to be the approximative *general* laws of those phenomena.

If the legitimacy of this inference be allowed, we are necessarily led to the conclusion, that the phenomena alluded to are referrible not to the particular and irregular action of merely local causes, but to the more widely diffused action of some simple cause, general in its nature with respect to every part of the globe, and general in its action at least with respect to the whole of each district throughout which the phenomena are observed to approximate, without interruption to the same geometrical laws. Between these phenomena and their actual causes necessary relations must of course exist; and in this paper I purpose to examine how far such relations do exist between our observed phenomena and a certain general cause to which they may be attributed. But in the first place it will be necessary to state distinctly the nature of the phenomena to which I refer, though without entering into more detail than may be necessary for my immediate object.

I. *Faults.*

a. In districts where faults abound, two distinct systems are usually found, in each of which the faults approximate to parallelism* with each other.

β. The common direction of one of these systems is approximately perpendicular to that of the other.

γ. The plane in which the dislocation at a fault has taken place is frequently somewhat inclined to the vertical; and it appears that the side of the fault on which the strata are most elevated, is more frequently that *towards* which the plane of dislocation inclines *from* a vertical through the lowest point of a section of the fault, by a vertical plane transverse to the plane of dislocation †.

II. *Mineral Veins.*

A distinct idea of a *mineral vein* is perhaps most easily formed by conceiving a vertical fissure, varying in width from a few inches to a few feet, to have been formed, extending downwards from the surface, and to have been subsequently filled up with matter in the midst of which the ore which properly constitutes the *mineral vein* ‡ is deposited,

* This term must in certain cases be taken in a modified sense, as will be explained hereafter, whatever may be the phenomena to which it is applied.

† See Encyclopedia Metropolitana, *Art.* Geology, p. 541.

‡ This is termed by miners in the Northern districts a *Rake-Vein.* In Cornwall the whole substance contained in the fissure is called a *Lode.*

sometimes in a regular vertical layer, and sometimes in irregular and detached masses. I shall therefore occasionally, without wishing to prejudge the question of the formation of veins, speak of the *fissures* in which they are deposited.

α. The *direction* of the intersection of a vein with a horizontal plane usually approximates to rectilinearity. It is not meant that every short portion of this intersection forms a straight line, but, when considered with reference to its whole extent, these variations are not for the most part considerable.

β. In every mining district the largest and most important veins are divided into two distinct groups, in each of which a very decided approximation to *parallelism* is observable, and of which the directions are nearly perpendicular to each other.

γ. When the veins occur in stratified masses, the *direction* of one of these systems usually coincides with that of the general dip of the strata, the other being consequently perpendicular to that direction*.

δ. A large proportion of the most productive mineral veins are found in the former of these systems. The latter (frequently termed by the miner *cross courses*) carry ore very irregularly.

ϵ. It seems doubtful whether any actual limits of a fissure containing a mineral vein were ever arrived at by the miner, though the division of a large fissure into several small ones not unfrequently seems to indicate a near approach to such a limit in the direction of its length. I know of no case, however, in which such indications have been observed of an approach to both extremities of a large vein. It is probable that their linear extent is frequently much greater than has yet been, or in many cases ever can be, observed. In numberless instances they

* I first observed this relation between the general direction of the mineral veins and that of the dip of the strata in the mining district of Derbyshire. I find on enquiry that the same relation holds in the Alston-moor district, and in Flintshire. In Cornwall also, when the lodes are in stratified rock, I apprehend this is generally the case, assuming the killas formation in the immediate vicinity of the granite to be stratified.

have been traced for four or five miles in the mining districts of this country, and in some cases to the distance of eight or ten miles.

ζ. Their depth appears to be uniformly greater than that to which man has been able to penetrate.

η. The width of the fissures in that system of the two above mentioned which contains the most productive and the most continuous mineral veins, varies in general from a few inches to about 12 feet. In the same vein the width will frequently vary, and sometimes suddenly along the same vertical line. In passing through a horizontal bed of clay the fissure will be sometimes almost entirely closed; and the toad-stone of Derbyshire produces the same effect, frequently closing the fissure so effectually that it can only be traced through it by means of small ramifying veins of calcareous spar. The average width however does not appear at all to diminish as we descend*. The strata through which the fissure penetrates generally form well defined though uneven walls bounding it on either side, and perfectly firm and solid, except where the strata themselves cease to be so.

θ. The width of the cross courses is frequently greater than that above stated, and generally much more irregular.

ι. The fact of the strata in one wall of a fissure being higher than the same strata in the opposite one, has been recognized by all miners in some parts of almost every vein of consequence that has been explored, when existing in a distinctly stratified mass. This difference in general does not exceed a few feet, though it has not unfrequently been found to be many fathoms, in which case the vein of course coincides with a *fault*. This is sometimes termed by miners the *throw*† of the vein.

* In the mining district about Alston-moor there appears to be a few exceptions to this rule, as well as to the assertion of the preceding paragraph (ζ), in what are termed *gash veins*. These are comparatively wide at the top, and become gradually narrower as they descend, till they appear to terminate. (See Forster's account of this district, p. 186.) They are probably *rents* the formation of which began at the surface, but are hardly worthy of notice as exceptions to our general rules.

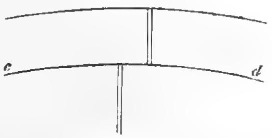
† A *throw* is in fact a small fault.

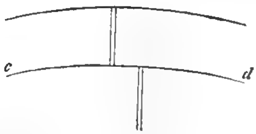
κ. The *inclination* of the plane of the fissure to a vertical plane, which is frequently termed by the miners of the more northern districts the *hade* of the vein, and by the Cornish miners its *underlie*, is very uncertain, amounting not unfrequently to as much perhaps as 20°, generally, however, to considerably less, though in particular cases to considerably more. It will sometimes vary at different depths along the same vertical line, so that in some instances, when the *hade* is small, it will be in one direction in the upper, and in the contrary direction in the lower part of the vein. Upon the whole, however, the hade is not very great, and tolerably regular in each vein*.

μ. Masses of the adjoining rock, more or less perfectly detached from it, are frequently found imbedded in the matter which occupies the fissure†.

ν. Apparent or real displacements in the position of a vein are frequently observed at its intersection with another vein, or with some particular stratified bed, which is generally found to be a bed of moist slimy clay. These intersections are of various kinds.

ο. First, that of a vertical or nearly vertical vein, with a clay bed horizontal, or nearly so. The displacements in this case are shewn in the figures annexed, which represent vertical sections perpendicular to the plane of the vein.





It is manifest that here either the part of the vein above the stratum *cd* has been moved, or that below it, or both, if the two portions were ever in the same plane.

* The underlie of the Cornish lodes is frequently greater, I conceive, than in our other mining districts. It may possibly also be more irregular.

† These insulated masses are frequently termed by miners, *Riders*.

π. Secondly, we may have the intersection of two vertical veins, the planes of which are inclined to each other at any given angle. In such case it frequently happens, that while the continuity of one vein is preserved that of the other is broken, apparently by a relative displacement of the portions on opposite sides of the unbroken vein. This kind of displacement is exhibited in the annexed figures, which represent horizontal sections.

ρ. Thirdly, we may have the intersection of veins the planes of which are inclined, but at different angles, to a vertical plane. If such veins be near enough to each other, their intersection will take place sufficiently near the surface to be within the limits of observation, and if they meet the horizontal surface in parallel lines their line of intersection will be horizontal. If the subjoined figures represent vertical

sections perpendicular to this line, the displacements observed will be such as they exhibit.

These phenomena of faults, and mineral veins, are those which appear to approximate the most distinctly to well defined laws, and therefore afford the best means of testing the truth of any theory of elevation. The following phenomena also bear equally on the investigations contained in this paper, though their characters are in general much less distinct than those of the phenomena already cited.

III. *Anticlinal and Synclinal Lines.*

When two or more anticlinal lines, with the corresponding synclinal ones, are found in the same geological district*, their general directions frequently approximate to parallelism with each other†.

IV. *Longitudinal Valleys.*

α. Along the flanks of elevated ranges, longitudinal valleys are not unfrequently found running nearly parallel to the general axis of elevation ‡.

β. The partial elevations along the sides of an elevated range have usually these escarpments presented towards the central ridge ‖.

* I mean by a *geological district*, any tract of country throughout which the phenomena may be regarded as following the same laws without discontinuity.

† If we take two planes coinciding at any proposed point of an anticlinal line, with the portions of the surface of a stratified bed on opposite sides of that line, these planes of stratification will intersect in a straight line not necessarily horizontal; and the *direction* of the anticlinal line at the proposed point will be determined by the azimuth of a vertical plane drawn through this intersection, or the direction of the intersection of this vertical plane with the horizon. Again, if through the proposed point we draw vertical planes respectively perpendicular to the two planes of stratification above mentioned, their respective intersections with them will be the lines of greatest inclination of the strata, and consequently the azimuths of these vertical planes will determine the *directions of the dip*. The angles between these two latter vertical planes, and the one before mentioned as determining the direction of the anticlinal line, will not generally be equal; they will become so only when the inclination of the planes of stratification on either side of the line is the same; *i. e.* the directions of the dip on opposite sides of an anticlinal line at any proposed point of it will not generally make equal angles with that of the line itself, unless the dip on opposite sides be the same. There is however an exception to this rule, when the direction of the dip on each side of the anticlinal line is perpendicular to it. This will occur when the two planes of stratification first mentioned intersect in a horizontal line.

‡ Saussure, Voyages dans les Alpes, Vol. I. Chap. x.

‖ Traité de Geognosie, by D'Aubuisson, Vol. I. §. 24. p. 82.; and Saussure, Voyage dans les Alpes, Vol. III. Chap. x. This rule is probably very general.

V. *Transverse Valleys.*

Deep valleys are sometimes found of which the directions are nearly at right angles to that of the general elevation*.

VI. *Dykes or Veins, and Horizontal Beds of Trap.*

α. The dykes are usually found in nearly vertical planes, and, when they occur in the vicinity of each other, with a general tendency to parallelism.

β Extensive beds of trap are found apparently interstratified with the stratified rocks.

VII. *Granite Veins.*

The form of a vein of this kind is frequently very different from that of mineral or trap-veins, as above described, inasmuch as a section of it does not generally approximate in the same degree to rectilinearity †.

These approximations to general laws have been, I believe, very generally recognized by geologists, and more especially in faults and mineral veins, in almost all cases in which these phenomena exist *throughout districts of considerable extent;* and this appears unquestionably to justify the notion, that they are not to be referred to partial causes, but to some cause general at least with reference to the district throughout which the same laws are observed to hold without breach of continuity. Local and accidental causes may in some cases act with sufficient energy to obliterate all traces of general laws in phenomena such as those above mentioned; but still this will manifestly not invalidate our inference with respect to those districts in which such laws have been clearly recognized. We may moreover

* These valleys may frequently be due in great measure to the effects of erosion. In some instances, however, they appear to have been obviously formed by the elevation of the strata on either side of them. The valley of the Wye, in Derbyshire, offers a beautiful example of this kind of formation.

† Trap veins sometimes assume the tortuous form of a granite vein. See M'Culloch's Description of the Western Islands of Scotland. Vol. III. Pl. XXXIII.

observe, that the law of approximate parallelism which equally characterizes the phenomena of anticlinal lines, faults, and mineral veins, affords, *à priori*, a strong probability that they are all assignable to the same general cause. We may also further remark, that if, with the previous conviction that the stratified beds have been deposited from water, and with a knowledge of the physical impossibility of beds of uniform thickness being so deposited except on planes but little inclined to the horizon,—if, I say, under these circumstances, we examine many of the phenomena above mentioned, it seems impossible not to be struck with the idea of their being referrible to the action of some powerful elevatory force acting beneath the superficial crust of the globe, and thus producing those elevations and dislocations which we now witness. And, accordingly, such is the almost universal impression on the minds of geologists.

It appears, then, that we are arrived at that stage of geological science in which we are able to recognize certain well defined geological phenomena, distinctly approximating to geometrical laws; and we have also a distinct mechanical cause to which geologists, with almost one consent, have agreed in considering them to be assignable. The next step we are therefore called upon to take is obvious—it is to institute an investigation, founded on mechanical and physical principles, of the necessary relations which may exist between our observed phenomena and the general cause to which we attribute them. This investigation I have attempted, and now beg to lay it before the Society. I hope the nature of it will be deemed a justification of my introduction of a new term into the science, that of ***Physical Geology***.

I have conducted the investigation by the methods supplied by mathematical analysis. I am aware, however, that to some persons the application of these methods to geological problems may appear like an affectation of an accuracy which the nature of the subject may not be conceived to admit of; but from this opinion I dissent entirely. We have, as I have before remarked, observed phenomena approximating to well-defined laws, and which we are prepared to regard as the effects of an assigned and definite cause; and to shew that this hypothetical cause is the true one, we must shew that, supposing all local and partial causes with which we are acquainted to be removed, it would produce effects strictly in harmony with those laws to which the actual phenomena are observed to approximate. The most obvious cause of deviation in our phenomena from strict geometrical laws, is irregularity in the intensity of the elevatory forces, and in the constitution of the masses on which they are supposed to act. Abstracting these sources of uncertainty, we have before us a definite problem, viz., to determine the nature of the effects produced by a general elevatory force acting at any assigned depth on extended portions of the superficial crust of the earth, and with sufficient intensity to produce in it dislocations and sensible elevations. To this simple and definite form the problem may be reduced; and at least a correctly approximate solution of it must necessarily be obtained by some means or other, before we can pronounce on the adequacy of the assigned cause to produce the observed effects. The complete solution of the problem presents many difficulties, which, however, are avoided by restricting ourselves to a *first approximation*, which will amply suffice for all practical applications of our results. This approximate solution is what I have now to offer; and I may be allowed to observe, that those who may object to the mathematical resources of which I have availed myself, are at least bound to offer a solution equally conclusive and available by some method more adapted to the general reader. A slight examination however of the problem will suffice to shew that it can admit of no accurate solution independently of reasoning too intricate to be clearly embodied in any language but that of mathematical analysis.

The hypotheses from which I set out, with respect to the action of the elevatory force, are, I conceive, as simple as the nature of the subject can admit of. I assume this force to act under portions of the earth's crust of considerable extent at any assignable depth, either with uniform intensity at every point, or in some cases with a somewhat greater intensity at particular points; as for instance, at points along the line of maximum elevation of an elevated range, or at other points where the actual phenomena seem to indicate a more than ordinary energy of this subterranean action. I suppose this elevatory force, whatever may be its origin, to act upon the lower surface of the uplifted mass through the medium of some fluid, which may be conceived to be an elastic vapour, or in other cases a mass of matter in a state of fusion from heat. Every geologist, I conceive, who admits the action of elevatory forces at all, will be disposed to admit the legitimacy of these assumptions.

The first effect of our elevatory force, will of course be to raise the mass under which it acts, and to place it in a state of *extension*, and consequently of *tension*. The increase of intensity in the elevatory force might be so rapid as to give it the character of an impulsive force, in which case it would be impossible to calculate the dislocating effects of it. This intensity and that of the consequent tensions will therefore be always assumed to increase *continuously*, till the tension becomes sufficient to rupture the mass, thus producing fissures and dislocations, the nature and position of which it will be the first object of our investigation to determine. These will depend partly on the elevatory force, and partly on the resistance opposed to its action by the cohesive power of the mass. Our hypotheses respecting the constitution of the elevated mass, are by no means restricted to that of perfect homogeneity; on the contrary, it will be seen that its cohesive power may vary in general, according to any continuous law; and moreover, that this power, in descending along any vertical line, may vary according to any discontinuous law, so that the truth of our general results will be independent, for example, of any want of cohesion between contiguous horizontal beds of a stratified portion of the mass. Vertical or nearly vertical planes, however, along which the cohesion is much less than in the mass immediately on either side of them, may produce considerable modifications in the phenomena resulting from the action of an elevatory force. The existence of joints for instance, or planes of cleavage in the elevated mass, supposing the regularly jointed or slaty structure to prevail in it previously to its elevation, might affect in a most important degree, the character of these phenomena. To a mass thus constituted, these investigations must not be considered as generally applicable. Vertical or highly inclined *planes of less resistance*, will only be assumed to exist partially and irregularly in the elevated mass.

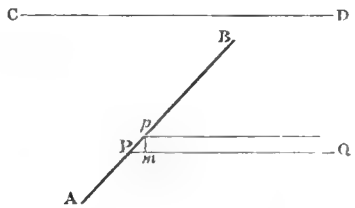
With these hypotheses then respecting the nature of the elevatory force, and the constitution of the elevated mass, I shall proceed in the next section to investigate the directions in which fissures will be formed in it when subjected to given internal tensions sufficiently great to overcome the cohesive power which binds together its component particles. These tensions, so far as this investigation is concerned, may either be supposed to be produced by external forces causing an *extension* of the mass, or by such as prevent that *contraction* of it which might be conceived to result from the loss of moisture or of temperature. It must be understood however that these internal forces are quite distinct from that sort of molecular action on which any kind of laminated or crystalline arrangement of the component particles may depend.

SECTION I.

1. THE simplest form of the mass in which we have to consider the formation of fissures, is obviously that of a thin lamina. The investigations therefore of this section will be applied directly to this case, from which the results applicable to a mass of three dimensions are immediately deducible. It will appear that its cohesive power may vary according to any continuous law.

§. *Lamina subjected to one System of Tensions.*

2. Suppose the lamina acted on by external forces, which shall place it in a state of tension, such that the direction of the tension at every point shall be parallel to a given line CD*. Let AB be any



proposed line in the lamina; P any point in this line. Also assume F to be the tension at P, estimated by the force which the tension at that point would produce, if it acted uniformly on a line of which the length should be unity, and which should be perpendicular to CD, the common direction of the forces of tension. Then if we take Pp a small and given element of the line AB, and draw PQ parallel to CD, and pm perpendicular to PQ, the force of tension on Pp in the direction PQ will be measured by $F.pm$, or $F.Pp.\sin\psi$ ($BPQ = \psi$); or the

* The reference will always be made to the figure in the same page, unless stated to the contrary.

tendency of the forces of tension to separate the particles which are contiguous, but on opposite sides of the geometrical line Pp, by causing them to move parallel to CD, will be measured by

$$pm \, . \, F \sin \psi,$$
$$= \delta x \, . \, F \sin \psi \ \ (\text{if } AP = x).$$

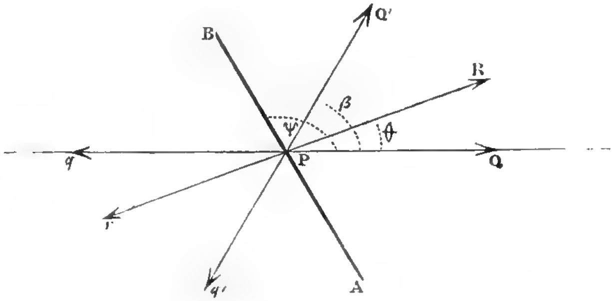
The tendency to separate the particles at P, by causing them to move in a direction making an angle θ with CD or PQ, will be estimated by

$$\delta x \, . \cos \theta \, . \sin \psi \, . \, F.$$

This is greatest when $\theta = 0$, and $\psi = \frac{\pi}{2}$, as of course it ought to be, AB being then perpendicular to CD.

§. *Lamina acted on by two or more Systems of Tensions—Direction in which their tendency to produce a fissure is greatest.*

3. Let us next suppose a second system of parallel tensions superimposed on the former, their common direction making an angle β with that of the first system. Let PQ, PQ', in the following figure, be the directions of the tensions acting on the element δx, of the line AB at



P, and therefore $QPQ' = \beta$; and let the intensities of these tensions (estimated as in Art. 2.) be represented by F and f. Then if $QPB = \psi$, and therefore $Q'PB = \psi - \beta$, we shall have the forces $\delta x \, . \, F \, . \sin \psi$, and $\delta x \, . f \, . \sin (\psi - \beta)$ acting on the element δx; and to find the tendency of these forces to separate the contiguous particles on opposite sides of the elementary portion δx, of the geometrical line AB, estimated by their tendency to give an opposite motion to these particles along any assigned line rPR, we must resolve the forces in the direction of that line. Let $RPQ = \theta$; then will the sum of the resolved parts of our forces in the directions PR and Pr be

$$\delta x \,.\, F \sin\psi \cos\theta + \delta x \,.\, f \sin(\psi - \beta) \cos(\beta - \theta) \dots\dots\dots\dots (A).$$

If the value of this expression, considered as a function of the independent variables ψ and θ be made a maximum, we shall manifestly obtain from the corresponding value of ψ that angular direction of the line AB along which the two sets of tensions we are considering have the greatest tendency to form a fissure.

Differentiating the expression with regard to θ, we have

$$\delta x \,.\, F \sin\psi \sin\theta - \delta x \,.\, f \sin(\psi - \beta) \sin(\beta - \theta) = 0.$$

The left-hand side of this equation is the expression for the sum of the resolved parts of the forces $\delta x \,.\, F \sin\psi$ and $\delta x f \sin(\psi - \beta)$, perpendicular to the line PR. Consequently the equation expresses the condition that PR must coincide with the direction of the resultant of the above forces.

Again, differentiating with respect to ψ, we obtain

$$F \cos\psi \,.\, \cos\theta + f \cos(\psi - \beta) \cos(\beta - \theta) = 0.$$

From the above equations we must determine ψ and θ. If we put $\frac{f}{F} = \mu$, we obtain from thence

$$1 + \mu (\cos\beta - \sin\beta \cot\psi)(\cos\beta - \sin\beta \,.\, \cot\theta) = 0,$$

$$1 + \mu (\cos\beta + \sin\beta \,.\, \tan\psi)(\cos\beta + \sin\beta \tan\theta) = 0,$$

or, putting $\cos\beta = c$, $\sin\beta = s$, $\cot\theta = x$, $\cot\psi = z$,

$$1 + \mu (c - sz)(c - sx) = 0,$$

$$1 + \mu \left(c + \frac{s}{z}\right)\left(c + \frac{s}{x}\right) = 0.$$

From the inspection of these equations it is manifest that $z = -\frac{1}{x}$; for putting this value for z, and therefore also $-\frac{1}{z}$ for x, the two equations will only be converted into each other. Substituting in the first equation, we have

$$1 + \mu (c - sz) \frac{cz + s}{z} = 0,$$

which gives

$$z^2 - \frac{1 + \mu (c^2 - s^2)}{\mu c s} z - 1 = 0,$$

$$\text{or } \cot^2 \psi - 2 \cdot \frac{1 + \mu \cos 2\beta}{\mu \cdot \sin 2\beta} \cdot \cot \psi - 1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

4. Let ψ_1, ψ_2 be the two values of ψ given by this equation. Then, since the last term is -1,

$$\cot \psi_1 \cot \psi_2 = -1;$$

which shews that the difference between ψ_1 and ψ_2 is $\frac{\pi}{2}$, or

$$\psi_1 = \frac{\pi}{2} + \psi_2;$$

and if θ_1, θ_2 be the values of θ corresponding respectively to ψ_1 and ψ_2, we have

$$\cot \theta_1 = -\frac{1}{\cot \psi_1}$$

$$= \cot \psi_2;$$

$$\therefore \theta_1 = \psi_2 = \psi_1 - \frac{\pi}{2}$$

$$\text{and } \theta_2 = \psi_1 = \psi_2 + \frac{\pi}{2}.$$

The angle BPR is consequently a right angle.

5. Since PR coincides, as shewn above, with the direction of the forces of tension acting on the element δx of the line AB, the expression (A) is the value of that resultant. Consequently ψ_1 and ψ_2, which correspond to the maximum and minimum values of the quantity (A), determine the position of the line AB in which the resultant of the above-mentioned forces of tension is a maximum or minimum.

6. If the two systems of tensions be equal and perpendicular to each other, equation (1) becomes

$$\sin\pi \,.\, \cot^2\psi - 2\,(1+\cos\pi)\cot\psi - \sin\pi = 0,$$

and is satisfied independently of particular values of ψ. In this case, therefore, there is no greater tendency to form a fissure in one direction than another. If F be greater than f, the equation becomes

$$\sin\pi \,.\, \cot^2\psi - 2\left(\frac{1}{\mu} - 1\right)\cot\psi - \sin\pi = 0,$$

of which the two roots are 0 and ∞, which shews that the greatest tendency is to form a fissure in a direction perpendicular to that of F.

7. The above investigation easily admits of generalization for any number of systems of parallel tensions superimposed upon each other. Let F denote, as before, the intensity of the tension in the direction from which θ and ψ are measured; f_1, f_2, &c. the tensions in directions making respectively angles β_1, β_2, &c. with the direction of the tension F. Then shall we have

$$\delta x \,.\, \{F\sin\psi\cos\theta + f_1\sin(\psi-\beta_1)\cos(\theta-\beta_1) + f_2\sin(\psi-\beta_2)\cos(\theta-\beta_2) + \&c.\} = \text{max.};$$

and proceeding exactly in the same manner as in the previous investigation, and adopting an analogous notation, we shall manifestly obtain the following equations:

$$1 + \mu_1(c_1 - s_1 z)(c_1 - s_1 x) + \mu_2(c_2 - s_2 z)(c_2 - s_2 x) + \&c. = 0,$$

$$1 + \mu_1\left(c_1 + \frac{s_1}{z}\right)\left(c_1 + \frac{s_1}{x}\right) + \mu_2\left(c_2 + \frac{s_2}{z}\right)\left(c_2 + \frac{s_2}{x}\right) + \&c. = 0,$$

and putting, for the same reason as before, $x = -\dfrac{1}{z}$, we obtain

$$1 + \mu_1(c_1 - s_1 z)\frac{c_1 z + s_1}{z} + \&c. = 0;$$

$$\therefore\ z + \mu_1\{c_1 s_1 + (c_1^2 - s_1^2)z - c_1 s_1 z^2\} + \&c. = 0;$$

$$\text{or}\ \ -\{\mu_1 c_1 s_1 + \mu_2 c_2 s_2 + \&c.\}z^2$$

$$+\{1 + \mu_1(c_1^2 - s_1^2) + \mu_2(c_2^2 - s_2^2) + \&c.\}z + \mu_1 c_1 s_1 + \mu_2 c_2 s_2 + \&c. = 0;$$

$$\text{or } z^2 - \frac{1 + \Sigma\mu\,(c^2 - s^2)}{\Sigma\mu c s}\, z - 1 = 0\,;$$

$$\therefore\ \cot^2\psi - 2\,.\,\frac{1 + \Sigma\mu\cos 2\beta}{\Sigma\mu\sin 2\beta}\cot.\psi - = 0 \ldots\ldots\ldots\ldots (2).$$

The same remarks will apply to this equation as to equation (1).

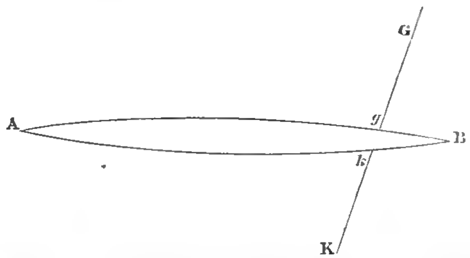
Hence, then, when the directions of the different tensions to which a lamina is subjected, and the ratios of their intensities, are known, this equation will determine that position of the line AB passing through any proposed point P, in the direction of which there is the maximum or minimum tendency to cause a fissure to *begin* at that point. If β be less than a right angle, it is manifest by inspection that the negative root will correspond to the former, and the positive root to the latter case.

8. The actual direction in which the fissure will begin to be formed at P, may, however, be different from that in which the tensions have the greatest tendency to form it; for if there be any particular line through that point, along which the cohesive power of the lamina is less than in any other, the fissure may begin to be formed in that direction, though it may not coincide with that of the maximum resultant tension. If however the cohesive power at the proposed point be equal in every direction, *i. e.* if it vary *continuously* in passing from one point to another, and not suddenly as at a line of less resistance, the direction in which the fissure will begin to be formed, will coincide with that of the maximum resultant tension determined by equation (2). This observation respecting the constitution of the mass to which the investigations of the previous articles are applicable, is important. *The cohesive power may vary according to any continuous law,* as was before stated. (Introd. p. 11).

Direction in which the Fissure will be continued.—Partial System of Tensions imposed on the Lamina about the extremities of the Fissure.—Direction of the Fissure not affected by it in the case proposed.

9. In the preceding investigations the tensions have not been considered necessarily sufficient to produce a fissure. Let us now suppose

their intensity to increase till the resultant tension becomes greater than the cohesive power at any proposed point P. A fissure will then begin to be formed in the direction determined by equation (2), in which the values of μ_1, μ_2, &c. express the ratios of the different tensions at P, at the instant the fissure begins to be formed there. Let us suppose the fissure AB to have been thus formed, and that the cohesive power



of the lamina beyond A and B is sufficient to prevent its further propagation, and let us then consider whether any modification of the tensions will be produced immediately beyond A and B, which may possibly influence the direction in which there will be the greatest tendency to continue the fissure.

10. Let GK be any physical line broken by the fissure. It is obvious that if it pass near the extremity of the fissure, its extension, and therefore its tension, will not be very much diminished; but since this tension is no longer counteracted at g and k by an equal and opposite tension, as in its unbroken state, it is manifest that the force exerted by each portion Gg, Kk, must produce an increased stress upon the portions of the lamina, immediately contiguous to and beyond the extremity of the fissure; and since a similar effect, differing only in degree, will be produced by each physical line broken by the fissure, it is possible that the intensity of the whole additional tension, thus partially superimposed upon the lamina, may be very considerable in comparison with the general tensions impressed upon it.

Now it is manifest, that the direction in which there is the greatest tendency to continue the fissure from A or B, under the circumstances we are supposing, will be determined by the whole tension contiguous to those points, consisting of that superimposed as above described, as well as of that impressed generally on the lamina; and consequently, if we conceive the latter of these tensions, (and therefore also the former) to increase till the resultant tension is sufficient to overcome the cohesive power at A or B, the fissure will not necessarily be continued in the same direction, as if its continuation were independent of the partial system superimposed about its extremities.

It will be observed, however, that in the case just considered, in which the forces are not producing motion in the mass, the whole force exerted by gG, and kK, and similar lines is effective in producing the superimposed system of tension about the extremity of the fissure. We shall shew however, that such is not generally the case during the propagation of the fissure, if propagated in the manner we shall suppose it to be, and that consequently this force will have no material effect on the direction in which the fissure will be continued, and which will therefore be very *approximately* determined by equation (2).

11. For this purpose, let us suppose in the first place, any systems of tensions impressed on the lamina, of which the resultant tension (R) shall be less than the cohesive power (Π), at any proposed point P; and let us then conceive subsequently superimposed on these another system of which the direction is different to that of R, and of which the intensity Φ shall increase continuously with the time t, till the resultant of R and Φ shall be equal to Π, so that a fissure shall then begin to be formed at P. Its direction will evidently depend on R, *and the value* (Φ_t), *which* Φ *shall have acquired at the instant the fissure commences.* If R differ but little from Π, Φ_t will be generally small*, and cannot (however the forces producing Φ may subsequently act on the lamina), produce any material influence on the direction of the fissure, which will therefore, in such case, nearly coincide with the direction in which the tensions whose resultant is R may have the

* If the direction of Φ coincided with that of R, the fissure would manifestly begin to be formed when $R+\Phi_t$ should $=\Pi$, or $\Phi_t=\Pi-R$, which by hypothesis is small. If the angle between the directions of R and Φ be not too near a right angle, it is equally manifest that Φ_t must be small. In the actual case considered in the text, this angle obviously cannot be very considerable.

greatest tendency to form it, *i. e.* it will be nearly perpendicular to the direction of that resultant.

12. Let us now suppose P_1 to designate a point in the lamina, at which a fissure shall begin, and P_2 another point through which it shall be subsequently propagated; and let Π_1, Π_2, denote the cohesive powers of the lamina at those points respectively, Π_1 being the least. It has been already stated, (Introd. p. 11.) that in the case to which these investigations are to be applied, the intensity of the elevatory force, and therefore, of the tensions produced by them, will be assumed to increase *continuously* from the commencement of the action of this force, to the formation of the fissures; we shall here also make an additional assumption, viz., that this intensity shall increase *rapidly*, so that a very small time shall elapse between the commencement of the elevatory action, and the instant when the fissures shall begin to be formed*. The tensions therefore to which our lamina is subjected, will be assumed to increase in the same manner. Let R_t denote the intensity of their resultant at the time t; then if t_1 be the time when the fissure begins at P_1, $R_{t_,}$ must be equal to the cohesive power at $P_1 = \Pi_1$. When the fissure is thus begun to be formed, the partial system of tensions described in Art. 9., will be superimposed about its extremities. Let Φ_t denote its intensity at the time t, and at any proposed point. As the fissure in its progressive formation approaches P_2, this force will be superimposed on the lamina there, in addition to the force R_t *previously acting there*, so that if $t_{,,}$ be the time when the fissure is first formed at P_2, we must have at P_2, the resultant of $R_{t_{,,}}$, and of $\Phi_{t_{,,}} = \Pi_2$. Now, if during the time $t_{,,} - t_,$, R_t increases from $R_{t_,}$, or Π_1, so that $R_{t_{,,}}$ nearly $= \Pi_2$, $\Phi_{t_{,,}}$ must be small at P_2, and therefore can have but little influence on the direction of the fissure through that point, whatever be the direction of that tension, or the intensity it might acquire if the cohesive power at P_2 were sufficient to prevent the propagation of the fissure beyond that point (Art. 11.) In such case therefore the direction of the fissure will be at least very approximately determined by equation (2), p. 18, in which the values of μ do not include the tension Φ,

* This assumption is not absolutely necessary for the truth of the approximation we have to establish or for the proof of it. It renders however the approximation more accurate, and the proof much more simple.

but only the values F, f_1, f_2, &c. of the general tensions, at the instant when the fissure is propagated through the proposed point*.

13. Under the circumstances here supposed, the fissure will be propagated from P_1 to P_2, nearly in the time $t_{\prime\prime} - t_{\prime}$, during which R_t increases from Π_1 to Π_2. Consequently, if the difference between these latter quantities be not great, *i. e.* if the cohesive power do not vary rapidly; or if R_t (heretofore assumed to be the same at the same time at different points of the lamina) increase with rapidity, it follows that the velocity of propagation will be extremely great, becoming infinite, when the cohesive power, and the tension R_t are accurately uniform throughout the lamina.

If R_t be not uniform, it is easy to see that reasoning similar to the above will hold equally true, with respect to the progressive formation of any fissure.

14. The fissure will be propagated in a straight line, if the values of μ in equation (2) remain the same, *i. e.* if the ratios of the tensions at different points be the same at the instant the fissure is propagated through them. If these ratios be different for different points, the fissure will generally be curvilinear; there is, however, an important exception to this rule, when there are only two systems of tension, of which the directions are perpendicular to each other; for in this case it appears by Art. 6, that the direction of the fissure will always be perpendicular to that of the greater of these two tensions.

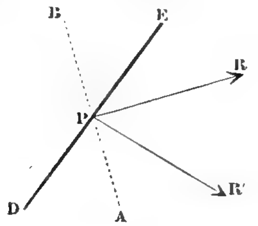
Effect of Lines of Less Resistance on the Direction of a Fissure. Permanent Direction of Cleavage.

15. In the preceding articles, we have supposed the cohesive power of the lamina to vary according to some continuous law. Let us now

* When the cohesive power of the lamina is not sufficient to prevent the propagation of the fissure, the problem presented to us is no longer a statical one. In the case above considered, a small portion only of the extraneous forces producing the tension Φ, is effective in causing an additional tension of the lamina before the formation of the fissure. The greater part is effective in communicating motion to those parts of the mass, the receding of which from each other causes the opening of the fissure. On the contrary, when the formation of the fissure is arrested, the whole of these forces is effective in producing this partial system of tensions.

consider the effect of the existence of *lines of less resistance* in the lamina, in which case the continuity above assumed will no longer exist along these lines.

Let DE be a line of this description, along which the cohesive power estimated in a direction perpendicular to it $= \Pi'$, that of the lamina near to DE being $= \Pi$. Also let R_t, acting in the direction



PR, be as before, the resultant at the time t and at the point P, of the general systems of tensions impressed upon the lamina; and let R_t' denote the tension along PR' perpendicular to DE at the time t. Then if

$$\frac{R_t'}{\Pi'} \text{ be } > \frac{R_t}{\Pi},$$

it is manifest that the fissure will begin to be formed along the line DE, rather than in a direction perpendicular to R_t, in which it would be formed in the absence of a line of less resistance*.

16. Let us now suppose this line to terminate at D and E. When the fissure has been propagated to those points, its progress will be arrested till the tension R_t and that superimposed just beyond the extremities of the fissure, and before denoted by Φ_t (Art. 11), produce a resultant tension greater than the cohesive power Π. The direction in which the fissure will be then immediately continued, will not be known, Φ_t being unknown; but without staying to enquire what this may be, we may observe, that the fissure must very soon in its pro-

* It is assumed in the above condition, that if the fissure be formed along DE, the particles on opposite sides of the fissure in separating would move in lines perpendicular to DE. This would be only approximately true.

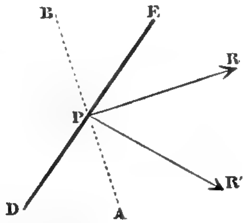
gressive formation, arrive at a point at which R_t will be very nearly equal to the cohesive power, {since that force by hypothesis increases rapidly with t, (Art. 12)}, and where, consequently, the direction of the fissure must necessarily be very approximately that determined by equation (2), as explained in Art. 12. Hence then we may conclude, that under the hypotheses we are taking, whatever may be the direction first given to the fissure by any local cause, its subsequent direction will soon become independent of that cause.

17. If the fissure, instead of beginning at some point in a line of less resistance meet it, in its progressive formation, it will pass along it, or will cross it, according as a condition exactly similar to that given above (Art. 15), be satisfied or not. At the termination of this line, the fissure will soon resume the direction given to it by the general systems of tensions to which the lamina is subjected, as just explained. Such also will be the case at the point at which the line of less resistance, should it be a curved or broken line, may assume a direction in which the condition just referred to is no longer satisfied.

18. The condition given in Art. 15 gives us

$$\frac{R_t'}{R_t} > \frac{\Pi'}{\Pi}.$$

The first of these ratios will in each particular case be a function of the angle RPR' or EPB, the angle between the line of less resistance and the direction AB, (perpendicular to PR) in which the

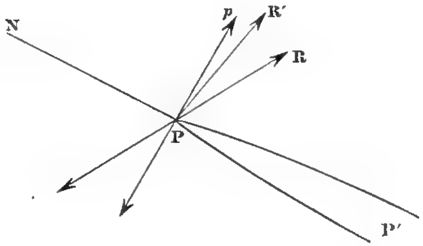


general tensions tend to form the fissure, the value of the function decreasing as RPR' or EPB increases from zero to a right angle, since the resultant tension is a maximum in the direction PR, and a minimum in that perpendicular to PR. (Art. 5). Consequently, the greater the ratio which the former of these resultants bears to the latter, the more rapidly will R_t' decrease while RPR' increases, and the smaller will be the angle EPR, within which the above condition will be satisfied, and the narrower therefore will the angular limits, within which a line of less resistance must be situated, in order that it may cause a fissure proceeding in any assigned direction to deviate from its course. A line through P perpendicular to PR, may be termed a *permanent line of cleavage*. If the ratios $\frac{f_1}{F}$, $\frac{f_2}{F}$, &c. be the same at every point of the lamina, all such lines will be straight lines (Art. 14) and parallel to each other. A fissure will always have a tendency to resume this direction, when made by any partial cause to deviate from it, and will resume it {taking our assumptions respecting the impressed tensions, (Art. 12)} almost immediately after the cessation of such cause. It will be well to examine this tendency in a few particular cases. It may be considered as measuring what may be termed the *permanence* of the fissure's general direction.

19. Let there be two systems of tensions, the directions of which are perpendicular to each other, and of which the intensities are F and f respectively, at any proposed point, when they become sufficient to form the fissure there. The greatest of these (F) will be the maximum, and f the minimum resultant tension, (Art. 6), and therefore the less f is, the greater will be the *permanence* of the permanent direction, perpendicular to that of F. If $f = F$, there will be no permanence in any particular direction. We have already seen (Art. 6), that there is, in fact, no greater tendency in this case to form a fissure in one direction than another.

20. Again, let us suppose in addition to the systems of tensions, of which the intensities are f_1, f_2, &c., and which have determinate directions, a force acting within the fissure perpendicularly to its direction, and with equal intensity on its opposite sides, exactly as a fluid would act when forcibly injected into a fissure formed in a solid mass.

Let *P′P* be the fissure. It is manifest that this force (p) will produce a tension on the mass contiguous to the extremity of the fissure, in a direction *Pp* perpendicular to *P′P*, and must therefore tend to propagate the fissure along *P′P* produced. Hence it will follow that such

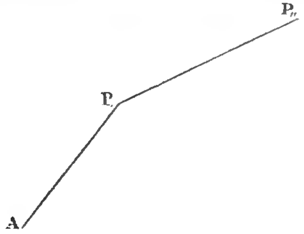


a force cannot affect the permanent direction of cleavage as determined by the tensions f_1, f_2, &c. alone. For, suppose *PR* the direction of the maximum resultant (R) of these tensions, it is manifest that the whole resultant tension (including that produced by p) immediately beyond the extremity *P* of the fissure, must be in a direction *PR′* between *Pp* and *PR*; consequently, the direction of propagation from *P* will deviate from *P′PN*, and approximate more nearly to perpendicularity with *PR′*, and therefore also with *PR*. For the same reason, the direction of its further propagation will approximate still more nearly to a line perpendicular to *PR*, till it coincide with it. The permanent direction will therefore be the same as if the force p did not exist.

If however p be large compared with R, it is manifest that the angle *pPR′* will be very small, and that the tendency to resume the permanent direction, when the fissure has been obliged by any partial cause to deviate from it, will be much less than if p were relatively smaller.

21. If the lamina be subjected to no tension, and the fissure be produced entirely by p, the tendency will be to propagate the fissure in the direction in which it may originally be formed. Suppose $AP_{,}$ to be its original direction, but that from $P_{,}$ it follows a line $P_{,}P_{,,}$

of less resistance; then if we suppose the force p not to act effectively in propagating the fissure, except near its extremity*, its action will not extend beyond the portion $P_{,}P_{,,}$ of the fissure, and consequently



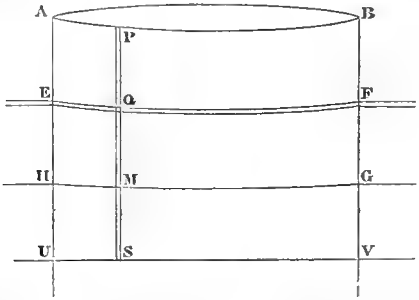
its tendency will be to propagate it in the direction of $P_{,}P_{,,}$ produced, after it has reached the termination of the line of less resistance. There will be no tendency, as in the former cases, to resume any particular direction.

§. *Modification of the Tensions in the vicinity of a Fissure.*

22. Let us now suppose a fissure to have been formed in the manner above described, and extending between two points in the lamina, where we may conceive its propagation to have been arrested either by an increased cohesive power, or by a diminution of intensity in the tensions. It is manifest that the state of tension in the vicinity of this fissure, will become entirely different from that which existed previously to its formation; and that the subsequent formation of any other fissure not very remote from the first, must therefore be influenced by the modification of the original tensions thus produced. It will now therefore be our object to examine this consequence of the existence of a fissure. For the greater simplicity, we may suppose it to be rectilinear. It will also suffice for our immediate purpose, to suppose the lamina subjected to two sets of tensions acting perpendicularly to each other, the direction of the fissure being perpendicular to that of the system of the greater intensity.

* This will be true in the actual case to which it is intended to apply this part of the investigation.

23. Let AB represent the fissure, AU, BV, the physical lines perpendicular to it, and passing close to its extremities, and for the greater distinctness, let us suppose the boundary of the lamina along UV to be parallel to the fissure. Let EF be a physical line originally parallel to the straight line AB. After the formation of the fissure,



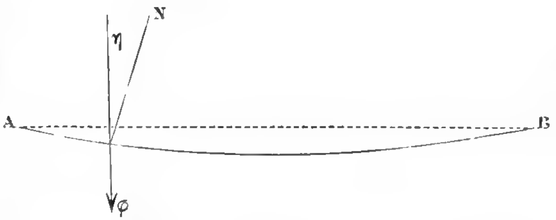
it will evidently assume a curved form resembling that of APB; but its curvature will be less than that of the latter line, since the curvature of all such lines must obviously be smaller the nearer they are situated to the fixed straight line UV, along which it becomes evanescent. If, however, the length of the fissure be considerable, the curvature of APB will be very small, and therefore the variation of curvature in successive physical lines such as EF, will be extremely slow, AU being very large*.

Also, let PQS be a physical line, parallel before the opening of the fissure, to AU. If the form of every such line as EQF were exactly the same, this line would still be accurately straight and parallel to AU, and consequently in the case we are supposing it will be approximately so. The tension of all such lines will evidently be much affected by the opening of the fissure. Since there is no force acting at P, the tension of SP in the direction of its length, will, at that

* If the boundary of the lamina be not parallel to the fissure, UV may be conceived to be a physical line in the lamina, *very distant* from and parallel to the line AB, previously to the formation of the fissure, since the position or rectilinearity of such a line will not be sensibly affected by the opening of the fissure, as appears from the text.

extremity, become evanescent; but since the line is extended, though not by a force at its extremity *P*, it must at every other point be subjected to a certain tension, and our object will be to compare this tension at any point *Q* with that acting in the direction *EQF* at the same point, with the view of ascertaining within what limits another fissure might be formed subsequently to the formation of *AB*, and parallel to it between the lines *AU* and *BV*. Such a fissure could not be formed through *Q*, by the tensions to which we are supposing the lamina subjected, if the tension in the direction *EQF* at that point should be greater than that in the direction *PS*, since the fissure must necessarily be formed perpendicular to the greater of these tensions (Art. 6).

24. In the first place, let us suppose a physical line of indefinitely small width to be attached at its extremities to the fixed points *A*, *B*, and then conceive parallel forces to act on each element of this line,



with the same or different intensities at different points, and in directions perpendicular to *AB*. The line will thus be made to assume a curvilinear form, and if the extensibility be small, as we shall suppose it to be, the curvature will be small, so that if $AQ=s$, and x be the original length of AQ, x and s may be considered as very approximately equal. Let τ denote the tension at *Q*, ρ the radius of curvature, and ϕ the intensity of the force at that point, ϕ being any function of x. Then the force on the element δs, will be $\phi . \delta x$, and the normal force produced by the tension τ, will $= \frac{\tau}{\rho}$ estimated by the effect it would produce, if it acted uniformly on a unit of the line, so that the normal force acting on the element δs, will $= \frac{\tau}{\rho} . \delta s$, or $\frac{\tau}{\rho}\ \delta x$ very approximately. Consequently, if the normal make an angle $\frac{\pi}{2} - \eta$ with *AB*, we shall have for the conditions of equilibrium of δs,

$$\frac{\tau}{\rho} . \delta s - \phi . \delta x \cos \eta = 0,$$

$$\delta \tau + \phi . \delta x \sin \eta = 0,$$

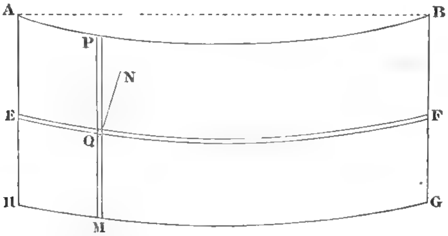
$$\text{or } \frac{\tau}{\rho} - \phi \cos \eta = 0.$$

$$\frac{d\tau}{dx} + \phi \sin \eta = 0,$$

Again, let us suppose another physical line exactly similar and equal to the former, with its extremities fixed to two other points in lines through *A* and *B* respectively, and perpendicular to *AB*, and so that the two lines shall be in contact, when not acted on by any force. When the force ϕ acts in exactly the same manner on both, they will assume exactly similar positions, and those elements of the two lines respectively which were in contact when the lines were straight, will remain so when they have assumed their curvilinear form, and will be in exactly the same relative positions with respect to each other, as if the lines had been united into one previously to their becoming curved. Whence it follows, that there can be no more action between these lines when united, as we have just supposed, than if they were perfectly independent, and therefore the tension of each must remain the same as if this independence existed. If we conceive any number of lines to be united in a similar manner, so as to form a lamina, the same conclusion will apply to each.

25. Let us now take then a rectangular lamina *ABGH*, which we may conceive to be formed in this manner, and which we will suppose to be brought into the position represented in the annexed figure, by the force ϕ acting perpendicularly to *AB*, and in the plane of the lamina. *EF* represents a physical line originally parallel to *AB*; and *PM* another originally straight and parallel to *AH*, and therefore, still evidently remaining so, though in a different position, in the curved form of the lamina. Let x be the original distance of *PM* from *AH*,

which will be approximately $= AP$, or EQ; then will δx be the original width of the element PM; and if AE, or $PQ = y$, δy will be the width of EQF. Also, if T denote the tension of the lamina at Q, (estimated as in Art. 2.), in the direction of a tangent to EQF at



that point, it is evident that $T.\delta y$ will equal the tension above denoted by τ. Therefore the force produced by this tension in the direction of the normal to EQF at Q, will $= \frac{T}{\rho}.\delta x.\delta y$, acting on the element common to the two physical lines PM and EQF at Q.

Now it is manifest, that the tension T, and $\frac{T}{\rho}$, will remain unaltered so long as the position of every element of the lamina remains so, whatever be the forces by which it is kept in that position. The action of ϕ will be the same at any point Q in PM as at P, since Q and P are similarly situated points in EQF and APB, and by hypothesis this force acts in the same manner upon each physical line, similar to APB. Consequently, the whole force on $PM = \phi.PM.\delta x$. Let us suppose this force instead of acting on each element of PM, to be applied entirely at its extremity M. If this be done to every such line as PM, and the lamina be sensibly inextensible in the direction of these lines, the position will remain undisturbed, and the normal force $\frac{T}{\rho}\delta x.\delta y$, at Q will not be altered. Hence, if T' denote the tension of the lamina at Q in the direction PM, and therefore $T'\delta x$ the tension of PM at that point, and η the angle which the normal there to EQF makes with PM, we shall have for the conditions of equilibrium of the element common to PM and EQF,

$$\frac{T}{\rho}.\delta x.\delta y - \delta(T'.\delta x)\cos\eta = 0,$$

$$\delta(T.\delta y) + \delta(T'.\delta x)\sin\eta = 0,$$

or since $\delta(T'.\delta x) = \dfrac{dT'}{dy}\delta y.\delta x$, and $\delta(T.\delta y) = \dfrac{dT}{dx}\delta x.\delta y$,

and η is by hypothesis very small,

$$\frac{T}{\rho} - \frac{dT'}{dy} = 0,$$

$$\frac{dT}{dx} + \frac{dT'}{dy}.\eta = 0,$$

neglecting terms involving η^2.

In the case we are considering, $\dfrac{T}{\rho}$ is a function of x alone, and therefore the first of the above equations gives

$$T' = \frac{T}{\rho}.y + C$$

$$= \frac{T}{\rho}y \text{......................(1)}$$

since $T' = 0$, when $y = 0$. This is subject to the condition $T'.\delta x =$ force at $M = \phi.PM.\delta x$, or $T' = \phi.PM$.

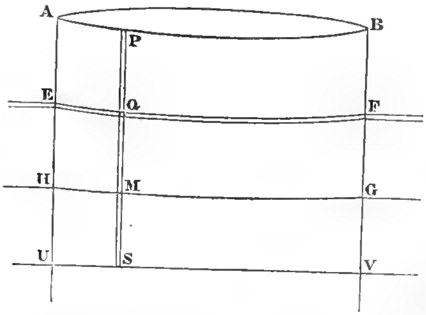
The second equation gives

$$T = \text{const. nearly} \text{ ....(2).}$$

26. If instead of supposing the lamina inextensible in the direction PM, we suppose it capable of small extension in that direction as well as in that parallel to AB, and still assume it to be acted on by forces applied at each point of HMG, so as to keep that extreme boundary in the same position as before, the physical line EF will assume a position differing in a small degree from its former one. Since the angle η will still be very small, we shall still have $T =$ const. nearly. The curvature at Q will no longer be the same as that at M, and ρ will therefore be a function of y, as well as of x. Consequently equation (1) of the

previous Article will no longer be accurately true; but since the variation of ρ as a function of y will be very slow, $\frac{T}{\rho}$ may still, for a first approximation, be considered constant from $y = 0$ to $y =$ a considerable value. Consequently both the equations (1) and (2) of Art. 25 may in our present case be considered as approximately true.

27. The case at which we have last arrived is exactly similar to that of Art. 23, which it is our object to investigate. For a portion $ABGH$ of the lamina, bounded by a line HMG, similar to EQF,



may be considered as being retained in its actual position, by the tensions acting parallel to AU and BV, at every point of HG, exactly in the same manner as that in which we have supposed the lamina represented in the figure in p. 31, to be kept in its position by forces acting at each point of HG in that figure. Also it has been shewn (Art. 23,) that the curvature of any such line as EQF, varies very slowly with its distance from AB. Consequently the variation of ρ, the radius of curvature at Q, is extremely small, considered as a function y (AE). This being the case, it is manifest likewise (assuming the original system of tensions parallel to AB, to have been uniform)* that T (the tension of EF) will vary very slowly with AE; and that therefore $\frac{T}{\rho}$ as a function of y, may approximately be considered constant. Consequently we shall have in this case

$$T' = \frac{T}{\rho} \cdot y, \text{ nearly.}$$

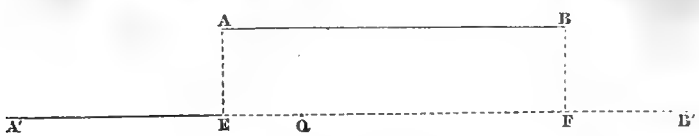
* This is not essential to the truth of our general conclusions.

If the fissure be of considerable length, ρ will be extremely large, and this equation will hold approximately for large values of y, and if y be less than ρ, T' will be less than T.

28. *Hence then it appears, that if the fissure be such that the curvature of its sides is extremely small, the greatest tension at any point within the lines* AU *and* BV, *and not extremely remote from* AB, *will be in a direction parallel to* AB; *and that consequently, if any fissure were propagated through* Q, *by the tension there, it must necessarily be in a direction perpendicular to that line.*

§. *On the Formation of Systems of Fissures.*

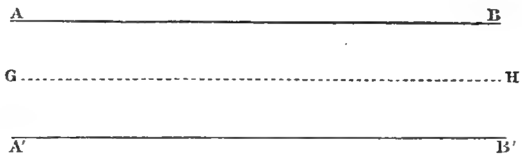
29. The result enunciated in the last Article is important, as shewing the impossibility of forming in succession parallel fissures not far distant from each other in a mass subjected to such tensions as we have supposed. Let us suppose, for instance, a fissure AB to have been formed in a lamina subjected to two systems of tensions, of which the directions are perpendicular to each other. The



propagation of the fissure beyond A and B, may be conceived to have been prevented by a greater cohesive power of the lamina there, or by a diminished intensity of the tensions perpendicular to AB. Let us also suppose another fissure to commence at A', subsequently to the formation of AB, and not remote from it, from the increased intensity of the tensions perpendicular to AB. Its direction AE will be parallel to AB, but it cannot be propagated in that direction from E to F; for the tension at Q along EF (as above stated) will be greater than that in a direction perpendicular to it, and therefore if a fissure be formed at all through that point, it must be perpendicular to EF. Nor would the formation of a fissure from E to F be rendered the more possible by the existence of this fissure through Q perpendicular to AB;

for it will be immediately seen, that this latter fissure together with AB, would destroy all tension at Q, and would of course prevent the possibility of the formation of any other fissure through that point*.

30. Hence it follows, that in any system of parallel fissures which are not remote from each other, *the fissures could not be formed in succession*. It will be easy however to understand how, in the case above assumed, of two systems of tension perpendicular to each other, any number of parallel fissures may be formed *simultaneously*. Let AB, $A'B'$ be two such fissures, and let GH be parallel to and equidistant from them.

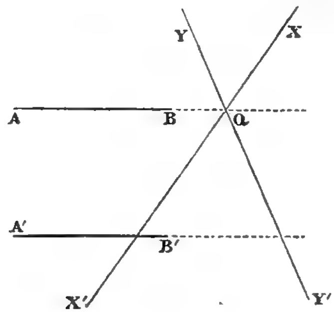


Now if the two fissures begin simultaneously at A and A', (the line AA' being perpendicular to the direction of propagation,) and be propagated with equal velocity, it is obvious that no point in the physical line GH will have any motion communicated to it by the relaxation of the portion of the lamina between the fissures. Hence, if the line GH were to become absolutely fixed, the formation of the fissures would not be affected; but in this case the portions of the lamina on opposite sides of GH might be regarded as two absolutely distinct laminæ, having that line for a common fixed boundary. Consequently it is as easy to understand the simultaneous formation of any number of parallel fissures, under the circumstances supposed, as that of a single fissure.

31. Let us assume the two systems of tension not to be perpendicular to each other, and suppose AB, $A'B'$, two parallel fissures of which the directions are perpendicular to the maximum resultant tension. These fissures would not necessarily be continued parallel to each other.

* It must be recollected that the impossibility here spoken of assumes the tensions not to be produced by *impulsive forces* acting on the mass, the intensity of these tensions being always supposed to increase *continuously*, till sufficient to produce the fissure, and not to acquire that requisite intensity *instantaneously*, as previously stated in the Introduction, p. 11.

For let YY' be parallel to the direction of one system, XX' (meeting the fissure $A'B'$) to that of the other. The opening of $A'B'$ will relax



the tension along XX', while that along YY' will not be affected. Consequently the ratio of the tensions at Q will not be the same as originally, when AB, $A'B'$ began to be formed. The direction of propagation of the former will evidently deviate towards perpendicularity with YY', and that of the latter in the same manner more nearly to perpendicularity with XX'. They will not therefore in such case preserve their parallelism.

A finite time, however, will be necessary to produce the relaxation at Q, after the opening of $A'B'$, and therefore if the distance between the fissures be not too small, and the velocity of propagation very great, as we have shewn it may be (Art. 13) AB may be propagated through Q before the relaxation is produced there, and the fissures might under such circumstances preserve, at least approximately, their parallelism.

32. It is evident, however, that in whatever manner a system of parallel fissures may be produced, that, after their formation, the only tension of the mass between them must be in a direction parallel to them. Consequently, *should any other system be subsequently formed, it must necessarily be in a direction perpendicular to that of the first system. No two systems of parallel fissures, not perpendicular to each other, could be formed by causes similar to those of which we have been investigating the effects.* It will appear also, as in Art. 30, that this second system must be of simultaneous origin.

33. From our assumptions respecting the variable cohesive power of the mass, it is manifest that different fissures might commence simultaneously at different points, and be propagated in opposite directions.

Thus, suppose the fissure *CD* to commence at *D*, when *AB* and *EF* commence at *A* and *E* respectively. When the first of these arrives at *C*, as the two others arrive respectively at *B* and *F*, the further propagation of each of them may be prevented by the relaxation of the mass. Consequently a system of fissures might thus be formed similar to that represented in the above figure.

§. *Application of the previous Propositions to a Mass of three dimensions.*

34. These investigations have been applied immediately to the case of a thin lamina, to avoid the complexity which would necessarily have been introduced in their immediate application to a mass of three dimensions. The extension of the preceding propositions, however, to this latter case is sufficiently obvious to require little more than an enunciation of the results, which may also serve as a summary of the most important of those at which we have arrived in this section.

A slight inspection of what has been advanced in Art. 15, will shew that the existence of a line of less resistance in a thin lamina, will have no effect on the propagation of a fissure in a direction perpendicular to it; and similarly, if we suppose any mass acted on by horizontal tensions, it is manifest that a horizontal plane of less resistance will have no effect on the verticality or horizontal direction of the vertical fissures resulting from such tensions. Consequently, the tensions being horizontal, the cohesive power of the mass may be supposed to vary continuously or discontinuously along any vertical line, and, as explained in Art. 8, it may vary according to any continuous law in any horizontal lamina of the mass. The same assumptions are made respecting the continuous but rapid increase of the tensions, as in Art. 12.

I. If this mass be acted on by a single system of horizontal parallel tensions, a fissure beginning at any point will be propagated in a vertical plane perpendicular to the direction of the system. (Art. 2).

II. If the mass be subjected to any number of systems of parallel tensions, the fissure will be propagated through any point in a direction perpendicular to the maximum resultant tension at that point, at the instant the fissure reaches it, (Art. 12.) the horizontal direction being determined by equation (2), (Art. 7). If the ratios of the tensions at each point at the instant of propagation through it be the same, the fissure will, in general, be formed in one vertical plane. (Art. 14.)

III. If there be only two systems of horizontal tensions, and these be perpendicular to each other, the fissure will lie in one vertical plane perpendicular to the direction of the system of the greatest intensity, whatever be the ratio of the tensions at each point in the two systems, provided the tension at each point always remain the greatest in the same system. (Arts. 6, 14.)

IV. Each fissure under the conditions assumed, will be propagated with extreme velocity. (Art. 13.)

V. The tendency of the tensions to propagate the fissure in one particular direction rather than in any other, or the *permanence* of the permanent direction of cleavage, depends on the rapidity with which the magnitude of the resultant tension, estimated in a particular direction, decreases as that direction deviates from that of the maximum resultant tension; or generally, on the ratio which the maximum bears to the minimum resultant tension, which is perpendicular to it. (Art. 18.)

VI. If in addition to a system of horizontal tensions, there be also a force acting on the opposite sides of the fissure, perpendicularly to its direction, and tending to increase its width*, the *permanence* of direction in the progressive formation of the fissure will be diminished, but the permanent direction will remain the same as if there were no other force than the system of horizontal tensions, *i. e.* if the direction in which the propagation of the fissure is taking place be disturbed by any partial cause, it will still constantly tend again to perpendicularity with the directions of the system of tensions; but this tendency will be less than if the force always acting perpendicularly to the fissure did not exist. (Art. 20.) Consequently, deviations from the permanent direction of cleavage will, in the case we have supposed, be greater than if the sides of the fissure were not subjected to the action of this last-mentioned force.

VII. If there be no tension acting on the mass, and a fissure be formed solely by this force, acting perpendicularly to its sides, the fissure will be propagated in the plane in which it begins to be formed, if the cohesive power of the mass vary according to any continuous law. There will be, however, but little permanence in its direction, so that if it be turned from its original direction by planes of less resistance, there will be little tendency to resume that direction, and the fissure may thus assume any form of irregular curvature. (Art. 20.)

VIII. If a fissure commence at, or in the course of its progressive formation meet, a partial plane of less resistance at an acute angle, it will, under certain conditions, be propagated along it; but when from any cause this ceases to be the case, the fissure will almost immediately resume a direction parallel to its original one, supposing it produced by tensions, which, independently of the existence of planes of less resistance, would produce rectilinear fissures. (Arts. 17, 18.)

IX. If the mass be subjected to two systems of parallel tensions, of which the directions are perpendicular to each other, two systems of

* This will be the case if the fissure be filled with any kind of fluid subjected to a great pressure from some external cause.

parallel fissures may be produced, of which the directions will be perpendicular to each other. No two systems of parallel fissures could be thus formed, of which the directions should not be perpendicular to each other. (Art. 32.)

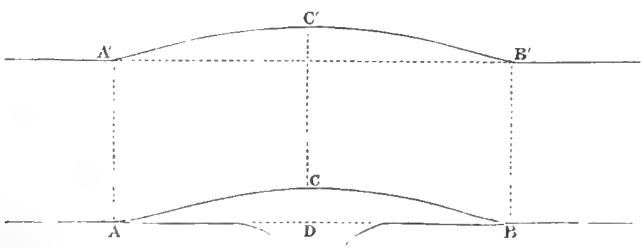
X. If the fissures in either of these systems be near to each other, they could not be formed by such tensions as we have been considering, *in succession*. They must be formed *simultaneously* in each system. One system, however, might be formed at any time subsequently to the other. (Art. 30, 32.)

SECTION II.

35. LET us now proceed to apply the results obtained in the last section to the actual case of a portion of the earth's crust, under the hypotheses respecting the action of the elevatory forces and the cohesive power of the mass, which have been already stated, (Introd. p. 11, and Art. 12.) And, first, let us suppose, for the greater simplicity, the surface of the mass acted on to be of indefinite length, and bounded laterally by two parallel lines. If we first suppose the elevatory force to be uniform, it is manifest that the extension, and therefore the tension, will be entirely in a direction perpendicular to the length; so that its whole tendency will be to produce *longitudinal* fissures, or such as are parallel to the axis of elevation.

§. *Formation of Longitudinal Fissures—Their Position and Width—Complete and Incomplete Fissures.*

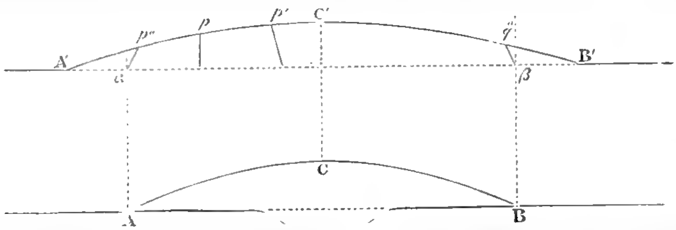
36. Let the annexed diagram represent a transverse section of the elevated mass, and let us suppose it symmetrical with respect to the line CC', and also that the mass below the horizontal line AB remains



perfectly undisturbed. The cavity $ACBD$, containing the fluid through the medium of which the elevatory force is supposed to act on the lower surface of the elevated mass, (see p. 10), may either be supposed to have existed previously to the action of the elevatory forces, or to have been partly produced by them.

If we suppose the mass not to become compressed, and the disturbance not to extend beyond the vertical lines AA', BB', it is manifest that the lengths of the lines ACB, $A'C'B'$ will be equal; and since their original lengths were so, their *extension* will be the same.

It is evident, however, that the force required to elevate the mass $ABB'A'$ will be much greater than that just necessary to overcome its weight, on account of the forces called into action at the extremities of the elevated mass, and that some degree of compression of the mass will consequently exist, which will render the vertical line



CC' shorter than its original length. It is also evident that the disturbance of the upper part of the mass will extend laterally beyond the verticals through A and B, as above represented.

The compression of CC' will clearly make the curvature of $A'C'B'$ less than that of ACB, and will consequently render its *extension* less than it would otherwise be. The greater extent of lateral disturbance in the upper portion will also produce the same effect. For let us suppose the portion $A'p$ of the upper curve exactly similar, and equal in length to pC', then is it easily seen (assuming the extension of $A'B'$ to be uniform throughout) that the line joining the physical point p, and its undisturbed position will be vertical, while similar lines for $p_{,}$, $p_{,,}$ and $q_{,,}$ will be inclined, as in the figure. Hence it immediately appears that the difference between the lengths $p_{,,}q_{,,}$ and $\alpha\beta$ will be less in this case than if $p_{,,}$ and $q_{,,}$ were in the verticals through A and B respectively. We may therefore infer that the same will hold generally, since the condition of the similarity of $A'p$ and pC' will be approximately satisfied when the tangents at A' and C' are parallel, and the curvature small, as we may here assume it to be.

Hence, then, we may conclude that the *extension* of the physical line *ACB*, under the circumstances supposed, will be at least equal, and generally greater, than that of any similar line in the higher portions of the uplifted mass. It seems also probable, that in cases occurring in nature the *extensibility* will be less in the lower portion of the elevated mass (at least to a certain depth) than in that which constitutes its upper surface.

Now the tendency of any horizontal portion of the mass to separate, so as to form a vertical fissure, will vary directly as the *extension*, and inversely as the *extensibility*. We may therefore safely conclude, that when a mass has been elevated as above supposed, the greatest tendency to rupture will not be in its upper portion; and consequently, that if any fissure be produced, whether by a gradual increase of the horizontal tension, or by any more sudden impulsive action on the mass in its state of tension, *such fissure will not commence at the surface, but at some lower part of the mass.*

37. It appears, from what has been proved in the previous Section, that if we suppose the fissure produced solely by the tensions to which the mass is subjected, the plane in which it will lie will be perpendicular to the direction of the single system of tensions which, in this case, act upon the mass, and will consequently decline as much from a vertical plane as that direction deviates from horizontality. According to the hypothesis we have made, however, of the force acting on the elevated mass through the medium of an elastic vapour, this vapour will necessarily ascend into the fissure, and exert a fluid pressure on its sides, in a direction perpendicular to them, and of which the intensity may bear a considerable ratio to that of the tension. To form a rough estimate of this intensity, let r be the radius of the circle which shall most nearly coincide with the curve *ACB* (Fig. p. 41), p the pressure of the fluid on a unit of surface, T the intensity of the tension (supposed uniform) of the elevated mass estimated as in the previous section, and b the thickness of the mass. Then the whole tension exerted on a portion of the mass included between two vertical planes perpendicular to the axis of elevation, at a distance unity from each other, will $= bT$, and we shall therefore have

$$p = \frac{b}{r} . T.$$

The value of r, according to the same rough approximation, will be nearly $= \frac{AD^2}{2CD}$, which will always be very large; but as b also is probably large, p may bear a very considerable ratio to T.

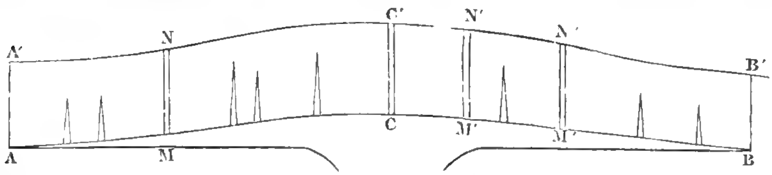
Here then we have the case which has been anticipated in the investigation of Art. 20; and it appears that the action of this force p will greatly tend to increase the effect of any local causes in producing partial deviations in the plane of the fissure from a vertical plane, but that it will not alter generally its position when considered with reference to its whole extent.

38. Again, with respect to the comparative width of the fissure at different depths, it is manifest, taking the case of the Fig. p. 41, where the extension of each lamina is the same, that if the mass, when relieved from its tension by the rupture, return to its original horizontal length, the width of the fissure will be the same throughout its whole depth; and in the case of the Fig. p. 42, the same conclusion might be considered as very approximately true under the same hypothesis. If, however, the different laminæ, which I have supposed to have different powers of cohesion, have also different degrees of elasticity, this difference may materially affect any approximation to this uniformity of width. It seems probable, however, that the *mean* width (at least within certain limits) will rather increase than decrease with the depth.

39. Any number of these fissures might thus be formed simultaneously, (Art. 30.); and this simultaneous formation would be very much facilitated by the action of the pressure p in the interior of the fissure. If it be supposed, however, that partial causes prevent the commencement of the formation of each fissure at the same instant, exactly equal forces will not be exerted in the production of each, and consequently they will not be propagated with the same velocity. Some therefore will reach the exterior surface sooner than others; and when a certain number have thus been formed from the lower to the upper

surface of the mass, the tension of it may become so far relaxed that the further formation of the others shall cease. We may therefore suppose it highly probable that the number of fissures formed in the inferior parts of the elevated mass, will be considerably greater than the number which reach the surface.

40. The phenomena, then, to which our investigation at present extends, may be represented as in the annexed diagram, a few of the fissures being *complete* ones, or running up to the external surface of



the mass, and the others being *incomplete* ones, or rising to different heights, without reaching the surface.

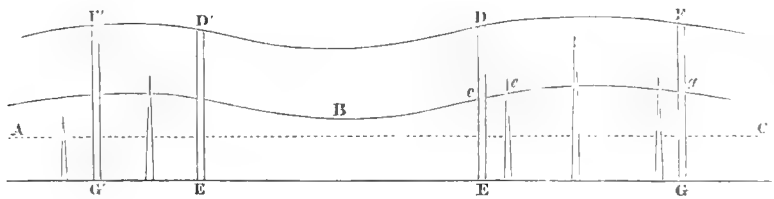
41. If we recur to what has been previously advanced respecting the depths of veins, (Introd. II. ρ.), we shall see the importance of the fact established above, that the formation of fissures produced by the causes we have supposed must necessarily begin in some lower portion, and not at the upper surface of the mass, where it might perhaps at first sight be supposed more probable that they would begin.

42. We may also see, in what has been above stated, one cause of the inclination or *hade* of a fissure. (See Introd. II. κ.)

§. *Formation of Transverse Fissures—Fissures of a Conical Elevation—Modification in the Position of Longitudinal Fissures.*

43. In the case we have been considering, the whole tendency of the elevatory force, acting with perfect uniformity, will be, as we have before remarked, to produce longitudinal fissures; and a vertical

section of the elevated mass parallel to the general axis of elevation, will be bounded above and below by straight horizontal lines. If, however, we now conceive this force to act with greater intensity at particular points along the general line of elevation, the section just mentioned will present such an appearance as represented in the annexed diagram,



in which the line *ABC*, previously to the elevation, was horizontal. In such case we shall have longitudinal extension, (equal to the difference between the line *ABC* and the dotted line *AC*), which, if sufficiently great, will necessarily produce transverse fissures, similar to the longitudinal ones already described, and such as represented in the above section.

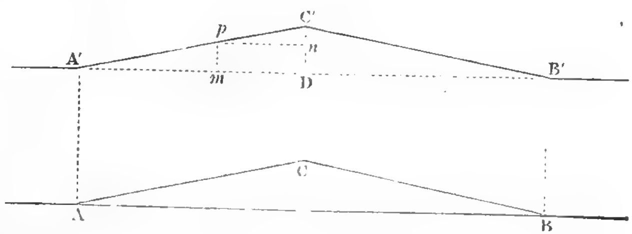
44. We may represent to ourselves this more intense action at particular points, by conceiving an additional force superimposed on a uniform force producing the general elevation independently of the irregularities resulting from this partial action. It is manifest therefore that the tension perpendicular to the line of elevation will result from the sum of these forces, while the longitudinal tension will be produced by the superimposed force alone. The former will therefore, when the partial force is not great, be much the greatest; and we may consequently conclude, that the longitudinal fissures may in such case be formed first, during the continuous though rapid increase of intensity in the elevatory forces, according to the assumption we have made respecting them, (Art. 12.); and when this system is once formed (the fissures in it not being remote from each other), the transverse system must necessarily be approximately perpendicular to it, whether it be formed at the next instant, or at any succeeding epoch, and notwithstanding any irregularity in the forces producing it, provided they do not act impulsively. In this

manner it is easy to understand the formation of a transverse system of fissures approximating to the law of parallelism, though resulting from forces which, acting partially, and under other circumstances, would produce the most irregular phenomena.

45. If however this more intense action at particular points be sufficiently great, and exactly simultaneous with that of the general elevatory force, it may modify materially the position of the longitudinal fissures. To determine the nature of this modification, we must consider the directions of the tensions which would be produced by an elevatory force, acting solely in the vicinity of any proposed point of a mass; because such tensions superimposed upon those produced by a force acting uniformly along the whole range, will be very nearly equivalent to the tensions produced by the simultaneous action of two forces such as those just mentioned.

46. For the greater simplicity, we may take a cone as the approximate type of the partial elevation we have to consider.

Let $A'C'B'$ represent this cone, $C'D$ its axis. Then if we assume the physical line $A'pC'$ to be equally extended, and AD to be its original length, we have



The original length of $A'p : A'p :: A'D : A'C'$,

and therefore,

$$\text{The original length of } A'p = A'p \cdot \frac{A'D}{A'C'}$$

$$= A'm,$$

mp being parallel to DC'. Consequently, the distance of the physical point p from the axis of the cone, will not be altered by the elevation; and since the same holds for every physical point in the circumference of the horizontal circle whose radius is pn, there can be no tension at any point of the physical line forming that circumference, in the direction of its tangent at that point. This is consistent with our assumption of the equable extension of every part of the line $A'C'$, which will therefore be true*. Similarly, if we conceive the whole mass $AA'B'B$ to be formed by the superposition of similar conical shells, it is easily seen that the same result will hold for every horizontal circle concentric about the axis of the cone. Hence it follows, that if any vertical plane be drawn through the axis of the cone, there will be no tension at any point of the mass in this plane in a direction perpendicular to it. The tension will be entirely in the plane, and parallel to the slant side of the cone.

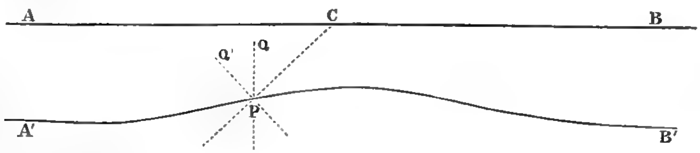
If, then, a fissure which should pass through any proposed point P, were formed according to the greatest tendency of the tensions of the unbroken mass to form it, it would manifestly coincide with the surface of an inverted cone, whose base would be the circle of which the radius is pn, and whose axis would coincide with that of the elevated cone. If p should coincide with C', an orifice would be formed along the axis $C'C$; and if we consider that the force will act, according to our hypothesis, with the greatest intensity at C, it seems highly probable that the first dislocation will usually take place along, or very near to that axis. For the greater distinctness, suppose this to be the case.

47. The instant this has occurred, the conditions of the problem will be entirely altered. The force at C' maintaining every such line as $A'C'$ and $B'C'$ in its state of tension, being now destroyed, the

* Suppose a tension T to exist along the physical line forming the circumference of the circle whose radius is pn. This would produce a force $\frac{T}{pn}$ acting at p in the direction pn, the resolved part of which in the direction pC' would increase the tension of $A'p$. In such case the extension of $A'C'$ would be greatest at A', and our assumption of the uniform extension of that line would not be true.

extremities of those lines at C' will separate from each other by the contraction of $A'C'$ and $B'C'$; and the same will be true for every similar pair of lines. An extension of the orifice at C' will thus be produced, and consequently a tension of the mass contiguous to it in the direction of a tangent to a horizontal section of it, while the tension in the direction of such lines as $C'A'$ will be entirely destroyed near to C', and much lessened at lower points. The whole tension therefore in the upper part of the mass, will be in the directions of the tangents of horizontal circles concentric about the axis; and the tendency to form a fissure there, will be entirely in a vertical plane passing through the axis of the cone. It is easily seen also that the tension at the vertex will be greater than in any other part. Consequently, if fissures be formed under these circumstances, they will commence at the vertex, and be in positions such as that just mentioned.

48. Let us now suppose the elevatory force to act with additional intensity beneath the point C of the annexed diagram, (which represents a horizontal section,) so as to superimpose on the general elevation



a conical one, having its apex at C. In addition to the tension (F) acting at any point P within the bounds of the cone, and in the direction perpendicular to the general axis of elevation, we shall also have another tension (f) acting at P, in the direction PQ' perpendicular to CP, (taking the case of Art. 47.) and the tendency of these tensions will be to form a fissure deviating from perpendicularity with PQ, in a degree depending on the relative intensities of f and F. Consequently, a fissure $A'PB'$ will deviate from parallelism with the line of general elevation, approximating towards C in the manner above represented.

49. If the partial elevation instead of approximating to the conical form, be more nearly spherical, without any such rupture at C, as

above supposed, the principal tension due to it will be in the direction CP, instead of being perpendicular to that line, in which case the deviation in the direction of the fissure will be the contrary of that above represented.

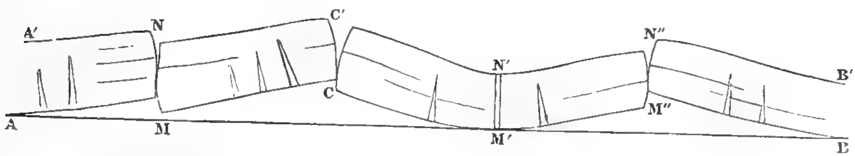
§. *Formation of Longitudinal and Transverse Faults—Anticlinal Lines—Longitudinal Valleys—Transverse Valleys—Comparative Effects of subsequent Movements on the Width of Longitudinal and Transverse Fissures—Throw of a Vein.*

It appears then, that in the case we have considered, and under the conditions assumed, the elevating forces will produce two systems of fissures with a general approximation (subject to certain modifications) to rectilinearity, and perpendicular to each other. Let us further consider what positions the different portions of the mass may assume subsequently to the formation of these fissures.

50. The diagram in page 45, represents a transverse section of the elevated range, immediately after the contemporaneous formation of the complete fissures MN, CC', &c. It does not appear probable that the effects of the continued action of the elevatory force will afterwards follow any general law; for the subsequent movements of the different portions of the mass, now rendered in some degree independent of each other by the fissures which separate them, must be constantly influenced by that irregularity in the action of the elevatory force, and those accidental and local causes of which it is now impossible to form any estimate. If the elevatory force be produced by an expansive vapour, or act through the medium of any fluid, as we have supposed it to do, its intensity must decrease after a certain time, thus causing subsidencies in the elevated mass, the degree of which in different portions will probably be in general determined by accidental circumstances. One consequence, however, of these irregular causes, would appear to be necessarily a very general one, viz. a difference of elevation in the adjoining parts of different portions of the mass separated by the fissures, whether longitudinal or trans-

verse, thus producing systems of *longitudinal and transverse faults*, such as described in the Introduction, (I. a, β.)

51. Sections of longitudinal faults which may be thus produced, are shewn in the annexed diagram, which represents one of the forms which, it is manifest, the uplifted mass represented in page 45, may ultimately assume from the causes above mentioned (Art. 50). In such case we shall have an *anticlinal line* through N'', running parallel to the general one through C' in the central part of the elevation; and a synclinal line through N' parallel to the two former ones. The existence also of these longitudinal fissures and consequent irregularities of surface, will obviously tend to direct the action of superficial



agents of denudation along longitudinal courses, and thus to facilitate the formation of *longitudinal valleys*, particularly in the case in which the relative elevation of two adjoining portions of the mass is such as represented at N. If this kind of elevation be continued for a considerable distance longitudinally, a distinct longitudinal valley must be the necessary consequence.

52. It not unfrequently happens that we observe in anticlinal lines a degree of deviation from approximate rectilinearity, which might at first sight appear inconsistent with the mode of formation which this theory would assign to them, assuming that great predominance of general over partial and accidental causes, throughout an extensive area, with which very irregular deviations in the direction of a fissure would not be accordant. It seems, however, highly probable, that this character of anticlinal lines would not, in fact, be the unfrequent consequence of the general causes we are considering. In the first place, we may observe that longitudinal fissures are not necessarily continuous for any great distance, as we have explained in Art. 33, and

therefore an anticlinal line formed along one fissure, may easily be conceived to be continued along another, not exactly in the same line. If we conceive several transferences of this kind to take place from one fissure to another, we shall have a discontinuous anticlinal line, each portion of which will be as rectilinear as the fissure with which it coincides; but if the physical structure of the mass should be placed under that disguise so frequently spread over it by superficial agencies, the geologist, instead of detecting this discontinuous line, consisting of a number of straight ones having parallel directions, will probably only recognize a somewhat ill defined anticlinal line of irregular curvature, and apparently destitute, in a considerable degree, of those characters of rectilinearity and parallelism with the general axis of elevation which this theory might appear to assign to such lines. It may also be observed, that since on the opposite sides of a transverse fissure the movements of the adjoining masses will be in some degree independent of each other, it is easy to conceive that this cause also may sometimes facilitate the transference of an anticlinal line from one longitudinal fissure to another, and thus destroy its apparent rectilinearity.

Similar observations will equally apply to the directions of longitudinal valleys, as far as their formation may be referrible to the causes above mentioned.

53. It has been stated how much the ultimate position of the dislocated mass may generally depend on accidental causes. In particular cases however, and especially with respect to those portions of the mass adjoining the lateral boundaries of the general elevation, there appears reason to expect that the phenomena would, according to our theory, frequently follow a certain law. Suppose the diagram, page 51, to represent the portion of the mass bounded by two parallel transverse fissures, produced as described in Art. 43, by a greater intensity of the elevatory force acting at the point *C*. For the greater simplicity, we may also suppose this force to act symmetrically with respect to the two transverse bounding fissures. Then, after the general elevation has proceeded as far as represented in the diagram, page 45, and the fissures have been formed,

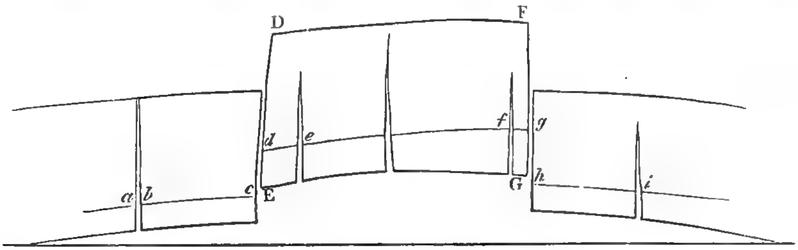
if the elevatory force act at C with a considerably greater intensity than at M, it will communicate to the mass $CC'NM$, together with its general upward motion, a rotatory one, of which the axis will be horizontal and perpendicular to the transverse boundaries. This motion will tend to depress the extremity M, particularly if CM be of sufficient length. No such cause will exist in the adjoining mass $AA'NM$ to lower its extremity N; and moreover it may be remarked, that this mass once elevated is more likely to be supported by the debris produced by a convulsive movement such as we are supposing, and therefore its extremity N will be less likely to subside than the adjoining extremity of the contiguous mass. From these causes it would seem highly probable that these two portions of the general mass should assume the relative positions above represented. A partial elevation and escarpment may thus be produced in accordance with the general fact stated in the Introduction, (IV. β. p. 7.)

We may also observe, that the fault thus formed at N must very generally possess the character mentioned in the Introduction, (I. γ. p. 2.)

54. In the diagram, page 46, $DEFG$ may represent a section parallel to the general axis of elevation of the portion of the mass which we have supposed, in the preceding article, to be subsequently elevated in a greater degree than the portions contiguous to it on either side, as represented in the diagram of the following page. If we conceive the portion also of which the section is $F'D'E'G'$ (p. 46.) to be raised in the same manner, it is obvious that a *transverse valley* will thus be formed between these two partial elevations, such as described in the Introduction. (V. p. 8.)

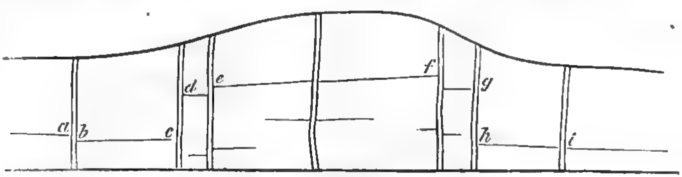
55. A section of one of our partial elevations above mentioned, by a vertical plane parallel to the axis of the general elevation and the longitudinal fissures, will now present an appearance (taking the phenomena as far as we have yet investigated them) similar to that of the annexed diagram, in which $DEFG$ represents the portion of the mass defined by the same letters in the diagram of page 46. The broken

line *cdgh*, supposed to be originally horizontal, indicates the faults along *DE* and *FG*. We may easily conceive, however, a further modification of the phenomena from any irregularity in the action of the elevating force, or in the resistance opposed to it, in adjoining portions of the mass on opposite sides of any one of the incomplete transverse



fissures, similar to that which we have assumed to produce the faults *DE, FG*, at complete fissures; for if this inequality of action on two such portions of the mass be sufficient, it may evidently convert the incomplete fissure into a complete one, provided the fissure extend near enough to the surface to weaken the mass so much as to render it unable to counteract the tendency of this unequal action, to give a greater elevation to the portion on one side of the fissure than to that on the other. In such case a fault would almost necessarily be produced, but probably smaller than that which would be produced by the same cause at a complete fissure. In either case, however, the fault may of course be of any magnitude, depending on the intensity of the action producing it.

If then we conceive the phenomena represented in the preceding diagram to be thus modified, and the superficial elevations to have been partially removed by denudation, the actual phenomena may be represented as in the annexed section. The broken line *abcdefghi* is as



before, supposed to have been originally continuous and horizontal, or if the mass be stratified, to represent a line of stratification. *cd* and *gh* are *faults;* the differences of elevation *ab*, *de*, *fg*, are supposed too small to be so designated. Small relative elevations of this kind constitute what is frequently termed the *throw* of the vein. (Introd. II. ι. p. 4.)

56. It is important to observe the different effects which will be produced on the form of the longitudinal and transverse fissures by the movements above described. It has been shewn (Art. 38.) that a fissure immediately after its formation, and before any subsequent movement of the mass has taken place, must offer a certain approximation to uniformity of width; but an inspection of the diagram in page 51, will make it appear very evident, that this subsequent movement must in general destroy, in great measure, this character in the longitudinal fissures, since it must almost necessarily close them in some parts and open them considerably in others; while a movement similar to that described in Art. 53, and represented in the figure, page 54, will not necessarily produce any derangement in this respect in a perfectly uniform fissure, because the motion of one wall of the fissure is parallel, or nearly so, to the other. We should expect therefore, as a necessary consequence of this view of the subject, a much nearer approximation to uniformity of width in the transverse, than in the longitudinal fissures. This is strikingly in accordance with what has been stated in the Introduction (I. θ. p. 4.) a rule to which, I believe, there are comparatively few exceptions.

§. *Proper signification of the term "System of Fissures"—Simultaneous Formation of Systems of Fissures.*

57. I have hitherto spoken of systems of parallel fissures, as if the parallelism of the fissures constituted the essential characteristic of each system; and in the case we have been considering of an elevation of indefinite length, and of which the axis is rectilinear, this parallelism will characterize the two systems at right angles to each other, and which I have designated as longitudinal and transverse. If, however, the axis of the general elevation of indefinite length be not in a right line, the fissures of the longitudinal system (assuming them to be produced in

the manner I have indicated,) will be still parallel to this axis (in the sense in which one curve line may be said to be parallel to another) and every fissure of the transverse system will be perpendicular to each fissure of the former system at the points of their intersections, and consequently the fissures in this transverse system will not be parallel. Again, if we suppose the superficies of our elevated mass to be of finite length, and to be bounded for instance by a line approximating to the form of an elongated ellipse, the directions of the fissures in the transverse system, as we approach towards either extremity of the elevated range, will gradually change from perpendicularity with the major axis (the axis of elevation) till they become parallel to it, at the extremities of the ellipse, always preserving their approximate coincidence with the directions of the lines of greatest inclination of the general surface of the mass. The fissures of the other system will be approximately perpendicular to these lines. In this case then, the two systems will be no longer characterized by any constant relations which their directions bear to that of the axis of elevation, and therefore the terms *longitudinal* and *transverse* will cease to designate them so correctly as in other cases; and still more is this the case, where the elevation approximates to the conical form, in which all the fissures analogous to those we have termed transverse, diverge from the vertex of the cone. I have not, however, thought it necessary to supersede these terms by others, since they are very generally applicable with great propriety. It is highly important, however, as respects the application of this theory of elevation, to distinguish these two systems carefully from each other. It has been pointed out (Art. 56) how much the *transverse* fissures exceed the others in regularity of formation, and it seems not improbable, that this fact may be in some way connected with that of their containing mineral veins, so much more continuous than those found in the more irregular fissures of the other system, (Introd. II. δ. p. 3.) The most general rule will probably be, whatever be the form of the elevated mass, that the direction of a *transverse* fissure approximates to that of the *dip* of the strata, (supposing the mass stratified) the direction of a *longitudinal* one, consequently, approximating to that of the *strike* of the stratified beds. It should be observed, however, that the present form of the elevated mass may in some cases differ materially from that which was originally given to it, by the movement to which the formation of the principal fissures must be referred. The rule would probably be more applicable immediately after this first elevation, than after the modifications in the position of the mass, which may possibly have been produced by subsequent ones.

It will be observed that the law of parallelism, which characterizes alike the phenomena of anticlinal lines, faults, mineral veins, &c., is to be traced, according to the view we are taking of the subject, to the same origin; viz. the formation of the two great systems of fissures, which have been shewn to be, under certain simple conditions, the necessary effects of the elevatory force to which they have been referred. The term *parallelism*, therefore, when used as characterizing systems of any of the above phenomena, must be equally regarded as subject in its interpretation to the exceptions or modifications pointed out in the last paragraph. In fact, if the extent of the mass be comparatively small, and its boundary irregular, this property would cease altogether to characterize the phenomena. If the elevated mass be of great superficial extent, partial irregularities in its boundary will have no appreciable effect on the directions of the fissures; and though two remote fissures of the same system might, in such case, (as appears from the preceding paragraph), be inclined at any angle to each other, any two adjoining fissures would in general be approximately parallel. The law of parallelism, however, in the strict acceptation of the term, could only hold through the whole extent of the elevated mass, in the case above considered of a rectilinear elevation of indefinite length. In other cases, the law must be subject to the modifications indicated above.

58. If the approximate accuracy of our assumptions be allowed, as applied to the crust of the globe, it appears, from our investigations, that an elevated range characterized by continuous systems of longitudinal and transverse fissures, referrible to the causes to which we have been assigning such phenomena, could not be produced by successive elevations of different points, by the partial action of an elevatory force. It has been shewn (Art. 46) that in such elevations

fissures would necessarily diverge in all directions from the central points, so that parallel systems such as above mentioned could not possibly be thus produced. It has moreover been shewn, (Art. 30.) that every system of parallel fissures in which no two consecutive fissures are remote from each other, must necessarily have had one simultaneous origin. Subsequent efforts of the subterranean forces may enlarge these fissures, and propagate some of them to the surface, converting incomplete into complete fissures, but it would seem essential, according to our view of the subject, that their *positions* in the lower portion of the mass, where their formation will commence, (Art. 36.) should be determined contemporaneously.

§. *Formation of Riders—Explanation of the Phenomena at the Intersections of Mineral Veins.*

59. If two systems of fissures were formed by forces acting in the manner we have supposed on a mass without vertical or nearly vertical planes of less resistance, these systems would present to us cases of *intersection* only of nearly vertical fissures with horizontal beds, or with other vertical fissures at right angles to the intersecting ones. It is manifest, however, that the existence of planes of less resistance, combined with an irregularity of intensity in the elevatory force such as we have assumed, may produce some fissures irregular both in direction and inclination to the horizon, though the general phenomena may still present that distinct approximation to the laws we have indicated, which would be the necessary consequence of the great predominance of general over local causes. It is at the intersections of the two perpendicular systems of veins (metalliferous veins and cross courses) that the most important of the phenomena we are about to consider are found, while others occur at the intersections of veins of more irregular formation.

60. Before we proceed to examine these phenomena more particularly, we may notice one probable consequence of this occasional irregularity in the formation of veins, viz., the production of what are usually termed *riders*. If a fissure be propagated through a point in which

two planes of less resistance meet, it is very possible that it may be propagated simultaneously along these planes. These diverging branches may continue separate, and present themselves at the surface as two distinct fissures, or they may meet again, and thus including a portion of the mass in which they are formed, produce the phenomenon above mentioned. If the insulation be perfect, the mass, if not too large, will of course fall, and may descend to any unknown depth; and possibly this may be one cause of the partial irregularities in the width of the fissures of mineral veins. If the insulation be imperfect, or the width of the mass be greater than that of the fissure immediately beneath it, it will be supported in its original position, or it may under other circumstances lodge at a certain depth below it. In either case if such a mass come within the sphere of the miner's observation, he terms it a *rider*, (Introd. II. μ).

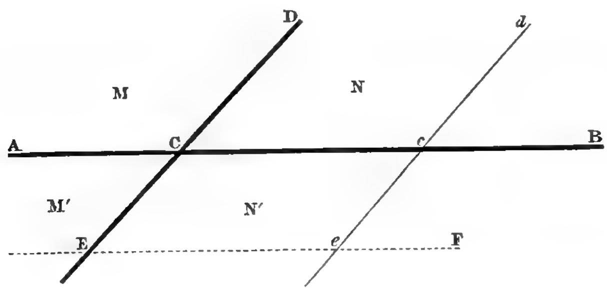
If the rider be originally supported as above suggested, till a sufficient quantity of matter shall have been deposited in the fissure, to afford a support to it independent of its contact with the walls, and the fissure be then increased in width by any renewed action similar to that which originally produced it, the rider may present itself to us supported by the *vein-stuff*, in a state of perfect insulation from the solid mass on either side of the vein*.

61. In the phenomena attending the intersections of veins, described in the Introduction (II. o, π, ρ,) the broken veins are generally supposed to have been originally continuous, and to have been broken by a relative movement of the portions of the mass on opposite sides of the unbroken vein. Adopting this hypothesis, we have not the smallest difficulty in accounting for the appearance represented in the figures, p. 6. (Introd. II. ρ,) since our elevatory force must necessarily produce in many cases that relative elevation of different sides of a fissure, which at once accounts for the phenomena in question. The other two cases above

* This perfect insulation of riders has been recently urged as an objection of the most serious weight against the mechanical origin of veins. It appears to me, on the contrary, to be an almost necessary consequence of the causes we are considering acting on a mass constituted like the crust of the earth.

alluded to, (Introd. II. o, π,) presenting apparent *horizontal* displacements of the mass on one side of the unbroken veins, is not at first sight so easily accounted for, since it can hardly be regarded perhaps as physically possible that any horizontal pressure can have acted on the mass with sufficient intensity to produce an absolute displacement equal in many instances to the apparent one. A very ingenious mode has, however, been suggested* of explaining phenomena of this kind, by referring them to relative *vertical* movements of the masses in which the fissures have been formed. It will not be difficult to convey an idea of the manner in which this may be effected.

62. Let the annexed figure (1) represent a horizontal section at the



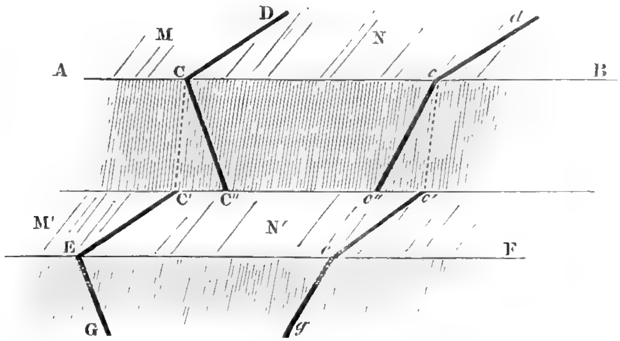
(1)

surface, of two veins which intersect, both being somewhat inclined to vertical planes through AB, ED respectively. Now suppose the portion of the mass bounded by the horizontal surface MN, and the nearly vertical plane ABC' (Fig. 2.)† of the vein AB, to be elevated (or the opposite portion to subside), so that the surface $M'N'$ may be at a lower level than MN. If this change be effected by a movement parallel to the plane ABC' of the vein AB, CE (Fig. 1.) will assume the position $C'E$ (Fig. 2.); and if EFG be a plane parallel to ABC', and intersecting the vein DCE (Fig. 1.) in EG (Fig. 2.) $C'EG$ will be the plane of the vein in the subsided mass, and it will no longer coincide with the plane DCC'', the original plane of the fissure DCE. If we now conceive the higher portion of

* By the late M. Smidt.

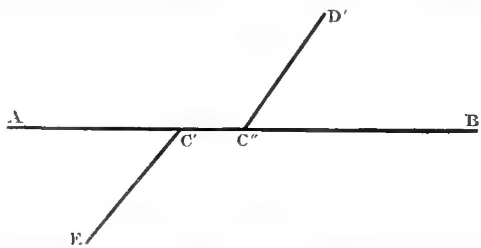
† The same letters denote the same points of the mass in the diagrams (1), (2), (3), (4).

the mass to be removed by denudation, the general surface will coincide



(2)

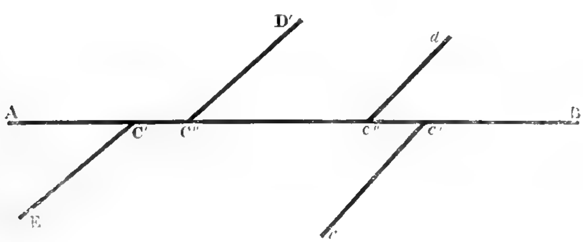
with $M'N'$, and the broken plane of the vein will no longer intersect it in a continuous line, but as represented in (Fig. 3), along the broken



(3)

line $EC'C''D'$; thus producing the appearance of a horizontal movement of the mass on one side of the vein AB, relatively to that on the other.

63. That these phenomena cannot, in some cases, have been pro-



(4)

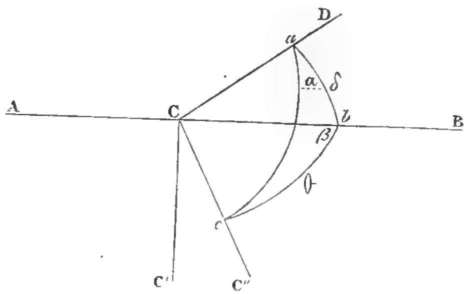
duced by actual horizontal movements, appears to admit of the most demonstrative proof; for it is sometimes found that when two veins intersect a third, both are apparently shifted horizontally, but in opposite directions, presenting the appearance represented in the preceding diagram (4), (a horizontal section), where $C''D'$ and $c''d'$ are apparently so shifted, though it is manifestly impossible that they should be so heaved by any horizontal displacement of the mass containing them.

This case admits, however, of a perfectly simple explanation on the hypothesis of a vertical motion, provided the two veins, which are apparently shifted, *hade or underlie in different directions.* This will be immediately seen by a reference to the diagram (2), where dcc'' represents the plane of the second vein intersected by AB in the higher portion of the mass, and $c'eg$ in the lower. The line cc' being parallel to CC', it is manifest that when C' coincided with C, c' would coincide with c; and consequently, after the denudation above supposed, the intersections of these veins with the exterior surface will present the appearance represented in (Fig. 4).

64. The case just described is admirably calculated to afford a decisive test, as to whether these phenomena have, or have not been produced by vertical movements, or rather by *upward movements parallel to the plane of the unbroken vein.* It is manifest that the explanation above given depends on the fact of the veins CD, cd, inclining in opposite directions, or more correctly, upon their intersecting the plane of the vein AB, in lines inclining towards each other from the parallel lines CC', cc' respectively. Consequently, it may be stated in general terms, that if the two shifted veins incline in the same direction, the above explanation is inadmissible; but if, on the contrary, it be found that these displacements in opposite directions occur only in veins which hade in opposite directions, the truth of the explanation can no longer admit of a reasonable doubt.

65. Other cases also of the apparent displacement of a single vein, may afford most valuable evidence respecting the fact of the kind of elevation of which we have spoken. It is manifest, that whatever the case of displacement may be, the horizontal extent of it must depend on the following quantities: the inclinations of the planes of the broken and unbroken veins to the horizon (the complement of the angles which

measure the hades), the angle *DCB* (Fig. 1.) between their intersections with the horizontal surface, and the length of the line *CC'*, which evidently measures the *throw* of the unbroken vein *AB*, produced by the supposed movement. To express the horizontal displacement of the vein in terms of these quantities, suppose a sphere described with center *C* in the previous diagram (2), {or in the following one in which the same letters denote the same points as in (2)}, and any radius so as to



form the spherical triangle *abc*, by its intersections with the planes of the veins and the horizontal plane. Let

α = angle *bac*, the inclination of the plane *DCC''* of the broken vein to the horizon.

β = *abc*, the inclination of the unbroken vein to the horizon.

δ = *ab* = *DCB* the angle between the intersections of the veins with the horizon.

bc = angle *BCC''*,

h = *CC'*, the throw of the unbroken vein.

Then shall we have

$$\cot\theta = \cot\alpha \, . \sin\beta \operatorname{cosec} . \delta + \cos\beta \, . \cot\delta;$$

and the apparent horizontal displacement $C'C''$

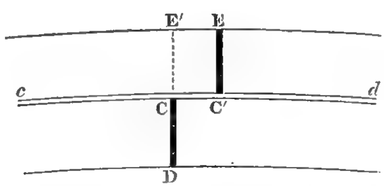
$$= h \, . \cot\theta$$

$$= h \, \{\cot\alpha \, . \sin\beta \operatorname{cosec} . \delta + \cos\beta \cot\delta\}.$$

The quantities $C'C''$, α, β and δ can generally be obtained with very considerable accuracy, as may *h* also, when the mass in which the veins are formed is distinctly stratified. In such cases therefore, by comparing our observed and computed values of $C'C''$, we might obtain very accurate tests of the truth of the explanation which has been given of these phenomena.

66. The value of the explanation which has been given above of the phenomena we are now considering, consists in the substitution of vertical for horizontal movements, and therefore depends on the approximate verticality of the unbroken vein, parallel to the plane of which the motion is assumed to take place. It not unfrequently happens, however, that a horizontal displacement of a vertical vein takes place at the thin horizontal beds of moist clay, of which so considerable a number is found interstratified with the mountain limestone. The slimy nature of these beds undoubtedly affords a great facility for a relative movement of the masses respectively above and below them; and therefore where the displacement is small, there seems no difficulty in accounting for it on the supposition of this relative motion. In other cases a more probable cause may be found in the following considerations.

67. In the annexed figure let *cd* represent a thin stratum of clay,



of such a nature as to give a considerable facility to a relative horizontal motion of the masses above and below it, and suppose a fissure to have been propagated upwards by the action of horizontal tensions, from D to C. If there were no cohesion whatever between the upper and lower divisions of the mass, it is manifest that the position of DC would not in any degree influence the position of a fissure $C'E$, which might be produced in the same manner and at the same time in the upper portion of the mass, and consequently the point C' would then be determined by the constitution of the upper mass, or some circumstance not immediately depending on the position of DC. In such a case, therefore, there might be an *apparent* horizontal shift of any magnitude. If, however, a certain force, arising from cohesion and friction, should oppose a relative horizontal movement of the upper and lower portions of the mass, a limit will be imposed on the extent of the apparent shift, for it is obvious that this force must be called into action in the formation of $C'E$, (in the progressive formation upwards of the fissure) by the opposite motions of the upper surface of the lower mass, and lower one of the upper mass between C and C', and in no other part. Consequently, if the resistance at C' to the formation of a fissure in the upper mass, together with the lateral force just mentioned, be greater than the resistance to the continuation of the fissure from C towards E', the former fissure cannot be formed in preference to the latter, and thus a limit will be imposed on the distance CC'. It is easy, however, to conceive, from the known constitution of the beds which appear to give rise to phenomena of this kind, that this distance may be sufficient to account very easily for all such appearances of displacement as we are now considering.

68. If we conceive the figure in page 64 to represent a horizontal instead of a vertical section of the mass, and cd to represent a fissure, then, if a fissure $DCC'E$ be propagated across it, it is manifest that considerations exactly similar to the above would enable us to account for the apparent displacement CC' in this as well as in the former case, and it appears highly probable that such appearances may have been not unfrequently thus produced. We may also observe, that if the fissure cd has not been completely filled, and its sides again cemented together, the movements of the masses on opposite sides of it will be in a certain degree independent of each other, so that a fissure DC propagated so as to meet cd at C, might be continued on the other side of cd, from a point C' quite remote from C. In such case DC would appear to terminate at C, and this, in fact, (DC being a small, and cd a large vein) is not of unfrequent occurrence.

69. There is also another manner somewhat different from the above, in which an apparent displacement of a fissure may be produced. It

has been already shewn (Art. 17), that if a fissure in its progressive formation meet with any line of less resistance, it will under certain conditions be propagated along it for a certain distance, and then resume its original direction. If AB (Fig. 3, p. 61) be a line of less resistance, $EC'C''D'$ would represent a horizontal section of the fissure formed in the manner just supposed, and thus presenting the apparent displacement $C'C''$.

It must be remarked, however, that an apparent displacement due to this cause must necessarily be such as represented in the figure just referred to, viz. on the side of the obtuse angle $EC'C''$, or $D'C''C'$, and not on that of the acute angle $ec'c''$, or $d'c''c'$ (Fig. 4, p. 61); and we may also observe, that neither this cause, nor that pointed out in the previous article, appear sufficient to account for the fact, which has been frequently recognized, of two or more adjoining veins being apparently displaced, or heaved, to the *same extent* and in the *same direction* by the same cross course. We see no reason why the apparent displacements of two such veins should be related in either of these particulars, when produced by the cause indicated in Art. 68; and if produced by that mentioned in the preceding paragraph, though the apparent displacements would necessarily be in the same direction, there seems to be no reason why they should be of the same extent. When the heaves, therefore, of adjoining veins appear to be related to each other both in extent and direction, the above two causes do not appear to offer an adequate explanation of the phenomena.

70. It was a notion first propagated, I believe, by Werner, and subsequently adopted by many other geologists and miners, that when two veins meet each other, of which one is heaved, and the other unbroken, the formation of the latter must necessarily have been posterior to that of the former. The theory of elevation, however, which we have been discussing, will not authorize this conclusion. If we assume the modes of producing apparent displacements considered in Arts. 68 and 69, it is evident that we must adopt a rule exactly the reverse of the one just stated; and if we suppose the displacements to be real, it is manifest from what has been advanced in this and

the previous section, that the formation of one or both of the fissures may have been either contemporaneous with, or anterior to that movement of the mass which produced these displacements; and consequently the existence of the heave in the one or the other of two intersecting veins, can afford no test of their relative ages. In cases, however, where several veins are found to have been heaved in the immediate vicinity of each other (as in some of the Cornish veins) indications may be obtained of their relative ages from the phenomena they exhibit, assuming them to have been produced in the manner just supposed.

71. It has been stated (Introd. η, p. 4), that the fissure of a vein is frequently almost entirely closed in passing through a thin stratum of clay. This fact may, I conceive, be easily accounted for from the greater *extensibility*, and less *elasticity* of this stratum, as compared with the masses with which it is interstratified. The former quality would allow it to remain unbroken, with an extension which the general mass could not but yield to, or if broken, it would from the latter property have little tendency to recede to its original extent.

72. It is not my intention to enter into any discussion on the mode in which the fissures of mineral veins have been filled*; but I would remark, that the frequent occurrence of the fact above mentioned seems equally unfavorable to the hypothesis of this process having taken place by superficial agency, or by any species of injection from beneath. The difficulty, however, assumes a far more formidable character when considered with reference to the toadstone of Derbyshire, which, as I have already stated (Introd. II. η.), produces the same effect, in nearly destroying the continuity of the fissure, as the clay beds above mentioned. But in this case, instead of a bed of a few inches in thickness, we find a bed of toadstone of from ten to forty fathoms, through which the vein can sometimes be traced only by mere threads of calcareous spar. How then can we conceive the upper part of such a fissure

* I do not here allude merely to the process by which the *mineral vein* properly so called, (see p. 2.) has been deposited, but that by which the whole fissure may have been filled with the *vein stuff* which now occupies it. The fissure may be several feet wide, while the mineral vein is not an inch in width.

to have been filled from below, or the lower part filled from above? Either the one hypothesis or the other appears totally inadmissible, unless we suppose the communication between the upper and lower parts of the vein to have been formerly very much more perfect than at present. This hypothesis would, perhaps, present no very serious difficulty, because it is very possible to conceive the toadstone to have been so imperfectly solidified at the time of the formation of these fissures, as afterwards to diminish their width, by yielding in some measure under the pressure of the superincumbent mass. But if we suppose the portions of the fissure both above and below the toadstone to have been filled either from above or below, while there existed a wider fissure connecting them through the toadstone, this fissure in the toadstone must also have been filled before its ultimate degree of contraction, in which case it appears almost impossible that there should not be a much more determinate trace of a vein through the toadstone, than is at present observed to exist. We seem almost necessarily driven in these cases to the hypothesis of some process of segregation or infiltration *into fissures previously formed for the reception of the segregated or infiltrated matter.*

§. *On the Formation of Granite Veins.*

73. These veins have been described (Introd. VII.) as distinguished in general by the absence of that tendency to rectilinearity and parallelism in their directions which so distinctly characterize the principal mineral veins in each mining district. The fact of these veins being found only at the junction of masses of granite with other masses of different mineralogical constitution, has naturally suggested the idea of these veins being *veins of injection;* the granite being assumed to be of igneous origin. This opinion seems strictly in accordance with the views which we have been developing. The rectilinearity of mineral veins is due, according to this theory, to the predominance of tensions acting in a particular direction, whereas fissures formed in great measure by the hydrostatic pressure of injected fluid matter, in a mass subjected to no tension very determinate in its direction, might assume

any tortuous course. The irregular and violent action, also, to which the mass through which, according to this view of the subject, the granite is supposed to have been protruded, would have a great tendency, independently of the hydrostatic pressure just mentioned, to form in the broken mass irregular fissures, which would facilitate the injection of the fluid matter, and increase the irregularity of the form of the injected veins.

§. *On the Formation of Trap-Dykes and Veins.*

74. The results above obtained respecting the formation of fissures in the crust of the globe will manifestly hold equally, whether we suppose the uplifted mass acted upon immediately through the medium of an elastic vapour, or by matter in a state of fusion in immediate contact with its lower surface. In the latter case, however, this fused matter will necessarily ascend into the fissures, and if maintained there till it cools and solidifies, will present such phenomena as we now recognize in dykes and veins of trap. The same phenomena would result from the injection of the fluid matter at any period posterior to that of the formation of the fissures as above described. To represent to ourselves, therefore, the phenomena of trap-veins, as referred to the causes to which we are referring them, we have only to conceive the fissures previously described filled with trap. The larger ones will thus form dykes, and the smaller ones veins of that rock.

75. It has been observed by geologists, and particularly by M'Culloch, that a large proportion of trap-dykes have been formed without producing any sensible disturbance in the ends of the stratified masses abutting against them. And this is precisely what we might expect, if we suppose such dykes to have been injected without excessive violence into fissures formed as above described, whether that injection be supposed to have taken place after the formation of the fissures, or contemporaneously with it. Where injection, however, has taken place in great abundance, and with great violence, corresponding degrees of disturbance might of course be expected to attend it.

The geologist to whom I have just referred, in speaking of the trap-veins of the Isle of Sky, observes: "It is necessary to point out one extraordinary effect which must have resulted from the intrusion of these veins. Whatever proportion, collectively taken, they may bear in breadth to the lateral dimension of the strata which they intersect, it is plain that the whole mass of strata must have undergone a lateral extension equal to that quantity; a motion so great as not to be easily reconciled with the present regularity of the whole. It is also a singular circumstance, that on the opposed shore of Sleat a different effect takes place, and proportioned, it would here seem, to the number of veins; the red-sandstone strata of this coast being often turned from a slightly inclined into a nearly vertical direction, with other considerable marks of disturbance. It is impossible to account for these apparently capricious differences, and we must for the present be content to rank them among the numerous unexplained phenomena in which the science abounds."

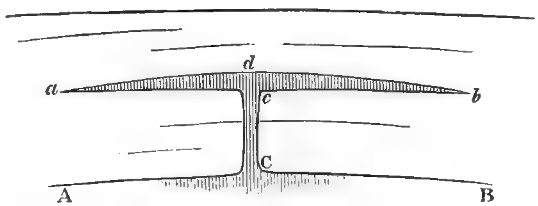
These phenomena present no difficulty except in the apparent lateral displacement of the stratified beds, without any other appearance of disturbance; and if this effect is to be referred to the lateral pressure of the injected matter, it does indeed present a difficulty no less, I conceive, than a physical impossibility. In the first place, it appears inconceivable how sufficient resistance could be obtained from above to produce the enormous lateral fluid pressure necessary to cause this lateral movement, as we have before remarked respecting the horizontal heaves of mineral veins; and in the next place, it is still more inconceivable how this force could have been exerted without indications of such violent action. Under the point of view, however, in which I have regarded the subject the difficulty no longer exists; for it must be recollected that the aggregate width of the veins, or apparent lateral displacement, is not to be taken with reference to the breadth of the mass in which the veins immediately exist, but with reference to the whole extent of the mass, the tension of which may have been relieved by the formation of these fissures. No rational account can be given, I conceive, of such lateral movements of extensive masses, except by referring them to the horizontal tension produced by vertical forces, and the consequent contraction when the mass becomes fractured by too great an extension.

§. *On the Formation of Horizontal Beds of Trap—By Ejection—By Injection—Remarks on some Phenomena observed by M'Culloch—Effect of imperfect Fluidity in Horizontal Injections.*

76. If the quantity of fluid matter forced into these fissures be more than they can contain, it will of course be *ejected* over the surface; and if this ejection take place from a considerable number of fissures, and over a tolerably even surface, it is easy to conceive the formation of a bed of the ejected matter of moderate and tolerably uniform thickness, and of any extent. If the ejection take place over a level surface, these properties of the resulting bed would seem to require a number of points or lines of ejection as a necessary condition, on account of the imperfect fluidity, which, according to analogy, we ought probably to assign to the ejected matter. If there were only a single center of eruption, a bed of such matter approximating to uniformity of thickness, could only be produced on a surface of a conical form, having the point of eruption at its vertex, and an angular elevation depending on the degree in which the fluidity of the ejected mass should differ from perfect fluidity. Where no such tendency to this conical structure can be traced, it would probably be in vain to look for any single *center* of ejection. On the supposition too, of ejection through continued fissures, or from a number of points, that minor unevenness of surface which must probably have existed under all circumstances during the formation of the earth's crust, would not necessarily destroy the continuity of a comparatively thin extensive bed of the ejected matter, in the same degree in which it would inevitably produce that effect in the case of central ejection.

77. I will now proceed to consider the formation of a horizontal bed by *injection*; what limits may be imposed on the probable or possible extent of it, and with what phenomena it may be accompanied, which may serve as tests for distinguishing a bed so formed from one formed by *ejection* over the external surface.

Let us suppose then, that the fluid mass has risen through the fissure of which *Cc* is the section, till it has reached the stratum *adb*. If this stratum have sufficient tenacity and extensibility, and but little adhesion to that on which it reposes, it is easy to conceive that it may be elevated without being broken, if the fluid mass be impelled upwards with sufficient force to overcome the weight of the superincumbent mass. In this case the fluid will necessarily be *injected* horizontally, as represented in the figure, and so long as the lower surface of the uplifted stratum remains perfectly continuous and unbroken, it is very possible that this injection may extend to any assignable distance *without the*

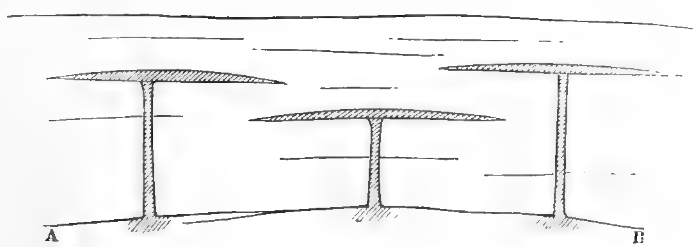


production of vertical dykes, on veins branching from the upper surface of the injected bed. In this case there would appear to be no indications of mechanical action from which the geologist of the present day could ascertain whether such bed had been *injected* among the beds associated with it, or *ejected* over the surface *acb* at a period anterior to the formation of the superincumbent strata.

The most favorable case we can conceive for the kind of injection we are considering, without the production of the vertical veins above mentioned, is that in which we assume the absence of all adhesion between the uplifted bed and that immediately beneath it; but even in this case the condition of unbroken continuity in the lower surface of the superincumbent mass, must be satisfied, not approximately, but accurately; for if the smallest crevice existed in the uplifted portion, the injected matter would be impelled into it with a force proportional to the enormous pressure to which it would be subjected from the weight of the superincumbent beds; and if the injection should take place under the weight also of a deep sea, the probability of this effect would be exceedingly increased by the consequent additional pressure, while the process of injection would not be in the smallest degree facilitated by it. Trap-veins would thus be produced, affording indubitable evidence of injection.

Again, the hypothesis we have made above of the entire absence of adhesion between two contiguous beds, though it may in some cases be true for limited spaces, cannot be uniformly so in cases in nature for spaces of considerable extent. Now in those instances, in which the force of adhesion between the two beds, bears any kind of ratio to that which holds together the component particles of the uplifted portion, an enormous force will be required to overcome this adhesion. And how are we to conceive such a force applied without producing the smallest rupture in the lower surface of the uplifted mass? If there be no adhesion between the beds, no considerable horizontal tension will be produced in this mass; but if the adhesion be considerable, such a tension will be produced, proportional to the increased force of injection called into action. Under these circumstances the smallest break or crevice will be torn open, the fluid matter will enter it, and acting on its vertical sides with an enormous pressure, and with the mechanical advantage of a wedge, will add immensely to the tendency of the horizontal tension to produce a vertical fissure.

78. It may perhaps be thought that the difficulty of conceiving the process of horizontal injection of considerable extent, without the production of vertical veins, may be obviated by supposing the fluid



matter injected from many points simultaneously between the same two horizontal beds. But this hypothesis appears extremely improbable, unless it be also assumed that the want of cohesion between these beds is co-extensive with the injected bed, an assumption, which as I have before remarked, must probably be in general considered as totally inadmissible. The probable consequence of simultaneous injection from different fissures, (supposing the injected matter not in too great quantity), would be the formation of partial and unconnected beds as represented in the annexed diagram.

For these reasons then we cannot hesitate, I think, to conclude, when we consider the general structure of stratified masses, that the absence of numerous trap-veins and dykes, *having their origin in the upper surface of a horizontal bed of trap*, with the want also of very frequent indications of violent mechanical action in the lower portion of the *superincumbent mass*, affords indubitable proof of the fact of such horizontal bed having been ejected over the exterior surface existing at the time of its eruption.

79. The existence of a single vein or dyke such as above described, in rocks incumbent on a horizontal bed of trap, is clearly an indubitable proof of injection; but it must not therefore be concluded, that every trap-vein or dyke in the superincumbent strata affords this unequivocal testimony, since it is manifest that such a vein or dyke might possibly be produced by injection, subsequently to the formation of the horizontal bed, which it may have traversed exactly in the same manner as any other stratum*. The decisive character of the evidence of injection afforded by a vein, consists in its originating in the upper surface of the injected bed. We may also remark, that indications of mechanical action on the beds *beneath* a bed of trap will not necessarily afford conclusive testimony as to the fact of injection, because such appearances might be produced, to a certain extent, by the force of an *ejected*, as well as of an *injected* bed. It is in the superincumbent beds that we must seek for the evidence in question.

80. It is not my object to enter into any detailed comparison between observed facts, and these theoretical deductions, but I think it necessary

* Many instances are given by M'Culloch of veins of trap existing in trap. See "Description of the Western Islands."

to allude to some cases of injection described by M'Culloch in the Western Islands, in which the injected beds assume for considerable distances the appearance of being regularly interstratified, thus seeming, it might be thought, to offer exceptions to the rule I have deduced from theoretical considerations. Four or five only of these exceptions I think have been expressly mentioned by that author. Those on the coast of Trotternish in the Isle of Sky, which appear to be the most striking, are described as follows:

"In one case, which occurs not far from Holme, there is a bed extending for a great way, surmounted by a parallel series of the secondary strata in contact with it; but on a narrow inspection, innumerable veins are seen branching into the strata in every possible direction, illustrating in a very perfect manner the origin of at least one order of veins. In a second case, three beds of trap can be traced in a parallel direction for a considerable space, separated by the regular strata, when suddenly the whole unite into one mass. Had not this occurrence at length betrayed the true nature of these beds, there would have been no hesitation, from a limited observation, in describing them as unquestionable instances of alternation. In the last case which I shall mention, one regular bed of trap may be traced for more than a mile, lying in a parallel and undisturbed continuity between the secondary rocks. On a sudden, however, it bends downwards so as to pass through the strata immediately in contact, and then continues to hold its regular course for a space equally great, with a thickness and parallelism as unaltered as before*."

The first of these instances presents in its branching veins, exactly the phenomena which, I have been contending, must necessarily attend any extensive horizontal injection of a fluid mass. The others seem to indicate the possibility of this injection without such phenomena, for at least the extent of a mile. Nor am I disposed to doubt this possibility, though I should in general consider a horizontal injection of that extent without ramifying veins, as extremely improbable, and especially if the injected bed were not a very thin one. In fact, however, there

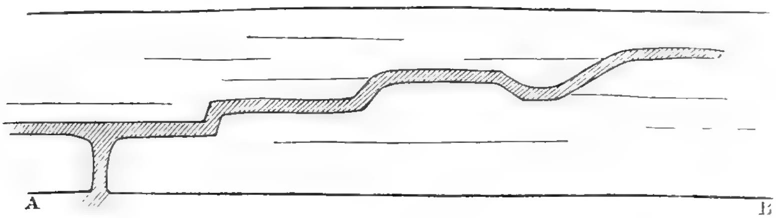
* Description of the Western Islands of Scotland, Vol. I. p. 382.

appears no reason to conclude that such has been the case. The whole of Trotternish is described as consisting of an enormous overlying mass of trap, which appears to have risen in numberless places through the stratified rocks on which it reposes. It extends (if I understand the description rightly) quite to the coast, so that scarcely any stratified rocks are visible, except in the vertical section formed by the steep cliffs along the beach, and in which the appearances above described are observed. Hence it is probable that these horizontal beds are connected with vertical masses of trap, at distances from the visible sections of them, small in comparison with their apparent range along the cliffs, and consequently it is very possible that the extent of horizontal injection may have been much less than at first sight it appears to have been.

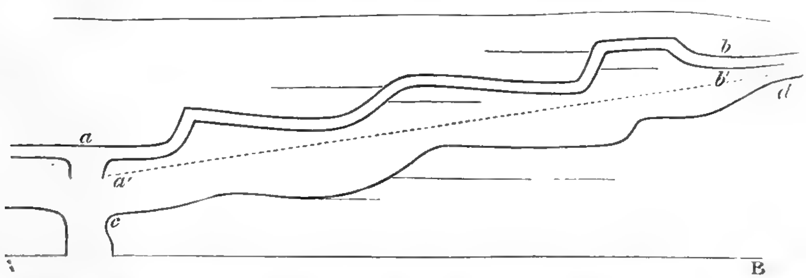
The same observation will apply to the other phenomena of the same kind as described by the author just quoted; and so far from offering any thing opposed to the theoretical views I have been explaining, they may, I think, be considered, when taken in conjunction with numberless cases of vertical dykes and veins, as strongly corroborative of them; since the comparatively insignificant number of these injected horizontal beds, clearly proves them to offer only so many exceptions to the very general rule of verticality in trap-veins, so frequently recognized by M'Culloch himself.

81. In speaking of horizontal injection, I have not yet alluded to the consequences of imperfect fluidity in the injected matter. If we may be allowed to judge of the degree of this fluidity from the analogy which the injected matter may be presumed to have borne to modern lava in its eruption, we may conclude it to have fallen considerably short of that of perfect fluidity. Consequently the lateral pressure communicated by the fluid, would never be equal to the direct pressure impressed upon it, and this, it is evident, would increase the difficulty of horizontal injection in the cases which I have already considered. The most important consideration, however is, I conceive, that this property of imperfect fluidity, would thus impose a limit to the possible extent of lateral injection, supposing the injected matter not to form a bed lying in one plane, but to form an irregular surface, such

that the following diagram may represent a vertical section of it. For suppose the fluid capable of transmitting the $\left(\frac{m}{n}\right)^{\text{th}}$ of a force impressed upon it, in a direction perpendicular to that of the impressed force; then if the pressure be transmitted along a broken line consisting of straight lines at right angles to each other, it is clear that the force transmitted along the first straight portion (supposed horizontal), will be $\frac{m}{n}\cdot p$, p being the impressed force and acting vertically. Along the second portion of the broken line the transmitted force will be $\left(\frac{m}{n}\right)^2 \cdot p$, and generally



along the r^{th} portion it will be $\left(\frac{m}{n}\right)^r \cdot p$. If the different portions of the broken line be not at right angles to each other, or instead of being straight be curved, the diminution of the transmitted pressure must still be calculated on the same principle. It is important, however, to observe that the thickness of the injected bed would probably influence this diminution very materially, as may be illustrated by the following figure. If the section of the bed be represented by the space between



the lines ab and cd, a straight line may be drawn in it from one extremity to the other, and therefore the transmitted pressure at one extremity would nearly equal that impressed at the other. On the contrary, if the space between the lines *ab* and $a_{\prime}b_{\prime}$ represent the section of the bed, it is manifest that the smallest number of straight lines which could be drawn entirely within this space, so as to form a continued but broken line between *a* and *b*, would be considerable, and that consequently the loss of transmitted pressure would be considerable. The magnitude of the impressed pressure at *a* is limited by the power belonging to the incumbent mass of resisting dislocation there; and when the loss of pressure by transmission is so great, that there is no longer sufficient force to cleave the mass into which the injected matter is penetrating, the horizontal injection will cease. I think it very probable that the limits thus imposed on the extent of possible injection, in the case of a thin bed like that just described, may be much narrower than some geologists seem to have conceived.

§. *Effect of Joints in determining the Directions of Fissures.*

I have stated (Introd. p. 11), that the investigations of Sect. I., are not to be considered as applicable to a mass in which the jointed structure should prevail generally, because the cohesion of the mass being in great measure, or altogether destroyed along the joints, the fissures resulting from any external force, would of course be formed along them. If, however, there should be two systems of joints existing previously to the action of the elevatory forces, in directions respectively parallel and perpendicular to the general axis of elevation, it is evident that the systems of fissures produced by this force, as well as all the phenomena resulting from them, would be exactly the same as those already described. If the direction of these systems should be only approximately parallel, and perpendicular to the axis of elevation, the same would still be true as respects the distinctive characters of longitudinal and transverse fissures, (see Art. 56). If, however, the directions of these two systems of joints should not have approximately these relations to that of the axis of elevation, or should not be nearly at right angles to each other, systems of fissures will result different from those which we have already described as the consequence of a general elevatory force.

Since the existence of joints in rocks appears to be very general, it becomes a matter of interest to enquire what effect they may possibly have had in determining the positions of the lines of dislocation, which we at present observe in the crust of the globe, as already described. Our present limited knowledge of the extent of joints, horizontally and vertically, and of their relative directions, will not enable us to return any direct and definite answer to this enquiry. We may however, observe, (and the observation is important as respects the applicability of this theory) that in those districts where the directions of faults, mineral veins, cross courses, &c., bear those relations to a well defined axis of elevation, which would exist according to these theoretical views, and which observation, so far as it has proceeded, has shewn to hold very generally, it would appear absurd to assign those directions to the influence of joints, unless some cause can also be assigned why the elevatory force should act in such a manner as to give to the axis of elevation, a direction bearing a necessary relation to that of any previously existing system of joints. As it appears almost impossible to conceive any such cause, we may, I think, without hesitation, in the cases above-mentioned, reject the hypothesis of any extensive influence of a jointed structure upon the phenomena in question. Should a general coincidence be hereafter observed in the directions of joints, and those lines of dislocation which follow the laws before mentioned, it would seem far more probable, that the former had been influenced by the latter, than the latter by the former phenomena.

In asserting the generality of the laws above mentioned, it must not be supposed that we are assuming the absence of all exceptions, or that the directions of mineral veins may not, in some instances, have been determined by causes different from those we have been considering. This, I think, has been unquestionably the case in the veins or lodes of St Austle moor, in Cornwall, where we recognize systems of lodes forming acute angles with each other, and obviously referrible to some cause totally distinct from the action of extraneous forces on the general mass. This, however, forms no argument against our theory, as applied to those cases in which the phenomena present to us features entirely different from those just mentioned, and in perfect accordance with our theoretical deductions.

With the causes which may have superinduced the jointed structure in rocks, I have at present no concern, except so far as it might possibly be influenced by the action of extraneous forces. It has been shewn, however, (Art. 32), that such forces could only tend to produce systems of fissures crossing each other at right angles, whereas regular systems of joints appear to meet each other frequently at acute angles, and consequently, must necessarily have been owing to some different cause. I do not therefore conceive that any general tension of the mass produced by extension from elevation, or contraction in the course of solidification, can have had any material effect on the formation of joints. It is probably, I think, to be referred entirely to some kind of internal molecular action.

THOUGH the law of approximate parallelism has long been recognized by geologists as characterizing mineral veins, faults, &c., I am not aware that any attempt has hitherto been made to deduce this important law from the causes to which these phenomena have been referred. In the preceding investigations, however, I have shewn, that under certain simple conditions, such a law is the necessary consequence of a general elevatory force acting in the manner I have supposed; and I have moreover shewn, that this law is entirely inconsistent with the partial action of such a force; because an elevatory force acting thus partially at a particular point, would necessarily produce fissures diverging from that point, so that in a general elevated range produced by the elevation of different portions in succession, there could be no general system of parallel fissures. This deduction appears to me perfectly conclusive as to the respective claims of two theories, one of which should assign the phenomena of elevation, in which the law of parallelism is observable, to the *partial*, and the other to the *general* action of an elevatory force, the terms *general*, and *partial* being taken in the sense in which I have heretofore used them, (see p. 1.) It must not, however, be supposed that our theory would lead us to the conclusion, that the whole elevation of any elevated range must have been communicated to it at once. It requires only that the *first* movement should have been general, and sufficient to produce at least the commencement of the systems of fissures, by which the range may subsequently be characterized, (Art. 58). Elevations, partial or general, may afterwards take place without producing other fissures following any law different from that of the preceding ones.

In the present state of geological theory, this deduction will not, I conceive, be deemed unimportant. It forms no part, however, of my present purpose to examine the merits of the different theories of elevation, which have been propounded by geologists; nor have I entered into these investigations in the spirit of advocacy of any peculiar and preconceived notions. My object has been simply to develope the necessary or probable consequences of certain definite hypothetical causes, and to compare them with those results which appear to be at present best established by observation; but, at the same time, leaving the theory of elevation founded upon our hypotheses, open to that refutation, or more complete verification, which must arise from the comparison of the results of more extended and accurate geological research with those of theory, deduced not by vague and indeterminate methods, from assumptions still more vague and indeterminate, but by accurate methods, from hypotheses the most simple and definite, which the nature of the subject will admit of.

In our own country the elevated range extending from Derbyshire to Northumberland, seems peculiarly calculated to afford us an opportunity of comparing the results of observation with those of the theory we have been investigating. On the slightest inspection of a map of this portion of the island, the direction of the central line of elevation is indicated to us by the sources of the rivers, which pursue their courses from it respectively to the eastern and western coasts. This line appears to be almost straight, running nearly north from its southern extremity to the valley of the Eden, where the well defined ridge of Cross Fell commences, in a direction almost north-west and south-east. On the eastern side of this range, the different formations succeed each other with a general regularity in the order of their superposition, which would appear to indicate the absence of any comparatively irregular action of the elevatory forces in that region; and the existence of extensive mining and coal districts along this range, afford the surest means of ascertaining with accuracy the exact positions of the fissures and lines of dislocations which exist in it. Hitherto these phenomena have not, however, been made the objects of sufficiently careful examination, and if these observations should have the effect of leading to a more detailed investigation of them, one object of my entering into these researches will be accomplished. According to our theory the mineral veins in the southern part of the range above mentioned ought to run east and west, while in the Cross Fell part we should expect them to assume a direction more nearly north-east and south-west. From my own observation I have ascertained that in the mining district in Derbyshire, the phenomena are in this respect as well as in others strikingly accordant with theory, and I have reason to believe that in the coal district lying along the eastern boundary of that country they will be found so likewise. I hope, however, shortly to bring the details of this district under the notice of geologists.

The northern and southern portions of this range present us also with the important and interesting phenomena of extensive horizontal beds of trap, (the toadstone of Derbyshire, and the whinsill of the north) apparently interstratified with the sedimentary rocks with which they are associated. In the preceding investigations, I have entered with considerable detail into the subject of the formation of such beds, from the conviction that the notion of injection with reference to them has been carried by some geologists much too far, and that conclusions have been adopted without a due regard to the necessary effects on the contiguous beds, of that enormous hydrostatic pressure, which the process of injection of an extensive horizontal bed would necessarily call into action. That the toadstone of Derbyshire is not an injected bed, admits, I think, of the most indubitable proof from observation; and if the interstratification of the whinsill of the north, with comparatively thin beds of limestone and shale, be as regular as it is represented to be, I should have no hesitation in coming to the same conclusion with respect to that bed, for the reasons which have been heretofore mentioned, (Art. 77).

