THE

TRANSACTIONS

OF THE

ROYAL IRISH ACADEMY.

VOL. XVII.

DUBLIN:
PRINTED BY P. DIXON HARDY, 3, CECILIA STREET,
PRINTER TO THE ROYAL IRISH ACADEMY.

1837.

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CONTENTS.

PART I.

SCIENCE.

CONTENTS.

PART II.

SCIENCE.

CONTENTS.

PART III.

SCIENCE.

ANTIQUITIES.

Third Supplement to an Essay on the Theory of Systems of Rays. By WILLIAM R. HAMILTON, A. B., M. R. I. A., *M. R. Ast. Soc. London, M. G. Soc. Dublin, Hon. M. Soc. Arts for Scotland, Hon. M. Portsmouth Lit. and Phil. Soc., Member of the British Association for the Advancement of Science, Fellow of the American Academy of Arts and Sciences, Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.*

Read January 23, 1832, and October 22, 1832.

ERRATA.

Page 60, for Partridge, (Perdix *coturnix,*) read (Perdix *cinerea.*)
Page 67, for *pittock*, read *piltock.*
Page 68, for *Insecta*, read *Crustacea.*
Page 69, for *accidentally*, read *incidentally.*

Of these the theory of external and internal conical refraction, deduced by my general methods from the principles of FRESNEL, will probably be thought the least undeserving of attention. It is right, therefore, to state that this theory had been deduced, and was communicated to a general meeting of the Royal Irish Academy, not at the earlier, but at the later of the two dates prefixed to the present Supplement. After making this communication to the Academy, in October, 1832, I requested Professor LLOYD to examine the question experimentally, and to try whether he could perceive any such phenomena in biaxal crystals, as my theory of conical refraction had led me to expect. The experiments of Professor LLOYD, confirming my theoretical expectations, have been published by him in the numbers of the London and Edinburgh Philosophical Magazine, for the months of February and March, 1833; and they will be found with fuller details in the present Volume of the Irish Transactions.

I am informed that James Mac Cullagh, Esq. F.T.C.D. who published in the last preceding Volume of these Transactions a series of elegant Geometrical Illustrations of Fresnel's theory, has, since he heard of the experiments of Professor Lloyd, employed his own geometrical methods to confirm my results respecting the existence of those conoidal cusps and circles of contact on Fresnel's wave, from which I had been led to the expectation of conical refraction. And on my lately mentioning to him that I had connected these cusps and circles on Fresnel's wave, with circles and cusps of the same kind on a certain other surface discovered by M. Cauchy, by a general theory of reciprocal surfaces, which I stated last year at a general meeting of the Royal Irish Academy, Mr. Mac Cullagh said that he had arrived independently at similar results, and put into my hands a paper on the subject, which I have not yet been able to examine, but which will (I hope) be soon presented to the Academy, and published in their Transactions.

I ought also to mention, that on my writing in last November to Professor Airy, and communicating to him my results respecting the cusps and circles on Fresnel's wave, and my expectation of conical refraction which had not then been verified, Professor Airy replied that he had long been aware of the existence of the conoidal cusps, which indeed it is surprising that Fresnel did not perceive. Professor Airy, however, had not perceived the existence of the circles of contact, nor had he drawn from either cusps or circles any theory of conical refraction.

This latter theory was deduced, by my general methods, from the hypothesis of transversal vibrations in a luminous ether, which hypothesis seems to have been first proposed by Dr. Young, but to have been independently framed and far more perfectly developed by Fresnel; and from Fresnel's other principle, of the existence of three rectangular axes of elasticity within a biaxal crystallised medium. The verification, therefore, of this theory of conical refraction, by the experiments of Professor Lloyd, must be considered as affording a new and important probability in favour of Fresnel's views: that is, a new encouragement to reason from those views, in combining and predicting appearances.

The length to which the present Supplement has already extended, obliges me to reserve, for a future communication, many other results deduced by my general methods from the principle of the characteristic function: and especially a general theory of the focal lengths and aberrations of optical instruments of revolution.

WILLIAM R. HAMILTON.

OBSERVATORY, *June*, 1833.

CONTENTS OF THE THIRD SUPPLEMENT.

THIRD SUPPLEMENT.

Fundamental Formula of Mathematical Optics. Design of the present Supplement.

1. WHEN light is considered as propagated, according to that known general law which is called the law of least action, or of swiftest propagation, along any curved or polygon ray, ordinary or extraordinary, describing each element of that ray $ds = \surd(dx^2 + dy^2 + dz^2)$ with a molecular velocity or undulatory slowness v, which is supposed to depend, in the most general case, on the nature of the medium, the position and direction of the element, and the colour of the light, having only a finite number of values when these are given, and being therefore a function of the three rectangular co-ordinates, or marks of position, x, y, z, the three differential ratios or cosines of direction,

$$\alpha = \frac{dx}{ds}, \beta = \frac{dy}{ds}, \gamma = \frac{dz}{ds},$$

and a chromatic index or measure of colour, χ, the form of which function v depends on and characterises the medium; then if we denote as follows the variation of this function,

$$\delta v = \frac{\delta v}{\delta x}\delta x + \frac{\delta v}{\delta y}\delta y + \frac{\delta v}{\delta z}\delta z + \frac{\delta v}{\delta \alpha}\delta \alpha + \frac{\delta v}{\delta \beta}\delta \beta + \frac{\delta v}{\delta \gamma}\delta \gamma + \frac{\delta v}{\delta \chi}\delta \chi,$$

and if, by the help of the relation $\alpha^2 + \beta^2 + \gamma^2 = 1$, we determine

$$\frac{\delta v}{\delta \alpha}, \frac{\delta v}{\delta \beta}, \frac{\delta v}{\delta \gamma},$$

so as to satisfy the condition

$$\alpha\frac{\delta v}{\delta \alpha} + \beta\frac{\delta v}{\delta \beta} + \gamma\frac{\delta v}{\delta \gamma} = v,$$

namely, by making v homogeneous of the first dimension with respect to α, β, γ; it has been shown, in my First Supplement, that the variation of the definite integral $V = \int v\, ds$, considered as a function, which I have called the *Characteristic Function* of the final and initial co-ordinates, that is, the *variation of the action, or the time, expended by light of any one colour, in going from one variable point to another*, is

$$\delta V = (\delta \int v ds =) \frac{\delta v}{\delta \alpha} \delta x - \frac{\delta v'}{\delta \alpha'} \delta x' + \frac{\delta v}{\delta \beta} \delta y - \frac{\delta v'}{\delta \beta'} \delta y' + \frac{\delta v}{\delta \gamma} \delta z - \frac{\delta v'}{\delta \gamma'} \delta z' : \text{(A)}$$

the accented being the initial quantities. This general equation, (*A*), which I have called the *Equation of the Characteristic Function*, involves very various and extensive consequences, and appears to me to include the whole of mathematical optics. I propose, in the present Supplement, to offer some additional remarks and methods, connected with the characteristic function V, and the fundamental formula (A); and in particular to point out a new view of the auxiliary function W, introduced in my former memoirs, and a new auxiliary function T, which may be employed with advantage in many optical researches: I shall also give some other general transformations and applications of the fundamental formula, and shall speak of the connection of my view of optics with the undulatory theory of light.

Fundamental Problem of Mathematical Optics, and Solution by the Fundamental Formula. Partial Differential Equations, respecting the Characteristic Function V, and common to all optical combinations. Deduction of the Medium Functions Ω, v, from this Characteristic Function V. Remarks on the new symbols σ, τ, υ.

2. It may be considered as a *fundamental problem* in Mathematical Optics, to which all others are reducible, *to determine, for any proposed combination of media, the law of dependence of the two extreme directions of a curved or polygon ray, ordinary or extraordinary, on the positions of the two extreme points which are visually connected by that ray, and on the colour of the light:* that is, in our present notation, to determine the law of dependence of the extreme *direction-cosines* $\alpha\ \beta\ \gamma\ \alpha'\ \beta'\ \gamma'$, on the extreme co-ordinates $x\ y\ z\ x'\ y'\ z'$, and on the chromatic index χ. This fundamental problem is resolved by our fundamental formula (*A*); or by the six following equations into which (*A*) resolves itself, and which express the law of dependence required:

$$\left.\begin{array}{l} \frac{\delta V}{\delta x} = \frac{\delta v}{\delta \alpha}; \ \frac{\delta V}{\delta y} = \frac{\delta v}{\delta \beta}; \ \frac{\delta V}{\delta z} = \frac{\delta v}{\delta \gamma}; \\ -\frac{\delta V}{\delta x'} = \frac{\delta v'}{\delta \alpha'}; \ -\frac{\delta V}{\delta y'} = \frac{\delta v'}{\delta \beta'}; \ -\frac{\delta V}{\delta z'} = \frac{\delta v'}{\delta \gamma'}. \end{array}\right\} \quad \text{(B)}$$

These equations appear to require, for their application to any proposed combination, not only the knowledge of the form of the *Characteristic Function V*, that is, the law of dependence of the action or time on the extreme positions and on the colour, but also the knowledge of the forms of the functions v, v', that is, the optical properties of the final and initial media; but these final and initial *medium-functions* v, v', may themselves be deduced from the one characteristic function V, by reasonings of the following kind.

Whatever be the nature of the final medium, that is, whatever be the law of dependence of v on the position, direction, and colour, we have supposed, in deducing the general formula (A), that the expression of this dependence has been so prepared as to make the medium-function v homogeneous of the first dimension relatively to the direction-cosines α, β, γ; the partial differential co-efficients

$$\frac{\delta v}{\delta \alpha}, \frac{\delta v}{\delta \beta}, \frac{\delta v}{\delta \gamma},$$

of this homogeneous function, are therefore themselves homogeneous, but of the dimension zero; that is, they are functions of the two ratios

$$\frac{\alpha}{\gamma}, \frac{\beta}{\gamma},$$

involving also, in general, the co-ordinates $x\ y\ z$, and the chromatic index χ: if then we conceive the two ratios

$$\frac{\alpha}{\gamma}, \frac{\beta}{\gamma},$$

to be eliminated between the three first of the equations (B), and if, in like manner, we conceive

$$\frac{\alpha'}{\gamma'}, \frac{\beta'}{\gamma'},$$

to be eliminated between the three last equations (B), we see that such eliminations would give two partial differential equations of the first order, between the characteristic function V and the co-ordinates and colour, of the form

$$\left.\begin{aligned} 0 &= \Omega\left(\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z}, x, y, z, \chi\right), \\ 0 &= \Omega'\left(-\frac{\delta V}{\delta x'}, -\frac{\delta V}{\delta y'}, -\frac{\delta V}{\delta z'}, x', y', z', \chi\right), \end{aligned}\right\} \quad (C)$$

which both conduct to the following general equation, of the second order and third degree, common to all optical combinations,

$$\begin{aligned} &\frac{\delta^2 V}{\delta x \delta x'} \frac{\delta^2 V}{\delta y \delta y'} \frac{\delta^2 V}{\delta z \delta z'} + \frac{\delta^2 V}{\delta x \delta y'} \frac{\delta^2 V}{\delta y \delta z'} \frac{\delta^2 V}{\delta z \delta x'} + \frac{\delta^2 V}{\delta x \delta z'} \frac{\delta^2 V}{\delta y \delta x'} \frac{\delta^2 V}{\delta z \delta y'} \\ = &\frac{\delta^2 V}{\delta z \delta x'} \frac{\delta^2 V}{\delta y \delta y'} \frac{\delta^2 V}{\delta x \delta z'} + \frac{\delta^2 V}{\delta z \delta y'} \frac{\delta^2 V}{\delta y \delta z'} \frac{\delta^2 V}{\delta x \delta x'} + \frac{\delta^2 V}{\delta z \delta z'} \frac{\delta^2 V}{\delta y \delta x'} \frac{\delta^2 V}{\delta x \delta y'}. \end{aligned} \quad (D)$$

If now we put, for abridgment,

$$\left.\begin{aligned} &\frac{\delta V}{\delta x} = \sigma, \frac{\delta V}{\delta y} = \tau, \frac{\delta V}{\delta z} = \upsilon, \\ -&\frac{\delta V}{\delta x'} = \sigma', -\frac{\delta V}{\delta y'} = \tau', -\frac{\delta V}{\delta z'} = \upsilon', \end{aligned}\right\} \quad (E)$$

and if between the three first of these equations (E) we eliminate two of the three initial co-ordinates x' y' z', it is easy to perceive, by (C) or (D), that in every optical combination the third co-ordinate will disappear; and similarly that between the three last equations (E) we can eliminate all the three final co-ordinates, by eliminating any two of them; and that these eliminations will conduct to the relations (C) under the form

$$\left.\begin{array}{l} 0=\Omega\ (\sigma, \tau, \upsilon, x, y, z, \chi), \\ 0=\Omega'(\sigma,' \tau', \upsilon', x', y', z', \chi), \end{array}\right\} \quad \text{(F)}$$

which can thus be obtained, by differentiation and elimination, from the characteristic function V alone: and which, as we are about to see, determine the forms of v, v', that is, the properties of the extreme media. Comparing the differentials of the relations (F), with the following, that is, with the conditions of homogeneity of v, v', prepared by the definitions (E) and by the relations (B),

$$\left.\begin{array}{l} v = \alpha \dfrac{\delta v}{\delta \alpha} + \beta \dfrac{\delta v}{\delta \beta} + \gamma \dfrac{\delta v}{\delta \gamma} = \alpha\sigma + \beta\tau + \gamma\upsilon, \\ v' = \alpha' \dfrac{\delta v'}{\delta \alpha'} + \beta' \dfrac{\delta v'}{\delta \beta'} + \gamma' \dfrac{\delta v'}{\delta \gamma'} = \alpha'\sigma' + \beta'\tau' + \gamma'\upsilon', \end{array}\right\} \quad \text{(G)}$$

and with their differentials, that is with

$$\left.\begin{array}{l} \alpha\delta\sigma + \beta\delta\tau + \gamma\delta\upsilon = \dfrac{\delta v}{\delta x}\delta x + \dfrac{\delta v}{\delta y}\delta y + \dfrac{\delta v}{\delta z}\delta z + \dfrac{\delta v}{\delta \chi}\delta\chi, \\ \alpha'\delta\sigma' + \beta'\delta\tau' + \gamma'\delta\upsilon' = \dfrac{\delta v'}{\delta x'}\delta x' + \dfrac{\delta v'}{\delta y'}\delta y' + \dfrac{\delta v'}{\delta z'}\delta z' + \dfrac{\delta v'}{\delta \chi}\delta\chi, \end{array}\right\} \quad \text{(H)}$$

we find

$$\left.\begin{array}{l} \dfrac{\alpha}{v} = \dfrac{\delta\Omega}{\delta\sigma},\ \dfrac{\beta}{v} = \dfrac{\delta\Omega}{\delta\tau},\ \dfrac{\gamma}{v} = \dfrac{\delta\Omega}{\delta\upsilon}, \\ \dfrac{\alpha'}{v'} = \dfrac{\delta\Omega'}{\delta\sigma'},\ \dfrac{\beta'}{v'} = \dfrac{\delta\Omega'}{\delta\tau'},\ \dfrac{\gamma'}{v'} = \dfrac{\delta\Omega'}{\delta\upsilon'}, \end{array}\right\} \quad \text{(I)}$$

and also

$$\left.\begin{array}{l} -\dfrac{1}{v}\dfrac{\delta v}{\delta x} = \dfrac{\delta\Omega}{\delta x},\ -\dfrac{1}{v}\dfrac{\delta v}{\delta y} = \dfrac{\delta\Omega}{\delta y},\ -\dfrac{1}{v}\dfrac{\delta v}{\delta z} = \dfrac{\delta\Omega}{\delta z},\ -\dfrac{1}{v}\dfrac{\delta v}{\delta \chi} = \dfrac{\delta\Omega}{\delta \chi}, \\ -\dfrac{1}{v'}\dfrac{\delta v'}{\delta x'} = \dfrac{\delta\Omega'}{\delta x'},\ -\dfrac{1}{v'}\dfrac{\delta v'}{\delta y'} = \dfrac{\delta\Omega'}{\delta y'},\ -\dfrac{1}{v'}\dfrac{\delta v'}{\delta z'} = \dfrac{\delta\Omega'}{\delta z'},\ -\dfrac{1}{v'}\dfrac{\delta v'}{\delta \chi} = \dfrac{\delta\Omega'}{\delta \chi}, \end{array}\right\} \quad \text{(K)}$$

if we so prepare the expressions of the relations (F) as to have

$$\left.\begin{array}{l} \sigma\dfrac{\delta\Omega}{\delta\sigma} + \tau\dfrac{\delta\Omega}{\delta\tau} + \upsilon\dfrac{\delta\Omega}{\delta\upsilon} = 1, \\ \sigma'\dfrac{\delta\Omega'}{\delta\sigma'} + \tau'\dfrac{\delta\Omega'}{\delta\tau'} + \upsilon'\dfrac{\delta\Omega'}{\delta\upsilon'} = 1; \end{array}\right\} \quad \text{(L)}$$

which can be done by putting those relations under the form

$$\left.\begin{array}{l} 0 = (\sigma^2 + \tau^2 + \upsilon^2)^{\frac{1}{2}}\ \omega - 1 = \Omega, \\ 0 = (\sigma'^2 + \tau'^2 + \upsilon'^2)^{\frac{1}{2}}\ \omega' - 1 = \Omega'; \end{array}\right\} \quad \text{(M)}$$

in which ω, ω', that is, $(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$, and $(\sigma'^2+\tau'^2+\upsilon'^2)^{-\frac{1}{2}}$, are to be expressed as functions respectively of $\sigma\,(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$, $\tau\,(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$, $\upsilon\,(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$, x, y, z, χ, and of $\sigma'\,(\sigma'^2+\tau'^2+\upsilon'^2)^{-\frac{1}{2}}$, $\tau'\,(\sigma'^2+\tau'^2+\upsilon'^2)^{-\frac{1}{2}}$, $\upsilon'\,(\sigma'^2+\tau'^2+\upsilon'^2)^{-\frac{1}{2}}$, x', y', z', χ. After this preparation the partial differential coefficients

$$\frac{\delta\Omega}{\delta\sigma},\ \frac{\delta\Omega}{\delta\tau},\ \frac{\delta\Omega}{\delta\upsilon},$$

are homogeneous of dimension zero relatively to σ, τ, υ; and in like manner

$$\frac{\delta\Omega'}{\delta\sigma'},\ \frac{\delta\Omega'}{\delta\tau'},\ \frac{\delta\Omega'}{\delta\upsilon'},$$

are homogeneous of dimension zero relatively to $\sigma', \tau', \upsilon'$; if, therefore, between the three first equations (*I*), we eliminate any two of the three final quantities σ, τ, υ, the third will disappear; and similarly all the three initial quantities $\sigma', \tau', \upsilon'$, can be eliminated together, between the three last of the equations (*I*): and by these eliminations we shall be conducted to two relations of the form

$$\left.\begin{aligned} 0 &= \Psi\left(\frac{\alpha}{v}, \frac{\beta}{v}, \frac{\gamma}{v}, x, y, z, \chi\right), \\ 0 &= \Psi'\left(\frac{\alpha'}{v'}, \frac{\beta'}{v'}, \frac{\gamma'}{v'}, x', y', z', \chi\right), \end{aligned}\right\} \qquad \text{(N)}$$

which determine the forms of the final and initial medium-functions v, v'; so that these forms can be deduced from the form of the characteristic function V. We can therefore reduce to the study of this one function V, that general problem of mathematical optics which has been already mentioned.

The partial differential coefficients of the characteristic function V, taken with respect to the co-ordinates x, y, z, are of continual occurrence in the optical methods of my present and former memoirs; I have therefore thought it useful to denote them in this Supplement by separate symbols, σ, τ, υ, and I shall show in a future number their meanings in the undulatory theory: namely, that they denote, in it, the components of normal slowness of propagation of a wave.

Connection of the Characteristic Function V, with the Formation and Integration of the General Equations of a Curved Ray, Ordinary or Extraordinary.

3. It may be considered as a particular case of the foregoing general problem, to determine general forms for the differential equations of a curved ray, ordinary or extraordinary; that is, to connect the general changes of direction with those of position, in the passage of light through a variable medium. The following forms,

$$d\,\frac{\delta v}{\delta \alpha} = \frac{\delta v}{\delta x}\,ds,\ d\,\frac{\delta v}{\delta \beta} = \frac{\delta v}{\delta y}\,ds,\ d\,\frac{\delta v}{\delta \gamma} = \frac{\delta v}{\delta z}\,ds, \qquad \text{(O)}$$

(which are of the second order, because α, β, γ, α', β', γ', are defined by the equations

$$\left.\begin{aligned} \alpha &= \frac{dx}{ds}, \ \beta = \frac{dy}{ds}, \ \gamma = \frac{dz}{ds}, \\ \alpha' &= \frac{dx'}{ds'}, \ \beta' = \frac{dy'}{ds'}, \ \gamma' = \frac{dz'}{ds'}, \end{aligned}\right\} \qquad (P)$$

the symbol d referring, throughout the present Supplement, to motion along a ray, while δ refers to arbitrary infinitesimal changes of position, direction, and colour, and ds' being the initial element of the ray,) were deduced, in the First Supplement, by the Calculus of Variations, from the law of least action. The same forms (O), which are equivalent to but two distinct equations, may be deduced from the fundamental formula (A), by the properties of the characteristic function V. For, if we differentiate the first equation (C), (which involves the coefficients of this function V, and was deduced from the formula (A),) with reference to each of the three co-ordinates, x, y, z, considered as three independent variables, and with reference to the index of colour χ, we find, by the foregoing number,

$$\left.\begin{aligned} \alpha\frac{\delta^2 V}{\delta x^2} + \beta\frac{\delta^2 V}{\delta x \delta y} + \gamma\frac{\delta^2 V}{\delta x \delta z} &= \frac{\delta v}{\delta x}, \\ \alpha\frac{\delta^2 V}{\delta x \delta y} + \beta\frac{\delta^2 V}{\delta y^2} + \gamma\frac{\delta^2 V}{\delta y \delta z} &= \frac{\delta v}{\delta y}, \\ \alpha\frac{\delta^2 V}{\delta x \delta z} + \beta\frac{\delta^2 V}{\delta y \delta z} + \gamma\frac{\delta^2 V}{\delta z^2} &= \frac{\delta v}{\delta z}, \\ \alpha\frac{\delta^2 V}{\delta x \delta \chi} + \beta\frac{\delta^2 V}{\delta y \delta \chi} + \gamma\frac{\delta^2 V}{\delta z \delta \chi} &= \frac{\delta v}{\delta \chi}; \end{aligned}\right\} \qquad (Q)$$

and the three first of these equations (Q), by the help of the general relations (B), which were themselves deduced from (A), and by the meanings (P) of α, β, γ, may easily be transformed to (O). The differential equations (O) may also be regarded as the limits of the following,

$$\sigma - \sigma' = \left(\frac{\delta V}{\delta x}\right), \ \tau - \tau' = \left(\frac{\delta V}{\delta y}\right), \ \upsilon - \upsilon' = \left(\frac{\delta V}{\delta z}\right), \qquad (R)$$

in which

$$\left(\frac{\delta V}{\delta x}\right) \ \left(\frac{\delta V}{\delta y}\right) \ \left(\frac{\delta V}{\delta y}\right)$$

are obtained by differentiating V considered as a function of the seven variables x, y, z, Δx, Δy, Δz, χ, if $\Delta x = x - x'$, $\Delta y = y - y'$, $\Delta z = z - z'$; the variation of V, when so considered, being by (A), and by the definitions (E),

$$\delta V = (\sigma - \sigma')\delta x + (\tau - \tau')\delta y + (\upsilon - \upsilon')\delta z + \left(\frac{\delta V}{\delta \Delta x}\right)\delta \Delta x + \left(\frac{\delta V}{\delta \Delta y}\right)\delta \Delta y + \left(\frac{\delta V}{\delta \Delta z}\right)\delta \Delta z + \frac{\delta V}{\delta \chi}\delta \chi, (S)$$

in which

$$\left(\frac{\delta V}{\delta \Delta x}\right) = \sigma', \left(\frac{\delta V}{\delta \Delta y}\right) = \tau', \left(\frac{\delta V}{\delta \Delta z}\right) = \upsilon'. \qquad (T)$$

If we differentiate the first equation (*C*) relatively to x', y', z', we find, by the foregoing number,

$$\left.\begin{aligned}0 &= \alpha \frac{\delta^2 V}{\delta x \delta x'} + \beta \frac{\delta^2 V}{\delta y \delta x'} + \gamma \frac{\delta^2 V}{\delta z \delta x'} ,\\ 0 &= \alpha \frac{\delta^2 V}{\delta x \delta y'} + \beta \frac{\delta^2 V}{\delta y \delta y'} + \gamma \frac{\delta^2 V}{\delta z \delta y'} ,\\ 0 &= \alpha \frac{\delta^2 V}{\delta x \delta z'} + \beta \frac{\delta^2 V}{\delta y \delta z'} + \gamma \frac{\delta^2 V}{\delta z \delta z'} ,\end{aligned}\right\} \qquad \text{(U)}$$

of which, in virtue of (*D*), any two include the third, and which may be put by (*P*) under the form

$$0 = d \frac{\delta V}{\delta x'} ; \; 0 = d \frac{\delta V}{\delta y'} ; \; 0 = d \frac{\delta V}{\delta z'} ; \qquad \text{(V)}$$

and these differential equations (*V*) of the first order, in which the initial co-ordinates and the colour are constant, belong to the ray, and may be regarded as integrals of (*O*). They have, themselves, for integrals,

$$\frac{\delta V}{\delta x'} = \text{const.}, \; \frac{\delta V}{\delta y'} = \text{const.}, \; \frac{\delta V}{\delta z'} = \text{const.}, \qquad \text{(W)}$$

the constants being, by (*B*), the values of the initial quantities

$$-\frac{\delta v'}{\delta \alpha'}, \; -\frac{\delta v'}{\delta \beta'}, \; -\frac{\delta v'}{\delta \gamma'} .$$

In like manner, by differentiating the last equation (*C*), we find the following equations, which are analogous to (*Q*) and (*U*),

$$\left.\begin{aligned}\alpha' \frac{\delta^2 V}{\delta x'^2} + \beta' \frac{\delta^2 V}{\delta x' \delta y'} + \gamma' \frac{\delta^2 V}{\delta x' \delta z'} &= -\frac{\delta v'}{\delta x'} ,\\ \alpha' \frac{\delta^2 V}{\delta x' \delta y'} + \beta' \frac{\delta^2 V}{\delta y'^2} + \gamma' \frac{\delta^2 V}{\delta y' \delta z'} &= -\frac{\delta v'}{\delta y'} ,\\ \alpha' \frac{\delta^2 V}{\delta x' \delta z'} + \beta' \frac{\delta^2 V}{\delta y' \delta z'} + \gamma' \frac{\delta^2 V}{\delta z'^2} &= -\frac{\delta v'}{\delta z'} ,\\ \alpha' \frac{\delta^2 V}{\delta x' \delta \chi} + \beta' \frac{\delta^2 V}{\delta y' \delta \chi} + \gamma' \frac{\delta^2 V}{\delta z' \delta \chi} &= -\frac{\delta v'}{\delta \chi} ;\end{aligned}\right\} \qquad \text{(X)}$$

and

$$\left.\begin{aligned}0 &= \alpha' \frac{\delta^2 V}{\delta x \delta x'} + \beta' \frac{\delta^2 V}{\delta x \delta y'} + \gamma' \frac{\delta^2 V}{\delta x \delta z'} ,\\ 0 &= \alpha' \frac{\delta^2 V}{\delta y \delta x'} + \beta' \frac{\delta^2 V}{\delta y \delta y'} + \gamma' \frac{\delta^2 V}{\delta y \delta z'} ,\\ 0 &= \alpha' \frac{\delta^2 V}{\delta z \delta x'} + \beta' \frac{\delta^2 V}{\delta z \delta y'} + \gamma' \frac{\delta^2 V}{\delta z \delta z'} .\end{aligned}\right\} \qquad \text{(Y)}$$

The second members of the three first equations (*X*) vanish when the initial medium is uniform, and those of the three first equations (*Q*) when the final medium is so; and in this latter case, of a final uniform medium, the final portion of the ray is straight, and in its whole extent we have not only the equations (W) but also the following,

$$\frac{\delta V}{\delta x} = \text{const.}\,, \quad \frac{\delta V}{\delta y} = \text{const.}\,, \quad \frac{\delta V}{\delta z} = \text{const.}\,, \qquad (\text{Z})$$

the constants being by (B) those functions of the final direction-cosines and of the colour which we have denoted by

$$\frac{\delta v}{\delta \alpha}, \ \frac{\delta v}{\delta \beta}, \ \frac{\delta v}{\delta \gamma},$$

and which are here independent of the co-ordinates. In general, if we consider the final co-ordinates and the colour as constant, the relations (Z) between the initial co-ordinates are forms for the equations of a ray. And though we have hitherto considered rectangular co-ordinates only, yet we shall show in a future number that there are analogous results for oblique and even for polar co-ordinates.

Transformations of the Fundamental Formula. New View of the Auxiliary Function W; New Auxiliary Function T. Deductions of the Characteristic and Auxiliary Functions, V, W, T, each from each. General Theorem of Maxima and Minima, which includes all the details of such deductions. Remarks on the respective advantages of the Characteristic and Auxiliary Functions.

4. The fundamental equation (A) may be put under the form

$$\delta V = \sigma \delta x - \sigma' \delta x' + \tau \delta y - \tau' \delta y' + \upsilon \delta z - \upsilon' \delta z' + \frac{\delta V}{\delta \chi} \delta \chi, \qquad (\text{A}')$$

employing the definitions (E), and introducing the variation of colour; it admits also of the two following general transformations,

$$\delta W = x \delta \sigma + y \delta \tau + z \delta \upsilon + \sigma' \delta x' + \tau' \delta y' + \upsilon' \delta z' - \frac{\delta V}{\delta \chi} \delta \chi, \qquad (\text{B}')$$

and

$$\delta T = x \delta \sigma - x' \delta \sigma' + y \delta \tau - y' \delta \tau' + z \delta \upsilon - z' \delta \upsilon' - \frac{\delta V}{\delta \chi} \delta \chi, \qquad (\text{C}')$$

in which

$$W = -V + x\sigma + y\tau + z\upsilon, \qquad (\text{D}')$$

and

$$T = W - x'\sigma' - y'\tau' - z'\upsilon'. \qquad (\text{E}')$$

In the two foregoing Supplements, the quantity W was introduced, and was considered as a function of the final direction-cosines α, β, γ, the final medium being regarded as uniform, and the luminous origin and colour as given; we shall now take another and a more general view of this auxiliary function W, and shall consider it as depending, by (*B′*), for all optical combinations, on the seven quantities $\sigma\ \tau\ \upsilon\ x'\ y'\ z\ \chi$. In like manner, we shall consider the new auxiliary function T as depending, by the new transformation (*C′*), on the seven quantities $\sigma\ \tau\ \upsilon\ \sigma'\ \tau'\ \upsilon'\ \chi$. The forms of these auxiliary functions, W, T, are connected with each other, and with the characteristic function V, by relations of which the knowledge is important, in the theory of optical systems. Let us therefore consider how the form of each of the three functions, V, W, T, can be deduced from the form of either of the other two.

These deductions may all be effected by suitable applications of the three forms (*A′*) (*B′*) (*C′*), of our fundamental equation (*A*), together with the definitions (*D′*) (*E′*), as we shall soon see more in detail, by means of the following remarks.

When the form of the characteristic function V is known, and it is required to deduce the form of the auxiliary function W, we are to eliminate the three final co-ordinates, x, y, z, between the equation (*D′*) and the three first of the equations (*E*); and similarly when it is required to deduce the form of T from that of V, we are to eliminate the six final and initial co-ordinates $x\ y\ z\ x'\ y'\ z'$ between the six equations (*E*), (which are all included in the formula (*A′*),) and the following,

$$T = -V + x\sigma - x'\sigma' + y\tau - y'\tau' + z\upsilon - z'\upsilon' : \qquad \text{(F′)}$$

and if it be required to deduce the form of T from that of W, we are to eliminate the three initial co-ordinates $x'\ y'\ z'$, between the equation (*E′*) and the three following general equations,

$$\sigma' = \frac{\delta W}{\delta x'},\ \tau' = \frac{\delta W}{\delta y'},\ \upsilon' = \frac{\delta W}{\delta z'}. \qquad \text{(G′)}$$

But when it is required to deduce reciprocally V from T or from W, or W from T, we must distinguish between the cases of variable and of uniform media; because we must then use the equations into which (*B′*) and (*C′*) resolve themselves, and this resolution, when the extreme media are not both variable, requires the consideration of the connexion that then exists between the quantities $\sigma\ \tau\ \upsilon\ \sigma'\ \tau'\ \upsilon'\ \chi$: which circumstance also, of a connexion between these variable quantities, leaves a partial indeterminateness in the forms of T and W as deduced from V, and in the form of T as deduced from W, for the case of uniform media.

When the final medium is variable, then σ, τ, υ, χ, may in general vary independently, and the equation (*B′*) gives

$$\frac{\delta W}{\delta \sigma} = x,\ \frac{\delta W}{\delta \tau} = y,\ \frac{\delta W}{\delta \upsilon} = z,\ \frac{\delta W}{\delta \chi} = -\frac{\delta V}{\delta \chi}; \qquad \text{(H′)}$$

and, in this case, V can in general be deduced from W by eliminating σ, τ, υ, between the equation (*D′*), and the three first equations (*H′*). But if the final medium be uniform, then σ, τ, υ, χ, are connected by the first of the relations (*F*), from which, in this case, the final co-ordinates disappear; and instead of the four equations (*H′*) we have the three following

$$\frac{\frac{\delta W}{\delta\sigma}-x}{\frac{\delta\Omega}{\delta\sigma}}=\frac{\frac{\delta W}{\delta\tau}-y}{\frac{\delta\Omega}{\delta\tau}}=\frac{\frac{\delta W}{\delta\upsilon}-z}{\frac{\delta\Omega}{\delta\upsilon}}=\frac{\frac{\delta W}{\delta\chi}+\frac{\delta V}{\delta\chi}}{\frac{\delta\Omega}{\delta\chi}}; \qquad \text{(I')}$$

by means of the two first of which, combined with the relation already mentioned, namely,

$$0=\Omega\,(\sigma,\tau,\upsilon,\chi), \qquad \text{(K')}$$

which depends on, and characterises, the nature of the final uniform medium, we can eliminate σ, τ, υ, from the equation (D'), and so deduce V from W.

In like manner, if both the extreme media be variable, then the seven quantities $\sigma\,\tau\,\upsilon\,\sigma'\,\tau'\,\upsilon'\,\chi$ may in general vary independently, and the equation (C') resolves itself into the seven following,

$$\frac{\delta T}{\delta\sigma}=x, \frac{\delta T}{\delta\tau}=y, \frac{\delta T}{\delta\upsilon}=z, \frac{\delta T}{\delta\chi}=-\frac{\delta V}{\delta\chi}, \frac{\delta T}{\delta\sigma'}=-x', \frac{\delta T}{\delta\tau'}=-y', \frac{\delta T}{\delta\upsilon'}=-z', \qquad \text{(L')}$$

by the three first and three last of which we can eliminate $\sigma\,\tau\,\upsilon\,\sigma'\,\tau'\,\upsilon'$ from (F'), and so deduce V from T. And in the same case, or even in the case when only the initial medium is variable, the three last of the equations (L') are true, and suffice to eliminate σ', τ', υ', from (E'), and so to deduce W from T.

But if the final medium be uniform, the initial being still variable, then σ, τ, υ, χ, are connected by the relation (K'), while $\sigma'\,\tau'\,\upsilon'$ remain independent; and instead of the four first equations (L') we have the three following,

$$\frac{\frac{\delta T}{\delta\sigma}-x}{\frac{\delta\Omega}{\delta\sigma}}=\frac{\frac{\delta T}{\delta\tau}-y}{\frac{\delta\Omega}{\delta\tau}}=\frac{\frac{\delta T}{\delta\upsilon}-z}{\frac{\delta\Omega}{\delta\upsilon}}=\frac{\frac{\delta T}{\delta\chi}+\frac{\delta V}{\delta\chi}}{\frac{\delta\Omega}{\delta\chi}}; \qquad \text{(M')}$$

by the two first of which, combined with the relation (K'), and with the three last equations (L'), we can eliminate σ, τ, υ, σ', τ', υ', from (F'), and so deduce V from T.

If both the extreme media be uniform, we have then not only the relation (K') for the final medium, but also an analogous relation

$$0=\Omega'\,(\sigma',\tau',\upsilon',\chi) \qquad \text{(N')}$$

for the initial; and instead of the seven equations (L'), we have the two first of the equations (M'), and the two following,

$$\frac{\frac{\delta T}{\delta\sigma'}+x'}{\frac{\delta\Omega'}{\delta\sigma'}}=\frac{\frac{\delta T}{\delta\tau'}+y'}{\frac{\delta\Omega'}{\delta\tau'}}=\frac{\frac{\delta T}{\delta\upsilon'}+z'}{\frac{\delta\Omega'}{\delta\upsilon'}}, \qquad \text{(O')}$$

together with this equation,

$$\frac{\delta T}{\delta\chi}+\frac{\delta V}{\delta\chi}=\lambda\frac{\delta\Omega}{\delta\chi}+\lambda'\frac{\delta\Omega'}{\delta\chi}, \qquad \text{(P')}$$

in which λ is the common value of the three first equated quantities in (M'), and λ' is the common value of the three equated quantities in (O'). And in this case, by means of the two equations (O'), and the two that remain of (M'), combined with the two relations (K') (N'), we can eliminate $\sigma, \tau, \upsilon, \sigma', \tau', \upsilon'$, from (F'), and so deduce V from T: while, in the same case, or even if the initial medium alone be uniform, we are to deduce W from T, by eliminating $\sigma', \tau', \upsilon'$, between the equations $(E')(N')(O')$.

When all the media of the combination are not only uniform, but bounded by plane surfaces, which happens in investigations respecting prisms, ordinary or extraordinary, then of the seven quantities $\sigma, \tau, \upsilon, \sigma', \tau', \upsilon', \chi$, only three are independent; two other relations existing besides (K') and (N'), which may be thus denoted,

$$\left.\begin{array}{l} 0 = \Omega''\ (\sigma, \tau, \upsilon, \sigma', \tau', \upsilon', \chi), \\ 0 = \Omega'''\ (\sigma, \tau, \upsilon, \sigma', \tau', \upsilon', \chi); \end{array}\right\} \quad (Q')$$

because, in this case, the initial direction, and the colour, determine the final direction. In this case, we may still treat the variations of $\sigma, \tau, \upsilon, \sigma', \tau', \upsilon', \chi$, as independent, in δT, by introducing the variations of the four conditions (K') (N') (Q'), multiplied by factors $\lambda, \lambda', \lambda'', \lambda'''$, that is by putting

$$\begin{array}{l} \delta T = x\delta\sigma - x'\delta\sigma' + y\delta\tau - y'\delta\tau' + z\delta\upsilon - z'\delta\upsilon' - \dfrac{\delta V}{\delta\chi}\delta\chi \\ \qquad + \lambda\delta\Omega + \lambda'\delta\Omega' + \lambda''\delta\Omega'' + \lambda'''\delta\Omega''' : \end{array} \quad (R')$$

an equation which decomposes itself into the seven following,

$$\left.\begin{array}{l}
\dfrac{\delta T}{\delta\sigma} - x = \lambda\dfrac{\delta\Omega}{\delta\sigma} + \lambda''\dfrac{\delta\Omega''}{\delta\sigma} + \lambda'''\dfrac{\delta\Omega'''}{\delta\sigma}, \\
\dfrac{\delta T}{\delta\tau} - y = \lambda\dfrac{\delta\Omega}{\delta\tau} + \lambda''\dfrac{\delta\Omega''}{\delta\tau} + \lambda'''\dfrac{\delta\Omega'''}{\delta\tau}, \\
\dfrac{\delta T}{\delta\upsilon} - z = \lambda\dfrac{\delta\Omega}{\delta\upsilon} + \lambda''\dfrac{\delta\Omega''}{\delta\upsilon} + \lambda'''\dfrac{\delta\Omega'''}{\delta\upsilon}, \\
\dfrac{\delta T}{\delta\sigma'} + x' = \lambda'\dfrac{\delta\Omega'}{\delta\sigma'} + \lambda''\dfrac{\delta\Omega''}{\delta\sigma'} + \lambda'''\dfrac{\delta\Omega'''}{\delta\sigma'}, \\
\dfrac{\delta T}{\delta\tau'} + y' = \lambda'\dfrac{\delta\Omega'}{\delta\tau'} + \lambda''\dfrac{\delta\Omega''}{\delta\tau'} + \lambda'''\dfrac{\delta\Omega'''}{\delta\tau'}, \\
\dfrac{\delta T}{\delta\upsilon'} + z' = \lambda'\dfrac{\delta\Omega'}{\delta\upsilon'} + \lambda''\dfrac{\delta\Omega''}{\delta\upsilon'} + \lambda'''\dfrac{\delta\Omega'''}{\delta\upsilon'}, \\
\dfrac{\delta T}{\delta\chi} + \dfrac{\delta V}{\delta\chi} = \lambda\dfrac{\delta\Omega}{\delta\chi} + \lambda'\dfrac{\delta\Omega'}{\delta\chi} + \lambda''\dfrac{\delta\Omega''}{\delta\chi} + \lambda'''\dfrac{\delta\Omega'''}{\delta\chi},
\end{array}\right\} \quad (S')$$

between the six first of which, and the five equations marked (F') (K') (N') (Q'), we can eliminate the ten quantities $\sigma, \tau, \upsilon, \sigma', \tau', \upsilon', \lambda, \lambda', \lambda'', \lambda'''$, and thus deduce the relation between $V, x, y, z, x', y', z', \chi$, from that between $T, \sigma, \tau, \upsilon, \sigma', \tau', \upsilon', \chi$. It is easy to extend this method to other cases, in which there exists a mutual dependence, expressed by any number of equations, betwen the seven quantities $\sigma, \tau, \upsilon, \sigma', \tau', \upsilon', \chi$.

And all the foregoing details respecting the mutual deductions of the functions

V, *W*, *T*, may be summed up in this one rule or theorem: that each of these three functions may be deduced from either of the other two, by using one of the three equations (*D'*) (*E'*) (*F'*) and by making the sought function a maximum or minimum with respect to the variables that are to be eliminated. For example we may deduce *T* from *V*, by making the expression (*F'*) a maximum or minimum with respect to the initial and final co-ordinates.

An optical combination is more perfectly characterised by the original function *V*, than by either of the two connected and auxiliary functions *W*, *T*; because *V* enables us to determine the properties of the extreme media, which *W* and *T* do not; but there is an advantage in using these latter functions when the extreme media are uniform and known, because the known relations which in this case exist, of the forms (*K'*) and (*N'*), (together with the other relations (*Q'*) which arise when the combination is prismatic,) leave fewer independent variables in the auxiliary than in the original function. At the same time, as has been already remarked, and will be afterwards more fully shown, the existence of relations between the variables produces a partial indeterminateness in the forms of the auxiliary functions, from which the characteristic function *V* is free, but which is rather advantageous than the contrary, because it permits us to introduce suppositions and transformations, that contribute to elegance or simplicity.

General Transformations, by the Auxiliary Functions W, T, of the Partial Differential Equations in V. Other Partial Differential Equations in V, for Extreme Uniform Media. Integration of these Equations, by the Functions W, T.

5. Another advantage of the auxiliary functions *W*, *T*, is that they serve to transform, and in the case of extreme uniform media to integrate, the partial differential equations (*C*), which the characteristic function *V* must satisfy. In fact, if the final medium be variable, the first of the two partial differential equations (*C*) may be put by the foregoing number under the two following forms,

$$\left.\begin{aligned} 0 &= \Omega\left(\sigma, \tau, \upsilon, \frac{\delta W}{\delta \sigma}, \frac{\delta W}{\delta \tau}, \frac{\delta W}{\delta \upsilon}, \chi\right), \\ 0 &= \Omega\left(\sigma, \tau, \upsilon, \frac{\delta T}{\delta \sigma}, \frac{\delta T}{\delta \tau}, \frac{\delta T}{\delta \upsilon}, \chi\right); \end{aligned}\right\} \quad (\mathrm{T'})$$

and if the initial medium be variable, the second of the two partial differential equations (*C*) may be put under these two forms,

$$\left.\begin{aligned} 0 &= \Omega'\left(\frac{\delta W}{\delta x'}, \frac{\delta W}{\delta y'}, \frac{\delta W}{\delta z'}, x', y', z', \chi\right), \\ 0 &= \Omega'\left(\sigma', \tau', \upsilon', -\frac{\delta T}{\delta \sigma'}, -\frac{\delta T}{\delta \tau'}, -\frac{\delta T}{\delta \upsilon'}, \chi\right); \end{aligned}\right\} \quad (\mathrm{U'})$$

of which indeed the first is general. But if the final medium be uniform, then W remains an arbitrary function of the four variables σ, τ, υ, χ, which are in this case connected with each other by the relation (K'); and the two equations (D') (K'), together with the two first of those marked (I'), compose a system, which is a form for the integral of the partial differential equation

$$0 = \Omega\left(\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z}, \chi\right), \qquad (\mathrm{V}')$$

to which the first equation (C) in this case reduces itself. In like manner, if both the extreme media be uniform, in which case the second equation (C) reduces itself to the form

$$0 = \Omega'\left(-\frac{\delta V}{\delta x'}, -\frac{\delta V}{\delta y'}, -\frac{\delta V}{\delta z'}, \chi\right), \qquad (\mathrm{W}')$$

the system of the partial differential equations (V') (W') has for integral the system composed of the equations (F') (K') (N') (O'), and the two first equations (M'), in which T is considered an arbitrary function of σ, τ, υ, σ', τ', υ', χ. It will be found that these integrals are extensively useful, in the study of optical combinations.

The two partial differential equations, (V') (W'), of the first order, are themselves integrals of the two following, of the second order,

$$\frac{\delta^2 V}{\delta x^2}\,\frac{\delta^2 V}{\delta y^2}\,\frac{\delta^2 V}{\delta z^2} + 2\,\frac{\delta^2 V}{\delta x \delta y}\,\frac{\delta^2 V}{\delta y \delta z}\,\frac{\delta^2 V}{\delta z \delta x} =$$
$$\frac{\delta^2 V}{\delta x^2}\left(\frac{\delta^2 V}{\delta y \delta z}\right)^2 + \frac{\delta^2 V}{\delta y^2}\left(\frac{\delta^2 V}{\delta z \delta x}\right)^2 + \frac{\delta^2 V}{\delta z^2}\left(\frac{\delta^2 V}{\delta x \delta y}\right)^2, \qquad (\mathrm{X}')$$

and

$$\frac{\delta^2 V}{\delta x'^2}\,\frac{\delta^2 V}{\delta y'^2}\,\frac{\delta^2 V}{\delta z'^2} + 2\,\frac{\delta^2 V}{\delta x' \delta y'}\,\frac{\delta^2 V}{\delta y' \delta z'}\,\frac{\delta^2 V}{\delta z' \delta x'} =$$
$$\frac{\delta^2 V}{\delta x'^2}\left(\frac{\delta^2 V}{\delta y' \delta z'}\right)^2 + \frac{\delta^2 V}{\delta y'^2}\left(\frac{\delta^2 V}{\delta z' \delta x'}\right)^2 + \frac{\delta^2 V}{\delta z'^2}\left(\frac{\delta^2 V}{\delta x' \delta y'}\right)^2, \qquad (\mathrm{Y}')$$

which are obtained by elimination from (Q) and (X), after making

$$\left.\begin{aligned} &\frac{\delta v}{\delta x} = 0, \frac{\delta v}{\delta y} = 0, \frac{\delta v}{\delta z} = 0; \\ &\frac{\delta v'}{\delta x'} = 0, \frac{\delta v'}{\delta y'} = 0, \frac{\delta v'}{\delta z'} = 0. \end{aligned}\right\} \qquad (\mathrm{Z}')$$

The system of the three first of these six equations (Z'), or the partial differential equation of the second order (X'), or its integral of the first order (V'), expresses that the final medium is uniform; and the uniformity of the initial medium is, in like manner, expressed by the three last equations (Z'), or by the partial differential equation (Y'), or by its integral of the first order (W'). The integral systems of equations, also, which we have already assigned, express properties peculiar to optical combinations that have one or both of the extreme media uniform.

The first equation (U') has for transformation the second equation (U'), when the initial medium is variable; and it has for integral, when the initial medium is uniform, the system (E') (N') (O'), by which, in that case, W is deduced from the arbitrary function T: while, in the same case, of an initial uniform medium, the first equation (U') becomes of the form

$$0 = \Omega' \left(\frac{\delta W}{\delta x'}, \frac{\delta W}{\delta y'}, \frac{\delta W}{\delta z'}, \chi \right), \qquad (\mathrm{A}^2)$$

and is an integral of the following equation of the second order, analogous to (Y'),

$$\frac{\delta^2 W}{\delta x'^2} \frac{\delta^2 W}{\delta y'^2} \frac{\delta^2 W}{\delta z'^2} + 2 \frac{\delta^2 W}{\delta x' \delta y'} \frac{\delta^2 W}{\delta y' \delta z'} \frac{\delta^2 W}{\delta z' \delta x'} =$$
$$\frac{\delta^2 W}{\delta x'^2} \left(\frac{\delta^2 W}{\delta y' \delta z'} \right)^2 + \frac{\delta^2 W}{\delta y'^2} \left(\frac{\delta^2 W}{\delta z' \delta x'} \right)^2 + \frac{\delta^2 W}{\delta z'^2} \left(\frac{\delta^2 W}{\delta x' \delta y'} \right)^2. \qquad (\mathrm{B}^2)$$

When the final medium is variable, the function W satisfies the following partial differential equation, analogous to the general equation (D),

$$\frac{\delta^2 W}{\delta \sigma \delta x'} \frac{\delta^2 W}{\delta \tau \delta y'} \frac{\delta^2 W}{\delta \upsilon \delta z'} + \frac{\delta^2 W}{\delta \sigma \delta y'} \frac{\delta^2 W}{\delta \tau \delta z'} \frac{\delta^2 W}{\delta \upsilon \delta x'} + \frac{\delta^2 W}{\delta \sigma \delta z'} \frac{\delta^2 W}{\delta \tau \delta x'} \frac{\delta^2 W}{\delta \upsilon \delta y'}$$
$$= \frac{\delta^2 W}{\delta \upsilon \delta x'} \frac{\delta^2 W}{\delta \tau \delta y'} \frac{\delta^2 W}{\delta \sigma \delta z'} + \frac{\delta^2 W}{\delta \upsilon \delta y'} \frac{\delta^2 W}{\delta \tau \delta z'} \frac{\delta^2 W}{\delta \sigma \delta x'} + \frac{\delta^2 W}{\delta \upsilon \delta z'} \frac{\delta^2 W}{\delta \tau \delta x'} \frac{\delta^2 W}{\delta \sigma \delta y'}; \qquad (\mathrm{C}^2)$$

and when both the extreme media are variable, the function T satisfies the following analogous equation,

$$\frac{\delta^2 T}{\delta \sigma \delta \sigma'} \frac{\delta^2 T}{\delta \tau \delta \tau'} \frac{\delta^2 T}{\delta \upsilon \delta \upsilon'} + \frac{\delta^2 T}{\delta \sigma \delta \tau'} \frac{\delta^2 T}{\delta \tau \delta \upsilon'} \frac{\delta^2 T}{\delta \upsilon \delta \sigma'} + \frac{\delta^2 T}{\delta \sigma \delta \upsilon'} \frac{\delta^2 T}{\delta \tau \delta \sigma'} \frac{\delta^2 T}{\delta \upsilon \delta \tau'}$$
$$= \frac{\delta^2 T}{\delta \upsilon \delta \sigma'} \frac{\delta^2 T}{\delta \tau \delta \tau'} \frac{\delta^2 T}{\delta \sigma \delta \upsilon'} + \frac{\delta^2 T}{\delta \upsilon \delta \tau'} \frac{\delta^2 T}{\delta \tau \delta \upsilon'} \frac{\delta^2 T}{\delta \sigma \delta \sigma'} + \frac{\delta^2 T}{\delta \upsilon \delta \upsilon'} \frac{\delta^2 T}{\delta \tau \delta \sigma'} \frac{\delta^2 T}{\delta \sigma \delta \tau'}. \qquad (\mathrm{D}^2)$$

General Deductions and Transformations of the Differential and Integral Equations of a Curved or Straight Ray, Ordinary or Extraordinary, by the Auxiliary Functions W, T.

6. The auxiliary functions W, T, give new equations for the initial and final portions of a curved or polygon ray. Thus the function W gives generally the following equations, between the final quantities σ, τ, υ, analogous to the equations (W),

$$\frac{\delta W}{\delta x'} = \text{const.}, \frac{\delta W}{\delta y'} = \text{const.}, \frac{\delta W}{\delta z'} = \text{const.}, \qquad (\mathrm{E}^2)$$

in which x' y' z' are the co-ordinates of some fixed point on the initial portion, and the constants are, by (G'), the corresponding values of the initial quantities σ', τ', υ'. The equations (E^2) have for differentials the following,

$$\left.\begin{aligned}0&=\frac{\delta^2 W}{\delta\sigma\delta x'}d\sigma+\frac{\delta^2 W}{\delta\tau\delta x'}d\tau+\frac{\delta^2 W}{\delta\upsilon\delta x'}d\upsilon,\\0&=\frac{\delta^2 W}{\delta\sigma\delta y'}d\sigma+\frac{\delta^2 W}{\delta\tau\delta y'}d\tau+\frac{\delta^2 W}{\delta\upsilon\delta y'}d\upsilon,\\0&=\frac{\delta^2 W}{\delta\sigma\delta z'}d\sigma+\frac{\delta^2 W}{\delta\tau\delta z'}d\tau+\frac{\delta^2 W}{\delta\upsilon\delta z'}d\upsilon;\end{aligned}\right\}\qquad(F^2)$$

d still referring to motion along a ray: and if we combine these with the following,

$$\left.\begin{aligned}0&=\frac{\delta v}{\delta x}\frac{\delta^2 W}{\delta\sigma\delta x'}+\frac{\delta v}{\delta y}\frac{\delta^2 W}{\delta\tau\delta x'}+\frac{\delta v}{\delta z}\frac{\delta^2 W}{\delta\upsilon\delta x'},\\0&=\frac{\delta v}{\delta x}\frac{\delta^2 W}{\delta\sigma\delta y'}+\frac{\delta v}{\delta y}\frac{\delta^2 W}{\delta\tau\delta y'}+\frac{\delta v}{\delta z}\frac{\delta^2 W}{\delta\upsilon\delta y'},\\0&=\frac{\delta v}{\delta x}\frac{\delta^2 W}{\delta\sigma\delta z'}+\frac{\delta v}{\delta y}\frac{\delta^2 W}{\delta\tau\delta z'}+\frac{\delta v}{\delta z}\frac{\delta^2 W}{\delta\upsilon\delta z'},\end{aligned}\right\}\qquad(G^2)$$

which are obtained by differentiating the first equation (T') relatively to the initial co-ordinates $x'\ y'\ z'$, and by attending to the relations (K), we see that for a curved ray the differentials $d\sigma$, $d\tau$, $d\upsilon$, are proportional to

$$\frac{\delta v}{\delta x},\ \frac{\delta v}{\delta y},\ \frac{\delta v}{\delta z};$$

and from this proportionality, combined with the relation

$$\alpha d\sigma+\beta d\tau+\gamma d\upsilon=\left(\alpha\frac{\delta v}{\delta x}+\beta\frac{\delta v}{\delta y}+\gamma\frac{\delta v}{\delta z}\right)ds,\qquad(H^2)$$

which results from (H) and (P), we can easily infer the equations (O): these differential equations (O) for the final portion of a curved ray, which can be extended to the initial portion by merely accenting the symbols, may therefore be deduced from the consideration of the auxiliary function W. The equations (O) for a curved ray, may also be deduced from the function W, by combining the differentials d of the three first equations (H'), with the partial differentials of the first equation (T'), taken with respect to σ, τ, υ.

The same auxiliary function W gives for the final straight portion of a polygon ray, the two first equations (I'), which may be thus written,

$$\frac{1}{\alpha}\left(x-\frac{\delta W}{\delta\sigma}\right)=\frac{1}{\beta}\left(y-\frac{\delta W}{\delta\tau}\right)=\frac{1}{\gamma}\left(z-\frac{\delta W}{\delta\upsilon}\right):\qquad(I^2)$$

these equations may also be put under the form

$$\left.\begin{aligned}x\frac{\delta\sigma}{\delta\theta}+y\frac{\delta\tau}{\delta\theta}+z\frac{\delta\upsilon}{\delta\theta}&=\frac{\delta W}{\delta\theta},\\x\frac{\delta\sigma}{\delta\phi}+y\frac{\delta\tau}{\delta\phi}+z\frac{\delta\upsilon}{\delta\phi}&=\frac{\delta W}{\delta\phi},\end{aligned}\right\}\qquad(K^2)$$

if in virtue of (K'), we consider σ, τ, υ, as functions, each, of χ, and of two other independent variables denoted by θ, ϕ, and consider W as a function of the six independent variables θ, ϕ, χ, x', y', z'. We may choose σ, τ, for the independent variables θ, ϕ, considering υ as, by (K'), a function of σ, τ, χ, such that by (H),

$$\frac{\delta\upsilon}{\delta\sigma}=-\frac{\alpha}{\gamma},\ \frac{\delta\upsilon}{\delta\tau}=-\frac{\beta}{\gamma},\ \frac{\delta\upsilon}{\delta\chi}=\frac{1}{\gamma}\,\frac{\delta v}{\delta\chi}, \qquad (\mathrm{L}^2)$$

and considering W as a function of the six independent variables σ, τ, χ, x', y', z'; and then the equations (I^2) or (K^2), for the final straight portion of a polygon ray, ordinary or extraordinary, will take these simpler forms, which we shall have frequent occasion to employ,

$$x-\frac{\alpha}{\gamma}z=\frac{\delta W}{\delta\sigma}\,;\ y-\frac{\beta}{\gamma}z=\frac{\delta W}{\delta\tau}\,. \qquad (\mathrm{M}^2)$$

The other auxiliary function, T, gives the following equations between σ, τ, υ, for the final portion, straight or curved, when the initial medium is variable,

$$\frac{\delta T}{\delta\sigma'}=\text{const.},\ \frac{\delta T}{\delta\tau'}=\text{const.},\ \frac{\delta T}{\delta\upsilon'}=\text{const.}, \qquad (\mathrm{N}^2)$$

in which σ', τ', υ', belong to some point on the initial portion, and in which the constants are, by (L′), the negatives of the co-ordinates of that point; it gives, in like manner, for the initial portion, when the final medium is variable, the following equations between σ', τ', υ',

$$\frac{\delta T}{\delta\sigma}=\text{const.},\ \frac{\delta T}{\delta\tau}=\text{const.},\ \frac{\delta T}{\delta\upsilon}=\text{const.}, \qquad (\mathrm{O}^2)$$

σ, τ, υ, belonging to some point upon the final portion, and the constants being the co-ordinates of that point: and from these equations we might deduce the differential equations (O), by processes analogous to those already mentioned. When both the extreme media are uniform, and therefore both the extreme portions straight, we have, for these straight portions, the following equations, deduced from $(M')\,(O')\,(I)$,

$$\left.\begin{aligned}\frac{1}{\alpha}\left(x-\frac{\delta T}{\delta\sigma}\right)&=\frac{1}{\beta}\left(y-\frac{\delta T}{\delta\tau}\right)=\frac{1}{\gamma}\left(z-\frac{\delta T}{\delta\upsilon}\right),\\ \frac{1}{\alpha'}\left(x'+\frac{\delta T}{\delta\sigma'}\right)&=\frac{1}{\beta'}\left(y'+\frac{\delta T}{\delta\tau'}\right)=\frac{1}{\gamma'}\left(z'+\frac{\delta T}{\delta\upsilon'}\right);\end{aligned}\right\} \qquad (\mathrm{P}^2)$$

which may be thus transformed,

$$\left.\begin{aligned}0&=x\frac{\delta\sigma}{\delta\theta}+y\frac{\delta\tau}{\delta\theta}+z\frac{\delta\upsilon}{\delta\theta}-\frac{\delta T}{\delta\theta},\\ 0&=x\frac{\delta\sigma}{\delta\phi}+y\frac{\delta\tau}{\delta\phi}+z\frac{\delta\upsilon}{\delta\phi}-\frac{\delta T}{\delta\phi},\\ 0&=x'\frac{\delta\sigma'}{\delta\theta'}+y'\frac{\delta\tau'}{\delta\theta'}+z'\frac{\delta\upsilon'}{\delta\theta'}+\frac{\delta T}{\delta\theta'},\\ 0&=x'\frac{\delta\sigma'}{\delta\phi'}+y'\frac{\delta\tau'}{\delta\phi'}+z'\frac{\delta\upsilon'}{\delta\phi'}+\frac{\delta T}{\delta\phi'},\end{aligned}\right\} \qquad (\mathrm{Q}^2)$$

if, as before, by virtue of (K'), we consider σ, τ, v, as functions, each, of χ and of two other independent variables θ, ϕ, considering similarly σ', τ', v', as functions, each, by (N'), of three independent variables θ', ϕ', χ; and T as a function of the five independent variables θ, ϕ, θ', ϕ', χ. If we choose the independent variables θ, ϕ, so as to coincide with σ, τ, and if in like manner we take σ', τ', for the independent variables θ', ϕ', making, by (H),

$$\frac{\delta v'}{\delta \sigma'} = -\frac{\alpha'}{\gamma'}, \quad \frac{\delta v'}{\delta \tau'} = -\frac{\beta'}{\gamma'}, \quad \frac{\delta v'}{\delta \chi} = \frac{1}{\gamma'}\frac{\delta v'}{\delta \chi}, \qquad (\mathrm{R}^2)$$

and considering T as a function of the five independent variables σ, τ, σ', τ', χ, we have the following transformed equations for the extreme straight portions of a polygon ray, ordinary or extraordinary,

$$\left.\begin{aligned} 0 &= x - \frac{\alpha}{\gamma} z - \frac{\delta T}{\delta \sigma}; \quad 0 = y - \frac{\beta}{\gamma} z - \frac{\delta T}{\delta \tau}; \\ 0 &= x' - \frac{\alpha'}{\gamma'} z' + \frac{\delta T}{\delta \sigma'}; \quad 0 = y' - \frac{\beta'}{\gamma'} z' + \frac{\delta T}{\delta \tau'}: \end{aligned}\right\} \qquad (\mathrm{S}^2)$$

which are analogous to the equations (M^1) and, like them, will often be found useful.

It may be remarked here, that from the differential equations (O) of a curved ray, ordinary or extraordinary, to which, in the present and former numbers, we have been conducted by so many processes, the following may be deduced,

$$\left.\begin{aligned} &0 = \frac{d\sigma}{dV} + \frac{\delta \Omega}{\delta x}, \quad 0 = \frac{d\tau}{dV} + \frac{\delta \Omega}{\delta y}; \quad 0 = \frac{dv}{dV} + \frac{\delta \Omega}{\delta z}: \\ &dW = dT = \left(x\frac{\delta v}{\delta x} + y\frac{\delta v}{\delta y} + z\frac{\delta v}{\delta z}\right) ds = x\,d\sigma + y\,d\tau + z\,dv. \end{aligned}\right\} \qquad (\mathrm{T}^2)$$

We may also remark, that when the final medium is uniform, and when therefore the quantities σ, τ, v, χ, are connected by a relation (K'), the quantity

$$W(\sigma^2 + \tau^2 + v^2)^{-\frac{n}{2}}$$

may, in general, by means of this relation, be expressed as a function of

$$\frac{\sigma}{v}, \ \frac{\tau}{v}, \ x', y', z', \chi,$$

and that $T(\sigma^2 + \tau^2 + v^2)^{-\frac{n}{2}}$ may, in like manner, be expressed as a function of

$$\frac{\sigma}{v}, \ \frac{\tau}{v}, \ \sigma', \tau', v', \chi;$$

and that therefore W, T, may both be made homogeneous functions, of any assumed dimension n, relatively to σ, τ, v, so as to satisfy the following conditions

$$\left.\begin{aligned} \sigma\frac{\delta W}{\delta \sigma} + \tau\frac{\delta W}{\delta \tau} + v\frac{\delta W}{\delta v} &= n\,W, \\ \sigma\frac{\delta T}{\delta \sigma} + \tau\frac{\delta T}{\delta \tau} + v\frac{\delta T}{\delta v} &= n\,T. \end{aligned}\right\} \qquad (\mathrm{U}^2)$$

With this preparation, the two first equations (I'), and the two first equations (M'), which belong to the straight final portion of the ray, may be transformed by (L) to the following,

$$\left.\begin{aligned} x - \frac{\delta\Omega}{\delta\sigma}(\sigma x + \tau y + \upsilon z) &= \frac{\delta W}{\delta\sigma} - n\,W\frac{\delta\Omega}{\delta\sigma} = \frac{\delta T}{\delta\sigma} - n\,T\frac{\delta\Omega}{\delta\sigma}\,, \\ y - \frac{\delta\Omega}{\delta\tau}(\sigma x + \tau y + \upsilon z) &= \frac{\delta W}{\delta\tau} - n\,W\frac{\delta\Omega}{\delta\tau} = \frac{\delta T}{\delta\tau} - n\,T\frac{\delta\Omega}{\delta\tau}\,, \\ z - \frac{\delta\Omega}{\delta\upsilon}(\sigma x + \tau y + \upsilon z) &= \frac{\delta W}{\delta\upsilon} - n\,W\frac{\delta\Omega}{\delta\upsilon} = \frac{\delta T}{\delta\upsilon} - n\,T\frac{\delta\Omega}{\delta\upsilon}\,. \end{aligned}\right\} \qquad (V^2)$$

If then we make $n=1$, that is if we make W homogeneous of the first dimension relatively to σ, τ, υ, and if we attend to the relation (D'), we see that the equations of this straight final portion may be thus written,

$$x = \frac{\delta W}{\delta\sigma} + V\frac{\delta\Omega}{\delta\sigma}, y = \frac{\delta W}{\delta\tau} + V\frac{\delta\Omega}{\delta\tau}, z = \frac{\delta W}{\delta\upsilon} + V\frac{\delta\Omega}{\delta\upsilon}, \qquad (W^2)$$

of which any two include the third, and which we shall often hereafter employ, on account of their symmetry.

In like manner, when the initial medium is uniform, and therefore the initial portion straight, the equations (O') of this straight portion may be put under the form,

$$\left.\begin{aligned} x' - \frac{\delta\Omega'}{\delta\sigma'}(\sigma' x' + \tau' y' + \upsilon' z') &= -\frac{\delta T}{\delta\sigma'} + n' T\frac{\delta\Omega'}{\delta\sigma'}\,, \\ y' - \frac{\delta\Omega'}{\delta\tau'}(\sigma' x' + \tau' y' + \upsilon' z') &= -\frac{\delta T}{\delta\tau'} + n' T\frac{\delta\Omega'}{\delta\tau'}\,, \\ z' - \frac{\delta\Omega'}{\delta\upsilon'}(\sigma' x' + \tau' y' + \upsilon' z') &= -\frac{\delta T}{\delta\upsilon'} + n' T\frac{\delta\Omega'}{\delta\upsilon'}\,, \end{aligned}\right\} \qquad (X^2)$$

by making T homogeneous of dimension n' relatively to σ', τ', υ', so as to have

$$\sigma'\frac{\delta T}{\delta\sigma'} + \tau'\frac{\delta T}{\delta\tau'} + \upsilon'\frac{\delta T}{\delta\upsilon'} = n' T. \qquad (Y^2)$$

If both the extreme media be uniform, and if we make $n=0$, $n'=0$, that is if we express W as a function of

$$\frac{\sigma}{\upsilon},\ \frac{\tau}{\upsilon},\ x',\ y',\ z',\ \chi,$$

and T as a function of

$$\frac{\sigma}{\upsilon},\ \frac{\tau}{\upsilon},\ \frac{\sigma'}{\upsilon'},\ \frac{\tau'}{\upsilon'},\ \chi,$$

we find the following forms for the equations of the extreme straight portions of a a polygon ray, ordinary or extraordinary, less simple than (S^2), but more symmetric,

$$\left.\begin{array}{l} x - \frac{\delta\Omega}{\delta\sigma}(\sigma x + \tau y + \upsilon z) = \frac{\delta W}{\delta\sigma} = \frac{\delta T}{\delta\sigma}, \\ y - \frac{\delta\Omega}{\delta\tau}(\sigma x + \tau y + \upsilon z) = \frac{\delta W}{\delta\tau} = \frac{\delta T}{\delta\tau}, \\ z - \frac{\delta\Omega}{\delta\upsilon}(\sigma x + \tau y + \upsilon z) = \frac{\delta W}{\delta\upsilon} = \frac{\delta T}{\delta\upsilon}; \\ x' - \frac{\delta\Omega'}{\delta\sigma'}(\sigma' x' + \tau' y' + \upsilon' z') = -\frac{\delta T}{\delta\sigma'}, \\ y' - \frac{\delta\Omega'}{\delta\tau'}(\sigma' x' + \tau' y' + \upsilon' z') = -\frac{\delta T}{\delta\tau'}, \\ z' - \frac{\delta\Omega'}{\delta\upsilon'}(\sigma' x' + \tau' y' + \upsilon' z') = -\frac{\delta T}{\delta\upsilon'}. \end{array}\right\} \quad (Z')$$

The case of prismatic combinations may be treated as in the fourth number.

General Remarks on the Connexions between the Partial Differential Coefficients of the Second Order of the Functions V, W, T. General Method of investigating those Connexions. Deductions of the Coefficients of V from those of W, when the Final Medium is uniform.

7. It is easy to see, from the manner in which the equations of a ray involve the partial differential coefficients of the first order, of the functions V, W, T, that the partial differential coefficients of the second order, of the same three functions, must present themselves in investigations respecting the geometrical relations between infinitely near rays of a system; and that therefore it must be useful to know the general connexions between these coefficients of the second order. Connexions of this kind, between the coefficients of the second order of the characteristic function V, taken with respect to the final co-ordinates, and those of the auxiliary function W, considered as belonging to a final system of straight rays of a given colour, which issued originally from a given luminous point, were investigated in the First Supplement; but these connexions will now be considered in a more general manner, and will be extended to the new auxiliary function T, which was not introduced before: the new investigations will differ also from the former, by making W depend on the quantities σ, τ, υ, rather than on α, β, γ.

The general problem of investigating these connexions may be decomposed into many particular problems, according to the way in which we pair the functions, and according as we suppose the extreme media to be uniform or variable; but all these particular problems may be resolved by attending to the following general principle, that the connexions between the partial differential coefficients of the three functions, whether of the second or of higher orders, are to be obtained by differentiating and comparing the equations which connect the three functions themselves: that is, by differentiating and comparing the three forms (A') (B') (C') of the fundamental equation (A), and the equations into which these forms (A') (B') (C') resolve themselves.

Thus, to deduce the twenty-eight partial differential coefficients of the second order, of the characteristic function V, taken with respect to the extreme co-ordinates and the colour, from the coefficients of the same order of the auxiliary function W, or T, we are to differentiate the equations into which (B') or (C') resolves itself, together with the relations between the variables on which W or T depends, if any such relations exist; and then by elimination to deduce the variations of the first order of the seven coefficients of the variation (A') as linear functions of the seven variations of the first order of the extreme co-ordinates and the colour: these seven linear functions will have forty-nine coefficients, of which, however, only twenty-eight will be distinct, and these will be the coefficients sought.

More particularly, if the final medium be variable, and if it be required to deduce the coefficients of the second order of V from those of W, we first obtain expressions for $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, as linear functions of δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$, from the differentials of the three first equations (H'), deduced from (B'), expressions which will necessarily satisfy the first condition (H); we then substitute these expressions for $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, in the differentials of the three equations (G'), deduced from (B'), so as to get analogous expressions for $\delta\sigma'$, $\delta\tau'$, $\delta\upsilon'$, which must satisfy the second condition (H); and substituting the same expressions for $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, in the differential of the last equation (H'), also deduced from (B'), we get an expression of the same kind for $\delta\,\frac{\delta V}{\delta\chi}$: after which, we have only to compare the expressions so obtained, with the following, that is, with the differentials of the equations into which the formula (A') resolves itself,

$$\left.\begin{aligned}
\delta\sigma &= \frac{\delta^2 V}{\delta x^2}\delta x + \frac{\delta^2 V}{\delta x\delta y}\delta y + \frac{\delta^2 V}{\delta x\delta z}\delta z + \frac{\delta^2 V}{\delta x\delta x'}\delta x' + \frac{\delta^2 V}{\delta x\delta y'}\delta y' + \frac{\delta^2 V}{\delta x\delta z'}\delta z' + \frac{\delta^2 V}{\delta x\delta\chi}\delta\chi,\\
\delta\tau &= \frac{\delta^2 V}{\delta x\delta y}\delta x + \frac{\delta^2 V}{\delta y^2}\delta y + \frac{\delta^2 V}{\delta y\delta z}\delta z + \frac{\delta^2 V}{\delta y\delta x'}\delta x' + \frac{\delta^2 V}{\delta y\delta y'}\delta y' + \frac{\delta^2 V}{\delta y\delta z'}\delta z' + \frac{\delta^2 V}{\delta y\delta\chi}\delta\chi,\\
\delta\upsilon &= \frac{\delta^2 V}{\delta x\delta z}\delta x + \frac{\delta^2 V}{\delta y\delta z}\delta y + \frac{\delta^2 V}{\delta z^2}\delta z + \frac{\delta^2 V}{\delta z\delta x'}\delta x' + \frac{\delta^2 V}{\delta z\delta y'}\delta y' + \frac{\delta^2 V}{\delta z\delta z'}\delta z' + \frac{\delta^2 V}{\delta z\delta\chi}\delta\chi,\\
-\delta\sigma' &= \frac{\delta^2 V}{\delta x\delta x'}\delta x + \frac{\delta^2 V}{\delta y\delta x'}\delta y + \frac{\delta^2 V}{\delta z\delta x'}\delta z + \frac{\delta^2 V}{\delta x'^2}\delta x' + \frac{\delta^2 V}{\delta x'\delta y'}\delta y' + \frac{\delta^2 V}{\delta x'\delta z'}\delta z' + \frac{\delta^2 V}{\delta x'\delta\chi}\delta\chi,\\
-\delta\tau' &= \frac{\delta^2 V}{\delta x\delta y'}\delta x + \frac{\delta^2 V}{\delta y\delta y'}\delta y + \frac{\delta^2 V}{\delta z\delta y'}\delta z + \frac{\delta^2 V}{\delta x'\delta y'}\delta x' + \frac{\delta^2 V}{\delta y'^2}\delta y' + \frac{\delta^2 V}{\delta y'\delta z'}\delta z' + \frac{\delta^2 V}{\delta y'\delta\chi}\delta\chi,\\
-\delta\upsilon' &= \frac{\delta^2 V}{\delta x\delta z'}\delta x + \frac{\delta^2 V}{\delta y\delta z'}\delta y + \frac{\delta^2 V}{\delta z\delta z'}\delta z + \frac{\delta^2 V}{\delta x'\delta z'}\delta x' + \frac{\delta^2 V}{\delta y'\delta z'}\delta y' + \frac{\delta^2 V}{\delta z'^2}\delta z' + \frac{\delta^2 V}{\delta z'\delta\chi}\delta\chi,\\
\delta\frac{\delta V}{\delta\chi} &= \frac{\delta^2 V}{\delta x\delta\chi}\delta x + \frac{\delta^2 V}{\delta y\delta\chi}\delta y + \frac{\delta^2 V}{\delta z\delta\chi}\delta z + \frac{\delta^2 V}{\delta x'\delta\chi}\delta x' + \frac{\delta^2 V}{\delta y'\delta\chi}\delta y' + \frac{\delta^2 V}{\delta z'\delta\chi}\delta z' + \frac{\delta^2 V}{\delta\chi^2}\delta\chi.
\end{aligned}\right\}\ (\mathrm{A}^3)$$

But if the final medium be uniform, then $\sigma, \tau, \upsilon, \chi$, are not independent, but related by (K'); and the formula (B') resolves itself, not into the seven equations (G') and (H') but into the six equations (G') and (I'), the differentials of which are to be combined with the differential of the relation (K'), so as to give the expressions for $\delta\sigma, \delta\tau, \delta\upsilon, \delta\sigma', \delta\tau', \delta\upsilon', \delta\frac{\delta V}{\delta\chi}$, which are to be compared with (A^3) as before. And in this case, of a final uniform medium, we may employ, instead of the two first equations (I'), any of the transformations of those equations in the foregoing number; or we may employ the following transformations of (I'),

$$x + z\frac{\delta\upsilon}{\delta\sigma} = \frac{\delta W}{\delta\sigma};\; y + z\frac{\delta\upsilon}{\delta\tau} = \frac{\delta W}{\delta\tau};\; z\frac{\delta\upsilon}{\delta\chi} = \frac{\delta V}{\delta\chi} + \frac{\delta W}{\delta\chi}: \qquad (\mathrm{B}^3)$$

in which, W is considered as a function of the six independent variables $\sigma, \tau, \chi, x', y', z'$, obtained by substituting for υ its value as a function of σ, τ, χ; the form of which function υ depends on and characterises the properties of the final medium, and is deduced from the relation (K'). It may be useful here to go through the process last indicated, both to explain its nature more fully, and to have its results ready for future researches.

Differentiating therefore the two first equations (B^3), we obtain

$$\left.\begin{aligned} \delta x + \frac{\delta\upsilon}{\delta\sigma}\delta z - \delta'\frac{\delta W}{\delta\sigma} + z\frac{\delta^2\upsilon}{\delta\sigma\delta\chi}\delta\chi &= \left(\frac{\delta^2 W}{\delta\sigma^2} - z\frac{\delta^2\upsilon}{\delta\sigma^2}\right)\delta\sigma + \left(\frac{\delta^2 W}{\delta\sigma\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\delta\tau}\right)\delta\tau,\\ \delta y + \frac{\delta\upsilon}{\delta\tau}\delta z - \delta'\frac{\delta W}{\delta\tau} + z\frac{\delta^2\upsilon}{\delta\tau\delta\chi}\delta\chi &= \left(\frac{\delta^2 W}{\delta\sigma\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\delta\tau}\right)\delta\sigma + \left(\frac{\delta^2 W}{\delta\tau^2} - z\frac{\delta^2\upsilon}{\delta\tau^2}\right)\delta\tau, \end{aligned}\right\} \qquad (\mathrm{C}^3)$$

in which we have put for abridgment

$$\left.\begin{aligned} \delta'\frac{\delta W}{\delta\sigma} &= \frac{\delta^2 W}{\delta\sigma\delta x'}\delta x' + \frac{\delta^2 W}{\delta\sigma\delta y'}\delta y' + \frac{\delta^2 W}{\delta\sigma\delta z'}\delta z' + \frac{\delta^2 W}{\delta\sigma\delta\chi}\delta\chi,\\ \delta'\frac{\delta W}{\delta\tau} &= \frac{\delta^2 W}{\delta\tau\delta x'}\delta x' + \frac{\delta^2 W}{\delta\tau\delta y'}\delta y' + \frac{\delta^2 W}{\delta\tau\delta z'}\delta z' + \frac{\delta^2 W}{\delta\tau\delta\chi}\delta\chi, \end{aligned}\right\} \qquad (\mathrm{D}^3)$$

δ' referring only to the variations of the initial co-ordinates and of the colour: and if we put

$$w'' = \left(\frac{\delta^2 W}{\delta\sigma^2} - z\frac{\delta^2\upsilon}{\delta\sigma^2}\right)\left(\frac{\delta^2 W}{\delta\tau^2} - z\frac{\delta^2\upsilon}{\delta\tau^2}\right) - \left(\frac{\delta^2 W}{\delta\sigma\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\delta\tau}\right)^2, \qquad (\mathrm{E}^3)$$

the equations (C^3) give, by elimination,

$$\left.\begin{aligned} w''\delta\sigma = &\left(\frac{\delta^2 W}{\delta\tau^2} - z\frac{\delta^2\upsilon}{\delta\tau^2}\right)\left(\delta x + \frac{\delta\upsilon}{\delta\sigma}\delta z - \delta'\frac{\delta W}{\delta\sigma} + z\frac{\delta^2\upsilon}{\delta\sigma\delta\chi}\delta\chi\right)\\ &-\left(\frac{\delta^2 W}{\delta\sigma\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\delta\tau}\right)\left(\delta y + \frac{\delta\upsilon}{\delta\tau}\delta z - \delta'\frac{\delta W}{\delta\tau} + z\frac{\delta^2\upsilon}{\delta\tau\delta\chi}\delta\chi\right),\\ w''\delta\tau = &\left(\frac{\delta^2 W}{\delta\sigma^2} - z\frac{\delta^2\upsilon}{\delta\sigma^2}\right)\left(\delta y + \frac{\delta\upsilon}{\delta\tau}\delta z - \delta'\frac{\delta W}{\delta\tau} + z\frac{\delta^2\upsilon}{\delta\tau\delta\chi}\delta\chi\right)\\ &-\left(\frac{\delta^2 W}{\delta\sigma\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\delta\tau}\right)\left(\delta x + \frac{\delta\upsilon}{\delta\sigma}\delta z - \delta'\frac{\delta W}{\delta\sigma} + z\frac{\delta^2\upsilon}{\delta\sigma\delta\chi}\delta\chi\right); \end{aligned}\right\} \qquad (\mathrm{F}^3)$$

and hence by (A^3) we can deduce already, without any farther differentiation,

$$\left.\begin{aligned}
\frac{\delta^2 V}{\delta x^2} &= \frac{1}{w''}\left(\frac{\delta^2 W}{\delta\tau^2} - z\frac{\delta^2\upsilon}{\delta\tau^2}\right); & \frac{\delta^2 V}{\delta x\delta z} &= \frac{\delta\upsilon}{\delta\sigma}\,\frac{\delta^2 V}{\delta x^2} + \frac{\delta\upsilon}{\delta\tau}\,\frac{\delta^2 V}{\delta x\delta y};\\
\frac{\delta^2 V}{\delta x\delta y} &= -\frac{1}{w''}\left(\frac{\delta^2 W}{\delta\sigma\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\delta\tau}\right); & \frac{\delta^2 V}{\delta y\delta z} &= \frac{\delta\upsilon}{\delta\sigma}\,\frac{\delta^2 V}{\delta x\delta y} + \frac{\delta\upsilon}{\delta\tau}\,\frac{\delta^2 V}{\delta y^2};\\
\frac{\delta^2 V}{\delta y^2} &= \frac{1}{w''}\left(\frac{\delta^2 W}{\delta\sigma^2} - z\frac{\delta^2\upsilon}{\delta\sigma^2}\right); & \frac{\delta^2 V}{\delta z^2} &= \frac{\delta\upsilon}{\delta\sigma}\,\frac{\delta^2 V}{\delta x\delta z} + \frac{\delta\upsilon}{\delta\tau}\,\frac{\delta^2 V}{\delta y\delta z};
\end{aligned}\right\}\quad(\mathrm{G}^3)$$

observing, in deducing the sixth of these equations (G^3), that by the definitions (E), and by the dependence of υ on σ, τ, χ, we have

$$\delta\frac{\delta V}{\delta z} = (\delta\upsilon =)\,\frac{\delta\upsilon}{\delta\sigma}\,\delta\frac{\delta V}{\delta x} + \frac{\delta\upsilon}{\delta\tau}\,\delta\frac{\delta V}{\delta y} + \frac{\delta\upsilon}{\delta\chi}\,\delta\chi. \qquad (\mathrm{H}^3)$$

The equations (A^3) (F^3) (H^3) give also

$$\left.\begin{aligned}
\frac{\delta^2 V}{\delta x\delta x'} &= \frac{1}{w''}\,\frac{\delta^2 W}{\delta\tau\delta x'}\left(\frac{\delta^2 W}{\delta\sigma\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\delta\tau}\right) - \frac{1}{w''}\,\frac{\delta^2 W}{\delta\sigma\delta x'}\left(\frac{\delta^2 W}{\delta\tau^2} - z\frac{\delta^2\upsilon}{\delta\tau^2}\right);\\
\frac{\delta^2 V}{\delta x\delta y'} &= \frac{1}{w''}\,\frac{\delta^2 W}{\delta\tau\delta y'}\left(\frac{\delta^2 W}{\delta\sigma\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\delta\tau}\right) - \frac{1}{w''}\,\frac{\delta^2 W}{\delta\sigma\delta y'}\left(\frac{\delta^2 W}{\delta\tau^2} - z\frac{\delta^2\upsilon}{\delta\tau^2}\right);\\
\frac{\delta^2 V}{\delta x\delta z'} &= \frac{1}{w''}\,\frac{\delta^2 W}{\delta\tau\delta z'}\left(\frac{\delta^2 W}{\delta\sigma\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\delta\tau}\right) - \frac{1}{w''}\,\frac{\delta^2 W}{\delta\sigma\delta z'}\left(\frac{\delta^2 W}{\delta\tau^2} - z\frac{\delta^2\upsilon}{\delta\tau^2}\right);\\
\frac{\delta^2 V}{\delta y\delta x'} &= \frac{1}{w''}\,\frac{\delta^2 W}{\delta\sigma\delta x'}\left(\frac{\delta^2 W}{\delta\sigma\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\delta\tau}\right) - \frac{1}{w''}\,\frac{\delta^2 W}{\delta\tau\delta x'}\left(\frac{\delta^2 W}{\delta\sigma^2} - z\frac{\delta^2\upsilon}{\delta\sigma^2}\right);\\
\frac{\delta^2 V}{\delta y\delta y'} &= \frac{1}{w''}\,\frac{\delta^2 W}{\delta\sigma\delta y'}\left(\frac{\delta^2 W}{\delta\sigma\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\delta\tau}\right) - \frac{1}{w''}\,\frac{\delta^2 W}{\delta\tau\delta y'}\left(\frac{\delta^2 W}{\delta\sigma^2} - z\frac{\delta^2\upsilon}{\delta\sigma^2}\right);\\
\frac{\delta^2 V}{\delta y\delta z'} &= \frac{1}{w''}\,\frac{\delta^2 W}{\delta\sigma\delta z'}\left(\frac{\delta^2 W}{\delta\sigma\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\delta\tau}\right) - \frac{1}{w''}\,\frac{\delta^2 W}{\delta\tau\delta z'}\left(\frac{\delta^2 W}{\delta\sigma^2} - z\frac{\delta^2\upsilon}{\delta\sigma^2}\right);\\
\frac{\delta^2 V}{\delta z\delta x'} &= \frac{\delta\upsilon}{\delta\sigma}\,\frac{\delta^2 V}{\delta x\delta x'} + \frac{\delta\upsilon}{\delta\tau}\,\frac{\delta^2 V}{\delta y\delta x'};\\
\frac{\delta^2 V}{\delta z\delta y'} &= \frac{\delta\upsilon}{\delta\sigma}\,\frac{\delta^2 V}{\delta x\delta y'} + \frac{\delta\upsilon}{\delta\tau}\,\frac{\delta^2 V}{\delta y\delta y'};\\
\frac{\delta^2 V}{\delta z\delta z'} &= \frac{\delta\upsilon}{\delta\sigma}\,\frac{\delta^2 V}{\delta x\delta z'} + \frac{\delta\upsilon}{\delta\tau}\,\frac{\delta^2 V}{\delta y\delta z'};
\end{aligned}\right\}\quad(\mathrm{I}^3)$$

and

$$\left.\begin{aligned}
\frac{\delta^2 V}{\delta x\delta\chi} &= \frac{1}{w''}\left(\frac{\delta^2 W}{\delta\tau\delta\chi} - z\frac{\delta^2\upsilon}{\delta\tau\delta\chi}\right)\left(\frac{\delta^2 W}{\delta\sigma\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\delta\tau}\right) - \frac{1}{w''}\left(\frac{\delta^2 W}{\delta\sigma\delta\chi} - z\frac{\delta^2\upsilon}{\delta\sigma\delta\chi}\right)\left(\frac{\delta^2 W}{\delta\tau^2} - z\frac{\delta^2\upsilon}{\delta\tau^2}\right);\\
\frac{\delta^2 V}{\delta y\delta\chi} &= \frac{1}{w''}\left(\frac{\delta^2 W}{\delta\sigma\delta\chi} - z\frac{\delta^2\upsilon}{\delta\sigma\delta\chi}\right)\left(\frac{\delta^2 W}{\delta\sigma\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\delta\tau}\right) - \frac{1}{w''}\left(\frac{\delta^2 W}{\delta\tau\delta\chi} - z\frac{\delta^2\upsilon}{\delta\tau\delta\chi}\right)\left(\frac{\delta^2 W}{\delta\sigma^2} - z\frac{\delta^2\upsilon}{\delta\sigma^2}\right);\\
\frac{\delta^2 V}{\delta z\delta\chi} &= \frac{\delta\upsilon}{\delta\sigma}\,\frac{\delta^2 V}{\delta x\delta\chi} + \frac{\delta\upsilon}{\delta\tau}\,\frac{\delta^2 V}{\delta y\delta\chi} + \frac{\delta\upsilon}{\delta\chi}.
\end{aligned}\right\}(\mathrm{K}^3)$$

We have therefore found expressions (G^3) (I^3) (K^3), for eighteen out of the twenty-eight partial differential coefficients of V of the second order; and with respect to nine of the remaining ten, namely all except $\frac{\delta^2 V}{\delta \chi^2}$, we may obtain expressions for these by differentiating the three equations (G'), and comparing the differentials with (A^3); for thus we find,

$$\left.\begin{aligned}
\frac{\delta^2 V}{\delta x'^2} &= -\frac{\delta^2 W}{\delta x'^2} - \frac{\delta^2 W}{\delta\sigma\delta x'}\frac{\delta^2 V}{\delta x\delta x'} - \frac{\delta^2 W}{\delta\tau\delta x'}\frac{\delta^2 V}{\delta y\delta x'};\\
\frac{\delta^2 V}{\delta y'^2} &= -\frac{\delta^2 W}{\delta y'^2} - \frac{\delta^2 W}{\delta\sigma\delta y'}\frac{\delta^2 V}{\delta x\delta y'} - \frac{\delta^2 W}{\delta\tau\delta y'}\frac{\delta^2 V}{\delta y\delta y'};\\
\frac{\delta^2 V}{\delta z'^2} &= -\frac{\delta^2 W}{\delta z'^2} - \frac{\delta^2 W}{\delta\sigma\delta z'}\frac{\delta^2 V}{\delta x\delta z'} - \frac{\delta^2 W}{\delta\tau\delta z'}\frac{\delta^2 V}{\delta y\delta z'};\\
\frac{\delta^2 V}{\delta x'\delta y'} &= -\frac{\delta^2 W}{\delta x'\delta y'} - \frac{\delta^2 W}{\delta\sigma\delta x'}\frac{\delta^2 V}{\delta x\delta y'} - \frac{\delta^2 W}{\delta\tau\delta x'}\frac{\delta^2 V}{\delta y\delta y'};\\
\frac{\delta^2 V}{\delta y'\delta z'} &= -\frac{\delta^2 W}{\delta y'\delta z'} - \frac{\delta^2 W}{\delta\sigma\delta y'}\frac{\delta^2 V}{\delta x\delta z'} - \frac{\delta^2 W}{\delta\tau\delta y'}\frac{\delta^2 V}{\delta y\delta z'};\\
\frac{\delta^2 V}{\delta z'\delta x'} &= -\frac{\delta^2 W}{\delta z'\delta x'} - \frac{\delta^2 W}{\delta\sigma\delta z'}\frac{\delta^2 V}{\delta x\delta x'} - \frac{\delta^2 W}{\delta\tau\delta z'}\frac{\delta^2 V}{\delta y\delta x'};
\end{aligned}\right\} \quad (L^3)$$

and

$$\left.\begin{aligned}
\frac{\delta^2 V}{\delta x'\delta\chi} &= -\frac{\delta^2 W}{\delta x'\delta\chi} - \frac{\delta^2 W}{\delta\sigma\delta x'}\frac{\delta^2 V}{\delta x\delta\chi} - \frac{\delta^2 W}{\delta\tau\delta x'}\frac{\delta^2 V}{\delta y\delta\chi},\\
\frac{\delta^2 V}{\delta y'\delta\chi} &= -\frac{\delta^2 W}{\delta y'\delta\chi} - \frac{\delta^2 W}{\delta\sigma\delta y'}\frac{\delta^2 V}{\delta x\delta\chi} - \frac{\delta^2 W}{\delta\tau\delta y'}\frac{\delta^2 V}{\delta y\delta\chi},\\
\frac{\delta^2 V}{\delta z'\delta\chi} &= -\frac{\delta^2 W}{\delta z'\delta\chi} - \frac{\delta^2 W}{\delta\sigma\delta z'}\frac{\delta^2 V}{\delta x\delta\chi} - \frac{\delta^2 W}{\delta\tau\delta z'}\frac{\delta^2 V}{\delta y\delta\chi}:
\end{aligned}\right\} \quad (M^3)$$

the equations (G′) give also

$$\left.\begin{aligned}
\frac{\delta^2 V}{\delta x'\delta y'} &= -\frac{\delta^2 W}{\delta x'\delta y'} - \frac{\delta^2 W}{\delta\sigma\delta y'}\frac{\delta^2 V}{\delta x\delta x'} - \frac{\delta^2 W}{\delta\tau\delta y'}\frac{\delta^2 V}{\delta y\delta x'};\\
\frac{\delta^2 V}{\delta y'\delta z'} &= -\frac{\delta^2 W}{\delta y'\delta z'} - \frac{\delta^2 W}{\delta\sigma\delta z'}\frac{\delta^2 V}{\delta x\delta y'} - \frac{\delta^2 W}{\delta\tau\delta z'}\frac{\delta^2 V}{\delta y\delta y'};\\
\frac{\delta^2 V}{\delta z'\delta x'} &= -\frac{\delta^2 W}{\delta z'\delta x'} - \frac{\delta^2 W}{\delta\sigma\delta x'}\frac{\delta_2 W}{\delta x\delta z'} - \frac{\delta^2 W}{\delta\tau\delta x'}\frac{\delta^2 V}{\delta y\delta z'};
\end{aligned}\right\} \quad (N^3)$$

but these three expressions (N^3) agree with the corresponding expressions (L^3), because, by (I^3),

$$\left.\begin{aligned}
\frac{\delta^2 W}{\delta\sigma\delta x'}\frac{\delta^2 V}{\delta x\delta y'} + \frac{\delta^2 W}{\delta\tau\delta x'}\frac{\delta^2 V}{\delta y\delta y'} &= \frac{\delta^2 W}{\delta\sigma\delta y'}\frac{\delta^2 V}{\delta x\delta x'} + \frac{\delta^2 W}{\delta\tau\delta y'}\frac{\delta^2 V}{\delta y\delta x'};\\
\frac{\delta^2 W}{\delta\sigma\delta y'}\frac{\delta^2 V}{\delta x\delta z'} + \frac{\delta^2 W}{\delta\tau\delta y'}\frac{\delta^2 V}{\delta y\delta z'} &= \frac{\delta^2 W}{\delta\sigma\delta z'}\frac{\delta^2 V}{\delta x\delta y'} + \frac{\delta^2 W}{\delta\tau\delta z'}\frac{\delta^2 V}{\delta y\delta y'};\\
\frac{\delta^2 W}{\delta\sigma\delta z'}\frac{\delta^2 V}{\delta x\delta x'} + \frac{\delta^2 W}{\delta\tau\delta z'}\frac{\delta^2 V}{\delta y\delta x'} &= \frac{\delta^2 W}{\delta\sigma\delta x'}\frac{\delta^2 V}{\delta x\delta z'} + \frac{\delta^2 W}{\delta\tau\delta x'}\frac{\delta^2 V}{\delta y\delta z'}.
\end{aligned}\right\} \quad (O^3)$$

Finally, with respect to the twenty-eighth coefficient $\frac{\delta^2 V}{\delta \chi^2}$, this may be obtained by differentiating the third equation (B^3), which gives

$$\frac{\delta^2 V}{\delta\chi^2} = z\frac{\delta^2 \upsilon}{\delta\chi^2} - \frac{\delta^2 W}{\delta\chi^2} + \left(z\frac{\delta^2 \upsilon}{\delta\sigma\delta\chi} - \frac{\delta^2 W}{\delta\sigma\delta\chi} \right) \frac{\delta^2 V}{\delta x\delta\chi} + \left(z\frac{\delta^2 \upsilon}{\delta\tau\delta\chi} - \frac{\delta^2 W}{\delta\tau\delta\chi} \right) \frac{\delta^2 V}{\delta y\delta\chi}. \qquad (\mathrm{P}^3)$$

And if we would generalize the twenty-eight expressions (G^3) (I^3) (K^3) (L^3) (M^3) (P^3), so as to render them independent of the particular supposition, that W has been made, by a previous elimination of υ, a function involving only the six independent variables σ, τ, χ, x', y', z', we may do so by suitably generalising fifteen out of the twenty-one coefficients of W, of the second order, which result from the foregoing suppositions; that is by leaving unchanged the six that are formed by differentiating only with respect to x', y', z', but changing $\frac{\delta^2 W}{\delta\sigma^2}$, &c. to the following more general expressions $\left[\frac{\delta^2 W}{\delta\sigma^2}\right]$ &c.;

$$\left.\begin{aligned}
\left[\frac{\delta^2 W}{\delta\sigma^2}\right] &= \frac{\delta^2 W}{\delta\sigma^2} + 2\frac{\delta^2 W}{\delta\sigma\delta\upsilon}\frac{\delta\upsilon}{\delta\sigma} + \frac{\delta^2 W}{\delta\upsilon^2}\left(\frac{\delta\upsilon}{\delta\sigma}\right)^2 + \frac{\delta W}{\delta\upsilon}\frac{\delta^2\upsilon}{\delta\sigma^2};\\
\left[\frac{\delta^2 W}{\delta\tau^2}\right] &= \frac{\delta^2 W}{\delta\tau^2} + 2\frac{\delta^2 W}{\delta\tau\delta\upsilon}\frac{\delta\upsilon}{\delta\tau} + \frac{\delta^2 W}{\delta\upsilon^2}\left(\frac{\delta\upsilon}{\delta\tau}\right)^2 + \frac{\delta W}{\delta\upsilon}\frac{\delta^2\upsilon}{\delta\tau^2};\\
\left[\frac{\delta^2 W}{\delta\chi^2}\right] &= \frac{\delta^2 W}{\delta\chi^2} + 2\frac{\delta^2 W}{\delta\chi\delta\upsilon}\frac{\delta\upsilon}{\delta\chi} + \frac{\delta^2 W}{\delta\upsilon^2}\left(\frac{\delta\upsilon}{\delta\chi}\right)^2 + \frac{\delta W}{\delta\upsilon}\frac{\delta^2\upsilon}{\delta\chi^2};\\
\left[\frac{\delta^2 W}{\delta\sigma\delta\tau}\right] &= \frac{\delta^2 W}{\delta\sigma\delta\tau} + \frac{\delta^2 W}{\delta\tau\delta\upsilon}\frac{\delta\upsilon}{\delta\sigma} + \frac{\delta^2 W}{\delta\sigma\delta\upsilon}\frac{\delta\upsilon}{\delta\tau} + \frac{\delta^2 W}{\delta\upsilon^2}\frac{\delta\upsilon}{\delta\sigma}\frac{\delta\upsilon}{\delta\tau} + \frac{\delta W}{\delta\upsilon}\frac{\delta^2\upsilon}{\delta\sigma\delta\tau};\\
\left[\frac{\delta^2 W}{\delta\sigma\delta\chi}\right] &= \frac{\delta^2 W}{\delta\sigma\delta\chi} + \frac{\delta^2 W}{\delta\chi\delta\upsilon}\frac{\delta\upsilon}{\delta\sigma} + \frac{\delta^2 W}{\delta\sigma\delta\upsilon}\frac{\delta\upsilon}{\delta\chi} + \frac{\delta^2 W}{\delta\upsilon^2}\frac{\delta\upsilon}{\delta\sigma}\frac{\delta\upsilon}{\delta\chi} + \frac{\delta W}{\delta\upsilon}\frac{\delta^2\upsilon}{\delta\sigma\delta\chi};\\
\left[\frac{\delta^2 W}{\delta\tau\delta\chi}\right] &= \frac{\delta^2 W}{\delta\tau\delta\chi} + \frac{\delta^2 W}{\delta\chi\delta\upsilon}\frac{\delta\upsilon}{\delta\tau} + \frac{\delta^2 W}{\delta\tau\delta\upsilon}\frac{\delta\upsilon}{\delta\chi} + \frac{\delta^2 W}{\delta\upsilon^2}\frac{\delta\upsilon}{\delta\tau}\frac{\delta\upsilon}{\delta\chi} + \frac{\delta W}{\delta\upsilon}\frac{\delta^2\upsilon}{\delta\tau\delta\chi};\\
\left[\frac{\delta^2 W}{\delta\sigma\delta x'}\right] &= \frac{\delta^2 W}{\delta\sigma\delta x'} + \frac{\delta^2 W}{\delta\upsilon\delta x'}\frac{\delta\upsilon}{\delta\sigma}; \quad \left[\frac{\delta^2 W}{\delta\tau\delta x'}\right] = \frac{\delta^2 W}{\delta\tau\delta x'} + \frac{\delta^2 W}{\delta\upsilon\delta x'}\frac{\delta\upsilon}{\delta\tau};\\
\left[\frac{\delta^2 W}{\delta\sigma\delta y'}\right] &= \frac{\delta^2 W}{\delta\sigma\delta y'} + \frac{\delta^2 W}{\delta\upsilon\delta y'}\frac{\delta\upsilon}{\delta\sigma}; \quad \left[\frac{\delta^2 W}{\delta\tau\delta y'}\right] = \frac{\delta^2 W}{\delta\tau\delta y'} + \frac{\delta^2 W}{\delta\upsilon\delta y'}\frac{\delta\upsilon}{\delta\tau};\\
\left[\frac{\delta^2 W}{\delta\sigma\delta z'}\right] &= \frac{\delta^2 W}{\delta\sigma\delta z'} + \frac{\delta^2 W}{\delta\upsilon\delta z'}\frac{\delta\upsilon}{\delta\sigma}; \quad \left[\frac{\delta^2 W}{\delta\tau\delta z'}\right] = \frac{\delta^2 W}{\delta\tau\delta z'} + \frac{\delta^2 W}{\delta\upsilon\delta z'}\frac{\delta\upsilon}{\delta\tau};\\
\left[\frac{\delta^2 W}{\delta\chi\delta x'}\right] &= \frac{\delta^2 W}{\delta\chi\delta x'} + \frac{\delta^2 W}{\delta\upsilon\delta x'}\frac{\delta\upsilon}{\delta\chi};\\
\left[\frac{\delta^2 W}{\delta\chi\delta y'}\right] &= \frac{\delta^2 W}{\delta\chi\delta y'} + \frac{\delta^2 W}{\delta\upsilon\delta y'}\frac{\delta\upsilon}{\delta\chi};\\
\left[\frac{\delta^2 W}{\delta\chi\delta z'}\right] &= \frac{\delta^2 W}{\delta\chi\delta z'} + \frac{\delta^2 W}{\delta\upsilon\delta z'}\frac{\delta\upsilon}{\delta\chi}:
\end{aligned}\right\} \quad (\mathrm{Q}^3)$$

obtained by differentiating the three corresponding expressions of the first order,

$$\left[\frac{\delta W}{\delta\sigma}\right] = \frac{\delta W}{\delta\sigma} + \frac{\delta W}{\delta\upsilon}\frac{\delta\upsilon}{\delta\sigma}; \quad \left[\frac{\delta W}{\delta\tau}\right] = \frac{\delta W}{\delta\tau} + \frac{\delta W}{\delta\upsilon}\frac{\delta\upsilon}{\delta\tau}; \quad \left[\frac{\delta W}{\delta\chi}\right] = \frac{\delta W}{\delta\chi} + \frac{\delta W}{\delta\upsilon}\frac{\delta\upsilon}{\delta\chi}, \qquad (\mathrm{R}^3)$$

which are to be substituted in (B^3), in place of

$$\frac{\delta W}{\delta\sigma}, \frac{\delta W}{\delta\tau}, \frac{\delta W}{\delta\chi}.$$

Deduction of the Coefficients of W from those of V. Homogeneous Transformations.

8. Reciprocally, if it be required to deduce the partial differential coefficients of W, of the second order, from those of V, in the case of a final variable medium, we have only to compare the expressions for

$$\delta x, \ \delta y, \ \delta z, \ \delta\sigma', \ \delta\tau', \ \delta\upsilon', \ -\delta\frac{\delta V}{\delta\chi},$$

as linear functions of $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$, deduced from the equations (A^3), with those that are obtained by differentiating the seven equations (G') (H'), into which (B') resolves itself: that is with the developed expressions for the variations of

$$\frac{\delta W}{\delta\sigma}, \ \frac{\delta W}{\delta\tau}, \ \frac{\delta W}{\delta\upsilon}, \ \frac{\delta W}{\delta x'}, \ \frac{\delta W}{\delta y'}, \ \frac{\delta W}{\delta z'}, \ \frac{\delta W}{\delta\chi}.$$

But if the final medium be uniform, then (B') no longer furnishes the seven equations (G') (H'), nor can δx, δy, δz, themselves, but only certain combinations of them, be deduced from (A^3); and the auxiliary function W is no longer completely determined in form, by the mere knowledge of the form of the characteristic function V, with which it is connected; because, in this case, the seven variables on which W depends, are not independent of each other, four of them being connected by the relation (K'), by means of which relation the dependence of W on the seven may be changed in an infinite variety of ways, while the dependence of V on its seven variables, and the properties of the optical combination, remain unaltered. Accordingly this indeterminateness of W, as deduced from V, in the case of a final uniform medium, produces an indeterminateness, in the same case, in the partial differential coefficients of W; and whereas W, considered as a function of seven variables, has thirty-five partial differential coefficients of the first and second orders, we have only twenty-seven relations between these thirty-five coefficients, unless we make some particular supposition respecting the form of W; such as the supposition, already mentioned, that one of the related variables, for example υ, has been removed by a previous elimination, which gives the eight conditions,

$$\frac{\delta W}{\delta\upsilon}=0, \ \frac{\delta^2 W}{\delta\sigma\delta\upsilon}=0, \ \frac{\delta^2 W}{\delta\tau\delta\upsilon}=0, \ \frac{\delta^2 W}{\delta\upsilon^2}=0, \ \frac{\delta^2 W}{\delta\upsilon\delta\chi}=0, \ \frac{\delta^2 W}{\delta\upsilon\delta x'}=0, \ \frac{\delta^2 W}{\delta\upsilon\delta y'}=0, \ \frac{\delta^2 W}{\delta\upsilon\delta z'}=0. \qquad (S^3)$$

This last supposition removes the indeterminateness of W itself, and therefore of its partial differential coefficients; of which, for the two first orders, eight vanish by

(S^3), and the remaining twenty-seven are determined, (when the variables and coefficients of V are known,) by the six equations (G'), (B^3), the three lefthand equations (G^3), the six first (I^3), the two first (K^3), and the ten (L^3) (M^3) (P^3); in resolving which equations it is useful to observe, that by (E^3) and (G^3),

$$\frac{1}{w''} = \frac{\delta^2 V}{\delta x^2} \frac{\delta^2 V}{\delta y^2} - \left(\frac{\delta^2 V}{\delta x \delta y}\right)^2. \qquad (\mathrm{T}^3)$$

And the twenty-seven expressions thus found for the coefficients of W of the two first orders, on the supposition of a previous elimination of one of the seven related variables, may be generalised, by (Q^3) and (R^3), into the twenty-seven relations already mentioned as existing between the thirty-five coefficients on any other supposition; which supposition, if it be sufficient to determine the form of W, will give the eight remaining conditions analogous to the conditions (S^3), that are necessary to determine the coefficients sought.

If, for example, we determine W by supposing it made homogeneous of the first dimension with respect to σ, τ, υ, we shall have the eight following conditions,

$$\sigma \frac{\delta W}{\delta \sigma} + \tau \frac{\delta W}{\delta \tau} + \upsilon \frac{\delta W}{\delta \upsilon} = W, \qquad (\mathrm{U}^3)$$

and

$$\left.\begin{aligned}
\sigma \frac{\delta^2 W}{\delta \sigma^2} + \tau \frac{\delta^2 W}{\delta \sigma \delta \tau} + \upsilon \frac{\delta^2 W}{\delta \sigma \delta \upsilon} &= 0, \\
\sigma \frac{\delta^2 W}{\delta \sigma \delta \tau} + \tau \frac{\delta^2 W}{\delta \tau^2} + \upsilon \frac{\delta^2 W}{\delta \tau \delta \upsilon} &= 0, \\
\sigma \frac{\delta^2 W}{\delta \sigma \delta \upsilon} + \tau \frac{\delta^2 W}{\delta \tau \delta \upsilon} + \upsilon \frac{\delta^2 W}{\delta \upsilon^2} &= 0, \\
\sigma \frac{\delta^2 W}{\delta \sigma \delta x'} + \tau \frac{\delta^2 W}{\delta \tau \delta x'} + \upsilon \frac{\delta^2 W}{\delta \upsilon \delta x'} &= \frac{\delta W}{\delta x'}, \\
\sigma \frac{\delta^2 W}{\delta \sigma \delta y'} + \tau \frac{\delta^2 W}{\delta \tau \delta y'} + \upsilon \frac{\delta^2 W}{\delta \upsilon \delta y'} &= \frac{\delta W}{\delta y'}, \\
\sigma \frac{\delta^2 W}{\delta \sigma \delta z'} + \tau \frac{\delta^2 W}{\delta \tau \delta z'} + \upsilon \frac{\delta^2 W}{\delta \upsilon \delta z'} &= \frac{\delta W}{\delta z'}, \\
\sigma \frac{\delta^2 W}{\delta \sigma \delta \chi} + \tau \frac{\delta^2 W}{\delta \tau \delta \chi} + \upsilon \frac{\delta^2 W}{\delta \upsilon \delta \chi} &= \frac{\delta W}{\delta \chi},
\end{aligned}\right\} \qquad (\mathrm{V}^3)$$

to be combined with the twenty-seven which are independent of the form of W, and are deduced by the general method already mentioned. But this supposition of homogeneity appears to deserve a separate investigation, on account of the symmetry of the processes and results to which it leads.

Let us therefore resume the equations

$$x = \frac{\delta W}{\delta \sigma} + V \frac{\delta \Omega}{\delta \sigma}, \; y = \frac{\delta W}{\delta \tau} + V \frac{\delta \Omega}{\delta \tau}, \; z = \frac{\delta W}{\delta \upsilon} + V \frac{\delta \Omega}{\delta \upsilon}, \tag{W^2}$$

which were deduced in the sixth number from the homogeneous form that we now assign to W, and which are to be combined with the following

$$0 = \frac{\delta W}{\delta \chi} + \frac{\delta V}{\delta \chi} + V \frac{\delta \Omega}{\delta \chi}, \tag{W^3}$$

and with the general equations of the fourth number,

$$\sigma' = \frac{\delta W}{\delta x'}, \; \tau' = \frac{\delta W}{\delta y'}, \; \upsilon' = \frac{\delta W}{\delta z'} : \tag{G$'$}$$

and let us eliminate

$$\delta x, \delta y, \delta z, \delta \sigma', \delta \tau', \delta \upsilon', \delta \frac{\delta V}{\delta \chi},$$

by (A^3), from the differentials of these seven equations, (W^2) (G') (W^3), that is from the seven following,

$$\left.\begin{aligned}
&\delta \frac{\delta W}{\delta \sigma} + V \delta \frac{\delta \Omega}{\delta \sigma} = \delta x - \frac{\delta \Omega}{\delta \sigma} \delta V, \\
&\delta \frac{\delta W}{\delta \tau} + V \delta \frac{\delta \Omega}{\delta \tau} = \delta y - \frac{\delta \Omega}{\delta \tau} \delta V, \\
&\delta \frac{\delta W}{\delta \upsilon} + V \delta \frac{\delta \Omega}{\delta \upsilon} = \delta z - \frac{\delta \Omega}{\delta \upsilon} \delta V, \\
&\delta \frac{\delta W}{\delta x'} = \delta \sigma', \; \delta \frac{\delta W}{\delta y'} = \delta \tau', \; \delta \frac{\delta W}{\delta z'} = \delta \upsilon', \\
&\delta \frac{\delta W}{\delta \chi} + V \delta \frac{\delta \Omega}{\delta \chi} = - \delta \frac{\delta V}{\delta \chi} - \frac{\delta \Omega}{\delta \chi} \delta V.
\end{aligned}\right\} \tag{X^3}$$

This elimination gives

$$\left.\begin{aligned}
\lambda^{(1)} \delta \Omega = -\delta \sigma + \delta' \frac{\delta V}{\delta x} &+ \frac{\delta^2 V}{\delta x^2} \left(\delta \frac{\delta W}{\delta \sigma} + V \delta \frac{\delta \Omega}{\delta \sigma} \right) + \frac{\delta^2 V}{\delta x \delta y} \left(\delta \frac{\delta W}{\delta \tau} + V \delta \frac{\delta \Omega}{\delta \tau} \right) \\
&+ \frac{\delta^2 V}{\delta x \delta z} \left(\delta \frac{\delta W}{\delta \upsilon} + V \delta \frac{\delta \Omega}{\delta \upsilon} \right); \\
\lambda^{(2)} \delta \Omega = -\delta \tau + \delta' \frac{\delta V}{\delta y} &+ \frac{\delta^2 V}{\delta x \delta y} \left(\delta \frac{\delta W}{\delta \sigma} + V \delta \frac{\delta \Omega}{\delta \sigma} \right) + \frac{\delta^2 V}{\delta y^2} \left(\delta \frac{\delta W}{\delta \tau} + V \delta \frac{\delta \Omega}{\delta \tau} \right) \\
&+ \frac{\delta^2 V}{\delta y \delta z} \left(\delta \frac{\delta W}{\delta \upsilon} + V \delta \frac{\delta \Omega}{\delta \upsilon} \right); \\
\lambda^{(3)} \delta \Omega = -\delta \upsilon + \delta' \frac{\delta V}{\delta z} &+ \frac{\delta^2 V}{\delta x \delta z} \left(\delta \frac{\delta W}{\delta \sigma} + V \delta \frac{\delta \Omega}{\delta \sigma} \right) + \frac{\delta^2 V}{\delta y \delta z} \left(\delta \frac{\delta W}{\delta \tau} + V \delta \frac{\delta \Omega}{\delta \tau} \right) \\
&+ \frac{\delta^2 V}{\delta z^2} \left(\delta \frac{\delta W}{\delta \upsilon} + V \delta \frac{\delta \Omega}{\delta \upsilon} \right);
\end{aligned}\right|$$

$$\left.\begin{aligned}
\lambda^{4}\ \delta\Omega &= \delta\frac{\delta W}{\delta x'} + \delta'\frac{\delta V}{\delta x'} + \frac{\delta^2 V}{\delta x\delta x'}\left(\delta\frac{\delta W}{\delta\sigma} + V\delta\frac{\delta\Omega}{\delta\sigma}\right) + \frac{\delta^2 V}{\delta y\delta x'}\left(\delta\frac{\delta W}{\delta\tau} + V\delta\frac{\delta\Omega}{\delta\tau}\right)\\
&\quad + \frac{\delta^2 V}{\delta z\delta x'}\left(\delta\frac{\delta W}{\delta\upsilon} + V\delta\frac{\delta\Omega}{\delta\upsilon}\right);\\
\lambda^{(5)}\ \delta\Omega &= \delta\frac{\delta W}{\delta y'} + \delta'\frac{\delta V}{\delta y'} + \frac{\delta^2 V}{\delta x\delta y'}\left(\delta\frac{\delta W}{\delta\sigma} + V\delta\frac{\delta\Omega}{\delta\sigma}\right) + \frac{\delta^2 V}{\delta y\delta y'}\left(\delta\frac{\delta W}{\delta\tau} + V\delta\frac{\delta\Omega}{\delta\tau}\right)\\
&\quad + \frac{\delta^2 V}{\delta z\delta y'}\left(\delta\frac{\delta W}{\delta\upsilon} + V\delta\frac{\delta\Omega}{\delta\upsilon}\right);\\
\lambda^{(6)}\ \delta\Omega &= \delta\frac{\delta W}{\delta z'} + \delta'\frac{\delta V}{\delta z'} + \frac{\delta^2 V}{\delta x\delta z'}\left(\delta\frac{\delta W}{\delta\sigma} + V\delta\frac{\delta\Omega}{\delta\sigma}\right) + \frac{\delta^2 V}{\delta y\delta z'}\left(\delta\frac{\delta W}{\delta\tau} + V\delta\frac{\delta\Omega}{\delta\tau}\right)\\
&\quad + \frac{\delta^2 V}{\delta z\delta z'}\left(\delta\frac{\delta W}{\delta\upsilon} + V\delta\frac{\delta\Omega}{\delta\upsilon}\right);\\
\lambda^{7}\ \delta\Omega &= \delta\frac{\delta W}{\delta\chi} + V\delta\frac{\delta\Omega}{\delta\chi} + \delta'\frac{\delta V}{\delta\chi} + \frac{\delta^2 V}{\delta x\delta\chi}\left(\delta\frac{\delta W}{\delta\sigma} + V\delta\frac{\delta\Omega}{\delta\sigma}\right) + \frac{\delta^2 V}{\delta y\delta\chi}\left(\delta\frac{\delta W}{\delta\tau} + V\delta\frac{\delta\Omega}{\delta\tau}\right)\\
&\quad + \frac{\delta^2 V}{\delta z\delta\chi}\left(\delta\frac{\delta W}{\delta\upsilon} + V\delta\frac{\delta\Omega}{\delta\upsilon}\right);
\end{aligned}\right\}\ (\mathrm{Y}^3)$$

if we put for abridgment

$$\left.\begin{aligned}
\delta'\frac{\delta V}{\delta x} &= \frac{\delta^2 V}{\delta x\delta x'}\delta x' + \frac{\delta^2 V}{\delta x\delta y'}\delta y' + \frac{\delta^2 V}{\delta x\delta z'}\delta z' + \frac{\delta^2 V}{\delta x\delta\chi}\delta\chi,\\
\delta'\frac{\delta V}{\delta y} &= \frac{\delta^2 V}{\delta y\delta x'}\delta x' + \frac{\delta^2 V}{\delta y\delta y'}\delta y' + \frac{\delta^2 V}{\delta y\delta z'}\delta z' + \frac{\delta^2 V}{\delta y\delta\chi}\delta\chi,\\
\delta'\frac{\delta V}{\delta z} &= \frac{\delta^2 V}{\delta z\delta x'}\delta x' + \frac{\delta^2 V}{\delta z\delta y'}\delta y' + \frac{\delta^2 V}{\delta z\delta z'}\delta z' + \frac{\delta^2 V}{\delta z\delta\chi}\delta\chi,\\
\delta'\frac{\delta V}{\delta x'} &= \frac{\delta^2 V}{\delta x'^2}\delta x' + \frac{\delta^2 V}{\delta x'\delta y'}\delta y' + \frac{\delta^2 V}{\delta x'\delta z'}\delta z' + \frac{\delta^2 V}{\delta x'\delta\chi}\delta\chi,\\
\delta'\frac{\delta V}{\delta y'} &= \frac{\delta^2 V}{\delta x'\delta y'}\delta x' + \frac{\delta^2 V}{\delta y'^2}\delta y' + \frac{\delta^2 V}{\delta y'\delta z'}\delta z' + \frac{\delta^2 V}{\delta y'\delta\chi}\delta\chi,\\
\delta'\frac{\delta V}{\delta z'} &= \frac{\delta^2 V}{\delta x'\delta z'}\delta x' + \frac{\delta^2 V}{\delta y'\delta z'}\delta y' + \frac{\delta^2 V}{\delta z'^2}\delta z' + \frac{\delta^2 V}{\delta z'\delta\chi}\delta\chi,\\
\delta'\frac{\delta V}{\delta\chi} &= \frac{\delta^2 V}{\delta x'\delta\chi}\delta x' + \frac{\delta^2 V}{\delta y'\delta\chi}\delta y' + \frac{\delta^2 V}{\delta z'\delta\chi}\delta z' + \frac{\delta^2 V}{\delta\chi^2}\delta\chi,
\end{aligned}\right\}\ (\mathrm{Z}^3)$$

using δ' as in the notation (D^3); and if we observe that the partial differential equation of the fifth number,

$$0 = \Omega\left(\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z}, \chi\right), \qquad (\mathrm{V}')$$

gives

$$\left.\begin{aligned}
0 &= \frac{\delta\Omega}{\delta\sigma}\frac{\delta^2 V}{\delta x^2} + \frac{\delta\Omega}{\delta\tau}\frac{\delta^2 V}{\delta x\delta y} + \frac{\delta\Omega}{\delta\upsilon}\frac{\delta^2 V}{\delta x\delta z},\\
0 &= \frac{\delta\Omega}{\delta\sigma}\frac{\delta^2 V}{\delta x\delta y} + \frac{\delta\Omega}{\delta\tau}\frac{\delta^2 V}{\delta y^2} + \frac{\delta\Omega}{\delta\upsilon}\frac{\delta^2 V}{\delta y\delta z},\\
0 &= \frac{\delta\Omega}{\delta\sigma}\frac{\delta^2 V}{\delta x\delta z} + \frac{\delta\Omega}{\delta\tau}\frac{\delta^2 V}{\delta y\delta z} + \frac{\delta\Omega}{\delta\upsilon}\frac{\delta^2 V}{\delta z^2},\\
0 &= \frac{\delta\Omega}{\delta\sigma}\frac{\delta^2 V}{\delta x\delta x'} + \frac{\delta\Omega}{\delta\tau}\frac{\delta^2 V}{\delta y\delta x'} + \frac{\delta\Omega}{\delta\upsilon}\frac{\delta^2 V}{\delta z\delta x'},\\
0 &= \frac{\delta\Omega}{\delta\sigma}\frac{\delta^2 V}{\delta x\delta y'} + \frac{\delta\Omega}{\delta\tau}\frac{\delta^2 V}{\delta y\delta y'} + \frac{\delta\Omega}{\delta\upsilon}\frac{\delta^2 V}{\delta z\delta y'},\\
0 &= \frac{\delta\Omega}{\delta\sigma}\frac{\delta^2 V}{\delta x\delta z'} + \frac{\delta\Omega}{\delta\tau}\frac{\delta^2 V}{\delta y\delta z'} + \frac{\delta\Omega}{\delta\upsilon}\frac{\delta^2 V}{\delta z\delta z'},\\
-\frac{\delta\Omega}{\delta\chi} &= \frac{\delta\Omega}{\delta\sigma}\frac{\delta^2 V}{\delta x\delta\chi} + \frac{\delta\Omega}{\delta\tau}\frac{\delta^2 V}{\delta y\delta\chi} + \frac{\delta\Omega}{\delta\upsilon}\frac{\delta^2 V}{\delta z\delta\chi}.
\end{aligned}\right\} \quad (A^4)$$

We have introduced, in the equations (Y^3), the terms $\lambda^{(1)}\delta\Omega,\ldots\lambda^{(7)}\delta\Omega$, that we may treat as independent the variations $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, $\delta\chi$, which are connected by the condition $\delta\Omega = 0$.

To determine the multipliers $\lambda^{(1)},\ldots\lambda^{(7)}$, we are to observe that in deducing the foregoing equations, the relation $\Omega = 0$ between the four variables σ, τ, υ, χ, has been supposed to have been so expressed, by the method mentioned in the second number, that the function Ω when increased by unity becomes homogeneous of the first dimension with respect to σ, τ, υ; in such a manner that we have identically, for all values of the four variables σ, τ, υ, χ,

$$\sigma\frac{\delta\Omega}{\delta\sigma} + \tau\frac{\delta\Omega}{\delta\tau} + \upsilon\frac{\delta\Omega}{\delta\upsilon} = \Omega + 1, \qquad (B^4)$$

and therefore,

$$\left.\begin{aligned}
\sigma\frac{\delta^2\Omega}{\delta\sigma^2} + \tau\frac{\delta^2\Omega}{\delta\sigma\delta\tau} + \upsilon\frac{\delta^2\Omega}{\delta\sigma\delta\upsilon} &= 0,\\
\sigma\frac{\delta^2\Omega}{\delta\sigma\delta\tau} + \tau\frac{\delta^2\Omega}{\delta\tau^2} + \upsilon\frac{\delta^2\Omega}{\delta\tau\delta\upsilon} &= 0,\\
\sigma\frac{\delta^2\Omega}{\delta\sigma\delta\upsilon} + \tau\frac{\delta^2\Omega}{\delta\tau\delta\upsilon} + \upsilon\frac{\delta^2\Omega}{\delta\upsilon^2} &= 0,\\
\sigma\frac{\delta^2\Omega}{\delta\sigma\delta\chi} + \tau\frac{\delta^2\Omega}{\delta\tau\delta\chi} + \upsilon\frac{\delta^2\Omega}{\delta\upsilon\delta\chi} &= \frac{\delta\Omega}{\delta\chi}.
\end{aligned}\right\} \quad (C^4)$$

Hence, and from the conditions (V^3), relative to the homogeneity of the function W, it is easy to infer that the multipliers have the following values;

$$\lambda^{(1)} = -\sigma\,;\ \lambda^{(2)} = -\tau\,;\ \lambda^{(3)} = -\upsilon\,;\ \lambda^{(4)} = \sigma'\,;\ \lambda^{(5)} = \tau'\,;\ \lambda^{(6)} = \upsilon'\,;\ \lambda^{(7)} = -\frac{\delta V}{\delta\chi}: \qquad (D^4)$$

attending to (G') and (W^3). If we substitute these values of the multipliers, in the seven equations (Y^3), we may decompose each of those equations into seven others, by treating the seven variations $\delta\sigma, \delta\tau, \delta\upsilon, \delta x', \delta y', \delta z', \delta\chi$, as independent; and thus obtain forty-nine equations of the first degree, of which however only twenty-eight are distinct, for the determination of the twenty-eight partial differential coefficients of the second order, of W considered as a function of $\sigma, \tau, \upsilon, x', y', z', \chi$, which relatively to σ, τ, υ, is homogeneous of the first dimension: the corresponding coefficients of the first order being determined by the seven equations (G') (W^2) (W^3).

Instead of calculating in this manner the coefficients of W of the second order, by eliminating between the equations into which the system (Y^3) may be decomposed, it is simpler to eliminate between the equations (Y^3) themselves, and thus to obtain expressions for the variations

$$\delta\frac{\delta W}{\delta\sigma}, \ldots \delta\frac{\delta W}{\delta\chi},$$

of the coefficients of the first order, from which expressions the coefficients of the second order will then immediately result. Eliminating, therefore, between the three first equations (Y^3), in order to get expressions for the three variations

$$\delta\frac{\delta W}{\delta\sigma}, \quad \delta\frac{\delta W}{\delta\tau}, \quad \delta\frac{\delta W}{\delta\upsilon},$$

we find, after some symmetric reductions,

$$\left.\begin{aligned}
\delta\frac{\delta W}{\delta\sigma} = - V\delta\frac{\delta\Omega}{\delta\sigma} &+ \frac{1}{v^2 V''}\left(\tau\frac{\delta^2 V}{\delta x\delta z} - \upsilon\frac{\delta^2 V}{\delta x\delta y}\right)\left\{\upsilon\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right) - \tau\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right)\right\}\\
-\frac{\delta\Omega}{\delta\sigma}\delta' V &+ \frac{1}{v^2 V''}\left(\tau\frac{\delta^2 V}{\delta y\delta z} - \upsilon\frac{\delta^2 V}{\delta y^2}\right)\left\{\sigma\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right) - \upsilon\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right)\right\}\\
&+ \frac{1}{v^2 V''}\left(\tau\frac{\delta^2 V}{\delta z^2} - \upsilon\frac{\delta^2 V}{\delta y\delta z}\right)\left\{\tau\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right) - \sigma\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right)\right\};\\
\delta\frac{\delta W}{\delta\tau} = - V\delta\frac{\delta\Omega}{\delta\tau} &+ \frac{1}{v^2 V''}\left(\upsilon\frac{\delta^2 V}{\delta x^2} - \sigma\frac{\delta^2 V}{\delta x\delta z}\right)\left\{\upsilon\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right) - \tau\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right)\right\}\\
-\frac{\delta\Omega}{\delta\tau}\delta' V &+ \frac{1}{v^2 V''}\left(\upsilon\frac{\delta^2 V}{\delta x\delta y} - \sigma\frac{\delta^2 V}{\delta y\delta z}\right)\left\{\sigma\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right) - \upsilon\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right)\right\}\\
&+ \frac{1}{v^2 V''}\left(\upsilon\frac{\delta^2 V}{\delta x\delta z} - \sigma\frac{\delta^2 V}{\delta z^2}\right)\left\{\tau\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right) - \sigma\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right)\right\};\\
\delta\frac{\delta W}{\delta\upsilon} = - V\delta\frac{\delta\Omega}{\delta\upsilon} &+ \frac{1}{v^2 V''}\left(\sigma\frac{\delta^2 V}{\delta x\delta y} - \tau\frac{\delta^2 V}{\delta x^2}\right)\left\{\upsilon\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right) - \tau\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right)\right\}\\
-\frac{\delta\Omega}{\delta\upsilon}\delta' V &+ \frac{1}{v^2 V''}\left(\sigma\frac{\delta^2 V}{\delta y^2} - \tau\frac{\delta^2 V}{\delta x\delta y}\right)\left\{\sigma\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right) - \upsilon\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right)\right\}\\
&+ \frac{1}{v^2 V''}\left(\sigma\frac{\delta^2 V}{\delta y\delta z} - \tau\frac{\delta^2 V}{\delta x\delta z}\right)\left\{\tau\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right) - \sigma\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right)\right\};
\end{aligned}\right\} (E^4)$$

in which,

$$\left.\begin{aligned} V''&=\frac{\delta^2 V}{\delta x^2}\,\frac{\delta^2 V}{\delta y^2}-\left(\frac{\delta^2 V}{\delta x\delta y}\right)^2+\frac{\delta^2 V}{\delta y^2}\,\frac{\delta^2 V}{\delta z^2}-\left(\frac{\delta^2 V}{\delta y\delta z}\right)^2+\frac{\delta^2 V}{\delta z^2}\,\frac{\delta^2 V}{\delta x^2}-\left(\frac{\delta^2 V}{\delta z\delta x}\right)^2, \\ \text{and}\qquad \frac{1}{v^2}&=\left(\frac{\delta\Omega}{\delta\sigma}\right)^2+\left(\frac{\delta\Omega}{\delta\tau}\right)^2+\left(\frac{\delta\Omega}{\delta\upsilon}\right)^2, \end{aligned}\right\}\quad (\mathrm{F}^4)$$

v having the same meaning as before: δ also referring, as before, to the variations of $x'\,y'\,z'\,\chi$ alone, and V'' having the same meaning as in the First Supplement. In effecting this elimination, we have attended to the forms of the functions W, Ω, which give

$$\sigma\left(\delta\frac{\delta W}{\delta\sigma}+V\delta\frac{\delta\Omega}{\delta\sigma}\right)+\tau\left(\delta\frac{\delta W}{\delta\tau}+V\delta\frac{\delta\Omega}{\delta\tau}\right)+\upsilon\left(\delta\frac{\delta W}{\delta\upsilon}+V\delta\frac{\delta\Omega}{\delta\upsilon}\right)=-\delta V; \quad (\mathrm{G}^4)$$

we have also employed the equations (A^4), which give, by (F^4),

$$\left.\begin{aligned} \frac{\delta^2 V}{\delta y^2}\,\frac{\delta^2 V}{\delta z^2}-\left(\frac{\delta^2 V}{\delta y\delta z}\right)^2 &= V''v^2\left(\frac{\delta\Omega}{\delta\sigma}\right)^2; & \frac{\delta^2 V}{\delta x\delta y}\,\frac{\delta^2 V}{\delta z\delta x}-\frac{\delta^2 V}{\delta x^2}\,\frac{\delta^2 V}{\delta y\delta z} &= V''v^2\,\frac{\delta\Omega}{\delta\tau}\,\frac{\delta\Omega}{\delta\upsilon}; \\ \frac{\delta^2 V}{\delta z^2}\,\frac{\delta^2 V}{\delta x^2}-\left(\frac{\delta^2 V}{\delta z\delta x}\right)^2 &= V''v^2\left(\frac{\delta\Omega}{\delta\tau}\right)^2; & \frac{\delta^2 V}{\delta y\delta z}\,\frac{\delta^2 V}{\delta x\delta y}-\frac{\delta^2 V}{\delta y^2}\,\frac{\delta^2 V}{\delta z\delta x} &= V''v^2\,\frac{\delta\Omega}{\delta\upsilon}\,\frac{\delta\Omega}{\delta\sigma}; \\ \frac{\delta^2 V}{\delta x^2}\,\frac{\delta^2 V}{\delta y^2}-\left(\frac{\delta^2 V}{\delta x\delta y}\right)^2 &= V''v^2\left(\frac{\delta\Omega}{\delta\upsilon}\right)^2; & \frac{\delta^2 V}{\delta z\delta x}\,\frac{\delta^2 V}{\delta y\delta z}-\frac{\delta^2 V}{\delta z^2}\,\frac{\delta^2 V}{\delta x\delta y} &= V''v^2\,\frac{\delta\Omega}{\delta\sigma}\,\frac{\delta\Omega}{\delta\tau}. \end{aligned}\right\}\quad (\mathrm{H}^4)$$

Having thus obtained expressions (E^4) for the three variations

$$\delta\frac{\delta W}{\delta\sigma},\quad \delta\frac{\delta W}{\delta\tau},\quad \delta\frac{\delta W}{\delta\upsilon},$$

it only remains to substitute these expressions in the four last equations (Y^3), and so to deduce, without any new elimination, the four other variations

$$\delta\frac{\delta W}{\delta x'},\quad \delta\frac{\delta W}{\delta y'},\quad \delta\frac{\delta W}{\delta z'},\quad \delta\frac{\delta W}{\delta\chi};$$

after which, we shall have immediately the twenty-eight coefficients of W, of the second order. The six coefficients, for example, of this order, which are formed by differentiating W with respect to σ, τ, υ, are expressed by the six following equations, deduced from (E^4);

$$\left.\begin{aligned} \frac{\delta^2 W}{\delta\sigma^2}&=-V\frac{\delta^2\Omega}{\delta\sigma^2}+\frac{1}{v^2V''}\left(\tau^2\frac{\delta^2 V}{\delta z^2}+\upsilon^2\frac{\delta^2 V}{\delta y^2}-2\tau\upsilon\frac{\delta^2 V}{\delta y\delta z}\right); \\ \frac{\delta^2 W}{\delta\tau^2}&=-V\frac{\delta^2\Omega}{\delta\tau^2}+\frac{1}{v^2V''}\left(\upsilon^2\frac{\delta^2 V}{\delta x^2}+\sigma^2\frac{\delta^2 V}{\delta z^2}-2\upsilon\sigma\frac{\delta^2 V}{\delta z\delta x}\right); \\ \frac{\delta^2 W}{\delta\upsilon^2}&=-V\frac{\delta^2\Omega}{\delta\upsilon^2}+\frac{1}{v^2V''}\left(\sigma^2\frac{\delta^2 V}{\delta y^2}+\tau^2\frac{\delta^2 V}{\delta x^2}-2\sigma\tau\frac{\delta^2 V}{\delta x\delta y}\right); \\ \frac{\delta^2 W}{\delta\sigma\delta\tau}&=-V\frac{\delta^2\Omega}{\delta\sigma\delta\tau}+\frac{1}{v^2V''}\left(-\upsilon^2\frac{\delta^2 V}{\delta x\delta y}+\tau\upsilon\frac{\delta^2 V}{\delta z\delta x}+\upsilon\sigma\frac{\delta^2 V}{\delta y\delta z}-\sigma\tau\frac{\delta^2 V}{\delta z^2}\right); \\ \frac{\delta^2 W}{\delta\tau\delta\upsilon}&=-V\frac{\delta^2\Omega}{\delta\tau\delta\upsilon}+\frac{1}{v^2V''}\left(-\sigma^2\frac{\delta^2 V}{\delta y\delta z}+\upsilon\sigma\frac{\delta^2 V}{\delta x\delta y}+\sigma\tau\frac{\delta^2 V}{\delta z\delta x}-\tau\upsilon\frac{\delta^2 V}{\delta x^2}\right); \\ \frac{\delta^2 W}{\delta\upsilon\delta\sigma}&=-V\frac{\delta^2\Omega}{\delta\upsilon\delta\sigma}+\frac{1}{v^2V''}\left(-\tau^2\frac{\delta^2 V}{\delta z\delta x}+\sigma\tau\frac{\delta^2 V}{\delta y\delta z}+\tau\upsilon\frac{\delta^2 V}{\delta x\delta y}-\upsilon\sigma\frac{\delta^2 V}{\delta y^2}\right): \end{aligned}\right\}\quad (\mathrm{I}^4)$$

which may be shown to agree with the less simple equations of the same kind in the First Supplement, and may be thus summed up,

$$\begin{aligned} v^2 V'' (\delta''^2 W + V \delta''^2 \Omega) = {} & \frac{\delta^2 V}{\delta x^2} (\tau\delta\upsilon - \upsilon\delta\tau)^2 + 2 \frac{\delta^2 V}{\delta y \delta z} (\upsilon\delta\sigma - \sigma\delta\upsilon)(\sigma\delta\tau - \tau\delta\sigma) \\ & + \frac{\delta^2 V}{\delta y^2} (\upsilon\delta\sigma - \sigma\delta\upsilon)^2 + 2 \frac{\delta^2 V}{\delta z \delta x} (\sigma\delta\tau - \tau\delta\sigma)(\tau\delta\upsilon - \upsilon\delta\tau) \\ & + \frac{\delta^2 V}{\delta z^2} (\sigma\delta\tau - \tau\delta\sigma)^2 + 2 \frac{\delta^2 V}{\delta x \delta y} (\tau\delta\upsilon - \upsilon\delta\tau)(\upsilon\delta\sigma - \sigma\delta\upsilon), \quad (\mathrm{K}^4) \end{aligned}$$

the mark of variation δ'' referring only to the variables σ, τ, υ, as δ' referred only to x', y', z', χ.

And the whole system of the twenty-eight expressions for the twenty-eight coefficients of W, of the second order, may be summed up in this one formula:

$$\begin{aligned} & v^2 V'' (\delta^2 W + V \delta^2 \Omega + 2\delta' V \delta \Omega + \delta'^2 V) = \frac{\delta^2 V}{\delta x^2} \left\{ \tau\left(\delta\upsilon - \delta' \frac{\delta V}{\delta z}\right) - \upsilon\left(\delta\tau - \delta' \frac{\delta V}{\delta y}\right) \right\}^2 \\ & + \frac{\delta^2 V}{\delta y^2} \left\{ \upsilon\left(\delta\sigma - \delta' \frac{\delta V}{\delta x}\right) - \sigma\left(\delta\upsilon - \delta' \frac{\delta V}{\delta z}\right) \right\}^2 + \frac{\delta^2 V}{\delta z^2} \left\{ \sigma\left(\delta\tau - \delta' \frac{\delta V}{\delta y}\right) - \tau\left(\delta\sigma - \delta' \frac{\delta V}{\delta x}\right) \right\}^2 \\ & + 2 \frac{\delta^2 V}{\delta x \delta y} \left\{ \tau\left(\delta\upsilon - \delta' \frac{\delta V}{\delta z}\right) - \upsilon\left(\delta\tau - \delta' \frac{\delta V}{\delta y}\right) \right\} \left\{ \upsilon\left(\delta\sigma - \delta' \frac{\delta V}{\delta x}\right) - \sigma\left(\delta\upsilon - \delta' \frac{\delta V}{\delta z}\right) \right\} \\ & + 2 \frac{\delta^2 V}{\delta y \delta z} \left\{ \upsilon\left(\delta\sigma - \delta' \frac{\delta V}{\delta x}\right) - \sigma\left(\delta\upsilon - \delta' \frac{\delta V}{\delta z}\right) \right\} \left\{ \sigma\left(\delta\tau - \delta' \frac{\delta V}{\delta y}\right) - \tau\left(\delta\sigma - \delta' \frac{\delta V}{\delta x}\right) \right\} \\ & + 2 \frac{\delta^2 V}{\delta z \delta x} \left\{ \sigma\left(\delta\tau - \delta' \frac{\delta V}{\delta y}\right) - \tau\left(\delta\sigma - \delta' \frac{\delta V}{\delta x}\right) \right\} \left\{ \tau\left(\delta\upsilon - \delta' \frac{\delta V}{\delta z}\right) - \upsilon\left(\delta\tau - \delta' \frac{\delta V}{\delta y}\right) \right\}; (\mathrm{L}^4) \end{aligned}$$

in which the symbols δ^2, δ'^2, are easily understood by what precedes, and in which the seven variations $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$, may be treated as independent of each other.

The formula (K^4) has an inverse, deduced from (X^3), namely

$$\begin{aligned} \frac{\delta'''^2 V}{v^2 V''} = {} & \left(\frac{\delta^2 W}{\delta\sigma^2} + V \frac{\delta^2 \Omega}{\delta\sigma^2}\right) \left(\frac{\delta\Omega}{\delta\tau} \delta z - \frac{\delta\Omega}{\delta\upsilon} \delta y\right)^2 \\ & + \left(\frac{\delta^2 W}{\delta\tau^2} + V \frac{\delta^2 \Omega}{\delta\tau^2}\right) \left(\frac{\delta\Omega}{\delta\upsilon} \delta x - \frac{\delta\Omega}{\delta\sigma} \delta z\right)^2 \\ & + \left(\frac{\delta^2 W}{\delta\upsilon^2} + V \frac{\delta^2 \Omega}{\delta\upsilon^2}\right) \left(\frac{\delta\Omega}{\delta\sigma} \delta y - \frac{\delta\Omega}{\delta\upsilon} \delta x\right)^2 \\ & + 2 \left(\frac{\delta^2 W}{\delta\sigma\delta\tau} + V \frac{\delta^2 \Omega}{\delta\sigma\delta\tau}\right) \left(\frac{\delta\Omega}{\delta\tau} \delta z - \frac{\delta\Omega}{\delta\upsilon} \delta y\right) \left(\frac{\delta\Omega}{\delta\upsilon} \delta x - \frac{\delta\Omega}{\delta\sigma} \delta z\right) \\ & + 2 \left(\frac{\delta^2 W}{\delta\tau\delta\upsilon} + V \frac{\delta^2 \Omega}{\delta\tau\delta\upsilon}\right) \left(\frac{\delta\Omega}{\delta\upsilon} \delta x - \frac{\delta\Omega}{\delta\sigma} \delta z\right) \left(\frac{\delta\Omega}{\delta\sigma} \delta y - \frac{\delta\Omega}{\delta\tau} \delta x\right) \\ & + 2 \left(\frac{\delta^2 W}{\delta\upsilon\delta\sigma} + V \frac{\delta^2 \Omega}{\delta\upsilon\delta\sigma}\right) \left(\frac{\delta\Omega}{\delta\sigma} \delta y - \frac{\delta\Omega}{\delta\tau} \delta x\right) \left(\frac{\delta\Omega}{\delta\tau} \delta z - \frac{\delta\Omega}{\delta\upsilon} \delta y\right), \qquad (\mathrm{M}^4) \end{aligned}$$

in which δ''' refers to x, y, z, and in which V'' may be deduced from W by the relation

$$\frac{\sigma^2+\tau^2+\upsilon^2}{V''\,\upsilon^2} = \left(\frac{\delta^2 W}{\delta\sigma^2} + V\frac{\delta^2\Omega}{\delta\sigma^2}\right)\left(\frac{\delta^2 W}{\delta\tau^2} + V\frac{\delta^2\Omega}{\delta\tau^2}\right) - \left(\frac{\delta^2 W}{\delta\sigma\delta\tau} + V\frac{\delta^2\Omega}{\delta\sigma\delta\tau}\right)^2$$
$$+ \left(\frac{\delta^2 W}{\delta\tau^2} + V\frac{\delta^2\Omega}{\delta\tau^2}\right)\left(\frac{\delta^2 W}{\delta\upsilon^2} + V\frac{\delta^2\Omega}{\delta\upsilon^2}\right) - \left(\frac{\delta^2 W}{\delta\tau\delta\upsilon} + V\frac{\delta^2\Omega}{\delta\tau\delta\upsilon}\right)^2$$
$$+ \left(\frac{\delta^2 W}{\delta\upsilon^2} + V\frac{\delta^2\Omega}{\delta\upsilon^2}\right)\left(\frac{\delta^2 W}{\delta\sigma^2} + V\frac{\delta^2\Omega}{\delta\sigma^2}\right) - \left(\frac{\delta^2 W}{\delta\upsilon\delta\sigma} + V\frac{\delta^2\Omega}{\delta\upsilon\delta\sigma}\right)^2 : \qquad (N^4)$$

and the more extensive formula (L^4) has an inverse also, namely,

$$\frac{1}{V''\upsilon^2}\left(\delta^2 V + V\delta'^2\Omega + 2\delta V\delta'\Omega + \delta^2 W\right) =$$
$$\left(\frac{\delta^2 W}{\delta\sigma^2} + V\frac{\delta^2\Omega}{\delta\sigma^2}\right)\left\{\frac{\delta\Omega}{\delta\tau}\left(\delta z - \delta'\frac{\delta W}{\delta\upsilon} - V\delta'\frac{\delta\Omega}{\delta\upsilon}\right) - \frac{\delta\Omega}{\delta\upsilon}\left(\delta y - \delta'\frac{\delta W}{\delta\tau} - V\delta'\frac{\delta\Omega}{\delta\tau}\right)\right\}^2$$
$$+ \left(\frac{\delta^2 W}{\delta\tau^2} + V\frac{\delta^2\Omega}{\delta\tau^2}\right)\left\{\frac{\delta\Omega}{\delta\upsilon}\left(\delta x - \delta'\frac{\delta W}{\delta\sigma} - V\delta'\frac{\delta\Omega}{\delta\sigma}\right) - \frac{\delta\Omega}{\delta\sigma}\left(\delta z - \delta'\frac{\delta W}{\delta\upsilon} - V\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\right\}^2$$
$$+ \left(\frac{\delta^2 W}{\delta\upsilon^2} + V\frac{\delta^2\Omega}{\delta\upsilon^2}\right)\left\{\frac{\delta\Omega}{\delta\sigma}\left(\delta y - \delta'\frac{\delta W}{\delta\tau} - V\delta'\frac{\delta\Omega}{\delta\tau}\right) - \frac{\delta\Omega}{\delta\tau}\left(\delta x - \delta'\frac{\delta W}{\delta\sigma} - V\delta'\frac{\delta\Omega}{\delta\sigma}\right)\right\}^2$$
$$+ 2\left(\frac{\delta^2 W}{\delta\sigma\delta\tau} + V\frac{\delta^2\Omega}{\delta\sigma\delta\tau}\right)\left\{\begin{array}{l} \frac{\delta\Omega}{\delta\tau}\left(\delta z - \delta'\frac{\delta W}{\delta\upsilon} - V\delta'\frac{\delta\Omega}{\delta\upsilon}\right) \\ -\frac{\delta\Omega}{\delta\upsilon}\left(\delta y - \delta'\frac{\delta W}{\delta\tau} - V\delta'\frac{\delta\Omega}{\delta\tau}\right)\end{array}\right\}\left\{\begin{array}{l} \frac{\delta\Omega}{\delta\upsilon}\left(\delta x - \delta'\frac{\delta W}{\delta\sigma} - V\delta'\frac{\delta\Omega}{\delta\sigma}\right) \\ -\frac{\delta\Omega}{\delta\sigma}\left(\delta z - \delta'\frac{\delta W}{\delta\upsilon} - V\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\end{array}\right\}$$
$$+ 2\left(\frac{\delta^2 W}{\delta\tau\delta\upsilon} + V\frac{\delta^2\Omega}{\delta\tau\delta\upsilon}\right)\left\{\begin{array}{l} \frac{\delta\Omega}{\delta\upsilon}\left(\delta x - \delta'\frac{\delta W}{\delta\sigma} - V\delta'\frac{\delta\Omega}{\delta\sigma}\right) \\ -\frac{\delta\Omega}{\delta\sigma}\left(\delta z - \delta'\frac{\delta W}{\delta\upsilon} - V\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\end{array}\right\}\left\{\begin{array}{l} \frac{\delta\Omega}{\delta\sigma}\left(\delta y - \delta'\frac{\delta W}{\delta\tau} - V\delta'\frac{\delta\Omega}{\delta\tau}\right) \\ -\frac{\delta\Omega}{\delta\tau}\left(\delta x - \delta'\frac{\delta W}{\delta\sigma} - V\delta'\frac{\delta\Omega}{\delta\sigma}\right)\end{array}\right\}$$
$$+ 2\left(\frac{\delta^2 W}{\delta\upsilon\delta\sigma} + V\frac{\delta^2\Omega}{\delta\upsilon\delta\sigma}\right)\left\{\begin{array}{l} \frac{\delta\Omega}{\delta\sigma}\left(\delta y - \delta'\frac{\delta W}{\delta\tau} - V\delta'\frac{\delta\Omega}{\delta\tau}\right) \\ -\frac{\delta\Omega}{\delta\tau}\left(\delta x - \delta'\frac{\delta W}{\delta\sigma} - V\delta'\frac{\delta\Omega}{\delta\sigma}\right)\end{array}\right\}\left\{\begin{array}{l} \frac{\delta\Omega}{\delta\tau}\left(\delta z - \delta'\frac{\delta W}{\delta\upsilon} - V\delta'\frac{\delta\Omega}{\delta\upsilon}\right) \\ -\frac{\delta\Omega}{\delta\upsilon}\left(\delta y - \delta'\frac{\delta W}{\delta\tau} - V\delta'\frac{\delta\Omega}{\delta\tau}\right)\end{array}\right\}, \quad (O^4)$$

δ' retaining its recent meaning, so that, as Ω does not contain x', y', z', we have, in the last formula,

$$\left.\begin{array}{l} \delta'\Omega = \frac{\delta\Omega}{\delta\chi}\delta\chi, \quad \delta'^2\Omega = \frac{\delta^2\Omega}{\delta\chi^2}\delta\chi^2, \\ \delta'\frac{\delta\Omega}{\delta\sigma} = \frac{\delta^2\Omega}{\delta\sigma\delta\chi}\delta\chi, \quad \delta'\frac{\delta\Omega}{\delta\tau} = \frac{\delta^2\Omega}{\delta\tau\delta\chi}\delta\chi, \quad \delta'\frac{\delta\Omega}{\delta\upsilon} = \frac{\delta^2\Omega}{\delta\upsilon\delta\chi}\delta\chi. \end{array}\right\} \quad (P^4)$$

If we do not choose to suppose W homogeneous of the first dimension with respect to σ, τ, υ, and if we put for abridgment

$$\sigma \frac{\delta W}{\delta \sigma} + \tau \frac{\delta W}{\delta \tau} + \upsilon \frac{\delta W}{\delta \upsilon} - W = w_1, \qquad (\text{Q}^4)$$

and denote by δW_1, $\delta^2 W_1$, the expressions already found on this particular supposition, for the variations of W, of the two first orders, so that, for the first order, by (G') (W^2) (W^3),

$$\delta W_1 = x\delta\sigma + y\delta\tau + z\delta\upsilon + \sigma'\delta x' + \tau'\delta y' + \upsilon'\delta z' - \frac{\delta V}{\delta\chi}\,\delta\chi - V\delta\Omega, \qquad (\text{R}^4)$$

and, for the second order, $\delta^2 W_1 =$ the value of $\delta^2 W$ assigned by the formula (L^4); we may generalise these particular values δW_1, $\delta^2 W_1$, by the following relations,

$$\left.\begin{aligned} &\delta W_1 = \delta W - w_1 \delta\Omega, \\ &\delta^2 W_1 = \delta^2 W - w_1 \delta^2\Omega - 2\delta w_1 \delta\Omega \\ &\qquad + \left(\sigma \frac{\delta w_1}{\delta\sigma} + \tau \frac{\delta w_1}{\delta\tau} + \upsilon \frac{\delta w_1}{\delta\upsilon}\right) \delta\Omega^2, \end{aligned}\right\} \qquad (\text{S}^4)$$

in which δW, $\delta^2 W$, are general expressions, independent of the condition of homogeneity $w_1 = 0$, and of every other particular supposition respecting the form of W. It is, however, here understood that the final medium is uniform, and that in forming the variations of the function W, the quantities σ, τ, υ, χ, x', y', z', on which it depends, are treated as if they were seven independent variables.

And if we would deduce expressions, δW_n, $\delta^2 W_n$, for the variations of W, of the two first orders, on the supposition that W is made, before differentiation, homogeneous of any dimension n, with respect to σ, τ, υ, we may put

$$\sigma \frac{\delta W}{\delta \sigma} + \tau \frac{\delta W}{\delta \tau} + \upsilon \frac{\delta W}{\delta \upsilon} - n W = w_n, \qquad (\text{T}^4)$$

and we shall have the following relations

$$\left.\begin{aligned} &\delta W_n = \delta W - w_n \delta\Omega, \\ &\delta^2 W_n = \delta^2 W - w_n \delta^2\Omega - 2\delta w_n \delta\Omega \\ &\qquad + \left(\sigma \frac{\delta w_n}{\delta\sigma} + \tau \frac{\delta w_n}{\delta\tau} + \upsilon \frac{\delta w_n}{\delta\upsilon} + w_n - n w_n\right) \delta\Omega^2, \end{aligned}\right\} \qquad (\text{U}^4)$$

which include the relations (S^4). The general analysis of these homogeneous transformations is interesting, but we cannot dwell upon it here.

Deductions of the Coefficients of T from those of W, and reciprocally.

9. The general principles of investigation, respecting the connexions between the partial differential coefficients of the second order, of the characteristic and auxiliary

functions, having been sufficiently explained by the remarks made at the beginning of the seventh number, and by the details into which we have since entered; we shall confine ourselves, in the remaining research of such connexions, for the new auxiliary function T, to the case of extreme uniform media. And having already treated of the mutual connexions between the coefficients of the two functions V and W, it will be sufficient now to connect the coefficients of either of these two, for example, the coefficients of W, with those of T, of the first and second orders: since the connexions between the coefficients of all three functions will thus be sufficiently known. We shall also suppose that W has been made, before differentiation, homogeneous of the first dimension with respect to σ, τ, υ, that our results may be the more easily combined with the symmetric expressions already deduced from this supposition, expressions which can be generalised in the manner that has been explained: and similarly we shall suppose that T is made homogeneous of the first dimension with respect to σ, τ, υ, and also with respect to $\sigma', \tau', \upsilon'$. Let us then seek to express the partial differential coefficients of the two first orders, of T, by means of those of W, both functions being thus symmetrically prepared.

In this inquiry, we have, as before, the conditions of homogeneity (U^3) (V^3), relative to the function W, and analogous conditions relative to T, namely, for the first order,

$$\left.\begin{aligned}
\sigma\frac{\delta T}{\delta\sigma} + \tau\frac{\delta T}{\delta\tau} + \upsilon\frac{\delta T}{\delta\upsilon} &= T,\\
\sigma'\frac{\delta T}{\delta\sigma'} + \tau'\frac{\delta T}{\delta\tau'} + \upsilon'\frac{\delta T}{\delta\upsilon'} &= T;
\end{aligned}\right\}\quad (\mathrm{V}^4)$$

and, for the second order,

$$\left.\begin{aligned}
0&=\sigma\frac{\delta^2 T}{\delta\sigma^2} + \tau\frac{\delta^2 T}{\delta\sigma\delta\tau} + \upsilon\frac{\delta^2 T}{\delta\sigma\delta\upsilon}; & 0 &= \sigma'\frac{\delta^2 T}{\delta\sigma'^2} + \tau'\frac{\delta^2 T}{\delta\sigma'\delta\tau'} + \upsilon'\frac{\delta^2 T}{\delta\sigma'\delta\upsilon'};\\
0&=\sigma\frac{\delta^2 T}{\delta\sigma\delta\tau} + \tau\frac{\delta^2 T}{\delta\tau^2} + \upsilon\frac{\delta^2 T}{\delta\tau\delta\upsilon}; & 0 &= \sigma'\frac{\delta^2 T}{\delta\sigma'\delta\tau'} + \tau'\frac{\delta^2 T}{\delta\tau'^2} + \upsilon'\frac{\delta^2 T}{\delta\tau'\delta\upsilon'};\\
0&=\sigma\frac{\delta^2 T}{\delta\sigma\delta\upsilon} + \tau\frac{\delta^2 T}{\delta\tau\delta\upsilon} + \upsilon\frac{\delta^2 T}{\delta\upsilon^2}; & 0 &= \sigma'\frac{\delta^2 T}{\delta\sigma'\delta\upsilon'} + \tau'\frac{\delta^2 T}{\delta\tau'\delta\upsilon'} + \upsilon'\frac{\delta^2 T}{\delta\upsilon'^2};\\
\frac{\delta T}{\delta\sigma'}&= \sigma\frac{\delta^2 T}{\delta\sigma\delta\sigma'} + \tau\frac{\delta^2 T}{\delta\tau\delta\sigma'} + \upsilon\frac{\delta^2 T}{\delta\upsilon\delta\sigma'}; & \frac{\delta T}{\delta\sigma} &= \sigma'\frac{\delta^2 T}{\delta\sigma\delta\sigma'} + \tau'\frac{\delta^2 T}{\delta\sigma\delta\tau'} + \upsilon'\frac{\delta^2 T}{\delta\sigma\delta\upsilon'};\\
\frac{\delta T}{\delta\tau'}&= \sigma\frac{\delta^2 T}{\delta\sigma\delta\tau'} + \tau\frac{\delta^2 T}{\delta\tau\delta\tau'} + \upsilon\frac{\delta^2 T}{\delta\upsilon\delta\tau'}; & \frac{\delta T}{\delta\tau} &= \sigma'\frac{\delta^2 T}{\delta\tau\delta\sigma'} + \tau'\frac{\delta^2 T}{\delta\tau\delta\tau'} + \upsilon'\frac{\delta^2 T}{\delta\tau\delta\upsilon'};\\
\frac{\delta T}{\delta\upsilon'}&= \sigma\frac{\delta^2 T}{\delta\sigma\delta\upsilon'} + \tau\frac{\delta^2 T}{\delta\tau\delta\upsilon'} + \upsilon\frac{\delta^2 T}{\delta\upsilon\delta\upsilon'}; & \frac{\delta T}{\delta\upsilon} &= \sigma'\frac{\delta^2 T}{\delta\upsilon\delta\sigma'} + \tau'\frac{\delta^2 T}{\delta\upsilon\delta\tau'} + \upsilon'\frac{\delta^2 T}{\delta\upsilon\delta\upsilon'};\\
\frac{\delta T}{\delta\chi}&= \sigma\frac{\delta^2 T}{\delta\sigma\delta\chi} + \tau\frac{\delta^2 T}{\delta\tau\delta\chi} + \upsilon\frac{\delta^2 T}{\delta\upsilon\delta\chi}; & \frac{\delta T}{\delta\chi} &= \sigma'\frac{\delta^2 T}{\delta\sigma'\delta\chi} + \tau'\frac{\delta^2 T}{\delta\tau'\delta\chi} + \upsilon'\frac{\delta^2 T}{\delta\upsilon'\delta\chi};
\end{aligned}\right\}\quad (\mathrm{W}^4)$$

together with the conditions relative to Ω, Ω', namely (B^4), (C^4), and the following,

$$\left.\begin{aligned}
\sigma'\frac{\delta\Omega'}{\delta\sigma'}+\tau'\frac{\delta\Omega'}{\delta\tau'}+\upsilon'\frac{\delta\Omega'}{\delta\upsilon'}&=\Omega'+1=1,\\
\sigma'\frac{\delta^2\Omega'}{\delta\sigma'^2}+\tau'\frac{\delta^2\Omega'}{\delta\sigma'\delta\tau'}+\upsilon'\frac{\delta^2\Omega'}{\delta\sigma'\delta\upsilon'}&=0,\\
\sigma'\frac{\delta^2\Omega'}{\delta\sigma'\delta\tau'}+\tau'\frac{\delta^2\Omega'}{\delta\tau'^2}+\upsilon'\frac{\delta^2\Omega'}{\delta\tau'\delta\upsilon'}&=0,\\
\sigma'\frac{\delta^2\Omega'}{\delta\sigma'\delta\upsilon'}+\tau'\frac{\delta^2\Omega'}{\delta\tau'\delta\upsilon'}+\upsilon'\frac{\delta^2\Omega'}{\delta\upsilon'^2}&=0,\\
\sigma'\frac{\delta^2\Omega'}{\delta\sigma'\delta\chi}+\tau'\frac{\delta^2\Omega'}{\delta\tau'\delta\chi}+\upsilon'\frac{\delta^2\Omega'}{\delta\upsilon'\delta\chi}&=\frac{\delta\Omega'}{\delta\chi};
\end{aligned}\right\}\qquad(\mathrm{X}^4)$$

we have also the general equations

$$\frac{\delta W}{\delta x'}=\sigma',\ \frac{\delta W}{\delta y'}=\tau',\ \frac{\delta W}{\delta z'}=\upsilon',\qquad(\mathrm{G}')$$

by combining which with the foregoing conditions and with the partial differential equation (A^2), we find the following, analogous to (A^4),

$$\left.\begin{aligned}
0&=\frac{\delta\Omega'}{\delta\sigma'}\frac{\delta^2W}{\delta x'^2}+\frac{\delta\Omega'}{\delta\tau'}\frac{\delta^2W}{\delta x'\delta y'}+\frac{\delta\Omega'}{\delta\upsilon'}\frac{\delta^2W}{\delta x'\delta z'},\\
0&=\frac{\delta\Omega'}{\delta\sigma'}\frac{\delta^2W}{\delta x'\delta y'}+\frac{\delta\Omega'}{\delta\tau'}\frac{\delta^2W}{\delta y'^2}+\frac{\delta\Omega'}{\delta\upsilon'}\frac{\delta^2W}{\delta y'\delta z'},\\
0&=\frac{\delta\Omega'}{\delta\sigma'}\frac{\delta^2W}{\delta x'\delta z'}+\frac{\delta\Omega'}{\delta\tau'}\frac{\delta^2W}{\delta y'\delta z'}+\frac{\delta\Omega'}{\delta\upsilon'}\frac{\delta^2W}{\delta z'^2},\\
\frac{\delta\Omega}{\delta\sigma}&=\frac{\delta\Omega'}{\delta\sigma'}\frac{\delta^2W}{\delta\sigma\delta x'}+\frac{\delta\Omega'}{\delta\tau'}\frac{\delta^2W}{\delta\sigma\delta y'}+\frac{\delta\Omega'}{\delta\upsilon'}\frac{\delta^2W}{\delta\sigma\delta z'},\\
\frac{\delta\Omega}{\delta\tau}&=\frac{\delta\Omega'}{\delta\sigma'}\frac{\delta^2W}{\delta\tau\delta x'}+\frac{\delta\Omega'}{\delta\tau'}\frac{\delta^2W}{\delta\tau\delta y'}+\frac{\delta\Omega'}{\delta\upsilon'}\frac{\delta^2W}{\delta\tau\delta z'},\\
\frac{\delta\Omega}{\delta\upsilon}&=\frac{\delta\Omega'}{\delta\sigma'}\frac{\delta^2W}{\delta\upsilon\delta x'}+\frac{\delta\Omega'}{\delta\tau'}\frac{\delta^2W}{\delta\upsilon\delta y'}+\frac{\delta\Omega'}{\delta\upsilon'}\frac{\delta^2W}{\delta\upsilon\delta z'},\\
\frac{\delta\Omega}{\delta\chi}-\frac{\delta\Omega'}{\delta\chi}&=\frac{\delta\Omega'}{\delta\sigma'}\frac{\delta^2W}{\delta\chi\delta x'}+\frac{\delta\Omega'}{\delta\tau'}\frac{\delta^2W}{\delta\chi\delta y'}+\frac{\delta\Omega'}{\delta\upsilon'}\frac{\delta^2W}{\delta\chi\delta z'};
\end{aligned}\right\}\qquad(\mathrm{Y}^4)$$

and if we combine the conditions of homogeneity of the two functions W, T, with the fundamental relation (E') between these two functions, and with the properties of Ω, Ω', and attend to (G'), we find the following expressions for the partial differential coefficients of T, of the first order,

$$\left.\begin{aligned}
\frac{\delta T}{\delta\sigma} &= \frac{\delta W}{\delta\sigma} + (T - W)\frac{\delta\Omega}{\delta\sigma}; \quad \frac{\delta T}{\delta\sigma'} = -x' + W\frac{\delta\Omega'}{\delta\sigma'};\\
\frac{\delta T}{\delta\tau} &= \frac{\delta W}{\delta\tau} + (T - W)\frac{\delta\Omega}{\delta\tau}; \quad \frac{\delta T}{\delta\tau'} = -y' + W\frac{\delta\Omega'}{\delta\tau'};\\
\frac{\delta T}{\delta\upsilon} &= \frac{\delta W}{\delta\upsilon} + (T - W)\frac{\delta\Omega}{\delta\upsilon}; \quad \frac{\delta T}{\delta\upsilon'} = -z' + W\frac{\delta\Omega'}{\delta\upsilon'};\\
\frac{\delta T}{\delta\chi} &= \frac{\delta W}{\delta\chi} + (T - W)\frac{\delta\Omega}{\delta\chi} + W\frac{\delta\Omega'}{\delta\chi}.
\end{aligned}\right\} \quad (Z^4)$$

Differentiating the expressions (Z^4), and eliminating $\delta x'$, $\delta y'$, $\delta z'$, by means of the differentials of the general equations (G), we obtain, by (Y^4), the following system, analogous to the system (Y^3);

$$\left.\begin{aligned}
\lambda_1\delta\Omega + \lambda'_1\delta\Omega' &= \frac{\delta^2 W}{\delta x'^2}\left(\delta\frac{\delta T}{\delta\sigma'} - W\delta\frac{\delta\Omega'}{\delta\sigma'}\right) + \frac{\delta^2 W}{\delta x'\delta y'}\left(\delta\frac{\delta T}{\delta\tau'} - W\delta\frac{\delta\Omega'}{\delta\tau'}\right) + \frac{\delta^2 W}{\delta x'\delta z'}\left(\delta\frac{\delta T}{\delta\upsilon'} - W\delta\frac{\delta\Omega'}{\delta\upsilon'}\right)\\
&\quad - \delta_{,}\frac{\delta W}{\delta x'} + \delta\sigma';\\
\lambda_2\delta\Omega + \lambda'_2\delta\Omega' &= \frac{\delta^2 W}{\delta x'\delta y'}\left(\delta\frac{\delta T}{\delta\sigma'} - W\delta\frac{\delta\Omega'}{\delta\sigma'}\right) + \frac{\delta^2 W}{\delta y'^2}\left(\delta\frac{\delta T}{\delta\tau'} - W\delta\frac{\delta\Omega'}{\delta\tau'}\right) + \frac{\delta^2 W}{\delta y'\delta z'}\left(\delta\frac{\delta T}{\delta\upsilon'} - W\delta\frac{\delta\Omega'}{\delta\upsilon'}\right)\\
&\quad - \delta_{,}\frac{\delta W}{\delta y'} + \delta\tau';\\
\lambda_3\delta\Omega + \lambda'_3\delta\Omega' &= \frac{\delta^2 W}{\delta x'\delta z'}\left(\delta\frac{\delta T}{\delta\sigma'} - W\delta\frac{\delta\Omega'}{\delta\sigma'}\right) + \frac{\delta^2 W}{\delta y'\delta z'}\left(\delta\frac{\delta T}{\delta\tau'} - W\delta\frac{\delta\Omega'}{\delta\tau'}\right) + \frac{\delta^2 W}{\delta z'^2}\left(\delta\frac{\delta T}{\delta\upsilon'} - W\delta\frac{\delta\Omega'}{\delta\upsilon'}\right)\\
&\quad - \delta_{,}\frac{\delta W}{\delta z'} + \delta\upsilon';\\
\lambda_4\delta\Omega + \lambda'_4\delta\Omega' &= \frac{\delta^2 W}{\delta\sigma\delta x'}\left(\delta\frac{\delta T}{\delta\sigma'} - W\delta\frac{\delta\Omega'}{\delta\sigma'}\right) + \frac{\delta^2 W}{\delta\sigma\delta y'}\left(\delta\frac{\delta T}{\delta\tau'} - W\delta\frac{\delta\Omega'}{\delta\tau'}\right) + \frac{\delta^2 W}{\delta\sigma\delta z'}\left(\delta\frac{\delta T}{\delta\upsilon'} - W\delta\frac{\delta\Omega'}{\delta\upsilon'}\right)\\
&\quad - \delta_{,}\frac{\delta W}{\delta\sigma} + \delta\left(\frac{\delta T}{\delta\sigma} - T\frac{\delta\Omega}{\delta\sigma}\right) + W\delta\frac{\delta\Omega}{\delta\sigma};\\
\lambda_5\delta\Omega + \lambda'_5\delta\Omega' &= \frac{\delta^2 W}{\delta\tau\delta x'}\left(\delta\frac{\delta T}{\delta\sigma'} - W\delta\frac{\delta\Omega'}{\delta\sigma'}\right) + \frac{\delta^2 W}{\delta\tau\delta y'}\left(\delta\frac{\delta T}{\delta\tau'} - W\delta\frac{\delta\Omega'}{\delta\tau'}\right) + \frac{\delta^2 W}{\delta\tau\delta z'}\left(\delta\frac{\delta T}{\delta\upsilon'} - W\delta\frac{\delta\Omega'}{\delta\upsilon'}\right)\\
&\quad - \delta_{,}\frac{\delta W}{\delta\tau} + \delta\left(\frac{\delta T}{\delta\tau} - T\frac{\delta\Omega}{\delta\tau}\right) + W\delta\frac{\delta\Omega}{\delta\tau};\\
\lambda_6\delta\Omega + \lambda'_6\delta\Omega' &= \frac{\delta^2 W}{\delta\upsilon\delta x'}\left(\delta\frac{\delta T}{\delta\sigma'} - W\delta\frac{\delta\Omega'}{\delta\sigma'}\right) + \frac{\delta^2 W}{\delta\upsilon\delta y'}\left(\delta\frac{\delta T}{\delta\tau'} - W\delta\frac{\delta\Omega'}{\delta\tau'}\right) + \frac{\delta^2 W}{\delta\upsilon\delta z'}\left(\delta\frac{\delta T}{\delta\upsilon'} - W\delta\frac{\delta\Omega'}{\delta\upsilon'}\right)\\
&\quad - \delta_{,}\frac{\delta W}{\delta\upsilon} + \delta\left(\frac{\delta T}{\delta\upsilon} - T\frac{\delta\Omega}{\delta\upsilon}\right) + W\delta\frac{\delta\Omega}{\delta\upsilon};\\
\lambda_7\delta\Omega + \lambda'_7\delta\Omega' &= \frac{\delta^2 W}{\delta\chi\delta x'}\left(\delta\frac{\delta T}{\delta\sigma'} - W\delta\frac{\delta\Omega'}{\delta\sigma'}\right) + \frac{\delta^2 W}{\delta\chi\delta y'}\left(\delta\frac{\delta T}{\delta\tau'} - W\delta\frac{\delta\Omega'}{\delta\tau'}\right) + \frac{\delta^2 W}{\delta\chi\delta z'}\left(\delta\frac{\delta T}{\delta\upsilon'} - W\delta\frac{\delta\Omega'}{\delta\upsilon'}\right)\\
&\quad - \delta_{,}\frac{\delta W}{\delta\chi} + \delta\left(\frac{\delta T}{\delta\chi} - T\frac{\delta\Omega}{\delta\chi}\right) + W\delta\left(\frac{\delta\Omega}{\delta\chi} - \frac{\delta\Omega'}{\delta\chi}\right):
\end{aligned}\right\} (A)$$

in which $\delta_,$ refers only to the four variations $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, $\delta\chi$, and in which we may treat the seven variations, $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, $\delta\chi$, $\delta\sigma'$, $\delta\tau'$, $\delta\upsilon'$, as independent, if we assign to the fourteen multipliers $\lambda_1, \ldots \lambda'_7$, the following values;

$$\left.\begin{array}{l}
\lambda_1 = \frac{\delta T}{\delta\sigma'}\frac{\delta^2 W}{\delta x'^2} + \frac{\delta T}{\delta\tau'}\frac{\delta^2 W}{\delta x'\delta y'} + \frac{\delta T}{\delta\upsilon'}\frac{\delta^2 W}{\delta x'\delta z'} - \frac{\delta W}{\delta x'}; \\
\lambda_2 = \frac{\delta T}{\delta\sigma'}\frac{\delta^2 W}{\delta x'\delta y'} + \frac{\delta T}{\delta\tau'}\frac{\delta^2 W}{\delta y'^2} + \frac{\delta T}{\delta\upsilon'}\frac{\delta^2 W}{\delta y'\delta z'} - \frac{\delta W}{\delta y'}; \\
\lambda_3 = \frac{\delta T}{\delta\sigma'}\frac{\delta^2 W}{\delta x'\delta z'} + \frac{\delta T}{\delta\tau'}\frac{\delta^2 W}{\delta y'\delta z'} + \frac{\delta T}{\delta\upsilon'}\frac{\delta^2 W}{\delta z'^2} - \frac{\delta W}{\delta z'}; \\
\lambda_4 = \frac{\delta T}{\delta\sigma'}\frac{\delta^2 W}{\delta\sigma\delta x'} + \frac{\delta T}{\delta\tau'}\frac{\delta^2 W}{\delta\sigma\delta y'} + \frac{\delta T}{\delta\upsilon'}\frac{\delta^2 W}{\delta\sigma\delta z'} - T\frac{\delta\Omega}{\delta\sigma}; \\
\lambda_5 = \frac{\delta T}{\delta\sigma'}\frac{\delta^2 W}{\delta\tau\delta x'} + \frac{\delta T}{\delta\tau'}\frac{\delta^2 W}{\delta\tau\delta y'} + \frac{\delta T}{\delta\upsilon'}\frac{\delta^2 W}{\delta\tau\delta z'} - T\frac{\delta\Omega}{\delta\tau}; \\
\lambda_6 = \frac{\delta T}{\delta\sigma'}\frac{\delta^2 W}{\delta\upsilon\delta x'} + \frac{\delta T}{\delta\tau'}\frac{\delta^2 W}{\delta\upsilon\delta y'} + \frac{\delta T}{\delta\upsilon'}\frac{\delta^2 W}{\delta\upsilon\delta z'} - T\frac{\delta\Omega}{\delta\upsilon}; \\
\lambda_7 = \frac{\delta T}{\delta\sigma'}\frac{\delta^2 W}{\delta\chi\delta x'} + \frac{\delta T}{\delta\tau'}\frac{\delta^2 W}{\delta\chi\delta y'} + \frac{\delta T}{\delta\upsilon'}\frac{\delta^2 W}{\delta\chi\delta z'} - T\frac{\delta\Omega}{\delta\chi} + W\frac{\delta\Omega'}{\delta\chi}; \\
\lambda'_1 = \sigma'; \quad \lambda'_2 = \tau'; \quad \lambda'_3 = \upsilon'; \\
\lambda'_4 = \frac{\delta W}{\delta\sigma} - W\frac{\delta\Omega}{\delta\sigma}; \; \lambda'_5 = \frac{\delta W}{\delta\tau} - W\frac{\delta\Omega}{\delta\tau}; \; \lambda'_6 = \frac{\delta W}{\delta\upsilon} - W\frac{\delta\Omega}{\delta\upsilon}; \; \lambda'_7 = \frac{\delta W}{\delta\chi} - W\frac{\delta\Omega}{\delta\chi}:
\end{array}\right\} (B^5)$$

the values of $\lambda_1 \ldots \lambda_7$ may also be thus expressed,

$$\left.\begin{array}{ll}
\lambda_1 = -\frac{\delta w'}{\delta x'}, & \lambda_4 = -\frac{\delta w'}{\delta\sigma} + (W-T)\frac{\delta\Omega}{\delta\sigma}, \\
\lambda_2 = -\frac{\delta w'}{\delta y'}, & \lambda_5 = -\frac{\delta w'}{\delta\tau} + (W-T)\frac{\delta\Omega}{\delta\tau}, \\
\lambda_3 = -\frac{\delta w'}{\delta z'}, & \lambda_6 = -\frac{\delta w'}{\delta\upsilon} + (W-T)\frac{\delta\Omega}{\delta\upsilon}, \\
 & \lambda_7 = -\frac{\delta w'}{\delta\chi} + (W-T)\frac{\delta\Omega}{\delta\chi},
\end{array}\right\} \quad (C^5)$$

if we put for abridgment

$$w' = x'\frac{\delta W}{\delta x'} + y'\frac{\delta W}{\delta y'} + z'\frac{\delta W}{\delta z'}, \qquad (D^5)$$

and consider w', like W, as a function of σ, τ, υ, χ, x', y', z', which, relatively to σ, τ, υ, is homogeneous of the first dimension. The four last equations (A^5) give, by addition, after multiplying them respectively, by $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, $\delta\chi$,

$$\left.\begin{aligned}
\delta^2 T = (T - W)\,\delta^2\Omega + W\,\delta^2\Omega' + (W - T)\,\delta\Omega^2 \\
+ (\delta T - \delta_{,} w')\,\delta\Omega + (\delta_{,} W - W\delta\Omega)\,\delta\Omega' + \delta_{,}^{2} W \\
- \left(\delta_{,}\frac{\delta W}{\delta x'} - \delta\sigma'\right)\left(\delta\frac{\delta T}{\delta\sigma'} - W\delta\frac{\delta\Omega'}{\delta\sigma'}\right) \\
- \left(\delta_{,}\frac{\delta W}{\delta y'} - \delta\tau'\right)\left(\delta\frac{\delta T}{\delta\tau'} - W\delta\frac{\delta\Omega'}{\delta\tau}\right) \\
- \left(\delta_{,}\frac{\delta W}{\delta z'} - \delta\upsilon'\right)\left(\delta\frac{\delta T}{\delta\upsilon'} - W\delta\frac{\delta\Omega'}{\delta\upsilon'}\right),
\end{aligned}\right\}\qquad (\mathrm{E}^5)$$

$\delta_{,}$ still referring only to the variations of σ, τ, υ, χ; and the three first equations (A^5) give, by elimination,

$$\left.\begin{aligned}
\delta\frac{\delta T}{\delta\sigma'} - W\delta\frac{\delta\Omega'}{\delta\sigma'} = \frac{\delta\Omega'}{\delta\sigma'}(\delta_{,}W - W\delta\Omega) + \frac{\delta T}{\delta\sigma'}\delta\Omega \\
+ \frac{1}{\upsilon'{}_{2}W'''}\left(\upsilon'\frac{\delta^2 W}{\delta x'\delta y'} - \tau'\frac{\delta^2 W}{\delta x'\delta z'}\right)\left\{\tau'\left(\delta_{,}\frac{\delta W}{\delta z'} - \delta\upsilon'\right) - \upsilon'\left(\delta_{,}\frac{\delta W}{\delta y'} - \delta\tau'\right)\right\} \\
+ \frac{1}{\upsilon'^2 W'''}\left(\upsilon'\frac{\delta^2 W}{\delta y'^2} - \tau'\frac{\delta^2 W}{\delta y'\delta z'}\right)\left\{\upsilon'\left(\delta_{,}\frac{\delta W}{\delta x'} - \delta\sigma'\right) - \sigma'\left(\delta_{,}\frac{\delta W}{\delta z'} - \delta\upsilon'\right)\right\} \\
+ \frac{1}{\upsilon'^2 W'''}\left(\upsilon'\frac{\delta^2 W}{\delta y'\delta z'} - \tau'\frac{\delta^2 W}{\delta z'^2}\right)\left\{\sigma'\left(\delta_{,}\frac{\delta W}{\delta y'} - \delta\tau'\right) - \tau'\left(\delta_{,}\frac{\delta W}{\delta x'} - \delta\sigma'\right)\right\}; \\
\delta\frac{\delta T}{\delta\tau'} - W\delta\frac{\delta\Omega'}{\delta\tau'} = \frac{\delta\Omega'}{\delta\tau'}(\delta_{,}W - W\delta\Omega) + \frac{\delta T}{\delta\tau'}\delta\Omega \\
+ \frac{1}{\upsilon'^2 W'''}\left(\sigma'\frac{\delta^2 W}{\delta x'\delta z'} - \upsilon'\frac{\delta^2 W}{\delta x'^2}\right)\left\{\tau'\left(\delta_{,}\frac{\delta W}{\delta z'} - \delta\upsilon'\right) - \upsilon'\left(\delta_{,}\frac{\delta W}{\delta y'} - \delta\tau'\right)\right\} \\
+ \frac{1}{\upsilon'^2 W'''}\left(\sigma'\frac{\delta^2 W}{\delta y'\delta z'} - \upsilon'\frac{\delta^2 W}{\delta x'\delta y'}\right)\left\{\upsilon'\left(\delta_{,}\frac{\delta W}{\delta x'} - \delta\sigma'\right) - \sigma\left(\delta_{,}\frac{\delta W}{\delta z'} - \delta\upsilon'\right)\right\} \\
+ \frac{1}{\upsilon'^2 W'''}\left(\sigma'\frac{\delta^2 W}{\delta z'^2} - \upsilon'\frac{\delta^2 W}{\delta x'\delta z'}\right)\left\{\sigma'\left(\delta_{,}\frac{\delta W}{\delta y'} - \delta\tau'\right) - \tau'\left(\delta_{,}\frac{\delta W}{\delta x'} - \delta\sigma'\right)\right\}; \\
\delta\frac{\delta T}{\delta\upsilon'} - W\delta\frac{\delta\Omega'}{\delta\upsilon'} = \frac{\delta\Omega'}{\delta\upsilon'}(\delta_{,}W - W\delta\Omega) + \frac{\delta T}{\delta\upsilon'}\delta\Omega \\
+ \frac{1}{\upsilon'^2 W'''}\left(\tau'\frac{\delta^2 W}{\delta x'^2} - \sigma'\frac{\delta^2 W}{\delta x'\delta y'}\right)\left\{\tau'\left(\delta_{,}\frac{\delta W}{\delta z'} - \delta\upsilon'\right) - \upsilon'\left(\delta_{,}\frac{\delta W}{\delta y'} - \delta\tau\right)\right\} \\
+ \frac{1}{\upsilon'^2 W'''}\left(\tau'\frac{\delta^2 W}{\delta x'\delta y'} - \sigma'\frac{\delta^2 W}{\delta y'^2}\right)\left\{\upsilon'\left(\delta_{,}\frac{\delta W}{\delta x'} - \delta\sigma'\right) - \sigma'\left(\delta_{,}\frac{\delta W}{\delta z'} - \delta\upsilon'\right)\right\} \\
+ \frac{1}{\upsilon'^2 W'''}\left(\tau'\frac{\delta^2 W}{\delta x'\delta z'} - \sigma'\frac{\delta^2 W}{\delta y'\delta z'}\right)\left\{\sigma'\left(\delta_{,}\frac{\delta W}{\delta y'} - \delta\tau'\right) - \tau'\left(\delta_{,}\frac{\delta W}{\delta x'} - \delta\sigma'\right)\right\};
\end{aligned}\right\}(\mathrm{F}^5)$$

in which

$$W''' = \frac{\delta^2 W}{\delta x'^2}\frac{\delta^2 W}{\delta y'^2} - \left(\frac{\delta^2 W}{\delta x'\delta y'}\right)^2 + \frac{\delta^2 W}{\delta y'^2}\frac{\delta^2 W}{\delta z'^2} - \left(\frac{\delta^2 W}{\delta y'\delta z'}\right)^2 + \frac{\delta^2 W}{\delta z'^2}\frac{\delta^2 W}{\delta x'^2} - \left(\frac{\delta^2 W}{\delta z'\delta x'}\right)^2, \qquad (\mathrm{G}^5)$$

and

$$\frac{1}{v'^2} = \left(\frac{\delta\Omega'}{\delta\sigma'}\right)^2 + \left(\frac{\delta\Omega'}{\delta\tau'}\right)^2 + \left(\frac{\delta\Omega'}{\delta\upsilon'}\right)^2, \qquad (\mathrm{H}^5)$$

v' having the same meaning as in the second number. In effecting the last elimination, we have attended to the relations (Y^4), which give

$$\left.\begin{aligned}
&\frac{\delta^2 W}{\delta y'^2}\,\frac{\delta^2 W}{\delta z'^2} - \left(\frac{\delta^2 W}{\delta y'\delta z'}\right)^2 = W'''v'^2\left(\frac{\delta\Omega'}{\delta\sigma'}\right)^2;\\
&\frac{\delta^2 W}{\delta z'^2}\,\frac{\delta^2 W}{\delta x'^2} - \left(\frac{\delta^2 W}{\delta z'\delta x'}\right)^2 = W'''v'^2\left(\frac{\delta\Omega'}{\delta\tau'}\right)^2;\\
&\frac{\delta^2 W}{\delta x'^2}\,\frac{\delta^2 W}{\delta y'^2} - \left(\frac{\delta^2 W}{\delta x'\delta y'}\right)^2 = W'''v'^2\left(\frac{\delta\Omega'}{\delta\upsilon'}\right)^2;\\
&\frac{\delta^2 W}{\delta x'\delta y'}\,\frac{\delta^2 W}{\delta z'\delta x'} - \frac{\delta^2 W}{\delta x'^2}\,\frac{\delta^2 W}{\delta y'\delta z'} = W'''v'^2\,\frac{\delta\Omega'}{\delta\tau'}\,\frac{\delta\Omega'}{\delta\upsilon'};\\
&\frac{\delta^2 W}{\delta y'\delta z'}\,\frac{\delta^2 W}{\delta x'\delta y'} - \frac{\delta^2 W}{\delta y'^2}\,\frac{\delta^2 W}{\delta z'\delta x'} = W'''v'^2\,\frac{\delta\Omega'}{\delta\upsilon'}\,\frac{\delta\Omega'}{\delta\sigma'};\\
&\frac{\delta^2 W}{\delta z'\delta x'}\,\frac{\delta^2 W}{\delta y'\delta z'} - \frac{\delta^2 W}{\delta z'^2}\,\frac{\delta^2 W}{\delta x'\delta y'} = W'''v'^2\,\frac{\delta\Omega'}{\delta\sigma'}\,\frac{\delta\Omega'}{\delta\tau'}.
\end{aligned}\right\} \qquad (\mathrm{I}^5)$$

And combining (E^5) (F^5), we obtain the following formula for $\delta^2 T$, analogous to the formula (L^4), which completes the solution of our present problem, because it is equivalent to twenty-eight expressions for the twenty-eight partial differential coefficients of T, of the second order, deduced from the coefficients of W;

$$\begin{aligned}
0 = v'^2 W''' &\left\{\delta^2 T + (W-T)\delta^2\Omega - W\delta^2\Omega' - 2\delta_{,}W.\delta\Omega' - \delta_{,}^2 W + 2(x'\delta\sigma' + y'\delta\tau' + z'\delta\upsilon')\delta\Omega\right\}\\
&+\frac{\delta^2 W}{\delta x'^2}\left\{\tau'\left(\delta\upsilon' - \delta_{,}\frac{\delta W}{\delta z'}\right) - \upsilon'\left(\delta\tau' - \delta_{,}\frac{\delta W}{\delta y'}\right)\right\}^2\\
&+\frac{\delta^2 W}{\delta y'^2}\left\{\upsilon'\left(\delta\sigma' - \delta_{,}\frac{\delta W}{\delta x'}\right) - \sigma'\left(\delta\upsilon' - \delta_{,}\frac{\delta W}{\delta z'}\right)\right\}^2\\
&+\frac{\delta^2 W}{\delta z'^2}\left\{\sigma'\left(\delta\tau' - \delta_{,}\frac{\delta W}{\delta y'}\right) - \tau'\left(\delta\sigma' - \delta_{,}\frac{\delta W}{\delta x'}\right)\right\}^2\\
&+2\,\frac{\delta^2 W}{\delta x'\delta y'}\left\{\tau'\left(\delta\upsilon' - \delta_{,}\frac{\delta W}{\delta z'}\right) - \upsilon'\left(\delta\tau' - \delta_{,}\frac{\delta W}{\delta y'}\right)\right\}\left\{\upsilon'\left(\delta\sigma' - \delta_{,}\frac{\delta W}{\delta x'}\right) - \sigma'\left(\delta\upsilon' - \delta_{,}\frac{\delta W}{\delta z'}\right)\right\}\\
&+2\,\frac{\delta^2 W}{\delta y'\delta z'}\left\{\upsilon'\left(\delta\sigma' - \delta_{,}\frac{\delta W}{\delta x'}\right) - \sigma'\left(\delta\upsilon' - \delta'\frac{\delta W}{\delta z'}\right)\right\}\left\{\sigma'\left(\delta\tau' - \delta_{,}\frac{\delta W}{\delta y'}\right) - \tau'\left(\delta\sigma' - \delta_{,}\frac{\delta W}{\delta x'}\right)\right\}\\
&+2\,\frac{\delta^2 W}{\delta z'\delta x'}\left\{\sigma'\left(\delta\tau' - \delta_{,}\frac{\delta W}{\delta y'}\right) - \tau'\left(\delta\sigma' - \delta_{,}\frac{\delta W}{\delta x'}\right)\right\}\left\{\tau'\left(\delta\upsilon' - \delta_{,}\frac{\delta W}{\delta z'}\right) - \upsilon'\left(\delta\tau' - \delta_{,}\frac{\delta W}{\delta y'}\right)\right\}. \quad (\mathrm{K}^5)
\end{aligned}$$

And if we denote by $\delta^2 T_{1,1}$ the value of the second differential $\delta^2 T$ assigned by the formula (K^5), and determined on the supposition that T has been made, before differentiation, homogeneous of the first dimension with respect to σ, τ, υ, and also with respect to σ', τ', υ', and denote by $\delta T_{1,1}$ the corresponding value of δT, determined by the coefficients (Z^4), we may generalise these values by means of the following relations, analogous to (S^4);

$$\left.\begin{array}{l} \delta T_{1,1} = \delta T - \delta\Omega . \nabla_1 T - \delta\Omega' . \nabla_1' T ; \\ \delta^2 T_{1,1} = \delta^2 T - \delta^2\Omega . \nabla_1 T - \delta^2\Omega' . \nabla_1' T \\ \qquad - 2\delta\Omega . \delta\nabla_1 T - 2\delta\Omega' . \delta\nabla_1' T \\ \quad + \delta\Omega^2 . \nabla_1(\nabla_1 + 1) T + 2\delta\Omega . \delta\Omega' . \nabla_1\nabla_1' T + \delta\Omega'^2 . \nabla_1'(\nabla_1' + 1) T : \end{array}\right\} \quad (L^5)$$

∇_1, ∇_1', being here characteristics of operation, defined by the following symbolic equations,

$$\left.\begin{array}{l} \nabla_1 = \sigma \frac{\delta}{\delta\sigma} + \tau \frac{\delta}{\delta\tau} + \upsilon \frac{\delta}{\delta\upsilon} - 1 ; \\ \nabla_1' = \sigma' \frac{\delta}{\delta\sigma'} + \tau' \frac{\delta}{\delta\tau'} + \upsilon' \frac{\delta}{\delta\upsilon'} - 1 . \end{array}\right\} \quad (M^5)$$

More generally, if we denote by $T_{n,n'}$ the function deduced from T by the homogeneous preparation mentioned in the sixth number, which coincides with T when the variables $\sigma\ \tau\ \upsilon\ \sigma'\ \tau'\ \upsilon'\ \chi$ are connected by the relations $\Omega = 0$, $\Omega' = 0$, and which is, for arbitrary values of those variables, homogeneous of the dimension n with respect to σ, τ, υ, and of the dimension n' with respect to σ', τ', υ', we have the following expressions, analogous to (U^4),

$$\left.\begin{array}{l} \delta T_{n,n'} = \delta T - \delta\Omega . \nabla_n T - \delta\Omega' . \nabla'_{n'} T ; \\ \delta^2 T_{n,n'} = \delta^2 T - \delta^2\Omega . \nabla_n T - \delta^2\Omega' . \nabla'_{n'} T - 2\delta\Omega . \delta\nabla_n T - 2\delta\Omega' . \delta\nabla'_{n'} T \\ \qquad + \delta\Omega^2 . \nabla_n (\nabla_n + 1) T + 2\delta\Omega . \delta\Omega' . \nabla_n \nabla'_{n'} T + \delta\Omega'^2 . \nabla'_{n'}(\nabla'_{n'} + 1) T : \end{array}\right\} \quad (N^5)$$

defining the characteristics ∇_n, $\nabla'_{n'}$, as follows,

$$\nabla_n = \sigma \frac{\delta}{\delta\sigma} + \tau \frac{\delta}{\delta\tau} + \upsilon \frac{\delta}{\delta\upsilon} - n ; \quad \nabla'_{n'} = \sigma' \frac{\delta}{\delta\sigma'} + \tau' \frac{\delta}{\delta\tau'} + \upsilon' \frac{\delta}{\delta\upsilon'} - n' . \qquad (O^5)$$

Reciprocally to deduce the coefficients of W, of the second order, from those of T, on the same suppositions of homogeneity, and with the same dimensions $n = 1$, $n' = 1$, we are to eliminate $\delta\sigma'$, $\delta\tau'$, $\delta\upsilon'$, between the differentials of (G') and (Z^4), and we find the following system,

$$\left.\begin{aligned}
\lambda''_1\delta\Omega &= \left(\frac{\delta^2 T}{\delta\sigma'^2} - W\frac{\delta^2\Omega'}{\delta\sigma'^2}\right)\delta\frac{\delta W}{\delta x'} + \left(\frac{\delta^2 T}{\delta\sigma'\delta\tau'} - W\frac{\delta^2\Omega'}{\delta\sigma'\delta\tau'}\right)\delta\frac{\delta W}{\delta y'} + \left(\frac{\delta^2 T}{\delta\sigma'\delta\upsilon'} - W\frac{\delta^2\Omega'}{\delta\sigma'\delta\upsilon'}\right)\delta\frac{\delta W}{\delta z'} \\
&\qquad + \delta_{,}\frac{\delta T}{\delta\sigma'} - W\delta_{,}\frac{\delta\Omega'}{\delta\sigma'} + \delta x' - \frac{\delta\Omega'}{\delta\sigma'}\delta W; \\
\lambda''_2\delta\Omega &= \left(\frac{\delta^2 T}{\delta\sigma'\delta\tau'} - W\frac{\delta^2\Omega'}{\delta\sigma'\delta\tau'}\right)\delta\frac{\delta W}{\delta x'} + \left(\frac{\delta^2 T}{\delta\tau'^2} - W\frac{\delta^2\Omega'}{\delta\tau'^2}\right)\delta\frac{\delta W}{\delta y'} + \left(\frac{\delta^2 T}{\delta\tau'\delta\upsilon'} - W\frac{\delta^2\Omega'}{\delta\tau'\delta\upsilon'}\right)\delta\frac{\delta W}{\delta z'} \\
&\qquad + \delta_{,}\frac{\delta T}{\delta\tau'} - W\delta_{,}\frac{\delta\Omega'}{\delta\tau'} + \delta y' - \frac{\delta\Omega'}{\delta\tau'}\delta W; \\
\lambda''_3\delta\Omega &= \left(\frac{\delta^2 W}{\delta\sigma'\delta\upsilon'} - W\frac{\delta^2\Omega'}{\delta\sigma'\delta\upsilon'}\right)\delta\frac{\delta W}{\delta x'} + \left(\frac{\delta^2 T}{\delta\tau'\delta\upsilon'} - W\frac{\delta^2\Omega'}{\delta\tau'\delta\upsilon'}\right)\delta\frac{\delta W}{\delta y'} + \left(\frac{\delta^2 T}{\delta\upsilon'^2} - W\frac{\delta^2\Omega'}{\delta\upsilon'^2}\right)\delta\frac{\delta W}{\delta z'} \\
&\qquad + \delta_{,}\frac{\delta T}{\delta\upsilon'} - W\delta_{,}\frac{\delta\Omega'}{\delta\upsilon'} + \delta z' - \frac{\delta\Omega'}{\delta\upsilon'}\delta W; \\
\lambda''_4\delta\Omega &= \left(\frac{\delta^2 T}{\delta\sigma\delta\sigma'} - \frac{\delta T}{\delta\sigma'}\frac{\delta\Omega}{\delta\sigma}\right)\delta\frac{\delta W}{\delta x'} + \left(\frac{\delta^2 T}{\delta\sigma\delta\tau'} - \frac{\delta T}{\delta\tau'}\frac{\delta\Omega}{\delta\sigma}\right)\delta\frac{\delta W}{\delta y'} + \left(\frac{\delta^2 T}{\delta\sigma\delta\upsilon'} - \frac{\delta T}{\delta\upsilon'}\frac{\delta\Omega}{\delta\sigma}\right)\delta\frac{\delta W}{\delta z'} \\
&\qquad - \delta\frac{\delta W}{\delta\sigma} + \delta_{,}\frac{\delta T}{\delta\sigma} + (W - T)\delta\frac{\delta\Omega}{\delta\sigma} + \frac{\delta\Omega}{\delta\sigma}(\delta W - \delta_{,}T); \\
\lambda''_5\delta\Omega &= \left(\frac{\delta^2 T}{\delta\tau\delta\sigma'} - \frac{\delta T}{\delta\sigma'}\frac{\delta\Omega}{\delta\tau}\right)\delta\frac{\delta W}{\delta x'} + \left(\frac{\delta^2 T}{\delta\tau\delta\tau'} - \frac{\delta T}{\delta\tau'}\frac{\delta\Omega}{\delta\tau}\right)\delta\frac{\delta W}{\delta y'} + \left(\frac{\delta^2 T}{\delta\tau\delta\upsilon'} - \frac{\delta T}{\delta\upsilon'}\frac{\delta\Omega}{\delta\tau}\right)\delta\frac{\delta W}{\delta z'} \\
&\qquad - \delta\frac{\delta W}{\delta\tau} + \delta_{,}\frac{\delta T}{\delta\tau} + (W - T)\delta\frac{\delta\Omega}{\delta\tau} + \frac{\delta\Omega}{\delta\tau}(\delta W - \delta_{,}T); \\
\lambda''_6\delta\Omega &= \left(\frac{\delta^2 T}{\delta\upsilon\delta\sigma'} - \frac{\delta T}{\delta\sigma'}\frac{\delta\Omega}{\delta\upsilon}\right)\delta\frac{\delta W}{\delta x'} + \left(\frac{\delta^2 T}{\delta\upsilon\delta\tau'} - \frac{\delta T}{\delta\tau'}\frac{\delta\Omega}{\delta\upsilon}\right)\delta\frac{\delta W}{\delta y'} + \left(\frac{\delta^2 T}{\delta\upsilon\delta\upsilon'} - \frac{\delta T}{\delta\upsilon'}\frac{\delta\Omega}{\delta\upsilon}\right)\delta\frac{\delta W}{\delta z'} \\
&\qquad - \delta\frac{\delta W}{\delta\upsilon} + \delta_{,}\frac{\delta T}{\delta\upsilon} + (W - T)\delta\frac{\delta\Omega}{\delta\upsilon} + \frac{\delta\Omega}{\delta\upsilon}(\delta W - \delta_{,}T); \\
\lambda''_7\delta\Omega &= \left(\frac{\delta^2 T}{\delta\sigma'\delta\chi} - W\frac{\delta^2\Omega'}{\delta\sigma'\delta\chi} - \frac{\delta T}{\delta\sigma'}\frac{\delta\Omega}{\delta\chi}\right)\delta\frac{\delta W}{\delta x'} + \left(\frac{\delta^2 T}{\delta\tau'\delta\chi} - W\frac{\delta^2\Omega'}{\delta\tau'\delta\chi} - \frac{\delta T}{\delta\tau'}\frac{\delta\Omega}{\delta\chi}\right)\delta\frac{\delta W}{\delta y'} \\
&\qquad + \left(\frac{\delta^2 T}{\delta\upsilon'\delta\chi} - W\frac{\delta^2\Omega'}{\delta\upsilon'\delta\chi} - \frac{\delta T}{\delta\upsilon'}\frac{\delta\Omega}{\delta\chi}\right)\delta\frac{\delta W}{\delta z'} \\
&\qquad - \delta\frac{\delta W}{\delta\chi} + \delta_{,}\frac{\delta T}{\delta\chi} + (W - T)\delta\frac{\delta\Omega}{\delta\chi} + \frac{\delta\Omega}{\delta\chi}(\delta W - \delta_{,}T) - W\delta_{,}\frac{\delta\Omega'}{\delta\chi} - \frac{\delta\Omega'}{\delta\chi}\delta W;
\end{aligned}\right\}(\mathrm{P}^5)$$

$\delta_{,}$ still referring only to the variations of σ, τ, υ, χ, and the values of the multipliers being,

$$\left.\begin{aligned}
\lambda''_1 &= -x'; & \lambda''_4 &= \frac{\delta W}{\delta\sigma} - T\frac{\delta\Omega}{\delta\sigma}; \\
\lambda''_2 &= -y'; & \lambda''_5 &= \frac{\delta W}{\delta\tau} - T\frac{\delta\Omega}{\delta\tau}; \\
\lambda''_3 &= -z'; & \lambda''_6 &= \frac{\delta W}{\delta\upsilon} - T\frac{\delta\Omega}{\delta\upsilon}; \\
& & \lambda''_7 &= \frac{\delta W}{\delta\chi} - T\frac{\delta\Omega}{\delta\chi}.
\end{aligned}\right\}\qquad(\mathrm{Q}^5)$$

Hence may be deduced, by reasonings analogous to those already employed, the following formula for $\delta^2 W$, which is equivalent to twenty-eight separate expressions for the partial differential coefficients of W, of the second order, considered as deduced from the coefficients of T, on the foregoing suppositions of homogeneity:

$$\begin{aligned}0=\frac{1}{v'^2 W'''}\Big\{&\delta^2 W-\delta_{,}^2 T+(T-W)\delta^2\Omega+W\delta_{,}^2\Omega'+2\delta W(\delta_{,}\Omega'-\delta\Omega)+2\delta_{,}W.\delta\Omega \Big\}\\
&+\Big(\frac{\delta^2 T}{\delta\sigma'^2}-W\frac{\delta^2\Omega'}{\delta\sigma'^2}\Big)D^2+2\Big(\frac{\delta^2 T}{\delta\tau'\delta\upsilon'}-W\frac{\delta^2\Omega'}{\delta\tau'\delta\upsilon'}\Big)D'D''\\
&+\Big(\frac{\delta^2 T}{\delta\tau'^2}-W\frac{\delta^2\Omega'}{\delta\tau'^2}\Big)D'^2+2\Big(\frac{\delta^2 T}{\delta\upsilon'\delta\sigma'}-W\frac{\delta^2\Omega'}{\delta\upsilon'\delta\sigma'}\Big)D''D\\
&+\Big(\frac{\delta^2 T}{\delta\upsilon'^2}-W\frac{\delta^2\Omega'}{\delta\upsilon'^2}\Big)D''^2+2\Big(\frac{\delta^2 T}{\delta\sigma'\delta\tau'}-W\frac{\delta^2\Omega'}{\delta\sigma'\delta\tau'}\Big)DD'; \qquad (\mathrm{R}^5)\end{aligned}$$

in which we have put for abridgment,

$$\left.\begin{aligned}D&=\frac{\delta\Omega'}{\delta\tau'}\Big(\delta z'+z'\delta\Omega+\delta_{,}\frac{\delta T}{\delta\upsilon'}-W\delta_{,}\frac{\delta\Omega'}{\delta\upsilon'}\Big)-\frac{\delta\Omega'}{\delta\upsilon'}\Big(\delta y'+y'\delta\Omega+\delta_{,}\frac{\delta T}{\delta\tau'}-W\delta_{,}\frac{\delta\Omega'}{\delta\tau'}\Big),\\
D'&=\frac{\delta\Omega'}{\delta\upsilon'}\Big(\delta x'+x'\delta\Omega+\delta_{,}\frac{\delta T}{\delta\sigma'}-W\delta_{,}\frac{\delta\Omega'}{\delta\sigma'}\Big)-\frac{\delta\Omega'}{\delta\sigma'}\Big(\delta z'+z'\delta\Omega+\delta_{,}\frac{\delta T}{\delta\upsilon'}-W\delta_{,}\frac{\delta\Omega'}{\delta\upsilon'}\Big),\\
D''&=\frac{\delta\Omega'}{\delta\sigma'}\Big(\delta y'+y'\delta\Omega+\delta_{,}\frac{\delta T}{\delta\tau'}-W\delta_{,}\frac{\delta\Omega'}{\delta\tau'}\Big)-\frac{\delta\Omega'}{\delta\tau'}\Big(\delta x'+x'\delta\Omega+\delta_{,}\frac{\delta T}{\delta\sigma'}-W\delta_{,}\frac{\delta\Omega'}{\delta\sigma'}\Big),\end{aligned}\right\}(\mathrm{S}^5)$$

and in which W''' can be deduced from T, by the relation

$$\begin{aligned}\frac{\sigma'^2+\tau'+\upsilon'^2}{v'^2 W'''}=&\Big(\frac{\delta^2 T}{\delta\sigma'^2}-W\frac{\delta^2\Omega'}{\delta\sigma'^2}\Big)\Big(\frac{\delta^2 T}{\delta\tau'^2}-W\frac{\delta^2\Omega'}{\delta\tau'^2}\Big)-\Big(\frac{\delta^2 T}{\delta\sigma'\delta\tau'}-W\frac{\delta^2\Omega'}{\delta\sigma'\delta\tau'}\Big)^2\\
&+\Big(\frac{\delta^2 T}{\delta\tau'^2}-W\frac{\delta^2\Omega'}{\delta\tau'^2}\Big)\Big(\frac{\delta^2 T}{\delta\upsilon'^2}-W\frac{\delta^2\Omega'}{\delta\upsilon'^2}\Big)-\Big(\frac{\delta^2 T}{\delta\tau'\delta\upsilon'}-W\frac{\delta^2\Omega'}{\delta\tau'\delta\upsilon'}\Big)^2\\
&+\Big(\frac{\delta^2 T}{\delta\upsilon'^2}-W\frac{\delta^2\Omega'}{\delta\upsilon'^2}\Big)\Big(\frac{\delta^2 T}{\delta\sigma'^2}-W\frac{\delta^2\Omega'}{\delta\sigma'^2}\Big)-\Big(\frac{\delta^2 T}{\delta\upsilon'\delta\sigma'}-W\frac{\delta^2\Omega'}{\delta\upsilon'\delta\sigma'}\Big)^2. \qquad (\mathrm{T}^5)\end{aligned}$$

General Remarks and Cautions, with respect to the foregoing deductions. Case of a Single Uniform Medium. Connexions between the Coefficients of the Function v, Ω, υ, for any Single Medium.