In the preceding investigations, I have spoken of the law of parallelism only as recognized in phenomena of faults, mineral veins, &c., comprized within narrow boundaries as compared with those to which it has been attempted to extend it, in the theory of Elie de Beaumont. It is very possible, however, that the physical causes to which I have referred this law, may have had a far more extensive operation than that I have ventured to assign to them. The parallelism of two mountain chains might thus be accounted for as simply as that of two neighbouring anticlinal lines; but it is obvious, that the more remote they should be from each other, the less would be the probability of the fissures to which our theory would refer them, belonging to the same system, and the less satisfactory would our solution become.

I have been anxious to avoid, for the present, any speculations respecting the interior constitution of our globe, beyond what is comprized in the simple assumptions on which these investigations have been founded; we may, however, include in those assumptions, the hypothesis of the elevatory forces having acted in different cases at different depths. The application of our theory, alluded to in the preceding paragraph, would perhaps require the hypothesis of these forces having acted at a much greater depth in such instances, than in those where the resulting phenomena are on a much smaller scale; and we may observe, that if the formation of the fissures should commence very far beneath the surface, it is extremely probable that very few would become complete fissures (see Art. 39), or would ever reach nearly to the surface, in comparison with those which would do so in cases where these fissures should originate at a much smaller depth. The complete fissures would consequently be distant from each other and very large, and all the phenomena of elevation resulting from them might be expected to be of proportionate magnitude. I have no intention, however, of insisting on this extended application of our theory, but merely to indicate its possible extension (should established geological facts appear hereafter to require it) to account for phenomena on a much larger scale than those to which I have considered it essential to refer in the preceding investigations.

W. HOPKINS.

ST PETER'S COLLEGE,
May 4, 1835.

II. *Investigation of the Equation to Fresnel's Wave Surface.* By

ARCHIBALD SMITH, ESQ., *Trinity College, Cambridge.*

[Read *May* 18, 1835.]

"THE mathematical difficulties under which the beautiful and interesting theory of Fresnel has hitherto laboured are well known, and have been regarded as almost insuperable. He tells us in his Memoir (see the Memoirs of the Royal Academy of Sciences of Paris, tom. VII. p. 136.) that the calculations by which he assured himself of the truth of his construction for finding the surface of the wave were so tedious and embarrassing, that he was obliged to omit them altogether. A direct demonstration has since been supplied by M. Ampère (Annales de Chimie et de Physique, tom. XXXIX. p. 113.); but his solution is excessively complicated and difficult." A geometrical demonstration of considerable simplicity has been given by Mr M^c Cullagh in a paper in the XVIth Volume of the Transactions of the Royal Irish Academy, from which the preceding paragraph has been quoted.

The difficulties which were experienced in this problem arose from two causes of the same nature:—want of symmetry in the fundamental equations, and the use of the essentially unsymmetrical method of differential coefficients. By putting the fundamental equations of Fresnel under a symmetrical form, and by the use of the Method of Multipliers as it is employed in the Mécanique Analytique, the eliminations may be effected without difficulty.

To render what follows more intelligible, and to show in what it differs from the other methods, I shall give the fundamental equations of Fresnel and Ampère, and state shortly the steps by which they are obtained.

In the Memoir of Fresnel referred to above it is shewn, that if a section of the "surface of elasticity", whose equation is

$$(x^2+y^2+z^2)^2 = a^2x^2+b^2y^2+c^2z^2 \text{ .... } (1)$$

be made by the plane

$$z = mx+ny \text{ .... } (2),$$

the greatest and least radii vectores of the section will be the values of v, derived from the equation

$$(a^2-v^2)(c^2-v^2)n^2+(b^2-v^2)(c^2-v^2)m^2+(a^2-v^2)(b^2-v^2)=0 \text{ ... } (3),$$

and that if a plane be taken, parallel to the former and whose distance from the origin is one of the values of v, this plane, whose equation is

$$z = mx+ny+v\sqrt{1+m^2+n^2} \text{ .... } (4)$$

will be a tangent to the wave surface.

To deduce the equation to the surface we may solve (3) to find v, and substitute this value in (4), which will give the equation

$$(z-mx-ny)^2 = \tfrac{1}{2}\{(c^2+b^2)m^2+(a^2+c^2)n^2+a^2+b^2$$
$$\pm\sqrt{[(c^2-b^2)m^2+(a^2-c^2)n^2+(a^2-b^2)]^2-4(c^2-b^2)(a^2-c^2)m^2n^2}\}.$$

And if we differentiate this equation first with regard to m and then n, and eliminate m and n between the three equations, we shall obtain the equation to the wave surface. This is the method which M. Ampère employed with success.

Instead of eliminating v at first, we may differentiate (4), considering v as a function of m and n determined by equation (3), we shall thus obtain the equations

$$(a^2-v^2)(c^2-v^2)n^2+(b^2-v^2)(c^2-v^2)m^2+(a^2-v^2)(b^2-v^2)=0 \text{ ... } (1),$$

$$(z-mx-ny)^2 = v^2(1+m^2+n^2) \text{ .... } (2),$$

$$(z - mx - ny)\,x + v^2 m + (1 + m^2 + n^2)\,v\frac{dv}{dm} = 0 \;\ldots\ldots\ldots\ldots\; (3).$$

$$v\frac{dv}{dm}\left\{(1+n^2)(a^2-v^2) + (1+m^2)(b^2-v^2) + (m^2+n^2)(c^2-v^2)\right\}$$
$$- m\,(b^2 - v^2)(c^2 - v^2) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (4).$$

$$(z - mx - ny)\,y + v^2 n + (1 + m^2 + n^2)\,v\frac{dv}{dn} = 0 \;\ldots\ldots\ldots\; (5).$$

$$v\frac{dv}{dn}\left\{(1+n^2)(a^2-v^2) + (1+m^2)(b^2-v^2) + (m^2+n^2)(c^2-v^2)\right\}$$
$$- n\,(a^2 - v^2)(c^2 - v^2) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (6).$$

Between these six equations the five quantities m, n, v, $\frac{dv}{dm}$, $\frac{dv}{dn}$, are to be eliminated, and the resulting equation will be that of the wave surface. These are the equations given by Fresnel, but he was not successful in effecting the requisite eliminations.

The fundamental equations may be put under a symmetrical form by the introduction of an additional symbol.

If for m and n we substitute respectively $-\frac{l}{n}$, and $-\frac{m}{n}$, and suppose l, m, n, connected by the equation

$$l^2 + m^2 + n^2 = 1,$$

we shall, instead of (3) and (4), have the three equations

$$lx + my + nz = v \;\ldots\ldots\ldots\ldots\; (1),$$

$$l^2 + m^2 + n^2 = 1 \;\ldots\ldots\ldots\ldots\; (2),$$

$$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0 \;\ldots\ldots\ldots\; (3).$$

Differentiate these equations with regard to l, m, n, and v.

$$x\,dl + y\,dm + z\,dn = dv \;\ldots\ldots\ldots\ldots\; (4),$$

$$l\,dl + m\,dm + n\,dn = 0 \;\ldots\ldots\ldots\ldots\; (5),$$

$$\frac{l}{v^2 - a^2}\,.\,dl + \frac{m}{v^2 - b^2}\,.\,dm + \frac{n}{v^2 - c^2}\,dn = \left\{\left(\frac{l}{v^2 - a^2}\right)^2 + \left(\frac{m}{v^2 - b^2}\right)^2 + \left(\frac{n}{v^2 - c^2}\right)^2\right\} v\,dv \ldots (6).$$

Multiply (5) by A, (6) by B, and subtract the sum from (4), making the coefficients of each differential equal to zero. We thus obtain

$$x = Al + \frac{Bl}{v^2 - a^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (7),$$

$$y = Am + \frac{Bm}{v^2 - b^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (8),$$

$$z = An + \frac{Bn}{v^2 - c^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (9),$$

$$1 = Bv \left\{ \left(\frac{l}{v^2 - a^2}\right)^2 + \left(\frac{m}{v^2 - b^2}\right)^2 + \left(\frac{n}{v^2 - c^2}\right)^2 \right\} \quad\ldots\ldots\quad (10),$$

$(7)\, l + (8)\, m + (9)\, n$ gives

$$lx + my + nz = A\,(l^2 + m^2 + n^2) + B\left(\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2}\right),$$

which by equations (1), (2), (3), is reduced to

$$v = A;$$

$(7)^2 + (8)^2 + (9)^2$ gives

$$x^2 + y^2 + z^2 = A^2\,(l^2 + m^2 + n^2) + 2AB\left(\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2}\right)$$

$$+ B^2 \left\{ \left(\frac{l}{v^2 - a^2}\right)^2 + \left(\frac{m}{v^2 - b^2}\right)^2 + \left(\frac{n}{v^2 - c^2}\right)^2 \right\},$$

which by (2), (3), (10) is reduced to

$$x^2 + y^2 + z^2 = A^2 + \frac{B}{v},$$

putting r^2 for $x^2 + y^2 + z^2$, and for A its value v, this becomes

$$B = v\,(r^2 - v^2).$$

Substituting these values of A and B, equation (7) becomes

$$x = lv + lv \,.\, \frac{r^2 - v^2}{v^2 - a^2},$$

$$\text{or } x\,(a^2 - v^2) = lv\,(a^2 - r^2).$$

Putting $\frac{1}{r^2}\{a^2(r^2-v^2)+v^2(a^2-r^2)\}$ for a^2-v^2, this becomes

$$a^2x(r^2-v^2)=(lvr^2-xv^2)(a^2-r^2);$$

$$\text{or}\quad \frac{a^2x}{a^2-r^2}=\frac{v}{r^2-v^2}(lr^2-xv).$$

Similarly
$$\frac{b^2y}{b^2-r^2}=\frac{v}{r^2-v^2}(mr^2-yv),$$

$$\frac{c^2z}{c^2-r^2}=\frac{v}{r^2-v^2}(nr^2-zv).$$

Multiplying these by x, y, z, respectively, and adding

$$\frac{a^2x^2}{a^2-r^2}+\frac{b^2y^2}{b^2-r^2}+\frac{c^2z^2}{c^2-r^2}=\frac{v}{r^2-v^2}\{r^2.(lx+my+nz)-v(x^2+y^2+z^2\}$$

$$=\frac{v}{r^2-v^2}(r^2v-vr^2)$$

$$=0.$$

This is the simplest form which the equation to the wave surface can assume.

If we clear it of fractions, and replace r^2 by $x^2+y^2+z^2$, we obtain the equation given by Fresnel, viz.:

$$(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2)-a^2(b^2+c^2)x^2-b^2(a^2+c^2)y^2-c^2(a^2+b^2)z^2+a^2b^2c^2=0.$$

ARCH^D. SMITH.

TRINITY COLLEGE,
May 8, 1835.

III. *On the Resolution of Equations in Finite Differences.*

By the Rev. R. Murphy, M.A. F.R.S. *Honorary Member of the Royal Cork Institution, Fellow of Caius College, and of the Cambridge Philosophical Society.*

[Read *Nov.* 15, 1835.]

When the degree of equations in Finite Differences does not exceed the first, whatever may be their order, methods for their solution in most cases have been furnished by analysts. With respect to those of higher degrees, scarcely any thing has been done to assist in obtaining explicitly an algebraical expression for the unknown quantity*. The utility of solutions for such equations, occurring, as they do, in the theory of chances, is more apparent by the proof which they afford of the expansibility of various kinds of *successive* functions on which some doubt has hitherto existed.

The difficulties which those have encountered who attempted to obtain expansions in an algebraical form, for functions which from their nature may be denominated *repeated* functions, are known, such are for instance

$$a^{a^{a^{a^{\cdot^{\cdot^{\cdot (x \text{ times})}}}}}}$$

$$\text{log. log. log.} \ldots\ldots\ldots (x \text{ times}) \{a\}$$

$$\text{sin. sin. sin.} \ldots\ldots\ldots (x \text{ times}) \{a\},$$

* In the great work of Lacroix this subject is entirely passed over.

in which x is to be integer or fractional, positive or negative, real or imaginary: so anomalous have they appeared as to induce a belief in some, that they did not admit of an algebraical expansion, and therefore might be supposed to affect some of the first principles of the Differential Calculus.

In fact, the application of Maclaurin's Theorem requires the knowledge of the differential coefficients, which can only be deduced *à priori*, in forms which leave them still unknown, while the application of Taylor's Theorem in Finite Differences introduces impracticable coefficients of a nature more complicated to value than the proposed functions themselves.

As an illustration, suppose we denote by u_x the successive function,

$$\epsilon^{\epsilon^{\epsilon^{\epsilon^{\cdot^{\cdot^{\cdot (x \text{ times})}}}}}}$$

ϵ being the base of Napier's Logarithms, then, to find its differential coefficient, we have the equation

$$u_{x+1} = \epsilon^{u_x}, \text{ and putting } \frac{du_x}{dx} = u'_x,$$

$$u'_{x+1} = \epsilon^{u_x} . u'_x;$$

$$\text{therefore } \frac{u'_x}{u'_{x-1}} = \epsilon^{u_{x-1}}.$$

To solve which, put $u'_x = \epsilon^{b_x}$,

$$b_x - b_{x-1} = u_{x-1};$$

therefore if x should be an integer,

$$b_x = \text{const} + u_0 + u_1 + u_2 \ldots\ldots + u_{x-1};$$

or more generally,

$$b_x = \Sigma . u_x;$$

$$\therefore \frac{du_x}{dx} = \epsilon^{\Sigma u_x},$$

which contains an arbitrary multiplier; but it is plain that the expansion of u_x being unknown, Σu_x is also unknown, and the successive differential coefficients would similarly be expressed in forms of unknown functions; and therefore the expansion by the immediate application of Maclaurin's Theorem would be impracticable.

Taylor's Theorem in Finite Differences gives the identity

$$u_x = u_0 + x\Delta u_0 + \frac{x.(x-1)}{1.2}.\Delta^2 u_0 + \frac{x.(x-1)(x-2)}{1.2.3}.\Delta^3 u_0, \text{ \&c.}$$

now to find $\Delta^n u_0$ in this case, we must have recourse to the theorem

$$\Delta^n u_0 = u_n - n u_{n-1} + \frac{n(n-1)}{1.2}.u_{n-2} - \frac{n(n-1)(n-2)}{1.2.3}.u_{n-3}, \text{ \&c.}$$

the coefficients thus found being obviously more complicated than the function itself.

Try again to find u_x by a series arranged according to the powers of x, and containing indeterminate coefficients, that is, put

$$u_x = A + Bx + Cx^2 + Dx^3 + \text{\&c.}$$

and since $u_x = \log. u_{x+1}$; $\therefore\ u_0 = \log. u_1 = \log. \epsilon = 1$; $\therefore\ A = 1$.

Then the equation $u_{x+1} = \epsilon^{u_x}$ becomes

$$1 + B(x+1) + C(x+1)^2 + D(x+1)^3, \text{ \&c.} = \epsilon.\epsilon^{Bx}.\epsilon^{Cx^2}.\epsilon^{Dx^3}. \text{\&c.}$$

and equating like powers of x we get,

$$1 + B + C + D + \text{\&c.} = \epsilon,$$

$$B + 2C + 3D + 4E + \text{\&c.} = B.\epsilon,$$

$$C + 3D + 6E + 10F + \text{\&c.} = \frac{B^2}{1.2}.\epsilon + C.\epsilon,$$

&c. &c.

which clearly show that the coefficients cannot be found but by the resolution of equations of infinitely high degrees.

Now similar difficulties opposing the expansion of other kinds of successive functions, these can only be removed by attending carefully to the equations in finite differences, by which the law of the formation of the functions is expressed. I therefore here propose a means for resolving such equations, of whatever order or degree, in an algebraical form.

FIRST CLASS OF EQUATIONS.

All successive functions such as those above mentioned, are represented by the equation $u_{x+1} = \phi(u_x)$, for this manifestly expresses the law of the successive functions,

$$u_0 = a, \quad u_1 = \phi(a), \quad u_2 = \phi\phi(a), \quad u_3 = \phi\phi\phi(a),$$
$$u_x = \phi\phi\phi\phi \ldots\ldots (x \text{ times}) \ldots\ldots \{a\};$$

hence the expansion of u_x in every such case depends on the solution of this equation.

Put $u_x = f(\gamma^x)$ the form of the function f, and the quantity γ remaining at present unknown, and also let $\gamma^x = z$.

Then, since $u_x = f(z)$, and $u_{x+1} = f(\gamma z)$, we have

$$f(\gamma z) = \phi(fz).$$

Suppose now $f(z) = A + Bz + Cz^2 + Dz^3 + \&c.$

the preceding equation becomes

$$A + B\gamma z + C\gamma^2 z^2 + D\gamma^3 z^3, \ \&c. = \phi\{A + Bz + Cz^2 + Dz^3, \ \&c.\}$$

and it remains to expand the latter function, in order to compare like powers of z, and thus obtain the assumed coefficients and the quantity γ.

Now by Taylor's Theorem the development will be of the following nature, viz.

$\phi(A) + Z_1\phi'(A) + Z_2\phi''(A) + Z_3\phi'''(A) + \&c$, where Z_1, Z_2, Z_3, &c. are functions of z completely independent of the form of the function ϕ, and $\phi'(A)$, $\phi''(A)$, &c. represent the successive differential coefficients of $\phi(A)$.

To determine Z_1, Z_2, &c. put $\phi(A) = \epsilon^{aA}$, a being an arbitrary quantity.

Hence by substitution we have

$$\epsilon^{a[A+Bz+Cz^2+Dz^3, \&c.]} = \epsilon^{aA}\{1 + Z_1 a + Z_2 a^2 + Z_3 a^3 + \&c.\}$$

Now the value of the left-hand member of this equation is also

$$\epsilon^{aA}\left\{1 + aBz + \frac{a^2B^2}{1.2}\cdot z^2 + \frac{a^3B^3}{1.2.3}z^3, \&c.\right\}.\{1 + aCz^2, \&c.\}.\{1 + aDz^3, \&c.\}.\&c.$$

And if we equate the coefficients of z^n in each we get

$$Z_n = \Sigma \frac{B^bC^cD^d\ldots\ldots}{1.2\ldots b \times 1.2\ldots c \times 1.2\ldots d \times \ldots}.z^m$$

subject to the two conditions

$$\begin{cases} a + b + c + d + \&c. = n \\ \quad\;\; b + 2c + 3d + \&c. = m. \end{cases}$$

These conditions being satisfied we obtain the identity

$$\phi(A + Bz + Cz^2 + Dz^3, \&c.) = \Sigma \frac{B^bC^cD^d\ldots\ldots}{1.2\ldots b \times 1.2\ldots c \times 1.2\ldots d}.\phi'''^{(n)}(A).z^m;$$

and comparing this result with the first expansion, viz.

$$A + B\gamma z + C\gamma^2z^2 + D\gamma^3z^3 + \ldots\ldots + M\gamma^m z^m + \&c.$$

$$\text{we have } M = \Sigma \frac{B^bC^cD^d\ldots\ldots}{1.2\ldots b \times 1.2\ldots c \times 1.2\ldots d \times \&c.} \times \phi'''^{(n)}A;$$

where it must be observed, that b, c, d, &c. having previously satisfied the equation $b + 2c + 3d$, &c. $= m$, the quantity n is then found by summing b, c, d, &c.

$$\text{Put } C = c_1B^2, \quad D = c_2B^3, \quad E = c_3B^4, \ \&c.$$

and making $m = 1, 2, 3$, &c. successively, we get the following identities, by which A, γ, c_1, c_2, c_3, &c. are completely known.

$$A = \phi(A), \quad \gamma = \phi'(A),$$

$$c_1\gamma^2 = \frac{\phi''(A)}{1.2} + c_1\phi'(A),$$

$$c_2\gamma^3 = \frac{\phi'''(A)}{1.2.3} + c_1\phi''(A) + c_2\phi'(A),$$

$$c_3\gamma^4 = \frac{\phi^{\text{IV}}(A)}{1.2.3.4} + \frac{c_1}{1.2}.\phi'''(A) + c_2\phi''(A) + \frac{c_1^2}{1.2}\phi''(A) + c_3\phi'(A),$$

&c. = &c.

The general law of which equations is thus expressed:

$$c_{m-1}\gamma^m = \Sigma \frac{(1)^b (c_1)^c (c_2)^d \ldots\ldots}{1.2\ldots b \times 1.2\ldots c \times 1.2\ldots d \times \&c.} . \phi'''^{(n)}(A),$$

b, c, d, &c. being regulated by the two conditions before mentioned.

From hence we obtain the complete integral of the proposed equation $u_{x+1} = \phi(u_x)$; for since

$$f(z) = A + Bz + Cz^2 + Dz^3 + \&c.$$

$$\therefore u_x = A + B\gamma^x + c_1(B\gamma^x)^2 + c_2(B\gamma^x)^3 + c_3(B\gamma^x)^4, \&c.$$

The arbitrary constant B is determined as usual by assigning a particular value to x, as

$$u_0 = A + B + c_1B^2 + c_2B^3 + c_3B^4, \&c.$$

by the reversion of which series B is found in a series arranged according to the powers of $u_0 - A$.

Before proceeding to any particular applications of this general solution, a few observations will be useful.

I. When $\phi(u_x)$ is of the form $u_x +$ const., then A becomes generally infinite, and γ becomes 1, the solution therefore fails in this case, but more generally it may be remarked, that it also fails when the value of A deduced from the equation $A = \phi(A)$ satisfies the equation $\phi'(A) = 1$; for this, by making $\gamma = 1$, renders infinite the coefficients c_1, c_2, &c. Such cases of failure will shortly be separately considered.

II. When the equation $u_{x+1} = \phi(u_x)$ is of the n^{th} degree, the equation for finding A is of the same degree, and therefore A has n values; then $\gamma = \phi'(A)$ has also n corresponding values, which, being represented by $\gamma_1, \gamma_2, \dots \gamma_n$, and putting for abridgment $F(B\gamma^x)$ for the series above found for u_x, we have

$$u_x - F(B_1\gamma_1^x) = 0, \quad u_x - F(B_2\gamma_2^x) = 0, \ \&c.;$$

and the complete solution is found by taking the product of the members on the left side and equating to zero: the result will only contain one arbitrary constant, since B_1, B_2, &c. are all found in terms of u_0, as before shewn.

III. u_0 is a known function of B, u_x is the same function of $B\gamma^x$.

IV. Since
$$\phi\phi\phi\dots\dots\{x \text{ times}\}\,(u_n) = u_{x+n};$$
$$\therefore\ \phi\phi\phi\dots\dots\{x \text{ times}\}\,(u_{-x}) = u_0.$$

Let ϕ^{-1} be the function which is inverse to ϕ, that is, such that $\phi^{-1}\phi(a) = a$, then it follows that

$$u_{-x} = \phi^{-1}\phi^{-1}\phi^{-1}\dots\dots\{x \text{ times}\}\,(u_0).$$

The same formula therefore which represents the x^{th} successive direct function of u_0, will also give the x^{th} successive inverse function by merely writing $-x$ in place of x.

V. Put $\gamma^x = \dfrac{1}{B}$, or $x = -\dfrac{\log. B}{\log. \gamma}$, hence

$$\phi\phi\phi\dots\dots\left\{-\frac{\log. B}{\log. \gamma}\text{ times}\right\}(u_0) = A + 1 + c_1 + c_2 + c_3, \ \&c.,$$

which is a known numerical quantity and may be represented by a_0;

$$\text{hence, } \phi\phi\phi\dots\dots\left\{\frac{\log. B}{\log. \gamma}\text{ times}\right\}(a_0) = u_0.$$

Thus the number of times it is necessary to take the successive functions ϕ of a_0 to arrive at u_0 as a result, determines $\dfrac{\log. B}{\log. \gamma}$, and since γ is known, B may be thus also determined.

EXAMPLES:

I. $u_{x+1} = a + bu_x = \phi(u_x)$.

Results ...... $\begin{cases} u_x = \dfrac{a}{1-b} + B \, . \, b^x \\ u_0 = \dfrac{a}{1-b} + B. \end{cases}$

II. Given $u_{x+1} = \dfrac{u_x^2 - a^2}{2b} = \phi(u_x)$.

Immediately applying the general formulæ above found, they give

$$A = b + \sqrt{b^2 + a^2}, \quad \gamma = 1 + \sqrt{\left(1 + \frac{a^2}{b^2}\right)},$$

$$c_1 = \frac{1}{2b\gamma(\gamma - 1)}, \quad c_2 = \frac{c_1}{b\gamma(\gamma^2 - 1)}, \quad c_3 = \frac{c^2 + \frac{c_1^2}{1.2}}{b\gamma(\gamma^3 - 1)}, \text{ \&c.};$$

and by substituting these values, we have

$$u_x = b\gamma + B\gamma^x + \frac{(B\gamma^x)^2}{2b\gamma(\gamma-1)} + \frac{(B\gamma^x)^3}{2b^2\gamma^2(\gamma-1)(\gamma^2-1)}$$
$$+ \frac{(B\gamma^x)^4 . (\gamma + 5)}{2b^3\gamma^3(\gamma-1)(\gamma^2-1)(\gamma^3-1)}, \text{ \&c.};$$

and B is known by reverting the series

$$u_0 - b\gamma = B + \frac{B^2}{2b\gamma(\gamma-1)} + \frac{B^3}{2b^2\gamma^2(\gamma-1)(\gamma^2-1)} + \text{\&c.}$$

To facilitate the determination of B in every case of the general solution, viz., $u_0 = A + B\gamma^x + c_1(B\gamma^x)^2 + c_2(B\gamma^x)^3$, &c, we may apply the general theorem

$$B = .\Sigma \frac{1.2\ldots(n-1)(u_0 - A)^{b_0}(-1)^{n-b_0} . c_1^{b_1} c_2^{b_2} \text{\&c.}}{1.2\ldots b_0 \times 1.2\ldots b_1 \times 1.2\ldots b_2 \times \text{\&c.}},$$

where the indeterminate indices are subject to the two conditions

$$b_0 + b_1 + b_2 + \&c. = n$$
$$2b_1 + 3b_2 + \&c. = n - 1,$$

which I have given in my Memoir on the Resolution of Algebraic Equations.*

With respect to the inverse function in this particular example, we have

$$u_x = \sqrt{(a^2 + 2bu_{x+1})}; \quad \therefore \; u_{-x} = \sqrt{\{a^2 + 2bu_{-(x-1)}\}};$$

and putting for x successively 1, 2, 3 ... x, we get

$$u_{-x} = \sqrt{\{a^2 + 2b\sqrt{\{a^2 + 2b\sqrt{\{a^2 + \ldots + 2b\sqrt{(a^2 + 2bu_0)}\}}},$$

the number of roots being x; and for the value of this successive function we have

$$u_{-x} = b\gamma + B\gamma^{-x} + \frac{(B\gamma^{-x})^2}{2b\gamma(\gamma - 1)} + \frac{(B\gamma^{-x})^3}{2b\gamma(\gamma - 1)(\gamma^2 - 1)} \; \&c.$$

Thus to find the value of $\sqrt{\{20 + \sqrt{\{20 + \sqrt{\{20 + \ldots\}}}}$ x times,

we have $a^2 = 20$, $2b = 1$, $\gamma = 10$,

$$\tfrac{1}{5}u_{-x} = 1 + \frac{\frac{1}{5}B}{10^x} + \frac{(\frac{1}{5}B)^2}{2 \,.\, 9 \,.\, 10^{2x}} + \frac{(\frac{1}{5}B)^3}{2 \,.\, 9 \,.\, 99 \,.\, 10^{3x}} \; \&c.$$

and B is very readily found by putting $x = 1$, and thence the required value of u_{-x} is obtained in a series extremely convergent.

As another numerical example, let $u_{x+1} = 2u_x^2 - 1$.

Here $\gamma = 4$, $b\gamma = 1$, and therefore

$$u_x = 1 + B \,.\, 2^{2x} + \frac{B^2 \,.\, 2^{4x}}{2(2^2 - 1)} + \frac{B^3 \,.\, 2^{6x}}{2(2^2 - 1)(2^4 - 1)}$$
$$+ \frac{9B^4 \,.\, 2^{8x}}{2(2^2 - 1)(2^4 - 1)(2^6 - 1)} + \&c.$$

for B write $-\dfrac{\theta^2}{1 \,.\, 2}$, and this series becomes

* Trans. of Camb. Phil. Soc. Vol IV. p. 144.

$$u_x = 1 - \frac{(2^x\theta)^2}{1\,.\,2} + \frac{(2^x\theta)^4}{1\,.\,2\,.\,3\,.\,4} - \frac{(2^x\theta)^6}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6} + \frac{(2^x\theta)^8}{1\,.\,2 \ldots 8} - \&c.$$

$$= \cos.\,(2^x\theta),$$

and the continued surd which is the inverse function may be similarly expressed.

$$a\,.\,a^{a\,.\,a^{\cdot^{\cdot^{\cdot^{a\,.\,a^{u_0}}}}}}$$

III. To expand the function $a\,.\,a$, the indices being continued x times.

Denoting this quantity by u_x, we have

$$u_{x+1} = a\,.\,a^{u_x} = \phi\,(u_x),$$

and adapting the general formulæ to this case, we have

$$A = a\,.\,a^A, \text{ whence } A \text{ is known,}$$

$$\gamma = A\,.\,\text{h. l.}\,(a),$$

$$c_1\gamma^2 = \frac{A\,(\text{h. l. } a)^2}{1\,.\,2} + c_1\gamma,$$

$$c_2\gamma^3 = \frac{A\,(\text{h. l. } a)^3}{1\,.\,2\,.\,3} + c_1 A\,.\,(\text{h. l. } a)^2 + c_2\gamma,$$

$$\&c. = \&c.;$$

and then $u_x = A + B\gamma^x + c_1(B\gamma^x)^2 + c_2(B\gamma^x)^3 + \&c.,$

$$u_0 = A + B + c_1 B^2 + c_2 B^3 + \&c.$$

For the inverse function if we put $a = \frac{1}{\beta}$, and take logarithms in the system of which the base is a, we have

$$u_x = \log.\,\beta + \log.\,u_{x+1} = \log.\,(\beta u_{x+1}),$$

$$u_{-x} = \log.\,\{\beta u_{-(x-1)}\},$$

$$u_{-x} = \log.\,\{\beta\,.\,\log.\,\{\beta \log.\,\beta \ldots\ldots \log.\,\beta u_0\},\ x \text{ times};$$

and for its expansion,

$$u_{-x} = A + B\gamma^{-x} + c_1(B\gamma^{-x})^2 + c_2(B\gamma^{-x})^3 \text{ \&c.},$$

the quantities A, B, γ, c_1, c_2, &c. being the same as before.

Before leaving this example, we may observe, that the most rapid way of finding A, is this, let $\log.\beta = b$, i.e. $a^b = \beta$; and since $A = \frac{1}{\beta}.a^A$, $\therefore A = b + \log.A$; take b then as a first approximation, take its log. and add to b for a second; take the log. of the second approximation, and add to b for a third; and when $b > 1$, we shall get a very converging series of values for A.

If we had applied Lagrange's theorem in this case to the equation $A = a^{A-b}$, we should have A expressed in a divergent series; it is necessary therefore to limit the announcement, that this general series gives the least root, to the case of real roots, for when there are some imaginary, we see that it may express one of these instead of the least real root.

IV. By a similar process we easily obtain

$$\sin.m\sin.m\sin....m\sin.m u_0 = Bm^x - \frac{m^2}{m^2-1}\cdot\frac{(Bm^x)^3}{1.2.3} + \frac{m^4(m^2+3^2)}{(m^2-1)(m^4-1)}\cdot\frac{(Bm^x)^5}{1.2.3.4.5}\text{ \&c.}$$

$$\frac{1}{m}\sin.^{-1}\frac{1}{m}\sin.^{-1}\frac{1}{m}\sin.^{-1}......\frac{1}{m}\sin.^{-1}u_0 = Bm^{-x} - \frac{m^2}{(m^2-1)}\cdot\frac{(Bm^{-x})^3}{1.2.3} + \frac{m^4(m^2+3^2)}{(m^2-1)(m^4-1)}\cdot\frac{(Bm^{-x})^5}{1.2.3.4.5}\text{ \&c.}$$

the value of B being the same in both, and found as before.

CASE OF FAILURE OF THE GENERAL SOLUTION.

When the equations $\phi(A) = A$, $\phi'(A) = 1$ are simultaneously true, the terms in the expansion of u_x become infinite, as before remarked, we shall therefore give a solution in a different form, for this case.

The equation $\phi(A) = A$ must here have two equal roots, suppose each $= c$; transform the proposed equation by putting $u_x = v_x + c$,

$$\therefore\ v_{x+1} = -c + \phi(v_x + c) = F(v_x),$$

the corresponding equation $A = F(A)$ has then two roots each equal to zero, and consequently $F(A)$ must be of the form

$$A + mA^2 + nA^3 + \&c.,$$

and accordingly $F(v_x)$ or v_{x+1} is of the form $v_x + mv_x^2 + nv_x^3 + \&c.$

Hence if v_0 vanish, v_1, v_2, v_3, &c. successively vanish, and therefore we put generally

$$v_x = A_x v_0 + B_x . v_0^2 + C_x . v_0^3 + D_x . v_0^4\ \&c.$$

in which series the coefficients are unknown functions of x, but independent of v_0.

Hence, $v_{x+1} = A_x . v_1 + B_x . v_1^2 + C_x . v_1^3 + D_x . v_1^4\ \&c.......(1)$,

and $v_1 = F(v_0) = F_1 . v_0 + F_2 . v_0^2 + F_3 v_0^3 + F_4 . v_0^4\ \&c.$

where F_n is put for $\dfrac{F^{\prime\prime\prime(n)}(0)}{1.2.3...n}$, by Maclaurin's Theorem.

Beside the foregoing form of expressing v_{x+1}, there are two others, viz.

$$F(v_x) = v_{x+1} = F_1 . v_x + F_2 . v_x^2 + F_3 . v_x^3 + F_4 . v_x^4\ \&c.......(2),$$

$$v_{x+1} = A_{x+1} . v_0 + B_{x+1} . v_0^2 + C_{x+1} . v_0^3 + D_{x+1} . v_0^4\ \&c.......(3),$$

and since F_1 is manifestly unity, if we compare the expressions 1 and 2, when the latter is arranged according to the powers of v_0, we obtain $A_x = 1$, which is obvious by the law of the successive formation of the quantities v_1, v_2, &c.; also putting $x = 0$ in the general value of v_x,

we have $v_0 = v_0 + B_0 . v_0^2 + C_0 . v_0^3 + D_0 v_0^4\ \&c.$,

which shews that $B_0 = 0$, $C_0 = 0$, &c.; this being premised, we have by comparing the expressions (1) and (3), the following identity;

$$v_1 + B_{x+1} . v_0^2 + C_{x+1} . v_0^3 + D_{x+1} . v_0^4\ \&c. = \{v_0 + F_2 . v_0^2 + F_3 v_0^3 + F_4 v_0^4\ \&c.\}$$

$$\begin{aligned}&+ B_x v_0^2 \{1 + F_2 v_0 + F_3 v_0^2 \text{ \&c.}\}^2\\ &+ C_x v_0^3 \{1 + F_2 v_0 \text{ \&c.}\}^3\\ &+ D_x v_0^4 \{1 + \text{\&c.}\}^4\\ &+ \text{\&c.}\end{aligned}$$

and comparing like powers of v_0, we have

$$\begin{aligned}\Delta B_x = F_2, \qquad &\Delta C_x = F_3 + 2F_2 B_x,\\ &\Delta D_x = F_4 + (F_2^2 + 2F_3) . B_x + 3F_2 . C_x,\\ &\text{\&c.} = \text{\&c.}\end{aligned}$$

whence integrating, so that each integral may vanish when $x = 0$, as has been proved to be necessary, we have

$$\begin{aligned}B_x = F_2 . x, \qquad &C_x = F_3 x + F_2^2 . x(x-1),\\ &D_x = F_4 x + (F_2^3 + 5F_2 F_3) . \frac{x(x-1)}{2} + F_2^3 . \frac{x(x-1)(x-2)}{3},\\ &\text{\&c.} = \text{\&c.}\end{aligned}$$

and $v_x = v_0 + B_x v_0^2 + C_x v_0^3 + D_x v_0^4$, &c. is completely known.

Ex.

$$\text{Sin. sin. sin......}\{x \text{ times}\} \text{ of } (\theta) = \theta - \frac{x}{1.2.3} . \theta^3 + \left\{\frac{x}{1.2.3.4.5} + \frac{x(x-1)}{1.2.3.4}\right\} . \theta^5 \text{ \&c.}$$

$$\sin.^{-1} \sin.^{-1} \sin.^{-1}\{x \text{ times}\} \text{ of } (\theta) = \theta + \frac{x}{1.2.3} . \theta^3 + \left\{\frac{x(x+1)}{1.2.3.4} - \frac{x}{1.2.3.4.5}\right\} . \theta^5 \text{ \&c.}$$

Before leaving this class of equations, we may remark a curious relation between the equations

$$\left.\begin{aligned}u_{x+1} &= \phi f(u_x)\\ \text{and } v_{x+1} &= f\phi(v_x)\end{aligned}\right\};$$

which is such that the solution of one leads to that of the other, for if we put $f(u_x) = v_x$, we have

$$\begin{aligned}u_{x+1} &= \phi(v_x);\\ \therefore f(u_{x+1}) &= f\phi(v_x),\\ \text{or } v_{x+1} &= f\phi(v_x);\end{aligned}$$

if then we determine u_0 so that $v_0 = f(u_0)$, v_x will be readily found from u_x.

SECOND CLASS OF EQUATIONS.

Given $\phi\{u_x,\ u_{x+1},\ u_{x+2}\ldots\ldots u_{x+n}\}=0.$

Put $u_x\ \ =A_1+\ B\gamma^x\ +\ c_1(B\gamma^x)^2\ +c_2(B\gamma^x)^3$ &c.

$u_{x+1}=A_2+\gamma.B\gamma^x+c_1\gamma^2.(B\gamma^x)^2+c_2\gamma^3(B\gamma^x)^3$ &c.

$u_{x+2}=A_3+\gamma^2.B\gamma^x+c_1\gamma^4(B\gamma^x)^2+c_2\gamma^6(B\gamma^x)^3$ &c.

&c. = &c.

and $\Phi=\phi\{A_1,\ A_2,\ A_3\ldots\ldots A_{n+1}\}$,

and ultimately make $A_1=A_2=A_3=\ldots\ldots A_{n+1}$.

Substituting in the proposed equation we have

$$\Phi+\Gamma_1.B\gamma^x+\Gamma_2.(B\gamma^x)^2+\Gamma_3.(B\gamma^x)^3+\&c.=0,$$

where $\Gamma_1=\dfrac{d\Phi}{dA_1}+\gamma\dfrac{d\Phi}{dA_2}+\gamma^2\dfrac{d\Phi}{dA_3}+\ldots\ldots+\gamma^n\dfrac{d\Phi}{dA_{n+1}}$,

$$\Gamma_2=\frac{1}{1.2}\left\{\frac{d^2\Phi}{dA_1^2}+2\gamma\frac{d^2\Phi}{dA_1dA_2}+\gamma^2\frac{d^2\Phi}{dA_2^2}\ldots\ldots+\gamma^{2n}\frac{d^2\Phi}{dA^2_{n+1}}\right\}$$

$$+c_1\left\{\frac{d\Phi}{dA_1}+\gamma^2.\frac{d\Phi}{dA_2}+\gamma^4\frac{d\Phi}{dA_3}+\ldots\ldots\gamma^{2n}\frac{d\Phi}{dA_{n+1}}\right\},$$

&c. = &c.

Hence $\Phi=0$, $\Gamma_1=0$, $\Gamma_2=0$, &c.

the equation $\Phi=0$ (putting $A_1=A_2=\ldots\ldots A_{n+1}$) determines A_1,
$\Gamma_1=0$ will give n different values of γ, any of which may be used,

$\Gamma_2=0$ will determine c_1,

$\Gamma_3=0$ .... c_2,

&c.

and B will remain an arbitrary constant.

Now in linear equations the sum of the particular solutions gives the complete integral, but this is not generally the case in other instances; but there is a method at once common to algebraic equations, differential equations, and those of finite differences, which leads from particular solutions to the general: this, however, more properly belongs to the

THIRD CLASS OF EQUATIONS.

The most general form of an equation in finite differences of any order and of any degree is represented by

$$\phi\,\{x,\ u_x,\ u_{x+1},\ u_{x+2}\ldots\ldots u_{x+n}\} = 0.$$

Put $u_x = f(\gamma^x) = f(z)$, and wherever x enters, let its value $\dfrac{\log. z}{\log. \gamma}$ be substituted, z consequently entering in a different form from x, the transformed function may be represented by

$$F\,\{z,\ f(z),\ f(\gamma z),\ f(\gamma^2 z)\ldots\ldots f(\gamma^n z)\} = 0.$$

$$\text{Putting } z = 0,\quad F\,\{0,\ f(0),\ f(0)\ldots\ldots\ldots f(0)\} = 0,$$

from whence $f(0)$ is known.

Differentiating relative to z, and then putting $z = 0$, the result is manifestly of the form

$$F_0 + F_1 . f'(0) + F_2 \gamma f'(0) + \ldots\ldots F_{n+1} \gamma^n f'(0) = 0,$$

from whence $f'(0)$ is known in terms of the indeterminate quantity γ, unless $F_0 = 0$ when γ becomes known, and $f'(0)$ remains the indeterminate constant.

The successive differentiations putting $z = 0$ after each, will determine $f''(0)$, $f'''(0)$, and thence by Maclaurin's Theorem,

$$f(z) = f(0) + f'(0) . z + f''(0) . \frac{z^2}{1 . 2} \text{ \&c.}$$

$$u_x = f(0) + f'(0) . \gamma^x + f''(0) . \frac{\gamma^{2x}}{1 . 2} \text{ \&c.}$$

which solution is particular, since it contains only one arbitrary constant, viz., $f''(0)$ or γ; denote this value of u_x by X_1, and put $u_x = X_1 + v_x$, and having found v_x with another arbitrary, from the transformed equation, put $v_x = X_2 + t_x$, X_2 containing two arbitraries, and by continuing this process, we shall obtain the complete solution, viz.,

$$u_x = X_1 + X_2 + X_3 + \ldots\ldots + X_n.$$

R. MURPHY.

CAIUS COLLEGE,
Nov. 1, 1835.

IV. *Geometrical Theorems, and Formulæ, particularly applicable to some Geodetical Problems.* *By* WILLIAM WALLACE, A.M. F.R.S. *Edin.*, F.R.A.S. *Lond.*, *Member of the Cambridge Philosophical Society, and Professor of Mathematics in the University of Edinburgh.*

[Read *Nov.* 30, 1835.]

ART. 1. THE Geometrical Theorems and Trigonometrical Formulæ which are given in this paper are peculiarly applicable to the solution of some Geodetical Problems, in particular to this which follows.

"Three stations being given in position, or else the angles made by the lines which join them; also the angles which these lines subtend at a fourth station in the plane of the others; to determine the position of that fourth station."

This problem is remarkable on account of its antiquity, and the object to which it was applied. Hipparchus made use of it to determine the position of the Moon's apogee and the radius of her epicycle, and Ptolemy actually resolved it by a trigonometrical computation in his Mathematical Syntaxis*. Vieta has given a geometrical construction in his *Apollonius Gallus*†. He had in view the solution of Hipparchus' problem; but the fiction of epicycles being now rejected, Ptolemy's application of the problem is merely an interesting fact in the history of the ancient Astronomy.

* Histoire de l'Astronomie Ancienne, par Delambre. T. II. p. 150—164.

† Vietæ Opera Mathematica, p. 344. Edit. 1646.

2. In comparatively modern times an interesting application of the problem was made to geodetical measurements. In the year 1620 Snellius, when ascertaining the distance between Bergen-op-zoom and Alcmaer, with a view to the determination of the magnitude of the Earth, employed it in finding the position of his Observatory. He assumed as given points three stations whose positions had been determined, and taking the angle which each two of them made at the Observatory, he was able to determine, by a trigonometrical computation, the distances of the stations, and thence its position*. The same problem was proposed by Richard Towneley as a chorographical problem†, and resolved trigonometrically in the Philosophical Transactions about the year 1670, by John Collins.

3. We are informed by Delambre‡ that Lalande wishing to compute some observations of the Moon which had been made at the Military School, Paris, proposed to find the longitude of the station where the observations had been made, by observing there the angles subtended by three steeples whose positions were known. He was thus led to the same application as had been made long before by Snellius, without knowing or without thinking of his solution. Lalande's patience was exhausted by the length of the calculations, and the slips he made in performing them: he therefore referred the problem to Delambre, who gave a solution which was printed in Cagnoli's Trigonometry (First Edition), and again, but with more detail, in his own treatise *Methodes Analytiques pour la Determination d'un Arc du Meridien.*

Delambre's solution, which is analytical, is good, his formulæ have however but little of that symmetry and simplicity which constitute elegance in a geometrical speculation, and make it easy to be comprehended and remembered.

4. In considering the problem I have found two Theorems; from one of them a particularly simple and elegant geometrical construction

* SNELLIUS, Erastosthenes Batavus, p. 203.

† Lowthorpe's Abridgement of Phil. Trans. Vol. I. p. 120.

‡ Histoire de l'Astronomie Moderne, T. II. p. 109.

is obtained, and both suggest various solutions, some lineo-angular, others trigonometrical, to this and other related problems.

THEOREM.

"Let AB, AC be two straight lines which meet in A; and AD, Ad other two, which make with the former equal angles BAD, CAd, these last being either both within (figs. 1, 2.) or both without the angle

Fig. 2.

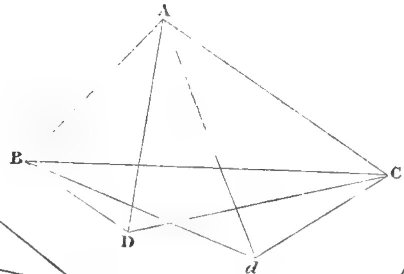


Fig. 1.

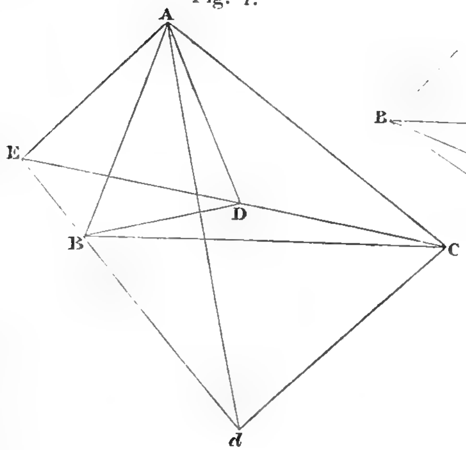
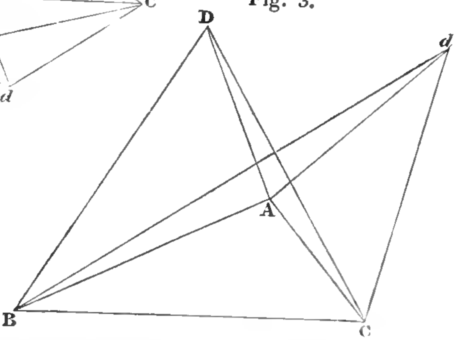


Fig. 3.



BAC (fig. 3). Let the lines be such that the rectangle $DA.Ad$ is equal to the rectangle $BA.AC$; draw lines from D and d to B and C: the triangles ADC, ABd, thus formed, are similar; and the triangles ADB, ACd are similar."

Demonstration. Because by hypothesis $AD.Ad=AB.AC$, therefore $AB : AD=Ad : AC$. Now the angles BAD, dAC are equal by hypothesis, therefore the triangles BAD, dAC are similar.

Also because $AB : Ad=AD : AC$, and the angles BAd, DAC are equal, for they are the sums or differences of the equal angles BAD,

CAd and the common angle DAd, therefore the triangles BAd, DAC are similar.

Corollary. Let ABC be any triangle, and let straight lines AD, BD, CD be drawn to D, any point in its plane; at the point B in the line BA make the angle ABd equal to the angle ADC, viz. that which the side opposite to B subtends at D; at the point C in the line CA make the angle ACd equal to the angle ADB, which AB the side opposite to C subtends at D; draw a line from the remaining angle A to d the intersection of the lines Bd, Cd; the triangles ADC, ABd are similar; and the triangles ADB, ACd are similar.

5. The truth of the Corollary may be inferred from the theorem: it may however be proved directly as follows.

Let E be the intersection of the lines dB, CD (fig. 1.); join AE. Because by construction the angles ADC, ABd are equal, the angles ADE, ABE are equal; therefore the points A, D, B, E are in the circumference of a circle; hence the angle AEB is either equal to the angle ADB, or is its supplement; now by hypothesis the angle ADB is equal to ACd; therefore AEB or AEd is either equal to ACd, or is its supplement; hence, in each case, the points A, d, C, E are in the circumference of a circle; and therefore the angle ACD or ACE is equal to AdB; now, by construction, the angle ADC is equal to the angle ABd; therefore the triangles ADC, ABd are similar; and since the angle CAD is equal to dAB; by adding or subtracting the angle dAD, we have the angle CAd equal to DAB; now the angle ACd is by construction equal to ADB; therefore the triangles ACd, ADB are similar.

6. In the demonstration, it was assumed that when the angles BAD, CAd are equal, then BAd, CAD are equal; this will always be true when the lines AD, Ad are either both within the angle BAC, or both without that angle, or, which is the same thing, when the similar triangles BAD, dAC are similarly situated; and the same is true of the similar triangles BAd, DAC.

7. I shall now apply the geometrical theorem to the construction of the problem enunciated in the beginning of this paper.

PROBLEM. (Figs. 1, 2, 3.)

Three stations *A, B, C* are given in position; also, there are given the angles *ADB, ADC* which the lines joining *A*, one of them, and *B, C*, the other two, subtend at a fourth station *D* in their plane; to determine the position of the fourth station, by a geometrical construction.

Solution. At the point *B*, in the line *BA*, make the angle *ABd* equal to *ADC* the angle which *AC* subtends at *D*; observing, that the line *Bd* must have such a position, that if the angle *ABd* were placed on *ADC*, so that *BA* lay along *DA*; then *Bd* would lie on *DC*. Also, at the point *C*, in the line *CA*, make the angle *ACd* equal to *ADB*, the angle which *AB* subtends at *D*; observing that the position of *Cd* must be such as to admit of the angle *ACd* being applied on *ADB*; join *A*, and *d* the intersection of the lines *Bd, Cd*. By the theorem, the triangles *ADC, ADB* will be similar to *ABd, ACd* respectively; and as all the angles of these last are manifestly given, because their sides are given in position, therefore all the angles of the triangles *ADC, ADB* will be known; the angle *DCA* being equal to *BdA*; *DBA* to *CdA*; *DAC* to *BAd*; and *DAB* to *CAd*.

Scholium. Since the angle *BdA* is equal to *DCA* (fig. 2.) and *CdA* to *DBA*, the angle *BdC* is equal to the sum of the angles *DCA, DBA*. Now it may happen that their sum is equal to two right angles; then, *Bd* and *Cd* will be in one straight line; and, there being no intersection, nothing can be determined in respect to the angles *BAD, CAD*. But in this case, since *DCA, DBA* make two right angles, the points *A, B, C, D* are in the circumference of a circle. Thus it appears, that when the point *D* is in the circumference of a circle which passes through *A, B, C*, the problem is indeterminate.

8. It is deserving of remark that this simple construction, by which the point *d* is found, has served to change the proposed geodetical problem into another which at first view appears easier of solution; for

since the angle $ABd = ADC$ is given (figs. 1, 2, 3.), and also the angle $ACd = ADB$; and moreover, because the angles which the lines AB, AC make with a line drawn from B to C may be considered as known, the angles dBC, dCB are known.

The proposed problem is now transformed to this:

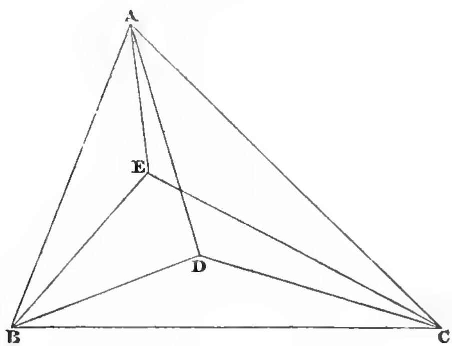
"Having given all the sides, or else all the angles of two triangles ABC, dBC which have a common base BC, it is required to find the angles which the line Ad joining their vertices makes with the sides."

This is the geometrical expression of a well known geodetical problem which is more frequently resolved than the other; probably, because its solution is supposed easier. The geometrical property of the figure by which the one problem is converted into the other, namely, the equality of the rectangles $AD.Ad$ and $AB.AC$ is easily remembered, a circumstance of considerable importance in practical applications of Geometry.

9. The following Theorem is a deduction from the proposition:

"If straight lines $BD.BE$ be drawn from B, one of the angles of a triangle ABC, making equal angles ABE, CBD with the sides about

Fig. 4.



that angle; and also straight lines CD, CE, making equal angles BCD, ACE with the sides about another of the angles, and meeting the former lines in D, E, then, straight lines drawn from the remaining angle A to D and E shall make equal angles BAD, CAE with the sides about the angle*."

The well known proposition, that the lines which bisect the three angles of a triangle meet in the same point is a particular case of this Theorem.

10. The Trigonometrical solution of the Geodetical Problem of Art. 1. which is deducible from the construction here given, is sufficiently obvious: I shall therefore, without at present entering into it, investigate another theorem which comprehends the former and various others.

Let the sides of any triangle be a, b, c, (fig. 5, 6).

The opposite angles .... A, B, C.

Fig. 5.

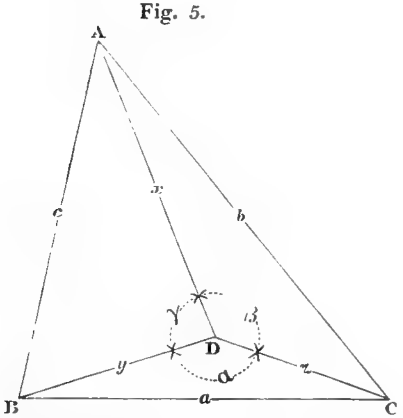
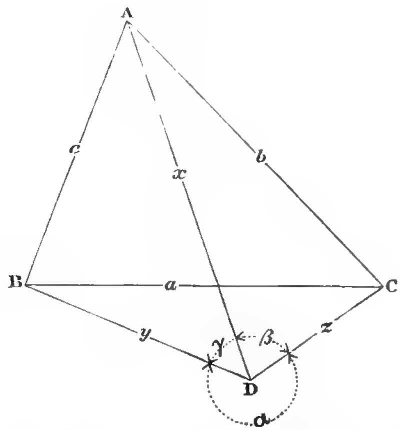


Fig. 6.



Let straight lines be drawn from any point D to the angles of the triangle, and put

$$AD = x, \quad BD = y, \quad CD = z.$$

Also the angles $\widehat{yz} = \alpha$, $\widehat{xz} = \beta$, $\widehat{xy} = \gamma$,

* I owe this elegant proposition to T. Galloway, Esq. F.R.S. to whom it occurred when considering the Theorem.

where it must be observed that the angles α, β, γ, must be so reckoned. that their sum is four right angles.

By a known Trigonometrical formula,

$$x^2 - 2xy \cos\gamma + y^2 = c^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$x^2 - 2xz \cos\beta + z^2 = b^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$y^2 - 2yz \cos\alpha + z^2 = a^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The condition that the triangle ABC is made up of the three triangles ADB, ADC, BDC (fig. 5), or else of the excess of two of them above the third (fig. 6), is expressed analytically by this other equation,

$$xy \sin\gamma + xz \sin\beta + yz \sin\alpha = bc \sin A \quad\ldots\ldots\ldots (4).$$

These hold true, whatever be the position of the point D, observing always that the angles α, β, γ must satisfy the condition of making up four right angles.

By adding the first and second equations, and subtracting the third, we obtain,

$$2x^2 - 2xy \cos\gamma - 2xz \cos\beta + 2yz \cos\alpha = b^2 + c^2 - a^2.$$

But $b^2 + c^2 - a^2 = 2bc \cos A$; therefore,

$$x^2 - xy \cos\gamma - xz \cos\beta + yz \cos\alpha = bc \cos A \quad\ldots\ldots (5).$$

Let equation (5) be multiplied by $\sin\alpha$, and equation (4) by $\cos\alpha$: the results are

$$x^2 \sin\alpha - xy \sin\alpha \cos\gamma - xz \sin\alpha \cos\beta + yz \sin\alpha \cos\alpha = bc \sin\alpha \cos A,$$

$$xy \cos\alpha \sin\gamma + xz \cos\alpha \sin\beta + yz \sin\alpha \cos\alpha = bc \cos\alpha \sin A.$$

By subtracting the second of these equations from the first, and observing that

$$\sin \alpha \cos \gamma + \cos \alpha \sin \gamma = -\sin \beta,$$

$$\sin \alpha \cos \beta + \cos \alpha \sin \beta = -\sin \gamma,$$

$$\sin \alpha \cos A - \cos \alpha \sin A = \sin (\alpha - A),$$

we obtain

$$x^2 \sin \alpha + xy \sin \beta + xz \sin \gamma = bc \sin (\alpha - A).$$

Hence we derive this elegant theorem,

$$x \sin \alpha + y \sin \beta + z \sin \gamma = \frac{bc}{x} \sin (\alpha - A).$$

11. From the form of the function which is the first member of this equation, it will remain the same, although we change the angles A and α into B and β; or into C and γ, provided corresponding changes are made in the lines a, b, c and x, y, z: so that, on the whole, we may conclude that

$$x \sin \alpha + y \sin \beta + z \sin \gamma \begin{cases} = \dfrac{bc}{x} \sin (\alpha - A) \\ = \dfrac{ac}{y} \sin (\beta - B) \\ = \dfrac{ab}{z} \sin (\gamma - C). \end{cases}$$

This is the property of the figure which I proposed to investigate; and it manifestly comprehends this other property,

$$\frac{ax}{\sin (\alpha - A)} = \frac{by}{\sin (\beta - B)} = \frac{cz}{\sin (\gamma - C)};$$

from which it also follows that

$$\frac{x}{y} = \frac{b}{a} \cdot \frac{\sin (\alpha - A)}{\sin (\beta - B)};$$

$$\frac{x}{z} = \frac{c}{a} \cdot \frac{\sin (\alpha - A)}{\sin (\gamma - C)};$$

$$\frac{y}{z} = \frac{c}{b} \cdot \frac{\sin (\beta - B)}{\sin (\gamma - C)}.$$

These formulæ, which are remarkable for their symmetry and simplicity, suggest various solutions to the problem enunciated in Art. 1. Their evident analogy to the property of a triangle "that the sines of the angles are proportional to the opposite sides", has suggested another form under which they may be put.

12. The hypothesis and notation of Art. 10. in regard to the triangle ABC (fig. 5. No. 1. and 6.) being retained, another triangle $A'B'C'$ (fig. 5. No. 2.) having remarkable relations with it, may be constructed as follows:

Let straight lines $D'A'$, $D'B'$, $D'C'$ meet in a point D', the angles $A'D'B'$, $B'D'C'$, $A'D'C'$ being equal to ADB, BDC, ADC respectively. At A' any point in $D'A'$ make the angles $D'A'B'$ equal to DBA, and $D'A'C'$ equal to DCA, thus forming two triangles $D'A'B'$, $D'A'C'$ (fig. 5. No. 2.) similar to DBA, DCA (fig. 5. No. 1). Join $B'C'$; because $DB : DA = D'A' : D'B'$ and $DA : DC = D'C' : D'A'$; therefore, *ex æq.* $DB : DC = D'C' : D'B'$; hence the triangles BDC, $C'D'B'$ are similar.

Let the lines and angles in the triangle $A'B'C'$ be expressed by the same letters as are used for the triangle ABC, with the distinction of an accent over such as differ in magnitude, so that

$$B'C' = a',\ A'C' = b',\ A'B' = c',\ D'A' = x',\ D'B' = y',\ D'C' = z'.$$

The angles about D in the two triangles being equal; viz. $\widehat{y'z'} = \alpha$, $\widehat{x'z} = \beta$, $\widehat{x'y'} = \gamma$.

13. The similarity of the partial triangles which constitute the two triangles ABC, $A'B'C'$, besides the equal ratios $x : y = y' : x'$, by which they were formed, give us also $x : c = y' : c'$, $y : c = x' : c'$; therefore $xx' = yy'$, $\frac{x}{y'} = \frac{c}{c'} = \frac{y}{x'}$; and a like result for each pair of triangles; hence the lines in the two triangles have the following properties:

$$\left.\begin{array}{l}\dfrac{x}{y'} = \dfrac{y}{x'} = \dfrac{c}{c'} \\[2ex] \dfrac{x}{z'} = \dfrac{z}{x'} = \dfrac{b}{b'} \\[2ex] \dfrac{y}{z'} = \dfrac{z}{y'} = \dfrac{a}{a'} \\[2ex] xx' = yy' = zz' \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ I.}$$

14. In the triangle ABC, the angle α is the sum of the three angles A, ABD, ACD, of which the last two are equal to the angles $B'A'D'$, $C'A'D'$ that make up A', hence we have this property:

$$A + A' = \alpha, \quad B + B' = \beta, \quad C + C' = \gamma \ \ldots\ldots\ldots\ldots \text{ II.}$$

The affinity of the two triangles in respect of these, and other properties which are to follow, may not improperly be indicated by calling them *Conjugate Triangles*.

15. Because $A = \alpha - A'$, and $A' = \alpha - A$, also, similarly,

$$B = \beta - B', \quad B' = \beta - B, \quad C = \gamma - C', \quad C' = \gamma - C,$$

the formulæ of Art. 11. gives us these,

$$\left.\begin{array}{l} x \sin\alpha + y \sin\beta + z \sin\gamma \left\{\begin{array}{l} = \dfrac{bc}{x} \sin A' \\[2ex] = \dfrac{ac}{y} \sin B' \\[2ex] = \dfrac{ab}{z} \sin C' \end{array}\right. \\[8ex] x' \sin\alpha + y' \sin\beta + z' \sin\gamma \left\{\begin{array}{l} = \dfrac{b'c'}{x'} \sin A \\[2ex] = \dfrac{a'c'}{y'} \sin B \\[2ex] = \dfrac{a'b'}{z'} \sin C \end{array}\right. \end{array}\right\} \ldots\ldots\ldots\ldots \text{ III.}$$

Hence we find $\frac{x}{y} = \frac{b}{a} \cdot \frac{\sin A'}{\sin B'} = \frac{b}{a} \cdot \frac{a'}{b'}$, and other formulæ of a like nature, which are comprehended in this table.

$$\left.\begin{aligned}
x : y &= \frac{a'}{a} : \frac{b'}{b} = \frac{a'}{b'} : \frac{a}{b} \\
y : z &= \frac{b'}{b} : \frac{c'}{c} = \frac{b'}{c'} : \frac{b}{c} \\
x : z &= \frac{a'}{a} : \frac{c'}{c} = \frac{a'}{c'} : \frac{a}{c} \\
x' : y' &= \frac{a}{a'} : \frac{b}{b'} = \frac{a}{b} : \frac{a'}{b'} \\
y' : z' &= \frac{b}{b'} : \frac{c}{c'} = \frac{b}{c} : \frac{b'}{c'} \\
x' : z' &= \frac{a}{a'} : \frac{c}{c'} = \frac{a}{c} : \frac{a'}{c'}.
\end{aligned}\right\} \quad \text{..................} \quad \text{IV.}$$

These formulæ give the ratios of the six lines $x, y, z;\ x', y', z'$, when the lines $a, b, c;\ a', b', c'$, or their ratios, are known; when the angles α, β, γ contained by these lines are given, the lines themselves may be found by known propositions in Trigonometry. To these I shall in the sequel add others.

16. We have found that

$$x \sin \alpha + y \sin \beta + z \sin \gamma = \frac{bc}{x} \sin A';$$

$$\text{Now } y = \frac{a}{b} \cdot \frac{b'}{a'} x, \qquad z = \frac{a}{c} \cdot \frac{c'}{a'} x.$$

These values of y and z being substituted in the above equation, it becomes

$$x \left\{\sin \alpha + \frac{a}{b} \cdot \frac{b'}{a'} \sin \beta + \frac{a}{c} \cdot \frac{c'}{a'} \sin \gamma\right\} = \frac{bc}{x} \sin A'.$$

Similar expressions may be had for y and z, and from these the following formulæ have been obtained:

$$\left.\begin{aligned} x^2\left\{\frac{a'}{a}\sin\alpha+\frac{b'}{b}\sin\beta+\frac{c'}{c}\sin\gamma\right\}&=\frac{a'}{a}bc\sin A'\\ y^2\left\{\frac{a'}{a}\sin\alpha+\frac{b'}{b}\sin\beta+\frac{c'}{c}\sin\gamma\right\}&=\frac{b'}{b}ac\sin B'\\ z^2\left\{\frac{a'}{a}\sin\alpha+\frac{b'}{b}\sin\beta+\frac{c'}{c}\sin\gamma\right\}&=\frac{c'}{c}ab\sin C'\end{aligned}\right\}\quad\ldots\ldots\ldots\ \text{V.}$$

By changing x, y, z, a, b, c, A' into x', y', z', a', b, c', A and the contrary, these formulæ serve for the conjugate triangle $A'B'C'$.

17. Another expression analogous to that found may be had by substituting for y and z their values in the formula

$$yz\sin\alpha+xz\sin\beta+xy\sin\gamma=bc\sin A.$$

By proceeding as before, we obtain

$$\left.\begin{aligned} x^2\left\{\frac{a}{a'}\sin\alpha+\frac{b}{b'}\sin\beta+\frac{c}{c'}\sin\gamma\right\}&=\frac{a'}{a}\cdot\frac{b}{b'}\cdot\frac{c}{c'}bc\sin A\\ y^2\left\{\frac{a}{a'}\sin\alpha+\frac{b}{b'}\sin\beta+\frac{c}{c'}\sin\gamma\right\}&=\frac{b'}{b}\cdot\frac{a}{a'}\cdot\frac{c}{c'}ac\sin B\\ z^2\left\{\frac{a}{a'}\sin\alpha+\frac{b}{b'}\sin\beta+\frac{c}{c'}\sin\gamma\right\}&=\frac{c'}{c}\cdot\frac{a}{a'}\cdot\frac{b}{b'}ab\sin C\end{aligned}\right\}\quad\ldots\ldots\ldots\ \text{VI.}$$

It is remarkable, that the coefficients of $\sin\alpha$, $\sin\beta$, $\sin\gamma$ in these formulæ are the reciprocals of their coefficients in the preceding.

18. Other values of x^2 may be obtained by putting the values of y and z in terms of x in the formulæ

$$x^2-2xy\cos\gamma+y^2=c^2,$$

$$x^2-2xz\cos\beta+z^2=b^2,$$

$$y^2-2yz\cos\alpha+z^2=a^2.$$

Of these I shall only put down that deduced from the last, as the most symmetrical,

$$\left.\begin{aligned} &\left(\frac{b'}{b}\right)^2 - 2\frac{b'}{b}\cdot\frac{c'}{c}\cos\alpha + \left(\frac{c'}{c}\right)^2 = \left(\frac{a'}{x}\right)^2 \\ \text{Similarly}\ &\left(\frac{a'}{a}\right)^2 - 2\frac{a'}{a}\cdot\frac{c'}{c}\cos\beta + \left(\frac{c'}{c}\right)^2 = \left(\frac{b'}{y}\right)^2 \\ \text{and}\ &\left(\frac{a'}{a}\right)^2 - 2\frac{a'}{a}\cdot\frac{b'}{b}\cos\gamma + \left(\frac{b'}{b}\right)^2 = \left(\frac{c'}{z}\right)^2 \end{aligned}\right\} \cdots\cdots\cdots\cdots \text{VII.}$$

The three sets of fumulæ V, VI, VII, are remarkable for their symmetry and simplicity, qualities of great importance in analysis; the last, viz. VII, seems however to be the most concise.

19. The computation of x from the expression

$$x^2\left\{\left(\frac{b'}{b}\right)^2 - 2\frac{b'}{b}\cdot\frac{c'}{c}\cos\alpha + \left(\frac{c'}{c}\right)^2\right\} = a'^2,$$

may be made by subsidiary angles; to determine these, let us assume that

$$x = \frac{b}{b'}v\sin\phi = \frac{c}{c'}v\cos\phi;\ \text{then, }\tan\phi = \frac{b'}{b}\cdot\frac{c}{c'};$$

$$\text{and } x^2\left\{\left(\frac{b'}{b}\right)^2 - 2\frac{b'}{b}\cdot\frac{c'}{c}\cos\alpha + \left(\frac{c'}{c}\right)^2\right\} = v^2(1 - 2\sin\phi\cos\phi\cos\alpha) = a'^2.$$

Let θ be such, that $\sin\theta = \sqrt{(2\sin\phi\cos\phi\cos\alpha)} = \sqrt{(\sin 2\phi\cos\alpha)}$;

$$\text{then } v\cos\theta = a', \text{ and } v = \frac{a'}{\cos\theta}.$$

If $\cos\alpha$ be negative, we must assume

$$\tan\theta = \sqrt{(2\sin\phi\cos\phi\cos\alpha)} = \sqrt{(\sin 2\phi\cos\alpha)},$$

and then $v\sec\theta = a'$ and $v = a'\cos\theta$.

To determine x from the expression

$$x^2\left\{\left(\frac{b'}{b}\right)^2 - 2\frac{b'}{b}\cdot\frac{c'}{c}\cos\alpha + \left(\frac{c'}{c}\right)^2\right\} = a'^2,$$

we have now these formulæ.

Case 1. When $\alpha \angle 90^\circ$: find ϕ and θ, such, that

$$\tan\phi = \frac{b'}{b}\cdot\frac{c}{c'};\quad \sin\theta = \sqrt{(2\sin\phi\cos\phi\cos\theta)} = \sqrt{(\sin 2\phi\cos\alpha)}.$$

Then, $$x = a'.\frac{b}{b'}\frac{\sin\phi}{\cos\theta} = a'\frac{c}{c'}\cdot\frac{\cos\phi}{\cos\theta}.$$

Case 2. When $\alpha > 90^\circ$; find ϕ and θ, such, that

$$\tan\phi = \frac{b'}{b}\cdot\frac{c}{c'};\quad \tan\theta = \sqrt{(2\sin\phi\cos\phi\cos\alpha)} = \sqrt{(\sin 2\phi\cos\alpha)}:$$

Then, $$x = a'\frac{b}{b'}\sin\phi\cos\theta = a'\frac{c}{c'}\cos\phi\cos\theta.$$

20. I shall now apply the formulæ to a case of Geodetic surveying, taking an example from Delambre's *Methodes Analytiques pour la Determination d'un Arc du Meridien*, (p. 141—2).

ABC (fig. 5.) is one of the triangles employed in measuring an arc of the Meridian in France; *A* is Villers-Bretonneux; *B Vignacourt*; *C* Sourdon; *D* is a station within the triangle. To determine its position, there are given,

	Log. Sines.		Logarithms.
$A = 99^\circ.\ 5'.\ 49''\cdot 2$	9·9945029	a	4·2734544.
$B = 49.\ 4.\ 13\cdot 0$	9·8782424	b	4·1571936.
$C = 31.\ 49.\ 57\cdot 8$	9·7221739	c	4·0011255.
$\alpha = 168.\ 43.\ 49\cdot 7$	9·2909798	a' (Assumed log.)	5·0000000.
$\beta = 130.\ 44.\ 16\cdot 5$	9·8794988	$b' = \frac{\sin B'}{\sin A'}a'$	5·0234267.
$\gamma = 60.\ 31.\ 53\cdot 8$	9·9398323	$c' = \frac{\sin C'}{\sin A'}a'$	4·7094635.

From these angles we find

$A' = (\alpha - A) = 69^\circ.\ 38'.\ 0''\cdot 5$	9·9719645	$\frac{a'}{a}$	0·7265456.
$B' = (\beta - B) = 81.\ 40.\ 3\cdot 5$	9·9953912	$\frac{b'}{b}$	0·8662328.
$C' = (\gamma - C) = 28.\ 41.\ 56\cdot 0$	9·6814280	$\frac{c}{c'}$	3·7083380.

Here a', b', c' are the sides, and A', B', C' the angles of the triangle (fig. 5. No. 2.) which is conjugate to ABC; as a', one of the sides of the conjugate triangle, may be any number, it may be considered as analogous to *radius* in Trigonometry, and its logarithm might be assumed = 10, or any other convenient number, by which negative or large indices may be avoided; it is here assumed to be 5. The lines to be determined are $AD = x$, $BD = y$, $CD = z$.

21. We shall begin with the formulæ

$$x^2\left\{\left(\frac{b'}{b}\right)^2 - 2\frac{b'}{b}\frac{c'}{c}\cos a + \left(\frac{c'}{c}\right)^2\right\} = a'^2, \text{ (Art. 18.)},$$

$$y = \frac{a}{a'}\cdot\frac{b'}{b}x, \qquad z = \frac{a}{a'}\cdot\frac{c'}{c}x. \quad \text{(Art. 15).}$$

The calculation may stand as follows:

$\frac{b'}{b}$		0·8662328
$\frac{c'}{c}$		0·7083380
$\left(\frac{b'}{b}\right)^2 =$	54·00892	1·7324656
$\left(\frac{c'}{c}\right)^2 =$	26·10213	1·4166760
	80·11105	
$\frac{b'}{b}\cdot\frac{c'}{c}$		1·5745708
$\cos a$ (negative)		9·9915445
2		0·3010300
$-2\frac{b'}{b}\cdot\frac{c'}{c}\cos a =$	73·64533	1·8671453
$\left(\frac{a'}{x}\right)^2$	= 153·75638	2·1868332
a'^2		10·0000000
x^2		7·8131668
x	= 8064·6	3·9065834

We have now x, and its logarithm; from which y and z are found as follows:

x	3·9065834		3·9065834
$\frac{a}{a'}$	$\bar{1}$·2734544		$\bar{1}$·2734544
$\frac{b'}{b}$	0·8662328	$\frac{c'}{c}$	0·7083380
$y = 11124·25$	4·0462706	$z = 7733·49$	3·8883758.

22. The computation by subsidiary angles from the formula for Case 2, Art. 19, may be as follows:

$\frac{b'}{b}$	0·8662328
$\frac{c'}{c}$	0·7083380
tan (ϕ = 55° . 11′.35 ·75)	10·1578948
sin (2ϕ = 110° . 23′.11″· 5)	9·9719082
cos a	9·9915445
	2)19·9634527
tan (θ = 43° . 47′.42″· 2)	9·9817263
cos θ	9·8584297
sin ϕ	9·9143866
$\bar{a}'$	5·0000000
$\frac{b}{b'}$	9·1337672
x = 8064·61	3·9065835.

This way of proceeding seems to have no advantage over the direct process farther than being a verification.

23. The analytical elegance and compactness of Formula V. Art. 16. has induced me to make the calculation which follows:

$\sin \alpha$		9·2909798
$\frac{a'}{a}$		0·7265456
$\frac{a'}{a} \sin \alpha =$	1·041179	0·0175254
$\sin \beta$		9·8794988
$\frac{b'}{b}$		0·8662328
$\frac{b'}{b} \sin \beta =$	5·568415	0·7457316
$\sin \gamma$		9·9398323
$\frac{c'}{c}$		0·7083380
$\frac{c'}{c} \sin \gamma =$	4·448056	0·6481703
$1 \div$	11·057650	$\bar{2}$·9563371
$\frac{a'}{a}$		0·7265456
b		4·1571939
c		4·0011255
$\sin A'$		9·9719645
x^2		7·8131666
$x =$	8064·61	3·9065833

This Formula employs more logarithms than Formula VII; therefore, the latter is better for practice than the former.

24. Formula VI. may be easily computed along with Formula V. as a verification of it, for the same logarithms serve to find the coefficients of sin α, sin β, sin γ, in both. In this way I have found

$$\frac{a}{a'}\sin\alpha = \cdot 0366804$$

$$\frac{b}{b'}\sin\beta = \cdot 1031017$$

$$\frac{c}{c'}\sin\gamma = \cdot 1704097$$

$\frac{a}{a'}\sin\alpha + \frac{b}{b'}\sin\beta + \frac{c}{c'}\sin\gamma = \cdot 3101918$	(Ar. Com.)	0·5083697
$\frac{a'}{a}$		0·7265456
$\frac{b}{b'}$		$\bar{1}$·1337672
$\frac{c}{c'}$		$\bar{1}$·2916620
b		4·1571939
c		4·0011255
sin A		9·9945029
x^2		7·8131668
$x = 8064{\cdot}61$		3·9065834.

This calculation involves more logarithms than either of the former; however, some of them occur twice.

25. The angular calculus applies with great advantage to the solution of problems of the kind which we have been considering. Resuming the consideration of the triangles ABC, $A'B'C$ (Fig. 5. No. 1. and 2), and the notation of Art. 10, we have by Trigonometry and Formulæ IV.

$$\frac{\sin \widehat{cx}}{\sin\gamma} = \frac{y}{c}, \quad \frac{c'}{b'}\frac{b}{c} = \frac{z}{y}, \quad \frac{\sin\beta}{\sin \widehat{bx}} = \frac{b}{z}.$$

Hence, by compounding these equal ratios,

$$\frac{\sin \widehat{cx}}{\sin \widehat{bx}} \cdot \frac{c'}{b'} \cdot \frac{\sin \beta}{\sin \gamma} = 1: \qquad \frac{\sin \widehat{cx}}{\sin \widehat{bx}} = \frac{b' \sin \gamma}{c' \sin \beta} = \frac{\sin B'}{\sin C'}\,\frac{\sin \gamma}{\sin \beta}.$$

Fig. 5. No. 1.

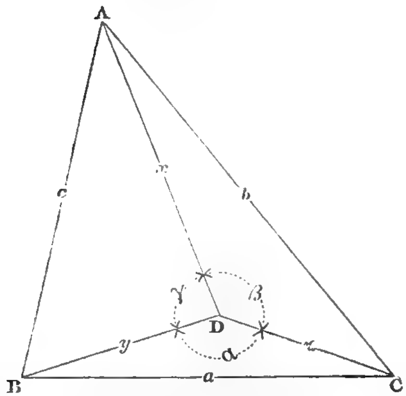
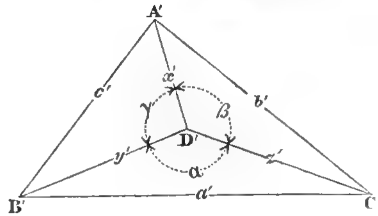


Fig. 5. No. 2.



Thus, to determine the angles cx, bx, we have their sum and the ratio of their sines.

Again, by Trigonometry and Formulæ IV,

$$\frac{\sin \widehat{ay}}{\sin \widehat{az}} = \frac{z}{y} = \frac{b}{c} \cdot \frac{c'}{b'} = \frac{b \sin C'}{c \sin B'}, \quad \text{also } \widehat{ay} + \widehat{az} = \pi - \alpha;$$

hence, as before, the sum of the angles $\widehat{ay}$, $\widehat{az}$ is given; also, the ratio of their sines, to determine the angles.

Lastly, since $\sin \widehat{cy} = \frac{x}{c} \sin \gamma$, and $\sin \widehat{bz} = \frac{x}{b} \sin \beta$;

$$\text{therefore } \frac{\sin \widehat{cy}}{\sin \widehat{bz}} = \frac{b}{c} \cdot \frac{\sin \gamma}{\sin \beta}, \quad \text{also } \widehat{cy} + \widehat{bz} = \alpha - A = A'.$$

By these cy and bz may be found.

26. The results which have been obtained may be applied to the pairs of like angles and tabulated for use as follows:

VIII.

$$\widehat{cx}+\widehat{bx}=A, \qquad \frac{\sin\widehat{cx}}{\sin\widehat{bx}}=\frac{\sin B'}{\sin C'}\cdot\frac{\sin\gamma}{\sin\beta} \quad \cdots\cdots\cdots (1),$$

$$\widehat{cy}+\widehat{ay}=B, \qquad \frac{\sin\widehat{cy}}{\sin\widehat{ay}}=\frac{\sin A'}{\sin C'}\cdot\frac{\sin\gamma}{\sin\alpha} \quad \cdots\cdots\cdots (2),$$

$$\widehat{bz}+\widehat{az}=C, \qquad \frac{\sin\widehat{bz}}{\sin\widehat{az}}=\frac{\sin A'}{\sin B'}\cdot\frac{\sin\beta}{\sin\alpha} \quad \cdots\cdots\cdots (3).$$

IX.

$$\widehat{ay}+\widehat{az}=\pi-\alpha, \qquad \frac{\sin\widehat{ay}}{\sin\widehat{az}}=\frac{b}{c}\cdot\frac{\sin C'}{\sin B'} \quad \cdots\cdots\cdots (1),$$

$$\widehat{bx}+\widehat{bz}=\pi-\beta, \qquad \frac{\sin\widehat{bx}}{\sin\widehat{bz}}=\frac{a}{c}\cdot\frac{\sin C'}{\sin A'} \quad \cdots\cdots\cdots (2),$$

$$\widehat{cx}+\widehat{cy}=\pi-\gamma, \qquad \frac{\sin\widehat{cx}}{\sin\widehat{cy}}=\frac{a}{b}\cdot\frac{\sin B'}{\sin A'} \quad \cdots\cdots\cdots (3).$$

X.

$$\widehat{cy}+\widehat{bz}=A', \qquad \frac{\sin\widehat{cy}}{\sin\widehat{bz}}=\frac{b}{c}\cdot\frac{\sin\gamma}{\sin\beta} \quad \cdots\cdots\cdots (1),$$

$$\widehat{cx}+\widehat{az}=B', \qquad \frac{\sin\widehat{cx}}{\sin\widehat{az}}=\frac{a}{c}\cdot\frac{\sin\gamma}{\sin\alpha} \quad \cdots\cdots\cdots (2),$$

$$\widehat{bx}+\widehat{ay}=C', \qquad \frac{\sin\widehat{bx}}{\sin\widehat{ay}}=\frac{a}{b}\cdot\frac{\sin\beta}{\sin\alpha} \quad \cdots\cdots\cdots (3).$$

27. The application of the formulæ in these tables, and many problems in Astronomy, require the solution of this problem in the Calculus of Sines:

Given the sum or difference of two angles, and the ratio of their sines, to find the angles.

This problem is identical with a case of Plane Trigonometry; but it may be elegantly resolved without any reference to Trigonometry, by means of subsidiary angles.

Let ϕ and ψ be the two angles: there are given $\phi+\psi$, or else $\phi-\psi$, and $\frac{\sin\psi}{\sin\phi}$, to determine ϕ and ψ.

First solution. Let ϵ be an angle, such that

$$\left.\begin{aligned} \sin\epsilon &= \frac{\sin\psi}{\sin\phi}; \\ \text{Then } \frac{\tan\frac{1}{2}(\phi-\psi)}{\tan\frac{1}{2}(\phi+\psi)} &= \tan^2(45^\circ-\tfrac{1}{2}\epsilon) \end{aligned}\right\} \cdots\cdots\cdots (1).$$

Second. Take ϵ such that

$$\left.\begin{aligned} \cos\epsilon &= \frac{\sin\psi}{\sin\Phi}; \\ \text{then } \frac{\tan\frac{1}{2}(\phi-\psi)}{\tan\frac{1}{2}(\phi-\psi)} &= \tan^2\tfrac{1}{2}\epsilon \end{aligned}\right\} \cdots\cdots\cdots (2).$$

Third. Find ϵ so that

$$\left.\begin{aligned} \tan\epsilon &= \frac{\sin\psi}{\sin\phi}; \\ \text{then } \frac{\tan\frac{1}{2}(\phi-\psi)}{\tan\frac{1}{2}(\phi-\psi)} &= \tan(45^\circ-\epsilon) \end{aligned}\right\} \cdots\cdots\cdots (3).$$

Fourth.

$$\left.\begin{aligned} \text{Take, } \tan\epsilon &= \sqrt{\frac{\sin\psi}{\sin\phi}}; \\ \text{then, } \frac{\tan\frac{1}{2}(\phi-\psi)}{\tan\frac{1}{2}(\phi-\psi)} &= \cos 2\epsilon \end{aligned}\right\} \cdots\cdots\cdots (4).$$

28. We may employ two subsidiary angles Δ and E, one of which may be taken at pleasure.

Fifth. Supposing E to be any known angle, let Δ be such that

$$\left.\begin{aligned} \sin\Delta &= \frac{\sin\psi}{\sin\phi}\sin E; \\ \text{then, } \frac{\tan\frac{1}{2}(\phi-\psi)}{\tan\frac{1}{2}(\phi+\psi)} &= \frac{\tan\frac{1}{2}(E-\Delta)}{\tan\frac{1}{2}(E+\Delta)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (5).$$

Sixth. Taking E as before any angle, take Δ such that

$$\left.\begin{aligned} \cos\Delta &= \frac{\sin\psi}{\sin\phi}\cos E; \\ \text{then, } \frac{\tan\frac{1}{2}(\phi-\psi)}{\tan\frac{1}{2}(\phi+\psi)} &= \tan\tfrac{1}{2}(E+\Delta)\tan\tfrac{1}{2}(E-\Delta) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (6).$$

Seventh. Assuming E any angle, find Δ so that

$$\left.\begin{aligned} \tan\Delta &= \frac{\sin\psi}{\sin\phi}\tan E; \\ \text{then, } \frac{\tan\frac{1}{2}(\phi-\psi)}{\tan\frac{1}{2}(\phi+\psi)} &= \frac{\sin(E-\Delta)}{\sin(E+\Delta)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (7).$$

29. These formulæ determine the sum or difference of the angles, the one from the other. I shall now investigate a formula that determines either angle by itself.

Let α, ϕ and θ be three angles such, that

$$\cot\theta = \cot\phi - \cot\alpha; \quad \text{then } \cot\phi = \cot\alpha + \cot\theta.$$

$$\text{Now } \cot\phi - \cot\alpha = \frac{\cos\phi}{\sin\phi} - \frac{\cos\alpha}{\sin\alpha} = \frac{\sin(\alpha-\phi)}{\sin\alpha\sin\phi};$$

$$\text{and } \cot\alpha + \cot\theta = \frac{\cos\alpha}{\sin\alpha} + \frac{\cos\theta}{\sin\theta} = \frac{\sin(\alpha+\theta)}{\sin\alpha\sin\theta}.$$

Hence, putting for $\cot\theta$ and $\cot\alpha$ their equivalents $\frac{1}{\tan\theta}$ and $\frac{1}{\tan\alpha}$, it appears that

$$\text{if } \tan\theta = \frac{\sin\alpha\sin\phi}{\sin(\alpha-\phi)}; \quad \text{then } \tan\phi = \frac{\sin\alpha\sin\theta}{\sin(\alpha+\theta)}.$$

By assuming that $\cot\theta = \cot\alpha - \cot\phi$, we find that

$$\text{if } \tan\theta = \frac{\sin\alpha\sin\phi}{\sin(\phi-\alpha)}; \quad \text{then } \tan\phi = \frac{\sin\alpha\sin\theta}{\sin(\theta-\alpha)}.$$

Now, make $\alpha = \phi + \psi$ in the first of these formulæ, and $\alpha = \phi - \psi$ in the second, and we get these other formulæ.

Eighth.

$$\left.\begin{array}{l} \text{If } \tan\theta = \dfrac{\sin\phi}{\sin\psi}.\sin(\phi+\psi); \\ \text{then } \tan\phi = \dfrac{\sin\theta}{\sin(\theta+\phi+\psi)}.\sin(\phi+\psi) \end{array}\right\} \ldots\ldots 8.\ \text{No. } 1.$$

$$\left.\begin{array}{l} \text{If } \tan\theta = \dfrac{\sin\phi}{\sin\psi}.\sin(\phi-\psi); \\ \text{then } \tan\phi = \dfrac{\sin\theta}{\sin\{\theta-(\phi-\psi)\}}.\sin(\phi-\psi) \end{array}\right\} \ldots\ldots 8.\ \text{No. } 2.$$

30. We may apply the eighth formula to the case of Trigonometry, in which there are given two sides a and b of a triangle, and C the angle they contain, to find A and B, the two remaining angles, of which A is opposite to A, and B to b.

$$\text{Find } \theta \text{ so that } \tan\theta = \frac{a}{b}\sin C.$$

$$\text{Then } \tan A = \frac{\sin\theta}{\sin(C-\theta)}\sin C.$$

This formula is particularly applicable when an angle and the logarithms of the sides containing it are given.

To verify the solution, try if $\dfrac{\sin A}{\sin B} = \dfrac{a}{b}$ by their logarithms.

31. I shall now apply to the example (Art. 20.) one of the formulæ for determining the angles which the lines x, y, z (Fig. 5.) make with the lines joining the stations: and for this purpose select No. 1. of Formulæ VIII.

Putting $\widehat{xb} = \phi$, $\widehat{xc} = \psi$, we have

$$\phi + \psi = A, \qquad \frac{\sin\phi}{\sin\psi} = \frac{\sin\beta \,.\, \sin(\gamma - C)}{\sin\gamma \,.\, \sin(\beta - B)}.$$

Again, to find ϕ and ψ we may use any one of the eight formulæ of Art. 27 and 28, the eighth for instance, viz.,

$$\text{find } \theta \text{ so that } \tan\theta = \frac{\sin\phi}{\sin\psi}\sin(\phi+\psi),$$

$$\tan\phi = \frac{\sin\theta}{\sin(\theta+\phi+\psi)}\sin(\phi+\psi),$$

$\sin\beta$	9·8794988
$\sin(\gamma - C)$	9·6814280
$1 \div \sin\gamma$	0·0601677
$1 \div \sin(\beta - B)$	0·0046088
$\sin A =$ 99° . 5′ . 49″·2)	9·9945029
$\tan(\theta =$ 22 . 38 . 21 ·9)	9·6202062
$\sin\theta$	9·5853822
$1 \div \sin(\theta + A =$ 121 . 44 . 11 ·1)	0·0703374
$\sin A$	9·9945029
$\tan(\phi = \widehat{xb} =$ 24 . 4 . 49 ·6)	9·6502225
$\psi = \widehat{xc} = (A - \phi) =$ 75 . 0 . 59 ·6)	

The angles $\widehat{xb}$, $\widehat{xc}$ being now known, all the angles of the triangles about the point D may be found by addition and subtraction.

To verify the solution we may try whether the angles ϕ and ψ satisfy the condition $\frac{\sin\phi}{\sin\psi} = \frac{\tan\theta}{\sin(\phi+\psi)}$. This will require no more logarithms than will be wanted to determine y and z.

$\sin(\phi = \hat{xb})$	9·6106802	$\tan\theta$	9·6202062
$\sin(\psi = \hat{xc})$	9·9849774	$\sin(\phi+\psi)$	9·9945029
	9·6257028		9·6257033

The near agreement of the logarithms shews that the angles ϕ, ψ are determined with sufficient accuracy.

The angles of the triangles are as below:

Triangle *ADB*.	Triangle *BDC*.	Triangle *ADC*.
$\hat{cx} = 75°.\ 0'.\ 59''\cdot6$	$\hat{ay} = 4°.\ 37'.\ 6''\cdot4$	$\hat{bx} = 24°.\ 4'.\ 49''\cdot6$
$\hat{cy} = 44.27.\ 6\cdot6$	$\hat{az} = 6.39.\ 3\cdot9$	$\hat{bz} = 25.10.53\cdot9.$
$\gamma = 60.31.53\cdot8.$	$a = 168.43.49\cdot7.$	$\beta = 130.44.16\cdot5.$

We may now find y and z for the computation of which we have all the logarithms.

In triangle *ADB*.		In triangle *ADC*.	
c	4·0011255	b	4·1571939
$\sin\hat{xc}$	9·9849774	$\sin\hat{xb}$	9·6106802
$1 \div \sin\gamma$	0·0601677	$1 \div \sin\beta$	0·1205012
$y = 11124\cdot25$	4·0462706.	$z = 7733\cdot484$	3·8883753.

From what has been done in this paper, it is manifest that the Geodetical problem which has suggested it may be resolved in a great variety of ways. Probably the solution by the Angular Calculus will be the most convenient in practice.

32. There is a geodetical problem akin to that which has been here so fully discussed, to which the propositions delivered in this paper apply elegantly. It is this:

"Given the elevation of an object above a plane as observed at three stations in that plane whose positions are known; to find the place of the object reduced to the plane."

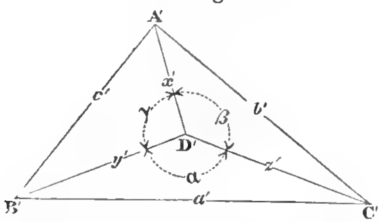
This is identical with the following geometrical problem:

"From three given points, to draw straight lines to a fourth in their plane, so that the lines may have to each other given ratios."

This problem may be easily constructed geometrically by the theorem of Art. 8. I shall here however give an analytical solution.

Let A, B, C the angles of the triangle ABC, be the given points,

Fig. 5. No. 2.



and D the point to be found; and employing the notation, of Art. 10,

Let the sides of the triangle be .... a, b, c,

the opposite angles being .... A, B, C,

the distances of D from the angles .... x, y, z,

the angles which the sides subtend at D ... α, β, γ.

There are given a, b, c, and therefore A, B, C, and the ratios $x : y$, $x : z$, $y : z$, to find x, y, z.

Let the sides of a triangle *conjugate* to ABC (Art. 14.) be

$$a', \quad b', \quad c',$$

the opposite angles A', B', C'.

The angles of the two triangles must satisfy the conditions,

$$A + A' = \alpha, \quad B + B' = \beta, \quad C + C' = \gamma, \quad \text{(Form. II. Art. 14).}$$

It has been found (Form. IV. Art. 15.) that

$$x : y = \frac{a'}{a} : \frac{b'}{b}, \quad x : z = \frac{a'}{a} : \frac{c'}{c}, \quad y : z = \frac{b'}{b} : \frac{c'}{c}.$$

Let p, q, r be given numbers which express the given ratios of x, y, z to one another, viz.,

$$x : y = p : q, \qquad x : z = p : r, \qquad y : z = q : r,$$

$$\text{then } a' : b' = pa : qb, \qquad a' : c' = pa : rc, \qquad b' : c' = qb : rc.$$

Since we may give one of the lines a', b', c' any magnitude, let us assume that

$$a' = pa; \text{ then } b' = qb \text{ and } c' = rc.$$

Thus a', b', c' the sides of a triangle are given, from which the opposite angles A', B', C' may be found, and therefore are known; and since A, B, C are known, $\alpha = A + A'$, $\beta = B + B'$, $\gamma = C + C'$ are known; the problem is now made identical with that of Art. 7. and the formulæ which apply to the one, resolve also the other.

Since the lines a', b', c' are the sides of a triangle, any two of them must be greater than the third; and unless this condition be satisfied, the problem will not admit of a solution in that particular case.

The angles of a triangle, whose sides a, b, c are given, may be most conveniently found from this formula.

$$\text{Find } s = \tfrac{1}{2}(a + b + c), \qquad R = \sqrt{\left\{\frac{(s-a)(s-b)(s-c)}{s}\right\}};$$

$$\text{then } \tan \tfrac{1}{2} A = \frac{R}{s-a}; \qquad \tan \tfrac{1}{2} B = \frac{R}{s-b}; \qquad \tan \tfrac{1}{2} C = \frac{R}{s-c}.$$

This way of finding the angles gives the advantage of a verification, viz. $\tfrac{1}{2}(A + B + C) = 90°$. The line R is the radius of a circle inscribed in the triangle*.

33. Since the function R may be either positive or negative, the angles A', B', C' may be either all positive or all negative; hence α, β, γ will have each two values, viz.,

$$\alpha_1 = A + A', \qquad \beta_1 = B + B', \qquad \gamma_1 = C + C',$$

$$\alpha_2 = A - A', \qquad \beta_2 = B - B', \qquad \gamma_2 = C - C'.$$

* There is a formula perfectly similar to this for spherical triangles.

Thus the problem has two solutions, which may be deduced from Art. 14. and Formula VII. Art. 18. as follows:

1. Find A, B, C the angles of the triangle whose sides are a, b, c.

2. Also A', B', C' the angles of a triangle whose sides are pa, qb, rc.

3. Then $x = \dfrac{pa}{\sqrt{\{q^2 - 2qr\cos(A \pm A') + r^2\}}}$.

There are like expressions for y and z, which are more simply.

$$y = \frac{q}{p}x, \qquad z = \frac{r}{p}x.$$

34. The value of x may be found from the formula by subsidiary angles. The radical $\sqrt{(q^2 - 2qr\cos\alpha + r^2)}$ being the base of a triangle whose sides are q and r, and the contained angle $\alpha = A \pm A'$, its value may be found by the formulæ of Trigonometry; for example, by one which I proposed in the Transactions of the Royal Society of Edinburgh, Vol. x. viz.,

Let a, b, c, be the sides, and A, B, C the opposite angles;

$$a+b : a-b = \tan\tfrac{1}{2}(A+B) : \tan\tfrac{1}{2}(A-B),$$

$$\cos\tfrac{1}{2}(A-B) : \cos\tfrac{1}{2}(A+B) = a+b : c;$$

$$\text{Also } \sin\tfrac{1}{2}(A-B) : \sin\tfrac{1}{2}(A+B) = a-b : c.$$

Hence, to find x, find E, such that

$$\text{cotan E} = \frac{q-r}{q+r}\cot\tfrac{1}{2}\alpha.$$

$$\text{Then } x \begin{cases} = \dfrac{pa}{q+r}\cdot\dfrac{\sin \text{E}}{\sin\frac{1}{2}\alpha} \\[2ex] = \dfrac{pa}{q-r}\cdot\dfrac{\cos \text{E}}{\cos\frac{1}{2}\alpha}. \end{cases}$$

35. We have seen that the first problem in this paper is convertible into another, in appearance easier. The solution of the latter is usually

given as an application of the rules of Trigonometrical calculation rather than as a distinct problem. But as it frequently occurs in Trigonometrical surveys it may be convenient to have formulæ for its solution in accordance with those which resolve the others.

PROBLEM. (Figs. 7, 8, 9).

Let *A, B, C, D* be four points or stations in a plane; there are given *BC*, the distance between two of them, and all the angles of

Fig. 8.

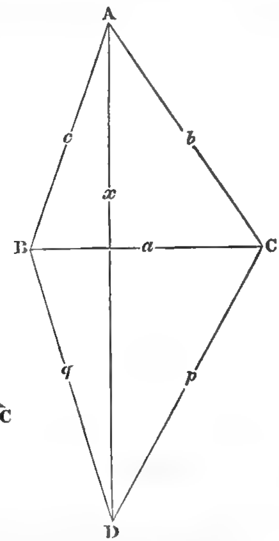


Fig. 7.

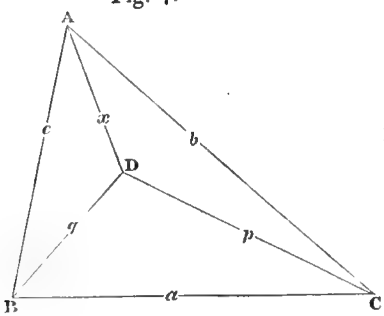
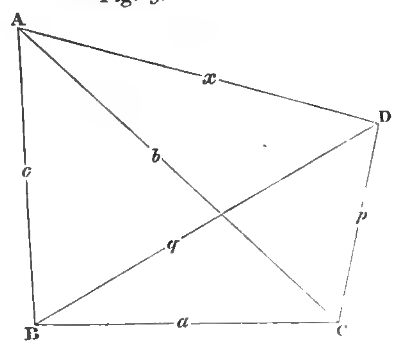


Fig. 9.



the triangles *ABC, DBC* which have *BC* as a common base; to find *AD*, the distance between their vertices.

Following the notation used in Art. 10. I shall express the lines by single letters, putting

$$BC=a,\ AC=b,\ AB=c; \quad DC=p,\ DB=q,\ AD=x.$$

By Trigonometry, in the triangles *ACD, ABD, BDC*;

$$\frac{\sin \widehat{bx}}{\sin \widehat{bp}}=\frac{p}{x}, \quad \frac{\sin \widehat{cq}}{\sin \widehat{cx}}=\frac{x}{q}, \quad \frac{\sin \widehat{ap}}{\sin \widehat{aq}}=\frac{q}{p}:$$

Hence compounding these ratios,

$$\frac{\sin \widehat{bx} \cdot \sin \widehat{cq} \cdot \sin \widehat{ap}}{\sin \widehat{bp} \cdot \sin \widehat{cx} \cdot \sin \widehat{aq}} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In like manner, in the triangles *ACD, ABD, ABC*;

$$\frac{\sin \widehat{px}}{\sin \widehat{bp}} = \frac{b}{x}, \quad \frac{\sin \widehat{cq}}{\sin \widehat{qx}} = \frac{x}{c}, \quad \frac{\sin \widehat{ab}}{\sin \widehat{ac}} = \frac{c}{b};$$

$$\text{therefore } \frac{\sin \widehat{px} \cdot \sin \widehat{cq} \cdot \sin \widehat{ab}}{\sin \widehat{bp} \cdot \sin \widehat{qx} \cdot \sin \widehat{ac}} = 1 \ldots\ldots\ldots\ldots\ldots (2).$$

The triangles *ACD, ABC, DBC* give these equal ratios,

$$\frac{\sin \widehat{bx}}{\sin \widehat{px}} = \frac{p}{b}, \quad \frac{\sin \widehat{ac}}{\sin \widehat{bc}} = \frac{b}{a}, \quad \frac{\sin \widehat{pq}}{\sin \widehat{aq}} = \frac{a}{p};$$

$$\text{therefore } \frac{\sin \widehat{bx} \cdot \sin \widehat{ac} \cdot \sin \widehat{pq}}{\sin \widehat{px} \cdot \sin \widehat{bc} \cdot \sin \widehat{aq}} = 1 \ldots\ldots\ldots\ldots\ldots (3).$$

And again the triangles *ABD, DBC, ABC* give

$$\frac{\sin \widehat{cx}}{\sin \widehat{qx}} = \frac{q}{c}, \quad \frac{\sin \widehat{ab}}{\sin \widehat{bc}} = \frac{c}{a}, \quad \frac{\sin \widehat{pq}}{\sin \widehat{pa}} = \frac{a}{q};$$

$$\text{therefore } \frac{\sin \widehat{cx} \cdot \sin \widehat{ab} \cdot \sin \widehat{pq}}{\sin \widehat{qx} \cdot \sin \widehat{bc} \cdot \sin \widehat{pa}} = 1 \ldots\ldots\ldots\ldots\ldots (4).$$

From the formulæ (1), (2), (3), (4), we form this table, from which it appears that the problem may be resolved in four different ways by the formulæ of Art. 28, which determine two angles whose sum or difference is given, and in addition, the ratio of their series.

$$\hat{cx} \pm \hat{bx} = \hat{cb}, \qquad \frac{\sin \hat{cx}}{\sin \hat{bx}} = \frac{\sin \hat{ap} \,.\, \sin \hat{cq}}{\sin \hat{aq} \,.\, \sin \hat{bp}} \quad \ldots\ldots\ldots\ldots (1).$$

$$\hat{px} \pm \hat{qx} = \hat{pq}, \qquad \frac{\sin \hat{px}}{\sin \hat{qx}} = \frac{\sin \hat{ac} \,.\, \sin \hat{bp}}{\sin \hat{ab} \,.\, \sin \hat{cq}} \quad \ldots\ldots\ldots\ldots (2).$$

$$\hat{yx} + \hat{bx} = \pi - \hat{pb}, \qquad \frac{\sin \hat{px}}{\sin \hat{bx}} = \frac{\sin \hat{ac} \,.\, \sin \hat{pq}}{\sin \hat{bc} \,.\, \sin \hat{aq}} \quad \ldots\ldots\ldots\ldots (3).$$

$$\hat{cx} + \hat{qx} = \pi - \hat{cq}, \qquad \frac{\sin \hat{cx}}{\sin \hat{qx}} = \frac{\sin \hat{bc} \,.\, \sin \hat{ap}}{\sin \hat{ab} \,.\, \sin \hat{pq}} \quad \ldots\ldots\ldots\ldots (4).$$

When the angles are known, the line x may be found from any one of the four angles $\hat{bx}$, $\hat{cx}$, $\hat{px}$, $\hat{qx}$ by two proportions which when united give these four values of x, viz.

$$x \begin{cases} = \dfrac{\sin \hat{bp} \,.\, \sin \hat{ac}}{\sin \hat{px} \,.\, \sin \hat{bc}}\, a \\[2ex] = \dfrac{\sin \hat{cq} \,.\, \sin \hat{ab}}{\sin \hat{qx} \,.\, \sin \hat{bc}}\, a \end{cases} \qquad x \begin{cases} = \dfrac{\sin \hat{bp} \,.\, \sin \hat{aq}}{\sin \hat{bx} \,.\, \sin \hat{pq}}\, a \\[2ex] = \dfrac{\sin \hat{cq} \,.\, \sin \hat{ap}}{\sin \hat{cx} \,.\, \sin \hat{pq}}\, a. \end{cases}$$

36. *Application to a problem in Algebra.*

The Angular Calculus was applied with advantage to the resolution of Quadratic Equations, first, I believe, by Dr Halley, in Lectures given at Oxford in 1704. From this it might be inferred that it may be applied to the solution of every Algebraic problem which produces a Quadratic Equation, without a previous reduction to that form, although I do not know that this application has been expressly treated of, and examples given. The formulæ which have been investigated in this paper apply with peculiar advantage to the solution of a known problem in Algebra, which appears at first sight to be by no means easy. It is this:

Find x, y, and z, from these equations

$$x^2 + xy + y^2 = c^2,$$

$$x^2 + xz + z^2 = b^2,$$

$$y^2 + yz + z^2 = a^2.$$

Here a, b, c are given numbers.

These equations become identical with the equations of Art. 10. or 18, if we suppose the angles α, β, γ to be all equal, and each 120°, because then $2\cos\alpha = -1$. Therefore A, B, C being the angles of a triangle whose sides are a, b, c, and

$$A' = 120^\circ - A, \qquad B' = 120^\circ - B, \qquad C' = 120^\circ - c.$$

In this case Formula v. becomes

$$x^2 \left\{\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c}\right\} \sin 60^\circ = \frac{a'}{a} bc \sin A'.$$

Here a' may be any number, provided we take

$$b' = \frac{\sin B'}{\sin A'} a', \qquad c' = \frac{\sin C'}{\sin A'} a'.$$

As a numerical example, let

$$x^2 + xy + y^2 = 193 = c^2,$$

$$x^2 + xz + z^2 = 219 = b^2,$$

$$y^2 + yz + z^2 = 271 = a^2.$$

		Logarithms.
In this case	$a = 16{\cdot}46208$	1·2164846,
	$b = 14{\cdot}79865$	1·1702226,
	$c = 13{\cdot}89244$	1·1427786.

And by the formula, Art. 32.

$A = 69^\circ.\ 56'.\ 12''$,	$A' = 50^\circ.\ 3'.\ 18''$	9·8846034,
$B = 57\ .\ 36\ .\ 46$,	$B' = 62\ .\ 23\ .\ 14$	9·9474829,
$C = 52\ .\ 26\ .\ 32$,	$C' = 67\ .\ 33\ .\ 28$	9·9657965.

Let us assume $\log a' = 2{\cdot}0000000$,

then $\log b' = 2{\cdot}0628795$,

$\log c' = 2{\cdot}0811931$.

With these logarithms, proceeding exactly as in the calculation Art. 23, we find $x=7$, and hence, as in Art. 21, $y=9$, $z=10$.

We might find x otherwise from formula VII. Art. 18, which may stand thus:

$$\left(\frac{b'}{b}\right)^2 + \frac{b'}{b}\cdot\frac{c'}{c} + \left(\frac{c'}{c}\right)^2 = \left(\frac{a'}{x}\right)^2,$$

and it is a remarkable property of this expression that it is exactly similar to the equation

$$y^2 + yz + z^2 = a^2,$$

where $\frac{b'}{b}$, $\frac{c'}{c}$ and $\frac{a'}{x}$ take the places of y, z, and a.

V. *Mathematical Considerations on the Problem of the Rainbow, shewing it to belong to Physical Optics.* *By* R. POTTER, ESQ. *of Queens' College.*

[Read *Dec.* 14, 1835.]

HAVING lately, in the course of my academical studies, had occasion to read more carefully the theory of the Rainbow, I was convinced of the inadequacy of the popular mode in which it is treated in elementary books (this is Sir Isaac Newton's explanation, see his Optics, p. 147). The reason which is given, why the various prismatic colours are seen, each, so brilliantly in the rainbow, is that the intensity of any colour fades away so rapidly from its maximum, that it does not prevent, in any great degree, the other colours being seen as such.

This is clearly an arbitrary assumption, which will not bear examination, for within the primary bow, the grayish light is of very considerable intensity when the display is a fine one, and in a like proportion in fainter displays. So that we should naturally expect the brilliant colours yellow and green (even laying aside the red and orange) to have still an intensity at the places of the indigo and violet, sufficient to drown the effect of colour in those weak shades.

We find that this is the fact in certain cases, as in Fog bows and similar appearances, when the size of the aqueous spheres is very small, but in other cases the violet especially is very bright. The popular theory, however, offers no reason why the *size* of the spherical drops should influence in any degree the colours of the bows, and it is only in referring the problem to Physical Optics, and considering the interference of the light which arises, that we understand how the size of the drops varying and determining the angular positions of the bright and dark fringes for any colour, causes the appearances to vary, by modifying the extent of the overlapping of the various colours.

The existence of the supernumerary bows furnishes a still more weighty objection to the common explanation; which supposes only one maximum of intensity for each colour, at the angle at which two consecutive rays emerge from the drop parallel to each other. But these supernumerary bows shew that the true explanation must furnish reasons for a succession of maxima and minima for each colour, and this the principle of interference does in a manner perfectly in accordance with recorded observations of the phenomena.

It was not until after I had finished the mathematical investigations, that I learned from Mr Whewell, that Dr Young had previously applied the principle of interferences to explain the supernumerary rainbows. If I had known this earlier, I probably should never have entered further into the subject; as it is, it will be found that I have shewn a method of obtaining the result, which I am not aware that he has any where given; and at any rate I shall have awakened the attention of mathematicians to this interesting phenomenon.

Dr Young's account is very concise, and insufficient as a mathematical explanation; he, however, notices the brilliancy of some of the colours being assisted by the interference of the others. At page 470 of his Lectures on Natural Philosophy, he says, "We have already seen that the light producing the ordinary rainbow is double, its intensity being only greatest at its termination where the common bow appears, while the whole light is extended much more widely. The two portions concerned in its production must divide this light into fringes; but unless almost all the drops of a shower happen to be of the same magnitude, the effects of these fringes must be confounded and destroyed; in general however they must at least co-operate more or less in producing one dark fringe, which must cut off the common rainbow much more abruptly than it would otherwise have been terminated, and consequently assist the distinctness of the colours. The magnitudes of the drops of rain, required for producing such of these rainbows as are usually observed, is between the 50th and 100th of an inch; they (i. e. *the supernumerary bows*) become gradually narrower as they are more remote from the common rainbows, nearly in the same proportions as the external fringes of a shadow, or the rings seen in a concave plate."

At page 643. Vol. II. in a reprint of a paper of his in the *Phil. Trans.* for 1803, he goes further into particulars, and says, "In order to understand the phenomenon, we have only to attend to the two portions of light which are exhibited in the common diagrams explanatory of the rainbow, regularly reflected from the posterior surface of the drop, and crossing each other in various directions, till, at the angle of greatest deviation, they coincide with each other, so as to produce, by the greater intensity of this redoubled light, the common rainbow of 41 degrees. Other parts of these two portions will quit the drop in directions parallel to each other; and these would exhibit a continued diffusion of fainter light for 25° within the bright termination which forms the rainbow, but for the general law of interference, which, as in other similar cases divides the light into concentric rings; the magnitude of these rings depending on that of the drop, according to the difference of time occupied in the passage of the two portions, which thus proceed in parallel directions to the spectator's eye, after having been differently refracted and reflected within the drop. This difference varies at first, nearly as the square of the angular distance from the primitive rainbow: and if the first additional red be at the distance of 2° from the red of the rainbow, so as to interfere a little with the primitive violet the fourth additional red will be at the distance of nearly 2° more, and the intermediate colours will occupy a space nearly equal to the original rainbow. In order to produce this effect the drops must be about $\frac{1}{76}$ of an inch or .013 in diameter: it would be sufficient if they were between $\frac{1}{70}$ and $\frac{1}{80}$." &c.

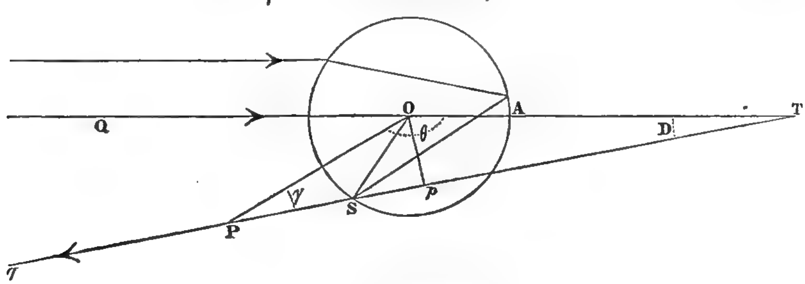
Dr Young does not explain the method by which he found the diameters of the rain-drops should be $\frac{1}{76}$ of an inch.

I regret my inability to deduce results perfectly rigorous by the method which I have followed on account of the complicated and transcendental nature of the relations between the quantities to be expressed, but I have pushed the mathematical part of the investigation to as close an approximation as the general discussion of the problem may require.

I have adopted the method, of first finding the caustic; because this and a very numerous class of interferences is produced, not by a separation of the original luminiferous surface into two separate surfaces, as in the cases ordinarily considered, but by a reduplication of the surface upon itself after reflection, or refraction, or both. In these cases, as I have shewn in a paper read before the Physical Section, at the meeting of the British Association at Cambridge, and published at Brussels in M. Quetelet's '*Correspondance Mathematique et Physique*,' there is an arête de rebroussement of the luminiferous surface at the caustic surface, or, in the usual sections a cusp in the section of the luminiferous surface at the caustic, and the former curve is always an involute of the latter. Having once found the caustic, this consideration enables us to proceed to the calculation of these complicated effects with a close approximation to the accurate result.

Proceeding first to find the expressions for the caustic when parallel rays have been twice refracted and once reflected in a transparent sphere, as in the primary rainbow, and using the ordinary mode of determining the caustic by considering it the locus of the intersections of consecutive rays.

Let $\phi = \angle$ of incidence,



$\phi' = \angle$ of refraction,

$D = \angle\, OTS$ = supplement of the angle of deviation of any ray (this angle is frequently itself called, incorrectly, the deviation),

ϕ_m and D_m the values of ϕ and D corresponding to the minimum deviation, or the maximum of its supplement D.

Let O be the centre of the sphere and origin of polar co-ordinates,

$$OP = \rho,$$

$$\angle\, POT = \theta;$$

let $QOAT$ be the ray incident perpendicularly on the sphere,

$qPSpT$ any other ray emergent at S,

$$OS = \text{radius} = r,$$

also $\angle\, OST = \angle$ of incidence,

$= \phi$, by property of a refracting sphere;

let also $\angle\, OPT = \gamma = \pi - (D + \theta)$,

then $Op = \rho \,.\, \sin\gamma = r \,.\, \sin\phi$;

$$\therefore\ \rho = \frac{r \,.\, \sin\phi}{\sin\gamma}.$$

Now when the point P is the intersection of consecutive rays, ρ and θ remain constant, whilst ϕ and γ vary;

$$\therefore\ 0 = r\,\frac{\sin\gamma\cos\phi - \sin\phi\cos\gamma\, d_\phi\gamma}{\sin^2\gamma},$$

and $d_\phi\gamma = -d_\phi D$;

$\therefore\ \tan\gamma = \tan\phi\, d_\phi\gamma$,

or $\tan(D+\theta) = \tan\phi\, d_\phi D$ .... (1).

But as shewn in elementary treatises on Optics,

$$D = 2(2\phi' - \phi),$$

also we have $\sin\phi = \mu\sin\phi'$;

$$\therefore\ d_\phi D = 2(2d_\phi\phi' - 1)$$

$$= 2\left(\frac{2}{\mu}\cdot\frac{\cos\phi}{\cos\phi'} - 1\right) \ \ldots\ldots\ (2).$$

Again, substituting for $\sin\gamma$ its value, we have

$$\rho = r.\sin\phi.\frac{\sqrt{1+\tan^2\phi\,(d_\phi\gamma)^2}}{\tan\phi d_\phi\gamma};$$

whence $\rho = r\sqrt{\dfrac{\cos^2\phi}{(d_\phi\gamma)^2}+\sin^2\phi}$

$$= r\sqrt{\frac{\cos^2\phi\cos^2\phi'}{\left\{2\left(\frac{2}{\mu}\cos\phi-\cos\phi'\right)\right\}^2}+\sin^2\phi} \quad \ldots\ldots\ldots\ldots\ldots (3).$$

This (3) with the equations (1) and (2) would suffice to eliminate ϕ and D, and give ρ in terms of θ and constants, if the transcendental relations of ϕ, ϕ' and θ did not prevent it: that expression would be the polar equation of the caustic for the primary bow: we may however trace this curve from equation (3).

Taking $\mu = \frac{4}{3}$, as is usually done for red light, we find two values of ϕ which give $\rho = r$,

namely, $\phi = 0$, and $\phi = \frac{\pi}{2}$.

Again ρ becomes $= \pm\infty$, when

$$\frac{2}{\mu}\cos\phi - \cos\phi' = 0,$$

$$\text{or } 4\cos^2\phi = \mu^2 - \sin^2\phi;$$

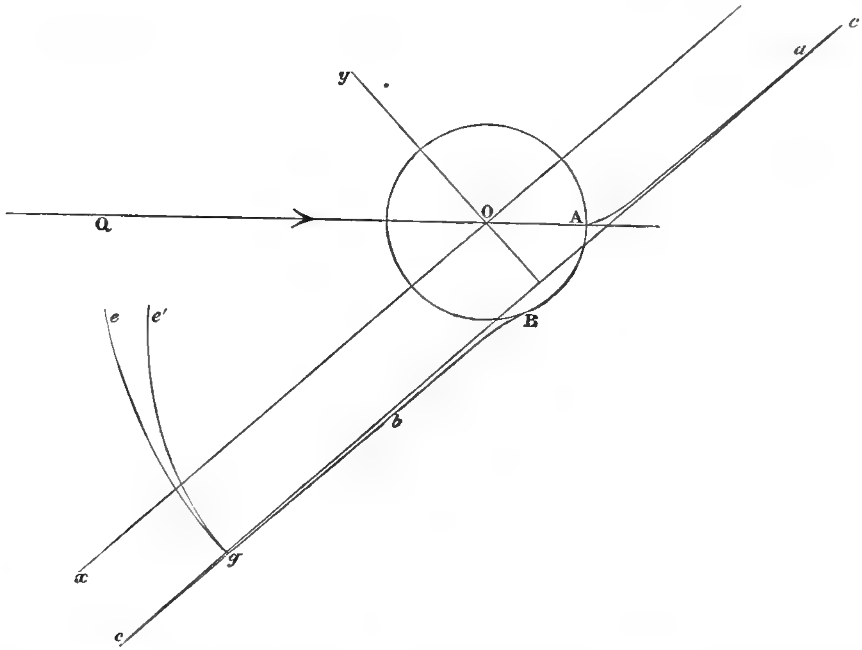
$$\therefore \cos\phi = \sqrt{\frac{\mu^2-1}{3}},$$

which, as seen in elementary books, is the angle of minimum deviation, the ray at this angle being an asymptote to the caustic.

The deviation diminishing from $\phi = 0$ to $\phi = \phi_m$ the intersections of consecutive rays are behind the sphere, but from $\phi = \phi_m$ to $\phi = \frac{\pi}{2}$ the deviation increases, and the intersections are in front. So that the two branches of the caustic are as at *Aa*, *Bb* in the Fig. 2. being perpendicular to the sphere at *A*, and tangential to it at *B*, and the line cc being the ray which has the minimum deviation and an asymptote to the two branches.

The section of the luminiferous surface at any position will be

Fig. 2.



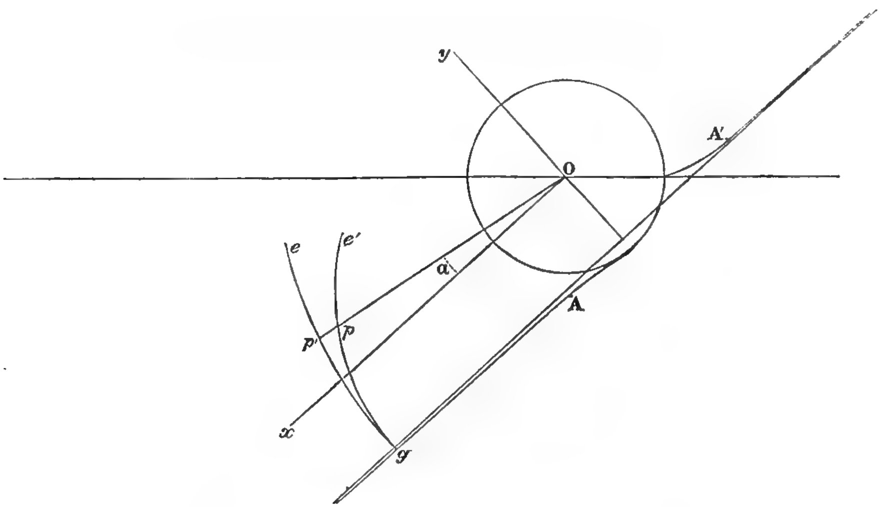
similar to ege', the branch $e'g$ being an involute of the caustic Bb, and eg of the caustic Aa, or more accurately the part of eg as far as the asymptote is the involute of the virtual branch Aa, and the remainder of the other branch.

For want of knowing the equation of the curve ege', I have used this approximation of considering the two branches near g for the small angular distances which we employ as coinciding with their osculating circles, at the given points.

Then finding ρ and ϕ, which correspond to this angular distance, and taking rectangular co-ordinates parallel and perpendicular to cc, we easily get the values of the co-ordinates (a, b) (a', b') of the centres of these osculating circles. Returning thence to polar co-ordinates for the branches *eg* and *e′g* to the *same* pole *O*, we find the space (pp') between the branches at the required point. The caustic in any observation of the rainbow may be considered at the eye as coinciding with its asymptote, as a spherule of water of $\frac{1}{70}$ inch diameter subtends no sensible angle at 1000 yards distance.

Let A be centre of circle osculating at p,

A' .... p',



a, b rectangular co-ordinates of A,

a', b' .... A',

Op = radius vector $= u$,

$Op' =$ .... $= u'$.

Also $\angle\, xOp = \alpha$.

Now if $(x-a)^2 + (y-b)^2 = R^2$ be equation of circle whose centre is A,

$(x'-a')^2 + (y'-b')^2 = R'^2$ .... A',

$$\text{then } x = u\cos\alpha, \qquad x' = u'\cos\alpha,$$

$$y = u\sin\alpha, \qquad y' = u'\sin\alpha,$$

$$\text{substituting } u^2 - 2(a.\cos\alpha + b.\sin\alpha)\,u + a^2 + b^2 = R^2;$$

$$\therefore\; u = a.\cos\alpha + b.\sin\alpha \pm \sqrt{R^2 - a^2 - b^2 + (a.\cos\alpha + b.\sin\alpha)^2}$$

$$= a\cos\alpha + b\sin\alpha \pm \left\{R - \frac{a^2 + b^2 - (a\cos\alpha + b\sin\alpha)^2}{2R}\right\} \text{ nearly.}$$

Similarly $u' = a'\cos\alpha + b'\sin\alpha \pm \left\{R' - \dfrac{a'^2 + b'^2 - (a'\cos\alpha + b'\sin\alpha)^2}{2R'}\right\}$;

$$\therefore\; pp' = u' - u = (a' - a)\cos\alpha + (b' - b)\sin\alpha \pm (R' - R + \&c.)$$

To establish the condition that the two branches are in contact at the cusp g, and remembering at the same time that, in an observation, the rain-drop is at a great distance, and that therefore R and R' are large quantities compared with a, b, a', b', we have

$$\text{when } \alpha = 0, \qquad u' - u = 0,$$

$$\text{and, } a' - a \pm (R' - R) = 0,$$

$$\text{substituting } u' - u = (a' - a)\cos\alpha + (b' - b)\sin\alpha - (a' - a) \text{ nearly.}$$

Or, since A' lies on the side of the negative x; therefore a' is negative; also $b > b'$, and they are also both negative, in our problem; therefore

$$u' - u = (a - a')(1 - \cos\alpha) - (b - b')\sin\alpha$$

$$= 2(a - a')\sin^2\frac{\alpha}{2} - (b - b')\sin\alpha.$$

To put this into a form for use in calculation, let

$$a = a_1, \qquad a' = -a_2, \qquad b = -b_1, \qquad b' = -b_2,$$

$$\text{then } u' - u = 2(a_1 + a_2)\sin^2\frac{\alpha}{2} + (b_1 - b_2)\sin\alpha.$$

The quantities $a_1\, a_2,\, b_1\, b_2$ are to be calculated from the values found for ρ and θ corresponding to any particular value of α, which I have taken, for example, as the angular distance from the red to the purple;

$$\therefore\; \alpha = 1^\circ.\,46', \text{ we have also } D = D_m - \alpha = 40^\circ.\,16'.$$

Finding the two values of ϕ, the one above and the other below the angle of minimum deviation for this value of a, and then deducing those corresponding for ρ and θ, I find

$$\phi = 50^\circ, \quad \rho = 1.961297r, \quad \theta = -17^\circ . 16^\circ . 33'',$$

$$\phi = 67^\circ . 55', \quad \rho = 1.272036r, \quad \theta = 92^\circ . 58' . 31'',$$

$$a_1 = .8996004r, \quad b_1 = .8993300r,$$

$$a_2 = 1.781031r, \quad b_2 = .8213489r.$$

Applying these to the above expression, we find

$$u' - u = .0036783r,$$

and that the second maximum of the red may occur at the place of the first violet, we must have, if λ be the interval of the luminiferous surfaces for red light,

$$u' - u = \lambda = .0000256 \text{ inch},$$

and the diameter of the drop $= 2r$,

$$= 2\,\frac{.0000256}{.0036783}$$

$$= \frac{1}{72} \text{ inch nearly},$$

which does not differ greatly from Dr Young's result $\frac{1}{76}$, for I have taken $a = 1^\circ . 46'$, and he has taken it $= 2^\circ$.

We see also that if r were very small, as in mist and in ordinary clouds not producing rain, probably much less than $\frac{1}{1000}^{\text{th}}$ of an inch, then the primary red would extend far beyond the violet's place, and so likewise with the other colours, and we ought to expect a bow with colour scarcely perceptible, and such is recorded as the fact.

In the case calculated the primary purple mixing with the second red would give a reddish purple, which agrees with an observation I made on a very splendid display on the 5th of June 1834, immediately after a heavy thunder-shower. I saw three sets of purples at the summit of the bow, and I have this memorandum: 'but the purple of the principal bow was evidently mingled with the red of the second, and I believed the purple of the second also to be mixed with the red, and perhaps the orange also of the third bow. The three bows thus considered were also of decreasing breadths as fringes in diffraction.'

But an observation by Dr Langwith, quoted by Dr Young, proceeds much more into details than the one I made as above, and the display must have been still more splendid. He says, "You see we had here four orders of colours, and perhaps the beginning of a fifth: for I make no question but that what I call the purple, is a mixture of the purple of each of the upper series with the red of the next below it and the green a mixture of the intermediate colours."

Again he has this important and philosophical remark: "There are two things which well deserve to be taken notice of, as they may perhaps direct us, in some measure, to the solution of this curious phenomenon. The first is, that the breadth of the first series so far exceeded that of any of the rest, that as near as I can judge, it was equal to them all taken together. The second is that I have never observed these inner orders of colours in the lower parts of the rainbow, though they have often been incomparably more vivid than the upper parts, under which the colours have appeared. I have taken notice of this so very often, that I can hardly look upon it to be accidental; and if it should prove true in general, it will bring the disquisition into a narrow compass; for it will shew that this effect depends upon some property which the drops retain, whilst they are in the upper part of the air, but lose as they come lower, and are more mixed with one another."

The first question, as to the decreasing breadths, is answered by Dr Young in the previous quotation, and is an effect familiar to those who have studied Physical Optics. The second also receives a complete solution from considering the expression we have obtained, and the state of rain in falling from a cloud. For though the drops were of small size on leaving the cloud, and such as to produce the supernumerary bows, yet as they fall down, having different velocities from the higher and lower parts of the cloud, they must come in contact, and gradually form large drops, and thus their diameters become at length too great to give an appearance of supernumerary bows. There are other points still, however, which theory will guide us to look for in future; thus if the drops are larger, the second maximum of the red may happen in the green's place, and thus the green be diluted with white light whilst the orange and yellow would be brilliant, but the second maxima of these latter falling in the blue and purple, these colours would again be diluted. In such bows the red, orange and yellow, would form the most striking part. I am not aware that there are any recorded observations relating to this or similar effects.

If we can judge by observation where the second series of maxima commence, we shall be able to calculate the size of the drops forming the bow.

There are observations on record of supernumerary bows attending the secondary rainbow: their solution is perfectly similar to the one given for the primary one.

The comparison of the results of interference with the common explanation of the rainbow, required that the plan followed should be in accordance with the undulatory theory of light. If the effects were considered to be those due to a difference of $\frac{1}{2}$ an interval in the paths of the rays at the cusp, the results would be similar, only modified a little in quantity.

I have also taken, as ordinarily is done, that $\mu = \dfrac{4}{3}$ for red rays, although Fraunhofer's observations shew it to belong to the letter *D*, nearly, and the middle of the orange.

Again, I have taken the interval λ, as given from Sir Isaac Newton's measures, although unpublished measures of my own confirm those of M. Fresnel in shewing that they are somewhat too small.

VI. *On the Dispersion of Light, as explained by the Hypothesis of Finite Intervals.* By P. Kelland, Esq., B.A. *Fellow of Queens' College.*

[Read *Feb.* 22, 1836.]

PRELIMINARY OBSERVATIONS.

There is no phenomenon in Optics more familiar and prominent than that a beam of solar light is composed of differently coloured rays, each endued with its own peculiar properties.

It was first satisfactorily proved by Newton, that the parts are distinct from each other, and are susceptible of separation and recomposition, so that any particular colour can be examined apart from the rest. At a very recent period Wollaston and Fraunhofer have examined more intimately the constitution of a beam of ordinary white light, and from the accurate measures of the latter, we are put in possession of a series of data by which, in a variety of substances, the position of each particular portion of the beam is accurately defined. Having then before us such observations, we are in a state to proceed to an explanation not merely of facts broadly and generally stated, but of the minutest details, and most trivial deviations from the rough outline.

It might perhaps be more easy to proceed on the hypothesis which Newton himself advanced, as it would be a matter of little difficulty to assign such forces or inertia to the particles of light, combined with the constant attractive or repulsive forces of the material particles composing a refracting substance, as should lead to results in unison with those of observation. There are, however, a variety of complex phenomena, to which scarcely any modification of Newton's hypothesis will apply, whilst that of undulations accounts for them in the clearest and most satisfactory manner. The phenomena of dispersion for a considerable time stood almost alone in the way of this theory, and appeared incompatible with its principles. It was assumed, and with good reason, that colour was dependent on the lengths of a wave, whilst the velocity of transmission determined the refractive index of the medium. It became then evident that the theory was at fault, unless the velocity of transmission within refracting media could be shewn to depend on the length of a wave. What was still worse, from the appearance of the stars we were forced to allow, that light of all colours was transmitted uniformly through vacuum.

Several suggestions were made, which, if they did not remove the difficulty, tended at least to clear the theory from suspicion of incapability, and to shift the ground of attack from the principles themselves to our power in applying them. Thus Mr Airy, reasoning from analogy, observes: "We have every reason to think that a part of the velocity of sound depends on the circumstance that the *law of elasticity* of the air is altered by the *instantaneous* developement of latent heat on compression, or the contrary effect on expansion. Now if this heat required *time* for its developement, the quantity of heat developed would depend on the time during which the particles remained in nearly the same relative state; that is, on the time of vibration. Consequently the law of elasticity would be different for different times of vibration, or for different lengths of waves: and therefore the velocity of transmission would be different for waves of different lengths. If we suppose some cause, which is put in action by the vibration of the particles, to affect in a similar manner the elasticity of the medium of light, and if we conclude the degree of developement of that cause to depend on time, we shall have a sufficient explanation of the unequal refrangibility of differently coloured rays."

These observations are important, inasmuch as they remove from the Undulatory Theory the imputation of being inadequate to account for dispersion, at the same time I think that simple as they may appear at a first glance, and satisfactory as they undoubtedly are to a certain extent, it will be found a difficult task to pursue them into detail, even in the case of sound. We know little or nothing of the laws which regulate the developement of heat, *which affect the velocity of light*, at least if we adopt the hypothesis of molecular radiation, and have thus only shifted our difficulty without removing it. If on the other hand, we choose to regard heat as an effect consequent on the alteration of the positions of the attractive or repulsive particles within a medium (which seems reasonable from some recent experiments on the Polarization, &c. of Heat), then, by analogy, Mr Airy's hypothesis amounts in fact to supposing the particles endued with attractive or repulsive energies, influenced by the particular positions into which they are thrown, and varying with the change of these positions, to the action of which all the effects are assigned.

The great obstacle to a simple explanation of this subject appears to have arisen from the fact, that theorists generally have not divested themselves of the idea of motion directly: *velocity* has been substituted for force, and *wave* for change of force.

It occurred to me about two years since, that if we could deduce a simple equation of motion on the supposition that the particles of a medium are at a finite distance from each other, we might arrive at results very different *in form* from those usually adopted. In fact it appeared probable that the velocity might depend on the positions into which the particles should arrange themselves, and thus might be affected by the length of a wave.

Such a formula I actually obtained, and deduced from it the necessary result, that the square of the velocity is represented by a series of terms of which $c\left(\frac{\sin\theta}{\theta}\right)^2$ is a type. There was, however, one point in my analysis which I regarded as fatal to the whole; namely, that having a function involving the distance between two consecutive particles, and the space through which a particle is disturbed, I had expanded it in terms of the ratio of the latter quantity to the former.

It appeared to me at the time doubtful whether this series might not be a diverging one, and thence it became extremely probable that the existence of the function in form above, was owing to the absence of terms omitted in this expansion. Lately M. Cauchy's Memoir on the same subject has fallen into my hands, and an opportunity has been afforded me of comparing his results with my own. The comparison has shewn me that although in some points we differ, in the essentials of principle, at least, we coincide.

Whatever difficulty may attach itself to my hypothesis as to the sphere of action of the particles, will attach itself equally to his, as they are identical: at least I have reduced mine to the same state as his.

Although there were many points of coincidence in our processes, there were not a few of difference, to many of which the present Memoir is indebted; as I have not scrupled to adopt anything which would tend either to simplify or generalize the results; my object being by no means to regard my formula as a re-discovery of what M. Cauchy had published in 1830, but rather to attempt an improvement on what is already known. I may be allowed to add, that M. Cauchy's equations, owing to his proceeding with great generality at first, and only adding new hypotheses to simplify them when they became perfectly unwieldy, are so buried in symbols, that a person must possess no ordinary sagacity to give to them *any* interpretation. And further, there are some points in which the result is more general than the hypothesis would render necessary.