10. We are then able, by combining the formulæ of the three preceding numbers, to deduce the partial differential coefficients of the two first orders, of any one of the three functions V, W, T, from those of either of the other two, when the extreme media are uniform and known: since we have expressed the coefficients of V by those of W, and the coefficients of W by those of T, and reciprocally, for this case

of uniform media. And if the extreme media be not uniform, but variable, that is, if they be atmospheres, ordinary or extraordinary, we can still connect the partial differential coefficients of the three functions, by the general method mentioned at the beginning of the seventh number: which method extends to orders higher than the second, without much additional difficulty of elimination, but with results of greater complexity, and of less interesting application.

This general method consists, as has been said, in differentiating and comparing the equations into which the general expressions (A') (B') (C') for the variations of the three functions resolve themselves: and *in making this preliminary resolution of the general expressions* (A') (B') (C'), *it is necessary to attend with care to the relations between the variables* $\sigma, \tau, \upsilon, \sigma', \tau', \upsilon', \chi$, or between $\sigma, \tau, \upsilon, x', y', z', \chi$, when any such relations exist. The investigations into which we have entered in the three last numbers, for the case of extreme uniform media, *suppose that the variables are connected only by the relations* $\Omega=0$, $\Omega'=0$, which arise from and express the optical properties of these media; and *other but analogous processes must be deduced from the general method, when any additional relations* $\Omega''=0$, $\Omega'''=0, \ldots$ between the variables of the question, arise from the particular nature of a combination which we wish to study. In the very simple case, for instance, of a single uniform medium, we have the three relations

$$\sigma'=\sigma, \quad \tau'=\tau, \quad \upsilon'=\upsilon, \qquad (\mathrm{U}^5)$$

which are to be combined with the relation $\Omega=0$; and with this combination of relations, the general expression (C') for the variation of T will no longer admit of being resolved in the same way as when more of the quantities on which T depends could vary independently of each other.

In the case last mentioned, of *a single uniform medium*, the characteristic function V involves the co-ordinates x, y, z, x', y', z', only by involving their differences $x-x'$, $y-y'$, $z-z'$, and is, with respect to these differences, homogeneous of the first dimension, being determined by an equation of the form

$$0=\Psi\left(\frac{x-x'}{V}, \frac{y-y'}{V}, \frac{z-z'}{V}, \chi\right), \qquad (\mathrm{V}^5)$$

which results from the equation (N) for the medium function v, by first suppressing in that equation the co-ordinates on account of the supposed uniformity, and then making

$$\frac{\alpha}{v}=\frac{x-x'}{V}, \quad \frac{\beta}{v}=\frac{y-y'}{V}, \quad \frac{\gamma}{v}=\frac{z-z'}{V}. \qquad (\mathrm{W}^5)$$

The relation (V^5) may also be deduced from the relation $\Omega=0$, by eliminating the ratios of σ, τ, υ, between the three following equations,

$$\frac{x-x'}{V}=\frac{\delta\Omega}{\delta\sigma}\,,\quad \frac{y-y'}{V}=\frac{\delta\Omega}{\delta\tau}\,,\quad \frac{z-z'}{V}=\frac{\delta\Omega}{\delta\upsilon}\,. \qquad (\mathrm{X}^5)$$

We have also, in this case of a single uniform medium,

$$V=\sigma(x-x')+\tau(y-y')+\upsilon(z-z'), \qquad (\mathrm{Y}^5)$$

and therefore, by (D') (E') (U^5),

$$\left.\begin{array}{l} W=\sigma x'+\tau y'+\upsilon z', \\ T=0: \end{array}\right\} \qquad (\mathrm{Z}^5)$$

the last of which results may be verified by observing that the general expression for the auxiliary function T may be put under the form

$$T= x\frac{\delta V}{\delta x}+y\frac{\delta V}{\delta y}+z\frac{\delta V}{\delta z}+x'\frac{\delta V}{\delta x'}+y'\frac{\delta V}{\delta y'}+z'\frac{\delta V}{\delta z'}-V, \qquad (\mathrm{A}^6)$$

so that T vanishes when V is homogeneous of the first dimension with respect to the six extreme co-ordinates. The formulæ of the last number, for the partial differential coefficients of T, all fail in this case of a single uniform medium, for the reason already assigned; but we may consider all these coefficients of T as vanishing, like T itself: we may however give any other values to these coefficients which when combined with the relations between the variables will make the variations of T vanish. The coefficients of W may be obtained by differentiating the expression (Z^5), which is of the homogeneous form that we have already found it convenient to adopt; they are, for the first two orders, included in the two following formulæ,

$$\left.\begin{array}{l} \delta W=x'\delta\sigma+y'\delta\tau+z'\delta\upsilon+\sigma\delta x'+\tau\delta y'+\upsilon\delta z', \\ \delta^2 W=2\delta\sigma\delta x'+2\delta\tau\delta y'+2\delta\upsilon\delta z', \end{array}\right\} \qquad (\mathrm{B}^6)$$

and they vanish for orders higher than the second. And the coefficients of V, of the two first orders, may be deduced from those of W by the formulæ of the eighth number, which are not vitiated by the existence of the relations (U^5), because those relations do not affect the variables that enter into the composition of V and W. The variation of V, of the first order, is

$$\delta V=\sigma(\delta x-\delta x')+\tau(\delta y-\delta y')+\upsilon(\delta z-\delta z')-V\frac{\delta\Omega}{\delta\chi}\,\delta\chi\,; \qquad (\mathrm{C}^6)$$

and that of the second order is given by the following equation, deduced from (O^4), (N^4), (B^6),

$$V\left\{\frac{\delta^2\Omega}{\delta\sigma^2}\,\frac{\delta^2\Omega}{\delta\tau^2}-\left(\frac{\delta^2\Omega}{\delta\sigma\delta\tau}\right)^2+\frac{\delta^2\Omega}{\delta\tau^2}\,\frac{\delta^2\Omega}{\delta\upsilon^2}-\left(\frac{\delta^2\Omega}{\delta\tau\delta\upsilon}\right)^2+\frac{\delta^2\Omega}{\delta\upsilon^2}\,\frac{\delta^2\Omega}{\delta\sigma^2}-\left(\frac{\delta^2\Omega}{\delta\upsilon\delta\sigma}\right)^2\right\}\left(\frac{\delta^2 V+V\delta'^2\Omega+2\delta V\delta'\Omega}{\sigma^2+\tau^2+\upsilon^2}\right)$$
$$=\frac{\delta^2\Omega}{\delta\sigma^2}\left\{\frac{\delta\Omega}{\delta\tau}\left(\delta z-\delta z'-V\delta'\frac{\delta\Omega}{\delta\upsilon}\right)-\frac{\delta\Omega}{\delta\upsilon}\left(\delta y-\delta y'-V\delta'\frac{\delta\Omega}{\delta\tau}\right)\right\}^2$$

$$+\frac{\delta^2\Omega}{\delta\tau^2}\left\{\frac{\delta\Omega}{\delta\upsilon}\left(\delta x-\delta x'-V\delta'\frac{\delta\Omega}{\delta\sigma}\right)-\frac{\delta\Omega}{\delta\sigma}\left(\delta z-\delta z'-V\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\right\}^2$$

$$+\frac{\delta^2\Omega}{\delta\upsilon^2}\left\{\frac{\delta\Omega}{\delta\sigma}\left(\delta y-\delta y'-V\delta'\frac{\delta\Omega}{\delta\tau}\right)-\frac{\delta\Omega}{\delta\tau}\left(\delta x-\delta x'-V\delta'\frac{\delta\Omega}{\delta\sigma}\right)\right\}^2$$

$$+\;2\frac{\delta^2\Omega}{\delta\sigma\delta\tau}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\tau}\left(\delta z-\delta z'-V\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\\-\frac{\delta\Omega}{\delta\upsilon}\left(\delta y-\delta y'-V\delta'\frac{\delta\Omega}{\delta\tau}\right)\end{array}\right\}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\upsilon}\left(\delta x-\delta x'-V\delta'\frac{\delta\Omega}{\delta\sigma}\right)\\-\frac{\delta\Omega}{\delta\sigma}\left(\delta z-\delta z'-V\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\end{array}\right\}$$

$$+\;2\frac{\delta^2\Omega}{\delta\tau\delta\upsilon}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\upsilon}\left(\delta x-\delta x'-V\delta'\frac{\delta\Omega}{\delta\sigma}\right)\\-\frac{\delta\Omega}{\delta\sigma}\left(\delta z-\delta z'-V\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\end{array}\right\}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\sigma}\left(\delta y-\delta y'-V\delta'\frac{\delta\Omega}{\delta\tau}\right)\\-\frac{\delta\Omega}{\delta\tau}\left(\delta x-\delta x'-V\delta'\frac{\delta\Omega}{\delta\sigma}\right)\end{array}\right\}$$

$$+\;2\frac{\delta^2\Omega}{\delta\upsilon\delta\sigma}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\sigma}\left(\delta y-\delta y'-V\delta'\frac{\delta\Omega}{\delta\tau}\right)\\-\frac{\delta\Omega}{\delta\tau}\left(\delta x-\delta x'-V\delta'\frac{\delta\Omega}{\delta\sigma}\right)\end{array}\right\}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\tau}\left(\delta z-\delta z'-V\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\\-\frac{\delta\Omega}{\delta\upsilon}\left(\delta y-\delta y'-V\delta'\frac{\delta\Omega}{\delta\tau}\right)\end{array}\right\}; \qquad (\mathrm{D}^6)$$

in which the symbol δ' has the same meaning as before, so that as $x'\,y'\,z'$ do not enter into the composition of the function Ω, δ' refers here to the variation of colour only. This equation (D^6) may be put under the following simpler form,

$$\begin{aligned}&\frac{V}{v}\left(\delta^2V+V\delta'^2\Omega+2\delta V\delta'\Omega\right)\\&=\frac{\delta^2v}{\delta\alpha^2}\left(\delta x-\delta x'-V\delta'\frac{\delta\Omega}{\delta\sigma}\right)^2\\&+\frac{\delta^2v}{\delta\beta^2}\left(\delta y-\delta y'-V\delta'\frac{\delta\Omega}{\delta\tau}\right)^2\\&+\frac{\delta^2v}{\delta\gamma}\left(\delta z-\delta z'-V\delta'\frac{\delta\Omega}{\delta\upsilon}\right)^2\\&+\;2\frac{\delta^2v}{\delta\alpha\delta\beta}\left(\delta x-\delta x'-V\delta'\frac{\delta\Omega}{\delta\sigma}\right)\left(\delta y-\delta y'-V\delta'\frac{\delta\Omega}{\delta\tau}\right)\\&+\;2\frac{\delta^2v}{\delta\beta\delta\gamma}\left(\delta y-\delta y'-V\delta'\frac{\delta\Omega}{\delta\tau}\right)\left(\delta z-\delta z'-V\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\\&+\;2\frac{\delta^2v}{\delta\gamma\delta\alpha}\left(\delta z-\delta z'-V\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\left(\delta x-\delta x'-V\delta'\frac{\delta\Omega}{\delta\sigma}\right),\qquad(\mathrm{E}^6)\end{aligned}$$

if we attend to the equations already established, in the second number,

$$\frac{\alpha}{v}=\frac{\delta\Omega}{\delta\sigma},\ \frac{\beta}{v}=\frac{\delta\Omega}{\delta\tau},\ \frac{\gamma}{v}=\frac{\delta\Omega}{\delta\upsilon},\ -\frac{1}{v}\frac{\delta v}{\delta\chi}=\frac{\delta\Omega}{\delta\chi},$$

$$\sigma=\frac{\delta v}{\delta\alpha},\ \tau=\frac{\delta v}{\delta\beta},\ \upsilon=\frac{\delta v}{\delta\gamma},$$

and to the relations which result from these, by differentiation and elimination. For thus we obtain

$$\left.\begin{aligned}
\delta\,\frac{a}{v} - \delta'\frac{\delta\Omega}{\delta\sigma} &= \frac{\delta^2\Omega}{\delta\sigma^2}\,\delta\,\frac{\delta v}{\delta a} + \frac{\delta^2\Omega}{\delta\sigma\delta\tau}\,\delta\,\frac{\delta v}{\delta\beta} + \frac{\delta^2\Omega}{\delta\sigma\delta\upsilon}\,\delta\,\frac{\delta v}{\delta\gamma},\\
\delta\,\frac{\beta}{v} - \delta'\frac{\delta\Omega}{\delta\tau} &= \frac{\delta^2\Omega}{\delta\sigma\delta\tau}\,\delta\,\frac{\delta v}{\delta a} + \frac{\delta^2\Omega}{\delta\tau^2}\,\delta\,\frac{\delta v}{\delta\beta} + \frac{\delta^2\Omega}{\delta\tau\delta\upsilon}\,\delta\,\frac{\delta v}{\delta\gamma},\\
\delta\,\frac{\gamma}{v} - \delta'\frac{\delta\Omega}{\delta\upsilon} &= \frac{\delta^2\Omega}{\delta\sigma\delta\upsilon}\,\delta\,\frac{\delta v}{\delta a} + \frac{\delta^2\Omega}{\delta\tau\delta\upsilon}\,\delta\,\frac{\delta v}{\delta\beta} + \frac{\delta^2\Omega}{\delta\upsilon^2}\,\delta\,\frac{\delta v}{\delta\gamma},\\
-\delta\left(\frac{1}{v}\,\frac{\delta v}{\delta\chi}\right) - \delta'\frac{\delta\Omega}{\delta\chi} &= \frac{\delta^2\Omega}{\delta\sigma\delta\chi}\,\delta\,\frac{\delta v}{\delta a} + \frac{\delta^2\Omega}{\delta\tau\delta\chi}\,\delta\,\frac{\delta v}{\delta\beta} + \frac{\delta^2\Omega}{\delta\upsilon\delta\chi}\,\delta\,\frac{\delta v}{\delta\gamma},
\end{aligned}\right\}\quad (F^6)$$

in which v is considered as a homogeneous function of the first dimension of a, β, γ, involving also the colour χ; and in which, although the three variations δa, $\delta\beta$, $\delta\gamma$, are connected by the relation $a\delta a + \beta\delta\beta + \gamma\delta\gamma = 0$, yet we may treat these variations as independent; because, if we introduced indeterminate multipliers of $a\delta a + \beta\delta\beta + \gamma\delta\gamma$, in ($F^6$), to allow for the relation, we should find that these multipliers vanish, on account of the conditions of homogeneity of v. And if we put for abridgment

$$\omega'' = \frac{\delta^2\Omega}{\delta\sigma^2}\,\frac{\delta^2\Omega}{\delta\tau^2} - \left(\frac{\delta^2\Omega}{\delta\sigma\delta\tau}\right)^2 + \frac{\delta^2\Omega}{\delta\tau^2}\,\frac{\delta^2\Omega}{\delta\upsilon^2} - \left(\frac{\delta^2\Omega}{\delta\tau\delta\upsilon}\right)^2 + \frac{\delta^2\Omega}{\delta\upsilon^2}\,\frac{\delta^2\Omega}{\delta\sigma^2} - \left(\frac{\delta^2\Omega}{\delta\upsilon\delta\sigma}\right)^2, \qquad (G^6)$$

the equations (F^6) give the following formula for $\delta^2 v$, that is, for the second variation of v, taken as if a β γ χ were four independent variables,

$$\begin{aligned}
\frac{v\omega''}{\sigma^2+\tau^2+\upsilon^2}\,(\delta^2 v + v\delta'^2\Omega + 2\delta v\delta'\Omega) = {} & \frac{\delta^2\Omega}{\delta\sigma^2}\left\{\frac{\delta\Omega}{\delta\upsilon}\left(\delta\beta - v\delta'\,\frac{\delta\Omega}{\delta\tau}\right) - \frac{\delta\Omega}{\delta\tau}\left(\delta\gamma - v\delta'\,\frac{\delta\Omega}{\delta\upsilon}\right)\right\}^2\\
& + \frac{\delta^2\Omega}{\delta\tau^2}\left\{\frac{\delta\Omega}{\delta\sigma}\left(\delta\gamma - v\delta'\,\frac{\delta\Omega}{\delta\upsilon}\right) - \frac{\delta\Omega}{\delta\upsilon}\left(\delta a - v\delta'\,\frac{\delta\Omega}{\delta\sigma}\right)\right\}^2\\
& + \frac{\delta^2\Omega}{\delta\upsilon^2}\left\{\frac{\delta\Omega}{\delta\tau}\left(\delta a - v\delta'\,\frac{\delta\Omega}{\delta\sigma}\right) - \frac{\delta\Omega}{\delta\sigma}\left(\delta\beta - v\delta'\,\frac{\delta\Omega}{\delta\tau}\right)\right\}^2\\
+ 2\frac{\delta^2\Omega}{\delta\sigma\delta\tau}\left\{\frac{\delta\Omega}{\delta\upsilon}\left(\delta\beta - v\delta'\,\frac{\delta\Omega}{\delta\tau}\right) - \frac{\delta\Omega}{\delta\tau}\left(\delta\gamma - v\delta'\,\frac{\delta\Omega}{\delta\upsilon}\right)\right\} & \left\{\frac{\delta\Omega}{\delta\sigma}\left(\delta\gamma - v\delta'\frac{\delta\Omega}{\delta\upsilon}\right) - \frac{\delta\Omega}{\delta\upsilon}\left(\delta a - v\delta'\frac{\delta\Omega}{\delta\sigma}\right)\right\}\\
+ 2\frac{\delta^2\Omega}{\delta\tau\delta\upsilon}\left\{\frac{\delta\Omega}{\delta\sigma}\left(\delta\gamma - v\delta'\,\frac{\delta\Omega}{\delta\upsilon}\right) - \frac{\delta\Omega}{\delta\upsilon}\left(\delta a - v\delta'\,\frac{\delta\Omega}{\delta\sigma}\right)\right\} & \left\{\frac{\delta\Omega}{\delta\tau}\left(\delta a - v\delta'\frac{\delta\Omega}{\delta\sigma}\right) - \frac{\delta\Omega}{\delta\sigma}\left(\delta\beta - v\delta'\frac{\delta\Omega}{\delta\tau}\right)\right\}\\
+ 2\frac{\delta^2\Omega}{\delta\upsilon\delta\sigma}\left\{\frac{\delta\Omega}{\delta\tau}\left(\delta a - v\delta'\,\frac{\delta\Omega}{\delta\sigma}\right) - \frac{\delta\Omega}{\delta\sigma}\left(\delta\beta - v\delta'\,\frac{\delta\Omega}{\delta\tau}\right)\right\} & \left\{\frac{\delta\Omega}{\delta\upsilon}\left(\delta\beta - v\delta'\frac{\delta\Omega}{\delta\tau}\right) - \frac{\delta\Omega}{\delta\tau}\left(\delta\beta - v\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\right\}; \quad (H^6)
\end{aligned}$$

which justifies the passage from (D^6) to (E^6), and expresses the law of dependence of the partial differential coefficients of the second order of the function v on those of Ω, for the case of a uniform medium.

If the medium be not uniform, and if we would still express the law of this dependence, we have onlp to change δ', in the four equations (F^6) to a new characteristic $\delta_{\prime\prime}$

referring to the variations of $x\ y\ z\ \chi$, and to combine the four thus altered with the three following,

$$\left.\begin{aligned}
-\delta\left(\frac{1}{v}\frac{\delta v}{\delta x}\right)-\delta_{\prime\prime}\frac{\delta\Omega}{\delta x}&=\frac{\delta^2\Omega}{\delta\sigma\delta x}\,\delta\frac{\delta v}{\delta a}+\frac{\delta^2\Omega}{\delta\tau\delta x}\,\delta\frac{\delta v}{\delta\beta}+\frac{\delta^2\Omega}{\delta\upsilon\delta x}\,\delta\frac{\delta v}{\delta\gamma},\\
-\delta\left(\frac{1}{v}\frac{\delta v}{\delta y}\right)-\delta_{\prime\prime}\frac{\delta\Omega}{\delta y}&=\frac{\delta^2\Omega}{\delta\sigma\delta y}\,\delta\frac{\delta v}{\delta a}+\frac{\delta^2\Omega}{\delta\tau\delta y}\,\delta\frac{\delta v}{\delta\beta}+\frac{\delta^2\Omega}{\delta\upsilon\delta y}\,\delta\frac{\delta v}{\delta\gamma},\\
-\delta\left(\frac{1}{v}\frac{\delta v}{\delta z}\right)-\delta_{\prime\prime}\frac{\delta\Omega}{\delta z}&=\frac{\delta^2\Omega}{\delta\sigma\delta z}\,\delta\frac{\delta v}{\delta a}+\frac{\delta^2\Omega}{\delta\tau\delta z}\,\delta\frac{\delta v}{\delta\beta}+\frac{\delta^2\Omega}{\delta\upsilon\delta z}\,\delta\frac{\delta v}{\delta\gamma},
\end{aligned}\right\}\qquad(\mathrm{I}^6)$$

in which $\delta_{\prime\prime}$ is the same new characteristic, and which are deduced from the equations already established for variable media,

$$-\frac{1}{v}\frac{\delta v}{\delta x}=\frac{\delta\Omega}{\delta x},\quad -\frac{1}{v}\frac{\delta v}{\delta y}=\frac{\delta\Omega}{\delta y},\quad -\frac{1}{v}\frac{\delta v}{\delta z}=\frac{\delta\Omega}{\delta z}:$$

and we are conducted to a formula for $\delta^2 v$, which no otherwise differs from (H^6) than by having $\delta_{\prime\prime}$ instead of δ' throughout.

And if, reciprocally, we would express the law of dependance of the coefficients of Ω of the second order, on those of v, we may do so by the following general formula,

$$\begin{aligned}
v''v^2(v\delta^2\Omega+\delta_{\prime\prime}{}^2v+2\delta_{\prime\prime}v\delta\Omega)=\;&\frac{\delta^2 v}{\delta a^2}\left\{\upsilon\left(\delta\tau-\delta_{\prime\prime}\frac{\delta v}{\delta\beta}\right)-\tau\left(\delta\upsilon-\delta_{\prime\prime}\frac{\delta v}{\delta\gamma}\right)\right\}^2\\
&+\frac{\delta^2 v}{\delta\beta^2}\left\{\sigma\left(\delta\upsilon-\delta_{\prime\prime}\frac{\delta v}{\delta\gamma}\right)-\upsilon\left(\delta\sigma-\delta_{\prime\prime}\frac{\delta v}{\delta a}\right)\right\}^2\\
&+\frac{\delta^2 v}{\delta\gamma^2}\left\{\tau\left(\delta\sigma-\delta_{\prime\prime}\frac{\delta v}{\delta a}\right)-\sigma\left(\delta\tau-\delta_{\prime\prime}\frac{\delta v}{\delta\beta}\right)\right\}^2\\
+2\frac{\delta^2 v}{\delta a\delta\beta}\left\{\upsilon\left(\delta\tau-\delta_{\prime\prime}\frac{\delta v}{\delta\beta}\right)-\tau\left(\delta\upsilon-\delta_{\prime\prime}\frac{\delta v}{\delta\gamma}\right)\right\}&\left\{\sigma\left(\delta\upsilon-\delta_{\prime\prime}\frac{\delta v}{\delta\gamma}\right)-\upsilon\left(\delta\sigma-\delta_{\prime\prime}\frac{\delta v}{\delta a}\right)\right\}\\
+2\frac{\delta^2 v}{\delta\beta\delta\gamma}\left\{\sigma\left(\delta\upsilon-\delta_{\prime\prime}\frac{\delta v}{\delta\gamma}\right)-\upsilon\left(\delta\sigma-\delta_{\prime\prime}\frac{\delta v}{\delta a}\right)\right\}&\left\{\tau\left(\delta\sigma-\delta_{\prime\prime}\frac{\delta v}{\delta a}\right)-\sigma\left(\delta\tau-\delta_{\prime\prime}\frac{\delta v}{\delta\beta}\right)\right\}\\
+2\frac{\delta^2 v}{\delta\gamma\delta a}\left\{\tau\left(\delta\sigma-\delta_{\prime\prime}\frac{\delta v}{\delta a}\right)-\sigma\left(\delta\tau-\delta_{\prime\prime}\frac{\delta v}{\delta\beta}\right)\right\}&\left\{\upsilon\left(\delta\tau-\delta_{\prime\prime}\frac{\delta v}{\delta\beta}\right)-\tau\left(\delta\upsilon-\delta_{\prime\prime}\frac{\delta v}{\delta\gamma}\right)\right\};\ (\mathrm{K}^6)
\end{aligned}$$

in which $\delta_{\prime\prime}$ refers still to the variations x, y, z, χ, and in which v'' has the same meaning as in the First Supplement, namely

$$v''=\frac{\delta^2 v}{\delta a^2}\frac{\delta^2 v}{\delta\beta^2}-\left(\frac{\delta^2 v}{\delta a\delta\beta}\right)^2+\frac{\delta^2 v}{\delta\beta^2}\frac{\delta^2 v}{\delta\gamma^2}-\left(\frac{\delta^2 v}{\delta\beta\delta\gamma}\right)^2+\frac{\delta^2 v}{\delta\gamma^2}\frac{\delta^2 v}{\delta a^2}-\left(\frac{\delta^2 v}{\delta\gamma\delta a}\right)^2:\qquad(\mathrm{L}^6)$$

this quantity v'' is also connected with the ω'' of (G^6) (H^6), by the relation

$$v''\,\omega''=\frac{\sigma^2+\tau^2+\upsilon^2}{v^4}.\qquad(\mathrm{M}^6)$$

The formula (K^6) is equivalent to twenty-eight separate expressions for the partial differential coefficients of Ω, of the second order, which extend to variable as well as to uniform media: the formula gives, for example, the six following general expressions, which enable us to introduce the coefficients of the function v, of the second order, instead of those of Ω, if it be thought desirable so to do, in many of the general equations of the present memoir, as the expressions contained in (H^6) would enable us to introduce Ω instead of v, in many of those of the First Supplement:

$$\left.\begin{aligned}
\frac{\delta^2\Omega}{\delta\sigma^2} &= \frac{1}{v''v^3}\left(\tau^2\frac{\delta^2 v}{\delta\gamma^2} + \upsilon^2\frac{\delta^2 v}{\delta\beta^2} - 2\tau\upsilon\frac{\delta^2 v}{\delta\beta\delta\gamma}\right);\\
\frac{\delta^2\Omega}{\delta\tau^2} &= \frac{1}{v''v^3}\left(\upsilon^2\frac{\delta^2 v}{\delta\alpha^2} + \sigma^2\frac{\delta^2 v}{\delta\gamma^2} - 2\upsilon\sigma\frac{\delta^2 v}{\delta\gamma\delta\alpha}\right);\\
\frac{\delta^2\Omega}{\delta\upsilon^2} &= \frac{1}{v''v^3}\left(\sigma^2\frac{\delta^2 v}{\delta\beta^2} + \tau^2\frac{\delta^2 v}{\delta\alpha^2} - 2\sigma\tau\frac{\delta^2 v}{\delta\alpha\delta\beta}\right);\\
\frac{\delta^2\Omega}{\delta\sigma\delta\tau} &= \frac{1}{v''v^3}\left(-\upsilon^2\frac{\delta^2 v}{\delta\alpha\delta\beta} + \tau\upsilon\frac{\delta^2 v}{\delta\gamma\delta\alpha} + \upsilon\sigma\frac{\delta^2 v}{\delta\beta\delta\gamma} - \sigma\tau\frac{\delta^2 v}{\delta\gamma^2}\right);\\
\frac{\delta^2\Omega}{\delta\tau\delta\upsilon} &= \frac{1}{v''v^3}\left(-\sigma^2\frac{\delta^2 v}{\delta\beta\delta\gamma} + \upsilon\sigma\frac{\delta^2 v}{\delta\alpha\delta\beta} + \sigma\tau\frac{\delta^2 v}{\delta\gamma\delta\alpha} - \tau\upsilon\frac{\delta^2 v}{\delta\alpha^2}\right);\\
\frac{\delta^2\Omega}{\delta\upsilon\delta\sigma} &= \frac{1}{v''v^3}\left(-\tau^2\frac{\delta^2 v}{\delta\gamma\delta\alpha} + \sigma\tau\frac{\delta^2 v}{\delta\beta\delta\gamma} + \tau\upsilon\frac{\delta^2 v}{\delta\alpha\delta\beta} - \upsilon\sigma\frac{\delta^2 v}{\delta\beta^2}\right).
\end{aligned}\right\}\quad (\mathrm{N}^6)$$

To make more complete this theory of the coefficients of the function Ω, which determines the nature of the final uniform or variable medium by the manner of its dependence on the seven variables $\sigma\ \tau\ \upsilon\ x\ y\ z\ \chi$, and is supposed to have been so prepared that $\Omega + 1$ is homogeneous of the first dimension relatively to $\sigma\ \tau\ \upsilon$, let us investigate the connexion of these coefficients of Ω with those of the simpler though less symmetric function υ, considered as depending on the six other variables $\sigma\ \tau\ x\ y\ z\ \chi$ by the relation $\Omega = 0$. For this purpose we are to combine the differentials of that relation with the conditions of homogeneity (B^4) (C^4), and with the following other conditions of the same kind, which are only useful in variable media,

$$\left.\begin{aligned}
\sigma\frac{\delta^2\Omega}{\delta\sigma\delta x} + \tau\frac{\delta^2\Omega}{\delta\tau\delta x} + \upsilon\frac{\delta^2\Omega}{\delta\upsilon\delta x} &= \frac{\delta\Omega}{\delta x},\\
\sigma\frac{\delta^2\Omega}{\delta\sigma\delta y} + \tau\frac{\delta^2\Omega}{\delta\tau\delta y} + \upsilon\frac{\delta^2\Omega}{\delta\upsilon\delta y} &= \frac{\delta\Omega}{\delta y},\\
\sigma\frac{\delta^2\Omega}{\delta\sigma\delta z} + \tau\frac{\delta^2\Omega}{\delta\tau\delta z} + \upsilon\frac{\delta^2\Omega}{\delta\upsilon\delta z} &= \frac{\delta\Omega}{\delta z}.
\end{aligned}\right\}\quad (\mathrm{O}^6)$$

In this manner we find, for the first order,

$$\delta\Omega = \lambda\left(\delta\upsilon - \frac{\delta\upsilon}{\delta\sigma}\delta\sigma - \frac{\delta\upsilon}{\delta\tau}\delta\tau - \frac{\delta\upsilon}{\delta x}\delta x - \frac{\delta\upsilon}{\delta y}\delta y - \frac{\delta\upsilon}{\delta z}\delta z - \frac{\delta\upsilon}{\delta\chi}\delta\chi\right); \qquad (\mathrm{P}^6)$$

that is

$$\left.\begin{aligned}
&\frac{\delta\Omega}{\delta\sigma} = -\lambda\frac{\delta\upsilon}{\delta\sigma};\ \frac{\delta\Omega}{\delta\tau} = -\lambda\frac{\delta\upsilon}{\delta\tau};\ \frac{\delta\Omega}{\delta\upsilon} = \lambda;\\
&\frac{\delta\Omega}{\delta x} = -\lambda\frac{\delta\upsilon}{\delta x};\ \frac{\delta\Omega}{\delta y} = -\lambda\frac{\delta\upsilon}{\delta y};\ \frac{\delta\Omega}{\delta z} = -\lambda\frac{\delta\upsilon}{\delta z};\ \frac{\delta\Omega}{\delta\chi} = -\lambda\frac{\delta\upsilon}{\delta\chi};
\end{aligned}\right\}\quad (\mathrm{Q}^6)$$

λ being a multiplier introduced for the purpose of treating the variations of $\sigma \tau \upsilon x y z \chi$ as independent; and to determine the value of this multiplier we have, by the condition of homogeneity (B^4),

$$\lambda \left(\upsilon - \sigma \frac{\delta \upsilon}{\delta \sigma} - \tau \frac{\delta \upsilon}{\delta \tau} \right) = \Omega + 1 = 1 : \qquad (\mathrm{R}^6)$$

the coefficients of Ω of the first order are therefore known, and we have for example,

$$\frac{\delta \Omega}{\delta \upsilon} = \lambda = \frac{1}{\upsilon - \sigma \frac{\delta \upsilon}{\delta \sigma} - \tau \frac{\delta \upsilon}{\delta \upsilon}} . \qquad (\mathrm{S}^6)$$

Again, for the second order,

$$\left. \begin{aligned}
\delta \frac{\delta \Omega}{\delta \sigma} &= - \delta . \lambda \frac{\delta \upsilon}{\delta \sigma} + \lambda_1 \left(\delta \upsilon - \frac{\delta \upsilon}{\delta \sigma} \delta \sigma - \&c. \right); \\
\delta \frac{\delta \Omega}{\delta \tau} &= - \delta . \lambda \frac{\delta \upsilon}{\delta \tau} + \lambda_2 \left(\delta \upsilon - \frac{\delta \upsilon}{\delta \sigma} \delta \sigma - \&c. \right); \\
\delta \frac{\delta \Omega}{\delta \upsilon} &= \quad \delta \lambda \quad + \lambda_3 \left(\delta \upsilon - \frac{\delta \upsilon}{\delta \sigma} \delta \sigma - \&c. \right); \\
\delta \frac{\delta \Omega}{\delta x} &= - \delta . \lambda \frac{\delta \upsilon}{\delta x} + \lambda_4 \left(\delta \upsilon - \frac{\delta \upsilon}{\delta \sigma} \delta \sigma - \&c. \right); \\
\delta \frac{\delta \Omega}{\delta y} &= - \delta . \lambda \frac{\delta \upsilon}{\delta y} + \lambda_5 \left(\delta \upsilon - \frac{\delta \upsilon}{\delta \sigma} \delta \sigma - \&c. \right); \\
\delta \frac{\delta \Omega}{\delta z} &= - \delta . \lambda \frac{\delta \upsilon}{\delta z} + \lambda_6 \left(\delta \upsilon - \frac{\delta \upsilon}{\delta \sigma} \delta \sigma - \&c. \right); \\
\delta \frac{\delta \Omega}{\delta \chi} &= - \delta . \lambda \frac{\delta \upsilon}{\delta \chi} + \lambda_7 \left(\delta \upsilon - \frac{\delta \upsilon}{\delta \sigma} \delta \sigma - \&c. \right);
\end{aligned} \right\} \qquad (\mathrm{T}^6)$$

in which, by (C^4) (O^6) (Q^6), the multipliers $\lambda_1 \ldots \lambda_7$ have the following values,

$$\left. \begin{aligned}
\lambda_1 &= \lambda \left(\sigma \frac{\delta}{\delta \sigma} + \tau \frac{\delta}{\delta \tau} \right) . \lambda \frac{\delta \upsilon}{\delta \sigma} ; \\
\lambda_2 &= \lambda \left(\sigma \frac{\delta}{\delta \sigma} + \tau \frac{\delta}{\delta \tau} \right) . \lambda \frac{\delta \upsilon}{\delta \tau} ; \\
\lambda_3 &= - \lambda \left(\sigma \frac{\delta}{\delta \sigma} + \tau \frac{\delta}{\delta \tau} \right) \lambda ; \\
\lambda_4 &= \lambda \left(\sigma \frac{\delta}{\delta \sigma} + \tau \frac{\delta}{\delta \tau} \right) . \lambda \frac{\delta \upsilon}{\delta x} - \lambda^2 \frac{\delta \upsilon}{\delta x} ; \\
\lambda_5 &= \lambda \left(\sigma \frac{\delta}{\delta \sigma} + \tau \frac{\delta}{\delta \tau} \right) . \lambda \frac{\delta \upsilon}{\delta y} - \lambda^2 \frac{\delta \upsilon}{\delta y} ; \\
\lambda_6 &= \lambda \left(\sigma \frac{\delta}{\delta \sigma} + \tau \frac{\delta}{\delta \tau} \right) . \lambda \frac{\delta \upsilon}{\delta z} - \lambda^2 \frac{\delta \upsilon}{\delta z} ; \\
\lambda_7 &= \lambda \left(\sigma \frac{\delta}{\delta \sigma} + \tau \frac{\delta}{\delta \tau} \right) . \lambda \frac{\delta \upsilon}{\delta \chi} - \lambda^2 \frac{\delta \upsilon}{\delta \chi} ;
\end{aligned} \right\} \qquad (\mathrm{U}^6)$$

λ, like υ, being here treated as a function of $\sigma, \tau, x, y, z, \chi$: and if we put, as usual,

$$\delta^2\Omega = \delta\sigma\delta\frac{\delta\Omega}{\delta\sigma} + \delta\tau\delta\frac{\delta\Omega}{\delta\tau} + \delta\upsilon\delta\frac{\delta\Omega}{\delta\upsilon} + \delta x\delta\frac{\delta\Omega}{\delta x} + \delta y\delta\frac{\delta\Omega}{\delta y} + \delta z\delta\frac{\delta\Omega}{\delta z} + \delta\chi\delta\frac{\delta\Omega}{\delta\chi}, \qquad (\mathrm{V}^6)$$

and similarly

$$\delta^2\upsilon = \delta\sigma\delta\frac{\delta\upsilon}{\delta\sigma} + \delta\tau\delta\frac{\delta\upsilon}{\delta\tau} + \delta x\delta\frac{\delta\upsilon}{\delta x} + \delta y\delta\frac{\delta\upsilon}{\delta y} + \delta z\delta\frac{\delta\upsilon}{\delta z} + \delta\chi\delta\frac{\delta\upsilon}{\delta\chi}, \qquad (\mathrm{W}^6)$$

we find

$$\begin{aligned}\delta^2\Omega = &-\lambda\delta^2\upsilon \\ &+ 2\delta\lambda.\left(\delta\upsilon - \frac{\delta\upsilon}{\delta\sigma}\delta\sigma - \frac{\delta\upsilon}{\delta\tau}\delta\tau - \frac{\delta\upsilon}{\delta x}\delta x - \frac{\delta\upsilon}{\delta y}\delta y - \frac{\delta\upsilon}{\delta z}\delta z - \frac{\delta\upsilon}{\delta\chi}\delta\chi\right) \\ &- \lambda^3\left(\sigma^2\frac{\delta^2\upsilon}{\delta\sigma^2} + 2\sigma\tau\frac{\delta^2\upsilon}{\delta\sigma\delta\tau} + \tau^2\frac{\delta^2\upsilon}{\delta\tau^2}\right)\left(\delta\upsilon - \frac{\delta\upsilon}{\delta\sigma}\delta\sigma - \&c.\right)^2, \qquad (\mathrm{X}^6)\end{aligned}$$

in which

$$\delta\lambda = \lambda^2\left(\sigma\delta\frac{\delta\upsilon}{\delta\sigma} + \tau\delta\frac{\delta\upsilon}{\delta\tau} - \frac{\delta\upsilon}{\delta x}\delta x - \frac{\delta\upsilon}{\delta y}\delta y - \frac{\delta\upsilon}{\delta z}\delta z - \frac{\delta\upsilon}{\delta\chi}\delta\chi\right), \qquad (\mathrm{Y}^6)$$

and which is equivalent to twenty-eight expressions for the partial differential coefficients of Ω of the second order: it gives, for example,

$$\frac{\delta^2\Omega}{\delta\upsilon^2} = \frac{\sigma^2\frac{\delta^2\upsilon}{\delta\sigma^2} + 2\sigma\tau\frac{\delta^2\upsilon}{\delta\sigma\delta\tau} + \tau^2\frac{\delta^2\upsilon}{\delta\tau^2}}{\left(\sigma\frac{\delta\upsilon}{\delta\sigma} + \tau\frac{\delta\upsilon}{\delta\tau} - \upsilon\right)^3}. \qquad (\mathrm{Z}^6)$$

And since the forms of the connected functions Ω, υ, r, of which each expresses the optical properties of the final medium, may be deduced, by the method of the second number, from the form of the characteristic function V, it evident that their partial differential coefficients also, of all orders, are not only related to each other, but may be deduced from the coefficients of that one characteristic function.

General Formula for Reflection or Refraction, Ordinary or Extraordinary. Changes of V, W, T. The Difference ΔV is $= 0$; $\Delta W = \Delta T =$ a Homogeneous Function of the First Dimension of the Differences $\Delta\sigma$, $\Delta\tau$, $\Delta\upsilon$, depending on the Shape and Position of the Reflecting or Refracting Surface. Theorem of Maxima and Minima, for the Elimination of the Incident Variables. Combinations of Reflectors or Refractors. Compound and Component Combinations.

11. Let us now endeavour to improve our theory of the characteristic and related functions, by applying the methods of the present memoir to improve the determina-

tion given in the First Supplement, of the sudden changes produced in these functions and in their coefficients, by reflexion or refraction, ordinary or extraordinary.

The general formula of such changes, which easily results from the nature of the characteristic function V, is

$$0=\Delta V=V_2-V_1; \qquad (\mathrm{A}^7)$$

V_1, V_2, being the two successive forms of the function V, before and after the reflexion or refraction; and the final co-ordinates x, y, z, in these forms, being connected by the equation

$$0=u\,(x, y, z) \qquad (\mathrm{B}^7)$$

of the reflecting or refracting surface. The formula (A^7) may be differentiated any number of times with reference to the final and initial co-ordinates and the colour, attending to the relation (B^7); and such differentiation, combined with the properties of the final uniform or variable media, conducts to the general laws of reflexion and refraction, and to all the conditions necessary for determining the changes of the coefficients of V, and therefore also of the connected coefficients of W and T, as well as to the laws of change of the functions V, W, T, themselves.

Thus, for the first order, we have the general formula

$$\delta V_2-\delta V_1=\delta\Delta V=\lambda\delta u, \qquad (\mathrm{C}^7)$$

which, on account of the multiplier λ, and the definitions (E), resolves itself into the seven following,

$$\left.\begin{array}{l}\Delta\sigma=\lambda\dfrac{\delta u}{\delta x};\ \Delta\tau=\lambda\dfrac{\delta u}{\delta y};\ \Delta\upsilon=\lambda\dfrac{\delta u}{\delta z};\\ \Delta\sigma'=0;\ \Delta\tau'=0;\ \Delta\upsilon'=0;\ \Delta\dfrac{\delta V}{\delta\chi}=0:\end{array}\right\} \qquad (\mathrm{D}^7)$$

the symbol Δ referring, as in (A^7), to the finite changes produced at the surface (B^7), so that $\Delta\sigma$, $\Delta\tau$, $\Delta\upsilon$, denote the differences $\sigma_2-\sigma_1$, $\tau_2-\tau_1$, $\upsilon_2-\upsilon_1$, between the new and the old values of σ, τ, υ, that is of the partial differential coefficients of the first order, of the characteristic function V, taken with respect to the final co-ordinates. The three first of the equations (D^7) contain the general laws of the sudden reflexion or refraction of a straight or curved ray, ordinary or extraordinary; because, when combined with the equation of the form (F),

$$0=\Omega_2\,(\sigma_2, \tau_2, \upsilon_2, x, y, z, \chi), \qquad (\mathrm{E}^7)$$

which expresses the nature of the final medium, they suffice, in general, when that final medium is known, to determine, or at least to restrict to a finite variety, the new values $\sigma_2, \tau_2, \upsilon_2$, of the quantities σ, τ, υ, on which the direction of the reflected or refracted ray depends, if we know the old values $\sigma_1\ \tau_1\ \upsilon_1$, which depend on the direction of the incident ray and on the properties of the medium containing it, and if we know also χ, x, y, z, and the ratios of $\frac{\delta u}{\delta x}$, $\frac{\delta u}{\delta y}$, $\frac{\delta u}{\delta z}$, that is the colour, the point of incidence, and the normal to the reflecting or refracting surface at that point. A remarkable case of indeterminateness, however, or rather two such cases, will appear, when we come to treat, in a future number, of external and internal conical refraction.

With respect to the new form V_2 of the characteristic function V, it is to be determined by the two following conditions; first, by the condition of satisfying, at the surface (B^7), the equation in finite differences (A^7), that is, by the condition of becoming equal to the value of the old form V_1, when the final co-ordinates x, y, z, are connected by the relation $u=0$; and secondly by the condition of satisfying, when the final co-ordinates are considered as arbitrary, the partial differential equation of the form (C),

$$0=\Omega_2\left(\frac{\delta V_2}{\delta x}, \frac{\delta V_2}{\delta y}, \frac{\delta V_2}{\delta z}, x, y, z, \chi\right), \qquad (\text{F}^7)$$

if the final medium be variable, or the simpler partial differential equation of the form (V'), if that final medium be uniform. And as it has been already shown that the partial differential equations relative to the characteristic function V, may be transformed, and in the case of uniform media integrated, by the help of the auxiliary functions W, T, it is useful to consider here the changes of those auxiliary functions, which are also otherwise interesting.

It easily follows from the definitions of W, T, that the increments of these two functions, acquired in reflexion or refraction, are equal to each other, and may be thus expressed,

$$\Delta W=\Delta T=x\Delta\sigma+y\Delta\tau+z\Delta\upsilon. \qquad (\text{G}^7)$$

And because the differences $\Delta\sigma$, $\Delta\tau$, $\Delta\upsilon$, are, by the general equations of reflexion or refraction (D^7), proportional to $\frac{\delta u}{\delta x}$, $\frac{\delta u}{\delta y}$, $\frac{\delta u}{\delta z}$, we may consider these differences as equal to the projections, on the rectangular axes of the co-ordinates x, y, z, of a straight line $= \surd(\Delta\sigma^2+\Delta\tau^2+\Delta\upsilon^2)$, perpendicular to the reflecting or refracting surface at the point of incidence, and making with the axes of co-ordinates angles of which the cosines may be called n_x, n_y, n_z; in such a manner that we shall have

$$\left.\begin{aligned}\Delta\sigma&=n_x \surd(\Delta\sigma^2+\Delta\tau^2+\Delta\upsilon^2);\\ \Delta\tau&=n_y \surd(\Delta\sigma^2+\Delta\tau^2+\Delta\upsilon^2);\\ \Delta\upsilon&=n_z \surd(\Delta\sigma^2+\Delta\tau^2+\Delta\upsilon^2);\\ \Delta W=\Delta T&=(xn_x+yn_y+zn_z)\surd(\Delta\sigma^2+\Delta\tau^2+\Delta\upsilon^2).\end{aligned}\right\} \qquad (\text{H}^7)$$

Now the quantity $xn_x+yn_y+zn_z$ is equal, abstracting from sign, to the perpendicular let fall from the origin of co-ordinates on the plane which touches the reflecting or refracting surface at the point of incidence; it is therefore constant if that surface be plane, and in general it may be considered as a function of the ratios of $\Delta\sigma$, $\Delta\tau$, $\Delta\upsilon$, because when those ratios are given we know the direction of the normal, and therefore, if the surface be curved and given, we know the point of incidence, or at least can in general restrict that point to a finite number of positions: we have therefore in general

$$\Delta W = \Delta T = f(\Delta\sigma, \Delta\tau, \Delta\upsilon), \qquad (\mathrm{I}^7)$$

the function f being homogeneous of the first dimension, and depending for its form on the shape and position of the reflecting or refracting surface, from the equation (B^7) of which surface it is to be deduced, by eliminating $x\ y\ z\ \lambda$ between the equations (B^7) (G^7) and the three first of those marked (D^7). We have also

$$\left.\begin{array}{l} \dfrac{f}{\Delta\upsilon} = \phi\left(\dfrac{\Delta\sigma}{\Delta\upsilon}, \dfrac{\Delta\tau}{\Delta\upsilon}\right); \ \dfrac{\Delta\sigma}{\Delta\upsilon} = -\dfrac{\delta z}{\delta x}; \ \dfrac{\Delta\tau}{\Delta\upsilon} = -\dfrac{\delta z}{\delta y}; \\ z - x\dfrac{\delta z}{\delta x} - y\dfrac{\delta z}{\delta y} = \phi\left(-\dfrac{\delta z}{\delta x}, -\dfrac{\delta z}{\delta y}\right); \end{array}\right\} \qquad (\mathrm{K}^7)$$

the form therefore of the homogeneous function f may easily be deduced from the equation of the surface (B^7), by so preparing that equation as to express $z - x\dfrac{\delta z}{\delta x} - y\dfrac{\delta z}{\delta y}$ as a function ϕ of $-\dfrac{\delta z}{\delta x}$, $-\dfrac{\delta z}{\delta y}$, which function ϕ reduces itself to a constant when the surface is plane: and we have a simple expression for the variation of the homogeneous function f, namely

$$\delta f = x\delta\Delta\sigma + y\delta\Delta\tau + z\delta\Delta\upsilon, \qquad (\mathrm{L}^7)$$

which, when the reflecting or refracting surface is curved, resolves itself into the following remarkable expressions for the co-ordinates of the point of incidence,

$$x = \frac{\delta f}{\delta\Delta\sigma}, \ y = \frac{\delta f}{\delta\Delta\tau}, \ z = \frac{\delta f}{\delta\Delta\upsilon}; \qquad (\mathrm{M}^7)$$

so that these co-ordinates, which, for a curved surface, we knew before to be functions of the ratios $\Delta\sigma$, $\Delta\tau$, $\Delta\upsilon$, are now seen to be, for such a surface, the partial differential coefficients of the homogeneous function f. When the surface (B^7) is plane, the differences $\Delta\sigma$, $\Delta\tau$, $\Delta\upsilon$, are no longer independent, since their ratios are then given; and although the expression (L^7) for δf still holds, it no longer resolves itself into the three equations (M^7).

Having thus studied some of the chief properties of the common increment f, which the functions W, T, receive, in the act of reflexion or refraction, we are prepared to investigate the new forms W_2, T_2, of these functions W, T, considered as depending on the new quantities σ_2, τ_2, υ_2, instead of the old σ_1, τ_1, υ_1. For this purpose we have first the equations

$$\left.\begin{array}{l} W_2 = W_1 + f(\sigma_2 - \sigma_1, \tau_2 - \tau_1, \upsilon_2 - \upsilon_1), \\ T_2 = T_1 + f(\sigma_2 - \sigma_1, \tau_2 - \tau_1, \upsilon_2 - \upsilon_1), \end{array}\right\} \qquad (\mathrm{N}^7)$$

by which W_2, T_2, at the reflecting or refracting surface, are expressed as explicit functions of $\sigma_1\ \tau_1\ \upsilon_1\ \sigma_2\ \tau_2\ \upsilon_2$; the expression of W_2 involving also $x'\ y'\ z'\ \chi$, and the expression of T_2 involving $\sigma'\ \tau'\ \upsilon'\ \chi$: and to eliminate from these expressions the incident quantities $\sigma_1\ \tau_1\ \upsilon_1$ we have, if the surface be curved, the following equations, in which the symbol $\delta_{\sigma_1, \tau_1, \upsilon_1}$ refers to the variations of those incident quantities,

$$\left.\begin{array}{ll} & \delta_{\sigma_1, \tau_1, \upsilon_1} f = -x\delta\sigma_1 - y\delta\tau_1 - z\delta\upsilon_1 \\ & \quad = -\delta_{\sigma_1, \tau_1, \upsilon_1} . W_1 = -\delta_{\sigma_1, \tau_1, \upsilon_1} . T_1 \,; \\ \text{and } \because & \delta_{\sigma_1, \tau_1, \upsilon_1} . W_2 = 0 \,; \;\; \delta_{\sigma_1, \tau_1, \upsilon_1} . T_2 = 0 \,; \end{array}\right\} \qquad (O^7)$$

we are therefore to disengage the incident quantities from the expressions for W_2, T_2, by making each of those expressions a maximum or minimum with respect to those quantities, attending to the relation $\Omega_1 = 0$, between them; the phrase *maximum or minimum* being employed with the usual latitude. For the case of a plane surface this method of elimination fails, the form of f becoming indeterminate, on account of the constant ratios which then exist, by (K^7) or (D^7), between $\Delta\sigma$, $\Delta\tau$, $\Delta\upsilon$; but these very ratios, combined with the relation $\Omega_1 = 0$, between the quantities $\sigma_1\ \tau_1\ \upsilon_1$, enable us in this case to eliminate those quantities from W_2, T_2. And when we have thus determined the new forms W_2, T_2, of the functions W, T, for the points of the reflecting or refracting surface, we may extend these forms to the other points of the final medium, if that medium be uniform, because then the final rays are straight, and for any one such ray the quantities $\sigma_2\ \tau_2\ \upsilon_2\ W_2\ T_2$ are constant; but if the final medium be variable, then the final rays are curved, and the general forms of W_2, T_2, for arbitrary points of the medium, are to be determined by combinations of partial differential equations and equations in finite differences, analogous to the combinations of such equations for V_2, and easily deduced from the principles already laid down.

It is easy to extend the foregoing remarks to any combination of reflexions or refractions, and to show, for example, that in the case of any combination of uniform media, producing any system of polygon rays, ordinary or extraordinary, the auxiliary function T is equal to the following expression,

$$T = \Sigma f\,(\Delta\sigma, \Delta\tau, \Delta\upsilon), \qquad (P^7)$$

that is, to the sum of all the homogeneous functions f of the differences of the quantities σ, τ, υ, obtained by considering the successive reflecting or refracting surfaces: from which expression the intermediate quantities of the form σ, τ, υ, are to be eliminated by making the expression a maximum or minimum with respect to those intermediate quantities, attending to the relations between them which result from the properties of the media, and using, for plane surfaces, the other method of elimination, founded on the ratios of $\Delta\sigma$, $\Delta\tau$, $\Delta\upsilon$. And when the function T is known, we can deduce from it, by the methods of the fourth number, the other auxiliary function W, and the characteristic function V.

In general for all optical combinations, whether with uniform or with variable media, we have, by the definitions of the functions V, W, T, and by the results of former numbers, the following expressions,

$$\left.\begin{aligned} V = \int_0^s v ds; \quad T &= \int_0^s \left(x \frac{\delta v}{\delta x} + y \frac{\delta v}{\delta y} + z \frac{\delta v}{\delta z} \right) ds \, ; \\ W = x'\sigma' + y'\tau' + z'\upsilon' &+ \int_0^s \left(x \frac{\delta v}{\delta x} + y \frac{\delta v}{\delta y} + z \frac{\delta v}{\delta z} \right) ds : \end{aligned}\right\} \qquad (\mathrm{Q}^7)$$

ds being, as before, the element of the curved or polygon ray; and hence it follows that if we consider any total combination, of $m + n - 1$ media, whether uniform or variable, as resulting from two partial combinations, of m and of n media respectively, combined so that the last medium of the one partial combination (m) is the first of the other partial combination (n), and so that the final rays of the one partial combination are the initial rays of the other, then the functions V, T, (but not in general W,) for the total combination, are the sums of the corresponding functions for the partial combinations: it follows also from the general expressions for the variations of these functions, that the intermediate variables, belonging to the last medium of the first partial combination, or to the first medium of the second, are to be eliminated from the sum, by the condition of making that sum a maximum or minimum with respect to them. Analogous remarks apply to compound combinations, composed of more than two component combinations. These properties of the functions V, T, for total or resultant combinations, will be found useful in the theory of double and triple object-glasses, and other compound optical instruments.

Changes of the Coefficients of the Second Order, of V, W, T, produced by Reflexion or Refraction.

12. With respect to the changes produced by reflexion or refraction in the coefficients of the second order, of the characteristic function V, and therefore also of the connected functions W, T, they may be deduced from the following formula, analogous to (C^7),

$$\delta^2 \Delta V = \delta^2 . \lambda u = \lambda \delta^2 u + 2 \delta\lambda \delta u \, ; \qquad (\mathrm{R}^7)$$

u, λ, having the same meanings as in (B^7) (C^7); and the multiplier λ, which was introduced also in the First Supplement, and was there regarded as a function of the final co-ordinates x, y, z, being now considered as involving also the initial co-ordinates x', y', z', and the chromatic index χ. The seven variations δx, δy, δz, $\delta x'$, $\delta y'$,

$\delta z'$, $\delta\chi$, may be treated as independent in (R^7), if we assign a proper value to $\delta\lambda$, as a linear function of these seven variations; so that we may deduce from (R^7) the seven following equations,

$$\left.\begin{array}{l}
\Delta\delta\,\frac{\delta V}{\delta x} = \lambda\delta\,\frac{\delta u}{\delta x} + \frac{\delta\lambda}{\delta x}\,\delta u + \frac{\delta u}{\delta x}\,\delta\lambda\,; \\
\Delta\delta\,\frac{\delta V}{\delta y} = \lambda\delta\,\frac{\delta u}{\delta y} + \frac{\delta\lambda}{\delta y}\,\delta u + \frac{\delta u}{\delta y}\,\delta\lambda\,; \\
\Delta\delta\,\frac{\delta V}{\delta z} = \lambda\delta\,\frac{\delta u}{\delta z} + \frac{\delta\lambda}{\delta z}\,\delta u + \frac{\delta u}{\delta z}\,\delta\lambda\,; \\
\Delta\delta\,\frac{\delta V}{\delta x'} = \frac{\delta\lambda}{\delta x'}\,\delta u\,;\ \Delta\delta\,\frac{\delta V}{\delta y'} = \frac{\delta\lambda}{\delta y'}\delta u\,;\ \Delta\delta\,\frac{\delta V}{\delta z'} = \frac{\delta\lambda}{\delta z'}\,\delta u\,; \\
\Delta\delta\,\frac{\delta V}{\delta\chi} = \frac{\delta\lambda}{\delta\chi}\,\delta u:
\end{array}\right\} \qquad (\mathrm{S}^7)$$

of which each may again be decomposed into seven others. But of the forty-nine expressions thus obtained for the changes of the twenty-eight coefficients of V of the second order, only twenty-eight expressions are distinct; and these involve seven multipliers as yet unknown, namely, the seven partial differential coefficients of λ: however we can determine these seven multipliers, and the twenty-eight coefficients of V_2 of the second order, by introducing the seven additional equations obtained by differentiating the partial differential equation (F^7), with respect to $x\ y\ z\ x'\ y'\ z'\ \chi$.

The differential of the equation (F^7), is

$$0 = \frac{\delta\Omega_2}{\delta\sigma_2}\,\delta\,\frac{\delta V_2}{\delta x} + \frac{\delta\Omega_2}{\delta\tau_2}\,\delta\,\frac{\delta V_2}{\delta y} + \frac{\delta\Omega_2}{\delta\upsilon_2}\,\delta\,\frac{\delta V_2}{\delta z} + \frac{\delta\Omega_2}{\delta x}\delta x + \frac{\delta\Omega_2}{\delta y}\delta y + \frac{\delta\Omega_2}{\delta z}\delta z + \frac{\delta\Omega_2}{\delta\chi}\delta\chi\,; \qquad (\mathrm{T}^7)$$

and this, when combined with the three first equations (S^7), conducts to the following formula,

$$\begin{array}{l}
0 = \frac{\delta\Omega_2}{\delta\sigma_2}\,\delta\,\frac{\delta V_1}{\delta x} + \frac{\delta\Omega_2}{\delta\tau_2}\,\delta\,\frac{\delta V_1}{\delta y} + \frac{\delta\Omega_2}{\delta\upsilon_2}\,\delta\,\frac{\delta V_1}{\delta z} + \frac{\delta\Omega_2}{\delta x}\delta x + \frac{\delta\Omega_2}{\delta y}\delta y + \frac{\delta\Omega_2}{\delta z}\delta z + \frac{\delta\Omega_2}{\delta\chi}\,\delta\chi \\
\quad + \lambda\ \left(\frac{\delta\Omega_2}{\delta\sigma_2}\,\delta\,\frac{\delta u}{\delta x} + \frac{\delta\Omega_2}{\delta\tau_2}\,\delta\,\frac{\delta u}{\delta y} + \frac{\delta\Omega_2}{\delta\upsilon_2}\,\delta\,\frac{\delta u}{\delta z}\right) \\
\quad + \delta u.\left(\frac{\delta\Omega_2}{\delta\sigma_2}\,\frac{\delta\lambda}{\delta x} + \frac{\delta\Omega_2}{\delta\tau_2}\,\frac{\delta\lambda}{\delta y} + \frac{\delta\Omega_2}{\delta\upsilon_2}\,\frac{\delta\lambda}{\delta z}\right) \\
\quad + \delta\lambda.\left(\frac{\delta\Omega_2}{\delta\sigma_2}\,\frac{\delta u}{\delta x} + \frac{\delta\Omega_2}{\delta\tau_2}\,\frac{\delta u}{\delta y} + \frac{\delta\Omega_2}{\delta\upsilon_2}\,\frac{\delta u}{\delta z}\right)\,; \qquad (\mathrm{U}^7)
\end{array}$$

which resolves itself into seven separate equations, sufficient to determine the seven multipliers

$$\frac{\delta\lambda}{\delta x},\ \frac{\delta\lambda}{\delta y},\ \frac{\delta\lambda}{\delta z},\ \frac{\delta\lambda}{\delta x'},\ \frac{\delta\lambda}{\delta y'},\ \frac{\delta\lambda}{\delta z'},\ \frac{\delta\lambda}{\delta\chi}.$$

Three of these seven equations into which (U^7) resolves itself, give, by a proper combination, a value for the trinomial

$$\frac{\delta\Omega_2}{\delta\sigma_2}\,\frac{\delta\lambda}{\delta x}+\frac{\delta\Omega_2}{\delta\tau_2}\,\frac{\delta\lambda}{\delta y}+\frac{\delta\Omega_2}{\delta\upsilon_2}\,\frac{\delta\lambda}{\delta z},$$

which enables us to eliminate that trinomial from (U^7) and so to deduce a value for $\delta\lambda$, which being combined with (R^7) gives,

$$\left\{\begin{array}{l}
\left(\frac{\delta\Omega_2}{\delta\sigma_2}\right)^2\left(\frac{\delta^2 V_1}{\delta x^2}+\lambda\,\frac{\delta^2 u}{\delta x^2}\right)+2\,\frac{\delta\Omega_2}{\delta\tau_2}\,\frac{\delta\Omega_2}{\delta\upsilon_2}\left(\frac{\delta^2 V_1}{\delta y\delta z}+\lambda\,\frac{\delta^2 u}{\delta y\delta z}\right)\\
+\left(\frac{\delta\Omega_2}{\delta\tau_2}\right)^2\left(\frac{\delta^2 V_1}{\delta y^2}+\lambda\,\frac{\delta^2 u}{\delta y^2}\right)+2\,\frac{\delta\Omega_2}{\delta\upsilon_2}\,\frac{\delta\Omega_2}{\delta\sigma_2}\left(\frac{\delta^2 V_1}{\delta z\delta x}+\lambda\,\frac{\delta^2 u}{\delta z\delta x}\right)\\
+\left(\frac{\delta\Omega_2}{\delta\upsilon_2}\right)^2\left(\frac{\delta^2 V_1}{\delta z^2}+\lambda\,\frac{\delta^2 u}{\delta z^2}\right)+2\,\frac{\delta\Omega_2}{\delta\sigma_2}\,\frac{\delta\Omega_2}{\delta\tau_2}\left(\frac{\delta^2 V_1}{\delta x\delta y}+\lambda\,\frac{\delta^2 u}{\delta x\delta y}\right)\\
+\frac{\delta\Omega_2}{\delta\sigma_2}\,\frac{\delta\Omega_2}{\delta x}+\frac{\delta\Omega_2}{\delta\tau_2}\,\frac{\delta\Omega_2}{\delta y}+\frac{\delta\Omega_2}{\delta\upsilon_2}\,\frac{\delta\Omega_2}{\delta z}.
\end{array}\right\}.\delta u^2$$

$$-\,2\left(\frac{\delta\Omega_2}{\delta\sigma_2}\,\frac{\delta u}{\delta x}+\frac{\delta\Omega_2}{\delta\tau_2}\,\frac{\delta u}{\delta y}+\frac{\delta\Omega_2}{\delta\upsilon_2}\,\frac{\delta u}{\delta z}\right)\delta u$$

$$\times\left\{\begin{array}{l}
\frac{\delta\Omega_2}{\delta\sigma_2}\left(\delta\,\frac{\delta V_1}{\delta x}+\lambda\delta\frac{\delta u}{\delta x}\right)+\frac{\delta\Omega_2}{\delta\tau_2}\left(\delta\,\frac{\delta V_1}{\delta y}+\lambda\delta\frac{\delta u}{\delta y}\right)+\frac{\delta\Omega_2}{\delta\upsilon_2}\left(\delta\,\frac{\delta V_1}{\delta z}+\lambda\delta\frac{\delta u}{\delta z}\right)\\
+\frac{\delta\Omega_2}{\delta x}\,\delta x+\frac{\delta\Omega_2}{\delta y}\,\delta y+\frac{\delta\Omega_2}{\delta z}\,\delta z+\frac{\delta\Omega_2}{\delta\chi}\,\delta\chi
\end{array}\right\}$$

$$=(\delta^2 V_2-\delta^2 V_1-\lambda\delta^2 u)\left(\frac{\delta\Omega_2}{\delta\sigma_2}\,\frac{\delta u}{\delta x}+\frac{\delta\Omega_2}{\delta\tau_2}\,\frac{\delta u}{\delta y}+\frac{\delta\Omega_2}{\delta\upsilon_2}\,\frac{\delta u}{\delta z}\right)^2: \qquad (V^7)$$

a formula that is equivalent to twenty-eight separate expressions for the twenty-eight coefficients of V_2, of the second order. This formula supposes the rays to be reflected or refracted into a variable medium; but it can be adapted to the simpler supposition of reflexion or refraction into an uniform medium, by merely making the quantities $\frac{\delta\Omega_2}{\delta x}$, $\frac{\delta\Omega_2}{\delta y}$, $\frac{\delta\Omega_2}{\delta z}$, vanish. Whether the last medium be variable or uniform, the formula (V^7) gives,

$$\delta'^2 V_2=\delta'^2 V_1; \qquad (W^7)$$

δ' referring, as in former numbers of this Supplement, to the variations of x', y', z', χ, alone, that is, to the variations of the initial co-ordinates and of the colour; and the final co-ordinates $x\,y\,z$ being those of any point on the reflecting or refracting surface. Thus the ten differential coefficients, of the second order, of the characteristic function V, like the four of the first order, taken with respect to the initial co-ordinates and the colour, undergo no sudden change by reflexion or refraction; but the differential coefficients of both orders, which involve the final co-ordinates, take suddenly new values which we have shown how to determine: and from these new coefficients of V, we can deduce those of W and T, by the methods of the foregoing numbers. The coefficients thus found, of W_2 and T_2, remain unchanged through the whole extent of the last reflected or refracted portion of the ray, when this last portion is straight, the final medium being uniform; but the coefficients of V_2, of the second order, change gradually in passing from one point to another, even of this straight portion, according to laws deducible from their connexion, already explained, with the constant coefficients of W_2.

The coefficients of W_2 and T_2 of the second and higher orders, may also be calculated, whether the last medium be uniform or variable, by differentiating the expressions (N^7), and eliminating the variations of $\sigma_1\ \tau_1\ \upsilon_1$ by the help of the conditions already mentioned, of maximum or minimum.

Another method of calculating the changes produced in the partial differential coefficients of V of the second order, by reflexion or refraction, ordinary or extraordinary, into a medium uniform or variable, is to develope the second differential of the general formula (A^7), considering ΔV as a function of the seven variables $x, y, z, x', y', z', \chi$, and considering x, y, z, as themselves functions of two independent variables; for example, considering z as a function of x, y, of which the form is determined by the equation of the reflecting or refracting surface. In this manner we obtain, besides the formula (W^7), which is equivalent to ten equations, the eleven following;

$$\left.\begin{array}{l}
0=\frac{\delta^2\Delta V}{\delta x^2}+2\frac{\delta^2\Delta V}{\delta x\delta z}\frac{\delta z}{\delta x}+\frac{\delta^2\Delta V}{\delta z^2}\left(\frac{\delta z}{\delta x}\right)^2+\frac{\delta\Delta V}{\delta z}\frac{\delta^2 z}{\delta x^2};\\
0=\frac{\delta^2\Delta V}{\delta y^2}+2\frac{\delta^2\Delta V}{\delta y\delta z}\frac{\delta z}{\delta y}+\frac{\delta^2\Delta V}{\delta z^2}\left(\frac{\delta z}{\delta y}\right)^2+\frac{\delta\Delta V}{\delta z}\frac{\delta^2 z}{\delta y^2};\\
0=\frac{\delta^2\Delta V}{\delta x\delta y}+\frac{\delta^2\Delta V}{\delta x\delta z}\frac{\delta z}{\delta y}+\frac{\delta^2\Delta V}{\delta y\delta z}\frac{\delta z}{\delta x}+\frac{\delta^2\Delta V}{\delta z^2}\frac{\delta z}{\delta x}\frac{\delta z}{\delta y}+\frac{\delta\Delta V}{\delta z}\frac{\delta^2 z}{\delta x\delta y};\\
0=\frac{\delta^2\Delta V}{\delta x\delta x'}+\frac{\delta^2\Delta V}{\delta z\delta x'}\frac{\delta z}{\delta x};\quad 0=\frac{\delta^2\Delta V}{\delta y\delta x'}+\frac{\delta^2\Delta V}{\delta z\delta x'}\frac{\delta z}{\delta y};\\
0=\frac{\delta^2\Delta V}{\delta x\delta y'}+\frac{\delta^2\Delta V}{\delta z\delta y'}\frac{\delta z}{\delta x};\quad 0=\frac{\delta^2\Delta V}{\delta y\delta y'}+\frac{\delta^2\Delta V}{\delta z\delta y'}\frac{\delta z}{\delta y};\\
0=\frac{\delta^2\Delta V}{\delta x\delta z'}+\frac{\delta^2\Delta V}{\delta z\delta z'}\frac{\delta z}{\delta x};\quad 0=\frac{\delta^2\Delta V}{\delta y\delta z'}+\frac{\delta^2\Delta V}{\delta z\delta z'}\frac{\delta z}{\delta y};\\
0=\frac{\delta^2\Delta V}{\delta x\delta\chi}+\frac{\delta^2\Delta V}{\delta z\delta\chi}\frac{\delta z}{\delta x};\quad 0=\frac{\delta^2\Delta V}{\delta y\delta\chi}+\frac{\delta^2\Delta V}{\delta z\delta\chi}\frac{\delta z}{\delta y};
\end{array}\right\}\quad (X^7)$$

which may be put under the form

$$\left.\begin{array}{l}
0=\Delta\left\{\frac{\delta^2 V}{\delta x^2}+2\frac{\delta^2 V}{\delta x\delta z}\frac{\delta z}{\delta x}+\frac{\delta^2 V}{\delta z^2}\left(\frac{\delta z}{\delta x}\right)^2+\frac{\delta V}{\delta z}\frac{\delta^2 z}{\delta x^2}\right\};\\
\&c.;
\end{array}\right\}\quad (Y^7)$$

and are deduced by differentiation from the analogous equations of the first order

$$0=\Delta\left(\frac{\delta V}{\delta x}+\frac{\delta V}{\delta z}\frac{\delta z}{\delta x}\right);\quad 0=\Delta\left(\frac{\delta V}{\delta y}+\frac{\delta V}{\delta z}\frac{\delta z}{\delta y}\right). \qquad (\mathrm{Z}^7)$$

And the eleven equations thus deduced, when combined with the ten given by (W^7), and with the seven into which (T^7) resolves itself, suffice, in general, to determine the twenty-eight coefficients of V_2 of the second order.

Changes produced by Transformation of Co-ordinates. Nearly all the foregoing Results may be extended to Oblique Co-ordinates. The Fundamental Formula may be presented so as to extend even to Polar or any other marks of position; and new Auxiliary Functions may then be found, analogous to, and including, the Functions W, T: together with New and General Differential and Integral Equations for Curved and Polygon Rays, Ordinary or Extraordinary.

13. In all the foregoing investigations, it has been supposed that the final and initial co-ordinates, x, y, z, x', y', z', were referred to one common set of rectangular axes. But since it may be often convenient to change the mode of marking the final and initial positions, let us now express the old rectangular co-ordinates as linear functions of new and more general co-ordinates $x_{,}$, $y_{,}$, $z_{,}$, and $x'_{,}$, $y'_{,}$, $z'_{,}$, which may or may not be rectangular, and may or may not be referred to one common set of final or initial axes. For this purpose we may employ the following formulæ,

$$\left.\begin{array}{l}
x=x_{\circ}+x_{x_{,}}\,x_{,}+x_{y_{,}}\,y_{,}+x_{z_{,}}\,z_{,};\\
y=y_{\circ}+y_{x_{,}}\,x_{,}+y_{y_{,}}\,y_{,}+y_{z_{,}}\,z_{,};\\
z=z_{\circ}+z_{x_{,}}\,x_{,}+z_{y_{,}}\,y_{,}+z_{z_{,}}\,z_{,};\\
x'=x'_{\circ}+x'_{x'_{,}}\,x'_{,}+x'_{y'_{,}}\,y'_{,}+x'_{z'_{,}}\,z'_{,};\\
y'=y'_{\circ}+y'_{x'_{,}}\,x'_{,}+y'_{y'_{,}}\,y'_{,}+y'_{z'_{,}}\,z'_{,};\\
z'=z'_{\circ}+z'_{x'_{,}}\,x'_{,}+z'_{y'_{,}}\,y'_{,}+z'_{z'_{,}}\,z'_{,};
\end{array}\right\} \qquad (\mathrm{A}^8)$$

in which each of the eighteen coefficients of the form $x_{x_{,}}$ is the cosine of the angle between the directions of the two corresponding semiaxes, so that these coefficients are connected by the six following relations, on account of the rectangularity of the old co-ordinates,

$$\left.\begin{array}{ll}
x_{x_{,}}^{2}+y_{x_{,}}^{2}+z_{x_{,}}^{2}=1; & x'^{2}_{x'_{,}}+y'^{2}_{x'_{,}}+z'^{2}_{x'_{,}}=1;\\
x_{y_{,}}^{2}+y_{y_{,}}^{2}+z_{y_{,}}^{2}=1; & x'^{2}_{y'_{,}}+y'^{2}_{y'_{,}}+z'^{2}_{y'_{,}}=1;\\
x_{z_{,}}^{2}+y_{z_{,}}^{2}+z_{z_{,}}^{2}=1; & x'^{2}_{z'_{,}}+y'^{2}_{z'_{,}}+z'^{2}_{z'_{,}}=1.
\end{array}\right\} \qquad (\mathrm{B}^8)$$

Let us also establish, according to the analogy of our former notation, the following definitions similar to (P),

$$\left.\begin{array}{l} \alpha_{,} = \dfrac{dx_{,}}{ds}\,,\ \beta_{,} = \dfrac{dy_{,}}{ds}\,,\ \gamma_{,} = \dfrac{dz_{,}}{ds}\,, \\ \alpha'_{,} = \dfrac{dx'_{,}}{ds'}\,,\ \beta'_{,} = \dfrac{dy'_{,}}{ds'}\,,\ \gamma'_{,} = \dfrac{dz'_{,}}{ds'}\,, \end{array}\right\} \quad (C^8)$$

and the following, similar to (E),

$$\left.\begin{array}{l} \sigma_{,} = \dfrac{\delta V}{\delta x_{,}}\,,\ \tau_{,} = \dfrac{\delta V}{\delta y_{,}}\,,\ \upsilon_{,} = \dfrac{\delta V}{\delta z_{,}}\,, \\ \sigma'_{,} = -\dfrac{\delta V}{\delta x'_{,}}\,,\ \tau'_{,} = -\dfrac{\delta V}{\delta y'_{,}}\,,\ \upsilon'_{,} = -\dfrac{\delta V}{\delta z'_{,}}\,: \end{array}\right\} \quad (D^8)$$

we shall then have

$$\left.\begin{array}{l} \alpha = \alpha_{,}\, x_{x_{,}} + \beta_{,}\, x_{y_{,}} + \gamma_{,}\, x_{z_{,}}\,; \\ \beta = \alpha_{,}\, y_{x_{,}} + \beta_{,}\, y_{y_{,}} + \gamma_{,}\, y_{z_{,}}\,; \\ \gamma = \alpha_{,}\, z_{x_{,}} + \beta_{,}\, z_{y_{,}} + \gamma_{,}\, z_{z_{,}}\,; \\ \alpha' = \alpha'_{,}\, x'_{x'_{,}} + \beta'_{,}\, x'_{y'_{,}} + \gamma'_{,}\, x'_{z'_{,}}\,; \\ \beta' = \alpha'_{,}\, y'_{x'_{,}} + \beta'_{,}\, y'_{y'_{,}} + \gamma'_{,}\, y'_{z'_{,}}\,; \\ \gamma' = \alpha'_{,}\, z'_{x'_{,}} + \beta'_{,}\, z'_{y'_{,}} + \gamma'_{,}\, z'_{z'_{,}}\,; \end{array}\right\} \quad (E^8)$$

and

$$\left.\begin{array}{ll} \sigma_{,} = \sigma x_{x_{,}} + \tau y_{x_{,}} + \upsilon z_{x_{,}}\,; & \sigma'_{,} = \sigma' x'_{x'_{,}} + \tau' y'_{x'_{,}} + \upsilon' z'_{x'_{,}}\,; \\ \tau_{,} = \sigma x_{y_{,}} + \tau y_{y_{,}} + \upsilon z_{y_{,}}\,; & \tau'_{,} = \sigma' x'_{y'_{,}} + \tau' y'_{y'_{,}} + \upsilon' z'_{y'_{,}}\,; \\ \upsilon_{,} = \sigma x_{z_{,}} + \tau y_{z_{,}} + \upsilon z_{z_{,}}\,; & \upsilon'_{,} = \sigma' x'_{z'_{,}} + \tau' y'_{z'_{,}} + \upsilon' z'_{z'_{,}}\,. \end{array}\right\} \quad (F^8)$$

And if, by substituting in the former homogeneous medium-functions, v, v', the expressions (E^8) for α, β, γ, α', β', γ', we obtain v under a new form, as a homogeneous function of $\alpha_{,}$, $\beta_{,}$, $\gamma_{,}$, of the first dimension, and v' as a homogeneous function of the same dimension of $\alpha'_{,}$, $\beta'_{,}$, $\gamma'_{,}$, and then differentiate these new forms of v, v', with reference to their new variables, we find, by (E^8), the following relations between the new and the old coefficients,

$$\left.\begin{array}{l} \dfrac{\delta v}{\delta \alpha_{,}} = \dfrac{\delta v}{\delta \alpha}\, x_{x_{,}} + \dfrac{\delta v}{\delta \beta}\, y_{x_{,}} + \dfrac{\delta v}{\delta \gamma}\, z_{x_{,}}\,; \\ \dfrac{\delta v}{\delta \beta_{,}} = \dfrac{\delta v}{\delta \alpha}\, x_{y_{,}} + \dfrac{\delta v}{\delta \beta}\, y_{y_{,}} + \dfrac{\delta v}{\delta \gamma}\, z_{y_{,}}\,; \\ \dfrac{\delta v}{\delta \gamma_{,}} = \dfrac{\delta v}{\delta \alpha}\, x_{z_{,}} + \dfrac{\delta v}{\delta \beta}\, y_{z_{,}} + \dfrac{\delta v}{\delta \gamma}\, z_{z'}\,; \\ \dfrac{\delta v'}{\delta \alpha'_{,}} = \dfrac{\delta v'}{\delta \alpha'}\, x'_{x'_{,}} + \dfrac{\delta v'}{\delta \beta'}\, y'_{x'_{,}} + \dfrac{\delta v'}{\delta \gamma'}\, z'_{x'_{,}}\,; \\ \dfrac{\delta v'}{\delta \beta'_{,}} = \dfrac{\delta v'}{\delta \alpha'}\, x'_{y'_{,}} + \dfrac{\delta v'}{\delta \beta'}\, y'_{y'_{,}} + \dfrac{\delta v'}{\delta \gamma'}\, z'_{y'_{,}}\,; \\ \dfrac{\delta v'}{\delta \gamma'_{,}} = \dfrac{\delta v'}{\delta \alpha'}\, x'_{z'_{,}} + \dfrac{\delta v'}{\delta \beta'}\, y'_{z'_{,}} + \dfrac{\delta v'}{\delta \gamma'}\, z'_{z'_{,}}\,; \end{array}\right\} \quad (G^8)$$

from which relations, combined with (D^8) (F^8), and with the equations (B) (E), of the second number, we obtain the following generalisations of the equations (B),

$$\left.\begin{array}{l} \dfrac{\delta V}{\delta x_{,}} = \dfrac{\delta v}{\delta \alpha_{,}}\,; \ \dfrac{\delta V}{\delta y_{,}} = \dfrac{\delta v}{\delta \beta_{,}}\,; \ \dfrac{\delta V}{\delta z_{,}} = \dfrac{\delta v}{\delta \gamma_{,}}\,; \\ -\dfrac{\delta V}{\delta x'_{,}} = \dfrac{\delta v'}{\delta \alpha'_{,}}\,; \ -\dfrac{\delta V}{\delta y'_{,}} = \dfrac{\delta v'}{\delta \beta'_{,}}\,; \ -\dfrac{\delta V}{\delta z'_{,}} = \dfrac{\delta v'}{\delta \gamma'_{,}}\,: \end{array}\right\} \qquad (\mathrm{H}^{8})$$

and therefore the following important generalisation of the fundamental formula (A),

$$\delta V = \frac{\delta v}{\delta \alpha_{,}}\,\delta x_{,} - \frac{\delta v'}{\delta \alpha'_{,}}\,\delta x'_{,} + \frac{\delta v}{\delta \beta_{,}}\,\delta y_{,} - \frac{\delta v'}{\delta \beta'_{,}}\,\delta y'_{,} + \frac{\delta v}{\delta \gamma_{,}}\,\delta z_{,} - \frac{\delta v'}{\delta \gamma'_{,}}\,\delta z'_{,}\,, \qquad (\mathrm{I}^{8})$$

which is thus shown to extend to oblique co-ordinates, and not even to require that the initial should coincide with the final axes.

We may adapt nearly all the foregoing reasonings and results, of the present Supplement, to this more general view. We have, for example, partial differential equations of the first order in V, analogous to the equations (C), and of the form

$$\left.\begin{array}{l} 0 = \Omega_{,}\left(\dfrac{\delta V}{\delta x_{,}}, \dfrac{\delta V}{\delta y_{,}}, \dfrac{\delta V}{\delta z_{,}}, x_{,}, y_{,}, z_{,}, \chi \right), \\ 0 = \Omega'_{,}\left(-\dfrac{\delta V}{\delta x'_{,}}, -\dfrac{\delta V}{\delta y'_{,}}, -\dfrac{\delta V}{\delta z'_{,}}, x'_{,}, y'_{,}, z'_{,}, \chi \right), \end{array}\right\} \qquad (\mathrm{K}^{8})$$

which conduct to a partial differential equation of the second order, analogous to (D): and if we put the equations (K^8) under the form

$$\left.\begin{array}{l} 0 = \Omega_{,}(\sigma_{,}, \tau_{,}, \upsilon_{,}, x_{,}, y_{,}, z_{,}, \chi), \\ 0 = \Omega'_{,}(\sigma'_{,}, \tau'_{,}, \upsilon'_{,}, x'_{,}, y'_{,}, z'_{,}, \chi), \end{array}\right\} \qquad (\mathrm{L}^{8})$$

and suppose them so prepared, by the method indicated in the second number, that the function $\Omega_{,} + 1$ shall be homogeneous of the first dimension with respect to $\sigma_{,}, \tau_{,}, \upsilon_{,}$, and that $\Omega'_{,} + 1$ shall be homogeneous of the same dimension with respect to $\sigma'_{,}, \tau'_{,}, \upsilon'_{,}$, we shall have

$$\left.\begin{array}{l} \dfrac{\alpha_{,}}{v} = \dfrac{\delta \Omega_{,}}{\delta \sigma_{,}}, \ \dfrac{\beta_{,}}{v} = \dfrac{\delta \Omega_{,}}{\delta \tau_{,}}, \ \dfrac{\gamma_{,}}{v} = \dfrac{\delta \Omega_{,}}{\delta \upsilon_{,}}, \\ \dfrac{\alpha'_{,}}{v'} = \dfrac{\delta \Omega'_{,}}{\delta \sigma'_{,}}, \ \dfrac{\beta'_{,}}{v'} = \dfrac{\delta \Omega'_{,}}{\delta \tau'_{,}}, \ \dfrac{\gamma'_{,}}{v'} = \dfrac{\delta \Omega'_{,}}{\delta \upsilon'_{,}}, \end{array}\right\} \qquad (\mathrm{M}^{8})$$

with many other relations, analogous to those of the second number. The differential equations of a curved ray, ordinary or extraordinary, in the third number, may be generalised as follows,

$$\frac{d}{ds}\,\frac{\delta v}{\delta \alpha_{,}} = \frac{\delta v}{\delta x_{,}}\,; \ \frac{d}{ds}\,\frac{\delta v}{\delta \beta_{,}} = \frac{\delta v}{\delta y_{,}}\,; \ \frac{d}{ds}\,\frac{\delta v}{\delta \gamma_{,}} = \frac{\delta v}{\delta z_{,}}\,; \qquad (\mathrm{N}^{8})$$

and their integrals may be extended to oblique co-ordinates, under the form,

$$\frac{\delta V}{\delta x_{,}'} = \text{const.} ; \; \frac{\delta V}{\delta y_{,}'} = \text{const.} ; \; \frac{\delta V}{\delta z_{,}'} = \text{const.} : \qquad (\text{O}^8)$$

while, if the final portion of the ray be straight, we have also, for that final portion,

$$\frac{\delta V}{\delta x_{,}} = \text{const.} ; \; \frac{\delta V}{\delta y_{,}} = \text{const.} ; \; \frac{\delta V}{\delta z_{,}} = \text{const.} \qquad (\text{P}^8)$$

The formula (A^7) of reflexion or refraction, ordinary or extraordinary, namely,

$$\Delta V = 0,$$

extends to oblique co-ordinates; and if we introduce new auxiliary functions $W_{,}$, $T_{,}$, analogous to W, T, and defined by the new equations

$$\left. \begin{aligned} W_{,} &= - V + x_{,} \sigma_{,} + y_{,} \tau_{,} + z_{,} \upsilon_{,}, \\ T_{,} &= = W_{,} - x_{,}' \sigma_{,}' - y_{,}' \tau_{,}' - z_{,}' \upsilon_{,}', \end{aligned} \right\} \qquad (\text{Q}^8)$$

analogous to the definitions (D') (E'), and attend to the meanings and properties of the symbols $\sigma_{,}$ $\tau_{,}$ $\upsilon_{,}$ $\sigma_{,}'$ $\tau_{,}'$ $\upsilon_{,}'$, we shall obtain the following expressions for the variations of V, $W_{,}$, $T_{,}$,

$$\left. \begin{aligned} \delta V &= \sigma_{,} \delta x_{,} - \sigma_{,}' \delta x_{,}' + \tau_{,} \delta y_{,} - \tau_{,}' \delta y_{,}' + \upsilon_{,} \delta z_{,} - \upsilon_{,}' \delta z_{,}' + \frac{\delta V}{\delta \chi} \delta \chi ; \\ \delta W_{,} &= x_{,} \delta \sigma_{,} + \sigma_{,}' \delta x_{,}' + y_{,} \delta \tau_{,} + \tau_{,}' \delta y_{,}' + z_{,} \delta \upsilon_{,} + \upsilon_{,}' \delta z_{,}' - \frac{\delta V}{\delta \chi} \delta \chi ; \\ \delta T_{,} &= x_{,} \delta \sigma_{,} - x_{,}' \delta \sigma_{,}' + y_{,} \delta \tau_{,} - y_{,}' \delta \tau_{,}' + z_{,} \delta \upsilon_{,} - z_{,}' \delta \upsilon_{,}' - \frac{\delta V}{\delta \chi} \delta \chi ; \end{aligned} \right\} \qquad (\text{R}^8)$$

which resemble the expressions (A') (B') (C'), and lead to analogous results. Thus, the partial differential coefficients of the new auxiliary functions $W_{,}$, $T_{,}$, may be deduced, by methods similar to those already employed, from the new coefficients of the characteristic function V, which may themselves be deduced from the old coefficients of that function, by the following general formula,

$$\left. \begin{aligned} &\left(\frac{\delta}{\delta x_{,}}\right)^{i} \left(\frac{\delta}{\delta y_{,}}\right)^{i'} \left(\frac{\delta}{\delta z_{,}}\right)^{i''} \left(\frac{\delta}{\delta x_{,}'}\right)^{k} \left(\frac{\delta}{\delta y_{,}'}\right)^{k'} \left(\frac{\delta}{\delta z_{,}'}\right)^{k''} \left(\frac{\delta}{\delta \chi}\right)^{l} V = \\ &\left(x_{x_{,}} \frac{\delta}{\delta x} + y_{x_{,}} \frac{\delta}{\delta y} + z_{x_{,}} \frac{\delta}{\delta z}\right)^{i} \left(x'_{x_{,}'} \frac{\delta}{\delta x'} + y'_{x_{,}'} \frac{\delta}{\delta y'} + z'_{x_{,}'} \frac{\delta}{\delta z'}\right)^{k} \\ &\left(x_{y_{,}} \frac{\delta}{\delta x} + y_{y_{,}} \frac{\delta}{\delta y} + z_{y_{,}} \frac{\delta}{\delta z}\right)^{i'} \left(x'_{y_{,}'} \frac{\delta}{\delta x'} + y'_{y_{,}'} \frac{\delta}{\delta y'} + z'_{y_{,}'} \frac{\delta}{\delta z'}\right)^{k'} \\ &\left(x_{z_{,}} \frac{\delta}{\delta x} + y_{z_{,}} \frac{\delta}{\delta y} + z_{z_{,}} \frac{\delta}{\delta z}\right)^{i''} \left(x'_{z_{,}'} \frac{\delta}{\delta x'} + y'_{z_{,}'} \frac{\delta}{\delta y'} + z'_{z_{,}'} \frac{\delta}{\delta z'}\right)^{k''} \frac{\delta l V}{\delta \chi^{l}} : \end{aligned} \right\} \qquad (\text{S}^8)$$

and the equations of a straight final ray may be put under the forms,

$$\left. \begin{aligned} \frac{1}{\alpha_{,}}\left(x_{,} - \frac{\delta W_{,}}{\delta \sigma_{,}}\right) &= \frac{1}{\beta_{,}}\left(y_{,} - \frac{\delta W_{,}}{\delta \tau_{,}}\right) = \frac{1}{\gamma_{,}}\left(z_{,} - \frac{\delta W_{,}}{\delta \upsilon_{,}}\right), \\ \frac{1}{\alpha_{,}}\left(x_{,} - \frac{\delta T_{,}}{\delta \sigma_{,}}\right) &= \frac{1}{\beta_{,}}\left(y_{,} - \frac{\delta T_{,}}{\delta \tau_{,}}\right) = \frac{1}{\gamma_{,}}\left(z_{,} - \frac{\delta T_{,}}{\delta \upsilon_{,}}\right), \end{aligned} \right\} \qquad (\text{T}^8)$$

while those of a straight initial ray may be put under these other forms,

$$\frac{1}{\alpha_,'}\left(x_,' + \frac{\delta T_,}{\delta \sigma_,'}\right) = \frac{1}{\beta_,'}\left(y_,' + \frac{\delta T_,}{\delta \tau_,'}\right) = \frac{1}{\gamma_,'}\left(z_,' + \frac{\delta T_,}{\delta \upsilon_,'}\right); \qquad (\mathrm{U}^8)$$

these new equations (T^8) (U^8) being analogous to (I^2) and (P^2). It is evident that the arbitrary constants introduced by these transformations of co-ordinates must often assist to simplify the solution of optical problems. In the comparison, for example, of a given polygon ray, ordinary or extraordinary, of any given system, with other near rays of the same system, it will often be found convenient to choose the final portion of the given polygon ray for the axis of $z_,$, and the initial portion for the the axis of $z_,'$, a choice which will make $\alpha_,$ $\beta_,$ $\alpha_,'$ $\beta_,'$ and many of the new partial differential coefficients vanish, without producing, by this simplification, any real loss of generality.

We may even carry these transformations farther, and introduce polar co-ordinates, or any other marks of initial and final position, and still obtain results having much analogy to the foregoing. For if we suppose that the final co-ordinates x, y, z are functions of any three quantities ρ, θ, ϕ, and that in like manner the initial co-ordinates x', y', z' are functions of any other three quantities ρ', θ', ϕ', so that

$$\left.\begin{aligned}
\delta x &= \frac{\delta x}{\delta \rho}\delta\rho + \frac{\delta x}{\delta \theta}\delta\theta + \frac{\delta x}{\delta \phi}\delta\phi, \quad dx = \frac{\delta x}{\delta \rho}d\rho + \frac{\delta x}{\delta \theta}d\theta + \frac{\delta x}{\delta \phi}d\phi, \\
\delta y &= \frac{\delta y}{\delta \rho}\delta\rho + \frac{\delta y}{\delta \theta}\delta\theta + \frac{\delta y}{\delta \phi}\delta\phi, \quad dy = \frac{\delta y}{\delta \rho}d\rho + \frac{\delta y}{\delta \theta}d\theta + \frac{\delta y}{\delta \phi}d\phi, \\
\delta z &= \frac{\delta z}{\delta \rho}\delta\rho + \frac{\delta z}{\delta \theta}\delta\theta + \frac{\delta z}{\delta \phi}\delta\phi, \quad dz = \frac{\delta z}{\delta \rho}d\rho + \frac{\delta z}{\delta \theta}d\theta + \frac{\delta z}{\delta \phi}d\phi, \\
\delta x' &= \frac{\delta x'}{\delta \rho'}\delta\rho' + \frac{\delta x'}{\delta \theta'}\delta\theta' + \frac{\delta x'}{\delta \phi'}\delta\phi', \quad dx' = \frac{\delta x'}{\delta \rho'}d\rho' + \frac{\delta x'}{\delta \theta'}d\theta' + \frac{\delta x'}{\delta \phi'}d\phi', \\
\delta y' &= \frac{\delta y'}{\delta \rho'}\delta\rho' + \frac{\delta y'}{\delta \theta'}\delta\theta' + \frac{\delta y'}{\delta \phi'}\delta\phi', \quad dy' = \frac{\delta y'}{\delta \rho'}d\rho' + \frac{\delta y'}{\delta \theta'}d\theta' + \frac{\delta y'}{\delta \phi'}d\phi', \\
\delta z' &= \frac{\delta z'}{\delta \rho'}\delta\rho' + \frac{\delta z'}{\delta \theta'}\delta\theta' + \frac{\delta z'}{\delta \phi'}\delta\phi', \quad dz' = \frac{\delta z'}{\delta \rho'}d\rho' + \frac{\delta z'}{\delta \theta'}d\theta' + \frac{\delta z'}{\delta \phi'}d\phi',
\end{aligned}\right\} \quad (\mathrm{V}^8)$$

we may consider V as a function of $\rho\ \theta\ \phi\ \rho'\ \theta'\ \phi'\ \chi$, obtained by substituting for $x\ y\ z\ x'\ y'\ z'$ their values; and if we substitute also the values of dx, dy, dz, in the differential dV, or vds, which was before a homogeneous function of the first dimension of dx, dy, dz, such that by our fundamental formula

$$\left.\begin{aligned}
\frac{\delta dV}{\delta dx} &= \frac{\delta . vds}{\delta dx} = \frac{\delta v}{\delta \alpha} = \frac{\delta V}{\delta x}, \\
\frac{\delta dV}{\delta dy} &= \frac{\delta . vds}{\delta dy} = \frac{\delta v}{\delta \beta} = \frac{\delta V}{\delta y}, \\
\frac{\delta dV}{\delta dz} &= \frac{\delta . vds}{\delta dz} = \frac{\delta v}{\delta \gamma} = \frac{\delta V}{\delta z},
\end{aligned}\right\} \quad (\mathrm{W}^8)$$

we may consider this differential $dV = vds$ as becoming now a homogeneous function of $d\rho$, $d\theta$, $d\phi$, of the first dimension, such that

$$\left.\begin{aligned}
\frac{\delta . vds}{\delta d\rho} &= \frac{\delta dV}{\delta d\rho} = \frac{\delta V}{\delta x}\frac{\delta x}{\delta \rho} + \frac{\delta V}{\delta y}\frac{\delta y}{\delta \rho} + \frac{\delta V}{\delta z}\frac{\delta z}{\delta \rho} = \frac{\delta V}{\delta \rho}, \\
\frac{\delta . vds}{\delta d\theta} &= \frac{\delta dV}{\delta d\theta} = \frac{\delta V}{\delta x}\frac{\delta x}{\delta \theta} + \frac{\delta V}{\delta y}\frac{\delta y}{\delta \theta} + \frac{\delta V}{\delta z}\frac{\delta z}{\delta \theta} = \frac{\delta V}{\delta \theta}, \\
\frac{\delta . vds}{\delta d\phi} &= \frac{\delta dV}{\delta d\phi} = \frac{\delta V}{\delta x}\frac{\delta x}{\delta \phi} + \frac{\delta V}{\delta y}\frac{\delta y}{\delta \phi} + \frac{\delta V}{\delta z}\frac{\delta z}{\delta \phi} = \frac{\delta V}{\delta \phi},
\end{aligned}\right\} \qquad (X^8)$$

the symbol d referring still to motion along a ray. In like manner we may consider the initial differential element of V, namely $v'ds'$, as a homogeneous function of the first dimension of $d\rho'$, $d\theta'$, $d\phi'$, and then we shall find that the partial differential coefficients of the first order of this function, are equal respectively to

$$-\frac{\delta V}{\delta \rho'},\ -\frac{\delta V}{\delta \theta'},\ -\frac{\delta V}{\delta \phi'};$$

we may therefore generalise the fundamental formula (A) as follows

$$\begin{aligned}
\delta V = {} & \frac{\delta . vds}{\delta d\rho}\delta\rho + \frac{\delta . vds}{\delta d\theta}\delta\theta + \frac{\delta . vds}{\delta d\phi}\delta\phi \\
& - \frac{\delta . v'ds'}{\delta d\rho'}\delta\rho' - \frac{\delta . v'ds'}{\delta d\theta'}\delta\theta' - \frac{\delta . v'ds'}{\delta d\phi'}\delta\phi' + \frac{\delta V}{\delta \chi}\delta\chi. \qquad (Y^8)
\end{aligned}$$

And the auxiliary functions W, T, correspond to the following more general functions,

$$-V + \rho\frac{\delta V}{\delta \rho} + \theta\frac{\delta V}{\delta \theta} + \phi\frac{\delta V}{\delta \phi}, \text{ and } -V + \rho\frac{\delta V}{\delta \rho} + \theta\frac{\delta V}{\delta \theta} + \phi\frac{\delta V}{\delta \phi} + \rho'\frac{\delta V}{\delta \rho'} + \theta'\frac{\delta V}{\delta \theta'} + \phi'\frac{\delta V}{\delta \phi'};$$

of which the first may be regarded as a function of

$$\frac{\delta V}{\delta \rho},\ \frac{\delta V}{\delta \theta},\ \frac{\delta V}{\delta \phi},\ \rho',\ \theta',\ \phi',\ \chi,$$

and the second as a function of

$$\frac{\delta V}{\delta \rho},\ \frac{\delta V}{\delta \theta},\ \frac{\delta V}{\delta \phi},\ -\frac{\delta V}{\delta \rho'},\ -\frac{\delta V}{\delta \theta'},\ -\frac{\delta V}{\delta \phi'},\ \chi.$$

It is easy also to establish the following general differential equations of a curved ray, ordinary or extraordinary, and the following general integrals analogous to and including those already assigned for rectangular and oblique co-ordinates,

$$\left.\begin{aligned}
& d\frac{\delta dV}{\delta d\rho} = \frac{\delta dV}{\delta \rho};\ d\frac{\delta dV}{\delta d\theta} = \frac{\delta dV}{\delta \theta};\ d\frac{\delta dV}{\delta d\phi} = \frac{\delta dV}{\delta \phi}: \\
& \frac{\delta V}{\delta \rho'} = \text{const.};\ \frac{\delta V}{\delta \theta'} = \text{const.};\ \frac{\delta V}{\delta \phi'} = \text{const.}
\end{aligned}\right\} \qquad (Z^8)$$

General geometrical Relations of infinitely near Rays. Classification of twenty-four independent Coefficients, which enter into the algebraical Expressions of these general Relations. Division of the general Discussion into four principal Problems.

14. It is an important general problem of mathematical optics, included in that fundamental problem which was stated in the second number, to investigate *the general relations of infinitely near rays,* or paths of light; and especially to examine *how the extreme directions change, for any infinitely small changes of the extreme points, and of the colour:* that is, in the notation of this Supplement, to examine the general dependence of the variations $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\delta\alpha'$, $\delta\beta'$, $\delta\gamma'$, on δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$. This important case of our fundamental problem is easily resolved by the application of our general methods, and by the partial differential coefficients, of the two first orders, of the characteristic and related functions: it may also be resolved by the partial differentials of the three first orders, of the characteristic function V alone. For from these we can in general deduce six linear expressions for $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\delta\alpha'$, $\delta\beta'$, $\delta\gamma'$, in terms of δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$, involving forty-two coefficients, of which however only twenty-four are independent, because they are connected by fourteen relations included in the formulæ $\alpha\delta\alpha+\beta\delta\beta+\gamma\delta\gamma=0$, $\alpha'\delta\alpha'+\beta'\delta\beta'+\gamma'\delta\gamma'=0$, and by four more included in the conditions that the final direction does not change when the initial point takes any new position on the given luminous path, nor the initial direction when the final point is removed to any new point on that given path.

Thus, if we employ the characteristic function V, and the final and initial medium-functions v, v', we have, by (B), the following general relations:

$$\left.\begin{array}{l} \delta\dfrac{\delta V}{\delta x}=\delta\dfrac{\delta v}{\delta\alpha}\,;\quad \delta\dfrac{\delta V}{\delta y}=\delta\dfrac{\delta v}{\delta\beta}\,;\quad \delta\dfrac{\delta V}{\delta z}=\delta\dfrac{\delta v}{\delta\gamma}\,; \\ -\,\delta\dfrac{\delta V}{\delta x'}=\delta\dfrac{\delta v'}{\delta\alpha'}\,;\quad -\,\delta\dfrac{\delta V}{\delta y'}=\delta\dfrac{\delta v'}{\delta\beta'}\,;\quad -\,\delta\dfrac{\delta V}{\delta z'}=\delta\dfrac{\delta v'}{\delta\gamma'}\,: \end{array}\right\}\qquad (A^9)$$

in which, by the last number, we are at liberty to assign different origins and different and oblique directions to the axes of the final and initial co-ordinates, if we assign new and corresponding values to the marks of final and initial direction, α, β, γ, α', β', γ', so as to have still the equations (P),

$$\alpha=\frac{dx}{ds},\quad \beta=\frac{dy}{ds},\quad \gamma=\frac{dz}{ds},\quad \alpha'=\frac{dx'}{ds'},\quad \beta'=\frac{dy'}{ds'},\quad \gamma'=\frac{dz'}{ds'},$$

ds being still the final, and ds' the initial element of the curved or polygon path. We may suppose, for example, that both sets of co-ordinates are rectangular, but that the origins of the final and initial co-ordinates are respectively the final and initial points of a given ordinary or extraordinary path, and that the positive semiaxes of z, z', coincide with the final and initial directions, so as to give

$$\left.\begin{aligned} &x=0,\ y=0,\ z=0,\ \alpha=0,\ \beta=0,\ \gamma=1,\ \delta\gamma=0;\\ &x'=0,\ y'=0,\ z'=0,\ \alpha'=0,\ \beta'=0,\ \gamma'=1,\ \delta\gamma'=0;\end{aligned}\right\}\qquad (\mathrm{B}^9)$$

and then the six equations (A^9), of which only four are distinct, reduce themselves to the four following,

$$\left.\begin{aligned} &\frac{\delta^2 v}{\delta\alpha^2}\,\delta\alpha+\frac{\delta^2 v}{\delta\alpha\delta\beta}\delta\beta=\frac{\delta^2 V}{\delta x\delta x'}\,\delta x'+\frac{\delta^2 V}{\delta x\delta y'}\,\delta y'+\left(\frac{\delta^2 V}{\delta x\delta\chi}-\frac{\delta^2 v}{\delta\alpha\delta\chi}\right)\delta\chi\\ &+\left(\frac{\delta^2 V}{\delta x^2}-\frac{\delta^2 v}{\delta\alpha\delta x}\right)\delta x+\left(\frac{\delta^2 V}{\delta x\delta y}-\frac{\delta^2 v}{\delta\alpha\delta y}\right)\delta y+\left(\frac{\delta v}{\delta x}-\frac{\delta^2 v}{\delta\alpha\delta z}\right)\delta z;\\ &\frac{\delta^2 v}{\delta\alpha\delta\beta}\,\delta\alpha+\frac{\delta^2 v}{\delta\beta^2}\,\delta\beta=\frac{\delta^2 V}{\delta y\delta x'}\,\delta x'+\frac{\delta^2 V}{\delta y\delta y'}\,\delta y'+\left(\frac{\delta^2 V}{\delta y\delta\chi}-\frac{\delta^2 v}{\delta\beta\delta\chi}\right)\delta\chi\\ &+\left(\frac{\delta^2 V}{\delta x\delta y}-\frac{\delta^2 v}{\delta\beta\delta x}\right)\delta x+\left(\frac{\delta^2 V}{\delta y^2}-\frac{\delta^2 v}{\delta\beta\delta y}\right)\delta y+\left(\frac{\delta v}{\delta y}-\frac{\delta^2 v}{\delta\beta\delta z}\right)\delta z;\\ &-\frac{\delta^2 v'}{\delta\alpha'^2}\,\delta\alpha'-\frac{\delta^2 v'}{\delta\alpha'\delta\beta'}\,\delta\beta'=\frac{\delta^2 V}{\delta x\delta x'}\delta x+\frac{\delta^2 V}{\delta y\delta x'}\,\delta y+\left(\frac{\delta^2 V}{\delta x'\delta\chi}+\frac{\delta^2 v'}{\delta\alpha'\delta\chi}\right)\delta\chi\\ &+\left(\frac{\delta^2 V}{\delta x'^2}+\frac{\delta^2 v'}{\delta\alpha'\delta x'}\right)\delta x'+\left(\frac{\delta^2 V}{\delta x'\delta y'}+\frac{\delta^2 v'}{\delta\alpha'\delta y'}\right)\delta y'-\left(\frac{\delta v'}{\delta x'}-\frac{\delta^2 v'}{\delta\alpha'\delta z'}\right)\delta z';\\ &-\frac{\delta^2 v'}{\delta\alpha'\delta\beta'}\,\delta\alpha'-\frac{\delta^2 v'}{\delta\beta'^2}\,\delta\beta'=\frac{\delta^2 V}{\delta x\delta y'}\delta x+\frac{\delta^2 V}{\delta y\delta y'}\,\delta y+\left(\frac{\delta^2 V}{\delta y'\delta\chi}+\frac{\delta^2 v'}{\delta\beta'\delta\chi}\right)\delta\chi\\ &+\left(\frac{\delta^2 V}{\delta x'\delta y'}+\frac{\delta^2 v'}{\delta\beta'\delta x'}\right)\delta x'+\left(\frac{\delta^2 V}{\delta y'^2}+\frac{\delta^2 v'}{\delta\beta'\delta y'}\right)\delta y'-\left(\frac{\delta v'}{\delta y'}-\frac{\delta^2 v'}{\delta\beta'\delta z'}\right)\delta z':\end{aligned}\right\}\quad (\mathrm{C}^9)$$

they give therefore, by easy eliminations, expressions for $\delta\alpha$, $\delta\beta$, $\delta\alpha'$, $\delta\beta'$, of the form

$$\left.\begin{aligned} &\delta\alpha=\frac{\delta\alpha}{\delta x}\,\delta x+\frac{\delta\alpha}{\delta y}\,\delta y+\frac{\delta\alpha}{\delta z}\,\delta z+\frac{\delta\alpha}{\delta x'}\,\delta x'+\frac{\delta\alpha}{\delta y'}\,\delta y'+\frac{\delta\alpha}{\delta\chi}\,\delta\chi,\\ &\delta\beta=\frac{\delta\beta}{\delta x}\,\delta x+\frac{\delta\beta}{\delta y}\,\delta y+\frac{\delta\beta}{\delta z}\,\delta z+\frac{\delta\beta}{\delta x'}\,\delta x'+\frac{\delta\beta}{\delta y'}\,\delta y'+\frac{\delta\beta}{\delta\chi}\,\delta\chi,\\ &\delta\alpha'=\frac{\delta\alpha'}{\delta x'}\,\delta x'+\frac{\delta\alpha'}{\delta y'}\,\delta y'+\frac{\delta\alpha'}{\delta z'}\,\delta z'+\frac{\delta\alpha'}{\delta x}\,\delta x+\frac{\delta\alpha'}{\delta y}\,\delta y+\frac{\delta\alpha'}{\delta\chi}\,\delta\chi,\\ &\delta\beta'=\frac{\delta\beta'}{\delta x'}\,\delta x'+\frac{\delta\beta'}{\delta y'}\,\delta y'+\frac{\delta\beta'}{\delta z'}\,\delta z'+\frac{\delta\beta'}{\delta x}\,\delta x+\frac{\delta\beta'}{\delta y}\,\delta y+\frac{\delta\beta'}{\delta\chi}\,\delta\chi,\end{aligned}\right\}\quad (\mathrm{D}^9)$$

which involve twenty-four coefficients, and enable us to determine the general geometrical relations between the final and initial tangents to the near luminous paths.

If the extreme media be ordinary, that is, if the functions v, v', be independent of the directions of the rays, we have

$$v=\mu\sqrt{(\alpha^2+\beta^2+\gamma^2)},\quad v'=\mu'\sqrt{(\alpha'^2+\beta'^2+\gamma'^2)},\qquad (\mathrm{E}^9)$$

μ, μ' being functions of the colour χ, of which μ involves also the final co-ordinates, and μ' the initial co-ordinates, when the extreme media are atmospheres: and then the equations (C^9) reduce themselves at once to the following expressions of the form (D^9),

$$\left.\begin{aligned}
\delta\alpha &= \frac{1}{\mu}\left(\frac{\delta^2 V}{\delta x^2}\,\delta x + \frac{\delta^2 V}{\delta x\,\delta y}\,\delta y + \frac{\delta\mu}{\delta x}\,\delta z + \frac{\delta^2 V}{\delta x\,\delta x'}\,\delta x' + \frac{\delta^2 V}{\delta x\,\delta y'}\,\delta y' + \frac{\delta^2 V}{\delta x\,\delta\chi}\,\delta\chi\right),\\
\delta\beta &= \frac{1}{\mu}\left(\frac{\delta^2 V}{\delta x\,\delta y}\,\delta x + \frac{\delta^2 V}{\delta y^2}\,\delta y + \frac{\delta\mu}{\delta y}\,\delta z + \frac{\delta^2 V}{\delta y\,\delta x'}\,\delta x' + \frac{\delta^2 V}{\delta y\,\delta y'}\,\delta y' + \frac{\delta^2 V}{\delta y\,\delta\chi}\,\delta\chi\right),\\
\delta\alpha' &= -\frac{1}{\mu'}\left(\frac{\delta^2 V}{\delta x'^2}\,\delta x' + \frac{\delta^2 V}{\delta x'\,\delta y'}\,\delta y' - \frac{\delta\mu'}{\delta x'}\,\delta z' + \frac{\delta^2 V}{\delta x\,\delta x'}\,\delta x + \frac{\delta^2 V}{\delta y\,\delta x'}\,\delta y + \frac{\delta^2 V}{\delta x'\,\delta\chi}\,\delta\chi\right),\\
\delta\beta' &= -\frac{1}{\mu'}\left(\frac{\delta^2 V}{\delta x'\,\delta y'}\,\delta x' + \frac{\delta^2 V}{\delta y'^2}\,\delta y' - \frac{\delta\mu'}{\delta y'}\,\delta z' + \frac{\delta^2 V}{\delta x\,\delta y'}\,\delta x + \frac{\delta^2 V}{\delta y\,\delta y'}\,\delta y + \frac{\delta^2 V}{\delta y'\,\delta\chi}\,\delta\chi\right).
\end{aligned}\right\}(\mathrm{F}^9)$$

In general we see that the twenty-four coefficients of the expressions (D^9) can easily be deduced, by (C^9), from the partial differentials of the two first orders of the characteristic function V, and of the extreme medium-functions v, v': we have for example

$$\left.\begin{aligned}
\frac{\delta\alpha}{\delta x} &= \frac{1}{v''}\,\frac{\delta^2 v}{\delta\beta^2}\left(\frac{\delta^2 V}{\delta x^2} - \frac{\delta^2 v}{\delta\alpha\,\delta x}\right) - \frac{1}{v''}\,\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\beta\,\delta x}\right),\\
\frac{\delta\alpha}{\delta y} &= \frac{1}{v''}\,\frac{\delta^2 v}{\delta\beta^2}\left(\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\alpha\,\delta y}\right) - \frac{1}{v''}\,\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\frac{\delta^2 V}{\delta y^2} - \frac{\delta^2 v}{\delta\beta\,\delta y}\right),\\
\frac{\delta\beta}{\delta x} &= \frac{1}{v''}\,\frac{\delta^2 v}{\delta\alpha^2}\left(\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\beta\,\delta x}\right) - \frac{1}{v''}\,\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\frac{\delta^2 V}{\delta x^2} - \frac{\delta^2 v}{\delta\alpha\,\delta x}\right),\\
\frac{\delta\beta}{\delta y} &= \frac{1}{v''}\,\frac{\delta^2 v}{\delta\alpha^2}\left(\frac{\delta^2 V}{\delta y^2} - \frac{\delta^2 v}{\delta\beta\,\delta y}\right) - \frac{1}{v''}\,\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\alpha\,\delta y}\right),
\end{aligned}\right\}\quad(\mathrm{G}^9)$$

v'' having the same meaning as in the tenth number. The same twenty-four coefficients of (D^9) may also be deduced (as we have said) from the partial differentials of the two first orders of the other related and auxiliary functions: or even from the partial differentials of the three first orders of the characteristic function V alone. Let us therefore suppose that these twenty-four coefficients of the expressions (D^9) are known, and let us consider their geometrical meanings and uses: that is, their connexions with questions respecting the infinitely small variations of the extreme directions or tangents of a luminous path, arising from variations of the extreme points and of the colour.

In discussing these connexions, it is evidently permitted, by the linear form of the differential expressions (D^9), to consider separately and successively the influence of the seven variations δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$, of the extreme co-ordinates and the colour, or the influence of any groupes of these seven variations, on the four variations $\delta\alpha$, $\delta\beta$, $\delta\alpha'$, $\delta\beta'$, of the extreme small cosines of direction. Thus, if it be required to compare the extreme directions of a given path of ordinary or extraordinary light of the colour χ, from a given initial point A to a given final point B, which path we shall denote as follows,

$$(A, B)_{\chi}, \qquad (\mathrm{H}^9)$$

with the extreme directions of an infinitely near path of infinitely near colour $\chi + \delta\chi$ from an infinitely near initial point A' to an infinitely near final point B', which near path we shall in like manner denote thus

$$(A', B')_{\chi + \delta\chi}, \qquad (\mathrm{I}^9)$$

we may do so by comparing separately the extreme directions of the given path $(A, B)_{\chi}$ with those of the three following other infinitely near paths;

$$\text{1st. } (A, B)_{\chi + \delta\chi}\ ; \quad \text{2d. } (A, B')_{\chi}\ ; \quad \text{3d. } (A', B)_{\chi} : \qquad (\mathrm{K}^9)$$

which are obtained by changing, successively and separately, the colour χ, the final point B, and the initial point A. We are therefore led, by this consideration, to examine separately and successively the meanings and uses of the three following groupes, out of the twenty-four coefficients of (D^9):

$$\left.\begin{array}{ll} \text{1st groupe} & \frac{\delta\alpha}{\delta\chi}, \frac{\delta\beta}{\delta\chi}, \frac{\delta\alpha'}{\delta\chi}, \frac{\delta\beta'}{\delta\chi}; \\ \text{2d groupe} & \frac{\delta\alpha}{\delta x}, \frac{\delta\alpha}{\delta y}, \frac{\delta\alpha}{\delta z}, \frac{\delta\beta}{\delta x}, \frac{\delta\beta}{\delta y}, \frac{\delta\beta}{\delta z}, \frac{\delta\alpha'}{\delta x}, \frac{\delta\alpha'}{\delta y}, \frac{\delta\beta'}{\delta x}, \frac{\delta\beta'}{\delta y}; \\ \text{3d groupe} & \frac{\delta\alpha}{\delta x'}, \frac{\delta\alpha}{\delta y'}, \frac{\delta\beta}{\delta x'}, \frac{\delta\beta}{\delta y'}, \frac{\delta\alpha'}{\delta x'}, \frac{\delta\alpha'}{\delta y'}, \frac{\delta\alpha'}{\delta z'}, \frac{\delta\beta'}{\delta x'}, \frac{\delta\beta'}{\delta y'}, \frac{\delta\beta'}{\delta z'}. \end{array}\right\} \quad (\mathrm{L}^9)$$

But we may simplify and improve the plan of our investigation, by means of the following considerations.

Of the three comparisons, of the given path (H^9) with the three near paths (K^9), the third is evidently of the same kind with the second, and need not be treated as distinct; because, of the two extreme points of a luminous path, it is indifferent which we consider as initial and which as final. We may therefore omit the third comparison (K^9), and confine ourselves to the first and second, that is, we may omit the consideration of the third groupe (L^9), in forming a theory of the general relations of infinitely near rays. For a similar reason we may omit the consideration of the two last coefficients of the first groupe (L^9), and so may reduce the study of the whole twenty-four to the study of half that number.

On the other hand, the second comparison (K^9) may conveniently be decomposed into two: for instead of the arbitrary infinitesimal line $\overline{BB'}$, connecting the given final point B with the near point B', we may conveniently consider the two projections of this line, on the final element or tangent of the given luminous path, and on the plane perpendicular to this element: that is, we may put

$$\overline{BB'}^2 = \overline{BB_d}^2 + \overline{BB_\delta}^2, \qquad (M^9)$$

$\overline{BB_d}$ being the projection on the element, and $\overline{BB_\delta}$ the projection on the perpendicular plane, and we may consider separately the two near points B_d, B_δ, upon this element and plane, and the two corresponding paths,

$$(A, B_d)_\chi, \ (A, B_\delta)_\chi, \qquad (N^9)$$

instead of considering the more general near point B', and the near path $(A, B')_\chi$. In this manner we are led to consider separately, as one subordinate class or set, suggested by the path $(A, B_d)_\chi$, the system of the two coefficients $\frac{\delta\alpha}{\delta z}$, $\frac{\delta\beta}{\delta z}$; distinguishing these from the eight other coefficients of the second groupe (L^9), which correspond to the other near path $(A, B_\delta)_\chi$; and these eight may again be conveniently divided into two distinct classes, according as we consider the changes of final or of initial direction.

We are then led to arrange the twelve retained coefficients of the expressions (D^9), in *four new sets* or classes, suggesting *four separate problems* :

$$\left.\begin{array}{llll} \text{First set} & \frac{\delta\alpha}{\delta\chi}, \frac{\delta\beta}{\delta\chi}; & \text{Second,} & \frac{\delta\alpha}{\delta z}, \frac{\delta\beta}{\delta z}; \\ \text{Third,} & \frac{\delta\alpha}{\delta x}, \frac{\delta\alpha}{\delta y}, \frac{\delta\beta}{\delta x}, \frac{\delta\beta}{\delta y}; & \text{Fourth,} & \frac{\delta\alpha'}{\delta x}, \frac{\delta\alpha'}{\delta y}, \frac{\delta\beta'}{\delta x}, \frac{\delta\beta'}{\delta y}. \end{array}\right\} \qquad (O^9)$$

In each of these four problems, the initial point is considered as given, and may be supposed to be a luminous origin, common to all the infinitely near paths of which we compare the extreme directions. In the first problem, the final point also is given, but the colour χ is variable; and we study the final chromatic dispersion of the different near paths of heterogeneous light, connecting the given final point with the given luminous origin: whereas, in the three remaining problems, the light is considered as homogeneous, but the luminous path varies by the variation of its final point. In the second problem, the new final point B_d is on the original path, or on that path prolonged; and we examine whether and in what manner the final direction varies, on account of the final curvature of that original path. In the third problem, the new final point B_δ is on an infinitely small line

$$\delta l = \overline{BB_\delta}, \qquad (P^9)$$

which is drawn from the given final point of the original path, perpendicular to the given final element of that path, namely to the element

$$ds = \overline{BB_d}; \qquad (Q^9)$$

and we inquire into the mutual arrangement and relations of the final system of right lines which coincide with and mark the final directions of the near luminous paths, at the several near points B_δ where they meet the given final plane perpendicular to the given element ds. In the fourth problem, we consider the initial system of right lines, which mark, at the luminous origin, the initial directions of the same near paths of homogeneous light; and we compare these initial directions with the positions of the points B_δ. Let us now consider separately these four principal problems, respecting the geometrical relations of infinitely near rays.

Discussion of the Four Problems. Elements of Arrangement of near Luminous Paths. Axis and Constant of Chromatic Dispersion. Axis of Curvature of Ray. Guiding Paraboloid, and Constant of Deviation. Guiding Planes, and Conjugate Guiding Axes.

15. The *first* of these four problems, namely that in which it is required to determine the final chromatic dispersion, by means of the two coefficients $\frac{\delta\alpha}{\delta\chi}$, $\frac{\delta\beta}{\delta\chi}$, is very easily resolved: since we have the following equations for the magnitude and plane of this dispersion,

$$\left.\begin{aligned} &\textit{Final angle of chromatic dispersion} = \xi\delta\chi\,;\ \ \xi = \sqrt{\left(\frac{\delta\alpha}{\delta\chi}\right)^2 + \left(\frac{\delta\beta}{\delta\chi}\right)^2}\,: \\ &\textit{Final plane of dispersion}\ldots\ldots\ldots\ldots\ldots\ \ y\,\frac{\delta\alpha}{\delta\chi} = x\,\frac{\delta\beta}{\delta\chi}\,. \end{aligned}\right\} \qquad (\mathrm{R}^9)$$

We may geometrically construct the effect of this dispersion, by making the given final line of direction of the original luminous path revolve through the small angle $\xi\delta\chi$, in which ξ may be called the *constant of final chromatic dispersion*, round the following line which may be called *the axis of final chromatic dispersion*,

$$x\,\frac{\delta\alpha}{\delta\chi} + y\,\frac{\delta\beta}{\delta\chi} = 0, \ \ z = 0. \qquad (\mathrm{S}^9)$$

The *second* problem, which relates to the final curvature of the given luminous path, is resolved by the analogous equations,

$$\left.\begin{aligned} &\textit{Final curvature of ray} = \sqrt{\left(\frac{\delta\alpha}{\delta z}\right)^2 + \left(\frac{\delta\beta}{\delta z}\right)^2}\,; \\ &\textit{Plane of curvature}\ldots\ldots\ldots\ \ y\,\frac{\delta\alpha}{\delta z} = x\,\frac{\delta\beta}{\delta z}\,; \end{aligned}\right\} \qquad (\mathrm{T}^9)$$

we have also the following equations for the axis of curvature, that is, for the axis of the circle of curvature, or of the final osculating circle to the given luminous path,

$$x\,\frac{\delta\alpha}{\delta z} + y\,\frac{\delta\beta}{\delta z} = 1, \ \ z = 0\,: \qquad (\mathrm{U}^9)$$

and in all these equations of curvature we may, consistently with the notation of the present Supplement, express the coefficients $\frac{\delta \alpha}{\delta z}$, $\frac{\delta \beta}{\delta z}$ by the symbols $\frac{d\alpha}{dz}$, $\frac{d\beta}{dz}$, because they relate to motion along a given luminous path. It is evident that these coefficients vanish, when the final portion of this path is straight. But when this final portion is curved, we may geometrically construct the effect of the curvature on the final direction, by making the final element *ds* revolve through an infinitely small angle round the final axis of curvature.

The two remaining problems are more complicated, because each involves two independent variations δx, δy, namely the two rectangular co-ordinates of the near point B_δ on the final plane of xy, which point is considered as the final point of a near luminous path. The equations of the right line, which is the final portion or final tangent of this near path, are,

$$\left.\begin{aligned} x &= \delta x + z\left(\frac{\delta \alpha}{\delta x}\,\delta x + \frac{\delta \alpha}{\delta y}\,\delta y\right), \\ y &= \delta y + z\left(\frac{\delta \beta}{\delta x}\,\delta x + \frac{\delta \beta}{\delta y}\,\delta y\right); \end{aligned}\right\} \qquad (\mathrm{V}^9)$$

and the equations of the right line which is the initial portion or the initial tangent of the same near path, are

$$\left.\begin{aligned} x' &= z'\left(\frac{\delta \alpha'}{\delta x}\,\delta x + \frac{\delta \alpha'}{\delta y}\,\delta y\right), \\ y' &= z'\left(\frac{\delta \beta'}{\delta x}\,\delta x + \frac{\delta \beta'}{\delta y}\,\delta y\right). \end{aligned}\right\} \qquad (\mathrm{W}^9)$$

Our *third* problem is to investigate the geometrical relations of the system of right lines (V^9), which we shall call *final ray-lines,* with each other, and with the co-ordinates δx, δy; and our *fourth* problem is to investigate the connexion of the same co-ordinates or variations with the right lines of the system (W^9), which may be called *initial ray-lines.*

The *third* problem may be considered as resolved, if we can assign any surface to which the final ray-lines (V^9) are normals, or with which they are determinately connected by any other known geometrical relation. Let us therefore examine whether the ray-lines of the system (V^9) are normals to any common surface, which passes through the given final point of the original luminous path. If so, this surface may be considered, in our present order of approximation, as perpendicular to the final rays themselves. Now, in general, when rays of a given colour diverge from a given luminous point, and undergo any number of ordinary or extraordinary and gradual or sudden reflexions or refractions, they are, or are not, perpendicular in their final state to a common surface, according as the following differential equation

$$\alpha \delta x + \beta \delta y + \gamma \delta z = 0 \qquad (\mathrm{X}^9)$$

is or is not integrable; and if there be any one surface perpendicular to all the final rays, there is also a series of such surfaces, represented by the integral of this equation. Hence, in the present question, the normal surface sought is such, if it exist at all, as to satisfy the conditions $\delta z = 0$, and

$$\delta^2 z + \delta\alpha\delta x + \delta\beta\delta y = 0; \qquad (Y^9)$$

that is, if it exist, it must touch the given final plane of xy, and must have contact of the second order with the following paraboloid, which may therefore in our present order of approximation be employed instead of it,

$$2z + \frac{\delta\alpha}{\delta x} x^2 + \left(\frac{\delta\alpha}{\delta y} + \frac{\delta\beta}{\delta x}\right) xy + \frac{\delta\beta}{\delta y} y^2 = 0. \qquad (Z^9)$$

The normals to this paraboloid, near its summit, that is, near the final point of the given luminous path, or the origin of the final co-ordinates, have for their approximate equations,

$$\left.\begin{aligned} x &= \delta x + z\left(\frac{\delta\alpha}{\delta x}\,\delta x + \frac{\delta\alpha}{\delta y}\,\delta y\right) + zn\delta y, \\ y &= \delta y + z\left(\frac{\delta\beta}{\delta x}\,\delta x + \frac{\delta\beta}{\delta y}\,\delta y\right) - zn\delta x, \end{aligned}\right\} \qquad (A^{10})$$

if we put for abridgment

$$n = \tfrac{1}{2}\left(\frac{\delta\beta}{\delta x} - \frac{\delta\alpha}{\delta y}\right); \qquad (B^{10})$$

they coincide therefore with the ray-lines (V^9) when the following condition is satisfied,

$$\frac{\delta\beta}{\delta x} = \frac{\delta\alpha}{\delta y}, \qquad (C^{10})$$

which is in fact the condition of integrability of the differential equation (X^9), because we have made α β vanish by our choice of the axis of z. The condition (C^{10}) is satisfied, by (F^9), when the final medium is ordinary; and in fact the final rays whether straight or curved are then perpendicular to the series of surfaces represented by the equation

$$V = \text{const.}: \qquad (D^{10})$$

which is, for ordinary rays, the integral of the equation (X^9), and gives, as an approximate equation of the normal surface at the origin, the following,

$$0 = \delta V + \tfrac{1}{2}\delta^2 V, \text{ or } 0 = \mu z + \tfrac{1}{2}\frac{\delta^2 V}{\delta x^2} x^2 + \frac{\delta^2 V}{\delta x \delta y} xy + \tfrac{1}{2}\frac{\delta^2 V}{\delta y^2} y^2; \qquad (E^{10})$$

agreeing, by (F^9), with the equation of the paraboloid (Z^9). In general, the condition (C^{10}) for the existence of a normal surface, may be put, by (G^9), under the form

$$\frac{\delta^2 v}{\delta\alpha^2}\left(\frac{\delta^2 V}{\delta x\delta y}-\frac{\delta^2 v}{\delta\beta\delta x}\right)-\frac{\delta^2 v}{\delta\alpha\delta\beta}\left(\frac{\delta^2 V}{\delta x^2}-\frac{\delta^2 v}{\delta\alpha\delta x}\right)$$

$$=\frac{\delta^2 v}{\delta\beta^2}\left(\frac{\delta^2 V}{\delta x\delta y}-\frac{\delta^2 v}{\delta\alpha\delta y}\right)-\frac{\delta^2 v}{\delta\alpha\delta\beta}\left(\frac{\delta^2 V}{\delta y^2}-\frac{\delta^2 v}{\delta\beta\delta y}\right): \qquad (F^{10})$$

and it is not satisfied by extraordinary rays, except in particular cases. We may however always consider the paraboloid (Z^9) as an auxiliary surface, with which the final ray-lines of the proposed system (V^9) are connected by a remarkable and simple relation. For if we take the rectangular planes of curvature of this paraboloid for the co-ordinate planes of xz, yz, and denote the two curvatures corresponding by r, t, so as to have the following form for the equation of the paraboloid

$$z=\tfrac{1}{2}rx^2+\tfrac{1}{2}ty^2, \qquad (G^{10})$$

we shall satisfy the condition

$$\frac{\delta\alpha}{\delta y}+\frac{\delta\beta}{\delta x}=0, \qquad (H^{10})$$

and may employ the following expressions for the four coefficients of our problem,

$$\frac{\delta\alpha}{\delta x}=-r,\ \frac{\delta\alpha}{\delta y}=-n,\ \frac{\delta\beta}{\delta x}=+n,\ \frac{\delta\beta}{\delta y}=-t: \qquad (I^{10})$$

the ray-lines of our system (V^9) may therefore be thus represented

$$\left.\begin{aligned} x&=\delta x-z(r\delta x+n\delta y),\\ y&=\delta y-z(t\delta y-n\delta x),\end{aligned}\right\} \qquad (K^{10})$$

while the normals to the paraboloid are represented by these equations

$$x=\delta x-zr\delta x\,,\quad y=\delta y-zt\delta y\,; \qquad (L^{10})$$

from which it follows that the angle $\delta\nu$ between a ray-line (K^{10}) and the corresponding normal (L^{10}) may be thus expressed

$$\delta\nu=n\delta l,\ \text{in which}\ \delta l=\sqrt{\delta x^2+\delta y^2}, \qquad (M^{10})$$

δl being the same small line $\overline{BB_{\delta}}$ as before; and that the plane of this angle $\delta\nu$, or in other words, the plane containing the ray-line and the normal, has for equation

$$x\delta x+y\delta y=\delta l^2-z(r\delta x^2+t\delta y^2): \qquad (N^{10})$$

this plane therefore contains also the right line having for equations

$$x\delta x+y\delta y=0,\ z=\frac{\delta l^2}{r\delta x^2+t\delta y^2}, \qquad (O^{10})$$

that is, the axis of the osculating circle of curvature of the normal or diametral section of the paraboloid, of which the line δl is an element; and *the normal may be brought to coincide with the ray-line by being made to revolve round the element δl, through an angle δv proportional to δl, and equal to that element multiplied by the constant n*: the direction of the rotation depending on the sign of the constant. On account of this simple law of deviation of the final ray-lines from the normals of the paraboloid, we shall call this paraboloid the *guiding surface*: and the constant n, we shall call the *constant of deviation.* And we may consider this theory, of the guiding paraboloid and the constant of deviation, as containing an adequate solution of our third general problem, in the discussion of the geometrical relations of infinitely near rays: since this theory shows adequately the general arrangement of the final system of ray-lines (V^9), and the geometrical meanings of the third set of coefficients (O^9), namely,

$$\frac{\delta\alpha}{\delta x}, \ \frac{\delta\alpha}{\delta y}, \ \frac{\delta\beta}{\delta x}, \ \frac{\delta\beta}{\delta y}.$$

The geometrical construction suggested by this theory may be still farther simplified by observing that the infinitely near normals to the guiding surface, all pass through two rectangular lines, namely, the axes of the two principal circles of curvature of the surface; it is therefore sufficient to draw through any proposed point B_δ two planes containing respectively these two given axes of curvature, and then to make the line of intersection of these two planes revolve round the proposed small line δl or $\overline{BB_\delta}$, through the same small angle $n\delta l$ as before, in order to obtain the sought final ray-line for the proposed final point.

Finally, to compare, as required in the *fourth* problem, the initial system of ray-lines (W^9) with the corresponding final points B_δ on the given final plane, we may denote these initial ray-lines by the equations

$$x' = z'\delta\theta'.\cos.\phi', \ y' = z'\delta\theta'.\sin.\phi', \qquad (\mathrm{P}^{10})$$

if we put

$$\delta\alpha' = \delta\theta'.\cos.\phi', \ \delta\beta' = \delta\theta'.\sin.\phi': \qquad (\mathrm{Q}^{10})$$

and if in like manner we put

$$\delta x = \delta l.\cos.\phi, \ \ \delta y = \delta l.\sin.\phi, \qquad (\mathrm{R}^{10})$$

we shall have the following relations, between ϕ, ϕ', δl, $\delta\theta'$, and the fourth set of partial differential coefficients (O^9),

$$\left.\begin{aligned} \delta\theta'.\cos.\phi' &= \left(\frac{\delta\alpha'}{\delta x}\cos.\phi + \frac{\delta\alpha'}{\delta y}\sin.\phi\right)\delta l, \\ \delta\theta'.\sin.\phi' &= \left(\frac{\delta\beta'}{\delta x}\cos.\phi + \frac{\delta\beta'}{\delta y}\sin.\phi\right)\delta l. \end{aligned}\right\} \qquad (\mathrm{S}^{10})$$

These relations give

$$\tan.\ \phi' = \frac{\frac{\delta\beta'}{\delta x} + \frac{\delta\beta'}{\delta y}\tan.\ \phi}{\frac{\delta\alpha'}{\delta x} + \frac{\delta\alpha'}{\delta y}\tan.\ \phi}; \qquad (\mathrm{T}^{10})$$

they enable us therefore to determine, for any given value of ϕ, that is, for any proposed direction of the small final line δl, or $\overline{BB_\delta}$, the corresponding value of ϕ', that is, the direction of the initial plane of ray-lines, having for equation

$$y' = x' \tan.\ \phi'. \qquad (\mathrm{U}^{10})$$

Thus the final line δl and initial plane ϕ' revolve together, but not in general with equal rapidity; and arbitrary rectangular directions of the one do not in general give rectangular directions of the other, because the conditions

$$\left.\begin{array}{l} \tan.\ \phi'_1 = \dfrac{\frac{\delta\beta'}{\delta x} + \frac{\delta\beta'}{\delta y}\tan.\ \phi_1}{\frac{\delta\alpha'}{\delta x} + \frac{\delta\alpha'}{\delta y}\tan.\ \phi_1},\quad \tan.\ \phi'_2 = \dfrac{\frac{\delta\beta'}{\delta x} + \frac{\delta\beta'}{\delta y}\tan.\ \phi_2}{\frac{\delta\alpha'}{\delta x} + \frac{\delta\alpha'}{\delta y}\tan.\ \phi_2}, \\ \phi_2 = \phi_1 + \frac{\pi}{2},\quad \phi'_2 = \phi'_1 + \frac{\pi}{2}, \end{array}\right\} \qquad (\mathrm{V}^{10})$$

(in which π is the semicircumference to the radius unity,) give the following formula for the angle ϕ_1,

$$2\left(\frac{\delta\beta'}{\delta x}\frac{\delta\beta'}{\delta y} + \frac{\delta\alpha'}{\delta x}\frac{\delta\alpha'}{\delta y}\right)\text{cotan. } 2\phi_1 =$$
$$\left(\frac{\delta\beta'}{\delta x}\right)^2 - \left(\frac{\delta\beta'}{\delta y}\right)^2 + \left(\frac{\delta\alpha'}{\delta x}\right)^2 - \left(\frac{\delta\alpha'}{\delta y}\right)^2, \qquad (\mathrm{W}^{10})$$

which is not in general satisfied by arbitrary values of that angle. There are however in general two rectangular final directions determined by this formula, which correspond to two rectangular initial planes; and if we take these rectangular directions and planes respectively for the directions of x, y, and for the planes of $x'\ z'$, $y'\ z'$, we shall have

$$\frac{\delta\alpha'}{\delta y} = 0,\quad \frac{\delta\beta'}{\delta x} = 0. \qquad (\mathrm{X}^{10})$$

We may also in general satisfy, at the same time, by a proper choice of the semiaxes of co-ordinates, the following other conditions,

$$\frac{\delta\beta'}{\delta y} > 0,\quad \frac{\delta\alpha'}{\delta x} > \frac{\delta\beta'}{\delta y}. \qquad (\mathrm{Y}^{10})$$

By this choice of co-ordinates, the relations (S^{10}) are simplified, and become

$$\left.\begin{array}{l} \delta\theta'.\ \cos.\ \phi' = \dfrac{\delta\alpha'}{\delta x}.\delta l.\cos.\ \phi\ ; \\ \delta\theta'.\ \sin.\ \phi' = \dfrac{\delta\beta'}{\delta y}.\delta l.\ \sin.\ \phi\ : \end{array}\right\} \qquad (\mathrm{Z}^{10})$$

while the equations (IV^9) of the initial ray-lines reduce themselves to the following,

$$x' = z'\,\frac{\delta\alpha'}{\delta x}\,\delta x\ ; \qquad y' = z'\,\frac{\delta\beta'}{\delta y}\,\delta y\ . \qquad (\mathrm{A}^{11})$$

If, then, these initial ray-lines form a circular cone having for equation

$$x'^2 + y'^2 = z'^2\delta\theta'^2, \qquad (\mathrm{B}^{11})$$

the corresponding locus of the final point B_δ, on the final plane of xy, will not in general be a circle, but an ellipse, having for its equation

$$\left(\frac{\delta\alpha'}{\delta x}\right)^2 \delta x^2 + \left(\frac{\delta\beta'}{\delta y}\right)^2 \delta y^2 = \delta\theta'^2, \qquad (\mathrm{C}^{11})$$

of which, by (Y^{10}), the axis of x coincides with the least and the axis of y with the greatest axis ; and reciprocally if the final locus be a circle having for equation

$$\delta x^2 + \delta y^2 = \delta l^2, \qquad (\mathrm{D}^{11})$$

the initial cone of ray-lines will have for equation

$$x'^2\left(\frac{\delta\alpha'}{\delta x}\right)^{-2} + y'^2\left(\frac{\delta\beta'}{\delta y}\right)^{-2} = z'^2\delta l^2, \qquad (\mathrm{E}^{11})$$

so that its perpendicular sections are ellipses, having their greater axes in the plane of $x'\,z'$, and their lesser axes in the plane of $y'\,z'$. It is evident that a circle equal to the final circle (D^{11}) may be obtained from the elliptic cone (E^{11}), by cutting that elliptic cone by any one of the four following planes,

$$z' = \pm\left(\frac{\delta\alpha'}{\delta x}\right)^{-1} \pm y'\sqrt{\left(\frac{\delta\alpha'}{\delta x}\right)^2\left(\frac{\delta\beta'}{\delta y}\right)^{-2} - 1}\ ; \qquad (\mathrm{F}^{11})$$

and in like manner the four elliptic sections of the circular cone (B^{11}), made by the same four planes, are all equal and similar to the final ellipse (C^{11}). In general it is easy to prove by the equations of the initial ray-lines (A^{11}), that whatever final locus we take for the point B_δ, represented by the equation

$$\delta y = f(\delta x), \qquad (\mathrm{G}^{11})$$

the corresponding initial cone

$$\frac{y'}{z'}\left(\frac{\delta\beta'}{\delta y}\right)^{-1} = f\left(\frac{x'}{z'}\left(\frac{\delta\alpha'}{\delta x}\right)^{-1}\right) \qquad (\mathrm{H}^{11})$$

will have four sections equal and similar to this final locus, namely, the sections by the four planes (F^{11}). We may therefore consider these as *four guiding planes* for the initial ray, since *each contains for any proposed final curve or locus* (G^{11}) *of the final point* B_δ, *an equal and similar guiding curve or locus, which is a section of the sought initial cone, and by which therefore that cone may be determined.* If, then, we know these four *guiding planes*, or any one of them, and the corresponding system of final and initial rectangular directions, or *conjugate guiding axes*, of which two are determined by a guiding plane, we shall be able to construct the initial ray-line or ray-cone corresponding to any final position or locus of the point B_δ. The fourth and last general problem of those proposed above, may therefore be considered as resolved, by this theory of the guiding planes and guiding axes.

We see then that in order to compare completely the extreme directions of any two near luminous paths

$$(A, B)_\chi, \ (A', B')_{\chi+\delta\chi},$$

in which A is the initial and B the final point of a given path, and A', B', are any other initial and final points infinitely near to these, the following geometrical *elements of arrangement*, or some data equivalent to them, are necessary and sufficient to be known.

First. The final axis, and the initial axis, of chromatic dispersion; and the corresponding final and initial constants ξ, ξ', with their proper signs, to indicate the directions, as well as the quantities of dispersion.

Second. The final axis, and the initial axis, of curvature of the given path.

Third. The final pair, and the initial pair, of axes of curvature of the guiding paraboloids, at the ends of this given path; and the final and initial constants of deviation n, n'.

Fourth. A guiding plane for the initial ray-lines, and a guiding plane for the final ray-lines; together with the final system and the initial system of rectangular directions, or conjugate guiding axes, connected with these guiding planes.

When these different elements of arrangement of the extreme ray-lines are known, we can deduce from them the dependence of $\delta\alpha$, $\delta\beta$, $\delta\alpha'$, $\delta\beta'$, and more generally of $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\delta\alpha'$, $\delta\beta'$, $\delta\gamma'$, on δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$; and reciprocally when this latter dependence has been deduced from the partial differential coefficients of the characteristic or related functions, we can deduce from it the geometrical elements above mentioned.

Application of the Elements of Arrangement. Connexion of the two final Vergencies, and Planes of Vergency, and Guiding Lines, with the two principal Curvatures and Planes of Curvature of the Guiding Paraboloid, and with the Constant of Deviation. The Planes of Curvature are the Planes of Extreme Projection of the final Ray-Lines.

16. To give now an example of the application of these geometrical elements of arrangement, let us employ them to determine the *conditions of intersection of two near final ray-lines*, corresponding to a given colour and to a given luminous origin; and let us suppose, for simplicity, that one of these two straight ray-lines being the final portion or final tangent of a given luminous path $(A, B)_{\chi}$, the other corresponds (as in the third of the foregoing problems) to a final point B_{δ} on the given final plane perpendicular to this given path at B. Then if the constant n of deviation vanishes, so that the final ray-lines are normals to the guiding paraboloid, the condition of intersection requires evidently that the near point B_{δ} should be in one of the two principal diametral planes, that is, on one of the two rectangular tangents to the lines of curvature on this surface; and the corresponding point of intersection must be one of the two centres of curvature. But when n does not vanish, the deviation of the ray-lines obliges us to alter this result. The intersection of the near ray-line with the given ray-line will not now take place for the directions of the lines of curvature; but for those other directions, if any, for which the angular deviation $n\delta l$ of the ray-line from the normal is equal and contrary to the angular deviation of the normal from the corresponding plane of normal section, that is, from the corresponding diametral plane of the guiding paraboloid. This latter deviation, abstracting from sign, is, by the general properties of normals, equal to the semidifference of curvatures multiplied by the element of the normal section δl, and by the sign of twice the inclination of this element to either of the lines of curvature; it cannot therefore destroy the deviation $n\delta l$ of the ray-line from the normal, unless the semidifference of the two principal curvatures of the paraboloid is greater, or at least not less, abstracting from sign, than the constant of deviation n; this then is a necessary condition for the possibility of the intersection sought. But when the semidifference of curvatures is greater (abstracting from sign) than n, then there are two distinct directions P_1, P_2, of the normal or diametral plane of section, symmetrically placed with respect to the two principal planes of curvature, and such that if the element of section δl be contained in either of these two planes, P_1, P_2, (but not if the element δl be in any other normal plane,) the corresponding ray-line from the extremity of that element will be contained in the same normal plane P_1 or P_2, and will intersect the given ray-line as required; and the point of intersection of these two near ray-

lines will be the centre of curvature of the corresponding normal section. We may therefore call the curvatures of these two diametral sections the *two vergencies* of the final ray-lines; and the two corresponding planes P_1 P_2 we may call the *two planes of vergency.*

The same conclusions may be deduced algebraically from the equations (K^{10}), which give the following conditions of intersection of a near ray-line with the given ray-line or axis of z,

$$0=(z^{-1}-r)\,\delta x-n\delta y\,;\quad 0=(z^{-1}-t)\,\delta y+n\delta x\,;\qquad (\mathrm{I}^{11})$$

z being the sought ordinate of intersection, and therefore z^{-1} the vergency: for thus we find by elimination the following quadratic to determine the ratio of δx, δy, that is the direction of δl,

$$(t-r)\,\delta x\delta y=n\,(\delta y^2+\delta x^2),\qquad (\mathrm{K}^{11})$$

which may be put under the form

$$\sin.\ 2\,\phi=\frac{2n}{t-r},\qquad (\mathrm{L}^{11})$$

the angle ϕ being, as in (R^{10}), the inclination of δl to the axis of x, that is, to one of the tangents of the lines of curvature, while r, t, are the two curvatures themselves, of the guiding paraboloid; there are therefore two real directions of δl, or one, or none, corresponding to the intersection supposed, according as we have

$$\left(\frac{t-r}{2}\right)^2 >,\ \text{or}\ =,\ \text{or}\ < n^2\,;\qquad (\mathrm{M}^{11})$$

so that we are thus conducted anew to the same conditions of reality, and to the same symmetric directions of the two planes of vergency, which we obtained before by a reasoning of a more geometrical kind. The same conditions may also be obtained by considering the quadratic for the vergency itself, namely

$$(z^{-1}-r)\,(z^{-1}-t)+n^2=0,\qquad (\mathrm{N}^{11})$$

which results from the equations (I^{11}) and shows that the sum and product of the two vergencies may be thus expressed, by means of the curvatures r, t, and the constant of deviation n,

$$z_1^{-1}+z_2^{-1}=r+t\,;\quad z_1^{-1}\,z_2^{-1}=rt+n^2.\qquad (\mathrm{O}^{11})$$

The equations (I^{11}) give also, by elimination of n,

$$z^{-1}=r\cos.\,\phi^2+t\sin.\,\phi^2\,;\qquad (\mathrm{P}^{11})$$

we see, therefore, as before, that the two vergencies, when real, of the final ray-lines, are the curvatures of the two corresponding sections of the guiding paraboloid. In general the centre of curvature of any section of this surface, made by a normal plane drawn through the given final ray-line, is the common *focus by projection* of all the near ray-lines from the points of that section; that is, the projections of these near ray-lines on this plane, all pass through this centre of curvature. *The two rectangular planes of curvature, or principal diametral planes, of the guiding paraboloid, may therefore be called the planes of extreme projection;* under which view they were considered in the First Supplement, for the case of an uniform medium, and were proposed as *a pair of natural co-ordinate planes* passing through any given straight ray. The two planes of vergency, for the case of straight final rays, were also considered in that First Supplement, in connexion with the two developable pencils or ray-surfaces which pass through a given straight ray, and of which the two tangent planes contain rays infinitely near, and therefore coincide with the two planes of vergency.

When the planes of vergency are real and distinct, then, whether the final rays are straight or curved, there exist *two guiding lines* perpendicular to the given final ray-line, which are both intersected by all the near final ray-lines from the points B_δ on the given final plane of xy, and which therefore suffice to determine the geometrical arrangement and relations of that system of final ray-lines. To prove the existence and determine the positions of these two guiding lines, let us examine what conditions are necessary and sufficient, in order that a right line having for equations

$$y = x \tan. \Phi, \ z = Z, \qquad (\mathrm{Q}^{11})$$

should be intersected by all the near final ray-lines of the system (K^{10}). These conditions are

$$Z^{-1} = r + n \text{ cotan. } \Phi = t - n \tan. \Phi ; \qquad (\mathrm{R}^{11})$$

they give

$$\sin. 2\Phi = \frac{2n}{t-r}, \qquad (\mathrm{S}^{11})$$

and

$$(Z^{-1} - r)(Z^{-1} - t) + n^2 = 0 : \qquad (\mathrm{T}^{11})$$

when therefore

$$(t-r)^2 > 4n^2, \qquad (\mathrm{U}^{11})$$

that is, when there are two real vergencies there are also two real guiding lines of the kind explained above; and these two guiding lines are contained in the two planes of vergency, and cross the final ray-line in the two corresponding points in which it is crossed by other ray-lines of the same system: the intersection of each guiding line with the given final ray-line being the point of convergence or divergence of the near ray-lines contained in that plane of vergency which contains the other guiding line. When the constant of deviation n vanishes, these guiding lines are necessarily real, and are the axes of the two principal circles of curvature of the guiding paraboloid. And when the final rays are straight, then, whether n vanishes or not, *the two guiding lines* (if real) *are tangents to the two caustic surfaces;* that is, to *the two surfaces which are touched by the final rays, and are the loci of the two points of vergency.* If the guiding lines are imaginary then the points of vergency are so too, and the final rays are not all tangents to any common surface. We shall have occasion to resume hereafter the theory of the caustic and developable surfaces.

If it happen that

$$t-r=\pm 2n, \qquad (\mathrm{V}^{11})$$

without $t-r$ and n separately vanishing, then the two planes of vergency close up into one plane, bisecting one pair of the right angles formed by the two principal planes of curvature of the guiding paraboloid; the two vergencies reduce themselves to a single vergency, corresponding to this single plane, and equal to the semisum of the two curvatures of the same surface: and the two guiding lines reduce themselves to a single guiding line, passing through the corresponding point of convergence or divergence, and having still the property of being intersected by all the near final ray-lines, although this property is not now sufficient to determine this system of ray-lines.

But if the two members of (V^{11}) vanish separately, that is, if the difference of curvatures and the constant of deviation are separately equal to zero, then the guiding paraboloid is a surface of revolution, having its summit at the given final point B, and all the near final ray-lines are normals to this paraboloid of revolution, and (with the same order of approximation) to the osculating sphere at its summit, and they all pass through the centre of this sphere. Reciprocally, if there be any one point 0, 0, Z, through which all the final ray-lines pass, the equations (K^{10}) give

$$n=0, \quad t=r=Z^{-1}: \qquad (\mathrm{W}^{11})$$

and the more general equations (V^{9}), in which the rectangular axes of x and y are arbitrary, give

$$\frac{\delta\alpha}{\delta x}=\frac{\delta\beta}{\delta y}=-Z^{-1}; \quad \frac{\delta\alpha}{\delta y}=0; \quad \frac{\delta\beta}{\delta x}=0; \qquad (\mathrm{X}^{11})$$

that is, by (G^{9}), or (C^{9}),

$$\left.\begin{aligned} &\frac{\delta^2 V}{\delta x^2}+Z^{-1}\frac{\delta^2 v}{\delta\alpha^2}=\frac{\delta^2 v}{\delta\alpha\,\delta x};\\ &\frac{\delta^2 V}{\delta x\,\delta y}+Z^{-1}\frac{\delta^2 v}{\delta\alpha\,\delta\beta}=\frac{\delta^2 v}{\delta\alpha\,\delta y}=\frac{\delta^2 v}{\delta\beta\,\delta x};\\ &\frac{\delta^2 V}{\delta y^2}+Z^{-1}\frac{\delta^2 v}{\delta\beta^2}=\frac{\delta^2 v}{\delta\beta\,\delta y}. \end{aligned}\right\} \qquad (\mathrm{Y}^{11})$$

When the final rays are straight, and satisfy these last conditions (Y^{11}), which then reduce themselves to the following,

$$\frac{\delta^2 V}{\delta x^2} + Z^{-1}\frac{\delta^2 v}{\delta \alpha^2} = 0, \quad \frac{\delta^2 V}{\delta x \delta y} + Z^{-1}\frac{\delta^2 v}{\delta \alpha \delta \beta} = 0, \quad \frac{\delta^2 V}{\delta y^2} + Z^{-1}\frac{\delta^2 v}{\delta \beta^2} = 0, \qquad (Z^{11})$$

the given final ray becomes one of those which we have called *principal rays* in former memoirs, and the point of convergence or divergence 0, 0, Z, is what we have called a *principal focus*.

Second Application of the Elements. Arrangement of the Near Final Ray-lines from an Oblique Plane. Generalisation of the Theory of the Guiding Paraboloid and Constant of Deviation. General Theory of Deflexures of Surfaces. Circles and Axes of Deflexure. Rectangular Planes and Axes of Extreme Deflexure. Deflected Lines passing through these Axes, and having the Centres of Deflexure for their respective Foci by Projection. Conjugate Planes of Deflexure, and Indicating Cylinder of Deflexion.

17. The foregoing theorems respecting the mutual relations of the final ray-lines, suppose that the near final point B_δ is on the given plane which is perpendicular to the given luminous path $(A, B)_\chi$ at its given final point B: but analogous theorems can be found for the more general case where the near final point B' is not in this given perpendicular plane, by combining the solutions of the second and third of the four problems lately discussed; that is, by considering jointly the second and third sets of coefficients (O^9), and therefore by employing the following equations for a final ray-line,

$$\left.\begin{aligned} x &= \delta x + z\left(\frac{\delta\alpha}{\delta x}\,\delta x + \frac{\delta\alpha}{\delta y}\,\delta y + \frac{\delta\alpha}{\delta z}\,\delta z\right), \\ y &= \delta y + z\left(\frac{\delta\beta}{\delta x}\,\delta x + \frac{\delta\beta}{\delta y}\,\delta y + \frac{\delta\beta}{\delta z}\,\delta z\right). \end{aligned}\right\} \qquad (A^{12})$$

If, in these equations, we establish no relation between δx, δy, δz, then the system of these final ray-lines (A^{12}) is what has been called (in my Theory of Systems of Rays) *a System of the Third Class*, because the equations of a ray-line in this system involve *three* arbitrary elements of position, namely, the co-ordinates δx, δy, δz, of the near point B'; but to study more conveniently the properties of this total system of the third class, we may decompose it into partial *systems of the second class*, that is, systems with only two arbitrary elements of position, by assuming some relation, with an arbitrary parameter, between the three co-ordinates δx, δy, δz, or, in other words, by assuming some arbitrary and variable surface, as a locus for the near point B'. For example we may assume, as this locus, an oblique plane passing through the given point B, and having for equation

$$\delta z = p\delta x + q\delta y, \qquad (B^{12})$$

in which one of the two parameters p, q, is arbitrary, and the other depends on it by some assumed law; and then, for every such assumed plane locus (B^{12}), we shall have to consider a partial system of the second class, deduced from and included in the total system of the third class (A^{12}); namely, a system in which the equations of a ray-line are follows,

$$\left.\begin{aligned} x &= \delta x + z\left(\frac{\delta\alpha}{\delta x} + p\frac{\delta\alpha}{\delta z}\right)\delta x + z\left(\frac{\delta\alpha}{\delta y} + q\frac{\delta\alpha}{\delta z}\right)\delta y; \\ y &= \delta y + z\left(\frac{\delta\beta}{\delta x} + p\frac{\delta\beta}{\delta z}\right)\delta x + z\left(\frac{\delta\beta}{\delta y} + q\frac{\delta\beta}{\delta z}\right)\delta y. \end{aligned}\right\} \qquad (\mathrm{C}^{12})$$

Let us therefore consider the geometrical arrangement and properties of this system of final ray-lines (C^{12}), corresponding to the oblique plane locus (B^{12}) of the final point B'.

The system (C^{12}), of ray-lines from the arbitrary oblique plane (B^{12}), includes, as a particular case, the system of ray-lines from the plane of no obliquity: that is, the system (V^9), considered in a former number. And as the ray-lines of that particular system (V^9) were found to have a remarkable connexion with the guiding paraboloid (Z^4), which touched the given perpendicular plane locus of the near final point B_δ, and which satisfied the differential condition of the second order (Y^9): so, the ray-lines of the more general system (C^{12}) may be shown to be connected in an analogous manner with the following more general paraboloid, which satisfies the same differential condition (Y^9), and touches the more general oblique plane locus (B^{12}) at the given final point B,

$$z = px + qy + \tfrac{1}{2}rx^2 + sxy + \tfrac{1}{2}ty^2; \qquad (\mathrm{D}^{12})$$

in which p, q, retain their recent meanings, and the coefficients r, s, t have the following values,

$$\left.\begin{aligned} r &= -\left(\frac{\delta\alpha}{\delta x} + p\frac{\delta\alpha}{\delta z}\right); \quad t = -\left(\frac{\delta\beta}{\delta y} + q\frac{\delta\beta}{\delta z}\right); \\ s &= -\tfrac{1}{2}\left(\frac{\delta\beta}{\delta x} + \frac{\delta\alpha}{\delta y} + p\frac{\delta\beta}{\delta z} + q\frac{\delta\alpha}{\delta z}\right). \end{aligned}\right\} \qquad (\mathrm{E}^{12})$$

But in order to develope this more general connexion, between the ray-lines (C^{12}), and the paraboloid (D^{12}), it will be useful previously to establish some general theorems respecting the deflexures of curved surfaces, which include some of the known theorems respecting their curvatures and planes of curvature.

Let us then consider the paraboloid (D^{12}), or any other curved surface which has, at the origin of co-ordinates, a complete contact of the second order therewith, and which is therefore approximately represented by the same equation: that is, (on account of the arbitrary position of the origin, and arbitrary values of the coefficients $p, q, r, s, t,$) any surface of continuous curvature, near any assumed point upon this surface. The tangent plane at this arbitrary point or origin, has for equation

$$z = px + qy ; \qquad (F^{12})$$

and the *deflexion* from this tangent plane, measured in the direction of the arbitrary axis of z, which we shall call the *axis of deflexion*, or in any direction infinitely near to this, is, for any point B' infinitely near to the point of contact B,

$$Deflexion = \tfrac{1}{2}\delta^2 z = \tfrac{1}{2} r\delta x^2 + s\delta x\delta y + \tfrac{1}{2} t\delta y^2. \qquad (G^{12})$$

This deflexion depends therefore on the perpendicular distance δl of the near point B' from the axis of deflexion, and on the direction of the plane containing this point and axis; in such a manner that if we put, as in (R^{10}),

$$\delta x = \delta l. \cos. \phi, \quad \delta y = \delta l. \sin. \phi,$$

and give the name of *deflexure* (after the analogy of the known name *curvature*) to the quotient $\frac{\delta^2 z}{\delta l^2}$, that is, to the double deflexion divided by the square of the perpendicular distance from the axis of deflexion, we shall have the following law of dependence of this *deflexure*, which we shall denote by f, on the angle ϕ,

$$Deflexure = f = \frac{\delta^2 z}{\delta l^2} = r \cos. \phi^2 + 2s \cos. \phi \sin. \phi + t \sin. \phi^2. \qquad (H^{12})$$

There are, therefore, *two rectangular planes of extreme deflexure*, corresponding to angles ϕ_1, ϕ_2, determined by the following formula,

$$\tan. 2\phi = \frac{2s}{r-t} ; \qquad (I^{12})$$

and if we take these for the co-ordinate planes of xz, yz, and denote the *two extreme deflexures* corresponding by f_1, f_2, we have

$$r = f_1, \ s = 0, \ t = f_2, \qquad (K^{12})$$

and the general formula for the deflexure becomes

$$f = f_1 \cos. \phi^2 + f_2 \sin. \phi^2 : \qquad (L^{12})$$

which is analogous to, and includes, the known formula for the curvature of a normal section. And as it is usual to consider a system of circles of curvature, for any given point of a curved surface, namely, the osculating circles of the normal sections of that surface, so we may now more generally consider a system of *circles of deflexure*: namely, in each plane of deflexure ϕ, a circle passing through the given point of the surface, and having its centre on the given axis of deflexion, and its curvature equal to the deflexure f; so that the radius of this circle, or the ordinate of its centre, which we may call the *radius of deflexure*, is $\frac{1}{f}$, and so that the equations of the circle of deflexure are,

$$y = x \tan. \phi, \quad x^2 + y^2 + z^2 = \frac{2z}{f}. \qquad (M^{12})$$

We may also give the name of *axis of deflexure*, to the axis of this circle, that is, to the right line having for equations

$$y = -x \text{ cotan. } \phi, \ z = \frac{1}{f} : \qquad (\text{N}^{12})$$

and we easily see that there are *two principal circles of deflexure*, analogous to the two principal circles of curvature, namely, the two circles having for equations

$$\left.\begin{array}{l} \text{First} \quad y=0, \ x^2 + z^2 = \frac{2z}{f_1}; \\ \text{Second } x=0, \ y^2 + z^2 = \frac{2z}{f_2}; \end{array}\right\} \qquad (\text{O}^{12})$$

and *two principal rectangular axes of deflexure*, namely,

$$\text{First} \quad x=0, \ z = \frac{1}{f_1}; \quad \text{Second} \quad y=0, \ z = \frac{1}{f_2}. \qquad (\text{P}^{12})$$

These principal axes of deflexure are analogous to the principal axes of curvature, that is, to the axes of the two principal osculating circles of the normal sections, in the less general theory of normals. And as, in that theory, the near normals all pass through the two principal axes of curvature, so we may now consider a more general system of right lines, which we shall call the *deflected lines*, all near the arbitrary axis of deflexion, and all passing through the two corresponding principal axes of deflexure, and therefore having for equations,

$$x = \delta x - z f_1 \delta x, \ y = \delta y - z f_2 \delta y, \qquad (\text{Q}^{12})$$

when the co-ordinates are chosen as before. These deflected lines are normals, in the present order of approximation, to the locus of the circles of deflexure (M^{12}), that is, to the surface of the fourth degree

$$x^2 + y^2 + z^2 = \frac{2z(x^2+y^2)}{f_1 x^2 + f_2 y^2}; \qquad (\text{R}^{12})$$

and they might be defined by this condition, or by the condition that they are normals, in the same order of approximation, to the following paraboloid,

$$z = \tfrac{1}{2}(f_1 x^2 + f_2 y^2), \qquad (\text{S}^{12})$$

which osculates to the locus (R^{12}), and has the property that its ordinates measure the deflexions (G^{12}) of the given surface.

A deflected line of the system (Q^{12}) is in the corresponding plane of deflexure

$$y \delta x = x \delta y, \qquad (\text{T}^{12})$$

if that plane coincide with either of those two principal rectangular planes of deflexure, which we have taken for co-ordinate planes; but otherwise the deflected line makes with the plane of deflexure an infinitesimal angle $\delta\psi$, expressed as follows,

$$\delta\psi = \tfrac{1}{2}(f_1 - f_2)\, \delta l . \text{ sin. } 2\phi : \qquad (\text{U}^{12})$$

this angle, therefore, is equal to the semidifference of the extreme deflexures multiplied by the infinitesimal perpendicular distance from the axis of deflexion, and by the sine of twice the inclination ϕ of this perpendicular (or of the plane of deflexure containing it) to one of the two rectangular planes of extreme deflexure. In this general case, the deflected line (Q^{12}) does not intersect the given axis of deflexion, which we have made the axis of z; but the deflected line (Q^{12}) always intersects its own axis of deflexure (N^{12}), in a point of which the co-ordinates may be thus expressed

$$x = \frac{-\delta\psi}{f} . \sin . \phi, \quad y = \frac{\delta\psi}{f} . \cos . \phi, \quad z = \frac{1}{f}, \qquad (\mathrm{V}^{12})$$

the symbols f, ϕ, and $\delta\psi$, retaining their recent meanings. It is easy also to see that if a near deflected line be projected on the corresponding plane of deflexure, the projection will cross the axis of deflexion in the centre of the circle of deflexure; and therefore that this centre of deflexure may be considered as a *focus by projection*, and that *the planes of extreme deflexure are planes of extreme projection.*

The foregoing results respecting the deflexures and deflected lines of a curved surface, near any given point upon that surface, and for any given axis of deflexion, may easily be expressed by general formulæ extending to an arbitrary origin and arbitrary axes of co-ordinates. If, for simplicity, we still suppose the co-ordinates rectangular, and still take the given point upon the surface for origin, and the given axis of deflexion for axis of z, but leave the rectangular co-ordinate planes of xz and yz arbitrary, so that the coefficient s in the equation of the surface shall not in general vanish, then the equations of a deflected line become

$$x = \delta x - z\,(r\delta x + s\delta y), \quad y = \delta y - z\,(s\delta x + t\delta y); \qquad (\mathrm{W}^{12})$$

since the equation of the paraboloid (S^{12}), to which they are nearly normals, and of which the ordinates measure the deflexions (G^{12}) of the given surface, becomes

$$z = \tfrac{1}{2} r x^2 + sxy + \tfrac{1}{2} t y^2. \qquad (\mathrm{X}^{12})$$

The deflexure for any plane ϕ is expressed by the general formula (H^{12}); and in like manner the general formulæ (M^{12}) (N^{12}) determine still the circle and axis of deflexure. The two principal planes of deflexure, ϕ_1, ϕ_2, are still determined by the formula (I^{12}), while the corresponding extreme deflexures, f_1, f_2, are the roots of the following quadratic

$$f^2 - f\,(r + t) + rt - s^2 = 0: \qquad (\mathrm{Y}^{12})$$

and the angular deviation $\delta\psi$ of a deflected line from the corresponding plane of deflexure, is thus expressed,

$$\delta\psi = \tfrac{1}{2}\,(f_1 - f_2) . \sin . (2\phi - 2\phi_1) . \delta l = \left(\frac{r - t}{2} . \sin .\ 2\phi - s . \cos .\ 2\phi\right) \delta l. \qquad (\mathrm{Z}^{12})$$

Before we proceed to apply these general remarks on the deflexures of surfaces to the optical question proposed in the present number, that is, to the study of the connexion of the ray-lines (C^{12}) with the paraboloid (D^{12}), we may remark that the theory which M. DUPIN has given, in his excellent *Développements de Géométrie*, of the *indicating curves* and *conjugate tangents* of a surface, may be extended from curvatures to deflexures. For if we consider the deflexion ($\frac{1}{2}\delta^2 z = \frac{1}{2} f \delta l^2$) in the given arbitrary direction of z as equal to any given infinitesimal quantity of the second order, that is, if we cut the given surface by a plane

$$z - px - qy = \tfrac{1}{2}\delta^2 z = deflexion = \text{const.}, \qquad (A^{13})$$

parallel and infinitely near to the given tangent plane (F^{12}), we obtain in general a plane curve of section which may be considered as of the second degree, namely, the *indicating curve* considered by M. DUPIN, of which the axes by their directions and values indicate the shape of the given surface near the given point, by indicating its curvatures and planes of curvature. This indicating curve is on the following *cylinder of the second degree*, which has for its indefinite axis the axis of deflexion, and which we shall call the *indicating cylinder of deflexion*,

$$rx^2 + 2sxy + ty^2 = \delta^2 z = \text{const.}; \qquad (B^{13})$$

and it is easy to see that the two principal planes of deflexure, ϕ_1, ϕ_2, are the principal diametral planes of this indicating cylinder, and that the two principal deflexures f_1, f_2, positive or negative, are equal respectively to the given double deflexion $\delta^2 z$ divided by the squares of the real or imaginary principal semidiameters or semiaxes of the cylinder, perpendicular to its indefinite axis. In general, the positive or negative deflexure f, corresponding to any plane of deflexure ϕ, is equal to the given double deflexion $\delta^2 z$ divided by the square of the real or imaginary semidiameter of the cylinder, contained in this plane of deflexure, and perpendicular to the axis of deflexion, that is, to the indefinite axis of the cylinder. Hence it follows, that if we consider any two conjugate diametral planes ϕ, $\phi_{,}$, which we shall call *conjugate planes of deflexure*, and which are connected by the relation

$$0 = r + s(\tan.\, \phi + \tan.\, \phi_{,}) + t.\, \tan.\, \phi\, \tan.\, \phi_{,}, \qquad (C^{13})$$

the sum of the two corresponding conjugate radii of deflexure, $\frac{1}{f} + \frac{1}{f_{,}}$, *is constant, and equal to the sum of the two extreme or principal radii:* that is, we have

$$\frac{1}{f} + \frac{1}{f_{,}} = \frac{1}{f_1} + \frac{1}{f_2}, \qquad (D^{13})$$

a relation which might also have been deduced from the general expression for the deflexure, without its being necessary to employ the indicating cylinder. We may remark that any two conjugate planes of deflexure, connected by the relation (C^{13}), intersect the tangent plane of the surface in two conjugate tangents of the kind considered by M. DUPIN.

Let us now resume the system of ray-lines (C^{12}), of which the equations may be put by (E^{12}) under the form

$$\left.\begin{aligned} x &= \delta x - z\,(r\delta x + s\delta y) - zn\delta y, \\ y &= \delta y - z\,(s\delta x + t\delta y) + zn\delta x, \end{aligned}\right\} \qquad (\mathrm{E}^{13})$$

if we make

$$n = \tfrac{1}{2}\left(\frac{\delta\beta}{\delta x} - \frac{\delta a}{\delta y} + p\,\frac{\delta\beta}{\delta z} - q\,\frac{\delta a}{\delta z}\right): \qquad (\mathrm{F}^{13})$$

and let us compare these ray-lines with the deflected lines from the auxiliary paraboloid (D^{12}), which have for equations

$$x = \delta x - z\,(r\delta x + s\delta y)\,,\ y = \delta y - z\,(s\delta x + t\delta y). \qquad (\mathrm{W}^{12})$$

We easily see, by this comparison, that the infinitesimal angle of deviation $\delta\nu$ of a ray-line (E^{13}) from the corresponding deflected line (W^{12}), is still determined by the same formula (M^{10})

$$\delta\nu = n\delta l,$$

as in the simpler theory of the guiding paraboloid explained in the fifteenth number; that is, this angular deviation $\delta\nu$ is still equal to the perpendicular distance δl of the near final point from the given final ray-line, multiplied by a constant of deviation n. The plane of this angle $\delta\nu$, that is, the plane containing the ray-line (E^{13}) and the deflected line (W^{12}), has for equation

$$x\delta x + y\delta y = \delta l^2 - z\,(r\delta x^2 + 2s\delta x\delta y + t\delta y^2), \qquad (\mathrm{G}^{13})$$

and therefore contains the right line having for equations

$$x\delta x + y\delta y = 0,\ \ z = \frac{\delta x^2 + \delta y^2}{r\delta x^2 + 2s\delta x\delta y + t\delta y^2}, \qquad (\mathrm{H}^{13})$$

that is, the axis of deflexure (N^{12}): results which are analogous to those of the fifteenth number, expressed by the equations (N^{10}) (O^{10}). And we may construct the final ray-line (E^{13}) by a process of rotation analogous to that already employed, namely, by making the deflected line (W^{12}), which passes through the two rectangular axes of deflexure of the auxiliary paraboloid (D^{12}), revolve round the perpendicular δl, through the infinitesimal angle $\delta\nu$, proportional to that perpendicular. The theory, therefore, of the guiding paraboloid and constant of deviation, which was given in the fifteenth number, for the ray-lines from the near points B_0 on the final perpendicular plane, extends with little modification to the ray-lines from the points B' on any final oblique plane locus passing through the given final point: namely, by employing a more general auxiliary paraboloid, and by considering deflexures and deflected lines, instead of curvatures and normals. And we may transfer to this more general auxiliary paraboloid, and to its connected constant of deviation, the reasonings of the sixteenth number, respecting the system of final ray-lines; for example, the reasonings respecting the foci by projection, and those respecting the condition of intersection of such ray-lines. And since for any given values of p, q, that is, for any given position of the oblique plane (B^{12}), we can construct the new auxiliary paraboloid (D^{12}), and its new constant of deviation (F^{13}), by the coefficients

$$\frac{\delta\alpha}{\delta x}, \frac{\delta\beta}{\delta x}, \frac{\delta\alpha}{\delta y}, \frac{\delta\beta}{\delta y}, \frac{\delta\alpha}{\delta z}, \frac{\delta\beta}{\delta z},$$

that is, by means of the former guiding paraboloid (Z^{9}) and the former constant of deviation (B^{10}), and by the magnitude and plane of curvature (T^{9}) of the final ray, we may be considered as having reduced the theory of the geometrical arrangement and relations of the system of final ray-lines (C^{12}), from an oblique plane (B^{12}), to the theory of the *elements of arrangement*, which was given in the fifteenth number.

Construction of the New Auxiliary Paraboloid, (or of an Osculating Hyperboloid,) and of the New Constant of Deviation, for Ray-lines from an Oblique Plane, by the former Elements of Arrangement.

18. To construct the new auxiliary paraboloid (D^{12}) by the former elements of arrangement, we may observe that this new paraboloid not only touches the given oblique plane (B^{12}) at the given final point B of the original luminous path, but osculates in all directions at that given point to a certain hyperboloid, represented by the following equation,

$$z = px + qy + \tfrac{1}{2}r_{\circ}x^2 + s_{\circ}xy + \tfrac{1}{2}t_{\circ}y^2 - \tfrac{1}{2}z\left(x\frac{\delta\alpha}{\delta z} + y\frac{\delta\beta}{\delta z}\right); \qquad (\mathrm{I}^{13})$$

in which $r_{\circ}$ $s_{\circ}$ $t_{\circ}$ are the particular values

$$r_{\circ} = -\frac{\delta\alpha}{\delta x}, \ s_{\circ} = -\tfrac{1}{2}\left(\frac{\delta\alpha}{\delta y} + \frac{\delta\beta}{\delta x}\right), \ t_{\circ} = -\frac{\delta\beta}{\delta y}, \qquad (\mathrm{K}^{13})$$

of the coefficients r s t, deduced from the general expressions (E^{12}) by making

$$p = 0, \ q = 0, \qquad (\mathrm{L}^{13})$$

that is, by passing to the case of no obliquity; so that the equation (Z^{9}) of the guiding paraboloid may be put under the form

$$z = \tfrac{1}{2}r_{\circ}x^2 + s_{\circ}xy + \tfrac{1}{2}t_{\circ}y^2, \qquad (\mathrm{M}^{13})$$

which includes the form (G^{10}). Reciprocally, the sought paraboloid (D^{12}) is the only paraboloid which has its indefinite axis parallel to the given final ray-line, and osculates in all directions at the given final point to the hyperboloid (I^{13}): it is therefore sufficient to construct this osculating hyperboloid, in order to deduce the sought paraboloid (D^{12}). We might even employ the hyperboloid as a new guiding surface for the ray-lines from the oblique plane, instead of employing the paraboloid, since thes two osculating surfaces have the same deflexures and deflected lines, near their given point of osculation.

Now to construct the osculating hyperboloid (I^{13}), by the oblique plane (B^{12}) or (F^{12}), and by the former elements of arrangement, that is, by the guiding paraboloid (M^{13}), and by the coefficients $\frac{\delta\alpha}{\delta z}, \frac{\delta\beta}{\delta z}$, which determine the magnitude and plane of curvature of the final ray, we may compare the sought hyperboloid (I^{13}) with the following new paraboloid

$$z = px + qy + \tfrac{1}{2} r_0 x^2 + s_0 xy + \tfrac{1}{2} t_0 y^2, \qquad (\mathrm{N}^{13})$$

which may be called *the guiding paraboloid removed*, since it is equal and similar to the guiding paraboloid (M^{13}), and may be obtained by transporting that guiding paraboloid without rotation to a new position such that it touches the given oblique plane at the given point. The intersection of the hyperboloid (I^{13}) and paraboloid (N^{13}), consists in general of an ellipse or hyperbola in the given plane

$$z = 0, \qquad (\mathrm{O}^{13})$$

perpendicular to the given final ray, and of a parabola in the plane

$$x \frac{\delta\alpha}{\delta z} + y \frac{\delta\beta}{\delta z} = 0, \qquad (\mathrm{P}^{13})$$

which contains the given final ray-line or ray-tangent, and is perpendicular to the final plane of curvature of the ray. If then, we make this final plane of curvature the plane of xz, so that its equation shall be

$$y = 0, \qquad (\mathrm{Q}^{13})$$

and so that, by (T^9),

$$\frac{\delta\beta}{\delta z} = 0, \qquad (\mathrm{R}^{13})$$

we shall have the following equations for the two curves of intersection; first, for the ellipse or hyperbola,

$$z = 0, \; px + qy + \tfrac{1}{2} r_0 x^2 + s_0 xy + \tfrac{1}{2} t_0 y^2 = 0; \qquad (\mathrm{S}^{13})$$

and secondly, for the parabola,

$$x = 0, \; z = qy + \tfrac{1}{2} t_0 y^2 : \qquad (\mathrm{T}^{13})$$

and these two curves may be considered as known, since they are the intersections of two known planes with the known guiding paraboloid removed to a known position. To examine now how far a surface of the second degree is restricted by the condition of containing these two known curves, and what other conditions are necessary, in order to oblige this surface to be the hyperboloid sought, let us employ the following general form for the equation of a surface of the second degree,

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + K = 0, \qquad (\mathrm{U}^{13})$$

and let us seek the relations which restrict the coefficients of this equation when the surface is obliged to contain the two known curves. The condition of containing the parabola (T^{13}), gives

$$K=0,\ H=-Iq,\ E=0,\ C=0,\ B=-\tfrac{1}{2}It_0; \qquad (\mathrm{V}^{13})$$

so that, by this condition alone, the general equation (U^{13}) is reduced to the following form,

$$z = qy + \tfrac{1}{2}t_0 y^2 - \frac{x}{I}(G + Fz + Dy + Ax). \qquad (\mathrm{W}^{13})$$

In order that this less general surface of the second degree, (W^{13}), should contain the ellipse or hyperbola (S^{13}), it is necessary and sufficient that we should have the relations,

$$G=-Ip,\ D=-Is_0,\ A=-\tfrac{1}{2}Ir_0: \qquad (\mathrm{X}^{13})$$

the general equation, therefore, of all those surfaces of the second degree which contain at once the two known curves (S^{13}) (T^{13}), involves only one arbitrary coefficient, and may be put under the form

$$z = px + qy + \tfrac{1}{2}r_0 x^2 + s_0 xy + \tfrac{1}{2}t_0 y^2 + \lambda xz. \qquad (\mathrm{Y}^{13})$$

This general equation, with the arbitrary coefficient λ, belongs to the guiding paraboloid removed, that is, to the surface (N^{13}), when we suppose

$$\lambda = 0; \qquad (\mathrm{Z}^{13})$$

and the same general equation belongs by (R^{13}) to the sought hyperboloid (I^{13}), when

$$\lambda = -\tfrac{1}{2}\frac{\delta a}{\delta z}. \qquad (\mathrm{A}^{14})$$

To put this last condition under a geometrical form, let us, as we have already considered the intersections of the hyperboloid with the two rectangular co-ordinate planes of xy and yz, consider now its intersection with the third co-ordinate plane of xz, that is, with the plane of curvature (Q^{13}) of the given final ray. This intersection is the following hyperbola,

$$y=0,\ z = px + \tfrac{1}{2}r_0 x^2 - \tfrac{1}{2}\frac{\delta a}{\delta z}xz, \qquad (\mathrm{B}^{14})$$

and the corresponding intersection for the surface (Y^{13}) is

$$y=0,\ z = px + \tfrac{1}{2}r_0 x^2 + \lambda xz; \qquad (\mathrm{C}^{14})$$

the condition (A^{14}) is therefore equivalent to an expression of the coincidence of these two intersections; and if we oblige the surface of the second degree (U^{13}) to contain the three curves (S^{13}) (T^{13}) (B^{14}), in the three rectangular co-ordinate planes, we shall thereby oblige it to become the sought hyperboloid (I^{13}). It is not necessary, however, though it is sufficient, to assign the hyperbola (B^{14}), as a third curve upon this hyperboloid. For, in general, if we know the intersections of a surface of the second degree with two known planes, there remains only one unknown quantity in the equation of that surface, and the intersection with a third known plane is more than sufficient to determine it. Thus, in the present question, if the intersection (C^{14}) be distinct from the following parabola

$$y=0, \ z=px+\tfrac{1}{2}r_{\circ}x^2, \qquad (\mathrm{D}^{14})$$

that is, if the surface (Y^{13}), containing the two known curves (S^{13}) (T^{13}), be distinct from the known guiding paraboloid removed, which also contains the same two curves, the intersection (C^{14}) with the plane of curvature of the ray is in general a hyperbola, which touches the known parabola (D^{14}) at the known origin of co-ordinates, and meets this parabola again in another known point on the axis of x, that is on the radius of curvature of the known final ray, namely, in the point

$$x=-\frac{2p}{r_{\circ}}, \ y=0, \ z=0; \qquad (\mathrm{E}^{14})$$

the hyperbola (C^{14}) has also one asymptote parallel to the known final ray-line or axis of z, namely, the asymptote having for equations

$$x=\frac{1}{\lambda}, \ y=0, \qquad (\mathrm{F}^{14})$$

and it will be entirely determined, if, in addition to the foregoing properties, we know also a line parallel to its other asymptote, namely, to that which has for equations

$$x=-2\left(\frac{\lambda}{r_{\circ}}\right) z-\frac{1}{\lambda}-\frac{2p}{r_{\circ}}, \ y=0: \qquad (\mathrm{G}^{14})$$

it will therefore be obliged to coincide with the hyperbola (B^{14}), if only we oblige its second asymptote (G^{14}) to be parallel to the following known right line,

$$x=\frac{z}{r_{\circ}}\frac{\delta a}{\delta z}, \ y=0, \qquad (\mathrm{H}^{14})$$

in which the coefficient

$$\frac{1}{r_{\circ}}\frac{\delta a}{\delta z}=\frac{\textit{curvature of final ray}}{\textit{deflexure of guiding paraboloid}}, \qquad (\mathrm{I}^{14})$$

the plane of the deflexure $r_{\circ}$ being the plane of curvature of the ray. We see, then, that this last condition, respecting the direction of the second asymptote (G^{14}) of the hyperbolic section (C^{14}), is sufficient, when combined with the conditions of containing the two known curves (S^{13}) (T^{13}), to determine completely the sought hyperboloid (I^{13}). Even the conditions of containing the two curves (S^{13}) (T^{13}) are not perfectly distinct and independent; nor would their coexistence be possible, in the determination of a surface of the second degree, if the two points in which the parabola (T^{13}) is intersected by the axis of y, that is, by the intersection-line of the planes of the two curves, namely, the origin and the point

$$x=0, \quad y=-\frac{2q}{t_0}, \quad z=0, \qquad (\mathrm{K}^{14})$$

were not also contained on the ellipse or hyperbola (S^{13}). But we may confine ourselves to the last chosen conditions, of having these two known curves as the intersections of the hyperboloid with two known planes, and of having known directions for the asymptotes of its hyperbolic curve of intersection with a third known plane, as adequate and sufficiently simple conditions for the construction of the sought hyperboloid, and thereby of the auxiliary paraboloid (D^{12}), to which that hyperboloid osculates. And with respect to the new constant of deviation n, connected with this auxiliary paraboloid, we may put its general value (F^{13}) under the form

$$n=n_0+\tfrac{1}{2}p\frac{\delta\beta}{\delta z}-\tfrac{1}{2}q\frac{\delta\alpha}{\delta z}, \qquad (\mathrm{L}^{14})$$

n_0 being the particular value

$$n_0=\tfrac{1}{2}\left(\frac{\delta\beta}{\delta x}-\frac{\delta\alpha}{\delta y}\right) \qquad (\mathrm{M}^{14})$$

for the plane of no obliquity, that is, the value (B^{10}) connected with the guiding paraboloid (Z^9) in the theory of the elements of arrangement which was given in a former number: we may therefore construct the new constant n, as the ordinate z of a plane

$$z=px+qy+n_0, \qquad (\mathrm{N}^{14})$$

which is parallel to the given oblique plane (B^{12}), and contains the point

$$x=0, \quad y=0, \quad z=n_0, \qquad (\mathrm{O}^{14})$$

so that it intersects the axis of z at a distance from the origin = the old constant of deviation n_0. The other co-ordinates x, y, to which the ordinate $z=n$ corresponds, are

$$x=\tfrac{1}{2}\frac{\delta\beta}{\delta z}, \quad y=-\tfrac{1}{2}\frac{\delta\alpha}{\delta z}, \qquad (\mathrm{P}^{14})$$

so that the corresponding line $\sqrt{x^2+y^2}$ is equal to half the curvature of the ray, and is perpendicular to the radius of that curvature.

The details of the present number have been given, in order to illustrate the subject, by combining it more closely with geometrical conceptions; but the new auxiliary paraboloid, and the new constant of deviation, might have been considered as sufficiently defined by their former algebraical expressions.

Condition of Intersection of Two Near Final Ray-lines. Conical Locus of the Near Final Points, in a variable medium, which satisfy this condition. Investigations of MALUS. *Illustration of the Condition of Intersection, by the Theory of the Auxiliary Paraboloid, for Ray-lines from an Oblique Plane.*

19. Returning now to the system of final ray-lines (C^{12}) from an oblique plane (B^{12}), let us consider the condition necessary in order that one of these near final ray-lines (C^{12}) may intersect the given final ray-line or axis of z. This condition may be at once obtained by making x and y vanish in the equations (C^{12}), and then eliminating z; it may therefore be thus expressed,

$$\delta x.\left\{\left(\frac{\delta\beta}{\delta x}+p\frac{\delta\beta}{\delta z}\right)\delta x+\left(\frac{\delta\beta}{\delta y}+q\frac{\delta\beta}{\delta z}\right)\delta y\right\}$$

$$=\delta y.\left\{\left(\frac{\delta\alpha}{\delta x}+p\frac{\delta\alpha}{\delta z}\right)\delta x+\left(\frac{\delta\alpha}{\delta y}+q\frac{\delta\alpha}{\delta z}\right)\delta y\right\}, \qquad (\mathrm{Q}^{14})$$

or more concisely thus, on account of the equation of the oblique plane (B^{12}),

$$\delta x.\left(\frac{\delta\beta}{\delta x}\delta x+\frac{\delta\beta}{\delta y}\delta y+\frac{\delta\beta}{\delta z}\delta z\right)$$

$$=\delta y.\left(\frac{\delta\alpha}{\delta x}\delta x+\frac{\delta\alpha}{\delta y}\delta y+\frac{\delta\alpha}{\delta z}\delta z\right), \qquad (\mathrm{R}^{14})$$

that is,

$$\delta x\ \delta\beta=\delta y\ \delta\alpha; \qquad (\mathrm{S}^{14})$$

it is therefore necessary and sufficient, for the intersection sought, that the near final point B′ should be on a certain conical locus of the second degree, determined by the equation (R^{14}), between the co-ordinates δx, δy, δz. A conical locus of this kind, appears to have been first discovered by MALUS. That excellent mathematician and observer had occasion, in his *Traité D'Optique,* to make some remarks on the general properties of a system of right-lines in space, represented by equations of the form

$$\frac{x-x'}{m}=\frac{y-y'}{n}=\frac{z-z'}{o},$$

in which m, n, o, are any given functions of the co-ordinates x', y', z', of a point through which the line is supposed to pass, and by which it is supposed to be determined; and he remarked that the condition of intersection of a line thus determined, with the corresponding near line from a point infinitely near, was expressed by an equation of the second degree between the differentials of the co-ordinates x', y', z', which might be considered as the equation of a conical locus of the second degree for the infinitely near point. The theory of systems of rays which was given by MALUS, differs much, in form and in extent, from that proposed in the present Supplement;

especially because, in the former theory, the coefficients which mark the direction of a ray were left as independent and unconnected functions, whereas, in the latter, they are shown to be connected with each other, and to be deducible by uniform methods from one characteristic function. But the mere consideration of the existence of some functional laws, whether connected or arbitrary, of dependence of the coefficients $m\,n\,o$ on the co-ordinates $x'\,y'\,z'$, or of $\alpha\,\beta\,\gamma$ on $x\,y\,z$, conducts easily, as we have seen, to a conical locus of the kind (R^{14}). This result may however be illustrated by the theory which we have given of the geometrical relations of the near final ray-lines from an oblique plane with the deflected lines of a certain auxiliary paraboloid, and with a certain law and constant of deviation.