The plan which I have pursued is to simplify the equations as I proceed, and not to retain any result which admits of reduction.

SECTION I.

Analytical Investigation.

THE problem about which we are to occupy ourselves, is the motion of any system of material particles, exerting on each other forces varying according to any function of the distance. It would be a useless generalization, in the present state of analysis, to proceed at once to the solution of this problem without any further restrictions, for even should we succeed in integrating the resulting equations, whether by approximation or any other method, we should at length be obliged to have recourse to particular hypotheses in order to interpret our results.

I propose then to make the following hypotheses:

1. That the distance between the particles is sufficiently large compared with their sphere of motion, to allow the square of the latter quantity to be omitted compared with that of the former.

2. That the disposition of the particles is a disposition of symmetry. It may serve to fix our ideas, if we consider them symmetrically situated with respect to the three co-ordinate planes; as, for instance, arranged in the angular summits of cubes, whose edges are parallel to the co-ordinate axes, and whose centre is the origin. This is, however, merely stated as something to guide us, since we *must* suppose, in whatever manner it can be accomplished, that the disposition is *perfectly* symmetrical. On these two hypotheses, which, virtually at least, are M. Cauchy's hypotheses, I shall now proceed to determine the equations of motion.

Let x, y, z be the co-ordinates of any particle P in its state of rest, the origin being taken at pleasure, and the axes any axes of symmetry. $x + \delta x$, $y + \delta y$, $z + \delta z$ those of any other particle Q, which lies within the sphere of sensible attraction to P; r the distance PQ; $x + \alpha$, $y + \beta$. $z + \gamma$ the co-ordinates of P after any time t from the beginning of the motion; $x + \alpha + \delta x + \delta\alpha$, $y + \beta + \delta y + \delta\beta$, $z + \gamma + \delta z + \delta\gamma$ those of Q at the same time; $r + \rho'$ the corresponding value of the distance PQ. Let the accelerating force of Q on P at the distance PQ be represented by the function $(r + \rho') . \phi(r + \rho')$. Resolving this attraction parallel to the axis of x, it gives $\phi(r + \rho') . (\delta x + \delta\alpha)$, whence we obtain

$$\frac{d^2\alpha}{dt^2} = \Sigma . \phi(r + \rho') (\delta x + \delta\alpha),$$

the symbol Σ having reference to the sum of similar expressions, taken for all the particles whose action on P is sensible. By expansion we obtain

$$\phi(r+\rho') = \phi(r) + F(r) \cdot \rho' + \ldots\ldots\ldots\ldots$$

$F(r)$ being the differential coefficient of $\phi(r)$ taken with respect to r,

$$\text{but } (r+\rho')^2 = (\delta x + \delta a)^2 + (\delta y + \delta \beta)^2 + (\delta z + \delta \gamma)^2;$$

$$\therefore\ r^2 + 2r\rho' = r^2 + 2\ (\delta x \delta a + \delta y \delta \beta + \delta z \delta \gamma),$$

omitting powers of ρ and δa, $\delta \beta$, $\delta \gamma$;

$$\therefore\ \rho' = \frac{1}{r}(\delta x \delta a + \delta y \delta \beta + \delta z \delta \gamma),$$

and by substituting this value in the above equation it gives

$$\frac{d^2 a}{dt^2} = \Sigma . \left\{\phi r + \frac{Fr}{r}(\delta x \delta a + \delta y \delta \beta + \delta z \delta \gamma) + \ldots\ldots\ldots\right\}(\delta x + \delta a)$$

but $\Sigma \phi(r) . \delta x$ is manifestly the accelerating force, resolved parallel to x, on the particle P in its state of rest, and consequently is equal to zero; we have then

$$\frac{d^2 a}{dt^2} = \Sigma . \left\{\phi r . \delta a + \frac{Fr}{r}(\delta x^2 \delta a + \delta x + \delta y \delta \beta + \delta x \delta z \delta \gamma)\right\}$$

which we will call equation (1).

Previous to the solution of this equation in its general form, let us examine what it becomes in that particular case where those particles only which are in the *immediate* vicinity of P sensibly affect its motion, an hypothesis which is tantamount to supposing all the particles very near each other, as it is manifest that on the latter supposition the sum of the forces exerted by those particles nearly in contact with P, is beyond all comparison greater than those of particles at a finite interval from it. Proceeding on this supposition, we obtain

$$\delta a = \frac{da}{dx}\delta x + \frac{da}{dy}\delta y + \frac{da}{dz}\delta z$$

$$+\frac{d^2a}{dx^2}\cdot\frac{\delta x^2}{2}+\frac{d^2a}{dy^2}\cdot\frac{\delta y^2}{2}+\frac{d^2a}{dz^2}\,\frac{\delta z^2}{2}$$

$$+\frac{d^2a}{dxdy}\delta x\delta y+\frac{d^2a}{dxdz}\delta x\delta z+\frac{d^2a}{dydz}\delta y\delta z+\ldots\ldots\ldots\ldots$$

and similar expressions for $\delta\beta$ and $\delta\gamma$.

But it is evident that the sum of a series of terms of which one factor in each is $\phi(r)$ and the other $\delta x^m.\delta y^n.\delta z^p$, where $m+n+p$ is an odd integer, will be identically equal to zero; since if m, for instance, is odd, we shall have, for any particular values of r, δy, δz, two equal values of δx, the one positive, and the other negative, and one of the quantities m, n, p *must* be an odd number. Substituting, therefore, the above values of δa, $\delta\beta$, $\delta\gamma$ in equation (1), and omitting quantities which vanish identically; we get

$$\frac{d^2a}{dt^2}=\tfrac{1}{2}\Sigma.\phi r\left\{\frac{d^2a}{dx^2}\delta x^2+\frac{d^2a}{dy^2}\delta y^2+\frac{d^2a}{dz^2}\delta z^2\right\}\ \ldots\ldots\ldots\ (2),$$

and again $\Sigma.\phi(r)\,\delta x^2=\Sigma\phi(r)\delta y^2=\Sigma\phi(r)\delta z^2=2n^2$,

writing n^2 for abbreviation;

$$\therefore\ \frac{d^2a}{dt^2}=n^2\left\{\frac{d^2a}{dx^2}+\frac{d^2a}{dy^2}+\frac{d^2a}{dz^2}\right\}$$

$$\frac{d^2\beta}{dt^2}=n^2\left\{\frac{d^2\beta}{dx^2}+\frac{d^2\beta}{dy^2}+\frac{d^2\beta}{dz^2}\right\}$$

$$\frac{d^2\gamma}{dt^2}=n^2\left\{\frac{d^2\gamma}{dx^2}+\frac{d^2\gamma}{dy^2}+\frac{d^2\gamma}{dz^2}\right\},$$

three equations of remarkable simplicity and elegance; of which the following are evidently solutions:

$$a=a\cos k\,\{nt-(ex+fy+gz)\},$$

$$\beta=b\cos k\,\{nt-(ex+fy+gz)\},$$

$$\gamma=c.\cos k\,\{nt-(ex+fy+gz)\},$$

subject to the restriction that $e^2+f^2+g^2=1$. We can easily get rid of this restriction by writing $\cos\theta$ for e, $\cos\phi$ for f, and $\cos\psi$ for g.

θ, ϕ and ψ being the angles which a straight line makes with the co-ordinate axes of x, y, z. For simplicity put

$$x \cos\theta + y \cos\phi + z \cos\psi = \rho,$$

ρ being the projection on the line making angles θ, ϕ, ψ with the axes, of the distance OP of the point P from the origin: the above equations then become

$$\alpha = a \cos k\,(nt - \rho),$$
$$\beta = b \cos k\,(nt - \rho),$$
$$\gamma = c \cos k\,(nt - \rho).$$

It is true a more general solution of these equations would have been obtained by assuming the values of k, θ, ϕ, ψ different for each of the three equations, but as the *complete* solution will consist of a *series* of terms similar in form to the above, it is sufficient for our purpose to exhibit that term above which has the same period for each of the three directions, and which consequently corresponds to one and the same undulation.

From these equations it is manifest that the same state of motion recurs when $k\rho$ is increased by $2n\pi$, and consequently $\frac{2\pi}{k}$ is the length of a wave; also the motion at the end of $t + \Delta t$ is the same at the point $\rho + \Delta\rho$ as at the time t at ρ, when $n\Delta t = \Delta\rho$, whence the velocity of transmission parallel to $\rho = \frac{d\rho}{dt} = n$.

It may be worth while to notice here that the proposition which we have considered assumes the velocity of transmission to be the same in all directions: in general, however, this will not be the case, the direction of transmission being defined by the simultaneous transmission of a *system* of waves, and the velocity will have reference to that direction only; but as $\frac{d\rho}{dt}$ is independent of any particular direction, and depends only on the nature of the substance, it must be either the velocity of transmission itself, or the velocity in a direction making a constant angle with that of transmission, and consequently varies as that velocity. And for the same reason ρ must be either in the direction of transmission, or making a *constant* angle with it, and as the introduction of a constant cannot in any manner affect our results, we may consider ρ and $\frac{d\rho}{dt}$ respectively as defining the place and velocity of the wave at the end of the time t.

Another remark is also important, that since from the constitution of the medium it is indifferent in what direction the axes of co-ordinates are taken, all the functions which we may introduce involving $\delta\rho$ must finally turn out independent of θ, ϕ and ψ, so that we might at once suppose the direction of transmission to be the axis of y, and put δy for $\delta\rho$; this, however, I shall not do, as it does not appear necessary, and it is convenient to retain the symbol ρ, on other accounts to be noticed hereafter. The above remark will be mainly useful in pointing out to us what are the quantities to be rejected in our equations of motion.

Let us now take as the solution of equation (1) the form we have obtained from equation (2), which is perfectly allowable, since the latter is only a particular case of the former: the quantities n and k are of course not necessarily the same for both.

Put the solution under the form

$$\alpha = a\cos(ct - k\rho);$$

$$\therefore\ \delta\alpha = a\cos(ct - k\rho - k\delta\rho) - a\cos(ct - k\rho)$$

$$= -a\cos(ct - k\rho)\,.\,(1 - \cos k\delta\rho) + a\sin(ct - k\rho)\sin k\delta\rho,$$

where $\delta\rho = \delta x\cos\theta + \delta y\cos\phi + \delta z\cos\psi$ is the projection of the distance PQ on the line OP.

By the substitution of this expression for $\delta\alpha$, and analogous ones for $\delta\beta$ and $\delta\gamma$; our equation (1) becomes

$$\frac{d^2\alpha}{dt^2} = -\,2\alpha\Sigma\phi(r)\sin^2\frac{k\delta\rho}{2} + a\sin(ct - k\rho)\Sigma\phi(r)\sin k\delta\rho$$

$$+\,\Sigma\frac{F(r)}{r}\,\delta x^2\left\{-\,2\alpha\sin^2\frac{k\delta\rho}{2} + a\sin(ct - k\rho)\sin k\delta\rho\right\}$$

$$+\Sigma \frac{F(r)}{r} \delta x \delta y \left\{-2\beta \sin^2 \frac{k\delta\rho}{2} + b \sin (ct-k\rho) \sin k\delta\rho\right\}$$

$$+\Sigma \frac{F(r)}{r} \delta x \delta y \left\{-2\gamma \sin^2 \frac{k\delta\rho}{2} + c \sin (ct-k\rho) \sin k\delta\rho\right\};$$

now it is manifest that $\Sigma\phi(r) \sin k\delta\rho = 0$; and also because $\sin (ct-k\rho)$ is independent of Σ, and the term $\Sigma \frac{F(r)}{r} \delta x \delta y \sin k\delta\rho$ is of an odd order, wherever there is a positive term, there will be a corresponding negative one: the whole expression denoted by the symbol Σ in this term will therefore be identically equal to zero.

Precisely the same reasoning applies to all similar expressions.

Further as regards the term $\Sigma \frac{F(r)}{r} \delta x \delta y \sin^2 \frac{k\delta\rho}{2}$, bearing in mind the remark above made with respect to ρ, it is manifest that the part $\Sigma \frac{F(r)}{r} \sin^2 \frac{k\delta\rho}{2} \delta y$ will have one and the same value, for two equal values of δx with different signs: thus, to assist the conception referring to the cube as at the commencement of this paper, the point P being at its centre, suppose a particle at each of the two upper corners of the face on which you are looking, and y vertical, then the expression $\Sigma \frac{F(r)}{r} \sin^2 \frac{k\delta\rho}{2} \delta y$ is the same for each of them, but δx in one corresponds to $-\delta x$ in the other, and the sum of the above function for two such particles vanishes. It is clear, therefore, that this expression $\Sigma \delta x \delta y \frac{F(r)}{r} \sin^2 \frac{k\delta\rho}{2}$, and all analogous ones are identically equal to zero.

Our equations then become much reduced, and assume the remarkably simple form,

$$\frac{d^2\alpha}{dt^2} = -\alpha\, 2\Sigma \left\{\phi r + \frac{F(r)}{r} \delta x^2\right\} \sin^2 \frac{k\delta\rho}{2} = -n^2\alpha,$$

$$\frac{d^2\beta}{dt^2} = -\beta\, 2\Sigma \left\{\phi r + \frac{F(r)}{r} \delta y^2\right\} \sin^2 \frac{k\delta\rho}{2} = -n^2\beta,$$

$$\frac{d^2\gamma}{dt^2} = -\gamma\, 2\Sigma\,\{\phi(r) + \frac{F(r)}{r}\,\delta z^2\}\,\sin^2\frac{k\delta\rho}{2} = -n^2\gamma,$$

$$\text{if } n^2 = 2\Sigma\,\{\phi(r) + \frac{F(r)}{r}\,\delta x^2\}\,\sin^2\frac{k\delta\rho}{2},$$

it being evident that

$$\Sigma\,\{\phi(r) + \frac{F(r)}{r}\,\delta x^2\} = \Sigma\,\{\phi(r) + \frac{F(r)}{r}\,\delta y^2\} = \Sigma\,\{\phi(r) + \frac{F(r)}{r}\,\delta z^2\}.$$

Now it must be observed that we have not deduced the above equations *directly* from the equations of motion; but have obtained them by first solving for one particular case, and assuming that the same *form* holds in the general one: our solution is

$$\alpha = a\cos(nt - k\rho)$$

$$\beta = b\cos(nt - k\rho),$$

$$\gamma = c\cos(nt - k\rho),$$

n being *now* that given by the equation

$$n^2 = 2\Sigma\,\{\phi(r) + \frac{F(r)}{r}\,\delta x^2\}.$$

These results appear to be very simple in their form, and recommend themselves from the readiness with which they can be applied. It is true, we have not obtained them on a general hypothesis, but I think we may venture to say they rest on one which carries with it an air of probability; and I confess there seems more difficulty to conceive an hypothesis different from this for uncrystallized media, than to concede this. It is, moreover, the same which M. Cauchy adopts, but the results obtained differ in one especial point, viz. that his assume, and are of so general a form, that little construction beyond the explanation of dispersion can be put upon them. Professor Powell has, it is true, deduced from them the expression $H\dfrac{\sin\theta}{\theta}$ for the velocity. I shall make no remark on this deduction, as it arose from the simple consideration of one attracting particle, which is too limited to be regarded as even an approximation to a general result. I shall merely observe that in the sequel, owing to the negative value of one of the terms (l) there adopted, it is clear from experiment that the above form is incorrect.

It is true, some subsequent hypothesis might be necessary to adapt the formula as we have it to all cases, but for the present we have a form as simple as possibly can be obtained, and whose interpretation will be a matter of little difficulty. Before, however, I proceed to such interpretation, it may be useful to examine how it applies to the known dispersions of a number of glasses and other substances, since, unless it has some pretensions to supply us with results coinciding with those of observation, it can have little claim on our notice.

SECTION II.

Examination and Illustration of the Formula.

LET λ represent the length of a wave; v the velocity of transmission.

Now from the equation

$$\alpha = a \cos (nt - k\rho) \text{ we deduce}$$

$$\alpha = a \cos \left\{nt - \left(k\rho + \frac{2\pi}{k}\right)\right\},$$

or the same state recurs after intervals of $\frac{2\pi}{k}$, which is consequently the length of a wave, or λ.

Also we obtain

$$\alpha = a \cos \{n(t + \Delta t) - k(\rho + \Delta\rho)\},$$

wherever $n.\Delta t = k.\Delta\rho$;

$$\therefore\ v^2 = \frac{n^2}{k^2} = 2\Sigma\left\{\phi(r) + \frac{F(r)}{r}\delta x^2\right\}\frac{\sin^2\frac{1}{2}k\delta\rho}{k^2}$$

$$= 2\Sigma\left\{\phi(r) + \frac{F(r)}{r}\delta x^2\right\}\frac{\sin^2\frac{\pi\delta\rho}{\lambda}}{\left(\frac{2\pi}{\lambda}\right)^2}.$$

The first remark suggested by this equation is, that when the distance between the particles is exceedingly small, the values of $\delta\rho$ for which $\Sigma\left(\phi r + \frac{Fr}{r}\delta x^2\right)$ will have a sensible magnitude, must be inconsiderable: if they are so small that $\left(\frac{\pi\delta\rho}{\lambda}\right)$ may be neglected, we shall have

$$v^2 = \frac{1}{2}\Sigma\left\{\phi(r) + \frac{F(r)}{r}\delta x^2\right\}\delta\rho^2,$$

which coincides with that obtained from equation (2), and the velocity is independent of the length of the wave.

If this reasoning do not appear entirely conclusive, I shall in the sequel offer another argument to shew that, assuming the velocity independent of the length of the wave, the particles of æther (by which name we designate the medium of light) are nearer each other than in dispersive media.

Now it is a well known fact, that the direction in which a star appears to the eye is that of the diagonal of a parallelogram whose sides represent in magnitude and direction the velocity of the Earth and of light. It is manifest then that if the velocity of light in vacuo varied considerably by the variation of the colour, a star would be stretched out into a spectrum and not appear a point as it actually does. This is one of the old objections to the theory, but it is immediately removed by the considerations above, introducing, however, as a necessary consequence, the condition that the particles of æther are very much nearer to each other *in vacuo* than in refracting media, and of course the density greater.

With respect to the velocity of transmission in this case, the second factor of v^2 is clearly a maximum, and as the function $\phi(r)$ is doubtless some *inverse* function of the distance, we have the first factor also very large, and have therefore reason to conclude that the velocity is a maximum.

This consideration then to which we are driven by the observed phenomena is perfectly consistent with itself, and it does not appear to bear about it any *à priori* absurdity. That it will be found a matter of some difficulty to adapt it to the explanation of certain phenomena which are reduced to mathematical computation by the contrary supposition, is no argument against its validity.

But let us examine how this result works when applied to the explanation of other phenomena. It is clearly reasonable to conclude, that since the action of a material substance on the particles of æther causes a diminution of their density, that diminution will increase in proportion as the number of material particles increases, and consequently for the same kind of material particles the density of the æther bears some inverse ratio to that of the substance.

Suppose the particles of glass through which light is transmitted, compressed by a force acting parallel to the axis of z: this, according to the above conclusion, would lead us to infer that the particles of æther were more dilated in the direction of z, than in other directions.

Now it is evident that whatever law of force we conceive as the true one, it must diminish as the distance increases, and consequently $F(r)$ must be negative.

Also if we consider the vibrations transversal (an hypothesis which will be noticed hereafter), we have the square of the velocity of transmission along the axis of x, of a vibration parallel to z, equal to

$$\tfrac{1}{2}\Sigma\left\{\phi(r)+\frac{F(r)}{r}\delta z^2\right\}\delta x^2 \text{ nearly,}$$

whilst the analogous expression for a wave transmitted along the axis of z is

$$\tfrac{1}{2}\Sigma\left\{\phi(r)+\frac{F(r)}{r}\delta x^2\right\}\delta z^2,$$

which, both on account of $F(r)$ being negative, and of every δz being greater than the corresponding δx or δy, is greater than the former; whence we conclude that a wave is transmitted with the greatest velocity when it passes in the direction of the pressure.

The piece of glass will consequently be analogous to a negative crystal, (Vid. *Ency. Metrop.* Light. Art. 803.) the direction of pressure corresponding to that of the axis of the crystal.

In like manner, had the glass been dilated the velocity would have been least along the axis, and the properties of a positive crystal would have been exhibited. Now by reference to the Transactions of the Royal Society for 1816, p. 158, Sir David Brewster informs us that he arrived at the following conclusion by experiment:

"When a piece of glass is under the influence of a compressing force, its structure is the same as that of one class of doubly refracting crystals, including calcareous spar, beryl, &c. (negative); but when it is under the influence of a dilating force, its structure is the same as that of the other class of doubly refracting crystals, including sulphate of lime, quartz, &c. (positive)."

Here then our formula gives results coincident with experiment.

This can, however, be considered only as a mere test of the general accuracy of our deductions; and a more satisfactory mode of examination will be obtained when we apply them to those substances of which we know accurately the refractive indices for different colours.

We have seen that the square of the velocity is equal to

$$\tfrac{1}{2}\Sigma\left\{\phi(r)+\frac{F(r)}{r}\delta x^2\right\}\frac{\sin^2\dfrac{\pi\delta\rho}{\lambda}}{\dfrac{\pi^2}{\lambda^2}}$$

$$=\tfrac{1}{2}\Sigma\left\{\left(\phi(r)+\frac{F(r)}{r}\delta x^2\right)\left(\delta\rho-\frac{\pi^2}{6}\frac{\delta\rho^3}{\lambda^2}+\frac{\pi^4\delta\rho^5}{120\lambda^4}\right)^2\right\}$$

$$=\tfrac{1}{2}\Sigma\left\{\left(\phi(r)+\frac{F(r)}{r}\delta x^2\right)\left(\delta\rho^2-\frac{\pi^2}{3\lambda^2}\delta\rho^4+\frac{2}{45}\frac{\pi^4}{\lambda^4}\delta\rho^6\right)\right\}.$$

Now the refractive index μ varies inversely as the velocity; hence we may put $\frac{1}{\mu^2}$ under the form

$$\frac{1}{\mu^2} = p - \frac{100{,}000 \, . \, q}{\lambda^2} + \frac{1{,}000{,}000 \, . \, l}{\lambda^4} \text{ nearly,}$$

$$\text{where } p = A\Sigma \left\{\phi(r) + \frac{F(r)}{r}\,\delta x^2\right\}\delta\rho^2,$$

$$(100{,}000)^2\, q = \frac{\pi^2}{3} A\Sigma \left\{\phi(r) + \frac{F(r)}{r}\,\delta x^2\right\}\delta\rho^4,$$

$$(100{,}000)^4\, l = \frac{2\pi^4}{45} A\Sigma \left\{\phi(r) + \frac{F(r)}{r}\,\delta x^2\right\}\delta\rho^6,$$

A being some constant factor.

I have extracted from the article on Light in the Encyclopædia Metropolitana the values of λ for the seven fixed lines, as determined by M. Fraunhofer, and have also taken the values of μ given by the same author for seven kinds of glass and three fluids, water, solution of potash, and spirit of turpentine.

For each of these substances we shall have seven equations between p, q and l, with the known values of μ and λ. I have always determined p, q, l from the equations given by the lines (B), (E) and (H), and by substituting the values of the last two quantities, q and l in the other equations, have determined from them values of p.

The verification of our formula consists in the near coincidence of the values of p with each other. The rest will be easily understood from the following tables.

Table I. contains the values of λ, $\frac{100{,}000}{\lambda^2}$ and $\frac{1{,}000{,}000}{\lambda^4}$ for the different fixed lines expressed in parts of a Paris inch: they are multiplied by some power of 10 merely to avoid decimals.

	B	*C*	*D*	*E*	*F*	*G*	*H*
$\lambda \cdot 10^8$	2541	2422	2175	1945	1794	1587	1464
$\frac{1,0000}{\lambda^2}$	15488	17047	21139	26434	31071	39705	46657
$\frac{1,000,000}{\lambda^4}$	23987	29060	44685	69875	96541	157650	217690

The next Table contains the values of μ to four places of decimals. I have not considered the succeeding places determined with sufficient accuracy to admit of our reasoning on them, as will likewise strike any person who examines the Tables wherein the results of *two* experiments on the same substance are given.

Refracting Medium.	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Flint Glass, No. 13.	1.6277	1.6297	1.6350	1.6420	1.6483	1.6603	1.6710
Flint Glass, No. 23.	1.6266	1.6285	1.6337	1.6405	1.6468	1.6588	1.6697
Flint Glass, No. 30.	1.6236	1.6255	1.6306	1.6373	1.6435	1.6554	1.6660
Flint Glass, No. 3.	1.6020	1.6038	1.6035	1.6145	1.6200	1.6308	1.6404
Crown Glass, letter M.	1.5548	1.5559	1.5591	1.5631	1.5667	1.5735	1.5795
Crown Glass, No. 9.	1.5258	1.5268	1.5296	1.5330	1.5360	1.5416	1.5466
Crown Glass, No. 13.	1.5243	1.5253	1.5280	1.5314	1.5343	1.5399	1.5447
Oil of Turpentine.	1.4705	1.4715	1.4744	1.4783	1.4817	1.4882	1.4939
Solution of Potash.	1.3996	1.4005	1.4028	1.4056	1.4081	1.4126	1.4164
Water.	1.3310	1.3317	8.3336	1.3358	1.3378	1.3413	1.3442

The following Table contains the values of $\frac{1}{\mu^2}$.

Refracting Medium.	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Flint Glass, No. 13.	.37744	.37652	.37408	.37090	.36807	.36276	.35813
Flint Glass, No. 23.	.37795	.37707	.37467	.37158	.36874	.36342	.35869
Flint Glass, No. 30.	.37923	.37847	.37610	.37303	.37032	.36492	.36029
Flint Glass, No. 3.	.38965	.38877	.38651	.38364	.38104	.37601	.37162
Crown Glass, letter M.	.41367	.41308	.41139	.40929	.40740	.40389	.40083
Crown Glass, No. 9.	.42954	.42892	.42741	.42551	.42385	.42078	.41806
Crown Glass, No. 13.	.43039	.42983	.42831	.42646	.42479	.42171	.41909
Oil of Turpentine.	.46246	.46183	.46001	.45759	.45549	.45152	.44808
Solution of Potash.	.51050	.50984	.50817	.50615	.50435	.50114	.49846
Water.	.56456	.56388	.56228	.56043	.55875	.55584	.55344

Designating the values of $\frac{1}{\mu^2}$ by b, c, d, &c. we obtain the following equations:

$$b = p - 15488 . q + 23987 . l,$$

$$c = p - 17047 . q + 29060 . l,$$

$$d = p - 21139 . q + 44685 . l,$$

$$e = p - 26434 . q + 69875 . l,$$

$$f = p - 31071 . q + 96541 . l,$$

$$g = p - 39705 . q + 157650 . l,$$

$$h = p - 46657 . q + 217690 . l;$$

$$\therefore\ b-e=10946\,.\,q-\ \ 45888\,.\,l,$$

$$e-h=20223\,.\,q-147815\,.\,l;$$

$$\therefore\ 20233\ (b-e)-10946\ (e-h)=(147815\,.\,10946-45888\,.\,20223)\times l,$$

$$l=-\frac{10946\ (e-h)-20223\ (b-e)}{147815\,.\,10946-45888\,.\,20223};$$

$$\therefore\ q=\frac{b-e+45888\ l}{10946},$$

$$p=b+15488\,.\,q-23987\ l.$$

And four other values of p are to be found from the equations c, d, f, g.

The following Table contains the values of q and l deduced from equations b, e, h.

Refracting Medium.	Value of q.	Value of l.
Flint Glass, No. 13.	.000,000,551,79	− .000,000,010,898
Flint Glass, No. 23.	.000,000,513,47	− .000,000,016,333
Flint Glass, No. 30.	.000,000,481,15	− .000,000,020,391
Flint Glass, No. 3.	.000,000,48812	− .000,000,014538
Crown Glass, letter M.	.000,000,37568	− 000,000,0058347
Crown Glass, No. 9.	.000,000,367,88	− 000,000,000,071
Crown Glass, No. 13.	.000,000.351,77	− 000,000,001,733
Oil of Turpentine.	.000,000,44217	− .000,000,000,81361
Solution of Potash.	.000,000,42045	+ .000,000.005,4999
Water.	.000,000,41988	+ .000,000,010,356

The following Table contains the values of p deduced in the manner before mentioned.

Refracting Medium.	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Flint Glass, No. 13.	.38624	.38624	.38623	.38624	.38626	.38639	.38624
Flint Glass, No. 23.	.38629	.38629	.38626	.38629	.38627	.28637	.38629
Flint Glass, No. 30.	.38717	.38726	.38718	.38717	.38722	.38732	.38717
Flint Glass, No. 3.	.39755	.39751	.39747	.39755	.39760	.39755	.39755
Crown Glass, letter M.	.41963	.41965	.41959	.41963	.41963	.41971	.41963
Crown Glass, No. 9.	.43524	.43520	.43519	.43524	.43528	.43538	.53524
Crown Glass, No. 13.	.43588	.43588	.43582	.43588	.43588	.43594	.43588
Oil of Turpentine.	.46932	.46938	.46939	.46932	.46931	.46939	.46932
Solution of Potash.	.51688	.51685	.51681	.51688	.51689	.51696	.51688
Water.	.57082	.57076	.57070	.57082	.57082	.57090	.57082

The principal discrepancy in these results arises from the values of p given by the line G, they being in nearly every case too great.

I can only conclude from this, that our approximation ought to have been carried to another term, as for G and H the value which the third term introduces is considerable, and there can be little doubt but that the fourth would produce a sensible effect to the fourth place of decimals. It would, however, diminish the variation which the expression admits of, to proceed to other terms, and for that reason, considering our object merely to test the accuracy of our conclusions, no better plan has suggested itself than to leave the expressions in their present form.

If, however, it were requisite to determine *accurately* the values of $p, q \ldots$ of course the plan to be adopted would be, that of introducing *seven* constants, and determining their values from the seven given equations. With respect to the results we have deduced, there is little doubt but that the values of b are very far from correct, as indeed any person will perceive who will take the trouble to determine that quantity in any case from the *first three* fixed lines. As to *sign*, however, there can be little doubt of its correctness; and, taken as a *mean* value for the determining p and q, I have no reason to complain of the sufficiency of the approximation. I have indeed adopted a process not the *most* likely to give results widely inconsistent with each other, but at the same time sufficient latitude is allowed for discrepancies far greater than those which actually appear. We could not in reason expect a coincidence in the results greater than that in the bases from which they were deduced; and it appears to me, that as an approximation, we could not have anticipated results more nearly coinciding, had we known, *à priori*, that the formulæ from which they were deduced were accurate.

I will point out a few of the discrepancies (omitting the consideration of G).

In Flint Glass, No. 13, the greatest error from the result from B, E and H, which I shall call the mean result, is .00002, or about $\frac{1}{20,000}^{\text{th}}$ of the whole.

In No. 23. it is .00003.

In No. 30. it is .00009, or about $\frac{1}{4,000}^{\text{th}}$ of the whole.

In No. 3. it is .00005.

In Crown Glass, letter M, it is .00004.

In No. 9. it is .00005.

In No. 13. it is .00006.

In no case, then, except for water, does the error amount to a figure in the *fourth* place of decimals, and hence, by what I have remarked above, in no case is there *any* error.

With respect, however, to the line G, the result is almost always too great: the worst deviation, however, which is that for Flint Glass, No. 23. is .00015, or only an *unit* in the fourth place of decimals.

But since the two errors will be in opposite directions, let us examine the difference between the greatest and least result.

These differences will be found equal to .00016, .00011, .00015, .00013, .00012, .00019, .00012, .00008, .00015, .00020. With only one exception, then, this difference has unity in the fourth place of decimals, and for that exception, which is water, the error is .00020, or barely 2 in the fourth place of decimals.

Results more nearly agreeing might doubtless be obtained by proceeding to one place farther in the expansion of $\sin\frac{\pi\delta\rho}{\lambda}$, but the above will suffice to establish the general accuracy of the formula.

With respect to water, I should have been surprised had the results been more closely coincident, for the values of the refractive indices for B and C are respectively 1.330935, 1.331712 from one experiment, 1.330977, 1.331709 from another, the difference between these values being in one case .000777, and in the other .000732; the former we should have written .0008, in taking to the fourth place only. This difference arises, I suppose, from the different circumstances as regards temperature under which the experiments were performed.

SECTION III.

Deductions from the general expression not confined to the explanation of Dispersion.

If our results are founded on correct principles, and are in themselves right, it is natural to expect that many important and interesting conclusions will follow from them. Some of these I proceed briefly to notice.

We have already deduced from the expression for the velocity a reason for its uniformity in vacuo. The necessary condition was, that the distance between the particles in vacuo should be much smaller than in refracting media. We will now recur to this point, and consider the subject in rather a different light. However it may arise, this is certain, that if $\frac{1}{\mu^2}$ be expanded in a series of the form

$$p - \frac{q}{\lambda^2} + \ldots\ldots\ldots,$$

the quantity q vanishes with respect to p in vacuo, whilst it does *not* in refracting media. On no hypothesis that I can conceive, of the action of forces producing undulations, should we expand our functions in a series *descending* by powers of λ, and also by those of the distance between two particles, for we must approximate by considering one quantity small *compared* with another; nor can I consider a series ascending by powers of the ratio of λ to the distance between two particles, as it would involve an absurdity. There seems, then, every reason to suppose that the *form* at which we have arrived is the correct one. I can indeed conceive it possible, and not at all improbable, that the particles of which the *substance* is composed should influence the motions of the particles of æther. But should they do so, the *form* of our functions would not be affected; and the only difference would be, that p would equal $A\Sigma \{\phi(r) + \frac{F(r)}{r}\,\delta x\}\,\delta\rho^2$, a quantity independent of the distance between two particles of æther, and varying only by the peculiar constitution of the material particles composing the medium. This, then, cannot in the slightest degree affect our reasoning.

Having, then, sufficient ground for the adoption of the results in their present form, we will proceed to re-examine the expansion from which p, q and l were derived. The general form is

$$\frac{1}{\mu^2} = A\Sigma \{\phi(r) + \frac{F(r)}{r}\,\delta x^2\}\,\sin^2 \frac{\pi\delta\rho}{\lambda}.$$

Now in expanding the sine of an arc it is requisite that the arc itself be less than $\frac{\pi}{2}$; if the arc be greater than this $=\frac{\pi}{2}+i$ suppose, since

$$\sin\left(\frac{\pi}{2}+i\right)=\sin\left\{\pi-\left(\frac{\pi}{2}+i\right)\right\}=\sin\left(\frac{\pi}{2}-i\right),$$

the expansion instead of proceeding in terms of the arc itself must proceed in terms of its supplement, and an analogous rule applies to larger arcs:

as long then as $\frac{\delta\rho}{\lambda}$ is less than $\frac{1}{2}$,

$$\sin\pi\frac{\delta\rho}{\lambda}=\frac{\pi\delta\rho}{\lambda}-\frac{1}{6}\frac{\pi^3\delta\rho^3}{\lambda^3}+\ldots\ldots\ldots\ldots\ldots$$

when $\frac{\delta\rho}{\lambda}$ lies between $\frac{1}{2}$ and 1, we have

$$\sin\frac{\pi\delta\rho}{\lambda}=\sin\pi-\frac{\pi\delta\rho}{\lambda}=\pi\left(1-\frac{\delta\rho}{\lambda}\right)-\frac{\pi^3}{6}\left(1-\frac{\delta\rho^3}{\lambda}\right)+\&c.$$

and so on for other values.

The expansion, then, with this restriction, which is perhaps not requisite, as at any rate the principal terms are those which arise from particles near P, proceeds with powers of $\frac{\delta\rho}{\lambda}$; and we obtain

$$p=A_1\Sigma\left\{\phi(r)+\frac{F(r)}{r}\delta x^2\right\}\delta\rho^2,$$

$$q=B_1\Sigma\left\{\phi(r)+\frac{F(r)}{r}\delta x^2\right\}\delta\rho^4.$$

Suppose the function which expresses the mutual action of two particles on each other, some power of the distance, or let $r\phi(r)=\frac{1}{r^n}$;

$$\therefore\ \phi(r)=\frac{1}{r^{n+1}},$$

$$F(r)=-\frac{n+1}{r^{n+2}},$$

$$p = A_1 \Sigma . \left\{ \frac{1}{r^{n+1}} - \frac{(n+1)\,\delta x^2}{r^{n+3}} \right\} \delta\rho^2,$$

$$q = B_1 \Sigma . \left\{ \frac{1}{r^{n+1}} - \frac{(n+1)\,\delta x^2}{r^{n+3}} \right\} \delta\rho^4;$$

let the distance between two consecutive particles be denoted by ϵ, and for δx, δy, δz, $\delta\rho$ respectively write $\epsilon\xi$, $\epsilon\eta$, $\epsilon\zeta$, $\epsilon\sigma$: ξ being the number of particles along a line through Q perpendicular to the plane of yz, and similarly of η, ζ and σ.

By substituting these values

$$p = A_1 \Sigma \left\{ \frac{1}{(\xi^2+\eta^2+\zeta^2)^{\frac{n+1}{2}}} - \frac{(n+1).\xi^2}{(\xi^2+\eta^2+\zeta^2)^{\frac{n+3}{2}}} \right\} \frac{\epsilon^2\sigma^2}{\epsilon^{n+1}}$$

$$= \frac{A_1}{\epsilon^{n-1}} \Sigma \left\{ \frac{1}{(\xi^2+\eta^2+\zeta^2)^{\frac{n+1}{2}}} - \frac{(n+1).\xi^2}{(\xi^2+\eta^2+\zeta)^{\frac{n+3}{2}}} \right\} \sigma^2,$$

$$q = \frac{B_1}{\epsilon^{n-3}} \Sigma \left\{ \frac{1}{(\xi^2+\eta^2+\zeta^2)^{\frac{n+1}{2}}} - \frac{(n+1).\xi^2}{(\xi^2+\eta^2+\zeta)^{\frac{n+3}{2}}} \right\} \sigma^4.$$

Now each part of these expressions, with the exception of the factors $\frac{A}{\epsilon^{n-1}}$ and $\frac{B}{\epsilon^{n-3}}$, is a numerical quantity not dependent on the nature of the medium, except inasmuch as it requires the medium of symmetry, and $\frac{B_1}{A_1}$ is evidently some *number*: in fact it is equal to $\frac{\pi^2}{3 \times (100{,}000)^2}$ by page 28; hence the only possible mode of causing p to become *large* in vacuum whilst q is *small*, at the same time that p is *not large* and q *not small*, (I speak comparatively) in glass, will be by supposing $\frac{1}{\epsilon^{n-1}}$ large, and $\frac{1}{\epsilon^{n-3}}$ small, in the former, whilst the same quantities are not *so* widely different in glass: but ϵ is small in all cases; in order, then, that $\frac{1}{\epsilon^{n-1}}$ shall be large, $n-1$ must be *positive*, and that ϵ^{n-1} may be small *at the same time*, and *vice versâ*, ϵ^{n-3} must be negative; therefore $n > 1 < 3$, or if $n=2$, all the conditions are satisfied: this requires the further condition, that ϵ shall be much less in vacuum than in refracting media.

We have, then, clearly been led to the conclusion, not only that the density of the æther is greater in vacuum than in refracting media, but also that its particles act on each other with forces varying *inversely as the square of the distance.*

But I do not stop here; it is a remarkable fact, and one which demands particular attention, that various phenomena appear to indicate not merely that the motion is in *general* transversal, but that it is altogether so.

Let us consider this point a little more accurately. Suppose the forces which the particles of the medium exert to be repulsive, as those of air, from which arise the phenomena of sound. A series of particles constituting any vertical line being simultaneously impelled in a horizontal direction would, by virtue of their repulsion, cause a similar motion in those immediately in front of them, whilst the latter particles would tend to check the impetus of the former, and thus vibrations in the direction of transmission are simple to conceive, and easy to explain.

On the other hand were the particles *attractive*, no such motion would be possible, except under peculiar restrictions.

But suppose, notwithstanding, that the forces which the particles exert are attractive—Let the system of particles in a vertical line have a *vertical* motion, and the slightest consideration will shew us that the immediate consequence is the production of a vertical motion in the particles immediately in advance of them; whilst, as before, the reciprocal action of the latter particles tends to impede the motion of the former. Here, then, we have as clear a case as before, and our general conclusion from the whole is, that repulsive forces allow of *direct*, attractive, of *transversal* vibrations only. In the former case I would refer for a simple conception to the wind blowing over a field of corn, and to a rope held between two persons, and jerked by one of them; in the latter, as they illustrate both the mode of vibration and the action of the forces.

I do not mean, however, to consider the above reasoning as applicable to all possible arrangements of particles; it might happen, and very probably is the case, that a differently constituted medium from that which we, merely for the sake of simplicity, have imagined, would lead us to results different from the above. As my object is not now to examine every possible circumstance attending the motion, but merely to illustrate one particular view, I shall proceed on the hypothesis of the arrangement in cubical forms. We will then endeavour to ascertain, from an examination of our mathematical results, what assistance analysis affords us in the investigation of the law of transversality of vibration. I shall here *assume* that the law of the inverse square of the distance has been proved; and shall adopt the same notation which I applied to the investigation of that property. Suppose, then, to fix the ideas that the wave is transmitted along the axis of y.

If v, v', v'' be the velocities of transmission respectively of vibrations whose motion is parallel to x, y, z, we have

$$v^2 = 2A . \Sigma . \left(\frac{\delta x^2 + \delta y^2 + \delta z^2 - 3\delta x^2}{r^5} \sin^2 \frac{\pi \delta \eta}{\lambda}\right)$$

$$= \frac{2A}{\epsilon^3} . \Sigma . \frac{\xi^2 + \eta^2 + \zeta^2 - 3\xi^2}{r^5} \sin^2 \frac{\pi \epsilon \eta}{\lambda}$$

$$= \frac{2A}{\epsilon^3} . \Sigma . \frac{\eta^2 + \zeta^2 - \xi^2}{r^5} \sin^2 \frac{\pi \epsilon \eta}{\lambda}$$

$$v'^2 = \frac{2A}{\epsilon^3} . \Sigma . \frac{\xi^2 + \eta^2 + \zeta^2 + 3\eta^2}{r^5} \sin \frac{\pi \epsilon \eta}{\lambda}$$

$$= \frac{2A}{\epsilon^3} . \Sigma . \frac{\xi^2 + \zeta^2 - 2\eta^2}{r^5} \sin^2 \frac{\pi \epsilon \eta}{\lambda},$$

$$v''^2 = \frac{2A}{\epsilon^3} . \Sigma . \frac{\xi^2 + \eta^2 + \zeta^2 - 3\zeta^2}{r^5} \sin^2 \frac{\pi \epsilon \eta}{\lambda}$$

$$= \frac{2A}{\epsilon^3} . \Sigma . \frac{\xi^2 + \eta^2 - 2\zeta^2}{r^5} \sin^2 \frac{\pi \epsilon \eta}{\lambda}.$$

Now, from the symmetry of the medium, we have

$$\Sigma . \xi^2 \frac{\sin^2 \frac{\pi \epsilon \eta}{\lambda}}{r^5} = \Sigma . \xi^2 \frac{\sin^2 \frac{\pi \epsilon \eta}{\lambda}}{r^5};$$

$$\therefore v^2 = \frac{2A}{\epsilon^3} . \Sigma \frac{\eta^2 - \xi^2}{r^5} \sin^2 \frac{\pi \epsilon \eta}{\lambda} = v''^2,$$

$$\text{and } v'^2 = \frac{2A}{\epsilon^3} \Sigma \frac{2\xi^2 - 2\eta^2}{r^5} \sin^2 \frac{\pi \epsilon \eta}{\lambda} = - \frac{4A}{\epsilon^3} \Sigma \frac{\eta^2 - \xi^2}{r^5} \sin^2 \frac{\pi \epsilon \eta}{\lambda}$$

$$= -2v^2 = -2v''^2.$$

But by reference to the preceding part of the investigation it will be found, that the forces have been considered positive when they acted in the direction in which the disturbing particle lay, and *vice versâ*; that is, they have been considered attractive. It appears, then, that such a supposition makes v and v'' possible and equal, but v' impossible, and of a different magnitude.

If, on the other hand, we had considered the forces repulsive, the factor A would have been negative, we should also have had v' possible, whilst v and v'' would have been impossible. Hence attractive forces give rise to *transversal* vibrations only, repulsive to *direct* vibrations only. The latter corresponds, both as to forces and vibrations, to the particles of air, the former then may be reasonably supposed true for light; and hence it follows, that from a comparison of our formula with observed facts, the forces are found to be *attractive*. I must, however, observe, that the equations deduced as I have obtained them will but very imperfectly apply to sound; there seems, however, a great probability that the general form will be an analogous one; and should it be found to be the same, then since all the waves of sound of different lengths travel with an equal velocity, the conclusions which we have deduced as to the forces varying inversely as the square of the distance might hold equally in air, a conclusion to which I hope shortly to recur in a separate Memoir, and for that reason shall defer any further consideration of the subject until then.

Another observation that suggests itself is, that since p is a factor of $\frac{1}{\epsilon}$, and q of ϵ, the product fq is independent of ϵ, and consequently would, were the action of the particles of æther alone influential, be the same for all substances. By a reference to the numerical values above given, it will appear that this only holds as a very rude approximation for the *glasses*, and altogether fails for the fluids. Thus much, however, may be gathered from an inspection of the tables, that there is a tendency to verify the result, and that we should not be induced to regard the effects of the material particles of such considerable magnitude, as to vitiate the *general* conclusions. When, however, our results are pursued into details, the action of the material particles (or whatever other actions we choose to consider, if these be rejected) produces a sensible effect. For it will be observed, that all the solids have l negative; whilst, on the contrary, water and solution of potash make it positive. Now the distances between the particles cannot affect these signs, nor can the *absolute* forces, as long as these forces are supposed all of the same nature. This, then, is a difficulty in our way which it would be well to remove. I cannot, however, enter into this subject further than to observe, that if there were *no* extraneous forces, the quantity l would undoubtedly be positive; and that, as the action of the particles on each other is *attractive*, an alteration in sign must arise from an addition of *repulsive* effects; and that since these effects are not particularly great in affecting p and q, the function to which they will give rise will be a series not so *rapidly converging* as that which expresses the velocity due to the actions of the particles of æther on each other.

I ought, however, to state that it is not impossible that this effect would have been explained, had we taken into account the terms in the expansion of $(r+\rho).\phi(r+\rho)$, &c. to the third order. This explanation I have not been able to succeed in deducing, as the equation of motion then assumes the following form:

$$\frac{d^2 a}{dt^2} = \Sigma . \left(\frac{1}{r^3} - \frac{3\delta x^2}{r^5}\right) \delta a$$

$$+ \Sigma . \{- \frac{3}{2r^5} + \frac{3.5}{r^7} \delta x^2\} (\delta a^2 + \delta \gamma^2) \delta a$$

$$- \frac{3.5.7}{6.r^9} \Sigma \{\delta x^4 \delta a^2 + 3\delta x^2 \delta z^2 \delta \gamma^2\} \delta a,$$

which would seem to indicate that the velocity is not altogether independent of the *extent* of vibration.

It would, however, lead us too far into speculations, which, after all, may have little grounds to rest upon, should we pursue what I have here barely alluded to.

We shall obtain a value of ϵ by dividing p by q, for supposing the wave transmitted parallel to the axis of y,

$$p = A_2 \Sigma \left\{\frac{1}{(\xi^2 + \eta^2 + \zeta^2)^{\frac{3}{2}}} - \frac{3\xi^2}{(\xi^2 + \eta^2 + \zeta^2)^{\frac{5}{2}}}\right\} . \frac{\eta^2}{\epsilon},$$

$$q = \frac{A_2 \pi^2}{3} \left\{\frac{1}{(\xi^2 + \eta^2 + \zeta^2)^{\frac{3}{2}}} - \frac{3\xi^2}{(\xi^2 + \eta^2 + \zeta^2)^{\frac{5}{2}}}\right\} \eta^4 . \epsilon;$$

$$\therefore \epsilon^2 = \frac{3}{\pi^2} \frac{\Sigma \left\{\frac{1}{(\xi^2 + \eta^2 + \zeta^2)^{\frac{3}{2}}} - \frac{3\xi^2}{(\xi^2 + \eta^2 + \zeta^2)^{\frac{5}{2}}}\right\} \eta^2}{\Sigma \left\{\frac{1}{(\xi^2 + \eta^2 + \zeta^2)^{\frac{3}{2}}} - \frac{3\xi^2}{(\xi^2 + \eta^2 + \zeta^2)^{\frac{5}{2}}}\right\} \eta^4},$$

which will serve to determine ϵ, if the above numerical quantities can be assigned.

The numerator can be determined without any considerable difficulty, but owing to the very slow convergence of the denominator, I have not been able to assign its value to any degree of accuracy, I shall consequently content myself with proving (what is essential to my remarks on the transversality of the vibrations) that each of the quantities is *positive*.

Let $\eta_1\ \xi_1$ be any values of η and ξ, then the position of the numerator, omitting the known factor, for these particular values of the η and ξ is

$$\Sigma\left\{\frac{1}{(\xi^2+\eta_1^2+\zeta_1^2)^{\frac{3}{2}}}-\frac{3\xi_1^2}{(\xi^2+\eta_1^2+\zeta_1^2)^{\frac{5}{2}}}\right\}\eta_1^2=\Sigma.\frac{(\eta_1^2-\zeta_1^2)\,\eta_1^2}{(\xi^2+\eta_1^2+\zeta_1^2)^{\frac{5}{2}}}.$$

This sum being taken for all values of ξ from 0 to infinity. But in the limits of a wave, η and ζ will have equal corresponding values, so that there will be a term $\frac{\xi_1^2-\eta_1^2)\,\zeta_1^2}{(\xi^2+\eta_1^2+\zeta_1^2)^{\frac{5}{2}}}$, the ζ_1 being here a particular value of η.

And for the next wave the expression becomes $\Sigma.\frac{(\eta_1^2-\zeta_1^2)\,(h-\eta_1)^2}{(\xi^2+\eta_1^2+\zeta_1^2)^{\frac{5}{2}}}$, which gives a result of the same form. Hence the sum of these two terms is $\Sigma\,\frac{(\eta_1^2-\zeta_1^2)^2}{(\xi^2+\zeta_1^2+\eta_1^2)^{\frac{5}{2}}}$ taken only on one side, which is an essentially positive quantity; and this is true for every particular value of η and ζ, and is therefore true for the sum of all the values; whence the numerator above is a positive quantity. Similar to the corresponding term of the denominator is $\Sigma\,\frac{(\eta_1^2+\zeta_1^2)\,(\eta_1^2-\zeta_1^2)^2}{(\xi^2+\eta^2+\zeta_2)^{\frac{5}{2}}}$, which is also essentially positive.

This result is necessary to the reasoning I adduced above, in order to shew that the forces which the particles exert on each other are attractive.

I wish it were in my power to offer any considerations relative to the phenomena of polarization by reflexion from the surface of glass, and so on. There appears to be little doubt of the truth of the results which have been deduced by M. Fresnel relatively to the coefficients of the intensity of reflected and transmitted light produced by the different vibrations. I cannot however think that the hypothesis of the æther within the glass being more dense than that without in the ratio of μ : 1 is altogether satisfactory, but I forbear making any remarks on that subject further than to shew what is the corresponding relation of the densities deducible from our hypothesis.

$$\text{We have } \frac{1}{\mu^2} = \frac{P}{\epsilon} - \frac{Q \cdot \epsilon}{\lambda^2} + \dots\dots\dots$$

and taking only a very approximate value, we obtain

$$\mu^2 = \frac{\epsilon}{P},$$

but the density evidently varies as the cube of the reciprocal of ϵ,

$$\text{or density} \propto \frac{1}{\mu^6}.$$

In conclusion, I would remark, that although what has here been treated of has been but roughly and approximately developed, there is good reason for supposing that the laws we have arrived at are the correct ones, not only as regards the action of the particles of æther, but as regards those of air also.

The law of the inverse square of the distance has always appeared to me a *necessary* law; necessary, I mean, as regards the actual state of the constitution of the Universe: and although I could allow that the particles of matter might have been impressed with any law at their creation, I cannot, in consistence with the *simplicity* of all *known* actions, conceive any other than Newton's law. It is true, the phenomena of Capillary Attraction seem to militate decidedly against it, but no person that I am aware of has proved that the phenomena could not arise from *discontinuing* the fluidity, and until that has been done, I think (I speak with deference to others far more capable of judging) we ought not to be too hasty in adopting a law of force, however simply it may account for the particular phenomena in question, which we have no reason to suppose is applicable to any others.

But I fear I am trespassing beyond the proper limits of my subject, and shall therefore proceed no further than merely to observe, that the farther we proceed in our investigations, the more simple do our conclusions become, and that from the apparent discrepancies, as, for instance, in the lateral spread of sound passing through an aperture, which is not the case for light, in general arise the strongest confirmations of the unity of the whole.

VII. *Sketch of a Method of Introducing Discontinuous Constants into the* Arithmetical *Expressions for Infinite Series, in cases where they admit of several Values. In a Letter to the Rev. George Peacock, &c. &c. By* AUGUSTUS DE MORGAN, *of Trinity College. Fellow of the Society, and Secretary of the Royal Astronomical Society.*

[Read *May* 16, 1836.]

DEAR SIR,

Two years ago, I presented to the Society through yourself, the detail of some anomalies which I had observed to exist in certain series which I then produced. They arose out of investigations connected with Functions, and since published in my Treatise on that subject in the Encyclopædia Metropolitana. But on further consideration, I find that I have not distinctly expressed the method by which the anomalies of the series in question may be reconciled, or rather by which the series may be so obtained that the difficulties shall not appear.

I beg leave therefore, to request that you will lay the following view of the subject before the Society.

The assumption of a given form for a development amounted to an express exclusion of several considerations, which, so it happened, did not affect the results of ordinary operations, in cases where the form assumed was that of development in whole powers of a variable. Among the exclusions, was that of the possibility of a *discontinuous constant,* which was never considered, I believe, until the errors which the omission of it created in the inversion of periodic developments forced attention to the subject. And even then, the discontinuous constant was only a new fundamental symbol, inserted in its proper place, in such form and manner as what I may call *discontinuous* investigations shewed to be necessary. In the method which I propose to explain, discontinuity not only appears in its proper place, but with its proper symbol.

When n terms of the series can be expressed in terms of n, the supposition $n = \infty$ will generally point out, in one way or another, whether any, and what, discontinuity exists. The method which I proceed to explain, while it depends for its strictness upon the passage from a finite to an infinite number of terms, does not require the actual expression of n terms as a condition of practicability.

As usual, let ϕx, $\phi^2 x$, &c. represent the results of successive functional operations; the symbol ϕx admits of two distinct characters, in the *periodic* and *non-periodic* cases. Either $\phi^n x = x$, for a finite whole value of n, or for no whole value whatsoever, except in the extension $n = 0$. In cases where ϕx is not periodic, it has this peculiarity; that $\phi^n x$, whatever may be the value of x, will either increase (with x) without limit, or will, for successive whole values of n, give a series of approximations to m different limits which are severally roots of $\phi^m x = x$. I am speaking of positive or negative functions, and of real roots. With this proposition my only concern here is as to the case where $\phi^n x$ has one limit, in which case it evidently must give $\phi L = L$, L being the limit in question. And this proposition is already well known in every part of mathematics. For instance, most direct methods of *successive* approximation depend upon the use of Taylor's Theorem, in a manner which will be recognized in the following particular case. If

$$\psi x = x - \frac{\phi x}{\phi' x} \text{ where } \phi' x = \frac{d\phi x}{dx},$$

then the limit of successive operations gives a root of $\psi x = x$; that is, either of $\phi x = 0$, or of $\frac{1}{\phi' x} = 0$.

But $\phi^\infty x$, considered as the limit of $\phi^n x$, may be one root of $\phi x = x$, for values of x intermediate to one set of limits, and another for another. For instance, let p be greater than unity, and let $\phi x = x^p$. Then we have

$$\phi^\infty x = x^{p^\infty} = 0 \text{ when } x < 1 = 1 \text{ when } x = 1 = \infty \text{ when } x > 1.$$

We might generalize the theorem, by a supposition which common algebra would admit. An equation of any degree, considered as one of a higher degree with evanescent terms, has *infinite* roots. In the common mode of speaking, we must say that $\phi^\infty x$ is either infinite, or a root of $\phi x = x$. In that just alluded to, we should simply say that $\phi^\infty x$ is a root of $\phi x = x$.

As a second example, let $\phi x = a(x + m) - m$.

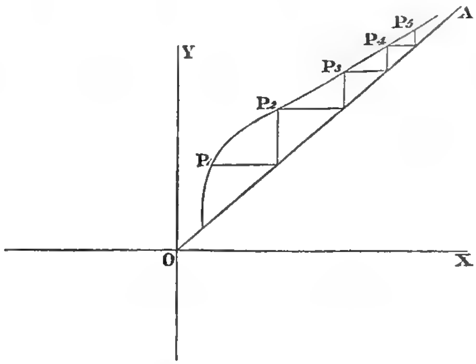
$$\text{Then } \phi^\infty x = a^\infty (x + m) - m,$$

and is infinite for all values of x except $-m$, when a is greater than 1, and $= -m$, for all values of x, when a is less than 1. But $-m$ is the root of $\phi x = x$.

It must, I suppose, be well known that successive approximation will not be vitiated by any error introduced into the approximate results, unless that error be so great that the process is made to tend towards another solution, or to increase without limit. For instance, in the solution of

$$x = \frac{1}{1 + x} \text{ by a continued fraction.}$$

The value of x may be what we please at the commencement, or the obtained value may be altered; and the attainment of any degree of accuracy, though retarded, is not rendered impossible. In a similar manner, a purely graphical process will lead to information upon the value of $\phi^\infty x$ in particular cases, such as with a little care may be made equivalent to demonstration. Let OA be a line equally inclined to OX and OY, rectangular axes to which the curve $y = \phi x$ is referred. Taking any point P_1, which has the abscissa required, x, proceed alternately from the curve to OA, parallel to the axis of x,



and of OA to the curve parallel to the axis of y. The ordinates of the successive points of the curve P_1, P_2, P_3, &c. are the values of ϕx, $\phi^2 x$, $\phi^3 x$, &c. for the given value of x.

The general expression $\phi^\infty x$, is then one which requires a development of the following kind,

$$x_1 C_{a,b} + x_2 C_{b,c} + x_3 C_{b,a} + \ldots$$

where x_1, x_2, x_3, &c. are specific quantities depending upon the function in question, and $C_{a,b}$ means 1 when x lies between a and b, and 0 for all other values, &c.

$$\text{Let} \qquad \phi x = \beta x + \gamma x \,.\, \phi a x \ldots\ldots (1),$$

where ax, βx, γx, are given functions, not periodic, and ϕx is to be found. In my treatise on the Calculus of Functions already alluded to, it is shewn that the complete solution of the preceding is

$$\phi x = \mu x + \nu x \,.\, \xi a x,$$

where μx is any particular solution, νx any particular solution of $\phi x = \gamma x \, \phi a x$, and $\xi a x$ the general solution of $\phi x = \phi a x$.

Among the solutions of (1) is the series

$$\beta x + \gamma x \,.\, \beta a x + \gamma x \,.\, \gamma a x \,.\, \beta a^2 x + \gamma x \,.\, \gamma a x \,.\, \gamma a^2 x \, \beta a^3 x + \ldots\ldots$$

obtained as follows:

$$\phi x = \beta x + \gamma x \,.\, \phi a x$$

$$\phi a x = \beta a x + \gamma a x \,.\, \phi a^2 x$$

$$\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots$$

$$\phi a^n x = \beta a^n x + \gamma a^n x \,.\, \phi a^{n+1} x,$$

whence, for a finite number of terms,

$$\phi x - \gamma x \,.\, \gamma a x \dots \gamma a^n x \, \phi a^{n+1} x = \beta x + \gamma x \,.\, \beta a x + \dots + \gamma x \,.\, \gamma a x \dots \gamma a^{n-1} x \,.\, \beta a^n x.$$

The first rule of which is a case of

$$\mu x + \nu x \,.\, \xi a x - \gamma x \,.\, \gamma a x \,.\, \dots \gamma a^n x \, (\mu a^{n+1} x + \nu a^{n+1} x \,.\, \xi a^{n+2} x).$$

In which $\nu x = \gamma x \,.\, \nu a x = \gamma x \,.\, \gamma a x \,.\, \dots \gamma a^n x \, \nu a^{n+1} x$,

and $\xi a x = \xi a^2 x = \dots = \xi a^{n+2} x$,

whence the expression in question becomes

$$\mu x - \gamma x \,.\, \gamma a x \,.\, \dots \gamma a^n x \,.\, \mu a^{n+1} x,$$

which cannot, as might appear at first sight, give a different value for every different value of μx: for, since two values of ϕx can only differ by some solution of $\phi x = \gamma x \,.\, \phi a x$, the preceding expression is the same whatever value of ϕx be adopted.

For an infinite number of terms of the preceding series, we have

$$\mu x - \gamma x \,.\, \gamma a x \,.\, \dots \gamma a^{\infty} x \,.\, \mu a^{\infty} x.$$

And the equation $\phi x = \gamma x \,.\, \phi a x$, if νx be one solution, can have no others, except of the form $\nu x \,.\, \xi a x$. But

$$\gamma x \,.\, \gamma a x \,.\, \dots \gamma a^{\infty} x$$

evidently satisfies $\phi x = \gamma x \, \phi a x$; or if we take $\nu x = \gamma x \,.\, \nu a x$, we find the preceding product to be $\nu x \div \nu a^{\infty} x$. Consequently, the expression for the series in question is

$$\mu x - \frac{\nu x}{\nu a^{\infty} x} \,.\, \mu a^{\infty} x, \text{ or } \mu x - \frac{\mu a^{\infty} x}{\nu a^{\infty} x} \, \nu x.$$

If therefore, the expression for $\alpha^{\infty} x$ be

$$x_1 C_{a_1 b} + x_2 C_{b_1 c} + \dots$$

that for the sum of the series is discontinuous, and represented by

$$\left(\mu x - \frac{\mu x_1}{\nu x_1} \nu x\right) C_{a_1 b} + \left(\mu x - \frac{\mu x_2}{\nu x_2} \nu x\right) C_{b_1 c} + \dots$$

I shall take the two instances given in my former paper, which will of course be the most satisfactory, as the difficulty was prior to the explanation. The first was the series

$$\frac{x}{1+x^2} + \frac{x}{1+x^2} \cdot \frac{x^2}{1+x^4} + \frac{x}{1+x^2} \cdot \frac{x^2}{1+x^4} \cdot \frac{x^4}{1+x^8} + \dots$$

$$\text{in which } \beta x = \gamma x = \frac{x}{1+x^2}, \quad \alpha x = x^2$$

$$\alpha^{\infty} x = x^{2^{\infty}} = \infty C_{-\infty, -1} + C_{-1} + 0 C_{-1, +1} + C_{+1} + \infty C_{1, \infty}.$$

The equation of the series is

$$\phi x = \frac{x}{1+x^2} + \frac{x}{1+x^2} \phi(x^2),$$

a particular solution of which is $\phi x = x = \mu x$,

a particular solution of $\phi x = \dfrac{x}{1+x^n} \phi(x^2)$ being $\phi x = x - \dfrac{1}{x} = \nu x$;

whence the expression for the series in question is

$$x - \frac{x^{2^{\infty}}}{x^{2^{\infty}} - x^{-2^{\infty}}} \left(x - \frac{1}{x}\right),$$

which, if x lie between $-\infty$ and -1, or $+\infty$ and $+1$, is $\dfrac{1}{x}$,

if x lie between -1 and $+1$, is x.

To explain the cases where $\alpha x = x$, return to the expression for the series, which then becomes

$$\mu x \{1 - (\gamma x)^{\infty}\},$$

giving in this case for $x = 0$, the value 0,

and for $x = 1$ the value 1,

The result may be easily verified. The given series may be thrown successively into the forms

$$1.\quad (x<1)\qquad (1-x^2)\left\{\frac{x}{1-x^4}+\frac{x^3}{1-x^8}+\frac{x^7}{1-x^{16}}+\ldots\right\},$$

$$\text{or } (1-x^2)\{x+x^3+x^5+\ldots\},\ \text{that is, } x.$$

$$2.\quad (x>1)\qquad (x^2-1)\left\{\frac{x}{x^4-1}+\frac{x^3}{x^8-1}+\frac{x^7}{x^{16}-1}+\ldots\right\},$$

$$\text{or } (x^2-1)\left\{\frac{1}{x^3}+\frac{1}{x^5}+\frac{1}{x^7}+\ldots\right\},\ \text{that is, } \frac{1}{x}.$$

The second example in my former paper was the series

$$\frac{x}{(1+x)(1+ax)}+\frac{ax}{(1+ax)(1+a^2x)}+\frac{a^2x}{(1+a^2x)(1+a^3x)}+\ldots$$

$$\beta x=\frac{x}{(1+x)(1+ax)},\quad \gamma x=1,\quad \alpha x=ax,\quad \alpha^\infty x=a^\infty x.$$

The equation of the series is

$$\phi x=\frac{x}{(1+x)(1+ax)}+\phi(ax),$$

a particular solution of which is $\mu x=\dfrac{1}{(a-1)(x-1)}$,

and a particular solution of $\phi x=\phi(ax)$ is $x=C=\nu x$,

$$\mu x-\frac{\mu a^\infty x}{\nu a^\infty x}\nu x=\frac{1}{(a-1)(x+1)}-\frac{1}{(a-1)(a^\infty x+1)},$$

$$\text{when } a>1=\frac{1}{(a-1)(x+1)},$$

$$\text{when } a<1=\frac{1}{(a-1)(x+1)}-\frac{1}{a-1}=\frac{x}{(1-a)(x+1)}.$$

This result may also be verified; for the original series developed term by term in powers of $\dfrac{1}{a}$ and $\dfrac{1}{x}$, and corresponding series of powers of $\dfrac{1}{a}$ collected gives

$$\frac{a^{-1}x^{-1}}{1-a^{-1}} - \frac{a^{-1}x^{-2}}{1-a^{-1}} + \frac{a^{-1}x^{-3}}{1-a^{-1}} - \ldots = \frac{1}{a-1} \cdot \frac{1}{x+1},$$

and resolved into series of positive powers of a and x, with a similar subsequent process, it yields

$$\frac{x}{1-a} - \frac{x^2}{1-a} + \frac{x^3}{1-a} - \ldots = \frac{x}{(1-a)(1+x)}.$$

Let us now apply this method to examine some of the more common series of analysis: let us take

$$1 + a + a^2 + a^3 + \ldots$$

Multiply every term by x, and it will then appear to be

$$\mu x - \frac{\mu a^{\infty} x}{\nu a^{\infty} x} \nu x, \text{ where } ax = ax,$$

and μx and νx are any particular solutions of

$$\phi x = x + \phi(ax), \qquad \phi x = \phi(ax),$$

$$\text{let } \mu x = \frac{x}{1-a}, \qquad \nu x = C,$$

then the series is
$$\frac{x}{1-a} - \frac{a^{\infty} x}{1-a}.$$

Like all other results of strict methods of passing from the sum of n terms to the sum of an infinite series, this expression is infinite when the series is infinite. But my object here is to remark, that owing to $a^m x = x$ having only the root $x = 0$, there can be no discontinuity among the values which correspond to arithmetical values of the above series.

If we consider the series

$$1 + a + \frac{a^2}{2} + \frac{a^3}{2.3} + \ldots$$

as a case of

$$a^x + \frac{a^{x+1}}{x+1} + \frac{a^{x+2}}{(x+1)(x+2)} + \ldots$$

we find the value of the preceding to be as follows,

$$\beta x = a^x, \quad \gamma x = \frac{1}{x+1}, \quad \phi x = a^x + \frac{1}{x+1}\phi(x+1).$$

A particular solution of $\phi x = \dfrac{\phi(x+1)}{x+1}$ is $\nu x = \Gamma(x+1)$.

No more simple value of μx can be found than the finite integral of

$$\Delta \frac{\mu x}{1.2.3\ldots x} = -\frac{a^x}{1.2.3\ldots x}.$$

If μx could be expressed, the value of the preceding series would be

$$\mu x - \frac{\mu(\infty)}{1.2.3\ldots\infty}(1.2.3\ldots x).$$

What we have here to observe is, that in consequence of $ax = x+1$, $a^{\infty} x$ increases without limit for all values of x, and there is no discontinuity.

I shall only further remark, that the preceding results confirm, so far as they go, an opinion which I have long entertained, namely, that series which may be divergent, or which may be brought as near to divergency as we please, such as that for ϵ^a, require much less circumspection than those which can never be made to diverge. In the first, generally speaking, the arithmetical value (between the limits of convergency) is the analytical value throughout; in the second, there is frequently discontinuity in the arithmetical values, and the general equivalent of analysis is not easily expressed. I will not however enlarge upon so general a topic, but beg to remain,

Dear Sir,

Yours very truly,

AUGUSTUS DE MORGAN.

5 Upper Gower Street,
April 30, 1836.

P. S. Some time ago, I communicated to the Society what I consider a failure in the proof of the celebrated theorem of M. Abel, on the expressibility of the roots of equations which are values of a periodic function. As I have since printed my objection in the Calculus of Functions alluded to in the preceding paper (§. 90, 302, 303,) I take this opportunity of referring to the subject.

VIII. *Piscium Maderensium Species quædam novæ, vel minus rite cognitæ, breviter descriptæ. Auctore* R. T. Lowe, A.M. *Iconibus illustravit* M. Young.

[Read *November* 10, 1834.]

Vel prudentissime cunctanti fugit inexorabile tempus; et qui rem dubiam, nimio suadente metu erroris, semper in crastinam horam differt, superbiæ forsan suæ potiusquam scientiæ commodo consulit. Piscium nempe Maderensium species quasdam insigniores, pro novis habitas, prorsus stabilire, aliis comparatis speciebus affinibus in Museis Britannicis tam servatis quam editis in libris, ipse de die in diem frustra cunctatus speravi. Quum autem rei certioris me fefellit spes, icones perpulchras saltem, cura vel exquisitissima pictas, pro erroris culpa in re qualibet momenti levioris indulgentiam impetraturas credens, animum recepi. Si enim opiniones et nomina falsa, veteresque pro novis species ponuntur, icones bonæ nunquam *non* utiles; *minus* tantum quam ex votis auctoris evadunt. Eum si culpa rodit, scientia vix ulla, leviore certe, afficitur injuria: immo ipsius periculo potius augeatur!

Ord. ACANTHOPTERYGIANÆ.

Fam. Percidæ.

Gen. SERRANUS, *Cuv.* et *Val.*

Sp. 1. Serranus fimbriatus, *Nob.*

1. S. fusco-nigricans, luteo maculatus, maculis evanescentibus: pinna caudali, dorsalisque analisque parte molli, postice rotundatis, nigris, candido fimbriatis: spinis pinnæ dorsalis analisque distincte filamentosis:

operculo spinis tribus latis, distinctis: præoperculo deorsum subsinuato, denticulato: osse intermaxillari esquamoso.

D. 11+15 *v.* 16; A. 3+8; P. 18; V. 1+5; C. $\frac{3\ v.\ 4+\text{VIII}}{3\ v.\ 4+\text{VII}}$; M. B. 7.

Tab. I. f. 1. e juniore hujusce magnitudinis picta.

f. 2. squama ejusdem, lente vitrea aucta.

S. marginatus nob. in Proceed. Zool. Soc. 1833. I. p. 142. "*Mero,*" Lusitanice.

Rarior. Ad 2 pedum longitudinem crescit, habitu et colore *Tincam vulgarem* Cuv. quodammodo referens.

Nomen mutare ægre et quasi coactus decrevi, ob *Serranum marginalem* Cuv. et Val. (*Holocentrum marginatum* Lacep.)

Sp. 2. Serranus fuscus, *Nob.*

2. S. fusco-nigricans, maculis griseis, obscuris, confluentibus subvariegatus s. marmoratus: pinna caudali truncata, supra sublobata, s. subemarginata; dorsalique postice angulata, analique postice truncata, nigris: spinis pinnæ dorsalis analisque simplicibus s. exappendiculatis: operculo spinis tribus; duobus inferioribus angustis; superiore obsoleta, rudimentali, squammiformi: præoperculo deorsum subsinuato, obsolete denticulato: osse intermaxillari deorsum squamoso.

D. 11+15 *v.* 16; A. 3+11; P. 16; V. 1+5; C. $\frac{3+\text{VII}}{3+\text{VI}}$; M. B. 7; Vert[æ]. 24.

"*Badeijo*" v. "*Badeija,*" Lusitanice.

Priori, quamvis distinctissima, simillima.

Gen. PRIACANTHUS, *Cuv.*

Sp. P. fulgens, *Nob.*

P. cauda integra, truncata: pinna dorsali et anali postice rotundatis; ventralibus abdomini adnatis.

D. 9 *v.* 10+13; A. 3+14 *v.* 15; P. 17–19; V. 1+5; C. 18 *v.* 19; M. B. 6; Vert[æ]. 23.

TAB. II.

An Pr. macropththalmus, Cuv. et Val. iii. 97? Potuisve Pr. boops Eorund. iii. 103?

Serranus rufus, Bowd. Exc. in Mad. p. 122, note? sed nomine lusitanico falso, ad Squalum quemdam pertinente.

Rarior. Variat corpore toto splendide rubro, dorso fuscescente; pallidove, rubro maculato.

Species haud forsan nova. Quum vero nihil in re tam incerta affirmare ausim, genus quoad specierum distinctiones veras adeo confusum, icone bona illustrare, Ichthyologicis haud ingratum fore speravi.

GEN. BERYX, *Cuv.*

Sp. B. splendens, *Nob.*

B. ruber, squamis muriculatis scaber: pinnis ventralibus radiis mollibus duodecim: membrana branchiostega novem-radiata.

D. 4+14 *v.* 15; A. 4+30; P. 1+17; V. 1+12; C. $\frac{5 + \text{x}}{5 + \text{ix}}$; M. B. 9; Vertæ. 24.

TAB. III. magnitudinis ad normam reductæ.

B. splendens nob. in Proceed. Zool. Soc. 1833. I. p. 142.

Ab uno ad duos pedes longus evadit. Oculi maximi, æneo-fulgentes. Squamæ asperæ, s. superficie dimidii posterioris sub lente retrorsum muriculato vel spinuloso, margine denticulato. Piscis in Madera, vernali præsertim tempore vulgatissimus: ob oculorum magnitudinem, et colorem pulcherrimum conspicuus.

Fam. Bramidæ, *Nob.*

Caput declive; rostro brevissimo, truncato. Pinnæ verticales (basi saltem) squamosæ; dorsalis unica; spinis tenuibus, paucis. Vomer, ossa palati, et intermaxillares plerumque dentibus scobinati, rarius nudi.

Obs. Huc referenda genera *Brama* Bl., *Polymixia* nob., et *Leirus* nob.; quamvis hoc, dentibus formaque pinnarum verticalium aberrans, characterem haud paullulum turbat. An charactere simpliciore facto, genera *Pimelepterus*, *Dipterodon*, *Pempheris*, et *Toxotes* Cuv. huc quoque referenda; familia *Chætodontidarum* Cuv. jam vastissima simul reducta, magisque definita facta?

Gen. POLYMIXIA, *Nob.*

Corpus elliptico-oblongum, compressum; squamis asperrimis, sat magnis. Caput parvum, declive, nuchaque squamosum, epunctatum. Rostrum brevissimum, obtusum, nudum: maxilla inferiore squamosa, cirrisque geminis longis symphysi subtus affixis. Ossa intermaxillaria, omnia palati, dentaria, linguaque dentibus minutis creberrimis scabra. Operculum inerme, rotundatum, squamosum. Præoperculum squamosum; limbo inferiore anguloque nudo, striato, margine eroso-denticulato. Interoperculum nudum, minutissime denticulatum. Pinna dorsalis analisque nudæ, antice elevatæ, spinis debilibus, inconspicuis, brevibus, paucis; basi in sulco sita, squamisque marginalibus sulci elevatis utrinque celata. Pinnæ ventrales septem-radiatæ; radio primo simplici, ut molli, articulato. Cauda furcata. Membrana branchiostega quadri-radiata.

Sp. Polymixia nobilis, *Nob.*

D. 5+36; A. 4 + 16; P. 1 + 16 *v.* 17; V. 1 + 6; C. $\frac{5 + \text{IX}}{4 + \text{VIII}}$; M. B. 4; Vert$^{\text{æ}}$. 29.

Tab. IV. f. 1. Magnitudinis ad normam reductæ.
f. 2. Squama, lente vitrea aucta.

Hab. rarior in alto, prope Maderam.

Gulosorum Maderensium deliciæ, Συστηματο-φιλων crux. Characteribus plurimis gravioribus *Percidarum* familiam omnino referens; sed habitu et affinitate generi *Bramæ* Bl. procul dubio revera quoque proximum; inter utramque hærens: *Chætodontidis* autem veris, quibuscum *Brama* Bl. a cl. et defl. Cuviero relegatur, conjungi prorsus abhorret. Ideoque, ni fallor, *Bramam* Bl., cum generibus quibusdam forsan affinibus ab aliis jam stabilitis, familiam constituere novam *Chætodontidas* et *Percidas* utrinque osculantem, genera duo Maderensia *Polymixia* et *Leirus*, illa *Percidis*, hæc *Chætodontidis* a *Brama* quasi centro utrinque tendens, necnon sibi invicem ac *Bramæ* affinia, satis superque edocent.

Præter affinitatem *Bramæ* supra indicatam, proxima inter *Percidarum* genera affinitas est *Polymixiæ:* hinc *Beryci* Cuv.; ob squamas asperas, ob pinnarum ventralium numerum radiorum anomalum, ob spinas pinnæ dorsalis analisque tenues et paucas, ob formam illarum triangularem, antice elevatam, situmque: hinc *Mullis;* ob cirros geminos ad symphysin maxillæ inferioris, numerumque radiorum membranæ branchiostegæ. Rectius forsan dices, *Polymixiam Beryci* affinem, *Mullo* analogam esse. Analogiam hanc spectat nomen piscis vernaculum Lusitanico-maderense, "*Salmoneta do alto;*" i. e. *Mullus Surmuletus* L. alticolens. Nomen *Polymixia* a πολυς, *multus*, et μιξια, *mistura*, multiplicem generis relationem refert. Charactere scilicet a plurimis generibus commixto, sive conficto, gaudet.

Gen. LEIRUS, *Nob.*

Corpus ellipticum, compressum; squamis deciduis, lævibus, parvis. Caput parvum, declive, nuchaque nudum, punctato-gelatinosum. Rostrum brevissimum, nudum, truncatum.

Os parvum: maxilla superior obtusissima, inferiore brevior, truncata. Dentes minuti, simplices, in utraque maxilla uniseriati: palatini nulli. Opercula inermia, squamosa, marginibus serratis. Pinna dorsalis analisque squamosæ, postice latiores. Cauda subfurcata. Membrana branchiostega septemradiata.

Obs. Genus inter *Bramidas* et *Chætodontidas* revera osculans. Quamvis *Bramæ* Bl. habitu et affinitate proximum, dentibusa berrat, Chætodontidis veris propior: dum squamis lævibus &c., necnon dentibus, a Percidis longius quam *Polymixia*, vel etiam *Brama* ipsa, recedit.

Sp. Leirus Bennettii, *Nob.*

D. 6–8+30 *v.* 31; A. 3+21 *v.* 22; P. 1+21; V. 1+5; C. $\frac{3+\text{I}+\text{VIII}}{3+\text{I}+\text{VII}}$; M. B. 7; Vert$^{\text{æ}}$. 25.

Leirus Bennettii Nob. in Proceedings of the Zool. Soc. 1833. I. p. 143.

Tab. V. f. 1. Piscis, magnit$^{\text{s}}$. ad normam reductæ.

f. 2. Squama, lente vitrea aucta.

Rarior. Pisces admodum deliciosus, nulli nisi præcedenti sapore cedens. Nomen ferat in honorem amici E. T. Bennett, Ichthyologi summi, qui affinitatem cum *Brama* Bl. primus indicavit.

TABULA SYNOPTICA COMPARATIVA.

POLYMIXIA.	BRAMA.	LEIRUS.
(P. Nobilis, Nob.)	(B. Raii Bl.)	(L. Bennettii, Nob.
Habitus et forma *Bramæ*		Habitus et forma *Bramæ*.
Dentes: Maxillæ superioris — Intermaxillares: Scobinati in fascia lata utrinque. Palati Vomerisque: Copiosi. Maxillæ inferioris: Scobinati, in fascia lata utrinque.	Dentes: Maxillæ superioris — Intermaxillares: Exteriores uniseriati. Interiores scobinati in fascia angusta. Palati: In fascia angusta. Vomeris: Nulli. Maxillæ inferioris: Biseriati, fascia intermedia angusta.	Dentes: Maxillæ superioris — Inter maxillares: Uniseriati minutissimi. Palati Vomerisque: Nulli. Maxillæ inferioris: Uniseriati, minutissimi.
Lingua aspera	Lingua lævis	Lingua lævis.
Præoperculum squamosum, limbo denticulato, nudo.	Præoperculum squamosum, limbo integro, nudo.	Præoperculum squamosum, limbo denticulato, nudo.
Operculum squamosum, integrum, inerme.	Operculum squamosum, integrum, inerme.	Operculum squamosum, denticulatum, inerme.
Interoperculum nudum, minutissime denticulatum.	Interoperculum squamosum, integrum.	Interoperculum squamosum, minutissime denticulatum.
Caput epunctatum.	Occiput, nucha, ambitusque oculorum minutissime punctati.	Occiput, nucha, ambitusque oculorum punctato-gelatinosi.
Squamæ asperæ	Squamæ læves	Squamæ læves.
Cirri duo ad symphysin maxillæ inferioris.	Cirri nulli	Cirri nulli
Suborbitaria eroso-dentata.	Suborbitaria integra	Suborbitaria integra.
Pinna Dorsalis et Analis nudæ, basi in sulco sita, antice elevatæ: spinis tenuibus, paucis.	Pinna D. et A. squamosæ, antice elevatæ: spinis tenuibus, paucis.	Pinna D. and A. squamosæ, postice elevatæ: spinis tenuibus, paucis.
Pinna caudalis furcata, squamosa.	Pinna caudalis furcata, squamosa.	Pinna caudalis subfurcata, squamosa.
Pinnarum Ventralium radio primo molli, flexili, articulato nec spinoso-pungente; ultimo libero.	Pinnarum Ventralium radio primo (spina) brevi, tenui, vix pungente, haud articulato; ultimo libero.	Pinnarum Ventralium radio primo (spina) tenui, pungente, nec articulato; ultimo corpori adnato.
Cæca numerosissima, parva, s. tenuia, densissime fasciculata.	Cæca quinque; duobus longis, tribus brevibus.	Cæca quinque, palmatim fasciculata; magna, duobus sublongioribus.
Vesica natatoria mediocris, simplex, elliptica.	Vesica natatoria nulla.	Vesica natatoria magna, simplex, elliptica.
D. 5+36	D. 4+32	D. 6–8+30 *v.* 31.
A. 4+16	A. 2+30	A. 3+21 *v.* 22.
P. 1+16 *v.* 17.	P. 2+19	P. 1+21.
V. 1+6	V. 1+5	V. 1+5.
C. $\frac{5+\mathrm{IX}}{4+\mathrm{VIII}}$	C. $\frac{5+\mathrm{VIII}}{6+\mathrm{VII}}$	C. $\frac{3+\mathrm{I}+\mathrm{VIII}}{3+\mathrm{I}+\mathrm{VII}}$.
M. B. 4	M. B. 7	M. B. 7.
Vertæ. 29.	Vertæ. 43. (41, *Cuv.* et *Val.*)	Vertæ. 25.

ORD. MALACOPTERYGIANÆ.

Fam. Pleuronectidæ ("Poissons plats," *Cuv.*)

GEN. RHOMBUS, *Cuv.*

SS. ii. Oculi remoti; superiore subpostico.

Sp. R. maderensis, *Nob.*

R. corpore ovali; latere sinistro scabriusculo, etuberculato, olivaceo-fusco, ferruginascente, annellis punctorum albidorum ocellatim picto: pinnæ dorsalis analisque radiis inclusis, indivisis: dentibus minutis, uniseriatis: maxillæ superiore ambituque oculorum antice tuberculato-cornutis.

D. 91 – 95; A. 69 – 71; C. 15 – 17.

P. sinistra 10 v. 11; dextra 9 v. 10.

V. sin. 6; dext. 5 *v.* 6.

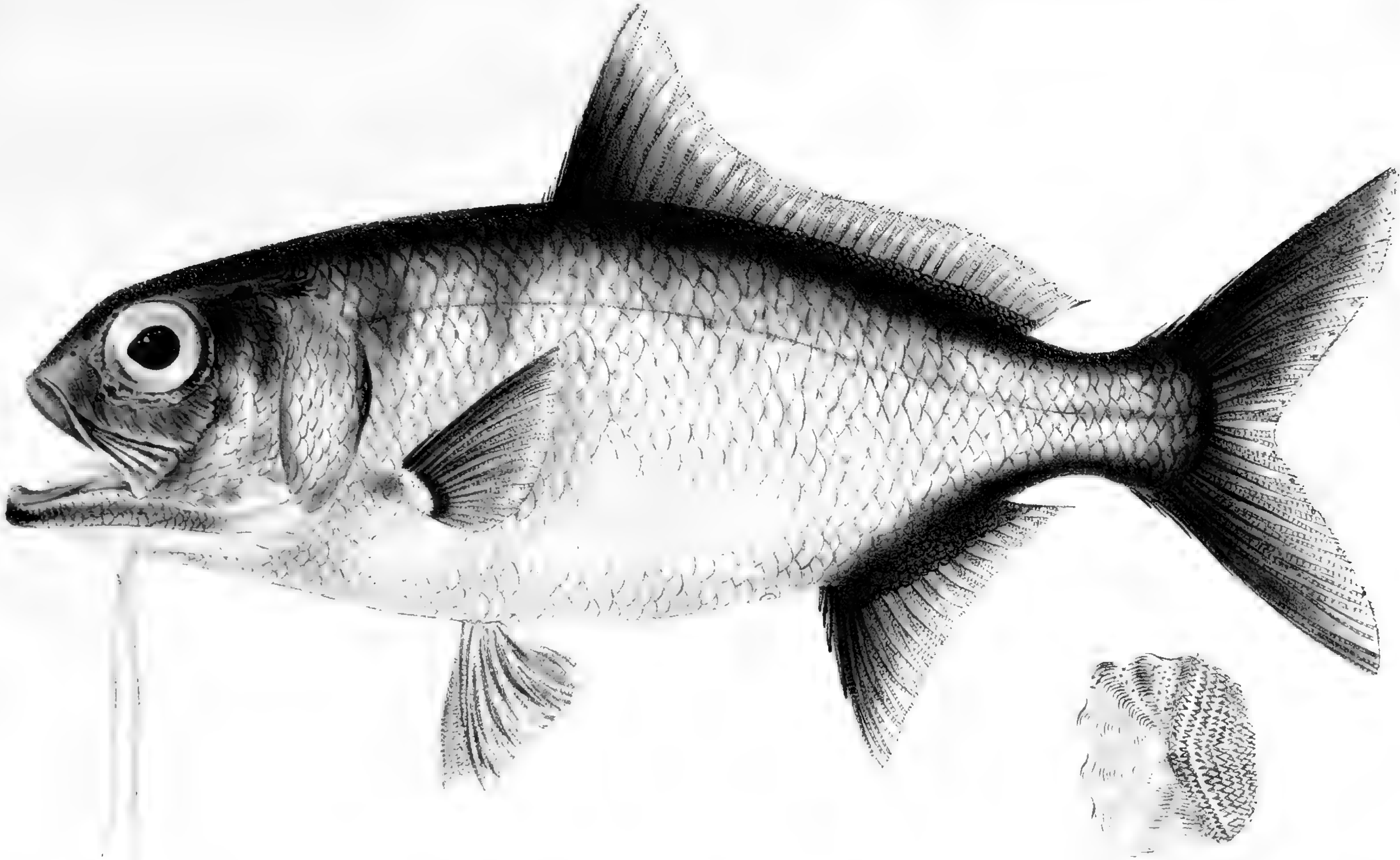
R. maderensis, nob. in Proceedings of the Zool. Soc. 1833. I. p. 143.

TAB. VI. f. 1. magn^s. nat^s.

f. 2. Ejusdem pars anterior lateris dextri.

Hab. rarior in statione navium prope urbem Funchalensem. Ad Insulam Portûs Sancti frequentior dicitur.

FUNCHAL, MADEIRA,
July 24, 1834.



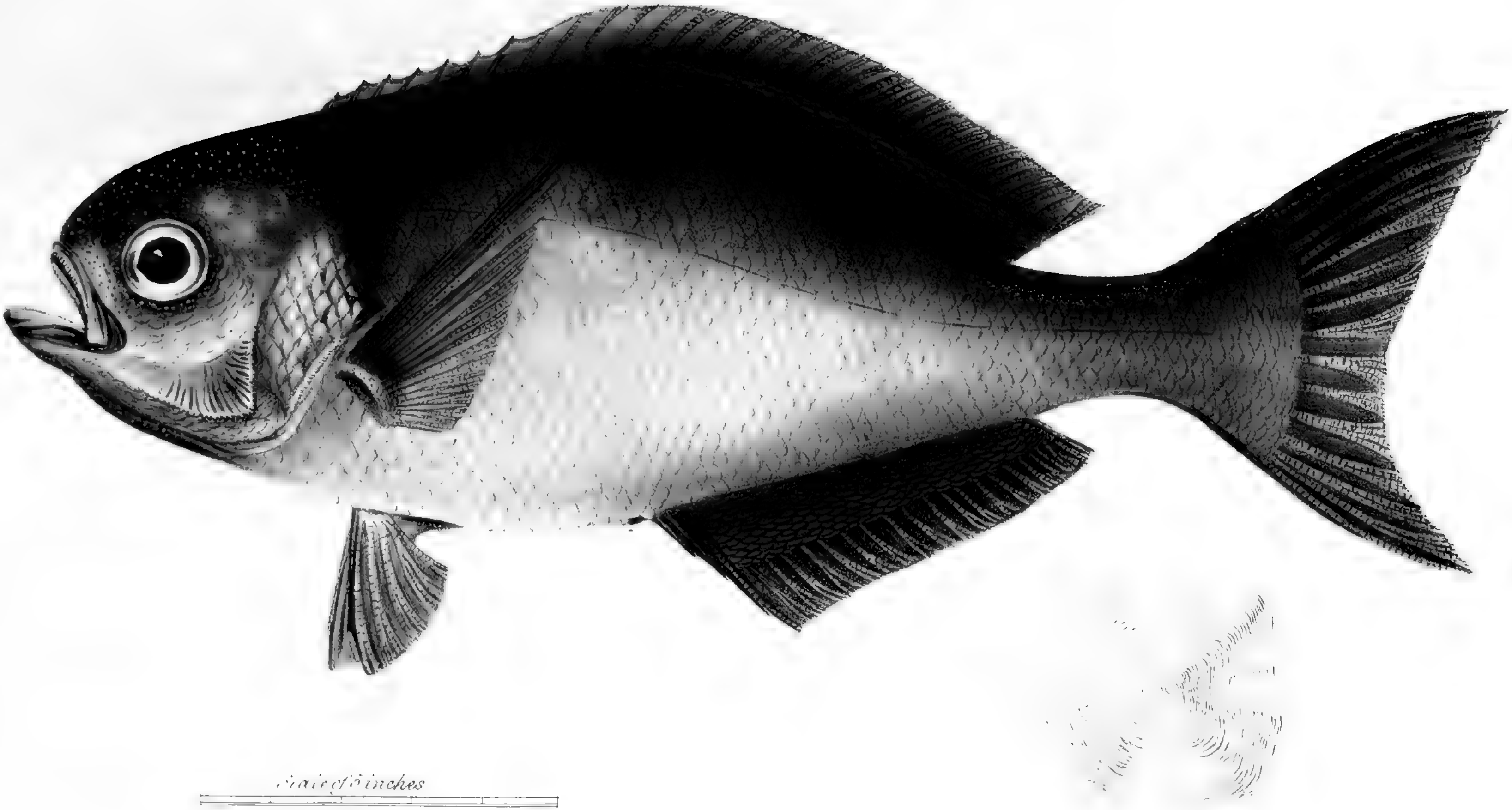
Scale of 5 inches.

M. Young del.t

M. Gauci lith

Polymixia nobilis.

Printed by C. Hallmandel.

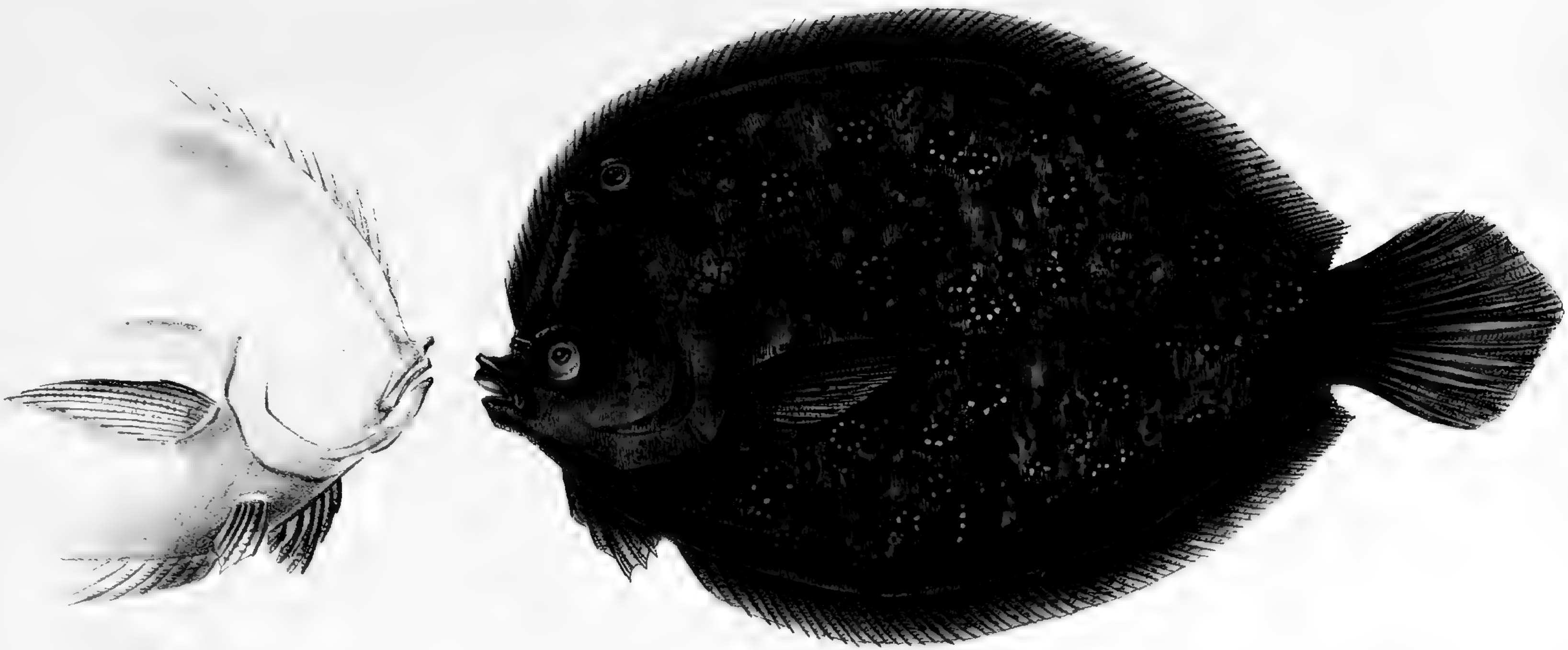


Scale of 5 inches

J. Yung delt

M. Ganci lithog

Leirus Bennettii



Rhombus. Maderensis

Printed by C. Hullmandel

TRANSACTIONS

OF THE

CAMBRIDGE

PHILOSOPHICAL SOCIETY.

VOLUME VI. PART II.

CAMBRIDGE:
PRINTED AT THE UNIVERSITY PRESS;
AND SOLD BY
JOHN WILLIAM PARKER, WEST STRAND, LONDON;
J. & J. J. DEIGHTON; AND T. STEVENSON,
CAMBRIDGE.

M.DCCC.XXXVII.

IX. *On Fluid Motion, so far as it is expressed by the Equation of Continuity.* *By* S. EARNSHAW, M.A. *of St John's College.*

[Read *March* 21, 1836.]

THE difficulty of this subject is so universally admitted, that I hope it will be received as a sufficient excuse for bespeaking the reader's indulgence should any thing occur, in the course of this paper, which he may judge not sufficiently borne out by the arguments on which it is sought to be established.

Though the subject of this communication is by no means new, yet what is brought forward in it will be found to possess some novelty both as to the results obtained, and the manner of treating the subject. Hitherto, nothing more could be done, beyond investigating the differential equations of fluid motion, than to endeavour to generalize the results obtained from a particular integral of the equation of continuity $d_x^2\phi + d_y^2\phi + d_z^2\phi = 0$. It is manifest, however, that the results of such generalization from a particular case, how skilfully soever deduced, must at least be clogged with some degree of uncertainty, and be therefore in some measure unsatisfactory. But in consequence of the discovery of the *general* integral of the equation

$$d_x^2\phi + d_y^2\phi + d_z^2\phi = 0,$$

not only is the difficulty of the subject shifted farther from the threshold of our researches, being reduced to that of interpreting this integral, but we are able to proceed with a much greater degree of generality.

GENERAL PROPERTY.

1. *If in a fluid medium we describe a surface whose differential equation is*

$$u\,dx + v\,dy + w\,dz = 0,$$

the motion of each particle through which this surface passes is in the direction of the normal at the point where the particle is situated.

u, v, w are the velocities, estimated parallel to the co-ordinate axes, of the particle whose co-ordinates are x, y, z. Let there be another particle in the surface very near to this; and let its co-ordinates be $x+dx$, $y+dy$, $z+dz$; and let ds be their distance from each other; by α, β, γ denote the inclinations of ds to the axes. Let also V be the velocity of the former particle, and by α', β', γ' denote the inclinations of the direction in which V takes place to the three axes.

Then,
$$\begin{aligned} 0 &= u\,dx + v\,dy + w\,dz \\ &= V ds.\left(\frac{u}{V}.\frac{dx}{ds} + \frac{v}{V}.\frac{dy}{ds} + \frac{w}{V}.\frac{dz}{ds}\right) \\ &= V ds.(\cos\alpha' \cos\alpha + \cos\beta' \cos\beta + \cos\gamma' \cos\gamma): \end{aligned}$$

and since, from the nature of the case, neither V nor ds is equal to zero,

$$\therefore \cos\alpha' \cos\alpha + \cos\beta' \cos\beta + \cos\gamma' \cos\gamma = 0.$$

But the left hand member expresses the cosine of the inclination of V to ds, which being equal to zero, V and ds must be at right angles to each other; that is, the motion of the particle whose co-ordinates are x, y, z, takes place in a direction perpendicular to the surface whose equation is

$$u\,dx + v\,dy + w\,dz = 0.$$

We may simultaneously draw surfaces of this nature through all parts of the fluid in motion, and shall thus obtain the direction of the motion of every particle. It is manifest that these surfaces are very analogous to the *level-surfaces*, which occur in investigations concerning the equilibrium of heterogeneous fluids.

2. If the expression $udx + vdy + wdz$ be integrable either immediately or by a multiplier, the integral of the equation

$$udx + vdy + wdz = 0,$$

will be of the form

$$f(x, y, z, t) = 0,$$

which will furnish the surfaces alluded to in last article. But if the above expression should neither be integrable at once nor by a multiplier, the integral of the above equation will be of the form

$$\left.\begin{array}{r} f(x, y, z, t) = 0 \\ \psi(x, y, z) = 0 \end{array}\right\},$$

and will denote, not a series of surfaces, but a series of curve lines.

3. It appears that all the particles through which the surface passes whose equation is

$$udx + vdy + wdz = 0,$$

are connected by the common property proved in Art. 1; and as we have no other idea of a *wave-surface* than that it is the locus of particles in a similar state of disturbance, we may be permitted to take the above equation as the expression of that similarity which constitutes a wave-surface; or in other words, we may assume the equation

$$udx + vdy + wdz = 0,$$

as the *mathematical definition of a wave-surface*, or *of a wave-line*, as the case may be.

By the assistance of this definition we may enunciate the proposition of Art. 1 in these terms;—

The motion of every particle of the fluid is perpendicular to the wave-surface in which it is situated.

4. It is proved by Pontécoulant in his "Théorie Analytique du Système du Monde," Tom. I. p. 163, and by most other writers on Hydrodynamics, that if $udx + vdy + wdz$ be at any one instant a complete

differential, it will be so as long as the motion lasts; this is the mathematical expression of the following physical fact;—

If, at any one instant, the motion of the fluid be in wave-surfaces, each surface will travel unbroken through the medium independently of all the rest; that is, as if the others did not exist.

Or, in other words, if the motion at any one instant be in *wave-lines* (Art. 2), then the motion can never resolve itself into *wave-surfaces;* and, conversely, if the motion at any one instant be in *wave-surfaces,* it can never break up into *wave-lines.*

5. If it happen that a particle be situated in two or more wave-surfaces at once, either the particle must be at rest, or the surfaces must have a contact at that point; for, if in motion, its direction must be perpendicular to all the wave-surfaces.

However complicated the motion of the fluid may be, it will always take place either in *wave-lines* or *wave-surfaces.* For the former will be the case when $udx + vdy + wdz$ is not integrable *per se* or by a multiplier, and the latter when this expression is integrable.

Some of these remarks are illustrated in the following example.

Ex. Suppose the motion of the fluid to be such that

$$udx + vdy + wdz = \omega^2 (ydx - xdy).$$

In this case the differential equation of the wave surfaces is

$$ydx - xdy = 0;$$

and therefore, $y = f(t) \,.\, x$

is the general equation of a wave-surface in such a motion of the fluid.

Hence, all the wave-surfaces are planes passing through the axis of z, and the motion of the particles, being at right angles to them, will be in circular arcs parallel to the plane of xy.

All the particles in the axis of z will be at rest, for there the wave-surfaces intersect each other.

6. It does not appear possible to carry these investigations much farther in a perfectly general form; it will be necessary therefore to introduce the hypothesis of the expression $udx + vdy + wdz$ being integrable *per se*. Denote its integral by ϕ, then

$$\phi = \text{constant} = f(t).$$

will be the equation of a wave-surface.

The effect of this hypothesis will be, to exclude from our researches many cases of motion in wave-surfaces, and all motion in wave-lines.

FLUID MOTION OF TWO DIMENSIONS.

7. I have preferred commencing my investigations with this simple case because the results more frequently admit of perfect investigation, and are more easily and briefly expressed in words than in the case of three dimensions.

The equation of continuity now to be considered is

$$d_x^2\phi + d_y^2\phi = 0,$$

and its integral is

$$\phi = F\,\{f(x-\alpha) + g(y-\beta), f, g, t\}$$
$$+ F_1\,\{f(x-\alpha) - g(y-\beta), f, g, t\};$$

subject to the following condition between the arbitrary constants f and g.

$$f^2 + g^2 = 0.$$

In this integral the forms of the functions F and F_1 are perfectly arbitrary, to be adapted in any example to express the law of sequence (as to space) of coexistent wave surfaces, according to the nature of the original disturbance. The arbitrariness of these functions shews that the fluid can transmit a disturbance of any kind which does not violate the continuity of the fluid. α, β are arbitrary constants enabling us to fix the origin of co-ordinates in the most convenient position: they may besides contain functions of t, which depend upon the nature of the original agitation. The functions of f, g, t, which enter under

F and F_1 enable us arbitrarily to fix the epoch from which the time is reckoned, and further to accommodate the wave-surfaces to any proposed form.

These observations will be fully illustrated in a subsequent part of this paper.

8. The object to which it will be necessary first to turn our attention in the above integral is the discovery of the meaning of the constants f, g. Whatever forms be given to F, F_1, whatever origin be taken for co-ordinates, whatever epoch for the time, still f and g are unaffected: and as an infinite number of quantities fulfilling the condition $f^2 + g^2 = 0$ may be invented, and any one set will satisfy the equation $d_x^2\phi + d_y^2\phi = 0$, which in a general view of the question is the only further condition to which they can be subjected, it follows that all imaginable values of f and g ought equally to appear in the general integral (see Art. 27); one set of values giving only a partial solution of the proposed differential equation. Hence the general integral of the equation of continuity of a moving fluid of two dimensions is

$$\begin{aligned}\phi = {} & F_1\{f_1(x-\alpha_1)+g_1(y-\beta_1), f_1, g_1, t\} + F_1'\{f_1(x-\alpha_1)-g_1(y-\beta_1), f_1, g_1, t\} \\ & + F_2\{f_2(x-\alpha_2)+g_2(y-\beta_2), f_2, g_2, t\} + F_2'\{f_2(x-\alpha_2)-g_2(y-\beta_2), f_2, g_2, t\} \\ & + \&c. \ldots\ldots\ldots\ldots\end{aligned}$$

the quantities of $f_1^2, f_2^2, f_3^2 \ldots\, g_1^2, g_2^2, g_3^2 \ldots$ embracing all values from $-\infty$ to $+\infty$. It is manifest, however, that inasmuch as each set of values can be separately made to satisfy the equation of continuity, each set will represent a possible motion, *i. e.* a motion of such a nature that the fluid can transmit it. Hence the general integral just exhibited furnishes us with the following physical fact, which I believe has never* before been fully accounted for;—

* It has been remarked that ϕ may be represented by $F_1(x+y\sqrt{-1}) + F_2(x+y\sqrt{-1}) + \ldots$
$+f_1(x-y\sqrt{-1}) + f_2(x-y\sqrt{-1}) + \ldots$
and thence the superposition of disturbances has been inferred: but before this principle can be inferred, is it not necessary to shew that $F_1(x+y\sqrt{-1}) + F_2(x+y\sqrt{-1}) + \ldots$

Any number of disturbances separately, though simultaneously, excited in a fluid medium, will be separately, independently and simultaneously transmitted through the fluid, each as perfectly as though the others did not exist. See Art. 4.

9. Having ascertained this to be the meaning of the integral in its general form, it will be sufficient now to consider the transmission of one disturbance only; and if this investigation be carried on upon the general hypothesis of a single disturbance of *any* kind affecting the fluid, the results will be of a general character also. This point will be gained by keeping our integral under the form

$$\phi = F\,\{f(x-a) + g(y-\beta), f, g, t\}$$
$$+ F_1\,\{f(x-a) - g(y-\beta), f, g, t\}.$$

From this we shall proceed to deduce the following results.

I. Motion cannot be represented by *one* of these functions alone. For, if possible, let motion be represented by

$$\phi = F\,\{f(x-a) + g(y-\beta), f, g, t\},$$

or, for brevity, by $\phi = F$.

Then the (velocity)^2 of the particle whose co-ordinates are (x, y) would be

$$= (d_x\phi)^2 + (d_y\phi)^2$$
$$= f^2 . F'^2 + g^2 . F'^2$$
$$= (f^2 + g^2) . F'^2 = 0:$$

that is, the medium is at rest. F' is used to denote the differential coefficient of F with regard to the quantity $f(x-a) + g(y-\beta)$.

is a more general expression than *one* function $F(x+y\sqrt{-1})$? In the *Integral Calculus*, we know that $C_1 + C_2 + C_3 \ldots$ represents only one constant C: are we certain that $F_1 + F_2 + \ldots$ represents more than F? are we sure that $F_1, F_2, F_3 \ldots$ are so essentially distinct that they *cannot* be united in one function?

II. Sometimes the disturbance may be such as to render it possible to introduce t entirely into the parts $f(x-\alpha) \pm g(y-\beta)$, so that the integral may be written

$$\phi = F\{f(x-\alpha-T) + g(y-\beta-\tau), f, g\}$$
$$+ F_1\{f(x-\alpha-T) - g(y-\beta-\tau), f, g\},$$

T and τ being functions of t.

Let us in this case refer the motion of the fluid to the moveable origin, in the plane of xy, whose co-ordinates are $T+\alpha$, $\tau+\beta$; which will be done by writing $x'+T+\alpha$, $y'+\tau+\beta$ for x and y; then the state of the fluid is expressed by

$$\phi = F(fx'+gy', f, g) + F_1(fx'-gy', f, g);$$

an equation which does not involve t; the state of the medium is therefore perfectly *invariable* with respect to the moveable origin. The original disturbance, then, of what kind soever it may be, is transmitted through the medium *unaltered in all respects,* with a velocity equal to that of the origin of co-ordinates, that is, of the point $T+\alpha$, $\tau+\beta$; hence the velocity of transmission, in the direction of x, $= d_t T$, and in the direction of y, $= d_t\tau$.

III. It may happen that it will be impossible to introduce t *entirely* within the parts $f(x-\alpha) \pm g(y-\beta)$; let it however be done as nearly as possible, so that the equation may be written

$$\phi = F\{f(x-\alpha-T) + g(y-\beta-\tau), f, g, t\}$$
$$+ F_1\{f(x-\alpha-T) - g(y-\beta-\tau), f, g, t\}.$$

After transposing the origin as before, this becomes

$$\phi = F\{fx'+gy', f, g, t\} + F_1\{fx'-gy', f, g, t\}.$$

Whence it appears that the *forms* of the equations of the wave-surfaces will remain unchangeable; but t entering into the parameters, shews that the magnitudes of the wave-surfaces will change with the time. In this case therefore the wave-surfaces will be transmitted through the medium, unchanged as to their *nature* only, but not as to their *magnitude*. Thus, if the original disturbance produced cylindrical wave-surfaces of concentric circular bases which at a given instant had certain magnitudes, the wave-surfaces would continue cylindrical throughout the motion, but the common centre of the bases, or the common axes of the cylinders would be transmitted in the direction of x with the velocity $d_t T$, and in the direction of y with the velocity $d_t \tau$, and the radii of the cylinders would constantly undergo variation of magnitude.

10. These are the principal results of a general character which I have been able to obtain. There is yet to be considered a certain integral of the equation of continuity; namely,

$$\phi = C \log r + C',$$

where $r^2 = (x - \alpha)^2 + (y - \beta)^2$; which has hitherto formed the basis of investigations in this part of fluid motion.

Now in the general integral for a single disturbance, namely,

$$\phi = F\{f(x-\alpha) + g(y-\beta), f, g\} + F_1\{f(x-\alpha) - g(y-\beta), f, g\},$$

let the forms of F and F_1 be assumed to be logarithmic, then

$$\begin{aligned}\phi &= \frac{C}{2} \log\{f(x-\alpha) + g(y-\beta)\} + \frac{C}{2} \log\{f(x-\alpha) - g(y-\beta)\} \\ &= \frac{C}{2} \cdot \log\{f^2(x-\alpha)^2 - g^2(y-\beta)^2\} \\ &= \frac{C}{2} \cdot \log(f^2 r^2); \quad \because f^2 = -g^2 \\ &= C \log r + C \log f \\ &= C \log r + C'.\end{aligned}$$

From this investigation it appears, that this integral is not equivalent to the general integral found by the usual process of integration, but is in reality a very particular integral, applicable only to those disturbances in which the sequence of the wave-surfaces can be expressed by logarithms.

11. By assigning other forms to F and F_1, other, and some of them very simple*, expressions for ϕ may be obtained, and as in Art. 5 the character of the motion may be determined. But as results so obtained would only be of a partial application, it is unnecessary to pursue the idea further. There is one form however which seems deserving of some consideration, as it presents us with a species of fluid motion entirely distinct from that denoted by the integral

$$\phi = C \log r + C';$$

and of such a nature as was supposed by Euler to make $udx + vdy$ not a complete differential; it is the following,

$$\phi = a \tan^{-1}\left(\frac{y - \beta}{x - \alpha}\right).$$

The equation of a wave-surface being in general (Art. 6) ϕ=constant, will in the present case be reduced to

$$y - \beta = f(t) . (x - \alpha),$$

which is that of planes intersecting each other in a line, parallel to the axis of z, which passes through the point (α, β), which is moveable or fixed according as α, β are or are not functions of t. And since the motion of every particle is perpendicular to its wave-surface, the general motion of the fluid at any instant will be in arcs of circles having their centres in the line which passes through (α, β): and the velocity of a particle being

$$= \sqrt{(d_x\phi)^2 + (d_y\phi)^2} = \pm \frac{a}{r},$$

in this case varies inversely as its distance from the centre.

The law of velocity in this instance is therefore the same as in the one last considered; but while that velocity carried the particles *towards* or *from* a fixed or moveable centre, it here carries them *round* that centre in such a manner that all of them describe equal areas in equal times about it.

* A very simple one is $\phi = a . \epsilon^{b(x-\alpha)} . \cos b(y - \beta)$.

MOTION OF THREE DIMENSIONS.

12. The differential equation for this case is

$$d_x^2\phi + d_y^2\phi + d_z^2\phi = 0,$$

and its integral is

$$\begin{aligned}\phi = &F_1\{f(x-\alpha) + g(y-\beta) + h(z-\gamma), f, g, h, t\}\\ &+ F_2\{-f(x-\alpha) + g(y-\beta) + h(z-\gamma), f, g, h, t\}\\ &+ F_3\{f(x-\alpha) - g(y-\beta) + h(z-\gamma), f, g, h, t\}\\ &+ F_4\{f(x-\alpha) + g(y-\beta) - h(z-\gamma), f, g, h, t\},\end{aligned}$$

f, g, h being constants, subject only to the condition

$$f^2 + g^2 + h^2 = 0.$$

It appears that in this case each set of values of f, g, h furnishes *four* independent arbitrary functions in the value of ϕ; and without repeating the reasoning of Art. 8, we may at once state, that the general value of ϕ will consist of the sum of an infinite number of such sets of values as the one above exhibited: and the same physical inferences may also be made here for three dimensions, as there for two; namely, that each set of values of f, g, h furnishes a distinct wave-surface, which is transmitted independently of all the others.

It may also be mentioned, that if these distinct wave-surfaces should be so situated as to be geometrically describable according to the same law, that is, if their equations be of the same form, and differ only in the value of the parameters which enter into them, and if those values are consecutive, then we are not to take the separate surfaces, but the surface which touches them all, as that form of the wave which in such a case is denoted by the general integral. Instances of this process will be given afterwards, see Art. 28. This property will enable us to deduce integrals of the equation of continuity adapted to wave-surfaces of any proposed form. See Art. 29.

13. By reasoning precisely similar to that employed in Art. 9, we may arrive at the following results which are general.

I. Motion cannot be represented by *one* of the functions alone, which enter into the value of ϕ.

II. Under certain conditions a disturbance may be transmitted through a medium unchanged in form and intensity, and the velocity and direction of transmission seems to be arbitrary: that is, to depend on the manner in which the disturbance is excited.

III. In other cases the velocity and direction of the transmission may be arbitrary as before; but the form and intensity of the disturbance will undergo continual change with the time.

IV. The proper motion of every particle is in the direction of a normal to the wave-surface in which it is situated.

ON THE MOTION OF ELASTIC FLUIDS.

14. The general equation of continuity for this case is

$$d_x(\rho u) + d_y(\rho v) + d_z(\rho w) + d_t(\rho) = 0.$$

But as it is impossible to enter upon the discussion of this equation in the general state in which it now stands, we shall, as in Art. 6, be under the necessity of introducing some hypothesis.

First. We may suppose the expression

$$\rho u dx + \rho v dy + \rho w dz \pm a^2 \rho dt,$$

a complete differential of x, y, z, t: in which case

$$\rho (u dx + v dy + w dz),$$

will be a complete differential of x, y, z: and therefore

$$u dx + v dy + w dz$$

will be integrable by a multiplier: wherefore the properties of wave-surfaces proved in the first five Articles will be applicable to this case.

Now putting ϕ for the integral of the above differential of x, y, z, t, we have

$$\rho u = d_x\phi, \quad \rho v = d_y\phi, \quad \rho w = d_z\phi, \quad \pm a^2\rho = d_t\phi:$$

and consequently the equation of continuity becomes

$$d_t^2\phi \pm a^2 (d_x^2\phi + d_y^2\phi + d_z^2\phi) = 0,$$

which will furnish two cases, according as a^2 is positive or negative.

Secondly. We may limit ourselves to those cases in which the proper motions of the elastic medium are very small: which limitation will enable us to neglect $d_x\rho$, $d_y\rho$, $d_z\rho$, inasmuch as the variation of density produced by such small changes of the relative position of the particles, will be too trifling to require attention when multiplied by the small quantities u, v, w; and consequently the equation of continuity may be written

$$\frac{d_t\rho}{\rho} + d_x u + d_y v + d_z w = 0.$$

Now as it has been proved* for this case that $udx + vdy + wdz$, is a complete differential ($= d\phi$ suppose), this will become

$$\frac{d_t\rho}{\rho} + d_x^2\phi + d_y^2\phi + d_z^2\phi = 0:$$

in which we may write $-\frac{d_t^2\phi}{a^2}$ for $\frac{d_t\rho}{\rho}$, as is shewn by all writers on this subject, a^2 being equal to the fraction

$$\frac{\text{pressure}}{\text{density}}.$$

Wherefore by help of this hypothesis, we are able to present the equation of continuity under the following form

$$d_t^2\phi = a^2 (d_x^2\phi + d_y^2\phi + d_z^2\phi).$$

* Pontécoulant, *Théor. Anal.* Vol. I. p. 164.

15. If we take the equation

$$0 = d_t^2\phi + a^2(d_x^2\phi + d_y^2\phi + d_z^2\phi),$$

its integral is

$$\begin{aligned}\phi = &\ F_1\{f(x-\alpha) + g(y-\beta) + h(z-\gamma) + at, f, g, h\}\\ &+ \mathrm{F}_1\{f(x-\alpha) + g(y-\beta) + h(z-\gamma) - at, f, g, h\}\\ &+ F_2\{-f(x-\alpha) + g(y-\beta) + h(z-\gamma) + at, f, g, h\}\\ &+ \mathrm{F}_2\{-f(x-\alpha) + g(y-\beta) + h(z-\gamma) - at, f, g, h\}\\ &+ \&c. \ldots\end{aligned}$$

subject to the condition

$$f^2 + g^2 + h^2 + 1 = 0.$$

Each set of values of f, g, h will furnish eight arbitrary functions in the value of ϕ. As there are no arbitrary functions of t to be added to complete the integral, as was necessary in incompressible fluids, t enters only in the form above exhibited; and it is evident that the integral is of such a nature as to render it impossible to make t disappear by changing the origin (as in Art. 9, II.): wherefore the reasoning of (III. Art. 9) can be applied here; from which we infer, that the extent and intensity of the disturbance are continually changing, inasmuch as the equation of the wave-surface does not change its form, but only the magnitude of its parameters which are functions of t. As a wave-surface expands, a point which has a certain relation to it remains fixed in the medium: so that the expansion may be said to take place about this point.

16. The following is also a solution of the differential equation of last Art.

$$r\phi = F(r + at\sqrt{-1}) + f(r - at\sqrt{-1});$$

r being $= \sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}$.

And by assigning particular forms to F and f, we shall obtain cases of possible motion *ad libitum*.

Thus, let F and f be logarithmic, as in Art. 10.

$$\therefore\ r\phi = C\log(r + at\sqrt{-1}) + C\log(r - at\sqrt{-1})$$
$$= C\log(r^2 + a^2t^2);$$
$$\therefore\ \phi = \frac{C\log(r^2 + a^2t^2)}{r}.$$

This will denote spherical waves converging towards their common centre with the velocity

$$\frac{a^2t}{r}.$$

17. The following is also a solution:

$$r\phi = C\tan^{-1}\left(\frac{r}{at}\right);$$

which denotes spherical waves diverging from a common centre with a velocity a.

18. I come now to by far the most interesting case; namely, that wherein the velocities of the particles of the fluid are small in comparison of a, and where

$$d\phi = udx + vdy + wdz.$$

The differential equation for this kind of motion is,

$$d_t^2\phi = a^2(d_x^2\phi + d_y^2\phi + d_z^2\phi),$$

and its integral is

$$\phi = F_1\{f(x-\alpha) + g(y-\beta) + h(z-\gamma) + at,\ f,\ g,\ h\}$$
$$+ \mathrm{F}_1\{f(x-\alpha) + g(y-\beta) + h(z-\gamma) - at,\ f,\ g,\ h\}$$
$$+ \&c. \ldots$$

the form of the integral being precisely the same as in Art. 15, but the equation of condition among the constants f, g, h is here

$$f^2 + g^2 + h^2 = 1.$$

We observe that f, g, h, may all be possible, and in what follows they will be supposed possible quantities. This circumstance will alone completely distinguish the fluid motion now under consideration from all that has yet been investigated.

Each set of values of f, g, h will furnish eight independent arbitrary functions in the expression for ϕ, but for reasons precisely similar to those advanced in Art. 8, we need only consider one set at a time, which we know denotes a possible motion; the possibility of the coexistence of any number of such sets in the value of ϕ being physically interpreted by saying,—that the fluid can simultaneously transmit any number of disturbances, each as though the others did not exist.

In Art. 13 it was mentioned that one arbitrary function alone could not represent fluid motion; but in the case before us, each function represents a possible motion, and it will be shewn hereafter that the eight arbitrary functions furnished by one set of values of f, g, h denote as many independent waves moving in different directions: and therefore no generality will be lost by employing only one of the eight functions; and the reasoning upon it will be applicable to the other seven; and the full effect denoted by one set of values of f, g, h will be seen by supposing the eight waves corresponding to the eight functions to coexist and to be transmitted simultaneously in the medium.

19. We are at liberty to take any one of the eight functions as a general representative of all, suppose then that

$$\phi = F\{f(x-\alpha) + g(y-\beta) + h(z-\gamma) - at, f, g, h\}\text{......}(1)$$

The origin of co-ordinates is a *fixed* point in space; but let it be transferred to the point X, Y, Z, which will be done by writing $X+x'$, $Y+y'$, $Z+z'$ for x, y, z respectively, and then

$$\phi = F\{fx' + gy' + hz' + f(X-\alpha) + g(Y-\beta) + h(Z-\gamma) - at, f, g, h\}.$$

Now as the new origin is arbitrary, we will suppose it *moveable* instead of fixed, and that it moves in such a manner as to satisfy the equation

$$f(X-\alpha) + g(Y-\beta) + h(Z-\gamma) = at\text{......}(2)$$

which supposition is allowable because f, g, h are possible quantities.

When referred to this new origin, the state of the fluid is represented by the equation

$$\phi = F(fx' + gy' + hz', f, g, h) \ldots\ldots (3).$$

Now the co-ordinates of the moveable origin are X, Y, Z, amongst which quantities there is no relation but that which is expressed by equation (2); wherefore, though the origin must be some where in the plane whose equation is (2), its position in that plane is perfectly indeterminate. And what point soever in this plane we take for origin, the state of the fluid in reference to that point is expressed by (3), an equation which does not involve t. Hence the state of the fluid is *invariable* with respect to the plane (2); *i. e.* to whatever point in this plane we transpose the origin, the same values of x', y', z' will give the same value of ϕ, or denote a point in the same wave-surface. Hence the wave-surfaces denoted by (1) are always parallel to the plane (2), and preserve an invariable distance from it. Consequently the wave denoted by (1) is a plane wave travelling parallel to, and at the same rate as, the plane (2), which may be called the *plane of origins.*

Now because $f^2 + g^2 + h^2 = 1$, f, g, h denote the cosines of the angles of inclination of a line to the axes of co-ordinates; the line itself is, as is well known from the principles of Analytical Geometry, perpendicular to the plane (2), and if drawn from the origin of the co-ordinates X, Y, Z, that is, from the original origin fixed in space, its length is equal to at. Wherefore the plane of origins travels in such a manner that at is the length of the perpendicular upon it from a fixed point: and consequently it moves with the uniform velocity a. As the wave-surfaces are always parallel to it, and preserve their distances from it unchanged, they are transmitted through the medium with the same uniform velocity a.

20. If the distance of a wave-surface from the plane of origins be denoted by p,

$$p = fx' + gy' + hz',$$

and therefore the state of the fluid particles composing that wave-surface, is represented by

$$\phi = F(p, f, g, h).$$

Wherefore the state of the particles composing a wave-surface is a function only of its distance from the plane of origins. By varying p we pass from one wave-surface to another; and a corresponding change takes place in the value of $F(p, f, g, h)$ or ϕ; the degree or law of this variation in ϕ entirely depends upon the form of the function F, which is arbitrary. If now we enquire what is meant by the fact of ϕ being denoted by an *arbitrary* function, it is clear we learn from that arbitrariness, that wave-surfaces may follow each other according to any arbitrary law of sequence; if that law be continuous (that is, if F be a continuous function) it will furnish us with an infinite number of wave-surfaces following close upon each other and composing a simple *wave*, due to an original single disturbance of a continuous nature. If F be a discontinuous function, the law of sequence of the wave-surfaces will be discontinuous also, and therefore in this case we shall have a wave composed of the broken parts of several waves, joined together, or separated by finite intervals, as the case may be. All this will be very evident, by considering p as the abscissa, and ϕ, which represents the state of the medium, as the ordinate of a curve whose equation is

$$\phi = F(p, f, g, h).$$

Upon the whole, then, equation (1) represents a *plane-wave*, (made up of plane wave-surfaces), which is uniformly transmitted in a direction *from* the origin with the velocity a, unchanged in all respects, through the medium; in such a manner that it continues parallel to itself during the transmission, and the perpendicular upon it from the fixed origin falls within that part of space where x, y, z are all positive.

21. By similar investigations which it is not necessary to enter upon, it may be shewn that, the seven other functions involved in the expression for ϕ, denote respectively seven plane waves, each transmitted

uniformly with the velocity a. And perpendiculars let fall upon them from the fixed origin will lie in the seven other parts into which space is divided by the co-ordinate planes, one in each. All the properties which have been shewn to belong to the plane-wave in the last Article, may be shewn to belong also to each of these.

22. Without further proof it will be sufficiently evident that if we suppose ϕ to take its most general shape, that is, to be the sum of an infinite number of arbitrary functions, involving all possible values of f, g, h, it would represent an infinite number of independent plane-waves, each transmitted with the same velocity a: and each moving parallel to itself. These waves would be inclined at all possible angles to the axes. This corresponds to the most general form of the integral. Any number of values of f, g, h might be omitted in the integral, and then the corresponding plane-waves would be deficient: or, f, g, h might vary according to some continuous law. Two cases therefore ought to be considered,

1. When f, g, h vary in passing from one function to another in the value of ϕ *independently*.

2. When f, g, h vary *continuously*.

The former of these cases has already been shewn to belong to independent plane-waves: but the latter, which is most important, will be considered presently. (Art. 26.)

23. When the disturbing cause gives rise to a *plane*-wave, the expression for the state of the medium must be

$$\phi = F\{f(x-\alpha) + g(y-\beta) + h(z-\gamma) - at\}.$$

Now it has been shewn for this case (Art. 19) that the state of the fluid may be represented by

$$\phi = F(fx' + gy' + hz'),$$

x', y', z' not involving t, and being measured from an origin situated any where in the plane of origins whose equation is

$$f(x-\alpha) + g(y-\beta) + h(z-\gamma) = at.$$

Now this equation denotes a plane of *infinite* extent, and as the wave-surface may be referred to any point in it, without altering the equation

$$\phi = F(fx' + gy' + hz'),$$

which expresses the state of the fluid, the same values of x', y', z', which, measured from a given origin, give the value of ϕ at a disturbed point, will give, when measured from every point in the plane of origins, a series of other points in the fluid, at which the value of ϕ is the same as before; therefore the plane of origins and the wave would seem to be of equal extent. What are we then to infer from this circumstance? The plane of origins is manifestly infinite, but the wave cannot stretch out beyond the limits of the medium. As far, however, as the fluid extends, so far we can prove the wave-surface to extend; for nothing prevents the equation

$$\phi = F(fx' + gy' + hz')$$

from being true for an infinite plane, but the fact of there being, beyond certain limits, no fluid medium: so far, therefore, as there is fluid, so far the wave-surface extends. Hence,

A plane-wave cannot be transmitted through any fluid unless it extend completely across the medium, from boundary to boundary.

Hence, if the medium be divided into two parts by a fixed screen, in which is a finite aperture, the fragment of the wave-surface (supposed parallel to screen) which passes through the opening cannot continue plane; but it must somehow or other abut upon the back of the screen. What is the precise form which it takes is very difficult to determine. The problem, however, does not seem to be absolutely impossible; and appears to depend entirely upon the discovery of some means of introducing the condition of a limit to the medium into the mathematical expression for ϕ.

24. I think it is sufficiently clear that the expression for a plane-wave necessarily supposes the extent of the medium to be *infinite*. When, therefore, the fluid is of finite extent, we must suppose it infinite,

and then introduce fixed screens so as to inclose a portion of any proposed shape. Now the effect of a fixed screen is simply to prevent the particles in contact with it taking any *normal* motion; and this being the only effect, we may suppose the screen removed, providing we introduce into the expression for ϕ such a function as shall represent the condition of the normal velocities being zero. In one case this can be very simply done, and may serve as an example of what is to be effected, and the process to be followed in other cases.

Suppose the medium bounded by a plane whose equation is

$$x = c \ldots\ldots\ldots\ldots\ldots (1),$$

and let a plane-wave be transmitted through this medium, such that

$$\phi = F(fx + gy + hz - at) \ldots\ldots\ldots (2).$$

This equation supposes the medium infinite; and we must now introduce the condition, that at every point in a plane whose equation is (1) the velocity in the direction of x is zero. Let, therefore, the value of ϕ which fulfils this condition be expressed by the equation

$$\phi = F(fx + gy + hz - at) + F_1\{f'(x-\alpha) + g'(y-\beta) + h'(z-\gamma) - at\}:$$

this is the proper* form of assumption, for otherwise the equation of continuity would not be satisfied. The velocity in direction of x is

$$d_x\phi = f.F'(fx + gy + hz - at) + f'F_1'\{f'(x-\alpha) + g'(y-\beta) + h'(z-\gamma) - at\},$$

which must be zero when $x = c$.

$$\therefore\ 0 = f.F'(fc + gy + hz - at) + f'F_1'\{f'(c-\alpha) + g'(y-\beta) + h'(z-\gamma) - at\}.$$

Now it is impossible that this equation should always be true, seeing

* The general integral of the equation of continuity shews that ϕ must consist of the sum of a series of functions of the form

$$F\{f(x-\alpha) + g(y-\beta) + h(z-\gamma) - at\}. \quad \text{(Art. 26.)}$$

Hence we might add to the right-hand member of equation (2) several functions, but as we have only one condition to satisfy, one additional function is sufficient. Hence, that assumed for ϕ in the text is the proper form for the problem under consideration.

that y, z, and t are indeterminate, unless the forms of F' and F_1' be the same: hence we may write

$$f . F'(fc + gy + hz - at) = -f' . F'\{f'(c - a) + g'(y - \beta) + h'(z - \gamma) - at\};$$

$$\therefore f = -f':$$

$$\text{and } fc + gy + hz - at = f'(c - a) + g'(y - \beta) + h'(z - \gamma) - at;$$

$$\therefore g' = g$$

$$h' = h$$

$$c = -(c - a) \text{ or } a = 2c$$

$$\beta = 0$$

$$\gamma = 0.$$

$$\therefore \phi = F(fx + gy + hz - at) + F\{f(2c - x) + gy + hz - at\}.$$

This expression represents two plane-waves: the latter of which is the effect of the screen, and is that which is usually called the reflected-wave. It is evident that the angles of incidence and reflection are equal.

25. When $f = 0$, there is no reflected-wave. Now when $f = 0$, the wave and the screen are at right angles to each other. Hence, a wave-surface may be cut into any number of parts by fixed *normal* screens, without affecting the motion. Consequently a plane-wave may be transmitted as perfectly through a prismatic tube, as through an infinite medium.

If the boundaries of the medium are not normals to the wave there will of necessity be reflection. This circumstance produces nodes and loops, and probably affects the timbre of the notes, and the tone of musical instruments.

26. I come now to consider the second case of Art. 22, viz. Where f, g, h in the value of ϕ, vary *continuously* in passing from one function to another.

In this case, that ϕ may be complete, f, g, h must have all values from -1 to $+1$, and therefore in assigning to f, g, h, all possible values in the function

$$F\{f(x - a) + g(y - \beta) + h(z - \gamma) - at\},$$

we shall obtain all possible functions, and amongst others 7 functions similar to those mentioned before (Art. 21). This function, therefore, may be considered as the representative of all, and consequently ϕ is now properly represented by

$$\phi = \Sigma F\{f(x-a) + g(y-\beta) + h(z-\gamma) - at, f, g, h\}.$$

This equation is the representative of any continuous wave, whatever be its form. The integral of the equation of continuity furnishes no other general means of representing a continuous wave. Hence, whatever be the nature of the original disturbance which gives rise to a continuous wave, the above equation teaches, that the wave may be hypothetically resolved into an infinite number of plane-waves moving with the same velocity a.

Plane-waves are therefore shewn to be the proper components of curvilinear waves.

And since the values of f, g, h are the cosines of the inclinations of these component-waves to the co-ordinate axes, if the original disturbance produce a single wave, f, g, h will follow some *law*, that *law* being in fact the condition that the original disturbance may be single.

The disturbance being thus resolved into component plane-waves, each component is to be supposed transmitted parallel to itself with the velocity a, and at any time we may compound them into a single wave, by finding the surface to which they are all simultaneously tangent planes.

27. In confirmation of these views, I shall make a few observations upon *general* and *singular* solutions of common differential equations of two variables, the theory of which is well understood and allowed.

Suppose, for instance, it were required to find a curve, such that the rectangle of the perpendiculars drawn from two given points upon any tangent shall be constant. This problem produces a differential equation whose general solution is

$$y = fx \pm \sqrt{a^2 f^2 + b^2},$$

f being an arbitrary constant. This quantity gives rise to a difficulty. For though the proposed problem is sufficiently specific, yet it affords no data for the determination of f, and consequently the curve sought remains as undetermined, and the problem apparently as unsolved, as ever; all that we can gather from the above integral being, that a straight line whose position depends on the value of f will fulfil the proposed conditions.

It may be said, however, that any value given at pleasure to f will determine a line answering the conditions of the question; but it is clear that a line so found can only be considered as a *partial* solution; inasmuch as the fixing upon a particular value of f tacitly implies the possession of data enabling us to decide upon that value in preference to all others. Now such a decision cannot be received, unless the data which led to it are supplied by the conditions of the proposed problem; and, as we have seen, no such data exist; consequently no particular value of f can be received. Thus it appears, that though the integral

$$y = fx \pm \sqrt{a^2 f^2 + b^2}\,*$$

furnishes partial solutions of the proposed problem without number, it does not present us with the required curve. The only resource left is, to employ equally all possible values of f from $-\infty$ to $+\infty$; all having an equal claim to have weight in the general interpretation of the above integral. This will present us with an infinite number of straight lines, not drawn at random, but according to a law expressed by

$$y = fx + \sqrt{a^2 f^2 + b^2},$$

and by the intersections of consecutive lines forming a curve to which they all have an equal relation, being tangents, which is allowed to be the curve required.

* Since f admits the sign $-$ as well as $+$, we have, by taking all the variations of sign, four straight lines; one in each portion of space comprehended between the co-ordinate axes. See Arts. 18, 21.

The general inference from this reasoning appears to be this: that when the integral of a differential equation contains constants (introduced by integration), the values of which the proposed conditions of the problem are not sufficient to determine, the singular solution* is the proper integral for that particular problem.