For, according to the theory of these relations, the ray-line from a near final point B' on a given oblique plane drawn through the given point B, will or will not intersect the given final ray-line from B, according as its deviation $\delta\nu$ from its own deflected line does or does not compensate for the deviation $\delta\psi$ of that deflected line from the corresponding plane of deflexure, by these two deviations being equal in magnitude but opposite in direction; the condition of intersection may therefore be thus expressed,

$$\delta\nu + \delta\psi = 0\,; \qquad (\mathrm{T}^{14})$$

or, by the values of the deviations $\delta\nu$, $\delta\psi$, established in the seventeenth number,

$$n = \frac{t-r}{2} \,.\, \sin.\, 2\phi + s \,.\, \cos.\, 2\phi, \qquad (\mathrm{U}^{14})$$

that is,

$$n\,(\delta x^2 + \delta y^2) = (t-r)\,\delta x\,\delta y + s\,(\delta x^2 - \delta y^2): \qquad (\mathrm{V}^{14})$$

and the condition of intersection thus obtained, by the consideration of two equal and opposite deviations, is, on account of the meanings $(E^{12})\,(F^{13})$ of n, r, s, t, equivalent to (Q^{14}), and therefore to the equation (R^{14}) of the cone of the second degree. In this manner, then, as well as by the former less geometrical process, we might perceive that the two planes of vergency for the ray-lines from an oblique plane, (determined by (U^{14}) or (V^{14}), and analogous to the two less general planes of vergency considered in the sixteenth number,) intersect the oblique plane in the same two lines in which that plane intersects a certain cone of the second degree, through the centre of which cone it passes; and that the planes of vergency are imaginary when the oblique plane does not intersect this cone. We may remark that the intersection of the oblique plane with the cone, or of a near final ray-line from the oblique plane with the given final ray-line, is impossible, when the constant of deviation corresponding to the oblique plane is greater (abstracting from its sign) than the semidifference of the extreme deflexures of the auxiliary paraboloid: for then the compensation of the two deviations $\delta\nu$, $\delta\psi$, is impossible, the near ray-line always deviating more from the corresponding deflected line of the auxiliary paraboloid, than this deflected line from the corresponding plane of deflexure. And when the compensation and therefore the intersection becomes possible, by the constant of deviation being less than the semidifference of the two extreme deflexures, then the two real planes of vergency of the near final ray-lines from the oblique plane are symmetrically situated with respect to the two rectangular planes of extreme deflexure: which latter planes may also, for a reason already alluded to, be called the planes of extreme projection of the final ray-lines.

Other Geometrical Illustrations of the Condition of Intersection, and of the Elements Arrangement. Composition of Partial Deviations. Rotation round the Axis of Curvature of a Final Ray.

20. The condition of intersection of two near final ray-lines may also be illustrated, and might have obtained, by other geometrical considerations, on which we shall dwell a little, because they will help to illustrate and improve the theory of the elements of arrangement.

It was remarked, in the fourteenth number, that the general comparison of a given luminous path $(A, B)_\chi$ with a near path $(A', B')_{\chi + \delta\chi}$ might be decomposed into several particular comparisons, such as the comparisons with the less general near paths $(A, B_d)_\chi$, $(A, B_\delta)_\chi$, and others, on account of the linear form of the expressions (D^9) for the variations $\delta\alpha$, $\delta\beta$, $\delta\alpha'$, $\delta\beta'$, of the extreme small cosines of direction, which form permits us to consider separately and successively the influence of the variations of the extreme co-ordinates and colour, or the influence of any groupes of these variations. Accordingly, by an *Analysis* founded on this remark, we decomposed the general discussion of the geometrical relations of infinitely near rays into four less general problems, which were treated of, in the fifteenth number. The applications, in the sixteenth number, to questions respecting the mutual intersections of the final ray-lines from the final perpendicular plane, may be considered as only illustrations and corollaries of the third of those four problems: but the questions since discussed, respecting the ray-lines from an oblique plane, require a combination of the solutions of the second and third of the four problems, and furnish therefore, an example of the *Synthesis* of those elements of arrangement of near rays, to which the former *Analysis* had conducted. This synthesis, however, has in the foregoing numbers been itself *algebraically* performed, (namely, by the algebraical addition of certain partial variations,) although many of the results were enunciated geometrically, and combined with geometrical conceptions: but a *geometrical* idea and method, of the Synthesis of the Elements of Arrangement, may be obtained by considering, in a general manner, the geometrical composition of partial deviations.

To understand more fully the occasion of such composition, let us remember that our theory of the Elements of Arrangement enables us to pass from the extreme directions of a given luminous path $(A, B)_{\chi}$, to the four following sets of near extreme directions, by the solution of the four problems considered in the fifteenth number.

First. The extreme directions of the near path $(A, B)_{\chi+\delta\chi}$, which has the same extreme points A, B, but differs by chromatic dispersion.

Second. The final direction of $(A, B_{d})_{\chi}$, that is, of the original path prolonged at the end, and the initial direction of $(A_{d}, B)_{\chi}$, that is, of the same path prolonged at the beginning; these near extreme directions being in general affected by curvature.

Third. The final direction of the path $(A, B_{\delta})_{\chi}$, and the initial direction of $(A_{\delta}, B)_{\chi}$; the small lines $\overline{AA_{\delta}}$, $\overline{BB_{\delta}}$, being perpendicular to the given path at its extremities.

Fourth. The initial direction of $(A, B_{\delta})_{\chi}$, and the final direction of $(A_{\delta}, B)_{\chi}$.

We saw also that the initial direction of $(A, B_{d})_{\chi}$ and the final direction of $(A_{d}, B)_{\chi}$ do not differ from the corresponding extreme directions of the original luminous path.

If then we would apply this theory to determine the final direction of an arbitrary near path $(A', B')_{\chi+\delta\chi}$, we have to consider and compound, algebraically or geometrically, the following partial deviations from the given final direction of the given path $(A, B)_{\chi}$: first, the chromatic deviation of the final direction of the near path $(A, B)_{\chi+\delta\chi}$ from that given final direction; second, the deviation of curvature of the final direction of $(A, B_{d})_{\chi}$; third, the final deviation of the path $(A, B_{\delta})_{\chi}$, to be determined by the theory of the final guiding paraboloid; and fourth, the deviation of the final direction of $(A_{\delta}, B)_{\chi}$, to be found by the theory of the guiding planes and conjugate guiding axes. A similar composition of four partial deviations is required for the determination of the initial direction of the same arbitrary near path $(A', B')_{\chi+\delta\chi}$.

Now to compound in a geometrical manner the four preceding partial deviations of the final ray-line, we may proceed as follows. We may construct each partial deviation, by drawing the deviated final ray-line corresponding, or a line parallel thereto, through the given final point B; the line thus drawn will differ little in direction from the given final ray-line or axis of z, and if we take its length equal to unity, then its small projection on the given final plane of xy, to which it is nearly perpendicular, will measure the magnitude and will indicate the direction of the deviation: and if we compound all these projections according to the usual geometrical rule of composition of forces, the result will be the projection of the equal line which represents in direction the resultant or total deviation. And similarly we may compound the four partial deviations of a near initial ray-line.

The geometrical synthesis of the partial deviations may also be performed in other ways. For example, we may consider each partial deviation as arising from a partial or component rotation, and we may compound these several rotations by the geometrical methods proper for such composition.

In particular, we may compound the final deviation of curvature with any of the other partial deviations, by making the deviated ray-line, obtained without considering the final curvature of the ray, revolve through an infinitely small angle round the axis of final curvature, that is, round the axis of the final osculating circle of the given final ray. By this rotation, the projection B_δ of a near final point B' on the final perpendicular plane, will be brought into the position B'; and, by the same rotation, the near final ray-line, which had been obtained by abstracting from the final curvature, and by considering B_δ as the final point, will be brought, at the same time, into the position of the sought ray-line, which corresponds to a final point at B'.

Applying now these general principles to the particular question respecting the condition of intersection of two near final ray-lines, from two near final points B, B', (the colour χ and the initial point A being considered as common and given,) we see that if the projection B_δ of B' be given, the small projecting perpendicular $\overline{B'B_\delta}$ or δz and therefore also the near point B' itself may in general be determined so as to satisfy the condition of intersection: for the final ray-line from B_δ may in general be brought to intersect the given final ray-line, by revolving through an infinitesimal angle round the axis of curvature of the given final ray. We see also that the angular quantity of rotation and therefore the length $\delta z = \overline{B_\delta B'}$ depends on the position of the projection B_δ, that is, on the co-ordinates δx, δy; and therefore that there must be some determined surface as the locus of the near final point B', when the final ray-line from that point is supposed to intersect the given final ray-line.

To investigate the form of this locus, by the help of the foregoing geometrical conceptions, we may observe that the only point, on the near ray-line from B_δ, which is brought by the supposed rotation to meet the given final ray-line, is the point contained in the final plane of curvature of the given final ray; and that if we call this point where the ray-line from B_δ intersects the given plane of curvature the point P, the angle of rotation required is the angle between the line $\overline{BP}$ and the given final ray-line; because the same infinitesimal rotation which brings the near ray-line from B_δ, that is, the line $\overline{B_\delta P}$, into a new position in which it intersects the given final ray-line, brings also the line $\overline{BP}$ into the position of the given final ray-line itself. Translating now these geometrical results into algebraical language, and taking the given final plane of curvature for the plane of xz, so as to satisfy the condition (R^{13}), we find the following co-ordinates of the point P of intersection of this plane of curvature with the ray-line (V^9) from B_δ,

$$x = \delta x - \frac{\delta y . \left(\frac{\delta \alpha}{\delta x}\,\delta x + \frac{\delta \alpha}{\delta y}\,\delta y\right)}{\frac{\delta \beta}{\delta x}\,\delta x + \frac{\delta \beta}{\delta y}\,\delta y}\,;\; y = 0\,;\; z = \frac{-\delta y}{\frac{\delta \beta}{\delta x}\,\delta x + \frac{\delta \beta}{\delta y}\,\delta y} : \qquad (\mathrm{W}^{14})$$

so that the angle between the line $\overline{BP}$ which connects this point with the origin of co-ordinates, and the given final ray-line or axis of z, is

$$\delta\theta = \left(-\frac{x}{z} =\right) \frac{1}{\delta y} . \delta x . \left(\frac{\delta \beta}{\delta x}\,\delta x + \frac{\delta \beta}{\delta y}\,\delta y\right) - \left(\frac{\delta \alpha}{\delta x}\,\delta x + \frac{\delta \alpha}{\delta y}\,\delta y\right); \qquad (\mathrm{X}^{14})$$

and this being equal to the infinitesimal angle of rotation, that is, to the small line δz or $\overline{B_\delta B'}$ multiplied by $\frac{\delta \alpha}{\delta z}$ or by the final curvature of the given ray taken with its proper sign, we have the following equation for the locus of the near point B', when the condition of intersection is to be satisfied,

$$\frac{\delta \alpha}{\delta z}\,\delta z = \frac{1}{\delta y}\,\delta x \left(\frac{\delta \beta}{\delta x}\,\delta x + \frac{\delta \beta}{\delta y}\,\delta y\right) - \left(\frac{\delta \alpha}{\delta x}\,\delta x + \frac{\delta \alpha}{\delta y}\,\delta y\right) : \qquad (\mathrm{Y}^{14})$$

which is, accordingly, the equation of the former conical locus (R^{14}), only simplified by the condition (R^{13}), arising from a choice of co-ordinates. Without making that choice, we might easily have deduced in a similar manner the equation (R^{14}), under the form

$$\delta z = \frac{\delta x \left(\frac{\delta \beta}{\delta x}\,\delta x + \frac{\delta \beta}{\delta y}\,\delta y\right) - \delta y \left(\frac{\delta \alpha}{\delta x}\,\delta x + \frac{\delta \alpha}{\delta y}\,\delta y\right)}{\frac{\delta \alpha}{\delta z}\,\delta y - \frac{\delta \beta}{\delta z}\,\delta x}\,, \qquad (\mathrm{Z}^{14})$$

in which each member is an expression for the infinitesimal angle of rotation divided by the curvature of the ray.

Another way of applying the foregoing geometrical principles to investigate the condition of intersection of two near final ray-lines, is to consider the infinitesimal angle by which the ray-line from B_δ deviates from the plane containing the given final ray-line and the near point B_δ. This angular deviation is expressed by the numerator of the fraction (Z^{14}), divided by δl, that is, divided by the small line $\overline{BB_\delta}$; and the denominator of the same fraction (Z^{14}), divided also by δl, is equal to the final curvature of the ray multiplied by the sine of the inclination of the line δl to the radius of this final curvature: and hence it is easy to see, by geometrical considerations, that the fraction in the second number of (Z^{14}) is equal to the infinitesimal angle of rotation required for destroying the last mentioned deviation, divided by the curvature of the ray, and therefore equal to the ordinate δz of the sought locus of the near point B', as expressed by the first member. We might therefore easily have obtained, by calculations founded on this other geometrical view, the same condition of intersection as before, and the same conical locus.

Relations between the Elements of Arrangement, depending only on the Extreme Points, Directions, and Colour, of a Given Luminous Path, and on the Extreme Media. In a Final Uniform Medium, Ordinary or Extraordinary, the two Planes of Vergency are Conjugate Planes of Deflexure of any Surface of a certain class, determined by the Final Medium; and also of a certain Analogous Surface, determined by the whole combination. Relations between the Visible Magnitudes and Distortions of any two small objects, viewed from each other through any Optical Combination. Interchangeable Eye-axes and Object-axes of Distortion. Planes of No Distortion.

21. It was shown in the fourteenth number, and the result has since been developed in detail, that the general geometrical relations between the extreme directions of infinitely near rays are determined by the co-efficients of the linear variations $\delta\alpha, \delta\beta, \delta\gamma, \delta\alpha', \delta\beta', \delta\gamma'$, of the six marks of extreme direction, considered as functions of the six extreme co-ordinates and of the colour; and that, between the forty-two general coefficients of these six linear variations, there exist eighteen general relations, leaving only twenty-four coefficients arbitrary, if we suppose for simplicity that the final and initial co-ordinates are referred to rectangular axes. But besides these eighteen general relations which are common to all optical combinations, there arise certain other relations between the coefficients, when the extreme media are considered as given, and when the extreme points, directions, and colour, of any one luminous path, are also supposed to be known. For, if we then employ the general equations (A^9), we may consider the extreme medium functions v, v', and their partial differentials, as known, and may deduce general expressions for the coefficients before mentioned of the linear variations of the extreme cosines of direction, involving only, as unknown quantities, twenty-seven partial differentials of the second order of the characteristic function V, namely, all of this order, which are not relative to the variation of colour only; but these twenty-seven are connected by the fourteen general relations (Q) (U) (X) (Y), deduced in the third number, of which however only thirteen are distinct, because the two systems (U) (Y) conduct both to one common equation (D); there remain, therefore, as independent quantities, only fourteen of the partial differentials of V, in the general expressions of those twenty-four coefficients of the linear variations of the extreme direction-cosines, which had before been considered as independent, when the extreme medium-functions v, v' were supposed unknown and arbitrary: and if we eliminate the fourteen independent differentials of V between the expressions of these twenty-four coefficients, we shall obtain *ten general relations, between the elements of arrangement of infinitely near rays,*

involving only the extreme points, directions, and colour, of the given luminous path, and the properties of the extreme media.

The simplest manner of obtaining these ten general relations, is to eliminate the fourteen differentials of V which enter into the twenty-four expressions, deducible from (C^9), from the twenty-four coefficients (D^9). The ten relations thus obtained, may be arranged in three different groupes: the first groupe containing the two following

$$\left.\begin{aligned} \frac{\delta^2 v}{\delta\alpha^2}\,\frac{\delta\alpha}{\delta z} + \frac{\delta^2 v}{\delta\alpha\delta\beta}\,\frac{\delta\beta}{\delta z} + \frac{\delta^2 v}{\delta\alpha\delta z} = \frac{\delta v}{\delta x}, \\ \frac{\delta^2 v}{\delta\alpha\delta\beta}\,\frac{\delta\alpha}{\delta z} + \frac{\delta^2 v}{\delta\beta^2}\,\frac{\delta\beta}{\delta z} + \frac{\delta^2 v}{\delta\beta\delta z} = \frac{\delta v}{\delta y}, \end{aligned}\right\} \qquad (\mathrm{A}^{15})$$

and two others similar to these, but with accented or initial symbols; the second groupe containing the final relation

$$\frac{\delta^2 v}{\delta\alpha^2}\,\frac{\delta\alpha}{\delta y} + \frac{\delta^2 v}{\delta\alpha\delta\beta}\,\frac{\delta\beta}{\delta y} + \frac{\delta^2 v}{\delta\alpha\delta y} = \frac{\delta^2 v}{\delta\alpha\delta\beta}\,\frac{\delta\alpha}{\delta x} + \frac{\delta^2 v}{\delta\beta^2}\,\frac{\delta\beta}{\delta x} + \frac{\delta^2 v}{\delta\beta\delta x}, \qquad (\mathrm{B}^{15})$$

and a similar initial relation; and the third groupe comprising the four following,

$$\left.\begin{aligned} \frac{\delta^2 v}{\delta\alpha^2}\,\frac{\delta\alpha}{\delta x'} + \frac{\delta^2 v}{\delta\alpha\delta\beta}\,\frac{\delta\beta}{\delta x'} + \frac{\delta^2 v'}{\delta\alpha'^2}\,\frac{\delta\alpha'}{\delta x} + \frac{\delta^2 v'}{\delta\alpha'\delta\beta'}\,\frac{\delta\beta'}{\delta x} = 0, \\ \frac{\delta^2 v}{\delta\alpha^2}\,\frac{\delta\alpha}{\delta y'} + \frac{\delta^2 v}{\delta\alpha\delta\beta}\,\frac{\delta\beta}{\delta y'} + \frac{\delta^2 v'}{\delta\alpha'\delta\beta'}\,\frac{\delta\alpha'}{\delta x} + \frac{\delta^2 v'}{\delta\beta'^2}\,\frac{\delta\beta'}{\delta x} = 0, \\ \frac{\delta^2 v}{\delta\alpha\delta\beta}\,\frac{\delta\alpha}{\delta x'} + \frac{\delta^2 v}{\delta\beta^2}\,\frac{\delta\beta}{\delta x'} + \frac{\delta^2 v'}{\delta\alpha'^2}\,\frac{\delta\alpha'}{\delta y} + \frac{\delta^2 v'}{\delta\alpha'\delta\beta'}\,\frac{\delta\beta'}{\delta y} = 0, \\ \frac{\delta^2 v}{\delta\alpha\delta\beta}\,\frac{\delta\alpha}{\delta y'} + \frac{\delta^2 v}{\delta\beta^2}\,\frac{\delta\beta}{\delta y'} + \frac{\delta^2 v'}{\delta\alpha'\delta\beta'}\,\frac{\delta\alpha'}{\delta y} + \frac{\delta^2 v'}{\delta\beta'^2}\,\frac{\delta\beta'}{\delta y} = 0. \end{aligned}\right\} \qquad (\mathrm{C}^{15})$$

The two first relations of the first groupe, namely, the equations (A^{15}), are equivalent to the two first differential equations (O) of a curved ray, and express that the magnitude and plane of final curvature of a luminous path, in a final variable medium, are determined, in general, by the properties of that medium, the colour of the light, the position of the final point, and the direction of the final tangent. And the two other relations of the same groupe express, in like manner, a dependence of the initial magnitude and plane of curvature of a luminous path, on the initial medium, colour, point, and tangent.

The equation (B^{15}), belonging to the second groupe, is a relation between the four coefficients $\frac{\delta\alpha}{\delta x}$, $\frac{\delta\alpha}{\delta y}$, $\frac{\delta\beta}{\delta x}$, $\frac{\delta\beta}{\delta y}$, and therefore a relation between the guiding paraboloid and constant of deviation for the final ray-lines, depending on the final medium, colour, point, and tangent. And similarly the other equation of the second groupe expresses an analogous relation for the initial medium.

In the extensive case of a final uniform medium, the equation (B^{15}) reduces itself to the following,

$$0 = \frac{\delta^2 v}{\delta\alpha^2}\frac{\delta\alpha}{\delta y} + \frac{\delta^2 v}{\delta\alpha\delta\beta}\left(\frac{\delta\beta}{\delta y} - \frac{\delta\alpha}{\delta x}\right) - \frac{\delta^2 v}{\delta\beta^2}\frac{\delta\beta}{\delta x}; \qquad (\mathrm{D}^{15})$$

and, in the same case, the general conical locus of the second degree (R^{14}), connected with the condition of intersection of the final ray-lines, reduces itself to two real or imaginary planes of vergency, represented by the quadratic

$$0 = \frac{\delta\alpha}{\delta y}\delta y^2 + \left(\frac{\delta\alpha}{\delta x} - \frac{\delta\beta}{\delta y}\right)\delta x\delta y - \frac{\delta\beta}{\delta x}\delta x^2, \qquad (\mathrm{E}^{15})$$

and coinciding with the two planes of vergency considered in the sixteenth number: attending therefore to (C^{13}), the relation (D^{15}) may be geometrically enunciated by saying, that *in a final uniform medium the two planes of vergency are conjugate planes of deflexure of any surface of a certain class determined by the nature of the medium*, namely, that class for which, at the origin of co-ordinates,

$$\frac{\delta^2 z}{\delta x^2} = \lambda\frac{\delta^2 v}{\delta\alpha^2}, \quad \frac{\delta^2 z}{\delta x\delta y} = \lambda\frac{\delta^2 v}{\delta\alpha\delta\beta}, \quad \frac{\delta^2 z}{\delta y^2} = \lambda\frac{\delta^2 v}{\delta\beta^2}, \qquad (\mathrm{F}^{15})$$

and therefore nearly, for points near to this origin,

$$z = px + qy + \frac{\lambda}{2}\left(x^2\frac{\delta^2 v}{\delta\alpha^2} + 2xy\frac{\delta^2 v}{\delta\alpha\delta\beta} + y^2\frac{\delta^2 v}{\delta\beta^2}\right), \qquad (\mathrm{G}^{15})$$

the given final ray or axis of z being taken as the axis of deflexion, and the constants p, q, λ, being arbitrary. This relation may be still farther simplified, by choosing the arbitrary constants as follows,

$$p = -\frac{1}{v}\frac{\delta v}{\delta\alpha}, \quad q = -\frac{1}{v}\frac{\delta v}{\delta\beta}, \quad \lambda = \frac{1}{vZ}, \qquad (\mathrm{H}^{15})$$

Z being any constant ordinate; for then, (by the theory of the characteristic function V for a single uniform medium, which was given in the tenth number,) the surface (G^{15}) acquires a simple optical property, and becomes, in the final uniform medium, the approximate locus of the points x, y, z, for which

$$V_{,} = \int v ds = v\rho = \text{const.}, \qquad (\mathrm{I}^{15})$$

the integral $V_{,} = \int v ds$ being taken here, in the positive direction, along the variable line ρ, from the fixed point $0, 0, Z$, to the variable point x, y, z, or from the latter to the former, according as Z is negative or positive. And though the equation (G^{15}) is only an approximate representation of the medium-surface (I^{15}), which was called in the First Supplement a *spheroid of constant action*, and which is in the undulatory theory a curved *wave* propagated from or to a point in the final medium, yet since the equation (G^{15}) gives a correct development of the ordinate z of this surface as far as terms of the second dimension inclusive, when the constants are determined by

(H^{15}), the conclusion respecting the deflexures applies rigorously to the surface (I^{15}); and *the two planes of vergency* (E^{15}), *in a final uniform medium, are conjugate planes of deflexure of the spheroid or wave* (I^{15}). We shall soon resume this result, and endeavour to illustrate and extend it. In the mean time we may remark that the same planes of vergency (E^{15}) are also conjugate planes of deflexure of a certain analogous surface, determined by the whole combination, and not merely by the final uniform medium, namely, the surface (D^{10}), for which

$$\int v ds \, (= V) = \text{const.}, \qquad (\mathrm{K}^{15})$$

the integral being here extended to the whole luminous path, and being therefore equal to the characteristic function V of the whole optical combination; an additional property of the planes of vergency, which is proved by the following relation, analogous to (D^{15}), and deducible from (C^{9}) or (G^{9}),

$$0 = \frac{\delta^2 V}{\delta x^2} \frac{\delta \alpha}{\delta y} + \frac{\delta^2 V}{\delta x \delta y} \left(\frac{\delta \beta}{\delta y} - \frac{\delta \alpha}{\delta x} \right) - \frac{\delta^2 V}{\delta y^2} \frac{\delta \beta}{\delta x} . \qquad (\mathrm{L}^{15})$$

Finally, with respect to the four remaining equations, of the third groupe (C^{15}), it is evident that they express certain general relations depending on the extreme media, between the coefficients which determine the guiding planes and conjugate guiding axes, for the final and initial ray-lines. In the extensive case of extreme ordinary media, they reduce themselves to the four following, which may also be deduced from (F^{9}),

$$\left. \begin{array}{ll} \mu \dfrac{\delta \alpha}{\delta x'} + \mu' \dfrac{\delta \alpha'}{\delta x} = 0, & \mu \dfrac{\delta \alpha}{\delta y'} + \mu' \dfrac{\delta \beta'}{\delta x} = 0, \\ \mu \dfrac{\delta \beta}{\delta x'} + \mu' \dfrac{\delta \alpha'}{\delta y} = 0, & \mu \dfrac{\delta \beta}{\delta y'} + \mu' \dfrac{\delta \beta'}{\delta y} = 0, \end{array} \right\} \qquad (\mathrm{M}^{15})$$

μ, μ' being the *indices* of the media; and they conduct to some simple conclusions, respecting the general relations between the visible magnitudes and distortions of a small plane object, placed alternately at each end of any given luminous path, and viewed from the other end, through any ordinary or extraordinary combination: at least so far as we suppose these distortions and magnitudes to be measured by the shape and size of the initial and final ray-cones. For then the conjugate guiding axes, initial and final, perpendicular to the given path at its extremities, and determined in the fifteenth number, may be called the *eye-axes and object-axes of distortion,* for a small object placed in the final perpendicular plane, and viewed from the initial point; and if we take these for the axes of initial and final co-ordinates, so as to have, by (X^{10}) (Y^{10}),

$$\frac{\delta \alpha'}{\delta y} = 0, \ \frac{\delta \beta'}{\delta x} = 0, \ \frac{\delta \beta'}{\delta y} > 0, \ \frac{\delta \alpha'}{\delta x} > \frac{\delta \beta'}{\delta y} ,$$

we shall then have also, by (M^{15}), (the extreme media being supposed ordinary, and their indices μ, μ' positive,)

$$\frac{\delta \alpha}{\delta y'} = 0,\ \frac{\delta \beta}{\delta x'} = 0,\ -\frac{\delta \beta}{\delta y'} > 0,\ -\frac{\delta \alpha}{\delta x'} > -\frac{\delta \beta}{\delta y'}\ ; \qquad (\mathrm{N}^{15})$$

that is, in this case, *the guiding axes for the initial ray-lines are also the guiding axes of the same kind for the final ray-lines measured backward ;* which is already a remarkable relation, and may be enunciated by saying that *the eye-axes and object-axes of distortion are interchangeable, when the extreme media are ordinary :* that is, for such extreme media, *the eye-axes of distortion become object-axes, and the object-axes become eye-axes, when the object is removed from the final to the initial perpendicular plane, and is viewed from the final instead of the initial point.* And while the equations of the fifteenth number,

$$x' = z' \frac{\delta \alpha'}{\delta x}\, \delta x,\ \ y' = z' \frac{\delta \beta'}{\delta y}\, \delta y, \qquad (\mathrm{A}^{11})$$

represent the initial visual ray-line corresponding to a final visible point B' which has for co-ordinates δx, δy, δz, the following other equations,

$$x = -\, z . \frac{\mu'}{\mu} \frac{\delta \alpha'}{\delta x} \delta x',\ \ y = -z . \frac{\mu'}{\mu} \frac{\delta \beta'}{\delta y}\, \delta y', \qquad (\mathrm{O}^{15})$$

will represent by (M^{15}) the final visual ray-line corresponding to an initial visible point A' which has for co-ordinates $\delta x'$, $\delta y'$, $\delta z'$; the initial visual ray-cone corresponding to any small object

$$\delta y = f\,(\delta x) \qquad (\mathrm{G}^{11})$$

in the final perpendicular plane is therefore represented by the equation

$$\frac{y'}{z'} \left(\frac{\delta \beta'}{\delta y}\right)^{-1} = f\left(\frac{x'}{z'} \left(\frac{\delta \alpha'}{\delta x}\right)^{-1}\right), \qquad (\mathrm{H}^{11})$$

and the final visual ray-cone corresponding to any small object

$$\delta y' = f'\,(\delta x') \qquad (\mathrm{P}^{15})$$

in the initial perpendicular plane is represented by the following analogous equation

$$-\frac{y}{z}\, \frac{\mu}{\mu'} \left(\frac{\delta \beta'}{\delta y}\right)^{-1} = f'\left(-\frac{x}{z}\, \frac{\mu}{\mu'} \left(\frac{\delta \alpha'}{\delta x}\right)^{-1}\right); \qquad (\mathrm{Q}^{15})$$

if therefore these two small objects, (G^{11}) (P^{15}), at the ends of a given luminous path, be equal and similar and similarly placed with respect to the conjugate axes of distortion, that is, if the final and initial functions $f\, f'$ be the same, and if we cut the two ray-cones (H^{11}) (Q^{15}) respectively by perpendicular planes having for equations

$$z' = \mu' R,\ \ z = -\mu R, \qquad (\mathrm{R}^{15})$$

in which R is any constant length, while μ, μ' are the same constant indices as before of the extreme ordinary media, the two perpendicular sections thus obtained will be equal and similar to each other; and if, besides, we put, by (Y^{10}),

$$\frac{\delta\beta'}{\delta y}=\frac{\delta\alpha'}{\delta x}\cos.\, G, \qquad (\mathrm{S}^{15})$$

(G being by (F^{11}) the inclination of an initial guiding plane to the plane perpendicular to the given initial ray-line,) and determine also the arbitrary quantity R as follows,

$$R=\frac{1}{\mu'}\left(\frac{\delta\alpha'}{\delta x}\right)^{-1}=-\frac{1}{\mu}\left(\frac{\delta\alpha}{\delta x'}\right)^{-1}, \qquad (\mathrm{T}^{15})$$

the perpendicular sections of the initial and final ray-cones may then be represented as follows,

$$y'=\cos.\, G.f(x'), \quad z'=\left(\frac{\delta\alpha'}{\delta x}\right)^{-1}, \qquad (\mathrm{U}^{15})$$

and

$$y=\cos.\, G.f(x), \quad z=\left(\frac{\delta\alpha}{\delta x'}\right)^{-1}: \qquad (\mathrm{V}^{15})$$

the visible distortions therefore, depending on the inclination G, *are the same for any two small equal objects, thus perpendicularly and similarly placed at the ends of any given luminous path, and viewed from each other along that path, through any optical combination.*

The distortion here considered will in general change, if the object at either end of the given luminous path be made to revolve in the perpendicular plane at that end, so as to change its position with respect to the axes of distortion. For example, if the object be a small right-angled triangle in the final perpendicular plane, having the summit of the right angle at the given final point B of the path, we know, by the theory given in the fifteenth number, that the right angle will appear right to an eye placed at the initial point A, when the rectangular directions of its sides ϕ'_1, ϕ'_2, coincide with those of the final guiding axes, or object axes of distortion; but that otherwise the right angle $\phi'_2-\phi'_1$ will appear acute or obtuse, its apparent magnitude $\phi_2-\phi_1$ being determined by the formula

$$-\tan.\left(\phi_2-\phi_1-\frac{\pi}{2}\right)=\frac{\left(\frac{\delta\alpha'}{\delta x}\right)^2-\left(\frac{\delta\beta'}{\delta y}\right)^2}{2\,\frac{\delta\alpha'}{\delta x}\,\frac{\delta\beta'}{\delta y}}\,.\sin.\, 2\phi'_1, \qquad (\mathrm{W}^{15})$$

which may, by (S^{15}), be reduced to the following,

$$-\tan.\left(\phi_2-\phi_1-\frac{\pi}{2}\right)=\tfrac{1}{2}\sin.\, G.\tan.\, G.\sin.\, 2\phi'_1. \qquad (\mathrm{X}^{15})$$

The law of change of the distortion, corresponding to a rotation in the final perpendicular plane, may also be deduced from the theory of the guiding planes, explained in the fifteenth number.

The distortion will also change, if the small plane object be removed into an oblique instead of a perpendicular plane. In this case we may still employ the equations (A^{11}) (O^{15}) for the initial and final ray-lines, and may still represent the initial and final ray-cones by the equations (H^{11}) (Q^{15}); but we are now to consider the equations (G^{11}) (P^{15}), for the final and initial objects, as representing the projections of those objects on the extreme perpendicular planes; or rather the *projecting cylinders*, which contain the objects, and which determine their visible magnitudes and distortions, by determining the connected ray-cones. For example, the equation (C^{11}) may be considered as representing a final elliptic cylinder, of which any section near the final point B of the given luminous path will correspond to an initial circular ray-cone (B^{11}), and will therefore appear a circle to an eye placed at the initial point A; while on the other hand we may regard the equation (D^{11}) as respecting a final circular cylinder, such that any section of this cylinder, near the final point B, will give an initial elliptic ray-cone (E^{11}), and will appear an ellipse at A. And as the elliptic ray-cone (E^{11}) conducted, by its circular sections, to the guiding planes (F^{11}) for the initial ray-lines, so, for small plane final objects, the planes

$$z = \pm x \tan. G, \qquad (Y^{15})$$

namely, by (S^{15}), *the planes of circular section of the elliptic cylinder* (C^{11}), *are planes of no distortion;* in such a manner that not only, by what has been said, the circular sections themselves in these two planes appear each circular, but every other small final object in either of the same two planes appears with its proper shape to an eye placed at the initial point A of the given luminous path; the angular magnitude of the final object thus placed, being the same as if it were viewed perpendicularly by straight rays, without any refracting or reflecting surface or medium interposed, from a final distance $= \left(\frac{\delta\beta'}{\delta y}\right)^{-1}$. In like manner, the planes

$$z' = \pm x'. \tan. G, \qquad (Z^{15})$$

which are the planes of circular section of an analogous initial elliptic cylinder, are *initial planes of no distortion*, of the same kind as the final planes (Y^{15}); since any small initial object, placed in either of these two initial planes (Z^{15}), and viewed from the final point B of the given luminous path, will appear with its proper shape, and with the same angular magnitude as if it were viewed directly from an initial distance $= -\left(\frac{\delta\beta}{\delta y'}\right)^{-1} = \frac{\mu}{\mu'}\left(\frac{\delta\beta'}{\delta y}\right)^{-1}$.

This theory of the *planes of no distortion* gives a simple determination of the visible shape and size of any small object placed in any manner near either end of a given luminous path; since we have only to project the object on one of the two planes of no distortion at that end, by lines parallel to the corresponding extreme direction of the path, and then to suppose this projection viewed directly from a final or initial distance determined as above. We might, for example, deduce from this theory the property of the guiding planes, the circular and elliptic appearances (B^{11}) (E^{11}) of the ellipse and circle (C^{11}) (D^{11}), and the acute or obtuse appearance (X^{15}) of a right angle in the final perpendicular plane, when the directions of the sides of this angle are different from those of the object-axes of distortion. And the relations (M^{15}) for extreme ordinary media may be expressed by the following theorems: first, that *the angle* $(2G)$ *between the final pair of planes of no distortion* (Y^{15}), *is equal to that between the initial pair* (Z^{15}); second, *the visible angular magnitudes of any small and equal linear objects in final and initial planes of no distortion, are proportional to the indices of the final and initial media,* when the objects are viewed along a given luminous path, from the initial and final points; and third, *the two intersection-lines of the two pairs of planes of no distortion coincide each with the visible direction of the other,* when viewed along the path.

Calculation of the Elements of Arrangement, for Arbitrary Axes of Co-ordinates.

22. In the foregoing formulæ for the elements of arrangement of near rays, we have chosen for simplicity the final and initial points of a given luminous path, as the respective origins of two sets of rectangular co-ordinates, final and initial, and we have made the final and initial ray-lines, or tangents to the given path, the axes of z and z'; a choice of co-ordinates which had the convenience of reducing to zero eighteen of the forty-two general coefficients in the expressions of $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\delta\alpha'$, $\delta\beta'$, $\delta\gamma'$, as linear functions of δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$. The twenty-four remaining coefficients (D^9) may however be easily deduced, by the methods already established, and by the partial differential coefficients of the characteristic and related funtions, from other systems of final and initial co-ordinates, for example, from any other rectangular sets of final and initial axes.

In effecting this deduction, it will be useful to distinguish by lower accents the particular co-ordinates and cosines of direction, which enter into the expressions (D^9), and are referred to particular axes of the kind already described; and then we may connect these particular co-ordinates and cosines with the more general analogous quantities $x\ y\ z\ x'\ y'\ z'\ \alpha\ \beta\ \gamma\ \alpha'\ \beta'\ \gamma'$, by the formulæ of transformation given in the thirteenth number, which may easily be shown to extend to the case of two distinct rectangular sets of given or unaccented co-ordinates. In this manner the axes of $z_{,}$ and $z'_{,}$, considered in the thirteenth number, become the final and initial ray-lines, and we have, by (A^8),

$$\left.\begin{aligned}
\delta x &= x_{x_{,}}\ \delta x_{,} + x_{y_{,}}\ \delta y_{,} + \alpha\ \delta z_{,},\\
\delta y &= y_{x_{,}}\ \delta x_{,} + y_{y_{,}}\ \delta y_{,} + \beta\ \delta z_{,},\\
\delta z &= z_{x_{,}}\ \delta x_{,} + z_{y_{,}}\ \delta y_{,} + \gamma\ \delta z_{,},\\
\delta x' &= x'_{x'_{,}}\ \delta x'_{,} + x'_{y'_{,}}\ \delta y'_{,} + \alpha'\ \delta z'_{,},\\
\delta y' &= y'_{x'_{,}}\ \delta x'_{,} + y'_{y'_{,}}\ \delta y'_{,} + \beta'\ \delta z'_{,},\\
\delta z' &= z'_{x'_{,}}\ \delta x'_{,} + z'_{y'_{,}}\ \delta y'_{,} + \gamma'\ \delta z'_{,},
\end{aligned}\right\}\qquad (A^{16})$$

because

$$\left.\begin{aligned}
&x_{z_{,}} = \alpha,\ \ y_{z_{,}} = \beta,\ \ z_{z_{,}} = \gamma,\\
&x'_{z'_{,}} = \alpha',\ \ y'_{z'_{,}} = \beta',\ \ z'_{z'_{,}} = \gamma'\ ;
\end{aligned}\right\}\qquad (B^{16})$$

we have also

$$\left.\begin{aligned}
&\alpha_{,} = 0,\ \ \beta_{,} = 0,\ \ \gamma_{,} = 1,\ \ \delta\gamma_{,} = 0,\\
&\alpha'_{,} = 0,\ \ \beta'_{,} = 0,\ \ \gamma'_{,} = 1,\ \ \delta\gamma'_{,} = 0,
\end{aligned}\right\}\qquad (C^{16})$$

and therefore, by (E^{8}),

$$\left.\begin{aligned}
&\delta\alpha = x_{x_{,}}\ \delta\alpha_{,} + x_{y_{,}}\ \delta\beta_{,}\ ;\quad \delta\alpha' = x'_{x'_{,}}\ \delta\alpha'_{,} + x'_{y'_{,}}\ \delta\beta'_{,}\ ;\\
&\delta\beta = y_{x_{,}}\ \delta\alpha_{,} + y_{y_{,}}\ \delta\beta_{,}\ ;\quad \delta\beta' = y'_{x'_{,}}\ \delta\alpha'_{,} + y'_{y'_{,}}\ \delta\beta'_{,}\ ;\\
&\delta\gamma = z_{x_{,}}\ \delta\alpha_{,} + z_{y_{,}}\ \delta\beta_{,}\ ;\quad \delta\gamma' = z'_{x'_{,}}\ \delta\alpha'_{,} + z'_{y'_{,}}\ \delta\beta'_{,}\ ;
\end{aligned}\right\}\qquad (D^{16})$$

and substituting these values (A^{16}) (D^{16}) for the twelve variations δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\delta\alpha'$, $\delta\beta'$, $\delta\gamma'$, in the general linear relations (A^{9}) between these twelve variations and the variation of colour $\delta\chi$, or in any other linear relations of the same kind, deduced from the characteristic and related functions, and referred to arbitrary rectangular co-ordinates, we shall easily discover the particular dependence, of the form (D^{9}), of $\delta\alpha_{,}$, $\delta\beta_{,}$, on $\delta x_{,}$, $\delta y_{,}$, $\delta z_{,}$, $\delta x'_{,}$, $\delta y'_{,}$, $\delta\chi$, and of $\delta\alpha'_{,}$, $\delta\beta'_{,}$, on $\delta x_{,}$, $\delta y_{,}$, $\delta x'_{,}$, $\delta y'_{,}$, $\delta z'_{,}$, $\delta\chi$.

We seem, by this transformation, to introduce twelve arbitrary cosines or coefficients, namely,

$$x_{x_{,}},\ y_{x_{,}},\ z_{x_{,}},\ x_{y_{,}},\ y_{y_{,}},\ z_{y_{,}},\ x'_{x'_{,}},\ y'_{x'_{,}},\ z'_{x'_{,}},\ x'_{y'_{,}},\ y'_{y'_{,}},\ z'_{y'_{,}}\ ;$$

but these twelve coefficients are connected by ten relations, arising from the rectangularity of each of the four sets of co-ordinates, and from the given directions of the semiaxes of $z_{,}$ and $z'_{,}$; so that there remain only two arbitrary quantities, corresponding to the arbitrary planes of $x_{,}\,z_{,}$, $x'_{,}\,z'_{,}$, of which planes we often, lately, disposed at pleasure, so as to make them coincide with certain given planes of curvature, or otherwise to simplify the recent geometrical discussions. Thus, although we may assign to the semiaxis of $x_{,}$ any position in the given final plane perpendicular to the luminous path, and therefore may assign to its cosines of direction, $x_{x_{,}}$, $y_{x_{,}}$, $z_{x_{,}}$, any values consistent with the first equation (B^8), namely,

$$x_{x_{,}}^2 + y_{x_{,}}^2 + z_{x_{,}}^2 = 1,$$

and with the following

$$\alpha x_{x_{,}} + \beta y_{x_{,}} + \gamma z_{x_{,}} = 0, \qquad (\mathrm{E}^{16})$$

yet when the axis of $x_{,}$ has been so assumed, the perpendicular axis of $y_{,}$ in the final perpendicular plane is determined, and we have

$$\left.\begin{aligned} x_{y_{,}} &= \pm(\beta z_{x_{,}} - \gamma y_{x_{,}}), \\ y_{y_{,}} &= \pm(\gamma x_{x_{,}} - \alpha z_{x_{,}}), \\ z_{y_{,}} &= \pm(\alpha y_{x_{,}} - \beta x_{x_{,}}), \end{aligned}\right\} \qquad (\mathrm{F}^{16})$$

the upper or lower signs being here obliged to accompany each other: and similarly for the initial axes of $x'_{,}$ and $y'_{,}$.

The characteristic and related functions give immediately, by their partial differentials of the first order, the dependence of the quantities which we have denoted by σ, τ, υ, σ', τ', υ', rather than that of α, β, γ, α', β', γ', on the extreme co-ordinates and the colour; and therefore the same functions give immediately, by their partial differentials of the second order, the variations $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, $\delta\sigma'$, $\delta\tau'$, $\delta\upsilon'$, rather than $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\delta\alpha'$, $\delta\beta'$, $\delta\gamma'$, in terms of δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$. But we can easily deduce the variations of $\alpha\,\beta\,\gamma\,\alpha'\,\beta'\,\gamma'$ from those of $\sigma\,\tau\,\upsilon\,\sigma'\,\tau'\,\upsilon'$ and of $x\,y\,z\,x'\,y'\,z'\,\chi$, by differentiating the relations

$$\sigma = \frac{\delta v}{\delta\alpha}, \quad \tau = \frac{\delta v}{\delta\beta}, \quad \upsilon = \frac{\delta v}{\delta\gamma},$$

$$\sigma' = \frac{\delta v'}{\delta\alpha'}, \quad \tau' = \frac{\delta v'}{\delta\beta'}, \quad \upsilon' = \frac{\delta v'}{\delta\gamma'},$$

which have often been employed already in the present Supplement; for thus we obtain

$$\left.\begin{aligned}
&\frac{\delta^2 v}{\delta\alpha^2}\,\delta\alpha + \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\,\delta\beta + \frac{\delta^2 v}{\delta\alpha\,\delta\gamma}\,\delta\gamma = \delta\sigma - \delta_{,,}\frac{\delta v}{\delta\alpha}, \\
&\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\,\delta\alpha + \frac{\delta^2 v}{\delta\beta^2}\,\delta\beta + \frac{\delta^2 v}{\delta\beta\,\delta\gamma}\,\delta\gamma = \delta\tau - \delta_{,,}\frac{\delta v}{\delta\beta}, \\
&\frac{\delta^2 v}{\delta\alpha\,\delta\gamma}\,\delta\alpha + \frac{\delta^2 v}{\delta\beta\,\delta\gamma}\,\delta\beta + \frac{\delta^2 v}{\delta\gamma^2}\,\delta\gamma = \delta\upsilon - \delta_{,,}\frac{\delta v}{\delta\gamma}, \\
&\frac{\delta^2 v'}{\delta\alpha'^2}\,\delta\alpha' + \frac{\delta^2 v'}{\delta\alpha'\,\delta\beta'}\,\delta\beta' + \frac{\delta^2 v}{\delta\alpha'\,\delta\gamma'}\,\delta\gamma' = \delta\sigma' - \delta'\frac{\delta v'}{\delta\alpha'}, \\
&\frac{\delta^2 v'}{\delta\alpha'\,\delta\beta'}\,\delta\alpha' + \frac{\delta^2 v}{\delta\beta'^2}\,\delta\beta' + \frac{\delta^2 v'}{\delta\beta'\,\delta\gamma'}\,\delta\gamma' = \delta\tau' - \delta'\frac{\delta v'}{\delta\beta'}, \\
&\frac{\delta^2 v'}{\delta\alpha'\,\delta\gamma'}\,\delta\alpha' + \frac{\delta^2 v'}{\delta\beta'\,\delta\gamma'}\,\delta\beta' + \frac{\delta^2 v'}{\delta\gamma'^2}\,\delta\gamma' = \delta\upsilon' - \delta'\frac{\delta v'}{\delta\gamma'},
\end{aligned}\right\} \qquad (\mathrm{G}^{16})$$

$\delta_{\prime\prime}$ referring, as in former numbers, to the variations of x, y, z, χ, and δ' to those of x', y', z', χ: and hence we have, by some symmetric eliminations,

$$\left.\begin{aligned}
v''\delta\alpha &= \left(\frac{\delta^2 v}{\delta\beta^2}+\frac{\delta^2 v}{\delta\gamma^2}\right)\left(\delta\sigma-\delta_{\prime\prime}\frac{\delta v}{\delta\alpha}\right)-\frac{\delta^2 v}{\delta\alpha\delta\beta}\left(\delta\tau-\delta_{\prime\prime}\frac{\delta v}{\delta\beta}\right)-\frac{\delta^2 v}{\delta\gamma\delta\alpha}\left(\delta\upsilon-\delta_{\prime\prime}\frac{\delta v}{\delta\gamma}\right),\\
v''\delta\beta &= \left(\frac{\delta^2 v}{\delta\gamma^2}+\frac{\delta^2 v}{\delta\alpha^2}\right)\left(\delta\tau-\delta_{\prime\prime}\frac{\delta v}{\delta\beta}\right)-\frac{\delta^2 v}{\delta\beta\delta\gamma}\left(\delta\upsilon-\delta_{\prime\prime}\frac{\delta v}{\delta\gamma}\right)-\frac{\delta^2 v}{\delta\alpha\delta\beta}\left(\delta\sigma-\delta_{\prime\prime}\frac{\delta v}{\delta\alpha}\right),\\
v''\delta\gamma &= \left(\frac{\delta^2 v}{\delta\alpha^2}+\frac{\delta^2 v}{\delta\beta^2}\right)\left(\delta\upsilon-\delta_{\prime\prime}\frac{\delta v}{\delta\gamma}\right)-\frac{\delta^2 v}{\delta\gamma\delta\alpha}\left(\delta\sigma-\delta_{\prime\prime}\frac{\delta v}{\delta\alpha}\right)-\frac{\delta^2 v}{\delta\beta\delta\gamma}\left(\delta\tau-\delta_{\prime\prime}\frac{\delta v}{\delta\beta}\right),\\
v'''\delta\alpha' &= \left(\frac{\delta^2 v'}{\delta\beta'^2}+\frac{\delta^2 v'}{\delta\gamma'^2}\right)\left(\delta\sigma'-\delta'\frac{\delta v'}{\delta\alpha'}\right)-\frac{\delta^2 v'}{\delta\alpha'\delta\beta'}\left(\delta\tau'-\delta'\frac{\delta v'}{\delta\beta'}\right)-\frac{\delta^2 v'}{\delta\gamma'\delta\alpha'}\left(\delta\upsilon'-\delta'\frac{\delta v'}{\delta\gamma'}\right),\\
v'''\delta\beta' &= \left(\frac{\delta^2 v'}{\delta\gamma'^2}+\frac{\delta^2 v'}{\delta\alpha'^2}\right)\left(\delta\tau'-\delta'\frac{\delta v'}{\delta\beta'}\right)-\frac{\delta^2 v'}{\delta\beta'\delta\gamma'}\left(\delta\upsilon'-\delta'\frac{\delta v'}{\delta\gamma'}\right)-\frac{\delta^2 v'}{\delta\alpha'\delta\beta'}\left(\delta\sigma'-\delta'\frac{\delta v'}{\delta\alpha'}\right),\\
v'''\delta\gamma' &= \left(\frac{\delta^2 v'}{\delta\alpha'^2}+\frac{\delta^2 v'}{\delta\beta'^2}\right)\left(\delta\upsilon'-\delta'\frac{\delta v'}{\delta\gamma'}\right)-\frac{\delta^2 v'}{\delta\gamma'\delta\alpha'}\left(\delta\sigma'-\delta'\frac{\delta v'}{\delta\alpha'}\right)-\frac{\delta^2 v'}{\delta\beta'\delta\gamma'}\left(\delta\tau'-\delta'\frac{\delta v'}{\delta\beta'}\right),
\end{aligned}\right\}(\mathrm{H}^{16})$$

v'' having the meaning (L^6), and v''' the analogous meaning

$$v''' = \frac{\delta^2 v'}{\delta\alpha'^2}\,\frac{\delta^2 v'}{\delta\beta'^2}-\left(\frac{\delta^2 v'}{\delta\alpha'\delta\beta'}\right)^2+\frac{\delta^2 v'}{\delta\beta'^2}\,\frac{\delta^2 v'}{\delta\gamma'^2}-\left(\frac{\delta^2 v'}{\delta\beta'\delta\gamma'}\right)^2+\frac{\delta^2 v'}{\delta\gamma'^2}\,\frac{\delta^2 v'}{\delta\alpha'^2}-\left(\frac{\delta^2 v'}{\delta\gamma'\delta\alpha'}\right)^2. \qquad (\mathrm{I}^{16})$$

We might also deduce the variations of $\alpha\ \beta\ \gamma\ \alpha'\ \beta'\ \gamma'$ from those of $\sigma\ \tau\ \upsilon\ \sigma'\ \tau'\ \upsilon'\ x\ y\ z\ x'\ y'\ z'\ \chi$, by differentiating the equations (I) of the second number, and by employing the functions Ω, Ω', instead of v, v'.

The general Linear Expressions for the Arrangement of Near Rays, fail at a Point of Vergency. Determination of these Points, and of their Loci, the Caustic Surfaces, in a Straight or Curved System, by the Methods of the present Supplement.

23. We have hitherto supposed that the infinitesimal or limiting expressions of the variations of the extreme cosines of direction of a luminous path, are linear functions of the variations of the extreme co-ordinates and colour. But although this supposition is in general true, it admits of an important and extensive exception; for the linear form becomes inapplicable when the given luminous path $(A, B)_\chi$, with which other near paths are to be compared, is intersected in its initial and final points A, B, by another path infinitely near, and having the same colour χ: since then the extreme directions may undergo certain infinitesimal variations, while the extreme positions A, B, and the colour χ, remain unaltered. It is therefore an important general problem of mathematical optics, to determine, for any proposed optical combination, *the relations between the extreme co-ordinates and the colour of a luminous path*

which is intersected in its extreme points by another infinitely near path of the same colour. This general problem, of which the solution includes the general theory of the caustic surfaces touched by the straight or curved rays of any proposed optical system, may easily be resolved by the methods of the present Supplement.

In applying these methods to the present question, we are to differentiate the general equations which connect the extreme directions with the extreme positions and colour, by the partial differential coefficients of the first order of the characteristic and related functions, and then to suppress the variations of $x\ y\ z\ x'\ y'\ z'\ \chi$. And of the partial differential coefficients of the second order, introduced by such differentiation, it is easy to see by (A^9) that those of the characteristic function V, or at least some of them, are infinite in the present research: it is therefore advantageous here to employ one of the auxiliary functions W, T, combined if necessary with the functions υ, υ', or Ω, Ω', which express by their form the properties of the extreme media.

Thus, when the final medium is uniform, and therefore the final rays straight, we may conveniently employ the following equations, which involve the coefficients of the functions W, Ω, and were established in the sixth number,

$$x = \frac{\delta W}{\delta\sigma} + V\frac{\delta\Omega}{\delta\sigma}, \quad y = \frac{\delta W}{\delta\tau} + V\frac{\delta\Omega}{\delta\tau}, \quad z = \frac{\delta W}{\delta\upsilon} + V\frac{\delta\Omega}{\delta\upsilon}. \qquad (\mathrm{W}^2)$$

Differentiating these equations with respect to $\sigma\ \tau\ \upsilon$ as the only variables, and suppressing the variation of the first order of V, as well as those of $x\ y\ z\ x'\ y'\ z'\ \chi$, we obtain

$$\left.\begin{aligned}
0&=\Big(\frac{\delta^2 W}{\delta\sigma^2}+V\frac{\delta^2\Omega}{\delta\sigma^2}\Big)\delta\sigma+\Big(\frac{\delta^2 W}{\delta\sigma\delta\tau}+V\frac{\delta^2\Omega}{\delta\sigma\delta\tau}\Big)\delta\tau+\Big(\frac{\delta^2 W}{\delta\sigma\delta\upsilon}+V\frac{\delta^2\Omega}{\delta\sigma\delta\upsilon}\Big)\delta\upsilon,\\
0&=\Big(\frac{\delta^2 W}{\delta\sigma\delta\tau}+V\frac{\delta^2\Omega}{\delta\sigma\delta\tau}\Big)\delta\sigma+\Big(\frac{\delta^2 W}{\delta\tau^2}+V\frac{\delta^2\Omega}{\delta\tau^2}\Big)\delta\tau+\Big(\frac{\delta^2 W}{\delta\tau\delta\upsilon}+V\frac{\delta^2\Omega}{\delta\tau\delta\upsilon}\Big)\delta\upsilon,\\
0&=\Big(\frac{\delta^2 W}{\delta\sigma\delta\upsilon}+V\frac{\delta^2\Omega}{\delta\sigma\delta\upsilon}\Big)\delta\sigma+\Big(\frac{\delta^2 W}{\delta\tau\delta\upsilon}+V\frac{\delta^2\Omega}{\delta\tau\delta\upsilon}\Big)\delta\tau+\Big(\frac{\delta^2 W}{\delta\upsilon^2}+V\frac{\delta^2\Omega}{\delta\upsilon^2}\Big)\delta\upsilon,
\end{aligned}\right\}\quad(\mathrm{K}^{16})$$

and hence, by a symmetric elimination, and by the forms of W, Ω,

$$\begin{aligned}
0 = {}&\Big(\frac{\delta^2 W}{\delta\sigma^2}+V\frac{\delta^2\Omega}{\delta\sigma^2}\Big)\Big(\frac{\delta^2 W}{\delta\tau^2}+V\frac{\delta^2\Omega}{\delta\tau^2}\Big)-\Big(\frac{\delta^2 W}{\delta\sigma\delta\tau}+V\frac{\delta^2\Omega}{\delta\sigma\delta\tau}\Big)^2\\
&+\Big(\frac{\delta^2 W}{\delta\tau^2}+V\frac{\delta^2\Omega}{\delta\tau^2}\Big)\Big(\frac{\delta^2 W}{\delta\upsilon^2}+V\frac{\delta^2\Omega}{\delta\upsilon^2}\Big)-\Big(\frac{\delta^2 W}{\delta\sigma\delta\tau}+V\frac{\delta^2\Omega}{\delta\tau\delta\upsilon}\Big)^2\\
&+\Big(\frac{\delta^2 W}{\delta\upsilon^2}+V\frac{\delta^2\Omega}{\delta\upsilon^2}\Big)\Big(\frac{\delta^2 W}{\delta\sigma^2}+V\frac{\delta^2\Omega}{\delta\sigma^2}\Big)-\Big(\frac{\delta^2 W}{\delta\upsilon\delta\sigma}+V\frac{\delta^2\Omega}{\delta\upsilon\delta\sigma}\Big)^2: \qquad (\mathrm{L}^{16})
\end{aligned}$$

which is a form of the condition required, for the final and initial intersections of two near luminous paths, of any common colour, the final medium being uniform. The condition (L^{16}) is quadratic with respect to V, and determines, for any final system of straight rays, corresponding to any given luminous or initial point A, and to any given colour χ, two real or imaginary points of vergency B_1, B_2, on any one straight final ray, that is, two points in which this ray is intersected by infinitely near rays of the same final system; and the joint equation in $x\ y\ z$, (involving also $x'\ y'\ z'\ \chi$ as parameters,) of the *two caustic surfaces* which are touched by all the final rays and are the loci of the points of vergency, may be obtained by eliminating $\sigma\ \tau\ \upsilon$ between the equations (W^2) and the quadratic (L^{16}): which quadratic, by the homogeneity of the functions W and $\Omega + 1$, may be put under the following simpler form,

$$\left(\frac{\delta^2 W}{\delta\sigma^2} + V\frac{\delta^2\Omega}{\delta\sigma^2}\right)\left(\frac{\delta^2 W}{\delta\tau^2} + V\frac{\delta^2\Omega}{\delta\tau^2}\right) - \left(\frac{\delta^2 W}{\delta\sigma\delta\tau} + V\frac{\delta^2\Omega}{\delta\sigma\delta\tau}\right)^2 = 0, \qquad (\mathrm{M}^{16})$$

and admits of several other transformations. When V has either of the two values determined by this quadratic, that is, when the final point B of the luminous path has any position B_1 or B_2 on either of the two caustic surfaces, then the equations deduced from (W^2) by differentiating with respect to $x\ y\ z$ as well as $\sigma\ \tau\ \upsilon$, namely,

$$\left.\begin{aligned}
\delta x - \frac{\delta\Omega}{\delta\sigma}(\sigma\delta x + \tau\delta y + \upsilon\delta z) &= \delta\frac{\delta W}{\delta\sigma} + V\delta\frac{\delta\Omega}{\delta\sigma},\\
\delta y - \frac{\delta\Omega}{\delta\tau}(\sigma\delta x + \tau\delta y + \upsilon\delta z) &= \delta\frac{\delta W}{\delta\tau} + V\delta\frac{\delta\Omega}{\delta\tau},\\
\delta z - \frac{\delta\Omega}{\delta\upsilon}(\sigma\delta x + \tau\delta y + \upsilon\delta z) &= \delta\frac{\delta W}{\delta\upsilon} + V\delta\frac{\delta\Omega}{\delta\upsilon},
\end{aligned}\right\} \qquad (\mathrm{N}^{16})$$

conduct to a linear relation between δx, δy, δz, which may be put under several forms, for example under the following,

$$\begin{aligned}
\frac{1}{\lambda}\left\{\delta x - \frac{\delta\Omega}{\delta\sigma}(\sigma\delta x + \tau\delta y + \upsilon\delta z)\right\} &= \frac{1}{\lambda'}\left\{\delta y - \frac{\delta\Omega}{\delta\tau}(\sigma\delta x + \tau\delta y + \upsilon\delta z)\right\}\\
&= \frac{1}{\lambda''}\left\{\delta z - \frac{\delta\Omega}{\delta\upsilon}(\sigma\delta x + \tau\delta y + \upsilon\delta z)\right\}, \qquad (\mathrm{O}^{16})
\end{aligned}$$

in which we may assign to $\lambda\ \lambda'\ \lambda''$ any of the following systems of values,

$$\left.\begin{aligned}
&\text{First } \lambda = \frac{\delta^2 W}{\delta\sigma^2} + V\frac{\delta^2\Omega}{\delta\sigma^2}, \quad \lambda' = \frac{\delta^2 W}{\delta\sigma\delta\tau} + V\frac{\delta^2\Omega}{\delta\sigma\delta\tau}, \quad \lambda'' = \frac{\delta^2 W}{\delta\sigma\delta\upsilon} + V\frac{\delta^2\Omega}{\delta\sigma\delta\upsilon};\\
&\text{Second } \lambda = \frac{\delta^2 W}{\delta\sigma\delta\tau} + V\frac{\delta^2\Omega}{\delta\sigma\delta\tau}, \quad \lambda' = \frac{\delta^2 W}{\delta\tau^2} + V\frac{\delta^2\Omega}{\delta\tau^2}, \quad \lambda'' = \frac{\delta^2 W}{\delta\tau\delta\upsilon} + V\frac{\delta^2\Omega}{\delta\tau\delta\upsilon};\\
&\text{Third } \lambda = \frac{\delta^2 W}{\delta\sigma\delta\upsilon} + V\frac{\delta^2\Omega}{\delta\sigma\delta\upsilon}, \quad \lambda' = \frac{\delta^2 W}{\delta\tau\delta\upsilon} + V\frac{\delta^2\Omega}{\delta\tau\delta\upsilon}, \quad \lambda'' = \frac{\delta^2 W}{\delta\upsilon^2} + V\frac{\delta^2\Omega}{\delta\upsilon^2}:
\end{aligned}\right\} \qquad (\mathrm{P}^{16})$$

and it is easy to see that the linear relation thus deduced, between δx, δy, δz, is the differential equation, or equation of the tangent plane, of the caustic surface at the point of vergency $x\ y\ z$. The same linear equation represents also the plane of vergency, or the tangent plane to the developable pencil of straight rays, corresponding to the other or conjugate point of vergency on the given final ray.

When the final medium is variable, the three first equations (H'), namely,

$$x = \frac{\delta W}{\delta \sigma}, \quad y = \frac{\delta W}{\delta \tau}, \quad z = \frac{\delta W}{\delta \upsilon},$$

are to be differentiated with respect to σ, τ, υ; and thus we obtain

$$\left.\begin{aligned} \frac{\delta^2 W}{\delta \sigma^2}\,\delta\sigma + \frac{\delta^2 W}{\delta\sigma\delta\tau}\,\delta\tau + \frac{\delta^2 W}{\delta\sigma\delta\upsilon}\,\delta\upsilon = 0, \\ \frac{\delta^2 W}{\delta\sigma\delta\tau}\,\delta\sigma + \frac{\delta^2 W}{\delta\tau^2}\,\delta\tau + \frac{\delta^2 W}{\delta\tau\delta\upsilon}\,\delta\upsilon = 0, \\ \frac{\delta^2 W}{\delta\sigma\delta\upsilon}\,\delta\sigma + \frac{\delta^2 W}{\delta\tau\delta\upsilon}\,\delta\tau + \frac{\delta^2 W}{\delta\upsilon^2}\,\delta\upsilon = 0, \end{aligned}\right\} \qquad (Q^{16})$$

and consequently, by elimination,

$$\frac{\delta^2 W}{\delta\sigma^2}\,\frac{\delta^2 W}{\delta\tau^2}\,\frac{\delta^2 W}{\delta\upsilon^2} + 2\,\frac{\delta^2 W}{\delta\sigma\delta\tau}\,\frac{\delta^2 W}{\delta\tau\delta\upsilon}\,\frac{\delta^2 W}{\delta\upsilon\delta\sigma} = \frac{\delta^2 W}{\delta\sigma^2}\left(\frac{\delta^2 W}{\delta\tau\delta\upsilon}\right)^2 + \frac{\delta^2 W}{\delta\tau^2}\left(\frac{\delta^2 W}{\delta\upsilon\delta\sigma}\right)^2 + \frac{\delta^2 W}{\delta\upsilon^2}\left(\frac{\delta^2 W}{\delta\sigma\delta\tau}\right)^2 : \qquad (R^{16})$$

this equation, therefore, (which may be put under other forms,) takes the place, when the final medium is variable, of the quadratic (L^{16}) for a final uniform medium; and if we eliminate from it $\sigma\ \tau\ \upsilon$ by (H'), it will give, for any proposed initial point and colour, the equation of the *single or multiple caustic surface, touched by the curved rays* of the corresponding final system.

The auxiliary function T may also be employed for the case of curved rays, but it is chiefly useful when both the extreme media are uniform. In that case the extreme portions of a luminous path are straight, and we may employ for these extreme straight portions the equations (S^2) under the form

$$x = \frac{\delta S}{\delta\sigma}, \quad y = \frac{\delta S}{\delta\tau}, \quad x' = -\frac{\delta S}{\delta\sigma'}, \quad y' = -\frac{\delta S}{\delta\tau'}, \qquad (S^{16})$$

in which we have put, for abridgment,

$$S = T - z\upsilon + z'\upsilon', \qquad (T^{16})$$

and in which we consider υ as a function of σ, τ, χ; υ' as a function of σ', τ', χ; T as a function of σ, τ, σ', τ', χ; and S as a function of z, z', σ, τ, σ', τ', χ. Differentiating these equations (S^{16}) with respect to σ, τ, σ', τ', we find that if the extreme straight portions, ordinary or extraordinary, of two infinitely near paths of light of the same colour, intersect in an initial point $x'\ y'\ z'$, and in a final point $x\ y\ z$, the final and initial variations $\delta\sigma$, $\delta\tau$, $\delta\sigma'$, $\delta\tau'$, and the final and initial ordinates of intersection z, z', must satisfy the four following conditions,

$$\left.\begin{aligned}
0&=\frac{\delta^2 S}{\delta\sigma^2}\,\delta\sigma+\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\,\delta\tau+\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\,\delta\sigma'+\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\,\delta\tau',\\
0&=\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\,\delta\sigma+\frac{\delta^2 S}{\delta\tau^2}\,\delta\tau+\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\,\delta\sigma'+\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\,\delta\tau',\\
0&=\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\,\delta\sigma+\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\,\delta\tau+\frac{\delta^2 S}{\delta\sigma'^2}\,\delta\sigma'+\frac{\delta^2 S}{\delta\sigma'\,\delta\tau'}\,\delta\tau',\\
0&=\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\,\delta\sigma+\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\,\delta\tau+\frac{\delta^2 S}{\delta\sigma'\,\delta\tau'}\,\delta\sigma'+\frac{\delta^2 S}{\delta\tau'^2}\,\delta\tau';
\end{aligned}\right\}\quad(\mathrm{U}^{16})$$

which give, by eliminating between the two first,

$$\left.\begin{aligned}
&\Big(\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\,\frac{\delta^2 S.}{\delta\tau\,\delta\tau'}-\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\,\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\Big)\delta\sigma'=\Big(\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\,\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}-\frac{\delta^2 S}{\delta\sigma^2}\,\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\Big)\delta\sigma+\Big(\frac{\delta^2 S}{\delta\tau^2}\,\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}-\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\,\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\Big)\delta\tau;\\
&\Big(\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\,\frac{\delta^2 S}{\delta\tau\,\delta\tau'}-\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\,\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\Big)\delta\tau'=\Big(\frac{\delta^2 S}{\delta\sigma^2}\,\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}-\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\,\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\Big)\delta\sigma+\Big(\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\,\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}-\frac{\delta^2 S}{\delta\tau^2}\,\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\Big)\delta\tau;
\end{aligned}\right\}(\mathrm{V}^{16})$$

and therefore, by substituting these values of $\delta\sigma'$, $\delta\tau'$, in the two last,

$$\begin{aligned}
0=\delta\sigma\Big\{&\frac{\delta^2 S}{\delta\sigma'^2}\Big(\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\,\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}-\frac{\delta^2 S}{\delta\sigma^2}\,\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\Big)+\frac{\delta^2 S}{\delta\sigma'\,\delta\tau'}\Big(\frac{\delta^2 S}{\delta\sigma^2}\,\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}-\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\,\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\Big)\\
&+\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\Big(\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\,\frac{\delta^2 S}{\delta\tau\,\delta\tau'}-\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\,\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\Big)\Big\}\\
+\ \delta\tau.\Big\{&\frac{\delta^2 S}{\delta\sigma'^2}\Big(\frac{\delta^2 S}{\delta\tau^2}\,\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}-\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\,\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\Big)+\frac{\delta^2 S}{\delta\sigma'\,\delta\tau'}\Big(\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\,\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}-\frac{\delta^2 S}{\delta\tau^2}\,\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\Big)\\
&+\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\Big(\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\,\frac{\delta^2 S}{\delta\tau\,\delta\tau'}-\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\,\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\Big)\Big\};\\
0=\delta\sigma.\Big\{&\frac{\delta^2 S}{\delta\sigma'\,\delta\tau'}\Big(\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\,\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}-\frac{\delta^2 S}{\delta\sigma^2}\,\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\Big)+\frac{\delta^2 S}{\delta\tau'^2}\Big(\frac{\delta^2 S}{\delta\sigma^2}\,\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}-\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\,\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\Big)\\
&+\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\Big(\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\,\frac{\delta^2 S}{\delta\tau\,\delta\tau'}-\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\,\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\Big)\Big\}\\
+\ \delta\tau.\Big\{&\frac{\delta^2 S}{\delta\sigma'\,\delta\tau'}\Big(\frac{\delta^2 S}{\delta\tau^2}\,\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}-\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\,\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\Big)+\frac{\delta^2 S}{\delta\tau'^2}\Big(\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\,\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}-\frac{\delta^2 S}{\delta\tau^2}\,\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\Big)\\
&+\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\Big(\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\,\frac{\delta^2 S}{\delta\tau\,\delta\tau'}-\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\,\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\Big)\Big\};\quad(\mathrm{W}^{16})
\end{aligned}$$

so that by a new elimination we obtain, between the final and initial ordinates z, z', the following equation, which, by the form of S, is quadratic with respect to each ordinate separately, and involves the product of their squares:

$$\begin{aligned}
0=&\Big(\frac{\delta^2 S}{\delta\sigma^2}\,\frac{\delta^2 S}{\delta\tau^2}-\Big(\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\Big)^2\Big)\Big(\frac{\delta^2 S}{\delta\sigma'^2}\,\frac{\delta^2 S}{\delta\tau'^2}-\Big(\frac{\delta^2 S}{\delta\sigma'\,\delta\tau'}\Big)^2\Big)+\Big(\frac{\delta_2 S}{\delta\sigma\,\delta\sigma'}\,\frac{\delta_2 S}{\delta\tau\,\delta\tau'}-\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\,\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\Big)^2\\
&-\frac{\delta^2 S}{\delta\sigma'^2}\Big\{\frac{\delta^2 S}{\delta\sigma^2}\Big(\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\Big)^2-2\,\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\,\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\,\frac{\delta^2 S}{\delta\tau\,\delta\tau'}+\frac{\delta^2 S}{\delta\tau^2}\Big(\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\Big)^2\Big\}
\end{aligned}$$

$$+\ 2\,\frac{\delta^2 S}{\delta\sigma'\delta\tau'}\left\{\frac{\delta^2 S}{\delta\sigma^2}\,\frac{\delta^2 S}{\delta\tau\delta\sigma'}\,\frac{\delta^2 S}{\delta\tau\delta\tau'}-\frac{\delta^2 S}{\delta\sigma\delta\tau}\left(\frac{\delta^2 S}{\delta\sigma\delta\sigma'}\,\frac{\delta^2 S}{\delta\tau\delta\tau'}+\frac{\delta^2 S}{\delta\sigma\delta\tau'}\,\frac{\delta^2 S}{\delta\tau\delta\sigma'}\right)+\frac{\delta^2 S}{\delta\tau^2}\,\frac{\delta^2 S}{\delta\sigma\delta\sigma'}\,\frac{\delta^2 S}{\delta\sigma\delta\tau'}\right\}$$

$$-\ \frac{\delta^2 S}{\delta\tau'^2}\left\{\frac{\delta^2 S}{\delta\sigma^2}\left(\frac{\delta^2 S}{\delta\tau\delta\sigma'}\right)^2-2\,\frac{\delta^2 S}{\delta\sigma\delta\tau}\,\frac{\delta^2 S}{\delta\sigma\delta\sigma'}\,\frac{\delta^2 S}{\delta\tau\delta\sigma'}+\frac{\delta^2 S}{\delta\tau^2}\left(\frac{\delta^2 S}{\delta\sigma\delta\sigma'}\right)^2\right\}. \qquad (X^{16})$$

When the point of intersection of the infinitely near initial rays removes to an infinite distance, this equation reduces itself to the following,

$$0=\frac{\delta^2 S}{\delta\sigma^2}\,\frac{\delta^2 S}{\delta\tau^2}-\left(\frac{\delta^2 S}{\delta\sigma\delta\tau}\right)^2$$

$$=\left(\frac{\delta^2 T}{\delta\sigma^2}-z\,\frac{\delta^2 v}{\delta\sigma^2}\right)\left(\frac{\delta^2 T}{\delta\tau^2}-z\,\frac{\delta^2 v}{\delta\tau^2}\right)-\left(\frac{\delta^2 T}{\delta\sigma\delta\tau}-z\,\frac{\delta^2 v}{\delta\sigma\delta\tau}\right)^2: \qquad (Y^{16})$$

and when in like manner the two infinitely near final rays become parallel it gives the following quadratic to determine the two corresponding positions of the point of initial intersection,

$$0=\frac{\delta^2 S}{\delta\sigma'^2}\,\frac{\delta^2 S}{\delta\tau'^2}-\left(\frac{\delta^2 S}{\delta\sigma'\delta\tau'}\right)^2$$

$$=\left(\frac{\delta^2 T}{\delta\sigma'^2}+z'\,\frac{\delta^2 v'}{\delta\sigma'^2}\right)\left(\frac{\delta^2 T}{\delta\tau'^2}+z'\,\frac{\delta^2 v'}{\delta\tau'^2}\right)-\left(\frac{\delta^2 T}{\delta\sigma'\delta\tau'}+z'\,\frac{\delta^2 v'}{\delta\sigma'\delta\tau'}\right)^2. \qquad (Z^{16})$$

The caustic surfaces of straight systems, ordinary or extraordinary, were determined in the First Supplement: but it seemed useful to resume the subject in a more general manner here, and to treat it by the new methods of the present memoir.

Connexion of the Conditions of Initial and Final Intersection of two Near Paths of Light, Polygon or Curved, with the Maxima or Minima of the Time or Action-Function $V+V_{,}=\Sigma\int v ds$. Separating Planes, Transition Planes, and Transition Points, suggested by these Maxima and Minima. The Separating Planes divide the Near Points of less from those of greater Action, and they contain the Directions of Osculation or Intersection of the Surfaces for which V and $V_{,}$ are constant; the Transition Planes touch the Caustic Pencils, and the Transition Points are on the Caustic Curves. Extreme Osculating Waves, or Action-Surfaces: Law of Osculation. Analogous Theorems for Sudden Reflexion or Refraction.

24. The conditions of initial and final intersection of two near luminous paths, have a remarkable connexion with the maxima and minima of the integral in the law of least action, that is, with those of the characteristic function V, or rather with those of the sum of two such integrals or functions, which may be investigated in the following manner.

Let A, B, C, be three successive points, at finite intervals, on one common luminous path. Let the rectangular co-ordinates of these three points be x', y', z' for A; x, y, z for B; and $x_,$, $y_,$, $z_,$ for C. Let $V(A, B)$ denote the integral $\int v ds$ taken from the first point A to the second point B; let $V(B, C)$ denote the same integral, taken from the second point B to the third point C; and similarly, let $V(A, C)$ be the integral from A to C, which is evidently equal to the sum of the two former,

$$V(A, C) = V(A, B) + V(B, C), \qquad (A^{17})$$

so that, if we put for abridgment

$$V(A, B) = V, \qquad V(B, C) = V_,, \qquad (B^{17})$$

we shall have, by the continuity of the integral,

$$V(A, C) = V + V_,. \qquad (C^{17})$$

If we do not suppose that the intermediate point B is a point of sudden reflexion or refraction, the final direction of the part (A, B) will coincide with the initial direction of the part (B, C), and the final direction-cosines α β γ of the one part will be equal to the initial direction-cosines of the other; considering V therefore, as usual, as a function of $x\,y\,z\,x'\,y'\,z'\,\chi$, and $V_,$ as a function of $x_,\,y_,\,z_,\,x\,y\,z\,\chi$, we have, by our fundamental formula (A),

$$\frac{\delta V}{\delta x} = \frac{\delta v}{\delta \alpha} = -\frac{\delta V_,}{\delta x}, \quad \frac{\delta V}{\delta y} = \frac{\delta v}{\delta \beta} = -\frac{\delta V_,}{\delta y}, \quad \frac{\delta V}{\delta z} = \frac{\delta v}{\delta \gamma} = -\frac{\delta V_,}{\delta z}; \qquad (D^{17})$$

that is, we have

$$\delta V + \delta V_, = 0, \qquad (E^{17})$$

for any infinitesimal variations of the co-ordinates $x\,y\,z$, and therefore, to the accuracy of the first order,

$$V(A, B') + V(B', C) = V(A, B) + V(B, C) = V(A, C), \qquad (F^{17})$$

B' being any new intermediate point infinitely near to B, and the path (B', C) being not in general a continuation of the path (A, B'). If therefore we regard the extreme points A, C, as fixed, but consider the intermediate point B as variable and as not necessarily situated on the path (A, C), the function $V + V_,$, or $\Sigma \int v ds$, composed of the two partial and now not necessarily continuous integrals (B^{17}), will acquire what may be called a *stationary value*, when the paths (A, B) (B, C) become continuous, that is, when the intermediate point B takes any position on the path (A, C) from one given extreme point to the other: since then the change of this function will be infinitely small of the second order, for any infinitely small alteration $\overline{BB'}$, of the first order, in the position of the point B. The stationary value thus determined, namely, $V(A, C)$, might be called, by that customary latitude of expression which leads to the received phrase of *least action*, a *maximum* or *minimum* of the function $V + V_{,}$; but in order that this value should really be greater than all the neighbouring values, or less than all, a new condition is necessary. To find this new condition, we may observe that the relations

$$\left.\begin{aligned}
\alpha\frac{\delta^2 V}{\delta x^2} + \beta\frac{\delta^2 V}{\delta x\delta y} + \gamma\frac{\delta^2 V}{\delta x\delta z} = \frac{\delta v}{\delta x} = -\left(\alpha\frac{\delta^2 V_{,}}{\delta x^2} + \beta\frac{\delta^2 V_{,}}{\delta x\delta y} + \gamma\frac{\delta^2 V_{,}}{\delta x\delta z}\right),\\
\alpha\frac{\delta^2 V}{\delta x\delta y} + \beta\frac{\delta^2 V}{\delta y^2} + \gamma\frac{\delta^2 V}{\delta y\delta z} = \frac{\delta v}{\delta y} = -\left(\alpha\frac{\delta^2 V_{,}}{\delta x\delta y} + \beta\frac{\delta^2 V_{,}}{\delta y^2} + \gamma\frac{\delta^2 V_{,}}{\delta y\delta z}\right),\\
\alpha\frac{\delta^2 V}{\delta x\delta z} + \beta\frac{\delta^2 V}{\delta y\delta z} + \gamma\frac{\delta^2 V}{\delta z^2} = \frac{\delta v}{\delta z} = -\left(\alpha\frac{\delta^2 V_{,}}{\delta x\delta z} + \beta\frac{\delta^2 V_{,}}{\delta y\delta z} + \gamma\frac{\delta^2 V_{,}}{\delta z^2}\right),
\end{aligned}\right\}\qquad (G^{17})$$

which result from the third number, give

$$\delta^2 V + \delta^2 V_{,} = \left(\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V_{,}}{\delta x^2}\right)\left(\delta x - \frac{\alpha}{\gamma}\,\delta z\right)^2 + \left(\frac{\delta^2 V}{\delta y^2} + \frac{\delta^2 V_{,}}{\delta y^2}\right)\left(\delta y - \frac{\beta}{\gamma}\,\delta z\right)^2$$
$$+ 2\left(\frac{\delta^2 V}{\delta x\delta y} + \frac{\delta^2 V_{,}}{\delta x\delta y}\right)\left(\delta x - \frac{\alpha}{\gamma}\,\delta z\right)\left(\delta y - \frac{\beta}{\gamma}\,\delta z\right); \qquad (H^{17})$$

the *condition of existence of a maximum or minimum, properly so called, of the function* $V + V_{,}$, is therefore,

$$Q > 0, \text{ if } Q = \left(\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V_{,}}{\delta x^2}\right)\left(\frac{\delta^2 V}{\delta y^2} + \frac{\delta^2 V_{,}}{\delta y^2}\right) - \left(\frac{\delta^2 V}{\delta x\delta y} + \frac{\delta^2 V_{,}}{\delta x\delta y}\right)^2. \qquad (I^{17})$$

When we have on the contrary

$$Q < 0, \qquad (K^{17})$$

the variation of the second order $\delta^2 V + \delta^2 V_{,}$ admits of changing sign, in passing from one set of values of δx, δy, δz to another, that is, in passing from one near point B' to another; and since, to the accuracy of the second order,

$$V(A, B') + V(B', C) - V(A, C) = \tfrac{1}{2}\left(\delta^2 V + \delta^2 V_{,}\right), \qquad (L^{17})$$

we shall have the one or the other of the two following opposite inequalities

$$V(A, B') + V(B', C) > \text{ or } < V(A, C), \qquad (M^{17})$$

according as the near point B' is in one or the other pair of opposite diedrate angles formed by *two separating planes* P' P'' determined by the following equation

$$\delta^2 V + \delta^2 V_{,} = 0, \qquad (N^{17})$$

which is, by (H^{17}), quadratic with respect to the ratio

$$\frac{\delta y - \frac{\beta}{\gamma}\,\delta z}{\delta x - \frac{\alpha}{\gamma}\,\delta z}.$$

These two separating planes P' P'' contain each the ray-line or element of the path (A, B, C) at B; and they divide the near points of less from those of greater action, or those of shorter from those of longer time, when the continuous integral $V + V_{,} = V(A, C)$ is not greater than all, or less than all, the adjacent values of the sum $\Sigma f v ds$. The directions of these planes depend on the positions of the points A, B, C; so that if we consider A and B as fixed, but suppose C to move along the prolongation (B, C) of the path (A, B), the separating planes P', P'', will in general revolve about the ray-line at B. They will even become imaginary, when by this motion of C the quantity Q becomes $>$ instead of < 0, so as to satisfy the condition of existence of a maximum or minimum of the function $V + V_{,}$; and in this transition from the real to the imaginary state the two separating planes P' P'' will close up into one real *transition-plane* P, determined by either of the two following equations,

$$\left.\begin{aligned} 0 &= \left(\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V_{,}}{\delta x^2}\right)\left(\delta x - \frac{\alpha}{\gamma}\,\delta z\right) + \left(\frac{\delta^2 V}{\delta x \delta y} + \frac{\delta^2 V_{,}}{\delta x \delta y}\right)\left(\delta y - \frac{\beta}{\gamma}\,\delta z\right), \\ 0 &= \left(\frac{\delta^2 V}{\delta x \delta y} + \frac{\delta^2 V_{,}}{\delta x \delta y}\right)\left(\delta x - \frac{\alpha}{\gamma}\,\delta z\right) + \left(\frac{\delta^2 V}{\delta y^2} + \frac{\delta^2 V_{,}}{\delta y^2}\right)\left(\delta y - \frac{\beta}{\gamma}\,\delta z\right), \end{aligned}\right\} \qquad (\mathrm{O}^{17})$$

while the corresponding position of the point C, which we may call by analogy a *transition-point*, will satisfy the condition

$$Q = 0, \text{ that is, } \left(\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V_{,}}{\delta x^2}\right)\left(\frac{\delta^2 V}{\delta y^2} + \frac{\delta^2 V_{,}}{\delta y^2}\right) = \left(\frac{\delta^2 V}{\delta x \delta y} + \frac{\delta^2 V_{,}}{\delta x \delta y}\right)^2. \qquad (\mathrm{P}^{17})$$

We are now prepared to perceive a remarkable connexion between the transition-planes and transition-points to which we have been thus conducted by the consideration of the maxima and the minima of the function $V + V_{,}$, and the condition of final and initial intersection of two near luminous paths. For these conditions of intersection may be obtained by supposing that not only the point B, having for co-ordinates $x\ y\ z$, is on a given path (A, C), so as to satisfy the equations (D^{17}), but that also an infinitely near point B', having for co-ordinates $x + \delta x$, $y + \delta y$, $z + \delta z$, is on another path of the same colour connecting the same extreme points A and C, so as to give the differential equations

$$\delta\,\frac{\delta V}{\delta x} = -\,\delta\,\frac{\delta V_{,}}{\delta x},\ \delta\,\frac{\delta V}{\delta y} = -\,\delta\,\frac{\delta V_{,}}{\delta y},\ \delta\,\frac{\delta V}{\delta z} = -\,\delta\,\frac{\delta V_{,}}{\delta z}: \qquad (\mathrm{Q}^{17})$$

and since these last equations may be reduced, by the relations (G^{17}), to the forms (O^{17}), we see that when the conditions of initial and final intersection of a given path (A, B, C) with a near path (A, B', C) are satisfied, and when we consider the initial point A as fixed, the near intermediate point B' must be in a transition-plane P of the form (O^{17}), and the final point of intersection C must be a transition-point of the form (P^{17}). Continuing therefore to regard the initial point A as the fixed origin of a system of luminous paths, polygon or curved, of any common colour, which undergo any number of refractions or reflexions, ordinary or extraordinary, and gradual or sudden, it is easy to see that we may consider these paths as touching a certain set of *caustic curves*, in the final state of the system, and therefore as grouped into certain sets of consecutively intersecting paths, and as having for their loci certain corresponding sets of ray-surfaces, which may be called *caustic pencils:* and that *these caustic pencils are touched by the transition planes* (O^{17}), while *the transition-points* (P^{17}) *are on the caustic curves*, and therefore on their loci the caustic surfaces. The transition-points are also evidently the points of consecutive intersection, or of vergency, of the luminous paths from A, in the final state of the system. And it is manifest, from the foregoing remarks, that these final points of intersection are also transition-points in the following other sense, that when the point C, in moving along the prolongation of the path (A, B), arrives at any one of these positions of intersection, the condition of existence of maximum or minimum of the function $V + V_{,}$ begins or ceases to be satisfied.

The separating planes P' P'', have, when real, another remarkable property, namely, that of containing the directions of mutual osculation, at the point B, of the two action-surfaces or waves determined by the equations

$$V = \text{const.}, \quad V_{,} = \text{const.}; \qquad (\text{R}^{17})$$

for these equations may be put approximately under the following forms, (when we choose the point B for origin and the final direction of the path (A, B) for the positive semiaxis of z, so as to have $\alpha = 0$, $\beta = 0$, $\gamma = 1$,)

$$\left.\begin{aligned} z &= px + qy + \tfrac{1}{2} rx^2 + sxy + \tfrac{1}{2} ty^2, \\ z_{,} &= p_{,}x + q_{,}y + \tfrac{1}{2} r_{,}x^2 + s_{,}xy + \tfrac{1}{2} t_{,}y^2, \end{aligned}\right\} \qquad (\text{S}^{17})$$

in which the coefficients have the following relations,

$$\left.\begin{aligned} & p_{,} = p, \quad q_{,} = q, \\ & r_{,} - r = \frac{1}{v}\left(\frac{\delta^2 V_{,}}{\delta x^2} + \frac{\delta^2 V}{\delta x^2}\right), \\ & s_{,} - s = \frac{1}{v}\left(\frac{\delta^2 V_{,}}{\delta x \delta y} + \frac{\delta^2 V}{\delta x \delta y}\right), \\ & t_{,} - t = \frac{1}{v}\left(\frac{\delta^2 V_{,}}{\delta y^2} + \frac{\delta^2 V}{\delta y^2}\right), \end{aligned}\right\} \qquad (\text{T}^{17})$$

and therefore the planes

$$0 = (r_{,} - r)\, x^2 + 2(s_{,} - s)\, xy + (t_{,} - t)\, y^2, \qquad (\text{U}^{17})$$

which pass through the given ray-line at the point B, and contain the directions of osculation of the second order of the two touching surfaces (R^{17}) or (S^{17}), are the separating planes (N^{17}). We might also characterise these separating planes, or planes of osculation, as containing the directions of mutual intersection of the same two touching surfaces for which V and $V_{,}$ are constant; or as the planes in which the deflexures of these two surfaces are equal, the ray-line at B being made the axis of deflexion.

The comparison of the same two waves or action-surfaces (R^{17}) gives a new property of the planes and points of transition; for the equations which determine a plane and point of this kind may be put under the form

$$(r-r_{,})x+(s-s_{,})y=0, \quad (s-s_{,})x+(t-t_{,})y=0, \quad \text{or,}\ \delta p_{,}=\delta p, \quad \delta q_{,}=\delta q: \qquad (\mathrm{V}^{17})$$

they express, therefore, that when C is a transition point, the two surfaces (R^{17}) touch one another not only at the point B, but in the whole extent of an infinitely small arc contained in the transition-plane.

The point C may be called the *focus* of the second wave or action-surface $V_{,}$, since all the corresponding paths of light (B', C) are supposed to meet in it; and in like manner the point A may be called the focus of the first surface V of the same kind, since all the paths (A, B') are supposed to diverge from A. The focus A and the point of osculation B remaining fixed, we may change the focus C, and thereby the directions of osculation; but there are, in general, certain *extreme or limiting positions for the osculating focus* C, *corresponding to extreme osculating waves or action-surfaces* $V_{,}$, and it is easy to show that *these extreme osculating foci coincide with the transition-points or points of vergency:* and that *the transition-planes or tangent-planes of the caustic pencils contain the directions of such extreme or limiting osculation.*

These theorems of intersection and osculation include several less general theorems of the same kind, assigned in former memoirs. It is easy also to see that they extend to the case when the order of the points $A\ B\ C$ on a luminous path is different, so that B is not intermediate between A and C, and so that the paths (A, B) (A, B'), which go from A to the points B and B', coincide at those points with the paths (C, B) (C, B'), and not with the opposite paths (B, C) (B', C), that is, tend *from* the point C, not *to* it; observing only that we must then employ the *difference* instead of the *sum* of the two integrals $\int v ds$, or of the two functions V and $V_{,}$.

When the point C is on a given straight ray in a given uniform medium, we can easily prove, by the theory of the partial differential coefficients of the second order of the characteristic and related functions which was explained in former numbers, that the equation (P^{17}) becomes quadratic with respect to $z_{,}$ or $V_{,}$, and assigns, in general, two or real imaginary positions C_1, C_2, for the transition-point, or point of vergency; and that the equations (O^{17}) assign two corresponding real or imaginary transition-planes P_1 P_2, or tangent planes of caustic pencils. And when, besides, the points B, C, are both in one common uniform medium, so that the paths (B, C) (B', C) are straight, then each of the caustic pencils, or ray-surfaces, composed of such straight paths consecutively intersecting each other and touching one caustic curve, becomes a *developable pencil,* and its tangent plane becomes a *plane of vergency,* of the kind considered in the sixteenth number. The relations also between the two planes of vergency in a final uniform medium, which were pointed out in the twenty-first number, may easily be deduced from the present more general view and from the recent theorems of osculation ; for thus we are led to consider a series of waves or action-surfaces $V_{,}$, similar and similarly placed, and determined in shape but not in size or focus by the uniform medium, and then to seek the extreme or limiting surfaces of this set which osculate to the given surface V at the given point B ; and since it can be shown that *in general among any series of surfaces, similar and similarly placed, but having arbitrary magnitudes, and osculating to a given surface at a given point, there are two extreme osculating surfaces,* real or imaginary, and that *the tangents which mark the two corresponding directions of osculation are conjugate tangents* (of the kind discovered by M. DUPIN) *on each surface of the osculating series, and also on the given surface,* it follows as before that the conjugate planes of vergency in a final uniform medium are conjugate planes of deflexure of each medium-surface $V_{,}$, and also of the surface V determined by the whole combination. When the final medium is ordinary as well as uniform, then the osculating surfaces $V_{,}$ are spheres, and the directions of extreme osculation are the rectangular directions of the lines of curvature on the surface V, which is now perpendicular to the rays ; in this case, therefore, and more generally when a given final ray in a final uniform medium corresponds to an *umbilical point* or point of spheric curvature on the medium-surface $V_{,}$, the planes of vergency cut that surface, and the surface V to which it osculates, in two rectangular directions, because two conjugate tangents at an umbilical point are always perpendicular to each other : and, in like manner, the planes of vergency being conjugate planes of deflexure will (by the seventeenth number) be themselves rectangular, if the final ray whether ordinary or extraordinary be such that taking it for the axis of deflexion of the medium-surface $V_{,}$ the indicating cylinder of deflexion is circular.

The foregoing principles give also the *law of osculation* of the variable medium-surface $V_{,}$ between its extreme positions, in a final uniform medium, namely, that *the distances of the variable osculating focus from the two points of vergency, are proportional to the squares of the sines of the inclinations of the variable plane of osculation to the two planes of vergency, multiplied respectively by certain constant factors.* A formula expressing this law was deduced in the First Supplement ; but the constant and in general unequal factors, (in the formula ζ and 1,) for the squares of the sines of the inclinations, were inadvertently omitted in the enunciation. Our present methods would enable us to investigate without difficulty the law for the more complicated case, when the osculating focus C being still in a uniform medium, the point of osculation B is in another uniform medium, or even in an atmosphere ordinary or extraordinary.

We might extend the reasonings of the present number to the case of sudden reflexion or refraction, ordinary or extraordinary, and obtain analogous results, which would include, in like manner, the results of former memoirs. In this case we should find a certain analogous condition for the existence of a maximum or minimum of the function $\Sigma \int v ds$; and when this condition is not satisfied, we should have to consider *two pairs of separating planes*, which cross the tangent plane of the reflecting or refracting surface in one common pair of *separating lines:* the two pairs of planes passing together from the real to the imaginary state, and in this passage closing up into *two transition-planes, which touch the caustic pencil before and after the sudden reflexion or refraction*, and intersect in one common *transition-line, on the tangent plane of the reflector or refractor*, connected with *a transition-point upon the caustic curve of the pencil*, and with certain *extreme osculating waves or action-surfaces and focal reflectors or refractors*, of a kind easily discovered from the analogy of the foregoing results.

Formulæ for the Principal Foci and Principal Rays of a Straight or Curved System, Ordinary or Extraordinary. General method of investigating the Arrangement and Aberrations of the Rays, near a Principal Focus, or other point of vergency.

25. Among the various points of consecutive intersection of the rays of an optical system, there are in general certain eminent points of vergency, in which certain particular luminous paths are intersected each by all the infinitely near paths of the system. These eminent points and paths have been pointed out in my former memoirs, and have been called *principal foci*, and *principal rays*. They may be determined for straight final systems, by the characteristic function V, and by any three of the six following equations,

$$\left.\begin{array}{ll}
\dfrac{\delta^2 V}{\delta x^2} + \dfrac{1}{R}\dfrac{\delta^2 v}{\delta \alpha^2} = 0, & \dfrac{\delta^2 V}{\delta x \delta y} + \dfrac{1}{R}\dfrac{\delta^2 v}{\delta \alpha \delta \beta} = 0, \\
\dfrac{\delta^2 V}{\delta y^2} + \dfrac{1}{R}\dfrac{\delta^2 v}{\delta \beta^2} = 0, & \dfrac{\delta^2 V}{\delta y \delta z} + \dfrac{1}{R}\dfrac{\delta^2 v}{\delta \beta \delta \gamma} = 0, \\
\dfrac{\delta^2 V}{\delta z^2} + \dfrac{1}{R}\dfrac{\delta^2 v}{\delta \gamma^2} = 0, & \dfrac{\delta^2 V}{\delta z \delta x} + \dfrac{1}{R}\dfrac{\delta^2 v}{\delta \gamma \delta \alpha} = 0,
\end{array}\right\} \qquad (\mathrm{W}^{17})$$

x, y, z being the co-ordinates of any point on a principal ray, and $x + \alpha R$, $y + \beta R$, $z + \gamma R$ being the co-ordinates of the principal focus; they may also be deduced from the auxiliary function W, when made homogeneous of the first dimension with respect to σ, τ, υ, by the equations

$$\left.\begin{array}{ll} \frac{\delta^2 W}{\delta\sigma^2}+V\frac{\delta^2\Omega}{\delta\sigma^2}=0, & \frac{\delta^2 W}{\delta\sigma\delta\tau}+V\frac{\delta^2\Omega}{\delta\sigma\delta\tau}=0, \\ \frac{\delta^2 W}{\delta\tau^2}+V\frac{\delta^2\Omega}{\delta\tau^2}=0, & \frac{\delta^2 W}{\delta\tau\delta\upsilon}+V\frac{\delta^2\Omega}{\delta\tau\delta\upsilon}=0, \\ \frac{\delta^2 W}{\delta\upsilon^2}+V\frac{\delta^2\Omega}{\delta\upsilon^2}=0, & \frac{\delta^2 W}{\delta\upsilon\delta\sigma}+V\frac{\delta^2\Omega}{\delta\upsilon\delta\sigma}=0, \end{array}\right\} \quad (X^{17})$$

of which only three are distinct, and in which V corresponds to the focus: or from the function T, when expressed as depending on σ, τ, σ', τ', χ, by the following,

$$\left(\frac{\delta^2 S}{\delta\sigma'^2}\right)^{-1}\left\{\frac{\delta^2 S}{\delta\sigma^2}\left(\frac{\delta^2 S}{\delta\tau\delta\sigma'}\right)^2-2\frac{\delta^2 S}{\delta\sigma\delta\tau}\frac{\delta^2 S}{\delta\sigma\delta\sigma'}\frac{\delta^2 S}{\delta\tau\delta\sigma'}+\frac{\delta^2 S}{\delta\tau^2}\left(\frac{\delta^2 S}{\delta\sigma\delta\sigma'}\right)^2\right\}$$

$$=\left(\frac{\delta^2 S}{\delta\sigma'\delta\tau'}\right)^{-1}\left\{\frac{\delta^2 S}{\delta\sigma^2}\frac{\delta^2 S}{\delta\tau\delta\sigma'}\frac{\delta^2 S}{\delta\tau\delta\tau'}-\frac{\delta^2 S}{\delta\sigma\delta\tau}\left(\frac{\delta^2 S}{\delta\sigma\delta\sigma'}\frac{\delta^2 S}{\delta\tau\delta\tau'}+\frac{\delta^2 S}{\delta\sigma\delta\tau'}\frac{\delta^2 S}{\delta\tau\delta\sigma'}\right)+\frac{\delta^2 S}{\delta\tau^2}\frac{\delta^2 S}{\delta\sigma\delta\sigma'}\frac{\delta^2 S}{\delta\sigma\delta\tau'}\right\}$$

$$=\left(\frac{\delta^2 S}{\delta\tau'^2}\right)^{-1}\left\{\frac{\delta^2 S}{\delta\sigma^2}\left(\frac{\delta^2 S}{\delta\tau\delta\tau'}\right)^2-2\frac{\delta^2 S}{\delta\sigma\delta\tau}\frac{\delta^2 S}{\delta\sigma\delta\tau'}\frac{\delta^2 S}{\delta\tau\delta\tau'}+\frac{\delta^2 S}{\delta\tau^2}\left(\frac{\delta^2 S}{\delta\sigma\delta\tau'}\right)^2\right\}$$

$$=\frac{\delta^2 S}{\delta\sigma^2}\frac{\delta^2 S}{\delta\tau^2}-\left(\frac{\delta^2 S}{\delta\sigma\delta\tau}\right)^2, \quad (Y^{17})$$

in which as before, $S=T-z\upsilon+z'\upsilon'$.

When the final medium is variable, we may employ the following equations,

$$\left.\begin{array}{l} \frac{\frac{\delta^2 W}{\delta\sigma^2}}{\left(\frac{\delta\Omega}{\delta\sigma}\right)^2}=\frac{\frac{\delta^2 W}{\delta\tau^3}}{\left(\frac{\delta\Omega}{\delta\tau}\right)^2}=\frac{\frac{\delta^2 W}{\delta\upsilon^2}}{\left(\frac{\delta\Omega}{\delta\upsilon}\right)^2}=\frac{\frac{\delta^2 W}{\delta\sigma\delta\tau}}{\frac{\delta\Omega}{\delta\sigma}\frac{\delta\Omega}{\delta\tau}}=\frac{\frac{\delta^2 W}{\delta\tau\delta\upsilon}}{\frac{\delta\Omega}{\delta\tau}\frac{\delta\Omega}{\delta\upsilon}}=\frac{\frac{\delta^2 W}{\delta\upsilon\delta\sigma}}{\frac{\delta\Omega}{\delta\upsilon}\frac{\delta\Omega}{\delta\sigma}} \\ \qquad =-\left(\frac{\delta\Omega}{\delta\sigma}\frac{\delta\Omega}{\delta x}+\frac{\delta\Omega}{\delta\tau}\frac{\delta\Omega}{\delta y}+\frac{\delta\Omega}{\delta\upsilon}\frac{\delta\Omega}{\delta z}\right)^{-1}, \\ \text{or, } \frac{1}{\alpha^2}\frac{\delta^2 W}{\delta\sigma^2}=\frac{1}{\beta^2}\frac{\delta^2 W}{\delta\tau^2}=\frac{1}{\gamma^2}\frac{\delta^2 W}{\delta\upsilon^2}=\frac{1}{\alpha\beta}\frac{\delta^2 W}{\delta\sigma\delta\tau}=\frac{1}{\beta\gamma}\frac{\delta^2 W}{\delta\tau\delta\upsilon}=\frac{1}{\gamma\alpha}\frac{\delta^2 W}{\delta\upsilon\delta\sigma} \\ \qquad =\left(\alpha\frac{\delta\upsilon}{\delta x}+\beta\frac{\delta\upsilon}{\delta y}+\gamma\frac{\delta\upsilon}{\delta z}\right)^{-1}, \end{array}\right\} \quad (Z^{17})$$

of which only three are distinct, but which are sufficient to determine the *principal foci and principal rays of a curved system, ordinary or extraordinary*, by the auxiliary function W, considered as depending on σ, τ, υ, x', y', z', χ, in conformity to the new view of that function, proposed in the present Supplement. The new function T might also be employed for the same purpose, but with somewhat less facility.

It was remarked, in a former number, that at a point of vergency the general linear expressions for the relations of near rays fail; but the more complex expressions by which these linear forms must be replaced at a principal focus or other point of vergency, and generally when it is proposed to determine the aberrational corrections of the first approximate or limiting relations, can always be obtained without difficulty by developing to the required order of accuracy the general and rigorous equations which we have given for a luminous path. An example of such deduction will occur, when we come to consider the theory of *instruments of revolution,* which on account of its extent and importance must be reserved for a future occasion.

Combination of the foregoing View of Optics with the Undulatory Theory of Light.

The quantities σ, τ, υ, or $\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z}$, that is, the Partial Differential Coefficients of the First Order of the Characteristic Function V, taken with respect to the Final Co-ordinates, are, in the Undulatory Theory of Light, the Components of Normal Slowness of Propagation of a Wave. The Fundamental Formula (A) may easily be explained and proved by the principles of the same theory.

26. It remains, for the execution of the design announced at the beginning of this Supplement, to illustrate the mathematical view of optics proposed in this and in former memoirs, by connecting it more closely with the undulatory theory of light.

For this purpose we shall begin by examining the undulatory meanings of the symbols σ, τ, υ, of which, in the present Supplement, we have made so frequent a use, and which we have defined by the equations (E),

$$\sigma = \frac{\delta V}{\delta x}, \quad \tau = \frac{\delta V}{\delta y}, \quad \upsilon = \frac{\delta V}{\delta z},$$

V being the undulatory time of propagation of light of some given colour, from some origin x', y', z', to a point x, y, z, through any combination of media. It is evident that these quantities σ, τ, υ are proportional to the direction-cosines of the normal to the wave for which the time V is constant, and which has for its differential equation

$$0 = \delta V = \sigma \delta x + \tau \delta y + \upsilon \delta z; \qquad (A^{18})$$

and if, as in the second number, we denote $(\sigma^2 + \tau^2 + \upsilon^2)^{-\frac{1}{2}}$ by ω, these direction-cosines themselves will be $\sigma\omega$, $\tau\omega$, $\upsilon\omega$; and ω will be the *normal velocity,* because the infinitesimal time δV, during which the wave propagates itself in the direction of its own normal through the infinitesimal line δl, from the point x, y, z, to the point $x + \sigma\omega.\delta l$, $y + \tau\omega.\delta l$, $z + \upsilon\omega.\delta l$, is

$$\delta V = \sigma.\sigma\omega.\delta l + \tau.\tau\omega.\delta l + \upsilon.\upsilon\omega.\delta l = \frac{1}{\omega}\,\delta l: \qquad (B^{18})$$

we may therefore call the quantities σ, τ, υ, *the components of normal slowness*, because they are equal to the reciprocal of the normal velocity, that is, to *the normal slowness, multiplied respectively by the direction-cosines of the normal*, that is, by the cosines of the angles which it makes with the rectangular axes of co-ordinates.

Such then may be said to be the optical meaning of our quantities σ, τ, υ, in the theory of the propagation of light by waves. And we might easily deduce from this meaning, and from the first principles of the undulatory theory, the general expression (A) for the variation of the characteristic function V, which has been proposed in the present and former memoirs, as fundamental in mathematical optics. For it is an immediate consequence of the dynamical ideas of the undulatory theory of light, that for a plane wave of a given direction and colour, in a given uniform medium, the normal velocity of propagation is determined, or at least restricted to a finite variety of values; so that this normal velocity may be considered as a function of its cosines of direction, involving also the colour, and depending for its form on the nature of the uniform medium, and on the positions of the axes of co-ordinates, to which the angles of direction are referred: and if the medium be variable instead of uniform, and the wave curved instead of plane, we must suppose that the normal velocity ω is still a function of its direction-cosines $\sigma(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$, $\tau(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$, $\upsilon(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$, and of the colour χ, involving also, in this more general case, the co-ordinates x, y, z. In this manner we are conducted, by the principles of the undulatory theory, to a relation between σ, τ, υ, x, y, z, χ, of the kind already often employed in the present Supplement, namely,

$$0=\Omega=(\sigma^2+\tau^2+\upsilon^2)^{\frac{1}{2}}\omega-1, \qquad \text{(M)}$$

$\Omega+1$ being a homogeneous function of σ, τ, υ, of the first dimension, which satisfies therefore the condition

$$\sigma\frac{\delta\Omega}{\delta\sigma}+\tau\frac{\delta\Omega}{\delta\tau}+\upsilon\frac{\delta\Omega}{\delta\upsilon}=\Omega+1,$$

and which involves also in general the co-ordinates x, y, z, and the colour χ, and depends for its form on the optical properties of the medium in which the point $x\,y\,z$ is placed. *To connect now*, for any given point and colour, the *velocity and direction of the ray with the direction of the normal of the wave*, we may suppose, at first, that the medium is uniform, and that the wave is plane. The two positions of this plane wave, at the time V, and at the time $V+\Delta V$, may be denoted by the equations

$$\left.\begin{array}{ll}\text{First} & \sigma x+\tau y+\upsilon z=V+W,\\ \text{Second} & \sigma\Delta x+\tau\Delta y+\upsilon\Delta z=\Delta V,\end{array}\right\} \qquad (\text{C}^{18})$$

in which σ, τ, υ, W, are constants; and by the principles of the same undulatory theory, if the point $x+\Delta x$, $y+\Delta y$, $z+\Delta z$, on the second plane wave, corresponding to the time $V+\Delta V$, be upon the ray that passes through the point $x\ y\ z$ of the first plane wave, it will be also on all the other infinitely near plane waves which correspond to the same time $V+\Delta V$, these other waves having passed through the point $x\ y\ z$ at the time V, and having made infinitely small angles with the first plane wave; we are therefore to find the co-ordinaets $x+\Delta x$, $y+\Delta y$, $z+\Delta z$, of the second point upon the ray, by seeking the intersection of the second wave (C^{18}) with all those other waves which are obtained from it by assigning to σ, τ, υ, any infinitely small variations consistent with the relation

$$0=\delta\Omega=\frac{\delta\Omega}{\delta\sigma}\,\delta\sigma+\frac{\delta\Omega}{\delta\tau}\,\delta\tau\;+\frac{\delta\Omega}{\delta\upsilon}\,\delta\upsilon\,;$$

and thus we find

$$\frac{\alpha}{v}=\frac{\Delta x}{\Delta V}=\frac{\delta\Omega}{\delta\sigma}\,,\quad\frac{\beta}{v}=\frac{\Delta y}{\Delta V}=\frac{\delta\Omega}{\delta\tau}\,,\quad\frac{\gamma}{v}=\frac{\Delta z}{\Delta V}=\frac{\delta\Omega}{\delta\upsilon}\,,\qquad(\mathrm{D}^{18})$$

as in the second number of this Supplement, and therefore

$$v\;=\alpha\sigma\;+\beta\tau\;+\gamma\upsilon,$$

$$0\;=\alpha\delta\sigma+\beta\delta\tau+\gamma\delta\upsilon,$$

$$\delta v=\sigma\delta\alpha+\tau\delta\beta+\upsilon\delta\gamma,$$

and finally

$$\frac{\delta v}{\delta\alpha}=\sigma,\quad\frac{\delta v}{\delta\beta}=\tau,\quad\frac{\delta v}{\delta\gamma}=\upsilon\,,\qquad(\mathrm{E}^{18})$$

if we denote by v the reciprocal of the undulatory velocity with which the light is propagated along the ray, and by α, β, γ, the cosines of the angles which the ray makes with the axes of co-ordinates. We see, therefore, by the foregoing reasoning, which it is easy to extend to the case of curved waves and of variable media, that *the components* σ, τ, υ, *of normal slowness of a wave, or the partial differential coefficients of the first order of the time-function* V, *are equal to the partial differential coefficients of the first order,* $\frac{\delta v}{\delta\alpha}$, $\frac{\delta v}{\delta\beta}$, $\frac{\delta v}{\delta\gamma}$, *of the undulatory slowness* v *of propagation along the ray, when this latter slowness is expressed as a homogeneous function of the first dimension of the direction-cosines* $\alpha\ \beta\ \gamma$ *of the ray:* which is the general theorem of mathematical optics, expressed by our fundamental formula (A).

That general theorem does not appear to have been perceived by other writers; nor do they seem to have distinctly thought of the components of normal slowness, nor of the function of which these components are partial differential coefficients, that is, the time V of propagation of light from one variable point to another, through any combination of uniform or variable media, considered as depending on the final and initial co-ordinates and on the colour: much less do those who have hitherto written upon light, appear to have thought of *this time-function* V *as a* CHARACTER-

ISTIC FUNCTION, *to the study of which may be reduced all the problems of mathematical optics.* But the problem of connecting by general equations the direction and velocity of a ray with the direction and with the law of normal velocity of a wave, has been elegantly resolved by M. CAUCHY, in the 50th *Livraison* of the *Exercices de Mathématiques :* and the formulæ which have been there deduced by considering the normal velocity as a homogeneous function of the first dimension of its three cosines of direction, may easily be shown to agree with the equations (D^{18}).

Theory of FRESNEL. *New Formulæ, founded on that theory, for the Velocities and Polarisations of a Plane Wave, or Wave-Element. New method of deducing the Equation of* FRESNEL'S *Curved Wave propagated from a Point in a Uniform Medium with Three Unequal Elasticities. Lines of Single Ray-Velocity, and of Single Normal-Velocity, discovered by* FRESNEL.

27. Let us now consider more particularly the undulatory theory of FRESNEL.

In that theory, the small displacements of the vibrating etherial points are confined to the surface of the wave, the ether being supposed to be sensibly incompressible, and so to resist and prevent any sensible normal vibration : and the tangential forces, which regulate the tangential or transversal vibrations, result in general from the elasticity of the ether, combined with this normal resistance. It is also supposed that the etherial medium has in general three principal unequal elasticities, corresponding to displacements in the directions of three rectangular *axes of elasticity ;* in such a manner that if we take these for the axes of co-ordinates, any small component displacements δx, δy, δz parallel to these three axes will produce elastic forces $-a^2\delta x$, $-b^2\delta y$, $-c^2\delta z$ parallel to the same axes, and equal to the displacements taken with contrary signs and multiplied by certain constant positive factors a^2, b^2, c^2 : and any small resultant displacement, δl, in any other direction, having δx, δy, δz for its components or projections, will produce a corresponding elastic force $-E\delta l$, of which the components are $-a^2\delta x$, $-b^2\delta y$, $-c^2\delta z$, and which has not in general the same direction as the displacement δl, nor a direction exactly opposite to that. Light, polarised in any plane P, is supposed to correspond to vibrations perpendicular to that plane, and propagated without change of direction ; and in order that a vibration should thus preserve its direction unchanged, while the plane wave or wave-element to which it belongs is propagated through the uniform medium with a normal velocity ω, it is necessary and sufficient that the elastic force $-E\delta l$, when combined with a normal resistance arising from the incompressibility of the ether, should produce a tangential force $-\omega^2\delta l$, in the direction opposite to the displacement δl, and equal to this displacement taken with a contrary sign, and multiplied by the square of the normal velocity of propagation, so that its components are $-\omega^2\delta x$, $-\omega^2\delta y$, $-\omega^2\delta z$: that is, we must have the equations

$$\frac{1}{\sigma}(\omega^2-a^2)\,\delta x=\frac{1}{\tau}(\omega^2-b^2)\,\delta y=\frac{1}{\upsilon}(\omega^2-c^2)\,\delta z, \qquad (\mathrm{F}^{18})$$

in which σ, τ, υ, are, as before, the components of normal slowness, so that the equation of the wave-element containing the transversal vibration is

$$\sigma\delta x+\tau\delta y+\upsilon\delta z=0. \qquad (\mathrm{A}^{18})$$

These equations (A^{18}) (F^{18}) *suffice in general to determine, on* FRESNEL'S *principles, the velocities of propagation and the planes of polarisation for any given wave-element in any known crystallised medium.*

Thus, eliminating the components of displacement δx, δy, δz, between the equations (A^{18}) (F^{18}), we find the following *law of the normal velocity* ω, considered as depending on the normal direction, that is, on the ratios of σ, τ, υ,

$$\frac{\sigma^2}{\omega^2-a^2}+\frac{\tau^2}{\omega^2-b^2}+\frac{\upsilon^2}{\omega^2-c^2}=0. \qquad (\mathrm{G}^{18})$$

To deduce hence the *direction and velocity of a ray, for any given normal direction and normal velocity,* compatible with the foregoing law, that is, for any given values of the components of normal slowness σ, τ, υ, compatible with the relation (G^{18}), we are to make, by (M),

$$\omega^2=\frac{(\Omega+1)^2}{\sigma^2+\tau^2+\upsilon^2}, \qquad (\mathrm{H}^{18})$$

and we then find, by (I), or by (D^{18}), the following expressions for the components of the velocity of the ray,

$$\left.\begin{aligned}\frac{\alpha}{\upsilon}&=\frac{\delta\Omega}{\delta\sigma}=\frac{\sigma\omega^2}{\Omega+1}\,\frac{\lambda^2-a^2}{\omega^2-a^2},\\ \frac{\beta}{\upsilon}&=\frac{\delta\Omega}{\delta\tau}=\frac{\tau\omega^2}{\Omega+1}\,\frac{\lambda^2-b^2}{\omega^2-b^2},\\ \frac{\gamma}{\upsilon}&=\frac{\delta\Omega}{\delta\upsilon}=\frac{\upsilon\omega^2}{\Omega+1}\,\frac{\lambda^2-c^2}{\omega^2-c^2},\end{aligned}\right\} \qquad (\mathrm{I}^{18})$$

if we put for abridgment

$$\lambda^2=\frac{\left(\frac{a^2\sigma}{\omega^2-a^2}\right)^2+\left(\frac{b^2\tau}{\omega^2-b^2}\right)^2+\left(\frac{c^2\upsilon}{\omega^2-c^2}\right)^2}{\left(\frac{a\sigma}{\omega^2-a^2}\right)^2+\left(\frac{b\tau}{\omega^2-b^2}\right)^2+\left(\frac{c\upsilon}{\omega^2-c^2}\right)^2}. \qquad (\mathrm{K}^{18})$$

And to deduce the *law of the velocity* $\frac{1}{\upsilon}$ *of the ray, considered as depending on its own direction,* that is, on the cosines α β γ of its inclinations to the semiaxes a b c of elasticity, we are to eliminate (according to the general method of the second number) the ratios of $\sigma\ \tau\ \upsilon$ between the three expressions (I^{18}), and so to deduce the relation between the three components of velocity $\frac{\alpha}{v}, \frac{\beta}{v}, \frac{\gamma}{v}$; now the equations ($I^{18}$) give evidently, by ($K^{18}$),

$$\frac{a^2\alpha^2}{\lambda^2-a^2}+\frac{b^2\beta^2}{\lambda^2-b^2}+\frac{c^2\gamma^2}{\lambda^2-c^2}=0; \qquad (L^{18})$$

they give also, when we attend to (G^{18}),

$$\left(\frac{\alpha}{v}\right)^2+\left(\frac{\beta}{v}\right)^2+\left(\frac{\gamma}{v}\right)^2=\lambda^2: \qquad (M^{18})$$

λ therefore is the velocity of the ray, or the radius vector of the curved *unit-wave*, propagated in all directions from the origin of co-ordinates during the unit of time; and the *equation of the wave* in rectangular co-ordinates $x\ y\ z$, parallel to the axes of elasticity, is

$$\frac{a^2x^2}{x^2+y^2+z^2-a^2}+\frac{b^2y^2}{x^2+y^2+z^2-b^2}+\frac{c^2z^2}{x^2+y^2+z^2-c^2}=0, \qquad (N^{18})$$

or, when freed from fractions,

$$\begin{aligned}&(x^2+y^2+z^2)\,(a^2x^2+b^2y^2+c^2z^2)+a^2b^2c^2\\&=a^2(b^2+c^2)\,x^2+b^2(c^2+a^2)\,y^2+c^2(a^2+b^2)\,z^2.\end{aligned} \qquad (O^{18})$$

This method of determining the equation of FRESNEL's *Wave*, will perhaps be thought simpler than that which was employed by the illustrious discoverer, and than others which have since been proposed.

Reciprocally to determine by our general methods the normal direction and velocity, or the components of normal slowness σ, τ, υ, for any proposed direction and velocity of a ray compatible with this form of the wave, that is, for any values of $\alpha\ \beta\ \gamma\ \lambda$ compatible with the relation (L^{18}), we are to substitute for the ray-velocity λ in that relation its value (M^{18}), and we find, by (E^{18}),

$$\left.\begin{aligned}\sigma&=\frac{\delta v}{\delta\alpha}=\frac{\alpha}{v}\cdot\frac{1-a^2\nu^2}{\lambda^2-a^2},\\ \tau&=\frac{\delta v}{\delta\beta}=\frac{\beta}{v}\cdot\frac{1-b^2\nu^2}{\lambda^2-b^2},\\ \upsilon&=\frac{\delta v}{\delta\gamma}=\frac{\gamma}{v}\cdot\frac{1-c^2\nu^2}{\lambda^2-c^2},\end{aligned}\right\} \qquad (P^{18})$$

if we put for abridgment

$$\nu^2=\frac{\left(\frac{\alpha}{\lambda^2-a^2}\right)^2+\left(\frac{\beta}{\lambda^2-b^2}\right)^2+\left(\frac{\gamma}{\lambda^2-c^2}\right)^2}{\left(\frac{a\alpha}{\lambda^2-a^2}\right)^2+\left(\frac{b\beta}{\lambda^2-b^2}\right)^2+\left(\frac{c\gamma}{\lambda^2-c^2}\right)^2}. \qquad (Q^{18})$$

It is easy to see that the value of ν thus determined is the normal slowness, or reciprocal of ω, because the expressions (P^{18}) give, by (L^{18}),

$$\sigma^2 + \tau^2 + \upsilon^2 = \nu^2 ; \qquad (\mathrm{R}^{18})$$

and since the same expressions give also evidently, by (Q^{18}),

$$\frac{\sigma^2}{1-a^2\nu^2} + \frac{\tau^2}{1-b^2\nu^2} + \frac{\upsilon^2}{1-c^2\nu^2} = 0, \qquad (\mathrm{S}^{18})$$

we easily deduce the law (G^{18}) of dependence of the normal velocity on the normal direction, from the form of Fresnel's wave, as we had deduced the latter from the former.

The equations (L^{18}) (M^{18}) which gave us the equation of the wave in rectangular co-ordinates, give also the following polar equation for the reciprocal of its radius-vector, that is, for the slowness v of the ray,

$$\begin{aligned} 0 = v^4 - v^2\{\alpha^2(b^{-2}+c^{-2}) + \beta^2(c^{-2}+a^{-2}) + \gamma^2(a^{-2}+b^{-2})\} \\ + (\alpha^2+\beta^2+\gamma^2)(\alpha^2 b^{-2} c^{-2} + \beta^2 c^{-2} a^{-2} + \gamma^2 a^{-2} b^{-2}), \end{aligned} \qquad (\mathrm{T}^{18})$$

and therefore the following double expression for the square of this slowness,

$$\left.\begin{aligned} v^2 = \tfrac{1}{2}(c^{-2}+a^{-2})(\alpha^2+\beta^2+\gamma^2) \\ + \tfrac{1}{2}(c^{-2}-a^{-2})\{A'A'' \pm \sqrt{\alpha^2+\beta^2+\gamma^2-A^2}\ \sqrt{\alpha+\beta^2+\gamma-A''^2}\}, \end{aligned}\right\} \qquad (\mathrm{U}^{16})$$

if we put for abridgment

$$\left.\begin{aligned} A' = \alpha\sqrt{\frac{b^{-2}-a^{-2}}{c^{-2}-a^{-2}}} + \gamma\sqrt{\frac{c^{-2}-b^{-2}}{c^{-2}-a^{-2}}}, \\ A'' = \alpha\sqrt{\frac{b^{-2}-a^{-2}}{c^{-2}-a^{-2}}} - \gamma\sqrt{\frac{c^{-2}-b^{-2}}{c^{-2}-a^{-2}}}: \end{aligned}\right\} \qquad (\mathrm{V}^{18})$$

supposing therefore $a^2 > b^2 > c^2$, the *polar equation of the wave* may be put under the form

$$\rho^{-2} = \tfrac{1}{2}(c^{-3}+a^{-2}) + \tfrac{1}{2}(c^{-2}-a^{-2}) \cos.\left((\rho\rho') \pm (\rho\rho'')\right), \qquad (\mathrm{W}^{18})$$

ρ being the radius-vector or velocity, and $(\rho\rho')$ $(\rho\rho'')$ being the angles which this radius ρ makes with two constant radii ρ', ρ'', determined by the following cosines of their inclinations to the semiaxes of $x\ y\ z$, or of $a\ b\ c$,

$$\rho'_a = \rho''_a = \sqrt{\frac{b^{-2}-a^{-2}}{c^{-2}-a^{-2}}},\ \rho'_b = \rho''_b = 0,\ \rho'_c = -\rho''_c = \sqrt{\frac{c^{-2}-b^{-2}}{c^{-2}-a^{-2}}}. \qquad (\mathrm{X}^{18})$$

The expression (W^{18}), for the reciprocal of the square of the velocity of a ray, has been assigned by Fresnel, who has also remarked that it gives always two unequal velocities unless the direction ρ of the ray coincide with some one of the four directions $\pm\rho'$, $\pm\rho''$, which are opposite two by two, and situated in the plane $a\ c$ of the extreme axes of elasticity. FRESNEL has shown in like manner that any given normal direction corresponds to two unequal normal velocities, except four particular directions, which we may call $\pm\omega'$, $\pm\omega''$, and which are determined by the following cosines of direction,

$$\omega'_a = -\omega''_a = \sqrt{\frac{a^2-b^2}{a^2-c^2}}, \quad \omega'_b = \omega''_b = 0, \quad \omega'_c = \omega''_c = \sqrt{\frac{b^2-c^2}{a^2-c^2}}: \qquad (\mathrm{Y}^{18})$$

and in fact it is easy to establish the following expression for the double value of the square of the normal velocity, analogous to the expression (W^{18}),

$$\omega^2 = \tfrac{1}{2}(a^2+c^2) + \tfrac{1}{2}(a^2-c^2)\ \mathrm{cos.}\left((\omega\omega') \pm (\omega\omega'') \right), \qquad (\mathrm{Z}^{18})$$

which cannot reduce itself to a single value, unless the sine of $(\omega\omega')$ or of $(\omega\omega'')$ vanishes. FRESNEL has given the name of *optic axes* sometimes to the one and sometimes to the other of the two sets of directions (X^{18}) (Y^{18}); but to prevent the confusion which might arise from this double use of a term, we shall, for the present, call the set $\pm\rho'$, $\pm\rho''$, by the longer but more expressive name of the directions or *lines of single ray-velocity:* and similarly we shall call the set $\pm\omega'$, $\pm\omega''$, the directions or *lines of single normal velocity.*

New Properties of FRESNEL's *Wave. This Wave has Four Conoidal Cusps, at the Ends of the Lines of Single Ray-Velocity: it has also Four Circles of Contact, of which each is contained on a Touching Plane of Single Normal-Velocity. The Lines of Single Ray-Velocity may therefore be called Cusp-Rays; and the Lines of Single Normal-Velocity may be called Normals of Circular Contact.*

28. The reasonings of the foregoing number suppose that the axes of co-ordinates coincide with the axes of elasticity; but it is easy to extend the results thus obtained, to any other axes of co-ordinates, by the formulæ of transformation which were given in the thirteenth number. We shall content ourselves at present with considering two remarkable transformations of this kind, suggested by the two foregoing sets of lines of single velocity, which conduct to some new properties of FRESNEL's wave, and to some new consequences of his theory.

The polar equation (W^{18}) of the wave may be put under the form

$$1 = \tfrac{1}{2}(c^{-2}+a^{-2})\,\rho^2 + \tfrac{1}{2}(c^{-2}-a^{-2})\left\{ r'r'' \pm \sqrt{\rho^2-r'^2}\ \sqrt{\rho^2-r''^2} \right\}, \qquad (\mathrm{A}^{19})$$

if we put for abridgment

$$r' = A'\rho = x\rho'_a + z\rho'_c, \quad r'' = A''\rho = x\rho''_a + z\rho''_c, \qquad (\mathrm{B}^{19})$$

so that r', r'', are the projections of the radius-vector ρ on the directions ρ', ρ'', of single ray-velocity; and if we take new rectangular co-ordinates $x_{,}\, y_{,}\, z_{,}$, such that the plane of $x_{,}\, z_{,}$ is still the plane $a\, c$ of the extreme axes of elasticity, but that the positive semi-axis of $z_{,}$ coincides with the line ρ', we may employ the following formulæ of transformation

$$x = x_{,}\rho'_c + z_{,}\rho'_a, \quad y = y_{,}, \quad z = -x_{,}\rho'_a + z_{,}\rho'_c, \qquad (C^{19})$$

which give

$$\rho^2 = x_{,}^2 + y_{,}^2 + z_{,}^2, \quad r' = z_{,}, \quad r'' = x_{,} \sin.(\rho'\rho'') + z_{,} \cos.(\rho'\rho''), \qquad (D^{19})$$

and change the equation (A^{19}) of the wave to the form

$$\begin{aligned} 1 = b^{-2} z_{,}^2 + \tfrac{1}{2} z_{,}x_{,}(c^{-2} - a^{-2}) \sin.(\rho'\rho'') + \tfrac{1}{2}(c^{-2} + a^{-2})(x_{,}^2 + y_{,}^2) \\ \pm \tfrac{1}{2}(c^{-2} - a^{-2}) \sqrt{x_{,}^2 + y_{,}^2} \sqrt{(z_{,} \sin.(\rho'\rho'') - x_{,} \cos.(\rho'\rho''))^2 + y_{,}^2}. \qquad (E^{19}) \end{aligned}$$

This equation enables us easily to examine the shape of the wave near the end of the radius ρ', that is, near the point having for its new co-ordinates

$$x_{,} = 0, \; y_{,} = 0, \; z_{,} = b; \qquad (F^{19})$$

for it takes, near that point, the following approximate form,

$$z_{,} = b - \tfrac{1}{2} b^2 \sqrt{c^{-2} - b^{-2}} \sqrt{b^{-2} - a^{-2}} \, (x_{,} \pm \sqrt{x_{,}^2 + y_{,}^2}), \qquad (G^{19})$$

which shows that at the point (F^{19}) *the wave has a conoidal cusp*, and is touched not by one determined tangent plane but by *a tangent cone of the second degree*, represented rigorously by the equation (G^{19}). FRESNEL does not appear to have been aware of the existence of this tangent cone to his wave; he seems to have thought that at the end of a radius ρ' of single ray-velocity, the wave was touched only by two right lines, contained in the plane of $a\,c$, namely, by the tangents to a certain circle and ellipse, the intersections of the wave with that plane: but it is evident from the foregoing transformation that every other section of the wave, made by a plane containing the radius-vector ρ', is touched, at the end of that radius, by two tangent lines, contained on the cone (G^{19}). It is evident also that there are *four such conoidal cusps*, at the ends of the four lines of single ray-velocity, $\pm\rho'$, $\pm\rho''$. They are determined by the following co-ordinates, when referred to the axes of elasticity,

$$x = \pm c \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \quad y = 0, \quad z = \pm a \sqrt{\frac{b^2 - c^2}{a^2 - c^2}}; \qquad (H^{19})$$

and they are the four intersections of FRESNEL's circle and ellipse, in the plane of $a\,c$, which have for their equations in that plane

$$x^2 + z^2 = b^2, \quad a^2x^2 + c^2z^2 = a^2c^2. \qquad (I^{19})$$

Again, if we employ the following new formulæ of transformation,

$$x = x_{,,}\omega'_c + z_{,,}\omega'_a, \quad y = y_{,,}, \quad z = -x_{,,}\omega'_a + z_{,,}\omega'_c, \qquad (K^{19})$$

so as to pass to a new system of rectangular co-ordinates such that the plane of $x_{,,}\, z_{,,}$ coincides with the plane of $a\, c$, and the positive semiaxis of $z_{,,}$ with the line ω' of single normal velocity, we find a new transformed equation of the wave, which may be thus written,

$$(x_{,,}^2 + y_{,,}^2 + x_{,,} z_{,,} b^{-2} \sqrt{a^2 - b^2} \sqrt{b^2 - c^2})^2 = Q\,(1 - z_{,,}^2 b^{-2}), \qquad (\mathrm{L}^{19})$$

if we put for abridgment

$$Q = (a^2 + c^2)\,\rho^2 + (a^2 - c^2)\, r' r'' - a^2 c^2 (1 + z_{,,}^2 b^{-2})\,; \qquad (\mathrm{M}^{19})$$

and hence it is easy to prove that *the plane*

$$z_{,,} = b, \qquad (\mathrm{N}^{19})$$

which is perpendicular to the line ω' *at its extremity, touches the wave in the whole extent of a circle ;* the equation of this circle of contact being, in its own plane,

$$x_{,,}^2 + y_{,,}^2 + x_{,,} b^{-1} \sqrt{a^2 - b^2} \sqrt{b^2 - c^2} = 0. \qquad (\mathrm{O}^{19})$$

It is evident that there are *four such circles of plane contact at the ends of the four lines* $\pm\omega'$, $\pm\omega''$, *of single normal-velocity.* They are all equal to each other, and the common magnitude of their diameters is $b^{-1} \sqrt{a^2 - b^2} \sqrt{b^2 - c^2}$. The same conclusions may be drawn from FRESNEL's equation of the wave in co-ordinates $x\, y\, z$ referred to the axes of elasticity : the equations of the *four planes of circular contact* being, in these co-ordinates,

$$z \sqrt{b^2 - c^2} \pm x \sqrt{a^2 - b^2} = \pm b \sqrt{a^2 - c^2}. \qquad (\mathrm{P}^{19})$$

FRESNEL however does not appear himself to have suspected the existence of these circles of contact, nor do they seem to have been since perceived by any other person. We shall find that the circles and cusps, pointed out in the present number, conduct to some remarkable theoretical conclusions respecting the laws of refraction in biaxal crystals.

New Consequences of FRESNEL's *Principles. It follows from those Principles, that Crystals of sufficient Biaxal Energy ought to exhibit two kinds of Conical Refraction, an External and an Internal : a Cusp-Ray giving an External Cone of Rays, and a Normal of Circular Contact being connected with an Internal Cone.*

29. The general formulæ for reflexion or refraction, ordinary or extraordinary, which we have deduced from the nature of the characteristic function V, become simply

$$\Delta\sigma=0,\ \Delta\tau=0, \qquad (Q^{19})$$

when we take for the plane of $x\,y$ the tangent plane to the reflecting or refracting surface; they show therefore that *the components of normal slowness parallel to this tangent plane are not changed*, which is a new and general form for the laws of reflexion and refraction. It is easy to combine this general theorem with FRESNEL'S law of velocity, and so to deduce new consequences from that law with respect to biaxal crystals.

For this deduction, our theorem may be expressed as follows,

$$0=\Delta\left(a_t\frac{\delta v}{\delta\alpha}+b_t\frac{\delta v}{\delta\beta}+c_t\frac{\delta v}{\delta\gamma}\right), \qquad (R^{19})$$

in which v is the undulatory slowness of a ray considered as a homogeneous function of the first dimension of the cosines $\alpha\,\beta\,\gamma$ of its inclinations to any three rectangular semiaxes $a\,b\,c$, while Δ refers to the changes produced by reflexion or refraction, the unaltered trinomial to which it is prefixed being the component of normal slowness in the direction of any line t on the tangent plane of the reflecting or refracting surface, and $a_t\,b_t\,c_t$ being the cosines of the inclinations of this line to the semiaxes $a\,b\,c$: and in order to combine this theorem with the principles of FRESNEL, we have only to suppose that the rectangular semiaxes $a\,b\,c$ in each medium are the semiaxes of elasticity of that medium, and that the form of the function v is determined as in the twenty-seventh number.

Thus, to calculate the refraction of light on entering from a vacuum into a biaxal crystal $a\,b\,c$ bounded by a plane face F, we may denote by $\alpha_0\,\beta_0\,\gamma_0$ the cosines of the inclinations of the external or incident ray to two rectangular lines s, t upon the face F, and to the inward normal, and we shall have the two equations following,

$$\left.\begin{aligned}\alpha_0&=a_s\frac{\delta v}{\delta\alpha}+b_s\frac{\delta v}{\delta\beta}+c_s\frac{\delta v}{\delta\gamma}\ (=\sigma a_s+\tau b_s+\upsilon c_s),\\ \beta_0&=a_t\frac{\delta v}{\delta\alpha}+b_t\frac{\delta v}{\delta\beta}+c_t\frac{\delta v}{\delta\gamma}\ (=\sigma a_t+\tau b_t+\upsilon c_t),\end{aligned}\right\} \qquad (S^{19})$$

which contain the required connexions between $\alpha_0\,\beta_0\,\gamma_0$ and $\alpha\,\beta\,\gamma$, that is, between the external and internal directions. In this manner we find in general two incident rays for one refracted, and two refracted for one incident; because a given system of values of $\alpha\,\beta\,\gamma$, that is, a given direction of the internal ray, corresponds in general to two systems of values of the internal components of normal slowness $\sigma\,\tau\,\upsilon$, and therefore to two systems of values of $\alpha_0\,\beta_0\,\gamma_0$, that is, to two external directions; while, reciprocally, a given system of two linear relations between σ, τ, υ, deduced by (S^{19}) from a given external direction, corresponds in general to two directions of the internal ray. But there are two remarkable exceptions, connected with the two sets of lines of single velocity, and with the conoidal cusps and circles of contact on FRESNEL'S wave.

For we have seen that at a conoidal cusp the tangent plane to the wave is indeterminate; it is evident therefore that a *cusp-ray* must correspond to an infinite variety of systems of components of normal slowness σ, τ, υ, within the biaxal crystal, and therefore also to an infinite variety of systems of direction-cosines $\alpha_0\ \beta_0\ \gamma_0$ of the external ray; so that *this one internal cusp-ray must correspond to an external cone of rays, according to a new theoretical law of light*, which may be called EXTERNAL CONICAL REFRACTION.

And again, at a circle of contact, the wave has one common tangent plane for all the points of that circle, and therefore the infinite variety of internal rays which correspond to these different points have all one common wave-normal, which may be called a *normal of circular contact*, and all these internal rays have one common system of components of normal slowness $\sigma\ \tau\ \upsilon$ within the crystal, and consequently correspond to one common external ray: so that *this one external ray is connected with an internal cone of rays, according to another new theoretical law of light*, which may be called INTERNAL CONICAL REFRACTION.

To develope, somewhat more fully, these two new consequences from FRESNEL's principles, let us begin by considering *external conical refraction:* and let us seek the equation of the external cone of rays, corresponding to the internal cusp-ray ρ'. The approximate equation (G^{19}) of the wave, near the end of this cusp-ray, in the transformed co-ordinates $x_{,}\ y_{,}\ z_{,}$, gives the following approximate expression for the undulatory slowness v of a near ray, considered as a homogeneous function of the first dimension of the cosines $\alpha_{,}\ \beta_{,}\ \gamma_{,}$ of its inclinations to the positive semiaxes of these co-ordinates $x_{,}\ y_{,}\ z_{,}$,

$$v = b^{-1}\,\gamma_{,} + r_{,}\,(\alpha_{,} \pm \sqrt{\alpha_{,}^{2} + \beta_{,}^{2}}), \qquad (\mathrm{T}^{19})$$

in which

$$r_{,} = \tfrac{1}{2}\,b\,\sqrt{c^{-2} - b^{-2}}\ \sqrt{b^{-2} - a^{-2}}\,; \qquad (\mathrm{U}^{19})$$

it gives therefore by our general method, the following components of normal slowness parallel to the same semiaxes of $x_{,}\ y_{,}\ z_{,}$,

$$\left.\begin{aligned} \sigma\rho'_{c} - \upsilon\rho'_{a} = \sigma_{,} &= \frac{\delta v}{\delta \alpha_{,}} = r_{,} \pm \frac{r_{,}\alpha_{,}}{\sqrt{\alpha_{,}^{2} + \beta_{,}^{2}}}\,, \\ \tau = \tau_{,} &= \frac{\delta v}{\delta \beta_{,}} = \pm \frac{r_{,}\beta_{,}}{\sqrt{\alpha_{,}^{2} + \beta_{,}^{2}}}\,, \\ \sigma\rho'_{a} + \upsilon\rho'_{c} = \upsilon_{,} &= \frac{\delta v}{\delta \gamma_{,}} = b^{-1}, \end{aligned}\right\} \qquad (\mathrm{V}^{19})$$

the expressions for $\sigma_{,}\ \tau_{,}$ becoming indefinitely more accurate as $\alpha_{,}\ \beta_{,}$ diminish, that is, as the near internal ray approaches to the cusp-ray ρ', and the expression for υ' being rigorous: *the relations between the components of normal slowness* $\sigma\ \tau\ \upsilon$ *of the cusp-ray* ρ' are therefore

$$(\sigma\rho'_c - \upsilon\rho'_a)^2 + \tau = 2\,r_{,}(\sigma\rho'_c - \upsilon\rho'_a),\ \ \sigma\rho'_a + \upsilon\rho'_c = b^{-1}, \qquad (\mathrm{W}^{19})$$

and *the equation* (in $\alpha_{\circ}\ \beta_{\circ}$) *of the external cone of rays corresponding to the one internal cusp-ray* ρ' *is to be found by eliminating these three internal components* $\sigma\,\tau\,\upsilon$ *between the two relations* (W^{19}) *and the two equations of refraction* (S^{19}).

For example, if the internal cusp-ray ρ' coincide with the inward normal to the refracting face F of the crystal, we may take, for the semiaxes s, t upon that face, the projection of a, and the semiaxis b of elasticity; and then the equations of refraction (S^{19}) becoming

$$\alpha_{\circ} = \sigma\rho'_c - \upsilon\rho'_a,\ \ \beta_{\circ} = \tau, \qquad (\mathrm{X}^{19})$$

we have, by (W^{19}), the following polar equation of the external cone of rays,

$$\alpha_{\circ}^2 + \beta_{\circ}^2 = 2r_{,}\,\alpha_{\circ}\,; \qquad (\mathrm{Y}^{19})$$

or, in rectangular co-ordinates, an equation of the fourth degree,

$$(x_{\circ}^2 + y_{\circ}^2)^2 = 4\,r_{,}^2 x_{\circ}^2 (x_{\circ}^2 + y_{\circ}^2 + z_{\circ}^2). \qquad (\mathrm{Z}^{19})$$

This cone is nearly circular in all the known biaxal crystals, because the coefficient $r_{,}$ is small, by (U^{19}), when the biaxal energy is weak, that is, when the semiaxes of elasticity $a\ b\ c$ are nearly equal to each other: and rigorously the external cone (Z^{19}) meets the concentric sphere of radius unity in a curve contained on a circular cylinder of radius $= r_{,}$, one side of this cylinder coinciding with a ray of the cone.

With respect to the *internal conical refraction*, the *equation of the internal cone of rays corresponding to the internal wave-normal* ω', or *normal of circular contact*, is always, by (N^{19}) (O^{19}),

$$x_{,,}^2 + y_{,,}^2 + 2\,r_{,,}\,x_{,,}\,z_{,,} = 0,\ \ \text{if } r_{,,} = \tfrac{1}{2}\,b^{-2}\sqrt{a^2 - b^2}\ \ \sqrt{b^2 - c^2}, \qquad (\mathrm{A}^{20})$$

when referred to the rectangular co-ordinates $x_{,,}\ y_{,,}\ z_{,,}$ by the transformation (K^{19}); and in the simpler rectangular co-ordinates $x\ y\ z$ which are parallel to the axes of elasticity the equation of this cone is

$$(x\omega'_c - z\omega'_a)^2 + y^2 + 2\,r_{,,}\,(x\omega'_c - z\omega'_a)\,(x\omega'_a + z\omega'_c) = 0, \qquad (\mathrm{B}^{20})$$

in which we may change the co-ordinates $x\ y\ z$ to the direction-cosines $\alpha\ \beta\ \gamma$ of an internal ray of the cone: while the one external ray corresponding is determined by the following direction-cosines

$$\alpha_{\circ} = b^{-1}\,\omega'_s,\ \ \ \beta_{\circ} = b^{-1}\,\omega'_t\,; \qquad (\mathrm{C}^{20})$$

or by the ordinary law of proportional sines, since the internal wave-normal of circular contact ω', which is one ray of the internal cone, is connected with the external ray by this ordinary law, if we take as the refracting index of the crystal the reciprocal b^{-1} of the mean semiaxis of elasticity. It is evident hence that if the internal cone emerge at a new plane face, it will *emerge a cylinder*, whether the two faces be parallel or inclined, that is, whether the crystal be a plate or a prism.

Theory of Conical Polarisation. Lines of Vibration. These Lines, on FRESNEL'S *Wave, are the Intersections of Two Series of Concentric and Co-axal Ellipsoids.*

30. A given direction of a wave-normal in a biaxal crystal corresponds in general to two directions of vibration, and therefore to two planes of polarisation, determined by the equations (F^{18}), namely one for each of the two values ω_1^2, ω_2^2 of the square of the normal velocity deduced by (G^{18}) from the given system of ratios of σ, τ, υ; and these two directions of vibration, or the two planes of polarisation, that is, the two normal planes of the wave perpendicular to these vibrations, are perpendicular to each other, since we can easily deduce from (G^{18}) the following relation between ω_1^2, ω_2^2,

$$\frac{\sigma^2}{(\omega_1^2-a^2)(\omega_2^2-a^2)}+\frac{\tau^2}{(\omega_1^2-b^2)(\omega_2^2-b^2)}+\frac{\upsilon^2}{(\omega_1^2-c^2)(\omega_2^2-c^2)}=0: \qquad (\mathrm{D}^{20})$$

which general rectangularity of the two vibrations on any one plane wave has been otherwise established by FRESNEL, and is an important result of his theory. But besides this general *double polarisation* connected with the general *double refraction* in biaxal crystals, we may consider two other kinds which may be called *conical polarisation*, connected with the two kinds of *conical refraction*, which were pointed out in the foregoing number.

To examine the law of the conical polarisation connected with the internal conical refraction, and therefore with the planes of circular contact, we may employ the co-ordinates $x_{\prime\prime}\, y_{\prime\prime}\, z_{\prime\prime}$ defined by (K^{19}), and thus transform the general equations of polarisation (A^{18}) (F^{18}) into the following equally general,

$$\left.\begin{array}{c}\dfrac{\omega'_c\,\delta x_{\prime\prime}+\omega'_a\,\delta z_{\prime\prime}}{\omega'_c\,\sigma_{\prime\prime}+\omega'_a\,\upsilon_{\prime\prime}}\,(\omega^2-a^2)=\dfrac{\delta y_{\prime\prime}}{\tau_{\prime\prime}}\,(\omega^2-b^2)=\dfrac{-\omega'_a\delta x_{\prime\prime}+\omega'_c\,\delta z_{\prime\prime}}{-\omega'_a\sigma_{\prime\prime}+\omega'_c\,\upsilon_{\prime\prime}}\,(\omega^2-c^2),\\ \sigma_{\prime\prime}\delta x_{\prime\prime}+\tau_{\prime\prime}\delta y_{\prime\prime}+\upsilon_{\prime\prime}\delta z_{\prime\prime}=0\,;\end{array}\right\}\qquad (\mathrm{E}^{20})$$

which give, for the projection of a vibration on the plane $x_{\prime\prime}\, y_{\prime\prime}$ of single normal velocity, the rigorous formula

$$\frac{\delta y_{\prime\prime}}{\delta x_{\prime\prime}}=\frac{(\omega^2-a^2)(\omega^2-c^2)}{\omega^2-b^2}\;\frac{\tau_{\prime\prime}}{\upsilon_{\prime\prime}\sqrt{a^2-b^2}\;\sqrt{b^2-c^2}+\sigma_{\prime\prime}(\omega^2+b^2-a^2-c^2)}\,, \qquad (\mathrm{F}^{20})$$

and for any plane wave slightly inclined to this plane of $x_{\prime\prime}\, y_{\prime\prime}$ the following approximate relation between the components of normal slowness,

$$v_{\prime\prime} = b^{-1} + r_{\prime\prime}\,(\sigma_{\prime\prime} \pm \sqrt{\sigma_{\prime\prime}{}^2 + \tau_{\prime\prime}{}^2}), \qquad (\mathrm{G}^{20})$$

retaining the meaning (A^{20}) of $r_{\prime\prime}$; and if we attend to the general connexions, established in this Supplement, between the direction-cosines of a ray and the components of normal slowness of a wave, we easily deduce from (G^{20}), by differentiation, the following other relations,

$$\frac{\alpha_{\prime\prime}}{\gamma_{\prime\prime}} = -\frac{\delta v_{\prime\prime}}{\delta \sigma_{\prime\prime}} = -\,r_{\prime\prime}\left(1 \pm \frac{\sigma_{\prime\prime}}{\sqrt{\sigma_{\prime\prime}{}^2 + \tau_{\prime\prime}{}^2}}\right),\ \frac{\beta_{\prime\prime}}{\gamma_{\prime\prime}} = -\frac{\delta v_{\prime\prime}}{\delta \tau_{\prime\prime}} = \frac{\mp r_{\prime\prime}\tau_{\prime\prime}}{\sqrt{\sigma_{\prime\prime}{}^2 + \tau_{\prime\prime}{}^2}}; \qquad (\mathrm{H}^{20})$$

and finally for the vibrations of a near wave

$$\frac{\delta y_{\prime\prime}}{\delta x_{\prime\prime}} = \frac{\tau_{\prime\prime}}{\sigma_{\prime\prime} \pm \sqrt{\sigma_{\prime\prime}{}^2 + \tau_{\prime\prime}{}^2}} = \frac{\beta_{\prime\prime}}{\alpha_{\prime\prime}}. \qquad (\mathrm{I}^{20})$$

This formula contains the theory of the conical polarisation connected with internal conical refraction. It shows that *the vibrations at the circle of contact on* FRESNEL'S *wave, are in the chords of that circle drawn from the extremity of the normal* ω' *of single velocity;* and therefore that *the corresponding planes of polarisation all pass through another parallel normal at the opposite point of the circle.* The plane of polarisation, therefore, in passing from one position to another, *revolves only half as rapidly* as the revolving radius, so that the angle between any two planes of polarisation is only *half* the angle between the two corresponding radii of this circle on FRESNEL'S wave. And if we suppose that the direction of the external incident ray coincides with the wave-normal ω', and therefore also with the normal to the refracting face of the crystal, then the small internal components of normal slowness, $\sigma_{\prime\prime}\, \tau_{\prime\prime}$, parallel to this refracting face, are equal (by our general theorem of refraction) to the small external direction-cosines $\alpha_0\, \beta_0$ of the inclinations of a near incident ray to the semiaxes of $x_{\prime\prime}$ and $y_{\prime\prime}$; from which it follows, by (I^{20}), that *the plane of external incidence containing this near incident ray revolves twice as rapidly as the corresponding plane of refraction.*

For the other kind of conical polarisation, connected with the external conical refraction, and therefore with the conoidal cusps on FRESNEL'S wave, we find by a similar process,

$$\frac{\delta y_{\prime}}{\delta x_{\prime}} = \frac{\tau_{\prime}}{\sigma_{\prime}} = \frac{\beta_{\prime}}{\alpha_{\prime} \pm \sqrt{\alpha_{\prime}{}^2 + \beta_{\prime}{}^2}}, \qquad (\mathrm{K}^{20})$$

and

$$\delta z_{\prime} = -2\,b\,r_{\prime}\,\delta x_{\prime}, \qquad (\mathrm{L}^{20})$$

$r_{\prime}$ having the meaning (U^{19}). The formula (K^{20}) shows that the normal plane to the wave, containing any vibration near the cusp, contains either the cusp-ray itself, or a line parallel to this ray; so that the direction of any near vibration coincides with or is parallel to the projection of the cusp-ray on the corresponding tangent plane of the wave, or of the cone which touches it at the cusp: and the formula (L^{20}) shows that all these near vibrations are parallel to one common plane, which is easily seen to be perpendicular to the plane of $a\ c$, and to contain the tangent at the cusp to the elliptic section (I^{19}) of the wave, made by this latter plane; so that *all the planes of polarisation near the cusp, contain, or are parallel to, the normal of this elliptic section.* And the direction of any near vibration on the wave, or on its tangent cone, may be obtained by cutting the corresponding tangent plane of this wave or cone by a plane perpendicular to this elliptic normal.

If the cusp-ray be incident perpendicularly on a refracting face of the crystal, then the internal components $\sigma_{,}\ \tau_{,}$ are equal to the direction-cosines $\alpha_{\circ}\ \beta_{\circ}$ of the corresponding ray of the emerging external cone; and therefore, by (K^{20}), the plane of refraction of this external ray contains the internal vibration, and therefore also, by FRESNEL'S principles, the external vibration corresponding: so that, *in the external conical polarisation, produced by the perpendicular internal incidence of a cusp-ray, the plane of polarisation of an external ray is perpendicular to its plane of refraction; and therefore revolves about half as rapidly as the plane containing this emergent ray and passing through the approximate axis of the nearly circular emergent cone, when the biaxal energy is small.* We see also, by (K^{20}), that the plane containing the cusp-ray and containing or parallel to a near internal ray, revolves with double the rapidity of the plane containing the cusp-ray and parallel to the near wave-normal; and therefore, in the case of perpendicular incidence of the cusp-ray, the plane of incidence of a near internal ray revolves with double the rapidity of the plane of external refraction, which, as we have seen, contains here the external vibrations.

In general, the equations of polarisation (F^{18}), which we have deduced from FRESNEL'S principles, conduct, by (I^{18}) (L^{18}), to the following simple formula

$$a^2\alpha\delta x + b^2\beta\delta y + c^2\gamma\delta z = 0, \qquad (\mathrm{M}^{20})$$

δx, δy, δz being still the components of displacement parallel to the semiaxis a, b, c, and α, β, γ being still the cosines of the inclinations of the ray to the same semiaxes of elasticity: and this formula (M^{20}), when combined with the equation of transversal vibrations,

$$\delta V = 0, \text{ or, } \sigma\delta x + \tau\delta y + + \upsilon\delta z = 0, \qquad (\mathrm{A}^{18})$$

determines easily the direction of vibration for any given direction and velocity of a ray, that is, for any point of FRESNEL'S curved wave propagated from a luminous origin within a biaxal crystal. And we easily see that *on any wave in a biaxal crystal*, whether propagated from within or from without, the differential equation (M^{20}) determines *a series of lines of vibration*, having the property that at any point of such a line the vibration is in the direction of the line itself. To find these lines on FRESNEL's wave (O^{18}), we may change $\alpha \beta \gamma$ to $x y z$ in the differential equation (M^{20}), and we then find, by integration,

$$a^2 x^2 + b^2 y^2 + c^2 z^2 = \varepsilon^4, \qquad (N^{20})$$

ε being an arbitrary constant; and since this integral, when combined with the equation (O^{18}) of the wave itself, gives

$$(a^4 + \varepsilon^4)\, x^2 + (b^4 + \varepsilon^4)\, y^2 + (c^4 + \varepsilon^4)\, z^2 = (a^2 + b^2 + c^2)\, \varepsilon^4 - a^2\, b^2\, c^2, \qquad (O^{20})$$

we see that *the lines of vibration on* FRESNEL's *wave, propagated from a point in a biaxal crystal, are the intersections of two series* (N^{20}) (O^{20}) *of concentric and co-axal ellipsoids.*

By this general integration, extending to the whole wave, or by integrating the approximate equations for vibrations near the conoidal cusps and circles of contact, obtained from (K^{20}) (I^{20}) by changing the direction-cosines of a ray to the proportional co-ordinates of the wave, we find that near a cusp the lines of vibration coincide nearly with small parabolic arcs on the tangent cone of the wave, in planes perpendicular to the elliptic normal already mentioned; and that in crossing a circle of contact the course of each line of vibration is directed towards that point of the circle which is the end of the corresponding wave-normal of single velocity, that is, towards the foot of the perpendicular let fall from the centre of the wave on the plane of circular contact.

In any Uniform Medium, the Curved Wave propagated from a point is connected with a certain other surface, which may be called the surface of components, by relations discovered by M. CAUCHY, *and by some new relations connected with a General Theorem of Reciprocity. This new Theorem of Reciprocity gives a new construction for the Wave, in any Undulatory Theory of Light: and it connects the Cusps and Circles of Contact on* FRESNEL's *Wave, with Circles and Cusps of the same kind on the Surface of Components.*

31. The theory of the wave propagated from a point in any uniform medium, may be much illustrated by comparing this wave with a certain other surface which appears to have been first discovered by M. CAUCHY, who has pointed out some of its properties in the *Livraison* already referred to. In that *Livraison*, M. CAUCHY has treated

of the propagation of plane waves in a system of mutually attracting or repelling particles; and has been conducted to a relation between the normal velocity of propagation, which he calls s, and the cosines of its inclinations to the positive semiaxes of x, y, z, which cosines he denotes by a, b, c. The relation thus found being expressed by equating to zero a certain homogeneous function (of the sixth dimension) of s, a, b, c, it has suggested to M. CAUCHY the consideration of s as a homogeneous function of the first dimension of the cosines a, b, c, whereas we have preferred to treat the normal velocity (denoted in this Supplement by ω) as a homogeneous function of its cosines of direction of the dimension zero; a difference in method which makes no real difference in the results, because the relation existing between the cosines (namely, that the sum of their squares is unity,) permits us to transform in an infinite variety of ways any equation into which they enter. M. CAUCHY deduces from his view of the relation between the normal velocity and cosines of normal direction, the following equations between the time t and the co-ordinates $x\ y\ z$ of a ray from the origin of co-ordinates,

$$\frac{x}{t} = \frac{ds}{da}, \quad \frac{y}{t} = \frac{ds}{db}, \quad \frac{z}{t} = \frac{ds}{dc},$$

which were alluded to in the twenty-sixth number of the present Supplement, as substantially equivalent to our equations (D^{19}). He deduces also an equation of the form

$$F\left(\frac{a}{s}, \frac{b}{s}, \frac{c}{s}\right) = 0,$$

which he constructs by a surface having $\frac{a}{s}$, $\frac{b}{s}$, $\frac{c}{s}$, for its co-ordinates. Our methods suggest immediately the same surface, as the construction of the same equation under the form

$$\Omega\,(\sigma, \tau, \upsilon) = 0,$$

which has been so frequently employed in this Supplement; and from the optical meanings that we have pointed out for the co-ordinates σ, τ, υ, of this surface $\Omega = 0$, we shall call it the surface of components of normal slowness, or simply *the surface of components*. M. CAUCHY shows that this surface is connected with the curved wave propagated from the origin of co-ordinates in the unit of time, (which we have called the *unit-wave* and may denote by the equation

$$V = 1,)$$

by two remarkable relations, which can easily be deduced from our formulæ, and may be thus enunciated: first, *the sum of the products of their corresponding co-ordinates*, or, in other words, *the product of any two corresponding radii multiplied by the cosine of the included angle, is unity*; and secondly, *the wave is the enveloppe* *of the planes which cut perpendicularly the radii of the surface of components at distances from the centre equal to the reciprocals of those radii.*

To these two relations, discovered by M. CAUCHY, we may add a third, not less remarkable, which he does not seem to have perceived: namely, that *the surface of components is the enveloppe of the planes which cut perpendicularly the radii of the wave at distances from its centre equal to the reciprocals of those radii,* that is, equal to the slownesses of the rays. For it is a general theorem of reciprocity between surfaces, which can easily be deduced from the evident coexistence of the three equations

$$\left.\begin{array}{l} xx' + yy' + zz' = 1, \\ x\delta x' + y\delta y' + z\delta z' = 0, \\ x'\delta x + y'\delta y + z'\delta z = 0, \end{array}\right\} \qquad (\mathrm{P}^{20})$$

that *if one surface B be deduced from another A by drawing radii vectores to the latter from an arbitrary origin O, and altering the lengths of these radii to their reciprocals without charging their directions, and seeking the enveloppe B of the planes perpendicular at the extremities to these altered radii of A, then reciprocally, the surface A may be deduced from B by a repetition of the same construction,* employing the same origin *O*, and the same arbitrary unit of length. For example, if the surface *A* be formed by the revolution of an ellipse about its greater axis, and if we place the arbitrary origin *O* at one focus of this ellipsoid *A*, and take the arbitrary unit equal to the semiaxis minor, the enveloped surface *B* will be a sphere, having its diameter equal to the axis major of the ellipsoid, and its centre on that axis major, the interval between the centres of the two surfaces being bisected by the origin *O*; and if from this excentric origin we draw radii to the sphere *B*, and change these unequal radii to their reciprocals, and draw perpendicular planes at the extremities of these new radii, the enveloppe of the planes so drawn will be the ellipsoid *A*. Another particular case of this general theory of *reciprocal surfaces*, namely, the case of two concentric and co-axal ellipsoids, referred to their centre as origin, and having the semiaxes of one equal to the reciprocals of those of the other, has been perceived by Mr. MACCULLAGH, and elegantly proved by him, in the Second Part of the Sixteenth Volume of the Transactions of the Royal Irish Academy.

This general theorem of reciprocity, when applied to the unit-wave and surface of components, gives *a new construction for the unit-wave in any uniform medium, and for any law of velocity:* namely, that *the wave is the locus of the points obtained by letting fall perpendiculars from the centre on the tangent planes of the surface of components, and then altering the lengths of these perpendiculars to their reciprocals, without altering their directions.*

It follows also from this general theory of *reciprocal surfaces*, that *a conoidal cusp on any surface A corresponds in general to a curve of plane contact on the reciprocal* *surface B,* and reciprocally; and, accordingly the cusps and circles on FRESNEL's wave are connected with circles and cusps on the corresponding surface of components, which latter surface is indeed deducible from the former by merely changing the semiaxes of elasticity $a\,b\,c$ to their reciprocals. And it was in fact by this general theorem that I was led to discover the four circles of contact on FRESNEL's wave, by concluding that this wave must touch four planes in curves instead of points of contact, as soon as I had perceived the existence of four conoidal cusps on the surface of components, by obtaining (in some investigations respecting the aberrations of biaxal lenses) the formula (G^{20}), which is the approximate equation of such a cusp. I easily found also that there were *only four* such cusps on each of the two reciprocal surfaces, and therefore concluded that there were *only four* curves of plane contact on each. I may mention that though I have taken care to attribute to M. CAUCHY the discovery of the surface of components, yet before I met the *Exercices de Mathématiques,* I was familiar, in my own investigations, with the existence and with the foregoing properties of this surface: it is indeed immediately suggested by the first principles of my view of optics, since it constructs the fundamental partial differential equation

$$\Omega\left(\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z}\right) = 0$$

which my characteristic function V must satisfy in a final uniform medium.

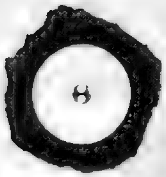
The surface of components possesses many other interesting properties, for example the following, that in a final uniform medium any two conjugate planes of vergency (E^{15}) are perpendicular to two conjugate tangents on it: which is analogous to the less simple relations considered in the twenty-first number. But the length to which this Supplement has extended, confines me here to remarking, that the general equations of reflexion or refraction,

$$\Delta\sigma = 0, \ \Delta\tau = 0, \qquad (Q^{19})$$

may be thus enunciated; *the corresponding points* (σ, τ, υ, and $\sigma + \Delta\sigma$, $\tau + \Delta\tau$, $\upsilon + \Delta\upsilon$) *upon the surface or surfaces of components* ($0 = \Omega$, $0 = \Omega + \Delta\,\Omega$,) *before and after any reflexion or refraction ordinary or extraordinary, are situated on one common perpendicular to the plane which touches the reflecting or refracting surface at the point of reflexion or refraction;* a new geometrical relation, which gives a new and general construction to determine a reflected or refracted ray, simpler in many cases than the construction proposed by HUYGHENS.

CONICAL REFRACTION

a



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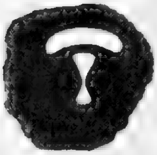
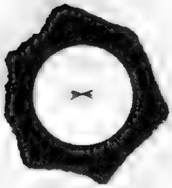


d

e

b



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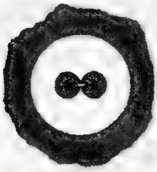
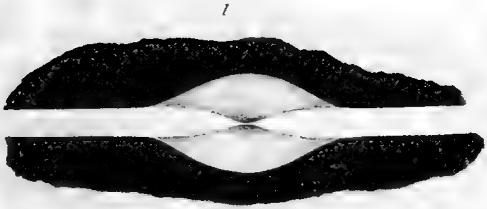


g

h

i



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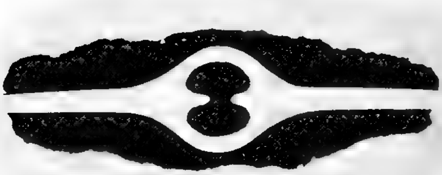
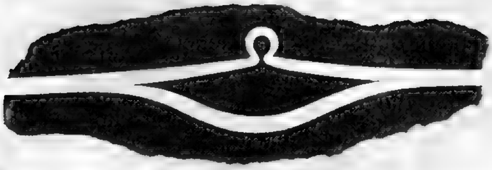


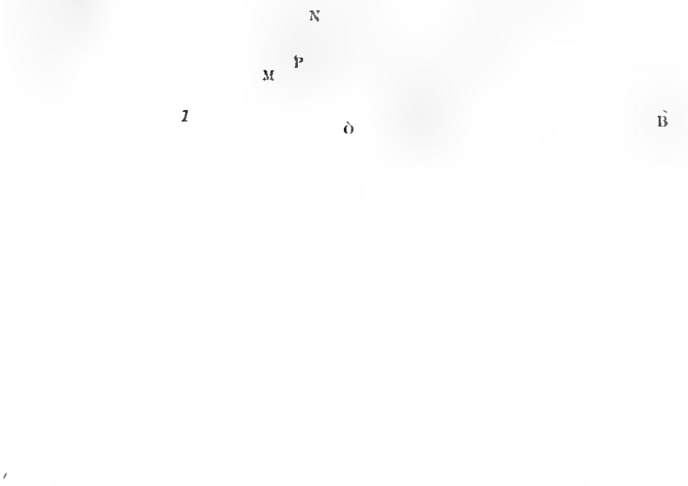
l



m

CONICAL REFRACTION.

1



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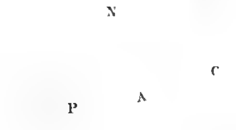
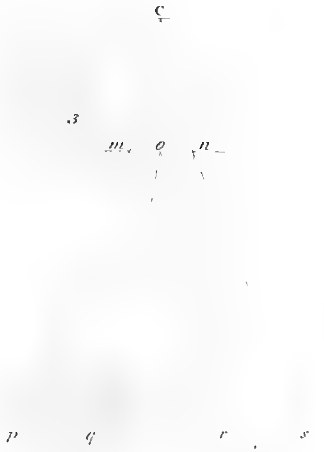


3



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5

J. Kirkwood Sc. Dublin

On the Phenomena presented by Light in its passage along the Axes of Biaxal Crystals. By the Rev. HUMPHREY LLOYD, A. M., M. R. I. A., *Fellow of Trinity College, and Professor of Natural and Experimental Philosophy in the University of Dublin.*

Read January 28, 1833.

It is well known that when a ray of light is incident upon certain crystals, such as Iceland spar and quartz, it is in general divided into two pencils, of which one is refracted according to the known law of the sines, while the direction of the other is determined by a new and extraordinary law, first assigned by HUYGHENS.

These laws were long supposed to apply to all doubly refracting substances; and it was not until the subject was examined by the ablest advocate of the undulatory theory, that the problem of double refraction was solved in all its generality. Setting out from the hypothesis, that the elasticity of the vibrating medium within the crystal is unequal in three rectangular directions, FRESNEL has shown that the surface of the wave is not, in general, either a sphere or spheroid, as in the Huyghenian law, but a surface of the fourth order, consisting of two sheets; and that the directions of the two refracted rays are determined by tangent planes drawn to these surfaces under known conditions. Such crystals have, in general, two optic axes, and are thence denominated *biaxal.* When the elasticity of the medium is the same in two of the three directions, the equation of the wave-surface is resolvable into two, which represent the sphere and spheroid of the Huyghenian law. The two optic axes in this case coincide; and the law of HUYGHENS is thus proved to be a case of a more general law, and shown to belong to *uniaxal* crystals only. Finally, when the elasticity is the same in all the three directions, the wave-surface becomes a sphere; and the refraction is single, and takes place according to the ordinary law of the sines. This case comprises a few of the crystallized, and most uncrystallized substances.

There are two remarkable cases, however, in this elegant and profound theory, which its author seems to have overlooked, if not to have misapprehended. In a communication made to the Academy at its last meeting, Professor HAMILTON has supplied these omissions in the theory of FRESNEL, and has been thus led to results in the highest degree novel and remarkable.

To understand these conclusions, it may be useful to revert for a moment to the original theory of FRESNEL. The general form of the wave-surface is determined by the equation

$$(a^2 \cos.^2 \alpha + b^2 \cos.^2 \beta + c^2 \cos.^2 \gamma)\, r^4$$
$$-[a^2(b^2+c^2)\cos.^2 \alpha + b^2(a^2+c^2)\cos.^2 \beta + c^2(a^2+b^2)\cos.^2 \gamma]\, r^2$$
$$+ a^2 b^2 c^2 = 0\,;$$

in which α, β, γ, denote the angles made by the radius-vector with the three axes, and a^2, b^2, c^2, the elasticities of the medium in these directions. If now we make $\cos.\beta = 0$, in this equation, so as to obtain the section of the surface made by the plane of xz, the result is reducible to the form

$$(r^2 - b^2)\,[(a^2 \cos.^2 \alpha + c^2 \sin.^2 \alpha)\, r^2 - a^2 c^2] = 0.$$

So that the surface intersects the plane of xz in a *circle* and *ellipse*, whose equations are

$$r = b, \quad (a^2 \cos.^2 \alpha + c^2 \sin.^2 \alpha)\, r^2 = a^2 c^2.$$

Now b, the radius of the circle, being intermediate between a and c, the semiaxes of the ellipse, it is obvious that the two curves must intersect in four points, or cusps, as represented in (fig. 1); and the angle which the radius-vector OP, drawn to the cusp, makes with the axis of x, is found by eliminating r between the two equations, by which means we obtain

$$\sin.\alpha = \pm \frac{a}{b}\sqrt{\frac{b^2 - c^2}{a^2 - c^2}}.$$

At each of the points thus determined there will be two tangents to the plane section; and consequently the ray OP, proceeding within the crystal to one of these points, might be supposed to be divided at emergence into two, whose directions are determined by those of the tangents.

Such seems to have been FRESNEL's conception of this case. Professor HAMILTON has shown, however, that there is a cusp at each of these points, not only in this particular section, but in every section of the wave surface passing through the line OP; or, in other words, that there is a *conoidal cusp* on the general wave-surface at the four points of intersection of the circle and ellipse; so that there must be an infinite number of tangent planes at each of these points, and consequently *a single ray*, such as OP, proceeding from a point within the crystal to one of these points, must be divided into an *infinite number of emergent rays*, constituting a conical surface.

It is evident further, that the circle and ellipse will have four common tangents, such as MN (fig. 1.) The planes passing through these tangents, and parallel to the third or mean axis, are parallel to the circular sections of the *surface of elasticity* of FRESNEL's theory, or perpendicular to the optic axes. FRESNEL seems to have concluded that these planes touched the wave-surface only in the two points just mentioned; and, consequently, that a single ray, incident upon a biaxal crystal in such a manner that one of the refracted rays should coincide with an optic axis, would be divided into two, determined by the points of contact. This result, if verified by experience, would place a remarkable distinction between the phenomena of uniaxal and biaxal crystals; but though the case was examined by M. Biot, no corresponding appearances were observed.

Professor Hamilton has shown that the four planes of which we have spoken touch the wave-surface, not in two points only, but in an infinite number of points, constituting each a small circle of contact, whose plane is parallel to one of the two circular sections of the surface of elasticity; and that, consequently, *a single ray* of common light, incident externally in the above-mentioned direction, should be divided within the crystal into an *infinite number of refracted rays*, constituting a conical surface.

Here, then, are two singular and unexpected consequences of the undulatory theory, not only unsupported by any facts hitherto observed, but even opposed to all the analogies derived from experience. If confirmed by experiment, they would furnish new and almost convincing proofs of the truth of that theory; and if disproved, on the other hand, it is evident that the theory must be abandoned or modified.

Being naturally anxious to submit the undulatory hypothesis to this delicate test, and to establish or disprove these new results of theory, Professor Hamilton requested me to institute a series of experiments with that view. I accordingly applied myself to this interesting research with all the attention which the subject so well merited, and have fortunately succeeded in verifying both cases of conical refraction. The substance I employed in these experiments was *arragonite*, which is well known to be a biaxal crystal, whose axes are inclined at an angle of nearly 20°. I selected it partly on account of the magnitude of its biaxal energy, and partly also because the optical elements of this mineral have been determined, apparently with great care, by Professor Rudberg; and therefore the results of theory could be applied to it at once without further examination. The specimen I used was one of considerable size and purity, procured for me by Mr. Dollond, and cut with its parallel faces perpendicular to the line bisecting the optic axes.

The first-mentioned species of conical refraction, it has been observed, takes place *in air*, when a ray of common light is transmitted within the crystal in the direction of the line joining two opposite cusps of the wave. If we suppose such a ray to pass in both directions out of the crystal, it is evident that it must emerge similarly at both surfaces; consequently, the rays which are transmitted along this line within the crystal, and form a diverging cone at emergence at the second surface, must be incident in a converging cone at the first. Having therefore nearly ascertained the required direction by means of the system of rings in polarized light, I placed a lens of short focus at its focal distance from the first surface, and in such a position that the central part of the pencil might have an incidence nearly corresponding to the cusp-ray within. Then looking through the crystal at the light of a lamp placed at a considerable distance, I observed in the expected direction a point more luminous than the space immediately about it, and surrounded by something resembling a stellar radiation. Fearing that this singular appearance might have arisen from some imperfection in the crystal, I transmitted the light in the same manner through several different parts of its substance, and always with the same result. The connexion of the phenomenon with the optic axis was proved by the system of rings which appeared in the same direction when the light was examined with a polarizing and analyzing plate.

This result is of some interest in itself, independently of its connexion with theory. It has been hitherto supposed that the only method of determining experimentally the direction of the optic axes, in most doubly refracting substances, consisted in observing the system of coloured rings, which appear around them when the incident and emergent light is polarized. Here, however, we find that common, or unpolarized light, undergoes such modifications in the neighbourhood of one of the optic axes, that the apparent direction of that axis may be at once determined, and with the aid of the simplest contrivance.*

But to examine the emergent cone, it was necessary to exclude the light which passed through the crystal in all but one direction. For this purpose, a plate of thin metal, having a minute aperture, was placed on the surface of the crystal next the eye, and the position of the aperture so adjusted, that the line connecting it with the luminous point on the first surface might be, as nearly as possible, in the direction of the cusp-ray. The exact adjustment to this direction was made by subsequent trial. The phenomenon which presented itself when this disposition was complete, was in the highest degree curious. There appeared at first a luminous circle, with a small dark space in the centre; and in this dark central space were two bright points, separated by a narrow and well-defined dark line. These appearances are represented in (figures *a* and *b*.) When the aperture in the plate was slightly shifted, the phenomena rapidly changed, assuming in succession the forms represented in (figs. *c*, *d*, *e*.) In the first stage of its change, the central dark space became greatly enlarged, and a double sector appeared in the centre. The circle was reduced to about a quadrant, and was separated by a dark interval from the sector just mentioned. This is represented in (fig. *c*.) The remote sector then disappeared, and the circular arch diminished, as in (fig. *d*);

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* This fact is here mentioned, rather as a matter of curiosity than as one likely to be of practical value in determining the optical elements of crystals. It is to be observed, moreover, that the direction thus determined is that of the normal to the circular section of the *ellipsoid* of Fresnel's theory; while the rings (there is strong reason to believe) are related to the normals to the circular sections of the *surface of elasticity*.

and as the inclination of the internal ray to the cusp-ray was further increased, these two luminous portions merged gradually into the two pencils, into which a single ray is divided in the other parts of the crystal. This change is represented in (fig. *e.*)

Similar observations were made without the lens, by bringing the flame of the lamp near the first surface of the crystal, and forming the converging cone by covering that surface also with a thin metallic plate, perforated with a minute aperture. In this case the line connecting the two minute apertures was adjusted as before, and the phenomena were the same as in the former instance, the rays which passed along this line within the crystal forming a diverging cone at emergence.

In all these experiments the emergent rays were received directly by the eye placed close to the aperture on the second surface. It was obviously desirable, however, to receive them on a screen, and thus to observe the section of the cone at different distances from its summit. After some trials, I effected this with the sun's light, the light of a lamp being too weak for the purpose. The emergent cone being made to fall on a screen of roughened glass, I was enabled to observe its sections at various distances, and therefore with all the advantages of enlargement. The light was sufficiently bright, and the appearance distinct, when the diameter of the section was between one and two inches.

On examining the emergent cone with a tourmaline plate, I was surprised to observe that one radius only of the circular section* vanished in a given position of the axis of the tourmaline, and that the ray which disappeared ranged through 360° as the tourmaline plate was turned through 180°. Thus it appeared that all the rays of the cone are polarized in different planes.

On examining this curious phenomenon more attentively, I discovered the remarkable law, "*that the angle between the planes of polarization of any two rays of the cone is half the angle between the planes containing the rays themselves and the axis.*"

Having assured myself of the near truth of this law by experiment, I was naturally led to inquire how far it was in accordance with theory; and on examining Fresnel's theory with this view, I was gratified to find that it led to the very same result.

According to the known rule, the plane of polarization of any one ray of the emergent cone must bisect the angle contained by the planes passing through the corresponding normal to the front of the wave and the two optic axes. Now, it can be easily shown that the normals to the wave, at the cusp, surround one of the optic axes, and are inclined to it all round at small angles. For the tangent of the angle

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* These sections are not mathematically circular, the line being, in fact, one of the fourth order.

which the normals to the circle and ellipse in the plane of $x z$ make with one another is

$$\frac{\sqrt{a^2-b^2}\,\sqrt{b^2-c^2}}{a\,c};$$

and it can be easily shown that the tangent of the angle which the optic axis makes with the normal to the circle, or the cusp-ray, is

$$\frac{\sqrt{a^2-b^2}\,\sqrt{b^2-c^2}}{b^2+a\,c}.$$

Now, this is about half the former, since $b^2=a\,c$, nearly; and consequently the optic axis nearly bisects the angle contained by the extreme normals in the plane of $x z$. Hence if A and B be the intersections of the two optic axes with the sphere whose centre is at the cusp, and N the intersection of one of the normals at that point with the same (fig. 2), the angle $N A C$ ranges through every magnitude between 0 and 360°, the arch $N A$ being all the time very small. Let the angle $N A C$ be denoted by α, and $N P C$ by ω, $N P$ being the arch bisecting the angle N; then in the triangle $A P N$, we have

$$\cos.\ \omega=\cos.\ A N.\ \sin.\ \alpha\ .\ \sin.\ \tfrac{1}{2}\,N+\cos.\ \alpha\ .\ \cos.\ \tfrac{1}{2}\,N;$$

or, since $A N$ is very small, and therefore $\cos.\ A N=1$, nearly,

$$\cos.\ \omega=\cos.\ (\alpha-\tfrac{1}{2}\,N),\ \text{and}\ \omega=\alpha-\tfrac{1}{2}\,N,\ \text{nearly}.$$

But, when any side of a spherical triangle is very small in comparison with the other two, the adjacent angles are together equal to 180° q. p. Consequently,

$$N=\alpha,\ \text{and}\ \omega=\tfrac{1}{2}\,\alpha,\ \text{nearly}.$$

From this it appears that the angle which the plane of polarization of any ray makes with the plane of the optic axes, is half the angle which the plane passing through the normal and the near axis makes with the same plane. But this latter angle, it may be easily shown, is very nearly the same as that which the plane passing through the emergent ray and the axis of the cone makes with the plane of the optic axes. Consequently, the angle which the plane of polarization of any ray of the emergent cone makes with the plane of the optic axes is half of that which the plane containing that ray and the axis of the cone forms with the same plane.

The general phenomena being observed, it remained to examine the magnitude and position of the emergent cone, and to compare the results with those furnished by theory. For this purpose I viewed the aperture in the second plate through a small telescope, which was moved in a plane nearly perpendicular to the axis of the emergent cone; and by noting the points at which the light failed, I obtained the magnitude of the section of the cone made by that plane. The distance of this section from the crystal being then measured, the angle of the cone was obtained from the trigonometrical tables; and was found to be very nearly 6 . I then placed the flame of a wax taper at the centre of the section, and removing the plate from the second surface of the crystal, found the direction of the ray reflected from the surface. A well defined mark was then placed on this line, at a considerable distance, and the angular distance between the centre of the flame and the mark measured by a sextant, whose centre was brought exactly to the place of the crystal. This angle was found to be 31° 56′; and consequently the angle of emergence corresponding to the central rays of the cone was 15° 58′.

Now to compare these results with those of theory.—It is a well-known principle of the theory of waves, that the direction of a ray incident upon, or emergent from, a crystal, and the normal to the front of the wave, are always in the same plane perpendicular to the surface of incidence or emergence; and the angles which these two lines make with the perpendicular to the surface, are connected by the known law of the sines; the index of refraction being the reciprocal of the normal velocity of the wave, or of the perpendicular upon the tangent plane. Now, at the cusp, there are an infinite number of normals to the wave, and consequently an infinite number of corresponding emergent rays. Of these the two rays in the plane of the optic axes form the greatest angle, and their directions are determined by those of the normals to the circle and ellipse, which constitute the section of the wave-surface in that plane. If then ρ' and ρ'' denote the angles of emergence of these rays, ι the angle which the normal to the circle, or cusp-ray, makes with the perpendicular to the surface, a the angle contained by the normals to the circle and ellipse, and p the perpendicular from the centre on the tangent to the ellipse at the cusp, we have

$$\sin. \rho' = \frac{1}{b} \sin. \iota, \quad \sin. \rho'' = \frac{1}{p} \sin. (\iota - a);$$

In which

$$\frac{1}{p} = \frac{\sqrt{a^2+c^2-b^2}}{a\,c}, \quad \tan. a = \frac{\sqrt{a^2-b^2}\,\sqrt{b^2-c^2}}{a\,c}.$$

Now in Arragonite, according to the determination of M. Rudberg,

$$\frac{1}{a} = 1.5326, \quad \frac{1}{b} = 1.6863, \quad \frac{1}{c} = 1.6908;$$

And substituting these values we find

$$\frac{1}{p} = 1{,}68708, \quad a = 1°.44'.48''.$$

These values being introduced in the first two equations, ρ' and ρ'' will be determined for any given surface of emergence. In this manner Professor Hamilton has found that when $\iota = 0$, or the surface of emergence perpendicular to the cusp-ray, $\rho' = 0$, and $\rho' = 2°. 56'\text{-}51''$. And when $\iota = 9°. 56'\text{-}27''$, or the surface perpendicular to the line bisecting the optic axes, $\rho' = 16^\circ\, 55'\, 27''$, and $\rho'' = 13^\circ\, 54'\, 49'$. Accordingly, the difference of these angles, $\rho' - \rho''$, which is the extreme angle of the emergent cone, is in the former case $2^\circ\, 56'\, 51''$,* and in the latter $3^\circ\, 0'\, 38''$. Also, half the sum of these angles, which is the angle of emergence corresponding to the axis of the cone, is $15^\circ\, 25'\, 8''$.

Comparing these with the results of observation, it will be seen that they agree nearly with respect to the mean angle of emergence, the difference amounting only to 33′; whereas the angle of the cone determined by experiment is about double of that furnished by calculation.

I also measured the angle of the cone by tracing the outline of its section on a screen of roughened glass, when the sun's light was employed instead of that of a lamp. The mean diameter of this section being then accurately ascertained, and the distance of the screen from the aperture measured, the angle was given by the tables. Measurements taken in this manner gave for the value of the angle, 6° 24′, 5° 56′, 6° 22′, respectively; and the mean of these is 6° 14′, which, like the former measurement, differs very little from the double of the calculated angle.

The results of observation thus appeared to be at variance with those of theory in two important particulars. In the first place, the emergent rays appeared to form a *solid cone*, instead of a conical surface; and in the next, the magnitude of this cone was about double of the expected magnitude. Conceiving that these discrepancies might probably be owing to the rays which are inclined to the cusp-ray at small angles, aud which pass by the edge of the aperture, I determined to ascertain the fact by trying the effects of apertures of various sizes.

I found accordingly that when the aperture was at all considerable, such as that formed by a large-sized pin, two concentric circles were seen to surround the axis, the interior of which had about double the brightness of the exterior annulus. And it was remarkable that the light of the interior circle was unpolarized, while that of the surrounding annulus was polarized according to the law already explained. When smaller apertures were used, the inner circle contracted, the breadth of the exterior annulus remaining nearly the same; until the former was finally reduced to a point in the centre of a fainter circle. When the aperture was still further diminished, a dark space sprung up in the centre, enlarging as the aperture decreased; until finally, with a very minute aperture, the breadth of this central space increased to about $\frac{3}{4}$ths of the entire diameter.

The phenomena exhibited in these cases assumed the forms represented in figures

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* It is easily shown that the sine of the angle of the cone, in this case, is generally expressed by the formula $\frac{\sqrt{a^2 - b^2}\ \sqrt{b^2 - c^2}}{a\, b\, c}$.

(*f*) and (*g*). (Fig. *h*) represents the appearance of the section when the line connecting the aperture with the luminous point on the first surface was slightly inclined to the cusp-ray.

It is easy to render an account of these various appearances. When the aperture *m n*, (fig. 3.) is at all considerable, the rays *c m*, *c n*, proceeding to its circumference from a point on the first surface, will be sensibly inclined to the cusp-ray, which we shall suppose to be the line *c o*, connecting the point on the first surface with the centre of the aperture. Consequently the interior refracted rays, *m q*, *n r*, as well as the exterior, *m p*, *n s*, will be inclined *outwards ;* and it is obvious that there will be a central bright space, limited by the lines *m q*, *n r*, each point of which will be illuminated by one interior and one exterior ray. The light in this space, therefore, will have double the intensity of that of the surrounding space ; and as the rays which combine to form it are polarized in planes at right-angles to one another, *the resulting light will be unpolarized.* When the aperture is diminished, the inclination of the rays *m q*, *n r*, to one another is lessened, until finally they are reduced to *parallelism*, and the central bright space contracts to a point. This is represented in (fig. 4.) When the aperture is still further diminished, the rays *m q*, *n r*, become inclined *inwards*, and cross (fig. 5.) It is obvious that beyond the point of intersection there will be a dark space illumined by no ray whatever ; and as in the surrounding annulus there is no meeting of rays oppositely polarized, *the whole of the light will be polarized*, and according to the law already explained. With a yet diminished aperture, the rays *m q*, *n r*, approach to parallelism with the exterior rays, *n s*, *m p ;* and the central dark space enlarges, and approaches to equality with the outer and limiting cone. Thus the annulus of light in any section is diminished indefinitely in breadth, and the cone approaches to a mathematical surface.

Now if we assume that the divergence of the two refracted rays in this plane, corresponding severally to the rays *c m*, *c o*, *c n*, is the same, as must be nearly the case, it will follow that the angle of the true cone, which would arise from the single ray *c o*, is half the sum of the angles of the exterior and interior surfaces of the conical annulus ; and that when a bright circle appears in the centre, as is the case with a considerable aperture, the dark space must be considered as negative, and the true angle is half the difference of the observed angles.

From this it follows that when the central bright space is reduced to a point, the true angle is just half the observed. Now this was very nearly the case in the experiments from which the measures were taken ; consequently the corrected angle, deduced from these measures, coincides very nearly with that assigned by theory.

Two other measurements, taken since with a more direct reference to this correction, were as follows :—

1. Distance of screen from the aperture on the second surface of the crystal = 19.3 half inches. Mean diameter of section of exterior cone = 1.27. Mean diameter of interior = 0.55. Corrected angle of cone thence computed = 2° 44′.

2. Distance of screen = 11.9. Mean diameter of section of exterior cone = 0.93. Mean diameter of interior = 0.41. Computed angle of cone = 3° 14′.

The mean of these two measurements is 2° 59′.

Inasmuch as the cusp-ray, within the crystal, corresponds to a cone of rays without, it is evident that there must be a *converging* cone incident on the first surface, equal to that which diverges from the second. With a view to determine its magnitude, I placed a kind of rough micrometer, consisting of two moveable metallic plates, immediately before the lens; and closed the plates until, on looking through the aperture on the second surface, I could see them touching the circumference of the annular section. The diameters of the interior and exterior circumferences of this section, at the distance of the lens, being thus ascertained, and the focal length of the lens measured, the corrected angle of the cone was found. The mean of three measurements taken in this manner gave for this angle 3° 47′. But the methods by which this last result was obtained, do not seem susceptible of much accuracy.

Before I conclude this part of the subject, I may observe that an interesting variation in the phenomena is obtained by substituting a narrow linear aperture for the small circular one, in the plate which covers the first surface of the crystal—that surface being close to the lamp. The linear aperture is to be so fixed, that the plane passing through it and the aperture in the plate next the eye, shall be the plane of the optic axes. In this case, according to the received theory, all the rays transmitted through the two apertures should be refracted doubly in the plane of the optic axes, so that no part of the line should appear enlarged in breadth on looking through the second aperture; whereas, according to Professor Hamilton's beautiful conclusion from the same theory, the cusp-ray should be refracted in every possible azimuth. I found accordingly that the luminous line was undilated, except in the direction corresponding to that of the cusp-ray; and that in the neighbourhood of this direction its boundaries were no longer rectilinear, but swelled out in the form of an oval curve (fig. *i.*)

When a very minute aperture was used on the surface next the eye, in this experiment, the phenomenon was rendered much more remarkable. The swelling curves in this case were separated by a considerable dark interval, and the luminous line was prolonged into this dark space, terminating abruptly near its centre. This appearance is represented in (fig. *k.*) When the plate next the eye was slightly shifted, so that the plane passing through the two apertures no longer coincided accurately with the plane of the optic axes, the curves rapidly changed, preserving, however, in all cases, the form of the *conchoid,* whose pole was the projection of the axis of the emergent cone, and asymptot the line on the first surface—(figs. *l, m.*) It is easy to show that these results are in accordance with theory.

The second kind of *conical refraction,* whose existence has been anticipated by Professor Hamilton, depends (it will be remembered) on the mathematical fact, that the wave-surface is touched in an infinite number of points, constituting a small *circle of contact,* by a single plane parallel to one of the circular sections of the surface of elasticity. It takes place when a *single external ray* falls upon a biaxal crystal in such a manner, that one refracted ray may coincide with an optic axis. When this is the case, there will be a *cone of rays* within the crystal, determined by lines connecting the centre of the wave with the points of the periphery of the circle of contact. The angle of this cone is equal to

$$\text{tang.}^{-1}\ \frac{\sqrt{a^2-b^2}\,\sqrt{b^2-c^2}}{b^2}\,;$$

and its numerical value in the case of Arragonite is 1° 55′, assuming the values of the three indices as determined for the ray E by Professor Rudberg; (see page 151.)

As the rays constituting this cone will be refracted at emergence in a direction parallel to the incident ray, they will form a small cylinder of rays in air. This cylinder, it will be seen, is in all cases extremely small; for the diameter of its section made by the surface of emergence subtends an angle of 1° 55′ only, at a distance equal to the thickness of the crystal. Hence the experiments required to detect its existence and measure its magnitude, demand more care and precision than those already described. The incident light was that of a lamp placed at some distance; and in order to reduce as much as possible the breadth of the incident beam, it was constrained to pass through two small apertures, the first of which was in a screen placed near the flame, and the second perforated in a thin metallic plate adjoining to the first surface of the crystal. Under ordinary circumstances, it is obvious, the incident ray will be divided into two within the crystal, and these will emerge parallel from the second surface. I was able to distinguish these two rays by the aid of a lens; and turning the crystal slowly, so as to vary the incidence gradually, I at length observed that there was a position in which the two rays changed their relative places rapidly, on any slight change of incidence, and appeared at times to revolve round one another, as the incidence was altered. Being convinced that the ray was now near the critical incidence, I changed the position of the crystal, with respect to the incident ray, very slowly; and after much care in the adjustment, I at last saw the two rays spread into a continuous circle, whose diameter was apparently equal to their former interval.

This phenomenon was exceedingly striking. It looked like a small ring of gold viewed upon a dark ground; and the sudden and almost magical change of the appearance from two luminous points to a perfect luminous ring, contributed not a little to enhance the interest.

The emergent light, in this experiment, being too faint to be reflected from a screen, I repeated the experiment with the sun's light, and received the emergent cylinder upon a small piece of silver-paper. I could detect no sensible difference in the magnitude of the circular sections at different distances from the crystal.

When the adjustment was perfect, the light of the entire annulus was white, and of equal intensity throughout. But when there was a very slight deviation from the exact position, two opposite quadrants of the circle appeared more faint than the other two, and the two pairs were of complementary colours.* The light of the circle was polarized, according to the law which I had before observed in the other case of conical refraction. In this instance, however, the law was anticipated from theory by Professor Hamilton.

I measured the angle of incidence by a method similar to that already employed for the emergent ray in the former case; and found it to be 15° 40′. This determination is, for many reasons, capable of much greater accuracy than the other; and was probably, in this instance, very near the truth.

In order to compare it with the result of theory, it is to be observed that the optic axis is a normal to the wave-surface, and therefore the corresponding incident ray will be given by the ordinary law of the sines, the index of refraction being the *mean index* of the crystal. Now the angle which the normal to the circular section of the surface of elasticity, or the optic axis, makes with the axis of x, or the perpendicular to the surface, is equal to $\text{tang.}^{-1}\sqrt{\frac{b^2-c^2}{a^2-b^2}}$; and its numerical value in the case of Arragonite, is 9° 1′. We have then

$$\sin. \iota = 1.6863, \ \sin. (9^\circ \ 1');$$

from which we find $\iota = 15^\circ \ 19'$. The difference between this and the observed angle is 21′.

In order to measure the angle of the cone, I was compelled to employ a method somewhat indirect, but (I think) susceptible of considerable accuracy. As the aperture on the first surface of the crystal must have some physical magnitude, it is obvious that instead of a cone of mathematical rays within the crystal, there will be in all cases a cone of cylindrical pencils, overlapping one another near the point of divergence; and that the diameter of these pencils will be equal to the diameter of the aperture. Now I tried a number of apertures, until I found one with which these cylindrical pencils just separated at the second surface of the crystal. It is evident that, in this case, the interval between the axes of the cylinders at the surface of

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* This part of the phenomenon appears to be explained by the non-coincidence of the optic axes for the rays of different colours.

emergence, is just equal to the diameter of one of them, or to the diameter of the aperture. I had then only to measure the aperture itself. This was effected by the aid of a micrometer divided to the 1-500th of an inch, placed along with the aperture before a compound microscope; and it was found to be .016 of an inch. This therefore was the diameter of the oblique section of the cone made by the surface of emergence; and the diameter of the circular section at the same distance was .016 cos. 9°, since the axis of the cone makes an angle of 9° with the normal to the faces of the crystal. The perpendicular thickness of the crystal was .49 of an inch; and therefore the thickness estimated in the direction of the axis of the cone was $\frac{.49}{\cos. 9^\circ}$. From these data the angle of the cone was calculated by the tables, and found to be 1° 50′; a result which differs from the theoretical angle by 5′ only.

SCIENCE.

An attempt to facilitate Observations of Terrestrial Magnetism.

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Read October 28, 1833.

SINCE the important researches of Baron HUMBOLDT, Captain SABINE, and Professor HANSTEEN, much interest has been awakened on the subject of Terrestrial Magnetism. The intensity of the magnetic force, and the dip, have been determined at various parts of the earth's surface, and are found to follow laws agreeing remarkably with the results of theory. But the experimental data necessary for the complete solution of this important problem are, as yet, far from adequate; and accordingly, the multiplication of observations in remote quarters of the globe, has engaged the zeal of many a scientific individual, and even called forth the resources of more than one state of Europe.

The imperfections of our instruments of observation have long presented a formidable obstacle to the advancement of these inquiries. Many of these have indeed been guarded against by the skill of the observer, and by the application of suitable corrections. There are still, however, some inherent sources of fallacy in the instruments themselves, which seem almost inseparable from them in their ordinary form. The defects of the common dipping needle have been long acknowledged, and still continue to embarrass observers. The chief of these defects arises from the nature of the suspension. The directive force, it is well known, diminishes with the angular distance from the true line of the dip, varying as the sine of that distance. Hence when the needle is near the position which it would assume under the influence of the terrestrial action, if free to move, the directive tendency may be so small as to be more than balanced by the friction of the axle against its supports; and in this case the needle will rest out of the position in which the earth's force would place it. This evil will of course be increased by any irregularity in the form of the axle itself, and such irregularities can never be wholly removed. To this is added another source of error, arising from the non-coincidence of the axle with the centre of gravity of the needle. Where this fault exists, gravity unites with magnetism in determining the position of equilibrium of the needle, and it will assume a direction different from that of the resultant of the terrestrial magnetic forces. The error arising from this cause, it is true, is in a great degree corrected by the process of reversing the poles of the needle, in the usual mode of observation; but still it cannot be said to be wholly removed, nor are the methods of correction altogether unobjectionable.