28. Upon these grounds I state the following general principle,

A wave may, at any moment, be resolved into plane component waves, each of which is a tangent to the original wave. These components may be supposed to be uniformly transmitted with the velocity a, *and at any time they may be compounded into a single wave by taking that surface to which they are all simultaneously tangents.*

Hence if the form of a wave-surface at any one instant be known, its form at any other time will be determinable from geometrical principles.

The thickness (or, as it is sometimes called, the breadth) of a wave is never altered by transmission.

Ex. 1. In a quiescent medium let us suppose one of its particles to expand, pushing equally from its centre on every side the adjacent particles. The effect of such a disturbance will be a sudden condensation in its neighbourhood, which we may divide into concentric spherical wave-surfaces, for each one of which ϕ is constant; though from surface to surface ϕ may vary. Now resolve any one of these surfaces into its components by drawing an infinite number of tangent planes to it; each one is transmitted with the velocity a parallel to its edge, and thus at the end of any period they will be equidistant from the centre of original disturbance, and be tangent planes to a spherical surface, which is therefore the form of the wave at any moment.

Ex. 2. In a quiescent medium let all the particles situated in a given

* Or rather the solution determined by the usual method of finding the singular solution, for as is well known such a one may not happen to be a *singular* solution but a *particular* integral.

straight line suddenly expand equally. In this case each surface of equal density is a cylinder with hemispherical ends; the given line being the axis, and the extremities of the line being the centres of the hemispheres.

By resolving each wave-surface into component waves, by drawing tangent planes to the cylinder and hemispheres, and supposing these components transmitted, each parallel to itself with the velocity a, it will appear that at any time each wave-surface is of the form of a cylinder with hemispherical ends. The radius of the cylinder increases with the uniform velocity a.

29. It is sufficiently manifest, from what has already been done, that the form of a wave-surface depends only on the form of it at any given moment, or indeed only on its initial form; its magnitude, but not its nature, depends on the time. This is equivalent to saying, that the *nature* of the equation of a wave-surface depends only on the form of the given disturbance, while the parameters of that equation depend upon the time. This will sometimes enable us to determine the properties of the wave when it is curved, without employing the general integral

$$\phi = \Sigma F\{f(x-\alpha)+g(y-\beta)+h(z-\gamma)-at,\ f,\ g,\ h\}.$$

Having determined the form of the wave-surface upon the principles of Art. 28, let its equation be $\chi(x, y, z;\ A, B, C\ldots)=0$; in which $A, B, C\ldots$ are parameters depending on the time only; also let the state of the fluid be expressed by the equation

$$\phi = \psi(x, y, z;\ A, B, C\ldots)$$

Now for any point in a wave-surface ϕ = constant; that is, any values of x, y, z which make $\chi = 0$, will make ψ = constant. ψ therefore can only be made to change in value by writing different values of A, B, C. We may therefore say that ψ is a function of $A, B, C\ldots$ only, which are connected by the equation $\chi = 0$. We may therefore consider x, y, z as functions of $A, B, C\ldots$ by virtue of $\chi = 0$; and having found the values of $d_x^2\phi$, $d_y^2\phi$, $d_z^2\phi$ in terms of the partial differential coefficients of ϕ with regard to A, B, C... we may change the equation

$$d_t^2\phi = a^2(d_x^2\phi + d_y^2\phi + d_z^2\phi)$$

into another not containing x, y, z. It is to be observed that A, B, C... are dependent only on t and quantities which are absolutely constant; B, C,... are therefore expressible in terms of A, and consequently there is only one independent parameter A. Hence the process above pointed out will give us an equation between $d_t^2\phi$. $d_A^2\phi$, $d_A\phi$ and A: that is, ϕ will be a function of A and t only.

30. As one of the simplest examples of this process, let us suppose the form of the waves to be spherical. Then $\chi = 0$ is

$$x^2 + y^2 + z^2 = r^2,$$

r being the parameter A. Now ϕ being a function of t and r only, and x, y, z being functions of r only, we have

$$d_x^2\phi = \frac{x^2}{r^2} . d_r^2\phi + \left(\frac{1}{r} - \frac{x^2}{r^3}\right) d_r\phi,$$

$$d_y^2\phi = \frac{y^2}{r^2} . d_r^2\phi + \left(\frac{1}{r} - \frac{y^2}{r^3}\right) d_r\phi,$$

$$d_z^2\phi = \frac{z^2}{r^2} . d_r^2\phi + \left(\frac{1}{r} - \frac{z^2}{r^3}\right) d_r\phi;$$

$$\therefore\ d_t^2\phi = a^2\left(d_r^2\phi + \frac{2}{r} . d_r\phi\right) = \frac{a^2}{r} . d_r^2(r\phi);$$

$$\therefore\ d_t^2(r\phi) = a^2 d_r^2(r\phi).$$

This equation being integrated gives

$$r\phi = F(r - at) + f(r + at),$$

which expresses the state of the fluid when the motion is in spherical waves: the former term $F(r - at)$ shews that spherical waves may diverge from a fixed centre; and the latter $f(r + at)$ that they may converge towards a fixed centre. The velocity of either wave is a. The motion of each particle is directed towards or from the fixed centre Art. 3, and its velocity $= d_r\phi$

$$= -\frac{F(r-at)}{r^2} + \frac{F'(r-at)}{r} \text{ for the diverging wave,}$$

$$\text{and } = -\frac{f(r+at)}{r^2} + \frac{f'(r+at)}{r} \text{ for the converging wave.}$$

31. Since in general the motion of each particle (Art. 3) is perpendicular to the wave-surface in which it is situated, the motions will be directed to or from focal lines, which may be fixed or moveable: and if the motion of the wave were perpendicular to its front we might always deduce the law of variation of the velocity of a particle; but inasmuch as the direction of a wave's motion entirely depends on its form, and is never in the direction of a normal except at those points where the curvature is a maximum or minimum, or when the waves are spherical, *no general law of the velocity can be deduced;* but we must first find the value of ϕ by the method of Art. 29, and then the velocity may be obtained.

32. When the form of the waves is spherical the law of variation of density may be found.

For if ρ' be the equilibrium density, and ρ the density at a point in the wave-surface,

$$a^2 \int_\rho . \frac{1}{\rho} = - d_t\phi = \frac{a}{r} . F'(r-at) \text{ for the diverging wave;}$$

$$\therefore \log_\epsilon \left(\frac{\rho}{\rho'}\right) = \frac{1}{ar} . F'(r-at);$$

$$\therefore \rho = \rho' . \epsilon^{\frac{F'(r-at)}{ar}}.$$

For a given part of this wave (as the *front* or the *middle*, ...) $r-at$ is constant, and therefore $F'(r-at)$ is constant $= Aa$ suppose,

$$\therefore \rho = \rho' . \epsilon^{\frac{A}{r}};$$

wherefore as the wave travels through the medium, the density of a given part of it varies as $\epsilon^{\frac{A}{r}}$, which rapidly diminishes.

Similarly it may be shewn, that the density of a given part of the converging wave varies as $\epsilon^{\frac{n}{r}}$, which rapidly increases.

33. If a plane-wave be transmitted through the medium, the particles as it successively reaches them are displaced with the same velocity; and there is a certain relation between the density and the velocity of displacement which holds good for all plane-waves.

For, denoting the velocity of displacement of a particle by v, and the density as before,

$$\phi = F(fx + gy + hz - at),$$

$$\text{and } v^2 = (d_x\phi)^2 + (d_y\phi)^2 + (d_z\phi)^2$$

$$= (f^2 + g^2 + h^2)\,\{F'(fx + gy + hz - at)\}^2;$$

$$\therefore\ v = \pm F'(fx + gy + hz - at);$$

and by referring this to a moveable origin (as in Art. 19), we have

$$v = \pm F'(fx' + gy' + hz') = \pm F'(p);$$

which is independent of t, and is constant for all particles situated in the same part of the wave, because for such particles p is constant.

Again,

$$a^2 \log_\epsilon \frac{\rho}{\rho'} = - d_t\phi = aF'(fx + gy + hz - at) = \pm av;$$

$$\therefore\ \rho = \rho' . \epsilon^{\pm\frac{v}{a}}.$$

If v be reckoned positive in the direction of the wave's motion, and negative when in the opposite direction,

$$\rho = \rho' \epsilon^{\frac{v}{a}}.$$

Hence in plane-waves all points of equal velocity are points of equal density.

Again, when ρ is $> \rho'$, v is positive,

when $\rho = \rho'$, v is evanescent,

when ρ is $< \rho'$, v is negative.

Hence, at points of mean density the particles are stationary; at points of condensation the particles are moving forwards; and at points of rarefaction they are moving backwards.

34. The property which was proved in Art. 23, respecting plane-waves, may be extended to curvilinear waves; and we may shew generally,

That no wave-surface, terminated abruptly by sharp edges, can be transmitted through a medium unless its edges rest upon the boundaries of the medium.

For, the only expression for a curvilinear wave which the equation of continuity furnishes is

$$\phi = \Sigma . F\{f(x-\alpha)+g(y-\beta)+h(z-\gamma)-at, f, g, h\};$$

which, as before observed, teaches that we may suppose the wave composed of plane-waves, which we may suppose transmitted through the medium, and then we shall have the true wave-surface by taking that to which they are all tangents.

Now, suppose a wave-surface terminated abruptly to be by some means or other excited in a medium. Upon referring to the above expression for ϕ, we should find that the tangent-planes at the edges of the wave-surface, or rather the component waves, represented by these tangent-planes, and expressed by terms of the form

$$F . \{f'(x-\alpha)+g'(y-\beta)+h'(z-\gamma)-at, f', g', h'\},$$

stretch out indefinitely beyond the boundary of the wave-surface into the medium: and when these components are transmitted and afterwards compounded into one wave, the portions of these waves which (as it were) hung over the proper wave-surface must remain. Hence it appears,

First, That a wave-surface terminated abruptly by sharp edges cannot be excited in a medium: and

Secondly, That if such a wave were excited, it could not be transmitted in that form.

Hence, if a curvilinear wave in traversing a medium meet with a fixed screen in which is an orifice, the part of the wave which passes through the orifice must afterwards abut with its edges upon the back of the screen.

35. In the Undulatory Theory of Light, each point in the front of a wave is considered as the origin of an indefinitely small wave. (See Airy's *Tracts*, 2d Edit. p. 267. Art. 21). This hypothesis, however, as is well known, is affected by a very troublesome difficulty. "What is to limit the waves diverging from each of these small sources of motion? The disturbance spreads generally in a spherical form, so that the front of each little wave is a sphere: are we to suppose the sphere complete, so that each small undulation is propagated backwards as well as forwards?" (Airy, Art. 22.)

It will have been perceived from what has been done in this paper, that in the transmission of waves by pressure through an elastic medium, the tangent-planes are to be taken, and that these tangent-planes move only in the direction of the wave's motion. Might not the same hypothesis be applied in the Undulatory Theory of Light, in which case the above difficulty would be avoided?

X. *On the Motion of a System of Particles, considered with reference to the Phenomena of Sound and Heat.* *By* PHILIP KELLAND, B.A. *Fellow and Tutor of Queens' College, Cambridge.*

[Read *May* 16, 1836.]

INTRODUCTION.

IN a former Memoir, it was my endeavour to simplify the equations of motion of a system of particles attracting each other with forces varying according to any law. The discussion of these equations was restricted to their bearing on the phenomena of Light, on which account one of the three was left untouched.

It appeared that the hypothesis of attractive forces led to the result that two of the equations corresponding to the motion in a plane perpendicular to the direction of transmission, indicated vibratory motion, whilst the third assumed a form altogether different, shewing that, as far as it was concerned, the motion was not vibratory.

On the other hand, the hypothesis of repulsive forces would give the motion in the direction of transmission vibratory, whilst the contrary would be the case in a plane perpendicular to this direction.

The discussion of the equations corresponding to motion in the direction of transmission is the object of the present memoir.

It is not improbable that to the action of forces, such as those of which we are treating, a considerable number of the phenomena of nature may be referred; but on account of our imperfect knowledge of the analogies subsisting between phenomena which apparently differ widely from each other in some essential points, we are obliged to restrict ourselves to the most simple, or to those which have been the most carefully examined.

Instances offer themselves in the cases of sound and light, since both have, for a long period, been referred to vibrations, though the difference in the nature of these vibrations had to be pointed out before it was admitted that a complete parallel was not to be expected between them. The same observation is applicable to the theories of light and heat. Remarkable as are the analogies between them, demanding as it would seem from their very nature the same mode of explanation in each, there are nevertheless peculiarities in the latter which seem to strike at the very foundation of the theory, and to require the construction of another on totally new principles.

On whatever grounds then a theory be raised, we must not be discouraged if some succeeding facts appear for the moment to militate against it, and particularly, when that theory is one in which the action of force in its different modifications plays a conspicuous part, for there we are presented with a range so wide that facts, almost antagonist to each other, are brought together in the interpretations of the various kinds of motion which occur.

In the present memoir, I have ventured to push these interpretations to a considerable extent, from a conviction that the explanation of many phenomena is contained in them, and the hope that in some cases at least the real explanation may coincide with, or at any rate bear a close resemblance to, those which I have attempted.

I have adopted the hypothesis that the medium, whose motion we have under consideration, is not composed of particles of one nature, but of a regularly distributed series of particles of two kinds, of which each is endued with forces and inertia differing from those of the other. For the sake of distinction, in forming the equations, I have called these media A and B, which when applied to sound signify air and vapour, when applied to light and heat, ether and caloric or those substances, by whatever names they may be designated, which serve for the propagation of these respectively.

We will assume that the *law* of force is the same in both cases, an assumption which there appears no reason to suspect.

SECTION I.

Interpretation of the Equations corresponding to Vibratory Motion.

1. LET the mass of a particle of medium A, estimated by its repulsion at the distance unity, be represented by P; that of a particle of medium B, estimated in like manner, by Q; and the moving force of a particle of A on a particle of B by M, which is also the moving force of a particle of B on one of A.

We will first consider the motion of a particle of medium A.

Let x, y, z be its co-ordinates when at rest,

$x + \delta x,\ y + \delta y,\ z + \delta z$ those of another particle of the same medium,

$x + \Delta x,\ y + \Delta y,\ z + \Delta z$ those of a particle of B;

$$r = \sqrt{\delta x^2 + \delta y^2 + \delta z^2},$$

$$R = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}.$$

Let the same quantities at the end of the time t become

$$x + a,\ y,\ z.$$

$$x + a + \delta x + \delta a,\ y + \delta y,\ z + \delta z.$$

$$x + a + \Delta x + \Delta a,\ y + \Delta y,\ z + \Delta z.$$

$$r + \delta r.$$

$$R + \Delta R.$$

(the motion being in the direction of the axis of x)

and let the function which expresses the force be $r\phi r$.

The action of the particles of A on the particle in question parallel to x, is evidently the sum of all such expressions as the following,

$$P^2 . \phi (r + \delta r) (\delta x + \delta a);$$

and that of the particles of B on the same particle the sum of

$$M . \phi (R + \Delta R) (\Delta x + \Delta a),$$

hence, $\frac{d^2 a}{dt^2} = - \sigma . P\phi(r + \delta r).(\delta x + \delta a) - \frac{M}{P} \Sigma \phi(R + \Delta R)(\Delta x + \Delta a)$;

σ and Σ indicating the respective sums taken for all the particles which are in motion.

2. In order to reduce these expressions to an integrable form, it is requisite to adopt some process of approximation. Suppose then we omit δa compared with δx: this appears at the first glance a doubtful process, for we cannot here suppose, as we did in the case of light, that the particles have a very small motion; we know, in fact, that this is not the case for sound; but all scruple will be removed when we reflect that any particular δa is the approach of two particles to each other, whilst the corresponding δx is their original distance; and from the fact of the repulsive nature of the forces, we cannot, even in the most unfavourable case, suppose the approach other than a small fraction of the original distance. Or again, if δa be the amount of *recess* of the particles, since any recess must, in the case of vibrations, have a corresponding *approach*, the same reasoning applies.

3. Now $\phi(r + \delta r) = \phi r + Fr . \delta r$ nearly, where $F(r)$ stands for the differential coefficient of $\phi(r)$, taken with respect to r,

$$(r + \delta r)^2 = (\delta x + \delta a)^2 + \delta y^2 + \delta z^2;$$

$$\therefore\ 2r\delta r = 2\delta x \delta a$$

$$\delta r = \frac{\delta x}{r} . \delta a;$$

$$\therefore\ \phi(r + \delta r).(\delta x \div \delta a) = (\phi r + \frac{Fr}{r}\delta x \delta a)(\delta x + \delta a)$$

$$= \phi r . \delta x + (\phi r + \frac{Fr}{r}\delta x^2)\delta a.$$

Similarly, $\phi(R + \Delta R).(\Delta x + \Delta a) = \phi R . \Delta x + (\phi R + \frac{FR}{R}\Delta x^2)\Delta a$;

$$\therefore\ \frac{d^2 a}{dt^2} = - P\sigma . \{\phi r . \delta x + (\phi r + \frac{Fr}{r}\delta x^2).\delta a\}$$

$$- \frac{M}{P}\Sigma\{\phi R . \Delta x + (\phi R + \frac{FR}{R}\Delta x^2)\Delta a\},$$

but $P^2.\sigma\phi r.\delta x + M.\Sigma\phi R.\Delta x$ is the force which acts on the particle at rest; and consequently is identically equal to zero;

$$\therefore \frac{d^2\alpha}{dt^2} = -P.\sigma.\{\phi r + \frac{Fr}{r}\delta x^2\}.\delta\alpha$$

$$-\frac{M}{P}\Sigma.\{\phi R + \frac{FR}{R}\Delta x^2\}\Delta\alpha,$$

assume $\alpha = a\cos(ct - kx)$,

then $\delta\alpha = a\cos(ct - kx - k\delta x) - a\cos(ct - kx)$

$$= a\{\cos(ct-kx)\cos k\delta x + \sin(ct-kx)\sin k\delta x - \cos(ct-kx)\}$$

$$= a\sin(ct-kx).\sin k\delta x - 2\alpha.\sin^2\frac{k\delta x}{2}.$$

In the same manner, if both particles vibrate in the direction of transmission,

$$\Delta\alpha = a\sin(ct-kx)\sin k\Delta x - 2\alpha\sin^2\frac{k\Delta x}{2},$$

and, by the hypothesis that each medium is a medium of symmetry, we shall, by reasoning precisely the same as that which I adopted on a previous occasion, [Part I. p. 156. of this Vol.], arrive at the following result:

$$\frac{d^2\alpha}{dt^2} = \Big\{2P\sigma(\phi r + \frac{Fr}{r}\delta x^2)\sin^2\frac{k\delta x}{2}$$

$$+\frac{2M}{P}\Sigma(\phi R + \frac{FR}{R}\Delta x^2)\sin^2\frac{k\Delta x}{2}\Big\}.\alpha \text{ .... (1).}$$

$$= -c^2\alpha \text{ suppose,}$$

where $\alpha = a\cos(ct - kx)$.

4. It is clear that, in order to effect this, we must suppose

$$\sigma(\phi r + \frac{Fr}{r}\delta x^2)\delta x^2$$

a negative quantity. Now if $\phi r = \frac{1}{r^{n+1}}$, or the force vary as the inverse n^{th} power of the distance, this can be accomplished;

$$\text{for } \frac{Fr}{r} = -\frac{n+1}{r^{n+3}};$$

$$\therefore \ (\phi r + \frac{Fr}{r}\delta x^2)\sin^2\frac{k\delta x}{2} = \frac{\delta x^2 + \delta y^2 + \delta z^2 - (n+1).\delta x^2}{r^{n+3}} \sin^2\frac{k\delta x}{2}$$

$$= \frac{\delta y^2 + \delta z^2 - n\delta x^2}{r^{n+3}} . \sin^2\frac{k\delta x}{2},$$

and $\sigma\,\delta y^2 fr = \sigma\,\delta z^2 fr$;

$$\text{hence, } \sigma(\phi r + \frac{Fr}{r}\delta x^2)\sin^2\frac{k\delta x}{2} = \sigma\frac{2\delta y^2 - n\delta x^2}{r^{n+3}}\sin^2\frac{k\delta x}{2}.$$

Now on expanding the sine, it is clear that when $\frac{k\delta x}{2}$ becomes greater than unity, or which is the same thing, when δx becomes greater than $\frac{\lambda}{2}$, we must put the supplement instead of the arc in the expansion; but we saw reason in the case of light to suppose that the expression for the whole force has the same sign, as the expression for the force exerted by those particles only which lie within the range of the first half wave.

In fact, if the different half waves give different signs, it is evident that they must give them *alternately;* and thus the above hypothesis would be confirmed; we shall therefore retain only the first term in the expansion of $\sin^2\frac{k\delta x}{2}$, and reduce our investigation to the consideration of the sign of $\sigma\,\frac{2\delta y^2 - n\delta x^2}{r^{n+3}}\,\delta x^2$, taken within the range of the first half wave.

For every value of δy within this range there is a corresponding equal value of δx, and *vice versâ:* so that we may write the above expression as follows;

$$\sigma\,\frac{(2\delta y^2 - n\delta x^2)\,\delta x^2 + (2\delta x^2 - n\delta y^2).\,\delta y^2}{2r^{n+3}}$$

$$= -\sigma\,\frac{n.(\delta y^4 + \delta x^4) - 4\,\delta y^2\delta x^2}{2r^{n+3}};$$

from which it appears that if $n = 2$ our expression is $-\sigma \frac{(\delta y^2 - \delta x^2)^2}{r^5}$ an essentially negative quantity. This conclusion was seen to be requisite, in order to satisfy the conditions of direct vibrations with repulsive forces.

It is also evident that the same conditions would be fulfilled by making n greater then 2, whereas if n be zero or negative this will not be the case. The hypothesis, which makes n equal to unity, we shall examine hereafter.

5. By integration of the equation (1) and reduction, we obtain for the square of the velocity of transmission,

$$v^2 = \frac{P}{2}\sigma \cdot \frac{n\delta x^2 - 2\delta y^2}{r^{n+3}} \cdot \left(\frac{\sin\frac{\pi\delta x}{\lambda}}{\frac{\pi}{\lambda}}\right)^2 + \frac{M}{2P}\Sigma \cdot \frac{n\Delta x^2 - 2\Delta y^2}{R^{n+3}}\left(\frac{\sin\frac{\pi\Delta x}{\lambda}}{\frac{\pi}{\lambda}}\right)^2;$$

which, if ϵ be put for the distance between two consecutive particles of A, and E for that between two of B, and we make

$$\delta x = \xi\epsilon, \quad \delta y = \eta\epsilon, \quad \delta z = \zeta\epsilon,$$
$$\Delta x = XE, \quad \Delta y = YE, \quad \Delta z = ZE$$

becomes

$$v^2 = \tfrac{1}{2}\sigma\frac{P}{\epsilon^{n-1}}\left\{\frac{n\xi^2 - 2\eta^2}{(\xi^2+\eta^2+\zeta^2)^{\frac{n+3}{2}}}\left(\xi^2 - \frac{\pi^2}{3\lambda^2}\xi^2\epsilon^2 + \ldots\ldots\right)\right\}$$

$$+ \frac{M}{2P}\Sigma\frac{1}{E^{n-1}}\left\{\frac{nX^2 - 2Y^2}{(X^2+Y^2+Z^2)^{\frac{n+3}{2}}}\left(X^2 - \frac{\pi^2}{3\lambda^2}X^2E^2 + \ldots\ldots\right)\right\}.$$

For the present, we will omit that part of the expression which depends on the length of a wave; hence we have

$$v^2 = \tfrac{1}{2}\frac{P}{\epsilon^{n-1}}\sigma\left\{\frac{n\xi^2 - 2\eta^2}{(\xi^2+\eta^2+\zeta^2)^{\frac{n+3}{2}}}\cdot\xi^2\right\} + \tfrac{1}{2}\frac{M}{P.E^{n-1}}\Sigma\left\{\frac{nX^2 - 2Y^2}{(X^2+Y^2+Z^2)^{\frac{n+3}{2}}}X^2\right\}.$$

6. In the same manner, denoting by accented letters the quantities corresponding to a, &c. for the motion of a particle of medium B, we obtain

$$\frac{d^2\alpha'}{dt^2} = +\left\{2Q\Sigma\left(\phi R + \frac{FR}{R}\Delta x^2\right)\sin^2\frac{k\Delta x}{2}\right.$$

$$\left. + \frac{2M}{Q}\sigma\left(\phi r + \frac{Fr}{r}\delta x^2\right)\sin^2\frac{k\delta x}{2}\right\}.\alpha'\ldots\ldots\ldots(2),$$

$$\text{whence } v'^2 = \tfrac{1}{2}\sigma\frac{M}{Q\epsilon^{n-1}}\left\{\frac{n\xi^2 - 2\eta^2}{(\xi^2+\eta^2+\zeta^2)^{\frac{n+3}{2}}}\left(\xi^2 - \frac{\pi^2}{3\lambda^2}\xi^2\epsilon^2 + \&c.\right)\right\}$$

$$+ \tfrac{1}{2}Q\Sigma\frac{1}{E^{n-1}}\left\{\frac{nX^2 - 2Y^2}{(X^2+Y^2+Z^2)^{\frac{n+3}{2}}}\left(X^2 - \frac{\pi^2}{3\lambda^2}X^2E^2 + \&c.\right)\right\}$$

which independently of λ is equal to

$$\tfrac{1}{2}\frac{M}{Q\epsilon^{n-1}}\sigma.\left\{\frac{n\xi^2 - 2\eta^2}{(\xi^2+\eta^2+\zeta^2)^{\frac{n+3}{2}}}.\xi^2\right\} + \tfrac{1}{2}\frac{Q}{E^{n-1}}\Sigma\left\{\frac{nX^2 - 2Y^2}{(X^2+Y^2+Z^2)^{\frac{n+3}{2}}}.X^2\right\}.$$

7. If we were to suppose every part of medium A to be mixed with a portion of medium B according to a given law, all that we should require would be the direct integration of these equations, considering M a constant quantity depending on the relative natures of the media: but it will be more analogous to the nature of the question when applied to air or ether, if we suppose a want of uniformity in the mixture. Conceive, for example, that a given mass of medium A, impregnated with medium B, is enclosed by other portions of the same medium not thus impregnated.

Let ϵ_1 be the distance between two consecutive particles in the latter mass: then it is easily seen, that the attraction of the mass of A, in the mixture, on a particle at its confines is $C.\frac{P^2}{E^3}$; the quantity C depending on the mass so impregnated,

and that of the mass of B is $C\frac{M}{E^3}$;

hence the action on a particle of A is $C\left(\frac{P^2}{\epsilon^3}+\frac{M}{E^3}\right)$, but this is retained at rest by a corresponding quantity of A at distances ϵ_1^3;

$$\therefore\ C\left(\frac{P^2}{\epsilon^3}+\frac{M}{E^3}\right)=D\frac{P^2}{\epsilon_1^3}.$$

Similarly, the action on a particle of B at the confines of the medium gives us

$$C\left(\frac{Q^2}{E^3}+\frac{M}{\epsilon^3}\right)=D\frac{M}{\epsilon_1^3},$$

$$\text{hence}\ \frac{MP^2}{\epsilon^3}+\frac{M^2}{E^3}=\frac{P^2Q^2}{E^3}+\frac{MP^2}{\epsilon^3},$$

$$\text{or}\ M^2=P^2Q^2$$

$$M=PQ;$$

and, by substitution in the above approximative equations,

$$v^2=A\,.\left\{\frac{P}{\epsilon^{n-1}}+\frac{Q}{E^{n-1}}\right\},$$

the integrals being evidently equal

$$v'^2=A\,.\left\{\frac{P}{\epsilon^{n-1}}+\frac{Q}{E^{n-1}}\right\}$$

$$=v^2$$

or the velocity with which the motion of a particle of B is transmitted, is equal to the same quantity for a particle of A.

8. Thus far we have neglected the influence, which the particles of matter with which the media are united, (such are the elements of solid bodies) exert directly on the motion. Such influence will be calculated by resuming our equations, and supposing, in addition to the forces exerted by the particles which have motion, other forces produced by particles at rest.

Let Δx, Δy, Δz be the co-ordinates of a material particle, measured from the place of rest of the particle under consideration;

R its distance from that point: then it is manifest that the only difference in the form of this term will be, that in the present case the difference of the co-ordinates of the two particles parallel to the axis of x, instead of being $\Delta x + a + \delta a - a$, is $\Delta x - a$; whence we obtain, supposing the force to vary inversely as the square of the distance,

$$\frac{d^2 a}{dt^2} = -Q\Sigma \frac{\Delta x - a}{(R + \Delta R)^3} + \&c.$$

$$= -Q\Sigma \frac{(\Delta x - a)}{R^3}\left\{1 - \frac{3\Delta R}{R}\right\} + \&c.$$

$$\text{but } (R + \Delta R)^2 = (\Delta x - a)^2 + \Delta y^2 + \Delta z^2$$

$$R^2 + 2R\Delta R = R^2 - 2\Delta x \,.\, a;$$

$$\therefore \Delta R = -a\frac{\Delta x}{R},$$

whence by substitution we get

$$\frac{d^2 a}{dt^2} = -Q\Sigma \left(\frac{\Delta x - a}{R^3}\right)\left(1 + 3a\frac{\Delta x}{R^2}\right) + \&c.$$

$$= -Q\Sigma \left(\Delta x - a + 3a\frac{\Delta x^2}{R^2}\right)\frac{1}{R^3} + \&c.$$

$$= \quad Q\Sigma \frac{1}{R^5}(R^2 - 3\Delta x^2)\,.\,a + \&c.$$

$$= \quad Q\Sigma \frac{a}{R^5}(\Delta y^2 + \Delta z^2 - 2\Delta x^2) + \&c.$$

$$\text{Now } \Sigma\frac{\Delta y^2}{R^5} = \Sigma\frac{\Delta z^2}{R^5} = \Sigma\frac{\Delta x^2}{R^5},$$

hence the expression $\Sigma\frac{1}{R^5}(\Delta y^2 + \Delta z^2 - 2\Delta x^2)$ is zero,

and the velocity is independent of Q; the only effect which is produced by the material particles being an *indirect* one, arising from the alteration of distance which they produce in the particles themselves.

9. In the preceding investigation, it has been supposed that the force, which a particle exerts on another particle of the same medium, as well as that which it exerts on a particle of the other medium, is repulsive.

Had we adopted the contrary course, considering the action either attractive or repulsive as it might happen to be, we should clearly have arrived at the following conclusions, the force varying inversely as the square of the distance.

$$\frac{d^2\alpha}{dt^2} = P\sigma\left(\frac{1}{r^3} - \frac{3\delta x^2}{r^5}\right)\delta\alpha + \frac{M}{P}\Sigma\left(\frac{1}{R^3} - \frac{3\Delta x^2}{R^5}\right)\Delta\alpha,$$

$$\frac{d^2\beta}{dt^2} = P\sigma\left(\frac{1}{r^3} - \frac{3\delta x^2}{r^5}\right)\delta\beta + \frac{M}{P}\Sigma\left(\frac{1}{R^3} - \frac{3\Delta x^2}{R^5}\right)\Delta\beta,$$

$$\frac{d^2\gamma}{dt^2} = P\sigma\left(\frac{1}{r^3} - \frac{3\delta x^2}{r^5}\right)\delta\gamma + \frac{M}{P}\Sigma\left(\frac{1}{R^3} - \frac{3\Delta x^2}{R^5}\right)\Delta\gamma;$$

$$\frac{d^2\alpha'}{dt^2} = Q\Sigma\left(\frac{1}{R^3} - \frac{3\Delta x^2}{R^5}\right)\Delta'\alpha' + \frac{M}{Q}\sigma\left(\frac{1}{r^3} - \frac{3\delta x^2}{r^5}\right)\Delta\alpha',$$

$$\frac{d^2\beta'}{dt^2} = Q\Sigma\left(\frac{1}{R^3} - \frac{3\Delta x^2}{R^5}\right)\Delta'\beta' + \frac{M}{Q}\sigma\left(\frac{1}{r^3} - \frac{3\delta x^2}{r^5}\right)\Delta\beta',$$

$$\frac{d^2\gamma'}{dt^2} = Q\Sigma\left(\frac{1}{R^3} - \frac{3\Delta x^2}{R^5}\right)\Delta'\gamma' + \frac{M}{Q}\sigma\left(\frac{1}{r^3} - \frac{3\delta x^2}{r^5}\right)\Delta\gamma';$$

where $\delta\alpha$, $\delta\beta$, $\delta\gamma$ represent the variation of motion for two particles of A estimated in the directions x, y and z.

$\Delta\alpha$, $\Delta\beta$, $\Delta\gamma$ represent the variation of motion between the given one of A, and one of B.

$\Delta'\alpha'$, $\Delta'\beta'$, $\Delta'\gamma'$ represent the variation of motion between two of B.

$\Delta\alpha'$, $\Delta\beta'$, $\Delta\gamma'$ represent the variation of motion between the given one of B, and one of A.

10. Now, by reference to (7), it will be seen that the conditions of equilibrium will be satisfied by making $M = - PQ$, or supposing, in analogy with Coulomb's hypothesis, that if the mutual action of similar particles be of one nature, that of opposite ones will be of another.

The two equations in α, α' then become

$$\frac{d^2\alpha}{dt^2} = P\sigma A \,.\, \delta\alpha - Q\Sigma B \,.\, \Delta\alpha,$$

$$\frac{d^2\alpha'}{dt^2} = - P\sigma A \,.\, \Delta\alpha' + Q\Sigma B \,.\, \Delta'\alpha',$$

writing A and B for the functions of x.

Now if both series of particles vibrate, we must have

$$\alpha = a \cos (ut - kx),$$

$$\alpha' = a' \cos (u't - kx + c);$$

$$\therefore\ \delta\alpha = - 2\alpha \,.\, \sin^2 \frac{k\delta x}{2} \text{ (omitting terms which vanish)},$$

$$\Delta'\alpha' = - 2\alpha' \,.\, \sin^2 \frac{k\delta x}{2};$$

$$\Delta\alpha = a' \cos \{u't - k(x + \delta x) + c\} - a \cos (ut - kx)$$

$$= a' \{\cos (u't - kx + c) \cos k\delta x + \sin (u't - kx + c) \sin k\delta x\}$$

$$- a \cos (ut - kx) = \alpha' \cos \,.\, k\delta x - \alpha,$$

$$\Delta\alpha' = a \cos (ut - kx - k\delta x) - a' \cos (u't - kx + c)$$

$$= a \{\cos (ut - kx) \cos k\delta x + \sin (ut - kx) \sin k\delta x\}$$

$$- a' \cos (u't - kx + c) = \alpha \cos k\delta x - \alpha';$$

hence by substitution:

$$\frac{d^2\alpha}{dt^2} = - 2P\sigma A \,.\, \sin^2 \frac{k\delta x}{2} \,.\, \alpha - Q\Sigma B \,.\, \cos k\delta x \,.\, \alpha' + \alpha Q\Sigma B$$

$$= - 2P\sigma A \,.\, \sin^2 \frac{k\delta x}{2} \,.\, \alpha - Q\Sigma B \,.\, \alpha' + 2Q\Sigma B \,.\, \sin^2 \frac{k\delta x}{2} \,.\, \alpha' + \alpha Q\Sigma B$$

$$= - 2P\sigma A \,.\, \sin^2 \frac{k\delta x}{2} \,.\, \alpha + 2Q\Sigma B \,.\, \sin^2 \frac{k\delta x}{2} \,.\, \alpha',$$

since $\Sigma B = 0$. (see page 244.)

Similarly, $\dfrac{d^2\alpha'}{dt^2} = -\,2Q\Sigma B\,.\,\sin^2\dfrac{k\delta x}{2}\,.\,\alpha' + 2P\sigma A\,.\,\sin^2\dfrac{k\delta x}{2}\,.\,\alpha$

$$= -\frac{d^2\alpha}{dt^2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(1).$$

In the same manner $\dfrac{d^2\beta'}{dt^2} = -\dfrac{d^2\beta}{dt^2}$,

$$\frac{d^2\gamma'}{dt^2} = -\frac{d^2\gamma}{dt^2}.$$

11. The integral of one of these equations is

$$\frac{d\alpha'}{dt} + \frac{d\alpha}{dt} = M \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2),$$

$$\alpha' + \alpha = Mt + N \;\ldots\ldots\ldots\ldots\; (3),$$

but the circumstance that we have proceeded by supposing

$$\alpha = a\cos(ut - kx),$$

$$\alpha' = a'\cos(u't - kx + c),$$

gives $u^2\alpha + u'^2\alpha' = 0$, by (1),

also $ua\sin(ut - kx) + u'a'\sin(u't - kx + c) = -M.$

Thus $\alpha + \alpha' = Mt + N$ becomes

$$\alpha - \frac{u^2}{u'^2}\alpha = Mt + N,$$

$$\left(1 - \frac{u^2}{u'^2}\right)\frac{d^2\alpha}{dt^2} = 0;$$

hence, $u = u'$,

$$\alpha = -\alpha',$$

or, $a'\cos c = -a$,

$$a'\sin c = 0;$$

and $c = \pi$,

$$a' = a,$$

therefore the following solutions

$$\alpha = a \cos(ut - kx),$$

$$\alpha' = - a \cos(ut - kx),$$

appear to be the only possible way of satisfying the equations so as to retain the form.

12. Had we, however, taken the more general expression for α, viz.

$$\alpha = a \cos(ut - kx) + b \sin(ut - kx),$$

it is not impossible that we should have obtained other modes of solution. The above is sufficiently general for my present purpose. Recurring again to the differential equations it will be found that they become

$$\frac{d^2\alpha}{dt^2} = - \left(2P\sigma A \sin^2 \frac{k\delta x}{2} + 2Q\Sigma B \sin^2 \frac{k\delta x}{2}\right) \alpha,$$

$$\frac{d^2\alpha'}{dt^2} = - \left(2P\sigma A \sin^2 \frac{k\delta x}{2} + 2Q\Sigma B \sin^2 \frac{k\delta x}{2}\right) \alpha',$$

so that when the forces, which the particles of the same kind exert on each other, are repulsive, we have vibrations in the direction of the motion; when attractive, transverse to that direction.

13. This conclusion is not altogether uninteresting, as it leads us to the possibility of a *transfer* of particles, owing to vibratory motion, which we shall discuss in the sequel. Suppose, for instance, the solution of one equation made it appear, that the particles, of which it represented the motion, had an uniform velocity of transmission along a certain line, then the corresponding equation for the other series of particles would shew us, that these particles were transmitted with the same velocity, along the same line, but in an opposite direction: thus giving us a transfer forwards of one series of particles to supply the places of the other series of which the transfer would be backwards.

For the present then we may confine ourselves to the consideration of *one* medium, as none of its results, so far as regards its own motion, have a different form on account of the interposition of another.

14. Let us proceed, in the next place, to examine some of the results which we have above deduced, and to apply them to the case of sound.

We suppose, in this application, that the two media are air and vapour, and we have seen that, on the hypothesis of a repulsive energy (7) in each, the velocity of transmission of the vibration of vapour is equal to that of air, not only when the two media are united in an uniform mixture, but even when they are not so, provided they are so distributed, as to be in equilibrium when not under the influence of an external disturbance.

This would probably be the case during a day, in which the fluctuations of the barometer were inconsiderable; on other occasions we might expect a slight difference in the velocities, and thus might be explained the circumstance, mentioned by Mr Herschel in the Encyclopædia Metropolitana, of the double report of a gun. I should scarcely, however, consider the above results consistent with the fact, did it not appear probable, that such phenomena occur at a time, when either the vapour is *passing* into water, or some other change is taking place which destroys the equilibrium of the mass.

15. When the vapour has actually become water, if it act on air by an attractive or repulsive force, its effect on the transmission of sound will disappear altogether: the formula for this case is that which we have investigated (8) for a mixture with air of material particles, or particles whose magnitude or inertia is such, that whilst they affect the motion of air, themselves are not sensibly affected by its vibrations. We found that such a system introduced no additional force to that which the particles of air exert on each other.

16. On examining the expression for the velocity, it is evident that unless λ be supposed very great, we should expect to find the velocity influenced by the length of the wave, so as to increase with it. It is generally assumed that no such influence is sensible, but no very conclusive experiments appear to have been undertaken for the express purpose of ascertaining what the fact really is. That of

M. Biot carries with it the greatest weight, yet it is far from decisive, and the subject appears to be still open to investigation.

If, indeed, experimental enquiries had completely set this point at rest, it would remain for us to determine whether the hypothesis which we have adopted was applicable to sound. We should have, if we did not entirely reject it, to introduce a modification, the effect of which would be to cause the second term in the expansion of the velocity to vanish. The same modification would probably apply to the motion of light *in vacuo*.

17. Regarding this part of our subject then as probably not opposed to facts, although not deducible from them, let us recur to our equations.

We have seen that vibrations in the direction of transmission are possible for any law of force, which can be expressed in terms of powers of the reciprocal of the distance, greater than the first (4).

Now if we suppose n some large quantity, since $\frac{1}{\epsilon^{n-1}}$ enters as the coefficient of the square of the velocity, P must be excessively small, for the velocity itself is not large, and, in fact, bears a very small ratio to that of light, which is expressed in an analogous form.

We are then limited in the choice of our power on both sides—first, n must be positive, secondly, not large, and we shall see presently it cannot be equal to unity.

Suppose the particles acted on in such a manner that the moving force of one on another is a quantity bearing a given finite ratio to the velocity of transmission: this gives

$$\frac{P}{\epsilon^{\frac{n}{2}}} = C,$$

but $\frac{P}{\epsilon^{n-1}}$ varies as $(\text{velocity})^2 = C'$, hence $\frac{P}{\epsilon^{\frac{n}{2}}} \propto \frac{P}{\epsilon^{n-1}}$;

$$\therefore \frac{n}{2} = n - 1 \text{ or } n = 2.$$

18. This argument, undoubtedly, does not carry with it much weight, although it appears to a certain extent plausible. It will be extremely difficult to attempt a formal proof of the law after which we are seeking. Experimenters have had their attention drawn to any thing rather than this; for it must be remembered that it is not by the examination of broad facts and familiar phenomena that we can generally construct a Theory, but by the intricate pursuit of the slight deviations from, and apparent exceptions to, the general rule.

Supposing such a law established as we have given above, it is manifest, that the velocity must be decreased in the higher regions of the atmosphere, whilst the variation of the velocity, due to an alteration of the length of the wave, is increased in the same ratio.

We might then hope to arrive at satisfactory conclusions, and indeed completely establish a law of repulsion, could we make observations in the higher regions of the atmosphere, as for instance, on or near the summits of considerable mountains.

At the same time, supposing such experiments have been made, and that the apparent result of them is contrary to that obtained, on the above supposition, I should still hesitate before I reject a law of force which is, as far as we at present know, the universal law for all particles not in *actual* contact.

The case here is widely different from that of light. In the latter the length of a wave is only about the $\frac{1}{50,000}$ part of an inch; in the former, it is several inches; calling to mind then that even for light the difference of velocity bears but a small ratio to the whole, we should not expect, unless some of the other quantities proportionally increase, that a difference so minute would be sensible to the ear. Should this be the case it is unfortunate, as it deprives us of a ready mode of ascertaining the fundamental property which pervades the whole.

19. But in the absence of the observations requisite for obtaining accurate results, we may still discover, by some indirect process,

relations which shall furnish arguments in favour of any particular assumption. With this object let us compare the equations for the motion of light with those before us, limiting their interpretation by the phenomena of which they are the mathematical expressions.

In both cases the variation of velocity due to the length of a wave must depend on the magnitude of the term $\left(\frac{\epsilon}{\lambda}\right)^2$, and, from the smallness of this variation in the case of sound, (if it does vary at all), we conclude that ϵ cannot be *large* in sound compared with its value in light, yet the *velocity* of sound is very small compared with that of light, whence it appears that the term $\frac{P}{\epsilon^{n-1}}$ is small, not by reason of the comparative increase of ϵ^{n-1}, but by the diminution of P.

We will then endeavour to ascertain the value of n which renders the expression

$$\Sigma \,.\, \frac{2\eta^2 - n\xi^2}{(\xi^2 + \eta^2 + \zeta^2)^{\frac{n+3}{2}}} \,.\, \sin^2 \frac{\pi\xi\epsilon}{\lambda},$$

(taken from 0 to infinity with respect to each of ξ, η and ζ) a small quantity.

20. It is obvious that this may be effected by making n very large, but it is doubtful whether this will be the mode by which it actually becomes small from the circumstance that ϵ^{n-1} becomes proportionally small by the same hypothesis.

In order to find another value of n, which will satisfy the same conditions, we pursue the following process. After writing x, y, z instead of ξ, η and ζ for convenience, let the above function be expanded in a series ascending by powers of z; then the finite sum of the expansion with respect to z, will be the value of the expression corresponding to given values of x and y.

$$\text{But } \Sigma(z+1)^n = \frac{z^{n+1}}{n+1} + \frac{z^n}{2} + \&c.$$

hence it is evident that, omitting all the terms of the summation except those respectively which involve the highest power of z, we have, as the *sum* with respect to z, nothing more than the integral with respect to z; and, by the integral Calculus, we know that

$$\int_{z=0}^{z=\infty} \frac{dz}{(x^2+y^2+z^2)^{\frac{n+3}{2}}} = \frac{n.(n-2)\ldots 2}{(n+1).(n-1)\ldots 3} \cdot \frac{1}{(x^2+y^2)^{\frac{n+2}{2}}}.$$

Pursuing an analogous process for y, we obtain

$$\int_{y=0}^{y=\infty} \frac{dy}{(x^2+y^2)^{\frac{n+2}{2}}} = \frac{(n-1).(n-3)\ldots 1}{n.(n-2)\ldots 2} \cdot \int \frac{dy}{(x^2+y^2)\,x^{n-1}}$$

$$= \frac{(n-1).(n-3)\ldots 1}{n.(n-2)\ldots 2} \cdot \frac{1}{x^{n+1}} \cdot \frac{\pi}{2};$$

$$\int_{y=0}^{y=\infty} \frac{y^2\,dy}{(x^2+y^2)^{\frac{n+2}{2}}} = \frac{(n-3).(n-5)\ldots 1}{(n-2)\ldots 2}\,\frac{1}{x^{n-1}} \cdot \frac{\pi}{2}$$

$$- \frac{(n-1).(n-3)\ldots 1}{n.(n-2)\ldots 2}\,\frac{1}{x^{n-1}} \cdot \frac{\pi}{2}$$

$$= \frac{(n-3).(n-5)\ldots 1}{n.(n-2)\ldots 2} \cdot \frac{1}{x^{n-1}} \cdot \frac{\pi}{2};$$

hence $$\int_y \int_z \frac{1}{(x^2+y^2+z^2)^{\frac{n+3}{2}}} = \frac{n.(n-2)\ldots 2}{(n+1)\ldots 3} \cdot \frac{(n-1)\ldots 1}{n.(n-2)\ldots 2} \cdot \frac{\pi}{2x^{n-1}}$$

$$= \frac{1}{n+1} \cdot \frac{1}{x^{n+1}} \cdot \frac{\pi}{2}$$

and $$\int_y \int_z \frac{y^2}{(x^2+y^2+z^2)^{\frac{n+3}{2}}} = \frac{n.(n-2)\ldots 2}{(n+1)\ldots 3} \cdot \frac{(n-3)\ldots 1}{n.(n-2)\ldots 2} \cdot \frac{\pi}{2x^{n-1}}$$

$$= \frac{1}{(n+1).(n-1)} \cdot \frac{1}{x^{n-1}} \cdot \frac{\pi}{2},$$

whence the above expression becomes

$$C \cdot \int_x \left(\frac{n}{n+1} - \frac{2}{(n+1)\cdot(n-1)}\right) \frac{\pi}{2} \cdot \frac{1}{x^{n-1}} \cdot \sin^2 \frac{\pi x \epsilon}{\lambda},$$

the coefficient in which contains $\frac{1}{\epsilon^{n-1}}$.

This owes its value to the factor $\left(n - \frac{2}{n-1}\right)\frac{1}{\epsilon^{n-1}}$.

We could not, consistently with the restriction before imposed on this quantity that it shall be small, suppose $n = 1$, as that hypothesis would make it very large, and, what is worse, would affect it with the negative sign.

If n be a considerable quantity, the factor of $\frac{1}{\epsilon^{n-1}}$ is not large, but $\frac{1}{\epsilon^{n-1}}$ is itself considerable.

If $n = 2$ the expression is reduced to zero, and the velocity will be found by extending the expansions in finding the finite integrals, and retaining the smaller terms, and will consequently be very small.

The condition which we required, therefore, that the velocity should be very small is satisfied, and, apparently, only satisfied by the hypothesis of the force varying inversely as the square of the distance.

21. It may not then be uninteresting to examine this law of the force a little more closely, although it appears by no means probable that the statical condition of the pressure varying as the density, can be reduced to this law, or indeed to any other, properly so called.

The investigation which follows, of the statical condition of a system of particles exerting repulsive energies of this nature, will sufficiently prove this, if proof be necessary.

SECTION II.

PRESSURE.

22. CONCEIVE a vertical tube of given dimensions filled with air, of which the distance between two consecutive particles is ϵ, and whose law of action is that of the inverse square of the distance: the particles in the tube being acted on as well by the external air as by that in the tube. It is clear that the force on any particle will be, that of the air in the tube, diminished by the force of an equal volume of air of the density of the atmosphere.

Let $2a$ = the length of a side of the section, which, for convenience, suppose square; and consider the action on each unit of the base to be equal to that on the central unit, which is, however, by no means accurate; then, if δx, δy, δz be the co-ordinates of any other particle measured from this, we have the pressure due to the air in the tube

$$= \mu \Sigma \frac{\delta z}{(\delta x^2 + \delta y^2 + \delta z^2)^{\frac{3}{2}}};$$

δz being in the direction of the length of the tube:

but if $\delta x = x\epsilon$......

$$\text{this is } \frac{4\mu}{\epsilon^2} \; \Sigma \frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}};$$

x, y, z being respectively the number of particles by which the particle under consideration is distant from the point in the base.

The limits of this sum are $x = \frac{a}{\epsilon}$, $y = \frac{a}{\epsilon}$, and $z = \frac{h}{\epsilon}$.

Now whilst y and z have any one particular value respectively, x will go through *all* its values; if then we expand the above function and take the *finite integral* of each term separately, we shall, by summing the resulting series, have the repulsion corresponding to any value of y and z.

Thus $\dfrac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \dfrac{z}{(y^2+z^2)^{\frac{3}{2}}}\left\{1 - \dfrac{3}{2}\,\dfrac{x^2}{y^2+z^2} + \dfrac{3.5}{2.4}\,\dfrac{x^4}{(y^2+z^2)^2} - \&c.\right\},$

and $\Sigma_x . 1 = x,$

$$\Sigma . (x+1)^2 = \frac{x^3}{3} + \frac{x^2}{2} + \dots$$

$$\Sigma . (x+1)^n = \frac{x^{n+1}}{n+1} + \frac{x^n}{2} + \dots$$

Substituting and, for the present, retaining only the first term, which amounts to supposing $\dfrac{a}{2}$ very large, we have

$$\Sigma_x \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{xz}{(y^2+z^2)(x^2+y^2+z^2)^{\frac{1}{2}}}.$$

Proceeding with this in like manner for z, we obtain

$$\Sigma_{x,z} \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \tfrac{1}{2}\log \frac{\sqrt{x^2+y^2+z^2} - x}{\sqrt{x^2+y^2+z^2} + x} \cdot \frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}-x},$$

and applying the same process to y, we obtain finally

$$\Sigma_{x,y,z} \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$= y\log\frac{\sqrt{x^2+y^2}+x}{y} + x\log\frac{\sqrt{x^2+y^2}+y}{x}$$

$$- y\log\frac{\sqrt{x^2+y^2+z^2}+x}{\sqrt{y^2+z^2}} - x\log\frac{\sqrt{x^2+y^2+z^2}+y}{\sqrt{x^2+z^2}}$$

$$+ z.\cot^{-1}\frac{z\sqrt{x^2+y^2+z^2}}{xy}$$

$$= \frac{1}{\epsilon}\left\{h\cot^{-1}\frac{h\sqrt{2a^2+h^2}}{a^2} + 2a\log\frac{2+\sqrt{2}}{\sqrt{3}+1}\right\}.$$

If we further suppose the height of the tube considerable, compared with the side of the section, we may omit $\dfrac{a^4}{h^2}$ compared with a:

hence pressure from the air in the tube

$$= \frac{4\mu}{\epsilon^3}\left\{2a \,.\, \log\frac{2+\sqrt{2}}{\sqrt{3}+1} + \frac{a^2}{h}\right\}$$

$$= C \,.\, D\left\{1 + \frac{a^2}{h}\right\};$$

C depending on the size of the tube.

23. If it be allowable thus to suppose an action on the *tube*, this gives the elastic force

$$E = C \,.\, D\left\{1 + \frac{a^2}{h}\right\}.$$

Let now a *given mass* of air be supposed compressed, as in Professor Robison's experiments, then since the mass $\propto \frac{a^2 h}{\epsilon^3}$

$$\frac{1}{h} \propto D.$$

The conclusion then at which we have arrived is that $E = CD(1 + qD)$; q being small.

Now the external air, acting in the same manner, produces a force which is $C'D'(1 + qD)$; subtracting these we get for the pressure

$$p = CD(1 + \alpha D) - C'D'.$$

$$\text{Now if } D' = D = 1, \qquad p = 0;$$

$$\therefore \; C(1 + \alpha) = C';$$

$$p = C\{D(1 + \alpha D) - (1 + \alpha)\}.$$

$$\text{Also if } D = 0, \quad p = -1; \qquad \therefore \; C = \frac{1}{1 + \alpha},$$

$$p = D\{1 + \alpha(D - 1)\} - 1;$$

but $p + 1$ is the actually measured pressure, hence calling this P,

$$P = D\{1 + \alpha(D - 1)\} \quad (2)$$

which, however, supposes α to depend on the volume of the tube.

24. Let us apply this expression to Professor Robison's experiments for dry air, considering it merely empirical: from this we obtain

$$D = 2; \quad 1.957 = 2(1 + a);$$

$$\therefore \; a = -.022,$$

$$P = D\{1 - .022(D - 1)\};$$

$$D = 3, \qquad P = 3(1 - .044) = 2.868,$$

$$D = 4, \qquad P = 4(1 - .066) = 3.736,$$

$$D = 5.5, \qquad P = 5.5(1 - .099) = 4.955,$$

$$D = 6, \qquad P = 6(1 - .11) = 5.340,$$

$$D = 7.62, \qquad P = 7.62(1 - .1456) = 6.50.$$

If we arrange these in a table with the measured results, we have as follows:

Calculated.	Observed.
2.868	2.848
2.736	2.737
4.955	4.930
5.340	5.342
6.50	6.49

The first and third differ by .02, the rest are nearly accurate, so that the empirical formula is not far from the truth.

25. In the summation of the preceding series (22), we have only retained the first terms of the various integrals, which amounts, in fact, to substituting the fluent for the definite integral. On applying this process to the expression for the velocity, it was shewn to be

equal to zero (20). The process is, in fact, manifestly insufficient in that case. The objection, however, which there applies, does not affect the expression under consideration, as the latter consists of a single term, whilst the former is the difference of two terms, in each of which the principal part is the same. It is not impossible that some of the omitted terms, even in the present case, should be considerable. The first of these has been exhibited merely on account of the symmetry of its form.

26. Recurring to the preceding expression for the pressure, and taking as the limits $z = 1$, $z = \frac{h}{\epsilon}$, we get

$$\Sigma_z \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} = -\frac{1}{\left\{x^2+y^2+\left(\frac{h}{\epsilon}\right)^2\right\}^{\frac{1}{2}}} + \frac{1}{(x^2+y^2+1)^{\frac{1}{2}}} + \frac{1}{2}\frac{\frac{h}{\epsilon}}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$$

$$\Sigma_{x,z} \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} = -\log\frac{\frac{a}{\epsilon}+\sqrt{\left(\frac{a}{\epsilon}\right)^2+y^2+\left(\frac{h}{\epsilon}\right)^2}}{\sqrt{y^2+\left(\frac{h}{\epsilon}\right)^2}} + \log\frac{\frac{a}{\epsilon}+\sqrt{\left(\frac{a}{\epsilon}\right)^2+y^2+1}}{\sqrt{y^2+1}}$$

$$+\frac{1}{2}\frac{\frac{ah}{\epsilon^2}}{\left\{\left(\frac{a}{\epsilon}\right)^2+y^2+\left(\frac{h}{\epsilon}\right)^2\right\}^{\frac{1}{2}}\left\{y^2+\left(\frac{h}{\epsilon}\right)^2\right\}^{\frac{1}{2}}}$$

$$-\frac{1}{2}\frac{1}{\left\{\left(\frac{a}{\epsilon}\right)^2+y^2+\left(\frac{h}{\epsilon}\right)^2\right\}^{\frac{1}{2}}} + \frac{1}{2}\frac{1}{\left\{y^2+\left(\frac{h}{\epsilon}\right)^2\right\}^{\frac{1}{2}}} + \frac{1}{2}\frac{1}{\left\{\left(\frac{a}{\epsilon}\right)^2+y^2+1\right\}^{\frac{1}{2}}}$$

$$-\frac{1}{2}\frac{1}{(y^2+1)^{\frac{1}{2}}} \text{ from } x=0, \text{ to } x=\frac{a}{\epsilon}.$$

$$\Sigma_{x,y,z} \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{1}{\epsilon}\left\{h\cot^{-1}\frac{h\sqrt{2a^2+h^2}}{a^2} + 2a\log\frac{2+\sqrt{2}}{\sqrt{3}+1}\right\}$$

$$-\tfrac{1}{2}\log\left\{\frac{\frac{a}{\epsilon}+\sqrt{\left(\frac{a}{\epsilon}\right)^2+y^2+\left(\frac{h}{\epsilon}\right)^2}}{\frac{a}{\epsilon}+\sqrt{\left(\frac{a}{\epsilon}\right)^2+\left(\frac{h}{\epsilon}\right)^2}}\cdot\frac{\frac{a}{\epsilon}+\sqrt{\left(\frac{a}{\epsilon}\right)^2+1}}{\frac{a}{\epsilon}+\sqrt{\left(\frac{a}{\epsilon}\right)^2+y^2+1}}\cdot\frac{y+\sqrt{\left(\frac{a}{\epsilon}\right)^2+y^2+\left(\frac{h}{\epsilon}\right)^2}}{\sqrt{\left(\frac{a}{\epsilon}\right)^2+\left(\frac{h}{\epsilon}\right)^2}}\cdot\right.$$

$$\left.\frac{\frac{h}{\epsilon}}{y+\sqrt{y^2+\left(\frac{h}{\epsilon}\right)^2}}\cdot\frac{\sqrt{\left(\frac{a}{\epsilon}\right)^2+1}}{y+\sqrt{\left(\frac{a}{\epsilon}\right)^2+y^2+1}}\cdot\frac{y+\sqrt{y^2+1}}{1}\cdot\frac{\frac{h}{\epsilon}}{\sqrt{y^2+\left(\frac{h}{\epsilon}\right)^2}}\cdot\frac{\sqrt{y^2+1}}{1}\right\}$$

$$+\tfrac{1}{2}\cot^{-1}.\frac{\frac{h}{\epsilon}\sqrt{\left(\frac{a}{\epsilon}\right)^2+y^2+\left(\frac{h}{\epsilon}\right)^2}}{\frac{ay}{\epsilon}},$$

and putting $\frac{a}{\epsilon}$ for y, we find the pressure on an unit of surface due to the air in the tube

$$=\frac{4\mu}{\epsilon^2}\left\{\frac{2a}{\epsilon}\log\frac{2+\sqrt{2}}{\sqrt{3}+1}+\frac{h}{\epsilon}\cot^{-1}\frac{h\sqrt{2a^2+h^2}}{a^2}\right.$$

$$-\log\frac{a+\sqrt{2a^2+h^2}}{a+\sqrt{2a^2+\epsilon^2}}\cdot\frac{a+\sqrt{a^2+\epsilon^2}}{a+\sqrt{a^2+h^2}}\cdot\frac{\sqrt{a^2+\epsilon^2}}{\sqrt{a^2+h^2}}\cdot\frac{h}{\epsilon}$$

$$\left.+\tfrac{1}{2}\cot^{-1}.\frac{h\sqrt{2a^2+h^2}}{a^2}\right\}.$$

If we adopt the same process as before, (*i. e.*) suppose h large compared with a, we have the pressure

$$=\frac{4\mu}{\epsilon^2}\left\{\frac{2a}{\epsilon}\log\frac{2+\sqrt{2}}{\sqrt{3}+1}+\frac{a^2}{\epsilon h}-\log\frac{\frac{2a}{\epsilon}}{\sqrt{2}+1}\right\}.$$

We see then that the introduction of another term has not, in the slightest degree, helped to extricate us from our difficulty; the result still depends on a; nor can I see any mode of avoiding this, and must therefore content myself with leaving it as a difficulty, or rather come to the conclusion that it does depend on a, and that, consequently, pressure is not produced in this manner, except this mode of summation by integrating is not allowable, which I am rather inclined to think is the case.

27. This, however, would not deter me from still retaining the above theory for the *motion* of air producing sound.

Pressure may be produced, and I think there is good reason for supposing this the case, by the actual contact of particles. It appears to be the most simple hypothesis to which we can refer it. I do not conceive it necessary to such a supposition that an atom should be variable in its form and dimensions. It would be more simple, to consider each particle as a collection of atoms arranged about a central nucleus of attractive or repulsive *imponderable* matter. Thus the whole pressure on a *surface* would be the pressure produced by the material particles actually in contact with it, the only effect of the immaterial particles being to compress the material ones together, and their *sensible* effect being consequently zero. Dr Dalton, however, supposes each atom surrounded by an atmosphere (as he terms it) of its own, which hypothesis although far more simple, does not, I think, so readily solve the difficulties, as that which I have adopted.

28. We recur then to the *motion* of the particles on the hypothesis of a law of force: the transmission not being an effect of actual contact, but of agitation amongst the immaterial particles. Of course such an agitation will produce, or be produced by, variations in the actual pressure, and thus the pressures may in one sense be said to produce the motion.

Having given reasons above for supposing the particular form of the law to be that of the inverse square of the distance, we shall in future adopt it.

Let us now proceed to consider the effect of the factor $\frac{1}{\epsilon}$ on the velocity. Conceive an atmosphere *of one kind;* let r, r' be the distances of two strata from the centre of the Earth; ϵ, ϵ' the distances of two immaterial particles at those points, respectively; then, for the equilibrium of a small portion of air between them, if p, p' be the units of pressure; we have

$$\frac{pr^2}{\epsilon^2} = \frac{p'r'^2}{\epsilon'^2} - \frac{c}{r^2}.\frac{r^2dr}{\epsilon^3} \text{ (}c\text{ being due to gravity),}$$

$$\text{or } \frac{d.\frac{pr^2}{\epsilon^2}}{dr} = \frac{c}{\epsilon^3};$$

$$\text{hence } \epsilon.\frac{d.pr^2}{dr} - 2pr^2.\frac{d\epsilon}{dr} = c.$$

$$\text{Let } \epsilon = Ar^n + \ldots\ldots$$

then p is the force upwards produced by the coating of atmosphere, diminished by that of gravity acting downwards, and

$$= -\frac{a}{r^2} + \frac{1}{r^2}\int_\rho \frac{1}{\epsilon^3}\rho^2$$

$$= -\frac{a}{r^2} + \frac{1}{r^2}\int_\rho \frac{\rho^2}{A^3\rho^{3n}} + \ldots$$

$$= -\frac{a}{r^2} - \frac{1}{r^2}\left\{\frac{1}{A_1^3\rho^{3n-3}} + \ldots\right\}$$

$$= -\frac{a}{r^2} - \frac{1}{A_1^3 r^{3n-1}} + \ldots$$

Now if $n = 2$ this is $\frac{a}{r^2}$, also if $n = 1$ the same form is true; assume therefore

$$p = \frac{a}{r^2}; \quad \therefore pr^2 = a,$$

$$\text{hence } \frac{d}{dr}\frac{pr^2}{\epsilon^2} = \frac{c}{\epsilon^3} \text{ becomes}$$

$$\frac{d}{dr}\frac{a}{\epsilon^2} = \frac{c}{\epsilon^3};$$

$$\text{also } \epsilon = Ar + B.$$

Now the altitude of the thermometer diminishes (*cæteris paribus*) nearly uniformly as r increases, hence, putting $r = a - t$, we get

$$\frac{1}{\epsilon} = \frac{1}{Ar + B} = \frac{1}{a - bt},$$

$$\text{and } \frac{\mu}{\epsilon} = V'(1 + ct);$$

$$\therefore\ v^2 = V_0^2(1 + ct);$$

$$\therefore\ v = V_0\sqrt{1 + ct},$$

which is the common expression.

29. Newton's hypothesis, founded on the law of the pressure varying as the density, is that the particles exert on each other forces varying as the reciprocal of the distance between them, but that the sphere of influence of each particle is terminated in the particles immediately adjoining it. The latter clause has been overlooked by many writers on the subject, although, as Professor Robison observes, it is absolutely necessary to retain it; and, indeed, it appears from our foregoing conclusion (20), that direct vibrations are incompatible with the hypothesis of the law of force varying as the reciprocal of the distance.

There seems to have been a great disinclination manifested to the adoption of Newton's Theory. Even writers, since Professor Robison, who have amply extracted from his work, have omitted the *restriction* of the law to the adjoining particles. Nor is it very extraordinary that this should be the case, when we consider how little pretensions this has to be considered a *law* of force at all.

Robison mentions an hypothesis which would reduce the above to a more intelligible form, which is, that the particles be supposed elastic and in contact. At the same time he rejects it, alleging as his reason for doing so, that if the particles were originally spherical, they would, before the pressure was doubled, become cubes. If this be the only objection against it, I confess that it appears to me to stand on a firmer foundation than any other. The hypothesis I should adopt from the formula $p = D\{1 + c(D-1)\}$ would be that the air consists of aggregations of different kinds of particles about a fluid which i attractive or repulsive, or, perhaps, of aggregations about two fluids. The particles themselves might be indifferent to each other's action, and obey the pressure of this fluid. Thus we should reduce the problem of pressure to that of particles in contact.

30. The result of our foregoing calculations has gone to shew, that, if we may suppose the phenomenon of sound due to the action of the repulsive energies of particles varying according to the Newtonian Law, there is no necessity for introducing an auxiliary hypothesis of the developement of Heat, but that the modifications, which the forces themselves produce, supply the requisite change in the energies of the action. The fact, that the same supposition is inapplicable *directly* to Pressure, is no argument against its validity for Sound; as it is clear, that, if these attractive particles be not the particles of air itself, but of some one or two media, exerting pressures on those of air, we must refer the statical pressure of the atmosphere to that of the particles of air, uniformly acted on; but the motion, at least the vibratory one, must be attributed to the condensation or rarefaction of the attracting particles, and therefore to the variation of the action of these particles on each other, and on the air.

SECTION III.

Combination of the Vibratory Motion of Particles with the Motion of Translation.

31. In the preceding Section we have considered the forces *repulsive:* though, as far as their mutual action on each other is concerned, there appears to be no reason for so doing; indeed it appeared from considerations connected with light that the ether, at least, is an exception to this rule. The difficulty attending the hypothesis of attractive forces consists in the apparent instability of a system of such particles.

This difficulty, however, is readily obviated, as the following considerations sufficiently evince.

32. If the particles have the cubical arrangement which I have before adopted, it is clear that the action of the forces on any particle, moved slightly from its position of equilibrium, would tend to bring it back. And if a series of particles in a given plane be simultaneously moved, they, in like manner, would be brought back. But another case presents itself which is not so easily solved, viz., that, in which a series of particles in one plane are moved simultaneously at right angles to that plane. We will then endeavour to find what is the force exerted on a particle in these circumstances.

33. Take the position of equilibrium of any particle, so displaced, as the origin of co-ordinates: let x, y, z be the co-ordinates of any other particle, at the distance r, suppose the displacement to be through the space a, and that the force varies inversely as the square of the distance, then the attraction on this particle to carry it forward, is

$$\Sigma \frac{x-a}{\{(x-a)^2+y^2+z^2\}^{\frac{3}{2}}} \text{ taken to infinity,}$$

$$\text{but } \Sigma \frac{x-a}{\{(x-a)^2+y^2+z^2\}^{\frac{3}{2}}} = \Sigma \frac{x-a}{\{r^2-(2ax-a^2)\}^{\frac{3}{2}}}$$

$$= \Sigma \frac{x-a}{r^3} \left\{1 + \frac{3}{2}\left(\frac{2ax-a^2}{r^2}\right) + \frac{3.5}{2.4}\left(\frac{2ax-a^2}{r^2}\right)^2 + \frac{3.5.7}{2.4.6}\left(\frac{\dots}{\dots}\right)^3 + \dots\right\}$$

$$= \Sigma \frac{1}{r^3} \left\{x - a + \frac{3}{2}\,\frac{2ax^2-2a^2x+a^3}{r^2} \right.$$

$$\left. + \frac{3.5}{2.4}\left(\frac{4a^2x^3-8a^3x^2}{r^4}\right) + \frac{3.5.7}{2.4.6}\left(\frac{8a^3x^4}{r^6}\right) + \dots\right\}$$

$$= \Sigma \frac{1}{r^3} \left\{-a\left(1-\frac{3x^2}{r^2}\right) - a^2\left(\frac{9x}{2r^2} - \frac{15}{2}\frac{x^3}{r^4}\right)\right.$$

$$\left. + a^3\left(\frac{3}{2r^2} - 3.5\frac{x^2}{r^4} + \frac{5.7}{2}\frac{x^4}{r^6}\right) + \dots\right\}.$$

$$\text{Now } \Sigma\left(\frac{x^2+y^2+z^2}{r^2}\right) = 3\Sigma\frac{x^2}{r^2};$$

$$\therefore \Sigma\left(1-\frac{3x^2}{r^2}\right) = 0,$$

also it is evident that $2\Sigma\frac{1}{r^2}\left(\frac{3}{2} - \frac{15x^2}{r^2} + \frac{35}{2}\frac{x^4}{r^4}\right)$

$$= \Sigma\frac{3}{r^2}\left\{1 - \frac{10x^2}{r^2} + \frac{35}{3}\frac{x^4}{r^4}\right\}$$

$$= \Sigma\frac{3}{r^6}\left\{(x^2+y^2+z^2)^2 - 10x^2(x^2+y^2+z^2) + \frac{35}{3}x^4\right\}$$

$$= \Sigma\frac{3}{r^6}\left\{-7x^4 - 14x^2y^2 + \frac{35}{3}x^4\right\}$$

$$= \Sigma\frac{1}{r^6}\left\{14x^4 - 42x^2y^2\right\}$$

$$= 14\Sigma\frac{1}{r^6}\left\{x^4 - 3x^2y^2\right\};$$

so that the tendency will be to restore the equilibrium, if

$$\Sigma\frac{x^2}{r^9}(x^2-3y^2) \text{ is negative.}$$

In order to ascertain this, I have adopted an approximative process of finding and comparing the *integrals*, and the result being negative, we may conclude that the tendency is to restore the equilibrium in this case. It is true an objection may be adduced from the enormous complication of disturbances which must affect the equilibrium. The only answer I can offer to this, is, that the rapidity with which a disturbance is transmitted is enormous in the same proportion, and hence we may conclude that the disturbances are easily righted.

34. Adopting then a series of attractive particles, or two series, each of which attracts its own, but repels the others, we obtain, as our equations of motion, (the direction of transmission being that of the axis of x):

$$\frac{d^2 a}{dt^2} = -2P\sigma\left\{\frac{\delta x^2 + \delta y^2 + \delta z^2 - 3\delta x^2}{r^5}\right\}\sin^2\frac{k\delta x}{2}\,.\,a$$

$$= -\frac{2P}{\epsilon^3}\sigma\,.\left\{\frac{\xi^2 + \eta^2 + \zeta^2 - 3\xi^2}{r^5}\right\}\sin^2\frac{\pi\epsilon\xi}{\lambda}\,.\,a$$

$$= -\frac{2P}{\epsilon^3}\sigma\,.\left\{\frac{\eta^2 + \zeta^2 - 2\xi^2}{r^5}\right\}\sin^2\frac{\pi\epsilon\xi}{\lambda}\,.\,a,$$

$$\text{but } \sigma\left(\frac{\zeta^2\,.\sin^2\dfrac{\pi\epsilon\xi}{\lambda}}{r^5}\right) = \sigma\left(\frac{\eta^2\sin^2\dfrac{\pi\epsilon\xi}{\lambda}}{r^5}\right);$$

$$\therefore\ \frac{d^2 a}{dt^2} = -\frac{4P}{\epsilon^3}\sigma\,.\left\{\frac{\zeta^2 - \xi^2}{r^5}\,.\sin^2\frac{\pi\epsilon\xi}{\lambda}\right\}.\,a$$

$$= \frac{4P}{\epsilon^3}\sigma\,.\,\frac{\xi^2 - \zeta^2}{r^5}\sin^2\frac{\pi\epsilon\xi}{\lambda}\,.\,a,$$

$$\frac{a\ \beta}{dt^2} = -\frac{2P}{\epsilon^3}\sigma\left\{\frac{\xi^2 + \eta^2 + \zeta^2 - 3\eta^2}{r^5}\right\}\sin^2\frac{\pi\epsilon\xi}{\lambda}\,.\,\beta$$

$$= -\frac{2P}{\epsilon^3}\sigma\left\{\frac{\xi^2 + \zeta^2 - 2\eta^2}{r^5}\right\}\sin^2\frac{\pi\epsilon\xi}{\lambda}\,.\,\beta$$

$$= -\frac{2P}{\epsilon^3}\sigma\,.\,\frac{\xi^2 - \zeta^2}{r^5}\sin^2\frac{\pi\epsilon\xi}{\lambda}\,.\,\beta,$$

$$\frac{d^2\gamma}{dt^2} = -\frac{2P}{\epsilon^3}\sigma\frac{\xi^2-\zeta^2}{r^5}\sin^2\frac{\pi\epsilon\xi}{\lambda}\cdot\gamma,$$

and by what has been shewn in (4),

the quantity $\frac{4P}{\epsilon^2}\sigma\cdot\frac{\xi^2-\zeta^2}{r^5}\sin^2\frac{\pi\epsilon\xi}{\lambda}$ is positive:

let it be equal to $2u^2$, and we shall have

$$\frac{d^2a}{dt^2} = 2u^2a,$$

$$\frac{d^2\beta}{dt^2} = -u^2\beta,$$

$$\frac{d^2\gamma}{dt^2} = -u^2\gamma.$$

On examining the investigation for light, we perceive that the solution of the last two equations was suggested by that of the approximate ones

$$\frac{d^2\beta}{dt^2} = \frac{u^2}{k^2}\left(\frac{d^2\beta}{dx^2}\right),$$

$$\frac{d^2\gamma}{dt^2} = \frac{u^2}{k^2}\left(\frac{d^2\gamma}{dx^2}\right);$$

and applying the same suggestion to this case, we should have to solve the equation

$$\frac{d^2a}{dt^2} + \frac{2u^2}{k^2}\frac{d^2a}{dx^2} = 0,$$

which would arise from supposing the extent of influence of the particles small, or the length of a wave large.

This equation is nearly identical, in form, with that, which Fourier has so amply discussed in his treatise on Heat.

35. It may not be uninteresting to compare the results in the case of inelastic fluids with ours. I shall adopt the usual notation, and suppose the motion to take place in the following manner. A series of waves is transmitted along the axis of x, whilst the motion of an

individual particle is partly in the same direction and partly in a vertical direction, which we will call that of y. There is no motion in the direction of z.

It is evident that we are here considering the motion of waves on water or of the tide waves.

The equations of motion are

$$\frac{dp}{dy} = D\left\{-g - \frac{d(v)}{dt}\right\}$$

$$\frac{dp}{dx} = D\left\{-\frac{d(u)}{dt}\right\},$$

which give one as their result, viz.

$$\frac{d}{dx}\frac{d(v)}{dt} = \frac{d}{dy}\frac{d(u)}{dt} \;\ldots\ldots\ldots (a),$$

also we have the equation $\dfrac{du}{dx} + \dfrac{dv}{dy} = 0.$

36. Now $$\frac{d(u)}{dt} = \frac{du}{dx}u + \frac{du}{dy}v + \frac{du}{dt}$$

$$\frac{d(v)}{dt} = \frac{dv}{dx}u + \frac{dv}{dy}v + \frac{dv}{dt},$$

and the additional hypothesis that the motion is a *wave* motion, introduces the further conditions

$$u = a\sin\frac{2\pi}{\lambda}(ct - x)fy,$$

$$v = b\cos\frac{2\pi}{\lambda}(ct - x)F(y),$$

$f(y)$ and $F(y)$ being any functions of y.

By substitution in the equation

$$\frac{du}{dx} + \frac{dv}{dy} = 0,$$

we get

$$-\frac{2\pi}{\lambda}a\cos\frac{2\pi}{\lambda}(ct - x)fy + b\cos\frac{2\pi}{\lambda}(ct - x)F'y = 0;$$

$$\therefore\ afy = \frac{b\lambda}{2\pi} F'y,$$

$$\text{and } u = \frac{b\lambda}{2\pi} \sin \frac{2\pi}{\lambda}(ct - x) \,.\, F'y;$$

and representing $\frac{2\pi}{\lambda}(ct-x)$ by θ, we thus get

$$\frac{du}{dx} = -\, b \cos\theta \,.\, F'y,$$

$$\frac{du}{dy} = \frac{b\lambda}{2\pi} \sin\theta \,.\, F''y,$$

$$\frac{du}{dt} = cb\cos\theta \,.\, F'y,$$

$$\frac{dv}{dx} = b\frac{2\pi}{\lambda}\sin\theta \,.\, Fy,$$

$$\frac{dv}{dy} = b\cos\theta \,.\, F'y,$$

$$\frac{dv}{dt} = -\, b\frac{2\pi}{\lambda} c\sin\theta \,.\, Fy;$$

$$\therefore\ \frac{d(u)}{dt} = -\frac{b^2\lambda}{2\pi}\sin\theta\cos\theta\,(F'y)^2 + \frac{b^2\lambda}{2\pi}\sin\theta\cos\theta \,.\, Fy\,F''y$$

$$+\, cb\cos\theta \,.\, F'y$$

$$= \frac{b^2\lambda}{2\pi}\sin\theta\cos\theta\,\{Fy\,F''y - (F'y)^2\} + cb\cos\theta \,.\, F'y$$

$$\frac{d(v)}{dt} = b^2\sin^2\theta\, Fy\, F'y + b^2\cos^2\theta\, Fy\, F'y - \frac{2\pi c}{\lambda} b\sin\theta \,.\, Fy$$

$$= b^2\, Fy\, F'y - \frac{2\pi cb}{\lambda}\sin\theta \,.\, Fy;$$

and hence the equation (a) is reduced to

$$\frac{4\pi^2}{\lambda^2} cb\cos\theta \,.\, Fy$$

$$= \frac{b^2\lambda}{2\pi} \sin\theta \cos\theta \,(Fy\,F'''y - F'y\,F''y) + cb\cos\theta \,.\, F''y,$$

and dividing by $\cos\theta$, and equating coefficients of $\sin\theta$ and of unity to zero, we have

$$Fy\,F'''y - F'y\,F''y = 0,$$

$$F''y = \frac{4\pi^2}{\lambda^2} Fy.$$

The latter gives

$$Fy = e^{\frac{2\pi}{\lambda} y},$$

which also satisfies the former, and we obtain

$$u = b \sin \frac{2\pi}{\lambda}(ct - x)\, e^{\frac{2\pi}{\lambda} y},$$

$$v = b \cos \frac{2\pi}{\lambda}(ct - x)\, e^{\frac{2\pi}{\lambda} y}.$$

37. It is clear from these equations that if there be a motion vertically, there must be a corresponding horizontal motion, and that when the vertical motion is zero, as when $ct - x = \frac{\lambda}{4}$ or $\frac{3\lambda}{4}$, the horizontal is at its maximum, and *vice versâ*.

If we take the origin at a given depth h, and suppose the maximum values of u and v at that point $= m$, we have $m = b$, and the greatest velocity at the surface is

$$u = m e^{\frac{2\pi}{\lambda} h} = v.$$

38. Suppose now we have two sets of fluid, and that at the depth h in one, and h_1 in the other, the maximum velocity is m.

Let u, u_1 be the maximum velocities at the surface,

$$\text{then } u = m e^{\frac{2\pi}{\lambda} h},$$

$$u_1 = m e^{\frac{2\pi}{\lambda} h_1},$$

$$\text{then } \frac{u}{u_1} = e^{\frac{2\pi}{\lambda}(h - h_1)},$$

so that if h be greater than h_1 u is greater than u_1, or for the same length of wave (on this hypothesis) the velocities increase with the increase of the depth of fluid.

I return to the equations in (34).

39. It is possible, that the original disturbance may have been such as to make $\alpha = 0$, that is, entirely transversal; such an hypothesis will in no way affect our investigations, as it does not interfere with the other equations, and moreover in the case of light is probably correct.

Again, the original disturbance may have been such as to *impel forwards* the particles, at the same time that a transversal vibration is communicated to them. Integrating the first equation, we have

$$\left(\frac{d\alpha}{dt}\right)^2 = 4u^2\alpha^2 + C,$$

$$V^2 = C;$$

V being the velocity communicated; hence, we have

$$\alpha = Vt;$$

and if we assume that $\alpha = mt + ae^{ct}.\cos kx$, we have

$$\delta\alpha = -ae^{ct}.2\sin^2\frac{k\delta x}{2}\cos kx + t\delta m$$

$$= -2(\alpha - mt)\sin^2\frac{k\delta x}{2} + t\delta m;$$

$$\therefore \frac{d^2\alpha}{dt^2} = 2u^2(\alpha - mt) + tf(x),$$

the latter term arising from m being a function of x;

$$\text{but } \frac{d^2\alpha}{dt^2} = ac^2e^{ct}\cos kx = c^2(\alpha - mt);$$

$$\therefore c^2(\alpha - mt) = 2u^2(\alpha - mt) + t.f(x);$$

and by equating coefficients of α we get

$$c^2 = 2u^2,$$

from which it appears that $f(x) = 0$, or m is a constant $= V$; and hence $a = Vt + ae^{u\sqrt{2}.t}.\cos kx$ is a solution of the equation.

40. We conclude that a motion of translation is perfectly consistent with vibrations, and, from the form of the solution, it will be perceived that the transmission is not uniform, but proceeds as it were by fits; the uniform motion of transmission being combined with a variable one, depending on the lengths of the waves in the transversal vibration. In other words, at particular points the direct motion is accelerated or retarded by the effect of the transverse motion of the particle.

41. In order to ascertain the value of V it will be necessary to recur to the circumstances under which any particle started into motion.

Let the particles behind it be in motion according to the regular type, then clearly the force acting on this particle at the first moment is represented by

$$\Sigma . \frac{\delta x + a}{\{(\delta x + a)^2 + (\delta y + \beta)^2 + (\delta z + \gamma)^2\}^{\frac{3}{2}}} - \Sigma \frac{\delta x}{r^3},$$

taken only on one side of it with reference to a plane perpendicular to the axis of x.

The expression becomes by expansion,

$$\Sigma \frac{\delta x + a}{r^3} . \left\{1 - \frac{3}{2} . \frac{2(a\delta x + \beta\delta y + \gamma\delta z) + a^2 + \beta^2 + \gamma^2}{r^2}\right.$$

$$\left. + \frac{3.5}{2.4} . \frac{4(a^2\delta x^2 + \beta^2\delta y^2 + \gamma^2\delta z^2) + 8(\delta x\delta y a\beta + \dots)}{r^4}\right\} - \Sigma \frac{\delta x}{r^3}$$

$$= \Sigma \frac{1}{r^3}\left\{a - \frac{3}{2} . \frac{2(a^2\delta x + a\beta\delta y + a\gamma\delta z)}{r^2} - \frac{3}{2r^2}\delta x(\beta^2 + \gamma^2)\right.$$

$$+ \frac{3.5}{2.4} \frac{4(\delta x^3 . a^2 + \delta x\delta y^2 . \beta^2 + \delta x\delta z^2\gamma^2)}{r^4}$$

$$\left. - \frac{3}{2} . \frac{2.a\delta x^2 + (a^2 + \beta^2 + \gamma^2)\delta x}{r^2} + \dots\right\}$$

$$= \Sigma \frac{1}{r^3} \left\{ \alpha - \frac{9}{2r^2} \delta x \, . \, a^2 + \frac{3 \, . \, 5}{2 \, . \, 4} \frac{\delta x^3}{r^4} a^2 + \frac{3 \, . \, 5}{2} \, . \, \frac{\delta x \delta y^2}{r^4} (\beta^2 + \gamma^2) \right.$$

$$\left. - \frac{3 \delta x}{r^2} \, . \, (\beta^2 + \gamma^2). \right.$$

Suppose the particle under consideration situated at the confines of a medium, whose temperature is less than that of the medium in which the *uniform* motion has been transmitted; we have then $\alpha = 0$, and the force exerted

$$= A(\beta^2 + \gamma^2)$$

$$= A \, . \, (b^2 + c^2) \text{ suppose,}$$

omitting the variable part, so that the value of a in this instance depends on $b^2 + c^2$.

It will readily be perceived that the above investigation has reference to the possibility of a *transmission* of the particles whilst they are in a state of vibration. And since we have shewn that the tendency to motion forwards varies as the extent of the vibratory motion, we have stumbled upon an interpretation which coincides with the physical characters of Heat.

42. But all that we have hitherto done is to suppose the motion of a series of particles, symmetrically situated with respect to the axes. Such an investigation will probably be correct for Light, but when treating of the transmission of Heat, the contrary must undoubtedly be adopted.

Leaving out of the question then the above investigation which may apply to constant radiation, and which is nearly identical with Leslie's hypothesis, we come to the case of a medium having more particles on one side of a plane, parallel to that of yz, than on the other; and our object is to determine the initial velocity and motion of transmission of a particle along the axis of x.

We shall in this investigation suppose no reciprocating motion of the particles along the axis, and consider them to have had no velocity of transmission, which supposition will not affect the results due to that velocity if it exist, but merely introduce a portion to be added to the former.

Now $(r+\rho)^2 = \delta x^2 + (\delta y + \delta\beta)^2 + (\delta z + \delta\gamma)^2$

$$= r^2 + 2(\delta y\,\delta\beta + \delta z\,\delta\gamma) + \delta\beta^2 + \delta\gamma^2$$

$= r^2 + \rho'$ suppose;

$$\therefore \frac{1}{(r+\rho)^3} = \frac{1}{r^3}\frac{1}{\left(1+\frac{\rho'}{r^2}\right)^{\frac{3}{2}}}$$

$$= \frac{1}{r^3}\left\{1 - \frac{3}{2}\frac{\rho'}{r^2} + \frac{3.5}{2.4}\left(\frac{\rho'}{r^2}\right)^2\right\},$$

and $\rho'^2 = 4(\delta y^2\delta\beta^2 + \delta z^2\delta\gamma^2) + 4\delta y\,\delta z\,\delta\beta\,\delta\gamma$;

$$\text{hence } \frac{1}{(r+\rho)^3} = \frac{1}{r^3}\left\{1 - \frac{3}{2}\cdot\frac{2\delta y\,\delta\beta + 2\delta z\,\delta\gamma + \delta\beta^2 + \delta\gamma^2}{r^2} + \frac{3.5}{2.4}\frac{4\delta y^2\,\delta\beta^2 + 4\delta z^2\,\delta\gamma^2 + 4\delta y\,\delta z\,\delta\beta\,\delta\gamma}{r^4}\right\},$$

$$\text{hence } \frac{d^2a}{dt^2} = \Sigma\frac{\delta x}{(r+\rho)^3}$$

$$= \Sigma\left\{\frac{\delta x}{r^3} - \frac{3}{2}\frac{\delta x\,(\delta\beta^2 + \delta\gamma^2)}{r^5} + \frac{3.5}{2}\,\delta x\,\frac{\delta y^2\,\delta\beta^2 + \delta z^2\,\delta\gamma^2}{r^7}\right\},$$

$$\text{but } \Sigma\frac{\delta y^2}{r^7} = \frac{1}{3}\Sigma\frac{\delta x^2 + \delta y^2 + \delta z^2}{r^7} \text{ nearly,}$$

$$= \frac{1}{3}\Sigma\frac{1}{r^5};$$

$$\therefore \frac{d^2a}{dt^2} = \Sigma\frac{\delta x}{r^3} + \frac{3}{2}\Sigma\delta x\,\frac{5(\delta\beta^2 + \delta\gamma^2)}{3r^5} - \frac{3}{2}\Sigma\delta x\,\frac{\delta\beta^2 + \delta\gamma^2}{r^5}$$

$$= \Sigma\frac{\delta x}{r^3} + \Sigma\delta x\,\frac{\delta\beta^2 + \delta\gamma^2}{r^5},$$

or, omitting the first term on the principle above stated, and putting for $\delta\beta^2$ and $\delta\gamma^2$ their values $b\cos(ut-kx)$ $c.\cos(ut-kx+A)$, it becomes

$$\frac{d^2a}{dt^2} = R\{b^2\cos^2(ut-kx)+c^2\cos^2(ut-kx+A)\}$$

$$= R\frac{b^2+c^2}{2} + \frac{R}{2}\{b^2\cos 2(ut-kx)+c^2\cos 2(ut-kx+A)\};$$

$$\therefore \frac{da}{dt} = R.\frac{b^2+c^2}{2}.t + \frac{R}{4u}\{b^2\cos 2(ut-kx)+c^2\sin 2(ut-kx+A)\}$$

$$= R\frac{b^2+c^2}{2}.t \text{ omitting the reciprocating part,}$$

$$\text{and supposing } \frac{da}{dt}=0, \text{ when } t=0;$$

hence, when the medium is of this kind, there will be a transmission *depending on the motion of vibration.*

Thus it appears that the quantity of one substance *A*, which enters a medium, depends, *cæteris paribus*, on the *intensity* of the undulation excited in the particles without the medium. At the same time, a quantity of the other substance *B* is withdrawn from the medium, and retires to supply the place which the former has left.

43. The only doubt, which I can perceive, of this being a sufficient explanation of the coincidence of phenomena, which require considerations of an undulatory, as well as of a corpuscular, nature arises from the apparent smallness of the term and the consequent smallness of the transmission.

It would appear, from the recent experiments of Professor Forbes on the Polarization of Heat, that some such terms must enter from whatever considerations they may be deduced. It will, perhaps, with some shew of justice, be objected to the above statement, that all that has been attempted is, to shield ourselves from the necessity of accounting for Polarization of Heat, by referring to its parallel in the absorption of light.

I cannot, however, admit such an objection, as the great difficulty which hangs about absorption is its elective character, which is not at all necessary, as far as we know, for Heat.

44. But allowing all the difficulties, it is desirable to adopt an hypothesis which has some chance of meeting a few of them, and I hope it is not quite unimportant to shew that if the vibration in one plane be stopped (as for instance if $b=0$), the transmission is affected by a constant loss, and if another substance be so placed, that the corresponding planes in the two are parallel, c^2 will not be destroyed by it, whilst on the other hand, if they are perpendicular, it will; which is, in fact, such an explanation as will account for the phenomena.

It appeared to me utterly impossible to refer the phenomena of Heat to vibrations in the same manner as we do those of light, from the obvious circumstance that a new and permanent *force* seems necessary in the consideration, and this must plead my apology for bringing these speculations before the Society.

45. I would have it observed that although in the majority of cases I have spoken as though I consider *one* attractive medium (at least as far as the action of the particles on each other is concerned,) sufficient to account for the phenomena, it is not at all my wish to have it supposed that I am anxious to advocate such an hypothesis. If we choose rather, in the present state of our knowledge of the subject, to admit Coulomb's hypothesis of two media, whose particles, respectively, act repulsively on particles of the same medium, and attractively on those of the other, none of my conclusions will be in the slightest degree affected, provided we allow that the effects of vibration, and transfer of which we have been speaking, have reference to the medium A, for which the quantity $-\frac{P}{\epsilon}+\frac{Q}{\epsilon'}$ is positive, or $\frac{Q^2}{\epsilon'^2}>\frac{P^2}{\epsilon^2}$, so that the mutual action of two consecutive particles of the *moving* media is less than that of the other.

I should, however, prefer retaining the two media above considered, for reasons which I shall presently adduce.

46. The conclusion at which we have arrived will be strengthened, if we can shew that the value of b^2+c^2 is considerable, in cases in which we have good reason to attribute to it effects, the magnitude of which is determined by experiment. Such effects appear to me to take place in the case of light. I must here refer to my Memoir on that subject, as I am not aware that it has elsewhere been pursued into detail.

By a reference to that Memoir, it appears that, representing the square of the velocity by a series of terms of the form

$$p-\frac{q}{\lambda^2}+\frac{l}{\lambda^4}-\ldots,$$

the quantity l is for all excepting two of Fraunhofer's substances *negative*.

This curious result is by no means attributable to the sum of the series which represents it on the hypothesis of an ordinary vibration, viz. $\frac{d^2\beta}{dt^2}=-2\beta\Sigma\frac{1}{r^3}\sin^2\frac{k\delta x}{2}\left(1-\frac{3\delta x^2}{r^2}\right)$. The coefficients would thence be the same for all substances independently of a common factor.

47. The explanation which I ventured to suggest of this anomaly was one of which I have seen no reason since to repent. At the same time I fear in repeating it that it may be pronounced inconsistent with facts. It is that the remaining terms in the expansion must be retained: an hypothesis which seems to imply that the refraction will depend (though probably very slightly indeed,) on the *intensity* of the light.

This hypothesis is, I suppose, totally unsupported by any experimental evidence, but it so readily solves the difficulty that I should not be justified in leaving it unnoticed.

Suppose then we adopt the hypothesis and examine the equations of motion which thereby result.

By a similar process to (42), we have

$$\frac{1}{(r+\rho)^3} = \frac{1}{r^3}\left\{1 - \frac{3}{2}\,\frac{2(\delta x\,\delta a + \delta y\,\delta\beta + \delta z\,\delta\gamma + \delta a^2 + \delta\beta^2 + \delta\gamma^2)}{r^2}\right.$$

$$+ \frac{3\,.\,5}{2\,.\,4}\left(\frac{4(\delta x^2\,\delta a^2 + \delta y^2\,\delta\beta^2 + \delta z^2\,\delta\gamma^2) + 4(\delta x\,\delta a\,\delta y\,\delta\beta + \ldots)}{r^4}\right.$$

$$\left. + \frac{4(\delta a^2 + \delta\beta^2 + \delta\gamma^2)(\delta x\,\delta a + \delta y\,\delta\beta + \delta z\,\delta\gamma)}{r^4}\right)$$

$$\left. - \frac{3\,.\,5\,.\,7}{6r^6}(\delta x\,\delta a + \delta y\,\delta\beta + \delta z\,\delta\gamma)^3\right\},$$

and by multiplication and reduction, this gives us

$$\Sigma\frac{\delta y + \delta\beta}{(r+\rho)^3} = \Sigma\frac{1}{r^3}\left\{\delta\beta - \frac{3\delta y^2\,\delta\beta}{r^2} - \frac{3}{2}\,\delta\beta\,\frac{\delta a^2 + \delta\beta^2 + \delta\gamma^2}{r^2}\right.$$

$$+ \frac{3\,.\,5}{2}\,\delta\beta\,\frac{\delta x^2\,\delta a^2 + \delta y^2\,\delta\beta^2 + \delta z^2\,\delta\gamma^2 + \delta y^2(\delta a^2 + \delta\beta^2 + \delta\gamma^2)}{r^4}$$

$$\left. - \frac{3\,.\,5\,.\,7}{6}\,.\,\frac{\delta y\,\delta\beta^3}{r^6}\right\},$$

hence, writing δy for δz: also, putting $\delta a = 0$, this is reduced to

$$\Sigma\frac{\delta\beta}{r^3}\left\{1 - \frac{3\delta y^2}{r^2} - \frac{3}{2}\left(\frac{\delta\beta^2 + \delta\gamma^2}{r^2} - 10\,\delta y^2\,\frac{\delta\beta^2 + \delta\gamma^2}{r^4} + \frac{35}{3}\,\frac{\delta y^4\,\delta\beta^2}{r^6}\right)\right\}.$$

Now the integral of this is not a simple quantity, as in the former case, but some knowledge of it, at least, may be obtained by the assumption (which is perfectly allowable) that the same form still holds;

$$\text{let then } \beta = b\cos(ut - kx);$$

$$\therefore\ \delta\beta = -\,2\beta\sin^2\frac{k\delta x}{2};$$

$$\text{similarly } \delta\gamma = -\,2\gamma\sin^2\frac{k\delta x}{2}.$$

and we obtain

$$\frac{d^2\beta}{dt^2} = -\,2\beta\Sigma\,.\,\frac{1}{r^3}\,.\,\sin^2\frac{k\delta x}{2}\,.\left\{1 - \frac{3\delta y^2}{r^2}\right.$$

$$\left. - \frac{3}{2}\left(4\,\frac{(\beta^2+\gamma^2)}{r^2}\left(1 - \frac{10\delta y^2}{r^2}\right)\sin^4\frac{k\delta x}{2} + \frac{35}{3}\,4\beta^2\frac{\delta y^4}{r^4}\sin^4\frac{k\delta x}{2}\right)\right\}.$$

$$= -\,2\beta\Sigma\frac{1}{r^3}\left(1 - \frac{3\delta y^2}{r^2}\right)\sin^2\frac{k\delta x}{2} +$$

$$6\beta\Sigma\frac{1}{r^3}\left\{(\beta^2+\gamma^2)\left(1 - \frac{10\delta y^2}{r^2}\right)\sin^6\frac{k\delta x}{2} + \frac{35}{3}\beta^2\frac{\delta y^4}{r^4}\sin^6\frac{k\delta x}{2}\right\}.$$

Now a term of the form β^3 is expressed by

$$b^3\cos^3(ut - kx) = \frac{b^3}{4}\{3\cos.(ut - kx) + \cos 3(ut - kx)\}$$

$$= \frac{3}{4}b^2\beta + \ldots\ldots$$

but β is supposed, in fact, to consist of an infinite number of terms, the type of which is the above, hence, omitting all terms which do not come under this type, for β^3 we shall write $\frac{3}{4}b^2\beta$.

Similarly for $\beta\gamma^2$, we shall write $\dfrac{c^2\beta}{2}$; and we obtain by substitution, putting also $b = c$,

$$\frac{d^2\beta}{dt^2} = -\,2\beta\Sigma\frac{1}{r^3}\left\{\left(1 - \frac{3\delta y^2}{r^2}\right)\sin^2\frac{\pi\delta x}{\lambda}\right.$$

$$\left. - 3b^2\frac{5}{4}\left(1 - \frac{10\delta y^2}{r^2} + 7\frac{\delta y^4}{r^4}\right)\sin^6\frac{\pi\delta x}{\lambda}\right\},$$

so that on this hypothesis

$$v^2 = \frac{\Sigma\dfrac{1}{r^3}\left\{\left(1 - \dfrac{3\delta y^2}{r^2}\right).\sin^2\dfrac{\pi\delta x}{\lambda} - \dfrac{15b^2}{4}\left(1 - \dfrac{10\delta y^2}{r^2} + \dfrac{7\delta y^4}{r^4}\right)\sin^6\dfrac{\pi\delta x}{\lambda}\right\}}{\dfrac{2\pi^2}{\lambda^2}},$$

which shews that the introduction of this term will affect the *third* term of the expansion for v^2, but not the second.

In fact v^2 now takes the form

$$v^2 = p - \frac{q}{\lambda^2} + \left(\frac{r - Ab^2}{\lambda^4}\right),$$

$2A$ being the coefficient of $b^2 \sin^6 \frac{\pi\delta x}{\lambda}$ above.

48. Of course, I do not mean to offer this as a *solution* of the equation, but merely as a proof of the possibility of its taking the above form: the quantity A may, in fact, be very different indeed from that above exhibited.

49. Now we found in the case of the ten substances examined by Fraunhofer, that eight of them gave the coefficient $\frac{1}{\lambda^4}$ *negative*, whilst the other two gave it positive; the value of Ab^2 must then be considerable (supposing the explanation of the fact to be contained here), and greater in the cases of glass than of water.

Now b^2, in the view above given, will determine the quantity of *heat* transmitted with the spectrum under any given circumstances, it follows therefore that substances which transmit heat most freely give l small, and hence referring to the list in my former paper, I obtain the following order for transmission of Heat, beginning at that in which the freedom of transmission is the greatest.

Flint Glass, No. 30.
.... 23.
.... 3.
.... 13.

Oil of Turpentine.

Crown Glass, Let. M.
.... No. 13.
.... 9.

Water.
Solution of Potash.

50. I am not aware that the transmissibility of these substances has been accurately ascertained: it appears, as far as I have had opportunity of judging, that they exactly coincide with the above table.

But further, the velocity of transmission of the heat depends on the quantity b^2, according to the view of the subject which I have given above. We ought then to find the refractive index for heat, or rather that for the point of *greatest* heat in the spectrum, diminishing as b^2 increases, that is, as l in my former paper diminishes: the above table then must represent the order in which the points of greatest heat deviate from *direct* transmission, beginning with that of least deviation.

This subject has not been examined so extensively as to enable me to compare the results of theory with those of observation, numerically.

M. Seebeck's results for Water, Crown Glass and Flint Glass, coincide with the above, and they are the only ones which he has given for Fraunhofer's substances.

51. It would be leading me too far into loose speculation, were I to proceed to consider the effect produced in the refractive index by the increase of temperature. In fact, we have so few experiments, by which such speculations could be guided, that it would be almost impossible to enter upon this subject. If we had a variety of substances, whose specific heat was well determined, and refractive indices known very accurately, it might be possible to trace the analogy that exists between light and heat with considerable accuracy. Admitting that the two fluids (which we have, for the sake of distinction, designated ether and caloric) are what we usually mean by those terms, it appears from the investigation that a *transfer* of caloric corresponds to an expulsion of ether. Hence, if the temperature (*i. e.* the density of the surrounding ether), and also the density of the body remain the same, whilst the latent heat is increased, we should expect the ether proportionably diminished, and anticipate a corresponding increase of the refractive index.

Of course I am guided, in saying *increase* of the refractive index, by the hypothesis that the refraction increases as the density of the ether diminishes, which I have, elsewhere, shewn reasons for supposing true (*Trans. Camb. Phil. Soc.* Vol. VI. Part I. p. 165.); on this ground we may explain the high refractive power of water compared with that of ice.

52. Following this reasoning a little further, it is evident that the refractive indices for bodies should increase, *cæteris paribus*, with the *specific heat* corresponding to equal volumes. We will assume the ordinary expression for the refractive energies of the different gases, viz. $\frac{\mu^2 - 1}{s}$, s being the specific gravity, and compare the results with their specific heat.

In the following Table I have placed, on the left-hand side, the order of the refractive energies of eight different gases, calculated from the above formula, and, on the right, the order of their specific heat; in each beginning with the one lowest in the scale.

The Table of Refractive Energies has been derived principally from M. Biot's Précis Elémentaire; that of Specific Heat entirely from MM. de La Roche and Berard.

Refractive Energy.	*Specific Heat.*
Oxygen.	Oxygen.
Air.	Air.
Carbonic Acid.	Nitrogen.
Nitrogen.	Carbonic Oxide.
Carbonic Oxide.	Carbonic Acid.
Nitrous Oxide.	Nitrous Oxide.
Olefiant Gas.	Olefiant Gas.
Hydrogen.	Hydrogen.

The only want of coincidence in these two Tables occurs in the case of carbonic acid. It arises from the specific gravity being very great compared with those of the gases below it. Had we taken some root, as the cube root of the density instead of the simple power for our denominator in the formula for the refractive energy, which seems more correct, it is not improbable that all our results would have agreed.

The above, however, is sufficiently accurate for my present purpose, which is merely to give a colour to my investigations, and to shew that, at least, they tend towards the truth.

53. I do not suppose the same results would be applicable to solids, even if they are to fluids. For in solids the effect is modified or altogether destroyed by the action of the *material* particles. Indeed, the quantity of either kind of particles, and the arrangement of them within the body must depend so much on the constitution of the body, that, in many cases, I could imagine no ether, and, in others, no caloric, according as either from the disposition of the material particles or their peculiar nature, the forces which the one or the other exerts would keep up an equilibrium with the external forces of the mixed ether and caloric.

And even fluids from their greater or less fluidity would in like manner essentially modify the effects of transmission of vibrations through them; instances of the above we have seen in the case of light, to which I have before alluded. At the same time that I make these remarks, I have not attempted either to verify or disprove the above analogy.

54. The connexion which we thus establish between light and heat is of the most intimate description. I shall briefly mention one or two circumstances in the latter, which can be readily explained.

Reflexion of light must arise from the vibrations at the reflecting surface being stopped: it is evident then that the transmission, put in play by such vibrations, will also be stopped; hence if heat be in the act of emerging from a polished metal, when the pulsations reach the surface they will diminish greatly in magnitude, and thus the corresponding impulse of radiation will be small, whilst from an unpolished surface, the converse will be the case. The same is true of the *acquisition* of heat. This is abundantly confirmed by experiment. The same reasoning applies to total internal reflexion for heat as for light, with the exception that in the former the word *total* would refer only to such motion as is due to the action of the vibratory forces.

55. From what has been said, (10, &c.) it appears that if bodies be impregnated with two systems of particles which are endued with forces attractive in their action on particles similar to themselves, but mutually repulsive of each other, the following results will ensue.

That a transverse vibratory motion of the one must, in general, be accompanied by a transverse vibratory motion of the other, (10) (34); that a translation is generally consequent on, and varying in intensity with, this vibratory motion (36). That when one fluid moves forward the other moves backward (10). By reference to those substances in which it would appear that the vibrations of the internal ether were considerable, the velocity of transmission of heat was found (by means of the hottest point of the spectrum) to be also considerable. And further, that when a body contained a large quantity of caloric, it uniformly (with one exception, and that not a striking one,) has been found to contain a proportionably small quantity of ether; results in which theory is confirmed by experiment.

SECTION IV.

Combination of Vibratory Motion with Motion of Translation when the Forces are repulsive.

56. IN the last Section we were occupied with the interpretation of the equations resulting from supposing the particles attractive, or, at least, the force on any one of the same denomination as would result from this hypothesis. It is possible to conceive that in nature the particles are so mixed and so varied in their properties as to allow the above supposition, at the same time that the total action produced on another system is of the opposite denomination. We ought then to examine the nature of the motion on each supposition separately, and, finally, to combine them.

57. I should be trespassing beyond the bounds of my subject, were I to proceed to the consideration of the modification which such an hypothesis introduces.

I shall therefore content myself with a few observations.

If we examine the equations of motion of the medium acted on by repulsive forces, we find them assume the form

$$\frac{d^2 a}{dt^2} = -2n^2 a,$$

$$\frac{d^2 \beta}{dt^2} = n^2 \beta,$$

$$\frac{d^2 \gamma}{dt^2} = n^2 \gamma,$$

from which we conclude, as before, that there is a vibration *in the direction* of transmission, and that there is a transfer, the motion of any particle being *in the plane* of the front of a wave.

This seems to be connected with the fact of Electro-Magnetism, in which a current produces a force in a plane perpendicular to its direction. I said a force, but it will easily be conceived that if a system be put in motion, and some other body containing similar particles be within its sphere, the effect will be a repulsion in the direction of the motion, as a particle within the body will have a tendency to motion.

58. The conclusions at which we have arrived, as to the difference of motion in the two systems of fluids, seem also to bear upon this point; for, from our analysis, it appears that the motion of one fluid will be exactly the opposite of that of the other. The greatest difficulty that Electro-Magnetism presents, is the circumstance that each system acts always in *one* direction.

It is foreign to my present purpose to consider this point, but it seems probably connected with the circumstance that a flow of Galvanism necessarily *commences from* the source; whether that be the positive or negative, I am not prepared to say.

59. In conclusion I would observe, that nothing which I have advanced has the slightest tendency to invalidate the results to which M. Fourier and others have been led.

The equations which they obtain may be deduced from the principles of this method with the same facility as from their own. I regret that it is not in my power to multiply examples by which, not only the application of a process can be tried, but even the truth of the principles be tested. The relation between the sign of a small term in the expansion, when the coefficients are determined by observations of the places of the fixed lines in the spectrum, and the permeability of the substance to heat combined with the place of the hottest point in the spectrum, is of a kind that tends to strengthen our convictions in the truth of the principles when satisfactory; and, to help us to modify those principles when unsatisfactory: on which account, I cannot help expressing my regret that those Philosophers, who have so admirably shewn the intimate connexion between Heat and Light, should not have undertaken observations on substances such as to compare the gradations in the affection of the latter with those of the former.

The nature of the investigation I have here attempted must plead my excuse for having been rather discursive. My object has been so to consider the constitution of the atmosphere, that one single hypothesis shall suffice as a key by which to proceed to the examination and explanation of the varied phenomena which present themselves to our notice. I do not presume to suppose that I have succeeded, but the necessity of keeping the different kinds of phenomena in as intimate connexion as possible, has induced me to offer the above to the notice of the Society.

ERRATUM.

IN Part I, page 179, replace the seven lines from the bottom by the following:

$$v^2 = 2A\Sigma . \frac{\delta x^2 + \delta y^2 + \delta z^2 - 3\delta x^2}{r^5} \sin^2 \frac{\pi \delta y}{\lambda}$$

$$= \frac{2A}{\epsilon^3} \Sigma . \frac{\xi^2 + \eta^2 + \zeta^2 - 3\xi^2}{r^5} \sin^2 \frac{\pi \epsilon \eta}{\lambda}$$

$$= \frac{2A}{\epsilon^3} \Sigma . \frac{\eta^2 + \zeta^2 - 2\xi^2}{r^5} \sin^2 \frac{\pi \epsilon \eta}{\lambda},$$

$$v'^2 = \frac{2A}{\epsilon^3} \Sigma . \frac{\xi^2 + \eta^2 + \zeta^2 - 3\eta^2}{r^5} \sin^2 \frac{\pi \epsilon \eta}{\lambda}$$

$$= \frac{2A}{\epsilon^3} \Sigma . \frac{\xi^2 + \zeta^2 - 2\eta^2}{r^5} \sin^2 \frac{\pi \epsilon \eta}{\lambda},$$

$$v''^2 = \frac{2A}{\epsilon^3} . \Sigma . \frac{\xi^2 + \eta^2 + \zeta^2 - 3\zeta^2}{r^5} \sin^2 \frac{\pi \epsilon \eta}{\lambda}$$

$$= \frac{2A}{\epsilon^3} . \Sigma . \frac{\xi^2 + \eta^2 - 2\zeta^2}{r^5} \sin^2 \frac{\pi \epsilon \eta}{\lambda},$$

and at the top of the following page, read

$$\Sigma \xi^2 . \frac{\sin^2 \frac{\pi \epsilon \eta}{\lambda}}{r^5} = \Sigma \zeta^2 . \frac{\sin^2 \frac{\pi \epsilon \eta}{\lambda}}{r^5}.$$

XI. *On the Relative Quantities of Land and Water on the Surface of the Terraqueous Globe. By* S. P. RIGAUD, A.M. *Savilian Professor of Astronomy in the University of Oxford.*

[Read *Feb.* 13, 1837.]

FROM the constitution of the Earth it is obvious that the greatest part of it is unfitted for the habitation of human beings. This, however, has been well accounted for. The fertility of the land depends upon the moisture of the atmosphere, which could not be furnished in sufficient supply, except from a wide expanse of waters and with mountains which may assist in its condensation. The oceans, therefore, bear a large proportion to the continents; but the relative distribution of them still remains a subject of great difficulty. The more accurately we study nature, the more clearly we see the operation of final causes, and, as a general truth, there can be no doubt that some beneficial objects are attained by the relative situation of those different portions of land, which rise above the level of the waters. Future investigations may lead to the discovery of them, and the best assistance, which can at present be given to the inquiry, must depend upon obtaining an accurate view of the facts. Even if we can as yet advance no further than physical phænomena, it is well worth while to examine them with precision.

The irregular figures and sinuous outlines of the land are serious impediments to the common methods of measuring its extent. It has therefore been suggested, that by cutting out the delineations of it and weighing the several parts, an estimate might be made of its relative magnitude. Dr Halley, in 1693, published an account, which he had in this manner collected of the number of acres in each county of England, "having cut a six sheet map in pieces for that purpose*." He used "nice scales" but he did not consider that he arrived at more than a certain approximation, which however he chiefly attributed to the imperfection of the maps on which he had to work. The projection, likewise, that was used for them, obliged him to reduce all the parts to the same mean size of the acre, and consequently the experiment best known for the purpose, and described by Dr Long in his Astronomy†, possessed in this respect a decided advantage. He says that by "weighing thus the papers of Mr Senex's globe of 16 inches diameter, the weight of the paper whereon the sea was represented was 349 grains, that of the land 124 grains: so the surface of the sea is almost three times as great as that of the land hitherto discovered. I omitted," he adds, "weighing the parts contained within the polar circles, because it is not known to any degree of exactness how much of them is land and how much is sea." Mr Vince refers to this passage‡, and observes, that "the conclusion would be more accurate, if the land were cut out from the sea before the paper was put upon the globe;" and he gives his opinion that "after all our modern discoveries, this method would probably give the proportion of land to water, to a considerable degree of accuracy." Senex died in December 1740, and Dr Long published the first volume of his work (which contains the passage just quoted) in 1742: they may, therefore, be considered as contemporaries; and no difficulty can be well imagined to interfere with the plates being obtained before they had been used. But Mr Vince prints the word "before" in Italics, which seems to indicate that he alludes to some tradition which was credited in his time. Under such circumstances, however, the experiment must have been worth absolutely nothing. The varnish must have been broken off unequally, and the greatest care could not have been sufficient for taking off the paper without some of it remaining in adhesion to the substance of which the globe was formed, or other portions bringing off some of that substance with them. We have,

* See a Collection for the Improvement of Husbandry and Trade, by John Houghton, F.R.S. Nos. 25, 26.

† Article 580.

‡ Astronomy, 4to. Vol. II. p. 112.

however, no later accounts of such a determination of these quantities. Several persons are said to have repeated the experiment, but as they have not published its results, it seemed desirable to try it again with that care, which might at least ascertain the reliance which can be placed upon the method. The advanced state of modern geography affords a more reasonable expectation of accuracy than could, under the most favourable circumstances, have been attained in the time of Dr Long; and the beautifully distinct manner, in which globe-plates of the largest size have now been executed, gives great advantage to the trial. In 1823 Mr Carey allowed me, for this purpose, to make use of the plates of his 21-inch globe; and when I recently wished to check the results, at which I had then arrived, Mr Addison obliged me with those which he has had engraved for a globe three feet in diameter—he took the trouble, likewise, not only of inserting all the latest discoveries, but of having the impressions expressly worked off for me with every precaution and attention.

There are some difficulties in the pursuit of this inquiry, which make it necessary to proceed with great care. Dr Halley observes, "that the moisture of the air imbibed by the paper, did very notably increase its weight, which made me very well dry the pieces before I weighed them, that so I might be assured there was no error upon this account; and in so doing, I found that in a very few minutes of time, their weight would sensibly increase by their reimbibing the humidity of the air." This effect is indeed so rapid that artificial drying is possibly the worst thing that can be done; it will occasion the weights to vary while the paper is in the scale, and will thus destroy the precision of the ratio, which may be derived from the examination of parts of the same sheet. A much better method is to lay the paper out for some time in a large room, where there is no danger of much fluctuation in the state of the air, the materials will then reach a nearly saturated state, in which they will generally continue stationary during the time which they are in hand. Such an exposure will of course on different days produce different degrees of dampness; but uniformity in this respect, for any great length of time, is unattainable, and if it can be secured for the interval, which the immediate operation requires, all is gained that can be hoped for. The results, which are now about to be described, were all obtained during the summer in this manner: each piece was weighed before the land was cut from the water, the separate parts were then weighed independently; this was usually repeated, when the second was very seldom found to vary from the first determination by $\frac{1}{10}$ of a grain, and the sum of the parts most commonly made up exactly the weight which had at first been found to belong to the whole.

Dr Halley points out another difficulty, for he says "that the map consisting of several sheets of paper, they were found to be of different thickness or compactness, so as to make a sensible difference, which obliged me to examine the proportion between the weight and acre in each sheet." Dr Long refers to this where he observes that "the paper whereon it" [the engraving of the globe] "is printed should be of an equable thickness as near as possible." Mr Addison obligingly endeavoured to obviate any such cause of inaccuracy by taking care that the impressions should be worked off on paper of an uniform texture. It was not possible to succeed in this so far as to have all the equal gores of the same weight, but there were hardly any knots to produce partial inequalities, and, by working out the results separately for each part, as near an approximation to the truth was upon the whole arrived at as the method seemed capable of producing. Relative quantities are all that were required, and by this means they rested on the uniformity of paper only of a small comparative size, a quality which might be assumed without any material error. The plates of Mr Addison's globe cover a plane surface of $4071\frac{1}{2}$ square inches, and from the many parts, into which they are of necessity divided, there is a fair chance for compensation, because it may be presumed that if the land were on the thicker part in one instance, it might be on the thinner in another. This compensation, being a general effect, might at first sight appear to be best secured by weighing all the land of the globe together, and all the water; but in addition to other advantages in the different process, there are objections to this method, which make it inexpedient. It would require constant and long attention in keeping the respective parts together after they have been once separated from the whole. To cut up the plates with due care is the work of many days; if suspicion occurred of any part being mislaid, or lost, it would be, in such a mass, impossible to obtain any satisfaction of the truth; and when a number of small islands or lakes were to be cut out and distributed, they could hardly be recovered, if it should be wished, from the divisions in which they had once been placed. These are not imaginary difficulties. We must suppose that the precaution would in this case be taken of ascertaining in the first instance the weight of the whole, and if the sum of what was found for the parts should not be equal to it, there would remain no possibility of determining the cause of error: there might be a deficiency, and there would be no means of discovering whether it was to be assigned to the land or to the water, or to both. But even if all this could be provided against, there would still be an essential obstacle in the different degrees of humidity, which would be imbibed by the several parts of so many pieces of paper, which could not all be equally exposed to the air. These difficulties were almost entirely avoided by the careful and distinct examination of each piece, and the further advantage was gained, that not only the ratio might by this means be determined for the whole, but, as it had been settled in detail, the corrections from future discoveries may, at any time, be introduced, without the necessity of repeating the entire examination.

The gores of Mr Addison's globe are made each for 15 degrees of longitude, and there are five divisions of each for the five zones. The twenty-four for the torrid zone were cut into two at the equator, and examined in forty-eight portions, in order to have the quantities for each of the hemispheres. The forty-eight gores for the two temperate zones, when added to these, make up ninety-six, which may be considered as having been analysed with tolerable completeness. In one or two instances the precise terminations of land and water were of necessity assumed in an arbitrary manner, but this was to a very limited extent, and could not materially affect the general conclusion. In the polar circles there is a much greater degree of uncertainty, and for these it was necessary in some parts to have recourse to conjectural estimates. The southern was taken as consisting entirely of sea. The expedition, about to sail from the United States of America, will make us better acquainted with the constitution of these parts; and if it should discover any lands in them, the correction which this circumstance will require may easily be applied. In the Northern Polar Circle the parts which belong to Europe and Asia seem to be sufficiently distinct; but the land, in high latitude, of North America, admits as yet of no certain measure. It is impossible, in many parts, to tell what belongs to a continent and what are the boundaries of islands, of which seldom more than a portion of the coasts has been traced out. In this state of things nothing more was attempted than a very rough guess, that from 180° to 270° of longitude, one half of the American portion of the circle, might be considered as land, which also (on account of the probable extent of Greenland) might, from 270° to 360°, have to the water a ratio of 2 to 1.

Every care was taken to separate the land and sea with accuracy. All the bays, æstuaries, and indentations, were attended to, especially when the precise form of them appeared to indicate the representation of actual surveys. The several weights were taken, to the tenth of a grain, which was considered to be as minute a measure as was consistent with the nature of the experiment.

A table of versed sines gives the ratio of the spherical superficies, to any parts of it which are bounded by given circles. Hence the hemisphere being taken at 1, the portion between the æquator and the tropics will be $1 - 0.6012509 = 0.3987491$; that within the temperate zone will be $0.6012509 - 0.0829399 = 0.5183110$; and that within the polar circle 0.0829399: but for the immediate comparison of the results it seemed most convenient to suppose the surface of the globe to be divided into 1000 parts, and to reduce all the measures to this standard; consequently, under this condition, $0.3987491 \times 500 = 199.37455$ will give the relative magnitude of the half torrid zone, and $\frac{199.37455}{24} = 8.30727$ will give the magnitude of each of its gores; in the same manner for the temperate zones $0.518311 \times 500 = 259.1555$ is the quantity for the whole, and $\frac{259.1555}{24} = 10.79814$ for each of its parts;

and $0.0829399 \times 500 = 41.46995$ will be equal to the surface of the whole polar circle. The reduction of absolute weights to the proposed scale is dependent merely on the rule of three, and the numbers which were found from it may be seen in the following tables.

TORRID ZONE.

NORTHERN HALF.

Longitude.			Water.
0°... 15°	Africa.......	7.1391	1.1682
15... 30	Africa.......	8.3073	
30... 45	Africa.......	6.6296	0.7035
	Asia.........	0.9741	
45... 60	Africa.......	0.9585	4.8459
	Asia.........	2.5028	
60... 75	Asia.........	0.6116	7.6957
75... 90	Asia.........	2.8031	5.5042
90...105	Asia.........	3.7201	4.5872
105...120	Asia.........	2.1998	6.1074
120...135	Asia.........	0.7417	7.5656
135...150	Asia.........	0.0258	8.2815
150...165	....		8.3073
165...180	Asia.........	0.0249	8.2824
180...195	....		8.3073
195...210	N. America..	0.0548	8.2524
210...225	....		8.3073
225...240	....		8.3073
240...255	N. America..	0.1068	8.2004
255...270	N. America..	1.7969	6.5104
270...285	N. America..	1.2958	6.3635
	S. America...	0.6479	
285...300	N. America..	0.2314	4.3503
	S. America...	3.7255	
300...315	S. America...	1.3721	6.9352
315...330	....		8.3073
330...345	Africa.......	0.4646	7.8427
345...360	Africa.......	6.2240	2.0832
		52.5582	146.8162
			52.5582
			199.3744

SOUTHERN HALF.

Longitude.			Water.
0°... 15°	Africa.......	1.3426	6.9647
15... 30	Africa.......	8.3073	
30... 45	Africa.......	5.2567	3.0505
45... 60	Africa.......	0.8631	7.4442
60... 75	Asia.........	0.0251	8.2822
75... 90	....		8.3073
90...105	Asia.........	0.4512	7.8561
105...120	New Holland	0.3851	6.8769
....	Asia.........	1.0453	
120...135	New Holland	3.1152	4.4392
....	Asia.........	0.7528	
135...150	New Holland	2.5260	4.4531
....	Asia.........	1.3281	
150...165	Asia.........	0.2929	8.0144
165...180	Asia.........	0.0789	8.2284
180...195	S. America...	0.0261	8.2812
195...210	....		8.3073
210...225	....		8.3073
225...240	....		8.3073
240...255	....		8.3073
255...270	S. America...	0.0274	8.2799
270...285	S. America...	1.4617	6.8456
285...300	S. America...	7.5114	0.7959
300...315	S. America...	8.1095	0.1978
315...330	S. America...	3.2528	5.0544
330...345	....		8.3073
345...360	....		8.3073
		46.1592	153.2156
			46.1592
			199.3748

Northern half		Southern half	
Africa.	29.7231	Africa.	15.7697
Asia.	13.6039	Asia.	3.9743
N. America...	3.4857	New Holland.	6.0263
S. America. ..	5.7455	S. America...	20.3889
	52.5582		46.1592

TEMPERATE ZONE.

NORTH.

Longitude.			Water.
0°... 15°	Europe......	3.3066	3.5391
	Africa.......	3.9524	
15... 30	Europe......	4.5997	3.1976
	Africa.......	2.7303	
	Asia.........	0.2706	
30... 45	Europe......	3.6663	1.7829
	Asia.........	4.7210	
	Africa.......	0.6278	
45... 60	Europe......	2.1596	1.4748
	Asia.........	7.1636	
60... 75	Asia.........	10.3594	0.4388
75... 90	Asia.........	10.6468	0.1514
90...105	Asia.........	10.7214	0.0768
105...120	Asia.........	10.4398	0.3583
120...135	Asia.........	6.0740	4.7242
135...150	Asia.........	2.9054	7.8927
150...165	Asia.........	1.3755	9.4226
165...180	Asia.........	0.7510	10.0472
180...195	Asia.........	0.1616	10.5558
	N. America..	0.0808	
195...210	N. America..	1.2305	9.5677
210...225	N. America..	1.1998	9.5984
225...240	N. America..	3.5994	7.1988
240...255	N. America..	9.1017	1.6965
255...270	N. America..	9.1432	1.6550
270...285	N. America..	6.0535	4.7446
285...300	N. America..	3.5424	7.2558
300...315	N. America..	0.7422	10.0559
315...330	N. America..	0.3723	10.4258
330...345	Europe......	0.1670	10.6033
	Africa.......	0.0278	
345...360	Europe......	1.7997	6.0607
	Africa.......	2.9377	
		126.6308	132.5247
			126.6308
			259.1555

SOUTH.

Longitude.			Water.
0°... 15°	....		10.7981
15... 30	Africa.......	3.1713	7.6268
30... 45	Africa.......	0.5386	10.2595
45... 60	Africa.......	0.0977	10.7004
60... 75	Asia........	0.0227	10.7755
75... 90	....		10.7981
90...105	....		10.7981
105...120	New Holland	1.3656	9.4326
120...135	New Holland	3.1005	7.6977
135...150	New Holland	4.5622	6.2359
150...165	New Holland	0.6780	10.1201
165...180	Asia........	0.5081	10.2900
180...195	....		10.7981
195...210	....		10.7981
210...225	....		10.7981
225...240	....		10.7981
240...255	....		10.7981
255...270	....		10.7981
270...285	S. America...	0.0283	10.7698
285...300	S. America...	5.8762	4.9219
300...315	S. America...	2.5996	8.1986
315...330	....		10.7981
330...345	....		10.7981
345...360	....		10.7981
		22.5488	236.6060
			22.5488
			259.1548

North		South	
Europe.......	15.6989	Africa..........	3.8076
Africa........	10.2760	Asia...........	0.5308
Asia..........	65.5901	New Holland...	9.7063
N. America...	35.0658	S. America......	8.5041
	126.6308		22.5488

NORTH POLAR CIRCLE.

Longitude.		Water.	
0°... 15°	Europe. ... 0.9524 Asia. 1.2517	} 8.1634	Europe. ... 0.9524 Asia. ...... 5.0329 N. America. 12.0410
90...180	Asia. 3.6724	6.6951	
180...270	Asia. ...... 0.1088 N. America. 5.1293	} 5.1294	18.0263
270...360	N. America. 6.9117	3.4558	
	18.0263	23.4437 18.0263	
		41.4700	

From these numbers we have the following conclusions:—

	Land.	Water.		
For the north polar circle............	18.0263...	23.4437	or a ratio of	100 to 139
the north temperate zone	126.6308...	132.5247	....	100 to 105
the north half of the torrid zone	52.5582...	146.8162	....	100 to 279
the south half of the torrid zone	46.1592...	153.2156	....	100 to 332
the south temperate zone	22.5488...	236.6060	....	100 to 1049
the south polar circle	....	41.4700		
the whole sphere	265.9233...	734.0762	....	100 to 276
the northern hemisphere.........	197.2153...	302.7846	....	100 to 154
the southern hemisphere.........	68.7080...	431.2916	....	100 to 628
the whole torrid zone............	98.7171...	300.0318	....	100 to 304.

It has been mentioned that Dr Long, omitting the polar circles, found the ratio of land to water as 124 to 349, or 100 to 281; in the present case it comes out, under the same circumstances, as 100 to 270. This is not a greater difference than might be expected from the advance in geographical knowledge since the time of Senex; and it seems to relieve Dr Long from any suspicion of having neglected to take proper precautions in trying the experiment; but it must be acknowledged that the numbers just stated differ much from those which M. Malte Brun* has assigned to the same portions of

* Geography (Eng. Trans. Edinb. 1822) Vol. I. p. 159. His words are, "We have found, by a pretty exact computation, that the land in each hemisphere and the zone bears to the whole the following proportions:

"In

the Earth's surface. Whether his results are more to be depended upon than those which have been now obtained, must be left for others to decide; but there is a circumstance which seems to corroborate very strongly the dependence which may be placed on the present method, since there is a remarkable coincidence between the numbers, which were found from Mr Carey's globe-plates in 1823, and Mr Addison's in the more recent examination.

The proportionate quantity of land in the southern hemisphere was found in 1823 to be 69.58, which exceeds that which is now given by only 0.87. The extents of the continents, likewise, with the islands respectively belonging to them, came out as follows:

	1823.		1836.	
Europe	16.80		16.65	
Asia	89.21	104.48	88.73	104.46
New Holland	15.27		15.73	
Africa	59.14		59.58	
North America	46.99		50.59	
South America	35.36		34.64.	

"In the icy zone of the north	0.400"	(or the ratio of the land to the water	100 to 150)
"In the temperate zone of the north	0.559"	(................	100 to 79)
"In the northern part of the torrid zone	0.297"	(................	100 to 237)
"In the northern hemisphere	0.419"		
"In the icy zone of the south	0.000"		
"In the temperate zone of the south	0.075"	(................	100 to 1233)
"In the southern part of the torrid zone	0.313"	(................	100 to 219)
"In the southern hemisphere	0.129"		

As M. Malte Brun does not explain his method of computation for the several zones, no more can at present be said, in that respect, than is pointed out in the text: but it may be remarked that he is not correct in the quantity of land, which he has deduced from his own data for the two hemispheres, $\frac{0.400+0.559+0.297}{3}=\frac{1.256}{3}=0.419$ and $\frac{0.075+0.313}{3}=\frac{0.388}{3}=0.129$; but the fundamental decimals are parts of three different integers, and consequently the third of their sum will not give the true proportion of them when taken together, to the whole.

When it is recollected that these calculations were made from the works of two distinct geographers, the agreement is possibly as near as could be hoped for; but it is really closer than, at first sight, can appear; for some little variations were accidentally introduced by the first determinations not having been referred to, until the second had been completed. It was then found that Van Dieman's land had been weighed with New Holland in 1836, but that it had been taken with the other Asiatic islands in 1823. The interior of Africa is so imperfectly known, that in the last instance it was thought best to omit all attention to the inland lakes that were drawn upon this part of the plates, although they were cut out from the other quarters of the globe. This exception was not made in 1823, which will account for the somewhat smaller quantity which was then found for this continent. The difference in North America is to be attributed to the discoveries which have been made of late years, and the larger allowance which was in consequence assigned to the land within the North Polar Circle. The deficiency for South America is not great, but it was sufficient to make it desirable to ascertain, if possible, a cause by which it might be occasioned. The larger dimension of Mr Addison's plates afforded the means of cutting more deeply into the great rivers of this continent: and Mr Carey, having been consulted, pointed out another probable source of variation in the outline of the eastern coast, which, in consequence of recent surveys, is not now laid down exactly as it was in 1823. It certainly is remarkable that the deficiency, which in this case amounts to 0.72, is very nearly the same as occurs for the whole southern hemisphere; and it may also be stated, that the land of the northern hemisphere came out in 1823 as 193.19, which is 4.02 less than is now assigned for it, while the quantity (as may be seen above) which was then found for North America was also less by 4.60.

As the force of the present argument depends upon the two trials being not only distant in time, but in every other respect independent of each other, it may not be superfluous to mention that the numbers were also deduced from them in two different ways. On Mr Carey's globe the gores are 20° wide, and they extend each from the æquator to the pole, so then the land being separated from the water, their ratio could immediately be found for each of the thirty-six parts, without any reduction for the different magnitudes of the several zones.

*** The different portions of ink on the different parts of the paper may be thought to affect their weights. These are generally in larger quantities on the land than on the sea, but not always: there is uniformly a kind of shading, which extends to some distance from the several coasts, and when the interior of a country is little known, it is comparatively blank. There is reason also to believe, that when the ink is thoroughly dried, it adds very little to the weight. The difficulty of reducing the paper, at different times, with any certainty, to the same degree of dryness, prevents a direct trial of the alteration, which might be produced in printing, but workmen consider it to be very small. Two pieces as nearly as possible of the same size, having been cut out of the same gore, the one which was perfectly white weighed 8.2 grains, while the other which was covered with names weighed only 8.1. The difference must have been occasioned by some accidental circumstances; but the experiment, as far as it goes, will tend to shew that no sensible error was likely to be occasioned by the attempt not having been made to introduce an allowance for this particular.

XII. *On the Results of Observations made with a new Anemometer.* By the *Rev.* W. WHEWELL, M.A. *Fellow and Tutor of Trinity College.*

[Read *May* 1, 1837.]

IN the present paper I shall give an account of the mode which has been employed in using an Anemometer which I have invented and caused to be erected. By this account I hope to shew that such instruments may be made to give consistent and comparable results, and may lead to a more complete knowledge of the course of the winds than we yet possess.

It is not necessary to describe in detail the construction of the instrument here spoken of; its general principles may be easily explained. A fly (resembling the fly of a revolving ventilator or the sails of a windmill) is fixed to the small end of the vane of a weathercock, so as always to be turned with its circular disk to the wind; and it consequently revolves by the action of the wind with a rapidity increasing as the strength of the wind increases. The revolutions of the axis of this fly are converted, by a train of toothed wheels and screws, into a vertical motion, by which a pencil is carried downwards touching the surface of a vertical cylinder, the cylinder having the axis of the weathercock for its axis. As the vertical rod on which the pencil slides is attached to the vane of the weathercock, the *point of the compass* from which the wind blows is recorded by the *side* of the cylinder on which the mark is made, while the *quantity* of the wind is represented by the extent of the *descent* of the pencil.

In the instruments which I have had constructed, the pencil descended one-twentieth of an inch for ten thousand revolutions of the fly, and the cylinder on which the marks were made was sixteen or eighteen inches high. By this means the surface of the cylinder would contain the trace of the wind of one or two days when there was much wind, and of several days when the winds were lighter.

These instruments were made at first by Mr Newman of Regent Street, and since, by Mr Simms of Fleet Street. They have been erected and observed by Professor Forbes and Mr Ranken at Edinburgh, by Mr Southwood at Plymouth, and also at other places; but the observations of which I am able to give the most complete account are those made under the direction of Professor Challis at the Cambridge Observatory, and under my own direction at the house of this Society. The Anemometer at the Observatory was placed over the portico, in which situation it was free on the other sides, but considerably sheltered by the dome of the equatoreal, on the north side. The Anemometer placed on the top of the Society's house is favourably circumstanced, being higher than any neighbouring building which is near enough to intercept the wind. The observations were made with care and regularity by Mr Crouch the Society's housekeeper.

Various improvements in the instrument were suggested by using it; and as it had not been foreseen what strength of workmanship would be requisite to resist the weather, all the instruments were, at one time or other, disabled, so as to interrupt the observations.

One of the difficulties which most interfered with the precision of the observations, was that which arose from the *wavering* of the wind. The weathercock is in almost constant motion, swinging to and fro through an arc often not less than a quadrant, and the consequence is, that the pencil describes upon the cylinder, not a single line, but a broad path of irregular form, made up of the transverse lines which the oscillation of the vane occasions. It might at first be supposed that this oscillation arose from the momentum of the vane, and might be remedied by some contrivance which should cause the change of direction of the wind to come into effect more slowly; such for example as the *tail* of a windmill. But the cause of this oscillation is in reality almost entirely the constant shifting of the wind, as may be seen by examining the motions of the vane, for it often swings into a new position or stands still awhile before it swings back again.