These and other difficulties in the direct determination of the dip, have led to the occasional employment of *indirect* methods. Thus, if the total magnetic intensity be known, together with its vertical or its horizontal component, the dip is readily found by elimination. A method founded on this principle was suggested by LAPLACE, and practised by HUMBOLDT and Captain SABINE. The total intensity was determined by noting the time of vibration of a dipping needle, oscillating in the plane of the magnetic meridian; and the vertical component was similarly obtained from the time of vibration of the same needle in the plane at right angles to the former: the ratio of these two forces is equal to the sine of the dip. Captain SABINE has also employed another method derived from the same principle. The horizontal component of the terrestrial magnetic force was found by observing the time of vibration of the needle suspended horizontally: and this component, divided by the total intensity, is equal to the cosine of the dip. The dipping needle, however, seems to be badly adapted to observations of vibration. Owing to the friction of the axis, the needle is soon brought to rest, unless the initial arc of vibration be considerable; and, when this is the case, the observer is involved in many sources of error, against which it seems difficult to guard.

A method similar to these in principle, but free from the objections just stated, was adopted many years since by COULOMB. In this method, the elements obtained by observation were the vertical and horizontal components of the intensity, the ratio of which is equal to the tangent of the dip. The horizontal component was determined, as in the last case, by observing the number of oscillations performed in a given time by the needle suspended horizontally; while the vertical component was found by determining the weight of a counterpoise, placed at a given distance on the southern arm of the needle, and sufficient to bring it to the horizontal position. The moment of this counterpoise, or its weight multiplied by its distance from the axle, is obviously equal to the vertical component of the magnetic intensity. M. BIOT considers this as the most accurate method that has been ever applied to the determination of the dip. It seems, however, open to very serious objections, which have been pointed out by M. POUILLET. In order to determine the horizontal component from the time of vibration, so that it may be comparable with the other component determined *statically*, it is necessary that the moment of inertia of the needle should be known a priori; and that this should be, the needle must have a regular mathematical form.

That used by COULOMB was of uniform and very small dimensions throughout its entire length, and in the calculation was treated as a mathematical line. There are other difficulties connected with the practice of this method, on which it is unnecessary to dwell: so that, notwithstanding its high sanction, it has not, as far as I am aware, been adopted by any other observer.

I have alluded to the method of COULOMB in this place, because it suggested that which I am now about to propose. It occurred to me that the components of the magnetic intensity might *both* be determined *statically*, and by one and the same process. Thus all the objections, to which COULOMB's method is liable, would be avoided, and the process itself rendered much simpler of application, as well as more accurate in its results.

Let us conceive a magnetic needle suspended as the ordinary dipping needle, and placed in the magnetic meridian; and let us calculate the directive effect of the terrestrial magnetic force upon it in any position.

If q denote the quantity of the magnetic fluid in any point of the needle, and ϕ the terrestrial magnetic force acting on the unit of quantity, $\phi\, q$ will be the force exerted on the point in question. Let δ denote the angle which the direction of this force makes with the horizon, or the *dip*, and let θ be the angle formed by the needle itself with the horizon: then $\delta - \theta$ is the angle which the direction of the force makes with the needle; and consequently, the moment of the force tending to turn the needle round its axle is $\phi\, q\, r \sin(\delta - \theta)$; r being the distance of the point in question from the axle. Now, to obtain the total effect of these forces acting on all the points of the needle, we have only to multiply this moment by the element of the mass, dm, and integrate the result. The total directive effect of the earth upon the needle is therefore

$$\int \phi\, qrdm. \sin(\delta - \theta) = \phi \sin(\delta - \theta) \int qrdm.$$

The integral $\int qrdm$, in this expression, will depend on the form of the needle, and on the law of the magnetic distribution in it. Let its value, taken within the limits of the dimensions of the needle, be denoted by σ: then the moment of the terrestrial magnetic force, tending to turn the needle round its axle, is expressed by

$$\phi\, \sigma \sin(\delta - \theta).$$

Now, if a weight be placed on the southern arm of the needle, its effect to turn the needle in the opposite way is

$$\mu \cos \theta;$$

μ being the *moment* of the mass, or the weight multiplied by its distance from the axle. Accordingly, in the case of equilibrium, we have $\mu \cos \theta = \phi\sigma \sin(\delta - \theta)$

Or, $$\mu = \phi\, \sigma (\sin \delta - \cos \delta \tan \theta) \qquad (1)$$

Now, let us suppose two weights to be placed in succession on the southern arm of the needle, and let their moments, μ' and μ'', be adjusted so as to bring the needle into the positions forming half a right angle with the horizon, below and above; the corresponding values of θ then are $+45°$ and $-45°$, and substituting, we have the two equations following

$$\mu' = \phi\ \sigma\ (\sin\delta - \cos\delta), \qquad \mu'' = \phi\ \sigma\ (\sin\delta + \cos\delta).$$

Adding and substracting these equations, we have

$$\mu'' + \mu' = 2\ \phi\ \sigma\ \sin\delta, \qquad \mu'' - \mu' = 2\ \phi\ \sigma\ \cos\delta.$$

from which we obtain,

$$\tan\delta = \frac{\mu'' + \mu'}{\mu'' - \mu'}, \qquad \phi\ \sigma\ \sqrt{2} = \sqrt{\mu''^2 + \mu'^2} \qquad (2)$$

Accordingly, to determine the intensity of the terrestrial magnetic force, as well as the dip, we have only to find the moments of two counterpoises which will bring the needle into the two positions above mentioned; the sum of these divided by their difference is equal to the tangent of the dip, and the square root of the sum of their squares furnishes a measure of the intensity. Another set of results might be obtained by observing also the moment of the counterpoise which brings the needle to the horizontal position; for it will be easily seen that this moment is equal to the arithmetical mean between the two preceding, or equal to $\phi\ \sigma\ \sin\delta$.

In the practice of this method, it is obvious that a divided circle is not required; it is only necessary to place four lines on the frame of the instrument corresponding to the inclinations of 45°. It is requisite, however, to have some means of determining the moments of the counterpoises with facility. For this purpose I procured a needle furnished with a small sliding weight, and having a scale on its southern arm, divided to the 1-50th of an inch. I had then only to observe the division on the scale coinciding with a fixed mark on the sliding weight, when the needle was brought to the two required positions.

Simple as this method appears, a few trials were sufficient to convince me that it would prove a most inconvenient one in practice. The adjustment of the counterpoise, so as to bring the needle to the required inclinations, was found to be a tedious and delicate operation, requiring a frequent removal of the needle from its supports; and the repetition of this process, in the various positions of the needle and instrument in which an observation of dip is usually taken, multiplies, of course, the difficulty. For these, and other reasons, it appeared far preferable to observe the *positions* which the needle assumes with *fixed counterpoises*, rather than the counterpoises which bring it into fixed positions. It will be easily seen, that two such observations are sufficient to determine the dip and the magnetic intensity.

Let μ' and μ'' be the *moments* of the two fixed counterpoises, and θ' and θ'' the corresponding inclinations of the needle to the horizon: then, substituting in equation (1),

$$\mu' = \phi\ \sigma\ (\sin \delta - \cos \delta \tan \theta'), \qquad \mu'' = \phi\ \sigma\ (\sin \delta - \cos \delta \tan \theta'').$$

We have thus two equations containing ϕ and δ, from which these quantities may be obtained by elimination. To effect this, let the former be divided by the latter, and we find

$$\frac{\mu'}{\mu''} = \frac{\tan \delta - \tan \theta'}{\tan \delta - \tan \theta''}, \qquad \tan \delta = \frac{\mu' \tan \theta'' - \mu'' \tan \theta'}{\mu' - \mu''};$$

And denoting the constant factors $\frac{\mu'}{\mu' - \mu''}$ and $\frac{\mu''}{\mu' - \mu''}$ by ν' and ν'',

we have finally,

$$\tan \delta = \nu' \tan \theta'' - \nu'' \tan \theta'. \qquad (3)$$

If we eliminate δ between this equation and either of the two equations given above, we readily obtain an expression for ϕ in terms of μ', μ'', and θ', θ''; but as the resulting expression is somewhat complicated, it will be much simpler in practice to obtain the dip, in the first instance, from the equation last deduced, and to substitute its value in the formula

$$\phi = \frac{\mu \cos \theta}{\sigma \sin (\delta - \theta)}.$$

The quantity σ, in this formula, depending on the law of distribution of the magnetic fluid in the needle employed, is unknown; and, consequently, the *absolute intensity* of the terrestrial magnetic force cannot be determined without some other artifices. In general, however, the ratio of this force at different parts of the earth's surface is alone required; and for this purpose the force at some given place (usually the magnetic equator) is taken as unity, and the force at other places determined by comparison. Let $\phi_{,}$ be the force at the given place, and $\delta_{,}$ $\theta_{,}$ the corresponding values of δ θ, then

$$\phi_{,} = \frac{\mu \cos \theta_{,}}{\sigma \sin (\delta_{,} - \theta_{,})},$$

and dividing

$$\frac{\phi}{\phi_{,}} = \frac{\sin (\delta_{,} - \theta_{,}) \cos \theta}{\sin (\delta - \theta) \cos \theta_{,}}. \qquad (4)$$

The counterpoises I employed, in the practice of this method, were small pieces of brass wire inserted in holes which were drilled at different distances along the axis of the needle. Finding that the weights which were sufficient to balance the magnetic force, at a moderate distance from the axle, were unmanageably small, I used two counterpoises, one of which was a weight of one grain, placed on the southern arm, at the distance 1.38 inches from the axle; the other was a weight of .7 of a grain, placed on the northern arm, at the distances 1.1 and 1.38 inches successively. Thus the moments of the actual counterpoises were the differences of those on the two arms, added to the difference of the moments of the arms themselves.*

It was now necessary to determine with accuracy the ratio of these moments; and, as this could hardly be effected with the requisite precision by any direct measurements, I determined to use for the purpose the means suggested by the proposed method of observation itself. This method, it has been shown, determines the dip by means of the two positions of the counterpoised needle, when the ratio of the moments is known: therefore, conversely, if the dip be otherwise found, the same formulæ may be used to calculate the ratio of the moments. The following observation was accordingly taken for this purpose:

Killiney, Sept. 11. *Needle I.*†

Marked end	Limb	E marked side		W marked side	
		N	S	N	S
N	E	71° 7′	71° 15′	71° 15′	71° 30′
	W	70 45	70 45	70 50	70 50
S	E	71 10	71 20	71 30	71 45
	W	70 45	70 40	71 10	71 5
Mean		70° 56′.8	71° 0′	71° 11′.3	71° 17′.5
		Final Mean = 71° 6′.4.			

Killiney, Sept. 12. *Needle II.*†

Counterpoise	Limb	E marked side		W marked side	
		N	S	N	S
I	E	−18° 45′	−18° 37′	−19° 0′	−18° 52′
	W	−20 25	−20 15	−21 0	−21 0
II	E	+23 10	+23 20	+25 40	+25 55
	W	+22 30	+22 35	+25 20	+25 25
Mean		$\theta' = -19° 44'$		$\theta'' = +24° 14'$	

Substituting the values of δ, θ' and θ'', thus obtained, in the equation,

$$\frac{\mu'}{\mu''} = \frac{\tan \delta - \tan \theta'}{\tan \delta - \tan \theta''};$$

we find the ratio of the connterpoises to be 1.3274.

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* The moments of the two arms of the needle were rendered unequal by the weights which were unsymmetrically abstracted in the process of drilling the holes.

† Needle I. is the common needle—Needle II. the counterpoised needle.

It now remained to take a series of observations, with the common as well as the counterpoised needle, and to compare their results. The following tables contain the results of two such series. The first series of observations was taken at places differing in dip: the second series at the same place. The poles of the counterpoised needle were not reversed, as such a process would be incompatible with one of the objects of the needle—namely, the determination of the variations of terrestrial intensity; and it fortunately happens, that this troublesome operation is, in the present method, wholly unnecessary. The use of reversing the poles in the common needle, it is well known, is to correct for the moment of the needle itself. If the centre of gravity of the needle deviates at all from the centre of motion, *in the direction of its length*, the moment of the needle will increase or diminish its inclination to the horizon, according as the deviation is towards the north or south pole; and this error can only be corrected by reversing the poles of the needle, so that its moment may act in opposite directions in the two cases.* But, in the needle proposed in these pages, the centre of gravity of the whole is made intentionally to deviate from the axle, and the dip determined from the altered inclinations. In this case, accordingly, the moment of the needle, if any, simply increases or diminishes the moments of the actual counterpoises; and, as these are determined *a posteriori*, the moment of the needle itself will of course be included in the determination.

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* It may be readily shown, that the tangent of the true dip, is the arithmetical mean between the tangents of the observed inclinations of the needle, when the same end is a north and south pole. But when the difference of the observed inclinations is small, we may, without much error, suppose the same relation to hold amongst the angles themselves.

TABLE I.—*Containing Observations of the positions of the Common and Counterpoised Needle, at places differing from one another in Dip.*

			Observatory, Armagh, Sept. 16.				*Dundonnell, County of Down, Sept.* 23.			
	Marked end Limb		E	marked side	W		E	marked side	W	
			N	S	N	S	N	S	N	S
NEEDLE I	N	E	72° 5′	72° 5′	71° 55′	71° 55′	72° 0′	72° 5′	71° 55′	72° 0′
		W	71 22	71 10	71 25	71 15	71 15	71 7	71 30	71 20
	S	E	72 0	72 4	72 5	72 15	71 55	72 0	72 0	72 5
		W	71 25	71 10	71 20	71 5	71 20	71 15	71 22	71 15
	Mean		71 43	71 37.2	71 41.2	71 37.5	71 37.5	71 36.8	71 41.8	71 40
			Final mean = 71° 39′.7				Final mean = 71° 39′			
	Counter-poise Limb		E	marked side	W		E	marked side	W	
			N	S	N	S	N	S	N	S
NEEDLE II	I	E	−21° 20′	−21° 15′	−22° 10′	−22° 7′	−21° 10′	−21° 5′	−22° 50′	−22° 50′
		W	−21 25	−21 15	−22 5	−21 55	−22 0	−21 55	−22 52	−22 45
	II	E	+22 30	+22 30	+25 40	+25 35	+22 5	+22 5	+25 15	+25 20
		W	+22 0	+22 5	+23 45	+23 45	+22 7	+22 7	+24 40	+24 40
	Mean		θ′ = − 21° 41′.5		θ″ = + 23° 28′.8		θ′ = − 22° 11′		θ″ = +23° [illegible]2′	
			Calculated dip = 71° 26′				Calculated dip = 71° 38′			

			Museum, Belfast, Sept. 28.				*Garden, Trinity College, Oct.* 21.			
	Marked end Limb		E	marked side	W		E	marked side	W	
			N	S	N	S	N	S	N	S
NEEDLE I	N	E	72 20	72 22	72 15	72 15	71 16	71 16	71 10	71 21
		W	71 30	71 25	71 30	71 28	70 43	70 35	70 33	70 30
	S	E	72 15	72 20	72 10	72 15	71 15	71 27	71 18	71 22
		W	71 35	71 37	71 30	71 20	70 36	70 34	70 35	70 32
	Mean		71 55	71 56	71 51.2	71 49.5	70 57.5	70 58	70 54	70 56.2
			Final mean = 71° 52′.9				Final mean = 70° 56′.4			
	Counter-poise Limb		E	marked side	W		E	marked side	W	
			N	S	N	S	N	S	N	S
NEEDLE II	I	E	−21° 10′	−21° 0′	−21° 45′	−21° 35′	−23° 27′	−23° 30′	−23° 42′	−23° 41′
		W	−22 7	−22 7	−22 37	−22 30	−24 15	−24 17	−25 7	−25 7
	II	E	+22 40	+22 40	+25 30	+25 35	+20 27	+20 27	+21 40	+21 43
		W	+22 52	+22 45	+25 30	+25 40	+19 52	+19 52	+22 37	+22 32
	Mean		θ′ = − 21° 51′		θ″ = +24° 9′		θ′ = −24° 8′.2		θ″ = +21° 8′.8	
			Calculated dip = 71° 49′				Calculated dip = 71° 12′			

TABLE II.—*Containing Observations of the positions of the Common and Counterpoised Needle, taken in the Philosophy School of Trinity College.*

				October 22. Therm.=56½°			
NEEDLE I	Marked end Limb			E	marked side	W	
				N	S	N	S
		N	E	71° 20′	71° 20′	71° 15′	71° 20′
			W	70 40	70 30	70 25	70 30
		S	E	71 15	71 22	71 15	71 22
			W	70 35	70 30	70 50	70 50
	Mean			70 57.5	70 55.5	70 56.2	71 0.5
				Final mean=70° 57′.4			
NEEDLE II	Counter-poise Limb			E	marked side	W	
				N	S	N	S
		I	E	−27° 52′	−27° 52′	−29° 20′	−29° 22′
			W	−28 45	−28 45	−28 52	−28 52
		II	E	+16 15	+16 15	+20 15	+20 25
			W	+15 15	+15 15	+19 0	+19 0
	Mean			$\theta' = -28^\circ\ 42'.5$		$\theta'' = +17^\circ\ 42'.5$	
				Calculated dip = 71° 22′.5			

				October 23. Therm.=57°			
NEEDLE I	Marked end Limb			E	marked side	W	
				N	S	N	S
		N	E	71° 15′	71° 7′	71° 15′	71° 22′
			W	70 35	70 30	70 35	70 30
		S	E	71 15	71 20	71 15	71 25
			W	70 35	70 25	70 30	70 15
	Mean			70 55	70 50.5	70 53.8	70 53
				Final mean=70° 53′.1			
NEEDLE II	Counter-poise Limb			E	marked side	W	
				N	S	N	S
		I	E	−12° 30′	−12° 22′	−14° 15′	−14° 30′
			W	−13 37	−13 37	−15 55	−15 50
		II	E	+26 7	+26 15	+29 35	+29 45
			W	+26 0	+26 0	+30 10	+30 10
	Mean			$\theta' = -14^\circ\ 4'.5$		$\theta'' = +28^\circ\ 0'.2$	
				Calculated dip = 71° 6′			

				(1) *October* 24. Therm.=58½°			
NEEDLE I	Marked end Limb			E	marked side	W	
				N	S	N	S
		N	E	71° 15′	71° 22′	71° 7′	71° 22′
			W	70 40	70 30	70 30	70 22
		S	E	71 15	71 20	71 7	71 15
			W	70 35	70 25	70 30	70 30
	Mean			70 56.2	70 54.2	70 48.5	70 52.2
				Final mean=70° 52′.8			
NEEDLE II	Counter-poise Limb			E	marked side	W	
				N	S	N	S
		I	E	−35° 30′	−35° 45′	−38° 10′	−38° 15′
			W	−38 7	−38 0	−39 55	−39 55
		II	E	+ 6 35	+ 6 37	+ 8 45	+ 8 50
			W	+ 4 15	+ 4 20	+ 6 30	+ 6 30
	Mean			$\theta' = -37^\circ\ 57'$		$\theta'' = +6^\circ\ 32'.8$	
				Calculated dip = 70° 39′			

				(2) *October* 24. Therm.=59°			
NEEDLE I	Marked end Limb			E	marked side	W	
				N	S	N	S
		N	E	71° 0′	71° 15′	71° 10′	71° 25′
			W	70 45	70 45	70 30	70 30
		S	E	71 20	71 40	71 7	71 20
			W	70 35	70 30	70 35	70 30
	Mean			70 55	71 2.5	70 50.5	70 56,2
				Final mean=70° 56′			
NEEDLE II	Counter-poise Limb			E	marked side	W	
				N	S	N	S
		I	E	−10° 45′	−10° 40′	−12° 45′	−12° 45′
			W	−12 45	−12 45	−14 0	−14 0
		II	E	+26 30	+26 30	+30 30	+30 30
			W	+27 0	+27 0	+30 30	+30 30
	Mean			$\theta' = -12^\circ\ 33'$		$\theta'' = +28^\circ\ 37'.5$	
				Calculated dip = 70° 56′			

From the first of these Tables we see that the values of the dip, obtained by the common and counterpoised needle, do not differ from one another by 16′, at any of the four stations; while at two of the stations the difference is considerably within the limits of error of a single observation of dip, taken in the usual manner.

The general agreement of the results of the two methods being thus established, the second series of observations, given in Table II, enables us to compare the methods with respect to accuracy; for these observations being all taken at the same place, the consistency of the results will afford the means of estimating the probable limit of error in each method.

Comparing then, the results of Table II, it will be seen in the first place, that the mean of the four observations made with Needle I, gives 70° 55′.1 for the dip, in the Philosophy school of Trinity College; while the mean of the observations taken with Needle II, is 71° 0′.9, differing from the former by less than 6′.

The results of Needle II, are, however, by no means as consistent among themselves as those of Needle I. The greatest difference between any one result and the mean, with the common needle, is only 2½′; while, with the counterpoised needle, the corresponding difference amounts to 22.′ This, however, can scarcely be regarded as decisive against the accuracy of the method. The common needle was one of remarkable nicety of construction; while that used with the counterpoises, was obviously erroneous. It was necessary for the consistency of the results that the points of application of the counterpoises, should coincide exactly with the magnetic axis, or central line of the needle. Now, no very precise means were adopted to insure this coincidence, in drilling the holes for the counterpoises; and that the coincidence was in fact not effected, will readily appear on looking over the results. It will be seen that, when the second counterpoise is employed, there is a difference amounting sometimes to 4° between the angles read off with the face of the needle in opposite positions; and this plainly indicates an error in the place of the second hole on the northern arm of the needle. It is to be observed, further, that the magnetic state of the needle was widely different in the several observations, having been altered before each trial by a pair of bar magnets; so that the method was subjected to a more severe test than any to which it could be exposed in practice. Even with these disadvantages, however, the values of the dip obtained with this needle do not appear to differ from one another more than is common in such observations; so that the result of the trial cannot be regarded as unfavorable.

During the progress of the observations recorded in Table II, I made a cotemporaneous series of observations on the rate of vibration of Needle II, suspended horizontally; with the view of ascertaining how far the results obtained with this needle could be relied on for the determination of the force. For this purpose the magnetism of the needle was altered by a pair of bar magnets, and its rate of vibration ascertained in the usual manner, after each change in its magnetic condition, by a good chronometer. The observation of the angles of position of the same needle, when counterpoised, was taken immediately before or after, and, as nearly as possible, at the same temperature. But unfortunately the moment of inertia of the needle was unguardedly altered during the observations, so that no reliance could be placed on the results.

Since the preceding trials were made, it appeared to me desirable to examine theoretically the accuracy of the counterpoised needle in its different positions, and to ascertain, if possible, the most eligible. It will appear from the foregoing observations, that between two distant places, the *range*, or change of position of the counterpoised needle, is considerably greater than the corresponding change in the dip. This I had been previously led to expect from theory, and had found that the change of position, corresponding to a given change in the dip, would be greatest when $\tan 2\,\theta = \frac{1}{4} \tan \delta$. When $\delta = 70°$, this formula gives, for the *angle of maximum range*, $\theta = 17° \; 15'$; and the ratio of the range itself to the corresponding variation in the dip was found, in this case, to amount to 3.275. I was thus led to expect that changes of dip, which would be inappreciable by the common needle, might be detected with this. It appeared, however, on trial, that the *limits of error* were likewise increased in this needle; so that it was necessary to examine it in another point of view.

Resuming the original equation

$$\mu \cos \theta = \phi \, \sigma \, (\delta - \theta),$$

and differentiating and dividing by the equation itself, considering θ, δ and ϕ as all variable, we find

$$\cos \theta \, \sin (\delta - \theta) \frac{d\phi}{\phi} + \cos \theta \, \cos (\delta - \theta) \, d\delta - \cos \delta \, d\theta = 0.$$

This equation will give the error in the dip, or in the force, corresponding to a given error in the position of the counterpoised needle; for, making $d\phi$ and $d\delta$, successively, equal to nothing, there is

$$d\delta = \frac{\cos \delta \, d\theta}{\cos \theta \cos (\delta - \theta)}, \qquad \frac{d\phi}{\phi} = \frac{\cos \delta \, d\theta}{\cos \theta \sin (\delta - \theta)}. \tag{5}$$

Hence, supposing the error in the position of the needle, $d\theta$, to be given, we can find the directions of the needle, in which the *resulting error* in the determination of the dip, or of the force, shall be a *minimum*. It is easily seen that $d\delta$ is a minimum, when $\theta = \frac{1}{2} \delta$; and that $\frac{d\phi}{\phi}$ is least, when $\theta = \frac{1}{2}\delta - 45°$. Hence, in our latitudes, for the determination of the dip and of the force, the counterpoises should be such as to bring the needle into the positions forming the angles $+35°$ and $-10°$, respectively, with the horizon.

Such then should be the inclinations of the needle, in order that any *constant error of position*, $d\theta$, may have the smallest influence on the calculated dip and intensity. Of such constant errors the most obvious is the error to which the observer is subject in reading the angle on the limb, arising from the smallness or imperfection of the divisions. But the *error of reading* is not the only, or even the most important error, to which we are liable in determining the position of the needle. It has been already stated that, owing to the friction of the axle, the needle is often brought to rest out of its true direction : now the error of position arising from this cause is, in instruments of the usual size, of greater magnitude than the error of reading, and that magnitude is different in the different positions of the needle.

In order to determine the amount of this error, it will be necessary to consider the directive force, by which the needle is urged to its position of equilibrium. This force is obviously the difference between the magnetic moment, $\phi\,\sigma \sin(\delta-\theta)$, and the moment of the counterpoise, $\mu \cos\theta$; so that, if its magnitude be denoted by F,

$$F=\phi\,\sigma \sin(\delta-\theta)-\mu\cos\theta.$$

But, if $\theta_,$ be the position of equilibrium, there is $\phi\,\sigma\sin(\delta-\theta_,)-\mu\cos\theta_,=0$; and substituting the value of μ, obtained from this equation, in the preceding formula, and observing that

$$\sin(\delta-\theta)\cos\theta_,-\sin(\delta-\theta_,)\cos\theta=\cos\delta\,(\sin\theta_,\cos\theta-\cos\theta_,\sin\theta)=\cos\delta\sin(\theta_,-\theta),$$

we have finally

$$F=\phi\,\sigma\frac{\cos\delta}{\cos\theta_,}\sin(\theta_,-\theta).$$

Hence the directive force varies as the sine of the angular distance from the position of equilibrium. Accordingly, when that angular distance is reduced to a certain limit, the force becomes equal to the friction, and is balanced by it. Let ε be the magnitude of the angle, $\theta_,-\theta$, when the directive force becomes equal to the friction, f; then

$f=\phi\,\sigma\frac{\cos\delta}{\cos\theta}\sin\varepsilon=\phi\,\sigma\frac{\cos\delta}{\cos\theta}\varepsilon$, since ε is small; and therefore

$$\varepsilon=\frac{f\cos\theta}{\phi\,\sigma\cos\delta}\,. \tag{6}$$

The angle ε, thus found, is obviously the *limit of error* to which we are liable in determining the position of the needle, arising from friction. Its value depends, as we see, upon the force of friction, the intensity of the terrestrial magnetic force, the magnetic moment of the needle, and its position with relation to the dip. In order to determine the *resulting error* which it will produce, in the determination of the dip and intensity, we have only to substitute its value for $d\theta$, in the equations (5). We find, in this manner,

$$d\delta = \frac{f}{\phi\,\sigma\cos(\delta-\theta)}\ , \qquad \frac{d\phi}{\phi} = \frac{f}{\phi\,\sigma\sin(\delta-\theta)}\,;$$

or, denoting the limit of error arising from friction in the ordinary position of the needle by ε' ,

$$d\delta = \varepsilon' \sec(\delta-\theta)\ , \qquad \frac{d\phi}{\phi} = \varepsilon' \operatorname{cosec}(\delta-\theta)\,. \qquad (7)$$

We learn, then, that the error in the determination of the dip, arising from friction, is least when $\delta-\theta=0$; and that the smallest value of $d\delta$ is ε'.* The corresponding error in the determination of the force will be a minimum, when $\delta-\theta=90°$; in which case $\frac{d\phi}{\phi} = \varepsilon'$, ε' being expressed in parts of radius.

As far then, as friction is concerned, it would appear to be the most advantageous modification of the method suggested in the preceding pages, to observe the position of the needle, in the first instance, without any counterpoise, and, afterwards, with a counterpoise which will bring it into a position nearly perpendicular to the line of the dip. The former of these angles is the dip itself; and the two angles, when substituted in formula (4), furnish the measure of the intensity. But, in order to avoid the error arising from the non-coincidence of the centre of gravity with the axle, I think it would be far better to use a small counterpoise in the first instance; or even to consider the moment of the needle itself, (or its weight multiplied by the distance of its centre of gravity from the axle,) as a counterpoise acting with or against the magnetic moment. The ratio of this moment to that of the other counterpoise should, of course, be determined by the indirect method which has been already explained.†

We may now form an estimate of the accuracy of this method, as compared with the usual one, in the determination of the magnetic intensity. In the received method, it is well known, the horizontal component of the force is determined by observing the time of vibration of the needle suspended horizontally. Now let us suppose this portion of the force to be *completely* determined, and inquire how far the probable error in the dip will affect the total intensity, thence deduced. If h denote the horizontal component, we have

$$h = \phi\,\sigma\cos\delta\,;$$

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* We have here considered the effect of friction on the result, so far as it depends upon a single reading. The eight readings usually taken may undoubtedly diminish still further the resulting error of dip; but as these readings are taken in order to correct other errors, we have disregarded their effect here. If, however, the needle be so perfect in its construction, that the errors arising from the non-coincidence of the centre of gravity with the axle, and the deviation of the magnetic axis from the axis of the needle, &c. are less than the error of friction, then the multiplication of readings will have the effect of reducing the latter in the ultimate result.

† See page 6.

and differencing and dividing, h being considered as constant,

$$\frac{d\phi}{\phi} = d\delta \tan \delta.$$

Now, confining our consideration to the error arising from friction, $d\delta = \frac{f}{\phi\sigma} = \epsilon'$; so that the *limit of error* in the determination of the force, in the usual method, is $\epsilon' \tan \delta$; while, in that now proposed, it may be reduced to ϵ'. The limit of error in the common method, therefore, is to that of the method now proposed, in the ratio of the tangent of the dip to unity; that is, in our latitude, as 2.75 to 1, nearly.

In the instrument in my possession, constructed by Mr. Robinson, of Devonshire-street, London, I consider the limit of error arising from friction, in the ordinary position of the needle, not to exceed 5′. With this instrument, therefore, $\epsilon' = 5' = .0015$, nearly, radius being unity; so that the error in the determination of the force does not exceed the .0015 part of the entire quantity.

In order to satisfy myself more fully as to the accuracy of this method, the following series of observations was taken. The position of the needle (Needle I of the preceding experiments) was observed, first without any counterpoise, and secondly with a counterpoise which brought it into a position nearly perpendicular to the line of the dip. The observations were taken in the same spot of the room, and as nearly as possible at the same temperature; the magnetism of the needle was not interfered with during the observations.

TABLE III. *Containing the results of Four Observations taken with Needle I. in the Philosophy School of Trinity College. Therm. = 61°.5.*

Angle	Limb	I. E marked side		W		II. E marked side		W	
		N	S	N	S	N	S	N	S
δ	E	71° 7′	71° 18′	71° 12′	71° 25′	71° 5′	71° 10′	71° 10′	71° 25′
	W	70 35	70 30	70 25	70 22	70 30	70 22	70 30	70 28
θ	E	−17 30	−17 30	−15 15	−15 7	−17 0	−17 7	−15 15	−15 7
	W	−19 40	−19 37	−18 15	−18 5	−19 40	−19 45	−18 15	−18 7
Mean		δ = 70° 51′.8,		θ = − 17° 37′.4		δ = 70° 50′,		θ = −17° 32′	
Angle	Limb	III. E marked side		W		IV. E marked side		W	
		N	S	N	S	N	S	N	S
δ	E	71° 0′	71° 10′	71° 15′	71° 30	71° 5′	71° 15′	71° 10′	71° 20′
	W	70 30	70 25	70 30	70 25	70 30	70 22	70 30	70 30
θ	E	−16 55	−17 0	−15 15	−15 7	−17 22	−17 35	−15 20	−15 20
	W	−19 30	−19 30	−17 45	−17 45	−19 45	−19 45	−17 55	−17 55
Mean		δ = 70° 50′.6,		θ = − 17° 20′.9		δ = 70° 50′.2,		θ = − 17° 37′.1	

The mean values of δ and θ, as deduced from these four observations, are

$$\delta_{,} = 70^\circ.\ 50'.7\ ,\ \ \theta_{,} = -17^\circ.\ 31'.8\ ;$$

and if we take the force corresponding to these mean values as unity, the numerical values of the force, derived from each separate observation, will be obtained by substituting the corresponding values of δ and θ, given by the preceding table, in the formula

$$\log \phi = \log \cos \theta - \log \cos \theta_{,} + \log \sin (\delta_{,} - \theta_{,}) - \log \sin (\delta - \theta).$$

We find in this manner

$$\phi_1 = .9994\ ,\ \ \phi_2 = 1.0000\ ,\ \ \phi_3 = 1.0011\ ,\ \ \phi_4 = .9995.$$

The greatest deviation of these values from the mean is .0011; so that the limit of error in the determination of the force, would appear from these observations to be little more than the one-thousandth part of the entire quantity.

It appears then from the preceding, that the Terrestrial Magnetic Intensity may be determined, together with the Dip, with the aid of a single instrument, and by a process even somewhat less troublesome than that by which the dip alone is usually determined.* To any one who considers the numerous precautions required, in the common method of determining the magnetic intensity, the saving of time and labour thus effected will be abundantly obvious. But it is an advantage of much greater moment, that the results of the proposed method, so far as the intensity is concerned, will be less liable to error than those obtained in the usual manner, as long as the dip exceeds 45°; and that, in our latitudes, the accuracy of the new method is nearly three-fold that of the old.

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* The same number of readings is taken in the two cases, while in the proposed method the process of reversing the poles is dispensed with.

On a New Case of Interference of the Rays of Light. By the Rev. HUMPHREY LLOYD, A. M., M.R.I.A. *Fellow of Trinity College, and Professor of Natural and Experimental Philosophy in the University of Dublin.*

Read January 27, 1834.

THE experiment of FRESNEL, on the interference of the lights proceeding from the same origin, and reflected by two mirrors inclined at a very obtuse angle, has been justly regarded as one of the most important in the whole range of physical optics. The principle of interference itself had, indeed, been stated broadly by YOUNG, and supported by the evidence of phenomena, which, to the unbiassed inquirer, left little to desire. All these phenomena, however, admitted of other possible explanations; and the advocates of the corpuscular theory of light had recourse to these, rather than admit the truth of a law which afforded such strong support to the undulatory theory. In most of these phenomena, the light was in part intercepted by an obstacle, and it was conceived that, in passing by the edge, the molecular action, which might be supposed to exist between the particles of the body and those of light, was sufficient to account for the facts observed. But, in FRESNEL's experiment, the two lights which interfere are regularly reflected by the surfaces of the mirrors, according to the ordinary laws, and are divested of every extraneous circumstance which could, by possibility, be supposed to influence the result. This experiment, accordingly, has materially changed the character of the controversy respecting the nature of light; and the advocates of the Newtonian theory, of the present day, are forced to admit the principle thus rigidly established, and labour only to show how the theory and that principle may be reconciled.

While examining this important experiment—the adjustment of which is a matter of some delicacy—it occurred to me that the fact of direct interference might be shown in a yet simpler manner, by the mutual action of *direct* and reflected light. An interference of this kind was assumed by YOUNG to account for some of the phenomena of diffraction; but FRESNEL showed that the explanation was incomplete, and that the phenomena in question were caused merely by the interference of the secondary waves, reflexion playing no part in their production. Under these circumstances it is somewhat strange that the fact of the interference of direct and reflected lights should not have been itself submitted to the test of experiment; especially as the character of this interference, if it were found to exist, might be expected to throw some light upon the laws of reflexion itself.

The theory of such interference is easily deduced from the general principles. Let light proceeding from a single luminous origin fall upon a reflecting surface, at an incidence of nearly 90°: a screen placed at the other side of the reflector will be illuminated, throughout a certain extent, by both direct and reflected lights; and, if the difference of the paths traversed by these lights amount only to a small multiple of the length of an undulation, the two lights will form fringes by their interference.

Let the intensities of the direct and reflected lights be denoted by a^2 and a'^2, and that of the resulting light by A^2; then, by the theory of the composition of coexisting vibrations, we have

$$A^2 = a^2 + 2\,aa' \cos 2\pi \left(\frac{\delta' - \delta}{\lambda}\right) + a'^2 \;;$$

δ and δ' denoting the lengths of the paths traversed by the two waves, from their origin to any given point, and λ the length of an undulation.

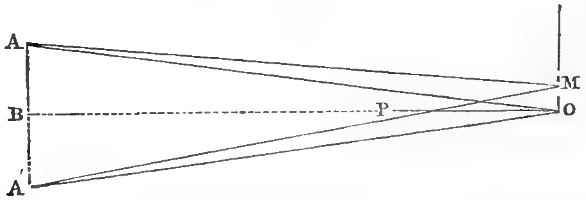
The intensity of the resulting light will be a *maximum*, and equal to $(a + a')^2$, at those points for which

$$\cos 2\pi \left(\frac{\delta' - \delta}{\lambda}\right) = + 1 \,, \qquad \text{or} \quad \delta' - \delta = 2n\frac{\lambda}{2} \;;$$

It will be a *minimum*, and equal to $(a - a')^2$, when

$$\cos 2\pi \left(\frac{\delta' - \delta}{\lambda}\right) = - 1 \,, \qquad \text{or} \quad \delta' - \delta = (2n + 1)\frac{\lambda}{2} \;;$$

n being any number of the natural series 0, 1, 2, 3, &c. *Bright fringes*, therefore, will be formed at all the points included in the former equation, and *dark* ones at the points corresponding to the latter.



Let OP be the reflector, OM the screen placed in contact with it, and perpendicular to its plane; and let A be the luminous origin, and A′ its reflected image at an equal distance below the line OB; then if M be any point, whose illumination is required, $\delta = \text{AM}$, $\delta' = \text{A}'\text{M}$.

Now if AB be denoted by p, BO by d, and OM by x, it is obvious that

$$\delta^2 = d^2 + (p-x)^2 \ , \quad \delta'^2 = d^2 + (p+x)^2 .$$

From these equations we have approximately

$$\delta = d\left\{1 + \tfrac{1}{2}\left(\frac{p-x}{d}\right)^2\right\}, \qquad \delta' = d\left\{1 + \tfrac{1}{2}\left(\frac{p+x}{d}\right)^2\right\} ;$$

and therefore

$$\delta' - \delta = \frac{2px}{d} = 2 \tan \alpha \ x,$$

denoting the angle AOB by α. Hence the general expression of the intensity of the light, at any point M, is

$$A^2 = a^2 + 2aa' \cos\left(4\pi \tan \alpha \frac{x}{\lambda}\right) + a'^2 .$$

Again, substituting for $\delta' - \delta$ its value just found, we see that the successive fringes will be formed at the distances given by the formula

$$x = \tfrac{1}{4}\, m\, \lambda \text{ cotan } \alpha ;$$

in which m is any number of the natural series, its even values giving the places of the *bright* fringes, and its odd values those of the *dark* ones. Accordingly, the bright fringes are formed at the distances 0, $2l$, $4l$, &c., and the dark ones at the distances intermediate, l, $3l$, $5l$, &c., l being equal to $\frac{1}{4}\ \lambda$ cotan α : the successive fringes, therefore, are equidistant. It is obvious that the angle α must be very small, or the incidence very oblique, in order that the fringes should have any sensible breadth.

We have hitherto assumed that the light has undergone no change by reflexion, excepting the change of direction. Let us now suppose that the *phase* of the vibration is *accelerated*, and let us examine the effect produced in the position of the fringes.

Let the amount of this acceleration be denoted by the angle $\mu\pi$; then the difference of the phases will be

$$2\pi\left(\frac{\delta' - \delta}{\lambda}\right) - \mu\pi = 2\pi\left(\frac{\delta' - \delta - \frac{1}{2}\mu\lambda}{\lambda}\right).$$

So that the successive fringes will be formed at the points for which

$$\delta' - \delta - \tfrac{1}{2}\mu\lambda = \tfrac{1}{2}m\lambda,$$

m being any number of the natural series. But we have already found that $\delta' - \delta = 2 \tan \alpha \ x$; so that the points in question are given by the formula,

$$x = \tfrac{1}{4}\,(m + \mu)\ \lambda \text{ cotan } \alpha;$$

the *even* values of m corresponding to the bright fringes, and the *odd* values to the dark ones. It is evident from this that the magnitude of the fringes will be unaltered; and that the only effect of the acceleration is to push the entire system from the edge, the amount of the shifting being equal to $\frac{1}{4}\mu\lambda$ cotan α.

In order to submit these results to the test of trial, I employed the apparatus consisting of two moveable metallic plates, which is of so much use in experiments of interference. The plates being closed, so as to form a narrow horizontal aperture, the flame of a lamp was placed behind; and the light thus diverging from the aperture was received, at the distance of about three feet, on a piece of black glass truly polished, and also horizontal. This reflector was then adjusted, so that its plane might pass a little below the aperture; or, in other words, that the light might be incident upon it at an angle of nearly 90°. It is evident, then, that the light thus obliquely reflected will meet the direct light diverging from the aperture under a very small angle, and with a difference in the lengths of their paths which is capable of indefinite diminution. The two lights, therefore, are in a condition to interfere; and I found, accordingly, that when they were received upon an eyepiece, placed at a short distance from the reflector, a very beautiful system of bands was visible, in every respect similar to one-half of the system formed by the two mirrors, in Fresnel's experiment.

The first band was a *bright* one, and *colourless*. This was succeeded by a very sharply defined black band; then followed a coloured bright band, and so on alternately. Under favorable circumstances I could easily count seven alternations; the breadth of the bands being, as far as the eye could judge, the same throughout the series, and increasing with the obliquity of the reflected beam. The first dark band was of *intense blackness;* but the darkness of the succeeding bands was less intense, as they were of higher orders; and after three or four orders, they were completely obliterated by the closing in of the bright bands. At the same time the coloration of the bright bands increased with the order of the band; until, after six or seven alternations, the colours of different orders became superimposed, and the bands were thus lost in a diffused light of nearly uniform intensity. All these circumstances are similar to those observed in Fresnel's experiment, and correspond exactly with the results of theory.

These bands are most perfectly defined when the eyepiece is close to the reflector. Their breadth and coloration increased with the distance of the eyepiece, but remained of a finite and very sensible magnitude, when the latter was brought into actual contact with the edge—a circumstance which distinguishes them altogether from the diffracted fringes formed on the boundary of the shadow.

These fringes appear to me to possess some interest in a theoretical point of view, independently of that which attaches to them as illustrations of an important general law. Depending on the interference of two lights, one of which proceeds directly from the luminous origin, while the other has undergone reflexion, they would seem to afford the means of detecting any difference which might exist in their condition when they meet, and therefore of tracing the modifications produced by reflexion.

There are two circumstances which chiefly demand our attention in the case of reflected light—namely: 1st, *the amplitude of the vibration,* on which the intensity of the light depends; and 2dly, *the phase.* The facts before us seem, to a certain extent, to bear on both these points.

The reasonings of FRESNEL with respect to the intensity of reflected light, are partly of an analogical nature, and very far indeed from being strictly demonstrative. Still, however, they have led to conclusions fully borne out by experience, and of the most interesting kind; and we can hardly refuse our assent to doctrines which bear with them such characters of truth. The formula which FRESNEL has obtained for the intensity of reflected light has not received any direct confirmation from experiment, except in the case of a few observations made by M. ARAGO. It results from this formula that the intensity of the reflected light must be equal to that of the incident, or the *whole* of the light reflected, at the limiting incidence of 90°. FRESNEL himself notices this consequence, and adds that we should doubtless find it to be experimentally true, if we could reach this limit. Now the present experiment affords the means of examining this conclusion, and seems fully to establish it. We have already alluded to the intense blackness of the first dark bar, in the phenomena now described. As far as the eye can judge, the intensity of the light is absolutely nothing, at the points corresponding to this bar; and as the intensity of the light in the dark bands is generally expressed by the formula $(a-a')^2$, we are forced to admit that $a=a'$, or that the intensities of the direct and reflected lights are equal at this extreme incidence.

With respect to the effect of reflexion upon the *phase of vibration,* there seems to be some uncertainty in the theory. The phenomena of thin plates compel us to admit that half an undulation is either lost or gained, by the wave reflected from the first or second surface; so that half an undulation must be added to, or subtracted from, the difference in the lengths of the paths traversed by the two waves. That such an effect should take place is in the highest degree probable from theoretical considerations. The light in the one case is reflected from the surface of a denser, in the other from that of a rarer medium; and the mechanical laws, on which FRESNEL has founded the doctrine of reflexion, lead us to the conclusion that the displacements of the ethereal particles, in the moment after reflexion, must be of opposite signs in the two cases. This difference in the phase of the vibration is equivalent to a difference of half an undulation in the length of the path.

But it does not seem to be clearly understood to which surface we are to attribute this physical change in the condition of the ray. Dr. YOUNG, indeed, who was the first to state this law, says expressly that where "light has been reflected at the surface of the *rarer* medium, it must be supposed to be retarded one-half of the appropriate interval." I cannot avoid thinking, however, that the very analogy by which he himself illustrates this point, and still more the reasonings of FRESNEL on the subject, lead to an opposite conclusion, and tend to ascribe the effect which is found to take place to reflexion at the surface of the *denser* medium. In fact, it would appear from Fresnel's conclusions, that the *sign* of the vibratory movement is in all cases changed by reflexion at the surface of the denser medium, the angle of incidence exceeding the polarizing angle; and it can readily be shown that this change of sign is equivalent to the addition of $\pm \pi$ to the phase.

The present case of interference seems to support this view. It follows, as we have seen, from theory, that if the light undergoes no change of phase by reflexion, the distances of the successive dark fringes from the edge of the shadow will be as the odd numbers 1, 3, 5, &c.; so that the distance of the first dark band from the edge will be half the interval between each succeeding pair of dark bands. But it appears, on the contrary, from the phenomena, that the distance is, as far as the eye can judge, exactly equal to the succeeding intervals; or that the bands are all shifted from the edge by the amount of *half an interval.* The phenomena, therefore, require us to suppose that the phase of the reflected wave is *accelerated*, and that the amount of this acceleration is exactly *half a phase*, or π. For the general expression for the shifting of the bands is $\frac{1}{4}\mu\lambda$ cotan α; and as this is found to be equal to $\frac{1}{4}\lambda$ cotan α, it follows that $\mu = 1$, or the acceleration equal to π. It appears then that when light is reflected at the surface of a denser medium, the wave—at the limiting incidence at least—gains half an undulation at the instant of reflexion.

In order to satisfy myself more fully of the effects of reflexion upon the phase, I repeated the experiment with polarized light. The light was polarized, before it reached the aperture in the screen, by transmission through a good tourmaline; and the fringes were observed in various positions of the plane of polarization with respect to the plane of reflexion. I could detect, however, no sensible difference in the position of the fringes under all these changes of circumstance; and, in particular, the distance of the first dark band from the edge of the shadow seemed, as before, to be precisely equal to the intervals of the succeeding bands, whether the light was polarized in the plane of incidence, or the plane perpendicular to it.

This result seems to be just what might be expected from Fresnel's theory of reflexion. From this theory it appears that if $+a$ be the coefficient of the displacement of the incident ray, or the amplitude of the vibration, and i and i' the angles of incidence and refraction, the coefficients of displacement of the reflected ray will be

$$-a\frac{\sin(i-i')}{\sin(i+i')}, \qquad \text{or} \qquad +a\frac{\tan(i-i')}{\tan(i+i')},$$

according as the plane of polarization coincides with the plane of reflexion, or is perpendicular to it. Now the former quantity is always *negative*, so long as i is greater than i', or the ray incident on the surface of the *denser* medium. Under the same circumstances, the latter quantity is *positive* or *negative*, according as $i + i'$ is less or greater than 90°, or the angle of incidence *below* or *above* the polarizing angle. For very oblique reflexion, then, both displacements are negative; and, therefore, whether the plane of polarization coincides with, or is perpendicular to, the plane of reflexion, the wave will undergo a change of half a phase at the instant of reflexion.

From Sir DAVID BREWSTER's important researches on the nature of metallic reflexion, it appears that a plane-polarized ray, which is incident upon a metallic reflector, becomes elliptically-polarized after reflexion; a result which indicates a difference in the phases of the two resolved vibrations. But it appears further, from the same researches, that this difference of phase varies with the incidence, and vanishes altogether at the extreme incidences; so that at the limiting incidence of 90°, there is either no alteration in the phase of vibration, whether parallel or perpendicular to the plane of reflexion, or that alteration is the same for the two vibrations. From some observation of the fringes produced by the interference of direct light with that reflected from speculum metal, I conclude that the former is the case.

An Essay on the Climate of Ireland. By JOSEPH M'SWEENY, M. D. *In answer to the question proposed by the Royal Irish Academy.*—"WHETHER WE HAVE REASON TO BELIEVE THAT A CHANGE HAS TAKEN PLACE IN THE CLIMATE OF IRELAND, AND IF SUCH CHANGE HAS OCCURRED, THROUGH WHAT PERIOD CAN WE TRACE IT, AND TO WHAT CAUSES SHOULD WE ASSIGN IT."

Read November 8th, 1830.

> Sæpe etiam steriles incendere profuit agros
> Atque levem stipulam crepitantibus urere flammis.
>
> VIRGIL. GEOR. LIB. 1.

DIFFICULT would be the task of treating of the climate of Ireland at different periods, and of drawing conclusions from data so scanty as we possess, but that general considerations of the immutable laws of nature aid us in the investigation, and we derive no small assistance from the more accurate observations on the climate of England, where the history of the weather has been preserved with more care, than in Ireland; analogy in this case lends us its powerful aid.

The comparison of the nature of the vegetation in the island in ancient and modern times, and the comparison also of the animal kingdom at different periods, give us general views as to the climate. The clothing of the ancient inhabitants, the nature of their habitations, the casual allusions to the weather made by historians in describing sieges and battles; all these matters must engage the attention of the person, who undertakes to treat of the climate of Ireland at remote periods, when no regular accounts of the weather were kept to be handed down to posterity.

If a stranger were to direct his attention to the subject of the climate of Ireland, he would be led to suppose from the situation of the island in a temperate zone, and from its situation with respect to the vast Atlantic, to the vapours of which it must be exposed, that the climate ought to be mild and moist; but on ascertaining that the island contained no mountains of extraordinary height, and on tracing on the map the number of rivers by which it is intersected, and amongst them such a river as the

Shannon, and knowing the effect of insular situation on climate, he could have no doubt on the subject.

Mr. Daniel, in one of his meteorological essays, compares the Caspian sea, which has no outlet for the rivers which it receives, with the lakes of another continent, North America, which send an immense volume of water to the ocean; he considers them as hygrometers on a large scale, by which we may judge of the state of saturation of the two atmospheres. *In Ireland we have also a hygrometer on a large scale, we have the Shannon, an index of the large quantity of rain that annually falls.* The fact of this fine river being in so small an island, "spreading like a sea," as the poet Spencer described it, is alone quite sufficient to prove the great humidity of the climate of Ireland.

As the causes of climate here glanced at, are of a permanent nature, and as the laws of nature are immutable, it is just to suppose that mildness and humidity were the characteristics of this climate always, when compared with the climate of other countries of Europe. The effects of draining and cultivation on climate, have been noticed in every quarter of the globe; it would be strange if draining and cultivation, which have gone on rapidly with an increasing population in this island, have not produced some effect on its climate. We will have to investigate in the course of this essay, whether Ireland be an exception to a general rule in this particular. Before entering on the task of tracing the history of the climate, it may be right to make some observations on climate in general, and to examine the assertion so boldly made, that the climate of Europe has greatly changed since the time of the Roman dominion.

Our knowledge of different agents, such as heat, light, and electricity, is yet limited, their mode of operation is not sufficiently understood, to enable us to speak decidedly on several meteorological points. The nature of the sun itself is only surmised, the question of its being a habitable globe, surrounded by a phosphorescent atmosphere, or of its being a body of fire, is one that admits of discussion. Philosophers are undecided about the nature of the solar spots; the question of their producing any effects on the weather is yet involved in doubt.

To Franklin we are indebted for the knowledge of the agency of electricity in the clouds. Scheele pointed out the action of solar light in blackening the nitrate of silver, but to modern research we owe the knowledge of the connexion between light, heat, electricity, and magnetism. The curious effect produced by light on a mixture of chlorine and hydrogen, excited attention, and the discovery of the polarisation of light by Malus, directed the first philosophers of Europe to the subject of light. The action of the violet rays in exciting the magnetic influence, noticed by Morichini, and the connexion between electricity and magnetism, discovered by Œrsted, have opened a new field for inquiry, which has been cultivated with assiduity and success.

The connexion between heat, light, electricity, and magnetism, fortunately, is engaging the attention of the first philosophers of the day. According to Baumgartner of Vienna, Ann. de Chimie, 1826, the direct white light of the sun, may be made to produce magnetism more rapidly, than the process of Morichini, or of Mrs. Somerville. The researches of De La Rive, of Œrsted, and of Ampere, the creator of the branch of physics, called electro dynamics, and of Faraday, on the movements of continued rotation discovered by him; and of Becquerel, and of other eminent philosophers, may help, when compared with the meteorological observations of such men as Humboldt, Daniel, Howard, and Flaugergues, to give insight into intricate points of meteorology. Much yet is to be done in the way of simultaneous observation, in different parts of the world, by philosophers capable of undertaking the task, and aided by the most *improved instruments*, before we will be warranted to speak dogmatically on several points.

A more intimate knowledge than we at present possess of the laws that govern light, heat, electricity, and magnetism, may be absolutely necessary for the explanation of several meteorological phenomena. It has been said that a second Newton is yet to come, whose comprehensive mind will be capable of arranging the multitude of facts and observations, and of deducing the general laws that govern climate. It is only necessary to study attentively, the most approved of works on meteorology, to know what little at present is understood, of the causes that make the winds to deviate at different times, and that influence temperature in different countries, and in different years.

The doctrine of central heat in our earth, has latterly been brought forward on the continent, to explain the difference of temperature in countries, where a similarity of temperature might be expected. The Neptunian theory of Werner, which had obtained ascendency in public opinion, cannot be well reconciled with the mass of evidence now before the scientific world, with regard to the increase of temperature, as we descend from the surface of the earth into mines. In the *Essai sur la temperature de l'interièur de la Terre, par M. Cordier*, will be found a deal of information on this curious subject.

If the opinion were admitted of our earth being a planet partially cooled, as Des Cartes, and Leibnitz supposed, and having its centre in a state of fluidity, there would still be a difficulty in reconciling the accounts of the very severe cold of Europe in former days, with its climate at present, and a cooling process constantly going on, by means of the conducting power of the crust of the earth.

Humboldt, perhaps the first opinion in the world on the subject of temperature, inclines to the belief of a central heat. After alluding to the observations of Arago on the temperature of water brought up from deep borings, he remarks, that these important observations show, that in the earliest state of our planet, tropical temperature, and tropical vegetation, could arise under every zone, and continue until the radiation of heat from the hardened surface of the earth, could cause it to cool. If this be admitted, it follows as a consequence, that Ireland must have experienced, at one period, a climate of a high temperature. In Jameson's Edinburgh Philosophical Journal, for 1828, vol. xviii., an extract of a lecture on climate, delivered by Humboldt, is given, in which are to be found many valuable observations. He observes, that the time is passed, when persons were satisfied with some undefined views of the difference of climates, and when all the modifications of temperature were ascribed either to the shelter afforded by ridges of mountains, or to the various elevations of the earth. He remarks, that remarkable differences of climates are perceived in large tracts of country under the same latitude, and on the same level above the surface of the sea, which do not arise from the trifling influence of individual localities; but are subject to general laws determined by the form of the continents in general, by their outlines, by the state of their *surface*, but particularly by their respective positions, and the proportion of their size to the neighbouring seas. From the proportion of the size of Ireland to the vast Atlantic, no wonder that our winters should be mild, and that our summers should not be very warm.

The influence of the sea on climate is a matter that admits of no dispute; but other matters relating to temperature are involved in obscurity—for instance, the occasional visitation of very severe winters. Are the supporters of the doctrine of central heat, prepared to maintain, that its agency in some winters is less than in others? Cordier supposes the thickness of the crust of the earth to vary in different places, and explains on the principle of central heat, the difference of climate in countries in the same latitude. This curious subject is engaging the attention of the scientific world, but it must be acknowledged, that the occasional return of very severe winters, and of extremely hot summers, does not at present admit of any satisfactory explanation.

Many writers, by selecting from the works of the ancients, passages which treat of severe cold, have led to an almost general belief, that the climate of Europe has very much changed: but a person residing in a distant country would form a very erroneous notion of the climate of England, if he were only to read a number of accounts of the Thames having been frozen over at different times, and the extraordinary years are these that are most likely to be recorded. Doctor Patterson had the courage, in his work on the climate of Ireland, to deny the asserted change in the climate of Italy.

It is a matter of importance in this inquiry to investigate, if any change of consequence has taken place in the climate of the continent; if it could be shown that an important change has taken place there, it would be but reasonable to suppose a change also in the climate of Ireland. But let the climate of Europe in general be what it may, at any time, it may be safely asserted, that Ireland, from not having mountains of great height when compared with those of other countries from its insular situation in a temperate zone, from its being exposed to the vapours of the Atlantic, must from the most remote periods have possessed a climate, mild and moist, when compared with that of the rest of Europe.

A large work has been written by M. Schow, professor of Botany in the University of Copenhagen, which treats of the climate of the earth during the existence of man on its surface; it treats of the climate of the antediluvian world, as far as can be ascertained by fossils.

A paper by Professor Schow on the supposed changes in the meteorological constitution of the different parts of the earth, during the historical period, was read before the Royal Society of Copenhagen, and has been translated and published in the 8th volume of Brewster's Edinburgh Journal of Science. He investigates the accounts given by the ancients, of animals and plants in different countries. He observes that the most rigorous criticism is required in such an inquiry, that persons should not be led into error, as the ancients are not very careful in their description of plants and animals, and matters considered essential in determining the species, were unknown to them; besides, their descriptions are not free from fabulous admixtures. He remarks that great caution must be observed in drawing conclusions as to climate, from animals and plants; for instance, it is not a higher temperature which has driven the beaver from the greater part of Europe, and which in North America compels it, more and more, to retire into the interior, but an increasing population.

With respect to the cultivation of plants, it is not enough to know that a plant was not cultivated by the ancients, but it should be ascertained that they attempted its cultivation in vain.

With regard to the freezing of the sea, a great difference must be made between that which is usual, and that which is extraordinary; and great allowance must be made for the weakness of human memory, which recollects much better the remarkable exceptions than the general rule of things.

He begins the investigation with Palestine, on the authority of the Bible, and treats of the existence of the date tree, and of the vine, about which there can be no question in ancient or modern times. The date tree was abundant, and principally in the southern part of the country.

Jerico was noted for palm trees; the people had palm branches in their hands. Pliny mentions the palm tree as abounding in Judea. Tacitus and Josephus, speak of woods of palm, as well as Strabo, Diodorus Siculus, and Theophrastus.—Among the Hebrew coins, those with date trees are by no means rare, and the tree is recognised as it is figured with its fruit. The vine was one of the plants most cultivated in Palestine. In many places vineyards and wine are spoken of. Strabo and Diodorus speak of the cultivation of the vine in Palestine. Both dates and grapes, together, are symbols on Hebrew coins.

Professor Schow, argues—that as the date tree, in order to bring its fruit to perfection, required a mean temperature of 21° centigrade, that the country about Jerusalem could not have a lower mean temperature than 21° centigrade.

He observes, that in Barbary the vine succeeds only on the coast, and even there, the north sides of the hills are chosen for its cultivation. The mean temperature of Algiers is 21° centigrade. In Egypt, the cultivation of the vine is insignificant; Cairo has a mean temperature of 22° centigrade. At Abusheer, in Persia, they are obliged to plant vines in ditches, to protect them from the heat of the sun.

From the successful cultivation of the date and vine in Palestine, its mean temperature could not have been above 22_0, probably not above 21° centigrade, and he concludes—" if there has been any difference between the mean temperature of Jerusalem, in ancient and modern times, it can hardly amount to one degree."

The time of harvest formerly in Palestine, compared with the time of harvest described by modern travellers, he shows, favours the same conclusions.

It follows from the observations of Theophrastus and Pliny, that as the olive tree was cultivated in Upper Egypt, the climate could not have been more warm, because the olive tree does not bear a great heat. Professor Schow thinks from comparing the accounts of the ancients with these of the French observers, that the rise of the Nile happens at the same period of the year as formerly, showing that the rainy season began in the tropical part of Africa at the same period that it does now.

The ancients spoke of the central part of Africa, as uninhabitable on account of the heat, but not from their own observation it is to be remarked.

From the careful study of the writings of the ancients, and from observation of the vegetation of Italy at the present day, he is against the idea of any change of consequence having taken place in the climate of Italy. He says that the passages of Virgil are taken from his description of pastoral life suited to the mountains, since in the lower plains, there is not sufficient grass on account of the heat. Myrtle and bay have grown near Rome since the earliest times; myrtle branches were made use of in the peace between the Romans and Sabines; and bay crowns were used in the time of their kings; he says that the climate could not be much colder, since myrtle and bay grew there.

That the climate of the south of Europe has not been more warm, is proved by the account which Theophrastus gives about the date tree in Persia, which, when brought to Greece, did not ripen the fruit. Schow carefully compares the times of the corn, and wine harvests, in ancient and modern times, and thinks that the climate of Greece and Italy, like that of Palestine and Egypt, has undergone no important change; but if on account of somewhat later harvests, and the possible growth of beech trees in the Roman plains, we might be led to the opinion, that formerly the climate had been a little colder than at present, the difference will hardly come up to one or two degrees, and will not be greater than might be occasioned by the improvement and cultivation of the north of Europe in modern times.

From Greece and Italy he passes to the countries on the Black and Caspian sea, here it has been pretended that the change of climate has been extraordinary. The Abbè Mann, who collected the accounts of the ancient writers, says that they concur in asserting that the climate there was such as is now hardly to be found in Lapland or Siberia. At present there grow, according to the accounts of travellers, olive trees, fig trees, bay trees, and most of these which grow in the south of Europe. Schow maintains, that a severe criticism will do away with a deal of these pretended changes. Herodotus says, that the Bosphorus freezes, over which the Scythians led their armies, and their waggons. Strabo speaks of the freezing of the sound, and adds, that it was reported thet Neoptolemus fought a battle with cavalry in winter, where in summer he had engaged in a sea-fight.

But Pallas, who in modern times has described those regions, informs us that the Bosphorus, even in moderately severe winters, is covered with ice, principally drift ice from the river Don; and that in severe winters, loaded waggons are carried over it. It is thus at present as it was in former times. To these remarks, adduced by professor Schow, may be added the severity of the winter on the Danube, which interfered with the operations of the Russian army in the late war; and the London papers at the time, gave us accounts from Odessa, of the 3d of January, 1829, which stated, that the sea, as far as the eye could reach, was frozen, and that vessels were prevented from going out or coming in.

Schow enumerates the fruits of these regions, described by the ancients, to prove that no material change has taken place.

In treating of the climate of France, he observes, that we are informed by Strabo, that in Gallia Narbonnensis, the same fruits are found as in Italy, but that in going farther north, the olive tree, and fig tree disappear. In comparing Decandolle's map to his *Flore Française*, the limit assigned by Strabo holds good, and it proves that the climate had not been colder formerly.

The high authority of Professor Schow is here adduced, to prove that no change of great consequence has taken place in the climate of the continent of Europe, in the historical period; it is, therefore, just to suppose, that during the same time, no very great change has taken place in the climate of Ireland; but it is also reasonable to suppose, that it has been modified in some degree by draining and cultivation.

In endeavouring to arrive at general conclusions, with regard to the climate of Ireland by analogical reasoning, the subject of the climate of England naturally suggests itself, for our investigation. It may be right here not to confine our attention to the historical period, but to endeavour to obtain some idea of the nature of the climate of England in antediluvian times.

The fossil remains in England enable us to form an opinion on the subject; not only in England but even in the northern parts of Europe, the remains of animals have been found, which prove that at some remote period, animals existed there in great numbers, which are now only to be met with in warm regions. In different parts of Europe also have been found the impressions of plants, so well defined, that they are easily ascertained to have been the produce of a very warm climate. Impressions of vegetables have been found in countries, the climate of which now could not produce such vegetable productions. It does not come within the plan of this Essay, to treat of the geological changes of our earth. A work has been written by Doctor Ure, of Glasgow, showing that the revolutions of our earth, and animated nature, can be reconciled to *modern science and sacred history.*

In this work he alludes to the futile attempts of Voltaire to explain the impressions of fishes, found on mountains. Doctor Ure, in the introduction to his work, truly observes: "As the stream of civilization advances towards the general diffusion of knowledge, truth, and piety, over the earth, new chambers of nature are unlocked, new scenes of instruction are disclosed, and new means and motives of intellectual and moral excellence, are presented to our view."

In treating of the subject of vegetation in Europe in antediluvian times, he observes: "There is no doubt, however, that palms with fan-shaped leaves, covered Europe with their lofty vegetation at this remote period, in regions where no species of these plants could now grow. The opinion of some writers that these vegetables may have been transported from remote climates into the places where they are actually deposited, appears at variance with every fact hitherto observed, and possesses in reality no solid foundation."—*Ure's Geol.* p. 452.

The fossil vegetables found at Newhaven, in England, agree with those found in the Paris basin; one was the fruit of the palm tree, an instance of the produce of a warm climate. In Doctor Ure's work on geology, there is given the figure of an impression of a vegetable in slate clay, from Lancashire, considered to be the production of a tropical climate.

At Kirkdale, have been discovered the remains of hyænas, and even of the hippopotamus, inhabitants of warm regions. Doctor Buckland in quoting Cuvier, to prove the dispersion of the remains of elephants over every country in Europe, combats the opinion, that the remains found in England, were of elephants imported by the Roman armies. He shows that the fossil elephant belongs to an extinct species of this genus. He observes, that the idea of their being drifted by the diluvian waters from the tropical regions must be abandoned, on the evidence afforded at the den of Kirkdale; and he adds, it remains only to admit that they must have inhabited the countries in which their bones are found.

If it be admitted, that the climate of England at any one period was capable of the growth of such vegetable productions as palms, and served as the abode of the elephant, hippopotamus, and hyæna, it follows as a natural consequence, that Ireland from its proximity, must have possessed somewhat of a similar climate at the same time. The evidence afforded by the den at Kirkdale, cannot be explained away. Doctor Ure remarks, that "there are few physical properties established on a larger or sounder induction than that the Kirkdale and Torquay caves having been dens occupied by hyænas in antediluvian times."—*Ure's Geol.* p. 574.

We shall now have to direct attention to the climate of England in the historical period. According to Doctor Halley, Cæsar landed in England in the latter end of August; an attempt has been made on this computation, to prove that the harvests were earlier at that time than at present, from a passage in his commentaries, where it is stated that the corn was all reaped except in one place. But by reading Cæsar attentively, it is easy to see that he alludes to the country near the camp, and that this sweeping conclusion cannot be admitted.

The Britons, who after the first battle had agreed to submit, no sooner learned that the Roman fleet had been damaged, than they resolved to break off negotiations, and to starve the Romans, "*frumento commeatuque nostros prohibere.*" They hoped by preventing the return of the Romans, that no one after, would attempt to pass into Britain for the purpose of waging war.

Under such circumstances, it is not to be supposed, that Cæsar would be very particular in waiting until the very last day for the ripening of the corn; on the contrary, he would be willing to lay his hands on any thing that might support his troops. "*At Cæsar etsi nondum eorum consilia cognoverat, tamen et ex eventu navium suarum, et ex eo quod obsides dare intermiserant, fore id quod accidit suspicabatur. Itaque ad omnes casus subsidia comparabat. Nam et frumentum ex agris quotidie in castra conferebat.*"

He also repaired his damaged fleet; and while these matters were going on, the soldiers stationed before the camp, informed him that an unusual dust was to be seen in the direction in which the legion had gone, which, according to custom, had been sent out to forage. He hastened to their assistance, and found them engaged with the Britons, who had formed an ambuscade for them in a place where the corn had remained uncut, and who attacked them while engaged in reaping—"*Nam quod omni, ex reliquis partibus demesso frumento pars una erat reliqua suspicati hostes huc nostros esse venturos.*" Surely it cannot be contended for with any reason, from this passage, that all the corn in Britain was reaped except in this place. The passage has reference only to the neighbourhood of the camp; all the corn on the ground within sight of the camp, was probably cut down by the Romans themselves, except in the spot where the ambuscade was laid for them: Cæsar informs us that they were daily employed about it. Its being not quite ripe would not prevent them, situated as they were; besides the number of days which elapsed from the time of their fleet being injured, until this engagement, is not mentioned.

We may infer from the dust seen from the camp, that the weather was at the time, dry. The account of the weather after the engagement, coincides with the variability of the climate of England in modern times—"*Secutæ sunt, continuos complures dies, tempestates quæ et nostros in castris continerent, et hostem a pugna prohiberent.*"

Thus this passage in the 4th Book of Cæsar, is far from proving that a change has taken place in the climate of England.

The general character which Cæsar gives of the climate of Britain, holds good to this day. He describes it as being more temperate than that of Gaul, the cold being less severe—

> "Loca sunt temperatiora quam in Gallia remissioribus frigoribus."
>
> *Cæsar de Bello Gal. Lib.* v.

The cold in the north of France in the winter, is much more severe than in England. Persons from England, who reside during the winter at Paris, are surprised at the cold of the weather.

Tacitus describes the British climate as foul, with frequent showers and clouds.

> "Cœlum crebris imbribus ac nebulis fœdum, asperitas frigorum abest."
>
> *Vita Agricolæ.*

This passage in Tacitus, is a difficulty not to be surmounted by those who maintain, that the great humidity of England is of recent origin.

The account which Tacitus gives us of the vegetation of Britain, answers perfectly at the present day.

> "Solum præter oleam vitemque et cetera, calidioribus terris oriri sueta patiens frugum fecundum."
>
> *Vita Agricolæ.*

If it should be maintained, that the climate of England at one period was well fitted for the cultivation of the vine, and for the production of wine; the supporters of this doctrine, to get rid of the quotation from Tacitus, ought to be able to prove that the climate ameliorated to the time of William of Malmesbury, and that subsequently it again grew cold.

It was probably a succession of favourable seasons that led the Romans to encourage attempts at cultivating the vine in Britain. Nothing was more likely to be fostered than the vine by religious establishments, after the introduction of Christianity.

Bede speaks of the vine growing in some places in England—

> "Vineas quibusdam in locis."

The same may be said now.

Camden, speaking of Gloucestershire, says: "The west part beyond the Severn, is covered with woods. But I need not spend much time on this head, William of Malmesbury, will save me this trouble, who is lavish of his praises and description of this country; take, therefore, his words: 'The country is called from its principal city, the vale of Gloucester, productive throughout of corn and fruits, either by the sole bounty of nature or the industry of art, so that it invites the most indolent persons to labour, when the product will return a hundred fold; you may see the high roads bedecked with fruit trees not planted by art, but natives of the soil. The ground spontaneously produces fruit in taste and colour far exceeding others, many of which will keep the year round, so as to serve their owners till others come in again. No county in England has more or richer vineyards, or which yield greater plenty of grapes, or of a more agreeable flavour. The wine has not a disagreeable sharpness to the taste, as it is little inferior to that of France in sweetness.'"

Camden comments on this passage thus: "What he says of the hundred-fold increase of the land, is a mistake. Not that I am of the opinion of those peevish lazy husbandmen, whom Columella complains of, that the soil is worn out and rendered barren by its excessive ancient plenty. But on this account, not to mention others, we need not wonder that so many places in this county were called vineyards from their vines, since wine was one of the productions of this county; and certainly it seems more owing to the indolence of the inhabitants than to the alteration of the climate, that it now yields none."—*Gough's Camden*, Vol. I. p. 379.

Here we have the opinion of Camden, that no alteration had taken place in the climate from the time of William of Malmesbury to his time; and the experience of latter days shows, that in favourable years, wine may be produced in England, which may be mistaken for continental wine by good judges, instead of being inferior, as William of Malmesbury described the wine in his time.

Mr. Williams,* who advocates the opinion of a change of climate, of course has not passed by the celebrated passage in Malmesbury; he thinks with Camden on the subject of the hundred-fold produce from the land, that the learned monk may have drawn rather too flattering a picture; if so, it can be immediately retorted, that the learned monk may have also drawn too flattering a picture of the fruit trees and vineyards.

The archives of the Church of Ely, give us positive information of the making of wine from a vineyard for some years, and we learn that in an unfavourable year no wine but verjuice was made.

Speechly, in his treatise on the culture of the vine, mentions a controversy between the Rev. Mr. Pegg, and another, on the subject of vineyards in England, formerly. The Rev. Mr. Pegg, after stating the evidence, observes, that it appears plainly, that at Ely grapes would sometimes ripen, and the convent made wine of them, and sometimes not, and then they were converted into verjuice.

Speechly gives an account of many successful attempts at raising vines in England, for the purpose of making wine at subsequent periods. He describes the vineyards at Pain's Hill, which belonged to the Hon. Charles Hamilton. This gentleman produced wine, which was supposed by good judges, to be superior to any champagne they ever drank, and which was sold for fifty guineas a hogshead; one merchant purchased £500 worth.—*Speechly on the Vine*, p. 213, 3d edition.

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* Williams on the Climate of Great Britain, p. 125.

Speechly says : " From the foregoing accounts it is evident, that good wine may be made in this country in a propitious season." There can be little doubt but that the opinion of Speechly would have been verified, with great profit to any individual, so fortunate as to possess a vineyard, in a favourable soil in England in the year 1826. But where the seasons are so uncertain, such a speculation would be hazardous indeed. Evelyn, in 1655, writes thus : " I went to see Colonel Blount's subterranean warren, and drank of the wine of his vineyard, which was good for little." Phillips in his Pomarium Britannicum, has the following very sensible remark on the subject of English wine. " We may conclude that as our intercourse increased with the continent, it was found more advantageous to import wine, than to depend on the product of our own crop, which must have been an uncertain one from the variableness of our climate."

When the English army assembled at York in the year 1327, to repel an invasion of the Scots, Froissart informs us that " good wines from Gascony, Alsace, and the Rhine, were in abundance, and reasonable."—*Froissart's Chronicles, by Johnes,* Vol. I. p. *55.*

The fact of wines from such distant places being conveyed to the centre of England, and sold at reasonable prices there, proves that the climate then was not suited to the cultivation of the vine.

The statement of the production of wine in England at a distant period, can be met by similar statements in modern times, and the description of the vale of Gloucester, by Malmesbury, bears with it the marks of being an exaggerated statement.

Where the climate is so variable, it would not be an easy matter to attempt to draw precise conclusions by means of a calendar of Flora ; in some years vegetation is more forward than in others ; and it has been remarked by close observers, that after a succession of favourable years, many plants acquire, as if by habit, the power of blossoming somewhat earlier for some time.

Lord Bacon, in his celebrated essays, gives us an account of gardening, and of the time several plants come into flower, near London ; probably the description was drawn up from the state of the gardens in the year the essay was written ; we shall compare some of his accounts with those of modern times.

" There followeth for the latter part of January and February, the mezereon tree which then blossoms."—*Bacon.*	" The mezereon sometimes blossoms as early as the end of January or beginning of February."—*Phillips's Sylva Florifera.*
" For March there come violets."—*Bacon.*	" Violet—This favourite flower is a native of Europe, flowering in March and April."—*Miller's Gardener's Dictionary by Martyn.*

"In April—the wall flower."—*Bacon.*	Wild wall flower, "its bright golden flowers are very ornamental in April and May."—*Sowerby's English Botany.*
In April, the cowslip."—*Bacon.*	Cowslip—April is commonly assigned as the month of flowering for all; but the primrose appears in March, and the cowslip in April."—*Miller's Dictionary by Martyn.*
"In May and June, apple tree in blossom."—*Bacon.*	"When about the end of May it is covered with bloom, few if any shrubs surpass the crab in beauty."—*Sowerby's English Botany.*
"In July—the lime tree in blossom."—*Bacon.*	Lime tree, "the flowers begin to open by the middle of May, but are not in their full beauty before the middle of July."—*Phillips's Sylva Florifera.*

From the mildness of some winters in England, furze is met with in blossom, sometimes about Christmas. In the calendar of Flora, in White's Natural History of Selborne, we find the primrose in flower on the 10th of November, and furze in blossom on the 21st of December.

It shows that where the years differ so much, and where the seasons are proverbially variable, complete uniformity cannot be expected in the accounts of the flowering of plants.

Phillips, in treating of the mulberry, observes: "The mulberry tree is stated to have been introduced into this Country (England) in the year 1548, and it is said that it was first planted at Sion House, where the *original trees* still thrive, and which we have seen since the first part of this work has been put to press."—*Phillips's Pomarium Britannicum,* p. 239, London 1823.

Although years may differ, yet on an average of a great number of years, the fact as told by Phillips of these mulberry trees standing the weather so long, tends to show that no great alteration has taken place in the climate of England since the time of their being planted.

There is every reason to think, that variability of seasons is not of modern date. Lord Bacon, in his essay on the vicissitude of things, says: "There is a toy which I have heard, and would not have it given over, but waited upon a little. They say it is observed in the Low Countries, I know not in what part, that every five and thirty years, the same kind and suit of years and weathers comes about again; as great frosts, great wet, great droughts, *warm winters, summers with little heat,* and the like, and they call it the *prime:* it is a thing I do the rather mention, because, computing backwards, I have found some concurrence."

We find White, in 1774, in his Natural History of Selborne, complaining of a run of wet seasons, and observing that there was no use in newspapers inflaming the public mind about combination, that plenty was not to be expected until Providence would send more favourable seasons.

Howard, in his treatise on the climate of London, treating of the popular opinion with regard to St. Swithin's-day, observes: "To do justice to popular observation, I may now state, that in a majority of our summers, a showery period which, with some latitude as to time and local circumstances, may be admitted to constitute rain for forty days, does come on about the time indicated by this tradition, not that any long space before is often so dry as to mark distinctly its commencement; the tradition is so far valuable, as it proves that the summers in this southern part of our island, were subject *a thousand years ago* to occasional heavy rains in the same way as at present." *Howard's Climate of London*, Vol. II. p. 198.

When the English army were searching for the army of the Scots, to bring them to an engagement, near New Castle upon Tyne, in July 1327, the English army suffered severely from rain. Froissart describes their situation thus: "To add to their unpleasant situation, it had rained all the week, by which their saddles and girths were rotted, and the greater part of their cavalry were worn down. They had not wherewithal to shoe their horses that wanted it, nor had they any thing to clothe themselves or preserve them from the rain and cold, but their jerkins or armour, and the green huts."—*Froissart's Chronicles by Johnes*, Vol. I. p. 54.

This quotation from Froissart is valuable, as it corroborates the statement of Howard. All the ancient accounts we have, tend to prove that the climate of England has not materially changed; the dress of the Britons, as described by Cæsar, is well fitted for a humid clime—

> "Pellibusque sunt vestiti."
>
> *Cæsar de Bello Gal.* Lib. V.

Skins were well adapted for keeping out rain and preserving the animal heat.

The description of the climate by Tacitus, would hold good for some of the worst years that are now experienced in England. It would be very difficult for Mr. Williams to get over the quotation from Tacitus. It may be objected to him at every turn. This gentleman thinks that the humidity of the summers in England, has greatly increased, owing to a change on the surface of the island from the increase of hedge rows, from the planting of trees, and from the extension of green crops shading the ground, and preventing its being parched up. There can be little doubt but that the state of the surface of the island, must have *some* effect on its temperature; but when it is remembered that hot and cold seasons arise from *general causes*, and that the *vast Atlantic is the grand source* of moisture; it is to be supposed that Mr. Williams attributed too much to modern improvements.

The very observation which he has quoted of a very old gentleman of Worcester, on the subject of drought, tells against him—"Never fear, I have often known England to suffer from too much *cold and wet,* but never from too much heat."—*Williams's Climate of Great Britain,* p. 227.

Howard, in the preface to the second volume of his Climate of London, published in 1820, says: "The result of my experience is, on the whole, unfavourable to the opinion of a permanent change having taken place of latter times, either for the better or the worse, in the climate of this country; our recollection of the weather, even at the distance of a few years, being very imperfect, we are apt to suppose that the seasons are not what they formerly were; while, in fact, they are only going through a series of changes such as we may have heretofore already witnessed and forgotten." Howard also thinks that in its great leading features the climate differs little from what it was at a remote period.

Doctor Rutty, in his Natural History of the County of Dublin, instituted a comparison between the climate of London and the climate of Dublin, by means of registries kept in both cities, from which he concluded that the winters in Dublin were warmer and moister, than in London.

We have the decided opinion of Howard, that no change has taken place in the climate of England, a man, who in knowledge of meteorological details, stands unrivalled.

The comparison which Cæsar made between the climate of Britain, and of Gaul, is what a foreigner would be apt to make at the present day.

The description of the climate and vegetation of Britain, by Tacitus, holds good at the present day.

From the proximity of Ireland to England, we may conclude that no great change has taken place in the climate of Ireland since the time of Cæsar.

That the temperature of Europe in antediluvian times, was greater than it is at the present day, is a subject that admits of no doubt. Attempts have been made, to explain away the fact of the bones of the elephant being found in cold countries; but the finding of the numerous impressions of plants, that now only thrive in tropical regions, sets the matter at rest.

The subject of fossil plants has been studied with great care by M. Brogniart.

Not only in England have been found the remains of the elephant, but also in Scotland.—*Buckland's Reliquiæ Diluvianæ,* p. 179.

From the bones and vegetable impressions found in England, we would from analogy be fully warranted in concluding, that the temperature of Ireland was also great in antediluvian times; but the bones of the elephant have been found in Ireland also.

In Grierson's edition of Boate's and Molyneux's Natural History of Ireland, there is an account from the Philosophical Transactions, of the remains of the elephant found in Ireland. The writer who describes the finding of them, is sadly at a loss to account for their being found in such a climate; he thinks that no ship of the ancient inhabitants could be capable of importing the elephant, and leaves the reader to imagine how the animal got there before the flood.

Molyneux, in commenting on this subject, thinks, that the elephant under consideration was not brought to Ireland by any industry of man. He supposes that the globe in the early ages of the world, before all records, differed materially from its present geography, as to the distribution of ocean, dry land, islands, and continents, so as to allow this beast a free and open passage from the continent. But a change of climate must be supposed too; the evidence of the Kirkdale cave leaves no doubt but that the climate of the British isles was once suited to animals, that are now only to be met with in warm regions. We may therefore conclude that the elephant in Ireland did not find its way there by mere chance, but that it inhabited the country, and that the climate of Ireland was suited to its existence and habits in *antediluvian times*.

The remains of the moose deer have been found in different places in Ireland; the bone, when treated with muriatic acid, has been rendered flexible. Dub. Phil. Journal, Vol. I. p. 484. The bones discovered at Kirkdale, when treated in the same manner, were made flexible; a proof that the gelatine had not been destroyed by time.

The great temperature of Europe at this period has been explained on the supposition of central heat in our globe: to this cause the writer of this essay attributes the temperature which once rendered Ireland a fit abode for the elephant.

We now come to the historical period, in endeavouring to trace the climate: the researches of Professor Schow, prove that there is no decisive evidence of a material change of climate on the continent of Europe, by records, during the historical period.

Although it is maintained in this essay, that the general character of the climate of Ireland, has been the same from a very early date; yet it is contended for, that the weather has been modified from local causes. The state of the surface of the island has been different at different times, at one time abounding with timber, at another time denuded; at present the surface is furrowed from the potatoe culture in every direction. An ancient name of Ireland was, the woody island.

Endeavours have been made to trace back the history of Ireland to remote antiquity, but on account of some evident fable mixed up with the accounts given, many think themselves warranted in rejecting the entire history of very remote periods altogether.

There is one point (to which if any credence be given) that would be decisive evidence of a change of climate in Ireland at a very distant period, namely, the appearance of rivers that did not exist before, and the formation of new lakes.

This would be evidence of the highest description to prove an increase of humidity in the climate at the time.

Keating, in his History of Ireland, says: "In the time of Partholanus, seven lakes broke out in the island;" again he informs us that "Partholanus found but nine rivers and three lakes in the island."

In the Annals of the Four Masters we have the following accounts:

"Ætas mundi IIDXXXII eruptio lacus con et lacus Techet anno hoc.

"Ætas mundi IIIDIII eruptio novem fluminum.

"Ætas mundi IIIDLXXXI eruptio novem lacuum.

"*Doctor O'Connor's Version of the Four Masters.*"

If these accounts are to be credited, there can be no question as to the increase of humidity of the climate. If we suppose them to be true, we may attempt to explain the previous aridity by the theory of central heat. M. Cordier thinks that the thickness of the crust of the earth varies in different places, and he explains on the principle of central heat, the difference of climate in countries in the same latitude. We know from the fact of volcanoes, that internal fire in some places is not far distant from the surface of our earth, and that its operation sometimes is more active than it is at other times.

We have only to suppose a cessation of activity in the operation of the internal heat in that part of the globe on which Ireland is situated, to account for the condensation of vapours into rain, which previously might be dissolved by the heated air. Indeed there is one point which appears to corroborate this explanation, it is the account in the Irish Annals of the formation of not only new rivers, but also of new lakes.

An increased quantity of rain might cause old rivers to be flooded, or new rivers to be formed, but it would not cause the formation of new lakes, unless the level of the ground had been disturbed by its sinking in some places, or by its elevation in other parts from the operation of an *earthquake,* a visitation universally attributed to the agency of internal fire.

This matter rests entirely on the authority of the old Irish records; where they assume more the shape of historical narrative, they give reason to believe that the climate in Ireland did not materially differ from the climate of the present day, and that remarkably wet, dry, cold, and mild seasons happened occasionally as at the present time.

We have a very early account of a mild climate in Ireland, in the Annals of the Four Masters; we have also an early account of snow. "Ætas mundi IIIDCCCLXVII Erat floribus æstivis ornatus omnis campus in Hibernia tempore Fiachi."—*Annales IV. Magistrorum, Dr. O'Connor's Version.*

Keating tells us in his History of Ireland, that Fionnachta, the son of Ollamh Fodhla, obtained the name by which he was distinguished, on account of the quantity of snow that fell upon the island in his reign. We have in the Annals of the Four Masters, an account of a mild winter at an early period.

"Ætas mundi IIIICLX Regnante Conario—oberrabant armenta absque custode in Hibernia in ejus regimine propter abundantiam pacis et concordiæ, non fuit tonitrale vel procellosum ejus regnum. Nam non fluxu afficiebat ventus asper, armenta a medio autumno ad medium veris. Videbantur sylvæ pendentes præ pondere suorum fructuum ejus tempore."—*O'Connor's Version.*

Here we have a description of a mild winter ; this and the account of the great fall of snow from which Fionnachta obtained his name, tend to prove the occasional return of mild and severe winters at a remote period.

In Jocelin's Life of St. Patrick, frost and snow are mentioned. The Annals of Ulster show us the recurrence of *wet summers*, of droughts, and of severe winters. Years, remarkable for the abundance of nuts, are mentioned; and we find the frequent recurrence of bowel complaints, which coincides with subsequent statements.

Subjoined is a list of some of the most remarkable years, from Doctor O'Connor's version of the Annals of Ulster :

"634 Nix magna occidit multos in campo Breg.

684 Ventus magnus terræ motus in insula.

713 Siccitas magna.

719 Æstas pluvialis.

747 Nix insolitæ magnitudinis ita ut pene pecora deleta sunt totius Hiberniæ et postea insolita siccitate mundus exarsit.

758 Æstas pluvialis.

761 Nix magna.

763 Nix magna tribus fere mensibus—siccitas magna ultra modum, fluxus sanguinis in tota Hibernia.

772 Insolita siccitas.

773 Eugan Mac Colmain a fluxu sanguinis moritur, et multi alii ex isto dolore mortui sunt.

776 Diluvia ventosa in æstate, i. e. inundatio magna imbrium et ventus magnus.

778 e, ventris profluvies per Hiberniam totam.

817 —— Gelu mirandum, et nix magna permanserunt a natali quasi usque ad Quinquagesimam. Transitus paludum pedibus siccis, et plurima flumina eodem modo gelata ac lacus.—Plurima materia ad construendas domos transvecta trans lacum Eirne e regionibus Connaciæ in regionem posterorum Crimthani.

821 Gelu mirabile gelavit *mare* et lacus et flumina ita ut conducerentur armenta equorum et greges et vectigalia ultra citraque.

855 Nix et gelu magnum.

911 Pluvialis tenebrosus annus.

912 Pluvialis tenebrosus annus.

916 Nix et frigus valde magnum et gelu mirabile hoc anno ita ut transgredirentur principaliores lacus et amnes Hiberniæ.

944 Gelu magnum mirabile, ita ut transgredirentur lacus et flumina.

1011 Mortalitas, fluxus sanguinis hoc anno in Ardmacha occidit plurimos.

1026 Exercitus cum Flahertaco O'Neill, abstulit obsides et profectus est supra glaciem, in insulam Mochtei et eam vastavit.

1047 Nix magna hoc anno a festo Mariæ ad festum Patricii, cujus non visa est similis ita ut inde mortalitas hominum plurimorum et armentorum et ferarum innumerabilium et volucrum.

1094 Inundationes ingentes in Hibernia tota.

1095 Nix magna cecidit die primi jejunii (i. e. die Mercurii) post Kalendas hujus anni, ita ut occiderit plurimos homines et volucres et armenta.

1107 Imbrium inundationes ingentes cum gelu et nive a xv Kal. Jan. ad xv Kal. Martii vel paulo plus, ita ut mortalitas esset volucrum et armentorum et hominum."

In the year 821, which, according to the computation in these Annals, agrees with the year 822 of the Christian era, it is recorded that the sea was frozen; but we find that in the year 822, the principal rivers of Europe, such as the Danube, the Elbe, and the Seine, were frozen so hard as to bear heavy waggons for a month. It may be boldly asserted, that Ireland, from its insular situation, suffered less by cold that year than the continent of Europe. But it is to be believed, that the state of the surface of the island, unimproved as it was, in comparison with its present state, must have greatly aggravated the cold of a severe winter, when it happened from a *general cause.* Swamps, and bogs, and pools, are soon frozen; at the present, draining, cultivation, and reclaiming, have made great progress.

By the aid of chemistry it is easy to prove that Ireland never could have just claims to the appellation of Glacialis Ierne:

> " Illa ego sum Graiis olim Glacialis Ierne
> Dicta, et Jasoniæ puppis bene cognita nautis."

The water of the ocean mitigates cold in this way; the upper particles of the water, when cooled by the air, sink, and allow a warmer stratum of water to come in contact with the atmosphere; this process goes on, until the water, by long exposure to the cold, acquires its maximum of density.

The vast body of water of the Atlantic must, therefore, at all times, have rendered the winters in Ireland, mild, when compared with the winters in other countries.

The appellation of " Glacialis Ierne," might have arisen on account of navigators from the Mediterranean, having been in the island during a severe winter, in former times. The crew of a vessel from a warm climate in the Mediterranean, would be apt now to form an erroneous opinion of the climate of Ireland, if they had been here in the winter of 1812.

The island described by Diodorus Siculus, is supposed by some to be Ireland, from the description; the soil fruitful, the country diversified with mountain and plain, watered by navigable rivers, abounding in woods, and orchards, and all the island watered by streams, and the summer season fitted for pleasure and amusement—*Diod. Sic. vers. Wesseling, t.* 1, *p.* 344.

Whether this island was the one described by him, or not, is a matter of surmise; however, this description may be received as a true one of Ireland, at the present day. The island inhabited by the Hyperborei, he tells us, was so fruitful, and the climate so temperate, that they mowed twice in the year.—*Diod. Sic. t.* 1, *p.* 158.

Festus Avienus, writing of that which was called the sacred island, says:

> " Hæc inter undas, multum cespitem jacit
> Eamque late, gens Hibernorum colit
> Propinqua, rursus, insula Albionum patet."—*De Oris Marit.*

The expression "multum cespitem" may be received now, as applicable to Ireland, in reference to a peaty soil, or to its grassy sward.

Much confidence is not to be placed in poetic description, with regard to a matter of philosophical inquiry.

We find Claudian also giving Ireland the title of "Glacialis Ierne"—

> "Scotorum cumulos flevit Glacialis Ierne."

It has been shown that Ireland never could have been remarkable for cold, in comparison with other countries.

Tacitus described the climate of Britain as being foul, with frequent showers and clouds, and stated the absence of severe cold; and he described the climate of Ireland as not differing much from it—

> "Solumque cœlum et ingenia cultusque hominum haud, multum a Britannia differunt."

Cæsar previously had described the cold of Britain as less than that of Gaul. Professor Schow has proved that there is no evidence to establish a material change in the climate of France; we here connect Ireland with the chain of his reasoning.

The comparison which Doctor Rutty made between the climate of Dublin, and the climate of London, shows, that in the grand leading features, there is a similarity in the climate of both places: thus his statement corroborates that of Tacitus.

In a work supposed to be written by Æthicus, the climate is described as superior to that of Britain. Orosius repeats this statement—"*Hæc propior Britanniæ, spatio terrarum angustior, sed cœli, solisque temperie magis utilis.*"—*Orosius, Lib.* 1, *Hist. cap.* 2.

Isidorus says that Ireland is more fertile than Britain.

The venerable Bede gives a decided preference to the climate of Ireland. "Hibernia autem salubritate ac serenitate aerum multum Britanniæ præstat."

The observations of Doctor Rutty show that the winters in Ireland are milder than in England."—*Rutty's Natural Hist. of the County of Dublin,* Vol. II. p. 466.

The description of the island by Donatus, has been often given:

> "Insula dives opum, gemmarum, vestis et auri
> Commoda corporibus, aere, sole, solo
> Melle fluit pulchris et lacteis Scotia campis
> Vestibus atque armis, frugibus, arte, viris
> Ursorum rabies nulla est ibi sæva leonum
> Semina nec unquam Scotica terra tulit
> Nulla venena nocent nec serpens serpit in herba
> Nec conquesta, canit, garrula rana lacu."

The picture which Cambrensis has drawn of the climate of Ireland, bears with it the marks of having been highly coloured; in the first place the style is poetical—"Terra autem terrarum temperatissima nec Cancri calor exæstuans compellit ad umbras, nec ad focos Capricorni rigor invitat, aeris amænitate temperieque tempora ferè cuncta tepescunt."—*Topog. Hib. dist.* 1, *cap.* 25.

Again, in another part he observes—"Aeris clementia tanta est ut nec nebula inficiens, nec spiritus hic pestilens nec aura corrumpens, medicorum opera parum indiget insula; morbidos enim homines præter moribundos paucos invenies."—*Topog. Hib. Dist.* 1, *cap.* 27.

The Abbè Ma-Geoghegan, in commenting on this description by Cambrensis, in his History of Ireland, remarks thus—"Cependant le temoignage de Cambrensis me paroit un peu suspect, parce qu'il est outré. En effet les pluies, les neiges et les gelées y sont assez frequentes en hyver." He might have shown by the Irish Annals, previous to the invasion, and by Ware's Annals, at a subsequent period, that cold winters have been often recorded. It is manifest, by these documents, that the inhabitants of this country were not entirely so free from disease, as Cambrensis described them to be.

In the Irish Annals, we find that bowel complaints were not unfrequent. In fact, distempers of this nature were called by the general name of the country disease.

Dermot Mac Murrough, the cause of the invasion in the time of Cambrensis, did not die of old age, but of disease. The soldiers of the English army were affected by sickness.

It is probable that the armour of the English adventurers, particularly of the chiefs, afforded protection, not only against the weapons of the natives, but also protected them in some measure, from the drenching rains of the island.

The expression of Cambrensis is very vague—"Morbidos enim homines præter moribundos paucos invenies." It might as well imply that the faculty in those days, made quick work with their patients.

Although the account by Cambrensis, of the climate, is exaggerated, still it may be received in evidence, as to the general mildness of the weather in Ireland.

Kirwan in his work* on the temperature of different latitudes, thinks that "the astronomical source of heat is permanent." If this be the fact on an average of years, and if it has held good in former times,† it must follow that less inconvenience was felt from heat in summer, at the time of Cambrensis, in Ireland, than at a subsequent period, when the woods were cut down. When woods abounded in Ireland, of course a great portion of the surface of the island was sheltered from the rays of the sun; therefore, moisture on the ground in the woods, could not be rapidly dried up.

Evaporation causes a depression of temperature; the constant evaporation from extensive woods must, independently of the shade afforded, have tended to keep the surface of the island cool in summer, at the time of Cambrensis.

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* Kirwan, page 107.

† There is every reason to think that it has, from the investigation of Professor Schow of Copenhagen.

A good deal of information relating to the weather in Ireland, may be collected from Ware's Annals. The following are extracts :

"A.D. 1171 This winter the English soldiers, by the scarcity of provision, and change of air and diet, contracted several distempers, and many died.

"1172 A very tempestuous winter, the king having stayed three months in Dublin.

"1192 This likewise may seem worth the remembering, that this year there were so great tempests in Desmond, that many houses and churches were beaten down, and much cattle and men destroyed.

"1209 The city of Dublin, by reason of some great mortality, being waste and desolate, the inhabitants of Bristol flocked thither to inhabit.

"1247 The sáme year, saith Florilegus, there was a marvellous and strange earthquake over England, but saith Feleon, over Ireland, and all the west of the world; and there followed immediately a continual intemperature of the air, with a filthy skurf, the winter stormy, cold, and *wet,* which continued until the 11th of July, and put the gardeners, fruiterers, and husbandmen, void of all hope, insomuch that they complained *that winter was turned to summer, and summer to winter*, and that they were like to lose all, and be undone.

"1326 The earth received fruitfulness, the air temperature, and the sea calmness.

"1348 This year there was great mortality in all places.

"1361 About Easter, began a great mortality of men, but few women in England and Ireland.

"1370 There was a third pestilence in Ireland.

"1383 The fourth great pestilence was in Ireland.

"1486 March, there happened so great a storm of wind and rain, that trees were pulled up by the roots, and many houses, and some churches, were blown down to the ground.

"1489 This summer proving very pestilent and feverish, many people died.

"1491 This year was commonly called by the natives, the dismal year, by reason of *the continual fall of rain all the summer and autumn,* which caused great scarcity of all sorts of grain throughout Ireland.

"About the latter end of December, after the appearance of a blazing star, which shone for some days, a certain grievous and pestilential sickness, commonly called the English sweat, began first to afflict this nation.

"1492 There was so great a drought this summer, throughout Ireland, that many rivers were almost dried up, the cattle dying every where with thirst ; also soon after the pestilence began to rage.