In consequence of this circumstance the direction of the wind cannot be ascertained with very great precision. By carefully taking the middle of the broad path, the direction may be read off to a single point of the compass; but in the observations at the Society, we contented ourselves, for the most part, with reading off to the double points (one *sixteenth* of the circumference).

The vertical scale is divided into tenths of inches, and read off by means of two indexes, which slide on the same vertical rod which guides the pencil. The detail of the process of observation will be best understood by attending to the following directions. It may be observed that the cylinder is of brass japanned white, on which common pencil marks can be rubbed out in the way described below.

Directions for observing with Whewell's Anemometer.

1. Place the instrument in a situation well exposed on all sides, and fix it so that when the wind is South, the pencil is on the line S on the barrel.

This may be done by clamping the weathercock part of the instrument with the pencil on the line S, then turning the box till the vane points due north, and then fixing it in that position.

2. Read off the instrument every day at a constant hour.

The pencil in descending will make a broad path, in consequence of the wavering of the wind. The darkest part of this path must be taken; and from this, the *direction* of the wind determined, by reference to the points of the compass marked at the bottom of the cylinder; and, as the wind changes, the directions of the successive strips of wind must be noted.

3. To read off the *amount* of the wind in each of these successive strips;—slide the lower index so that the point is upon the top of the first strip; then slide the upper index to touch the lower; then slide the lower index to the bottom of the first strip, or the top of the second strip of wind; then read off on the graduated rod, the interval (in tenths of inches,) through which the lower index has moved; then again slide down the upper index to touch the lower, slide down the lower index to the bottom of the second strip, and read off the interval;—and so on. Write down these intervals under the corresponding directions of the strips of wind, observed as above.

4. When the pencil has reached the bottom of the barrel, the instrument must be wound up, by unscrewing the clamping screw of the nut, removing it to the top of the barrel, and clamping it.

At the same time the barrel must be cleaned, by rubbing it with a soaped cloth enclosing a smooth wooden rubber.

5. The following is suggested as a simple way of marking the points of the compass; for example, from the North to the East the points may be

N. N*e*. NNE. NE*n*. NE. NE*e*. ENE. E*n*. E.

and so on for the other quadrants.

The only ambiguities which can arise by this method, are N*e*, N*w*, S*e*, S*w*; which must be distinguished from NE, NW, SE, SW.

N*e* is N *by* E; and so of the rest.

I shall now give the Register of the wind as observed for the months of January, February, March and April of the present year at the Society's house. I shall add also the observations made at the Observatory for a portion of the month of February. The readings are in tenths of inches on the scale.

* The asterisk indicates the times when the instrument was wound up.

INDICATIONS OF WHEWELL'S ANEMOMETER
AT THE HOUSE OF THE PHILOSOPHICAL SOCIETY.

JANUARY, 1837.

		Total.			Total.			Total.
1	N. 12	12	12	N. NE. 2 0	2	22	SSE. 45	45
*2	NNW. SW. 8 1	9	*13	SSW. SW. 45 9	54	*23	S. SSW. 22 34	56
3	W. 7	7	14	SW. NW. 11 58	69	24	SSW. SW. 44 10	54
4	SW. 11	11	*15	NNW. 56	56	25	SW. E. 7 8	15
5	SW. 11	11	16	NNW. WSW. 7 2	9	*26	ENE. 44	44
6	SSW. 28	28	17	SW. WNW. 2 1	3	27	ENE. 65	65
*7	SW. 51	51	18	NW. 0	0	*28	ENE. 48	48
8	WSW. 29	29	19	ENE. 0	0	29	NE. 34	34
9	SSW. 26	26	20	ENE. 12	12	30	ENE. SE. 10 15	25
*10	SSW. 76	76	21	SSE. 6	6	*31	SSE. 48	48
11	SW. N. WNW. 13 25 4	42						

FEBRUARY.

		Total.			Total			Total.
1	SSE. S. SSE. 5 5 8	18	11	S. 67	67	20	SSW. W. 40 8	48
2	S. 2	2	*12	S. SSW. 48 47	95	21	Under repair.	
3	SE. 14	14	13	SW. S. SE. WSW. 26 9 6 7	48	22	WSW. 50	50
4	SE. 30	30	*14	SW. SSW. SW. 18 24 6	48	*23	WSW. SW. 37 15	52
*5	SE. SSE. 17 10	27	15	SW. 18	18	24	S. SSW. WNW. 15 13 53	81
6	SSE. 29	29	16	SSW. 45	45	*25	NW. NNW. 23 6	29
7	SSE. S. 31 7	38	*17	SSW. 49	49	26	Under repair.	
*8	S. 41	41	18	W. SSE. 10 9	19	27	NE. N. WSW. 7 15 7	29
9	S. SSW. 37 8	45	19	S. WNW. SW. 32 3 8	43	28	NE. 2	2
*10	S. 41	41						

MARCH, 1837.

		Total.			Total.			Total.
1	NE. ENE. 18 12	30	12	WSW. 23	23	*22	SSW. SSE. 9 2	11
2	NE. 28	28	13	WSW. 4	4	23	SSE. 10	10
*3	N. NNE. 24 15	39	*14	NNE. N. 22 13	35	24	S. SSW. 4 5	9
4	NNE. 57	57	15	NE. ENE. 31 17	48	25	NNE. NNW. 4 15	19
*5	NNE. N. NNW. 53 11 5	69	16	ENE. NE. 23 10	33	26	NNW. WSW. WNW. 14 6 3	23
6	WNW. NNW. N. NE. 16 2 8 3	29	*17	NE. 9	9	27	NNW. NNE. NNW. 6 5 3	14
7	NNE. WNW. 15 2	17	18	NE. 13	13	28	NNW. WSW. 10 18	28
8	WNW. 12	12	19	NE. 21	21	*29	W. WSW. 18 28	46
9	W. WSW. 22 17	39	20	NE. N. 19 15	34	30	W. N. 18 2	20
*10	SW. 60	60	21	NNE. NE. N. 22 10 5	37	31	NNE. N. 13 3	16
11	SW. S. 63 2	65						

APRIL.

		Total.			Total.			Total.
1	SSW. WSW. WNW. 3 5 0	8	11	NW. Under repair.		21	WSW. WNW. WSW. SSW. 4 8 2 2	16
2	NW. 40	40	12	S. E. 3 1	4	*22	SSW. S. NNW. 16 13 0	29
*3	NNW. SSW. 6 23	29	13	NE. 29	29	23	NNE. WSW. 8 10	18
4	WNW. 36	36	14	NE. NNW. 23 2	25	24	WSW. 29	29
5	N. NNE. NE. 8 7 5	20	15	N. SSW. 7 4	11	25	WSW. 9	9
6	NE. NNE. 27 11	38	16	SSW. N. 7 12	19	26	SW. WSW. 35 6	41
*7	NE. N. 12 14	26	*17	NNW. 46	46	*27	WNW. WSW. 10 4	14
8	NE. 39	39	18	NNE. 46	46	28	SSW. WSW. 7 3	10
9	NE. 39	39	19	NNE. 18	18	29	WSW. 4	4
10	NW. Under repair.		20	WSW. 1	1	30	S. SW. 21 18	39

Several questions obviously offer themselves respecting the numbers thus registered. What is their real import? How far is each instrument consistent with itself? In what manner are two such instruments comparable?

I cannot at present answer these questions completely, but I will make a few observations on each.

As to the import of the indications of this Anemometer, it is evident, that their magnitude will increase with the force of the wind, and with the time to which each number refers. If we could assume that the velocity of revolution of the fly of the Anemometer is always *proportional* to the velocity of the wind, the space which the index passes over on the scale would be proportional to the velocity of the wind, and the time during which it has blown, jointly; that is, to the total quantity of the aerial current which has passed the point: and however the velocity of the wind might vary, the instrument would give the sum of all the elements of the current, or in other words, would *integrate* the velocity multiplied into the differential of the time. Hence I term the amount registered by this instrument the *Integral Effect* of the wind. That the velocity of the fly is thus proportional to that of the wind, I have not yet ascertained; and till that is done, I can only urge, that it appears highly probable that the instrument will afford at least some approximation to such a result; which no instrument hitherto erected, so far as I am aware, has ever pretended to do.

The question whether the instrument be consistent with itself, is one of considerable difficulty; for it does not readily appear how we are to obtain any permanent standard by which we may test its indications at different times, and thus ascertain whether its scale has varied. It is certainly very conceivable that the friction and other impediments to motion should alter considerably from month to month, so as to affect materially the rate at which the instrument would move with a given wind. We might however imagine means by which the actual velocity of the current of air which turns the instrument should be ascertained, and thus this difficulty overcome. For example, the Anemometer might be placed on some part of a large machine which moves for a long time with a known velocity; and thus the actual value of the indications of the instrument might be determined. And a small Comparative Anemometer, more easily transferable from place to place than the working instrument, might be employed to obtain the value of the scale of the instrument in this manner. This process might be performed at any time, and might therefore serve to compare the Anemometer with itself at different times. The relation between the velocity of rotation produced, in a wheel with oblique blades, and the velocity of a fluid which flows past it, is so steady, that the rotation of such a machine has already been used in measuring the velocity of the motion, in Masson's Patent Log, and Saxton's Current-meter.

The same process which would compare an instrument with itself, would also compare it with another instrument of the same kind. But, as we have not yet any such means of judging what is the comparative going of different Anemometers, we may say a word or two of the comparison of them by means of their results. The station at the Society's house and the Observatory are so near each other, that there can hardly be any great difference in the quantity of wind which blows at the two places. Assuming these quantities to be equal, it appears that the index at the Observatory moves nearly twice as fast as that at the Society's house. The equality of the wind at Cambridge and Edinburgh cannot so safely be assumed; but if we proceed upon the equality for March, as our only accessible basis, we shall find that the index of the Society's Anemometer moves more than twice as fast as that of the Edinburgh one. But I shall return to this comparison in another form.

In order to exhibit the general course of the winds at each place I have adopted the following graphical method.

Assuming, on a sheet of paper, the proper relative directions of the points of the compass, I begin from a point and draw a line in the direction of the first recorded wind, and of such a length as to represent this wind in magnitude on a scale of equal parts. From the extremity of this line, I draw another line representing in direction and magnitude in like manner the second recorded wind; and from the extremity of this line, a third; and so on. In this manner I obtain a continuous line, which represents the course of the wind as long as it is continued. Such lines were drawn for February and March 1837; those for March are exhibited in Plate VII, which represents the curves for March, drawn for the stations, at the Society, at the Observatory and at Edinburgh. In all the cases the observations experienced interruptions, which make it difficult to draw any general conclusions from them. But we may remark that in February the wind blew almost constantly from a more westerly point at the Observatory than at the Society. It is not difficult to conceive this result to be occasioned by the peculiar circumstances of the Anemometer at the Observatory: but it is also possible that it may be a general fact that such differences obtain at neighbouring places, in consequence of the direction of valleys, &c. Further observation alone can clear up this and similar points.

It has been deemed an important point by Meteorologists to obtain the *mean direction* of the wind at a given place for a given time, for instance, a year. Kämtz in his *Meteorologie*, Vol. II. p. 218, has collected several results of this kind. But in these researches the force of the wind has entirely been left out of the account, and each wind was reckoned according to the *number of days* which it blew. It is clear that such a procedure is entirely fallacious; for the high wind of one day may be greater, with regard to every possible effect, than the gentle breezes of a week. The mean annual direction is probably constant at each place within certain limits: and the mean directions at different places are perhaps connected by certain general relations, depending upon the quantity of fluid transferred, and upon other atmospherical conditions, which may hereafter be found to be important elements of meteorological speculation. But it is not at all likely that this will hold if the mean direction be taken without reference to the strength of the wind; and no mode of measurement can be good for this purpose which does not give the whole quantity of the aerial current, depending both upon velocity and upon time.

The Anemometer here referred to is, as I have said, the only one, so far as I know, which has been constructed with the view of thus registering both the quantity and direction of the wind; and however imperfect its construction may yet be, it must give some approximation to the quantity which it is our object to measure, and must thus afford the means of a better estimate of the mean direction for a year (or for any other time) than has hitherto been possible.

It is obvious that the mode of obtaining the mean direction of the wind for any time would be to resolve each partial wind into its component parts E. and W. and N. and S. The sum of all the west components, subtracting the east elements, give the effective west wind; and the sum of all the south elements, subtracting the north elements, give the effective south wind. The magnitude and proportion of these two effective winds compounded will give the magnitude and direction of the effective wind, between west and south, which belongs to the whole time. And the same may be said of any other cardinal points.

The reduction of any wind to these cardinal directions is of course to be performed by considering it as the hypotenuse of a right-angled triangle, and here the multipliers by which the reduction is to be performed are easily found. We may take fractions which are sufficiently accurate, and yet simple enough to be easily used. Thus the *inter-cardinal* winds, NE, SE, SW, NW, are reduced to the cardinal directions N, S, E, W, by multiplying by $\frac{7}{10}$. The *subordinate* winds NNE, ENE, ESE, SSE, SSW, WSW, WNW, NNW, are reduced to the cardinal directions by the multipliers $\frac{12}{13}$ and $\frac{4}{10}$; thus a wind NNE 65, is equivalent to N 60 and E 26. The *oblique* winds N by E, &c. might be reduced in the same manner by the multipliers $\frac{2}{10}$ and $\frac{98}{100}$ or $1-\frac{2}{100}$. But these last I have not used.

I annex calculations made in this manner for the months of January, February and March, 1837; in which I have resolved the days into periods during which a certain group of neighbouring winds were prevalent. Thus from January 1 to 14, the prevailing winds were SSW, SW and WSW; from January 26 to 30 they were NE, ENE and E.

It appears in this manner that on the scale of the Society's Anemometer the total wind for these three months was W. 321, S. 558.

It may be observed that the graphical method offers at once the mean direction of the wind, and the resolution of the winds into their cardinal parts. A straight line drawn from the beginning of the curve to its end is the direction and magnitude of the resulting wind; and if lines E and W and N and S be drawn from its extremities, they will give its component parts.

If we were to draw the graphical curve of the wind as registered by the Anemometer for a year, and were to do this for several years at the same place, beginning from the same point, we should probably have a set of curves in which a considerable resemblance might be traced; for there is a kind of annual cycle of the winds at each place. The mean of such curves for a sufficient time would be the *mean annual type* of the winds for that place. The mean annual type of the winds at different places would vary very much, as is clear from the materials which Kämtz has collected. Thus he finds (Vol. II. p. 223) that the mean direction at Paris is S 68° W, at Montmorenci N 48° W, at Utrecht N 85° W, and Amsterdam S 61° W. And though, as we have seen, his method of obtaining these results is very insufficient, it still serves to show that they would probably be various by any method.

If Anemometers of the kind now described were fixed in various parts of the world, and the annual type, and other circumstances of the wind thus obtained, it cannot be doubted but that this portion of meteorology, and probably other portions which are connected with this, would soon make great progress.

W. W.

TRINITY COLLEGE,
May 1, 1837.

ANEMOMETER, 1837. REDUCTION.

	N.	NNE.	NE.	ENE.	E.	ESE.	SE.	SSE.	S.	SSW.	SW.	WSW.	W.	WNW.	NW.	NNW.	N.
Jan. 1 to 14.	12		0							28	1	29	7	4		8	
1 14	25									26	11						
N. SW.	2									76	11						
12 11										45	51						
											13						
											9						
											11						
	39									175	107	29	7	4		8	
SSW.									162				70				
SW.									75				75				
WSW.									12				27				
WNW.	1												4				
NNW.	7												2				
	47								249			W.	185				
								N.	47								
								S.	202								
Jan. 14 to 20.				12							2	2		1	58	56	
14 20																7	
NW. ENE. 58 12				12							2	2		1	58	63	
ENE.	5				11				1				1				
SW.									0				2				
WSW.													1				
WNW.	0												41				
NW.	41												25				
NNW.	58																
	104												70				
S.	1											E.	11				
												W.	59				
N.	103																
Jan. 21 to 25.								6	22	34	10						
21 25								45		44	7						
SSE. SW. 6 7								51	22	78	17						
SSE.					20												
SSW.									47				31				
WSW.									72				16				
									7								
													47				
								S.	148			E.	20				
												W.	27				

	N.	NNE.	NE.	ENE.	E.	ESE.	SE.	SSE.	S.	SSW.	SW.	WSW.	W.	WNW.	NW.	NNW.	N.
Jan. 26 to 30.			34	44	8												
				65													
25 30				48													
E. ENE.				10													
8 10																	
			34	167	8												
NE.	24				24												
ENE.	65				154												
N.	89			E.	186												
Jan. 30.—Feb. 7.							15	48	5								
SE. S.							14	5	2								
15 7							30	8	7								
							17	10									
								29									
								31									
							76	131	14								
SE.					53				53								
SSE.					52				121								
					105			S.	188								
Feb. 8.—Feb. 20.							6	9	41	8	26	7	10	3			
									37	47	18		8				
									41	24	6						
									67	45	18						
									48	49	8						
									9	40							
									32								
									275	213	76	7	18	3			
SSW.									197				85				
SW.									30				30				
WSW.									3				6				
WNW.													3				
								S.	505			W.	142				
Feb. 22–25.									15	13	15	50		53	23	6	
												37					
									15	13	15	87		53	23	6	
								SSW.	12				5				
								SW.	11				11				
								WSW.	35				80				
												WNW.	49				21
									73			NW.	16				16
								N.	42			NNW.	2				5
								S.	31			W.	163			N.	42

	N.	NNE.	NE.	ENE.	E.	ESE.	SE.	SSE.	S.	SSW.	SW.	WSW.	W.	WNW.	NW.	NNW.	N.
Feb. 27—Mar. 8.	15	15	7	12								7		16		5	
	24	57	2											2		2	
	11	53	18											12			
	8	15	28														
			3														
	58	140	58	12								7		30		7	
NNE.	129				56			WSW.	3				6				
NE.	41				41								28				
	5				11								3				
	12																
	6																
N.	251			E.	108				3				37				
S.	3			W.	37												
N.	248			E.	71												
Mar. 9–13.									2		60	17	22				
											63						
											23						
											4						
									2		150	17	22				
								SW.	105				105				
								WSW.	7				16				
								S.	114			W.	143				
Mar. 14–21.	13	22	31	17													
	15	22	10	23													
	5		9														
			13														
			21														
			19														
			10														
	33	44	113	40													
	41				18												
	79				79												
	16				37												
N.	169			E.	134												
Mar. 22–24.								2	4	9							
								10		5							
								12	4	14							
					5		SSE		11				6				
									13			E.	5				
								S.	28			W.	1				

	N.	NNE.	NE.	ENE.	E.	ESE.	SE.	SSE.	S.	SSW.	SW.	WSW.	W.	WNW.	NW.	NNW.	N.
Mar. 25–31.	2	4										6	18	3		15	
	3	5										18	18			14	
		13										28				6	
																3	
																10	
	5	22										52	36	3		48	5
					9							WNW.	19				44
								SSW.	21				48				1
													3				20
												W.	106				70
												E.	9			S.	21
												W.	97			N.	49

SUMMARY.

	N.	E.	S.	W.
Jan. 1 to 14	......	......	202	185
... 14 ... 20	103	......	......	59
... 21 ... 25	......	......	148	27
... 26 ... 30	89	186		
Jan. 30 to Feb. 7	......	105	188	
Feb. 8 20	......	......	505	142
...... 22 25	......	......	31	163
...... 27 to Mar. 8	248	71		
Mar. 9 13	......	......	114	143
...... 14 21	169	134		
...... 22 24	......	......	28	1
...... 25 31	49	......	......	97
	658	496	1216	817
			658	496
			S. 558	W. 321

XIII. *On the Explanation of a Difficulty in Analysis noticed by Sir William Hamilton.* *By* ARTHUR AUGUSTUS MOORE, ESQ. *of Trinity College.*

[Read *May* 1, 1837.]

IN the Memoirs of the Royal Irish Society, Sir William Hamilton has made an important observation upon a general principle of Analysis, which has been used by La Grange as the basis of his Calculus of Functions. Sir W. H. remarks that there is a case in which this principle (which had till then been considered axiomatic and universally true) does not hold good. The case which Sir W. H. cites is the function $e^{-\frac{1}{x^2}}$, which M. Cauchy had already in his Calcul Différentiel shown to be an exception to another generally received principle of analysis*. M. Cauchy seems to be of opinion that the existence of this anomaly is a sufficient reason for rejecting the mode of exposition of the Differential Calculus of which La Grange is the author, and which is certainly based upon the assumption of both these principles, the latter however of which is comprised in the former as a particular case. But the function $e^{-\frac{1}{x^2}}$ is only one of a general class of functions which with another constitute the only known exceptions to La Grange's principle. The latter class has no apparent analogy with the former, but on examination we shall find that both these apparent anomalies are immediate consequences of the fundamental conditions of analytical developement, and that the only reason why they were not at once recognized, *à priori*, as exceptions to the general principle was that in the demonstration

* M. Cauchy remarks that this function and all it differential coefficients vanish for the particular value of the variable $x=0$, although the function itself does not vanish for any other value of the variable, thus constituting an exception to a generally received analytical principle.

of this principle itself we commit the error of passing directly from finite to infinite states of functions and variables, instead of estimating and comparing their relations in the different stages of their convergency. I have attempted in what follows to give a rigorous demonstration of the principle in question, at the same time fixing the precise limits of its application, and enumerating the different classes of functions to which it necessarily does not apply.

1. To effect this object I shall begin by explaining what is understood by infinitesimals of different orders. If a function of x converges indefinitely towards zero along with x, in such a manner that, for a very small value of x, $f(x)$ shall be less than any given magnitude, the function $f(x)$ at the limit of those values of x which converge indefinitely towards zero is called an infinitely small quantity or infinitesimal. But as for similar decreasing values of x the ratio of convergency may be much higher in one function than in another, we are led naturally to consider indefinitely decreasing quantities of different degrees or orders of convergency. And having fixed upon some one function whose ratio of decrease we assume as the unit of convergency, we call a second function which for similar decreasing values of x decreases in m times as fast a ratio as the first, an indefinitely decreasing quantity of the m^{th} order. We extend this definition to the infinitesimals which are the limits of these quantities, and call the infinitesimal which is the limit of the former of the quantities, an infinitesimal of the first, and the infinitesimal which is the limit of the latter, an infinitesimal of the m^{th} order. Choosing x itself for the function whose ratio of decrease is taken as the unit of convergency, we see clearly that when x is less than unity Ax^m is an indefinitely decreasing quantity of the m^{th} order, where m may be integer or fractional. From this we infer that Ax^m may represent an indefinitely decreasing quantity of any order, and that the limit of Ax^m for values of x which converge indefinitely towards zero may represent an infinitesimal of any order. This we shall designate by the notation $\lim_{x=0}(Ax^m)$. A very wide generalization, which only suggests itself from the study of the different analytical functions, is given to this definition by defining $\lim_{x=0}\{f(x)\}$ to be an infinitesimal of the m^{th} order, if any finite and positive value of m can

be found which will render $\lim_{x=0}\left\{\frac{f(x)}{x^m}\right\}$ a finite quantity. A corollary to this definition immediately offers itself, viz. that if $\lim_{x=0}\{f(x)\}$ is an infinitesimal of the m^{th} order, the function $\frac{f(x)}{x^{m_1}}$ increases or decreases indefinitely, while x converges indefinitely towards zero, according as m_1 is greater or less than m. For

$$\frac{f(x)}{x^{m_1}} = \frac{f(x)}{x^m} \cdot \frac{1}{x^{m_1-m}},$$

and as $\frac{f(x)}{x^m}$ converges towards a finite limit while x converges indefinitely towards zero, it depends upon the sign of $m_1 - m$ whether $\frac{f(x)}{x^{m_1}}$ increases or decreases indefinitely at the same time. But if no finite and positive value of m can be found which will render $\lim_{x=0}\left\{\frac{f(x)}{x^m}\right\}$ a finite quantity, there are two cases to be considered. 1st. If $\frac{f(x)}{x^m}$ converges indefinitely towards zero along with x however great m may be taken, it follows from the general definition of an infinitesimal of the m^{th} order that $\lim_{x=0}\{f(x)\}$ is an infinitesimal of an infinitely high order. 2d. If $\frac{f(x)}{x^m}$ increases indefinitely towards $\frac{1}{0}$ for values of x which converge indefinitely towards zero, however small m may be taken, it follows from the same general definition that $\lim_{x=0}\{f(x)\}$ is an infinitesimal of an infinitely low order.

2. Of infinitesimals in general I may enunciate the following theorem.

THEOREM.

If $\lim_{x=0}\{f(x)\}$ is an infinitesimal of the m^{th} order, and if $\lim_{x=0}\{\phi(x)\}$ is an infinitesimal of the m_1^{th} order, the equation $f(x) = \phi(x)$ cannot exist for the general value of x.

Dem. For if $f(x)$ can be equal to $\phi(x)$ for the general values of x, dividing by x^m, we find that $\frac{f(x)}{x^m}$ can be equal to $\frac{\phi(x)}{x^m}$, which is equal to $\frac{\phi(x)}{x^{m_1}} \cdot \frac{1}{x^{m-m_1}}$. Now by hypothesis $\frac{f(x)}{x^m}$ and $\frac{\phi(x)}{x^m}$ converge towards a finite limit as x converges indefinitely towards zero, whilst $\frac{1}{x^{m-m_1}}$ converges towards $\frac{1}{0}$ or zero at the same time, according as m_1 is greater or less than m. Therefore a quantity converging towards a finite limit can throughout be equal to another which converges towards $\frac{1}{0}$ or zero, which is absurd. Therefore $f(x)$ cannot be equal to $\phi(x)$ for the general value of x. Q. E. D.

COR. If $\lim_{x=0}\{f(x)\}$ is an infinitesimal of the m^{th} order, and $\lim_{x=0}\{\phi(x)\}$, $\lim_{x=0}\{\phi_1(x)\}$, $\lim_{x=0}\{\phi_2(x)\}$, $\lim_{x=0}\{\phi_{m-1}(x)\}$ infinitesimals of the m_1^{th}, m_2^{th}, m_{m-1}^{th} orders, $f(x)$ cannot be equal to

$$A\phi(x) + B\phi_1(x) + C\phi_2(x)$$

for the general value of x.

SCHOLIUM.

Hence we see that the law of homogeneity, which is so essential an element of all analytical developements, holds good at the limits of the functions and variables as well as for values varying between finite limits. We shall now see that this law is alone sufficient to demonstrate La Grange's principle within the proper limits of its application, as well as to indicate at once the cases in which it is necessarily inapplicable.

3. I shall now enunciate and demonstrate La Grange's principle.

THEOREM.

If $f(x)$ be a function of x continuous between the limits 0 and x_1, and if $\lim_{x=0}\{f(x)\}$ is an infinitesimal of a finite and positive order $\propto$, the function $f(x)$ for any value of x within those limits may be analytically represented by a series of terms of the forms $Ax^\alpha + Bx^\beta + Cx^\gamma +$ &c. where A, B, C are finite coefficients, and α, β, γ finite and positive exponents.

DEM. As $\lim_{x=0}\{f(x)\}$ is an infinitesimal of the finite and positive order $\propto$, it follows that the limit of the ratio $\frac{f(x)}{x^\alpha}$ for the values of x which converge indefinitely towards zero, is equal to a finite quantity. This finite quantity is the coefficient A. From this and from the continuity of the functions (the difference of two continuous functions being itself a continuous function) it follows that if we make x increase insensibly from zero to some finite quantity within the limit x_1 the values which the functions

$$x^\alpha\left\{\frac{f(x)}{x^\alpha}-A\right\},\quad x^\beta\left\{\frac{f(x)-Ax^\alpha}{x^\beta}-B\right\},\quad x^\gamma\left\{\frac{f(x)-Ax^\alpha-Bx^\beta}{x^\gamma}-C\right\}$$

successively assume, may be respectively represented by Bx^β, Cx^γ, Dx^δ where B, C, D are finite, and α, β, γ, δ finite and positive with the law $\beta>\alpha$, $\gamma>\beta$, $\delta>\gamma$. Therefore, reducing and transposing, we see that for any finite value of x within the limit x_1, $f(x)$ may be analytically represented by the series $Ax^\alpha+Bx^\beta+Cx^\gamma+Dx^\delta+$&c. Q. E. D.

COR. We may from the preceding proposition deduce a mode of finding successively the terms Ax^α, Bx^β, Cx^γ, and thus of actually effecting the developement of $f(x)$. This is best explained by an example. Let $f(x)$ be $\sin x$ and assume $\sin x = Ax + Bx^\beta + Cx^\gamma +$&c. Dividing by x^α we get $\frac{\sin x}{x^\alpha} = A + Bx^{\beta-\alpha} + Cx^{\gamma-\alpha} +$&c. Now making x converge indefinitely towards zero, as $\beta>\alpha$ and $\gamma>\alpha$, it is manifest that A is equal to $\lim_{x=0}\left(\frac{\sin x}{x^\alpha}\right)$ where α is that finite and positive number which can render $\lim_{x=0}\left(\frac{\sin x}{x^\alpha}\right)$ a finite quantity. But by the ordinary rules of the Differential Calculus for finding the values of fractions which for certain values of the variable become $\frac{0}{0}$, we find that

$$\lim_{x=0}\left(\frac{\sin x}{x^\alpha}\right)=\lim_{x=0}\left(\frac{\cos x}{\alpha x^{\alpha-1}}\right),$$

which for $\alpha=1$ becomes finite and equal to unity. Therefore

$$\sin x = x + Bx^\beta + Cx^\gamma + \&c.$$

Treating the function $\sin x - x$ in the same manner as we have just treated $\sin x$, we get $\beta = 3$ and $B = -\frac{1}{1.2.3}$. Similarly $\sin x - x + \frac{x^3}{1.2.3}$ gives $\beta = 5$ and $C = +\frac{1}{1.2.3.4.5}$, and thus finally

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} \text{ \&c.}$$

This process is general, and may be easily applied to demonstrate the theorems of Taylor and Maclaurin.

4. The theorem of the last article is La Grange's principle, and was used by that analyst as the fundamental principle of his Calculus of Functions. By the corollary to the theorem in Article 2, it is clear that it is inapplicable to functions whose limits for the values of x which converge indefinitely to zero are infinitesimals of an infinitely high or an infinitely low order, Ax^α, Bx^β, Cx^γ being at the same limit infinitesimals of finite orders. There are however only two known classes of functions which have this property, viz. $e^{-\frac{1}{x^n}}$ and $\frac{1}{\log x}$. The limit of the ratio of $e^{-\frac{1}{x^n}}$ to x^α is easily shown to be infinitely small, however great α may be taken. $\text{Lim}_{x=0}(e^{-\frac{1}{x^n}})$ is therefore an infinitesimal of an infinitely high order, and consequently $e^{-\frac{1}{x^n}}$ cannot be represented by a series like $Ax^\alpha + Bx^\beta + Cx^\gamma + \text{\&c.}$ On the contrary, the limit of the ratio of $\frac{1}{\log x}$ to x^α may be shown to be infinitely great, however small α may be taken. $\text{Lim}_{x=0}\left(\frac{1}{\log x}\right)$ is therefore an infinitesimal of an infinitely low order, and therefore cannot be represented by a series such as $Ax^\alpha + Bx^\beta + Cx^\gamma + \text{\&c.}$ In either case indeed, if we assumed the principle, we should, by passing to the limits of the equivalent series, find an infinitesimal of an infinitely high or infinitely low order, equal to a series of infinitesimals of finite orders, which would violate the principle of homogeneity which exists equally in finite and infinitesimal quantities.

XIV. *On the Transmission of Light in Crystallized Media.* By PHILIP KELLAND, B.A. *Fellow and Tutor of Queens' College, Cambridge.*

[Read *Feb.* 13, 1837.]

INTRODUCTION.

THE object which I have principally had in view in the Memoirs which I have hitherto laid before this Society, has been the development of the equations for the motion of a series of particles in a form calculated to lead to a simple and tangible interpretation.

The point of greatest interest connected with the subject, is the determination of the *law of force* by which the particles act on each other. The data for the investigation of this law are neither numerous nor well defined, and one difficulty in particular attaches itself to every part of it, arising from our uncertainty respecting the number and nature of the *causes* which may conspire to the production of any particular phenomenon.

In my first Memoir I discarded all complexity from my investigations, and conceived the whole effect to be due to the action of particles of the same kind: from a comparison of my results with those of observation, I was led to the conclusion that the law of force is that of the inverse square of the distance, and by means of that law was enabled to shew that the vibrations are necessarily transversal.

In my second Memoir I treated the subject in a more general manner, attributing the phenomena to the action not of one system of particles, but of two, which act mutually on each other. There appeared numerous coincidences, which, if they did not suffice perfectly to establish the law in question, afforded strong presumptive evidence in its favour; not confined to the action of the particles of ether, but extending to those of air, and giving *normal* vibrations in the latter instance as the cause of the phenomena of sound.

All the investigations were, however, confined to a *perfectly symmetrical* medium, on which account the results were limited to non-crystallized substances.

My object at present is to complete the view I have taken of the subject, by extending analogous artifices of simplification to particles arranged not in a *perfectly* symmetrical manner, but symmetrical only with respect to three planes at right angles to one another.

In entering on this subject, I must remind you that I take for granted the law of the inverse square of the distance as established; and the novelty which is presented by the present view of the subject arises from the difference in the *form* of the force corresponding to a disturbance in the *normal* direction, from that put in play by a disturbance in the *transverse* direction.

I have limited my operations to one series of particles, from the circumstance that the *form* is not altered by introducing another series, provided the latter act on the former, and are themselves subject to the action of the former. The results arising from the combination of two sets, I have proved to be the sums or differences of the results arising from each set respectively.

It is true that the action of *material* particles has been totally omitted, the material particles being supposed to exert on those of ether an influence by which they themselves are not reciprocally affected. My reason for this omission is, that such influence will not affect the motion in a non-crystallized medium, (see *Trans. Camb. Phil. Soc.* Vol. VI. p. 244.) and, consequently, will not materially affect it in a crystallized one. The charge which has lately been brought against the hypothesis which M. Cauchy and others have adopted, is, that it omits altogether the action of the particles of matter.

Now I conceive that this is by no means a fair charge, for if the material particles themselves vibrate, we have two systems of vibrating particles, the combined motion of which has been considered, and if they do not vibrate, they produce no effect.

As regards the law of force, a Memoir has lately been circulated, in which M. Cauchy arrives at the conclusion that it is the inverse fourth power of the distance. Adopting this law, Professor Lloyd has proved that the vibrations are transversal, in a paper read before the Irish Academy, in November last. In a short abstract of that paper, it is stated that the object of the Author has been simplification, and the mode of accomplishing that object is given. This mode is precisely that which I adopted, and some of the conclusions are apparently the same; as for instance, that the vibrations are transversal. This conclusion is stated as follows: "When this law of force (the inverse fourth power) is substituted in the corresponding relation for the *normal* vibration, the velocity of propagation is infinite; so that the normal disturbance is propagated instantaneously, and gives rise to no wave."

I do not think from this statement that the grounds on which the law of the inverse square stands, are less tenable than those which lead to the inverse fourth power, and shall not therefore consider it incumbent on me to change my views with respect to the law.

I have dwelt at considerable length on this point, as it is of essential importance to all my succeeding investigations that the law of the inverse square of the distance be not set aside; and I think it will be allowed, that as far as the above speculations are concerned, that of the inverse fourth power does not appear to be established.

In attempting to offer any investigations connected with the transmission of light through crystals, we are naturally prompted to recur, as to the established theory, to those of M. Fresnel, which stand prominent as an example of clearness of conception and distinctness of explanation. The agreement of the results with those of observation, the remarkable predictions which they have afforded of phenomena which have fully verified those predictions, the simplicity with which they explain a multitude of various and complex phenomena, have stamped them with a character so firm that it would be presumptuous to attempt to set them aside. Truth however compels me to state, that whilst I feel the highest admiration of M. Fresnel's theory, I am at the same time doubtful whether some of the points on which it rests are not defective, at least as commonly stated. I allude only to the *mechanical* part of it; nothing can be more complete or more elegant than the geometrical part. I trust I shall not be understood in anything which follows as endeavouring in the slightest to detract from M. Fresnel's fame. I mean far otherwise; but having advanced the opinion that some parts of the mechanical theory are inaccurate, it becomes incumbent on me to explain in what manner this inaccuracy is introduced, and how it happens that from imperfect premises accurate conclusions have been deduced.

It shall be my endeavour then to point out, as clearly as I am able, the circumstances in which the theory labours under a difficulty, and then to shew the cause of this difficulty.

SECTION I.

Remarks on M. Fresnel's Theory.

M. Fresnel in his Memoir on double refraction, p. 103, states the principle, that "the elasticity put in play by luminous vibrations depends solely on *their* direction and not on that of the waves." Of this principle he demonstrates, in a very satisfactory manner, the theoretic possibility, and there appears little room to doubt its truth. Taking it for granted then, he proceeds (p. 106.) to an application of it in the following manner.

If we have two displacements corresponding to *different* waves, we may consider each of them as belonging to a new wave, the front of which is the plane passing through them, and shall, if we wish to combine the two, have only to combine two vibrations in the front of a common wave. Thus far, I think, there can be no ground for the slightest objection. But the statement in p. 107 cannot, I think, lay claim to the same degree of evidence as this.

It would occupy too much space to give here the whole of this statement. It will be quite sufficient to give an abstract of it, which I copy from Professor Airy's Tracts, p. 343.

"If the displacement of a particle considered in any direction be resolved into three displacements in the directions of x, y, z, the variations of force in those directions produced by the alteration of a single particle (and consequently the force produced by the whole system) are the same as if the displacements in those directions had been made independently. From this it easily follows that the sum of any number of displacements causes forces equal to the sum of the forces corresponding to the separate displacements: and then any number of undulations, produced by vibrations in different directions, may coexist without destroying each other." It will be seen that this statement supposes the force put in play to depend only on the *displacement*, and not at all on the position of the front of the wave. If, indeed, it could restrict the hypothesis by adding that the force put in play by a displacement in any direction *in* the front of a wave is independent of the position of that front, or remains constant whilst that front is made to revolve about the line of displacement, it would coincide with what M. Fresnel had established above.

It will be seen then that I object not to the supposition that the force put in play is independent of the position of the plane of the wave, but to the converse, that if a displacement be resolved parallel to x, y, z, the forces put in play will be the same as if the wave was in each individual case perpendicular to yz, xz, xy respectively.

It is clear that such an hypothesis takes for granted, what I should not think Fresnel could mean, that the force on each particle in any direction is of the same form as if that particle alone were in motion in that direction.

It will not suffice to urge in answer to these objections, that however they might apply to motion in general, in the particular instance of vibrations the nature of the arrangement is such as to render them invalid. On no hypothesis, that I can conceive, would the force due to a displacement *in* the direction of transmission be the same as that in a perpendicular direction, and when the law of force is that of the inverse square of the distance, far from being identical, they are of a *directly* opposite character; and the effects which they produce, instead of being analogous, are totally different even in form; the one being an oscillation, the other a progression. But this is not the only objection which I adduce. M. Fresnel determines by his construction the two directions in which a vibration taking place the law of transmission is satisfied. He finds the force in each of these directions, by supposing the whole force put in play to be resolved into two; one in the direction itself, and the other, perpendicular to the front of the wave. The former of these he assumes as the force which produces vibration, the latter he omits altogether.

Now the fact of considering that the force in any direction is the resolved part in that direction of the whole force put in play, requires that the forces be all of the same nature: how does it happen then that a part of them may be *omitted altogether?* Should it be urged in reply, that the motion of a particle in a given direction is not affected by a force which acts always at right angles to that direction; I answer that this is not the solution of the real difficulty, though most persons appear perfectly satisfied with it. That the absolute motion of the particle will be such as continually to change the plane in which it moves is quite obvious. If then, as M. Fresnel supposes, the velocity depends on the position of this plane, the velocity itself must be continually varying for the same ray.

Nor has the plane in which the particle moves a reciprocating motion. The construction consisting of an ellipsoid cut by the plane of vibration through its centre sufficiently proves this; for it is found that the whole force due to a displacement in one of the axes of the elliptic section acts in the direction of a normal to the ellipsoid at the extremity of that axis. Suppose then the particle to be at its greatest distance from its position of rest; the action of the normal force causes it to return in a direction *above* the plane of its disturbance, (suppose). When it has reached the other extremity of its oscillation, the force tends to pull it below the line of its return: by each action, therefore, the change from its original line of motion is in the same direction, and this will take place continually, so that the plane of motion will continually vary, and the velocity of transmission constantly increase or constantly diminish.

These points appear to me weak points in the theory: the former is indeed of such magnitude, that were there nothing to limit its effects, the results would be very far from the truth. The error, however, which is committed by this step is exactly righted by the second, and thus two hypotheses which individually are erroneous, do, when combined, lead to correct results. Indeed it is manifest that whereas the former error arises from not giving to the front of the wave its due effect, the latter arises from giving it an effect which it could not produce: the former requires that the force should act out of the plane of the wave, the latter rejects the part which does; and these will right each other if we can shew (as I trust I have done in the sequel) that the actual vibratory force is *in* the front of the wave. I could have desired that my investigations should have assumed a more inviting form, but I have not the means at present of throwing them into a shape other than that under which they appear.

The first step I have taken is to prove the transversality of the vibration, and thus having established a direction in which vibrations do take place, I suppose that the forces put in play by a displacement may be determined (as far as their action in the direction of that displacement alone is concerned) in the same manner as Fresnel does. The modification then which I propose, consists in restricting the theorem of Fresnel, and reducing it to the following:

"That the whole *vibratory* force put in play by a displacement *in a direction which admits of a vibration*, is the sum of the resolved parts *along that direction* of the vibratory forces due to the resolved parts of the displacement along the axes of elasticity." With a direction normal to those of vibration I have nothing to do, except to prove that the force in that direction is *not* part of the vibratory force.

SECTION II.

Investigation of the Motion of a System of Particles within a Crystal.

1. We shall assume that the arrangement is an arrangement of symmetry with respect to three planes mutually at right angles to each other.

Let their lines of intersection be taken as the axes of co-ordinates.

x, y, z the co-ordinates of the particle under consideration in its position of rest.

$x+\alpha,\ y+\beta,\ z+\gamma$ its co-ordinates after the time t.

$x+\delta x,\ y+\delta y,\ z+\delta z$ the co-ordinates of another particle whose distance from the former is r.

Then, by pursuing a process precisely analogous to that which applies to non-crystallized media, (*Trans. Camb. Phil. Soc.* Vol. VI. Part I. page 162.) supposing the particle in *vibration*, we have the following equations of motion:

$$\frac{d^2\alpha}{dt^2} = -2\alpha\Sigma\left(\frac{1}{r^3}-\frac{3\,\delta x^2}{r^5}\right)\sin^2\frac{k\,\delta\rho}{2} - 6\beta\Sigma\frac{\delta x\,\delta y}{r^5}\sin^2\frac{k\,\delta\rho}{2} - 6\gamma\Sigma\frac{\delta x\,\delta z}{r^5}\sin^2\frac{k\,\delta\rho}{2},$$

$$\frac{d^2\beta}{dt^2} = -2\beta\Sigma\left(\frac{1}{r^3}-\frac{3\,\delta y^2}{r^5}\right)\sin^2\frac{k\,\delta\rho}{2} - 6\alpha\Sigma\frac{\delta x\,\delta y}{r^5}\sin^2\frac{k\,\delta\rho}{2} - 6\gamma\Sigma\frac{\delta y\,\delta z}{r^5}\sin^2\frac{k\,\delta\rho}{2},$$

$$\frac{d^2\gamma}{dt^2} = -2\gamma\Sigma\left(\frac{1}{r^3}-\frac{3\,\delta z^2}{r^5}\right)\sin^2\frac{k\,\delta\rho}{2} - 6\alpha\Sigma\frac{\delta x\,\delta z}{r^5}\sin^2\frac{k\,\delta\rho}{2} - 6\beta\Sigma\frac{\delta y\,\delta z}{r^5}\sin^2\frac{k\,\delta\rho}{2}$$

and calling

$$\begin{cases} 2\Sigma \dfrac{\delta x^2}{r^5} \sin^2 \dfrac{k\delta\rho}{2} = m, & 2\Sigma \left(\dfrac{1}{r^3} - \dfrac{3\delta x^2}{r^5}\right) \sin^2 \dfrac{k\delta\rho}{2} = a, \\ 2\Sigma \dfrac{\delta y^2}{r^5} \sin^2 \dfrac{k\delta\rho}{2} = n, & 2\Sigma \left(\dfrac{1}{r^3} - \dfrac{3\delta y^2}{r^5}\right) \sin^2 \dfrac{k\delta\rho}{2} = b, \\ 2\Sigma \dfrac{\delta z^2}{r^5} \sin^2 \dfrac{k\delta\rho}{2} = p, & 2\Sigma \left(\dfrac{1}{r^3} - \dfrac{3\delta z^2}{r^5}\right) \sin^2 \dfrac{k\delta\rho}{2} = c, \\ 6\Sigma \dfrac{\delta x\,\delta z}{r^5} \sin^2 \dfrac{k\delta\rho}{2} = Y, & \text{where } a + b + c = 0, \\ 6\Sigma \dfrac{\delta y\,\delta z}{r^5} \sin^2 \dfrac{k\delta\rho}{2} = X, & \\ 6\Sigma \dfrac{\delta x\,\delta y}{r^5} \sin^2 \dfrac{k\delta\rho}{2} = Z, & \end{cases}$$

multiplying the equations by P, Q and R respectively, and adding them, we may put the result under the form

$$P\frac{d^2\alpha}{dt^2} + Q\frac{d^2\beta}{dt^2} + R\frac{d^2\gamma}{dt^2} = -A(P\alpha + Q\beta + R\gamma), \quad (1),$$

provided we make

$$(a-A)P = QZ + RY,$$
$$(b-A)Q = PZ + RX,$$
$$(c-A)R = PY + QX.$$

By eliminating P, we obtain

$$\{(a-A)(b-A) - Z^2\}\, Q = \{YZ + (a-A)X\}\, R,$$
$$\{(a-A)(c-A) - Y^2\}\, R = \{YZ + (a-A)X\}\, Q,$$

and finally

$$(a-A)(b-A)(c-A) - X^2(a-A) - Y^2(b-A) - Z^2(c-A) - 2XYZ = 0,$$

$$\text{or}\quad (A-a)(A-b)(A-c) - X^2(A-a) - Y^2(A-b) - Z^2(A-c) + 2XYZ = 0, \quad (2);$$

an equation which it may be readily shewn gives three *possible* values of A.

2. These values are either two positive and one negative, or two negative and one positive, for if we write the equation under the form

$$A^3-(a+b+c)A^2+\{ab+ac+bc-(X^2+Y^2+Z^2)\}A$$
$$-abc+aX^2+bY^2+cZ^2+2XYZ=0,$$

it will be evident that the coefficient of A^2 is equal zero (1).

The roots then must assume the form $A_1 A_2 - (A_1 + A_2)$, in which $A_1 A_2$ may be either both positive or both negative: suppose the former.

3. Now, corresponding to any value of A, a value of P, Q, R, respectively can be determined; but A is the velocity of transmission of a vibration whose direction makes with the co-ordinate axes the angles $\cos^{-1} P$, $\cos^{-1} Q$, $\cos^{-1} R$ respectively, and which is transmitted in a direction making with the same axes other angles θ, ϕ and ψ.

We conclude then, that there are in general *two* directions and no more, in which a vibration taking place, the transmission along a given line is possible. A disturbance in a given direction being resolved into these two, will give rise to two different rays, transmitted with different velocities.

4. The third value of A which is negative, will not correspond to a vibration; the manner in which it may affect the motion, and the probable results to which it gives rise, I have fully discussed in a paper read before this Society a short time since, and shall leave it untouched in the present Memoir.

Discussion of the Equation for A.

5. As a preliminary step towards a complete discussion of this equation, we will first consider the medium *perfectly* symmetrical.

Transform the co-ordinates in such a manner that the axis of x' shall coincide with the direction of transmission, and that of y' lie in the plane of xy.

Denote the angle between the axes of x and x' by the symbol $(x'x)$, and so on for the others:

$$\text{then}\quad \cos(xx') = \cos\theta,$$

$$\cos(x'y) = \cos\phi,$$

$$\cos(x'z) = \cos\psi;$$

$$\text{and}\quad \cos(y'x)\cos\theta + \sin(y'x)\cos\phi = 0;$$

$$\therefore\quad \tan(y'x) = -\frac{\cos\theta}{\cos\phi},$$

$$\text{also}\quad \cos\theta\cos(z'x) + \cos\phi\cos(z'y) + \cos\psi\cos(z'z) = 0,$$

$$\cos(y'x)\cos(z'x) + \sin(y'x)\cos(z'y) = 0,$$

$$\cos^2(z'x) + \cos^2(z'y) + \cos^2(z'z) = 1;$$

which three equations give

$$\cos(z'x) = -\cot\psi\cos\theta,$$

$$\cos(z'y) = -\cot\psi\cos\phi,$$

$$\cos(z'z) = \sin\psi;$$

$$\text{and}\quad \delta\rho = \delta x' = \delta x\cos\theta + \delta y\cos\phi + \delta z\cos\psi,$$

$$\delta y'\sin\psi = -\delta x\cos\phi + \delta y\cos\theta,$$

$$\delta z' = -\delta x\cot\psi\cos\theta - \delta y\cot\psi\cos\phi + \delta z\sin\psi,$$

$$\delta x = \delta x'\cos\theta - \delta y'\frac{\cos\phi}{\sin\psi} - \delta z'\cot\psi\cos\theta,$$

$$\delta y = \delta x'\cos\phi + \delta y'\frac{\cos\theta}{\sin\psi} - \delta z'\cot\psi\cos\phi,$$

$$\delta z = \delta x'\cos\psi + \delta z'\sin\psi.$$

6. Making the substitutions, and calling $m + n + p = h$, we obtain

$$a = h - 6\Sigma\left(\delta x'\cos\theta - \delta y'\frac{\cos\phi}{\sin\psi} - \delta z'\cos\psi\cos\theta\right)^2 . \frac{\sin^2 . \dfrac{k\delta\rho}{2}}{r^5}$$

$$= h - 6\Sigma\left(\delta x'^2\cos^2\theta + \delta y'^2\frac{\cos^2\phi}{\sin^2\psi} + \delta z'^2\cot^2\psi\cos^2\theta\right) . \frac{\sin^2 . \dfrac{k\delta\rho}{2}}{r^5},$$

for by the symmetry of the medium

$$\Sigma \frac{\delta x' \delta y'}{r^5} \sin^2 . \frac{k \delta x'}{2},$$

and similar expressions for $\delta x'$, $\delta y'$ and $\delta z'$ must all equal zero;

$$\therefore\ a = h - 3\left(m' \cos^2\theta + n' \frac{\cos^2\phi}{\sin^2\psi} + p' \cot^2\psi \cos^2\theta\right),$$

$$b = h - 3\left(m' \cos^2\phi + n' \frac{\cos^2\theta}{\sin^2\psi} + p' \cot^2\psi \cos^2\phi\right),$$

$$c = h - 3\left(m' \cos^2\psi + p' \sin^2\psi\right),$$

adopting a notation similar to that in (1).

$$\text{Also } X = 6\Sigma . \left\{ \left(\delta x' \cos\phi + \delta y' \frac{\cos\theta}{\sin^2\psi} - \delta z' \cot\psi \cos\phi\right) \right.$$

$$\left. \times \left(\delta x' \cos\psi' + \delta z' \sin\psi\right) \frac{\sin^2 \frac{k\delta\rho}{2}}{r^5} \right\},$$

$$= 6\Sigma \left(\delta x'^2 \cos\phi \cos\psi - \delta z'^2 \cos\phi \cos\psi\right) \frac{\sin^2 . \frac{k\delta\rho}{2}}{r^5}$$

$$= 3\left(m' - p'\right) \cos\phi \cos\psi,$$

$$Y = 3\left(m' - p'\right) \cos\theta \cos\psi,$$

$$Z = 3\left(m' - \frac{n'}{\sin^2 . \psi} + p' \cot^2\psi\right) \cos\theta \cos\phi.$$

7. These values of a, b, &c. will be much simplified in an uncrystallized medium, for in that case it is evident that

$$\Sigma \delta y'^2 \frac{\sin^2 . \frac{k\delta\rho}{2}}{r^5} = \Sigma \delta z'^2 \frac{\sin^2 \frac{k\delta\rho}{2}}{r^5}; \text{ and } \therefore n' = p'.$$

This reduces the above equations to the following, in which I have suppressed the accents for the sake of brevity.

$$a = h - 3(m\cos^2\theta + p\sin^2\theta),$$
$$b = h - 3(m\cos^2\phi + p\sin^2\phi),$$
$$c = h - 3(m\cos^2\psi + p\sin^2\psi);$$
$$X = 3(m-p)\cos\phi\cos\psi,$$
$$Y = 3(m-p)\cos\theta\cos\psi,$$
$$Z = 3(m-p)\cos\theta\cos\phi.$$

8. To find the coefficient of A in equation (2).

$$ab + ac + bc = 3h^2 - 6h\{m(\cos^2\theta + \cos^2\phi + \cos^2\psi) + p(\sin^2\theta + \ldots)\}$$
$$+ 9\{m^2(\cos^2\theta\cos^2\phi + \cos^2\theta\cos^2\psi + \cos^2\phi\cos^2\psi) + p^2(\sin^2\theta\sin^2\phi + \ldots)\}$$
$$+ 9mp(\cos^2\theta\sin^2\phi + \ldots) + 9mp(\cos^2\phi\sin^2\theta + \ldots),$$

let $\cos^2\theta\cos^2\phi + \cos^2\theta\cos^2\psi + \cos^2\phi\cos^2\psi = f$

$$\cos^2\theta\cos^2\phi\cos^2\psi = g,$$

and $ab + ac + bc = 3h^2 - 6h^2 + 9\{m^2 f + p^2(1+f)\}$

$$+ 18mp(1-f)$$
$$= -3(m+2p)^2 + 9(2mp + p^2) + 9f(m^2 - 2mp + p^2)$$
$$= -3(m+2p)^2 + 9(2mp + p^2) + 9f(m-p)^2.$$

Again $X^2 + Y^2 + Z^2 = 9(m-p)^2 f$;

$\therefore$ coefficient of $A = ab + ac + bc - (X^2 + Y^2 + Z^2)$

$$= -3(m+2p)^2 + 9(2mp + p^2),$$
$$= -3m^2 + 6mp - 3p^2,$$
$$= -3(m-p)^2,$$

which is independent of f, or of the direction in which the particle vibrates.

9. To find the last term of the equation

$$abc = \{h - 3(m\cos^2\theta + p\sin^2\theta)\}\{h - 3(m\cos^2\phi + p\sin^2\phi)\}\{h - 3(m\cos^2\psi + p\sin^2\psi)\}$$

$$= 2h^3 + 9h\{(m-p)^2 f + 2mp + p^2\}$$

$$- 27\{m^3 g + p^3(f-g) + m^2 p(f-3g) + mp^2(1-2f+3g)\}$$

$$= -2(m+2p)^3 + 9(m+2p)\{(m-p)^2 f + p^2 + 2mp\}$$

$$- 27\{(m-p)^3 g + p(m-p)^2 f + mp^2\};$$

$$aX^2 + bY^2 + cZ^2 = 9(m-p)^2\{(m+2p)f - 3(3mg + p(f-3g)\}$$

$$= 9(m-p)^2\{(m-p).(f-9g)\}$$

$$= 9(m-p)^3(f-9g);$$

$$2XYZ = 54(m-p)^3.g;$$

$$\therefore\ -abc + aX^2 + bY^2 + cZ^2 + 2XYZ$$

$$= 2(m+2p)^3 - 9(m+2p)(p^2+2mp) + 27mp^2$$

$$+ 27(m-p)^2\{(m-p).g + pf\} - 9(m+2p)(m-p)^2 f$$

$$+ 9(m-p)^3(f-9g) + 54(m-p)^3.g$$

$$= (m+2p)(2m^2 - 10mp - p^2) + 27mp^2$$

$$= 2m^3 - 6m^2 p + 6mp^2 - 2p^3$$

$$= 2(m-p)^3,$$

which is also independent of the direction of vibration.

10. By the substitution of these values the equation in A is reduced to the simple form

$$A^3 - 3(m-p)^2 A + 2(m-p)^3 = 0,$$

the roots of which are evidently

$$m-p,\quad m-p \quad\text{and}\quad -2(m-p).$$

This result shews that the vibrations are transmitted with the *same* velocity in every direction in which such can possibly be transmitted.

11. We proceed now to the determination of these directions, and to commence with the last value of A, substituting in the equations of 1;

$$\therefore\ a - A = m + n + p - 3(m\cos^2\theta + p\sin^2\theta) + 2(m - p)$$

$$= 3m\sin^2\theta - 3p\sin^2\theta$$

$$= 3(m - p)\sin^2\theta;$$

$$b - A = 3(m - p)\sin^2\phi,$$

$$c - A = 3(m - p)\sin^2\psi;$$

$$\therefore\ \{9(m - p)^2\sin^2\theta\sin^2\phi - 9(m - p)^2\cos^2\theta\cos^2\phi\}\,Q$$

$$= \{9(m - p)^2\cos^2\theta\cos\phi\cos\psi + 9(m - p)^2\sin^2\theta\cos\phi\cos\psi\}\,R;$$

$$\text{or}\ (1 - \cos^2\theta - \cos^2\phi)\,Q = \cos\phi\cos\psi\,R,$$

$$\cos^2\psi\,.\,Q = \cos\phi\cos\psi\,R,$$

$$\cos\psi\,Q = \cos\phi\,R,$$

$$\frac{Q}{\cos\phi} = \frac{R}{\cos\psi}.$$

Similarly, from the other equations we deduce

$$P\sin^2\theta = Q\cos\theta\cos\phi + R\cos\theta\cos\psi$$

$$= Q\cos\theta\left(\cos\phi + \frac{\cos^2\psi}{\cos\phi}\right)$$

$$= Q\cos\theta\,\frac{\sin^2\theta}{\cos\phi},$$

$$\text{or}\ \frac{P}{\cos\theta} = \frac{Q}{\cos\phi},$$

which shews that P, Q, R have the ratio

$$\cos\theta,\ \cos\phi,\ \cos\psi;$$

$$\text{hence}\quad P = C.\cos\theta,$$
$$Q = C.\cos\phi,$$
$$R = C.\cos\psi,$$

C being some constant factor.

Substituting these values in the equation

$$P\frac{d^2\alpha}{dt^2} + Q\frac{d^2\beta}{dt^2} + R\frac{d^2\gamma}{dt^2} = -A(P\alpha + Q\beta + R\gamma),$$

$$\text{we obtain}\quad C\left(\cos\theta\frac{d^2\alpha}{dt^2} + \cos\phi\frac{d^2\beta}{dt^2} + \cos\psi\frac{d^2\gamma}{dt^2}\right)$$
$$= -AC(\cos\theta.\alpha + \cos\phi.\beta + \cos\psi.\gamma),$$

which by making $\alpha\cos\theta + \beta\cos\phi + \gamma\cos\psi = \delta$,

$$\text{gives}\quad \frac{d^2\delta}{dt^2} = -A\delta$$
$$= 2(m-p)\delta,$$

a form which *does not* correspond to a vibration.

The expression $\alpha\cos\theta + \beta\cos\phi + \gamma\cos\psi$ is evidently the resolved part of the disturbance parallel to the direction of transmission: it follows therefore, that there is *no* vibration in the direction of transmission, or in other words, that the vibration is entirely *transversal*, a result to which I also arrived in a former Memoir.

12. To determine the directions of vibration corresponding to the values $m-p$ of A.

$$a - A = m + 2p - 3(m\cos^2\theta + p\sin^2\theta) - (m-p)$$
$$= 3p\cos^2\theta - 3m\cos^2\theta$$
$$= -3(m-p)\cos^2\theta,$$
$$b - A = -3(m-p)\cos^2\phi,$$
$$c - A = -3(m-p)\cos^2\psi,$$

which values substituted in (1) give

$$(\cos^2\theta\cos^2\phi - \cos^2\theta\cos^2\phi)\,Q = (\cos^2\theta\cos\phi\cos\psi - \cos^2\theta\cos\phi\cos\psi)\,R,$$

an identical equation independently of Q and R: and the other equations give

$$-3(m-p)\cos^2\theta \,.\, P = 3(m-p)\cos\theta\cos\phi \,.\, Q + 3(m-p)\cos\theta\cos\psi R,$$

which is satisfied by making either $\cos\theta = 0$

$$\text{or} \quad P\cos\theta + Q\cos\phi + R\cos\psi = 0,$$

and the former is evidently impossible, wherefore the latter equation must be satisfied: and it is the *only* equation in P, Q, R.

Suppose now the direction of motion of the particle to make angles X, Y, Z with the axes of x, y, z, then the displacement Δ is

$$\Delta = \alpha\cos X + \beta\cos Y + \gamma\cos Z:$$

$$\text{but it is also } \frac{P\alpha + Q\beta + R\gamma}{C};$$

$$\therefore \quad \frac{P}{C} = \cos X, \quad \frac{Q}{C} = \cos Y, \quad \frac{R}{C} = \cos Z,$$

and from the above equation

$$\cos X\cos\theta + \cos Y\cos\phi + \cos Z\cos\psi = 0;$$

which shews that the directions of displacement, and of transmission are at right angles with each other, and since this is the only condition which exists amongst the quantities P, Q, R, the displacement may be in any direction in the plane perpendicular to the direction of transmission, and it is propagated with a velocity equal in all directions.

13. We will now return to our equations, and suppose the medium symmetrical with respect to each of the axes respectively, but not *absolutely* a medium of symmetry: suppose, for instance, that in passing from the plane of xy the distance between two consecutive

particles is ϵ; in passing from the plane of $xz-\epsilon'$ and in passing from the plane of $yz-\epsilon''$, our equation still retains the form

$$A^3+\{ab+ac+bc-(X^2+Y^2+Z^2)\}A-abc+aX^2+bY^2+cZ^2 +2XYZ=0,$$

which is evidently analogous both in its form and mode of derivation from the elimination of P, Q, R to that for determining the three principal axes of a solid: and may therefore be proved in the same manner to give three values of A corresponding to directions at right angles to each other.

From the form of the equation it is obvious that of the three directions two only correspond to a vibration.

14. For these two vibrations we shall manifestly have equations of motion analogous to those in other cases.

The plane in which they lie is called the front of the wave, and the values of A are the velocities of transmission in a direction perpendicular to the front.

Let the axes of x_1, y_1, z_1, be the three directions; x_1 being that of no vibration, and let them make angles $\theta_1\phi_1\psi_1$; $\theta_2\phi_2\psi_2$; $\theta_3\phi_3\psi_3$ with the axes of x, y, z, respectively.

Then we must have

$$\frac{d^2\alpha_1}{dt^2}=-2\Sigma\left(\frac{1}{r^3}-\frac{3\delta x_1^2}{r^5}\right)\sin^2\frac{k\delta x_1}{2},$$

$$\frac{d^2\beta_1}{dt^2}=-2\Sigma\left(\frac{1}{r^3}-\frac{3\delta y_1^2}{r^5}\right)\sin^2\frac{k\delta x_1}{2},$$

$$\frac{d^2\gamma_1}{dt^2}=-2\Sigma\left(\frac{1}{r^3}-\frac{3\delta z_1^2}{r^5}\right)\sin^2\frac{k\delta x_1}{2};$$

extending to α_1, β_1, γ_1, &c. the same meaning as to similar quantities along other axes.

15. Now if we had taken the three axes of x_1, y_1, z_1, as those of co-ordinates in the commencement, we should have had the extra term (supposing $\alpha_1 = 0$) $\Sigma \frac{\delta y_1 \delta z_1}{r^5} \sin^2 \frac{k \delta x_1}{2}$ in each.

Our equations then arise from making this quantity vanish.

Let v, v' be the velocities of transmission of the vibrations respectively perpendicular to the front of the wave.

$$\text{Then } v^2 = 2\Sigma \left(\frac{1}{r^3} - \frac{3\delta y_1^2}{r^5}\right) \sin^2 \frac{k\delta x_1}{2},$$

$$v'^2 = 2\Sigma \left(\frac{1}{r^3} - \frac{3\delta z_1^2}{r^5}\right) \sin^2 \frac{k\delta x_1}{2}.$$

These values of the squares of the velocity depend (it is supposed) only on the direction of vibration, provided that direction be perpendicular to the direction of transmission: the quantity

$$\Sigma \left(\frac{1}{r^3} - \frac{3\delta y_1^2}{r^5}\right) \sin^2 \frac{k\delta x_1}{2} \beta_1$$

is in fact the force due to the displacement β_1, and the position of β_1 in the front of the wave defines its value. Now this displacement may be resolved into three parallel respectively to x, y, z, and Fresnel's hypothesis is, that the force put in play by that resolved part bears the same ratio to it as it would were the vibration one simply in that direction.

16. This supposition amounts to the following:

That if we take the expression $\Sigma \frac{\delta y_1^2}{r^5} \sin^2 \frac{k\delta x_1}{2}$, and transform the expression δy_1^2 into an equivalent one in δx, δy, δz; placing before each of the terms, not the expression $\sin^2 \frac{k\delta x_1}{2}$, but an expression of the same form corresponding to a transmission at right angles to the axis belonging to that term; the expression will be unaltered.

Now $\delta y_1^2 = (\delta x \cos\theta_2 + \delta y \cos\phi_2 + \delta z \cos\psi_2)^2$;

$$\therefore \ \Sigma \frac{\delta y_1^2}{r^5} \sin^2 \frac{k\delta x_1}{2} = M \cos^2\theta_2 + N \cos^2\phi_2 + P \cos^2\psi,$$

and $\Sigma \dfrac{1}{r^3} \sin^2 \dfrac{k\delta x}{2} = c$;

when $M = \Sigma \dfrac{\delta x^2}{r^5} \sin^2 \dfrac{k\delta y}{2}$, $\quad N = \Sigma \dfrac{\delta y^2}{r^5} \sin^2 \dfrac{k\delta x}{2}$,

$$P = \Sigma \frac{\delta z^2}{r^5} \sin^2 \frac{k\delta x}{2};$$

and $\therefore$ $v^2 = 2\{(c - 3M) \cos^2\theta_2 + (c - 3N) \cos^2\phi_2 + (c - 3P) \cos^2\psi_2\}$

$$= a^2 \cos^2\theta_2 + b^2 \cos^2\phi_2 + c^2 \cos^2\psi_2 \text{*},$$

a^2, b^2, c^2 being respectively the squares of the velocity of transmission, when the vibration is simply in the direction of one of the axes.

Similarly $v'^2 = a^2 \cos^2\theta_3 + b^2 \cos^2\phi_3 + c^2 \cos\psi_3$.

17. Now we have seen that

$$\Sigma \frac{\delta y_1 \, \delta z_1}{r^5} \sin^2 \frac{k\delta x_1}{2} = 0.$$

Applying the same process to this equation, we obtain

$$M \cos\theta_2 \cos\theta_3 + N \cos\phi_2 \cos\phi_3 + P \cos\psi_2 \cos\psi_3 = 0,$$

which will determine the *directions* of vibration corresponding to the values of the squares of the velocities given above.

18. In order to determine these directions, we express the co-ordinates and angles by making ϵ the angle between the axis of x and the line of intersection of the planes of $y_1 z_1$ with xy; and calling μ the angle between this line and the axis of y, ψ the angle of inclination of $z_1 y_1$ to xy.

Thus $xPN = \epsilon$, (see note at end of Sup.)

$NPy_1 = \mu$.

* This equation I have obtained by a totally different process in a subsequent Memoir.

We shall find

$$\cos\theta_1 = \sin\epsilon \sin\psi,$$
$$\cos\phi_1 = -\cos\epsilon \sin\psi,$$
$$\cos\psi_1 = \cos\psi,$$
$$\cos\theta_2 = \cos\mu \cos\epsilon - \sin\mu \sin\epsilon \cos\psi,$$
$$\cos\phi_2 = \cos\mu \sin\epsilon + \sin\mu \cos\epsilon \cos\psi,$$
$$\cos\psi_2 = \sin\mu \sin\psi,$$
$$\cos\theta_3 = -\sin\mu \cos\epsilon - \cos\mu \sin\epsilon \cos\psi,$$
$$\cos\phi_3 = -\sin\mu \sin\epsilon + \cos\mu \cos\epsilon \cos\psi,$$
$$\cos\psi_3 = \cos\mu \sin\psi.$$

19. By these values of the angles, the equation in (17) is reduced to

$$-M(\cos\mu\cos\epsilon - \sin\mu\sin\epsilon\cos\psi)(\sin\mu\cos\epsilon + \cos\mu\sin\epsilon\cos\psi)$$
$$-N(\cos\mu\sin\epsilon + \sin\mu\cos\epsilon\cos\psi)(\sin\mu\sin\epsilon - \cos\mu\cos\epsilon\cos\psi)$$
$$+P\sin\mu\cos\mu\sin^2\psi = 0,$$

or $$-M\{\sin 2\mu(\cos^2\epsilon - \sin^2\epsilon\cos^2\psi) + \cos 2\mu \sin 2\epsilon \cos\psi\}$$
$$-N\{\sin 2\mu(\sin^2\epsilon - \cos^2\epsilon\cos^2\psi) - \cos 2\mu \sin 2\epsilon\cos\psi\}$$
$$+P\sin 2\mu \sin^2\psi = 0,$$

or $$\sin 2\mu\{(M-N)(\cos^2\epsilon - \sin^2\epsilon\cos^2\psi) + (N-P)\sin^2\psi\}$$
$$+\cos 2\mu (M-N)\sin 2\epsilon\cos\psi = 0.$$

20. The same substitutions reduce the value of v^2 to

$$v^2 = a^2\cos^2\theta_2 + b^2\cos^2\phi_2 + c^2\cos^2\psi_2$$
$$= a^2(\cos^2\mu\,.\cos^2\epsilon + \sin^2\mu\sin^2\epsilon\cos^2\psi - \tfrac{1}{2}\sin 2\mu\sin 2\epsilon\cos\psi)$$
$$+ b^2(\cos^2\mu\sin^2\epsilon + \sin^2\mu\cos^2\epsilon\cos^2\psi + \tfrac{1}{2}\sin 2\mu\sin 2\epsilon\cos\psi)$$
$$+ c^2\sin^2\mu\sin^2\psi;$$

$$v'^2 = a^2 (\sin^2 \mu \cos^2 \epsilon + \cos^2 \mu \sin^2 \epsilon \cos^2 \psi + \tfrac{1}{2} \sin 2\mu \sin 2\epsilon \cos \psi)$$
$$+ b^2 (\sin^2 \mu \sin^2 \epsilon + \cos^2 \mu \cos^2 \epsilon \cos^2 \psi - \tfrac{1}{2} \sin 2\mu \sin 2\epsilon \cos \psi)$$
$$+ c^2 \cos^2 \mu \sin \psi.$$

Hence $v^2 + v'^2 = a^2 (\cos^2 \epsilon + \sin^2 \epsilon \cos^2 \psi) + b^2 (\sin^2 \epsilon + \cos^2 \epsilon \cos^2 \psi)$
$$+ c^2 \sin^2 \psi \ldots\ldots(1);$$

$$v'^2 - v^2 = (a^2 - b^2) \{\sin 2\mu \sin 2\epsilon \cos \psi - \cos 2\mu (\cos^2 \epsilon - \sin^2 \epsilon \cos^2 \psi)\}$$
$$- (b^2 - c^2) \sin^2 \psi \cos 2\mu \ldots\ldots\ldots\ldots(2).$$

21. Now in the equation of condition (19) since

$$M = \frac{c}{3} - \frac{a^2}{6}, \quad N = \frac{c}{3} - \frac{b^2}{6}, \quad P = \frac{c}{3} - \frac{c^2}{6};$$

$$\therefore M - N = -\frac{(a^2 - b^2)}{6}, \quad N - P = -\frac{b^2 - c^2}{6};$$

by substituting these values, that equation becomes

$$\sin 2\mu \{(a^2 - b^2)(\cos^2 \epsilon - \sin^2 \epsilon \cos^2 \psi) + (b^2 - c^2) \sin^2 \psi\}$$
$$+ \cos 2\mu (a^2 - b^2) \sin 2\epsilon \cos \psi = 0;$$
$$\therefore (a^2 - b^2)(\cos^2 \epsilon - \sin^2 \epsilon \cos^2 \psi) + (b^2 - c^2) \sin^2 \psi$$
$$= - \cot 2\mu (a^2 - b^2) \sin 2\mu \cos \psi \ldots\ldots(a);$$

hence $v'^2 - v^2 = (a^2 - b^2) \sin 2\mu \sin 2\epsilon \cos \psi + (a^2 - b^2) \dfrac{\cos^2 2\mu}{\sin 2\mu} \sin 2\epsilon \cos \psi$

$$= (a^2 - b^2) \frac{\sin 2\epsilon \cos \psi}{\sin 2\mu}.$$

If $\psi = \dfrac{\pi}{2}$, it appears from the above equation for $\cot 2\mu\ (a)$, that $\mu = 0$, or $\dfrac{\pi}{2}$, and

$$\therefore \sin 2\mu = 0.$$

This value therefore does not make the expression vanish, as would appear at first sight; it can vanish only, when

$$\sin 2\epsilon = 0;$$

$$\therefore \quad \epsilon = 0,$$

$$\text{or} \quad \epsilon = \frac{\pi}{2},$$

and the vibration is either in the plane of xz or of yz.

And from equation (a), if $\epsilon = 0$,

$$a^2 - b^2 + (b^2 - c^2)\sin^2\psi = 0,$$

$$\sin\psi = \sqrt{\frac{b^2 - a^2}{b^2 - c^2}},$$

$$\text{and}\ \tan\psi = \sqrt{\frac{b^2 - a^2}{a^2 - c^2}};$$

therefore a is intermediate to b and c.

If $\epsilon = \frac{\pi}{2}$ we get $(a^2 - b^2)\cos^2\psi = (b^2 - c^2)\sin^2\psi$;

$$\therefore \tan\psi = \sqrt{\frac{b^2 - a^2}{c^2 - b^2}},$$

and b must be intermediate to a and c. Both these cannot be true for the same medium: let the latter only be true, then the transmission is in the plane of xz, and there are two directions, one on each side of the axis of z, for which the velocities of transmission of both vibrations are the same, which directions are the optic axes*.

We will call m the angle made by this optic axis with the axis of z, so that

$$\tan^2 m = \frac{b^2 - a^2}{c^2 - b^2} = \frac{a^2 - b^2}{b^2 - c^2}.$$

22. The equation (a) gives

$$-\cot 2\mu = \frac{(a^2 - b^2)(\cos^2\epsilon - \sin^2\epsilon\cos^2\psi) + (b^2 - c^2)\sin^2\psi}{(a^2 - b^2)\sin 2\epsilon\cos\psi}$$

$$= \frac{\cos^2\epsilon - \sin^2\epsilon\cos^2\psi + \cot^2 m\sin^2\psi}{\sin 2\epsilon\cos\psi}$$

* The optic axis here is not the same as that which Fresnel calls by the same name. This is determined by the direction of the wave, his by that of the ray. As I shall have to compare them, I will use the term *radial axis* instead of optic axis when speaking of the latter.

$$= \frac{\dfrac{(\cot m \sin \psi - \sin \epsilon \cos \phi)}{\cos \epsilon} \dfrac{(\cot m \sin \psi + \sin \epsilon \cos \psi)}{\cos \epsilon}}{\dfrac{2 \cos \psi \sin \epsilon}{\cos \epsilon}} + 1.$$

Now if O and R in the figure (18) be the two optic axes, it is evident that

$$zx_1 = \psi, \quad x_1 z O = \frac{\pi}{2} - \epsilon, \quad x_1 z R = \frac{\pi}{2} + \epsilon;$$

$$\therefore \cot O x_1 z = \cot m \sin \psi \sec \epsilon - \cos \psi \tan \epsilon,$$

$$\cot R x_1 z = \cot m \sin \psi \sec \epsilon + \cos \psi \tan \epsilon;$$

$$\therefore \; -\cot 2\mu = \cot (O x_1 z - R x_1 z),$$

$$\begin{aligned} \pi - 2\mu &= O x_1 z - R x_1 z \\ &= O x_1 y_1 - R x_1 y_1 + 2 z x_1 y_1, \\ &= O x_1 y_1 - R x_1 y_1 + 2 (z x_1 N - \mu) \\ &= O x_1 y_1 - R x_1 y_1 + \pi - 2\mu; \end{aligned}$$

$$\therefore \; O x_1 y_1 = R x_1 y_1;$$

therefore the plane which defines one vibration bisects the angle between the planes passing through the normal to the front of the wave and these two optic axes.

The plane which defines the other is manifestly at right angles to this.

23. We saw in (21) that the expression for the difference of the squares of the velocities is

$$v'^2 - v^2 = (a^2 - b^2) \frac{\sin 2\epsilon \cos \psi}{\sin 2\mu}.$$

$$\text{Now } \sin O x_1 = \frac{\cos \epsilon \sin m}{\sin O x_1 z},$$

$$\sin R x_1 = \frac{\cos \epsilon \sin m}{\sin R x_1 z};$$

$$\therefore \sin Ox_1 \sin Rx_1 = \frac{\cos^2\epsilon \sin^2 m}{\sin Ox_1 z \sin Rx_1 z},$$

and $\sin Ox_1 z \sin Rx_1 z = - \sin Ox_1 z \sin (2\mu + Ox_1 z)$

$$= - \sin^2 Ox_1 z \left(\sin 2\mu \frac{\cot m \sin\psi - \cos\psi \sin\epsilon}{\cos\epsilon} + \cos 2\mu\right)$$

$$= - \sin 2\mu \frac{\left(\dfrac{\cot m \sin\psi - \cos\psi \sin\epsilon}{\cos\epsilon} - \dfrac{\cos^2\epsilon - \sin^2\epsilon \cos^2\psi + \cot^2 m \sin^2\psi}{\sin 2\epsilon \cos\psi}\right)}{1 + \left(\dfrac{\cot m \sin\psi - \sin\epsilon \cos\psi}{\cos\epsilon}\right)^2}$$

$$= + \frac{\sin 2\mu \cos\epsilon}{2 \sin\epsilon \cos\psi};$$

$$\therefore \sin Ox_1 \sin Rx_1 = \frac{\sin 2\epsilon \sin^2 m \cos\psi}{\sin 2\mu}$$

$$= \frac{a^2 - b^2}{a^2 - c^2} \frac{\sin 2\epsilon \cos\psi}{\sin 2\mu}$$

$$= \frac{v'^2 - v^2}{a^2 - c^2};$$

$$\therefore v'^2 - v^2 = (a^2 - c^2) \sin Ox_1 \sin Rx_1.$$

24. From M. Fresnel's construction it appears that the sum of the squares of the velocities of the two waves perpendicular to their front, and travelling in the same line (we are speaking of vibrations not of the motion of the rays conveyed by them) is (*Ency. Met.* Light, p. 544.)

$$\frac{a^2 + b^2 + m^2(b^2 + c^2) + n^2(a^2 + c^2)}{1 + m^2 + n^2}$$

where $z = mx + ny$ is the equation to the plane in which the vibrations take place, so that

$$+ \frac{m}{\sqrt{1 + m^2 + n^2}} = \sin\epsilon \sin\psi$$

$$+ \frac{n}{\sqrt{1 + m^2 + n^2}} = - \cos\epsilon \sin\psi$$

$$\frac{1}{\sqrt{1 + m^2 + n^2}} = \cos\psi,$$

and the sum of the squares of the velocities is

$$(a^2+b^2)\cos^2\psi + (a^2+c^2)\cos^2\epsilon\sin^2\psi + (b^2+c^2)\sin^2\epsilon\sin^2\psi$$
$$= a^2(\cos^2\psi + \cos^2\epsilon\sin^2\psi) + b^2(\cos^2\psi + \sin^2\epsilon\sin^2\psi) + c^2\sin^2\psi$$
$$= a^2(\cos^2\epsilon + \sin^2\epsilon\cos^2\psi) + b^2(\sin^2\epsilon + \cos^2\epsilon\cos^2\psi) + c^2\sin^2\psi$$

which is the same expression as we deduced for the sum of the squares of the velocities in (20).

25. Having then results coinciding with those of M. Fresnel, I shall pursue the subject no further. The formula which I have given for the value of the difference of the squares of the velocities of the two vibrations, is a very elegant and useful one. Whether it had ever before been deduced from theory, or not, I cannot tell. Mr Herschel states that it has long been established by experiment. The only analogous one which I can find, is that of M. Fresnel, viz., "that the difference of the squares of the reciprocals of the velocities of the two rays is proportional to the product of the sines of the angles which their common direction makes with the optic axes of the crystal." M. Fresnel also defines "optic axes" as those in which the *rays* travel when their velocity is the same for both. I have preferred to retain the name of optic axes to those directions which are normals to the directions of waves which move with a common velocity perpendicular to their own front; and it is very evident that these are the optic axes of experiment.

I wish to add that, as far as I am aware, M. Fresnel's law, beautiful as it undoubtedly is, appears to me utterly incapable of being tested by experiment; so far as I can see, it requires a connexion with the index of refraction in order to apply experiment at all, and the index of refraction depends only on the *wave*. It must however be observed, that the older experimenters always use the word *ray*, but the slightest examination is sufficient to convince us that they mean, what we now call wave.

It is not then a matter of surprize, that modern writers should in some cases confound the two; and this particular formula has been differently enunciated by different writers.

Thus Mr M'Cullagh, in the Transactions of the Royal Irish Academy, enunciates Fresnel's proposition as follows:

"The difference of the squares of the reciprocals of the velocities of the two rays having a common direction in the crystal, is proportional to the product of the sines of the angles which that direction makes with the optic axes."

Mr Airy gives the following:

"The difference between the reciprocals of the squares of the velocities of the two rays is proportional to the product of the sines of the two angles made by the front of the wave with the two circular sections, or to the product of the sines of the angles made by the normal to the front with the two optic axes." The latter is, I have no doubt, incorrect.

Having then, in some instances, contradictory statements of the nature of the theory, I have, probably, here misled in some points. With respect to the mechanical part to which I object, all statements, which I have seen, coincide.

26. I refrain from making any extended application of the subject, but will only trouble you with one case, which I adduce on account of its great importance. The explanation of the lemniscates in biaxal crystals depends on the difference of the retardation of two vibrations which have a common normal to their front. The usual method of proceeding has been to find the retardation for uniaxal crystals, and from the circumstance of the retardation in that case being proportional to the difference of the squares of the velocities of the two *waves*, the same is true of the difference of the squares of the velocities of the two *rays* in biaxal crystals, and then, finally, to assume the difference of the reciprocals of the squares of the velocities of the rays to vary as the product of the sines of the angle made by the normal with the optic

axes. The expression which I have given above is remarkably elegant, and is evidently the one on which the differences of the refractions of the different rays depends, whilst M. Fresnel's formula is not susceptible, as far as I know, of any application, except in those numerous instances where, being incorrectly adopted, it still gives a result nearly correct. This arises from the difference of the reciprocals of the squares of the velocities of the *rays* varying as the difference of the squares of the velocities of the *vibrations* parallel to their fronts.

27. To apply the formula to the particular case in question:

Let T be the thickness of a plate of a biaxal crystal cut perpendicularly to the greatest or least axis of elasticity;

V the velocity in air;

v, v' those of the vibrations perpendicular to their front, the incidence being nearly perpendicular;

ϕ, ϕ' the angles which the perpendiculars to the fronts of our waves before and after incidence make with the normal;

Then the retardation of this wave may be easily shewn (Airy's *Tracts*, p. 376.) to equal

$$\frac{T}{\cos\phi'}\left\{\frac{V}{v} - \cos\phi\cos\phi' - \sin\phi\sin\phi'\right\}$$

$$= \frac{T}{\cos\phi'}\left\{\frac{V}{v} - \cos\phi'\sqrt{1 - \frac{V^2}{v^2}\sin^2\phi'} - \sin^2\phi'\frac{V}{v}\right\}$$

$$= \frac{T}{\cos\phi'}\left\{\frac{V}{v}\cos^2\phi' - \frac{\cos\phi'}{v}\sqrt{v^2 - V^2\sin^2\phi'}\right\}$$

$$= T\left\{\frac{V}{v}\cos\phi' - \frac{1}{v}\sqrt{v^2 - V^2\sin^2\phi'}\right\}$$

$$= T\left\{\frac{V}{v}\cos\phi' - 1\right\} \text{ nearly.}$$

Similarly the retardation for the other ray is

$$T\left\{\frac{V}{v'}\cos\phi' - 1\right\},$$

and the difference of these retardations is

$$TV\cos\phi'\left(\frac{1}{v} - \frac{1}{v'}\right)$$

$$= TV\cos\phi'\,\frac{v'^2 - v^2}{v'v\,(v'+v)}$$

$$= \frac{TV\cos\phi'}{v'v\,(v'+v)}\,.\,(a^2 - c^2)\sin Ox_1 \sin Rx_1 \quad (23).$$

Now $(a^2 - c^2)$ is a small quantity, hence, if the square of such a quantity be omitted,

$$v'v\,(v'+v) = 2a^3;$$

and difference of retardation becomes

$$\frac{TV}{2a^3}\cos\phi'\,(a^2 - c^2)\sin Ox_1 \sin Rx_1.$$

28. In conclusion, the principal point in which the present view of the subject differs from those which have gone before, is in the fact of the non-existence of a normal vibratory force, or, in other words, that there is no resolved part of the force perpendicular to the front of the wave. The greatest utility of this view of the subject will appear when we shall consider the effect which takes place at the confines of two media, for it is evident that in resolving our vibrations at the point of change, we shall be obliged to consider the whole resolved part as lying in the plane of the front of the new wave. The complete discussion of this point, however, involves considerable difficulty, and I must delay it for the present, hoping shortly to make it the subject of a separate communication.

XV. *Supplement to the Memoir on the Transmission of Light in Crystallized Media.* *By* PHILIP KELLAND, B.A. *Fellow and Tutor of Queens' College.*

[Read *May* 1, 1837.]

(BIOT's LAW.)

1. IN the latter part of this Memoir, I make an application of the formula which I had before deduced, viz. "that the difference of the squares of the velocities of two waves having a common normal, in the direction of that normal, is proportional to the product of the sines of the angles made by it with the two optic axes of the crystal."

As my object was merely to shew that it was a Theorem *wanted* for such considerations, I adopted all the approximations which I found in common use. On examining the subject more attentively, I find that some of them if allowable are superfluous, and that the same result is attained, by proceeding to work in a *direct* manner. I am not, it is true, quite sure that the authors of the investigations considered them as approximations; they make no remark to that effect, but assume at once that the ray and wave coincide.

2. In order to find the appearance presented on the transmission of polarized light through a plate of biaxal crystal, the most important point to be determined is, the difference of retardation of the two *waves*.

The want of a proposition, such as that which appears in (23), seems to have driven writers to adopt an approximative process of the following nature.

First, a ray is supposed nearly to coincide with a wave, and the theorem that the difference of the squares of the reciprocals of the velocities of the two *rays* is proportional to the product of the sines of the angles which their common *direction* makes with the optic axes suggested (apparently) that the same Theorem approximately held when *wave* was put for ray, and normal to front for direction, &c., and thus a Theorem which is in no way connected with the result, does from the circumstance of its close analogy to the true one, give correct results, or nearly so.

3. Let BC be the direction of one ray in the crystal; BE a normal to its front; CG perpendicular to BA; ϕ the angle of incidence; ϕ' the angle which BE makes with the normal to the plane surface of the crystal; BC makes θ with the same; T the thickness of the plate. (Note at end.)

Then if v be the velocity before incidence, v' the velocity perpendicular to the front after refraction,

$$\frac{v'}{v} = \frac{\sin\phi'}{\sin\phi},$$

and the ray has moved perpendicularly to its former front through a space

$$= BG = CB\cos(\phi - \theta)$$

$$= \frac{T}{\cos\theta}\cos(\phi - \theta),$$

since $\dfrac{T}{\cos\theta} = BC$; also $\dfrac{v'}{\cos(\theta - \phi')}$ is the velocity along BC;

$$\therefore \text{ time of describing } BC = \frac{T\cos(\theta - \phi')}{v'\cos\theta};$$

therefore the space which the wave would describe in the same time in air, is

$$\frac{Tv}{v'\cos\theta}\cos(\theta - \phi'), \text{ and the retardation is}$$

$$\frac{T}{\cos\theta}\left\{\frac{v}{v'}\{\cos(\theta - v')\} - \cos(\phi - \theta)\right\}$$

$$= \frac{T}{\cos\theta}\left\{\frac{v}{v'}(\cos\theta\cos\phi' + \sin\theta\sin\phi') - \cos\phi\cos\theta - \sin\phi\sin\theta\right\}$$

$$= \frac{T}{\cos\theta}\left\{\frac{v}{v'}\cos\theta\cos\phi' - \cos\phi\cos\theta\right\}$$

$$= T\left\{\frac{v}{v'}\cos\phi' - \cos\phi\right\}$$

$$= T\left\{\frac{v}{v'}\sqrt{1-\sin^2\phi'} - \cos\phi\right\}$$

$$= T\left\{\frac{v}{v'}\sqrt{1-\frac{v'^2}{v^2}\sin^2\phi} - \cos\phi\right\};$$

and if v_1 be the velocity of the other wave, its retardation is

$$T\left\{\frac{v}{v_1}\sqrt{1-\frac{v_1^2}{v^2}\sin^2\phi} - \cos\phi\right\}$$

the angle of emergence being supposed the same for both.

Hence the difference of retardation is

$$T\left\{\frac{v}{v_1} - \frac{v}{v'}\right\} = Tv\left\{\frac{1}{v_1} - \frac{1}{v'}\right\} \text{ nearly.}$$

4. This hypothesis that the two waves are moving parallel to each other at emergence, is clearly not compatible with the hypothesis that they have the same normal within the crystal.

If v_2 be the velocity of the wave which has a common normal with that whose velocity is v_1, we have

$$\text{retardation of this} = T\left\{\frac{v}{v_2}\cos\phi' - \cos\phi\right\}$$

$$= T\left\{\frac{v}{v_2}\cos\phi' - \sqrt{1-\frac{v^2}{v_2}\sin^2\phi'}\right\},$$

and the difference of retardation is

$$= Tv \cos\phi' \left\{\frac{1}{v_2} - \frac{1}{v'}\right\}$$

$$= Tv \sqrt{1 - \sin^2\phi'} \left\{\frac{1}{v_2} - \frac{1}{v'}\right\}$$

$$= Tv \left\{\frac{1}{v_2} - \frac{1}{v'}\right\} \text{ if } \phi' \text{ be very small;}$$

hence the hypothesis that the angle of incidence is small, reduces this case to the same form as the former, and we may in such circumstances consider the difference of the retardation as proportional to the difference between the two refractive indices.

5. In the applications of this formula, we must introduce the relations which are given by the constitution of the crystal determined by the passage of light through it. Such relations must, I conceive, depend on the refractive energies of the crystal in different directions.

Now the refractive energy has undoubtedly no connexion whatever with the velocities of transmission of the rays, since these velocities are merely nominal ones; that is, they are not estimated in the direction in which the effect is transmitted. Indeed, I do not suppose we have any notion of these velocities independent of theory, whilst the velocity of the wave is a physical motion, apart from the idea which is suggested by the expression.

I have been under the necessity of giving the term *radial* to M. Fresnel's axes, since they are not at all the same thing as the optic axes. M. Fresnel himself remarks, that "although the difference between them is very slight in almost all crystals, there are some where it becomes more sensible, and in which we must not confound the two."

6. We are concerned only with waves which have a common direction in air, and must consequently assume that the difference of the velocities of the two corresponding refracted waves, is very nearly the same as the difference of the velocities of two waves which travel

in the direction of one of them, omitting consequently the variation of velocity of one wave due to difference of its velocity from that of the other, or in other words, omitting the variation of the difference of the velocities, compared with that difference itself which is perfectly allowable.

Let m', n' be the angles made by the direction, which we consider common to the two, with the optic axes.

α, β the angles made in air by the incident ray, with rays, which, when they enter the crystal, move in rays; the normals to which are the axes.

$$\text{Then retardation} = Tv\left(\frac{1}{v_1} - \frac{1}{v'}\right)$$

$$= Tv\frac{v' - v_1}{v'v_1}$$

$$= Tv\frac{v'^2 - v_1^2}{v'v_1(v' + v_1)}$$

$$= Tv\frac{(a^2 - c^2)\sin m' \sin n'}{v'v_1(v' + v_1)} \quad (23).$$

And $v' + v_1 = \dfrac{2v}{\mu}$, μ being the mean index of refraction;

$$v'v_1 = \frac{v^2}{\mu^2},$$

so that $\sin\phi = \mu\sin\phi'$ as a factor of small terms.

$Pz = \phi'$, (see note at end)

$Qz = \phi$,

$Rz = m$,

$$\sin Sz = \frac{v}{b}\sin Rz = \frac{v}{b}\sin m = \mu\sin m,$$

$$Tz = Sz, \quad PzO = \theta,$$

O and R the optic axes.

$$\text{Then } \sin^2 m' = 1 - \cos^2 m'$$

$$= 1 - (\cos\phi' \cos m + \sin\phi' \sin m \cos\theta)^2$$

$$\sin^2 n' = 1 - (\cos\phi' \cos m - \sin\phi' \sin m \cos\theta)^2;$$

$$\therefore \sin^2 m' \sin^2 n' = (1 - \cos^2\phi' \cos^2 m - \sin^2\phi' \sin^2 m \sin^2\theta)^2$$

$$- 4 \sin^2\phi' \cos^2\phi' \sin^2 m \cos^2 m \cos^2\theta$$

$$= \{\sin^2\phi' + \sin^2 m - \sin^2\phi' \sin^2 m (1 + \sin^2\theta)\}^2$$

$$- 4 \sin^2\phi' \cos^2\phi' \sin^2 m \cos^2 m \cos^2\theta;$$

$$\sin^2\alpha \sin^2\beta = \{\sin^2\phi + \sin^2 Tz - \sin^2\phi \sin^2 Tz (1 + \sin^2\theta)\}^2$$

$$- 4 \sin^2\phi \cos^2\phi \sin^2 Tz \cos^2 Tz \cos^2\theta$$

$$= \mu^4 \{[\sin^2\phi' + \sin^2 m - \mu^2 \sin^2\phi' \sin^2 m (1 + \sin^2\theta)]^2$$

$$- 4 \sin^2\phi' \sin^2 m \cos^2\phi \cos^2 Tz \cos^2\theta\}.$$

Now if ϕ' and m be both small, this expression becomes

$$\sin^2\alpha \sin^2\beta = \mu^4 (\sin^4 m + \sin^4\phi' - 2 \sin^2 m \sin^2\phi' \cos 2\theta),$$

$$\text{and } \sin^2 m' \sin^2 n' = (\sin^4 m + \sin^4\phi' - 2 \sin^2 m \sin^2\phi' \cos 2\theta);$$

$$\therefore \sin^2\alpha \sin^2\beta = \mu^4 \sin^2 m' \sin^2 n'.$$

If m be very small compared with ϕ'

$$\sin^2\alpha \sin^2\beta = \mu^4 \sin^4\phi'$$

$$= \mu^4 \sin^2 m' \sin^2 n'.$$

If ϕ' be very small compared with m,

$$\sin^2\alpha \sin^2\beta = \mu^4 \sin^4 m = \mu^4 \sin^2 m' \sin^2 n'.$$

In all cases therefore, provided one of two, either ϕ' or m be small, or if they are both small, we have

$$\sin\alpha \sin\beta = \mu^2 \sin m' \sin n';$$

and by substitution we obtain, difference of retardation

$$= \frac{T}{2v}(a^2 - c^2)\frac{\mu^2 \sin m' \sin n'}{v_1}$$

$$= \frac{T(a^2 - c^2)\sin \alpha \sin \beta}{2vb}.$$

7. This formula for the retardation on which depends the explanation of the coloured lemniscates, is true it would appear even if the angle between the optic axes were considerable. I do not know whether this be true experimentally or not.

I forbear from proceeding further in the development of the lemniscates, as that has been already effected with a formula coincident with my own, or nearly so. As to the experimental verification of formulæ such as these, they involve so much of calculation that there is considerable difficulty in being able to form a correct judgment on their coincidence; as far as I am aware, Sir J. Herschel and Sir D. Brewster calculate the angles between the optic axes (see *Phil. Trans.* 1820.) from assuming the law of refraction to be the Snellian law; this is evidently not treating of *rays* but of waves; and consequently any law of velocity which would be by this means established, would be one relating to the velocity of a wave; and in the same manner, the directions within the crystal can be no other than the normals.

So that M. Biot's law when translated into the language of the undulatory theory, is precisely that which I have enunciated above. Indeed as far as I can collect, Sir J. Herschel appears to state it so in one place. (*Ency. Met.* 1812.)

Many writers make M. Fresnel's beautiful law of the reciprocals of the squares of the velocities of the rays, to be the same thing as this of Biot and Brewster. The cause appears to lie in the confusion of language which naturally has been fallen into by different writers, the one denoting by ray what the other denotes by wave, and so on.

I have dwelt a long time on this subject, from a wish rather to obtain information on the subject than to communicate it. It is no slight matter of astonishment to me, that a law so elegant as is that which we have been discussing, and one too, the necessity of which (or something analogous) must have been felt at every step which was taken in the development of the Biaxal Theory, has never been mentioned in connexion with this theory, in any writers that I have seen, whilst others apply M. Fresnel's law in its stead without stating their reasons. I do not presume to suppose that it had not been established, my object will be fully attained if I shall have succeeded in exhibiting its importance, and in obtaining for it its proper place in the theory which has been usurped by the no less elegant Theorem of M. Fresnel.

NOTE. The figure referred to in (18) is constructed by drawing three rectangular axes Px, Py, Pz, and Px_1, Py_1 inclined to these; PN being the line in which $y_1 z_1$ intersects xy.

O, R are two points in xz equally distant on opposite sides from Pz.

The figure of (4) in Supplement, is a broken line AB, BC, CD; as an incident, proceeding and emergent ray of common optics.

AB is produced to G

That of (6) is a series of spherical triangles, $SRzOT$ being a large arc, $Sz=zT$, $Rz=zO$.

TPQ, PR, PO, QS, QT all arcs.

XVI. *A Statistical Report of Addenbrooke's Hospital, for the Year* 1836. *By* Henry J. H. Bond, M.D.

[Read *March* 13, 1837.]

IN-PATIENTS, 1836.

Number of Beds:

66 in the general Wards.
12 in the Fever Wards.
Total......78

Number of Patients in the Hospital:

Maximum	78
Minimum	49
Mean	67

Length of Time Patients remained in the Hospital:

Mean duration $35\frac{1}{2}$ days.

Admissions from January 1st, 1836, to January 1st, 1837:

Male Patients	337
Female	260
Total ...	597

Previous Residence of the Patients:

226	(38 per cent.)	in the Town of Cambridge.
371	(62 per cent.)	in the Country, principally in Cambridgeshire and the Isle of Ely.

Description of In-Patients:

Male.

180	(53 per cent.)	Labourers, chiefly agricultural.
20	....	Boys of ten years of age or under.
8	....	Grooms.
7	....	Tailors.
7	....	Watermen.
6	....	Carpenters.
6	....	Servants.
6	....	Tramps.
97	....	Of forty-five different occupations, four being the largest number belonging to any one of them.
337		

Female.

100	(38 per cent.)	Servants.
90	....	Women occupied at home with the care of their families, or children above ten years of age, residing likewise at home.
22	....	Girls ten years of age or under, living at home.
3	....	Dressmakers.
3	....	Laundresses.
42	....	Occupations of, not registered.
260		

Ages of In-Patients (597).

	Male.	*Female.*	*Total.*	
	6	6	12	5 years or under.
	16	19	35	from 5 to 10 inclusive.
	22	32	54	... 10 ... 15 ......
	44	83*	127*	... 15 ... 20 ......
	39	47	86	... 20 ... 25 ......
	40	22	62	... 25 ... 30 ......
	29	9	38	... 30 ... 35 ......
	34	14	48	... 35 ... 40 ......
	14	2	16	... 40 ... 45 ......
	25	12	37	... 45 ... 50 ......
	14	5	19	... 50 ... 55 ......
	22	2	24	... 55 ... 60 ......
	8	3	11	... 60 ... 65 ......
	11	3	14	... 65 ... 70 ......
	2	0	2	... 70 ... 75 ......
	2	1	3	... 75 ... 80 ......
	328	260	588	
Ages not registered	9		9	
	337		597	

Results of the 597 In-Patient cases:

348	Recovered.
57	Benefitted.
17	Discharged at their own request.
2	.... for irregular conduct
19	.... as incurable†.
15	Died.
81	Made Out-Patients.
539	Discharged from the Hospital in 1836.
58	Remaining in the house at the end of the year.
597	

* 83 gives 31.9 per cent. on the whole number of female In-Patients (260).
127 ..... 21.5 .... In-Patients (588).

† Principally Phthisical.

Mean stay in the house of the 539 that were discharged was $35\frac{1}{2}$ days.

.... 348 discharged as recovered $33\frac{1}{2}$ ...

	PER CENT.		PER CENT.		WEEK.
Of the 539 total discharged ..	6.7	Of the 348 discharged as recovered, ...	4.6	were discharged in	1
....	16.6	....	19.9	....	2
....	15.3	....	16.4	....	3
....	16.2	....	15.8	....	4
....	11.7	....	11.4	....	5
....	6.7	....	7.9	....	6
....	5.7	....	6.7	....	7
....	4.6	....	4.1	....	8
....	4.2	....	4.1	....	9
....	2.8	....	2.3	....	10
....	8.8	....	6.3	....	from 10 to 42
	99.3		99.5		

The recoveries (348) were 64.5 per cent. on the whole number of those discharged or made Out-Patients (539).

The deaths (15) .... 2.7 (or 1 in 36 nearly) ... ...

Diseases, Ages, &c. of the 15 fatal cases.

	Males.	Females.	Total.	From 10 to 20 inclusive.	20 to 30.	30 to 40.	40 to 50.	50 to 60.	60 to 70.	January.	February.	March.	April.	May.	June.	July.	August.	September.	October.	November.	December.
Fractures.........	3	...	3	1	...	1	1	...	...	...	...	...	1	1	...	...	...	...	...	1	
Phthisis..........	1	2	3	2	...	1	...	...	...	...	...	...	...	...	...	1	...	...	1	–	1
Fever.............	2	1	3	2	...	...	...	1	...	...	...	...	...	...	1	...	...	...	1	1	
Pneumonia	2	...	2	...	1	...	...	1	...	...	...	1	...	...	...	...	...	...	1		
Apoplexy.........	1	...	1	...	...	1	...	...	...	...	...	...	...	...	...	...	...	...	...	1	
Erysipelas........	1	...	1	...	...	...	...	...	1	...	...	...	...	...	...	...	1				
Disease of Kidneys	1	...	1	...	...	...	...	...	1	...	...	...	...	...	...	...	1				
Syphilis..........	1	...	...	1	...	...	...	...	...	...	...	1									
Males.......	...	...	...	3	1	3	1	2	2	...	...	2	1	1	1	...	2	...	2	2	1
Females.....	...	...	...	3	...	...	...	...	...	...	...	...	...	...	...	1	...	...	1	1	
	12	3	15	6	1	3	1	2	2			2	1	1	1	1	2		3	3	1

Number of Operations:

Lithotomy	6
Amputations	8
Hernia	1
Artificial Pupil	1
Excision of large Tumor in Thigh	1
	17 (besides minor Operations.)

OUT-PATIENTS, 1836.

Admissions from January 1st, 1836, to January 1st, 1837:

Males	341
Females	516
Total	857

Previous Residence of Patients:

486 (57 in 100) in the Town of Cambridge.
365 (43 ....) in the Country, principally Cambridgeshire and Isle of Ely.
6 Residence not registered.
857

Ages of the 857 Out-Patients:

23 in 100 of the females were from 15 to 20 years of age: a larger proportion than that of any other quinquennial period.

15 in 100 of the males were from 20 to 25 years of age; a larger proportion than that of any other quinquennial period.

30 in 100 of the total 857 Out-Patients were from 15 to 25 years of age; a larger proportion than that of any other decennial period.

Results of the 857 Out-Patient cases:

374	Recovered.
44	Benefitted.
5	Discharged at their own request.
1	.... for irregular conduct.
8	.... as incurable*.
18	Died.
56	Made In-Patients.
231†	Discharged for having discontinued attendance.
737	Total of Patients discharged from the Out-Patients' Register.
120	Remaining under treatment as Out-Patients at the end of the year.
857	

The recoveries (374) amount to rather more than half the whole number of those discharged or made In-Patients.

The 18 registered deaths amount to 2.4 per cent. on the discharges, or 1 in 41.

Report of the In-Patients and Out-Patients combined.

Of the 597 In-Patients, 52 were previously Out-Patients, leaving	545
...... 857 Out-Patients, 81 .... In-Patients, ...	777
Total number of cases treated in 1836	1322

* These cases were chiefly phthisical.

† Of the 231 discharged from the registers for non-attendance, from the nature of the entries respecting them, two thirds, it seems, were at the time of their last attendance, either recovering, or not affected with complaints of a fatal or grave nature; but either neglected to present themselves to be discharged on their recovery, or for various reasons discontinued attendance. The remaining third were patients labouring apparently under disorders of a fatal nature (a large proportion being phthisical); and who, probably from the advanced stage of the disorders, were unable longer to come to the Hospital, or at the time of their discharge from the register had already died, but whose deaths were never reported to the medical officers.

Monthly admissions*—of the 1322 there were admitted:

January......	127	May.........	130	September.....	79
February....	144	June.........	93	October	98
March	128	July.........	105	November	99
April.........	126	August.......	112	December	81

Of the 1322, { 613 Male Patients
709 Female ......

656 came from the Town of Cambridge.

666 came from the Country, principally Cambridgeshire and the Isle of Ely.

Description of Patients:

Male.

307	(50 per cent.)	Labourers, chiefly agricultural.
78	....	Boys of ten years of age or under.
15	....	Shoemakers.
14	....	Tramps and Hawkers.
11	....	Servants.
11	....	Watermen.
9	....	Tailors.
9	....	Carpenters.
7	....	Printers.
6	....	Brewer's men.
6	....	Grooms.
5	....	Bricklayers.
4	....	Dyers.
4	....	Butchers.
127	....	Of fifty-eight different occupations, three being the greatest number belonging to any one of them.
613		

* The number of admissions in any one month is not determined solely by the greater or less prevalence of illness, but by other circumstances also, as the facility in obtaining recommendations for admissions, which is not the same at all seasons of the year, and the state of the weather being favourable or unfavourable for the conveyance of patients to the Hospital. The number of admissions is always lower in September, in consequence of the subscribers having then, for the most part, exhausted their recommendations; an unfortunate circumstance, since at this season so much illness usually prevails. In the present year, the admission of

Female.

245	(above one-third)	Women occupied at home with the care of their families, or children above ten years of age, living at home.
239	(one-third)	Servants.
80	....	Girls of ten years of age or under
14	....	Laundresses.
14	....	Sempstresses.
7	....	Of five different occupations.
110*	....	Occupations not registered.
709		

Ages of the In-Patients and Out-Patients combined (1322):

Male (598).		*Female* (713)†.		*Total.*		
44 or	7.35 per ct.	38 or	5.32 per ct.	82 or	6.25 per ct.	5 years or under.
36 ...	6.02	40 ...	5.61	76 ...	5.79	from 5 to 10 inclusive.
39 ...	6.52	70 ...	9.81	109 ...	7.13	10 ... 15
63 ...	10.53	186 ...	25.96	249 ...	18.99	15 ... 20
64 ...	10.60	116 ...	16.26	180 ...	13.72	20 ... 25
67 ...	11.20	59 ...	8.27	126 ...	9.61	25 ... 30
54 ...	9.03	42 ...	5.89	96 ...	7.32	30 ... 35
62 ...	10.68	46 ...	6.45	108 ...	8.23	35 ... 40
35 ...	5.85	25 ...	3.50	60 ...	4.57	40 ... 45
37 ...	6.18	45 ...	6.31	82 ...	6.25	45 ... 50
25 ...	4.18	17 ...	2.38	42 ...	3.20	50 ... 55
30 ...	5.01	8 ...	1.12	38 ...	2.89	55 ... 60
22 ...	3.67	9 ...	1.26	31 ...	2.36	60 ... 65
13 ...	2.17	7 ...	0.98	20 ...	1.52	65 ... 70
3 ...	0.50	2 ...	0.28	5 ...	0.38	70 ... 75
4 ...	0.66	3 ...	0.42	7 ...	0.53	75 ... 80 and upwards.
				1311		
				11	Ages not registered.	
				1322		

of so small a number in December was caused by the state of the weather at the end of the year, being such as to prevent the conveyance of patients from the country.

* Most of these probably should be added to those at the head of the list.

† The ages of four females are here accidentally included which are not included in the previous statements.

Results of the total 1322 cases admitted in 1836:

722	Recovered.
101	Benefitted.
22	Discharged at their own request.
3	.... for irregular conduct.
27	.... as incurable.
33	Died.
231	Discharged for non-attendance.
1139	Total number of Patients discharged.
183*	Remaining under treatment as In or Out-Patients at the end of the year.
1322	

The (722) recoveries were 63.38 per cent. on the total number discharged.
The (33) deaths .... 2.89 (or 1 in 34)† ....

Diseases, Ages, &c. of the 33 Fatal Cases.

	Males.	Females.	Total.	10 or under.	From 10 to 20 inclusive.	20 to 30.	30 to 40.	40 to 50.	50 to 60.	60 to 70.	January.	February.	March.	April.	May.	June.	July.	August.	September.	October.	November.	December.
Phthisis	3	8	11	...	3	3	2	3	...	...	...	...	...	...	3	...	2	...	2	2	1	1
Continued Fever ..	2	3	5	...	2	2	...	...	1	...	...	...	...	...	1	1	...	...	...	1	2	
Fractures	3	...	3	...	1	...	1	1	...	...	...	...	...	1	1	...	...	...	...	...	1	
Bronchitis	2	...	2	1	...	...	...	...	1	...	...	...	...	...	1	1						
Pneumonia	2	...	2	...	...	1	...	...	1	...	...	...	1	...	...	...	...	...	...	1		
Pertussis	...	1	1	1	...	...	...	...	...	...	...	...	...	...	...	1						
Puerperal Fever ..	...	1	1	...	1	...	...	...	...	...	...	...	...	...	...	...	1					
Disease of Heart ..	1	...	1	...	...	...	...	1	...	...	...	...	...	1								
Diarrhœa	1	...	1	...	...	...	...	...	...	1	...	...	...	...	...	...	...	...	...	1		
Apoplexy	1	...	1	...	...	...	1	...	...	...	...	...	...	...	...	...	...	...	...	...	1	
Cerebral Disease ..	1	...	1	...	...	...	...	1	...	...	...	...	...	...	...	...	...	...	...	1		
Erysipelas	1	...	1	...	...	...	...	...	...	1	...	...	...	...	...	...	...	1				
Rachitis	...	1	1	1	...	...	...	...	...	...	...	...	...	1								
Disease of Kidneys	1	...	1	...	...	...	...	...	...	1	...	...	...	...	...	...	...	1				
Syphilis	1	...	1	...	1	...	...	...	...	...	...	...	1									
Males	...	...	...	1	3	2	3	4	3	3	...	...	2	2	3	2	...	2	...	5	2	1
Females	...	...	...	2	5	4	1	2	...	...	...	...	...	1	3	1	3	...	2	1	3	
	19	14	33	3	8	6	4	6	3	3	...	...	2	3	6	3	3	2	2	6	5	1

* There is an error of 5 patients; 178 only appearing on the registers as remaining under treatment.

† Among those discharged for non-attendance or at their own request, it seemed, from the nature of the entries respecting them in the registers, that 107 were at the time of their discharge

The annexed table contains a list of the diseases or accidents of Out and In-Patients combined, and represents the frequency with which each of them occurred.

An attempt has been made to arrange them in a somewhat physiological order, for convenience of reference; and also that, with some approximation to truth, the comparative frequency of certain *classes* of diseases or accidents, or the relative frequency with which individual *systems* or *organs* were the seat of disorder, might appear.

It was conceived that it would render the table more serviceable, if the *proportion* to the annual total of cases were given, as well as the precise number of the cases of each class of disease, and of the most prevailing of the individual diseases.

The table likewise represents the *monthly* distribution of the cases of each disease. It was here still more necessary, when the cases of any disease or class of disease were sufficiently numerous to make them an object of analysis, to give their proportion to the amount of cases of *all kinds* admitted in each month respectively, as well as their absolute number; since, as has already been stated, the *number* of admissions in each month is not an accurate criterion of the prevalence of disease.

It is not, however, pretended that any general conclusions respecting the prevalence of diseases according to seasons can be drawn from the present table, which is constructed with a view chiefly to its connection with reports of ensuing years, by which some useful inferences may eventually, it is thought, be derived. At least, the remarks that are made upon it at present are intended only as provisional: indeed, had the numbers been much higher, the report of a single year, from January to January, would still have been insufficient for the purpose; since the year should be otherwise divided with the view of

discharge labouring under disorders of a fatal description (a large proportion being phthisical); these 107, added to 33 registered deaths and the 27 grave cases discharged as incurable, give an amount of 167 cases out of the 1139 patients discharged, which appeared likely to terminate fatally, i. e. 14.66 per cent., or 1 in 6.82.

SPITAL FOR 1836.

	...ne.	July.	August.	September.	October.	November.	December.
		...	...	...	...		
	...	2 ...	...	...	...	1 ...	3
Eye 99...(7,59)		...	...	1			
		1 ...	...	...	...	1	
		...	1 ...	...	2		
		1 ...	...	1			
		...	...	...	...	1	
	...	...	2				
	7,78)	9...(8,91)	8...(7,27)	9...(11,11)	7...(7,16)	9...(9,18)	7...(9,72)
Ear 4		1 ...	...	1 ...	1 ...	...	1
Nose 1		1					
	3,26)	2...(1,98)	2...(1,81)	2...(2,46)	2...(2,04)	3...(3,06)	3...(4,16)
		...	...	...	1		
xternal Local		2 ...	...	...	1		
Diseases		1 ...	3 ...	...	1 ...	1 ...	1
		...	2 ...	...	1		
		...	...	1 ...	2 ...	3 ...	2
		...	...	...	...	1	
ırsæ		...	1				
		1 ...	...	...	1		
		1 ...	1 ...	...	...	2 ...	1
nts...........		...	2 ...	1 ...	1 ...	2 ...	1
		1 ...	...	...	...	1	
		1					
		1 ...	1 ...	...	1		
es...........		2 ...	1 ...	...	...	1	
	..	1 ...	...	...	1 ...	2	
	...	6 ...	8 ...	7 ...	8 ...	5 ...	2
	...	...	1 ...	...	1 ...	1	
	...	1 ...	...	...	...	...	1
	.	1 ...	1 ...	...	1		
	...	...	...	...	1		
	...	...	1 ...	1 ...	...	...	1
ıts 11		...	...	...	1		
		...	...	...	...	2	
		...	...	...	1 ...	...	1
		...	...	1 ...	1		
		...	...	...	...	...	1
		...	...	1			
TOTAL .		9 / 101	11 / 110	10 / 81	15 / 98	10 / 98	6 / 72

Dr BOND'S STATISTICAL ACCOUNT OF ADDENBROOKE'S HOSPITAL FOR 1836.

Class		Diseases	In-Patients	Total
Mouth and Fauces	13 (0,99)	Ranula		2
		Inflammation of Fauces, &c		9
		Ptyalism		2
Stomach	86 (6,60)	Dyspepsia	[illegible]	68
		[illegible]		1
		Gastritis	1	9
		Disease of Pylorus	[illegible]	3
		Hæmatemesis		5
Intestines	97 (7,44)	Diarrhœa	2	14
		Dysentery		1
		Constipation	[illegible]	30
		Hæmorrhoids	2	[illegible]
		Vermes		1
		Fistula in Ano	6	[illegible]
		Colic	2	8
		Enteritis	2	[illegible]
		Peritonitis	1	[illegible]
		Tabes mesenterica		[illegible]
		Abdominal Tumor	1	1
		Hernia (1 strangulated)	4	5
Liver	11 (0,90)	Icterus	[illegible]	[illegible]
		Disease of Liver	[illegible]	1
Spleen	1	Enlargement of Spleen	1	1
Respiratory Organs	[illegible]	Catarrh		[illegible]
		Croup	1	1
		Chronic Laryngitis	1	1
		Hooping Cough		5
		[illegible]	[illegible]	[illegible]
		Chronic Cough	[illegible]	[illegible]
		Hæmoptysis	[illegible]	[illegible]
		Phthisis	[illegible]	[illegible]
		Pneumonia	[illegible]	[illegible]
		Pleurisy		[illegible]
Thyroid Gland	[illegible]	Bronchocele	1	[illegible]
Circulating System	[illegible] (2,70)	Pericarditis		
		Disease of Heart (organic)		[illegible]
		Palpitation		
		Epistaxis		1
		Phlebitis	2	[illegible]
		Varicose Veins	[illegible]	[illegible]
		Anæmia and Chlorosis	6	[illegible]
Fevers	[illegible]	Intermittent Fever	11	[illegible]
		Continued Fever	24	30 (2,30)
Dropsy	[illegible]	Dropsy (ascites [illegible])	14	26 (1,99)
Scrofula	[illegible]	Scrofulous Glands	13	25 (1,91)
Kidneys	[illegible]	[illegible]		
		[illegible]		
		Gravel		
Bladder	[illegible]	Enuresis	1	[illegible]
		Disease of Bladder	7	12
		Calculus	[illegible]	8
External Genital Organs	[illegible]	Gonorrhœa and Syphilis	31	41
		Hydrocele	1	2
		Disease of Testis	1	1
		Recto-vaginal Fistula		1
Uterus — Deranged Menstruation	[illegible]	Amenorrhœa	12	[illegible]
		Menorrhagia		8
		Dysmenorrhœa	1	[illegible]
Uterus — Uterine Disease	[illegible]	Leucorrhœa	[illegible]	[illegible]
		Uterine Hæmorrhage		1
		Prolapsus Uteri		
		Polypus Uteri		
		Carcinoma Uteri	1	1
		Disease of Uterus (?)	1	1
Mamma	[illegible]	Abscess of Mamma	1	1
		Carcinoma of Mamma	1	1
		Pregnancy (complicated amenorrh.)		
Muscular System, &c	[illegible]	Rheumatism	7	[illegible]
		Weakness	1	[illegible]
		Debility	1	
Nervous System	[illegible]	Irritation of Spinal Chord	1	
		Neuralgia		
		Convulsions (children)		
		Chorea	1	
		Epilepsy		
		[illegible]		
		Tremors		
		Paralysis [illegible]	1	
		Apoplexy		
		Delirium tremens		
		Mental Disorders [illegible]		
		Hypochondriasis		[illegible]
		Vertigo		[illegible]
		Cephalalgia and Hemicrania		
		Cerebral Congestion	1	
Skin	[illegible]	[illegible]		
		[illegible]		1
		[illegible]		1
		Lepra and Psoriasis	2	[illegible]
		Lupus Nodosum	1	1
		Chimney-sweeper's cancer	1	1
		Syphilitic and other cutaneous diseases	1	[illegible]
		Calvities*		1
Eye	[illegible]	Ophthalmia conjunctival	11	[illegible]
		Inflammation of Tarsi	[illegible]	[illegible]
		Opacities and ulcers of Cornea	7	12
		Staphyloma		1
		Adhesions of Iris	2	
		Cataract	2	[illegible]
		Amaurosis	1	[illegible]
		Fistula lacrymalis	2	[illegible]
		Injuries to Eye		[illegible]
Ear	[illegible]	Deafness		[illegible]
Nose	[illegible]	Polypus nasi	1	1
External Local	[illegible] (7,52)	Ulcers of Leg	22	28
		Syphilitic ulcer	1	1
		Ulcers	5	12
		Abscesses	13	20
		Tumors (external)	4	6
		External Inflammations	18	[illegible]
		Sphacelus of Foot	1	1
Bursæ	[illegible]	Inflammation of Bursæ	1	[illegible]
Joints	[illegible]	Inflammation of Knee Joint	18	26
		——— Hip	13	13
		——— Ankle	1	3
		Floating Cartilage in Knee Joint		1
Bones	[illegible] (1,22)	Disease of Vertebræ	5	
		——— other Bones	8	[illegible]
		Rachitis		1
		Congenital Deformity of Foot		1
Accidents	[illegible]	Burns and Scalds	7	8
		Lacerations and Bruises	[illegible]	12
		Fractures of Femur	4	[illegible]
		——— Fibula and Tibia	4	[illegible]
		——— Humerus	4	[illegible]
		——— Radius		[illegible]
		——— Clavicle		
		——— [illegible]	1	1
		——— [illegible]		1
		——— Metatarsus		1
		——— [illegible]	1	1
		——— Cranium	2	2
		——— Ribs	1	2
		Dislocations of Humerus	1	1
		——— Elbow	1	1
		——— Carpus		[illegible]
		——— Semilunar Cartilage of Knee		1
Total	[illegible]		[illegible]	1303
		Cases not recorded in the Table		19
			[illegible]	1322

Monthly totals (January–December): 125, 144, 131, 126, 127, 92, 101, 110, 81, 98, 98, 72

* Complete Baldness occurring twice after Ague

determining the prevalence of diseases according to *seasons;* the December of one year should be associated with the January and February of the ensuing year, to constitute the winter season.

The following results, extracted from the table, have been selected, as deserving some notice.

Arrangement of the different Classes *of Cases, or of the Affections (of whatever kind) of individual* Systems *or* Organs, *according to the respective frequency of their occurrence.*

	Proportion to Total of Cases.
Diseases of the Pulmonary Organs	10.74
Accidents	8.44
Diseases of Nervous System	8.13
.... Intestines	7.74
.... Eye	7.59
External local diseases	7.52
Fevers	7.06
Diseases of Stomach	6.60
Rheumatic complaints	5.27
Diseases of Menstruation	4.68
.... the Skin	4.37
.... Joints	3.45
.... Bones	1.22
	82.81
Diseases of other organs, &c. constituting 16 other classes	17.19
	100.00

Sixteen diseases out of 143 diseases or accidents enumerated in the table, furnished, it is seen by the subjoined list, nearly the moiety of the cases.

Phthisis*	6.89
Rheumatism	5.29
Dyspepsia	5.21
Ophthalmia	4.60
Ague	4.06
Amenorrhæa	3.53
Syphilis and Gonorrhœa	3.14
Continued Fever	2.99
Bronchitis	2.37
Constipation	2.30
Dropsy	1.99
Scrofulous Glands	1.91
Opacities or Ulcers of Cornea	1.30
Paralysis	1.22
Chlorosis	1.15
Diarrhœa	1.07
	49.02

The following seem to be the results most worthy of notice (as far as a single year can funish data) in relation to the *monthly* distribution of the cases. The numbers represent the proportion per cent. on the admissions of each month.

Diseases of the Stomach	Maximum	13.58	in September.
	Minimum	2.17	.. June.
.... Intestines	Maximum	12.34	.. September.
	Minimum	4.34	.. June.
Pulmonary Diseases	Maximum	13.60	.. January and April equal.
	Minimum	5.45	.. August.
Intermittent Fever	Maximum	7.20	.. January.
	Minimum	0.00	.. August.

* The numbers annexed to Phthisis, Hæmoptysis, and Tussis, are here all included under Phthisis, as it was nearly certain that they were all phthisical cases.

Continued Fever	Maximum	5.10	..	November.
	Minimum	1.57	..	May.
Diseases of Menstruation	Maximum	7.69	..	February.
	Minimum	1.23	..	September.
Rheumatism	Maximum	8.00	..	January.
	Minimum	2.04	..	November.
Cutaneous Diseases	Maximum	6.93	..	July and December equal.
	Minimum	1.60	..	April.
Diseases of the Eye	Maximum	11.11	..	September.
	Minimum	5.20	..	April.

The following results (not derived from the table) are added from an analysis of the cases of a few of the most prevailing diseases.

Of 57 cases, entered in registers as Phthisis or Hæmoptysis;—

28 were males, i. e. 4.56 in 100 of the male patients (613) were phthisical.
29 were females, i. e. 4.09 .... female .... (709) ....

Residing previously in the Town 33, i. e. 5.04 in 100 of Town Patients (654) were phthisical.
.... Country 24, i. e. 3.60 .... Country ... (666) ....

Ages of the 57 cases.

	From 10 to 15 inclusive.	15 to 20.	20 to 25.	25 to 30.	30 to 35.	35 to 40.	40 to 45.	45 to 50.
Males	1	4	7	10	2	2	0	2
Females	0	5	1	8	7	3	3	2
TOTAL	1	9	8	18	9	5	3	4

Hence, nearly a third of the consumptive patients were between 25 and 30 years of age, and of all the patients admitted between these limits of age, one in eight was consumptive.

One half of the female consumptive patients were between 25 and 35 years of age, and one in six of all the female patients admitted between these limits of age was consumptive.

One third of the male consumptive patients were aged between 20 and 30, and of all the male patients admitted between these limits of age one in 7.69 was consumptive.

Of the 29 female consumptive patients 20 were following domestic occupations, and 7 were in service.

Of the 28 male consumptive patients 13 were Labourers, 2 Watermen (number of Watermen admitted being 11), and the remaining 13 were distributed among nearly as many different occupations.

Results of the 24 In-Patient cases of Fever;—13 males, 11 females.

16 Recoveries.
2 Convalescent from fever, but continued under treatment for chronic disorders.
1 Left the house during a relapse.
3 Died, (one during a relapse), two males and one female.
2 Remaining under treatment at close of year.
—
24

Ages.

	From 10 to 15 inclusive.	15 to 20.	20 to 25.	25 to 30.	30 to 35.	55 to 60.
Males 13	2	6	2	2		1
Females..... 11	4	2	4		1	
TOTAL ... 24	6	8	6	2	1	1

Ages of the three fatal cases respectively, 58, 18, and 14.

Of 78 cases of Ophthalmia or Ulceration and opacity of Cornea (the greater proportion being of a scrofulous nature),

30 were males, i. e. 4.89 in 100 of the male patients.
48 ... females, i. e. 6.77 .... female ......

Ages of the 78 *cases.*

	5 years or under.	5 to 10.	10 to 15.	15 to 20.	20 to 25.	25 to 30.	30 to 35.	35 to 40.	40 to 45.	45 to 50.	50 to 55.	55 to 60.	75 to 80.
Male 30	7	6	5	3	...	2	1	2	2	1	1		
Female...... 48	10	8	14	7	2	1	2	2	...	...	...	1	1
TOTAL... 78	17	14	19	10	2	3	3	4	2	1	1	1	1

Residing previously in the Town... 42, i. e. 6.57 in 100 of the Town Patients.
.... Country 36, i. e. 5.40 .... Country ......

Of 47 cases of Amenorrhæa,

25 resided previously in the Town, i. e. 3.82 in 100 of Town Patients.
22 .... Country, i. e. 3.30 ...... Country ......

Ages of the 47 cases,

7 were 15 years or under.
29 from 15 to 20, or 15.58 in 100 of the female patients admitted between these limits of age.
9 ...... 20 .. 25
1 ...... 25 .. 30
1 ...... 40 .. 45

Of 59 cases of Rheumatism,

34 were males, i. e. 5.54 in 100 of male Patients were rheumatic.
25 ... females, i. e. 3.52 ...... female ....

These cases were distributed equally between the Town and Country Patients
The range of ages extended from 10 to 75 years.

Of 75 cases of Dyspepsia,

17 were males, i. e. 2.77 in 100 of male Patients.
58 ..... females, i. e. 8.18 .... female ......
35 Resided previously in the Town, i. e. 5.35 in 100 of Town Patients.
40 .... Country, i. e. 6.00 .... Country ......

Ages of the 8 cases of Calculus (6 of which had been operated on and had recovered) were respectively 4, 5, 6*, 7*, 10, 49, 55, 58.

VACCINATION.

333 individuals were vaccinated in the course of the year 1836 at the Hospital: of these,

Cases in which the vaccination succeeded	298
Cases in which the vaccination did not succeed †	9
Cases in which small-pox supervened during the progress of the vaccination	2
Cases which were not presented after vaccination	24
	333

* Remaining for operation.

† Seven of these had previously been vaccinated.

REGISTER OF BAROMETER AND THERMOMETER, 1836.

(*Computed from the* Register *at the* Philosophical Society.)

1836.	Barometer at 8 A.M.							Thermometer.							
	Mean.	Maximum.	Minimum.	Range.	Greatest Diurnal Range.	Least Diurnal Range.	Mean Diurnal Range.	Mean.	Maximum.	Minimum.	Range.	Greatest Diurnal Range.	Least Diurnal Range.	Mean Diurnal Range.	Number of Days Minimum at 32° or under.
January.....	29.60	30.65	28.98	1.67	.67	.00	.24	37.90°	52°	19°	33°	22°	3°	9.75°	14
February...	29.80	30.42	28.94	1.48	.85	.00	.22	37.89°	52°	25°	27°	19°	3°	10.79°	15
March.......	29.52	30.29	28.85	1.44	.66	.02	.22	44.29°	68°	30°	38°	23°	6°	12.22°	2
April	29.85	30.34	29.05	1.29	.49	.00	.18	44.95°	61°	32°	29°	23°	5°	13.98°	4
May........	30.21	30.61	29.75	.86	.36	.00	.11	54.29°	79°	36°	43°	34°	12°	19.30°	0
June	29.92	30.31	29.56	.75	.34	.00	.14	62.43°	86°	46°	40°	34°	5°	18.38°	0
July........	29.98	30.35	29.59	.76	.49	.01	.15	63.77°	93°	43°	50°	36°	5°	19.00°	0
August......	30.03	30.35	29.58	.77	.36	.00	.11	60.95°	80°	45°	40°	31°	10°	18.74°	0
September...	29.88	30.35	29.35	1.00	.52	.00	.14	53.90°	71°	36°	35°	26°	7°	13.80°	0
October.....	29.79	30.44	28.88	1.56	.92	.00	.25	48.19°	65°	28°	37°	22°	5°	13.16°	3
November...	29.60	30.12	29.05	1.07	.79	.01	.26	42.00°	56°	29°	27°	23°	4°	12.25°	8
December...	29.78	30.35	29.09	1.26	.67	.01	.21	39.38°	55°	24°	31°	20°	2°	8.12°	12
For the year	29.83	30.65	28.85	1.80	.92	.00	.18	49.16°	93°	19°	74°	36°	2°	14.12°	58

TRANSACTIONS

OF THE

CAMBRIDGE

PHILOSOPHICAL SOCIETY.

VOLUME VI. PART III.

CAMBRIDGE:

PRINTED AT THE PITT PRESS;

AND SOLD BY

JOHN WILLIAM PARKER, WEST STRAND, LONDON;

J. & J. J. DEIGHTON; AND T. STEVENSON,

CAMBRIDGE.

M.DCCC.XXXVIII.

XVII. *On the Intensity of Light in the neighbourhood of a Caustic.*

By GEORGE BIDDELL AIRY, Esq. A.M., *Astronomer Royal: Late Fellow of Trinity College, and Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge.*

[Read *May* 2, 1836, and *March* 26, 1838.]

WHEN a great physical theory has been established originally on considerations and experiments of a simple kind, which by degrees have been exchanged for comparisons of more distant results of the theory with more complicated cases of experiment, it has always been considered a matter of great interest, to trace out accurately by mathematical process the consequences, according to that theory, of different modifications of circumstances: which can then be compared with measures that have been made, or that may easily be made in future. It is with this view that I solicit the indulgence of the Society, for the following investigation of the Intensity of Light in the neighbourhood of a Caustic, as mathematically estimated from the Undulatory Theory.

The investigation which I present here belongs, ostensibly, only to the case of reflection. The introductory part of it will, however, (with the proper modifications) apply equally well to all cases of refraction and all combinations of reflection and refraction. There seems also to be no reason why the latter part (the estimation of the intensity of light, by considering the wave of light when it leaves the last surface to be divided into a great number of small parts, whose separate effects are then to be compounded,) should not apply to those cases. For though, strictly speaking, we ought to consider the wave to be thus broken up where it leaves the first surface, in order to find the intensity of vibration at every point of the second; yet it seems clear, that those reasonings which establish the definite reflection or refraction of a wave, (and which are founded upon the consideration above alluded to,) point out that there will be as to sense a mutual destruction of all vibrations at the second surface, (supposed to be not distant from the first,) excepting those which would be fully taken into account on the ordinary laws of Geometrical Optics. Where the light meets the second surface in the state of convergence, this conclusion perhaps is not so clear: but even there I believe that it may easily be shewn to be correct. I have mentioned these points because one of the most interesting cases of natural caustics (the rainbow) is affected by them; the exterior bow involving the first-mentioned condition, and the interior bow involving both the first and the second.

---

1. The notion of a caustic, and its mathematical definition, are essentially founded upon the laws of Geometrical Optics; and to these, therefore, we must refer in order to discover a representation of the conditions adapted to the investigations of Physical Optics. For simplicity we shall confine our diagrams to the plane of reflection, and shall consider the reflecting surface as symmetrical (to a sensible extent) with respect to that plane, so that the portion of the caustic formed by that part of the surface will be in the same plane.

2. In fig. 1., let the origin of light S be the origin of co-ordinates; x, y, the co-ordinates of a point X of the reflecting surface; p, q the co-ordinates of a point P in the reflected ray; V the length of the path of light from S to any point of the reflecting surface and thence to the point P. The ordinary law of reflection informs us that the angles of incidence and reflection are equal; and therefore, that, if we take a point X' on the reflecting surface very near to X, and join it with the origin and the point P, the lengthening ZX' of one

of these lines will be equal (ultimately) to the shortening XZ' of the other, and their sum (ultimately) will not be altered; or that, putting $\frac{d(V)}{dx}$ for the differential coefficient of V with regard to x, considering y also as a function of x, $\left(\text{which is otherwise written } \frac{dV}{dx} + \frac{dV}{dy} \cdot \frac{dy}{dx}\right)$, $\frac{d(V)}{dx} = 0$. This is the condition which holds at the point of reflection.

3. Now if p, q, be the co-ordinates of a focus, since in that case it is a point in the paths of rays reflected from every point of the surface, $\frac{d(V)}{dx} = 0$ at every point, and therefore V = constant, and $\frac{d^2(V)}{dx^2}$, $\frac{d^3(V)}{dx^3}$, &c. are $= 0$ at every point. This is the condition for the reflection of rays to a focus.

4. But though the condition $V = C$ and all its consequences are necessary for the convergence of reflected rays to a focus, yet this condition is not necessary for the convergence of a very small pencil of rays incident on the reflecting surface. It is only necessary for this, that the equations $\frac{d(V)}{dx} = 0$, and $\frac{d(V')}{dx'} = 0$ should hold at the same time, when $x' = x + \delta x$ and V' has the corresponding value; that is, that the following equations should be true at the same time,

$$\frac{d(V)}{dx} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad = 0,$$

$$\frac{d(V)}{dx} + \frac{d^2(V)}{dx^2} \cdot \frac{\delta x}{1} + \frac{d^3(V)}{dx^3} \cdot \frac{(\delta x)^2}{1.2} + \&c. = 0.$$

From this we obtain $\frac{d^2(V)}{dx^2} + \frac{d^3(V)}{dx^3} \cdot \frac{\delta x}{1.2} + \&c. = 0$, as the equation expressing that the rays incident at the points x and $x + \delta x$ intersect: and making δx indefinitely small, this reduces itself as nearly as we please to the equation $\frac{d^2(V)}{dx^2} = 0$. This then is the equation which must hold for the ultimate convergence of rays.

5. Now the definition of a caustic in Geometrical Optics, is "the locus of the ultimate intersections of reflected rays:" and therefore, for every point of a caustic, $\frac{d(V)}{dx} = 0$ and $\frac{d^2(V)}{dx^2} = 0$ when that value of x is used which corresponds to the point of the reflecting surface, from which the light is reflected to each particular point of the caustic. But $\frac{d^3(V)}{dx^3}$ is not necessarily $= 0$: and in general its value is finite. For if, in fig. 2, we take a point P' of the caustic nearer to the reflecting surface than P, and if X' is the corresponding point of the reflecting surface; then we know from the geometrical theory of caustics, that

$$SX' + X'P' + P'P = SX + XP.$$

Now if we join X' with P, it will be evident that

$$X'P < X'P' + P'P.$$

Therefore, $$SX' + X'P < SX + XP,$$

$$\text{or } V' < V.$$

Similarly, if we take a point P'' on the caustic further from the reflecting surface than P, and X'' for the corresponding point on the reflecting surface,

$$SX'' + X''P'' = SX + XP + PP''.$$

$$\text{But } X''P + PP'' > X''P''.$$

$$\text{Therefore } SX'' + X''P + PP'' > SX + XP + PP''.$$

$$\text{Or } SX'' + X''P > SX + XP,$$

$$\text{or } V'' > V;$$

consequently the first differential coefficient of V which has a finite value is of an odd order: and as, in the general case, we must, from the very meaning of the word *general*, take those conditions which require the smallest number of peculiar equations, we must fix on the first coefficient of an odd order which has not yet been fettered by any equation; and therefore in the general case, for a point in a caustic,

$$\frac{d^3(V)}{dx^3} \text{ has a finite value.}$$

It may possibly happen at singular points that $\frac{d^3(V)}{dx^3}$ and $\frac{d^4(V)}{dx^4}$ vanish, and that $\frac{d^5(V)}{dx^5}$ has a finite value: but of these peculiar cases I intend to take no further notice.

6. By pursuing this train of investigation we should find that at a cusp of a caustic $\frac{d(V)}{dx} = 0$, $\frac{d^2(V)}{dx^2} = 0$, $\frac{d^3(V)}{dx^3} = 0$, and $\frac{d^4(V)}{dx^4}$ has a finite value. I shall not however pursue this subject further.

7. The conditions then which hold, with reference to any point of a caustic in general, are these: If V be measured from the origin of light to any point of the reflecting surface and then to the given point of the caustic: in the case of the point of the reflecting surface coinciding with the corresponding point of reflexion,

$$\frac{d(V)}{dx} = 0,$$

$$\frac{d^2(V)}{dx^2} = 0,$$

$$\frac{d^3(V)}{dx^3} = C,$$

C being a finite function of x, y, p, and q. The sign of C may be thus found. In the case assumed in (5) and represented in fig. 2, V' was $< V$ and $V'' > V$; if then V' implies that x is diminished, or if x is measured from the convexity of the caustic, C is positive. If x is measured towards the convexity of the caustic, in that case C is negative.

8. The value of C may thus be found. Draw $P'Q'$ perpendicular to PX': then $Q'X'$ may be considered equal to $P'X'$ (it will differ from it only by quantities depending on the fourth power of PP' or XX');

and therefore $V - V'$, which in (5) was found $= X'P' + P'P - X'P$, is $= P'P - Q'P$. Let x be measured nearly perpendicular to the caustic at P; put ρ for the radius of curvature of the caustic at P, and ϕ for the small angle made by PX and $P'X'$. Then $\delta x = PX \cdot \phi$, therefore $\phi = \frac{\delta x}{PX}$. And $P'P - Q'P = \rho \cdot \frac{\phi^3}{1.2.3} = \frac{\rho}{(PX)^3} \cdot \frac{\delta x^3}{1.2.3}$. Making this equal to the corresponding term in Taylor's series for V', we find

$$C = \frac{d^3(V)}{dx^3} = \frac{\rho}{(PX)^3}.$$

9. Now take a point near the caustic, whose co-ordinates are $p + \delta p$ and q (δp being measured from the convexity of the caustic parallel to x, or nearly perpendicular to the caustic at P). Let $V_{,}$ be the length of the path of light from the origin to any point of the reflector and thence to the point $p + \delta p$, q. Then we have

$$V = \sqrt{x^2 + y^2} + \sqrt{(x-p)^2 + (y-q)^2},$$

$$V_{,} = \sqrt{x^2 + y^2} + \sqrt{(x-p-\delta p)^2 + (y-q)^2};$$

or $$V_{,} = V + \sqrt{(x-p-\delta p)^2 + (y-q)^2} - \sqrt{(x-p)^2 + (y-q)^2},$$

which, if we expand to the first power of δp, becomes

$$V_{,} = V - \frac{x-p}{\sqrt{(x-p)^2 + (y-q)^2}} \delta p,$$

and therefore, in the general case of measuring V through any point of the reflecting surface,

$$\frac{d(V_{,})}{dx} = \frac{d(V)}{dx} + A \cdot \delta p,$$

$$\frac{d^2(V_{,})}{dx^2} = \frac{d^2(V)}{dx^2} + B \cdot \delta p;$$

$$\frac{d^3(V_{,})}{dx^3} = \frac{d^3(V)}{dx^3} + D \cdot \delta p,$$

A, B, and D, being finite functions of x, y, p, and q.

In the particular case of measuring V through the point which reflects rays (according to the ordinary rules) to p, q,

$$\frac{d(V_{,})}{dx} = A\,.\,\delta p,$$

$$\frac{d^2(V_{,})}{dx^2} = B\,.\,\delta p;$$

$$\frac{d^3(V_{,})}{dx^3} = C + D\,.\,\delta p,$$

or, as we shall always suppose δp small,

$$\frac{d^3(V_{,})}{dx^3} = C.$$

10. Consequently, if $V_{,}'$ be put for the length of the path through $x + \delta x$, $y + \delta y$, to $p + \delta p$, q, (δp being entirely independent of δx),

$$V_{,}' = V_{,} + A\delta p\,\frac{\delta x}{1} + B\delta p\,.\,\frac{(\delta x)^2}{1.2} + C\,.\,\frac{(\delta x)^3}{1.2.3},$$

or, putting z for δx,

$$V_{,}' = V_{,} + A\delta p\,.\,\frac{z}{1} + B\delta p\,.\,\frac{z^2}{1.2} + C\,.\,\frac{z^3}{1.2.3},$$

omitting the following terms.

11. The value of C has been found: that of A (the only other quantity which interests us) will be obtained by actual differentiation of the expression for $V_{,}$: Thus we find

$$A\delta p = \delta p\,.\,\frac{(x-p)(y-q)\,.\,\frac{dy}{dx} - (y-q)^2}{\{(x-p)^2 + (y-q)^2\}^{\frac{3}{2}}}.$$

Or, as $x - p$ is supposed to be very small, $A = -\frac{1}{y-q} = -\frac{1}{PX}$.

12. There is one case so peculiar, and which seems so likely to cause a failure of these expressions, that it merits a particular investigation: the more so as it occurs in the rainbow. It is the case in

which, on ascertaining the deviation (from a fixed direction) of the rays reflected or refracted, we find, on proceeding in the same direction along the reflecting or refracting surface, that the deviation increases to a certain amount and then diminishes, or *vice versâ*. In this case the caustic consists of two unconnected infinite branches in opposite directions, with a common asymptote parallel to the position of maximum or minimum deviation of the rays. To investigate this case, we shall examine the form of the front of the wave immediately after leaving the reflecting or refracting surface, and shall measure the lengths of paths of light from that front. In fig. 3, let A be the point at which the asymptote intersects the front of the wave (which will be the same as the point of the front where the deviation is maximum or minimum) whose co-ordinates are 0 and b: let X be any other point in the front, whose co-ordinates are x and y, (x being measured from the asymptote and y parallel to it:) and p and q the co-ordinates, similarly measured, of any point P near the asymptote. If the length AX be called s, and the angle made by the tangent at X with the tangent at A be called θ, then the condition that the deviation of the direction of the rays from a fixed direction (or the deviation of the tangent to the front of the wave from another fixed direction,) is maximum or minimum gives $\theta = \frac{s^2}{a^2}$, a being some constant. Observing that $\frac{dx}{ds} = \cos\theta$, and $\frac{dy}{ds} = \sin\theta$, we get with sufficient approximation $x = s$, $y = b + \frac{s^3}{3a^2} = b + \frac{x^3}{3a^2}$: and the front of the wave is therefore a cubical parabola. The distance of the point P from X

$$= \sqrt{(x-p)^2+(y-q)^2} = \sqrt{p^2 - 2px + x^2 + (b-q)^2 + \frac{2(b-q)}{3a^2}x^3},$$

and, expanding this to the third power of x, and putting c^2 for $p^2+(b-q)^2$,

$$PX = c\left\{1 - \frac{p}{c^2}x + \frac{1}{2}\cdot\frac{c^2-p^2}{c^4}x^2 + \left(\frac{b-q}{3a^2c^2} + \frac{1}{2}\cdot\frac{p(c^2-p^2)}{c^6}\right)x^3\right\}.$$

If we take only the principal part of each coefficient (which for any practical case will be abundantly sufficient, observing that $\frac{p}{b-q}$ will probably never, in observation, amount to tan 2°), this becomes

$$PX = (b-q)\left\{1 - \frac{p}{(b-q)^2}\,x + \tfrac{1}{2}\cdot\frac{1}{(b-q)^2}\,x^2 + \frac{1}{3a^2(b-q)}\,x^3\right\},$$

$$= b - q - \frac{p}{b-q}\,x + \tfrac{1}{2}\cdot\frac{1}{b-q}\,x^2 + \frac{1}{3a^2}\,x^3.$$

In the applications of this, it will be important to notice that the coefficient of x^3 is independent of p and q, (depending only on the dimensions of the rain-drop or other refracting or reflecting body,) and that the coefficient of x depends only on the angle made by PX with the asymptote.

13. It appears, therefore, that in both the cases considered, the form of the expression for the length of the path of the wave to the point under consideration near the caustic, passing through the general point of the reflecting surface or of the front of the wave, is that of a general formula of the third order; in which the coefficient of the first power of the ordinate of the point on the front of the wave is proportional to the distance of the illuminated point from the caustic or the asymptote, and in which the coefficient of the third power is independent of that distance. If in the first instance we make $z + \frac{PX^3 . B . \delta p}{\rho} = z'$, the first expression becomes (putting E for a term independent of z, and in the coefficient of z omitting the term involving δp^2 in comparison with δp,)

$$V_{\prime} = E + \frac{\rho}{6PX^3}\left\{z'^3 - \frac{6.PX^2}{\rho}\,\delta p \,. z'\right\}.$$

And if in the second instance we make $x + \frac{1}{2}\cdot\frac{a^2}{b-q} = x'$, and observe that for the rainbow a is a very small fraction of an inch while p may be many feet, and that a may therefore be omitted in comparison with p,

$$PX = F + \frac{1}{3a^2}\left\{x'^3 - \frac{3a^2}{b-q}\,p \,. x'\right\}.$$

14. To proceed now with the intensity of light at the illuminated point. I shall omit entirely the integration for the ordinate perpendicular to the plane of x, y, because it would only introduce a factor common to every part, and therefore would not modify the proportion of intensity at different points. The great wave of light being supposed to be divided into indefinitely small parts, each of which is the origin of a small wave spreading in all directions: the disturbance of ether at the illuminated point produced by this small wave, on the Undulatory Theory, will be estimated by

$$\text{portion of surface of small wave} \times \sin \frac{2\pi}{\lambda}(vt - \text{whole path})$$

which in the first case becomes

$$\delta z' \times \sin \frac{2\pi}{\lambda}\left\{vt - E - \frac{\rho}{6PX^3}\left(z'^3 - \frac{6 . PX^2}{\rho}\,\delta p . z'\right)\right\},$$

and in the second case

$$\delta x' \times \sin \frac{2\pi}{\lambda}\left\{vt - F - \frac{1}{3a^2}\left(x'^3 - \frac{3a^2}{b-q}\,p . x'\right)\right\}.$$

15. In the first case therefore, the expression for the whole disturbance is

$$\int_{z'} \sin \frac{2\pi}{\lambda}\left\{vt - E - \frac{\rho}{6PX^3}\left(z'^3 - \frac{6 . PX^2}{\rho}\,\delta p . z'\right)\right\}.$$

The limits through which the integration is to be performed are from z' a sensible quantity negative to z' a sensible quantity positive, and on account of the minuteness of the divisor λ, and the inefficiency of the rays whose paths differ from E by many multiples of λ, this will be the same as taking it between the limits $-$ infinity, $+$ infinity. Now the integral is the same as

$$\sin \frac{2\pi}{\lambda}(vt - E) \int_{z'} \cos \frac{2\pi}{\lambda} . \frac{\rho}{6PX^3}\left(z'^3 - \frac{6 . PX^2}{\rho} . \delta p . z'\right)$$

$$- \cos \frac{2\pi}{\lambda}(vt - E) \int_{z'} \sin \frac{2\pi}{\lambda} . \frac{\rho}{6PX^3}\left(z'^3 - \frac{6 . PX^2}{\rho} . \delta p . z'\right).$$

But between $-$ infinity and $+$ infinity it is evident that

$$\int_{z'} \sin \frac{2\pi}{\lambda} . \frac{\rho}{6PX^3}\left(z'^3 - \frac{6 . PX^2}{\rho} . \delta p . z'\right) = 0,$$

because every positive value is balanced by an equal negative value; and therefore the expression for the disturbance of ether at the illuminated point is

$$\sin\frac{2\pi}{\lambda}(vt-E)\int_{z'}\cos\frac{2\pi}{\lambda}\cdot\frac{\rho}{6PX^3}\left(z'^3-\frac{6.PX^2}{\rho}.\delta p.z'\right),$$

the integral being taken between − infinity, + infinity;

$$\text{or,}\ \ 2\sin\frac{2\pi}{\lambda}(vt-E)\int_{z'}\cos\frac{2\pi}{\lambda}\cdot\frac{\rho}{6PX^3}\left(z'^3-\frac{6PX^2}{\rho}.\delta p.z'\right),$$

the integral being taken from 0 to infinity,

Making $\frac{2\pi}{\lambda}\cdot\frac{\rho}{6PX^3}\cdot z'^3=\frac{\pi}{2}w^3$, or $z'=PX\left(\frac{3\lambda}{2\rho}\right)^{\frac{1}{3}}.w$, and putting m for $\delta p\times\left(\frac{96}{\lambda^2\rho}\right)^{\frac{1}{3}}$, and omitting the constant factor, we find as the expression for the disturbance of ether at the illuminated point

$$\sin\frac{2\pi}{\lambda}(vt-E)\int_w\cos\frac{\pi}{2}(w^3-m.w),$$

and therefore the expression for the intensity of light is

$$\left[\int_w\cos\frac{\pi}{2}(w^3-m.w)\right]^2,$$

the integral being taken from $w=0$ to $w=$ infinity.

It will be observed that m is proportional to δp, and therefore the intensity of light at the Geometrical Caustic, or where $\delta p=0$, is found by making $m=0$ in this formula.

16. In the second case, the expression for the whole disturbance of ether is

$$\int_{x'}\sin\frac{2\pi}{\lambda}\left\{vt-F-\frac{1}{3a^2}\left(x'^3-\frac{3a^2}{b-q}p.x'\right)\right\},$$

which, as above, is shewn to be equal to

$$2\sin\frac{2\pi}{\lambda}(vt-F).\int_{x'}\cos\frac{2\pi}{\lambda}\cdot\frac{1}{3a^2}\left(x'^3-\frac{3a^2}{b-q}p.x'\right),$$

from $x'=0$ to $x'=$ infinity.

Making $\frac{2\pi}{\lambda}\cdot\frac{1}{3a^2}x'^3 = \frac{\pi}{2}w^3$, or $x' = \left(\frac{3a^2\lambda}{4}\right)^{\frac{1}{3}}.w$, and putting m for $\frac{p}{b-q}\times\left(\frac{48a^2}{\lambda^2}\right)^{\frac{1}{3}}$, and omitting the constant factor, the intensity of light, as above, is shewn to be

$$[\int_w \cos\frac{\pi}{2}(w^3 - m.w)]^2,$$

the integral being taken from $w = 0$ to $w =$ infinity.

It will be remarked that m, in this case, is proportional to $\frac{p}{b-q}$; it is therefore 0 for points in the direction of the asymptote, and for other points it is proportional to the angle made by the line from the center of the wave with the asymptote.

17. The values of $\int_w \cos\frac{\pi}{2}(w^3 - m.w)$, from $w = 0$ to $w =$ infinity, and the squares of these numbers, for every 0·2 from $m = -4·0$ to $m = +4·0$, are contained in the following table: for the calculation of which I refer to the Appendix.

Values of m	Corresponding values of $\int_w \cos\frac{\pi}{2}(w^3 - m.w)$ from 0 to $\frac{1}{0}$	Squares of the last Numbers	Values of m	Corresponding values of $\int_w \cos\frac{\pi}{2}(w^3 - m.w)$ from 0 to $\frac{1}{0}$	Squares of the last Numbers
−4·0	+0·00298	0·0000089	+0·6	+0·91431	0·83597
−3·8	+ ·00431	·0000186	+0·8	+0·97012	0·94114
−3·6	+ ·00618	·0000382	+1·0	+1·00041	1·00082
−3·4	+ ·00879	·0000773	+1·2	+0·99786	0·99572
−3·2	+ ·01239	·0001536	+1·4	+ ·95606	·91406
−3·0	+ ·01730	·000299	+1·6	+ ·87048	·75773
−2·8	+ ·02393	·000573	+1·8	+ ·73939	·54670
−2·6	+ ·03277	·001074	+2·0	+ ·56490	·31912
−2·4	+ ·04442	·00197	+2·2	+ ·35366	·12508
−2·2	+ ·05959	·00355	+2·4	+ ·11722	·013741
−2·0	+ ·07908	·00625	+2·6	− ·12815	·016422
−1·8	+ ·10377	·01077	+2·8	− ·36237	·13131
−1·6	+ ·13461	·01812	+3·0	− ·56322	·31721
−1·4	+ ·17254	·02977	+3·2	− ·70874	·50231
−1·2	+ ·21839	·04769	+3·4	− ·78018	·60868
−1·0	+ ·27283	·07444	+3·6	− ·76516	·58547
−0·8	+ ·33621	·11304	+3·8	− ·66054	·43631
−0·6	+ ·40839	·16678	+4·0	−0·47446	0·22511
−0·4	+ ·48856	·23869			
−0·2	+ ·57507	·33071			
0·0	+ ·66527	·44259			
+0·2	+ ·75537	·57059			
+0·4	+0·84040	0·70628			

The extent of this table for the positive values of m is not so great as I could wish; but it goes far enough to enable us to point out the most remarkable circumstances of the distribution of illumination.

18. From $m = -4{\cdot}0$ to $m = -1{\cdot}6$, the illumination is almost insensible. (In fact it appears to diminish, as the negative value of m increases, in a nearly geometrical or perhaps hyper-geometrical progression). It then increases rapidly, and acquires its maximum value when $m = +1{\cdot}08$ nearly; its value is then nearly 1·001. It then diminishes rapidly till $m = +2{\cdot}48$ nearly, when the illumination is zero. It then increases till $m = +3{\cdot}47$, when the illumination is nearly 0·615, or about three-fifths of its former maximum. It then diminishes rapidly to the end of the table: and appears likely to become zero for a value of m differing little from + 4·4.

19. One of the most important points to be remarked is, that the maximum illumination does not take place at the Geometrical Caustic, or where $m = 0$, but where $m = +1{\cdot}08$, that is, on the external side of the convexity of the caustic, or on the luminous side of the geometrical position of the rainbow, that is, (for the primary bow,) within it. The following rule derived from the numbers above, will suffice, in practice, to determine the geometrical position. When the first spurious bow is visible, measure the distance of its maximum intensity from that of the brilliant bow; then the geometrical bow is exterior to the brilliant bow by $\frac{11}{24}$ of this distance.

20. It is a matter of curiosity to ascertain the relation of the intensities, or at least of the places of maximum and minimum intensity, as determined thus by a complete investigation on the theory of undulations, with those which would be found on the imperfect theory that light proceeds in straight rays according to the laws of Geometrical Optics, and that rays of light are capable of interfering according to the simple rules of interference. We have first to discover the position of the two rays which interfere at any point. Now the

length of the path of any ray to the illuminated point is $E + \frac{\lambda}{4}(w^3 - m.w)$: and, by (2), the first differential coefficient of this quantity with regard to w will be zero for those rays which pass according to the ordinary rules of reflection and refraction. Performing this differentiation, $3w^2 - m = 0$: $w = \pm\sqrt{\frac{m}{3}}$: the lengths of path therefore of the two rays are $E - \frac{\lambda}{4}\sqrt{\frac{4m^3}{27}}$ and $E + \frac{\lambda}{4}\sqrt{\frac{4m^3}{27}}$: and the difference of these is $\frac{\lambda}{2}\sqrt{\frac{4m^3}{27}}$. The destruction of light would therefore take place, on this imperfect theory, when $\sqrt{\frac{4m^3}{27}} = 1$, or $= 3$, or $= 5$, &c.; that is, when $m = \sqrt[3]{\frac{27}{4}}$, or $= \sqrt[3]{\frac{27.9}{4}}$, &c.; or when $m = 1{\cdot}89$, $3{\cdot}93$, &c.: and there would be no light whatever for negative values of m. We have found above, on the complete theory, that there is sensible light for negative values of m, and that the destruction of light takes place when $m = 2{\cdot}48$, $4{\cdot}4$ (nearly). According to the imperfect theory, the intensity would be infinite when $m = 0$, and the next maximum would be nearer to $1{\cdot}89$ than to $3{\cdot}93$: perhaps when $m = 2{\cdot}7$: we have found above that the intensity is nowhere infinite, that the first maximum takes place when $m = 1{\cdot}08$, and the second when $m = 3{\cdot}47$.

21. In figure 4, I have represented the intensity of the light by the ordinates of a curve, of which the abscissa represents different values of m. The strong line corresponds to the determination of the complete theory: the dotted line, to that of the old theory of emission (supposing the intensity inversely as the square root of the distance from the caustic): and the faint line, to that of the imperfect theory of interference mentioned above, giving to the maxima values in some degree proportionate to the ordinates of the dotted line. The absolute values of the ordinates in the faint and the dotted line are not to be understood as necessarily referred to the same unit as those in the strong line; but the abscissæ correspond exactly in all.

APPENDIX.

On the numerical computation of the definite integral $\int_w \cos \frac{\pi}{2}(w^3 - m.w)$, between the limits 0 and $\frac{1}{0}$.

The simplicity of the form of this differential coefficient induces me to suppose that the integral may possibly be expressible by some of the integrals whose values have been tabulated. After many attempts however, I have not succeeded in reducing it to any known integral: and I have therefore computed its value by actual summation to a considerable extent and by series for the remainder.

The summation was continued {for each of the values of m in the table given in (17)}, as far as $w = 2$. This extent was divided into eight sections, corresponding nearly to quadrants of the circular function when $m = 0$, as follows:

1st section,	from $w = 0$	to $w = 1{\cdot}00$,
2nd	from $w = 1{\cdot}00$	to $w = 1{\cdot}26$,
3rd	from $w = 1{\cdot}26$	to $w = 1{\cdot}44$,
4th	from $w = 1{\cdot}44$	to $w = 1{\cdot}58$,
5th	from $w = 1{\cdot}58$	to $w = 1{\cdot}70$,
6th	from $w = 1{\cdot}70$	to $w = 1{\cdot}82$,
7th	from $w = 1{\cdot}82$	to $w = 1{\cdot}92$,
8th	from $w = 1{\cdot}92$	to $w = 2{\cdot}00$.

These sections were divided into small intervals corresponding to uniform increments in the value of w, as follows:

In the 1st section the increment of w was	0·04,
2nd	·02,
3rd	·018,
4th	·014,
5th	·012,
6th	·010,
7th	·010,
8th	·008.

As the increments in the 6th and 7th sections were the same, the calculations for these two sections were conducted without the interruption which was necessary at the separation of the other sections.

The values of w and $\frac{\pi}{2}(w^3 - m.w)$ corresponding to the middle of each interval were then computed, and the appropriate value of

$$\cos \frac{\pi}{2}(w^3 - m.w)$$

was formed. Thus, in the first section the computations were made for $w = {\cdot}02, {\cdot}06, {\cdot}10$, &c: in the 4th section, for $w = 1{\cdot}447, 1{\cdot}461, 1{\cdot}475$, &c. For a reason that will be shortly mentioned, the values of $\cos \frac{\pi}{2}(w^3 - m.w)$ were also computed for two values of w preceding the first, and two values following the last, of each section. These values were then differenced as far as the 4th order: which operation, besides giving a means of checking most severely the accuracy of the computations, supplied the numbers necessary (in the next process) for converting the *sum* into an *integral.*

Now, suppose, that u_x is a function of x, and that the quantity h is so small that the functions u_{x+h}, u_{x+2h}, admit of being expressed with sufficient accuracy by the formula $u_x + b.h + c.h^2 + d.h^3 + e.h^4$, $u_x + b.(2h) + c.(2h)^2 + d.(2h^3) + e.(2h)^4$. This assumption is abundantly accurate for the numbers of which we are treating here. Set down two values preceding u_x and two following it, and take their differences as far as the 4th order, thus

	1st Differences.	2d Differences.	3d Differences.	4th Diff.
$u_x - 2bh + 4ch^2 - 8dh^3 + 16ek^4$				
	$bh - 3ch^2 + 7dh^3 - 15eh^4$			
$u_x - bh + ch^2 - dh^3 + eh^4$		$2ch^2 - 6dh^3 + 14eh^4$		
	$bh - ch^2 + dh^3 - eh^4$		$6dh^3 - 12eh^4$	
u_x		$2ch^2 \ldots + 2eh^4$		$24eh^4$
	$bh + ch^2 + dh^3 + eh^4$		$6dh^3 + 12eh^4$	
$u_x + bh + ch^2 + dh^3 + eh^4$		$2ch^2 + 6dh^3 + 14eh^4$		
	$bh + 3ch^2 + 7dh^3 + 15eh^4$			
$u_x + 2bh + 4ch^2 + 8dh^3 + 16eh^4$				

These expressions correspond to the following quantities:

$$
\begin{array}{l|c|c|c|c|}
u_{x-2h} & & & & \\
 & \Delta'_{-\frac{3h}{2}} & & & \\
u_{x-h} & & \Delta''_{-h} & & \\
 & \Delta'_{-\frac{h}{2}} & & \Delta'''_{-\frac{h}{2}} & \\
u_x & & \Delta'' & & \Delta'''' \\
 & \Delta'_{\frac{h}{2}} & & \Delta'''_{\frac{h}{2}} & \\
u_{x+h} & & \Delta''_{h} & & \\
 & \Delta'_{\frac{3h}{2}} & & & \\
u_{x+2h} & & & &
\end{array}
$$

Equating the quantities on the middle line, or the sums of the quantities next above and next below,

$$eh^4 = \frac{\Delta''''}{24},$$

$$dh^3 = \frac{\Delta'''_{-\frac{h}{2}} + \Delta'''_{\frac{h}{2}}}{12},$$

$$ch^2 = \frac{\Delta''}{2} - eh^4 = \frac{\Delta''}{2} - \frac{\Delta''''}{24},$$

$$bh = \frac{\Delta'_{-\frac{h}{2}} + \Delta'_{\frac{h}{2}}}{2} - dh^3 = \frac{\Delta'_{-\frac{h}{2}} + \Delta'_{\frac{h}{2}}}{2} - \frac{\Delta'''_{-\frac{h}{2}} + \Delta'''_{\frac{h}{2}}}{12}.$$

Now the integral with respect to z of the function u_{x+z} or

$$u_x + bz + cz^2 + dz^3 + ez^4,$$

is

$$u_x . z + \frac{b}{2} . z^2 + \frac{c}{3} . z^3 + \frac{d}{4} . z^4 + \frac{e}{5} . z^5 :$$

and this between the limits $z = -\frac{h}{2}$ and $z = +\frac{h}{2}$ is

$$u_x . h + \frac{c}{12} h^3 + \frac{e}{80} h^5$$

$$= u_x . h + \frac{\Delta''}{24} h - \frac{\Delta''''}{288} h + \frac{\Delta''''}{1920} h$$

$$= h \left\{ u_x + \frac{\Delta''}{24} - \frac{17 . \Delta''''}{5760} \right\}.$$

This formula applies to the order of the calculations described above, because we have computed the value of u_x for the middle of each of the intervals into which w is divided. And as the same formula applies to every one of the intervals, it applies to the sum of all. And the sum of all the partial integrals through each section is the whole integral through each section. Thus we find,

$$\text{Integral through each section} = \text{interval} \times \left\{ \begin{array}{l} \text{sum of computed values of } \cos\frac{\pi}{2}(w^3 - m\,.\,w) \\ + \frac{1}{24} \text{ sum of corresponding 2d differences} \\ - \frac{17}{5760} \text{ sum of corresponding 4th differences} \end{array} \right\}$$

$$= \text{interval} \times \left\{ \begin{array}{l} \text{sum of computed values of } \cos\frac{\pi}{2}(w^3 - m\,.\,w) \\ + \frac{1}{24} \left\{ \begin{array}{c} \text{1st difference following the last term} - \text{1st difference} \\ \text{preceding the first term} \end{array} \right\} \\ - \frac{17}{5760} \left\{ \begin{array}{c} \text{3d difference following the last term} - \text{3d difference} \\ \text{preceding the first term} \end{array} \right\} \end{array} \right\}.$$

This process was used throughout. For the purpose of forming the 3rd difference following the last term and that preceding the first term, it was necessary to compute two values of $\cos\frac{\pi}{2}(w^3 - m\,.\,w)$ following the end of each section and two preceding its beginning.

The values of $\frac{\pi}{2}(w^3 - m\,.\,w)$ were computed by means of Delambre's *Tables Trigonométriques Décimales*. The centesimal division of the circle is, in every instance in which I have used it, far more convenient than the sexagesimal: but in an instance like the present, where there is continual addition or subtraction of arcs, and where the whole arc amounts to several circumferences, the labour and liability to error would be so great with the sexagesimal division, as to make the operation almost impracticable. The numbers were thus formed in each section. The first four values of $\frac{\pi}{2}w^3$ were computed independently and differenced, and with these differences the rest of the series of $\frac{\pi}{2}w^3$ was formed: the last was also computed independently as a check. The first term of each series of $\frac{\pi}{2}(w^3 - m.w)$ was formed by applying $-\frac{\pi}{2}m.w$ to the first term of the series of $\frac{\pi}{2}w^3$: and the first difference was formed, all through the series, by applying $-\frac{\pi}{2}m.\delta w$ to the corresponding 1st difference of $\frac{\pi}{2}w^3$. The last terms of all the series were compared together, as a check. Then for every term the arc less than $\frac{\pi}{4}$ was taken whose sine or cosine (with proper sign) represented $\cos\frac{\pi}{2}(w^3 - m.w)$. The natural numbers were taken to the 7th decimal place, and differenced, as has been mentioned. The number of arguments thus computed is 5166; and the number of natural terms for the summation is the same; and the whole of these have been differenced on paper to the third order, and mentally to the fourth order.

The integration as far as $w = 2{\cdot}00$ being thus completed, and with the utmost accuracy, the next step was to compute the integral from $w = 2{\cdot}00$ to $w =$ infinity. Let $u = \frac{\pi}{2}(w^3 - m.w)$: the problem is now to find $\int_w \cos u$. If we make $\frac{1}{\frac{du}{dw}} = v$, this integral

$$= \int_w v.\cos u\frac{du}{dw} = v\sin u - \int_w \sin u\frac{dv}{dw} = v\sin u - \int_w v\frac{dv}{dw}.\sin u\frac{du}{dw},$$

where the last term may be integrated by parts as before. Proceeding with this operation, and putting

$$v_0 = v,$$

$$v_1 = v\frac{dv}{dw},$$

$$v_2 = v\frac{d}{dw}\left(v\frac{dv}{dw}\right),$$

$$v_3 = v\frac{d}{dw}\left[v\frac{d}{dw}\left(v\frac{dv}{dw}\right)\right],$$

$$v_4 = v\frac{d}{dw}\left\{v\frac{d}{dw}\left[v\frac{d}{dw}\left(v\frac{dv}{dw}\right)\right]\right\},$$

and so on, we find for the integral generally

$$(v_0 - v_2 + v_4 - v_6 + \&c.) \sin u + (v_1 - v_3 + v_5 - v_7 + \&c.) \cos u.$$

The first limit of the integration being w, and the last being infinity, and the quantities v_0, v_1, &c. vanishing for w = infinity, the value of the integral between these limits is,

$$(-v_0 + v_2 - v_4 + v_6 - \&c.) \sin u + (-v_1 + v_3 - v_5 + v_7 - \&c.) \cos u.$$

It will be observed here that $v = \frac{2}{\pi} \cdot \frac{1}{3w^2 - m}$.

The following are the expressions for v_0, v_1, &c.

$$v_0 = \frac{2}{\pi} \cdot \frac{1}{3w^2 - m},$$

$$v_1 = -\frac{24}{\pi^2} \cdot \frac{w}{(3w^2 - m)^3},$$

$$v_2 = \frac{240}{\pi^3} \cdot \frac{1}{(3w^2 - m)^4} + \frac{288m}{\pi^3} \cdot \frac{1}{(3w^2 - m)^5},$$

$$v_3 = -\frac{11520}{\pi^4} \cdot \frac{w}{(3w^2 - m)^6} - \frac{17280m}{\pi^4} \cdot \frac{w}{(3w^2 - m)^7},$$

$$v_4 = \frac{253440}{\pi^5} \cdot \frac{1}{(3w^2 - m)^7} + \frac{725760m}{\pi^5} \cdot \frac{1}{(3w^2 - m)^8} + \frac{483840m^2}{\pi^5} \cdot \frac{1}{(3w^2 - m)^9},$$

$$v_5 = -\frac{21288960}{\pi^6} \cdot \frac{w}{(3w^2 - m)^9} - \frac{69672960m}{\pi^6} \cdot \frac{w}{(3w^2 - m)^{10}} - \frac{52254720m^2}{\pi^6} \cdot \frac{w}{(3w^2 - m)^{11}},$$

$$v_6 = \frac{723824640}{\pi^7} \cdot \frac{1}{(3w^2 - m)^{10}} + \frac{3413975040m}{\pi^7} \cdot \frac{1}{(3w^2 - m)^{11}} + \frac{4981616640m^2}{\pi^7} \cdot \frac{1}{(3w^2 - m)^{12}} + \frac{2299207680m^3}{\pi^7} \cdot \frac{1}{(3w^2 - m)^{13}},$$

$$v_7 = -\frac{86858956800}{\pi^8} \cdot \frac{w}{(3w^2 - m)^{12}} - \frac{450644705280m}{\pi^8} \cdot \frac{w}{(3w^2 - m)^{13}} - \frac{717352796160m^2}{\pi^8} \cdot \frac{w}{(3w^2 - m)^{14}} - \frac{358676398080m^3}{\pi^8} \cdot \frac{w}{(3w^2 - m)^{15}}.$$

Making w = 2·00, computing these expressions for every value of m, and substituting them in the expression for the integral from w = 2·00 to w = infinity, the numerical value of the integral for each value of m was found.

This process was exceedingly accurate for all the negative values of m, and for the positive values about as far as m = + 3·0, when a difficulty presented itself. It must be remarked that when $3w^2 - m$ (or in the present instance $12 - m$,) amounts to several integers, the values of the successive terms decrease at first with great rapidity: yet, in all cases, they increase at some part or other in hyper-geometrical proportion, and finally become greater than any assignable quantity. This will be seen most readily on observing the law of the terms when $m = 0$.

We have then $v_0 = \frac{2}{3\pi} \cdot \frac{1}{w^2}$: $v_1 = \left(\frac{2}{3\pi}\right)^2 \cdot \frac{-2}{w^5}$: $v_2 = \left(\frac{2}{3\pi}\right)^3 \cdot \frac{2.5}{w^8}$:

$$v_3 = \left(\frac{2}{3\pi}\right)^4 \cdot \frac{-2.5.8}{w^{11}} : \; v_4 = \left(\frac{2}{3\pi}\right)^5 \cdot \frac{2.5.8.11}{w^{14}} : \; \&c.$$

It is evident that these terms, however small may be the quantity $\frac{2}{3\pi . w^3}$, will at some stage receive in succession new multipliers, greater than $\frac{3\pi . w^3}{2}$; that after this they will increase, and that the rapidity of their proportionate increase will go on continually increasing. From this point then the magnitude of the terms will increase hyper-geometrically.

The value of the integral however will be finite; and a limit for the value remaining after the computation of any number of terms in the series v_0, v_1, &c. may be found. For, wherever we stop, the residual term will be of the form $\int_w \cos u . \frac{dv_n}{dw}$ or $\int_w \sin u . \frac{dv_n}{dw}$: where v_n is the term last found in the series. Now it is evident that either of these quantities is less than $\int_w \frac{dv_n}{dw}$; for the magnitudes of the quantities to be integrated are always smaller, except in the particular cases when $\cos u$ or $\sin u = \pm 1$; and their signs are constantly varying as the value of w varies: whereas the sign of $\frac{dv_n}{dw}$ is always the same. The residual integral, therefore, is certainly less than v_n, the last term found in the series, and is probably much less: and therefore, if the last term computed consist only of integers in the last place of decimals which we wish to retain, even though the divergence of the series be just beginning, the use of these terms will give the integral required with the utmost practical accuracy.

Now when $m = +3{\cdot}0$ nearly, it is found that the divergence commences at v_7, or before it; and that the term v_7 is not so small that a quantity which is likely to be a sensible portion of it can be safely neglected. To approximate here, I have used the following consideration. It is known that the slowly converging series

$$A - B + C - D + E - F + \&c.$$

may be converted into a series of the same kind with much smaller terms by putting it in this form,

$$\begin{aligned} &\tfrac{1}{2}A \\ &+ \tfrac{1}{2}(A-B) - \tfrac{1}{2}(B-C) + \tfrac{1}{2}(C-D) - \tfrac{1}{2}(D-E) + \tfrac{1}{2}(E-F) - \&c. \end{aligned}$$

In the instance before us, such a series would be produced by commencing the integration by parts with $\frac{1}{2}(v_0 - v_2)$ instead of v_0; and the residual term will be of the form of

$$\tfrac{1}{2}\textstyle\int_w \cos u \,.\, \dfrac{d}{dw}(v_n - v_{n+2}) \text{ or } \tfrac{1}{2}\int_w \sin u \, \dfrac{d}{dw}(v_n - v_{n+2}).$$

Now if the progression is stopped at such a point that $\frac{dv_n}{dw}$ is, for all the following values of w, greater than $\frac{dv_{n+2}}{dw}$, then the quantity $\frac{d}{dw}(v_n - v_{n+2})$ has always the same sign, and the reasoning above shews that the residual integral will be less than $\frac{1}{2}(v_n - v_{n+2})$. A close approximation therefore will be obtained by summing the series as if we had begun with $\frac{1}{2}(v_0 - v_2)$ instead of v_0: and this, it is easily seen, will be effected by taking half of the last term in each of the series multiplying sin u and cos u. The multipliers which I have used are, in fact,

$$-v_0 + v_2 - v_4 + \tfrac{1}{2}v_6 \ldots\ldots\ldots\ldots \text{for } \sin u,$$

$$-v_1 + v_3 - v_5 + \tfrac{1}{2}v_7 \ldots\ldots\ldots\ldots \text{for } \cos u.$$

The doubt which remains extends, I apprehend, to digits in the fifth place of decimals, but no higher.

The number of terms computed by logarithmic process for this part of the integral is about 900.

I could have wished to extend the computation as far as $m = +6{\cdot}0$, so as to include perhaps one or two more maxima of the values of intensity. Indeed, if it had been possible to foresee the approximate places &c. of maxima, I should probably have commenced the computations to that extent. The trouble however would be great, as the summation must be extended as far as $w = 2{\cdot}5$. Should any person be disposed to go to that extent, I would recommend that the summation of the values computed in this Paper should be also extended, for the values of m beginning with $+3{\cdot}0$.

I subjoin a Table of the different sections of the summation and remaining integration for all the values used in this Paper.

TABLE of the different parts of $\int_w \cos\frac{\pi}{2}(w^3 - m.w)$, from $x=0$ to $x=$ infinity.

Value of m.	Values of the integral found by actual summation.							Values of the remaining integral.		Sum.
	From x=0·00 to x=1·00.	From x=1·00 to x=1·26.	From x=1·26 to x=1·44.	From x=1·44 to x=1·58.	From x=1·58 to x=1·70.	From x=1·70 to x=1·92.	From x=1·92 to x=2·00.	Term depending on sin u.	Term depending on cos u.	
4·0	+·0929154	−·1621545	+·1280207	−·0757713	−·0036556	−·0150365	+·0374784	·0000000	+·0011788	+·0029754
−3·8	+·0899076	−·1537134	+·1071428	−·0318602	−·0507057	+·0210294	−·0021156	+·0236322	+·0009898	+·0043069
−3·6	+·0783010	−·1250013	+·0663779	+·0200012	−·0852129	+·0538698	−·0412717	+·0387234	+·0003926	+·0061800
−3·4	+·0591033	−·0787271	+·0125367	+·0683706	−·0979422	+·0721698	−·0655350	+·0392214	−·0004079	+·0087896
−3·2	+·0342832	−·0200615	−·0450110	+·1023750	−·0852309	+·0684171	−·0658255	+·0245557	−·0011101	+·0123920
−3·0	+·0066555	+·0439416	−·0959701	+·1141579	−·0500345	+·0418171	−·0418367	·0000000	−·0014269	+·0173039
−2·8	−·0202801	+·1051981	−·1309732	+·1006947	−·0012464	−·0008297	−·0022201	−·0252118	−·0012011	+·0239304
−2·6	−·0426027	+·1556300	−·1433068	+·0645600	+·0485494	−·0467686	+·0385325	−·0413456	−·0004776	+·0327706
−2·4	−·0562773	+·1882352	−·1301966	+·0134390	+·0863295	−·0810887	+·0653953	−·0419123	+·0004974	+·0444215
−2·2	−·0574983	+·1980667	−·0934064	−·0415027	+·1020423	−·0915539	+·0683497	−·0262631	+·0013569	+·0595912
−2·0	−·0430375	+·1829652	−·0390242	−·0880661	+·0912823	−·0729091	+·0461214	·0000000	+·0017488	+·0790805
−1·8	−·0105661	+·1439504	+·0235453	−·1157328	+·0565228	−·0291543	+·0067189	+·0270131	+·0014761	+·1037734
−1·6	+·0410858	+·0852018	+·0832149	−·1180453	+·0065563	+·0271234	−·0354520	+·0443405	+·0005884	+·1346138
−1·4	+·1117252	+·0136169	+·1291863	−·0941134	−·0457342	+·0783376	−·0648587	+·0449910	−·0006146	+·1725361
−1·2	+·1996990	−·0620069	+·1529169	−·0488959	−·0866990	+·1074010	−·0705636	+·0282196	−·0016814	+·2183897
−1·0	+·3019144	−·1320681	+·1497066	+·0078255	−·1054914	+·1034152	−·0502948	·0000000	−·0021732	+·2728342
−0·8	+·4139711	−·1873967	+·1196276	+·0635668	−·0969417	+·0656568	−·0113461	−·0290842	−·0018395	+·3362141
−0·6	+·5303921	−·2204699	+·0675952	+·1058949	−·0629782	+·0044341	+·0320494	−·0477897	−·0007356	+·4083923
−0·4	+·6449407	−·2264423	+·0025723	+·1252092	−·0122017	−·0616677	+·0639213	−·0485430	+·0007707	+·4885595
−0·2	+·7510400	−·2038501	−·0639887	+·1169208	+·0422963	−·1114242	+·0724424	−·0304813	+·0021152	+·5750704
0·0	+·8422086	−·1548861	−·1201662	+·0825417	+·0863120	−·1278013	+·0543205	·0000000	+·0027427	+·6652719
+0·2	+·9125150	−·0852037	−·1557205	+·0294203	+·1082374	−·1037569	+·0160657	+·0314867	+·0023295	+·7553735
+0·4	+·9570127	−·0032555	−·1639662	−·0308393	+·1021275	−·0450631	−·0283460	+·0517992	+·0009347	+·8404040
+0·6	+·9721117	+·0807460	−·1430312	−·0848876	+·0693085	+·0309509	−·0625827	+·0526798	−·0009826	+·9143128
+0·8	+·9558650	+·1561216	−·0962649	−·1206499	+·0181032	+·1005011	−·0739646	+·0331201	−·0027067	+·9701249
+1·0	+·9081446	+·2131038	−·0317049	−·1299775	−·0382828	+·1408098	−·0581618	·0000000	−·0035225	+1·0004087
+1·2	+·8306993	+·2441213	+·0393109	−·1105332	−·0851711	+·1375753	−·0208408	−·0343016	−·0030028	+·9978573
+1·4	+·7270641	+·2448168	+·1041461	−·0663663	−·1102398	+·0899919	+·0243670	−·0565062	−·0012095	+·9560641
+1·6	+·6023587	+·2146627	+·1511250	−·0070683	−·1067638	+·0115856	+·0608471	−·0575460	+·0012765	+·8704775
+1·8	+·4629601	+·1570802	+·1716481	+·0543138	−·0754212	−·0736006	+·0751107	−·0362310	+·0035295	+·7393896
+2·0	+·3160989	+·0790543	+·1617709	+·1041549	−·0241758	−·1383955	+·0617838	·0000000	+·0046115	+·5649030
+2·2	+·1693705	−·0097331	+·1229644	+·1312871	+·0337502	−·1611914	+·0256330	+·0376336	+·0039467	+·3536610
+2·4	+·0302520	−·0981192	+·0619006	+·1295037	+·0832918	−·1331501	−·0201404	+·0620890	+·0015960	+·1172234
+2·6	−·0944017	−·1748910	−·0107004	+·0989853	+·1114670	−·0615225	−·0587219	+·0633290	−·0016907	−·1281469
+2·8	−·1987892	−·2302290	−·0819605	+·0463043	+·1107817	+·0321479	−·0758654	+·0399339	−·0046921	−·3623684
+3·0	−·2785332	−·2569831	−·1391417	−·0169919	+·0812275	+·1185093	−·0651540	·0000000	−·0061508	−·5632179
+3·2	−·3309399	−·2516145	−·1719364	−·0769201	+·0303310	+·1696336	−·0304033	−·0416103	−·0052779	−·7087378
+3·4	−·3551338	−·2146670	−·1743344	−·1201569	−·0287648	+·1680727	+·0156965	−·0687550	−·0021380	−·7801807
+3·6	−·3520567	−·1507154	−·1457300	−·1370101	−·0807012	+·1128004	+·0562188	−·0702322	+·0022653	−·7651611
+3·8	−·3243480	−·0677906	−·0910614	−·1235960	−·1119003	+·0200155	+·0762169	−·0443470	+·0062719	−·6605390
+4·0	−·2761063	+·0236462	−·0199552	−·0827357	−·1141216	−·0815972	+·0682405	·0000000	+·0081698	−·4744595

G. B. AIRY.

ROYAL OBSERVATORY, GREENWICH,
March 12, 1838.

XVIII. *On the Reflexion and Refraction of Sound.* By G. GREEN, Esq., *of Caius College, Cambridge.*

[Read *December* 11, 1837.]

THE object of the communication which I have now the honour of laying before the Society, is to present, in as simple a form as possible, the laws of the reflexion and refraction of sound, and of similar phenomena which take place at the surface of separation of any two fluid media when a disturbance is propagated from one medium to the other. The subject has already been considered by Poisson, (*Mém. de l'Acad.*, &c. Tome X. p. 317, &c.) The method employed by this celebrated analyst is one that he has used on many occasions with great success, and which he has explained very fully in several of his works, and recently in a digression on the Integrals of Partial Differential Equations. (*Théorie de la Chaleur*, p. 129, &c.) In this way, the question is made to depend on sextuple definite integrals. Afterwards, by supposing the initial disturbance to be confined to a small sphere in one of the fluids, and to be everywhere the same at the same distance from its centre, the formulæ are made to depend on double definite integrals; from which are ultimately deduced the laws of the propagation of the motion at great distances from the centre of the sphere originally disturbed.

The chance of error in every very long analytical process, more particularly when it becomes necessary to use Definite Integrals affected with several signs of integration, induced me to think, that by employing a more simple method we should possibly be led to some useful result, which might easily be overlooked in a more complicated investigation. With this impression, I endeavoured to ascertain how a plane wave of infinite extent, accompanied by its reflected and refracted waves, would be propagated in any two indefinitely extended media of which the surface of separation in a state of equilibrium should also be in a plane of infinite extent.

The suppositions just made simplify the question extremely. They may also be considered as rigorously satisfied when light is reflected. In which case the unit of space properly belonging to the problem is a quantity of the same order as $\lambda = \frac{1}{50,000}$ inch, and the unit of time that which would be employed by light itself in passing over this small space. Very often too, when sound is reflected, these suppositions will lead to sensibly correct results. On this last account, the problem has here been considered generally for all fluids whether *elastic* or *non-elastic* in the usual acceptation of these terms; more especially, as thus its solution is not rendered more complicated. One result of our analysis is so simple that I may perhaps be allowed to mention it here. It is this: If A be the ratio of the density of the reflecting medium to the density of the other, and B the ratio of the cotangent of the angle of refraction to the cotangent of the angle of incidence. Then for all fluids

$$\frac{\text{the intensity of the reflected vibration}}{\text{the intensity of the incident vibration}} = \frac{A - B}{A + B}.$$

If now we apply this to the reflexion of sound at the surface of still water, we have $A > 800$, and the maximum value of $B < \frac{1}{4}$. Hence the intensity of the reflected wave will in every case be sensibly equal to that of the incident one. This is what we should naturally have anticipated. It is however noticed here because M. Poisson has inadvertently been led to a result entirely different.

When the velocity of transmission of a wave in the second medium, is greater than that in the first, we may, by sufficiently increasing the angle of incidence in the first medium, cause the refracted wave in the second to disappear. In this case the change in the intensity of the reflected wave is here shown to be such that, at the moment the refracted wave disappears, the intensity of the reflected becomes exactly equal to that of the incident one. If we moreover suppose the vibrations of the incident wave to follow a law similar to that of the cycloidal pendulum, as is usual in the Theory of Light, it is proved that on farther increasing the angle of incidence, the intensity of the reflected wave remains unaltered whilst the phase of the vibration gradually changes. The laws of the change of intensity, and of the subsequent alteration of phase, are here given for all media, *elastic* or *non-elastic*. When, however, both the media are *elastic*, it is remarkable that these laws are precisely the same as those for light polarized in a plane perpendicular to the plane of incidence. Moreover, the disturbance excited in the second medium, when, in the case of total reflexion, it ceases to transmit a wave in the regular way, is represented by a quantity of which one factor is a negative exponential. This factor, for light, decreases with very great rapidity, and thus the disturbance is not propagated to a sensible depth in the second medium.

Let the plane surface of separation of the two media be taken as that of (yz), and let the axis of z be parallel to the line of intersection of the plane *front* of the wave with (yz), the axis of x being supposed vertical for instance, and directed downwards; then, if Δ and Δ_1 are the densities of the two media under the constant pressure P and s, s_1 the condensations, we must have

$$\begin{cases} \Delta(1+s) = \text{density in the upper medium,} \\ \Delta_1(1+s_1) = \text{density in the lower medium.} \end{cases}$$

$$\begin{cases} P(1+As) = \text{pressure in the upper medium,} \\ P(1+A_1 s_1) = \text{pressure in the lower medium.} \end{cases}$$

Also, as usual, let ϕ be such a function of x, y, z, that the resolved parts of the velocity of any fluid particle parallel to the axes, may be represented by

$$\frac{d\phi}{dx}, \quad \frac{d\phi}{dy}, \quad \frac{d\phi}{dz}.$$

In the particular case, here considered, ϕ will be independent of z, and the general equations of motion in the upper fluid will be

$$0 = \frac{ds}{dt} + \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2},$$

$$0 = \frac{d\phi}{dt} + \gamma^2 s;$$

where we have

$$\gamma^2 = \frac{PA}{\Delta},$$

or by eliminating s

$$(1) \qquad \frac{d^2\phi}{dt^2} = \gamma^2 \left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2}\right).$$

Similarly, in the lower medium

$$(2) \qquad \frac{d^2\phi_{,}}{dt^2} = \gamma_{,}^2 \left(\frac{d^2\phi_{,}}{dx^2} + \frac{d^2\phi_{,}}{dy^2}\right),$$

where

$$s_{,} = \frac{-d\phi_{,}}{\gamma_{,}^2 dt}, \quad \text{and } \gamma_{,}^2 = \frac{PA_{,}}{\Delta_{,}}.$$

The above are the known general equations of fluid motion, which must be satisfied for all the internal points of both fluids; but at the surface of separation, the velocities of the particles perpendicular to this surface and the pressure there must be the same for both fluids. Hence we have the particular conditions

$$\left.\begin{aligned} \frac{d\phi}{dx} &= \frac{d\phi_{,}}{dx} \\ As &= A_{,}s_{,} \end{aligned}\right\} \text{ (where } x = 0),$$

neglecting such quantities as are very small compared with those retained, or by eliminating s and $s_{,}$, we get

$$(A) \qquad \left.\begin{aligned} \frac{d\phi}{dx} &= \frac{d\phi_{,}}{dx} \\ \Delta \frac{d\phi}{dt} &= \Delta_{,} \frac{d\phi_{,}}{dt} \end{aligned}\right\} \text{ (when } x = 0).$$

The general equations (1) and (2), joined to the particular conditions (A) which belong to the surface of separation (yz), only, are sufficient for completely determining the motion of our two fluids, when the velocities and condensations are independent of the co-ordinate z, whatever the initial disturbance may be. We shall not here attempt to give their complete solution, which would be complicated, but merely consider the propagation of a plane wave of indefinite extent, which is accompanied by its reflected and refracted wave.

Since the disturbance of all the particles, in any *front* of the incident plane wave, is the same at the same instant, we shall have for the incident wave

$$\phi = f(ax + by + ct),$$

retaining b and c unaltered, we may give to the *fronts* of the reflected and refracted waves, any position by making for them

$$\phi = F(a'x + by + ct),$$

$$\phi_, = f_,(a_,x + by + ct).$$

Hence, we have in the upper medium,

$$(4) \qquad \phi = f(ax + by + ct) + F(a'x + by + ct),$$

and in the lower one

$$(5) \qquad \phi_, = f_,(a_,x + by + ct).$$

These, substituted in the general equations (1) and (2), give

$$c^2 = \gamma^2(a^2 + b^2),$$

$$(6) \qquad c^2 = \gamma^2(a'^2 + b^2),$$

$$c^2 = \gamma_,^2(a_,^2 + b^2).$$

Hence, $a' = \pm a$, where the lower signs must evidently be taken to represent the reflected wave. This value proves, that the angle of incidence is equal to that of reflexion. In like manner, the value of $a_,$ will give the known relation of sines for the incident and refracted wave, as will be seen afterwards.

Having satisfied the general equations (1) and (2), it only remains to satisfy the conditions (A), due to the surface of separation of the two media. But these by substitution give

$$af'(by+ct) - aF'(by+ct) = a_{,}f'_{,}(by+ct),$$

$$\Delta\{f'(by+ct) + F'(by+ct)\} = \Delta_{,}f'_{,}(by+ct),$$

because $a' = -a$, and $x = 0$.

Hence by writing, to abridge, the characteristics only of the functions

$$(7)\qquad \begin{aligned} f' &= \tfrac{1}{2}\left(\frac{\Delta_{,}}{\Delta} + \frac{a_{,}}{a}\right)f'_{,}, \\ F' &= \tfrac{1}{2}\left(\frac{\Delta_{,}}{\Delta} - \frac{a_{,}}{a}\right)f'_{,}, \end{aligned}$$

or if we introduce θ, $\theta_{,}$, the angles of incidence and refraction, since

$$\cot\theta = \frac{a}{b},$$

$$\cot\theta_{,} = \frac{a_{,}}{b},$$

$$f' = \tfrac{1}{2}\left(\frac{\Delta_{,}}{\Delta} + \frac{\cot\theta_{,}}{\cot\theta}\right)f'_{,},$$

$$F' = \tfrac{1}{2}\left(\frac{\Delta_{,}}{\Delta} - \frac{\cot\theta_{,}}{\cot\theta}\right)f'_{,},$$

and therefore $$\frac{F'}{f'} = \frac{\dfrac{\Delta_{,}}{\Delta} - \dfrac{\cot\theta_{,}}{\cot\theta}}{\dfrac{\Delta_{,}}{\Delta} + \dfrac{\cot\theta_{,}}{\cot\theta}},$$

which exhibits under a very simple form, the ratio between the intensities of the disturbances, in the incident and reflected wave.

But the equations (6) give

$$\gamma^2\left(\frac{a^2}{b^2} + 1\right) = \gamma_{,}^2\left(\frac{a_{,}^2}{b^2} + 1\right);$$

and hence

$$\frac{\gamma}{\sin\theta} = \frac{\gamma_{,}}{\sin\theta_{,}},$$

the ordinary law of sines.

The reflected wave will vanish when

$$0 = \frac{\Delta_{,}}{\Delta} - \frac{\cot \theta_{,}}{\cot \theta};$$

which with the above gives

$$\cot \theta = \Delta \sqrt{\frac{\gamma^2 - \gamma_{,}^2}{(\gamma_{,} \Delta_{,})^2 - (\Delta \gamma)^2}}.$$

Hence the reflected wave may be made to vanish if $\gamma^2 - \gamma_{,}^2$ and $(\gamma \Delta)^2 - (\gamma_{,} \Delta_{,})$ have different signs.

For the ordinary elastic fluids, at least if we neglect the change of temperature due to the condensation, A is independent of the nature of the gas, and therefore

$$A = A_{,} \text{ or } \gamma^2 \Delta = \gamma_{,}^2 \Delta_{,}.$$

Hence

$$\tan \theta = \frac{\gamma}{\gamma_{,}},$$

which is the precise angle at which light polarized perpendicular to the plane of reflexion is wholly transmitted.

But it is not only at this particular angle that the reflexion of sound agrees in intensity with light polarized perpendicular to the plane of reflexion. For the same holds true for every angle of incidence. In fact, since

$$\gamma^2 \Delta = \gamma_{,}^2 \Delta_{,}; \quad \therefore \frac{\Delta_{,}}{\Delta} = \frac{\gamma^2}{\gamma_{,}^2} = \frac{\sin^2 \theta}{\sin^2 \theta_{,}},$$

and the formulæ (7) give

$$\frac{F'}{f''} = \frac{\dfrac{\sin^2 \theta}{\sin^2 \theta_{,}} - \dfrac{\tan \theta}{\tan \theta_{,}}}{\dfrac{\sin^2 \theta}{\sin^2 \theta_{,}} + \dfrac{\tan \theta}{\tan \theta_{,}}} = \frac{\tan (\theta - \theta_{,})}{\tan (\theta + \theta_{,})};$$

which is the same ratio as that given for light polarized perpendicular to the plane of incidence. (Vide Airy's *Tracts*, p. 356.)

What precedes is applicable to all waves of which the *front* is plane. In what follows we shall consider more particularly the case in which the vibrations follow the law of the cycloidal pendulum, and therefore in the upper medium we shall have,

$$(8) \qquad \phi = \alpha \sin (a x + b y + c t) + \beta \sin (-a x + b y + c t).$$

Also, in the lower one,

$$\phi_{,} = \alpha_{,} \sin (a_{,} x + b y + c t):$$

and as this is only a particular case of the more general one, before considered, the equation (7) will give

$$\alpha = \tfrac{1}{2}\left(\frac{\Delta_{,}}{\Delta} + \frac{a_{,}}{a}\right) \alpha_{,},$$

$$\beta = \tfrac{1}{2}\left(\frac{\Delta_{,}}{\Delta} - \frac{a_{,}}{a}\right) \alpha_{,}.$$

If $\gamma_{,} > \gamma$, or the velocity of transmission of a wave, be greater in the lower than in the upper medium, we may by decreasing a render $a_{,}$ imaginary. This last result merely indicates that the form of our integral must be changed, and that as far as regards the co-ordinate x an exponential must take the place of the circular function. In fact the equation,

$$\frac{d^2\phi_{,}}{dt^2} = \gamma_{,}^2 \left\{\frac{d^2\phi_{,}}{dx^2} + \frac{d^2\phi_{,}}{dy^2}\right\};$$

may be satisfied by

$$\phi_{,} = \epsilon^{-a_{,}' x} . B \sin \psi,$$

(where, to abridge, ψ is put for $b y + c t$) provided

$$c^2 = \gamma_{,}^2 (- a_{,}'^2 + b^2);$$

when this is done it will not be possible to satisfy the conditions (A) due to the surface of separation, without adding constants to the quantities under the circular functions in ϕ. We must therefore take, instead of (8), the formula,

$$(9) \quad \phi = \alpha \sin (a x + b y + c t + e) + \beta \sin (- a x + b y + c t + e_{,}).$$

Hence, when $x = 0$, we get

$$\frac{d\phi}{dx} = a\,\alpha \cos(\psi + e) - a\,\beta \cos(\psi + e_{,}),$$

$$\frac{d\phi}{dt} = c\,\alpha \cos(\psi + e) + c\,\beta \cos(\psi + e_{,}),$$

$$\frac{d\phi_{,}}{dx} = - a_{,}'\,B \sin\psi,$$

$$\frac{d\phi}{dt} = c\,B \cos\psi\,;$$

these substituted in the conditions (A), give

$$\alpha \cos(\psi + e) - \beta \cos(\psi + e_{,}) = - \frac{a_{,}'}{a} B \sin\psi,$$

$$\alpha \cos(\psi + e) + \beta \cos(\psi + e_{,}) = \frac{\Delta_{,}}{\Delta} B \cos\psi\,;$$

these expanded, give

$$\alpha \cos e - \beta \cos e_{,} = 0,$$

$$- \alpha \sin e + \beta \sin e_{,} = - \frac{a_{,}'}{a} B,$$

$$\alpha \cos e + \beta \cos e_{,} = \frac{\Delta_{,}}{\Delta} B,$$

$$\alpha \sin e + \beta \sin e_{,} = 0.$$

Hence, we get,

$$(10) \qquad 2\,\alpha \sin e = \frac{a_{,}'}{a} B,$$

$$2\,\alpha \cos e = \frac{\Delta_{,}}{\Delta} B,$$

$$2\,\beta \sin e_{,} = - \frac{a_{,}'}{a} B,$$

$$2\,\beta \cos e_{,} = \frac{\Delta_{,}}{\Delta} B,$$

and, consequently,

$$e = -e_{,}, \qquad \beta = \alpha,$$

and

$$\tan e = +\frac{a_{,}'\Delta}{a\Delta_{,}}.$$

This result is general for all fluids, but if we would apply it to those only which are usually called *elastic*, we have, because in this case $\gamma^2\Delta = \gamma_{,}^2\Delta_{,}$,

$$\tan e = \frac{a_{,}'\Delta}{a\Delta_{,}} = \frac{a_{,}'\gamma_{,}^2}{a\gamma^2}.$$

But generally

$$(11) \qquad c^2 = \gamma_{,}^2(-a_{,}'^2 + b^2) = \gamma^2(a^2 + b^2);$$

and therefore, by substitution,

$$\tan e = \frac{a_{,}'\gamma_{,}^2}{a\gamma^2} = \frac{\gamma_{,}\sqrt{\gamma_{,}^2 b^2 - (a^2 + b^2)\gamma^2}}{a\gamma^2} = \mu\sqrt{\mu^2\tan^2\theta - \sec^2\theta},$$

because $\mu = \frac{\gamma_{,}}{\gamma}$, and $\frac{b}{a} = \tan\theta$.

As $e = -e_{,}$ we see from equation (9), that $2e$ is the change of phase which takes place in the reflected wave; and this is precisely the same value as that which belongs to light polarized perpendicularly to the plane of incidence; (Vide Airy's *Tracts*, p. 362.) We thus see, that not only the intensity of the reflected wave, but the change of phase also, when reflexion takes place at the surface of separation of two elastic media, is precisely the same as for light thus polarized.

As $\alpha = \beta$, we see that when there is no transmitted wave the intensity of the reflected wave is precisely equal to that of the incident one. This is what might be expected: it is, however, noticed here because a most illustrious analyst has obtained a different result. (Poisson, *Memoires de l'Academie des Sciences,* Tome X.) The result which this celebrated mathematician arrives at is, That at the moment the transmitted wave ceases to exist, the intensity of the reflected becomes precisely equal to that of the incident wave. On increasing the angle of incidence this intensity again diminishes, until it vanish at a certain angle. On still farther increasing this angle the intensity continues to increase, and again becomes equal to that of the incident wave, when the angle of incidence becomes a right angle.

It may not be altogether uninteresting to examine the nature of the disturbance excited in that medium which has ceased to transmit a wave in the regular way. For this purpose, we will resume the expression,

$$\phi_{,} = B\epsilon^{-a_{,}'x} \sin \psi = B\,\epsilon^{-a_{,}'x} \sin (b\,y + c\,t)\,;$$

or if we substitute for B, its value given by the last of the equations (10); and for $a_{,}'$, its value from (11); this expression, in the case of ordinary elastic fluids where $\gamma^2 \Delta = \gamma_{,}^2 \Delta_{,}$, will reduce to

$$\phi_{,} = 2\,a\,\mu^2 \cos e \,.\, \epsilon^{-\frac{2\pi x}{\lambda}\sqrt{\frac{\mu^2 \sin^2\theta - 1}{\mu}}} \sin (b\,y + c\,t),$$

λ being the length of the incident wave measured perpendicular to its own front, and θ the angle of incidence. We thus see with what rapidity in the case of light, the disturbance diminishes as the depth x below the surface of separation of the two media increases; and also that the rate of diminution becomes less as θ approaches the *critical* angle, and entirely ceases when θ is exactly equal to this angle, and the transmission of a wave in the ordinary way becomes possible.

XIX. *On a new Genus of Fossil Multilocular Shells, found in the Slate-Rocks of Cornwall.* *By* D. T. Ansted, B.A. *of Jesus College; Fellow of the Society, and of the Geological Society.*

[Read *February* 26, 1838.]

It is not many years since the slate-rocks of Cornwall were described as contemporary in their formation with the Granite, and other igneous and altered rocks of that county. They were of course presumed at that time to be absolutely without trace of fossils; and when remains of organic life were first observed, the very possibility was questioned; but, after some doubt and sufficient inquiry, the fact was admitted, to the overthrow of the theory alluded to.

Since it has been granted that fossils may be expected in these beds, the search after them has not been unattended with success. Among others, Professor Sedgwick, during his geological researches in the South-west of England, has obtained many, in various states of preservation, which, with a few collected from the neighbourhood of Petherwyn, are now in the Woodwardian Museum; and it was during the temporary arrangement of these specimens, that I was struck with the occurrence of what seemed to me a new genus of multilocular shells, and induced to lay this paper before the Society.

The rest of the organic remains consist chiefly of fossil marine vegetables; many shells allied to terebratula; several species of orthoceratites; portions of the stems of radiated animals, and parts of some very small trilobites. Besides these, there are two species of a genus which has generally been considered identical with *goniatites*, although it appears to depart very widely from the type of that curious and well-known group.

In the *Annales des Sciences Naturelles*, for 1834*, will be found the translation of a paper by Count Münster, announcing the discovery of a new genus, which he calls *Clymenia*, and which he found among several new species of goniatites, in the transition limestone of the Fichtelgebirge†. It is to this genus, hitherto unknown in English Palæontology, that the newly-discovered Cornish fossils must be referred; although there appears to be so much difference in some respects, that they may possibly form a sub-genus, peculiar to the formation in which they occur.

The name Clymenia, however, is peculiarly unfortunate, both because it is already appropriated, (for it designates one of Cuvier's genera of Annelides,) and also from its entire want of analogy with all other names of fossil cephalopods. As it must be abandoned, I would propose to call the genus *Endosiphonites*, which has the advantage of indicating the most remarkable and important character—the ventral position of the siphuncle; while at the same time it sufficiently resembles the names of allied genera, and by a slight alteration of the first two syllables, might be applied to mark the different positions of the siphuncle which characterise ammonites, nautilites, &c.

The peculiar character of this genus is, as I have already remarked, the position of the siphuncle, in which it differs both from ammonite

* When my paper was read before the Society in February, I was not aware of the existence of this notice in the *Annales des Sciences*. It is to Professor Phillips that I am indebted for the reference; and he has already made use of it in the article "Goniatite," in the *Penny Cyclopedia*.

† These hills are situated in the South-eastern part of Germany, to the East of the Maine, and not very far North of Nuremberg. The central ridge is of granite and the transition limestone.

and nautilus. According to Count Münster, the species from the Fichtelgebirge admit of the following description:—The narrow siphuncle is constantly found on the ventral part of the spiral shell, where it passes through a succession of small funnel-shaped apertures in the chambers. The whorls of the spire are free, never entirely enveloping the inner ones; and the last, and part of the last but one, of the turns have no septa. The intersections of the septa with the shell form undulations, or simple lateral lobes, at oblique angles, and dorsal and lateral rounded saddles*; but the line of intersection is not denticulated as in goniatites, or marked in the more intricate lines which characterise the ammonite. The siphuncle not being generally visible. it is by means of the dorsal saddle that this new genus is distinguished from goniatite, which always has a dorsal lobe on the medial line of the back. Count Münster elsewhere observes, that it is so difficult to obtain specimens having the septa apparent, that without extreme care it is almost impossible to avoid error; and that it is still more rare to find the siphuncle visible, since in the new genus, as well as in goniatites, it is so slender and so close to the shell as to be usually invisible, even when the marble in which it is found is polished.

Now the condition of the Cornish specimens I have examined is very different from that of the German ones, and much more perfect in some respects than these seem to have been; but there are many points in the above description which do not at all agree with my observations. One of the most important of these is the nature of the siphuncle, which seems to be obscure in Count Münster's species, but is very prominent and easily seen in those which I have made out. But it is not only easily seen—it is decidedly large; and although near the shell cannot possibly be overlooked. In one species the diameter of the aperture in the septum is one-fifth of the extreme length of the septum; a proportion much larger than is commonly found in any species of nautilus, and which indeed is only paralleled in a

* The word *saddle* is here used to denote those separations between the lobes upon which the mantle of the animal is supposed to have rested. Dr Buckland has explained the language introduced by Von Buch on this subject, in a note, page 353, of his Bridgewater Treatise, to which I must refer for a more complete explanation.

few species of orthoceratites. If then, as there seems every reason to suppose, the siphuncle is the most important character in the shells of multilocular cephalopods, this very great difference would of itself warrant the formation of a separate group. All the species from Cornwall are provided with decidedly large apertures in the chambers, and in all, these funnel-shaped tubes are easily seen, produced beyond the septum about half way into the next chamber.

But again, the markings on the shell which seem so useful in determining Count Münster with regard to any doubtful cases, are in our English species apparently not to be depended on. Our fossils are in beautiful condition; the actual shell certainly remains in one specimen at least, and we can trace a succession of transverse striæ marked with great beauty and regularity upon it; but although the casts of the chambers may be separately examined, the nature and use of the lobes does not quite appear. One thing is certain, they do *not* correspond to the intersection of the septa and the shell, and in only one of three species do they occur at all. Some idea of the form of a septum will be obtained from Fig. 4, Plate VIII. which represents a side and front view of the cast of a chamber belonging to a species not determined.

The technical description of the genus will be thus expressed:—a discoid spiral multilocular shell; sides nearly simple; whorls contiguous, the last not enveloping the rest. Septa transverse, numerous, concave outwardly, and perforated on the ventral margin for a siphuncle.

In order to determine the place of this genus among other shells of cephalopods, it will be necessary to pay most attention to goniatite and nautilus, as it is to these that the nearest approximation is made. Von Buch gives as the character of the former group, the dorsal siphuncle of the ammonite, comparatively small and delicate; the lobes of the septa completely deprived of lateral denticulations, and the striæ of growth resembling those of nautilus, in not being directed forwards, as in ammonites, but reflected backwards. The nautilus is known by its usually central and comparatively large siphuncle, and the greater or less envelopement of the whorls of the spire by the last one formed.

The new genus Endosiphonites differs from goniatites and nautilus in the position of the siphuncle, and it agrees with both in the direction of the striæ of growth. The whorls, though contiguous, never envelope; and the septa seem usually very nearly simple. Taking D'Orbigny's classification, which being founded chiefly on the position of the siphuncle, must be preferred, we refer this genus to the family Nautilacea, and place it between Nautilus and Lituite.

We come now to consider the species, and the English localities not having been searched with a view to the discovery of these fossils, we have at present only three sufficiently well defined to admit of description. These are figured in Plate VIII. and I have named them as follows:—

(1). *Endosiphonites Münsteri nob.* PLATE VIII., Fig. 1. This being the largest species, I have named it in honour of the first discoverer of the genus. The individual figured is an extremely beautiful fossil, and, as will be seen from the figure, shows the siphuncle very clearly. It is partially burnt, having been obtained by the fracture of a lump which had passed through a kiln without being reduced to a calx; so that most of the striæ of growth are destroyed, although I have no doubt they were before the burning sufficiently clear. It is the only species in which there are decided lobes, and is remarkable for the very large size of the last whorl, the area of the aperture being more than five times as large as that of the corresponding chamber in the former whorl. It is also very flat, the length of the aperture being more than twice its width, and this without any appearance of the shell having been injured or crushed. It measures four inches across.

(2). *E. carinatus nob.* PLATE VIII., Fig. 2. This species is remarkable for its elliptical form, and for having a keel running along the dorsal margin. Its dimensions are fourteen-tenths by nine-tenths of an inch. It is marked by a series of fine but beautifully distinct striæ, which may be clearly seen in some parts of the specimen figured, where in all probability the original shell remains. This is the species alluded to above as having a very large siphuncle, seen in the figure at (*a*).

It is also one of those without the very slightest trace of the lobes alluded to by Count Münster.

(3). *E. minutus nob.* PLATE VIII., Fig. 3. twice the natural size. This very small and pretty species much resembles two of those described by Count Münster, except in the size of the siphuncle, but I have thought it better to give a new name, because of the great importance of this difference. Waved striæ may be observed on some parts of the only specimen I have examined, but they are very minute, and required the aid of the microscope to discover them.

It remains now that we consider, from analogy with known genera, how far the animal inhabitant of this new genus may have resembled, in its habits or locality, those of other multilocular shells most nearly allied to it.

What then are those points in the description of the shell that tell most of the history of the animal, and what light is thus thrown on the subject now under consideration?

It was the opinion of Von Buch, an opinion strengthened by the later researches of Dr Buckland, that the siphuncle must be considered as an all-important organ in the structure of a multilocular shell. It is true that the position of the tube has generally been considered much more than its magnitude, but the size must not be neglected; for assuming Dr Buckland's opinion of its use to be true, *viz.* that the whole mass of the animal and shell has its specific gravity changed by the pericardial fluid passing into the siphuncle, it is quite clear that the larger this tube in proportion to the area of the septa, the more sudden will be the change of specific gravity, and consequently the greater the facility with which the animal could alter its depth in the water.

Now in almost all the known species of the family nautilacea, this contrivance is large, well defended, and eminently adapted for resisting external injury, while on the other hand, it is comparatively rare to find a large siphuncle in an ammonite, or any allied genus, and sometimes the tube is found to have dwindled away and become a mere thread. While, however, the siphuncle diminishes in size and importance, the general shape of the shell and peculiar form of the septa indicate an increased capacity for resisting pressure and supporting the weight of a high column of water.

Perhaps, viewing the subject in this light, we may not be far wrong in assuming a natural ground of separation between these two families of cephalopods, since the one appears to have a contrivance for enabling it to swim freely in the ocean, and rise or sink at pleasure, while in the other, there is only as it were the rudimentary appearance of this contrivance; but, on the other hand, additional strength in its habitation, fitting it to dwell more at the bottom of the sea and at considerable depths, and there to keep within necessary limits those crustacea and molluscs, which might otherwise, by their rapid increase, have interfered with the established course of nature.

In applying this theory, if it may be called so, to our new genus, we must necessarily consider separately the group described by Count Münster as having a small siphuncle, and the species now introduced to your notice. In the former, there seems to be a provision for strength, without great power of locomotion; for the septa seem less simple than even in some goniatites, and the lobes must be supposed to increase the resisting power. In the latter there are no lobes, but the siphuncle being so much larger, we may reasonably suppose that the extent of the inhabitant's power of altering readily its depth in the water, must have been in a corresponding degree greater.

The study of comparative anatomy introduces to our notice, in a very striking manner, the strong resemblances in the structure of different animals, and the universal occurrence of what would seem *rudimentary attempts* at higher and more complete organisation. Such, for instance, are the rudimentary bones in the fins of swimming mammalia, which correspond to the bones of the extremities in man; and such would seem to be the case in this siphuncle, sometimes very large, then diminishing in size and importance, till it dwindles down to the merest thread, which can no longer be capable of performing any office in the animal economy.

As an extreme instance of this, I would refer to the fossil represented in Plate VIII., Figs. 5, 6. It is in extremely good preservation, but does not show the slightest appearance of a siphuncle on the dorsal margin, or elsewhere, although it resembles in some respects one of Count Münster's goniatites, named G. subsulcatus. I have had it figured, because it shows very beautifully the singular extent to which the envelopement of one whorl by the rest is sometimes carried, and the marked resemblance which the specimen bears to some of the microscopic genera of D'Orbigny's *Foraminifera*. Its shape is lenticular, and it measures more than three-quarters of an inch across. In the absence of better information, I am compelled to call it a goniatite, but I cannot help thinking, that for this and many other species also doubtful, it may be found necessary to establish a separate group, founded on the almost total absence of the siphuncle.

In conclusion, I would observe, that among the known, but as yet undescribed fossils of the *Silurian System*, there is no instance of any species referrible to our new genus; and thus we have another instance of the wide separation denoted even by the zoological character of these ancient formations, which are indeed sufficiently distinct by the known occurrence of intervening deposits. It is the opinion of Professor Sedgwick, that these Cornish rocks, which contain the organic remains described, are the lowest fossiliferous rocks of Devonshire and Cornwall, and far, very far removed in the order of their deposit from the mountain limestone, with which it has been attempted to identify them.

D. T. ANSTED.

JESUS COLLEGE,
18th *May*, 1838.

XX. *On a Question in the Theory of Probabilities.* By AUGUSTUS DE MORGAN, *of Trinity College, Professor of Mathematics in University College, London.*

[Read *February* 26, 1837.]

THE object of this paper is the correction of an oversight made both by Laplace and M. Poisson, in pages 279 and 209 of their respective works on The Theory of Probabilities.

The reputation of neither of those analysts requires an explanatory eulogium to accompany the detection of an error in their writings, particularly on a subject so liable to cause mistake as the theory in question: I shall therefore proceed at once to the point. Both arrive correctly at the conclusion, that*

$$\frac{2}{\sqrt{\pi}} \int_0^{l\sqrt{\frac{n}{2vw}}} \epsilon^{-t^2} dt + \frac{\epsilon^{-\frac{l^2 n}{2vw}}}{\sqrt{2\pi vw}} \qquad (1)$$

represents the probability that the number of arrivals of A shall fall between $v-l$ and $v+l$, both inclusive, where $n\,(=v+w)$ is the number of trials, and v and w are proportional to the chances of arrival or non-arrival in a single trial. That is, if the number of times which A will happen in n trials be called A_n, the preceding formula is the probability that u, as deduced from

$$A_n = np + u, \qquad \left(p = \frac{v}{n}\right),$$

will lie between $-l$ and $+l$: on the supposition that p is given, and A_n to be found by trial. And both Laplace and M. Poisson immediately infer that the preceding result *therefore* represents the same probability in the case where A_n has been observed, and p is to be

* See the Addition at the end.

inferred by reasoning from the observed event to the probability of its cause. That is, they assume in effect that the probability of the equation $\phi(x, y) = a$, where x is given and y presumed, must be the same as in the case where y is given and x presumed. The preceding formula is neither admissible upon the reasoning produced, nor in fact correct: as the following investigation will shew.

There having been made n (or $v + w$) trials, at each of which either A or B must have happened; and A having happened v times, and B w times: required the presumption that the probability of A happening lay between two given limits a and b ($b > a$).

The presumption that this probability lies between a and b, is

$$\frac{\int_a^b x^v (1-x)^w dx}{\int_0^1 x^v (1-x)^w dx}, \qquad (2)$$

to the approximate determination of which, when $v + w$ is a considerable number, I proceed to apply the method of Laplace.

Let y be a function of x which vanishes when $x = 0$ and when $x = 1$; and let Y, the intermediate maximum value of y, correspond to $x = X$. Assume $y = Y\epsilon^{-t^2}$, so that while x increases from 0 to X, and from X to 1, t shall increase from $-\infty$ to 0, and from 0 to $+\infty$. Let $x = X + \theta$, and determine θ from

$$Y + Y'' \frac{\theta^2}{2} + Y''' \frac{\theta^3}{2.3} + Y'''' \frac{\theta^4}{2.3.4} + \ldots\ldots = Y\epsilon^{-t^2}, \qquad (3)$$

Y' being $= 0$, since Y is a maximum value of y.

Let this process give

$$\theta = B_1 t + B_2 t^2 + B_3 t^3 + \ldots\ldots$$

and let $x = a$, and $x = b$, give $t = \mu$, and $t = \nu$.

Then, since $dx = d\theta$

$$\int_a^b y\,dx = Y\{B_1 \int_\mu^\nu \epsilon^{-t^2} dt + 2B_2 \int_\mu^\nu \epsilon^{-t^2} t\,dt + 3B_3 \int_\mu^\nu \epsilon^{-t^2} t^2 dt + \ldots\ldots\}$$

$$= Y\left(B_1 + \frac{3}{2} B_3 + \ldots\right) \int_\mu^\nu \epsilon^{-t^2} dt + Y\left(B_2 + \frac{3\mu}{2} B_3 + \ldots\right) \epsilon^{-\mu^2}$$

$$- Y\left(B_2 + \frac{3\nu}{2} B_3 + \ldots\right) \epsilon^{-\nu^2}. \qquad (4)$$

The general formula for the determination of B_n is

$$B_n = \left\{\frac{1}{2.3\ldots n}\frac{d^n}{dx^n}\left(\frac{x-X}{\sqrt{\log Y - \log y}}\right) \text{ when } x = X\right\},$$

but the first terms may be readily found by actual reversion of (3), which gives, if Y_n stand for $\frac{Y^{(n)}}{Y}$,

$$B_1 = \sqrt{-\frac{2}{Y_2}}, \qquad B_2 = \frac{1}{3}\frac{Y_3}{Y_2^2},$$

$$B_3 = \frac{5Y_3^2 - 3Y_2(Y_4 - 3Y_2^2)}{18Y_2^4}\cdot\sqrt{-\frac{Y_2}{2}}.$$

When $a = 0$, $b = 1$, we have $\mu = -\infty$, $\nu = +\infty$

$$\int_0^1 y\,dx = Y(B_1 + \tfrac{3}{2}B_3 + \ldots)\int_{-\infty}^{+\infty}\epsilon^{-t^2}dt,$$

whence the formula (2) becomes

$$\frac{1}{\sqrt{\pi}}\int_\mu^\nu \epsilon^{-t^2}dt + \frac{2B_2(\epsilon^{-\mu^2} - \epsilon^{-\nu^2}) + 3B_3(\mu\epsilon^{-\mu^2} - \nu\epsilon^{-\nu^2})}{(2B_1 + 3B_3)\sqrt{\pi}}, \qquad (5)$$

neglecting the terms not previously expressed.

Tables of the values of

$$\frac{1}{\sqrt{\pi}}\int_{-\mu}^{+\mu}\epsilon^{-t^2}dt, \text{ or } \frac{2}{\sqrt{\pi}}\int_0^\mu \epsilon^{-t^2}dt,$$

were published in the Berlin *Astronomisches Jahrbuch* for 1834, and are reprinted* at the end of my article on the Theory of Probabilities, in

* The increasing importance of this Table makes it worth while here to state that it will shortly appear in a comparatively popular Essay on Probabilities on which I am now engaged, and also, as I am informed, in the Article on the subject in the new edition of the *Encyclopaedia Britannica*.

the *Encyclopaedia Metropolitana.* By means of this table, and its differences, and the formulæ

$$\Delta x \frac{dy}{dx} = \Delta y - \tfrac{1}{2}\Delta^2 y + \tfrac{1}{3}\Delta^3 y - \dots\dots$$

$$(\Delta x)^2 \frac{d^2 y}{dx^2} = \Delta^2 y - \Delta^3 y + \dots\dots$$

we can also determine with sufficient accuracy

$$\frac{\epsilon^{-\mu^2}}{\sqrt{\pi}} \text{ and } \frac{\mu\epsilon^{-\mu^2}}{\sqrt{\pi}}.$$

Thus, u, Δu, &c. being taken positively as in the table (remembering that $\Delta^2 u$ is really negative, and that $\Delta x = \cdot 01$), we have

$$\frac{2}{\sqrt{\pi}} \int_0^\mu \epsilon^{-t^2} dt = u,$$

$$\left.\begin{aligned} \frac{2\epsilon^{-\mu^2}}{\sqrt{\pi}} &= 100\left(\Delta u + \tfrac{1}{2}\Delta^2 u + \tfrac{1}{3}\Delta^3 u\right) \\ \frac{4\mu\epsilon^{-\mu^2}}{\sqrt{\pi}} &= 10000\left(\Delta^2 u + \Delta^3 u\right) \end{aligned}\right\} \text{nearly.}$$

In the case before us we have, writing ϖ for X,

$$y = x^v (1-x)^w, \qquad \varpi = \frac{v}{v+w}.$$

Differentiate $\log y$ four times in succession;

$$\frac{y'}{y} = \left\{\frac{v}{x} - \frac{w}{(1-x)}\right\},$$

$$\frac{y''}{y} - \left(\frac{y'}{y}\right)^2 = -\left\{\frac{v}{x^2} + \frac{w}{(1-x)^2}\right\},$$

$$\frac{y'''}{y} - 3\frac{y'}{y}\frac{y''}{y} + 2\left(\frac{y'}{y}\right)^3 = 2\left\{\frac{v}{x^3} - \frac{w}{(1-x)^3}\right\},$$

$$\frac{y''''}{y} - 4\frac{y'}{y}\frac{y'''}{y} + 12\left(\frac{y'}{y}\right)^2\frac{y''}{y} - 3\left(\frac{y''}{y}\right)^2 - 6\left(\frac{y'}{y}\right)^4 = -6\left\{\frac{v}{x^4} + \frac{w}{(1-x)^4}\right\}.$$

From which, making $x = \frac{v}{v + w} = \varpi$,

$$Y_1 = 0, \quad Y_2 = -\frac{(v+w)^3}{vw} = -\frac{n}{\varpi(1-\varpi)},$$

$$Y_3 = -\frac{2(v+w)^4(v-w)}{v^2w^2} = -\frac{2(2\varpi - 1)n}{\varpi^2(1-\varpi)^2},$$

$$Y_4 - 3Y_2^2 = -\frac{6(v+w)^5(v^2 - vw + w^2)}{v^3w^3} = -\frac{6(3\varpi^2 - 3\varpi + 1)n}{\varpi^3(1-\varpi)^3};$$

$$B_1 = \sqrt{\frac{2vw}{(v+w)^3}} = \sqrt{\frac{2\varpi(1-\varpi)}{n}},$$

$$B_2 = -\frac{2}{3}\frac{v-w}{(v+w)^2} = \frac{2}{3}\frac{1-2\varpi}{n},$$

$$B_3 = \frac{v^2 + w^2 - 11vw}{9(v+w)^4} \cdot \sqrt{\frac{(v+w)^3}{2vw}} = \frac{13\varpi^2 - 13\varpi + 1}{9\sqrt{\varpi(1-\varpi)}} \frac{1}{\sqrt{2n^3}};$$

$$2B_1 + 3B_3 = \frac{12vw(v+w) + v^2 + w^2 - 11vw}{3(v+w)^4} \cdot \sqrt{\frac{(v+w)^3}{2vw}}$$

$$= \frac{12n\varpi(1-\varpi) + 13\varpi^2 - 13\varpi + 1}{3\sqrt{2n\varpi(1-\varpi)}},$$

whence the formula (5) becomes

$$\frac{1}{\sqrt{\pi}}\int_\mu^\nu \epsilon^{-t^2} dt$$

$$+ \frac{4\sqrt{2n}(1-2\varpi)\sqrt{\varpi(1-\varpi)}(\epsilon^{-\mu^2} - \epsilon^{-\nu^2}) + (13\varpi^2 - 13\varpi + 1)(\mu\epsilon^{-\mu^2} - \nu\epsilon^{-\nu^2})}{\{12n\varpi(1-\varpi) + 13\varpi^2 - 13\varpi + 1\}\sqrt{\pi}}.$$

The rejected terms are at lowest of the order $\frac{1}{n^2}$; and μ and ν are determined from

$$a^{n\varpi}(1-a)^{n(1-\varpi)} = \varpi^{n\varpi}(1-\varpi)^{n(1-\varpi)}\epsilon^{-\mu^2},$$

$$b^{n\varpi}(1-b)^{n(1-\varpi)} = \varpi^{n\varpi}(1-\varpi)^{n(1-\varpi)}\epsilon^{-\nu^2};$$

$$\left.\begin{aligned} \text{or, } \mu^2 &= n\varpi \log\frac{\varpi}{a} + n(1-\varpi)\log\frac{1-\varpi}{1-a} \\ \nu^2 &= n\varpi \log\frac{\varpi}{b} + n(1-\varpi)\log\frac{1-\varpi}{1-b} \end{aligned}\right\} \quad \ldots\ldots(7).$$

Let $a = \varpi - \theta$, $b = \varpi + \theta$, where θ is a small fraction both of ϖ and $1 - \varpi$: then

$$\mu = -\frac{\sqrt{n}\theta}{\sqrt{2\varpi(1-\varpi)}}, \quad \nu = +\frac{\sqrt{n}\theta}{\sqrt{2\varpi(1-\varpi)}}, \text{ nearly:}$$

and the presumption that p, the probability of the arrival of A, lies between

$$\varpi \pm \mu\sqrt{\frac{2\varpi(1-\varpi)}{n}}$$

$$\text{is } \frac{1}{\sqrt{\pi}}\int_{-\mu}^{+\mu} \epsilon^{-t^2}dt - \frac{(13\varpi^2 - 13\varpi + 1)\mu\epsilon^{-\mu^2}}{6n\varpi(1-\varpi)\sqrt{\pi}}.$$

In the formula of Laplace and M. Poisson, the result has

$$+\frac{\epsilon^{-\mu^2}}{\sqrt{2n\pi\varpi(1-\varpi)}} \text{ instead of } -\frac{(13\varpi^2 - 13\varpi + 1)\mu\epsilon^{-\mu^2}}{6n\varpi(1-\varpi)\sqrt{\pi}}.$$

The latter is of a lower order than the former, and may safely be rejected. By taking successively for the limits

$$\varpi - \theta \text{ and } \varpi, \quad \varpi - \theta \text{ and } \varpi + \theta, \quad \varpi \text{ and } \varpi + \theta,$$

we find that the presumption of p lying between the several pairs of limits stated in the first column is expressed by the formulæ in the second.

p lies between	Probability of the preceding.
$\varpi - \mu\sqrt{\frac{2\varpi(1-\varpi)}{n}}$ and ϖ	$\frac{1}{\sqrt{\pi}}\int_{-\mu}^{0}\epsilon^{-t^2}dt + \frac{\sqrt{2}(1-2\varpi)}{3\sqrt{\pi n\varpi(1-\varpi)}}\{\epsilon^{-\mu^2}-1\}$
$\varpi - \mu\sqrt{\frac{2\varpi(1-\varpi)}{n}}$ and $\varpi + \mu\sqrt{\frac{2\varpi(1-\varpi)}{n}}$	$\frac{1}{\sqrt{\pi}}\int_{-\mu}^{+\mu}\epsilon^{-t^2}dt$
ϖ and $\varpi + \mu\sqrt{\frac{2\varpi(1-\varpi)}{n}}$	$\frac{1}{\sqrt{\pi}}\int_{0}^{\mu}\epsilon^{-t^2}dt + \frac{\sqrt{2}(1-2\varpi)}{3\sqrt{\pi n\varpi(1-\varpi)}}\{1-\epsilon^{-\mu^2}\}$

the first and third become equal when $\varpi = \frac{1}{2}$; that is, when *A* happens as often as *B* in n trials: a result which we might have looked for *à priori*. It also appears that when ϖ is less than $\frac{1}{2}$, it is more likely that p should exceed ϖ than fall short of it; which is in accordance with another result of the theory, namely, that the chance of drawing *A* at the $(n+1)^{\text{th}}$ trial is $\dfrac{v+1}{v+w-2}$, which is nearer to $\frac{1}{2}$ than $\dfrac{v}{v+w}$.

Let $\varpi = 1 - \kappa$ where κ is small $\left(\text{not being less than } \dfrac{1}{n}\right)$.

Let the limits be $a = 1 - \lambda\kappa$ and $b = 1$; where λ is greater than unity, or μ is negative, and ν is positive and infinite. We have also

$$\mu^2 = n(1-\kappa)\log\frac{1-\kappa}{1-\lambda\kappa} + n\kappa\log\frac{1}{\lambda} = n\kappa(\lambda - 1 - \log\lambda) \text{ nearly:}$$

so that if *A* happen $n(1-\kappa)$ times out of n, the presumption that its probability lies between $1-\lambda\kappa$, and 1 is

$$\frac{1}{\sqrt{\pi}}\int_{\mu}^{\infty}\epsilon^{-t^2}dt - \frac{4\sqrt{2n\kappa} - \mu}{12n\kappa + 1}\,\frac{\epsilon^{-\mu^2}}{\sqrt{\pi}}\ \{\mu = -\sqrt{n\kappa}\,.\sqrt{\lambda - 1 - \log\lambda}\}.$$

Next, let $\varpi = \frac{1}{2} - \kappa$, where κ is small $\left(\text{not being less than } \dfrac{1}{n}\right)$, and let $a = 0$, $b = \frac{1}{2}$; that is, required the presumption that the less frequent (slightly) of two events is the less probable. Then μ is infinite and negative, and ν is positive and derived from

$$\nu^2 = n(\tfrac{1}{2} - \kappa)\log(1 - 2\kappa) + n(\tfrac{1}{2} + \kappa)\log(1 + 2\kappa)$$
$$= 2n\kappa^2 \text{ nearly.}$$

The presumption required is then

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\nu}\epsilon^{-t^2}dt \text{ or } \tfrac{1}{2} + \frac{1}{\sqrt{\pi}}\int_{0}^{\sqrt{2n}\,.\,\kappa}\epsilon^{-t^2}dt.$$

AUGUSTUS DE MORGAN.

UNIVERSITY COLLEGE, LONDON,
December 30, 1837.

Addition.

THE formula (1) at the beginning of this paper is unnecessarily complex, seeing that if $l+\frac{1}{2}$ were written instead of l in the limits of the integral, the increment of the latter would differ from the additional term only by a quantity of such an order as was rejected in the approximation.

If then v and w be the components of n (or $v+w$), which are as the probabilities of A and B in a single trial, the probability that A will happen a number of times between $v+l$ and $v-l$ in n trials is

$$\frac{2}{\sqrt{\pi}}\int_0^{(l+\frac{1}{2})\sqrt{\frac{n}{2vw}}} \epsilon^{-t^2}dt;$$

which is of the same order of exactness as the formula given by Laplace, and is somewhat more symmetrical and less difficult to calculate.

Perhaps it may not be here out of place to notice that the usual approximation to the product $1.2.3 \ldots\ldots x$ may be made very much more exact without being rendered materially more difficult to calculate. As follows: instead of

$$1.2.3\ldots\ldots x = \sqrt{2\pi}\, x^{x+\frac{1}{2}}\epsilon^{-x},$$

substitute

$$1.2.3\ldots\ldots x = \sqrt{2\pi}\, x^{x+\frac{1}{2}}\epsilon^{-x+\frac{1}{12x}}$$

this follows immediately from

$$1.2.3\ldots\ldots x = \sqrt{2\pi}\, x^{x+\frac{1}{2}}\epsilon^{-x}\left(1+\frac{1}{12x}+\frac{1}{288x^2}+\ldots\right)$$

since the third term of the series is half the square of the second. The approximation is so close that even if we take $x=1$, the error is very little more than the five hundredth part of the whole.

XXI. *On the Diffraction of an Object-glass with a triangular Aperture.* By S. Earnshaw, M.A. *Of St John's College, Cambridge.*

[Read *December* 12, 1836.]

The general adoption of Fresnel's theory seems to indicate that the scientific world is convinced that the Newtonian theory is inadequate to the explanation of the phænomena of Diffraction; and that the theory which ascribes them to reflexion at the edges of the obstacle is equally unsatisfactory; and hence it is that the phænomena of this class have been declared by Sir J. Herschel, to form the strongest points of the undulatory theory of light. Professor Airy also at the end of his paper on "The Diffraction of an Object-glass with a circular Aperture," (*Phil. Trans.* Vol. v.) has thus stated his opinion of the importance of these phænomena in the present state of science: "The investigation of cases of diffraction similar to that discussed here, appears to me a matter of great interest to those who are occupied with the examination of theories of light." This sentiment was expressed in 1834, and since that time I am not aware that any thing has been done in the comparison of theory with experiment in this class of phænomena. It is true, theory may have been applied to certain cases of diffraction, but it does not appear that the persons who have so applied it have ever contemplated more than merely to shew that theory gave a result *something like* the observed phænomenon. Such an inference being wholly useless in the present state of the theory of light, there is still need for the minute discussion of particular cases of diffraction, and for the impartial comparison of the results of theory and experiment. I have selected for this purpose, the diffraction of light at the object-glass of a telescope with a triangular aperture, for two reasons,—because the phænomenon is very singular and very beautiful;—and because Sir J. Herschel has declared that "to represent analytically the intensity of the light in one of the discontinuous rays, will call for the use of functions of a very singular nature and delicate management." His description of the phænomenon is as follows. (*Encyclop. Metrop.* Light. Art. 772.)

"When the object-glass of the telescope was limited by a diaphragm so that the aperture was in form of an equilateral triangle, the phænomenon seen by viewing a star through the telescope was extremely beautiful: it consisted of a perfectly regular, brilliant, six-rayed star, surrounding a well-defined circular disc of great brightness. The rays do not unite to the disc, but are separated from it by a black ring. They are very narrow and perfectly straight; and appear particularly distinct in consequence of the TOTAL destruction of all the diffused light, which fills the field when no diaphragm is used: a remarkable effect, and much more so than the mere proportion of the light stopped."

Let us suppose the aperture of the telescope an isosceles triangle, one of whose equal sides $= a$; the perpendicular from the vertical angle upon the base $= 3c$, and the inclination of either side to the base $= \alpha$. Let the image of the star be received upon a screen passing through the focus of the object-glass; and take the projection of the centre of gravity of the triangular aperture upon the screen for the origin of co-ordinates; the axes of x and y upon the screen being respectively perpendicular and parallel to the projection of the base of the triangular opening, and the axis of z coinciding with the axis of the telescope, and passing through the centre of gravity of the aperture. Suppose $b =$ focal length of the object-glass, and let x, y, z be co-ordinates of any point P in the wave surface, which emerges from the object-glass, and tends to the origin of co-ordinates as its focus;

$$\therefore \; x^2 + y^2 + z^2 = b^2, \quad \text{and} \quad 3c = a \sin \alpha.$$

Let p, q be co-ordinates of any point M in the screen: then the disturbance caused at M by the element $\delta x . \delta y$ of the wave surface may be represented by

$$\frac{\delta x . \delta y}{PM} . \sin \frac{2\pi}{\lambda} (vt - PM).$$

But as $\dfrac{\delta x . \delta y}{PM}$ refers only to the *intensity* of the light, which emanates from P, when it reaches M, it will not vary sensibly with the variation of PM in the problem under consideration, since both the triangle and its image are small. No appreciable error, therefore, will be committed if for the purpose of simplifying our calculations we suppose PM constant in this term, and assume the disturbance at M due to the element $\delta x . \delta y$ to be

$$\delta x . \delta y \sin \frac{2\pi}{\lambda} (vt - PM).$$

$$\begin{aligned} \text{Now} \quad PM^2 &= (x - p)^2 + (y - q)^2 + z^2 \\ &= x^2 + y^2 + z^2 - 2xp - 2yq + p^2 + q^2 \\ &= b^2 + p^2 + q^2 - 2px - 2qy; \end{aligned}$$

$$\begin{aligned} \therefore \; PM &= b + \frac{p^2 + q^2}{2b} - \frac{px}{b} - \frac{qy}{b} \text{ nearly} \\ &= B - \frac{px}{b} - \frac{qy}{b}. \end{aligned}$$

Hence the disturbance at M due to the element $\delta x . \delta y$ is represented by

$$\delta x . \delta y \sin \frac{2\pi}{\lambda} \left(vt - B + \frac{px}{b} + \frac{qy}{b} \right).$$

The whole disturbance at M will be found by integrating this expression, first with regard to y, between the limits $y = -(2c - x) \cot \alpha$, and $y = (2c - x) \cot \alpha$; and then with regard to x between the limits $x = -c$ and $x = 2c$.

The integral with regard to y is

$$- \delta x . \frac{b\lambda}{2q\pi} . \cos \frac{2\pi}{\lambda} \left(vt - B + \frac{px}{b} + \frac{qy}{b} \right);$$

which taken between the proper limits becomes

$$\delta x \cdot \frac{b\lambda}{2q\pi} \cdot \left\{ \cos \frac{2\pi}{\lambda} \left(vt - B - \frac{2cq \cot \alpha}{b} + \frac{p + q \cot \alpha}{b} \cdot x \right) \right.$$

$$\left. - \cos \frac{2\pi}{\lambda} \left(vt - B + \frac{2cq \cot \alpha}{b} + \frac{p - q \cot \alpha}{b} \cdot x \right) \right\};$$

and the integral of this, taken with regard to x between the limits before mentioned, is

$$\frac{b^2\lambda^2}{4q\pi^2} \cdot \left\{ \frac{\sin \frac{2\pi}{\lambda} \left(vt - B + \frac{2cp}{b} \right) - \sin \frac{2\pi}{\lambda} \left(vt - B - \frac{cp + 3cq \cot \alpha}{b} \right)}{p + q \cot \alpha} \right.$$

$$\left. - \frac{\sin \frac{2\pi}{\lambda} \left(vt - B + \frac{2cp}{b} \right) - \sin \frac{2\pi}{\lambda} \left(vt - B - \frac{cp - 3cq \cot \alpha}{b} \right)}{p - q \cot \alpha} \right\}.$$

Let us now refer the image on the screen to polar co-ordinates, which will be done by writing $r \cos \theta$, $r \sin \theta$ for p, q respectively. For brevity, also, write V for $\frac{2\pi}{\lambda} \left(vt - B + \frac{2cp}{b} \right)$, and $2m$ for $\frac{3cr}{b \sin \alpha} \cdot \frac{2\pi}{\lambda}$, or its equal $\frac{2ar\pi}{b\lambda}$; then the above expression for the disturbance at M may be written

$$\frac{a^2 \sin \alpha}{4m^2 \sin \theta} \cdot \left\{ \frac{\sin V - \sin \{V - 2m \sin (\alpha + \theta)\}}{\sin (\alpha + \theta)} - \frac{\sin V - \sin \{V - 2m \sin (\alpha - \theta)\}}{\sin (\alpha - \theta)} \right\}.$$

By expanding the numerators of these fractions, and arranging the result in two terms containing respectively $\sin V$ and $\cos V$, this expression for the disturbance at M may be written in the following form:

$$\frac{a^2 \sin \alpha}{4m^2 \sin \theta} \cdot \left\{ \frac{1 - \cos \{2m \sin (\alpha + \theta)\}}{\sin (\alpha + \theta)} - \frac{1 - \cos \{2m \sin (\alpha - \theta)\}}{\sin (\alpha - \theta)} \right\} \sin V$$

$$+ \frac{a^2 \sin \alpha}{4m^2 \sin \theta} \left\{ \frac{\sin \{2m \sin (\alpha + \theta)\}}{\sin (\alpha + \theta)} - \frac{\sin \{2m \sin (\alpha - \theta)\}}{\sin (\alpha - \theta)} \right\} \cos V.$$

The intensity of the light at the point M, as is well known, is equal to the sum of the squares of the coefficients of $\sin V$ and $\cos V$, and therefore calling it Z, we find

$$Z = \frac{a^4 \sin^2 \alpha}{8 m^4 \sin^2\theta} \cdot \left\{ \frac{1 - \cos\{2m \sin(\alpha+\theta)\}}{\sin^2(\alpha+\theta)} + \frac{1 - \cos\{2m\sin(\alpha-\theta)\}}{\sin^2(\alpha-\theta)} \right.$$

$$\left. - \frac{1 - \cos\{2m\sin(\alpha+\theta)\} - \cos\{2m\sin(\alpha-\theta)\} + \cos(2m \,.\, 2\cos\alpha\sin\theta)}{\sin(\alpha+\theta)\,.\,\sin(\alpha-\theta)} \right\}$$

$$= \frac{a^4 \sin^2\alpha}{4m^4\sin^2\theta} \cdot \left\{ \frac{\sin^2\{m\sin(\alpha+\theta)\}}{\sin^2(\alpha+\theta)} + \frac{\sin^2\{m\sin(\alpha-\theta)\}}{\sin^2(\alpha+\theta)} \right.$$

$$\left. - \frac{\sin^2\{m\sin(\alpha+\theta)\} + \sin^2\{m\sin(\alpha-\theta)\} - \sin^2(2m\cos\alpha\sin\theta)}{\sin(\alpha+\theta)\,.\,\sin(\alpha-\theta)} \right\}$$

$$= \frac{a^4 \sin^2\alpha \,.\, \cos\alpha}{2m^4 \sin\theta \,.\, \sin(\alpha+\theta)\,.\,\sin(\alpha-\theta)}$$

$$\times \left\{ \frac{\sin^2(2m\cos\alpha\sin\theta)}{2\cos\alpha\sin\theta} - \frac{\sin^2\{m\sin(\alpha+\theta)\}}{\sin(\alpha+\theta)} + \frac{\sin^2\{m\sin(\alpha-\theta)\}}{\sin(\alpha-\theta)} \right\}.$$

If in this expression we write $\frac{ar\pi}{b\lambda}$ for its equal m, we have the brightness at any point of the screen expressed in terms of its polar coordinates r and θ.

When the triangle is equilateral $\alpha = 60^\circ$, and the equation for the brightness assumes the very symmetrical form

$$Z = \frac{3a^4}{16m^4 \sin\theta \,.\, \sin(60^\circ+\theta)\,.\,\sin(60^\circ-\theta)}$$

$$\times \left\{ \frac{\sin^2(m\sin\theta)}{\sin\theta} - \frac{\sin^2\{m\sin(60^\circ+\theta)\}}{\sin(60^\circ+\theta)} + \frac{\sin^2\{m\sin(60^\circ-\theta)\}}{\sin(60^\circ-\theta)} \right\};$$

or more simply

$$Z = \frac{3a^4}{4m^4 \sin 3\theta}$$

$$\times \left\{ \frac{\sin^2(m\sin\theta)}{\sin\theta} - \frac{\sin^2\{m\sin(60^\circ+\theta)\}}{\sin(60^\circ+\theta)} + \frac{\sin^2\{m\sin(60^\circ-\theta)\}}{\sin(60^\circ-\theta)} \right\} \ldots\ldots (B.)$$

The Interpretation of the Formula for the Brightness.

It will be found upon trial that the value of Z is not altered when $-\theta$ is written for $+\theta$; and hence it follows that the light is symmetrically arranged with regard to the axis of x.

It will likewise be found that the value of Z is not affected when any one of these values, $\theta+60^\circ$, $\theta+120^\circ$, $\theta+180^\circ$, $\theta+240^\circ$, $\theta+300^\circ$, is substituted for θ; and hence it follows that if from the origin of co-ordinates, or centre of the screen, six lines be drawn upon it making respectively the angles 0°, 60°, 120°, 180°, 240°, 300° with the axis of x, or, which is the same, inclined at angles of 60° to each other, the light upon the screen is similarly and symmetrically arranged with regard to every one of them.

Wherefore, the light being symmetrically arranged about these six lines, it will only be necessary to examine our formula for Z between the values $\theta=0$ and $\theta=30^\circ$.

It will at once be seen from an inspection of the equation preceding the one marked (B), that the value of Z depends upon the three terms

$$\frac{\sin^2(m\sin\theta)}{\sin^2\theta},\quad \frac{\sin^2\{m\sin(60^\circ+\theta)\}}{\sin^2(60^\circ+\theta)},\quad \frac{\sin^2\{m\sin(60^\circ-\theta)\}}{\sin^2(60^\circ-\theta)},$$

each of which is precisely similar to the principal term in the expression for the intensity of light in the experiment of Fraunhofer's gratings; and at first sight it might be deemed sufficient to examine each of these terms separately, and thence judge of their united effect: but it will be found upon trial that the multipliers by which they are connected together exercise such an important influence upon their values, as to render this method utterly inapplicable in the present instance. Thus, if θ be very small, the second and third terms are very nearly equal, and having different signs their sum is very small; but being afterwards divided by $\sin\theta$, the quotient is large; and their united effect is as great as that of the first term.

From the necessity which thus exists of taking in at once the whole of the expression for Z at every step of our examination, we shall be obliged to feign several cases, and effect corresponding expansions and reductions for each: and from these particular results infer, in the best manner we are able, the general appearance and brightness of the image upon the screen.

1. Let us suppose r and therefore m extremely small. This will be true of parts very near the centre of the screen. In this case we must expand the expression for Z in a series of terms arranged according to the powers of m. This may be effected most readily as follows.

If $f_1 = \sin\theta$, $f_2 = \sin(60^\circ - \theta)$, and $f_3 = -\sin(60^\circ + \theta)$, f_1, f_2, f_3 will be the roots of the equation

$$x^3 - \tfrac{3}{4}x + \tfrac{1}{4}\sin 3\theta = 0.$$

And that part of the expression for Z which is enclosed within the brackets is equal to $\Sigma\left(\frac{\sin^2 mf}{f}\right)$.

$$\text{Now } \frac{\sin^2 mf}{f} = \frac{1 - \cos 2mf}{2f}$$

$$= fm^2 - \frac{2^2 f^3}{3\,.\,4}\,.\,m^4 + \frac{2^4 f^5}{3\,.\,4\,.\,5\,.\,6}\,.\,m^6 - \ldots$$

$$\therefore \quad \Sigma\left(\frac{\sin^2 mf}{f}\right) = m^2\,.\,\Sigma(f) - \frac{2^2 m^4}{3\,.\,4}\,.\,\Sigma(f^3) + \frac{2^4 m^6}{3\,.\,4\,.\,5\,.\,6}\,.\,\Sigma(f^5) - \ldots$$

By the usual method of finding the sums of the powers of the roots of equations, we easily find

$$\Sigma(f) = 0,$$

$$\Sigma(f^3) = -\frac{3}{4}\,.\,\sin 3\theta,$$

$$\Sigma(f^5) = -\frac{15}{16}\,.\,\sin 3\theta,$$

$$\Sigma(f^7) = -\frac{63}{64}\,.\,\sin 3\theta,$$

$$\Sigma(f^9) = -\frac{243}{256}\sin 3\theta - \frac{12}{256}\sin^3 3\theta,$$

&c. = &c.

Hence,

$$\Sigma\left(\frac{\sin^2 mf}{f}\right) = \frac{\sin 3\theta}{4}\,.\,m^4 - \frac{\sin 3\theta}{4\,.\,6}\,.\,m^6 + \frac{\sin 3\theta}{4\,.\,8\,.\,10}\,.\,m^8 - \frac{81\sin 3\theta + 4\sin^3 3\theta}{4\,.\,5\,.\,6\,.\,7\,.\,8\,.\,9\,.\,10}m^{10} + \ldots$$

Wherefore, by substitution, we finally obtain

$$Z = \frac{3a^4}{16}\left\{1 - \frac{m^2}{6} + \frac{m^4}{8\,.\,10} - \frac{(81 + 4\sin^2 3\theta)\,m^6}{5\,.\,6\,.\,7\,.\,8\,.\,9\,.\,10} + \ldots\right\}$$

$$= \frac{3}{16}.a^4\left\{1 - \frac{a^2\pi^2}{6.b^2}.\left(\frac{r}{\lambda}\right)^2 + \frac{a^4\pi^4}{8\,.\,10.b^4}\left(\frac{r}{\lambda}\right)^4 - \frac{(81 + 4\sin^2 3\theta)a^6\pi^6}{5.6.7.8.9.10.b^6}\left(\frac{r}{\lambda}\right)^6 + \ldots\right\}$$

A striking feature of this series is, that its leading terms are entirely independent of θ, and therefore while r is so small as to allow the series to be represented by its first three terms, the brightness will be independent of θ: and therefore consecutive circles of uniform brightness will surround the centre.

When r is $= 0$, the brightness $= \frac{3}{16}a^4$, which is independent of λ, and therefore the central point of the image is *white*.

When r is so large as to require the fourth term of the series to be taken notice of, the circles which correspond to those radii will have their brightness diminished by a term of the form $\sin^2 3\theta$: they will therefore be most bright when $\sin 3\theta = 0$, that is, where they are intersected by the rays drawn upon the screen as before mentioned; at points more remote from those rays the brightness will gradually diminish, and be least when $\sin 3\theta = 1$, that is, at those points which lie exactly between them.

2. Let us now examine the image in the neighbourhood of the six rays; for this purpose θ must be supposed small, and Z must be expressed in a series ascending by powers of θ. Upon this hypothesis we find

$$Z = \frac{a^4}{4m^4}\left\{m^2 - \frac{2m}{\sqrt{3}}\,.\,\sin(m\sqrt{3}) + \frac{4}{3}\sin^2\left(\frac{m\sqrt{3}}{2}\right)\right\} + \text{terms involving } \theta^2,\ \theta^4\ldots$$

For any of the six rays we may write $\theta = 0$, and therefore the brightness of any one ray is accurately expressed by

$$Z = \frac{a^4}{4m^4}.\left\{m^2 - \frac{2m}{\sqrt{3}}\sin(m\sqrt{3}) + \frac{4}{3}\sin^2\left(\frac{m\sqrt{3}}{2}\right)\right\}$$

$$= \frac{a^4}{4m^4}.\left\{\left(m - \frac{\sin(m\sqrt{3})}{\sqrt{3}}\right)^2 + \frac{4}{3}\sin^4\left(\frac{m\sqrt{3}}{2}\right)\right\}.$$

Since this expression for Z is the sum of two squares, Z can never $= 0$, and therefore the six rays cannot be interrupted by a perfectly black band or ring. Perhaps, however, there may be a ring of light of such feeble intensity, interrupting the rays, as to appear like a black band cutting off the rays from the central part in the manner described by Sir J. Herschel. To ascertain if this be the case, let us find the value of m which gives Z a minimum. By differentiation we obtain

$$d_m Z = -\frac{a'}{m^5}\left\{m\cos\left(\frac{m\sqrt{3}}{2}\right) - \frac{2}{\sqrt{3}}\sin\left(\frac{m\sqrt{3}}{2}\right)\right\}^2.$$

The only factor in this expression which can be equated to zero, for the purpose of finding the maximum and minimum values of Z, being an exact square, Z admits neither of a maximum nor minimum, but decreases perpetually from the centre of the screen, as the following Table will shew.

Values of $\frac{m\sqrt{3}}{2}$	Corresponding Brightness.	Values of $\frac{m\sqrt{3}}{2}$	Corresponding Brightness.	Values of $\frac{m\sqrt{3}}{2}$	Corresponding Brightness.
0	1·0000	$\frac{5\pi}{12}$	·6785	$\frac{10\pi}{12}$	·2011
$\frac{\pi}{12}$	·9849	$\frac{6\pi}{12}$	·5695	$\frac{11\pi}{12}$	·1425
$\frac{2\pi}{12}$	·9407	$\frac{7\pi}{12}$	·4617	$\frac{12\pi}{12}$	·1013
$\frac{3\pi}{12}$	·8719	$\frac{8\pi}{12}$	·3612	$\frac{15\pi}{12}$	·0504
$\frac{4\pi}{12}$	·7819	$\frac{9\pi}{12}$	·2730	$\frac{18\pi}{12}$	·0471

In this Table the central brightness $\left(=\frac{3}{16}a'\right)$ is taken as unity.

3. Let us now examine the part of the screen which is intermediate to two rays. The brightness of this part will be obtained by writing 30° for θ in equation (B); which by that means becomes

$$Z = \frac{3a^4}{16} \cdot \left(\frac{\sin \frac{m}{2}}{\frac{m}{2}} \right)^4,$$

or taking the central brightness as the unit

$$Z = \left(\frac{\sin \frac{m}{2}}{\frac{m}{2}} \right)^4.$$

When $m = 0$, $Z = 1$; and as m increases Z diminishes; at first rather slowly, but afterwards rapidly, so that when

$$\frac{m}{2} = \pi, \quad \text{or} \quad \frac{m\sqrt{3}}{2} = \pi\sqrt{3} = \frac{21\pi}{12} \text{ nearly},$$

there is perfect blackness; some time before this, however, the light will be too feeble for vision, and there the light on the screen will appear to terminate unless the star or original luminous point of light be very bright, for as m goes on increasing, Z never attains a value so great as $\frac{1}{400}$.

From this it follows that if there be a black ring surrounding the central disc of light as described by Sir J. Herschel, its radius will be such that $\frac{m\sqrt{3}}{2} = \frac{18\pi}{12}$ nearly. By reference to the above Table, we perceive that the six rays have, at that distance from the centre, a brightness of about $\frac{1}{21}$, the central brightness being represented by unity.

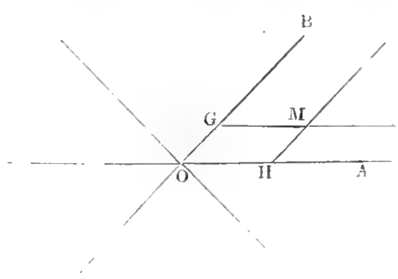
If this be considered sufficiently feeble to constitute a black ring, we are at a loss to account for the prolongation of the six bright rays mentioned by Sir J. Herschel, since their intensity has been shewn to decrease from the centre; and therefore when at any point in them the light is too feeble for vision, at every more distant point the light is still more feeble. Hence it would appear that, according to theory, the six rays are not interrupted by a dark ring, or band, in any part. In this particular, therefore, there is a decided disagreement between theory and the experiment recorded by Sir J. Herschel.

4. Let us now examine the intensity in that part of the screen which is situated between any two of the six rays. As we have already seen the results when $\theta = 0$, and when $\theta = 30^\circ$, we shall now suppose the values of θ to lie *between* 0° and 30°.

Since $\dfrac{1}{m^4 \sin 3\theta}$ is a factor of equation (*B*), it follows that the general brightness of the spectra will decrease very rapidly from the centre; and at a given distance from the centre the brightness is less the more θ differs from 0°, and is least when $\theta = 30^\circ$. The places and extent of the spectra are pointed out by the other factor of equation (*B*), viz.

$$\frac{\sin^2(m\sin\theta)}{\sin\theta} - \frac{\sin^2\{m\sin(60^\circ+\theta)\}}{\sin(60^\circ+\theta)} + \frac{\sin^2\{m\,.\,(60^\circ-\theta)\}}{\sin(60^\circ-\theta)} \dots\dots (C).$$

This expression vanishes entirely whenever m and θ are such that $(m\sin\theta)$ and $(m\sqrt{3}\cos\theta)$ are simultaneously both odd or both even multiples of π. If M be such a point, and MG, MH be drawn parallel to OA, OB, two of the six rays, then the distance of HM from OA, and the distance of MG from OB are both multiples of $\left(\dfrac{b\lambda}{a}\right)$. Hence there will be an infinite number of perfectly dark spots situated in the farther corner of parallelograms, such as $HMGO$, whose sides are parallel to OA, OB.



If the line HM be such that its distance from OA is an *even* multiple of $\dfrac{b\lambda}{a}$, then for every point in that line, the principal factor in the expression (C) is $\sin^2\left(\frac{m\sqrt{3}}{2}\cos\theta\right)$, which denotes spectra of the same character as are exhibited in Fraunhofer's gratings.

If the distance of HM from OA be an odd multiple of $\frac{b\lambda}{a}$, then the principal factor in the expression (C) is $\cos^2\left(\frac{m\sqrt{3}}{2}\cos\theta\right)$, which represents spectra of the same kind as before, but intermediate to them in position.

For a given value of m, r is greater for red than for violet coloured light, and consequently the spectra will have their red ends outwards, that is, farthest from the centre of the screen.

What is here said of MH referred to OA, is equally true of GM referred to OB: and what is said of the portion of the image within AOR, is true of the portion within BOR; the line OR bisecting the angle AOB.

In Sir J. Herschel's experiment no spectra of this nature were seen, but with strong sun light they are very distinctly visible, and form to the six bright silvery rays a very beautiful appendage. In fact, on account of the remarkable symmetry of its parts, and of the great extent and extreme narrowness and whiteness of its six principal rays, which stretch completely across the field of view; and on account of the number and geometrical arrangement of the coloured spectra, this experiment is inferior in beauty and splendour to very few of all those that have been exhibited in illustration of the science of Physical Optics.

S. EARNSHAW.

XXII. *On the Decrement of Atmospheric Temperature depending on the Height above the Earth's Surface. By the Rev.* J. CHALLIS, *Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge.*

[Read *February* 13, 1837.]

THE temperature at any height above a given place on the Earth's surface is here considered to be the mean which would be found by a great number of thermometrical observations, made at that elevation, for a time sufficiently long to eliminate the diurnal and annual variations and the more irregular changes from winds. This mean temperature, it is known, varies with the height, and the object of this Paper is to enquire respecting the law of the variation.

The causes which determine the temperature of the atmosphere at a given elevation, are probably of a very complicated nature, but among the principal may be reckoned the diminution of density in the higher regions. In the following reasoning it is assumed, that the temperature and density are functions of the height, and the effect of decrease of density will be considered apart from every other circumstance. If then θ be the temperature and ρ the density at the height z, we shall have

$$\left(\frac{d\theta}{dz}\right) = \frac{d\theta}{dz} + \frac{d\theta}{d\rho} \cdot \frac{d\rho}{dz}, \qquad (1).$$

in which equation $\frac{d\theta}{dz}$ expresses the variation of temperature corresponding to a change of height, so far as it varies independently of change of density. If also p be the pressure where the density is ρ, and g be the force of gravity, we have

$$\frac{dp}{dz} = -g\rho, \qquad (2).$$

Lastly, we have the known relation between the pressure, density, and temperature, given by the equation

$$p = a^2\rho\,(1 + \alpha\theta), \qquad (3).$$

in which the temperature θ is supposed to be reckoned in degrees of the centigrade thermometer, a^2 is the pressure where $\rho = 1$ and $\theta = 0$, and α is the numerical coefficient 0,00375. With respect to the equation (2) we may remark that though it is in strictness applicable only to the air at rest, it is very nearly true when the atmosphere is in motion; for the direction of winds is necessarily nearly parallel to the Earth's surface, and consequently the effective accelerative force in the vertical direction is very small. Hence $\frac{dp}{\rho dz}$ is nearly equal to the impressed accelerative force, that is, to the force of gravity.

The equation (3) differentiated gives,

$$\frac{dp}{dz} = a^2\,\frac{d\rho}{dz}\,(1 + \alpha\theta) + a^2\alpha\rho\left(\frac{d\theta}{dz}\right).$$

Hence by means of (2) we get,

$$\frac{d\rho}{dz} = -\,\frac{g\rho + a^2\alpha\rho\left(\frac{d\theta}{dz}\right)}{a^2\,(1 + \alpha\theta)}$$

and by substituting this value of $\frac{d\rho}{dz}$ in (1), it will be found that

$$\left(\frac{d\theta}{dz}\right) = \frac{\frac{d\theta}{dz} - \frac{g\rho}{a^2\,(1 + \alpha\theta)}\cdot\frac{d\theta}{d\rho}}{1 + \frac{\alpha\rho}{1 + \alpha\theta}\cdot\frac{d\theta}{d\rho}}, \qquad (4).$$

The solution of the problem requires, therefore, the knowledge of expressions for the partial differential coefficients $\frac{d\theta}{dz}$ and $\frac{d\theta}{d\rho}$. There are at present no means of finding these by a method entirely *a priori*; and recourse must consequently be had to experiment and observation. To obtain the value of $\frac{d\theta}{d\rho}$ we shall refer to the experimental determination of the velocity of sound, beginning, first, with some Propositions for finding the velocity theoretically.

Prop. I. *To find an expression for the velocity with which a given state of density is propagated in any medium.*

The motion is supposed to be in parallel lines. Take an axis parallel to the direction of motion, and let v, ρ, be the velocity and density of a particle in motion at the distance x from a fixed origin, and at the time t. Then we have the equation,

$$\frac{d\rho}{dt} + \frac{d \,.\, v\rho}{dx} = 0.$$

(*Poisson, Traité de Mecanique,* Tom. II. p. 674.)

The differential coefficients are partial with respect to time and space. Let now ρ' be the density at the same time at the distance $x + \delta x$. Then,

$$\rho' = \rho + \frac{d\rho}{dx}\,\delta x + \&c.$$

After the small time δt let the density at the distance $x + \delta x$ become ρ. Consequently,

$$\rho' = \rho - \frac{d\rho}{dt}\,\delta t + \&c.$$

By equating these two values of ρ' we obtain,

$$\frac{\delta x}{\delta t} = -\frac{\frac{d\rho}{dt}}{\frac{d\rho}{dx}} \text{ (taking the limits)}$$

$$= \frac{\frac{d.\rho v}{dx}}{\frac{d\rho}{dx}}, \text{ (by the foregoing equation,)}$$

$$= v + \frac{\frac{dv}{dx}}{\frac{d\rho}{\rho dx}}.$$

Now $\frac{\delta x}{\delta t}$ is the rate at which the density at the distance x is transferred to the distance $x + \delta x$, and is equal to the velocity of the particles + the velocity of propagation.

$$\text{Therefore the velocity of propagation} = \frac{\frac{dv}{dx}}{\frac{d\rho}{\rho dx}}.$$

If the given state of density be propagated with the uniform velocity b, it follows that

$$\frac{dv}{dx} = b . \frac{d\rho}{\rho dx},$$

an equation applicable to uniform propagation under whatever circumstances it takes place. By integration, $v = b$. Nap. log ρ, assuming that $v = 0$, when $\rho = 1$.

Supposing the medium to be such that $p = b^2 \rho$, the propagation is known to be uniform and equal to b. Therefore for this medium $\frac{dv}{dx} = b . \frac{d\rho}{\rho dx}$. Hence the relation between the velocity and density is the same in any medium which propagates a given state of density with the uniform velocity b, as in the medium defined by the equation $p = b^2\rho$, provided the motion in both be subject to the condition that $v = 0$ where $\rho = 1$.

By help of what precedes the following Propositions may be solved.

PROP. II. *To find the impressed accelerative force which will alter the rate of uniform propagation in a medium whose density varies as the pressure.*

Let the medium be such that $p = a^2\rho$, and let b be the altered rate of propagation. It has been shewn that the motion is the same as in a medium for which $p = b^2\rho$, no impressed force acting. Hence the effective accelerative force is the same. Hence

$$\frac{d^2x}{dt^2} = -\frac{b^2 d\rho}{\rho dx}.$$

But by the general equation,

$$\frac{dp}{\rho dx} = X - \frac{d^2x}{dt^2}.$$

$$\text{Hence, } \frac{a^2 d\rho}{\rho dx} = X + \frac{b^2 d\rho}{\rho dx},$$

$$\text{and } X = (a^2 - b^2)\frac{d\rho}{\rho dx}.$$

PROP. III. *To find the relation between the pressure and the density in a medium which propagates a given state of density uniformly.*

It is here supposed that there is no impressed force. The two equations following are therefore to be applied:

$$\frac{dp}{dx} = -\rho\left(\frac{dv}{dt}\right) = -\rho\frac{dv}{dt} - v\rho\frac{dv}{dx}, \qquad (a).$$

$$\frac{d\rho}{dt} = -\frac{d.\rho v}{dx} = -\frac{\rho dv}{dx} - \frac{v d\rho}{dx}, \qquad (b).$$

Now by supposition a given state of density is propagated with a uniform velocity. Hence if b equal the velocity of propagation, by Proposition I,

$$\frac{dv}{dx} = b \cdot \frac{d\rho}{\rho dx}.$$

Integrating,

$$v = b \text{ Nap. log } \rho + \phi(t);$$

and introducing the condition that $v = 0$ wherever $\rho = 1$, which can be satisfied when, as we suppose, the propagation is in a single direction only, it follows that $\phi(t) = 0$. Hence, differentiating with respect to time only,

$$\frac{dv}{dt} = b \cdot \frac{d\rho}{\rho dt}.$$

Substituting these values of $\frac{dv}{dx}$ and $\frac{dv}{dt}$ in the equations (a) and (b), we obtain,

$$\frac{d\rho}{\rho dt} + \frac{v d\rho}{\rho dx} + \frac{b d\rho}{\rho dx} = 0,$$

$$\frac{dp}{\rho dx} + \frac{b d\rho}{\rho dt} + bv \frac{d\rho}{\rho dx} = 0.$$

Multiplying the first of these by b and subtracting, the result is,

$$\frac{dp}{dx} = b^2 \cdot \frac{d\rho}{dx};$$

and integrating,

$$p = b^2 \rho + \psi(t).$$

We can now find an expression for the velocity of the propagation of sound in the atmosphere, assuming the velocity to be uniform. Let $\theta_{\prime}$ be the temperature of the air when at rest. Experiments shew that by sudden compression the temperature is increased, and by sudden dilatation diminished. Let $\theta_{\prime} + \phi$ be the temperature corresponding to any density ρ at a distance x from the origin, the air being in vibration. Then

$$p = a^2 \rho \{1 + \alpha (\theta_{,} + \phi)\},$$

$$\text{and } b^2 = \frac{\dfrac{dp}{dx}}{\dfrac{d\rho}{dx}} = a^2 \left\{ 1 + \alpha\theta_{,} + \alpha \, . \frac{\dfrac{d \, . \, \rho\phi}{dx}}{\dfrac{d\rho}{dx}} \right\}.$$

The same result may be obtained by means of Proposition II. For we may consider the effect of the heat developed or absorbed by the sudden condensation or rarefaction of the air in vibration to be the same as that of an impressed force, which alters the rate of uniform propagation. The velocity of propagation, supposing the temperature constant and equal to $\theta_{,}$, is $a\sqrt{1 + \alpha\theta_{,}}$. Hence, by what has been proved,

$$X = \{a^2 (1 + \alpha\theta_{,}) - b^2\} \frac{d\rho}{\rho dx}.$$

But the effective accelerative force which urges the element ρdx in the direction of x, is $-\dfrac{dp}{\rho dx}$; and

$$-\frac{dp}{\rho dx} = -\frac{a^2 d\rho}{\rho dx} (1 + \alpha\theta_{,}) - a^2 \alpha \, . \frac{d \, . \, \rho\phi}{\rho dx}.$$

The first term of the right-hand side of this equation is the accelerative force which would act supposing no change of temperature; the other is due to variation of temperature. Consequently,

$$\{a^2 (1 + \alpha\theta_{,}) - b^2\} \frac{d\rho}{\rho dx} = - a^2 \alpha \frac{d \, . \, \rho\phi}{\rho dx}.$$

This leads to the value of b^2 obtained above.

The vibrations which take place in the propagation of sound are so rapid as not to allow sufficient time for any sensible alteration of the difference of temperature of two contiguous portions of the air by communication of heat from one to the other. This difference may consequently be considered to be entirely owing to difference of density. We have therefore ϕ a function of ρ, and

$$\frac{d.\rho\phi}{dx} = \frac{d.\rho\phi}{d\rho}\cdot\frac{d\rho}{dx}.$$

Hence the value of b may be put under the form,

$$a\sqrt{1+a\theta_{\prime}}\left\{1+\frac{a}{1+a\theta_{\prime}}\cdot\frac{d.\rho\phi}{d\rho}\right\}^{\frac{1}{2}}.$$

The known facts of the transmission of articulate and musical sounds prove that different parts of the same aerial wave and waves of different magnitudes are propagated through air of given temperature with exactly the same velocity. It follows from this that $\frac{d.\rho\phi}{d\rho}$ is constant for a given value of $\theta_{\prime}$. Suppose

$$\frac{d.\rho\phi}{d\rho} = \frac{k}{a}(1+a\theta_{\prime}), \qquad (c).$$

Then,

$$b = a\sqrt{1+a\theta_{\prime}}\sqrt{1+k};$$

the numerical value of $1+k$ can therefore be obtained by an experimental determination of the velocity of sound. The mean value found by this method is 1,4152.*

The equation (c) gives,

$$k = \frac{a}{1+a\theta_{\prime}}\cdot\frac{d.\rho\phi}{d\rho}.$$

Hence,

$$1+k = \frac{(1+a\theta_{\prime})\,d\rho + a\,d.\rho\phi}{(1+a\theta_{\prime})\,d\rho}$$

$$= \frac{d.a^2\rho\{1+a(\theta_{\prime}+\phi)\}}{d.a^2\rho(1+a\theta_{\prime})}.$$

The expression under the latter form shews that $1+k$ is the ratio of the increment of pressure due to an increase of density produced suddenly and consequently accompanied by an increase of temperature, to

* From the experiments of Professor Moll. See *Phil. Trans. of the Royal Society*, 1824. p. 424, and 1830. p. 213.

the increment of pressure due to an *equal* increase of density without change of temperature. It may be supposed that the temperature is in both cases $\theta_{,}$ before the alteration of density, for ϕ in the above expression may be taken as small as we please. The experiment of Clement and Desormes, which is in fact a practical imitation, as near as may be, of what takes place in the sonorous vibrations of the air, may consequently be used for determining $1 + k$. In this experiment as described by Poisson (*Traité de Mécanique,* Tom. II. p. 641. 2d. Edit.) $p - p'$ is the difference of pressure due to a sudden alteration $\rho'' - \rho'$ of density, accompanied by a change of temperature; and $p'' - p'$ is the difference of pressure due to the same difference of density, the temperature being the same as it was before the sudden alteration in the first case. Hence,

$$1 + k = \frac{p - p'}{p'' - p'}.$$

The numerical values of p, p', and p'', given by the experiment, are respectively 0,7665, 0,7527, and 0,7629. The value of $1 + k$ derived from these is 1,3529. A similar experiment by Gay-Lussac, gives 1,3795. These values fall short of that derived from the observed velocity sound, probably because the experiment cannot be performed so exactly as to avoid all variation of temperature by communication with surrounding bodies. The above expression for $1 + k$ is a little different from that of Poisson, and something larger in numerical value.

It appears both from experiments of the same kind as that above mentioned, and from the observed velocity of sound in different temperatures and at different heights, that the constant k is independent of the temperature and density.

Recurring now to equation (c), we may derive from it,

$$\frac{d\phi}{d\rho} = \frac{k}{a\rho}(1 + a\theta_{,}) - \phi.$$

This equation gives the variation of temperature from one point to another at a given time, depending on variation of density only; or it gives the difference of temperature of two contiguous elements, which, but for difference of density would have the same temperature. Though derived from the consideration of fluid in motion, it may be extended to fluid at rest, if we take a case in which the effect of the motion is insensible. Thus supposing the velocity at every part of the wave to be exceedingly small, and consequently ϕ very small, and the density ρ to be very little different from $\rho_{,}$ the density the fluid would have at rest, the ratio $\frac{d\phi}{d\rho}$ approximates to $\frac{k}{a\rho_{,}}(1 + \alpha\theta_{,})$ as its limit, which is of finite magnitude. This limiting value must therefore express the ratio of the difference of temperature of two contiguous elements *at rest*, to their difference of density, supposing the variation of temperature to depend on nothing but variation of density.

Hence, θ being the temperature of the atmosphere at any altitude z, where the density is ρ,

$$\frac{d\theta}{d\rho} = \frac{k}{a\rho}(1 + \alpha\theta), \qquad (5).$$

We are thus conducted by reasoning, which, though indirect, appears to be exact, to an expression for $\frac{d\theta}{d\rho}$ proper for substitution in equation (4), and containing constants of known numerical value. By making the substitution,

$$\left(\frac{d\theta}{dz}\right) = \frac{1}{1+k} \cdot \frac{d\theta}{dz} - \frac{gk}{a^2\alpha(1+k)}, \qquad (5).$$

Neglecting for the present the first term on the right-hand side of the equation, and taking $g = 32\frac{1}{6}$ feet, $a\sqrt{1+k} = 1090$ feet, $k = ,415\dot{2}$, and $\alpha = ,00375$, it will be found that

$$\left(\frac{d\theta}{dz}\right) = -\frac{1}{334}.$$

$$\text{Hence } z = -334\,\theta,$$

supposing $\theta = 0$ where $z = 0$. Hence if $\theta = -1^{\circ}$, the height $= 334$ feet: that is, the centigrade thermometer falls 1° for an elevation of 334 feet above the Earth's surface. The fall is, consequently, 1° of Fahrenheit for an elevation of 186 feet. This, according to the theory, would be the case if the temperature were a function of the density only; and for every additional height of 186 feet there would be an equal fall of temperature. It appears, however, from the discussion of a large number of thermometrical observations at different elevations, contained in a Memoir by Mr Atkinson, in Vol. II. of the *Memoirs of the Royal Astronomical Society*, that the decrement of temperature is 1° of Fahrenheit at an elevation of 250 feet. Also, that the decrements are not exactly equal for equal increments of height, but rather for increments of height which increase in a slow arithmetical progression. The explanation of the difference in these two respects between the facts of experience and the above results of the theory, is to be sought for in the term involving $\frac{d\theta}{dz}$. By neglecting this term we, in fact, supposed the temperature to be given when the density is given, and consequently neglected the tendency which contiguous portions of air of different densities have to assume the same temperature. When a limited portion of air is suddenly rarefied, its temperature falls, in the first instance, but in a short time it assumes the temperature of the surrounding bodies, its density remaining the same, and the time required to produce this effect is greater as the portion of rarefied air is larger. A similar cause must be in operation in the atmosphere, tending to produce a nearer approximation to equality of temperature in the upper and lower regions, than would exist if the temperature were a function of the density merely: although the temperature of the different parts can never be entirely equalized, on account of the radiation from the cold parts above into the still colder spaces beyond the limits of the atmosphere. As the effect of the above-mentioned cause is to increase the temperature of the colder and diminish that of the warmer portions of the atmosphere, so far as this action is concerned, $\frac{d\theta}{dz}$ will be a positive quantity.

Again, in consequence of the unequal distribution of the Sun's heat, different columns of the air are differently heated. Motion consequently ensues, the warmer parts ascend and are continually being replaced by the descent of colder. The effect of this circulation is to make the gradation of mean temperature from the lower to the upper strata less rapid than it would otherwise be, and so far as this cause also operates, $\frac{d\theta}{dz}$ will be positive. Hence we may assume $\frac{d\theta}{dz}$ to be some positive function of z, and as we have no means of determining *à priori* the form of this function, we will assume that

$$\frac{d\theta}{dz} = A + Bz + Cz^2 + \&c.$$

Then substituting in equation (5), we have

$$\left(\frac{d\theta}{dz}\right) = \frac{A}{1+k} - \frac{gk}{a^2\alpha(1+k)} + \frac{B}{1+k}.z + \frac{C}{1+k}z^2 + \&c.$$

and integrating,

$$\theta = \frac{1}{1+k}\left(A - \frac{gk}{a^2\alpha}\right)z + \frac{Bz^2}{2(1+k)} + \frac{Cz^3}{3(1+k)} + \&c.,$$

supposing $\theta = 0$ when $z = 0$.

The empirical formula which Atkinson gives in his Memoir, for expressing the relation between the altitude (h) in English feet above any place on the Earth's surface, and the depression of temperature (n) in degrees of Fahrenheit at that elevation, is the following:

$$h = n\,\{251{,}5 + \tfrac{3}{2}(n-1)\}.$$

This in our notation is,

$$z = -450\,\theta + 4{,}86\,\theta^2,$$

from which it will be found, that

$$\theta = -\frac{z}{450} + \frac{z^2}{18750000} - \frac{z^3}{390625000000} + \&c.$$

The third term amounts to about 1° for an elevation of three miles, and may, within the heights to which observations can extend, be neglected in comparison of the others. Then finding the numerical values of A and B by comparing the two expressions for θ, we shall obtain,

$$\frac{d\theta}{dz} = \frac{1}{964} + \frac{z}{6624505} \text{ nearly.}$$

This result accords with the preceding theoretical considerations in giving a positive value to $\frac{d\theta}{dz}$. It also enables us to estimate to what amount the variation of the atmospheric temperature with the height above the Earth's surface is affected by causes distinct from that of variation of density. It appears, that for small altitudes the term in equation (5) involving $\frac{d\theta}{dz}$ is about one-fourth the other term. The formula of Atkinson from which these inferences are made, is strictly applicable only to the lower parts of the atmosphere where the grand aerial currents prevail, beyond which the law of the decrement of temperature probably undergoes some variation.

I have thus endeavoured to advance in the theoretical part of this problem, as far as the present state of our knowledge appears to admit, and to give as much exactness as possible to the mathematical reasoning. With respect to the latter, the course pursued in this paper may lay some claims to originality, but the fundamental principles regarding the atmosphere are not essentially different from those advanced by Dalton and Ivory in their writings on this subject.

XXIII. *On the Motion of Waves in a variable Canal of small Depth and Width.* By George Green, Esq. B.A. *of Caius College.*

[Read *May* 15, 1837.]

The equations and conditions necessary for determining the motions of fluids in every case in which it is possible to subject them to Analysis, have been long known, and will be found in the First Edition of the *Mec. Anal.* of Lagrange. Yet the difficulty of integrating them is such, that many of the most important questions relative to this subject seem quite beyond the present powers of Analysis. There is, however, one particular case which admits of a very simple solution. The case in question is that of an indefinitely extended canal of small breadth and depth, both of which may vary very slowly, but in other respects quite arbitrarily. This has been treated of in the following paper, and as the results obtained possess considerable simplicity, perhaps they may not be altogether unworthy the Society's notice.

The general equations of motion of a non-elastic fluid acted on by gravity (g) in the direction of the axis z, are,

$$(1).\qquad gz - \frac{p}{\rho} = \frac{d\phi}{dt}.$$

$$(2).\qquad 0 = \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}$$

supposing the disturbance so small that the squares and higher powers of the velocities &c. may be neglected. In the above formulæ p = pressure, ρ = density, and ϕ is such a function of x, y, z and t, that the velocities of the fluid particles parallel to the three axes are

$$u = \left(\frac{d\phi}{dx}\right), \quad v = \left(\frac{d\phi}{dy}\right), \quad w = \left(\frac{d\phi}{dz}\right).$$

To the equations (1) and (2) it is requisite to add the conditions relative to the exterior surfaces of the fluid, and if $A = 0$ be the equation of one of these surfaces, the corresponding condition is [Lagrange, *Mec. Anal.* Tom. II, p. 303. (I.)],

$$0 = \frac{dA}{dt} + \frac{dA}{dx}u + \frac{dA}{dy}v + \frac{dA}{dz}w.$$

Hence

$$(A) \quad 0 = \frac{dA}{dt} + \frac{dA}{dx} \cdot \frac{d\phi}{dx} + \frac{dA}{dy} \cdot \frac{d\phi}{dy} + \frac{dA}{dz} \cdot \frac{d\phi}{dz} \text{ (when } A = 0).$$

The equations (1) and (2) with the condition (A) applied to each of the exterior surfaces of the fluid will suffice to determine in every case the small oscillations of a non-elastic fluid, or at least in those where

$$udx + vdy + wdz$$

is an exact differential.

In what follows however, we shall confine ourselves to the consideration of the motion of a non-elastic fluid, when two of the dimensions, viz. those parallel to y and z, are so small that ϕ may be expanded in a rapidly convergent series in powers of y and z, so that

$$\phi = \phi_0 + \phi' \frac{y}{1} + \phi_{,} \frac{z}{1} + \phi'' \frac{y^2}{1.2} + \phi_{,}' yz + \phi_{,,} \frac{z^2}{1.2} + \&c.$$

Then if we take the surface of the fluid in equilibrium as the plane of (x, y,) and suppose the sides of the rectangular canal symmetrical with respect to the plane (x, z,) ϕ will evidently contain none but even powers of y, and we shall have

$$(3). \qquad \phi = \phi_0 + \phi_{,} z + \phi'' \frac{y^2}{1.2} + \phi_{,,} \frac{z^2}{1.2} + \&c.$$

Now if $$y = \pm \beta_x$$

represent the equation of the two sides of the canal, we need only satisfy one of them as

$$y - \beta_x = 0,$$

since the other will then be satisfied by the exclusion of the odd powers of y from ϕ.

The equation (A) gives, since here $A = y - \beta$

$$(a). \qquad 0 = \frac{d\phi}{dy} - \frac{d\beta}{dx} \cdot \frac{d\phi}{dx} \ldots\ldots \text{(when } y = \beta).$$

Similarly, if $z - \gamma_x = 0$ is the equation of the bottom of the canal,

$$(b). \qquad 0 = \frac{d\phi}{dz} - \frac{d\gamma}{dx} \cdot \frac{d\phi}{dx} \ldots\ldots \text{(when } z = \gamma).$$

If moreover, $z - \zeta_{x.t} = 0$ be the equation of the upper surface,

$$\left.\begin{array}{l} (c). \qquad 0 = \dfrac{d\phi}{dz} - \dfrac{d\zeta}{dx}\dfrac{d\phi}{dx} - \dfrac{d\zeta}{dt} \\ \text{But here } p = 0; \quad \therefore \text{ also by (2) } g\zeta = \dfrac{d\phi}{dt} \end{array}\right\} \ldots\ldots \text{(when } z = \zeta).$$

Substituting from (3) in (b) we get

$$0 = \phi_{,} + \phi_{,,}\gamma + \&c. - \frac{d\gamma}{dx}\left\{\frac{d\phi_0}{dx} - \frac{d\phi_{,}}{dx}\frac{\gamma}{1} + \&c. \right\};$$

or neglecting quantities of the order γ^2,

$$(b'). \qquad 0 = \phi_{,} + \phi_{,,}\gamma - \frac{d\gamma}{dx}\frac{d\phi_0}{dx}.$$

Similarly (a) becomes

$$(a'). \qquad 0 = \phi''\beta - \frac{d\beta}{dx} \cdot \frac{d\phi_0}{dx},$$

and (c) becomes, since ζ is of the order of the disturbance,

$$(c'). \qquad \left.\begin{array}{l} 0 = \phi_{,} - \dfrac{d\zeta}{dt} \\ g\zeta = \dfrac{d\phi_0}{dt} \end{array}\right\} \left.\begin{array}{c} \text{when } z = \zeta, \\ \text{or neglecting (disturbance)}^2\ z = 0 \end{array}\right\},$$

provided as above we neglect (disturbance)^2.

Again, the condition (2) gives by equating separately the coefficients of powers and products of y and z,

$$(2'). \qquad \left.\begin{aligned} 0 &= \frac{d^2\phi_0}{dx^2} + \phi'' + \phi_{,,} \\ 0 &= \&\text{c.} \end{aligned}\right\}.$$

If now by means of (a'), (b'), (c') we eliminate $\phi'' \, \phi_{,,}$ from $(2')$, there results

$$(4). \qquad 0 = \frac{d^2\phi_0}{dx^2} + \left\{\frac{d\beta}{\beta dx} + \frac{d\gamma}{\gamma dx}\right\} \frac{d\phi_0}{dx} - \frac{1}{g\gamma}\left(\frac{d^2\phi_0}{dt^2}\right).$$

It now only remains to integrate this equation.

For this we shall suppose β and γ functions of x which vary very slowly, so that if written in their proper form we should have

$$\beta = \psi(\omega x),$$

where ω is a very small quantity. Then,

$$\frac{d\beta}{dx} = \omega\psi'(\omega x), \quad \frac{d^2\beta}{dx^2} = \omega^2\psi''(\omega x), \ \&\text{c.}$$

Hence if we allow ourselves to omit quantities of the order ω^2, and assume, to satisfy (4),

$$\phi_0 = Af(t + X),$$

where A is a function of x of the same kind as β and γ, we have, omitting $\left(\frac{d^2A}{dx^2}\right)$,

$$\frac{d^2\phi_0}{dt^2} = Af'',$$

$$\frac{d\phi_0}{dx} = A\frac{dX}{dx}f' + \frac{dA}{dx}f,$$

$$\frac{d^2\phi_0}{dx^2} = A\left(\frac{dX}{dx}\right)^2 f'' + A\frac{d^2X}{dx^2}f' + 2\frac{dA}{dx}\cdot\frac{dX}{dx}f'.$$

Substituting these in (4), and still neglecting quantities of the order ω^2, we get

$$0 = \left\{A\left(\frac{dX}{dx}\right)^2 - \frac{A}{g\gamma}\right\}f''$$

$$+\left\{A\frac{d^2X}{dx^2} + 2\frac{dA}{dx}\frac{dX}{dx} + \left(\frac{d\beta}{\beta dx} + \frac{d\gamma}{\gamma dx}\right)A\frac{dX}{dx}\right\}f';$$

equating now separately the coefficients of f' and f'', we get

$$0 = \left(\frac{dX}{dx}\right)^2 - \frac{1}{g\gamma},$$

$$0 = \frac{\frac{d^2X}{dx^2}}{\frac{dX}{dx}} + 2\frac{dA}{Adx} + \frac{d\beta}{\beta dx} + \frac{d\gamma}{\gamma dx}.$$

The first, integrated, gives

$$X = \pm\int\frac{dx}{\sqrt{\gamma g}},$$

and the second

$$k = \frac{dX}{dx}A^2\beta\gamma = A^2\frac{\beta\gamma}{\sqrt{g\gamma}} = \frac{A^2\beta\gamma^{\frac{1}{2}}}{\sqrt{g}}.$$

Hence if we neglect the superfluous constant $k\sqrt{g}$, the general integral of (4) is, ($\because A = \beta^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}$),

$$\phi_0 = \beta^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}\left\{f\left(t + \int\frac{dx}{\sqrt{g\gamma}}\right) + F\left(t - \int\frac{dx}{\sqrt{g\gamma}}\right)\right\};$$

therefore, by (c'),

$$\zeta = \frac{d\phi_0}{gdt} = \frac{\beta^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}}{g}\left\{f'\left(t + \int\frac{dx}{\sqrt{g\gamma}}\right) + F'\left(t - \int\frac{dx}{\sqrt{g\gamma}}\right)\right\},$$

and the actual velocity of the fluid particles in the direction of the axis of x, is

$$u = \frac{d\phi}{dx} = \frac{d\phi_0}{dx} = \frac{\beta^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}}{\sqrt{g\gamma}}\left\{f'\left(t + \int\frac{dx}{\sqrt{g\gamma}}\right) - F'\left(t - \int\frac{dx}{\sqrt{g\gamma}}\right)\right\},$$

neglecting quantities which are of the order (ω) compared with those retained.

If the initial values of ζ and u are given, we may then determine f and F', and we thus see that a single wave, like a pulse of sound, divides into two, propagated in opposite directions. Considering, therefore, only that which proceeds in the direction of x positive, we have

$$(5).\qquad \zeta = \frac{\beta^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}}{g} F'\left(t - \int \frac{dx}{\sqrt{g\gamma}}\right).$$

$$(6).\qquad u = \frac{\beta^{-\frac{1}{2}}\gamma^{-\frac{3}{4}}}{g^{\frac{1}{2}}} F'\left(t - \int \frac{dx}{\sqrt{g\gamma}}\right).$$

Suppose now the value of $F'(x) = 0$, except from $x = a$ to $x = a + \alpha$, and δx to be the corresponding length of the wave, we have

$$t - \int \frac{dx}{\sqrt{g\gamma}} = a + \alpha,$$

$$\text{and}\quad t - \int \frac{dx}{\sqrt{g\gamma}} - \frac{\delta x}{\sqrt{g\gamma}} = a \text{ very nearly}.$$

Hence the variable length of the wave is

$$(7).\qquad \delta x = \alpha . \sqrt{g\gamma}.$$

Lastly, for any particular phase of the wave, we have

$$t - \int \frac{dx}{\sqrt{g\gamma}} = \text{const.};$$

therefore

$$(8).\qquad \frac{dx}{dt} = \sqrt{g\gamma},$$

is the velocity with which the wave, or more strictly speaking the particular phase in question progresses.

From (5), (6), (7), and (8) we see that if β represent the variable breadth of the canal and γ its depth,

ζ = height of the wave $\propto \beta^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}$,

u = actual velocity of the fluid particles $\propto \beta^{-\frac{1}{2}}\gamma^{-\frac{3}{4}}$,

δx = length of the wave $\propto \gamma^{\frac{1}{2}}$,

and $\frac{dx}{dt}$ = velocity of the wave's motion $= \sqrt{g\gamma}$.

XXIV. *On the Theory of the Equilibrium of Bodies in Contact.* By the Rev. H. MOSELEY, M.A., *of St. John's College, Professor of Natural Philosophy and Astronomy in King's College, London.*

[Read *May* 15, 1837.]

IN a paper on the Theory of the Arch, read before the Cambridge Philosophical Society in October 1833, and published in the fifth Volume of their Transactions, I have discussed the conditions of the equilibrium of a system of bodies in contact, on a principle referring it to the direction in respect to the surfaces of contact of a *certain line,* given in terms of the magnitudes and directions of the forces which compose the equilibrium. The condition that no one portion of the system shall *turn* on the edge of its surface of contact with another, being determined by the condition that the point at which the line leaves the surface of any one of the contiguous bodies, to enter the adjacent body, shall be within the boundary of the common surface of contact of the two; and the condition that no two contiguous bodies shall *slip* upon one another, by the condition that the direction in which this line intersects their common surface shall lie within a certain angle, which I have called the "limiting angle of resistance," and which is dependent for its magnitude on the circumstances of the friction of the two surfaces upon one another.

If a surface be imagined to intersect the system, and continually to change its position, and, if necessary, its form so as to coincide, in order, with all the surfaces of contact, and if, in each position, the resultant be taken, in respect to those forces which are impressed upon one of the two parts into which this surface divides the system; then the *locus of the consecutive intersections* of these resultants is that curved line to which I have assigned the properties of equilibrium described in the preceding page.

I wish now to correct this definition.

To the properties assigned to this line it is necessary that at each of the points where it intersects contiguous surfaces of the component masses, the whole pressure upon those surfaces should be supposed to be applied. Now, according to the definition given of it, this supposition is not, except under certain circumstances, admissible.

The resultant of the pressures upon each surface of contact is necessarily at some point or other a tangent to the locus of the *intersections* of the resultants, but it *may* be, and except in particular cases, *will* be, a tangent to it at a point *other* than that in which this line intersects the surface of contact itself.

The *point where the resultant intersects the dividing surface to which it corresponds*, is that element in the theory on which the condition, "that one portion of the system shall not *turn* over upon the boundary of its surface of contact with the adjacent portion," depends. I propose, therefore, in the following paper, to determine the line *which is the locus of intersections of the consecutive resultants, with the corresponding imaginary surfaces of division*, these surfaces being, here, supposed to be planes. This line I shall call the LINE OF RESISTANCE, including as it does the points of application of the resultants of all the *resistances* of the surfaces of contact.

The *direction* in which the resultant intersects two common surfaces of contact, is that on which the condition, "that these surfaces shall not *slip* upon one another," depends; moreover this *direction* is a tangent to the line which is the locus of the intersections of the consecutive resultants, drawn from the point where the *line of resistance* cuts the surface of contact. The determination of *this* line is therefore also an important feature in the theory. I propose that it should retain the name before given to it of the LINE OF PRESSURE.

One of these lines—the line of resistance—determining the *point* of application of the resultant of the pressures upon each of the surfaces of contact of the system, and the other—the line of pressure—the *direction* of that resultant, the determination of the two includes the whole theory of the equilibrium of the system.

In its application to the theory of the arch there belong to the line of *resistance* all those properties treated of in my former paper which have reference to the condition "that the voussoirs shall not turn upon the angles of one another."

It follows, therefore, on the principles established in that paper that this line *touches* the intrados of the arch at certain points equidistant from the crown, called points of rupture, and that the position of these points, and, consequently, that of the point of application of the resultant of the pressures upon the key-stone, are subject to the condition that this resultant is a minimum; and this condition being supposed, all the circumstances which connect themselves with the equilibrium of the circular arch, as a complete segment, and a broken or gothic arch, subjected to any variety of loading, are discussed and determined in the eleventh section of the following paper.

The condition, however, that the resultant pressure upon the key-stone is subjected in respect to the position of its point of application to the condition of a minimum, is dependant upon *hypothetical* qualities of the masonry. It supposes an unyielding material for the arch-stones, and a mathematical adjustment of their surfaces. These have no existence in practice. On the striking of the centers the arch invariably sinks at the crown, its voussoirs slightly opening there upon their lower edges, and thus pressing upon one another exclusively by their upper edges. Practically, the line of resistance then *touches the extrados* at the crown; whilst the condition of the minimum is satisfied by its contact with the intrados at the points of rupture in the haunches. This condition being assumed, all consideration of the yielding quality of the material of the arch or of its abutments is *eliminated.* It is thus discussed as a practical question in the twelfth section of this paper.

1. Let a continuous mass to which are applied certain forces of pressure, be supposed to be intersected by a *plane* whose equation is

$$z = Ax + By + C \text{......(I.)}$$

Let the sums of the forces impressed upon one of the parts and resolved in directions parallel to three rectangular axes, be respectively M_1, M_2, M_3, and the sums of their moments N_1, N_2, N_3.

Let, moreover, the position of the plane be such that these forces are reducible to a single resultant, a condition determined by the equation

$$M_1N_3 + M_2N_2 + M_3N_1 = 0 \text{......(II.)}$$

The equation to this single resultant will then be

$$\left.\begin{aligned} x &= \frac{M_1}{M_3}z + \frac{N_2}{M_3} \\ y &= \frac{M_2}{M_3}z + \frac{N_1}{M_3} \end{aligned}\right\} \text{.........(III.)}$$

If between the four preceding equations in which M_1, M_2, M_3, N_1, N_2, N_3 are functions of A, B, C, these three quantities A, B, C be eliminated, there will be obtained an equation in x, y, z, which is that to a surface of which this is the characteristic property; that it includes all the points of intersection of the resultant force with its corresponding intersecting plane in every position, which, according to the assumed conditions, this last may be made to take up.

This surface is the SURFACE OF RESISTANCE.

If to the preceding conditions there be added this, that in each two consecutive positions of the intersecting plane the corresponding resultants shall intersect, the surface of resistance will resolve itself into a line, which is the LINE OF RESISTANCE.

Differentiating on this hypothesis the equation III. in respect to A, B, C, we have

$$\left.\begin{aligned}
&\left\{\frac{d\left(\frac{M_1}{M_3}\right)}{dA}dA+\frac{d\left(\frac{M_1}{M_3}\right)}{dB}dB+\frac{d\left(\frac{M_1}{M_3}\right)}{dC}dC\right\}z+\left\{\frac{d\left(\frac{N_3}{M_3}\right)}{dA}dA+\frac{d\left(\frac{N_2}{M_3}\right)}{dB}dB+\frac{d\left(\frac{N_2}{M_3}\right)}{dC}dC\right\}=0\\
&\left\{\frac{d\left(\frac{M_2}{M_3}\right)}{dA}dA+\frac{d\left(\frac{M_2}{M_3}\right)}{dB}dB+\frac{d\left(\frac{M_2}{M_3}\right)}{dC}dC\right\}z+\left\{\frac{d\left(\frac{N_1}{M_3}\right)}{dA}dA+\frac{d\left(\frac{N_1}{M_3}\right)}{dB}dB+\frac{d\left(\frac{N_1}{M_3}\right)}{dC}dC\right\}=0
\end{aligned}\right\}\ldots(\text{IV.})$$

Eliminating z

$$\begin{aligned}
&\left\{\frac{d\left(\frac{M_2}{M_3}\right)}{dA}dA+\frac{d\left(\frac{M_2}{M_3}\right)}{dB}dB+\frac{d\left(\frac{M_2}{M_3}\right)}{dC}dC\right\}\left\{\frac{d\left(\frac{N_2}{M_3}\right)}{dA}dA+\frac{d\left(\frac{N_2}{M_3}\right)}{dB}dB+\frac{d\left(\frac{N_2}{M_3}\right)}{dC}dC\right\}\\
-&\left\{\frac{d\left(\frac{M_1}{M_3}\right)}{dA}dA+\frac{d\left(\frac{M_1}{M_3}\right)}{dB}dB+\frac{d\left(\frac{M_1}{M_3}\right)}{dC}dC\right\}\left\{\frac{d\left(\frac{N_1}{M_3}\right)}{dA}dA+\frac{d\left(\frac{N_1}{M_3}\right)}{dB}dB+\frac{d\left(\frac{N_1}{M_3}\right)}{dC}dC\right\}
\end{aligned}=0\ldots\ldots(\text{V.})$$

From the elimination of A, B, and C, between the five equations I, II, III, V, will result the two equations to the Line of Resistance; and from the elimination of the same three quantities between the five equations II, III, IV,* the two equations to the Line of Pressure.

The inclination ι of the resultant pressure to a perpendicular to the intersecting plane, in any of its positions, may be determined (see paper on *Equation of Arch*), independently of the line of pressure, from the equation

$$\cos\iota = -\frac{AM_1 + BM_2 + M_3}{\{(A^2+B^2+1)(M_1^2+M_2^2+M_3^2)\}^{\frac{1}{2}}}.$$

2. Let the mass be a Prism whose axis is horizontal, and the forces applied to which are, its weight and certain pressures, P, whose directions are in planes perpendicular to its axis and inclined at angles Φ to axis of z, and whose points of application are uniformly distributed along lines on the surface of the prism parallel to its axis; all those pressures which are applied in each line being equal to one another.

* It will be observed that the condition V. is included in these.

The relation of the forces which compose the equilibrium of the whole Prism, will then be the same with that of the forces impressed on any one of its sections perpendicular to the axis.

Let *CBD*, (Fig. 1,) represent any one of these sections. Suppose the mass to be intersected in any direction parallel to its axis by a plane, and let $N_1 \; N_2$ be the intersection of this plane with the section *CB* of the mass.

And *first*, let this intersecting plane in altering its position be supposed to remain always parallel to itself.

Take Az, the axis of z, perpendicular to $N_1 \; N_2$, and let it make an angle Θ with the vertical.

Let $MN_1 = y_1$, $MN_2 = y_2$, $AM = c$, $AK = k$.

$$M_1 = 0,$$

$$M_2 = \Sigma P \sin \Phi - \sin \Theta \int (y_1 - y_2)\, dC,$$

$$M_3 = \Sigma P \cos \Phi + \cos \Theta \int (y_1 - y_2)\, dC,$$

$$N_1 = \tfrac{1}{2} \cos \Theta \int (y_1^2 - y_2^2)\, dC + \sin \Theta \int C (y_1 - y_2)\, dC + \Sigma \pm Pk \cos \Phi,$$

$$N_2 = 0,$$

$$N_3 = 0.$$

This hypothesis with regard to the position of the axis of z, and these substitutions being made, all the equations of condition vanish except equation I, the second of equations III, and the second of equations IV. These resolve themselves into the following:—

$$z = C \qquad \text{(1)},$$

$$y = \frac{\{\Sigma P \sin \Phi - \sin \Theta \int (y_1 - y_2)\, dC\} z + \tfrac{1}{2} \cos \Theta \int (y_1^2 - y_2^2)\, dC + \sin \Theta \int C (y_1 - y_2)\, dC + \Sigma \pm Pk \cos \Phi}{\Sigma P \cos \Phi + \cos \Theta \int (y_1 - y_2)\, dC} \qquad \text{(2)},$$

$$y = \frac{z \dfrac{d \Sigma P \sin \Phi}{dC} dC + \dfrac{d \Sigma \pm Pk \cos \Phi}{dC} dC - (y_1 - y_2)\{(z - C) \sin \Theta - \tfrac{1}{2}(y_1 + y_2) \cos \Theta\}}{\dfrac{d \Sigma P \cos \Phi}{dC} dC + (y_1 - y_2) \cos \Theta} \qquad \text{(3)}.$$

The equation to the line of resistance is determined by eliminating C between the equations (1) and (2), and that to the line of pressure by eliminating it between (2) and (3).

If the first elimination be made, and it be observed, that

$$-z\int(y_1 - y_2)\,dz + \int z\,(y_1 - y_2)\,dz = -\iint(y_1 - y_2)\,dz^2,$$

there will be obtained the following general equation to the line of resistance,

$$y = \frac{z\Sigma P \sin\Phi + \frac{1}{2}\cos\Theta \int(y_1^2 - y_2^2)\,dz - \sin\Theta \iint(y_1 - y_2)\,dz^2 + \Sigma \pm Pk\cos\Phi}{\Sigma P\cos\Phi + \cos\Theta\int(y_1 - y_2)\,dz} \ldots (4).$$

The second elimination is greatly simplified in the case in which P, Φ, k are independent of C. Since in this case, equation (3) gives

$$y - \tfrac{1}{2}(y_1 + y_2) + (z - C)\tan\Theta = 0 \ldots\ldots\ldots\ldots (5).$$

If the intersections be supposed to be made horizontally, (Fig. 2,) Θ must be assumed $= 0$. If they be made vertically, (Fig. 3) $\Theta = \frac{\pi}{2}$. In the latter case, equation (5) gives $C = z$.

The elimination of C between (3) and (2) is therefore the same as that between (1) and (2), and the line of pressure in this case, coincides with the line of resistance.

3. Let the mass be a trapezoidal form. (Fig. 4.)

Let AB and CD be inclined to the axis of z at angles α_1, α_2, and assume $CA = a$; $\therefore\ y_1 = a + z\tan\alpha_1,\quad y_2 = z\tan\alpha_2.$

$$\left.\begin{aligned} \text{Hence}\quad \int(y_1^2 - y_2^2)\,dz &= az(a + z\tan\alpha_1) + \tfrac{1}{3}z^3(\tan^2\alpha_1 - \tan^2\alpha_2) \\ \int(y_1 - y_2)\,dz &= az + \tfrac{1}{2}z^2(\tan\alpha_1 - \tan\alpha_2) \\ \iint(y_1 - y_2)\,dz^2 &= \tfrac{1}{2}az^2 + \tfrac{1}{6}z^3(\tan\alpha_1 - \tan\alpha_2) \end{aligned}\right\}\ldots\ldots(6).$$

Therefore by substitution in equation (4) we have for the equation to the line of resistance

$$y = \frac{\frac{1}{6}z^3\{\tan\alpha_1 - \tan\alpha_2\}\{\tan\alpha_1 + \tan\alpha_2 - \tan\Theta\} + \frac{1}{2}az^2\{\tan\alpha_1 - \tan\Theta\} + z\{\sec\Theta\,\Sigma P\sin\Phi + \frac{1}{2}a^2\} + \sec\Theta\,\Sigma\pm Pk\cos\Phi}{\frac{1}{2}z^2\{\tan\alpha_1 - \tan\alpha_2\} + az + \sec\Theta\,\Sigma P\cos\Phi} \ldots (7).$$

This equation being of three dimensions in z, it follows that for *certain values* of y there are three possible values of z. The curve has therefore a point of contrary flexure, and is somewhat of the form shewn in the figure.

The LINE OF PRESSURE in a trapezoidal mass has been determined in my former paper. It is there shewn to be of three dimensions in z, and to have, like the line of resistance, a point of contrary flexure.

The POINTS OF RUPTURE being those where the line of resistance meets the *Intrados* or the *Extrados* of the mass may be determined by assuming in equation (7), $y = a + z_1 \tan \alpha$, and $y = z_2 \tan \alpha_2$; whence there is obtained

$$z_1^3 + \frac{3a}{\tan \alpha_1 - \tan \alpha_2} \cdot z_1^2 + 6 \frac{\sec \Theta \Sigma P \sin \Phi - \sec \Theta \tan \alpha_1 \Sigma P \cos \Phi - \frac{1}{2} a^2}{\{\tan \alpha_1 - \tan \alpha_2\}\{\tan \alpha_2 - 2 \tan \alpha_1 - \tan \Theta\}} z_1$$

$$+ 6 \frac{\sec \Theta \{\Sigma \pm Pk \cos \Phi - a \Sigma P \cos \Phi\}}{\{\tan \alpha_1 - \tan \alpha_2\}\{\tan \alpha_2 - 2 \tan \alpha_1 - \tan \Theta\}} = 0 \ldots\ldots(8),$$

$$z_2^3 + \frac{3a}{\tan \alpha_1 - \tan \alpha_2} \cdot z_2^2 + 6 \frac{\sec \Theta \Sigma P \sin \Phi - \sec \Theta \tan \alpha_2 \Sigma P \cos \Phi + \frac{1}{2} a^2}{\{\tan \alpha_1 - \tan \alpha_2\}\{\tan \alpha_1 - 2 \tan \alpha_2 - \tan \Theta\}} z_2^2$$

$$+ 6 \frac{\sec \Theta \Sigma \pm Pk \cos \Phi}{\{\tan \alpha_1 - \tan \alpha_2\}\{\tan \alpha_1 - 2 \tan \alpha_2 - \tan \Theta\}} = 0 \ldots\ldots(9).$$

If $\tan \alpha_2 - 2 \tan \alpha_1 = \tan \Theta$ there is but one point of rupture in the *Extrados.*

If $\tan \alpha_1 - 2 \tan \alpha_2 = \tan \Theta$ there is but one point of rupture in the *Intrados.*

These single points of rupture are determined in the two cases by the equations

$$z_1 = \frac{\Sigma \pm Pk \cos \Phi - a \Sigma P \cos \Phi}{\tan \alpha_1 \Sigma P \cos \Phi - \Sigma P \sin \Phi + \frac{1}{2} a^2 \cos \Theta} \ldots\ldots\ldots(10),$$

$$z_2 = \frac{\Sigma \pm Pk \cos \Phi}{\tan \alpha_2 \Sigma P \cos \Phi - \Sigma P \sin \Phi - \frac{1}{2} a^2 \cos \Theta} \ldots\ldots\ldots(11).$$

4. The Buttress.

If Θ be taken $= 0$ the trapezoidal mass will assume the position of the ordinary buttress, (Fig. 5), whose line of resistance is determined by the equation

$$y = \frac{\frac{1}{6} z^3 \{\tan^2 a_1 - \tan^2 a_2\} + \frac{1}{2} a z^2 \tan a_1 + z\{\Sigma P \sin \Phi + \frac{1}{2} a^2\} + \Sigma \pm Pk \cos \Phi}{\frac{1}{2} z^2 \{\tan a_1 - \tan a_2\} + az + \Sigma P \cos \Phi} \quad \ldots (12).$$

And its greatest possible height by the least root of the equation

$$z_1^3 + \frac{3a}{\tan a_1 - \tan a_2} \cdot z_1^2 + 6 \frac{\Sigma P \sin \Phi - \tan a_1 \Sigma P \cos \Phi - \frac{1}{2} a^2}{\{\tan a_1 - \tan a_2\}\{\tan a_2 - 2 \tan a_1\}} z_1$$
$$+ \frac{6\{\Sigma \pm Pk \cos \Phi - a \Sigma P \cos \Phi\}}{\{\tan a_1 - \tan a_2\}\{\tan a_2 - 2 \tan a_1\}} = 0 \ldots\ldots (13).$$

The best dimensions of the buttress would seem to be those which bring the line of pressure to the *center* of the base. These may be determined by assuming in equation 12, $y = \frac{1}{2}\{a + z_3(\tan a_1 + \tan a_2)\}$, whence,

$$z_3^3\{\tan^2 a_1 - \tan^2 a_2\} + 3 a z_3^2 \{\tan a_1 + \tan a_2\} - 6 z_3 \{2 \Sigma P \sin \Phi - (\tan_1 + \tan a_2) \Sigma P \cos \Phi\}$$
$$- 6\{2 \Sigma \pm Pk \cos \Phi - a \Sigma P \cos \Phi\} = 0 \ldots\ldots (14).$$

To determine the line of pressure in the buttress, assuming $\Theta = 0$ in equation (2), we have

$$y = \frac{\Sigma P \sin \Phi + \frac{1}{2} \int (y_1 - y_2)\, dC + \Sigma \pm Pk \cos \Phi}{\Sigma P \cos \Phi + \int (y_1 - y_2)\, dC} \ldots\ldots\ldots\ldots (15).$$

Also assuming P, k, Φ not to be functions of C, and taking $\Theta = 0$, $y_1 = a + C \tan a_1$, $y_2 = C \tan a_2$, we have, by equation (5),

$$C = \frac{2y - a}{\tan a_1 + \tan a_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16).$$

Performing the integrations indicated in equation (15), substituting this last value of C, and reducing,

$$z\Sigma P \sin\Phi = \tfrac{2}{3}\frac{\tan\alpha_1 - \tan\alpha_2}{(\tan\alpha_1 + \tan\alpha_2)^2}\cdot y^3 + 2a\frac{\tan\alpha_2}{(\tan\alpha_1 + \tan\alpha_2)^2}y^2$$

$$-\left\{\tfrac{1}{4}a^2\frac{\tan\alpha_1 + 3\tan\alpha_2}{(\tan\alpha_1+\tan\alpha_2)^2} - \Sigma P\cos\Phi\right\}y + \frac{1}{12}a^3\frac{\tan\alpha_1 + 2\tan\alpha_2}{(\tan\alpha_1+\tan\alpha_2)^2} - \Sigma \pm Pk\cos\Phi \ldots(17).$$

5. The Pier.

If $\alpha_1 = \alpha_2$, (see Fig. 6), the mass may be taken to represent a pier or a wall of uniform thickness, and the equation (7) to its line of *resistance* will become

$$y = \frac{\tfrac{1}{2}az^2\{\tan\alpha_1 - \tan\Theta\} + z\{\sec\Theta\Sigma P\sin\Phi + \tfrac{1}{2}a^2\} + \sec\Theta\Sigma\pm Pk\cos\Phi}{az + \sec\Theta\Sigma P\cos\Phi}\ldots(18).$$

Which is the equation to an hyperbola whose axis is inclined to the axis of z at an angle represented by the formula

$$\tfrac{1}{2}\tan^{-1}\left\{\frac{2}{\tan\alpha_1 - \tan\Theta}\right\}\ldots\ldots(19),$$

and the co-ordinates of whose center are

$$\frac{\Sigma P\sin\Phi - (\tan\alpha_1 - \tan\Theta)\,\Sigma P\cos\Phi + \tfrac{1}{2}a^2}{a\cos\Theta} \text{ and } -\frac{\Sigma P\cos\Phi}{a\cos\Theta}\ldots(20).$$

In the case in which $\Theta = 0$, or the intersections are horizontal, (Fig. 7), equation (17) gives for the equation to the line of *pressure*,

$$z\Sigma P\sin\Phi = \frac{1}{2}a\cot\alpha . y^2 - \left\{\frac{1}{4}a^2\cot\alpha - \Sigma P\cos\Phi\right\}y + \frac{1}{16}a^3\cot\alpha - \Sigma\pm Pk\cos\Phi\ldots(21),$$

which is the equation to a parabola whose axis is vertical, whose parameter is $\dfrac{1}{2}\dfrac{a\cot\alpha}{\Sigma P\sin\Phi}$, and the co-ordinates of whose vertex are

$$\frac{1}{2}\frac{a\cot\alpha}{\Sigma P\sin\Phi}\left\{(\tfrac{1}{4}a - \frac{1}{a}\tan\alpha\Sigma P\cos\Phi)^2 - \tfrac{1}{8}a^2 + \frac{1}{a}\tan\alpha\Sigma\pm Pk\cos\Phi\right\},$$

$$\text{and } \tfrac{1}{4}a - \frac{1}{a}\tan\alpha\Sigma P\cos\Phi.$$

The supposition $a_1 = \Theta = 0$ gives the case of an upright Pier with horizontal intersections (Fig. 8), and the equation (18) to the line of resistance becomes

$$y = \frac{z\{\Sigma P \sin \Phi + \frac{1}{2}a^2\} + \Sigma \pm Pk \cos \Phi^*}{az + \Sigma P \cos \Phi} \dots\dots\dots\dots(22),$$

the equation to a rectangular hyperbola, whose axis is by formula (19) inclined at an angle of 45° to the axis of z, whose asymptotes are therefore vertical, and the co-ordinates of whose center C are by formulæ (20),

$$AK = \frac{\Sigma P \sin \Phi}{a} + \tfrac{1}{2}a, \quad \text{and } KC = -\frac{\Sigma P \cos \Phi}{a}.$$

CE being an asymptote to the hyperbola, it is clear that if AK be less than AD, that is, if

$$\frac{\Sigma P \sin \Phi}{a} \text{ be less than } \tfrac{1}{2}a,$$

$$\text{or } 2\Sigma P \sin \Phi \text{ be less than } a^2,$$

the line of resistance will not meet the extrados of the pier however great may be its height. But that if

$$2\Sigma P \sin \Phi \text{ be greater than } a^2,$$

it will somewhere cut the extrados; there is, therefore, in this case, a certain height of the pier beyond which, if it be continued, it will be overthrown.

This maximum height of the pier is determined by the equation

$$z = \frac{\Sigma \pm Pk \cos \Phi - a\Sigma P \cos \Phi}{\frac{1}{2}a^2 - \Sigma P \sin \Phi} \dots\dots\dots\dots(23).$$

* This equation may be put under the following form, whence all the circumstances mentioned in the text are apparent,

$$\left\{y - \left(\frac{\Sigma P \sin \Phi}{a} + \tfrac{1}{2}a\right)\right\}\left\{z + \frac{\Sigma P \cos \Phi}{a}\right\} = \frac{1}{a}\Sigma \pm Pk \cos \Phi - \left\{\frac{\Sigma P \sin \Phi}{a^2} + \tfrac{1}{2}\right\}\Sigma P \cos \Phi.$$

6. The Straight Arch, or Plate Bande.

Let Θ be now assumed $=\frac{\pi}{2}$, the sections will then be vertical (Fig. 3), the line of resistance will coincide with the line of pressure (Art. 2), and the equation common to both will be

$$y=\frac{z\Sigma P\sin\Phi-\iint(y_1-y_2)\,dz^2+\Sigma\pm Pk\cos\Phi}{\Sigma P\cos\Phi}\ldots\ldots\ldots(24).$$

In the case of a trapezoidal mass (Fig. 9), this formula gives by equation (7)

$$y=\frac{-\frac{1}{6}z^3\{\tan\alpha_1-\tan\alpha_2\}-\frac{1}{2}az^2+z\Sigma P\sin\Phi+\Sigma\pm Pk\cos\Phi}{\Sigma P\cos\Phi}\ldots(25),$$

and the points of rupture are determined by the equation

$$z_1^3\{\tan\alpha_1-\tan\alpha_2\}+3az_1^2+6\{\tan\alpha_1\Sigma P\cos\Phi-\Sigma P\sin\Phi\}z$$
$$+\{a\Sigma P\cos\Phi-\Sigma\pm Pk\cos\Phi\}=0\ldots(26).$$

If $\alpha_1=\alpha_2$ (Fig. 10), the equation to the line of resistance and to the line of pressure becomes

$$y=\frac{-\frac{1}{2}az^2+z\Sigma P\sin\Phi+\Sigma\pm Pk\cos\Phi}{\Sigma P\cos\Phi}\ldots\ldots\ldots\ldots(27).$$

The equation to a parabola whose axis is vertical, whose parameter is $\frac{2\Sigma P\cos\Phi}{a}$, and the co-ordinates of whose vertex are

$$\frac{\Sigma P\sin\Phi}{a}\text{ and }\frac{a\Sigma\pm Pk\cos\Phi+\frac{1}{2}(\Sigma P\sin\Phi)^2}{a\Sigma P\cos\Phi}.$$

The elements of this parabola are thus independent of the position of the mass.

Let us suppose the impressed forces P to be placed symmetrically (Fig. 11), then will the vertex of the parabola manifestly be situated midway between the extremities of the mass. Let the length of it be $2b$;

$$\therefore\ \frac{\Sigma P\sin\Phi}{a}=b\ldots\ldots\ldots(28).$$

If, moreover, the forces P be supposed to be resistances, the line of resistance will *touch* the extrados (Fig. 12);

$$\therefore\ a = \frac{a\Sigma \pm Pk\cos\Phi + \tfrac{1}{2}(\Sigma P\sin\Phi)^2}{a\Sigma P\cos\Phi} \text{............}(29).$$

Suppose that there is only one force P applied at each extremity; therefore by equations (28) and (29),

$$P\sin\Phi = ab,\quad a^2 = \pm ak + \tfrac{1}{2}P\sin\Phi\tan\Phi;$$

$$\therefore\ \tan\Phi = \frac{2(a\mp k)}{b} \text{........................}(30).$$

$$P = ab\sqrt{1+\tfrac{1}{4}\left(\frac{b}{a\mp k}\right)^2} \text{...........}(31).$$

$$P\cos\Phi = \tfrac{1}{2}\frac{ab^2}{(a\mp k)} \text{........................}(32).$$

This last equation gives that portion of the force P which is resolved in the direction of a horizontal line. If $k=0$ or the force P be applied at the angle A of the mass,

$$\tan\Phi = \frac{2a}{b},\quad P = b\sqrt{a^2+\tfrac{1}{4}b^2},\quad P\cos\Phi = \tfrac{1}{2}b^2 \text{......}(33).$$

It is worthy of remark, that the last of these expressions is independent of a, the depth of the voussoirs.

Let a straight arch be supposed to be supported upon the edges of two vertical piers, (Fig. 12).

The point of rupture in the extrados of the pier, or its greatest height, so as to stand unsupported, will then be determined by the following equation derived from equation (23), by taking $k = 0$ and writing for Φ its complement, since in the straight arch Φ is measured from the horizontal axis of z, in the pier from the same axis in a vertical position, so that Φ in the one case is the complement of Φ in the other:

$$z = \frac{-Pa\sin\Phi}{\frac{1}{2}a^2 - P\cos\Phi}.$$

Eliminating therefore $P\sin\Phi$, and $P\cos\Phi$, between this equation and equations (33), and calling a_1 the width of the pier to distinguish it from the depth of the arch, we have

$$z = \frac{2aa_1b}{b^2 - a_1^2} \quad \text{.........(34).}$$

7. Let it now be supposed that the forces P are impressed upon all the points of the face BC (Fig. 2) of a mass, and that the plane of intersection is horizontal. Let moreover all these forces P be parallel to one another, and let them be represented respectively in magnitude by the values of a function P of z, continuously from Z to z;

$$\therefore\ \Sigma P\cos\Phi = \cos\Phi\int_Z^z P\,dz, \quad \Sigma P\sin\Phi = \sin\Phi\int_Z^z P\,dz,$$

$$\Sigma \pm Pk\cos\Phi = \int_Z^z P(y_2\cos\Phi - z\sin\Phi)\,dz.$$

These substitutions being made, the equation (4) to the line of resistance gives the following:

$$y = \frac{\frac{1}{2}\int_0^z (y_1^2 - y_2^2)\,dz + z\sin\Phi\int_Z^z P\,dz + \int_Z^z P(y_2\cos\Phi - z\sin\Phi)\,dz}{\cos\Phi\int_Z^z P\,dz + \int_0^z (y_1 - y_2)\,dz} \quad \text{......(35),}$$

$$\text{or } y = \frac{\frac{1}{2}\int_0^z (y_1^2 - y_2^2)\,dz + \sin\Phi\int_Z^z\int_Z^z P\,dz^2 + \cos\Phi\int_Z^z Py_2\,dz}{\cos\Phi\int_Z^z P\,dz + \int_0^z (y_1 - y_2)\,dz} \quad \text{..................(36).}$$

8. Dykes and Embankments.

Let the forces P be the pressures of a fluid mass upon the face of an embankment, supposed a *plane*, inclined to the vertical at the angle α_2, (see Fig. 13); Φ will then become $\frac{\pi}{2} + \alpha_2$, and P will vary as z; let it equal μz. Substituting in equation (35), and integrating,

$$y = \frac{\int_0^z (y_1^2 - y_2^2) - \frac{2}{3}\mu(z^3 - Z^3)\sec\alpha_2 + \mu z(z^2 - Z^2)\cos\alpha_2}{2\int_0^z (y_1 - y_2)\,dz - \mu(z^2 - Z^2)\sin\alpha_2} \quad \ldots\ldots\ldots(37).$$

From this equation the line of resistance may be determined for any given inclination of the internal face or form of the external face of the embankment; or conversely, these circumstances may themselves be determined according to a given equation to the line of resistance.

If the section of the embankment be of the form of a trapezoid (Fig. 14), by the integration (6) we have

$$y = \frac{z^3\{\frac{1}{3}(\tan^2\alpha_1 - \tan^2\alpha_2) - \mu(\frac{2}{3}\sec\alpha_2 - \cos\alpha_2)\} + az^2\tan\alpha_1 + z(a^2 - \mu Z^2\cos\alpha_2) + \frac{2}{3}\mu Z\sec\alpha_2}{z^2\{\tan\alpha_1 - \tan\alpha_2 - \mu\sin\alpha_2\} + 2az + \mu Z^2\sin\alpha_2} \ldots(38).$$

If $Z = 0$ or the fluid extend to the edge of the embankment, this becomes the equation to an hyperbola.

The point of rupture of the extrados, or the greatest possible height of the embankment, may be determined as before, by assuming

$$y = a + z\tan\alpha_1,$$

whence

$$\begin{aligned}\tfrac{1}{3} z^3\{(\tan\alpha_1 - \tan\alpha_2)(2\tan\alpha_1 + \tan\alpha_2) - \mu[3\cos(\alpha_1 - \alpha_2)\sec\alpha_1 - 2\sec\alpha_2]\} + az^2\{2\tan\alpha_1 - \tan\alpha_2 - \mu\sin\alpha_2\} \\ + z\left\{a^2 + \mu Z^2\frac{\cos(\alpha_1 - \alpha_2)}{\cos\alpha_1}\right\} + \mu Z^2\left\{\frac{2}{3}Z\sec\alpha_2 - a\sin\alpha^2\right\}\end{aligned} = 0 \ldots(39).$$

The supposition now about to be made, with regard to the line of resistance, is, that it traverses the center of the embankment. We have thence, by equation (37),

$$\tfrac{1}{2}(y_1 + y_2) = \frac{\int(y_1^2 - y_2^2)\,dz - \frac{2}{3}\mu(z^3 - Z^3)\sec\alpha_2 + \mu z(z^2 - Z^2)\cos\alpha_2}{2\int(y_1 - y_2)\,dz - \mu(z^2 - Z^2)\sin\alpha_2};$$

whence observing that

$$(y_1 + y_2)\int(y_1 \quad y_2)\,dz = \iint(y_1 - y_2)\,dz\,.\,d(y_1 + y_2) + \int(y_1^2 - y_2^2)\,dz$$

$$\iint(y_1 - y_2)\,dz\,.\,d(y_1 + y_2)$$

$$= \mu\{(z^2 - Z^2)[z\cos\alpha_2 + \tfrac{1}{2}(y_1 + y_2)\sin\alpha_2] - \tfrac{2}{3}(z^3 - Z^3)\sec\alpha_2\} \ldots\ldots(40).$$

Whence substituting for y_2 its value $z \tan \alpha_2$, and differentiating twice in respect to z, a differential equation will be obtained, the solution of which will determine the required relation of y_1 and z. If $\alpha_2 = 0$, or the intrados be a vertical plane, we obtain, by the first differentiation in respect to z,

$$\frac{1}{\mu}\int y_1 dy_1 = z^2 - Z^2; \quad \therefore \frac{1}{\mu}(y_1^2 - A^2) = z^2 - Z^2,$$

if A be the breadth of the embankment on the level of the surface of the fluid;

$$\therefore y_1^2 = \mu\{z^2 - (Z^2 - \frac{1}{\mu}A^2)\} \ldots\ldots(41).$$

The equation to an hyperbola, whose center is in the inner edge of the embankment A, the ratio of whose axes is $\sqrt{\mu}$, and whose semi-axis is $(Z^2 - \frac{1}{\mu}A^2)$.

9. The Arch.

The plane of intersection has hitherto been supposed, in its successive changes of position, to remain always parallel to itself. Let this hypothesis now be discarded, and as the simplest case of a variable inclination of the plane, let it be supposed to revolve about a given horizontal line or axis within itself. Let moreover the extrados and the intrados be supposed to be cylindrical surfaces, having this line for their common axis; and suppose this arch to have a load uniformly distributed along the extrados in a line parallel to the axis and at a horizontal distance from it equal to x.

Let ABD (Fig. 15), represent any section of this mass perpendicular to the axis C, and X the corresponding load. Let the horizontal force P be applied in AD at a vertical distance p from C; and let CT be any position of the intersecting plane, intersected by the resultant of the forces P and X, and the weight of the mass $ASTD$ in R,

$$PCA = \Theta, \quad PCR = \theta, \quad CT = R, \quad CS = r, \quad CR = \rho.$$

Therefore by the condition of the equality of moments,

$$\int_r^R \int_\Theta^\theta r^2 \sin\theta d\theta dr + Xx + Pp = \rho\{P\cos\theta + X\sin\theta + \sin\theta \int_r^R \int_\Theta^\theta r d\theta dr\} \ldots.(42).$$

At the *point of rupture* the line of resistance *meets* the intrados; therefore at this point

$$\rho = r \ldots\ldots\ldots(43).$$

Also, generally, sufficient dimensions of the arch being supposed, the line of resistance touches the intrados at this point;

$$\therefore \frac{d\rho}{d\theta} = \frac{dr}{d\theta} \ldots\ldots\ldots\ldots(44).$$

Moreover, $R = F\theta, \quad r = f\theta \ldots\ldots..(45),$

these being given functions of θ.

Assuming Ψ to be the value of θ at the point of rupture, and substituting it for θ in the five preceding equations, we may eliminate between them the four quantities ρ, R, r, p, or the four ρ, R, r, Ψ. There will result an equation involving the quantities P and Ψ in the one case, and P and p in the other.

Now if the force P be supplied by the pressure of another opposite and equal semi-arch, it has been shewn, (see *Memoir on the Theory of the Arch*, Vol. V. Part III.) that if the masonry be supposed perfect, P is a minimum in respect to the variable p; moreover if the masonry be supposed to possess those yielding properties which obtain in practice, and which shew themselves in the settlement of the arch, then $p = R$; according to either of these conditions, P and Ψ may therefore be determined.

The values of P and p becoming thus known, they may be substituted in equation (42), and the equation to the line of resistance will thus be completely determined.

10. The Circular Arch.

Let the intrados and extrados be circular cylindrical surfaces;

$$\therefore \int_r^R \int_\Theta^\theta r^2 \sin\theta d\theta dr = -\tfrac{1}{3}(R^3 - r^3)(\cos\theta - \cos\Theta),$$

$$\sin\theta \int_r^R \int_\Theta^\theta r d\theta dr = \tfrac{1}{2}\sin\theta(R^2 - r^2)(\theta - \Theta);$$

$$\therefore \tfrac{1}{3}(R^3 - r^3)(\cos\Theta - \cos\theta) + Xx + Pp$$
$$= \rho\{X\sin\theta + P\cos\theta + \tfrac{1}{2}(R^2 - r^2)(\theta - \Theta)\sin\theta\} \ldots\ldots(46).$$

Therefore, by equation (43),

$$\tfrac{1}{3}(R^3 - r^3)(\cos\Theta - \cos\Psi) + Xx + Pp$$
$$= r\{X\sin\Psi + P\cos\Psi + \tfrac{1}{2}(R^2 - r^2)(\Psi - \Theta)\sin\Psi\} \ldots\ldots(47).$$

By equation (44), observing that $\frac{dr}{d\theta} = 0$,

$$\tfrac{1}{3}(R^3 - r^3)\sin\Psi = r\{X\cos\Psi - P\sin\Psi + \tfrac{1}{2}(R^2 - r^2)(\Psi - \Theta)\cos\Psi + \tfrac{1}{2}(R^2 - r^2)\sin\Psi\};$$

hence, assuming $R = r(1 + a)$

$$\left\{\frac{6P}{r^2} + a^2(2a + 3)\right\}\tan\Psi = \left\{\frac{6X}{r^2} - 3a(a+2)\Theta\right\} + 3a(a+2)\Psi \ldots\ldots(48).$$

By equation (47),

$$P\{p - r\cos\Psi\}$$
$$= \{Xr + \tfrac{1}{2}r(R^2 - r^2)(\Psi - \Theta)\}\sin\Psi - Xx + \tfrac{1}{3}(R^3 - r^3)(\cos\Psi - \cos\Theta) \ldots\ldots(49).$$

Also, by equation (48),

$$Xr + \tfrac{1}{2}r(R^2 - r^2)(\Psi - \Theta) = \{Pr + \tfrac{1}{6}r^3a^2(2a + 3)\}\tan\Psi;$$

$$\therefore P\{p - r\cos\Psi\} = \{Pr + \tfrac{1}{6}r^3a^2(2a+3)\}\tan\Psi\sin\Psi$$
$$- Xx + r^3a(\tfrac{1}{3}a^2 + a + 1)(\cos\Psi - \cos\Theta);$$

hence, observing that $\tan\Psi\sin\Psi = \dfrac{\sec^2\Psi - 1}{\sec\Psi}$, we obtain the equation

$$\left\{\frac{P}{r^2} + a^2\left(\tfrac{1}{3}a + \tfrac{1}{2}\right)\right\}\sec^2\Psi - \left\{\frac{Xx+Pp}{r^3} + a\left(\tfrac{1}{3}a^2 + a + 1\right)\cos\Theta\right\}\sec\Psi = -a\left\{\tfrac{1}{2}a + 1\right\}\ldots(50).$$

11. The Equilibrium of the circular Arch, the Material being supposed unyielding and the contiguous Surfaces mathematically adjusted.

Let now the force P be supplied by the opposite pressure of an equal semi-arch, then on the hypothesis made, P is a minimum function of Ψ.

Therefore, by (48),

$$\sec^2\Psi = \frac{3a(a+2)}{\dfrac{6P}{r^2} + a^2(2a+3)} \ldots\ldots\ldots\ldots\ldots(51);$$

$$\therefore \tfrac{1}{2}\sin 2\Psi = \frac{\tan\Psi}{\sec^2\Psi} = \frac{\left\{\dfrac{6P}{r^2} + a^2(2a+3)\right\}\tan\Psi}{3a(a+2)};$$

therefore, by equation (48),

$$\sin 2\Psi = \left\{\frac{4X}{a(a+2)r^2} - 2\Theta\right\} + 2\Psi \ldots\ldots(52).$$

From which equation Ψ may be determined. Also by equation (50),

$$P = \frac{r^2}{6}\left\{3a(a+2)\cos^2\Psi - a^2(2a+3)\right\}\ldots\ldots\ldots(53).$$

whence P is known.

Also by equations (50) and (51),

$$\left\{\frac{Xx+Pp}{r^3} + a\left(\tfrac{1}{3}a^2 + a + 1\right)\cos\Theta\right\}\left\{\frac{\left(\tfrac{1}{2}a+1\right)a}{\dfrac{P}{r^2} + a^2\left(\tfrac{1}{3}a + \tfrac{1}{2}\right)}\right\}^{\frac{1}{2}} = a(a+2);$$

$$\therefore \left\{\frac{Xx+Pp}{r^3} + a\left(\tfrac{1}{3}a^2 + a + 1\right)\cos\Theta\right\} = \sqrt{\left\{\frac{2P}{r^2} + a^2\left(\tfrac{2}{3}a + 1\right)\right\}a(a+2)};$$

$$\therefore p = \frac{r^3}{P}\left\{\sqrt{a(a+2)\left\{\frac{2P}{r^2}+a^2(\tfrac{2}{3}a+1)\right\}}-a(\tfrac{1}{3}a^2+a+1)\cos\Theta-\frac{Xx}{r^3}\right\}\ldots\ldots(54).$$

Ψ, P, p are thus completely determined, and all the circumstances of the equilibrium of the circular arch, thus loaded, are known.

If there be no loading, and the two semi-arches be parts of the *same* continuous cylindrical mass, $X=0$, and $\Theta=0$.

Therefore, by equation (52), $\Psi=0$.

In this case, therefore, the point of rupture is in the crown of the arch (Fig. 18), at the intrados, also by equation (53),

$$P = r^2\{a-\tfrac{1}{3}a^3\};$$

therefore, by equation (54), $p=r$.

Substituting these values of P and p in equation (46), we obtain for the equation to the line of resistance in the unloaded circular arch,

$$\rho = r\frac{a+2-(\frac{1}{3}a^2+a+1)\cos\theta}{(\frac{1}{2}a+1)\theta\sin\theta+(1-\frac{1}{3}a^2)\cos\theta}\ldots\ldots\ldots(55).$$

Let ψ be the angular distance from the crown, at which the line of resistance meets the *extrados*, (Fig. 17), as Ψ is that at which it meets the intrados. Therefore, by the preceding equation,

$$1+a = \frac{a+2-(\frac{1}{2}a^2+a+1)\cos\psi}{(\frac{1}{2}a+1)\psi\sin\psi+(1-\frac{1}{3}a^2)\cos\psi};$$

$$\therefore (1+a)(\tfrac{1}{2}a+1)\psi\sin\psi+(2+2a-\tfrac{1}{3}a^3)\cos\psi = a+2\ldots\ldots\ldots(56).$$

ψ determined from this equation will measure the greatest semi-arch, which being unloaded, can be made to stand.

To determine the inclination Φ of the resultant P_1 to the vertical, corresponding to the angle θ, we have

$$P_1\sin\Phi = \text{horizontal force on segment} = P = r^2\{a-\tfrac{1}{3}a^3\},$$
$$P_1\cos\Phi = \text{vertical}\ \ldots\ldots\ldots\ldots\ldots\ldots = \text{mass of segment} = \tfrac{1}{2}r^2\{a^2+2a\}\theta;$$

$$\therefore \tan \Phi = \frac{2\{1 - \frac{1}{3}a^2\}}{\{a+2\}\theta} \text{.........}(57).$$

Suppose the arch to be supported upon upright piers of a given breadth a, (Fig. 18), and let it be required to determine what is the greatest height (z) to which they can be carried.

By equation (23),

$$z = \frac{P_1(k-a)\cos\Phi}{\frac{1}{2}a^2 - P_1 \sin\Phi},$$

where $k = \rho - r$.

Also by equation (55),

$$\rho - r = \frac{r(a+2)\left(\tan\frac{\theta}{2} - \frac{\theta}{2}\right)}{(\frac{1}{2}a+1)\theta + (1-\frac{1}{3}a^2)\cot\theta};$$

$$\therefore z = \left\{\frac{r^2(\frac{1}{2}a^2 + a)\theta}{\frac{1}{2}a^2 - r^2(a - \frac{1}{3}a^3)}\right\}\left\{\frac{r(a+2)\left(\tan\frac{\theta}{2} - \frac{\theta}{2}\right)}{(\frac{1}{2}a+1)\theta + (1-\frac{1}{3}a^2)\cot\theta} - a\right\}\text{...}(58).$$

If $\theta = \pi$, or the arch be a semi-circle,

$$z = \left\{\frac{\frac{1}{2}r^2(\frac{1}{2}a^2 + a)\pi}{\frac{1}{2}a^2 - r^2(a - \frac{1}{3}a^3)}\right\}\left\{r\left(\frac{1-\frac{\pi}{4}}{\frac{\pi}{4}}\right) - a\right\}\text{......}(59).$$

It is evident, from equation (50), that as X is increased Ψ increases; that is, the points of rupture descend continually upon the arch as it is more loaded. The experiments of Professor Robison on chalk models are explained by this fact*.

* Having constructed chalk models of the voussoirs of a circular arch, and put them together, he loaded the arch upon its crown, increasing the load until it fell. The first tendency in the chalk to crush was observed at points of the intrados, equidistant from the crown on either side, but near it; these points were manifestly those where the line of resistance first touched the intrados. As the load was increased, the tendency to crush exhibited itself continually at points *more distant* from the crown—that is, the points of rupture *descended.*

12. The Equilibrium of the circular Arch under the Conditions which obtain in its actual Construction.

The condition, taken as the basis of the conclusions arrived at in the last section, "that the resultant pressure P of the opposite semi-arch (see Fig. 15) is applied to that point in the depth AD of the key-stone which corresponds to its minimum value," true under an hypothetical perfection of the masonry, does not obtain as a practical condition.

It supposes a mathematical adjustment of the contiguous surfaces of the stones to one another, an immoveability of the abutments, and an unyielding quality of the arch-stones and cement, which have no practical existence.

Every arch, on the striking of the centers which have supported it whilst it was built up, sinks at the crown.

The effect of this sinking or *settlement* is to cause the voussoirs about the crown to separate slightly from one another at their lower edges, somewhat like the leaves of a book, and thus to throw the whole of their pressure, upon one another, on their *upper* edges.

However skilful may be the masonry of an arch, and however small comparatively may be its *first* settlement, some settlement always perceptibly takes place; and there can be little doubt that in every arch a transfer of the whole pressure upon the voussoirs at the crown to these upper edges, from the first, obtains.

Moreover it is certain, from numerous experiments of Gauthey and others, that when an arch is in the state bordering upon rupture by the yielding of its abutments, the direction of its pressure is through the superior edges of its voussoirs at the crown, and through the inferior edges of the voussoirs at its points of rupture, in the haunches. Now the great *practical* question is to determine the conditions of the pressure under those possible circumstances, which are most unfavourable to the stability of the arch; circumstances which manifestly occur in the state bordering upon its rupture. This question necessarily then supposes a direction of the pressure, and therefore of the line of resistance, touching the extrados at the crown, and the intrados at the haunches; and, this being supposed, all those conditions of the equilibrium which depend upon the nearer approach of the voussoirs after the first striking of the center, or which arise from the long continued pressure, or from the influence of changes in the temperature, are *eliminated.*

Let us then assume that the *line of resistance* touches the extrados at the crown, so that $p = r(1+a)\cos\Theta$; by equation (49),

$$P\{(1+a)\cos\Theta - \cos\Psi\} = \{X + r^2(\tfrac{1}{2}a^2 + a)(\Psi - \Theta)\}\sin\Psi$$
$$+ r^2(a + a^2 + \tfrac{1}{3}a^3)(\cos\Psi - \cos\Theta) - \frac{Xx}{r}.$$

By equation (48),

$$P\tan\Psi = X + r^2(\tfrac{1}{2}a^2 + a)(\Psi - \Theta) - r^2a^2(\tfrac{1}{3}a + \tfrac{1}{2})\tan\Psi;$$

$$\therefore \frac{(1+a)\cos\Theta - \cos\Psi}{\tan\Psi}$$

$$= \frac{\left\{\frac{X}{r^2} + (\tfrac{1}{2}a^2 + a)(\Psi - \Theta)\right\}\sin\Psi + (a + a^2 + \tfrac{1}{3}a^3)(\cos\Psi - \cos\Theta) - \frac{Xx}{r^3}}{\frac{X}{r^2} + (\tfrac{1}{2}a^2 + a)(\Psi - \Theta) - a^2(\tfrac{1}{3}a + \tfrac{1}{2})\tan\Psi}$$

Dividing this equation by $\sin\Psi$, subtracting unity from both sides, and reducing

$$\frac{(1+a)\cos\Theta - \cos\Psi - \sin\Psi\tan\Psi}{\tan\Psi}$$

$$= \frac{a^2(\tfrac{1}{3}a + \tfrac{1}{2})\tan\Psi\sin\Psi + (a + a^2 + \tfrac{1}{3}a^3)(\cos\Psi - \cos\Theta) - \frac{Xx}{r^3}}{\frac{X}{r^2} + (\tfrac{1}{2}a^2 + a)(\Psi - \Theta) - a^2(\tfrac{1}{3}a + \tfrac{1}{2})\tan\Psi}.$$

Now,

$$\frac{(1+a)\cos\Theta-\cos\Psi-\sin\Psi\tan\Psi}{\tan\Psi}=\frac{a\cos\Theta-(1-\cos\Theta)+1-\cos\Psi-\sin\Psi\tan\Psi}{\tan\Psi}$$

$$=a\cos\Theta\cot\Psi-\sin^2\frac{\Theta}{2}\left(\cot\frac{\Psi}{2}-\tan\frac{\Psi}{2}\right)-\tan\frac{\Psi}{2}$$

$$=a\cos\Theta\cot\Psi-\sin^2\frac{\Theta}{2}\cot\frac{\Psi}{2}-\cos^2\frac{\Theta}{2}\tan\frac{\Psi}{2}.$$

Substituting this value for the first member of the preceding equation; multiplying by the denominator of the fraction in the second member, and neglecting powers of a above the first,

$$\left.\begin{array}{l}\dfrac{Xa}{r^2}\cos\Theta\cot\Psi-\dfrac{X}{r^2}\left\{\sin^2\dfrac{\Theta}{2}\cot\dfrac{\Psi}{2}+\cos^2\dfrac{\Theta}{2}\tan\dfrac{\Psi}{2}\right\}\\ -a\left\{\sin^2\dfrac{\Theta}{2}\cot\dfrac{\Psi}{2}+\cos^2\dfrac{\Theta}{2}\tan\dfrac{\Psi}{2}\right\}\{\Psi-\Theta\}\end{array}\right\}=a(\cos\Psi-\cos\Theta)-\frac{Xx}{r^3}.$$

Whence, by transposition and reduction,

$$\sin^2\frac{\Theta}{2}\cot\frac{\Psi}{2}+\cos^2\frac{\Theta}{2}\tan\frac{\Psi}{2}=\frac{x}{r}$$

$$-\left\{(\cos\Psi-\cos\Theta)+\left(\sin^2\frac{\Theta}{2}\cot\frac{\Psi}{2}+\cos^2\frac{\Theta}{2}\tan\frac{\Psi}{2}\right)(\Psi-\Theta)-\frac{X}{r^2}\cos\Theta\cot\Psi\right\}\frac{ar^2}{X}\quad\ldots\ldots(60).$$

Assume $\cos^2\frac{\Theta}{2}\tan\frac{\Psi}{2}+\sin^2\frac{\Theta}{2}\cot\frac{\Psi}{2}=y$;

$$\therefore\ \cos^2\frac{\Theta}{2}\tan\frac{\Psi}{2}-\sin^2\frac{\Theta}{2}\cot\frac{\Psi}{2}=\sqrt{y^2-\sin^2\Theta};$$

$$\therefore\ \tan\frac{\Psi}{2}=\tfrac{1}{2}\{y+\sqrt{y^2-\sin^2\Theta}\}\sec^2\frac{\Theta}{2}\quad\ldots\ldots\ldots\ldots(61);$$

$$\therefore\ \frac{d\tan\frac{\Psi}{2}}{dy}=\frac{\tfrac{1}{2}\{y+\sqrt{y^2-\sin^2\Theta}\}\sec^2\frac{\Theta}{2}}{\sqrt{y^2-\sin^2\Theta}}=\frac{\tan\frac{\Psi}{2}}{\cos^2\frac{\Theta}{2}\tan\frac{\Psi}{2}-\sin^2\frac{\Theta}{2}\cot\frac{\Psi}{2}}.$$

Hence, by Lagrange's theorem, we have from equation (60), neglecting powers of a above the first,

$$\tan\frac{\Psi}{2} = \tan\frac{\Psi_0}{2}$$

$$+\left\{\frac{\left\{(\cos\Psi_0-\cos\Theta)+\left(\sin^2\frac{\Theta}{2}\cot\frac{\Psi_0}{2}+\cos^2\frac{\Theta}{2}\tan\frac{\Psi_0}{2}\right)(\Psi_0-\Theta)-\frac{X}{r^2}\cos\Theta\cot\Psi_0\right\}\tan\frac{\Psi_0}{2}}{\cos^2\frac{\Theta}{2}\tan\frac{\Psi}{2}-\sin^2\frac{\Theta}{2}\cot\frac{\Psi}{2}}\right\}\left\{\frac{ar^2}{X}\right\}.$$

Where Ψ_0 is taken to represent the value of Ψ when $a = 0$,* so that by equations (60) and (61),

$$\tan\frac{\Psi_0}{2} = \tfrac{1}{2}\left\{\frac{x}{r} + \sqrt{\frac{x^2}{r^2} - \sin^2\Theta}\right\}\sec^2\frac{\Theta}{2} \quad \text{.........(62).}$$

Now,

$$\frac{(\cos\Psi_0 - \cos\Theta)\tan\frac{\Psi_0}{2}}{\cos^2\frac{\Theta}{2}\tan\frac{\Psi_0}{2} - \sin^2\frac{\Theta}{2}\cot\frac{\Psi_0}{2}} = \frac{-2\sin\frac{1}{2}(\Psi_0+\Theta)\sin\frac{1}{2}(\Psi_0-\Theta)\sin^2\frac{\Psi_0}{2}}{\cos^2\frac{\Theta}{2}\sin^2\frac{\Psi_0}{2} - \sin^2\frac{\Theta}{2}\cos^2\frac{\Psi_0}{2}} = -2\sin^2\frac{\Psi_0}{2},$$

$$\frac{\cos^2\frac{\Theta}{2}\tan\frac{\Psi_0}{2}+\sin^2\frac{\Theta}{2}\cot\frac{\Psi_0}{2}}{\cos^2\frac{\Theta}{2}\tan\frac{\Psi_0}{2}-\sin^2\frac{\Theta}{2}\cot\frac{\Psi_0}{2}} = \frac{\tan^2\frac{\Psi_0}{2}+\tan^2\frac{\Theta}{2}}{\tan^2\frac{\Psi_0}{2}-\tan^2\frac{\Theta}{2}},$$

$$\frac{\cos\Theta\cot\Psi_0}{\cos^2\frac{\Theta}{2}\tan\frac{\Psi_0}{2}-\sin^2\frac{\Theta}{2}\cot\frac{\Psi_0}{2}} = \frac{\frac{1}{2}\left(1-\tan^2\frac{\Psi_0}{2}\right)\left(1-\tan^2\frac{\Theta}{2}\right)}{\tan^2\frac{\Psi_0}{2}-\tan^2\frac{\Theta}{2}};$$

* By Lagrange's Theorem, if $y = z + a\phi y$, then, neglecting powers of a above the first, and representing by fy any function of y,

$$fy = fz + \phi z \cdot \frac{dfz}{dz} \cdot a.$$

Let $y = F\Psi$, $\phi y = F_1\Psi$, $fy = F_2\Psi$; $\therefore z = F\Psi_0$, $\phi z = F_1\Psi_0$, $fz = F_2\Psi_0$.

If therefore $F\Psi = F\Psi_0 + aF_1\Psi$ and $F_2\Psi$ be any other function of Ψ, then, neglecting powers of a above the first,

$$F_2\Psi = F_2\Psi_0 + F_1\Psi_0\left(\frac{dF_2\Psi_0}{dF\Psi_0}\right)a.$$

$$\therefore \tan\frac{\Psi}{2} = \tan\frac{\Psi_0}{2} - \left\{2\sin^2\frac{\Psi_0}{2}\right.$$

$$\left. + \frac{\frac{1}{2}\left(1-\tan^2\frac{\Psi_0}{2}\right)\left(1-\tan^2\frac{\Theta}{2}\right)\frac{X}{r^2} - \left(\tan^2\frac{\Psi_0}{2}+\tan^2\frac{\Theta}{2}\right)(\Psi_0-\Theta)}{\tan^2\frac{\Psi_0}{2} - \tan^2\frac{\Theta}{2}}\tan\frac{\Psi_0}{2}\right\}\left\{\frac{ar^2}{X}\right\} \ldots (63).$$

By equation (48), omitting powers of a above the first,

$$P = \{X + ar^2(\Psi - \Theta)\}\cot\Psi \ldots\ldots\ldots (64);$$

$$\therefore \frac{dP}{d\Psi} = ar^2\cot\Psi - \{X + ar^2(\Psi-\Theta)\}\operatorname{cosec}^2\Psi,$$

also by equation (61), $\Psi = 2\tan^{-1}\left\{\frac{1}{2}(y + \sqrt{y^2 - \sin^2\Theta})\sec^2\frac{\Theta}{2}\right\}$;

$$\therefore \frac{d\Psi}{dy} = \frac{\{1 + y(y^2 - \sin^2\Theta)^{-\frac{1}{2}}\}\sec^2\frac{\Theta}{2}}{1 + \frac{1}{4}\{2y^2 + 2y\sqrt{y^2 - \sin^2\Theta} - \sin^2\Theta\}\sec^4\frac{\Theta}{2}}$$

$$= \frac{y + \sqrt{y^2 - \sin^2\Theta}}{\sqrt{y^2 - \sin^2\Theta}\left\{\cos\Theta + \frac{1}{2}y(y + \sqrt{y^2 - \sin^2\Theta})\sec^2\frac{\Theta}{2}\right\}}$$

$$= \frac{-2\tan\frac{\Psi}{2}\cos^2\frac{\Theta}{2}}{\left\{\sin^2\frac{\Theta}{2}\cot\frac{\Psi}{2} - \cos^2\frac{\Theta}{2}\tan\frac{\Psi}{2}\right\}\left\{\cos\Theta + \sin^2\frac{\Theta}{2} + \cos^2\frac{\Theta}{2}\tan^2\frac{\Psi}{2}\right\}}$$

$$= \frac{-2\tan\frac{\Psi}{2}\cos^2\frac{\Theta}{2}}{\left\{\sin^2\frac{\Theta}{2}\cot\frac{\Psi}{2} - \cos^2\frac{\Theta}{2}\tan\frac{\Psi}{2}\right\}\left\{\cos^2\frac{\Theta}{2} + \cos^2\frac{\Theta}{2}\tan^2\frac{\Psi}{2}\right\}} = \frac{-\sin\Psi}{\sin^2\frac{\Theta}{2}\cot\frac{\Psi}{2} - \cos^2\frac{\Theta}{2}\tan\frac{\Psi}{2}};$$

$$\therefore \frac{dP}{dy} = \frac{dP}{d\Psi} \cdot \frac{d\Psi}{dy} = -\frac{ar^2 \cos\Psi - \{X + ar^2(\Psi - \Theta)\} \operatorname{cosec}\Psi}{\sin^2\frac{\Theta}{2}\cot\frac{\Psi}{2} - \cos^2\frac{\Theta}{2}\tan\frac{\Psi}{2}}$$

$$= -\frac{X + ar^2\{\Psi - \Theta - \frac{1}{2}\sin 2\Psi\}}{2\left\{\cos^2\frac{\Theta}{2}\sin^2\frac{\Psi}{2} - \sin^2\frac{\Theta}{2}\cos^2\frac{\Psi}{2}\right\}}.$$

Whence, by Lagrange's Theorem, neglecting powers of a above the first, we have, by equation (60),

$$P = \{X + ar^2(\Psi_0 - \Theta)\}\cot\Psi_0$$

$$+\left\{\frac{(\cos\Psi_0 - \cos\Theta) + \left(\sin^2\frac{\Theta}{2}\cot\frac{\Psi_0}{2} + \cos^2\frac{\Theta}{2}\tan\frac{\Psi_0}{2}\right)(\Psi_0 - \Theta) - \frac{X}{r^2}\cos\Theta\cot\Psi_0}{2\left(\cos^2\frac{\Theta}{2}\sin^2\frac{\Psi_0}{2} - \sin^2\frac{\Theta}{2}\cos^2\frac{\Psi_0}{2}\right)}\right\}ar^2,$$

$$P = \{X + ar^2(\Psi_0 - \Theta)\}\cot\Psi_0$$

$$-\left\{1 + \frac{\frac{1}{2}\left(1 - \tan^2\frac{\Psi_0}{2}\right)\left(1 - \tan^2\frac{\Theta}{2}\right)\frac{X}{r^2} - \left(\tan^2\frac{\Psi_0}{2} + \tan^2\frac{\Theta}{2}\right)(\Psi_0 - \Theta)}{\left(\tan^2\frac{\Psi_0}{2} - \tan^2\frac{\Theta}{2}\right)\sin\Psi_0}\right\}ar^2,$$

$$P = X\cot\Psi_0 - \left\{1 + \frac{\frac{1}{2}\left(1 - \tan^2\frac{\Psi_0}{2}\right)\left(1 - \tan^2\frac{\Theta}{2}\right)\frac{X}{r^2} - \tan^2\frac{\Psi_0}{2}\sec^2\frac{\Theta}{2}(\Psi_0 - \Theta)}{2\left(\tan^2\frac{\Psi_0}{2} - \tan^2\frac{\Theta}{2}\right)\tan\frac{\Psi_0}{2}}\right\}ar^2 \ldots (65).$$

Equations (62), (63), (65) determine the values of P and Ψ, that is, the pressure upon the key-stone and the positions of the points of rupture, for every condition of loading, and every form of the gothic and circular arch.

Assuming $\Theta = 0$ and eliminating the value of $\tan\frac{\Psi_0}{2}$, we have, by equations (62), (63), and (65),

$$\tan\frac{\Psi}{2}=\frac{x}{r}-\left\{\frac{2rx}{X}\left[\left(\frac{r}{x}+\frac{x}{r}\right)^{-1}-\tan^{-1}\frac{x}{r}\right]+\frac{1}{2}\left(\frac{r}{x}-\frac{x}{r}\right)\right\}a \ldots\ldots\ldots\ldots(66).$$

$$P=\frac{1}{2}\left(\frac{r}{x}-\frac{x}{r}\right)-\left\{1+\frac{X}{2rx}\left(\frac{r^2}{x^2}-\frac{x^2}{r^2}\right)-2\frac{r}{x}\tan^{-1}\frac{x}{r}\right\}ar^2 \ldots\ldots\ldots(67).$$

It has been supposed that the load X is collected over a single point in the arch, or rather in a single line stretching across it in a direction parallel to the axis of the cylinder of which it forms part. Let it now be imagined to be distributed in any way over the extrados, but symmetrically on the two opposite semi-arches. Find the center of gravity of the load on either semi-arch, and let x be its distance from the vertical which passes through the center of the circle of which the semi-arch forms part. Imagine the whole load X to be collected in this point, and on this hypothesis determine Ψ and P; the values thus determined will evidently be their *true* values. To find the line of resistance, substitute in equation (46) the value of P, and for X and x substitute their values in terms of θ; that is, for X substitute the load incumbent upon that portion of the arch which subtends the angle θ, and for x the distance from the vertical through the center, of the center of gravity of that portion of the load. The resulting equation will be the true equation to the line of resistance. Thus the point where the resultant pressure of the arch intersects the supporting surface of the *abutment* will become known; and its direction being found, as in equation (57), all the circumstances which determine the equilibrium of the *abutment* will be known, and the conditions of its equilibrium may be determined by the equation given in section 5 of this paper. The analytical discussion of these conditions, and of that case in which the arch being overloaded at the haunches, its rupture takes place by the elevation of the *crown*, is yet wanting to complete the theory of the arch.

A very simple expression for P offers itself in the case in which $a=0$, or in which the thickness of the arch is considered evanescent in comparison with its radius. In this case

$$P=X\cot\Psi_0=X\left\{\left(\frac{x}{r}+\sqrt{\frac{x^2}{r^2}-\sin^2\Theta}\right)^{-1}-\frac{1}{2}\frac{x}{r}\sec^2\frac{\Theta}{2}\right\}\ldots(68).$$

When $\Theta = 0$ we obtain the *segmental* arch, and P then equals

$$\tfrac{1}{2} X \left(\frac{r}{x} - \frac{x}{r}\right).$$

If the weight of the arch itself be imagined to be included in that of its loading, that is, in X, and if x be determined on the same hypothesis; if, moreover, R be substituted for r, this expression for P will give *in every case* a useful approximation to its true value. It is a limit which the pressure on the key can never exceed, and to which it approximates more nearly as the radius of the arch is greater in comparison to its thickness. It possesses, moreover, this advantage to the practical man, that it admits of an easy geometrical construction.

13. Let us suppose that the arch were supported at its springing on the edge of its joint at the *extrados* (see Fig. 19). Instead of assuming ρ, in equation (46), equal to r, we must now assume it equal to R at the springing, since the line of resistance will manifestly pass through the point of support. By this supposition we obtain, taking $p = R \cos \Theta$,

$$P = \frac{X\{R\sin\Theta_1 - x\} - \frac{1}{3}\{R^3 - r^3\}\{\cos\Theta - \cos\Theta_1\} + \frac{1}{2}R\{R^2 - r^2\}\{\Theta_1 - \Theta\}\sin\Theta_1}{R\{\cos\Theta - \cos\Theta_1\}} \ldots (69).$$

In the case in which $\Theta = 0$ and $X = 0$,

$$P = \{R - r\}\left\{\tfrac{1}{2}(R + r)\frac{\Theta_1}{2}\cot\frac{\Theta_1}{2} - \tfrac{1}{3}\left(R + r + \frac{r^2}{R}\right)\right\}^* \ldots\ldots (70).$$

* The author has verified this formula, and a corresponding formula, for the case in which the arch is supported at its springing on the *inferior* edges of its extreme voussoirs, by experiments of which the results were communicated to the Mechanical Section of the British Association of Science, at their Meeting in 1837.

XXV. *Account of Observations of Halley's Comet.* *By* R. W. ROTHMAN, ESQ., M.A. *Fellow of Trinity College.*

[Read *December* 11, 1837.]

THE accompanying observations of Halley's Comet were made on the great tower of Trinity College, with a 30-inch achromatic telescope, of $2\frac{3}{4}$ inches aperture, to which was adapted a ring-micrometer. The power used with this micrometer was about 25: the radius of the inner ring was found by the transits of stars near the meridian = 1258″: of the outer = 1710″.

The observations here detailed form the whole of those that I was able to make on this Comet: I saw it on the 18th of Sept. and 1st of Nov., but was unable to get an observation. Whenever an observation has been rejected, the circumstance has been noted in its proper place and the reasons for doing so assigned.

The time employed is Greenwich mean solar time. The seconds watch I used was compared every evening with the clock by Molyneux in the Reading Room, and that again the next day with the transit-clock at the Observatory.

SEPT. 20.

Watch fast 3^{m} . 36^{s}.

	Ingress. h. m. s.	Egress. h. m. s.	
Star	11 . 38 . 37,0	11 . 41 . 31,0	*Inner Ring.*
Comet	39 . 54,0	42 . 21,0	Lower half of field.
Star	53 . 20,0	56 . 20,0	*Inner Ring.*
Comet	54 . 24,0	57 . 29,0	Upper half of field.

The Star is in the *Histoire Céleste*, p. 52, March 4, 1794, observed at 3rd wire 6^{h} . 8^{m} . 48^{s},5.

These are the first observations I ever made with the ring-micrometer: the fractions of seconds of time were not noted.

SEPT. 22.

Watch fast 3^{m} . 36^{s}.

	Ingress. h. m. s.	Egress. h. m. s.	
Comet	13 . 57 . 18,8	13 . 59 . 19,6	*Inner Ring.*
Star	59 . 11,6	14 . 0 . 55,8	Comet in the lower half of the field:
Comet	14 . 2 . 5,2	14 . 4 . 27,4	Star in the upper throughout.
Star	4 . 35,4	5 . 24,6	
Comet	15 . 48,4	17 . 53,6	
Star	17 . 49,6	19 . 17,0	

I imagined at first the Comet too far from the Star to be compared by means of the inner ring, and accordingly made one observation with the outer ring. Finding however measurements with the inner ring practicable, I proceeded to make them: always preferring to do so when possible, as more dependence can be placed on the perfect circularity of the latter. For this reason the single observation of the outer ring has been rejected.

The Star of comparison is ζ Aurigæ. Its Æ has been taken from Airy's *Cambridge Observations* for 1830, p. 103; its declination from Piazzi, vi. 98.

Sept. 25.

Watch fast $4^{m}.0^{s}$.

	Ingress. h. m. s.	Egress. h. m. s.	
Comet	10 . 48 . 54,0	10 . 51 . 38,2	*Inner Ring.*
Star	49 . 6,8	52 . 16,6	
Comet	11 . 4 . 34,8	11 . 6 . 54,8	Both in the same half of the field:
Star	4 . 44,0	7 . 39,8	Comet above throughout.
Comet	9 . 44,0	12 . 44,0	
Star	10 . 2,4	13 . 18,2	

The Star of comparison is *Histoire Céleste,* p. 212, Mars. 5. the first star: and also p. 273, Mars. 13. the first star.

The Comet to-night was faint, and appeared to have diminished in size since first seen. I was perplexed during the observations by a singular circumstance: a small star was so close to the brightest part of the nucleus, as to make it uncertain which point to observe. Owing to this cause, one observation, which on being reduced differed 10ˢ in Æ from the mean of the rest, and where clearly the small star had been taken by mistake, has been altogether set aside, and does not appear.

Sept. 28.

Watch fast $4^{m}.45^{s}$.

	Ingress. h. m. s.	Egress. h. m. s.	
Star	11 . 23 . 33,6	11 . 26 . 48,8	*Inner Ring.* Star in the lower half:
Comet	25 . 7,0	26 . 56,8	Comet in the upper.

I have been unable to find this Star either in the *Histoire Céleste* or Bessel's *Zones.*

SEPT. 30.

Watch fast 5m. 1s.

	Ingress. h. m. s.	Egress. h. m. s.	
Star (*a*)	10 . 49 . 40,0	10 . 52 . 31,0	*Inner Ring.*
Comet	50 . 50,4	53 . 24,8	
Star (*a*)	54 . 45,8	56 . 54,8	Comet below, Star above: in the same half of the field.
Comet	55 . 22,4	58 . 26,0	
Star (*a*)	10 . 59 . 39,0	11 . 2 . 8,6	
Comet	11 . 0 . 37,8	3 . 27,6	

	Ingress. h. m. s.	Egress. h. m. s.	
Comet	11 . 18 . 15,0	11 . 20 . 10,4	*Inner Ring.*
Star (*b*)	21 . 12,4	23 . 15,4	
Comet	24 . 35,6	26 . 47,4	Comet in the upper half: Star in the lower.
Star (*b*)	27 . 52,4	29 . 29,4	
Comet	30 . 39,4	31 . 54,4	
Star (*b*)	33 . 4,6	35 . 27,8	

The Star (*b*) is 61 Aurigæ. Piazzi, VI. 252. Unfortunately, I have been unable to find the Star (*a*) in the Catalogues.

OCT. 2.

Watch fast 5m. 18s.

	Ingress. h. m. s.	Egress. h. m. s.	
Comet	12 . 12 . 43,8	12 . 15 . 47,0	*Inner Ring.*
Star	14 . 19,2	15 . 31,0	
Comet	17 . 41,2	20 . 39,6	Star in the upper half of the field: Comet in the lower.
Star	19 . 23,2	20 . 20,0	
Comet	22 . 31,0	25 . 26,0	
Star	24 . 4,8	25 . 21,6	

One observation of the outer ring rejected, under circumstances similar to those of Sept. 30.

The Star is in *Histoire Céleste*, p. 208. A. Transit at $6^h . 49^m . 48^s,5$.

OCT. 4.

Watch fast $5^m . 37^s$.

	Ingress. h. m. s.	Egress. h. m. s.	
Star	11 . 59 . 53,0	12 . 3 . 14,8	
Comet	12 . 0 . 25,0	2 . 47,2	
Star	4 . 45,6	8 . 18,6	
Comet	5 . 9,6	7 . 57,0	*Inner Ring.*
Star	25 . 38,4	28 32,0	Both in the same half
Comet	26 . 10,4	28 . 12,0	of the field. The Comet above.
Star	30 . 11,6	33 . 10,8	
Comet	30 . 31,6	33 . 6,8	
Star	45 . 1,0	49 . 0,2	
Comet	45 . 28,0	48 . 50,2	

The Star is 58 Telescopii Herschel. Position for 1810 according to Groombridge, Æ = $7^h . 7^m . 30^s,5$. N.P.D. = $44^0 . 26' . 52''$.

The Comet to-night was extremely faint, and the instants of appearance and disappearance are little better than estimations.

OCT. 5.

Watch fast $5^m . 47^s$.

	Ingress. h. m. s.	Egress. h. m. s.	
Comet	11 . 17 . 55,4	11 . 21 . 25,4	*Inner Ring.* Comet in the upper half:
Star	17 . 55,4	20 . 53,6	Star in the lower.

They entered quite simultaneously. Tolerably good observation: the disappearance of the Comet may be marked a little too soon.

OCT. 7.

Watch fast 6m. 10s.

	Ingress. h. m. s.	Egress. h. m. s.	
Comet	11 . 19 . 41,4	11 . 23 . 16,6	*Inner Ring.* Star above: both in the same half of the field.
Star	19 . 41,4	22 . 16,8	
Comet	24 . 48,2	28 . 5,4	
Star	24 . 49,0	26 . 59,6	

One observation marked as doubtful in my journal has been rejected, though it agrees pretty well on being reduced. The Star is Piazzi, VIII. 15.

OCT. 10.

Watch fast 6m. 41s.

	Ingress. h. m. s.	Egress. h. m. s.	
Star	8 . 49 . 30,8	8 . 51 . 14,4	*Inner Ring.* Comet in the upper half: Star in the lower.
Comet	53 . 1,6	55 . 20,8	

Several comparisons were made with the outer ring this evening, before the Comet approached sufficiently near the Star to admit of the use of the inner ring. These have been rejected for reasons already explained.

The Star is in Groombridge Æ = 10h. 48m. 58s,14; D = 25°. 33′. 37″ } for 1810.

OCT. 11.

Watch fast 6m. 54s.

	Ingress. h. m. s.	Egress. h. m. s.	
Star	12 . 5 . 40,8	12 . 9 . 26,4	*Inner Ring.* Star in the upper half: Comet in the lower.
Comet	6 . 51,6	12 . 20,8	

Two observations have been excluded: one because the Star's entry was marked somewhat too late; the other, because the telescope received a slight blow during the observation which may have displaced it. Both of these on being reduced agree well, but I have thought it safer to reject them.

The Star is in the *Mémoires de l'Académie* for 1790, p. 383, A. Transit at $12^h.39^m.35^s,0$.

OCT. 18.

Watch fast $8^m.0^s$.

	Ingress. h. m. s.	*Egress.* h. m. s.	
Comet	7 . 48 . 39,2	7 . 50 . 52,8	
Star	52 . 48,4	53 . 56,6	
Comet	55 . 8,4	57 . 19,6	*Inner Ring.*
Star	59 . 1,6	8 . 0 . 29,6	Star in the upper half of the field:
Comet	8 . 1 . 42,6	4 . 2,0	Comet in the lower.
Star	5 . 37,8	7 . 11,2	
Comet	39 . 20,8	41 . 13,2	
Star	42 . 16,2	44 . 55,0	

The Star is in *Histoire Céleste*, p. 83, there called, but erroneously, 46 Herculis.

OCT. 19.

Watch fast $8^m.7^s$.

	Ingress. h. m. s.	*Egress.* h. m. s.	
Comet	7 . 5 . 20,4	7 . 7 . 31,2	*Inner Ring.* Both in the same half of the field.
Star	5 . 24,4	8 . 11,2	
Comet	9 . 0,4	10 . 58,0	
Star	8 . 56,8	11 . 41,6	
Comet	16 . 20,0	18 . 1,6	
Star	16 . 8,0	18 . 50,8	
Comet	25 . 38,8	27 . 17,2	
Star	25 . 22,0	28 . 6,6	
Comet	31 . 35,8	32 . 52,6	
Star	30 . 58,2	33 . 35,8	

The Star is ι Ophiuchi.

OCT. 23.

Watch slow $1^m.3^s$.

	Ingress. h. m. s.	*Egress.* h. m. s.	
Star	6 . 24 . 14,8	6 . 26 . 9,6	*Inner Ring.* Both in the same half of the field.
Comet	24 . 27,6	25 . 2,8	

I have been unable to find this Star in the Catalogues.

Oct. 27.

Watch slow $2^m.36^s$.

	Ingress. h. m. s.	*Egress.* h. m. s.	
Star	6 . 18 . 33,2	6 . 19 . 46,2	*Inner Ring.* Star in the upper half: Comet in the lower.
Comet	19 . 7,8	21 . 12,0	
Star	23 . 58,6	25 . 21,6	
Comet	24 . 42,8	26 . 41,6	
Star	27 . 59,4	29 . 7,0	
Comet	28 . 33,2	30 . 41,2	
Star	32 . 59,0	34 . 15,8	
Comet	33 . 37,4	35 . 45,4	
Star	35 . 55,8	38 . 18,2	
Comet	37 . 34,6	39 . 41,6	

The Star is in *Histoire Céleste*, p. 290, B. Transit at $17^h.8^m.32^s$.

To reduce these observations, it is necessary to allow for the differences of refraction in right ascension and declination, and also for the proper motion of the Comet. This proper motion was obtained, wherever it was possible, from the observations themselves: where this could not be done, as in the case of single observations, I made use of Mr Stratford's Ephemeris.

APPARENT PLACES OF THE STARS OF COMPARISON FOR THE RESPECTIVE TIMES OF OBSERVATION.

Date.	Name of Star.	Right Ascension.	Declination.	Authority.
		h. m. s.	° ′ ″	
Sept. 20	*Hist. Cél.*, p. 52. Mar. 4, 1794. transit at $6^h.8^m.48^s,5$.	6 . 11 . 30,9	+ 30 . 2 . 1	*Hist. Cél.*, one observation, 3d wire.
22	z Aurigæ....................	6 . 17 . 59,3	+ 30 . 35 . 19	Æ. Airy. Dec. Piazzi.
25	*Hist. Cél.*, p. 212, first star, and p. 273, Mars 13, first star.....................	6 . 22 . 53,8	+ 33 . 8 . 11	Mean of two observations, each 3d wire.
28	....	....	....	....
30 (*a*)	....	....	....	....
(*b*)	61 Aurigæ..................	6 . 42 . 39,7	+ 38 . 41 . 55	Piazzi VI. 252.
Oct. 2	*Hist. Cél.*, p. 208. A. $\alpha = 6^h.49^m.48^s,5$	6 . 52 . 29,8	+ 40 . 48 . 46	*Hist. Cél.*, middle and 3d wires.
4	58. Telescop. Herschel........	7 . 9 . 21,7	+ 45 . 30 . 38	Groombridge, unpublished Catalogue.
5	*Hist. Cél.*, p. 377. B. $\alpha = 7^h.19^m.13^s,4$	7 . 22 . 21,3	+ 48 . 32 . 17	*Hist. Cél.*, observed at three wires.
7	Piazzi VIII. 15..............	8 . 5 . 32,6	+ 54 . 38 . 45	Piazzi.
10	Æ. $=10^h.48^m.58^s.4$ N.P.D.= 25°.33′.37″ for 1810.	10 . 50 . 37,1	+ 64 . 18 . 12	Groombridge, unpublished Catalogue.
11	Mém. de l'Acad. for 1790. transit at $12^h.39^m.35^s$, p. 333. A................	12 . 41 . 24,3	+ 61 . 12 . 48	Observed at three wires.
18	*Hist. Cél.*, 46 Herculis.......	16 . 41 . 57,3	+ 13 . 34 . 42	Observed at middle and 3d wires.
19	ι Ophiuchi	16 . 46 . 12,7	+ 10 . 26 . 38	Pond, Catalogue for 1830.
23	....	....	....	....
27	*Hist. Cél.*, p. 290. B. transit at $17^h.8^m.32^s$.	17 . 11 . 9,6	− 5 . 43 . 52	Observed at 1st and middle wires.

The reduction of these observations, allowing for the watch error, gives:—

		Δα	Δδ	
	h. m. s	m s.	′ ″	
Sept. 20.	11 . 37 . 36	+ 1 . 3,5	+ 4 . 36	
	51 . 48	1 . 6,5	1 . 34	
22.	13 . 54 . 17	− 1 . 44,7	+ 34 . 12	
	59 . 18	1 . 43,9	34 . 43	
	14 . 12 . 51	1 . 42,5	35 . 5	
25.	10 . 46 . 16	− 25,3	− 5 . 19	
	11 . 1 . 45	27,1	5 . 1	
	7 . 14	26,3	4 . 54	
28.	11 . 21 . 17	+ 48,5	− 25 . 2	
30.	10 . 47 . 6	+ 1 . 1,3	+ 2 . 49	(a.)
	51 . 52	1 . 3,0	6 . 5	
	57 . 1	1 . 8,0	4 . 44	
	11 . 14 . 12	− 3 . 1,3	− 34 . 59	(b.)
	20 . 41	2 . 59,5	35 . 26	
	26 . 16	2 . 59,4	35 . 13	
Oct. 2.	12 . 8 . 57	− 39,3	+ 31 . 59	
	13 . 51	40,8	32 . 58	
	18 . 40	44,6	33 . 20	
4.	10 . 56 . 24	+ 2,0	− 1 . 33	
	11 . 0 . 56	0,9	5 . 31	
	21 . 2	5,4	12 . 4	
	26 . 24	7,7	0 . 26	
	47 . 9	8,1	11 . 16	
5.	11 . 13 . 53	+ 14,8	− 26 . 30	
7.	11 . 15 . 16	+ 27,7	+ 3 . 25	
	20 . 15	30,1	3 . 17	

		Δα	Δδ
d.	h. m. s.	m. s.	′ ″
Oct. 10.	8 . 47 . 26	+ 3 . 44,4	− 39 . 59
11.	12 . 2 . 7	+ 1 . 49,1	+ 26 . 19
18.	7 . 41 . 46	− 3 . 36,5	+ 32 . 43
	48 . 14	3 . 31,6	31 . 52
	54 . 51	3 . 32,2	30 . 11
	8 . 32 . 17	3 . 18,6	24 . 16
19.	6 . 58 . 19	− 22,4	− 9 . 2
	7 . 1 . 52	20,4	9 . 42
	7 . 9 . 4	18,9	10 . 33
	7 . 18 . 21	16,6	11 . 41
	7 . 23 . 57	13,0	11 . 47
23.	6 . 26 . 6	− 27,5	− 4 . 27
27.	6 . 22 . 46	+ 1 . 0,2	− 33 . 7
	28 . 18	1 . 2,1	33 . 6
	32 . 13	1 . 4,0	32 . 14
	36 . 40	1 . 4,0	32 . 17
	41 . 14	1 . 3,6	32 . 23

Having applied these differences to the places of the respective stars, as given in the accompanying Catalogue, I have obtained the apparent places of the Comet for the respective times of observation; and then applying the proper correction for parallax, I have got the geocentric places. In calculating the parallax, the Comet's distance from the Earth was deduced from Mr Stratford's Ephemeris.

APPARENT GEOCENTRIC PLACES OF THE COMET.

	Greenwich Mean Time.	Right Ascension.	Declination.	No. of Observations.
	d. h. m. s.	h. m. s.	° ′ ″	
Sept.	20 . 11 . 44 . 42	6 . 12 . 36,5	+ 30 . 5 . 15	2
	22 . 14 . 2 . 9	6 . 16 . 16,1	+ 31 . 10 . 7	3
	25 . 10 . 58 . 25	6 . 22 . 27,9	+ 33 . 3 . 14	3
	30 . 11 . 20 . 23	6 . 39 . 40,0	+ 38 . 6 . 56	3
Oct.	2 . 12 . 10 . 39	6 . 51 . 49,5	+ 41 . 21 . 46	3
	4 . 11 . 11 . 11	7 . 9 . 27,4	+ 45 . 25 . 45	4
	5 . 11 . 13 . 53	7 . 22 . 37,6	+ 48 . 6 . 4	1
	7 . 11 . 17 . 45	8 . 6 . 3,6	+ 54 . 42 . 27	2
	10 . 8 . 47 . 26	10 . 54 . 20,7	+ 63 . 38 . 59	1
	11 . 12 . 2 . 7	12 . 43 . 14,0	+ 61 . 39 . 48	1
	18 . 7 . 48 . 17	16 . 38 . 22,8	+ 14 . 6 . 38	3
	19 . 7 . 10 . 18	16 45 . 53,6	+ 10 . 16 . 24	5
	27 . 6 . 34 . 14	17 . 12 . 12,0	− 5 . 10 . 59	5

I think that no reliance is to be placed on the places for the 2nd and 4th of October. It has already been noticed how difficult and uncertain the observations of the latter day were: those of the former present an anomaly for which I am at a loss to account. Though they appear to agree with each other, they indicate a retrograde instead of a direct motion in right ascension. The cause of this error I cannot assign. Excluding these two places, I have compared the others with Mr Stratford's Ephemeris in the N.A. for 1839, in which the perturbations are taken into account. I have obtained the following results for the differences in right ascension and declination; where it will be observed that + indicates that the *observed* right ascension or declination *exceeds* that found by *calculation*, and also that the differences in right ascension have been reduced from the *equator to the parallel of the Comet.*

Date of Observation.	Difference in Right Ascension in Seconds of Space.	Difference in Declination.
Sept. 20	+ 6″	− 16″
22	− 5	− 14
25	− 13	+ 7
30	− 21	− 68
Oct. 5	− 24	− 72
7	+ 31	− 57
10	− 43	+ 6
11	− 16	+ 45
18	+ 29	+ 41
19	± 0	− 95
27	− 10	+ 20

These differences do not depend entirely, though of course they do partly, on errors of observation. They are, in several instances, largest on the places that are best determined. This is the case in a remarkable manner with the declination of the 19th October. This is the best determined place of the whole series. It rests upon five observations in good accordance with each other; and the star of comparison is in Mr Pond's last Catalogue, and consequently is known with great accuracy.

R. W. ROTHMAN.

LONDON, *Nov.* 23, 1837.

XXVI. *Mathematical Investigation of the Effect of Machinery on the Wealth of a Community in which it is employed, and on the Fund for the Payment of Wages.* By JOHN TOZER, ESQ. B.A. of Caius College.

[Read *May* 14, 1838.]

IN the third and fourth Volumes of the Society's Transactions are two papers by Mr Whewell, in which symbolical language is applied to the solution of some problems of Political Economy. In the following Paper another problem of the same science is subjected to a similar mode of investigation.

An opinion has been expressed that the term Political Economy has acquired an extent and a vagueness of meaning which in a high degree unfits it for the purposes of science. Certainly any attempt to apply mathematical reasoning to all the subjects which have by different writers, and at different times, been included in the name, must be altogether unsuccessful. Neither do we possess the data, nor has analysis the powers, necessary to such a task.

If, however, the investigation of the causes which affect the accumulation and distribution of wealth, be kept distinct from any considerations as to the effect of that accumulation, or the mode of its distribution on the happiness of mankind, and be also separated from any speculations or deductions as to the nature of those political and social institutions by which these causes may be modified or brought into action; the science that results, by whatever name it may be called, acquires an almost entirely demonstrative character—becomes a series of propositions which are logical deductions from assumed definitions, and form those properties of the things defined which furnish axiomatic truths, and is therefore a subject to which mathematical reasoning is not only proper but necessary.

The introduction of any process by which accuracy may be given to the reasonings of Political Economists, as tending, however remotely and indirectly, to place the science in this position, must be valuable; and even if we could suppose the principles to be as accurate in their enunciation, and as complete in their demonstrations as they can be rendered, there would still devolve on us the duty of rendering our deductions from them general, and of proving that they were necessary.

The particular problem under consideration, is of very limited extent, and of very easy solution. The method that has generally been employed has been to take particular numerical examples, and the results of these have frequently been assumed to lead legitimately to general conclusions. If the examples chosen had always been supplied by statistical facts, we should at least have been assured that the phenomena displayed in these results either had or might have happened, however unsatisfactory the general conclusions from them might have been.

This advantage, however, has not been afforded, the numbers have been generally assumed without reference to realities, and though it may sometimes have been carefully stated, that the conclusions could not possess a higher degree of truth than the premises, the impressions on the minds of general readers would be favourable to that particular conclusion which the example chosen tended to support.

PROP. A portion of capital, which either has been or would have been employed in the payment of wages, is used in the construction of machinery; to determine the effect on the wealth of the community, and on the fund for the payment of the labourer.

Let C be the capital,

$q-1$ the ordinary rate of profit.

If C be engaged in paying wages, it will yield annually a profit $= C(q-1)$; this, while the machinery is in progress, must be partly taken from capital.

Suppose the machinery to take b years to complete, and a part of $C = \frac{C}{x}$ to be expended every year in its construction till it is completed, then the productive capital will be diminished,

at the beginning of 1st year by $\frac{C}{x}$,

$$\ldots\ldots\ldots\ldots\ 2^{\text{nd}}\ \ldots\ldots\ \frac{C}{x}(1+q),$$

$$\ldots\ldots\ldots\ldots\ 3^{\text{rd}}\ \ldots\ldots\ \frac{C}{x}(1+q+q^2),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ b^{\text{th}}\ \ldots\ldots\ \frac{C}{x}(1+q+\ldots\ldots q^{b-1}) = \frac{C}{x}\frac{q^b-1}{q-1},$$

$$\ldots\ldots\ldots\ldots\ b+1^{\text{th}}\ldots\ldots\ \frac{C}{x}\cdot q\,\frac{q^b-1}{q-1},$$

and at this time the whole capital will have been expended on the machine;

$$\therefore\ C = \frac{C}{x}q\cdot\frac{q^b-1}{q-1}, \quad\text{and}\quad x = q\cdot\frac{q^b-1}{q-1}.$$

Let a fractional part, the m^{th} of the expenditure for machinery, go to the labour, and B_1, $B_2 \ldots\ldots B_\rho$ be the sums by which the fund for paying wages is diminished during the 1st, 2nd ...... ρ^{th} years of the progress of the machinery.

Thus

$$B_1 = \frac{C}{x}(1-m), \qquad B_2 = \frac{C}{x}(1+q-m);$$

$$B_3 = \frac{C}{x}(1+q+q^2-m),$$

$$B_\rho = \frac{C}{x}\left(\frac{q^\rho-1}{q-1} - m\right) = \frac{C}{q}\cdot\frac{q-1}{q^b-1}\left(\frac{q^\rho-1}{q-1} - m\right).$$

The total loss to the labour-fund while the machinery is in progress is

$$\Sigma B = \frac{C}{x}\{1 + \frac{(q^2-1) + q^3 - 1 + \dots + q^b - 1}{q-1} - bm\}$$

$$= \frac{C}{x}\left\{\frac{q + q^2 + \dots + q^b - b}{q-1} - bm\right\}$$

$$= \frac{C}{x}\left\{q \cdot \frac{q^b - 1}{(q-1)^2} - b\left(\frac{1}{q-1} + m\right)\right\}$$

$$= \frac{C}{x} \cdot \frac{q-1}{q^b-1}\left\{q \cdot \frac{q^b-1}{(q-1)^2} - b\left(\frac{1}{q-1} + m\right)\right\}.$$

And in the 1st year of its employment the whole capital C is abstracted.

Let the machinery be made to last d_1 years by the expenditure of a_1, a_2, $a_3 \dots\dots a_{d_1}$ in the 1st, 2nd, d^1 years respectively and be worth a when rejected: also let the expenditure and profit be equivalent when the machine continues d years unimpaired, and then becomes useless.

This condition gives

$$a_1 q^{d_1} + a_2 q^{d_1-1} + \dots\dots + a_{d_1} q - a = C(q^{d_1} - q^d),$$

$$\text{or} \quad \Sigma(a_t q^{d_1-t}) - a = C(q^{d_1} - q^d);$$

$$\text{and} \quad d = \log. \{(q^{d_1}) - \frac{1}{C}(\Sigma a_t q^{d_1-t} - a)\} \frac{1}{\log. q}.$$

Let V_e be the exchangeable value of the machinery,

A the annuity necessary to pay the profit on C and provide new machinery at the end of d^1 years, paying also for the necessary repairs during this interval,

r_1, the produce due to the use of the machinery,

r .... to the labour it displaces,

p, p_1, prices of produce before and after its employment,

G the whole annual gain to the community occasioned by its uses,

D_1, D_2, ... D_ρ sums by which labour-fund is diminished at the end of 1^{st}, 2^{nd} ...... ρ^{th} years after it comes into operation,

D diminution during the 1^{st} year.

$$\text{Then} \quad V_\epsilon = C,$$

$$A = C.\frac{q-1}{1-q^{-d}}.$$

Also, since the gain to the community may be measured by the price that would have been paid for the produce r, minus the price that is actually paid for it when the profits of the capitalist have reached the average rate,

$$G = Cq.\frac{r_1}{r} - A = Cq\left(\frac{r_1}{r} - \frac{1-q^{-1}}{1-q^{-d}}\right).$$

Or, this gain may be measured by the saving in expenditure, added to the cost of the additional produce enjoyed, reckoning that cost at the original price;

$$\therefore \quad G = Cq - A + \frac{r_1 - r}{r}.Cq = Cq.\frac{r_1}{r} - A \text{ as before.}$$

This will be an annuity in money or other produce, of which a k^{th} part may be consumed for immediate enjoyment and a $(1-k)^{th}$ part used as capital, also of the former an m^{th} part, and of the latter an $m_2{}^{th}$ part may go to the labourer, whilst an m^{th} part of the annuity A after deducting the profit $C(q-1)$ is employed in the same way. Before the use of the machinery the expenditure was C.

Hence, $D = C$,

$$D_1 = C - m\{A - (q-1)C\} - \{m_1 k + m_2(1-k)\}G,$$

$$D_2 = C - m\{A - (q-1)C\} - \{m_1 k + 2m_2(1-k)\}G,$$

....

....

$$D_\rho = C - m\{A - (q-1)C\} - \{m_1 k + \rho m_2(1-k)\}G$$

$$= C\left\{1 - m\frac{q-1}{q^d-1} - \{m_1 k + \rho m_2(1-k)\}q\left(\frac{r_1}{r} - \frac{1-q^{-1}}{1-q^{-d}}\right)\right\}.$$

When this becomes negative, the fund for the employment of labour will have become greater than it would have been if the machinery had not been constructed, that is, when

$$1 - m\frac{q-1}{q^d-1} - m_1 kq\left\{\frac{r_1}{r} - \frac{1-q^{-1}}{1-q^{-d}}\right\}$$

$$< \rho m_2 (1-k)\, q\left\{\frac{r_1}{r} - \frac{1-q^{-1}}{1-q^{-d}}\right\},$$

$$\text{or } \rho > \frac{1 - m\,.\,\dfrac{q-1}{q^d-1}}{m_2(1-k)\,q\left\{\dfrac{r_1}{r} - \dfrac{1-q^{-1}}{1-q^{-d}}\right\}} - \frac{m_1}{m_2}\cdot\frac{k}{1-k}.$$

This advantage will increase by the addition of

$$C\,.\,m_2(1-k)\,q\left\{\frac{r_1}{r} - \frac{1-q^{-1}}{1-q^{-d}}\right\}$$

annually, and will continue to increase indefinitely.

We may in these formulæ, substitute for the produce in terms of the price.

We have $pr = Cq$, $p_1 r_1 = A = Cq\,.\,\dfrac{1-q^{-1}}{1-q^{-d}}$;

$$\therefore\ \frac{r_1}{r} = \frac{p}{p_1}\cdot\frac{1-q^{-1}}{1-q^{-d}};$$

$$\therefore\ G = Cq\left\{\frac{p}{p_1} - 1\right\}\frac{1-q^{-1}}{1-q^{-d}},$$

$$D\rho = C\left\{1 - m\frac{q-1}{q^d-1} - \{m_1 k + m_2(1-k)\rho\}\, q\,.\left(\frac{p}{p_1} - 1\right).\frac{1-q^{-1}}{1-q^{-d}}\right\}.$$

It may be observed that p_1 cannot be $> p$; if it were, more than the ordinary profit would arise from employing labour, and the machine would be superseded.

In general the motive of the capitalist in supplanting labour by machinery, is to procure for his capital more than the ordinary profit. If, then, he can raise his rate of profit to q_1, $A^1 = C\,.\,\dfrac{q^1-1}{1-q_1{}^{-d}}$ will be the annuity which the produce must realize. The above values of G, however, are correct whatever q may become, since the gains of the capitalist are included in those of the community at large.

In a State where there is any number of machines at work, and also any number in progress, wealth will be increasing more rapidly than it would have been if all the capital had been employed in paying wages if ΣG be positive, and less rapidly if ΣG be negative, that is, more or less rapidly as,

$$\Sigma\frac{r_1}{r}.C > \text{ or } < \Sigma\frac{1-q^{-1}}{1-q^{-d}}.C, \text{ or as } \Sigma\frac{p.C}{p_1(1-q^{-d})} > \text{ or } < \Sigma\frac{C}{1-q^{-d}}.$$

If in any case the machinery lower prices, the community must gain by its use, for if $p_1 < p$, G is positive.

This supposes p_1 to give the capitalist no additional profit, *à fortiori*, therefore the community will gain, if, when the rate of profit is raised to $q'-1$, p_1 be $< p$.

Let p'' be the price that would pay ordinary profits $q-1$.

$$\text{Then } p''r_1 = C.\frac{q-1}{1-q^{-d}};\quad p_1r_1 = C.\frac{q_1-1}{1-q_1^{-d}},$$

$$p'' = p_1.\frac{q-1}{q^1-1}.\frac{1-q_1^{-d}}{1-q^{-d}}.$$

The community will gain by the use of machinery, if

$$p > p_1\frac{q-1}{q^1-1}.\frac{1-q_1^{-d}}{1-q^{-d}}.$$

If a portion of the dispossessed labourers, whose wages were paid by a part $l.C$ of the capital, become dependent on society for support, there will result the equation

$$G + lC = Cq\left\{\frac{r_1}{r} - \frac{1-q^{-1}}{1-q^{-d}}\right\},$$

$$G = C\left\{q\frac{r_1}{r} - \left(\frac{q-1}{1-q^{-d}} + l\right)\right\}.$$

And the community would gain or lose as

$$\frac{r_1}{r} > \text{or} < \frac{1-q^{-1}}{1-q^{-d}} + \frac{1}{q} l.$$

The common mode of calculating the annuity A has been used, but it does not appear to be strictly applicable; it makes no provision for paying machine makers in the first year after the machinery comes into use, and does provide for their payment in the $(d+1)$ year; that is, after the machinery is finished.

To commence the manufacture of new machinery immediately an additional capital must be employed, to advance the wages of its makers; and if the same sum is paid every year in this way, the additional capital must equal that part of the annuity A which is available for a similar purpose.

Let Cy be the additional capital.

Cy is the annuity, commencing at the beginning of 1^{st} year and ending at the beginning of d^{th} year, which pays for the machine.

Hence, since at the beginning of d^{th} year this amounts to

$$Cy\,(1+q+\dots\, q^{d-1}) = Cy\,.\,\frac{q^d-1}{q-1},$$

and at the end of d^{th} year this is worth $Cy\,.\,q\,.\,\dfrac{q^d-1}{q-1}$, which must $= C$,

$$y = \frac{1-q^{-1}}{q^d-1}.$$

Again, the produce must supply an annuity A, which pays the profits on $C(1+y)$, and accumulates by the end of d^{th} year to $C(1+y)$.

The accumulation is therefore worth

at end of 1^{st} year $A - C(1+y)(q-1)$,

.... 2^{nd} year $\{A - C(1+y)(q-1)\}(1+q)$,

....

.... d^{th} year $\{A - C(1+y)(q-1)\}\dfrac{q^d-1}{q-1}$,

which must $= C(1+y)$.

This gives

$$A = \frac{q-1}{q^d-1} \cdot C(1+y)\,q^d = C \cdot \frac{1-q^{-1}}{1-q^{-d}} \cdot \frac{q^{d+1}-1}{q^d-1},$$

$$G = Cq(1+y)\frac{r_1}{r} - A = Cq(1+y)\left\{\frac{r^1}{r} - \frac{1-q^{-1}}{1-q^{-d}}\right\}$$

$$= Cq \cdot \frac{1-q^{-(d+1)}}{1-q^{-d}}\left\{\frac{r_1}{r} - \frac{1-q^{-1}}{1-q^{-d}}\right\}.$$

The whole capital employed is

$$C(1+y) = C \cdot \frac{1-q^{-(d+1)}}{1-q^{-d}},$$

$$D\rho = C\frac{1-q^{-(d+1)}}{1-q^{-d}}\left\{1 - m\frac{q-1}{q^d-1} - \{m_1 k + m_2(1-k)\rho\}\,q\left(\frac{r_1}{r} - \frac{1-q^{-1}}{1-q^{-d}}\right)\right\},$$

$$D = C(1+y) - Cy = C.$$

Also

$$p_1 r_1 = A = C(1+y) \cdot \frac{q-1}{1-q^{-d}}; \quad pr = Cq(1+y);$$

$$\therefore \frac{r_1}{r} = \frac{p}{p_1} \cdot \frac{1-q^{-1}}{1-q^{-d}};$$

and therefore,

$$G = Cy \cdot \frac{1-q^{-(d+1)}}{1-q^{-d}} \cdot \frac{1-q^{-1}}{1-q^{-d}}\left\{\frac{p}{p_1} - 1\right\},$$

and $$D\rho = C\frac{1-q^{-(d+1)}}{1-q^{-d}}\left\{1 - m \cdot \frac{q-1}{q^d-1} - \{m_1 k + m_2(1-k)\rho\} \cdot \left(\frac{r^1}{r} - \frac{1-q^{-1}}{1-q^{-d}}\right)\right\}.$$

As before, the condition necessary that the community shall gain by the use of machinery is

$$p > p_1 \cdot \frac{q-1}{q_1-1} \cdot \frac{1-q_1^{-d}}{1-q^{-d}},$$

and since p^1 cannot be $> p$, the community cannot lose, unless q_1 be $< q$, or the capitalist also lose.

From the result of the above investigation it would seem that we are entitled to draw the following conclusions:

If we assume that a capitalist will employ machinery or labour as the one or the other will procure for him the highest rate of profit, then the employment of machinery will always increase the wealth of the community. Not only is the capitalist unable to secure his own advantage at the expense of any other class, he cannot even prevent a general participation in the benefit.

The operation on the labourer is to abstract a fund which has been or would have been annually employed in the payment of wages, and annually renewed by the produce due to his exertions, and to supply a new fund, by increasing the wealth of the community, a portion of which will in general be paid as wages; this portion is at first smaller than the fund abstracted, but it increases without any assignable limit, the rapidity of increase depending on the proportion in which the new fund is divided between the labourer and the other classes of society. Speaking with reference to the formulæ, the rapidity of increase depends on the values of the arbitrary quantities m, m_1, m_2, k, and these values can be assigned with greater or less exactness, as our statistical knowledge connected with the particular case is more or less accurate.

APPENDIX.

EXAMPLES extracted from writers on Political Economy.

M[c] CULLOCH, 2d Edition, 193.

Comparison of two machines, one of which lasts one year the other ten years, the cost of each being £.20000, and their produce the same, the ratio of profit 10 per cent.

When $d = 1$,

$$y = \frac{1-q^{-1}}{q-1} = q^{-1} = \frac{10}{11}, \quad A = C(q+1) = C\frac{21}{10} = 42000.$$

When $d = 10$, let y and A become y^1 and A_1,

$$y^1 = \frac{1-q^{-1}}{q^{10}-1} = \frac{1}{11} \cdot \frac{10^{10}}{11^{10}-10^{10}} = \frac{5}{88} \text{ nearly},$$

$$A^1 = C.\frac{1-q^{-1}}{1-q^{-10}} \cdot \frac{q^{11}-1}{q^{10}-1} = \frac{155}{844} C \text{ nearly} = 3672.$$

If G now represent the gain to the community by employing the more durable machine,

$$G = A - A_1 = \frac{8087}{4220} \cdot C = 38327.$$

Again, when the machine lasted 1 year there was paid in wages $mCy = \frac{10}{11} mC$. When 10 years, the amount paid in the ρ^{th} year of its being employed will be

$$mCy^1 + \{m_1 k + m_2 (1-k) \rho\} G = C\left\{\frac{5}{88} m + \{m_1 k + m_2 (1-k) \rho\} \frac{8087}{4220}\right\}.$$

Therefore if D_ϱ be the loss to the labour-fund at the end of the ρ^{th} year by employing the more durable machine,

$$D_\varrho = C\left\{m\left(\frac{10}{11} - \frac{5}{88}\right) - \frac{8087}{4270}\{m_1 k + m_2(1-k)\rho\}\right\},$$

$$D_0 = Cm\frac{75}{88}.$$

If $m_1 = m_2 = m = k = \frac{1}{2}$; D_1 will be negative, and the labour-fund will be increased at the end of the 1st year.

SISMONDI.

An improvement takes place in machinery which reduces prices 5 per cent.

Suppose the produce the same.

The condition gives $\dfrac{Cq}{A} = \dfrac{100}{95} = \dfrac{20}{19}$, $A = Cq.\dfrac{19}{20}$,

$$G = Cq - A = Cq.\frac{1}{20},$$

$$D_\varrho = C - m\{A - C(q-1)\} - G\{m_1 k + m_2(1-k)\}\rho$$

$$= C\left\{1 - m\left(\frac{19}{20}q - q + 1\right) - \frac{q}{20}[m_1 k + m_2(1-k)\rho]\right\}.$$

If $m = m_1 = m_2 = k = \frac{1}{2}$, and $q = \dfrac{11}{10}$.

$$D_\varrho = C\left\{1 - \tfrac{1}{2}.\frac{200-11}{200} - \frac{11}{200}.\frac{1}{4}(1+\rho)\right\}$$

$$= \frac{C}{800}\{411 - 11\rho\}.$$

If $\rho > \dfrac{411}{11}$ the labour-fund will have gained.

Hence, after 38 years the amount spent in wages is greater than it would have been if the improvement had not taken place, and increases beyond this period to an indefinite amount.

The values assigned to m_1 &c. are entirely arbitrary, if m, m_1, m_2, each $= \frac{5}{6}$, and $k = \frac{1}{2}$,

$$D\rho = \frac{C}{480}(91 - 11\rho);$$

and the labour-fund will be increased after 9 years.

In the value of $D\rho$ the whole of C is supposed to be taken from the labour-fund, the data necessary for calculating the effect of the change enunciated in the proposition are not given.

RICARDO, 3d Edition, 469.

A capitalist employs £.20000, £.7000 of which is vested in irremoveable capital, and £.13000 in labour, profits being 10 per cent. He then abstracts £.6500 to pay machine makers, and £.1000 for half the profits during the year the machinery is in progress. His capital is now £.6500 to pay wages, and £.7500 in machinery, in addition to the £.7000 irremoveable.

The machinery is supposed indestructible, and no more is produced by it than will pay the common profits on the £.7500.

Take the capital as £.13000, and suppose that with the aid of the £.7000 profits are raised on this from $q - 1$ to $q_1 - 1$.

$$\text{Then } q_1 - 1 = \frac{20000}{13000}(q - 1) = \frac{20}{13}(q - 1).$$

Now before the machinery was employed the produce of the £.7500 must have fetched $7500 . q$, and afterwards $7500 . (q_1 - 1)$;

$$\therefore \frac{r_1}{r} = \frac{q_1 - 1}{q_1} = 1 - q_1^{-1},$$

$$\text{and } G = Cq_1\left\{\frac{r_1}{r} - \frac{1 - q_1^{-1}}{1 - q_1^{-d}}\right\} = Cq_1\{1 - q_1^{-1} - (1 - q_1^{-1})\} = 0.$$

Since $d = D$,

$$D\rho = C\left(1 - m\frac{q - 1}{q^d - 1}\right) - \{m_1 k + m_2(1 - k)\rho\}.G = C.$$

If therefore it were possible that the capitalist could be influenced by a motive which would induce him to employ machinery under such circumstances as these, the community would gain nothing, and the labourer would lose the whole.

BARTON.

A manufacturer has £.1000, with which he pays 20 men £.50 each; his capital is suddenly increased to £.2000, when he lays out £.1500 in machinery, which takes 1 year to complete, and does the work of 15 men, but requires 1 man to keep it in repair, and also lasts 15 years, profits 10 per cent.

Here $C = 1500$, $b = 1$, $d = 15$, $q = \frac{11}{10}$;

$$\therefore x = q\frac{q^b - 1}{q - 1} = q = \frac{11}{10},$$

$$B = (1-m)\frac{C}{x} = (1-m)\frac{15000}{11}.$$

But £.500 only of the £.1500 was before employed in labour, and therefore the loss to the labour-fund during the year that the machine is making, is

$$(1-m)\frac{15000}{11} - 1000 = \frac{1000}{11}(4 - 15m),$$

which is negative if m be $> \frac{4}{15}$, or if the labourer receive more than $\frac{4}{15}$ of the price of the machine; the amount paid for wages will be increased.

After the completion, we have

$$A = 1500 \cdot \frac{\frac{11}{10} - 1}{1 - \left(\frac{10}{11}\right)^{15}} = 150 \cdot \frac{11^{15}}{11^{15} - 10^{15}} = 197,$$

and the machine does the work of 14 men, for whose labour the consumer paid $50 \cdot 14 \cdot \frac{11}{10} = 770$;

the gain in price is $770 - 197 = 573$,

and the capitalist gains 100;

$$\therefore \; G = 100 + 573 = 673.$$

There has been taken from the labour-fund £.500.

There is returned to it $m(197 - 150) + \{m_1 k + m_2(1-k)\rho\}\, 673$.

Whence $D_\rho = 500 - (47m + 673 m_1 k) - 673 m_2 (1-k)\rho$.

If $m = m_1 = m_2 = k = \frac{1}{2}$,

$$D_\rho = \frac{1233 - 673\rho}{4}.$$

The labour-fund gains after the 2nd year, and receives an addition of £ $\frac{673}{4}$ every subsequent year.

In this solution the value of A, taken by Mr M^c^Culloch (not B) in refuting Mr Barton's views, is employed.

Taking what I have assumed to be a more correct value, we get,

$$\text{since } \; y = \frac{1 - q^{-1}}{q^d - 1} = \frac{9}{75} \text{ nearly,}$$

$$\text{and } \; A = C . \frac{q - 1}{1 - q^{-d}} (1 + y) = 220.$$

The capital employed in the machinery $= C(1 + y) = 1680$; and therefore it must produce as much as £.680 paid in wages.

$$\text{Hence, gain in cost of produce} = \frac{11}{10} . 680 - 220 = 528,$$

and the capitalist gains 100;

$$\therefore \; G = 528 + 100 = 628.$$

Again, there has been taken from the labour-fund,

$$C(1 + y) - 1000 = 1500 . \frac{84}{75} - 1000 = £.680;$$

there is returned to it

$$mCy + \{m_1 k + m_2(1-k)\rho\}\, G = m180 + \{m_1 k + m_2(1-k)\rho\}\, 628$$
$$= 4\left[157m + \{m_1 k + m_2(1-k)\rho\}\, 157\right];$$

$$\therefore\ D_\rho = 4[170 - 45m - \{m_1 k + m_2(1-k)\rho\}\,157],$$

$$D_0 = 4(170 - 45m).$$

$$\text{If}\quad m = m_1 = m_2,$$

$$D_\rho = 4\{170 - m(45 + 157k) - m(1-k)\rho 157\};$$

and if each of these and also $k = \frac{1}{2}$,

$$D_\rho = 4(170 - \frac{247}{4} - \frac{157}{4}\rho) = 333 - 157\rho.$$

The labour-fund will have gained at the end of the 3rd year.

The produce is here supposed the same before and after the machine was employed. The machine must therefore do the work of as many men as the capitalist is deprived of the means of paying.

XXVII. *Novitiæ Floræ Maderensis: or Notes and Gleanings of Maderan Botany. By the Rev.* R. T. Lowe, M.A.

[Read *May* 28, 1838.]

FILICES.

1. *Acrostichum paleaceum*, Hook. et Grev. Icon. Fil. t. 235.

Identical I apprehend with *A. squamosum* of Swartz. Although his character of "frondes 1—2-pedales" certainly exceeds the average of Maderan specimens, I have lately seen some fully 18 inches long, without the stipes: and I am informed by my friend J. I. Bennett, Esq., that in "the Banksian Herbarium are barren fronds of 15 or 16 inches in length, in addition to the stipes (as by Swartz described) of 3 or 4; and some of them, which are abruptly mutilated, would, I think, justify the describing them as "1—2-pedales." In every other particular Swartz's description perfectly agrees; and was, I have little doubt, drawn up from the Maderan plant.

2. *Polypodium drepanum*, nob.

Aspidium drepanum Sw. (*Aspidium? drepanum* nob. Primit. p. 6. No. 3.), proves, as I have already stated in the Botanical Miscellany (New Series, I. p. 26.), to be a genuine *Polypodium;* not having the slightest trace of an indusium in any stage of growth. The following description of the fructification is derived both from abundant wild specimens, and from others cultivated in my garden, and watched carefully for several years.

Indusia nulla. Sori nudi, globosi, valde convexi, tumidi, distinctissimi, subconferti, biseriati, purpureo-nigri, capsulis nitidissimis; demum (sporis effusis) pallide ferruginei, minuti, punctiformes. *Polypodii* species vera.

3. *Asplenium productum*, nob.

A. fronde deltoidea, apice caudata s. longe acuminata, glaberrima, lucida, quadripinnatifida: pinnis primariis productis, acuminatis; ultimis oblongo-cuneatis, apice inciso-dentatis: soris confertis, mox confluentibus: stipite fusco, lævi, basi hirsutiusculo.

Aspl. acutum, Höll's List of Mad. Plants in Hook. Bot. Misc. New Series, I. p. 15; haud Bory!

Aspl. Adiantum nigrum var. nob. Ibid. p. 24; haud Linn.

Hab. in Madera, ab altitudine 1000 ad 3000 pedum ubique vulgatissimum.

This very common fern, the *Asplenium Adiantum nigrum* of most former lists of Maderan plants, I would now admit to be sufficiently distinct from the European species properly so called; the characters above enumerated proving permanent and uniform. With Höll and others I had long imagined it identical with *Aspl. acutum* Bory: but to my surprise, a specimen so ticketed, and obligingly communicated to me by its author the Baron himself, is a very different plant indeed: being undistinguishable from large narrow-leaved fruit-bearing Maderan specimens of my *Asplen. canariense* W.

Asplen. productum is distinguished from the true *Aspl. Adiantum nigrum* L. by its more compound, finely divided frond; the contour of which, as my friend Mr Arnott has well observed, is triangular or deltoid; while in the European plant, the shape is rather that of a rectangle or oblong, terminated by a triangle; the sides being parallel for some length from the base. But the chief character of the Maderan plant is found in the caudate or produced extremities of the primary divisions. The apex of the frond especially is gracefully attenuated.

With *Asplen canariense* W. as understood at least by me, (*Asplen. acutum* Bory!) *Aspl. productum* has very little indeed in common.

4. *Nephrodium fœnisecii* β. productum, Primit. p. 7.

A plant certainly bordering very closely upon the true *Aspidium spinulosum* W. and Sm. in Eng. Flora; but which, on account of the less degree of parallelism in the sides of the ultimate divisions, the smaller punctiform sori, and above all the fragrant scent, I still think best referred to *Nephrodium fœnisecii*. However this, rather than *Aspi-*

dium elongatum Sw., as formerly supposed, (See Hook. Bot. Misc. New Series, I. pp 25, 26.), may very possibly be *Aspidium spinulosum* of Herr Höll's List.

5. *Nephrodium affine*, nob. in Bot. Misc., New Series, I. 25.

N. fronde subtus hirsutiusculo, bipinnatifido; pinnis inferioribus brevioribus: pinnulis approximatis oblongis, subintegris; apice truncato-rotundatis, minutissime eroso-denticulatis; basi tota adnatis: incisuris deorsum acuminatis, sursum latioribus: pinnulis superioribus confluentibus: soris biseriatis distinctis: stipite rhachibusque densissime paleaceis.

Hab. rarior in Maderæ umbrosis, ab alt. 1500 ad 3500 pedum.

Fronds two or three feet long, disposed in a coronet or circle; beneath with scattered hairs, of a chaffy nature, at the margins and on the nerves of the pinnules. Stipes and rhachis throughout most densely chaffy; the chaffs or scales remarkably large as well as copious. Lower two to six pair of pinnæ smaller and shorter than the middle ones; the lowest very much so. Pinnules closer together than in *N. Filix mas:* all of them, even the uppermost rounded (not merely obtuse) at the apex; the lowest even truncate. In *N. Filix mas* the uppermost are acute, and the lower sometimes scarcely obtuse. The sides of all in *N. affine* are nearly entire, or with distant, scarcely perceptible, shallow teeth upwards; not in the least incised, or with any tendency to a higher degree of decomposition, like those of *N. Filix mas*, or of *N. elongatum;* their rounded apex irregularly notched with very minute shallow teeth, quite different from the obvious *serrated* teeth of *N. Filix mas*, and without a lens, scarcely perceptible; the pinnules appearing entire. *Incisures* (i. e. the spaces between the pinnules) acute at the bottom, i. e. towards the nerve, and widening upwards or rather outwards; evidently not reaching to the nerve or midrib, as they appear to do in *N. Filix mas*, though really they do not. Hence the base of all the pinnules is in no degree incised on either side, as in *N. elongatum;* but the pinnules are adnate by the entire breadth of their base: neither is their lower side arcuato-decurrent as in *N. Filix mas;* but the base is truncate and the opposite sides or margins of each pinnule are parallel to each other down to its very bottom, where they are both at right angles to the rhachis: whilst in *N. Filix mas*, the lower or inner margin, i. e. that towards the main rhachis, quits its condition of parallelism with its opposite upper or outer one towards the base, and forms a curve downwards towards the pinnule next below it; which is what I have called arcuato-decurrent. Thus the incisure in *N. Filix mas* is either irregular, or abrupt, open, and broad at the base; not regularly acuminate; so that the spaces between the pinnules being larger, or at least not regularly narrower downwards, these last appear more remote and distinct than in *N. affine*. In fact the incisures (not at all the pinnules) of *N. affine* rather resemble those of *Nephrodium* (*Aspidium* Auct.) *Oreopteris* than of *N. Filix mas*. The pinnules of the lower pinnæ, instead of having any tendency by incision to a farther degree of decomposition, as they have both in *N. Filix mas* and *N. elongatum*, are quite simple, and even more entire than the upper ones. Sori precisely similar in their arrangement and indusia to those of *N. Filix mas*.

I possess specimens of *N. affine* from various localities, differing in exposure, shadiness, and elevation; but all agree in the foregoing characters. By these, this fern approaches nearer to *N. elongatum* than to any other Maderan species: the true European *N. Filix mas* being the connecting link; from which it is curious to observe, these two Maderan ferns reciprocally recede in opposite directions: *N. elongatum* having the serratures much more aristate, and the stipes and rhachis, especially the latter, less chaffy than the European *N. Filix mas;* while *N. affine* has the serratures much less developed, but the stipes and rhachis much more copiously chaffy than the same.

The specimens of *N. Filix mas*, which I have particularly examined for comparison with *N. affine*, are British only: but my friend Mr J. Bennett has also compared specimens in the Banksian Herbarium, and noticed the same differences. Indeed his observations, exactly corresponding with my own, here made before and apart, have led me with considerable confidence to the conclusions and results here stated.

I subjoin, for facility of comparison, the specific characters of *N. Filix mas* and *elongatum;* eliminated however solely in reference to the three present species.

Nephrodium Filix mas.

N. fronde glabro, bipinnato: pinnis inferioribus brevioribus: pinnulis subremotis, oblongis, obtusis, serratis, adnatis, basi inferiore arcuato-decurrentibus; inferioribus serrato-incisis; superioribus confluentibus: soris biseriatis distinctis: stipite paleaceo; rhachibus sparsim paleaceo-hirtis.

Aspidium filix mas, Auct; Linn. Sm. Hook. &c. From English specimens, gathered at Dale Abbey in Derbyshire.

Main rhachis sparingly hairy rather than chaffy, and the nerves and margins of the pinnules beneath are not at all hairy. Pinnules subremote; so that the incisures are truncate or oblique at the base, or as wide at the bottom as at the top in the lower pinnules.

6. *Nephrodium elongatum.*

N. fronde rigido, glaberrimo, bipinnato: pinnulis oblongis, spinuloso-serratis; superioribus apice rotundatis, confluentibus; inferioribus sublanceolatis, acutiusculis, crenato-incisis, distinctis, subpetiolatis s. basi utrinque incisis; incisuris triangulari-decurrentibus: soris biseriatis, confertis, subimbricatis nervo approximatis; indusiis glanduloso-scabris: stipite elongato rhachibusque pallidis, paleaceis; paleis rhachidum raris, sparsis, distinctis, squamiformibus.

N. elongatum, Hook. et Grev. Icon. Fil. t. 234.

Aspidium elongatum, Sw., &c.

Hab. in Madera ab alt. 1500 ad 5000 ped. vulgaris.

Fronds not growing in a circle, of a much lighter and brighter green than in the two preceding species, and of a rigid brittle texture: with a greater tendency to a farther degree of decomposition than even *N. Filix mas.* Pinnæ, in full-sized specimens of 3 or 4 feet long, more remote and distinct; the lower ones not shorter than the rest, but rather the contrary. Upper pinnules close together; so that their edges often touch or even overlap each other, concealing the incisure; except at the base, where, by the incisure being produced downwards into the substance of the pinnule next below, and similarly, though in a less degree and sometimes not at all, into the pinnule next above, there is formed in all a kind of triangular hole, or open space, quite different from any thing in the two preceding species; though of the two, most like *N. Filix mas.* Hence the incisures, not the pinnules, are decurrent: the lower base of the pinnules, as well as the upper in a less degree, being notched by the incisure, and their margins being parallel at the base.

7. *Cheilanthes maderensis*, nob. in Bot. Misc. New Series, I. 26.

C. fronde oblongo-lanceolato vel ovato, bipinnato, glabro: pinnulis (s. laciniis secundi ordinis) oblongis, obtusissimis, adnatis, decurrentibus, sinuatis; inferioribus basi pinnatilobatis; superioribus confluentibus; omnibus vel omnino vel superne indivisis, foliiformibus, crenatis; lobis fructiferis rotundatis: indusiis interruptis, incisis; lobis rotundatis vel truncatis, margine integro: stipite rhachibusque paleaceo-hirtis.

Hab. in fissuris rupium prope urbem Funchalensem Maderæ; etiam in muris ipsius urbis.

Species cum aliis quibusdam diu confusa, revera distincta videtur. A *Ch. suaveolente* Sw. (*Polypod. fragrans* Desf. Fl. Atl. ii. 248. t. 257) prima facie differt fronde multo minus tenuiter diviso, nec leptophyllo; divisionibus sc. magis foliaceis, confluentibus nec distinctis; pinnulis oblongis, sinuatis crenatisve, multo majoribus, foliiformibus, omnino vel superne saltem indivisis; summis confluentibus; omnibus basi tota adnatis decurrentibusque, nec puncto tantum centrali baseos s. petiolulo rhachi affixis: soris indusiisque incisis, interruptis, nec continuis ut in figura 1ma. iconis jam citatæ delineantur. A *Ch. odora* Sw., planta sc. Helvetica et Pedemontana (*Adiantum pusillum* All.), iisdem characteribus, necnon fronde bipinnato nec tripinnato, indusiisque margine integris, nec "laceris, subciliatis" *Sw.*, satis superque distincta videtur: huic vero speciei procul dubio, monente amico J. I. Bennett, affinitate proxima. *Ch. fragrante* Sw., stirpe Indiæ Orientalis, cui cl. Swartzius olim dubio animo conjunxit, "*forsan speciem diversam*" tamen monens, magis ac magis recedit: quum ne alia dicam, illa pinnulis "*oppositis, ovatis, subpetiolatis, lacinulis 2—3-partitis,*" tenuibus; earum "*segmentis subacutis, apice soriferis*" (monosoris); "*soris minutissimis;*" indusiisque dentiformibus distinctissima est. In stirpe Maderensi pinnulæ haud raro alternæ; in tribus supra indicatis potius oppositæ videntur.

PHANEROGAMÆ.

GRAMINEÆ.

8. *Phragmites congesta*, nob.

P. panicula lanceolata, stricta, contracta, densa, subsecunda; spiculis 4—6-floris, glumis multo longioribus: culmis basi decumbentibus, ramosis; foliis planis mox convolutis, strictis, cuspidatis, glaucis.

Hab. in Maderæ ora maritima rariss.

Obs. *P. communi* Trin. (*Arundini Phragmitæ* L.) proxima, et forsan varietas tantum. Culmi basi ramosi, frutescentes, nudi, tenacissimi, late procumbentes; apice foliosi, ascendentes 2—4-pedales: foliorum margine subserrulato; vaginis apice, nodisque plerumque barbatis. Gluma inferior brevis, acuta, superior inferiore duplo longior, remota, acuminata. Paleæ inferiores florum inferiorum productæ, flores superiores longitudine subæquantes; omnes lanceolatæ, acuminatæ, glabræ. Rhachis supra florem inferiorem longissime sericeo-pilosa. Palea superior brevis, oblonga, plana, binervis, subciliata. Panicula erecta, arctissime glomerata, congesta, densissima, multiflora, 3—6-pollicaris longa, 1—2-lata, pallida, flavescens; demum albo-sericea. Radices repentes.

9. *Deschampsia argentea*, nob. (*Airæ argentea* nob. olim Prim. in Trans. Cam. Phil. Soc. IV. I. p. 9. No. 8.) Species distinctissima, *D. cespitosæ* Beauv. (*Airæ cespitosæ* L.) proxima. *Deschampsiæ* Beauv. species legitima. Pedicelli, floresque basi dense pilosi. Palea inferior apice 3—4-dentata.

10. *Avena marginata*, nob.

A. glaberrima, lævis: panicula simpliciuscula, coarctata, subsecunda; spiculis 4—5-floris, glumis sublongioribus; rhachi pedicelloque floris alterius superioris abortientis villosis; floribus omnino glabris s. nudis, scabriusculis; palea inferiore apice quadriseta; s. bifida, laciniis bifidis in setulas productis; dorsi medio arista geniculata: foliis distichis, brevissimis, obtusis, carinatis, marginatis, rigidis, glaucis; ligula lanceolata, producta: radice fibrosa, subcespitosa, perenni.

In rupibus Maderæ excelsis nuperrime invenit am. Car. Lemann, M. D.

Culmi bipedales et ultra, 2—3 ex uno cespite, vaginisque rhachique ramisque primordialibus paniculæ omnino lævibus, glabris. Pedicelli floresque scabriusculi. Panicula 4—5-pollicaris, suberecta, apice subnutante, subpauciflora. Spiculæ nitentes, semipollicares. Glumæ carinatæ, trinerviæ; nervis prominentibus. Flos summus spicularum minor, masculus, neuter, abortiensve, et ad pedicellum villosum reductus. Palea inferior nervosa, sulcato-striata; apice bifida; laciniis bifidis in setulas albidas hispidiusculas productis: arista dorsali, fusca, subsemipollicari, i. e. flore duplo longiore, infra medium tortili. Folia radicalia plano-carinata, arescentia conduplicata, lævia, margine elevato, conspicuo, discolore, serrulato-scabro, deorsum sæpe undulato-plicata, s. corrugata; apice mirandum in modum obtusa, 2—3-pollicaria; caulina superiora vix semipollicaria. Vaginarum ora omnino nuda. Ligula elongato-triangularis, acuminata, arcte amplexicaulis, integerrima.

11. *Cynosurus brizoides*, nob.

C. panicula lanceolata, gracili, coarctata; spiculis neutris amentaceis, distinctis; bracteis setaceis, aristisque confertis, rigidis, scabris.

Hab. in graminosis convallium Maderæ rarior.

Gramen annuum, exile, subinconspicuum. Culmi plures ex eodem radice, tenues, 1—2-pedales. Folia flaccida, brevia. Panicula parva, secunda, subpollicaris, simpliciuscula, æqualis; fructifera arcte contracta, sæpe violaceo-purpurascens; spiculis neutris spiculas *Festucæ*, *Poæ*, aut *Brizæ* referentibus. Bracteæ aristæque rectæ, floribus multo longiores.

12. *Festuca jubata*, nob.

F. cespitosa: culmo superne paniculæque lanceolatæ, abbreviatæ, subcoarctatæ, subsecundæ, rhachi ramulisque pedicellisque puberulis, haud scabris: spiculis lanceolatis, 3—5-floris, glabris; flosculis aristatis, sursum aristisque scabris: foliis culmos superantibus subsetaceis, subcanaliculatis, vaginisque striatis, glabris; ligula brevissima, abrupta: radice perenni.

Hab. rariss. in rupibus convallium Maderæ, cum *Deschampsia argentea* nob., cui habitu simillima, nascens. Primus invenit Car. Lemann M. D.

F. geniculatæ Willd. (*Bromo geniculato* L., *Festucæ stipoidei* Desf.) proxima. Differt radice perenni; culmis haud geniculatis, dense cespitosis; pedunculis subsimplicibus, &c.

Culmi pedales, tenues, teretes, erecti, glabri, paullo infra paniculam contractam puberuli, foliis breviores, haud geniculati, nodis demum fuscis. Folia numerosa, conferta, elongata, omnia subsetacea, tenuia, gracilia, rigidiuscula, glabra, striata; superiora sursum subplanata, subcanaliculata. Glumæ inæquales, læves; altera spiculæ subæquans, altera brevior. Flores glabri; basi tantum læves, sursum aristisque scabri; arista flore longior. Pedicelli simpliciusculi, sc. sursum vix subdilatati, cuneati, ancipites; haud vero magis quam in multis aliis.

13. *Festuca Donax*, nob. Prim. in Trans. Cam. Phil Soc. IV. I. p. 9. No. 9.

(Character auctus, emendatus).

F. paniculæ glabræ, largæ, laxæ, diffusæ, subsecundæ, nutantis ramis elongatis, flexuosis: spiculis trifloris, lineari-lanceolatis, compressis, glomeratis; flosculis muticis linearibus, angulatis, scabris; glumis subæqualibus, spiculam æquantibus: paleis apice membranaceis, obtusis, abruptis, subbifidis, nervis prominentibus; exteriore quinquenervia, nervis æquidistantibus; interiore binervia, dorso canaliculata: foliis omnibus planis, elongatis, acuminatis, striatis, marginibus serrulato-scabris; culmis vaginisque lævibus, glabris; ligula exserta, ovata: radice fibrosa, perenni.

The nearest ally of the species is *F. sylvatica* Vill. (*F. calamaria* Sm.)

14. *Festuca albida*, nob. Prim. in Cam. Trans. IV. I. p. 10. No. 10.

α. *longifolia;* foliis culmum æquantibus.

β. *brevifolia;* foliis culmo multum brevioribus. In rupibus nuper invenit C. Lemann, M. D.

CYPERACEÆ.

15. *Carex sagittifera*, nob.

C. spica solitaria, androgyna, (♀, superne ♂), subpauciflora: stigmatibus duobus: fructibus oblongis, utrinque attenuatis s. fusiformibus, planatis, glabris, nitidis, reflexis, squama acuta, oblonga, persistente longioribus: culmo subtereti, superne subcompresso, hinc subcanaliculato: foliis angustissimis, elongatis, superne canaliculatis, inferne carinatis; carina marginibusque serrulato-scabris.

Hab. in sylvis Convallium Maderæ, in declivibus prope rivulos rarior.

Dense cespitosa. Folia numerosa, conferta, culmos subæquantia s. excedentia, fere (pro latitudine) setacea s. filiformia. Fructus nitidissimi, magni, subremoti, squamisque fuscis, deflexis; inde sagittarum quasi cuspides plures, filo consertas, spica refert.

Cum *C. decipiente* Gay et La Perouse, monente am. Fr. Boott, M. D., conferenda. *C. pulicari* L. proxima, sed abunde distincta.

JUNCEÆ.

16. *Luzula elegans*, nob.

L. foliis lanceolatis, pilosis: corymbi erecti, supradecompositi ramis capillaribus, mox divaricatis, deflexis: pedunculis unifloris: bracteis sepalisque setaceo-acuminatis, capsula obtusa, mucronata longioribus: seminibus simplicibus, exappendiculatis.

Hab. in rupibus convallium Maderæ murisque rarior.

4—10-pollicaris; corymbo fructifero elongato, oblongo-angustato, semipedali; ramis ramulisque inferioribus elongatis, plerisque divaricato-refractis, tenuissimis, hinc inde pilosis. Flores solitarii, rufo-castanei, lucidi.

AMENTACEÆ.

17. *Quercus mitis*, Herb. Banks.

Since the publication of this, as a species, in the Cambridge Transactions (Vol. IV. I. p. 15. No. 21.), from the specimen preserved in the Banksian Herbarium, I am quite satisfied, from observation of two growing trees, evidently identical with the above, that it is nothing but a slight variety of *Q. Suber* L., with broader, more entire leaves than usual; such as might be expected in trees, growing like these, in cool, shady situations, at a considerable elevation. Both these trees are the inmates of gardens: and it is quite certain that Madera possesses no indigenous species of *Quercus* whatever.

URTICACEÆ.

18. *Parietaria gracilis*, nob. Prim. in Trans. Camb. Phil. Soc. IV. I. p. 16. No. 23.

(Character emendatus).

P. lucida, pubescens, ramosa: caule ramisque gracilibus, erectis: foliis rhombeo-ovatis, rotundatis, abbreviatis, obtusis, trinerviis, petiolatis; petiolis filiformibus, folia æquantibus: glomerulis axillaribus; floribus pedicellatis; pedicellis glomeratis, aliquando subcymosis, apice tribracteatis: bracteis (*involucri foliolis* Auct.) unifloris, lanceolato-linearibus, obtusis, calyce (quadrifido, glabro) brevioribus, glanduloso-pubescentibus; post anthesin inæqualibus, uno duobusve dilatatis, foliaceis, calycem superantibus, adpressis.

Hab. in Maderæ rupestribus declivibus umbrosis rariss.

Euphorbiaceæ.

19. *Euphorbia refracta*, nob.

E. annua, ramosa, pilosiuscula: ramis pubescentibus humifusis prostratisve, suffrutescentibus, geniculatis, flexuosis, refractis, alternis, apice dichotomis, ad genicula nodosis: foliis oppositis, oblique-oblongis, subrhomboideis, inæquilateralibus, argute serratis, hinc deorsum integerrimis, basi illinc auritis, semicordatis, subsexnerviis, brevissime petiolatis, utrinque pilosis: floribus pedicellatis, in dichotomia ramulorum solitariis, ad apices aggregatis; bracteis bractęolisque foliis conformibus, angustioribus: glandulis quatuor albidis, exappendiculatis, transverse ovalibus: capsulis triquetris, lævibus, glabris: seminibus minutis, nigrescentibus, subtetrahedris, ovalibus, punctato-rugulosis s. corrugatis.

Hab. in Maderæ regione inferiore et intermedio a mare usque ad 1800 ped. rarior.

Chenopodeæ.

20. *Suæda laxifolia*, nob.

S. fruticosa, glabra: ramis patulis, decumbentibus, laxis: foliis laxis, patentibus, linearibus, obtusiusculis, subcarnosis; supra depresso-planatis: floribus sessilibus, axillaribus, sub-solitariis; stigmate trifido, ante anthesin exserto; calycibus post anthesin clausis.

α. tenuifolia; foliis tenuibus; ramis debilibus, elongatis.

β. crassifolia; foliis carnosis obtusissimis, glaucescentibus; ramis validioribus.

Hab. in rupibus locisque saxosis maritimis Mad. et Portûs S[ti]; α vulgatiss.; ab incolis "*Barilla*" dicta.

Calyx simplex, *Suædæ* Mert. veræ.

21. *Suæda tomentosa*, nob.

S. fruticosa, ramis fragilibus, superne foliisque carnosis, subteretibus, obtusiusculis, fasciculatis, incano-tomentosis.

Hab. in collibus maritimis Promontorii P[ta] Saõ Lourenço dicti Maderæ rariss; etiam in Portu S[to].—In insulis Canariensibus invenit cl. P. B. Webb, arm. In Herbario Banksiano sunt exempla, a cl. Masson olim in "Insula Desertas prope Madeiram" et "Promont. S. Lourenço" sine floribus fructuve lecta, monente amico J. I. Bennett. Hieme (Dec. Jan.) floret. Calyx simplex; nec spinescens, nec membranaceo-dilatatus.

POLYGONEÆ.

22. *Rumex maderensis*, nob.

R. paniculis amplis, aphyllis, multifloris: floribus hermaphroditis: valvis nudis, intigerrimis, orbiculatis, reticulato-membranaceis: verticillis paucifloris: foliis hastatis, acutis, succulentis: caule frutescente.

α. *glauca.*

β. *virescens.*

Hab. in rupibus Maderæ ubique, vulgatiss.

Suffrutescens, 1—2-pedalis. Flores mense Junii rupes maritimas excelsiores colore pulchre lateritio ornantes. A. *R. scutato* L. distinctam primus admonuit Cl. Lemann.

PLUMBAGINEÆ.

23. *Armeria maderensis*, nob.

A. caule suffruticoso, simpliciusculo, humili, parum ramoso, ramisque brevissimis, subcespitosis: foliis fasciculatis, gramineis, latiusculis, lanceolato-linearibus, acuminatis, planiusculis, lævibus, lucidis, glabris, quinquenerviis, integerrimis: scapis teretibus, lævibus, glabris, foliis 3—4-plove longioribus: floribus pedicellatis, fastigiato-capitulatis; capitulis oblongis, truncatis, pedunculatis, umbellatis, bracteolatis; umbella laxiuscula, bracteata: bracteis lanceolatis; exterioribus acuminatis patentibus recurvisque; interioribus pellucido-membranaceo-marginatis: bracteolis magnis oblongo-ovalibus, latis, imbricatis, albo-membranaceis, obtusissimis, apice eroso-dentatis crenatisve, glabris; exteriore basi tantum pubescente: bracteis bracteolisque floribus brevioribus: pedunculis teretibus, scapique vagina striata, granulatis, glabris: pedicellis teretibus, glabris: calycis laciniis brevissimis, latis, acutis; costis pilosis: petalis retusis, subemarginatis: stylis (5) basi pilosis: ovario glabro.

Hab. in cacuminibus rupibusque præruptis montium excelsiorum Maderæ.

Armeriæ plantagineæ vel potius *A. scorzoneræfoliæ* Willd. Enum. et *Staticei plantagineæ* All. Fl. Pedem. et Lam. et D. C. Fl. Française videtur proxima. Sed in genere tot nubibus offuscato, confusionem minus nomine novo quam falso augitur.

Leaves bright, shining, rather dark green; not at all glaucous. Scapes from one to two feet high. Flowers rather large, deep rose-colour.

LABIATÆ.

24. *Sideritis candicans*, Ait.

α. *longifolia.* (S. candicans, Auct.)

β. *crassifolia;* foliis incrassatis, subcoriaceo-tomentosis, rotundato-ovatis, abbreviatis, obtusissimis: labio superiore plerumque integro.

Hab. in rupibus apricis maritimis Maderæ et Insularum Desertarum.

25. *Prasium medium*, nob.

P. hirsuto-pubescens: foliis ovato-oblongis, ovalibusque, acutis, basi in petiolum attenuatis, crenato-dentatis: dentibus calycis ovatis, acutis, mucronatis: corolla filamentis styloque glabris.

Hab. in Maderæ rupibus maritimis.

P. majus L. preserves all its characters in Maderæ; and the present plant appears a genuine species, though in some sort intermediate between *P. majus* and *minus;* having the mucronate calyxes of the former, and hairiness of the latter. In the shape of the leaves it seems to differ from both.

26. *Melissa rotundifolia*, Sol. MSS, Herb. Banks! et Von Buch.

M. hirsuto-pubescens: cymis subsimplicibus, laxis, paucifloris, folia ovato-rotundata, subserrata, superantibus: caule debili, ascendente, erectiusculo, hirsuto; basi lignoso, frutescente.

Thymus Calaminthoides, Reichb. (in Holl's List) Hook. Bot. Misc. 2d Series I. pp. 19, 38.

Planto 1—2-pedalis, perennis, suffruticulosa. Folia parva, subsemipollicaria, obsolete et remote serrata sive crenata. Cymi vix decompositi, 2—5-flores, floribusque plerumque folia excedentibus; pedunculis folii fere longitudine. Bracteæ minimæ, lineares Flores conspicui, majusculi, pallide purpurei, rarissime albi, pubescentes. Calycis subcylindrici, corolla fere triplo brevioris, fauce pilis inclusis, haud prominentibus, clauso; dentibus ciliato-hirtis; tribus superioribus ovatis, acutis; duobus inferioribus longioribus, lineari-acuminatis, porrectis, elongatis. Semina ovoidea vel globosa, sæpe triquetra, lævia, obsolete et minutissime punctulata.

Odor totius plantæ gravis, acris, subingratus, quodammodo *Menthæ Pulegii* L.

Cymi foliis longiores, ut in *Thymo Nepeta* L; subsimplices, pauciflores, ut in *T. Calamintha* L, cui certæ proxima. Pro mera varietate me diu habentem, summa tandem vincit cl. Solandri auctoritas. Characteres sane plantæ, per totam Insulam pervulgatissimæ, nunquam variantes inveni.

SOLANEÆ.

27. *Nycterium triphyllum*, nob.

N. herbaceum, inerme, viscoso-pubescens: ramis angulatis petiolisque pedicellisque subtomentosis: foliis utrinque molliter viscoso-pubescentibus, quibusdam simplicibus, plerisque ternato pinnatifidis; foliolis cordatis vel oblongo-ovatis, repando-dentatis, integriusculis: racemis sparsis, folio multum brevioribus, corymbosis; pedicellis secundis, nutantibus: bacca calyce dilatato-foliaceo tecto.

Hab. in Madera rariss: In parte Septentrionali prope *S. Vicente* secus vias invenit Car. Lemann M. D.: in orientali prope *Portella* serius detexit Lippold. Ex insulis Canariensibus, in Herbario Lemanniano, siccum quoque vidi.

Flores colore et magnitudine *S. tuberosi* L. Anthera rima longitudinali dehiscentia; 2—3-imis longe productis, cornutis. Bacca cerasi magnitudine, aurantiaca, globosa.

CONVOLVULACEÆ.

28. *Convolvulus solanifolius*, Prim. in Trans. Cam. Phil. Soc. supra IV. I. p. 22. No. 35.

In the first place, the discovery of the flowers of this rarest of Maderan plants, authorizes the removal of the mark of doubt before affixed to the generic name. They are white, and truly those of a *Convolvulus*. Hence therefore, no change of the specific name would be necessary on account of *Ipomæa solanifolia* L. But, though unfortunately I am unable to decide the matter by reference to the *Botanical Register*, t. 133, I have very little doubt the plant will prove identical with *Convolvulus Massoni* Dietrich (*C. suffruticosus* Ait., non Desf.)

JASMINACEÆ.

29. *Olea Europæa*, L. var.

Maderensis: foliis lineari-oblongis, angustis, mucronatis, integerrimis, utrinque subconcoloribus s. inferne nudiusculis: drupis subglobosis, purpurascentibus, demum nigris.

Hab. in rupibus apricis Maderæ, præsertim maritimis.

Specimen in Herbario Banksiano, a cl. Masson olim lectum, sub nomine *O. Europææ* a cl. R. Brown in "Von Buch's Catalogue," *O. glabellæ* Herb. Banks. (*O. exasperatæ* Jacq. Hort. Schoenbr. III. t. 1.) "valde simile" dicitur. Panicula vero terminali, ramisque tuberculatis hæc satis differe videtur.

Drupes about the size and shape of a small marble, half an inch in diameter, of a deep shining black, by no means constantly "crowned with the persistent style." Skin and flesh very thin and dry, rather bitter to the taste. Stone very large.

In drying, the plant gives out abundantly a whitish, powdery, granulated, sweetish substance (Manna?); resembling fine powder-sugar.

CAMPANULACEÆ.

30. *Prismatocarpus scaber*, nob.

P. scabro-pubescens: caule erecto, stricto, simplici, vel imo basi rarius ramoso; foliis radicalibus petiolatis, obovato-spathulatis oblongisve; superioribus sessilibus, lanceolatis; omnibus margine undulato-crenatis; summis integriusculis: floribus sessilibus, solitariis, per totum caulem axillaribus; sepalis linearibus, patentibus, corollam subæquantibus.

Hab. in Maderæ declivibus saxosis, rupestribus. Primus detexit amicus et plantarum indagator oculatissimus ac indefessus Car. Lemann, M. D.

Herba annua, subpedalis. Corolla, præsertim terminalis majuscula, subconspicua, violaceo-cærulea.

COMPOSITÆ.

31. *Senecio incrassatus*, nob.

S. herbacea, erecta, glaberrima, nitida: foliis auriculato-amplexicaulibus, carnosis, profunde sinuato-pinnatifidis; lobis integriusculis, æqualibus, remotis, obtusissimis: caule ramisque acutissime angulatis: floribus arcte corymbosis; pedicellis abbreviatis, sursum valde incrassatis, multibracteatis; bracteis adpressis, basi tumidis carnosis, apice marcidis, nigris; radio patente 7—8-ligulato: seminibus pubescentibus.

S. crassifolius W. var. D. C. in litt. 1834.

Hab. in collibus apricis aridis maritimis Maderæ rarior.

Herba annua, subspithamæa, ramosa, corymbo amplo, conspicuo, multifloro: sed magnitudine pro situ, aquæ copia, &c. valde varians; sc. caule sæpe 2—3-pollicari tantum, simplici, uni-pauci-floro. Flores aurei, conspicui, sat magni, diametro semipollicari, radio marcescenti modo revoluto.

32. *Helichrysum? obconicum*, D. C.

In the Botanical Miscellany, 2d Series, Vol. I. p. 35, I have very erroneously spoken of the common Maderan plant called by Holl and Reichenbach *Antennaria leucophylla*, and abounding on the rocky sea-cliffs and islets of these shores, as if it were *Gnaphalium crassifolium* L. Neither must it be confounded, I am advised by the Chevalier De Candolle, with the European (Majorca) plant, *Gn. crassifolium* Lam.: nor yet with a third species similarly named, *Gn. crassifolium* Willd. Its real affinity, the Professor writes, is with a fourth plant, which has also been confounded with *Gn. crassifolium* L., viz. *Gn. ovatum* Desf.; and with this, he is disposed to unite it into a genus, or at least a group, co-ordinate with the other Gnaphalian sections, or subgenera. That it is no true *Antennaria* either in characters or habit is most certain. Speaking strictly, it is perhaps intermediate between *Helichrysum* and *Gnaphalium*: though I could be well content to refer it simply to the former genus.

33. *Chrysanthemum dissectum*, nob.

C. fruticosum, glaberrimum: foliis profunde pinnatifidis; pinnis remotis, parallelis, æqualibus, linearibus, rectis, argute inciso-dentatis; laciniis omnibus acutis: floribus subsolitariis, vix corymbosis s. corymbo irregulari, paucifloro.

C. grandiflorum (W.) Spr. Syst iii. 584. No. 6?

Hab. in rupe quadam excelsa maritima Maderæ, Cabo Giram dicta.

Species habitu seminibusque *C. pinnatifidi* L. in Madera ubique obvii, sed distinctissima, floribusque minoribus.

34. *Calendula maderensis*, nob.

C. biennis subperennansve, basi suffrutescens, ramosa, viscosa: foliis semiamplexicaulibus, obovato-oblongis, repando-dentatis; junioribus ramulisque tomentosis: seminibus arcte inflexis, muricatis; exterioribus cymboideis, late triquetro-alatis; alis duabus dorsalibus expansis, inciso-dentatis.

C. maritima, nob. Bot. Misc. New Series, I. p. 36; haud Gussone.

C. amplexifolia, Reichb. in Holl's List?

Hab. in littore, rupibusque maritimis oræ præsertim septentrionalis Maderæ.

Flores majusculi, crocei, ligulis paucioribus, longioribus quam in *C. officinali* L. aut *arvensi* L.

Nomen mutavi ob *C. maritimam*, Gussone.

35. *Andryala robusta*, nob.

A. foliis crassissime molliterque incano-tomentosis, subintegerrimis; caulinis abbreviatis, lanceolato-ovatis: caule robusto, stricto, paniculæque ramis pedicellisque abbreviatis, anthodiisque magnis, hemisphæricis vel globoso-capitatis, densissime fulvo-glandulosis, villosis.

A. varia β. nob. MSS. olim.

Hab. in rupibus maritimis, præsertim oræ septentrionalis Maderæ et Insularum Desertarum: necnon in cacuminibus montium Insulæ Portûs S^ti^.

De *Andryalis* Maderensibus, ad unam speciem (*A. variam* nob.) olim redactis, diu vacillantem, formas tres insigniores (*A. variæ* α, β, γ. nob. olim) pro totidem speciebus melius habendas, observationes protractiores tandem suaserunt. Harum forma typica est *A. varia* nob (*A. varia*, α. nob. olim); cujus duæ adsunt varietates, α) *foliis integriusculis;* planta ubique obvia, vulgatissima, montana: et β) *foliis plus minus sinuato-pinnatifidis;* indigena, montana, rarior; culta in Europæ hortis subfrequentior (*A. cheiranthifolia* Herit.): hâc ad *A. crithmifoliam* Ait. (*A. variam* γ. nob. olim), illâ ad *A. robustam* nob. (*A. variam* β. nob. olim), utraque maritima, secedente.

36. *Carduus? squarrosus*, D. C. in litt^s^.

C. foliis decurrentibus, obovato-oblongis, indivisis, grosse serratis, spinellosis, subtus albo-tomentosis; floribus aggregato-glomeratis; anthodii squamis inermibus, scariosis, squarrosis, apice reflexis: pappo clavulato.

Hab. rariss. in Convallibus interioribus Maderæ.

Planta elatior, conspicua, floribus albis; a *C. clavulato* Link, planta Canariensi affinitate proxima, foliis indivisis, nec semipinnatifidis; squamisque anthodii squarroso-reflexis nec erectis, monente cl. De Candolle, distincta.

Plantas e seminibus a me ipso olim a Madera A. D. 1829, ad amicum Rev. M. J. Berkeley Angliam missis, in horto Barclayano ortas, cl. et am. Alph. De Candolle fil, vidit: ex quibus desiccatis, species characteribus plantæ indigenæ prorsus congruentibus a patre illustri in litteris stabilita est.

C. clavulatus Link verus in Maderæ nullibi obvenit.

37. *Cynara horrida*, Ait.

C. caule brevi, simplici, unifloro, incano, foliato: foliis pinnatifidis, ferocissime spinosis, subtus incanis: caulinis haud decurrentibus; pinnis distinctis, subremotis, angustis, longe acuminatis s. apice caudatis, inciso-laceris, spinis baseos pinnarum laciniarumque bi-pluri-aggregatis, basi connatis: anthodii squamis erectis, angusto-lanceolatis vel ensiformibus, acuminato-spinosis, rectis.

C. horrida, (Ait:) Spr. Syst. iii. 369. No. 3.

Hab. in collibus apricis graminosis Portûs S[ti] sat frequens, A. D. 1828: in Maderæ Promontorio Ponta S. Lourenço dicto solo rariss. nuperrime (A. D. 1837) invenit Lippold.

Radix magnus, crassus, perennis, cum capitulis apud accolas Portûs S[ti] edulis. Caulis subpedalis, strictus, firmus, erectus, rotundus. Capitulum sat magnum, sphæroideum, diametro fere bipollicari. Flosculi "cærulei" (*Lippold*); desiccati in ipso exemplo Lippoldiano purpurei; a me nunquam rite expansi visi, sed albi ab incolis Portûs S[ti] dicti.

Folia elegantissima; sed spinis rigidis, tenuissimis, acutissimis, numerosissimis pallide flavescentibus fulvisve horridissima tectis: radicalibus rosaceo-confertis, subpedalibus.

Rubiaceæ.

38. *Galium productum*, nob. Prim. p. 29. No. 50.

An a *G. cinereo* All., Sm., D. C. satis distincta? Confer etiam *G. erectum* Huds., Sm., D. C.

39. *Galium geminiflorum*, nob.

G. pumilum: caulibus tetragonis, gracilibus, lævibus, simpliciusculis, diffuso-erectis, dichotomis, subinermibus vel sparsim aculeolatis, aculeis subdeflexis: foliis 4—6-verticillatis, ovato-lanceolatis, cuspidatis, superne margineque aculeatis, aculeis antrorsum spectantibus: pedunculis geminatis, axillaribus, simplicibus, rarissime bifidis, unifloris, abbreviatis, s. folia vix superantibus: fructibus densissime uncinato-setosis hispidis.

Hab. in summis cacuminibus Insulæ Portûs S[ti] tantum.

Planta rarissima, parva, inconspicua, tenera, mox evanescens; *G. setaceo* Lam., Desf., D. C. (*G. capillari* Cav.) affinis.

UMBELLIFERÆ.

40. *Daucus neglectus*, nob.

D. caule superne aspero: foliis bi-tri-pinnatisectis, hirsutiusculis: foliolis (segmentis) omnibus ovatis, incisis; laciniis (segmentis ultimis) lanceolatis, acutis, cuspidatis: bracteis bipinnatifidis, umbella brevioribus; bracteolis pinnatifidis, umbellulas æquantibus: umbellæ amplæ radiis valde inæqualibus; umbellularum floribus externis radiantibus: fructus ovalis aculeis ejus latitudinem æquantibus, ad basin distinctis, apice glochidiatis.—

α. *asperocaulon:* hispidus; caule toto aspero, inferne præsertim retrorsum strigoso: foliis hirsutis.

β. *leiocaulon:* glabriusculus; caule fere nudo.

Hab. in saxosis apricis rupestribus regionis inferioris Maderæ.

Obs. *Dauco hispido* Desf. Fl. Atl. 1. 243. t. 63, præsertim foliorum habitu affinis. Laciniis vero foliorum acutis, imo cuspidatis, nec "obtusis;" bracteis (involucri foliolis) bipinnatifidis, umbellæ amplæ radiis exterioribus longe productis, floribusque albidis, anisopetalis s. exteriorum petalis extimis magnis, dilatatis, nec omnibus "minutis, sub-æqualibus, pallide flavis," mericarpiisque duplo majoribus, ovalibus, planiusculis nec "semiteretibus," satis differre videtur. Foliolis (segmentis) foliorum omnium conformibus a *D. maximo* Desf. distincta.

Planta annua; caule 1—2-pedali, erecto, parum ramoso. Flos umbellæ centralis magnus, carnosus, atropurpureus, abortivus. Umbellæ multiradiatæ; radiis defloratis incurvis.

41. *Melanoselinum decipiens*, (Hoffm.) D. C.

Hab. in convallibus umbrosis oræ septentrionalis Maderæ, ad altitudinem 2—3000 pedum.

The native country of this fine umbelliferous plant was unknown, till I discovered it, in the Autumn of the year 1829, growing plentifully high up the main, or central branch, of the Ribeira de Saõ Jorge; both among the rocks and stones, forming the bed of the ravine, and up the steep shady banks on each side. It has since occurred in others of the shady ravines of the North.

42. ***Bupleurum salicifolium***, Sol. MSS.

B. fruticosum, erectum, ramosum; ramis lævigatis, elongatis, gracilibus, inferne nudis: foliis angustis, lineari-lanceolatis, acuminatis, utrinque attenuatis, planis, coriaceis, pallide glaucis, obliquis, sessilibus, integerrimis, multinerviis: umbellis 5—10-radiatis; bracteis 4—5, reflexis, brevibus, lanceolatis; bracteolis 4—5 conformibus, umbellula multo brevioribus.

B. salicifolium, Sol. MSS. et Herb. Banks. (auct. J. I. Bennett.)

Hab. in rupibus convallium Maderæ.

A *B. gibraltarico* foliis multi- (nec uni-)nerviis prima fronte differt. *B. plantagineum* Desf., cui vero species nostra propior, foliis pro longitudine multo latioribus, obtusiusculis cum mucrone (nec acuminatis), concavis (nec planis), bracteisque adpressis (nec, ut in nostra æque ac in *B. gibraltarico*, reflexis) a *B. salicifolio* nob. satis superque distinctum videtur.

Frutex elegantior; ramis in rupibus declivibus sæpe pendentibus.

43. ***Bunium brevifolium***, nob.

B. (*Conopodium* D. C.) glabrum: radice subgloboso: caule simplici, striato, tereti, glauco: foliis petiolatis, rigidis, deltoideis, abbreviatis, omnibus 2—3-pinnatisectis; laciniis planis, tenuibus, remotis, pectinatis, oppositis; ultimis brevissimis, lineari-lanceolatis, acutis, integerrimis; vaginis amplis, elongatis, striatis, cum petiolo erecto-patentibus: bracteis bracteolisque nullis: stylopodiis tumidis, planatis; stylis rectis, invicem adpressis, demum subdivergentibus.

Hab. in regionibus excelsioribus graminosis montium Maderæ; jam rarissima, ob puerorum et porcorum prædationes.

A pube Maderensi monticolo sc in monte "*Pico Grande*" dicto degente, cui tubera escam gratissimam præbent, "Norsa" dicitur. Planta vero longe aliena, sc. *Tamnus Norsa* nob., ab incolis Portûs Caurum versus (Porto Moniz) etiam "Norsa" dicitur.

PARONYCHIEÆ.

44. ***Herniaria flavescens***, nob.

H. annua, herbacea, hirsuta, humifusa, flavescens: ramis ramulisque distichis, horizontalibus, cespitosis: foliis oblongo-lanceolatis, ciliatis: glomerulis frequentissimis, axillaribus, multifloris.

Hab. in collibus maritimis Maderæ rariss: Portûs S^{ti} frequentior.

Radix tenuis, subtenax, tortuosa, parum ramosa, annua. Caules humifusi, ramis ramulisque flabelliformibus s. concinne et creberrime distichis, horizontalibus, undique terræ arctissime per totam longitudinem adpressis; cespitem diametro 3—6-pollicari densam, pallide viridiflavescentem, hirsutam formantibus.

ROSACEÆ.

45. *Poterium megacarpon*, nob.

P. herbaceum: caulibus angulosis, deorsum hirsutis: rhachi foliorum foliolisque subtus hirtiusculis: foliolis inciso-dentatis; inferioribus subrotundis; superioribus ovalibus oblongisve: capitulorum solitariorum floribus omnibus hermaphroditis; inferioribus subabortientibus: filamentis abbreviatis: fructibus maximis, rugosis.

Hab. in collibus apricis graminosis saxosisque præsertim maritimis regionis inferioris Maderæ.

Habitus *P. Sanguisorbæ* L.: sed humilior, subpedalis, totaque insipida, inodora. Capitula longe pedunculata. Filamenta brevia, albida, s. pallide flavescentia, nec purpurascentia. Stigmata brevia, parva, læte coccinea. Sepala plerumque 4, lata, ovalia, magna. glabra, persistentia. Fructus quam in *P. Sanguisorba* L. duplo triplove major; pericarpio fungoso, laminoso-scrobiculato.

LEGUMINOSÆ.

46. *Vicia conspicua*, nob.

V. tenuis, gracilis; caulibus foliisque cirrosis tenellis, glabriusculis: foliolis subdenis (8—12), angustis, oblongis, mucronatis; foliorum inferiorum spathulatis vel obcordatis; summorum linearibus: stipulis inconspicuis, angustis, semisagittatis, paucidentatis; laciniis acuminatis, subtus ustulato-notatis: floribus subsessilibus, conspicuis, calyce 3—4-plo longioribus; inferioribus solitariis, summis 3—4-nis, plerisque (intermediis) binis: laciniis calycinis duobus superioribus longioribus, angusto-acuminatis tubo subæqualibus: leguminibus subcylindricis, vix compressis, rectis, angustis, puberulis, 9—12-spermis: seminibus parvis, sub-globosis, vix compressiusculis, fuscis, atro marmoratis, glabris.

Hab. ubique vulgatiss. inter vineta, segetes, graminosaque montana Maderæ.

Constantly distinct, and easily distinguished by its large and handsome flowers, delicate smooth foliage, fine narrow leaflets, and graceful habit, from *V. sativa* L., which preserves all its characters and much coarser habit in Madera. Neither is it to be confounded with *V. angustifolia* "Roth. and Willd." (*V. sativa* β. Sm. in E. Fl.); nor again with *V. Bobartii* Forst. (*V. angustifolia* Sm. in E. Fl. non Roth. et Willd.) It differs from the former in the much narrower leaflets of the upper, and inversely heart-shaped ones of the lower leaves: but more strikingly, in the much larger and conspicuous bright rich purple (approaching to crimson) flowers. In both these points it comes much nearer *V. Bobartii:* but still the flowers are larger; and those only which open first, at the beginning of the season, are solitary: later, they are two or three, and even sometimes quite the uppermost are four together: the leaflets are more numerous and smoother; and the whole plant is larger, with the stems from one or two to three feet long.

47. *Vicia capreolata*, nob.

V. subpubescens, ramis gracilibus, elongatis, filiformibus: foliis cirrosis; foliolis 5—11; lineari-olongis, remotis: stipulis parvis, lineari-oblongis, angustis, semi-sagittatis, simplicibusve, acuminatis vel apice bifidis, coloratis, marcescentibus: pedunculis submultifloris; floriferis folio æquantibus; fructiferis duplo longioribus: calyce puberulo $\frac{2}{3}$; dentibus duobus superioribus inter se arcuato-incurvis; tribus inferioribus longioribus, acuminatis: floribus secundis, subdenis: leguminibus oblongis, subfalcatis, glaberimis, obsolete reticulatis, 4—5-spermis: seminibus 3—4, subtetrahedris, oblongis, subcompressis, glabris, nigrescentibus; funiculo magno hiloque longo.

Hab. in rupibus umbrosis convallium Maderæ. Florentem primus detexit. cl. Car. Lemann, M. D.

Flores eorum *V. Craccæ* L. fere magnitudine, ochroleuci. Videtur *V. ochroleucæ* Ten. affinis. An *V. parviflora* Cav., Brouss?

48. *Biserrula Pelecinus* L.

a. pubescens.—B. Pelecinus L. D. C. Prodr. ii. 307.

β. *glabra*, nob.

Hab. ambæ varietates in apricis maritimis incultis Maderæ et Portûs S^ti^, una nascentes: sed β. nostra, quamvis prima fronte valde distincta, nullo modo nisi glabritie differt; an vero species?

49. *Lotus pisifolius*, nob.

L. herbaceus, glaucus, glaber: caule flexuoso ramisque divaricatis, crassis, fistulosis, firmis, erectis vel subdeclinatis: foliolis obovatis stipulisque subcordatis maximis: capitulis multifloris, longe pedunculatis: calycibus campanulatis; laciniis ciliato-pilosis, æstivatione stellatis: leguminibus......seminibus......

Hab. in humidis graminosis Montis excelsi Pico Grande dicti: semel tantum lecta.

An varietas luxurians monstrosa *L. majoris* Sm. e solo pinguiore orta? Sed habitu, colore, magnitudine toto cœlo differt. Foliola stipilæque 1—2-pollicariæ; juniora ad margines pilis raris sparsa. Pedunculi 4—6-pollicares. Capituli 12—15-flores, basi folio ternato bracteati. Alabastra floresque ut in *L. majore* Sm. Tota planta eximie glauca 2—3-pedalis, ramis flabellatim expansis, suberecta.

50. *Lotus macranthus*, nob.

L. subcinereo-glaucescens, sericeo-pubescens: radice annua aut bienni; caule basi lignoso, frutescente: ramis diffuso-prostratis, patulisve: stipulis subsessilibus, transverse ovalibus; foliolisque rotundato-obovatis minimis: floribus solitariis, versicoloribus, breviter pedicellatis: leguminibus lomentaceis, cylindricis, longissimis, rectis, glaberrimis, polyspermis: seminibus 30—40 minutis, orbicularibus, compressis, lævibus, glabris, fuscis.

Hab. in apricis maritimis Maderæ et Portûs S^ti^ rarior.

Habitus *L. glauci* Ait; sed minus incana. Pubescentia totius plantæ brevissima, inconspicua, arctissime adpressa. Flores maximi, pollicares, cernui, pallide citrino-virescentes, mox fusco-purpurei; carinæ apice semper atro-purpureo. Legumina rectissima, subbipollicaria. Species notabilior, distinctissima.

51. *Lotus divaricatus*, Sol. MSS.

L. annuus, pilosus, subcespitosus: caulibus prostratis, intricatis, numerosis, elongatis, flexuosis, ramosis, pallidis: foliolis obovato-cuneatis stipulisque ovatis subcordatis acutis: capitulis trifloris: leguminibus turgidis, crassis, brevibus.

L. divaricatus, Sol. MSS. et Herb. Banks.

Hab. incultis graminosisque montanis Maderæ sat frequens.

A. Loto diffuso, Sol., in Madera æque vulgari, capitulis normaliter tri- nec bi-floris, floribus aurantiacis, nec citrinis neque flavis, leguminibusque multo brevioribus, pinguibus, crassis, statim dignoscitur.

52. *Medicago pulchella*, nob.

M. subsericeo-pubescens, cinerascens: caulibus patulis prostratisve gracilibus: foliolis obcordatis vel obovatis, basi cuneatis, integris, sursum subdentatis, apiceque argute tridentato: stipulis integriusculis, ovato-lanceolatis, inferioribus acuminatis: pedunculis sub-bifloris, abbreviatis: leguminibus cochleatis, villosulis, parvis, inermibus, globosis; cyclis 3—4, angustissimis; margine simplici, angusto, lineari, utrinque costis prominentibus, oblique deflexo-arcuatis, grosse dentato: seminibus compressis, reniformibus, flavis.

Hab. in collibus apricis saxosisque maritimis Maderæ et Portûs S[ti] rarior.

53. *Ononis micrantha*, nob.

O. herbacea, annua, prostrata procumbensve foliosa, glanduloso-pubescens: foliis (præter summa) trifoliolatis; foliolis ovalibus, argute serratis: stipulis amplis, foliaceis, oblongo-ovatis, mucronatis, subserrulatis integrisve: floribus (purpureis) inconspicuis, sparsis, axillaribus, subsessilibus, solitariis, folio brevioribus: calycibus amplis, foliaceis, post anthesin dilatatis; laciniis æquis, acuminatis, integris, corollam subæquantibus, unicostatis: legumine erecto, turgido, ovali, brevi sc. laciniis calycinis breviore, dispermo: seminibus rufo nigroque marmoratis, compresso-rotundatis, minutissime granulato-scabris s. verruculatis.

"O. arthropodia Br. Fl. Lus. 2. 94," Herb. Banks. quoad exemplar unicum Gibraltaricum, a cl. Broussonet lectum aut communicatum!—haud Broteri in Fl. Lus. l. c.

An. O. parviflora, Brot. Fl. Lus. 2. p. 96?

Hab. in collibus apricis maritimis Maderæ et Portus S^{ti}.

Ab. O. *villosissima* Desf. Fl. Att. 2. p. 147. T. 192. vix nisi floribus sessilibus, sparsis, nec confertis, nec racemosis, stipulis calycibusque magnis, hirsutieque parciore differt.

54. *Ononis dentata* Sol. (Prim. p. 34. No. 59. t. 4.)

α. tridentata: laciniis calycinis 4 superioribus apice plerumque tridentatis.

O. dentata Sol. MSS. nob. l. c.

Hab. in Portu S^{to} et Insulis Desertis.

β. simplex: laciniis calycinis simplicibus, conformibus, acuminatis.

Hab. in Promontorio Maderæ *Ponta Saõ Lourenço* dicto, Dr C. Lemann; et in Insulis Desertis cum *α* mixta, Dr Lippold.

Quoad cetera, plantæ omnino conveniunt: quoad imo calycis lacinias, status intermedii, sc. lacinia uni-bi-dentata facile adsunt.

HYPERICINEÆ.

55. *Hypericum nubigenum*, nob. in Bot. Misc. 2d Series, I. p. 43.

H. glabrum: caulibus simplicibus, erectis, strictis, ancipitibus, basi suffrutescentibus: foliis epunctatis, erectis, lineari-oblongis, obtusissimis vel retusis, amplexicaulibus, margine revolutis: panicula terminali, corymbosa: sepalis ovatis, æqualibus, dentato-glandulosis petalisque nigropunctatis: floribus trigynis; antheris epunctatis s. eglandulosis.

H. angustifolium, Primit. p. 35. No. 61; haud Lam.

Hab. in Maderæ editioribus.

***H.** angustifolium* Lam. jam adest: ideoque nomen mutetur.

MALVACEÆ.

56. *Sidæ maderensis*, nob Prim. p. 35. No. 62.

Sidæ canariensis W. mera varietas statusve videtur: qualis *Sidæ* carpinifoliæ L. est verosimiliter *Sidæ carpinioides*, D. C.

CARYOPHYLLEÆ.

57. *Cerastium vagans*, nob.

C. viscoso-pubescens, perennis: caulibus diffusis, deorsum suffrutescentibus, apice ascendentibus, paniculisque pedicellisque calycibusque dense fulvo-glandulosis: foliis angustis, sublanceolatis, acuminatis: petalis bifidis, sepalis subduplo longioribus, conspicuis: capsulis ovatis, calyce sublongioribus.

α. *fulva;* dense fulvo-tomentosa.

β. *subnuda;* glabriuscula; foliis inferioribus glabris.

Hab. rarior sparsimque in rupibus siccis excelsioribus Maderæ.

Habitus quodammodo *Stellariæ*. Pedicelli semper erecti. Capsulæ fructiferæ calyce tectæ, abbreviatæ, obovatæ vel ovales.

58. *Arenaria serpyllifolia* L.

γ. *depauperata;* viscida, glanduloso-pubescens.

Hab. in Portu Sancto: etiam in cacuminibus summis Maderæ.

59. *Silene filiformis*, nob.

S. annua, pubescens: caule erecto: ramis divaricatis, filiformibus, gracilibus, strictis; internodiis sæpe viscidis: foliis angusto-lanceolatis, acutis; infimis obtusiusculis; summis raris linearibus: floribus solitariis, inconspicuis pedunculatis; pedunculis glabriusculis; calycibus oblongis; petalis linearibus, capsulisque cylindricis, sepala subæquantibus: anthophoro capsulæ dimidium vix æquante.

S. inaperta, Hort. Reid. quoad saltem stirpem Maderensem: haud Linn.

Hab. in sterilibus apricis, alveisque siccis convallium Maderæ.

Flores inaperti: petala viridi-fusca. *S. inapertæ* L. proxima; sat vero distincta.

60. *Silene ignobilis*, nob.

S. annua, glabriuscula, dichotome ramosa, erecta: foliis inferioribus subciliolatis, spathulatis, superioribus lanceolatis: floribus solitariis, inconspicuis, pedunculatis; calycibus oblongis, haud inflatis, mox ventricosiusculis basique coarctatis, reticulatis; petalis inconspicuis, calycem vix superantibus: capsulis ventricosis, doliiformibus, calycem æquantibus; anthophoro brevissimo.

Hab. inter segetes Maderæ rariss.

Viridis, vix glaucescens. Variat plus minus velutino-pubescens; plerumque fere glabra. Petala apice purpurascentia.

61. *Silene inflata*, Sm., D. C., &c.

Var. *intricata* nob.; vix glaucescens; caulibus ramosissimis, dense intricatis, elongatis, pendulis, basi suffrutescentibus.

Hab. in rupibus excelsis declivibus Convallium Maderæ.

FRANKENIACEÆ.

62. *Frankenia cespitosa*, nob.

F. caulibus fruticulosis, ramosissimis, densissime cespitosis, humilibus, humifusis, calycibusque basi velutino-pubescentibus: foliis sessilibus, basi connatis, linearibus, glabris, margine revolutis, basi breviter et parce ciliolatis: floribus in capitulos terminales congestis, subcymosis, foliis multo longioribus.

Hab. in collibus maritimis sterilibus aridisque Promontorii Ponta S. Lourenço Maderæ; etium Portûs S^{ti}.

F. ericifoliæ C. Sm., necnon *F. corymbosæ* Desf. nimis forsan affinis.

VIOLARIEÆ.

63. *Viola paradoxa*, nob.

V. suffruticosa, e basi ramosa; ramis subproductis, elongatis, simpliciusculis; inferne nudis, stipulisque simplicibus linearibus integris minutis sæpe obsoletis, petiolisque elongatis marginatis ternato-fasciculatis, foliisque rotundato-spathulatis crenatis basi abruptis cordatisve, glaberrimis: foliis summis cuneato-elongatis, in petiolum attenuatis, apice subtridentatis, petiolisque caulibusque subpuberulis: sepalis oblongis integris, bracteisque pedicelloque pubescentibus: calcare obtuso, calyce longiore; nectario......... capsula obsolete hexagona, glabra; seminibus pallide flavescentibus, paucis (15—20), ovatis.

Hab. rariss. in summis cacuminibus montium excelsiorum Maderæ, in fissuris rupium. Invenit cl. Car Lemann, M.D.

Obs. Cum *V. calcarata* L. conferenda. Flos aureo-flavus. Stylus ab apice ad basin attenuatus. Stigma urceolatum, utrinque fasciculato-pilosum, ore magno, expanso dilatato, inferne in labellum producto. Folia ad apices ramorum sterilium conferta. Pedunculi solitarii, axillares, subpollicares. Capsulæ abbreviatæ, obtusæ. Semina, præter colorem, fere ut in *V. tricolore* L.

CRUCIFERÆ.

64. *Sinapidendron salicifolium*, Prim. p. 37. No. 65.

Syn. Sinapis angustifolia, D. C. Prodr. I. 220.

Hab. in rupe quadam excelsa maritima, "*Cabo Giram*" dicta, prope vicum *Camera de lobos* Maderæ, nuperrime ab amico Rev°. M. Tucker, botanophilo vel oculatissimo, detecta. Species genuina videtur.

65. *Matthiola maderensis*, nob.

M. biennis: caule herbaceo, erecto, elato, ramoso: foliis oblongis, integerrimis, incano-tomentosis; radicalibus densissime rosaceo-confertis: siliquis compressis, glanduloso-muricatis.

Hab. in rupibus maritimis Maderæ et Portus S^{ti}. ubique vulg.

Flores pallide violacei, vespere præsertim odori, rarissime albi. Species intermedia, habitu foliisque *M. incanæ* R. Br.; siliquis, 3—5 poll. longis, *M. sinuatæ*, ejusd.

MADERA, *October* 1837.

SUPPLEMENTUM.

CHARACEÆ.

66. *Chara atrovirens*, nob.

C. atroviridis, pellucida, lucida, gracilis, fœtens: caule ramisque tenacibus, tenuibus, flexilibus, hinc inde articulatis superne sparsim minutissime retrorsum papilloso-spinellosis, contorto-striatis: ramulis verticillatis, 5—6-articulatis; articulis inferioribus striatis; summis simplicibus, bracteisque 2—4 inarticulatis, cylindrico-setaceis, nucula triplo quadruplove longioribus, lævibus, haud striatis, glabris.

Hab. in rivulis aquarum fluentium Madera rariss.

C. vulgari Ag. proxima. Differt colore, scabritieque ramorum, radicellos nascentes deflexos, sparsos æmulante.

MADERA, *May* 1838.

XXVIII. *On a New Correction in the Construction of the Double Achromatic Object-glass.* By RICHARD POTTER, ESQ. B.A. *Queens' College.*

[Read *April* 30, 1838.]

THE achromatism of the compound object-glass of telescopes has never, that I am aware of, been investigated otherwise than for very small pencils.

Sir John Herschel, in his elaborate and excellent paper on the aberrations of compound lenses and object-glasses, published in the Transactions of the Royal Society for 1821, when noticing the investigations of Clairaut, Euler, and D'Alembert, makes no mention of any higher approximation having been attempted; and he himself follows no other method for the chromatic dispersion, although he has pursued the subject of the spherical aberration so far as to render the object-glass free from it, for astronomical and terrestrial objects at the same time. Indeed, from the following passage in the same paper, it is clear that he did not suspect the existence of any unconsidered residual dispersion, of the magnitude of that which I am about to discuss; for he says, "The simplest considerations, indeed, suffice for the correction of that part of the aberration which arises from the different refrangibility of the differently coloured rays; and accordingly this part of the mathematical theory of refracting telescopes was soon brought to perfection, and has received no important accession since the original invention of the achromatic object-glass."

It must have struck most persons conversant with the subject, that the effect of the lenses, in an achromatic combination of two or three lenses in contact, must be sensibly different near their edges, on account of the oblique passage of the rays, from what it is near their centers; and this difference will be the more important, as the area of that part of the surface of the lens, with this unconsidered effect, is so much more than that part of the surface near the center for which the common theorem is accurate, or nearly so.

In the present paper, I have investigated the conditions of achromatism in a double object-glass for a ray passing through it at a distance from the center of its aperture, on the supposition that we may neglect powers, of the small quantities which enter the expressions, above the first, and also their products. It is also necessary to consider the thicknesses of the lenses, as that of their edges, for all parts at which the new correction rises to any important magnitude.

I have arrived, by two different methods, at the same result, which involves the expression obtained by the ordinary mode, together with others depending on the thicknesses of the lenses. The spaces, through which a ray has passed within the lenses, have on the achromatism an effect which is precisely similar to that of the distance of the lenses in achromatic eye-pieces. If the lens have great thickness, a ray of light after an oblique passage through the glass, will meet the second surface at a different angle to what it would have done if that thickness had been small; and hence, if we consider a virtual prism to be formed by the tangent planes to the surfaces of the lens, at the points at which the ray is incident and emergent, the angle of this virtual prism will depend on the thickness of the lens, as well as on the radii of the surfaces and the distance of the point of incidence from the center of its aperture. We may easily conceive, that this variable angle of the virtual prism will need more accurate consideration when we pass beyond the ordinary first approximation.

The new correction, which we thus arrive at, supposing the thicknesses of the lenses in a double object-glass such as might arise in practice, is however not very large in magnitude. But nevertheless, if it is important to correct as accurately as possible the spherical aberration, an error, which amounts to any sensible part of it, must also merit discussion.

Of the two methods before spoken of, I shall here detail the one which appears the more regular process, and only indicate the other, which however has the advantage of being more continuous in the working out.

Let $R_1 A R_2$ be a double convex lens, whose axis is $q_1\ O_1\ O_2$, (see Fig. 1), O_1 and O_2 being respectively the centers of its spherical surfaces,

$$\text{and} \quad \left.\begin{aligned} O_1R_1 &= r_1 \\ O_2R_2 &= r_2 \end{aligned}\right\} \text{ their radii.}$$

Let QR_1 be a ray of a pencil incident parallel to the axis, and let it be refracted at R_1 in the direction $R_1\ R_2\ q_1$ meeting the axis in q_1.

Let IR_1T_1, IR_2T_2 be tangents at R_1 and R_2 respectively, draw R_1M_1, R_2M_2 and Im, perpendicular to the axis;

$$\text{and put } R_1M_1 = y_1$$
$$R_2M_2 = y_2,$$
$$\text{also put the distance } R_1R_2 = t_1,$$
$$\text{and let } \mu = \text{refractive index.}$$

Then $T_1R_1IR_2T_2$ represents the virtual prism by which the ray is refracted, and its angle I is thus found:

$$\begin{aligned} \angle I &= \angle T_1Im + \angle T_2Im \\ &= \angle R_1O_1M_1 + \angle R_2O_2M_2 \\ &= \frac{R_1M_1}{O_1R_1} + \frac{R_2M_2}{O_2R_2} \\ &= \frac{y_1}{r_1} + \frac{y_2}{r_2}, \text{ nearly,} \end{aligned}$$

since these angles are always small.

$$\text{But}\quad y_2 = y_1\left(\frac{q_1R_2}{q_1R_1}\right)$$

$$= y_1\left(\frac{q_1R_1 - t_1}{q_1R_1}\right)$$

$$= y_1\left(1 - \frac{t_1}{q_1R_1}\right),$$

$$\text{and}\quad \frac{\mu}{q_1R_1} = \frac{\mu - 1}{r_1},$$

since the incident rays are parallel, and neglecting the aberration, &c.:

$$\therefore\ y_2 = y_1\left(1 - t_1\,.\,\frac{\mu-1}{\mu r_1}\right);$$

$$\text{and}\ \therefore\ \angle I = y_1\left(\frac{1}{r_1} + \frac{1}{r_2} - t_1\,.\,\frac{\mu-1}{\mu r_1 r_2}\right);$$

$$\text{or, putting}\ \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{\rho_1},$$

$$\angle I = y_1\left(\frac{1}{\rho_1} - t_1\,.\,\frac{\mu-1}{\mu r_1 r_2}\right);$$

and since the angle of incidence and the angle of the prism are always small, if D = the angular deviation of the ray, after emergence in the direction R_2q, we have

$$D = (\mu - 1) \times \angle I$$

$$= (\mu - 1)\,.\,y_1\left\{\frac{1}{\rho_1} - t_1\,.\,\frac{\mu-1}{\mu r_1 r_2}\right\}.$$

Again, if we consider the last lens as the outer lens of a double achromatic object-glass, and that the ray emerging immediately into a double concave lens is again refracted by it in the direction $R_2\ R_3\ q_2$, as in Fig. 2, the contiguous surfaces of the lenses having the same radius, and q_1, R_2, M_2, O_2 being the same points as in Fig. 1, we shall have $R_2\ q_1$ the direction of the ray, when it meets the concave lens at R_2. Also let

$R_2\ R_3\ q_2$ be the direction after refraction at R_2, (see Fig. 2.)

$R_3\ q_3$ .... emergence at R_3, then let the radii $O_2\, R_2 = r_2$,

$O_3\, R_3 = r_3$,

and $R_2\, I'$, $R_3\, I'$, the tangents at R_2 and R_3,

$R_2\, M_2 = y_2$,

$R_3\, M_3 = y_3$.

Distance $R_2\, R_3 = t_2$.

Also let $\mu' =$ index of refraction.

Then, as before, the angle of the virtual prism $= \angle I$

$$= \angle R_2\, O_2\, M_2 + \angle R_3\, O_3\, M_3$$

$$= \frac{y_2}{r_2} + \frac{y_3}{r_3},$$

and $$y_3 = y_2 \left(\frac{q_2 R_3}{q_2 R_2}\right)$$

$$= y_2 \frac{(q_2 R_2 - t_2)}{q_2 R_2}$$

$$= y_2 \left(1 - \frac{t_2}{q_2 R_2}\right);$$

also $$\frac{\frac{\mu'}{\mu}}{q_2 R_2} = -\frac{\frac{\mu'}{\mu} - 1}{r_2} + \frac{1}{q_1 R_2}, \text{ nearly.}$$

And $$\frac{1}{q_1 R_2} = \frac{1}{q_1 R_1 - t_1}$$

$$= \frac{1}{q_1 R_1} + \frac{t_1}{(q_1 R_1)^2} \text{ nearly}$$

$$= \frac{\mu - 1}{\mu r_1} + t_1 \,.\left(\frac{\mu - 1}{\mu r_1}\right)^2;$$

$$\therefore \frac{\frac{\mu'}{\mu}}{q_2 R_2} = -\frac{\frac{\mu'}{\mu} - 1}{r_2} + \frac{\mu - 1}{\mu r_1} + t_1 \,.\left(\frac{\mu - 1}{\mu r_1}\right)^2;$$

or $$\frac{1}{q_2 R_2} = -\frac{\mu' - \mu}{\mu' r_2} + \frac{\mu - 1}{\mu' r_1} + t_1 \,.\frac{(\mu - 1)^2}{\mu' \mu r_1^2}$$

$$= \frac{\mu - 1}{\mu' \rho_1} - \frac{\mu' - 1}{\mu' r_2} + \&c.;$$

$$\therefore\ y_3 = y_2 \left\{1 - t_2 \cdot \left(\frac{\mu - 1}{\mu' \rho_1} - \frac{\mu' - 1}{\mu' r_2}\right)\right\};$$

or, substituting for y_2 its value before found,

$$y_3 = y_1 \left\{1 - t_1 \cdot \frac{\mu - 1}{\mu r_1}\right\} \times \left\{1 - t_2 \cdot \left(\frac{\mu - 1}{\mu' \rho_1} - \frac{\mu' - 1}{\mu' r_2}\right)\right\}$$

$$= y_1 \left\{1 - t_1 \cdot \frac{\mu - 1}{\mu r_1} - t_2 \cdot \left(\frac{\mu - 1}{\mu' \rho_1} - \frac{\mu' - 1}{\mu' r_2}\right)\right\},$$

and $\angle I' = \frac{y_2}{r_2} + \frac{y_3}{r_3}$

$$= \frac{y_1}{r_2}\left(1 - t_1 \cdot \frac{\mu - 1}{\mu r_1}\right) + \frac{y_1}{r_3}\left\{1 - t_1 \cdot \frac{\mu - 1}{\mu r_1} - t_2 \cdot \left(\frac{\mu - 1}{\mu' \rho_1} - \frac{\mu' - 1}{\mu' r_2}\right)\right\}$$

$$= y_1 \left\{\frac{1}{r_2} + \frac{1}{r_3} - t_1 \cdot \left(\frac{1}{r_1 r_2} + \frac{1}{r_1 r_3}\right)\left(\frac{\mu - 1}{\mu}\right) - t_2 \cdot \left(\frac{\mu - 1}{\mu' \rho_1 r_3} - \frac{\mu' - 1}{\mu' r_2 r_3}\right)\right\}.$$

Let $\frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{\rho_2}$,

$$\text{then } \angle I' = y_1 \left\{\frac{1}{\rho_2} - t_1 \cdot \frac{\mu - 1}{\mu r_1 \rho_2} - t_2 \cdot \left(\frac{\mu - 1}{\mu' \rho_1 r_3} - \frac{\mu' - 1}{\mu' r_2 r_3}\right)\right\};$$

and, if D' be the angular deviation caused by this virtual prism, we have

$$D' = (\mu' - 1) \cdot y_1 \left\{\frac{1}{\rho_2} - t_1 \cdot \frac{\mu - 1}{\mu r_1 \rho_2} - t_2 \cdot \left(\frac{\mu - 1}{\mu' r_3 \rho_1} - \frac{\mu' - 1}{\mu' r_2 r_3}\right)\right\}.$$

Now when the dispersion by one of these virtual prisms is equal and opposite to that of the other, we shall have

$$\delta(D - D') = \delta D - \delta D' = 0.$$

Before however we find the actual value of this expression, we must shew that we may take t_1 and t_2 as constant quantities, namely, the thicknesses of the lenses at their edges.

In Fig. 3, let R_1, R_2, q_1, M_1, M_2, represent the same points as in Fig. 1;

$$\text{and } \therefore\ R_1 M_1 = y_1,$$
$$R_2 M_2 = y_2,$$
$$\&c. \qquad \&c.$$

Also let $mM_2 = \frac{1}{2}$ the aperture of the lens
$= a.$

Calling AB the thickness of the lens at its edge $= T_1$, we shall have the central thickness

$$CD = \left(T_1 + \frac{a^2}{2r_1} + \frac{a^2}{2r_2}\right) \text{ nearly,}$$

$$\text{and } R_1 R_2 = t_1 = \left\{T_1 + \frac{a^2 - y_1^2}{2r_1} + \frac{a^2 - y_2^2}{2r_2}\right\} \cdot \frac{1}{\cos R_2 q_1 M_2}$$

$$= \left\{T_1 + \frac{a^2 - y_1^2}{2r_1} + \frac{a^2 - y_2^2}{2r_2}\right\} \cdot \frac{1}{1 - \dfrac{y_1^2}{1.2.(q_1 M_1)^2}} + \&c.$$

$$= \left\{T_1 + \frac{a^2 - y_1^2}{2r_1} + \frac{a^2 - y_2^2}{2r_2}\right\} \left(1 + \frac{y_1^2}{1.2.(q_1 M_1)^2} + \&c.\right)$$

Now if it were necessary to retain the variable quantities y_1 and y_2 in this expression of the value of t_1, and similarly y_2 and y_3 in that of t_2, we should, from $\delta(D - D') = 0$, have the radii r_1, r_2, r_3, to be expressed in terms of constants, and y_1, y_2, y_3, or the surfaces would be surfaces of revolution generated by curves of variable curvature, in place of portions of surfaces of spheres.

But the dispersion near the edge of a lens is always very great in amount compared with that near the center of its aperture; and it becomes proportionally more important that it should be accurately corrected for the edge, and especially as the enlargement of the apertures of our telescopes may depend upon it. In the example I have calculated, further on in the paper, $\frac{a^2}{2r_1}$ and $\frac{a^2}{2r_2}$ are considerably smaller than T_1, and accordingly $\frac{a^2 - y_1^2}{2r_1}$ and $\frac{a^2 - y_2^2}{2r_2}$ are very small compared with T_1 for a ray which passes near the edges, and are therefore negligible; and also $\frac{y_1^2}{1.2.(q_1M_1)^2}$ is again a much smaller quantity, and therefore not needing attention: so that we shall consider t_1 and t_2 as the thicknesses of the lenses *at or near* their edges, and as constants.

Differentiating the expressions for D and D', and substituting in the expression $\delta(D - D') = 0$, we find, after the reductions,

$$0 = \frac{\delta\mu}{\rho_1} - \frac{\delta\mu'}{\rho_2} + t_1\left\{\left(\frac{\mu - 1}{\mu}.\delta\mu' + \frac{\mu' - 1}{\mu^2}.\delta\mu\right)\frac{1}{r_1\rho_2} - \frac{\mu^2 - 1}{\mu^2}.\frac{\delta\mu}{r_1r_2}\right\}$$
$$+ t_2\left\{\left(\frac{\mu' - 1}{\mu'}.\delta\mu + \frac{\mu - 1}{\mu'^2}.\delta\mu'\right)\frac{1}{r_3\rho_1} - \frac{\mu'^2 - 1}{\mu'^2}.\frac{\delta\mu'}{r_2r_3}\right\}.$$

In the ordinary formula, the condition that a double object-glass shall be achromatic, is

$$\frac{\delta\mu}{\rho_1} - \frac{\delta\mu'}{\rho_2} = 0.$$

So that the expression we have obtained consists of the common one together with other terms involving the thicknesses of the lenses.

The other method which I have mentioned will be easily comprehended from Fig. 4.

If *abcd* in this figure represent a double object-glass, and R_1, R_2, R_3, M_1, M_2, M_3, q_1, q_2, q_3, represent the same points as in the two first figures; $R_1 R_2 R_3 q_3$ being the path of the ray incident at R_1, then we have, tangent of the angle $R_3 q_3 M_3$, in which the emergent ray meets the axis at q_3,

$$= \frac{R_3M_3}{q_3M_3},$$

$$= \frac{y_3}{q_3M_3};$$

and, in order that the combination may be achromatic, we must have the variation of this $\angle R_3q_3M_3$ (or its tangent since it is small) $= 0$, whilst μ and μ', the refractive indices, vary for the different colours of the spectrum. For this purpose, expressing y_3 in terms of y_1 and the radii of the surfaces, and distances R_1R_2, R_2R_3, in the lenses; and q_3M_3

in terms of the radii and distances; and then performing the required differentiation, we find, for $\delta.\tan R_{.}q_{3}M_{3}$, the same expression as we have just obtained for $\delta(D-D')$, as we clearly ought to do.

To enable us to judge of the value of the new correction, it is necessary to apply it to a case which may arise in practice. For this purpose I have chosen the third case in Sir John Herschel's table in the paper before referred to; as the dispersive ratio in that case is what he considers the mean value for such glass as is usually obtained in England. I have also considered the radii of the interior surfaces to be the same, their difference being little more than a fiftieth of an inch in three feet, so that we have for our data for a telescope of ten feet focal length, as follows:—

$$\pi = 0.6 \doteq \frac{\delta\mu}{\delta\mu'} \cdot \frac{\mu'-1}{\mu-1},$$

$$\mu = 1.524,$$

$$\mu' = 1.585;$$

whence $$\frac{\delta\mu}{\delta\mu'} = .53743,$$

$$r_1 = 6.7069 \text{ feet},$$

$$r_2 = \tfrac{1}{2}\{3.0488 + 3.0640\},$$

$$= 3.0564,$$

$$r_3 = -14.2937.$$

If we take, now, the same dimensions, for our example, as those of the Northumberland telescope recently put up at the Cambridge Observatory, in which the focal length is 19 feet, the aperture $11\frac{1}{2}$ inches, the thickness of the double convex crown lens $\frac{3}{4}$ inch at the edge, and that of the concave flint lens 1 inch, we must take the radii of the surfaces in proportion to the focal length, and thus have

$$r_1 = 12.74311 \text{ feet},$$

$$r_2 = 5.80716,$$

$$r_3 = -27.15803,$$

$$y_1 = \frac{11.5}{2} \text{ inches,}$$
$$= .4791\dot{6} \text{ feet,}$$
$$t_1 = .03125,$$
$$t_2 = .08\dot{3},$$
$$\delta\mu' = .03 \text{ nearly, from Sir D. Brewster's tables.}$$

With these we obtain, by substituting in the formula before investigated, as follows:

$$\delta(D - D') = y_1 . \delta\mu' \{t_1 . 0009555 + t_2 . 00006055\}$$
$$= .000000501759375.$$

To find the diameter of the least circle of dispersion,

Let R_3M_3 be as in the last figure, (see Fig. 5.) qR_3q' the angle of residual dispersion

$$= \delta(D - D');$$

then ab, the diameter of the least circle into which all the coloured rays are collected,

$$= qq' \times \frac{R_3M_3}{qM_3};$$

and in the triangle qR_3q', we have $qq' = \dfrac{qR_3 . \sin qR_3q'}{\sin R_3q'M_3}$

$$= \frac{qM_3 . \delta(D - D')}{\dfrac{R_3M_3}{q'R_3}} \text{ nearly;}$$

$$\therefore \; ab = qR_3 \times \delta(D - D')$$
$$= \text{focal length} \times \delta(D - D');$$

and thus, in our example, the diameter of circle of residual dispersion

$$= .000009533 \text{ of a foot.}$$

The angle which this subtends at the object-glass is $\delta(D - D')$; and, measured in seconds of a degree, becomes

$$= \frac{.0000005017}{.0000048481}$$
$$= 0''.103 \text{ nearly.}$$

A residual circle of dispersion of this magnitude is such as would never be tolerated in a telescope like the Northumberland one, from which we have taken our example. The observations made by Professor Struve with the Dorpat telescope, on most difficult double stars, shew that no uncorrected dispersion to an amount like the above could exist in that telescope; and the Northumberland telescope may be reasonably expected to be no ways inferior.

From this we are led to conclude, that practical opticians have through experience adopted curvatures for their lenses of much greater accuracy than those given by any theoretical computations hitherto published, and the production of critical defining power in an object-glass must be left to their skill and patience in finding the forms which produce the desired effect.

To shew the effect of our correction on the radius of any one of the surfaces, I shall now give, as example, a case in which the convex crown lens is taken of a greater thickness than would occur in any modern object-glass, namely

$t_1 = \frac{1}{2}$ inch, $t_2 = \frac{5}{8}$ inch for an aperture of 6 inches, and focal length 10 feet.

Calculating with these, we find

$$\delta(D - D') = .0000011629,$$

and the diameter of the least circle of residual dispersion

$$= .000011629 \text{ of a foot.}$$

Now the diameter of the least circle of spherical aberration in a crossed lens of plate-glass, refractive index $= 1.5$, is for the same focal length and aperture $= .00008370$;

so that the former correction would amount to about one-seventh of the spherical aberration in an equivalent lens of the best form, and yet to correct this large residual dispersion would require only a very small alteration in the radius of one of the surfaces. To find this alteration, we must now take the value of $\delta(D - D') = 0$, and as both the terms in the residual value are positive, we cannot fulfil this condition whilst $\left(\frac{\delta\mu}{\rho_1} - \frac{\delta\mu'}{\rho_2}\right)$ is separately $= 0$, but must make the whole expression $=0$, when the values of t_1 and t_2 are given for any particular lens. I have performed the calculation to this effect for the above example, supposing the last surface of the flint-lens only to be altered, and I find that its radius (r_3) must be increased by $\frac{1}{30}$th of an inch nearly, which is a very small quantity in 14 feet, and shews us that it is scarcely to be hoped, that we can obtain a very fine object-glass by trusting to theoretical computations solely; but that, after the general forms have been investigated for the optician, we must rely on his experience to vary his curvatures slowly until he has obtained the maximum effect of distinctness.

With respect to the actual thicknesses adopted in England, I am indebted for information to the liberality of Mr Tully and Mr Robinson; and as it is important that such information should be recorded in print, I shall not hesitate to give the full extract from Mr Robinson's reply to my letter to him requesting such information. He says: "Thinking that the best information might be obtained from Mr Tully, I called on him, and not being so fortunate as to find him at home, I left your letter, with the request that he would furnish the information you required; he has just now called on me and tells me, that there is no absolute rule for the thickness of either the concave or convex lens; that great thickness for the convex lens, if it be of crown glass, is considered objectionable on account of its colour occasioning loss of light; and its being thin is objected to, but merely because if it be ground to a sharp edge, there is danger of the edge being broken in polishing: he has just made an object-glass of $5\frac{1}{2}$ inches diameter, the thickness of the crown glass is at the edge $\frac{1}{6}$ of an inch, and that of the concave $\frac{5}{8}$, and these he thinks very proper thicknesses and would not wish they should be thicker; but had these disks been thinner, they, (being of good glass,) would not have been rejected on that account; and in general, the only rule for thickness is, that it be such that the edge of the convex be not splintered in working, and the centre of the concave be not so thin as to change its form in polishing."

XXIX. *A Statistical Report of Addenbrooke's Hospital, for the Year* 1837. *By* Henry J. H. Bond, M.D.

[Read *April* 30, 1838.]

IN-PATIENTS, 1837.

Number of Beds have been increased since Oct. 1, 1837, to 101.

Number of Patients in the Hospital:

Maximum...	94
Minimum...	55
Mean........	71

Length of time Patients remained in the Hospital:

Mean duration $38\frac{1}{2}$ days.

Admissions from January 1st, 1837, to January 1st, 1838:

Male Patients	374
Female	311
Total...	685

Previous residence of Patients:

Males.	*Females.*	*Total.*	
164 (43 per c^t^.)	162 (52 per c^t^.)	326 (47 per c^t^.)	in Towns, Cambridge principally.
209 (57 per c^t^.)	149 (48 per c^t^.)	358 (53 per c^t^.)	in the Country.
1	...	1	residence not registered.

Number of In-Patients in 1837 discharged, including the 58 remaining from the year 1836:

Males.	*Females.*	*Total.*	*Results.*
209 (58.2 per c^t^. on total of discharges.)	179 (59.4 per c^t^.)	388 (58.7 per c^t^.)	Recovered.
38	17	55	Benefitted.
15	14	29	Discharged at their own request.
3	3	6	.... for irregular conduct.
7	5	12	.... as incurable.
22 (6.1 per c^t^.)	10 (3.3 per c^t^.)	32 (4.8 per c^t^.)	Died.
65	73	138	Made Out-Patients.
359	301	660	

Mean stay in the house of the 388 discharged as recovered was $33\frac{3}{4}$ days.

	PER CENT.		PER CENT.		WEEK.
Of the 660 total discharged...	7.72	Of the 388 discharged as recovered	6.70	were discharged in	1
....	11.96	....	13.40	....	2
....	15.30	....	17.52	....	3
....	15.75	....	16.75	....	4
....	12.42	....	13.65	....	5
....	8.63	....	9.79	....	6
....	6.06	....	4.89	....	7
....	4.69	....	3.60	....	8
....	3.48	....	2.83	....	9
....	2.27	....	2.57	....	10
....	11.66	....	8.24	.........from	10 to 48

In-Patients remaining under treatment. Dec. 31, 1837.

Males.........	40
Females	43
Total	83

Number of Operations;

Lithotomy	2
Amputations	5
Hernia	1
Cataract	2
Excision of Mammary Gland	2
.... Tumour	1
Removal of Testicle	2
	15

OUT-PATIENTS, 1837.

Admissions from January 1st, 1837, to January 1st, 1838.

Males	359
Females	562
Total	921

Previous Residence of Patients:

Males.	*Females.*	*Total.*	
208 (57 per c^t.)	315 (56 per c^t.)	523 (56 per c^t.)	in towns, Cambridge principally.
151 (43 per c^t.)	246 (44 per c^t.)	397 (44 per c^t.)	in the Country.
	1	1	residence not registered.

Out-Patients discharged in 1837, including 118 of 120 Patients remaining from 1836.

Males.	*Females.*	*Total.*	*Results.*
160 (45.9 per c^t.)	246 (45.5 per c^t.)	406 (45.7 per c^t.)	Recovered.
24	26	50	Benefitted.
3	2	5	Discharged as incurable.
...	4	4	.... irregular.
1	7	8	.... at their own request.
6 (1.7 per c^t.)	4 (0.7 per c^t.)	10 (1.1 per c^t.)	Died.
22	47	69	Made In-Patients.
129	204	333	Discharged for non-attendance.
3	...	3	Result not registered.
348	540	888	

Out-Patients remaining under treatment, Dec. 31, 1837, males 49, females 102, total 151.

REPORT OF THE IN-PATIENTS AND OUT-PATIENTS COMBINED.

Number of Cases treated in 1837.

Deducting 41 from the 685 In-Patients who were previously under treatment as Out-Patients, and 137 from the 921 Out-Patients who were previously under treatment as In-Patients,

The number of cases treated in 1837 were 653 males, 775 females, total 1428.

Of these, there were admitted in

January......	107.	May.........	133.	September...	103.
February....	108.	June........	129.	October......	114.
March........	119.	July........	124.	November...	128.
April.........	127.	August......	132.	December...	104.

335 males, 428 females, total 763, came from towns, Cambridge principally.
318 ......, 347 ...., ... 665 .... the country.

Description of Patients admitted:

Male.		
264	(40 per cent.)	Labourers, chiefly agricultural.
110	(16 per cent.)	Children living at home.
28	....	Shoemakers.
17	....	Bricklayers.
17	....	Carpenters.
16	....	Servants.
15	....	Grooms.
15	....	Porters.
15	....	Printers.
13	....	Tailors.
12	....	Watermen.
9	....	Gardeners.
8	....	Smiths.
7	....	Waggoners.
6	....	Brewers.
6	....	Butchers.
6	....	Tramps and Hawkers.
5	....	Bakers.
84	....	Of 51 different occupations.
653		

Female.

337	(43 per cent.)	Women occupied at home with the care of their families.
264	(32 per cent.)	Servants.
86	(11 per cent.)	Children at home.
22	....	Needlewomen.
19	....	Laundresses.
22	....	Of various occupations.
25	....	Occupations not registered.
775		

Ages of the 1428 Patients admitted in 1837:

Male (653).		*Female* (775).		*Total* (1428).		
32 or	4.90 per c^t^.	34 or	4.38 per c^t^.	66 or	4.62 per c^t^.	5 years or under.
50 ...	7.65	52 ...	6.70	102 ...	7.14 ...	from 5 to 10 inclusive.
49 ...	7.50	60 ...	7.74	109 ...	7.63	.........10 ... 15
70 ...	10.71	203 ...	26.19	273 ...	19.11	.........15 ... 20
107 ...	16.38	127 ...	16.38	234 ...	16.38	.........20 ... 25
70 ...	10.71	75 ...	9.67	145 ...	10.15	.........25 ... 30
38 ...	5.81	50 ...	6.45	88 ...	6.16	.........30 ... 35
64 ...	9.80	68 ...	8.77	132 ...	9.24	.........35 ... 40
35 ...	5.35	23 ...	2.97	58 ...	4.06	.........40 ... 45
45 ...	6.89	30 ...	3.87	75 ...	5.25	.........45 ... 50
28 ...	4.27	14 ...	1.80	42 ...	2.94	.........50 ... 55
29 ...	4.44	19 ...	2.45	48 ...	3.36	.........55 ... 60
24 ...	3.67	12 ...	1.54	36 ...	2.52	.........60 ... 65
5 ...	.76	3 ...	.38	8 ...	.56	.........65 ... 70
6 ...	.91	3 ...	.38	9 ...	.63	.........70 ... 75
1 ...	.15	2 ...	.25	3 ...	.21	.........75 ... 80 and upward

Total number of Cases discharged in 1837 from both In-Patient and Out-Patient Registers:

Males.	*Females.*	*Total.*	*Results.*
369 (59.5 per c^{t}.)	425 (58.9 per c^{t}.)	794 (59.4 per c^{t}.)	Recovered.
62	43	105	Benefitted.
10	7	17	Discharged as incurable.
16	21	37	.... at their own request.
3	7	10	.... for irregular conduct.
28 (4.5 per c^{t}.)	14 (1.9 per c^{t}.)	42 (3.1 per c^{t}.)	Died.
129	204	333	Discharged for non-attendance as Out-Patients.
3		3	Results not registered.
620	721	1341	Total number discharged.

Diseases, Ages, &c. in the 42 Fatal Cases.

	Males.	Females	Total.	10 or under.	From 10 to 20 inclusive.	20 to 30.	30 to 40.	40 to 50.	50 to 60.	60 to 70.	70 to 80.	January.	February.	March.	April.	May.	June.	July.	August.	September.	October.	November.	December.
Phthisis	10	5	15	1	3	4	4	2	...	1	...	2	...	1	2	2	...	5	1	...	2		
Pneumonia	3	...	3	...	1	...	...	1	...	1	...	1	...	...	...	...	...	1	...	...	...	...	1
Disease of Heart	3	...	3	...	1	1	1	...	...	...	...	...	...	...	1	...	...	...	1	1			
.... Brain	2	1	3	...	...	1	1	1	...	...	...	...	...	...	...	1	...	...	...	...	2		
Fever	1	1	2	...	2	...	...	...	...	...	...	...	1	...	...	...	1						
Dropsy	...	2	2	...	...	1	1	...	...	...	...	1	...	...	1								
Fractures	1	1	2	...	...	...	...	...	...	1	1	...	...	1	...	...	1						
Scrofula	1	...	1	...	1	...	...	...	...	...	...	1											
Pleurisy	1	...	1	...	...	...	...	1	...	...	...	...	...	...	...	...	...	...	1				
Bronchitis	1	...	1	...	...	...	...	...	...	1	...	1											
Peritonitis	1	...	1	...	...	1	...	...	...	...	...	...	1										
Disease of Liver		1	1	...	...	...	...	...	1	...	...	...	...	...	...	...	...	...	...	...	...	...	1
Paralysis	1	...	1	...	...	...	...	1	...	...	...	...	...	1									
Debility with ulcers	...	1	1	...	...	...	...	1	...	...	...	...	...	...	...	...	...	...	1				
Disease of Hip-Joint	1	...	1	...	...	1	...	...	...	...	...	...	...	...	...	...	1						
Carcinoma Mammæ	...	1	1	...	...	...	...	1	...	...	...	...	1										
.... Uteri	...	1	1	...	...	...	1	...	...	...	...	...	1										
Sphacelus of Foot	1	...	1	...	...	...	...	...	...	...	1	1											
Burn	1	...	1	1	...	...	...	...	...	...	...	...	...	...	1								
Males	...	...	...	1	5	7	5	5	...	4	1	6	2	1	4	2	2	3	3	1	3	...	1
Females	...	...	...	1	3	2	3	3	1	...	1	1	2	2	1	1	1	3	1	...	1	...	1
	28	14	42	2	8	9	8	8	1	4	2	7	4	3	5	3	3	6	4	1	4	...	2

Dr BOND'S STATISTICAL ACCOUNT OF ADDENBROOKE'S HOSPITAL FOR 1837.

Diseases	Male	Female	Total	January	February	March	April	May	June	July	August	September	October	November	December
	(per cent.)	(per cent.)	(per cent.)	(per cent.)	per cent.	per cent.	per cent.	per cent.	(per cent.)	per cent.	(per cent.)	per cent.	(per cent.)	(per cent.)	per cent.
Carcinoma of Lip and Tongue	3		3 (0,21)						1		1	1	1		
Ulcer of Tongue		1	1						1						
Ptyalism		1	1											1	
Enlarged submaxillary gland	1		1									1		1	
Cynanche tonsillaris	1 (0,15)	3 (0,38)	4 (0,28)	1	1		1								
Disease of Esophagus	1	1	2		1										
DISEASES OF MOUTH AND FAUCES	6 (0,91)	6 (0,77)	12 (0,84)	4 (3,73)	5 (4,62)	11 (9,24)	4 (3,14)	6 (4,51)	6 (4,63)	7 (5,64)	10 (10,57)	7 (6,79)	7 (6,14)	8 (6,25)	5 (4,80)
Dyspepsia	21	59 (7,61)	80 (5,60)												
[illegible]	1 (0,15)	8 (1,03)	9 (0,63)												
[illegible]	1 (0,15)	4 (0,51)	5 (0,35)												
[illegible]	(0,15)	2 (0,25)	3 (0,21)												
[illegible]		1													
DISEASES OF STOMACH	24 (3,67)	74 (9,54)	98 (6,86)	5 (4,67)	8 (7,40)	11 (9,24)	4 (3,14)	7 (5,26)	9 (6,97)	9 (7,25)	10 (10,57)	8 (7,76)	9 (7,89)	10 (7,81)	8 (7,69)
Diarrhœa	4 (0,61)	12 (1,54)	16 (1,12)												
Constipation	2 (0,30)	28 (3,61)	30 (2,10)												
Intestinal Hæmorrhage	1	1	2												
Hæmorrhoids		3 (0,38)	3 (0,21)												
Worms	1	7 (0,90)	8 (0,56)												
[illegible]	2 (0,30)	2 (0,25)	4 (0,28)												
[illegible]			1												
[illegible]			1												
[illegible]	1 (0,15)	2 (0,25)	3 (0,21)												
[illegible]		5 (0,64)	5 (0,35)												
[illegible]	6 (0,91)	5 (0,64)	11 (0,77)												
Colic	2 (0,30)	7 (0,90)	9 (0,63)												
Hernia	6 (0,91)	2 (0,25)	8 (0,56)												
DISEASES OF INTESTINES	26 (3,98)	75 (9,67)	101 (7,07)	8 (7,47)	5 (4,62)	6 (5,04)	12 (9,44)	11 (8,27)	9 (6,97)	9 (7,25)	11 (8,33)	8 (7,76)	4 (3,50)	10 (7,81)	8 (7,69)
Icterus	2 (0,30)	3 (0,38)	5 (0,35)												
Hepatitis	1	1	2												
Disease of Liver	5 (0,76)	3 (0,38)	8 (0,56)												
DISEASES OF LIVER	8 (1,22)	7 (0,90)	15 (1,05)												
*[illegible] Spleen			1												
[illegible]															
Influenza	6 (0,91)	4 (0,51)	10 (0,70)	6 (5,60)	4 (3,70)										
Chronic Laryngitis	1 (0,15)	6 (0,77)	7 (0,49)												
Hooping Cough		2 (0,25)	2												
Bronchitis	18 (2,75)	25 (3,22)	43 (3,01)	10 (9,34)	8 (7,40)	4 (3,36)	5 (3,93)	1 (0,75)	2 (1,55)	3 (2,41)	1 (0,75)	2 (1,94)	2 (1,75)	3 (2,34)	2 (1,92)
Chronic Cough	7 (1,07)	8 (1,03)	15 (1,05)												
[illegible]	2 (0,30)	3 (0,38)	5 (0,35)												
[illegible]	(5,97)	26 (3,35)	65 (4,55)	2 (5,60)	4 (3,70)	3 (2,52)	7 (5,51)	7 (5,26)	5 (3,87)	8 (6,44)	6 (4,53)	9 (8,73)	3 (2,61)	3 (2,34)	4 (3,84)
[illegible]	(1,07)	3 (0,38)	10 (0,70)												
[illegible]	2 (0,30)		2 (0,14)												
[illegible] Disorders not specified	3 (0,45)	7 (0,90)	10 (0,70)												
DISEASES OF PULMONARY ORGANS	86 (13,16)	84 (10,83)	170 (11,90)	26 (24,29)	19 (17,59)	13 (10,92)	16 (12,59)	12 (9,02)	12 (9,30)	17 (13,70)	8 (6,06)	14 (13,58)	9 (7,89)	11 (8,59)	13 (12,5)
*Bronchocele		4 (0,51)	4 (0,28)												
Pericarditis	1	1	2												
Organic disease of Heart	5 (0,76)	2 (0,25)	7 (0,49)												
Palpitation	1 (0,15)	3 (0,38)	4 (0,28)												
Varicose Veins	1 (0,15)	4 (0,51)	5 (0,35)												
Melanosis	1		1												
Phlegmasia dolens		1	1												
Anæmia and Chlorosis		6 (1,35)	2 (1,8)												
Epistaxis	1	1	2												
DISEASES OF CIRCULATING SYSTEM	10 (1,29)	38 (4,90)	[illegible]												
Intermittent Fever	18 (2,7)	12	[illegible]												
Continued Fever	10 (1,58)	13	20												
Measles	2 (0,30)		(0,14)												
FEVERS	30	2	(3,64)												
*Dropsy	5 (0,76)	17	(1,61)												
*[illegible] Glands	[illegible]	21	[illegible]												
*[illegible] of Absorbents	1		1												
[illegible]	1														
[illegible]	1														
[illegible] Diabetes	1		1												
DISEASES OF KIDNEYS	[illegible]														
Lithiasis	5 (0,76)	2	7 (0,49)												
Enuresis		1	1												
Retention of Urine	3 (0,45)	1 (0,12)	4 (0,28)												
Paralysis of Bladder	1		1												
[illegible] of Neck of Bladder, &c.	5 (0,76)		(0,35)												
DISEASES OF BLADDER	14	4	(1,20)												
Gonorrhœa and Syphilis	24	11	(2,45)												
Stricture of Urethra	4		4 (0,28)												
Hydrocele	2		2												
Hernia [illegible]	6		6												
Disease of Testis	[illegible]		(0,35)												
Fistula in perineo	1		1												
[illegible] of Vagina		1	1												
DISEASES OF EXTERNAL GENITAL ORGANS	(5,07)	[illegible]	[illegible]												
[illegible]		9 (7,35)	9 (4,00)												
[illegible]		[illegible]	[illegible]												
[illegible]															
[illegible]		[illegible] (8,00)	60 (4,81)												
[illegible]			[illegible]												
[illegible]		1	(0,14)												
[illegible]			3 (0,21)												
[illegible]		1	1												
[illegible]		1	7 (0,49)												
[illegible]															
*[illegible]		2 (0,25)	(0,14)												
[illegible]		6 (0,77)	6 (0,42)												
[illegible]															
[illegible]															
*[illegible] Lactation		5 (0,64)	(0,35)												
Rheumatism	60	30 (3,87)	90												
DISEASES OF MUSCULAR SYSTEM, &c.	[illegible]	[illegible]	91												
[illegible]		1	1												
Chorea	1	9 (1,16)	10 (0,70)												
Hysteria		19	[illegible]												
Tetanus	1		1												
[illegible]		1													
Paralysis	6	1	7												
[illegible]															
[illegible] tremens	(0,30)		2 (0,14)												
[illegible]		1	1												
Hypochondriasis	7 (1,07)	9	16 (1,12)												
NERVOUS DISORDERS	[illegible] (1,53)	10	(1,80)												
[illegible]	1 (0,15)	[illegible]	6 (0,42)												
[illegible] and Hemicrania	10	12	22 (1,54)												
[illegible] Disorders	4 (0,61)	[illegible]	1 (0,77)												
[illegible] CEREBRAL AFFECTIONS	18 (2,75)	21 (2,70)	[illegible]												
Erysipelas	6	[illegible]	[illegible]												
Purpura		1	1												
Eczema		6 (0,77)	6 (0,42)												
Herpes	1	1	2												
Urticaria febrilis		1	1												
Scabies	7 (1,07)	4 (0,51)	11 (0,77)												
Porrigo	3 (0,45)	6 (0,77)	9 (0,63)												
Impetigo	1 (0,15)	4 (0,51)	5 (0,35)												
Ecthyma	1		1												
Sycosis	1		1												
Lepra and Psoriasis	2 (0,30)	5 (0,64)	7 (0,49)												
Icthyosis	1		1												
Erythema nodosum	1 (0,15)	2 (0,25)	3 (0,21)												
Cutaneous Diseases not specified		11	11												
DISEASES OF SKIN	22 (3,36)	46 (5,93)	68 (4,76)	8 (7,47)	5 (4,62)										
Conjunctival Ophthalmia	27 (4,13)	28 (3,61)	55 (3,85)												
Purulent Ophthalmia	2 (0,30)	2 (0,25)	4 (0,28)												
Inflammation of Tarsi	1 (0,15)	5 (0,64)	6 (0,42)												
Opacities and ulcers of Cornea	3 (0,45)	5 (0,64)	8 (0,56)												
[illegible]		(0,25)	2 (0,14)												
Staphyloma		1	1												
Cataract	1	1	2												
Amaurosis		3 (0,38)	[illegible]												
Fistula lachrymalis		2 (0,25)	2 (0,14)												
Fungus of Eye		1	1												
Obstructed Puncta		1	1												
Other diseases of Eye	1	4 (0,51)	5 (0,35)												
DISEASES OF EYE	35 (5,35)	55 (7,09)	90 (6,30)	(4,67)	5 (4,62)	7 (5,88)	10 (7,87)	12 (9,02)	12 (9,30)	7 (5,64)	11 (8,33)	3 (2,91)	6 (5,26)	5 (3,90)	7 (6,73)
Deafness	1		1												
Otitis		1	1												
DISEASES OF EAR															
*Polypus of nose	2 (0,30)		2												
Ulcers of Leg	31 (4,74)	21 (2,70)	52 (3,64)												
Ulcers	9 (1,37)	4 (0,51)	13 (0,91)												
Abscesses	13 (1,98)	10 (1,29)	23 (1,61)												
External Tumors	3 (0,45)	3 (0,38)	6 (0,42)												
External Inflammations	7 (1,07)	11 (1,41)	18 (1,26)												
Sphacelus of Feet	1		1												
EXTERNAL LOCAL DISEASES	64 (9,80)	49 (6,32)	113 (7,91)												
*Inflammation of Bursæ and Ganglion		2 (0,25)	2 (0,14)												
*Relaxation of Ligamentum Patellæ		1	1												
*Periostitis	2 (0,30)		2 (0,14)												
Inflammation of Hip-Joint	10 (1,53)	3 (0,38)	13 (0,91)												
——— Knee	13 (1,99)	13 (1,67)	26 (1,82)												
——— Ankle	1 (0,15)	2 (0,25)	3 (0,21)												
——— Elbow	2 (0,30)		2 (0,14)												
DISEASES OF JOINTS	26 (3,98)	18 (2,32)	44												
Exostoses		3 (0,38)	3 (0,21)												
Rachitis	1		1												
Disease of Vertebræ	5 (0,76)	5 (0,64)	10 (0,70)												
Disease of other Bones	6 (0,91)	8 (1,03)	14 (0,98)												
DISEASES OF BONES	12 (1,83)	16 (2,06)	28 (1,96)												
Burns and Scalds	2 (0,30)	3 (0,38)	5 (0,35)												
Lacerations and Bruises	80 (12,25)	14 (1,81)	94 (6,58)												
Fractures of Ribs	1		1												
——— Clavicle	4 (0,61)	2 (0,25)	6 (0,42)												
——— Humerus	9 (1,37)	1 (0,12)	10 (0,70)												
——— Radius	2 (0,30)		2 (0,14)												
——— Femur	2 (0,30)		2 (0,14)												
——— Tibia and Fibula	6 (0,91)	1 (0,12)	7 (0,49)												
——— Patella	1		1												
Compound Fracture of Arm	1		1												
——— Leg	3 (0,45)		3 (0,21)												
Luxation of Humerus	4 (0,61)		4 (0,28)												
——— Radius	2 (0,30)		2 (0,14)												
——— Carpus	4 (0,61)	1 (0,12)	5 (0,35)												
——— Knee	1		1												
——— Ankle	1		1												
ACCIDENTS	123 (18,83)	22 (2,83)	145 (10,15)	5	11	12	16	15	15	17	16	13	6	11	11
*Assumed illnesses (2 pregnancy)	1 (0,15)	2 (0,25)	3 (0,21)												
*Intoxication	1		1												
*Diseases not specified	5	4	9												
	651	775	1429	107	108	119	127	133	129	124	132	103	114	128	104

The opposite table is constructed similarly to that of the report of the preceding year*, but additional columns have been placed which contain the relative distribution of each disease between the two sexes, and the proportion of the male and female cases of each to the total number of male and female patients respectively.

Arrangement of the different Classes *of Cases, or of the Affections (of whatever kind) of individual* Systems *or* Organs, *according to the respective frequency of their occurrence, as seen in the foregoing Table.*

	Proportion to Total of Cases.
Diseases of the Pulmonary Organs	11.90
Accidents	10.15
External local diseases	7.91
Diseases of Intestines	7.07
.... Stomach	6.86
Rheumatic Complaints	6.37
Diseases of Eye	6.30
.... Nervous System	6.23
.... Menstruation	4.83
.... Skin	4.76
Fevers	3.64
Diseases of Circulating System	3.36
.... Joints	3.08
.... Bones	1.96
	84.42
Diseases of other organs, &c. constituting 25 other classes	15.58
	100.00

Out of the 161 diseases and accidents enumerated in the Table, 21 have been selected as the most prevailing, furnishing in the aggregate 54 per cent of the whole number of cases, and particulars respecting them arranged in a tabular form.

* Vide Explanation of Table, p. 370. Vol. VI. Part II. of the Transactions.

Explanation of the Table.

The number of cases *admitted* of the 21 selected diseases, and their proportion to the *total* of admissions, their distribution between the sexes and in decimal periods of age, the proportion of town and country cases to the respective totals of town and country patients, and the months in which the maximum and minimum of cases of each disease occurred, are here tabulated. Also the recoveries, benefits, and deaths, combining the total of the cases *discharged* in which the results were known, are added; with the average number of days during which the patients that recovered were under treatment.

TABLE OF PRINCIPAL DISEASES, &c.

No. of Cases.		CASES ADMITTED. Proportion to Annual Total of Cases. Total.	Males.	Females.	Distribution of Ages. From birth to 10 years inclusive.	From 10 to 20.	From 20 to 30.	From 30 to 40.	From 40 to 50.	From 50 to 60.	From 60 to 70.	From 70 to 80.	Proportion of Town & Country Cases (1). Town.	Country.	Monthly Distribution of Cases (2). Maximum.	Minimum.	CASES DISCHARGED (3). Recovered.	Benefitted.	Died.	Total.	*Mean number of days, during which the cases that recovered were under treatment.*
90	Rheumatism	6.30	9.18	3.87	2	17	23	17	13	10	5	3	5.63	7.06	October 11.40	February 1.48	46	11	...	57	36
80	Dyspepsia	5.60	3.21	7.61	1	13	19	27	13	5	2	...	5.89	5.41	August 10.57	April 3.14	40	5	...	45	48
65	Phthisis	4.55	5.97	3.35	4	9	29	14	4	4	1	...	3.66	5.56	September 8.73	March 2.52	...	14	15	29	..
58	Amenorrhœa	4.06	...	7.48	...	41	16	1	...	...	...	...	4.32	3.75	January 8.41	December 0.96	39	2	...	41	48
55	Ophthalmia (conjunctival)	3.85	4.13	3.61	20	18	7	1	4	4	1	...	2.75	5.11	July 8.06	October 1.75	36	2	...	38	50
52	Ulcers of Leg	3.64	4.74	2.70	...	8	20	4	7	7	4	2	4.06	2.15	September 7.76	June & Nov. 0.00	42	1	1*	44	36
43	Bronchitis	3.01	2.75	3.22	3	2	14	5	10	3	6	...	3.93	1.95	January 9.34	May & Aug. 0.75	21	3	1	25	39
35	Syphilis and Gonorrhœa	2.45	3.67	1.41	...	7	22	1	4	1	...	...	3.53	1.20	...	...	20	3	...	23	39
30	Constipation	2.10	0.30	3.61	6	10	9	4	...	1	...	...	2.75	1.35	...	...	22	1	...	23	48
30	Ague	2.10	2.75	1.54	8	5	2	6	5	1	3	...	0.91	3.45	March 4.20	Aug. & Sept. 0.00	21	2	...	23	31
30	Scrofulous Glands	2.10	1.37	2.70	6	13	6	3	1	1	...	...	2.22	1.95	August 4.53	Feb. & Sept. 0.00	7	4	...	11	34
26	Anæmia	1.82	...	3.35	2	17	7	...	...	...	...	...	1.57	2.10	...	...	16	1	...	17	37
26	Inflammation of Knee	1.82	1.99	1.67	2	17	5	1	...	1	...	...	1.44	2.25	...	...	17	1	...	18	47
23	Abscesses	1.61	1.99	1.29	4	10	5	2	...	2	...	...	1.31	1.95	...	...	15	1	...	16	24
22	Dropsy	1.54	0.76	2.10	...	7	5	7	...	1	...	2	1.56	1.50	...	...	21	...	2	23	66
22	Cephalalgia	1.54	1.53	1.54	...	4	11	5	...	2	...	...	2.21	0.75	...	...	16	...	...	16	44
20	Fever	1.40	1.53	1.29	1	10	4	4	...	...	...	1	1.70	1.05	May 3.75	Mar. & Sept. 0.00	15	...	2	17	23
19	Hysteria	1.33	...	2.45	...	10	4	3	2	...	...	...	1.44	1.20	...	...	14	2	...	16	46
18	External Inflammations	1.26	1.07	1.41	1	9	2	1	1	3	1	...	1.18	1.35	...	...	15	2	...	17	38
16	Hypochondriasis	1.12	1.53	1.29	...	1	3	5	4	2	1	...	0.65	1.35	...	...	5	1	...	6	38
16	Diarrhœa	1.12	0.61	1.54	7	2	4	2	...	...	...	1	1.18	1.03	...	...	8	...	...	8	33
776		54.32	49.08	59.03													436	56	21	513	

(1) The figures in each column represent respectively the proportion per cent. of the Town or Country Cases of each disease to the total number of Town (763), or Country (665) patients.

(2) The figures represent the proportion per cent. of the cases of each disease to the total of admissions in the month in which the maximum or minimum occurred, respectively.

(3) Those cases are omitted which were discharged for non-attendance, irregularity, &c., their results not being determined.

* Attended with extreme debility.

The cases of the diseases and accidents not included in the preceding Table, were, for each disease respectively, too few, to render the statistical particulars respecting them confined to a single year, of any importance,—but the materials from which the reports of this and the preceding year are derived, are so arranged, and the method of keeping the hospital registers so contrived, that the particulars and results of any one of the diseases enumerated in the tables may be hereafter generalized for any given number of years.

The influenza, it may be remarked, attacked the greater part of the patients under treatment for other diseases, during its prevalence in January and February, and may have indirectly contributed to increase the mortality,—but few cases comparatively were admitted *originally* for that disorder.

VACCINATION. 1837.

	Vaccinated.	*Taken.*	*Not taken.*	
February	4	4		
July	6	6		
August	7	1	6	
September	13	13		
October	15	15		
November	4	4		
December	17	13		4 not presented after Vaccination.
	66	56	6	4

REGISTER OF BAROMETER AND THERMOMETER, 1837.

(*Computed from the* Register *at the* Philosophical Society.)

1837.	Barometer at 8 A.M.							Thermometer.							
	Mean.	Maximum.	Minimum.	Range.	Greatest Diurnal Range	Least Diurnal Range.	Mean Diurnal Range.	Mean.	Maximum.	Minimum.	Range.	Greatest Diurnal Range.	Least Diurnal Range.	Mean Diurnal Range.	Number of Days Minimum at 32° or under.
January	29.93	30.51	29.39	1.12	.70	.00	.18	36.64°	57°	21°	36°	25°	1°	9.48°	21
February ...	29.58	30.42	29.19	1.23	.74	.02	.20	39.30°	55°	27°	28°	23°	1°	12.87°	15
March	30.00	30.39	29.32	1.07	.54	.00	.14	35.79°	49°	12°	37°	27°	4°	13.95°	23
April	29.80	30.38	29.41	.97	.48	.00	.15	40.25°	61°	22°	39°	30°	7°	17.70°	18
May	29.95	30.32	29.63	.69	.30	.00	.11	48.90°	71°	31°	40°	33°	12°	21.11°	3
June	29.99	30.37	29.63	.74	.27	.00	.15	60.10°	83°	32°	51°	37°	11°	25.68°	1
July	29.99	30.38	29.41	.97	.41	.00	.08	63.20°	85°	40°	45°	38°	13°	22.12°	0
August	30.01	30.41	29.39	1.02	.50	.01	.14	62.20°	84°	41°	43°	32°	4°	20.75°	0
September ..	29.89	30.32	29.18	1.14	.58	.00	.11	56.60°	73°	39°	34°	27°	9°	18.73°	0
October	30.01	30.71	29.31	1.40	.54	.00	.11	51.77°	70°	32°	38°	25°	7°	17.32°	1
November ..	29.82	30.43	29.01	1.42	.65	.01	.17	40.73°	58°	26°	32°	25°	7°	14.50°	17
December ..	29.97	30.47	29.36	1.11	.56	.00	.17	42.20°	57°	24°	33°	21°	3°	10.41°	9
For the year	29.90	30.71	29.01	1.70	.74	.00	.14	48.14°	85°	12°	73°	38°	1°	17.05°	108

Fig I

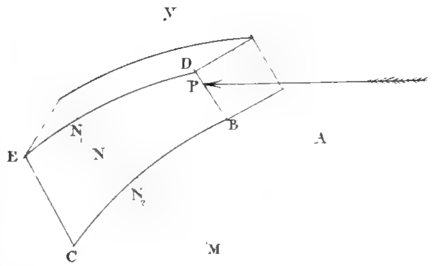


Fig II

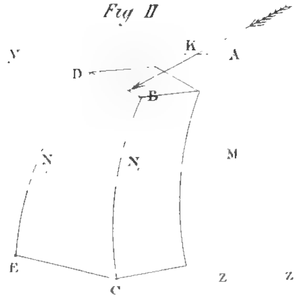


Fig III

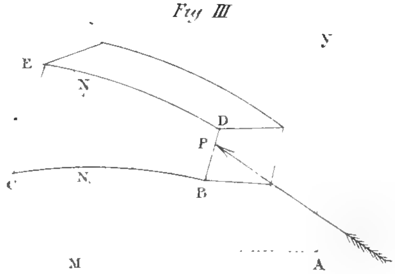


Fig IV

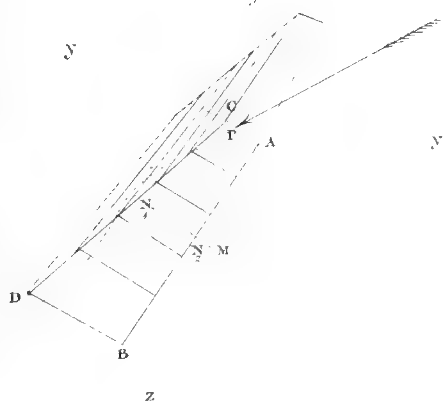


Fig V

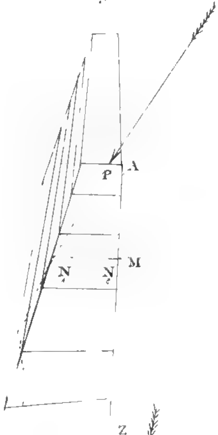


Fig VI

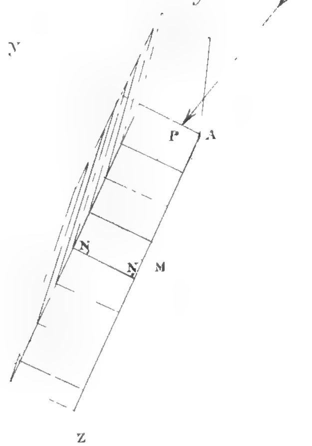


Fig VII

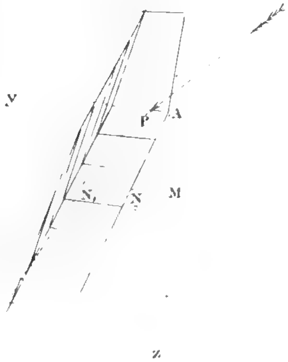


Fig VIII

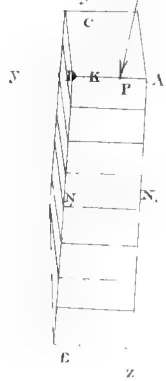


Fig IX

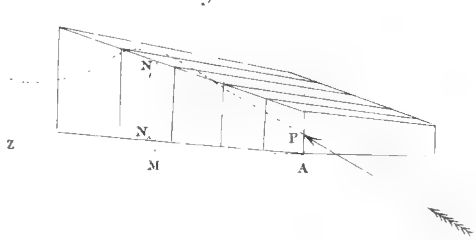


Fig X

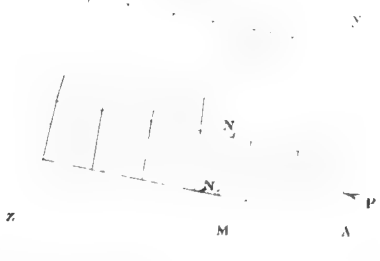


Fig XI

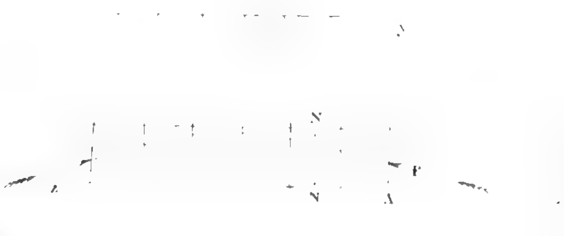


Fig. XII

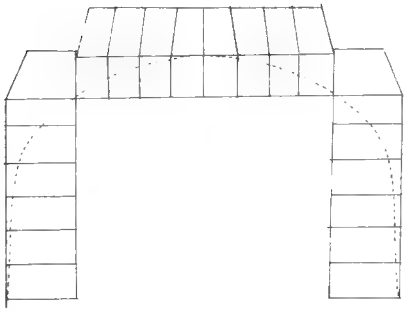


Fig. XIV

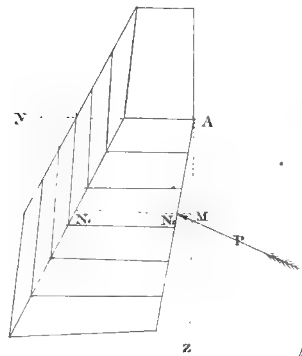


Fig. XIII

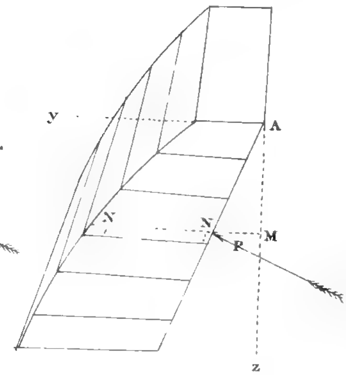
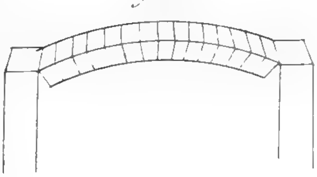


Fig. XIX



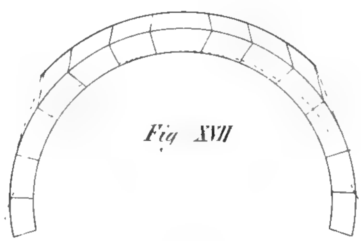


Fig. XVII

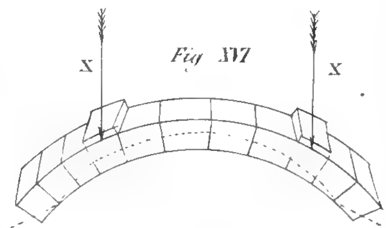


Fig. XVI

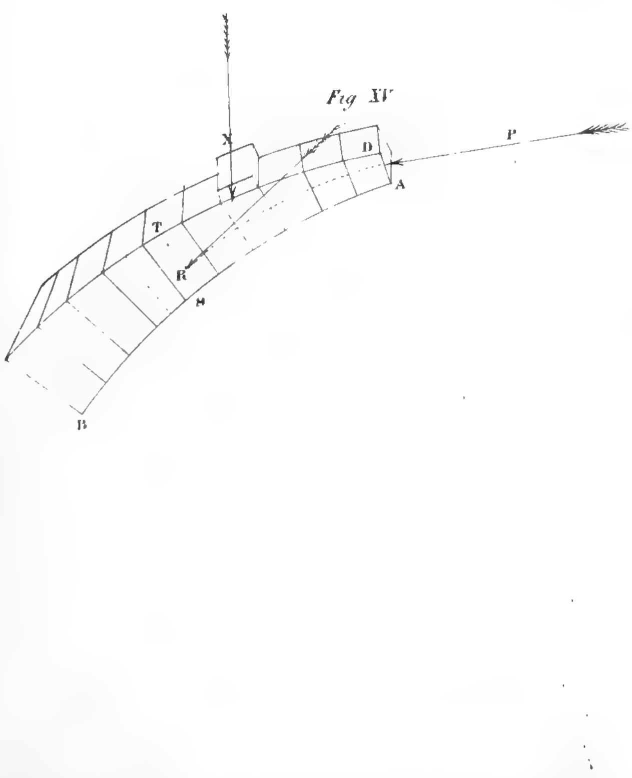
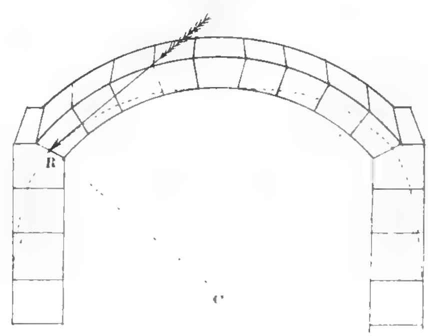
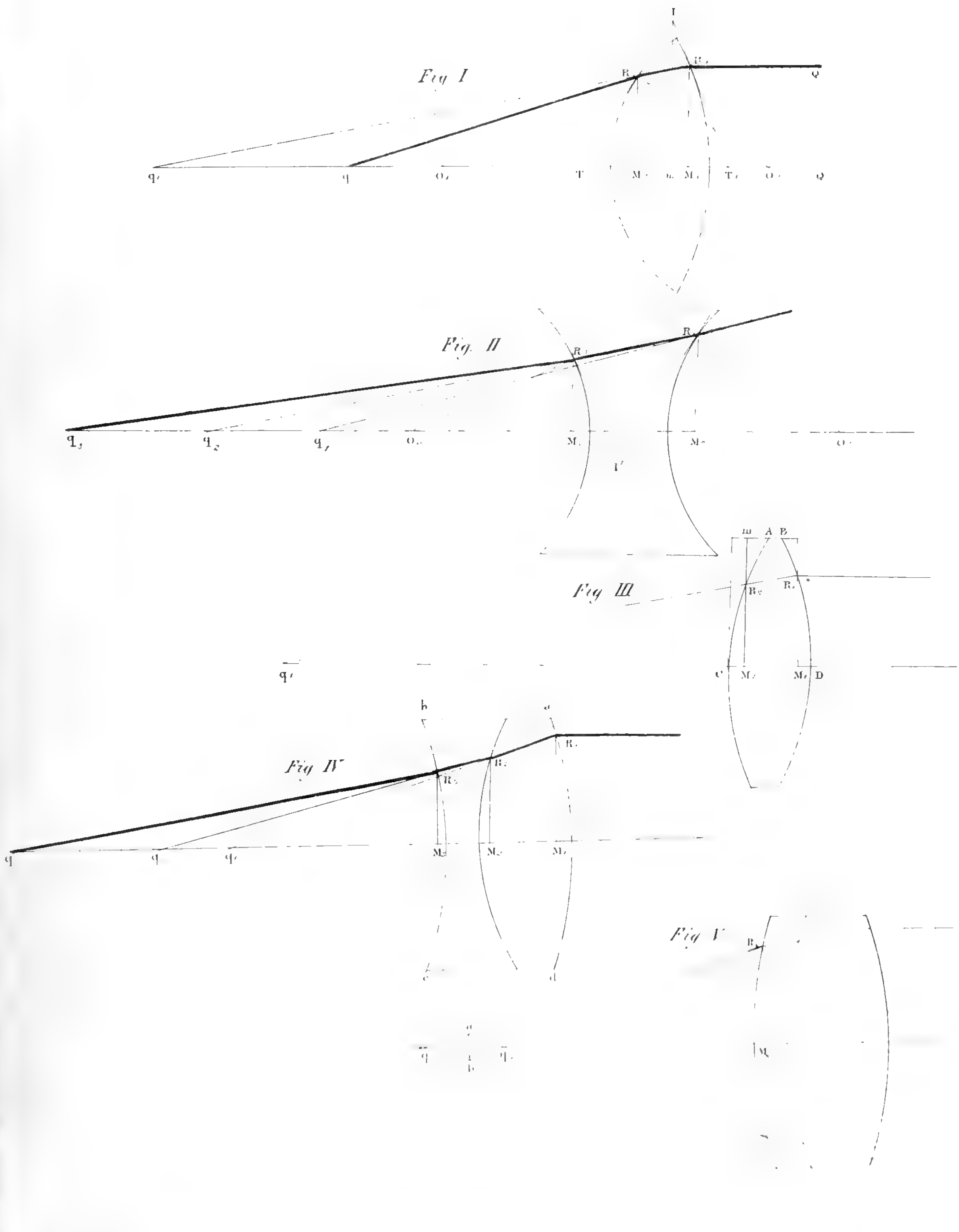


Fig. XV

Fig. XVIII





Metcalfe & Palmer lithog. Cambridge

Fig. III

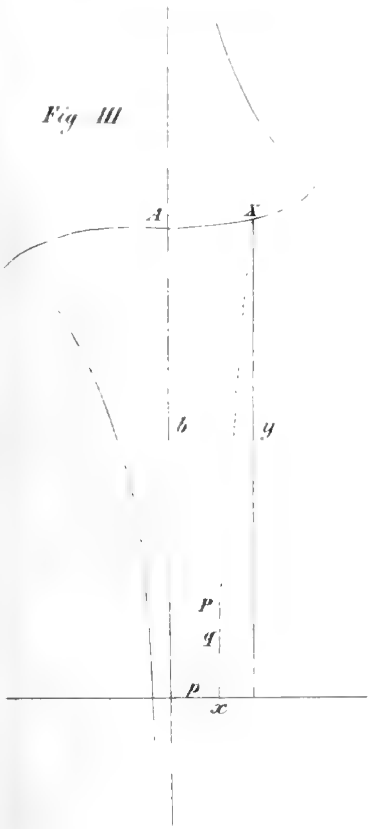
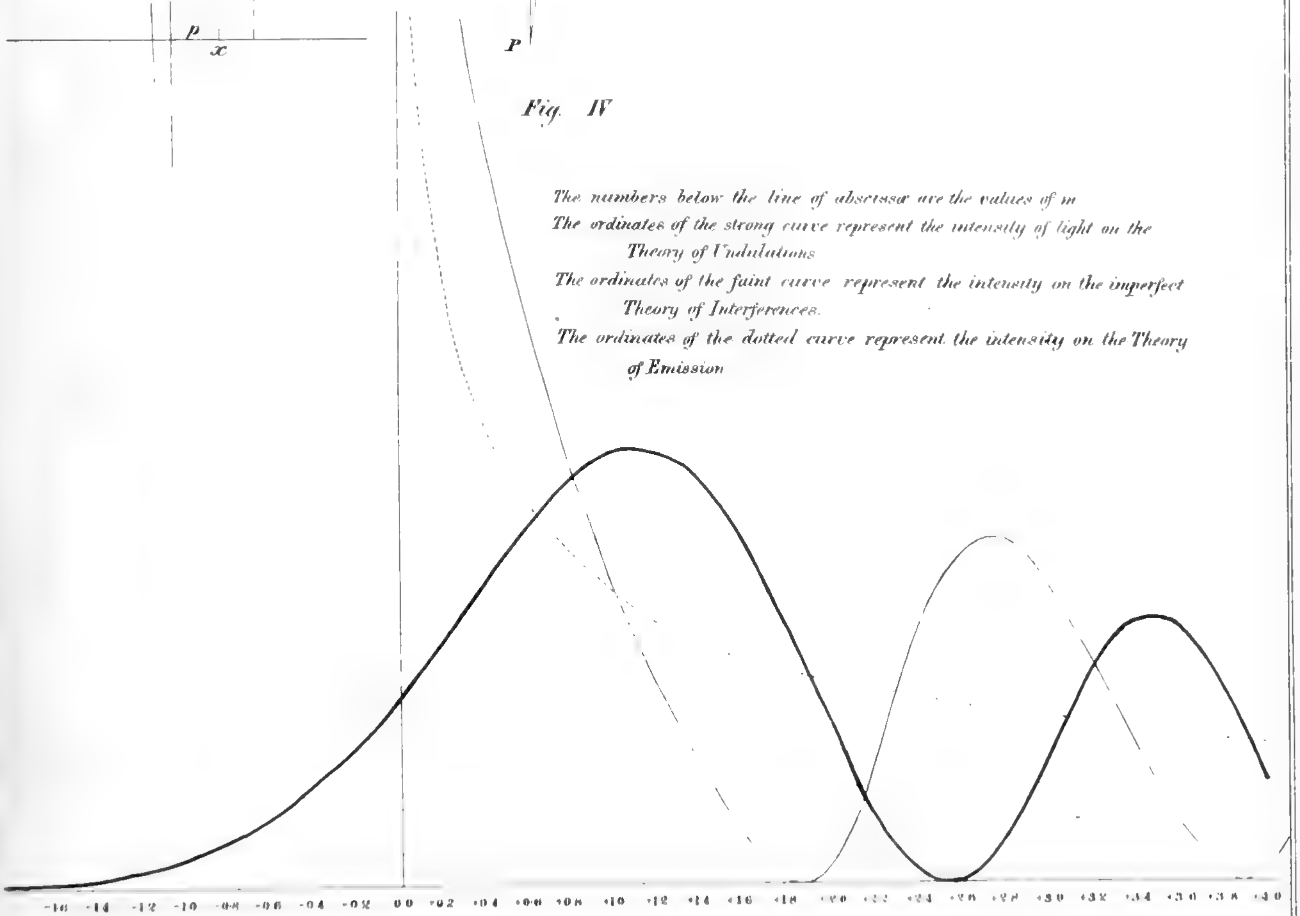


Fig. I

Fig. II

Fig. IV



The numbers below the line of abscissæ are the values of m

The ordinates of the strong curve represent the intensity of light on the Theory of Undulations

The ordinates of the faint curve represent the intensity on the imperfect Theory of Interferences.

The ordinates of the dotted curve represent the intensity on the Theory of Emission

Metcalfe & Palmer Lith. Cambridge

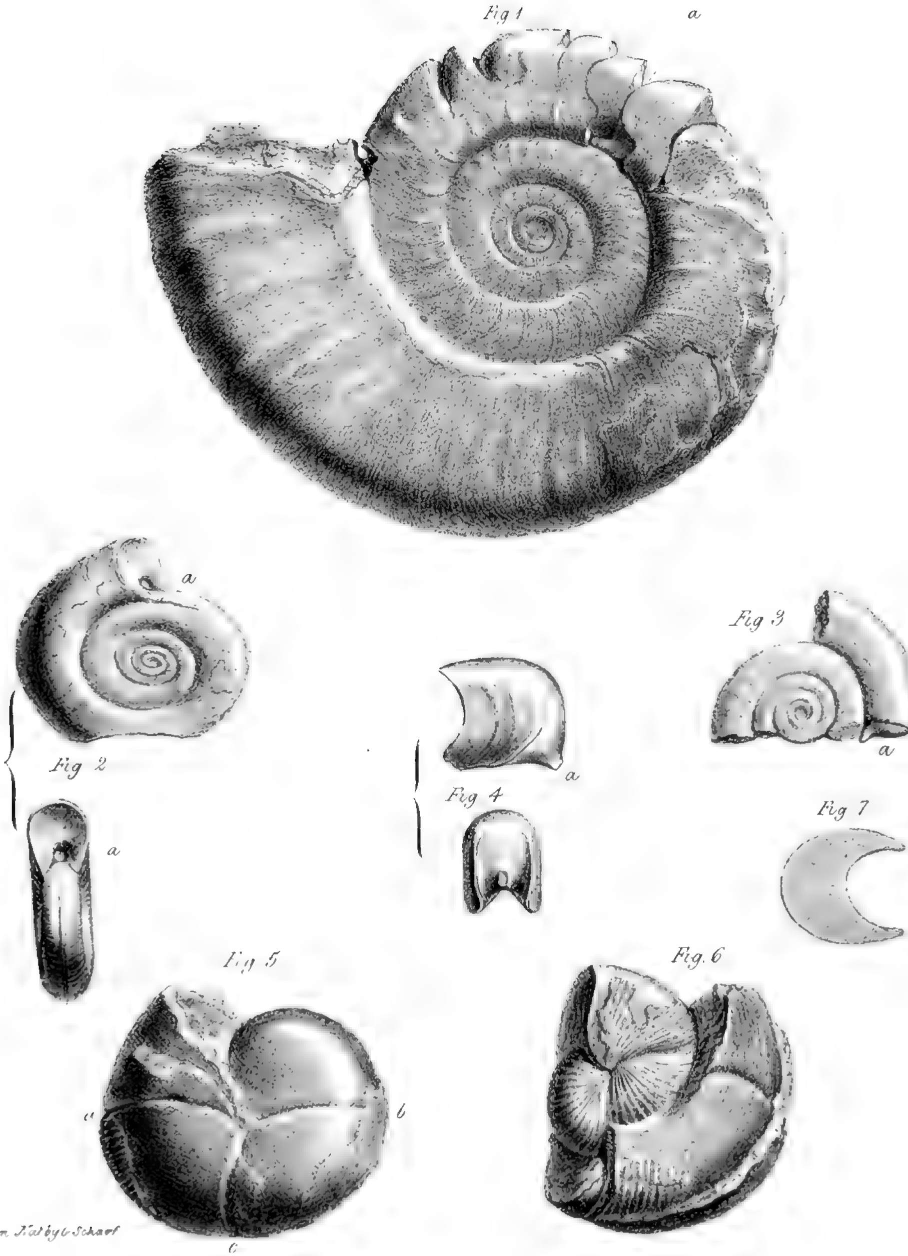


Fig 1 Endosiphonites Münsteri. } Nat! Size
2 .... carinatus
3 .... minutus. } Mag! twice
4. Portions of Endosiphonites.
5 6. Goniatite (?)