"1500 This year from the middle of September, till the end of winter, Ireland endured *continual rains,* and many tempests.

"1504 This year the pestilence swept away many people, almost every where, but especially in Ulster.

"1505 The plague not yet ceasing, did even this year also, grievously afflict Ireland, a great dearth of corn following it by reason of the *continual rains* that fell in summer and harvest.

"1510 This year, in the month of April, did happen great *inundations*, which overturned trees, houses, and bridges.

"1517 In this year was a very hard winter, so that the ice of the rivers did not only for a long season bear up men upon it, but also loaded carts.

"1522 The city of Limerick was sadly visited with the plague.

"1523 There was great scarcity of corn this year in Ireland, by reason of the *continual rains* in summer.

"1525 The pestilence was rife all this autumn, especially at Dublin.

"1528 This year a certain grievous and pestilential disease, commonly called the English sweat, did overspread a great part of Ireland.

"1534 An earthquake happened at Dublin, which accident is so rare in Ireland, that when it falls out so, it is esteemed as a prodigy.

"1535 A raging pestilence did this year sweep away many, especially in Ulster.

"1539 This summer so great a drought was in Ireland, that many rivers were almost dried up. The autumn also was very sickly, fevers and bloody fluxes, being rife every where, whereof many died. An extreme hard winter followed, insomuch that store of cattle perished in many places.

"1548 February, there happened such a strange violent tempest, or rather hurricane, in most parts of Ireland, that by the force of it, trees were rooted up, and churches and other edifices, quite blown down.

"1552 In this year there was such a scarcity of corn in Ireland, that a peck of wheat (which contains four bushels of English measure) was sold in Dublin for twenty-four shillings; but the following year carried such plenty with it, that a peck of pure wheat was sold for five shillings.

"1554 This year there was a very sad winter, especially from the 21st of December, to the end of the following spring, either *perpetual rain*, hail, or tempest.

"1574 This summer the plague raged in Dublin for several months.

"1599 The Lord Lieutenant, Earl of Essex 'towards the end of July, returned to Dublin, his army being much diminished in number, fatigued, and in a sickly condition.'"

These annals show the occasional occurrence of very dry summers, of very severe winters, and of seasons so wet as to cause a scarcity of corn in Ireland. It does not follow by any means, that all the remarkable years are included in these annals; for instance, Ware had to collect the accounts of the weather from books, written without any particular view to meteorology.

Necessity is justly called the mother of invention; where the mere whim of fashion does not influence, we may discover in the dress of particular nations, some indications of the nature of the climate.

The large shading hats of the Spaniards, bespeak a sunny clime.

The conical caps of the ancient Irish, were admirably adapted for protecting the head against rain, and may be received as collateral evidence of a moist climate.

Cambrensis describes the Irish as dressed in woollens, and the mantle as a protection against the rain, is mentioned by Spencer.

The account of the Irish in the reign of James I. as given by Morryson, is scarcely worthy of notice ; he says—" In the remote parts where the English laws and manners are unknown, the very chiefs of the Irish, as well men as women, go naked in the winter time." Dr. Leland gives no credit to this account of Morryson ; he remarks—" The fact is totally incredible, the climate must at all times have forced the most barbarous to some covering in their retired chambers." Walker, in his essay on the dress of the ancient Irish, agrees with Leland on this point ; and it is worthy of observation that Morryson speaks of the remote parts, with which of course he was the least acquainted ; besides, and what has not been remarked by Leland or Walker, Morryson had been present at the celebrated siege of Kingsale, at the time of the Spanish invasion, and therefore was a witness of the severe weather that prevailed during that siege.

In his history of Ireland, L. Abbè Ma Geoghegan, thus writes of the dress of the Irish—" Les manufactures de toiles d' etoffes, de tout ce qui etoit necessaire pour les couvrir et garantir de l' intemperie de l' air etoient connues aux anciens Irlandois."

The abundance of timber in former times, must have led the inhabitants, in Ireland, as it does in America at present, to construct habitations of that material. In Hanmer's Chronicle, we have the reasons assigned by Mac Mahon, an Irish chieftain, for not residing in a castle. Hanmer informs us, that Mac Mahon levelled two castles bestowed on him by Sir John De Courcy, a short time after the coming of the English. He said that he had promised not to hold stones, but land, and that it was contrary to his nature to couch himself within cold stones, the woods being so nigh.

The desire of the ancient Irish to reside in woods, no doubt arose from the shelter afforded against the winds, from the proximity of timber for the construction of habitations, and for fuel, and probably from the facility of enclosing, by means of stakes between the trees, during the night, their cattle, always a desirable prey amongst a pastoral people, divided into a number of septs, frequently in a state of hostility with each other.

Cambrensis tells us that woods were inhabited as places of defence : " Hibernicus enim populus castella non curat, sylvis namque pro castris, paludibus utitur pro fossatis."—*Top. Hib. Dist. 3, c. 37.*

In the Dublin Philosophical Journal, there is an account of the finding of the body of a man preserved in a peat bog, and dressed in a singular costume. The dress was composed of the skin of some animal, laced in front with thongs of the same material, and having the hairy side inwards. The writer who describes it, thinks that it belonged to a period antecedent to Cambrensis, as he described the Irish dressed in woollen garments. However this may be, the dress is well adapted for keeping out rain, and may be received in evidence of a rainy climate when the wearer lived.

The woods in winter afforded protection to the inhabitants against high winds, and in warm summers yielded a pleasant shade; but military defence was probably the great inducement for their choosing such places of residence.

In the Speculum Regale, a treatise written in the twelfth century, the inhabitants of Ireland are described as being well clothed in winter and summer.—*Antiquarian Repertory*, Vol. II. page 336.

The loose coat, a garment so much worn by our peasantry, is supposed to be a remnant of the old Irish mantle.

This garment, the great coat, worn in winter and summer, is often valuable in affording protection against rain in a variable climate.

However mild the climate of Ireland is to persons who can have the shelter of a house when necessary, still to troops in the field, obliged to march at all hours, and, of course, exposed to wet, it must be any thing but agreeable.

Therefore, it is not strange that we find, in the history of Irish warfare, complaints of the weather, and of sickness amongst the troops, particularly amongst those newly arrived.

It is extraordinary how well the Irish peasantry bear the drenching rains of this climate—they travel, and frequently work, in weather that would prove destructive to strangers, or even to men from the cities or towns in Ireland.

The sufferings of different armies, at different periods, tend to prove, therefore, that the general character of the climate has been the same.

The troops of Henry the Second were affected with sickness.

The army of Bruce, when he invaded Ireland, suffered from the weather.

The army of the Earl of Essex, in the reign of Elizabeth, was diminished by sickness; indeed the English troops in Ireland, in her reign, suffered dreadfully from the climate.

The sufferings of the soldiers of Cromwell are well known. It would appear that officers of rank, at this period, had sometimes recourse to oil cloth, as a protection against the weather. Ludlow, in his memoirs, says, "I clothed myself as warm as I could, putting on a fur coat over my buff, and an oiled one over that, by which means I prevented the farther increase of my distemper."

Who has not heard of the sufferings of the army of William the Third in Ireland from the climate?

In the Pacata Hibernia we have a good deal of detail given of the weather in Ireland, in the reign of Elizabeth.

June is stated to be a convenient time to be in the camp. "Whereas, if the service should be deferred until winter, difficulties should they find in the foulness of the weather, and deepness of the way." In July, the Lord President left Limerick, to relieve a place in Kerry, "and set forward the three and twentieth of July; but, whereas by reason of *continual rain*, that had lately fallen in great abundance, it was thought that the mountain of Sleulogher was impassable for carriages, was constrained to take the way of Thomond." In January, the Lord Dunboyne forced Redmond Burke's forces into the Nore, where seventy of his men were drowned, "the river Nore being at that time very high. The ninth of August Sir Francis and his troops lodged at Alphine, in the County Roscommon; the morning following was dark and misty."

(September.) And no sooner could there a ship appear upon the coast, but presently it was supposed to be a Spaniard, but there none appeared before the seventeenth of the same month, which the Lord President perceiving, and that the winds still were contrary, and the weather very stormy and tempestuous."

(October.) It is stated that in this month some ships with provisions were detained in Waterford, "enforced to stay there, the wind being southerly."

The early part of the month is described as *so wet*, that it was unfit for the army to take the field.

A short time after the commencement of the siege of Kingsale, the weather is described "falling out very foul." And again, "We attended all that day for the landing of the artillery, and perfected the entrenchment about the army, which was left unperfected the day before, through the extreme foulness of the weather."

(November.) "For the mountain Slewphelim, which, in summer time, is a good ground to pass over, was, by reason of *great rains, so wet and boggy*, as that no carriage or horse could pass over." The writer of the Pacata Hibernia next speaks of the frost that enabled O'Donnell's army to cross this mountain, on their way to assist the Spaniards: "There happened a great frost, the like whereof hath been seldom seen in Ireland."

An account is given, that ships, with supplies from England to the siege of Kingsale, were driven to the "southermost part of Ireland" by the foulness of the weather. The besiegers were prevented on the 17th from attempting any thing, on account of the weather, but at night, "when the storm was somewhat appeased," they caused some officers to view the ground of Castle Ny Parke. Next, the extreme frost is spoken of as a difficulty in making approaches: "Continued to work all night, and although the ground was extremely hard, and the night very light, yet they brought the work to very good perfection."

The variable nature of the climate is well marked by the next quotation—"The enemy sallied about eight of the clock in the night, being extreme dark and rainy, with about two thousand men."

(December.) Sir Richard Levison returned into the harbour of Kingsale, and reported to the Lord Deputy the damage done to the Spanish fleet at Castlehaven: "the seventh of December, the wind being extremely at south-east, he rode still at Castlehaven, the night following, with wind at west-south-west, he warped out with the ships." While Sir Richard was at Castlehaven he was exposed for some time to the fire of cannon from the shore, "being by no industry able to avoid it until some calmer weather came."

"The thirteenth, the weather fell out to be extreme foul and stormy. The fourteenth, foul weather, wherein nothing was performed. The seventeenth, foul and stormy weather. The nineteenth, by reason of stormy and foul weather, nothing on either side was performed. This morning the ordnance played oftener." By the context it appears to have been the twentieth. "The next morning that work was brought to great perfection, though the night fell out stormy, with great abundance of thunder and lightning, to the wonder of all men, considering the season of the year." The latter end of December is described as being extremely tempestuous, *cold, and wet,* at the time an attack was expected from O'Neal's army.

In February Captain Flower was obliged to put back, in an attempt to reach the castle of Dunboy, "by reason of foul weather and contrary winds."

In a letter, dated the 15th of February, 1601, O. S. the wind is described so westerly, as to prevent the arrival of shipping to carry away the Spaniards that surrendered.

"The eighth of March, Don Juan being at Kingsale, hourly expecting a wind to be gone, and, finding a flattering gale, went aboard, but, for want of a fair wind, departed not from Kingsale until the sixteenth of the same month."

(May.) The army is described as on its way to besiege the Castle of Dunboy. "The fifth and sixth the weather was so tempestuous, that we could not stir out of quarters. The thirteenth, unseasonable weather. From the seventeenth to the six and twentieth nothing happened worthy of notice, only we were detained in our camp with contrary winds, and with strange, unseasonable, and tempestuous weather. The six and twentieth the wind turned fair, and the shipping drew forth, but immediately the weather proved so tempestuous, they were constrained to return to their former road. The seven and twentieth, the eight and twentieth, the nine and twentieth, and the thirtieth, we were detained with like contrary winds, and unseasonable, foul, and stormy weather. The one and thirtieth the weather grew fair, and we took advantage thereof."

(June.) "The sixth being Sunday, a foul and stormy morning." It is to be supposed that the rest of this month was favourable, as no complaint is made of the weather during the siege of the castle of Dunboy.

(July.) We find, by the Pacata Hibernia, that in this month Sir Charles Willmot was sent into Kerry, to remove all the inhabitants, with their goods and cattle, into the County Limerick, and to destroy such corn as could not be presently reaped. "But in effecting hereof, the governor found great difficulty, for the harvest, by reason of the *winter-like summer, was very backward.*"

It ought to be remarked here, that barley was very much cultivated at this period in Ireland, and in treating of the vegetation of the island, it can be shown that barley in modern times has been reaped very early in Kerry.

(October) "Easterly winds are so seldom upon this coast, as it would ask a long time to transport victuals and munitions by sea."

(January) "The sharpness of this winter journey, did exceedingly weaken our companies, for the mountains of Beare, being at that time quite covered with snow, tasted the strong bodies, whereby many returned sick; and some, unable to endure the extremity, died standing sentinel."

Snow on the mountains, at this time of the year, is not of uncommon occurrence at present; and we have evidence from the same work, the Pacata Hibernia, to render it probable that the cold was not so severe as to freeze the rivers; for, on the 5th of January, in an account of a fight between the troops of Captain Taffe, and those of Owen Mac Eggan, the troops of the latter were driven into the river Bandon: "leaped into the river Bandon, hoping by that means to escape; but that little availed them, for they all for the most part, were either killed or drowned in the river.

(March) "After the Lord Deputy departed, by reason of easterly winds, the President was stayed about three weeks in Dublin, during which time, every day, posts were employed between them."

Here we have an account of the prevalence of easterly winds in the spring, which is well known to hold good at present. This prevalence of easterly winds, in the spring, was also remarked at the time of Doctor Boate, in Ireland. Doctor Rutty informs us, that the easterly winds in spring, are nearly double to what they are in autumn and winter, and that the North East wind in spring, is double to what it is in autumn and winter.—*Natural History of the County of Dublin, Vol. II. p.* 457.

Indeed, taking in general, the details of the weather in the reign of Elizabeth, as they are to be gathered from the Pacata Hibernia, we find a similarity between them and the observations of Doctor Rutty. June is stated to be a convenient time to be in camp, in the above cited work: by Rutty's statements, there would be a good chance of fair weather in this month. By the Pacata Hibernia, it appears that abundance of rain fell in July. During the space of forty-three years in Dublin, in seven years only was the month of July fair and dry.—*Natural History of the County of Dublin, Vol. II. p.* 462.

We have it expressly stated in the Pacata Hibernia, under the head of October, that easterly winds were seldom on the south coast of Ireland.

There is reason to think from the context, that the frost, which was of unusual severity, at the time of the siege of Kingsale, was ushered in by a north wind, because the ships, with supplies from England, were driven to the southermost part of Ireland. Rutty observes, that the great frost in 1739-1740, was attended with an unusual suspension of our trade winds of the west and south west.

The frost at the siege was put to an end, most likely by a change of the wind to the south. The enemy are stated to have sallied out on a dark, and rainy night; soon after, we find by a quotation, the wind to be at south east.

Campion, in his History of Ireland, says—"The soil is low and waterish, and includeth divers little islands, surrounded with bogs and marshes—highest hills have standing pools on their top. Inhabitants, especially new come, are subject to distillations, rhums, and flixes, for remedy whereof they use an ordinary drink of aqua vitæ, so qualified in the making, that it drieth more and inflameth less than other hot confections. The air is *wholesome, not altogether so clear and subtle as ours of England;* of bees, good store—*no vineyards,* contrary to the opinion of some writers, who both in this and other errors, touching the land, may be easily excused, as those that wrote of hearsay. Cambrensis, in his time, complaineth that Ireland had excess of wood, and very little champaign ground, but now the English pale is too naked."

This observation of Campion, respecting vineyards, may be considered as an answer to Bede's statement of the vine being found in Ireland. Indeed the expression of Bede is not very strong; he says "*nec vinearum expers.*" The same may be said of Ireland now, where the vine is cultivated for ornament.

Spencer describes the island adorned with woods, the heavens as most mild and temperate, though somewhat moist; but in another passage, in discussing the origin of the Irish mantle, and in maintaining that it was introduced by invaders; he says—"And coming lastly into Ireland, they found there more special use thereof, by reason of the raw cold climate." On this passage it may be observed, that it is well known to philosophers, that our sensations from cold are not always in proportion to the degree indicated by the thermometer; a certain degree of cold, combined with moisture, will produce on our frames very chilling effects. There is nothing in those observations of Spencer, that ought to induce us to think that any great change has taken place in the seasons, since his time. Moisture, combined with a certain degree of cold, is a sufficient inducement to the use of warm clothing. In describing the various uses of the mantle, he adds—"When it raineth it is his pent house, when it bloweth it is his tent, when it freezeth it is his tabernacle."

In Spencer's account of his plan for putting an end to the disturbances in Ireland, we have some insight into the nature of the winters in his time; he observes—"It is not with Ireland as it is with other countries, where the wars flame most in summer, and the helmets glitter brightest in the fairest sunshine. But in Ireland the winter yieldeth the best services, for then the trees are bare and naked, which use both to clothe and house kern; the ground is cold and wet, which useth to be his bedding; the air is sharp and bitter to blow through his naked sides and legs; the kyne are barren and without milk, which useth to be his only food; neither if he kill them, will they yield him flesh; nor if he keep them will they give him food: besides being all with calf for the most part, they will, through much chasing and driving, cast all their calves, and lose their milk which should relieve him next summer."

It ought here to be particularly remarked, that no allusion is made to severe frosts, as a hard frozen state of the ground would be of the greatest consequence to the pursuers, in enabling them to follow through bogs and marshes, in the most direct line, those who in other circumstances might escape by their knowledge of the country. It is, therefore, just to infer, that severe frosts were not of frequent occurrence in Ireland at the time. We have the ground described *as cold and wet*, such as it is commonly with us in winter.

It does not follow but that severe winters occasionally occurred as they doin modern times.

In Sir W. Betham's Antiquarian Researches, Life of O'Donnel, the frost is mentioned by which young O'Donnel lost some of his toes, in escaping from Dublin. In like manner, an unusually severe frost is described in the Pacata Hibernia, by which the same O'Donnel, at a subsequent period, was enabled to cross a swampy mountain with his army, on his way to assist the Spaniards, besieged in Kingsale. But, if a hard frozen state of the ground was of common occurrence in winter, it is difficult to suppose that the acute Spencer would not have alluded to it.

The remarks of Spencer are worthy of every attention; from his long residence in Ireland, he had ample opportunities of observing the general state of the island, and of making a comparison between its climate, and that of England. He, no doubt, contrasts the Irish climate with the English, when he says that it is most temperate, though somewhat moist.

In modern times, another Englishman, Arthur Young, came to the same conclusion from his own observation.

We shall now have to direct attention to the character of the climate, as given by Sir James Ware; he says—"Pomponius Mela affirms that the temperature of Ireland is unfit to bring seeds to maturity. But more particularly, Giraldus Cambrensis, says: 'Thus corn promises much in the grass, more in the straw, but least in the ear; for the grains of wheat are so small, that they can scarce be cleansed by the help of a fan.' Let us hear now what others of the ancients have written to the contrary. Thus, therefore, Orosius: 'It lies nearer,' says he, 'to Britain; is less in extent, but of a more temperate air and profitable soil.' Likewise, Isidore: 'The next island to Britain, less in extent of land, but more fertile;' and Bede: 'Ireland,' says he, 'both in healthfulness, and also serenity of the air, much excels Britain. But to speak my opinion: if these comparisons relate to the south part of Britain, which we call England, they are not to be allowed, yet we grant that Ireland is of so temperate an air, that *we see the fields green and flourishing in the midst of winter, and cattle put daily to grazing, unless in time of snow, which is rarely of two or three days continuance.*

Many boggy and fennish places being also now drained, the temperature of the air has been much *improved.* As to the grains of corn, they are not *generally* so small as Giraldus and his followers say; for in very few of the neighbouring countries, fairer or larger corn is to be found, than in Ireland. Nor can we allow of the opinion of Raphael Maffeus Volateranus, that Ireland produces nothing but corn and horses. The error likewise of Ranulphus Higden, that Ireland has no pheasants, partridges, deer, nor hedgehogs, is to be corrected. We might observe many things that are fabulously delivered by Giraldus Cambrensis, concerning Ireland; and the reader is to take notice that Giraldus's Topography is to be read with caution, as Giraldus himself in a manner acknowledges in the apology which he makes in his preface to his book of the conquest of Ireland."—*Ware's Antiquities of Ireland, chap.* 23.

Here we have the testimony of Sir James Ware, that in his time, the fields were green in the midst of winter, and that cattle were not prevented from grazing, *except in case of snow, which rarely lasted two or three days.*

In the Speculum Regale, a work supposed to be written in the twelfth century; we are informed that oxen and sheep were continually fed out of doors in Ireland.—*Antiquarian Repertory, Vol. II. p.* 336.

Petrus Lombardus, in his book de De Regno Hiberniæ Sanctorum Insula, stated that the inhabitants neglected to make hay. "Hic plerique negligunt resecare fœnum ob summam temperiem aeris." The mildness of the climate is here given by him as the cause; but though this neglect of saving hay might have been very common in his time, yet the word "plerique" shows that it was not universal.

Patterson thinks it may be accounted for, by the plenty of ground they had in proportion to the stock of cattle.

The evidence of Lombard and of Sir James Ware, may be put in opposition to the statement of Hamilton, who thinks that the great mildness of our winters is of recent date. He says in his memoir on the Climate of Ireland—"Winter has likewise felt the general influence of this Atlantic temperature, our grasses scarcely droop beneath the frost." When he penned this he certainly could not have recollected, that Boate, in his work on Ireland, had also mentioned, *that in his time, cattle fed out in the fields, day and night in winter, and were seldom troubled with great frost.* Thus the statement of Hamilton himself may be used *now* to show that no great change has taken place in the climate.

Some particulars relating to the climate, may be gleaned from the history of the civil war, at the time of Charles I. Sir J. Temple describes the weather as very severe on the breaking out of the rebellion in 1641—"Most bitter cold and frosty." He describes it as the severest year in the memory of man. Among other reasons for sending an army of Scots into Ireland, one was, "that their bodies would better sort with the climate."—*Sir J. Temple's History of the Rebellion of* 1641.

About the middle of March, 1643, the Marquis of Ormonde's army at the siege of Ross, suffered from "*continual rains.*"—*Warner's Civil Wars, Vol. I. p.* 252.

In the middle of June, 1643, some cavalry under Lord Castlehaven, "being favoured by the rain," succeeded in a charge, in routing the troops under Sir Charles Vavasour.—*Warner, Vol. I. p.* 271.

The forces of the council of Kilkenny, in 1645, laid siege to the fort of Duncannon "in January, and in extreme bad weather."—*Warner.*

When O'Neil and Preston, advanced in the winter of 1646, to Dublin, to besiege it, the bad weather and a flood in the Liffey, which carried away some bridges, interfered with their operations.—*Warner.*

In September, 1649, the English fleet, with an army, and Cromwell aboard, were put into Dublin by a strong gale from the south.—*Ludlow's Memoirs.*

The English army, shortly after their arrival, were affected with flux.—*Ludlow.*

Cromwell laid siege to Wexford, in October 1649, and took it after a short time. The Marquis of Ormonde was greatly disappointed, for he had flattered himself that it "would hold Cromwell long enough in play until his forces, which were unused to the climate of Ireland, would be so considerably reduced by the fatigues of a siege at such a season."—*Warner, Vol. II. p.* 188.

"Though the siege of Wexford had been very short, yet Cromwell's army were not all pleased with a winter campaign, and complaining of great hardships, began to mutiny."—*Ibid, p.* 189.

A. D. 1650 "The English army were much wasted with sickness and hard duty, as well as the plague, and the greatest part of those he (Cromwell) had brought with him, had perished; but the fatal revolt of the Munster forces, had recruited him with men, habituated to the climate, and inured to the hardships of an Irish war."—*Ibid, p.* 208.

A. D. 1651, Siege of Limerick—"Ireton lost many men by hard service, change of food, and alteration of the climate."—*Ibid, p.* 243.

Cromwell's army being attacked with flux soon after their arrival, coincides with the account of the climate which had been given by Campion.

We can judge of the general character of the climate of Ireland, on an average of years from the work of Doctor Boate. He observes—"So that the Irish air is greatly defectuous in this part, and too much subject *to wet and rainy weather*, wherein if it were of somewhat a better temperature, and as free from too much wet, as it is from excessive cold, it would be one of the sweetest and pleasantest in the whole world; and very few countries could be named that might be compared with Ireland for agreeable temperature. Although it is unlikely that any revolution of times will produce *any considerable alteration* in this, (the which indeed in some other countries, hath caused wonderful changes) because that those who, many years ago, have written of this island, *do witness the self same things of it in this particular, as we do find* *in our time:* there is, nevertheless, great probability, that this defect *may in part,* be amended by the industry of man, if the country, being once inhabited throughout by a civil nation, care were taken every where to diminish and take away the superfluous and excessive wetness of the ground, in all the watery and boggy places, whereby this too great moistness of the air is greatly increased, and also occasioned.

"This opinion is not grounded upon some uncertain speculation, but upon assured experience, for several knowing and credible persons have affirmed to me, that already some years since, good beginnings have been seen of it, and that in some parts of the land, well inhabited with English, and where great extents of bogs have been drained and reduced to dry land, it hath been found by the observation of some years, one after another, that they have had a drier air, and much less troubled with rain than in former times."

The number of rivers and brooks in Ireland is the best proof of its great moisture. Boate says: "No country in the world is fuller of brooks than Ireland, where the same be numberless, and water all the parts of the land on all sides."

On the subject of cold, he remarks there are commonly three or four frosts in one winter, but they are very short, seldom lasting longer than three or four days together, and withal at their very worst, nothing near so violent as in most other countries. "There hath been," he observes, "some winters wherein it hath frozen ten or twelve days together, so as the Liffie, and other the like rivers, were quite frozen, and might be gone upon by man and beast; but those are altogether extraordinary, and do come very seldom, hardly once in the space of ten or twelve years."

Here we have some evidence to show, that the extension of cultivation had, up to the time of Doctor Rutty, *some effect* in mitigating the cold of severe winters when they did happen. We do not find the Liffey, on an average of years, so often frozen over, that it might be gone upon by man and beast, as was described by Boate in his time.

Kirwan's observations on the weather, correspond in the general features with the accounts handed down by Rutty.

Rutty's descriptions are, as Kirwan remarks, merely popular; they therefore cannot be accurately compared with more precise accounts in latter days; but, on the other hand, these have been made at different periods, in different places, and by different persons.

They lose much of their value, if the opinion maintained by many, be correct, namely, that the seasons go through a cycle; therefore it would be necessary, that observations made in any one place, should be continued for a very long time, before we would be warranted in attempting to draw very precise conclusions.

The unconnected accounts we have, answer, however, to show the general nature of the climate, which agrees in its principal points with ancient accounts, and with those of Rutty and Kirwan. We have mostly a prevalence of south and south west winds in the winter, and of north and north east winds in the spring, the wetness of winter, the humidity of the summer, particularly about the time of July, and the variability of different years, when closely compared with each other. *On the supposition of a cycle, variability of years ought to be expected, and a variation is manifest in the accounts we have.*

When we have a severe winter in Ireland, it is the effect of a general cause, acting with greater effect on the continent. Ancient and modern accounts agree as to the usual mildness of our winters.

Boate observes—" For the most part there falleth no great store of snow in Ireland, and some years, none at all, especially in the plain countries. In the *mountains*, there is commonly greater plenty of snow than in other parts, so that all kinds of cattle do, all winter long, remain there abroad, being seldom troubled with very great frost or snow, and do feed in the fields, night and day."

The very attempt of Hamilton, in modern times, to prove a change of climate, by describing its great mildness, only corroborates Boate's statement, and must inevitably lead the impartial reader, who compares the two accounts, to conclude that *no great* change has taken place.

Boate describes the heat of summer thus—" The which is seldom so great, even in the hottest times of the year, as to be greatly troublesome. And it falleth out often enough in the very summer months, that the weather is more inclinable to cold, than to heat, so as one may very well endure to come near a good fire. And this cometh to pass only during the wet weather, for else, and whilst it is fair, it is very warm all summer long, albeit seldom over hot."

There is a strong similarity in the description of the spring of the year, given by Boate, to that given by Rutty in his time. Rutty showed that the winds from the rainy points were not prevalent in the spring. Doctor Boate, says—" And it raineth there very much all the *year long, in the summer as well as in the winter ;* commonly in the spring of the year it is very fair weather, with clear sunshine from morning till night, for the space of five or six weeks together, with very little or no interruption, which fair weather beginneth commonly in the month of March, some years in the beginning, other years in the midst, and sometimes in the latter end of it. But the same being once passed, it raineth afterwards very much all the summer long, so as it is a rare thing to see a whole week pass without it, *and many summers it is never dry weather two or three days together.* Which inconstancy of the weather, is not only troublesome to men, but also hurtful to all things growing out of the earth for man's behoof."

What will the advocates for an increase of humidity in summer, in modern times, say to this ?

The above quotation from Boate, is enough of itself, to upset the doctrine maintained in Hamilton's Memoir on the Climate of Ireland.

That close observer, Doctor Rutty, tells us, that "a series of hot and dry weather, even in summer, is what the farmer ought not to expect, but to provide for the contrary."—*Natural History of the County of Dublin, Vol. II. p.* 281.

In the Introduction to Cox's History of Ireland, the comparison is made between the climate of England and Ireland. The summers are stated to be warmer, and the winters colder in England, than in Ireland. He adds, thus—"It may be expected, that as the bogs are drained, and the country grows populous, the Irish air will meliorate, since it is already brought to that pass, that fluxes and dysenteries, which are the country diseases, are neither so rife, nor so mortal, as they have been heretofore."

In the reign of William III. the potatoe culture was extending itself in some degree in Ireland, the country was denuded of timber, and was therefore less shaded in summer, than it was in the time of Cambrensis. By Cox's statement, bowel complaints had become less prevalent than before.

The trenches in the potatoe culture, were admirably adapted for quickly removing superfluous water, as the best mode of forming them is in the direction of the summit of a range of hills at right angles with it. It was, therefore, no wonder, that in the places where the culture of the potatoe was commenced, *some improvement* should be experienced in the *time of Cox, in the reign of William the Third.*

As the country was denuded of trees, and as the surface was not shaded in every direction by luxuriant stalks of potatoes in summer, as it is at present, when a warm summer occurred, the inhabitants must have experienced some inconvenience from heat. Leland says of the garrison, during the celebrated siege of Derry—"The heats of summer proved even pestilential to men fatigued and confined; and their scanty, and unwholesome diet, inflamed their disorders."

The history of the war at this period, in Ireland, affords ample testimony of the general moist character of the climate.

Some extracts from Leland may answer better than a long commentary. Speaking of the sufferings of Schomberg's army, he says—"His men had already experienced the hardships of their present service, wasted by a fatiguing march in rain and tempest, in cold and hunger, through a country, dispiriting by its aspect, and by the inclemency of the season rendered still more dreary and distressing."

When Schomberg halted, waiting for the arrival of artillery and provisions, the situation of the army is thus described by Leland—"His soldiers in a confined and unwholesome situation, in the midst of damps and winter showers, without sufficient food, fuel, or covering, an unfriendly climate and inclement season, soon weakened the whole army by fluxes, and a burning fever was caught from the garrison of Derry. The English, unaccustomed to severities, confined to a low and moist situation, drenched with perpetual showers, without the means of health, or the relief necessary in sickness, died daily, in great numbers."

When strong efforts were made for the removal of the sick of the army, to places of safety, and of shelter, Leland informs us, that the general, at the age of four score, afflicted with the scene of wretchedness, exposed to the violence of a dreary and tempestuous season, stood for hours at the bridge of Dundalk, directing every means for alleviating the miseries of his men.

James's council of officers, before the battle of the Boyne, advised him to decline an engagement with the army of William, and to maintain a defensive war, as the one most likely to destroy men in an unfriendly climate, in want of provisions, and succours.—*Leland's History of Ireland.*

The sickness of the English army, corroborates the statement of Campion who, when writing of the island at a previous period, said that persons newly arrived, were particularly liable to be affected with bowel complaints.

The comparison which Cox made between the winters and summers of England and Ireland, agrees with that which has been given to us by Doctor Rutty.—*Natural History of the County of Dublin, Vol. II. p.* 469.

In the Pacata Hibernia, we find by a letter, dated August 1602, the Lord President giving his opinion against ever undertaking a winter siege in Ireland, "for Kingsale was bought at so dear a rate, as while I live, I will protest against a winter siege, if it may be avoided."—*Pacat. Hiber. p.* 631.

In the reign of William III. at the siege of Kinsale, as it is now called, the army suffered from the weather. The garrison surrendered upon conditions which would not have been granted, but that the weather was very bad, provisions scarce, and the army very sickly.—*Smith's History of the County of Cork, Vol. II. p.* 210.

It is difficult to ascertain the precise condition of the weather in distant periods, the invention of the thermometer, by Sanctorio, being comparatively of modern date; and a long time elapsed before the instrument was reduced to a correct standard by Fahrenheit Great allowance must be made for the accounts in old chronicles, and it is possible that extraordinary years happened, accounts of which have not been handed down to posterity in Irish, or continental annals.

In the Philosophical Magazine for 1820, Vol LV. is given a list of extraordinary years for a long period, chiefly from a work by Pilgram, in the German language, published at Vienna, in 1788.

This list of years is valuable, as it shows the occasional return of very severe winters in modern times, and may be used as an answer to those persons who would attempt to prove, by quotations from the classics, the greater cold of Europe in the time of the Roman dominion; it is also valuable, as it proves that one of the coldest years recorded in the Annals of Ulster, arose from a general cause, by which the great rivers of Europe were frozen so hard as to bear waggons for a month.

A. D. 401. The Black Sea was entirely frozen over.

462 The Danube was frozen, so that an army marched over the ice.

545 The cold was intense.

763!! The Black Sea, and the Straits of the Dardanelles, were frozen over.

800 The winter was intensely cold.

822!! The great rivers of Europe such as the Danube, the Elbe, and the Seine, *were frozen so hard as to bear heavy waggons for a month.*

860 The Adriatic was frozen.

874 Snow from the beginning of November to the end of March.

In 991 The vines were killed by the frost; and again in 993, cattle perished in their stalls.

1067 The cold was so intense that travellers in Germany were frozen to death on the roads.

1124 The winter was uncommonly severe.

1133!! In Italy, the Po was frozen from Cremona to the sea, the snow rendered the roads impassable, and wine casks were burst by the frost.

1179 In Austria the snow lay on the ground until Easter, and the crops failed.

1216!! The Po was frozen fifteen ells deep, and wine burst the casks.

1234!! The Po was again frozen, and loaded waggons crossed the Adriatic to Venice.

1236 The Danube was frozen to the bottom.

1261 The frost was intense in Scotland, and the Cattegat at Jutland, was frozen over.

1281 A vast quantity of snow fell in Austria.

1292 The Rhine was frozen over at Brisach, and bore loaded waggons.

1323 The winter was so severe that both horse and foot passengers crossed on the ice from Denmark to Dantzic.

1344!! All the rivers in Italy were frozen over.

1392 The vineyards and orchards were destroyed by frost and the trees torn to pieces.

1408 One of the coldest winters ever remembered. Not only the Danube was frozen over, but the sea between Norway and Denmark, so that wolves driven from their forests, came over the ice into Jutland. In France the vineyards and orchards were destroyed.

1423 Travellers passed on foot from Lubec to Dantzic, on the ice.

The successive winters of 1432–1433–1434 were uncommonly severe. All the rivers in Germany were frozen over.

1460 Horse and foot passengers crossed the ice from Denmark to Sweden, and the vineyards in Germany were destroyed.

1468 The winter was so severe in Flanders, that wine was cut in pieces with hatchets.

1544 The same thing happened again, the wine being frozen into solid lumps.

1548 Between Denmark and Rostock, sledges drawn by horses, travelled over the ice.

In 1564 and in 1565, the Scheld was frozen so as to support loaded waggons.

1571!! All the rivers in France were covered with ice, and fruit trees, even in Languedoc, were killed by the frost.

1594 The Rhine and the Scheld were frozen, and even the sea at Venice.

1608 The snow lay of an immense depth even at Padua.

In 1621 and 1622!! All the rivers of Europe were frozen, and even the Zuyder Zee, a sheet of ice covered the Hellespont, and the Venetian fleet was choaked up in the Adriatic.

1655 The winter was very severe.

The winters of 1658, 1659, 1660!! were intensely cold. *The rivers in Italy bore heavy carriages*, and so much snow had not fallen at Rome for several centuries. It was in 1658, Charles X. of Sweden, crossed the Little Belt, over the ice into Denmark, with his whole army.

1670 In Denmark, both the Little and Great Belt were frozen over.

1684 The winter was excessively cold, even oak trees in England were split by the frost, and coaches were driven along the Thames.

1691 The cold was so excessive, that the wolves entered Vienna, and attacked the men and cattle.

1695 The frost in Germany began in October, and continued until April; many persons were frozen to death.

The years 1697 and 1699, were nearly as cold.

In 1709!!! occurred the famous winter called by distinction the cold winter. In the south of France the olive plantations were almost entirely destroyed. The Adriatic was quite frozen over, and the citron and orange groves suffered in the finest parts of Italy.

1716 On the Thames booths were erected, and fairs held.

1726 People travelled from Copenhagen to Scania, in Sweden.

1729 Much injury done by the frost in Scotland, multitudes of cattle buried in the snow.

The successive winters of 1731—1732 were extremely cold.

1740!! The cold was scarcely inferior to that of 1709, the snow lay on the ground eight or ten feet deep in Spain and Portugal, and the Zuder Zee was frozen over.

1744 The winter was again very cold, the Mayne was covered with ice seven weeks.

The winters during the five successive years 1745–1746–1747–1748–1749, were all of them very cold.

In 1754 and again in 1755, the winters were particularly cold.

In England strong ale exposed to the air, in a quarter of an hour, was covered with a film of ice.

The winters of 1766–1767–1768, were very cold all over Europe.

In France the thermometer fell six degrees below the zero of Fahrenheit's scale. The thermometer laid on the snow at Glasgow, fell two degrees below zero.

1771 The Elbe was frozen to the bottom.

1776 The Danube was frozen that it had ice five feet thick below Vienna. Wine was frozen in the cellars in France.

The successive winters of 1784 and 1785, were so severe, that the Little Belt was frozen over.

In 1789 The cold was excessive, and again in 1795, when the republican armies overran Holland.

The successive winters of 1799 and 1800, were both very cold.

In 1809, and again in 1812, the winters were remarkably cold.

The following list is of years, the summers of which were remarkable for being hot and dry, from the same work, (the Philosophical Magazine, Vol. *55* :)

A.D.	A.D.	A.D.	A.D.	A.D.	A.D.	A.D.	A.D.	A.D.
763	1000	1171			1473	1556	1646	1718
860	1022	1232	1293	1333	1474	1615	1652	1723
993	1130	1260	1294	1393	1538	1616	1660	1724
994	1159	1276	1303	1394	1539			
		1277	1304		1540			
					1541			

1745	1754	1763	1779
1746	1760	1774	1788
1748	1761	1778	1811

As a succession of severe winters and of hot summers, have occasionally occurred, it is no wonder that an opinion of change of climate should have prevailed at different times.

In the time of the Hon. Robert Boyle, it was supposed, that the climate of Russia had changed. Boyle says in his Treatise on Cold—" The Czar's physician tells me by letter, that the winter he spent at Vologda, proved much less severe than usual, for as it happened, they had not three days of what they there call winter weather. He adds, that the cold which is thought to be excessive, hath been rare of late years, for some English who have lived upon the spot thirty years declare, that during their time, winters are become so mild, that the extreme cold which used to freeze people on the road in several postures, hath not been felt as formerly."—*Shaw's Edition of Boyle's Works, Vol. I. p.* 661.

Bonaparte, if he were alive, would not be very much inclined to subscribe to the doctrine of a change of climate in Russia.

In Lowthorp's Abridgment of the Philosophical Transactions, Vol. II. p. 42, we have a paper on the alteration of the climate of Ireland. It tends to show that the idea of change of climate has not been confined to modern days, and that a succession of favourable years has led to the belief of a change. The writer observes, " every one almost begins to take notice that this country becomes every year more and more temperate."

"It was not unusual to have frosts and deep snows of a fortnight and three weeks continuance, and that twice or thrice, sometimes oftener in a winter; nay, we have had great rivers and lakes frozen all over, whereas of late, especially these two or three years last past, we have had scarce any frost or snow at all. Neither can I impute this extraordinary alteration to any fortuitous circumstances, because it is manifest that it hath succeeded gradually, every year becoming more temperate than the year preceding.—This winter 1675, I have kept an exact account of wind and weather. To transcribe my journal here, would be too tedious, let it suffice therefore to tell, that it hath been a very fair and warm, or rather no winter at all."

The language used in 1675 is very like that of Hamilton in his Memoir on the Climate of Ireland.

In Rutty's time, complaints of the inversion of the seasons were common—"wicked exclamations we hear against the inclemency of the climate, our changeable, and particularly our moist and windy weather, *an inversion of the seasons, &c.*, which are owing to a want of due attention to this branch of natural history, for those changes are common to us."—*Natural History of the County of Dublin, Vol. II. p.* 281.

The writer in 1675, described the prevalence of south and west winds, and stated that persons sometimes had to wait three months for a fair wind to come to Ireland. He gives the usual height of the barometer in Ireland as at "29 inches 4 tenths."

Though our climate is variable, yet in its chief leading features its history shows it to be the same. It is well known to us all, that September and October are generally agreeable months; Rutty, in describing this time of the year, calls it *our little summer.*" *Ibid, p.* 465.

Doctor Boate says—"In the latter end of autumn, weather is commonly fair again, for some weeks together, in the same manner as in the spring, but not so long, which as it doth serve for to dry up and to get in the corn, hay which till then hath remained in the fields, the too much wet having hindered it from being brought away sooner; so it giveth the opportunity of ploughing the ground, and sowing the winter corn, the which otherwise would very hardly be done. For that season being once passed, you have very little dry weather the rest of the autumn, and during all the winter."

Boate speaks of thunder being of rare occurrence in Ireland, and says when it does happen, it is in the summer season.

In the reign of Elizabeth, at the siege of Kinsale, there was thunder in December "to the wonder of all men, considering the season of the year."—*Pacat. Hiber.*

In speaking of dry summers, Boate observes—"But as winters cruelly cold, so likewise over dry summers do in this island hardly come once in an age, and it is a common saying in Ireland, that the very driest summers never hurt the land: for although the corn and grass upon the high and dry grounds may get harm, nevertheless the country in general gets more good, than hurt by it; and when any dearths fall out to be in Ireland, they are not caused through immoderate heat and drought as in most other countries, but through too much wet and excessive rain."

In 1826 a famine was apprehended in Ireland, yet the potatoe crop turned out much better than could be expected after so very dry a summer.

In the Phil. Trans. No. 220, there is an account of a substance resembling butter, noticed in Ireland in November, 1695. A similar substance was remarked on grass in the autumn of 1826. The country people employed it to grease the wheels of carts.

Boate remarked, that often in Ireland after days of rain, the nights were fair and clear; the writer of this essay has frequently made a similar observation.

The researches of Doctor Wells explain the formation of dew; the cloudy sky of Ireland interferes with radiation, and is one of the chief causes of the mild temperature of our nights. Boate fell into an error, when he stated that there was as much dew in Ireland as there was in hotter and drier countries; but he may be refuted by his own words—"It is found ordinarily, that in a clear night following rain, the which is very ordinary, the dew cometh as liberally, as if it had not rained the day before." But the nights in Ireland are not in general so clear as the nights in drier countries; and Boate remarked that before rain, little or no dew was to be found, and he described the climate as subject to rain, therefore the formation of dew must have been frequently interfered with. His words are—"When it is towards any great rain, little or no dew doth fall." In another part he observes—"We have showeth how much Ireland is subject to rain, and it is likewise to dark weather and overcasting of the air, even when it raineth not, which continueth sometimes many days together, especially in winter time."

In Kirwan's work on the Temperature of Different Latitudes, under the head of Stockholm, we are informed that Wargentin, in examining a series of 39 years, could not find that any one year resembled another. When a person examines the details of the years such as they are registered in Ireland, he sees such a variation, that he must be convinced of the folly of attempting to draw precise conclusions, unless he had a series of accurate observations made in the same place for a very long period. Thus if the seasons go through a cycle of fifty-four years, he should have observations for 108 years, to be able to compare two cycles.

Mr. Howard, in treating of the mean temperature of London, observes: "The mean temperature of the year is found to vary in different years to the extent of full four and a half degrees, and this variation is periodical. The extent of the periods for want of a sufficient number of years of accurate observations cannot at present be fully determined, but they have the appearance of being completed in seventeen years."

He ventured to make predictions of some succeeding years, and he has failed.

Let us hear a prediction from him—"The year 1816 which was the coldest of a cycle, appears to have had its parallels in 1799 and 1782, and there is every reason to conclude from present appearances, that the warm temperature of 1806 will reappear in 1823."—*Howard on the Climate of London, Vol. II. p.* 289.

Let us now see the character of the year 1823 in the Philosophical Magazine, Vol. LXIII, p. 77—"The mean annual temperature fully confirms what has been before advanced, that wet summers are generally cold. The whole of the monthly means, with the exception of May and December, are unusually low, indeed the actual deficiency as to the annual amount exceeds 2½ degrees." Howard predicted also that the year 1821 would prove an extremely dry one.—*Climate of London, Vol. II. p.* 294.

Dr. Burney describes the ground in 1821 to be in a very moist state.—*Phil. Mag. Vol. LIX. p.* 278.

The failure of such a man as Howard in predicting a year, is the best possible proof of the variableness of the climate of the British isles.

Kirwan endeavoured to form rules of prognostication from the observations of Rutty. In describing the year 1792, in the 5th Vol. of the Transactions of the Royal Irish Academy, he admits that the autumn turned out wet, the least probable event. The autumn of 1794 turning out wet, he admits to be contrary to the rules of prognostication.—*Trans. Roy. Irish Acad. Vol. VI. p.* 171.

It would not be an easy matter to draw up rules, or to talk dogmatically on the minute details of the weather from the scanty materials we have in Ireland. In the diary of the weather for the year 1802 in the Transactions of the Dublin Society for 1803, we are informed that "the thermometer is noted at 8 morning, 12 noon, till the month of May. Then it is noted at 8 morning, 12 noon, and 4 afternoon, the remainder of the year. In some instances it is noted at 8 morning, 12 noon, 4 afternoon, 8 at night, *and at other hours.*" In the diary of the weather for the year 1806, we are informed in the same work that "the thermometer is noted at noon, and four o'clock, and occasionally at other times." Can any thing be more vague than this?

In noticing some difference in the mean temperature of some years in Dublin, or in other parts of Ireland, we are not to conclude that a change of climate has taken place; the mean temperature of London, according to Howard, varies to the extent of four and a half degrees, therefore the mean temperature in Ireland ought to vary also. The same reasoning will hold good with regard to the quantity of rain in different years.

Doctor Patterson's work on the climate of Ireland,* is evidently for the purpose of combating the opinion of Hamilton in the Transactions of the Royal Irish Academy, that trees fail now in situations where they once flourished, owing to a change of

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* Observations on the Climate of Ireland, by William Patterson, M. D. Dublin, 1804.

climate. But as it has been well shown, allowance has not been made for the shelter afforded by large trees.

In a long course of time, young trees may gradually extend up the sides of mountains, protected from the wind by the older and higher trees, until at length they may crown the very summits.

Patterson might have also easily refuted the opinion of Hamilton, by showing from Irish history, that in former times mountains were not the places remarkable for the growth of timber.

To establish this point, is a matter of some importance; if it can be satisfactorily proved, it will tend to overthrow the doctrine of Hamilton, with regard to a change of climate.

Jocelin, in his life of St. Patrick, makes the distinction between woods and mountains,

> "For he abode in the mountains, and in the woods."
>
> *Swift's Jocelin,* Chap. XIII.

Cambrensis describes the mountains for the pasturing of cattle:

> "Frugibus arva pecori montes."

Spencer informs us of the Irish holding meetings on the mountains. He says, speaking of the Irish—"There is one use amongst them, to keep their cattle and to live themselves the most part of the year in boolies, pasturing upon the mountain and waste wild places, and removing still to fresh land as they have depastured the former."

In another place he observes thus—"Yet it is very behoofeful in this country of Ireland, where there are great mountains and waste deserts full of grass, that the same should be eaten down and nourish many thousands of cattle for the good of the whole realm."

The distinction between woods and mountains, is of frequent occurrence in accounts of Irish warfare.

In Ware's Annals, we have an account given of a disaster, which befel the forces of Lord Grey in the county of Wicklow, in the reign of Elizabeth—"Marched with a good force to attack, and ordered his foot to enter into the woods, whilst he with the horse remained on the mountains hard by."—*Ware's Annals.*

Here the distinction between woods and mountains, is well marked; the mountains on which cavalry could manœuvre, agree with the account Spencer has given of the mountains fitted for the pasturing of cattle.

Sir John Davis, in his book to explain the reason why Ireland was never entirely subdued until the reign of James I. says that the English settlers erected their castles and habitations in the plains and open countries, and forced the Irish into the woods and mountains. Again he observes:

"The over large grants of land and liberties to the English, the plantations made by the English in the plains and open countries, leaving the *woods and mountains* to the Irish, were great defects in the civil policy, and hindered the perfection of the conquest very much."—*Sir J. Davis, Quarto Edition, p.* 36.

In 1586, when Sir Richard Bingham marched to put down an insurrection of the Burks, the distinction is made between the mountains and the woods, thus—"he immediately marched to the Abbey of Balintubber, from whence he sent his foot and kerns into the mountains and woods."—*Ware's Annals.*

In the Pacata Hibernia, Desmond is described as being a desolate country—"the whole country being nothing else but mountains, woods, and bogs."—*Pacat. Hiber. p.* 538.

Patterson has given plenty of instances of the growth of trees in exposed and high situations in modern times. Templeton, in the 8th Vol. Transactions of the Royal Irish Academy, says—"The Laurustinus is one of those plants that were introduced to Ireland before green houses were known, consequently planted in the open ground, and experience shows that it is seldom hurt by frost."

In the same volume he also states, that at Fair Head, the northern extremity of Ireland, the mountain ash, birch, and oak, grow luxuriantly within fifteen or twenty yards of high water mark.

In the 4th number of the Dublin Philosophical Journal, there is an account given of a number of plants naturalized under the climate of Ireland, by James Townsend Mackay. It would take up too much space to enumerate them, but the paper shows the great mildness of our climate, and proves that it is not becoming more ungenial, as a person might be led to think by reading Hamilton's Memoir on the Climate of Ireland. It is not to be supposed that the great mildness of our winters is of any recent origin, although it has been promoted by draining and cultivation.

Patterson, in describing the celebrated Arbutus at Mount Kennedy, states that in 1773, its age then exceeded one hundred years.

Some suppose that the arbutus which grows in such abundance at Killarney, was introduced by the Spaniards in the reign of Elizabeth. It was probably introduced by the monks at a much earlier period. Smith, in describing Innisfallen in the Lake of Killarney, says—"There are besides timber trees, the remains of several fruit trees, as plums, pears, &c. which have outlived the desolation that hath seized on the cells of those recluses who first planted them."

There can be little doubt but that apple trees were cultivated in Ireland, before the time of Henry II. An apple tree is mentioned in the life of St. Columba. The story of St. Kevin and the apples may be cited; but one of the most authentic documents we have relating to Ireland, St. Bernard's Life of Malachy, proves that there were apples in Ireland—

> "Accelera inquit fer illi tria poma."
>
> *Vita Malac. cap.* 23.

The account which Mela has given of the vegetation in the island, although exaggerated, yet it has a tendency to prove the moist and mild nature of the climate in his time. He states that the climate is unfit to bring grain to maturity, and that cattle, if not restrained from feeding, would be in danger of bursting from the luxuriant herbage.—*Mela, Lib.* 3. *c.* 6.

Some agriculturists maintain now, that corn is liable to degenerate in this moist climate, and they advise the importation of seed corn from a more congenial country. Cattle have been often injured by feeding on clover. Spencer says, speaking of corn in Ireland—"as for corn, it is nothing natural, save only for barley and oats, and some places for rye." Arthur Young, in modern times, gives a decided preference to English grain in comparison to Irish.

Petrus Lombardus imagined that the vine could be cultivated with success in Ireland; this ought to be looked on as a speculation, probably encouraged by a succession of favourable seasons about the period in which he wrote. He was an ecclesiastic who had spent a good deal of his time on the continent. Let us hear Camden on the subject of Ireland. "It has also vines, but more *for shade* than fruit, for when the sun quits Leo, cool breezes ensue in this our climate, and the afternoon heats in autumn, are too weak and short both here, and in Britain, to bring grapes to perfection."—*Gough's Camden.*

In an Irish almanack of the fourteenth century, the time of gathering grapes and of drinking new made wine, is pointed out.—*Anthol. Hiber. Vol. I. p.* 130.

This ought to be looked on in the light of a modern gardener's book. Although directions may be given in such a book how to cultivate the fig tree, we would not be led to suppose that the climate was fitted for it; yet in 1826, in the south of Ireland, figs in some favourable situations, came to perfection in the open air.

There is an ancient canon which imposes penalties on the owners of hens that damage vines.—*Dacherii Spicil, tom.* ix. *p.* 46.

Ecclesiastical communities might have raised vines for shade and ornament as at present, or for making verjuice.

It has been supposed that yew trees did not abound in Ireland in the middle ages, from an act being passed to oblige merchants to import bows. This may be accounted for; the Irish probably destroyed the yew tree wherever they met it, for two reasons, first, because it was poisonous to their cattle, secondly, because it afforded their enemies a destructive weapon.

Although Patterson has, in his treatise on the Climate of Ireland, refuted Hamilton, yet he has propagated an error relating to the quantity of rain in Ireland. He supposed that more rain fell in England, than in Ireland, and he has led others into the same mistake. We find in the Encyclopædia Metropolitana article, Ireland, the following:

"It is probable that the quantity of rain which falls annually in Ireland, is less than that which falls in England; but it is evidently impossible to arrive at certain results on this question, from the partial observations hitherto made on local climates." The writer of this essay has no hesitation in saying, that Patterson and his followers are wrong. The great quantity of rain that falls at Kendal, from its peculiar locality, deceived him as to the average quantity of rain in England and Scotland.

Dr. Campbell of Lancaster, observes, that the influence of hills in attracting clouds is no where more conspicuous, than at Kendal; that one third more rain falls at Kendal, than at Lancaster, a distance of only twenty miles, and that it is by no means unusual, to see from the church-yard of Lancaster, the hills about Kendal envolved in thick clouds, while the sky at the Lancaster side of Farlton Knott, appears perfectly clear.—*Memoirs of the Lit. and Phil. Soc. of Manchester, Vol. IV. part 2, p. 635.*

Dr. Campbell informs us in the same work, that the clouds from the South and South-west, are attracted by the hills which divide Yorkshire from Westmorland, and that while the western side of these hills is deluged with rain, frequently on the Yorkshire side, the weather is dry. Doctor Garnett says—"The summer of 1792 was remarkably dry in Yorkshire, and all the eastern side of the English Appenine was burnt up for want of rain, while on the western they had plenty of rain and abundant crops of grass."—*Ibid. p.* 634.

Doctor Patterson should have had observations made on the western side of the high grounds in the centre of Ireland, or at a remarkably rainy spot, such as Killarney, to institute a comparison with the very moist part of England.

No person, a priori, would suppose that more rain could fall in England, than in Ireland. In the first place, Ireland is nearer to the Atlantic, and in the second place, it has more mountains than England to attract clouds.

Long since, Boate remarked, that no country in the world was fuller of brooks than Ireland. The number of rivers, is the best proof of greater humidity. The high grounds in the centre of the island, arrest the clouds loaded with moisture, which did not deposit their burthen on the western coast; hence the magnificent Shannon, swelled by tributary streams, rolls its vast volume of water to the ocean.

Any person may point to the Shannon, and laugh at meteorological registries; here is the hygrometer of nature which does not err, in pointing out the greater humidity of Ireland. We have in the Derry Survey as follows:

"Taking the annual quantity of rain that falls in the east of England, which rarely is less than 18 inches, and the max, of the west of that country, the average will exceed 51 inches, and we cannot suppose that Scotland would produce a lower result."

A comparison between the quantities of rain at Derry and Edinburgh, will show that Patterson was wrong.

From the Derry Survey,		From Brewster's Encyclop. article Scotland.
year	inches	inches
1795	32-861	35-7
1796	25-718394	19-4
1797	30-821272	25-9
1798	33-2310176	23-9
1799	31-7709468	25-9

In one year the quantities were nearly alike in both places ; in the other years, the rain at Derry far exceeded that of Edinburgh. The quantity of rain that fell at Glasgow, on an average of thirty years, is marked at 29 inches in the same work.—*Brewster's Encyclopædia.*

The average quantity of 51 inches, which Patterson attributed to England, is entirely too much.* Howard, in his work on the climate of London, states that the greatest quantity of rain during twenty-three years, fell in 1816 at London, and he gives the amount as being 32 inches. He gives the general average for a period of twenty years at 25-179 inches, and the means taken on the ground.—*Howard, Vol. II. p.* 185.

Wakefield quotes Doctor Young, to show, that the average quantity of rain for England and Wales is 31 inches.

The average quantity during eighteen years at Liverpool, which is not a very great distance from Kendal, was 34 inches.—*Memoirs of the Lit. and Phil. Society of Manchester, Vol. IV. part 2, p. 575.*

Arthur Young observes—" I have known gentlemen in Ireland deny their climate being moister than England ; but if they have eyes let them open them, and see the verdure that clothes their rocks, and compare it with ours in England, where the rocky soils are of a russet brown, however sweet the feed for the sheep."

In another place he remarks—" If as much rain fell upon the clays of England, as falls upon the rocks of her sister island, those lands could not be cultivated."—*Tour in Ireland, Vol. II. p.* 74.

The prevalence of winds which waft vapours to the island, is from an early date. The prevailing winds in the time of Camden, were the same in the time of Cambrensis. Camden observes, that Giraldus said, not without reason—nature beheld the realm of zephyr, with an uncommonly favourable eye.

Solinus described the Irish sea as being stormy—" Mare quod Britanniam et Hiberniam interluit, undosum et inquietum toto in anno, non nisi aestivis pauculis diebus est navigabile, navigant autem vimineis alveis quos circumdant ambitione tergorum bubulinorum—*Solinus, c.* 35.

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* Williams quotes Hales, who estimated the annual quantity of rain in England at 22 inches.—*Williams's Climate of Great Britain, p.* 79.

It may be said that it would be dangerous to cross the sea, in such craft at present. He qualified the description by the words " navigant autem."

Boate gives a passage from Giraldus on the Irish sea, but he does not give his meaning correctly in the translation—" Hibernicum mare, concurrentibus fluctibus undosissimum fere semper est, inquietum ita, ut vix etiam aestivo tempore, paucis diebus se navigantibus tranquillum prœbeat."

Boate translates this passage thus—" The Irish sea being very boisterous through the concourse of the waves, is almost always restless, so as even in the summer time, it is hardly for a few days quiet enough to be sailed on."

Surely this is not the meaning of Cambrensis. His meaning in this passage is, that scarcely in the summer time, is it calm for a few days.

A perfect calm is not a frequent occurrence at the present day in the Irish sea.

Boate, speaking of the want of east wind to bring ships from England, to Ireland, observes—" But in the summer time, and chiefly in the spring, and in the months of March, April, and May, one is not so much subject to that incommodity, as in the other times of the year."

Boate's observations agree with these of Rutty.

It would be difficult from the natural history of Ireland, at least during the historical period, to prove a change of climate; wolves have been exterminated, the employment of guns has tended to banish birds from countries more than any change of climate.

In the list of birds by Cambrensis, are to be found cygni. Smith informs us that wild swans were common in the north of Ireland, but were only observed in the south in the great frost of 1739.

Wild swans were shot in the south of Ireland, in the winter of 1829.

It is allowed that magpies were driven here by a storm, at a period subsequent to Cambrensis, but many birds might have escaped his observation. The increase of population, and the use of fire-arms, no doubt, banished storks. They were in Ireland in the reign of Henry II. " We have seen," says the Irish king Dermot, in a letter preserved by Cambrensis, " the storks and the swallows. The birds of the spring have paid us their annual visit, and at the warning of the blast, have departed to other climes. But our best friend has hitherto disappointed our hopes. Neither the breezes of the summer, nor the storms of winter, have conducted him to these shores."—*Lingard's History of England.*

Ireland, in ancient and modern times, is similar in being free from serpents

Spencer described the Irish as being tormented by gnats, in the woods. Arthur Young says that the number of flies which devour one in a wood, prove the great humidity of Ireland."—*Young's Tour in Ireland, Vol. II. p.* 77.

In every period of the history of the climate of Ireland, we find evidence of its mildness.

Sir John Davis in his time, stated as follows :—" During the time of my service in Ireland, I have visited all the provinces of that kingdom, in sundry journies and circuits, wherein I have observed the good temperature of the air, and the fruitfulness of the soil."

We find its character at different periods to agree with its character of the present day, if we except the title of 'glacialis Ierne,' given in poetic description; to this it has been shown, it never could have just claims.

Cambrensis says—" Pascuis tamen quam frugibus, gramine quam grano fœcundior est insula."

Wakefield observes, that in the south of Ireland, the value of the mountains of Tipperary, Cork, and Kerry, was frequently mentioned to him, as the climate allowed them to be grazed throughout the whole year; a statement which agrees with the following—" sicut aestivo, sic hiemali tempore, herbosa virescunt pascua, unde nec ad pabula fœna secari nec armentis unquam stabula parari solent, aeris amœnitate temperieque, tempora fere cuncta tepescunt."

The woods have been cut down since the time of Cambrensis, with the exception of the woods; this description of the country may be received as applicable at the present day.

" Hibernia quidem terra inaequalis est, mollis, et aquosa, sylvestris, et paludosa."

The names of the letters in the Irish alphabet, show that a vegetation familiar to the present generation, was known to the inhabitants of the island, at a distant period.

Ledwich, on the authority of Lombard, says :—" About 1632, artichokes, collyflowers, pompions, and hops, seem to have been first introduced and grew very well."

These vegetables growing in Ireland at that time, do not prove any change of climate, if they had been introduced at an earlier period, there is little doubt but that they would have succeeded as well. The same reasoning holds good for the tobacco, cultivated at present to such an extent in the county of Wexford.

In the Preface to the Translation of Dandolo, on the silk-worm, the writer states, that " during the last century, some French refugees in the south of Ireland, made considerable plantations of the mulberry, and had begun the cultivation of silk with every appearance of success; but since their removal, the trees have been cut down."

An attempt has been made of late years in the south of Ireland, to produce silk on the estate of the Earl of Kingston.

The following queries were transmitted to a person in the neighbourhood.

Have the mulberry plants thriven or died?

Have the silk worms died; the cause?

Are any attempts continued to rear plants or worms?

To which the following was received :—" The plants did not thrive, the silk worms died, the climate did not appear congenial. No attempts are now made, the ground has been let to tenants. Lord Kingston went to much expense, in this attempt to establish a silk factory; the first season it appeared to have gone on well, and it was imagined that it would have been successful; however, the following season came wet, and the worms perished."

A great deal depends on favourable seasons. In 1768, a bounty of twenty guineas was given to a Mrs. Gregg, for having raised a considerable quantity of silk in the county Clare.—*Transactions of Dublin Society for* 1799.

In the Pacata Hibernia, it is stated, that they were prevented from reaping in Kerry in the month of July, the crops being backward on account of an unfavourable season. Barley was much cultivated in Ireland at that time, the country was but thinly inhabited, and of course the best lands were selected for cultivation, and the computation of time was according to the old style. Arthur Young, in his tour in Ireland, describing the rotation of crops in the Mahagree islands, near Tralee, in the County Kerry, observes—"All grain is remarkably early, they have sown English barley, and made bread of the crop in six weeks. I was assured, that in these islands, they have known two crops of barley gained from the same land, in one year, and the second better than the first. They sowed the first in April, and reaped the middle of May, and immediately sowed a second, which they reaped the end of August."—*Vol. I. p.* 472.

It would be a desirable thing, if we had an exact account of the weather in the south of Ireland for a long time. There is no registry of the weather made at the Royal Cork Institution, on Sundays. When the writer of this essay visited that establishment, to compare the statements in Smith's History of the County Cork, with a considerable series of years in modern times; the officers of that establishment could not tell what was become of the registry, previous to the year 1825.

In Smith's time the rain was as follows in Cork:—In 1738, 54 inches 5 tenths—the same nearly in 1739—in 1740, but 21 inches 5 tenths—in 1741, 33 inches 6 tenths—in 1742, 38 inches 1 tenth—in 1743, 39 inches 3 tenths—in 1744, 33 inches 6 tenths—in 1745, 48 inches 4 tenths—in 1746, 30 inches; the same nearly in 1747—and in 1748, 37 inches 4 tenths.—*Smith's Cork, Vol. II. p.* 404.

The quantity of rain at the Royal Cork Institution, was in round numbers as follows:

years	inches
1825	32
1826	28
1827	31
1828	40
1829	39

Hamilton gives the mean temperature of different parts of the City of Cork in 1788, at *52—5* to *53—5*.—*Transactions Royal Irish Academy, Vol. II.*

The mean temperature at the Royal Cork Institution on the average of five years is about 55. This does not prove a change of climate; in the first place the average is swelled by the warm year of 1826, and the thermometer is kept in the centre of the city, in a situation surrounded by high buildings, where it must be affected by radiation. Howard has shown, that the mean temperature of London varies to the extent of 4½ degrees in different years; therefore Hamilton was not warranted, in deducing the mean temperature of Ireland from the observations of a few years.

Smith's account of the winds in his day, agrees well enough with the average of five years in modern times in Cork. No precise conclusions can be drawn from comparisons between broken fragments of cycles of the weather.

Kirwan remarks—" Among all the years observed by Dr. Rutty from 1725 to 1765, there occurs but one similar to 1792, the year 1755, in that the three seasons, spring, summer, and autumn, were wet."—*Transactions of Royal Irish Academy, Vol. V. p.* 240.

It is a generally received opinion, that within the memory of our old peasants, the winters have become milder, and the summers less warm; in this essay it is contended for, that the general character of the climate has from a very early period been the same; yet it is certain that there is some good reason for this popular opinion, on account of a modification of the climate, from the general extension of the potatoe culture.

That the old people should imagine that the summers were warmer formerly, ought not to surprise us, as the buoyancy of youth, and warm blood in young days, cause a warm glow from moderate exercise; but, on account of the greater liability of old people, to be affected by cold, they ought to feel more severely now, winters of the same temperature. If no change has taken place, we should expect to hear from them complaints of the cold; but on the contrary, the old peasants maintain firmly, that the winters formerly were colder.

Popular and general opinion is not to be slightly passed over. The peasantry of France, obtained a signal triumph over the philosophers of their country, with regard to the fact of the fall of meteoric stones.

The potatoe culture has extended in almost every direction, even up to the tops of mountains in some places. *The paring and burning* of rough ground, makes it smooth, and allows the sun to exert its full influence along the surface in winter. On rough mountain ground, where there are inequalities, and tufts of heath, and furrows worn by the rain, snow in such places is liable to remain a long time undissolved. *Boate remarked in his time, that there was a greater plenty of snow on the mountains, than in other parts.*

In summer the shade of the stalks of potatoes protect the ground from the sun, and the trenches, which might serve in winter as fit receptacles for keeping snow undissolved a considerable time, are obliterated by the digging out of the potatoe crop, on the approach of severe weather. If the trenches remained during the winter, it is evident that after a fall of snow, some of it would lie a considerable time undissolved in these trenches.

The appearance of frost is a signal for the Irish peasant to dig out his potatoes, and consequently to obliterate the trenches. The level dark-coloured ground, which remains after the potatoe crop, is well adapted for melting snow. The sun exerts its influence to great advantage *on a dark surface*, according to the experiments of Franklin; and at night, fallow ground of this description, is, according to the experiments of Dr. Wells, particularly unfavourable for the production of *hoar frost*. Thus modern science is in favour of popular opinion.

The potatoe culture shades the ground in summer also, in the following manner: A number of the hills of Ireland lie east and west; the potatoe trenches, which are so many drains, run at right angles to the tops of these hills, for the purpose of conveying off superfluous water; the rays of the sun from east to west do not therefore traverse directly these trenches, and thus the beds, independently of the stalks, cause a shade. In this way a great portion of the surface of Ireland is shaded in summer, but particularly by the luxuriant stalks of potatoes, that meet the eye of the traveller in every direction.

The potatoe culture has also wonderfully increased the number of enclosures, and hedge rows, and has consequently added to the shading of the ground in summer, in every direction, even up to the tops of mountains. Hedge rows also afford shelter to cattle in winter. Smith, in his History of Kerry, stated that cattle in his time sometimes perished on the mountains in severe winters.

Before the general cultivation of the potatoe for the food of the people, (in the recollection of many, oaten bread constituted a considerable portion of their diet,) large tracts of pasture ground denuded of timber, and not intersected by hedges, must have been liable to be parched in summer. The bed and trench plan of culture, the favourite system with our peasantry, is admirably adapted for draining this moist island, and for mixing clay with a peaty surface.

In winter, bogs and shallow pools, were easily frozen at night, and served as reservoirs of cold on the following day. According to the experiments of Doctor Wells, grass is particularly liable to be covered with *dew and hoar frost*. The extensive system of pasture formerly followed in Ireland, must have often presented a large surface of hoar frost to the action of the morning's sun.

A considerable portion of the heat of the morning's sun must have been therefore expended, in thawing the ice on shallow pools, and in bogs, and in melting the hoar frost, formed on the grass during the night.

If, notwithstanding the draining of swamps, the reclaiming of bogs, and the amelioration of the soil by manures, and by more judicious cultivation, it should be contended, that on an average of years, the winters at present are exactly as cold as they were previous to the general cultivation of the potatoe; it would imply, that the power of the sun was then greater than it is at present, as it had then more obstacles to overcome in warming the surface of the island.

The vast quantity of manured fallow ground, of *a colour dark* in proportion as it is not exhausted by severe cropping, now materially aids the sun, to warm the surface of Ireland in the winter. It is well known, that the exhausting of ground by repeated corn crops, causes its colour to become lighter. This injudicious system, was much more practised formerly, than at present. Landlords everywhere endeavour to prevent it. The old country people are positive in asserting, that the snow lay longer on the ground when they were young, than it now is observed to remain.

If, notwithstanding luxuriant crops from an improved soil, and the shading of the surface by the general cultivation of the potatoe, and by the number of hedge rows, it should be contended, that on an average of years, the summers now are *exactly* as warm as they were formerly, it would imply, that the power of the sun's rays is greater for warming the island now, as its surface is better shaded than it was in the period subsequent to the destruction of the woods.

The greater power in the rays of the sun, on an average of years cannot be admitted, as there is no evidence to prove it; and the occasional return of very hot summers, and of very severe winters, is attributed to causes at present not perfectly understood.

If this reasoning be allowed, it must be admitted, that the modification of climate must have kept pace with agricultural improvement, on an average of years, and it explains, and corroborates popular opinion on the subject.

The vast increase of the potatoe culture, and the general use of this vegetable as the entire dependence of the peasantry, have been within the memory of the old persons of the present generation; therefore it is just to believe, that a modification of climate from local causes, has taken place within their recollection.

The general character of the climate has been the same from a very early period; hot summers, and cold winters arise from *general*, not from local causes; but when they do happen, the temperature must be influenced by the state of the surface in some degree.

The testimony of the peasantry, that the snow does not remain now so long on the ground as formerly, must be received. The experience of old sportsmen, who had been in the habit of traversing tracts of country now reclaimed, corroborates the evidence.

In Rutty's time, the cultivation of the potatoe was making progress on the rough grounds, in the county Dublin; he admits that his account of frost and snow in the city of Dublin, was too little when compared with *the accounts of country parts.*

On account of the scanty state of data on the weather in Ireland, it is impossible to put popular opinion to a severe test, by scientific records.

In fact if it be true, as was supposed by Lord Bacon in his time, and as is imagined by Howard and others, that the seasons go through a cycle, it is evident that we should have the weather accurately observed during two complete cycles, one at a late, another at a distant period, so as to be able to compare them, before we would be warranted in attempting to draw precise conclusions.

The potatoe culture is well fitted for draining the moist surface of the island, the trenches run from the summits of the hills to carry off superfluous water.

Not only have bogs been reclaimed, but in some districts they have been absolutely removed, and the peat which they afforded consumed as fuel. In some places, a bog is the most valuable part of an estate, where fuel is dear. Great masses, therefore, of this *vegetable sponge*, retentive of moisture, and liable to be quickly frozen, have been removed or reclaimed, and mixed with clay in the recollection of our peasantry. These places must have been fertile sources of vapour; and in time of frost, when once frozen, they must have been magazines of cold for a considerable time.

We have the testimony of Sir James Ware, of Boate, and of Cox, that good effects on the climate from cultivation, were experienced in Ireland when they lived; then why should we reject the testimony of the old peasants who are yet alive, particularly when it is consistent with science?

Smith, in his History of the County of Kerry, predicted that the culture of potatoes would render the country more wholesome, and stated that enclosures sheltered the land, and improved it, and kept it warm in winter.—*Smith's Kerry, p.* 159.

The process of adding calcareous, vegetable, and animal manures to the soil is constantly going on in Ireland, year after year. Sir H. Davy ascertained by experiment, that a dark-coloured soil, containing animal or vegetable matter, if heated within the common limits of solar heat, will cool more slowly, than a wet pale soil, composed entirely of earthy matter. Therefore it is what ought to be expected, that snow should not remain on the ground now, so long as formerly.—*Agricul. Chemis. p.* 156.

The few observations made with instruments in Ireland, have been made in towns, not in the country parts. Towns are not the fit places for observations; the heat from fires, the number of inhabitants and of domestic animals crowded together, the friction of vehicles and of machinery, the dark colour of the streets from animal and vegetable manures, the process of fermentation going on, all these matters tend to raise the temperature of towns.

Registries of the weather, kept for a very long period, and in country parts, only could disprove popular opinion on the subject of snow remaining on the ground.

The observations should be made with *great care* and for an extended series of years.

We have the authority of Howard for thinking, that implicit reliance is not to be placed on the registry, even of the Royal Society of London.—*Howard on the climate of London, Vol. II. p.* 190.

The name of mountain ground is frequently given in Ireland, to rough grounds producing heath, and such kind of vegetation of little value. In such places, not only in the time of Boate, but also in the recollection of our own country people, snow, when it fell, was apt to remain a considerable time. The inequalities on such ground, protect the snow from the rays of the sun.

By *paring and burning*, lands of this description are every year brought into cultivation.

The history of the weather in Ireland shows its general mild and moist nature, what might be expected from the island's locality, in regard to the Atlantic. The vapours of this ocean produce frequent rain; this produces rivers and verdure.

Well may Ireland be called—

> Land of brooks, and murmuring streams
> Of rivers wide, and verdant plains.

Sir James Ware, in his Antiquities of Ireland, quotes the following character of the island from Alexander Nechamus.

> " Fluminibus magnis lætatur Hibernia."

Spencer, in his Fairy Queen, has described our rivers.

> " There was the Liffie rolling down the lea,
> The sandy Slane, the stony Au-brian
> The spacious Shenan spreading like a sea,
> The pleasant Boyne, the fishy fruitful Ban,"

The general character of humidity and mildness of our climate, cannot be disproved by details of portions of cycles. The same causes always produced the same effects. Ireland in every age excelled other countries in mildness of climate and in verdure.

On Differences and Differentials of Functions of Zero. By WILLIAM R. HAMILTON, *Royal Astronomer of Ireland.*

Read June 13, 1831.

THE first important researches on the differences of powers of zero, appear to be those which Dr. BRINKLEY published in the Philosophical Transactions for the year 1807. The subject was resumed by Mr. HERSCHEL in the Philosophical Transactions for 1816; and in a collection of Examples on the Calculus of Finite Differences, published a few years afterwards at Cambridge. In the latter work, a remarkable theorem is given, for the development of any function of a neperian exponential, by means of differences of powers of zero. In meditating upon this theorem of Mr. HERSCHEL, I have been led to one more general, which is now submitted to the Academy. It contains three arbitrary functions, by making one of which a power and another a neperian exponential, the theorem of Mr. HERSCHEL may be obtained.

Mr. HERSCHEL's Theorem is the following:

$$f(e^t)=f(1)+t\,f(1+\Delta)\,o^1 + \frac{t^2}{1.2}\,f(1+\Delta)\,o^2 + \&c. \qquad \text{(A)}$$

$f(1+\Delta)$ denoting any function which admits of being developed according to positive integer powers of Δ, and every product of the form $\Delta^m\, o^n$ being interpreted, as in Dr. BRINKLEY's notation, as a difference of a power of zero.

The theorem which I offer as a more general one may be thus written:

$$\phi(1+\Delta)\,f\,\psi(o)=f(1+\Delta')\,\phi(1+\Delta)\,(\psi(o))^{o'}; \qquad \text{(B)}$$

or thus

$$F(D)\,f\psi(o)=f(1+\Delta')\;F(D)\cdot(\psi(o))^{o'}. \qquad \text{(C)}$$

In these equations, f, ϕ, F, ψ, are arbitrary functions, such however that $f(1+\Delta')$, $\phi(1+\Delta)$, $F(D)$, can be developed according to positive integer powers of Δ' Δ D; and after this development, Δ' Δ are considered as marks of differencing, referred to the variables o' o, which vanish after the operations, and D as a mark of derivation by differentials, referred to the variable o. And if in the form (C) we particularise the functions F, ψ, by making F a power, and ψ a neperian exponential, we deduce the following corollory :

$$D^x f(e^o) = f(1+\Delta') D^x e^{o'} = f(1+\Delta') o'^x ;$$

that is, the coefficient of $\frac{t^x}{1.2..x}$ in the development of $f(e^t)$ may be represented by $f(1+\Delta) o^x$; which is the theorem *(A)* of Mr. Herschel.

June 13, 1831.

ADDITION.

The two forms *(B) (C)* may be included in the following :

$$\nabla' f \psi(o') = f(1+\Delta) \nabla' (\psi(o'))^o. \qquad \text{(D)}$$

To explain and prove this equation, I observe that in Maclaurin's series,

$$f(x) = f(o) + \frac{D f(o)}{1} x + \frac{D^2 f(o)}{1.2} x^2 + \ldots + \frac{D^n f(o)}{1.2..n} x^n + \ldots$$

we may put $x = (1+\Delta) x^o$ and therefore may put the series itself under the form

$$f(x) = f(o) + \frac{D f(o)}{1}. (1+\Delta) x^o + \frac{D^2 f(o)}{1.2}. (1+\Delta)^2 x^o + \&c.$$

or more concisely thus

$$f(x) = f(1+\Delta) x^o : \qquad \text{(E)}$$

which latter expression is true even when Maclaurin's series fails, and which gives, by considering x as a function ψ of a new variable o' and performing any operation ∇' with reference to the latter variable,

$$\nabla' f \psi(o') = \nabla' f(1+\Delta) (\psi(o'))^o. \qquad \text{(F)}$$

If now the operation ∇' consist in any combination of differencings and differentiatings, as in the equations *(B)* and *(C)*, and generally if we may transpose the symbols of operation ∇' and $f(1+\Delta)$, which happens for an infinite variety of forms of ∇', we obtain the theorem *(D)*. It is evident that this theorem may be extended to functions of several variables.

June 20, 1831.

On a difficulty in the Theory of the Attraction of Spheroids.

By JAMES M'CULLAGH, A.B.

Read May 28, 1832.

AN approximate theorem, discovered by LAPLACE, and relating to the attraction of a solid slightly differing from a sphere, on a point placed at its surface, has given rise to many disputes among mathematicians.* I hope the question will be set in a clear light by the following remarks.

Let us consider the function which expresses the sum of every element of a solid divided by its distance from a fixed point, and let us denote it, as LAPLACE has done, by the letter V. It is necessary to find the value of V for a pyramid of indefinitely small angle, the fixed point being at its vertex. Calling ϕ the small solid angle of the pyramid (or the area which it intercepts on the surface of a sphere whose radius is unity and centre at the vertex), it is manifest that the element of the pyramid at the distance r from the vertex is $\phi r^2 dr$; dividing therefore by r, and integrating, we have $\frac{1}{2}\phi r^2$, or ϕ multiplied into half the square of the length, for the value of V.

Again, supposing the force to vary inversely as the square of the distance—the only hypothesis that can be of use in the present inquiry—the attraction of the same pyramid on a point at its vertex, and in the direction of its length, is manifestly equal to ϕr.

Let us now consider a solid of any shape, regular or irregular, terminated at one end by a plane to which the straight line PQ (*Fig.* 1,) is perpendicular at the point P; and let there be a sphere of any magnitude, whose diameter $P'Q$ is parallel to PQ. Let P'' be a fixed point, and from the points P, P', P'', draw three parallel straight lines Pp, $P'p'$, $P''p''$, the first two terminated by the surfaces of the solid and of the sphere, the third $P''p''$ in the same direction with them and equal to their difference, without regarding which of them is the greater, and suppose all the

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* See Pontécoulant, *Théorie analytique du système du monde,* Tome II. p. 380; with the references there given.

points p', taken according to the same law, to trace the surface of a third solid. Let Pp, $P'p'$, $P''p''$, be edges of three small pyramids with their other edges proceeding from $\boldsymbol{P}$, $\boldsymbol{P}'$, $\boldsymbol{P}''$, parallel, and having of course the same solid angle which we shall call ϕ, denoting by r, r', r'', their respective lengths, and by V, V', V'', the values of the function V for each of them. Drawing pR perpendicular to $\boldsymbol{PQ}$, the attraction of the pyramid $\boldsymbol{Pp}$ in the direction of $\boldsymbol{PQ}$ will be equal to $\phi \times \boldsymbol{PR}$; call this attraction A, and let a be the radius of the sphere.

Since r'' is the difference of r and r', we have $r^2 + r'^2 - r''^2 = 2\,rr' = 2\,\boldsymbol{PR} \times \boldsymbol{P'Q'}$, and multiplying by $\frac{1}{2}\,\phi$ we find $\frac{1}{2}\,\phi r^2 + \frac{1}{2}\,\phi r'^2 - \frac{1}{2}\,\phi r'^2 = 2a\phi \times \boldsymbol{PR}$, that is $V + V' - V'' = 2aA$. The same thing is true for any other three pyramids similarly related to each other, throughout the whole extent of the three solids which are exhausted by them at the same time; and hence, if we now denote by V, V', V'', the *whole* values of the function V for the three solids, and by A the *whole* attraction of the first of them parallel to $\boldsymbol{PQ}$ on a point at $\boldsymbol{P}$, we shall still have $V + V' - V'' = 2aA$.

To express this general theorem in the notation of LAPLACE, we have merely to observe that the attraction A is synonymous with $-\left(\frac{dV}{dr}\right)$, and that the quantity V' for the sphere is equal to $\frac{4}{3}\pi a^2$. Substituting these values, we find

$$V + 2a\left(\frac{dV}{dr}\right) = -\ \tfrac{4}{3}\pi a^2 + V'';$$

Cur.a

an exact equation, differing from the approximate one of LAPLACE only in containing the quantity V'', and totally independent of the nature of the surface or of the magnitude of the sphere; the only things supposed being that all the lines drawn from P meet the surface again but once, and that no part of it passes beyond a plane through P at right angles to PQ.

With respect to the limit of the quantity V'', it is obvious that if a hemisphere be described from $\boldsymbol{P}''$ as a centre with a radius equal to the greatest difference δ between the lines Pp, $P'p'$, the solid $\boldsymbol{P}''p''$ will lie wholly within this hemisphere, and consequently V'' will be less than the value of V for the hemisphere, that is, less than $\pi\delta^2$; for here all the little pyramids from the centre have the same length δ, and their bases are spread over the hemispherical surface; wherefore $V'' = 2\pi \times \frac{1}{2}\delta^2 = \pi\delta^2$. All this is independent of any thing but the suppositions just mentioned.

If now PQ be supposed to be a spheroid of any sort, slightly differing from the sphere $\boldsymbol{P'Q'}$, and such that the line $\boldsymbol{PQ}$, perpendicular to the surface at $\boldsymbol{P}$, passes nearly through the centre, than all the differences, of which δ is the greatest, being of the first order, the quantity V'', which is less than $\pi\delta^2$, will be of the second order; and therefore neglecting, as LAPLACE has done, the quantities of that order, we get the theorem in question.

It may be well to apply the general theorem to the simple case in which the first solid is a sphere of the radius a', because both Lagrange and Ivory have used this case to show that the reasonings of Laplace are incorrect. In this instance, then, the surface described by the point p'' is that of a sphere whose radius is the difference between a and a'; and the values of V, V', V'', and A, are $\frac{4}{3}\pi a'^2$, $\frac{4}{3}\pi a^2$, $\frac{4}{3}\pi(a'-a)^2$ and $\frac{4}{3}\pi a'$, respectively.

Substituting these values in the equation $V+V'-V''=2aA$, and omitting the common factor $\frac{4}{3}\pi$, the resulting equation

$$a'^2+a^2-(a'-a)^2=2aa'$$

ought to be identical;—and so it manifestly is.

November, 1831.

Plate


Fig. 1
Q R P Q' P' p p' p''
2
R' P' O q
N T
L M
T S
3
B R P Q C M
4
R P O N Q
5
Q N
6
S' d a C A d S O C a
S' p m g h
F O R I A
G H S
7
M P
6
S' O A M P m S M P

Geometrical Propositions applied to the Wave Theory of Light.

By James M'Cullagh, F.T.C.D.

Read June 24, 1833.

Part I.—Geometrical Propositions.

1. Theorem I. Conceive a curved surface B to be generated from a given curved surface A in the following manner: having assumed a fixed origin O, apply a tangent plane at any point Q of the given surface, and perpendicular to this plane draw a right line OPR cutting the plane in P, and terminated in R, so that OP and OR may be reciprocally proportional to each other, their rectangle being equal to a constant quantity k^2, and let all the points R taken according to this law generate the second surface B. *Then the relation between these two surfaces, and between the points Q and R, will be reciprocral;* that is to say, if a tangent plane be applied at the point R of the second surface, a perpendicular ON to this plane will pass through the point Q of the first surface, and ON and OQ will be reciprocally proportional to each other, the rectangle under them being also equal to k^2.

2. To prove this theorem, take a point q, in the tangent plane of the surface A, and near the point of contact Q. (*Fig.* 2.) Through q let several other planes be drawn touching the surface A in points Q', Q'', Q''', &c. and draw the perpendiculars $OP'R'$, $OP''R''$, $OP'''R'''$, &c. according to the same law as OPR. The points R, R', R'', R''', &c., will thus be upon the second surface B, and they will moreover be all in the same plane; for from any one of them R' let $R'n$ be drawn perpendicular to the right line Oq and meeting Oq in n; then on account of the similar right-angled triangles $OP'q$ and OnR', the rectangle nOq will be equal to the rectangle $R'OP'$, or to the constant quantity k^2, so that the point n, or the foot of the perpendicular let fall upon Oq, will be the same for all the points R, R', R'', R''', &c., and consequently all these points will lie in a plane cutting the right line Oqn perpendicularly in n, so as to make the rectangle nOq equal to k^2. Now while the point Q remains fixed, let the point q approach to it without limit in the tangent

plane at Q; and the points R', R'', R''', &c. will in like manner approach without limit to the fixed point R; the plane which contains all those neighbouring points having for its limiting position the tangent plane at R. Also the point n will ultimately coincide with N. It follows therefore that the tangent plane at R cuts the right line OQ perpendicularly in N, so as to make the rectangle NOQ equal to k^2.

3. *Corollary.* If any point Q upon the surface A should be a point of intersection, where the surface admits an infinite number of tangent planes, the perpendiculars from O upon these planes will form a conical surface having O for its vertex. In OQ take, as before, a point N, so that $ON \times OQ = k^2$, and let a plane passing through N at right angles to OQ cut the conical surface. The intersection will be a certain curve. From the preceding demonstration it is evident that every point of this curve belongs to the surface B, and that the plane which touches this surface at any point of the curve cuts OQ perpendicularly in N; or, in other words, that *the same plane touches the surface B through the whole extent of the curve.*

4. Two surfaces related to each other like A and B in the preceding theorem may be called *reciprocal surfaces*, and points like Q and R *reciprocal points*; the *radii* OQ and OR may likewise be termed reciprocal. A familiar example of such surfaces is afforded, as I have shown on a former occasion*, by two ellipsoids having a common centre at the point O, and their semi-axes coincident in direction, and connected by the relation $aa' = bb' = cc' = k^2$; where a, b, c, are the semi-axes of one ellipsoid in the order of their magnitude, a being the greatest; and a', b', c', those of the other ellipsoid, a' being the least. The mean semi-axes b and b' coincide, and the circular sections of both ellipsoids pass through the common direction of b and b'.

5. It has also been shown with regard to those ellipsoids, that if Q and R be reciprocal points on the surfaces of abc and $a'b'c'$ respectively, and if a right line Oqr, perpendicular to the plane QOR, cut the first ellipsoid in q and the second in r, the lines OQ and Oq will be the semi-axes of the section made in the ellipsoid abc by a plane passing through them; and the lines OR and Or, in like manner, will be the semi-axes of the section made in the other ellipsoid $a'b'c'$ by the plane in which they lie.

6. It may further be remarked, that if the radius OQ in one of the reciprocal ellipsoids describe a plane, the corresponding radius OR will describe another plane. For the planes touching the ellipsoid abc in the points Q will all be parallel to a certain right line, and therefore the perpendiculars OR to these tangent planes will all lie in a plane perpendicular to that right line. These two planes, containing the reciprocal radii, may, for brevity, be called *reciprocal planes.*

When two reciprocal radii lie in a principal plane, at right angles to a semi-axis of

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* Transactions of the Royal Irish Academy, Vol. XVI. Part II. pp. 67, 68.

the ellipsoids, it is evident that two planes intersecting in this semi-axis and passing through the reciprocal radii, are reciprocal planes.

7. THEOREM II. If three right lines at right angles to each other pass through a fixed point O, so that two of them are confined to given planes; the plane of these two, in all its positions, touches the surface of a cone whose sections, parallel to the given planes, are parabolas; while the third right line describes another cone, whose sections parallel to the given planes are circles.

Let the plane of the figure, (*Fig.* 3) supposed parallel to one of the given planes, be intersected by the other given plane in the right line MN; and let OQ be perpendicular to the latter plane, while OP is perpendicular to the former and to the plane of the figure, so that PQ being joined will meet MN at right angles in R. Let OA, OB, OC, be the three perpendicular lines, of which OA is parallel to the plane of the figure; this plane will be intersected by the plane of OA and OB in a right line BT parallel to OA, and therefore perpendicular to both OB and OP, and to the plane BOP, and to the line BP. Thus the angle PBT is always a right angle, and therefore BT always touches the parabola whose focus is P and vertex R; or, which comes to the same thing, the plane $AOBT$ always touches the cone which has O for its vertex, and the parabola for its section.

Again, since OB, OP, OC, are all at right angles to OA, they are in the same plane, and therefore the points B, P, C, are in the same straight line; and as BOC is a right angle, the rectangle under BP and PC is equal to the square of the perpendicular OP; but QOR is also a right angle, and therefore $QP \times PR = OP^2$; whence $BP \times PC = QP \times PR$, and therefore the points B, R, C, Q, are in the circumference of a circle, so that the angle at C is a right angle, being in the same segment with the angle at R. Thus the point C describes the circle whose diameter is PQ, and OC describes the cone of which this circle is the section.

8. Of the two right lines OP and OQ perpendicular to the given planes, one is also perpendicular to the plane of the section. That one is OP. Its extremity P is the focus of the parabola. The extremities of both are the extremities of the diameter PQ of the circle. The vertex of the parabola is the point R where the diameter of the circle intersects that given plane to which the plane of section is *not* parallel.

9. THEOREM III. In a straight line at right angles to any diametral section QOq of an ellipsoid *abc* whose centre is O, let OT and OV be taken respectively equal to OQ and Oq the semi-axes of the section, and imagine the double surface which is the locus of all the points T and V; then if OS be perpendicular to the plane which touches the surface in T, and OP to the plane which touches the ellipsoid in Q, the lines OP and OS will be equal and perpendicular to each other, and the four straight lines OP, OQ, OS, OT, will lie in the same plane at right angles to Oq.

10. This theorem is taken from a former communication to the Academy*. The surface to which it relates, being the *wave surface* of Fresnel, is one of frequent occurrence in optical inquiries, and it is therefore desirable to give it a distinctive name not derived from any physical hypothesis. I shall call it a *biaxal surface*, from the circumstance implied in its construction, and adopted as the definition on which the preceding theorem is founded;—namely, that any pair of its coincident diameters are equal to the *two axes* of a central section made in the *generating ellipsoid abc*, by a plane perpendicular to the common direction of the two diameters. The name, perhaps, may appear the more appropriate, as it reminds us of the place which the surface holds in the optical theory of biaxal crystals.

11. Theorem IV. The biaxal surfaces generated by two reciprocal ellipsoids are themselves reciprocal.

For if $\boldsymbol{Q}$ and $\boldsymbol{R}$ (*Fig.* 4.) be reciprocal points on the two ellipsoids, abc and $a'b'c'$, a tangent plane at $\boldsymbol{Q}$ will cut $\boldsymbol{OR}$ perpendicularly in $\boldsymbol{P}$; a tangent plane at $\boldsymbol{R}$ will cut $\boldsymbol{OQ}$ perpendicularly in N; and the rectangles $\boldsymbol{ROP}$ and $\boldsymbol{NOQ}$ will be equal to each other and to k^2 (*Art.* 4). Also if the straight line Oqr, at right angles to the plane of the figure, cut the first ellipsoid in q and the second in r, then (5) the elliptic section $\boldsymbol{QO}q$ will have $\boldsymbol{OQ}$ and $\boldsymbol{O}q$ for its semi-axes, and the lines $\boldsymbol{OR}$ and $\boldsymbol{O}r$ will be the semi-axes of the other section $\boldsymbol{RO}r$. Draw therefore, in the plane of the figure, the right lines $\boldsymbol{OTL}$ and $\boldsymbol{OSM}$ perpendicular to the right lines $\boldsymbol{OQN}$ and $\boldsymbol{OPR}$, making $\boldsymbol{OT}$, $\boldsymbol{OL}$, $\boldsymbol{OS}$, $\boldsymbol{OM}$, equal to $\boldsymbol{OQ}$, $\boldsymbol{ON}$, $\boldsymbol{OP}$, $\boldsymbol{OR}$, respectively; the angles at S and L being of course right angles. Then it is evident that the point $\boldsymbol{T}$ is on the biaxal surface generated by the ellipsoid abc, because $\boldsymbol{OT}$ is perpendicular to the plane of the ellipse $\boldsymbol{QO}q$ and equal to the semi-axis $\boldsymbol{OQ}$; and by Theorem III. it appears that $\boldsymbol{OS}$ is perpendicular to the tangent plane at T. In like manner, the point $\boldsymbol{M}$ is on the biaxal surface generated by the other ellipsoid $a'b'c'$, and $\boldsymbol{OL}$ is perpendicular to the tangent plane at M. Moreover, the rectangles $\boldsymbol{MOS}$ and $\boldsymbol{LOT}$, being equal to the rectangles $\boldsymbol{ROP}$ and $\boldsymbol{NOQ}$, are each equal to k^2. Hence the proposition is manifest.

12. As the ellipsoid whose semi-axes are a, b, c, may be called the ellipsoid abc, so the biaxal surface generated by this ellipsoid may be called the biaxal abc; and that which is generated by the ellipsoid $a'b'c'$ may be called the biaxal $a'b'c'$.

13. Proposition V. To find what properties of biaxal surfaces are indicated by the cases wherein one of the two sections $\boldsymbol{QO}q$, $\boldsymbol{RO}r$, in the preceding theorem, is a circle.

Case 1. When $\boldsymbol{QO}q$ is a circular section of the ellipsoid abc, the points $\boldsymbol{T}$ and V, (9) in the description of the biaxal surface abc, coincide in a single point n. At this

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* Transactions of the Royal Irish Academy, Vol. XVI. Part II. pp. 67, 68.

point there are an infinite number of tangent planes; because the semi-axes of the circular section **QOq** being indeterminate, any two perpendicular radii of the circle may take the place of **OQ**, Oq, in the general construction. The point n is therefore a point of intersection (3), where the two biaxal sheets cross each other, and it may be called a *nodal point*, or simply a *node*. As **OQ** always lies in the plane of the circle **QOq**, the line **OR**, which is reciprocal to **OQ**, must lie (6) in a given plane reciprocal to the plane of the circle. And as Oq lies in the plane of the circle, we have three right lines **OR**, Oq, OS, which are at right angles to each other, and of which the first two are confined to given planes. Therefore by Theorem II. the third line **OS** describes a cone whose sections parallel to the given planes are circles. Now TS—or in the present case nS—is parallel to the fixed plane which contains **OR**, and therefore the point S describes a circle; or, in other words, the feet of the perpendiculars **OS**, let fall from **O** on the nodal tangent planes, occupy the circumference of a circle passing (8) through the nodal point.

14. Parallel to the plane of the circle and to its reciprocal plane, conceive two planes passing through the node, and call them the *principal tangent planes* at n. The plane of the circle and its reciprocal plane are intersected in the right lines **Oq**, **OR**, by the plane qOR which is parallel to a tangent plane at n. Consequently this tangent plane at n intersects the two principal tangent planes in lines that are parallel to **Oq**, **OR**; and as **Oq**, **OR** are perpendicular to each other, it follows that every nodal tangent plane intersects the two principal tangent planes in lines that are at right angles.

Hence again, the nodal tangent planes touch (7) the surface of a cone whose sections, parallel to the principal tangent planes, are parabolas. As this cone touches the biaxal surface all round the point n, it may be called the *nodal tangent cone.*

15. *Case* 2. When **ROr** is a circular section of the ellipsoid $a'b'c'$, any two perpendicular radii of the circle may be taken for **OR**, **Or**: and because **$OR = b'$**, and **$OR \times OP = k^2 = bb'$**, we have **OP** or **OS** equal to b, the mean semiaxis of the ellipsoid abc. Hence **OS** is given both in position and length; for it is perpendicular to the fixed plane **ROr**, and it is equal to b. Now a plane cutting **OS** perpendicularly at S, is a tangent plane to the biaxal abc; and we have just seen that this tangent plane remains the same, whatever pair of rectangular radii are taken for **OR**, **Or**. But the point of contact **T** is variable, for the plane **ROS** in which it lies changes with **OR**. Therefore as **OR** revolves, the point **T** describes a *curve of contact* on the tangent plane of the biaxal abc.

The lines **OR**, **Or**, are in the fixed plane **ROr**; and as **OQ** is reciprocal to **OR**, it lies in a fixed plane reciprocal to the plane **Ror** (6). Therefore the first two of the three perpendicular right lines **Or**, **OQ**, **OT**, are confined to fixed planes. Hence the third line **OT** describes a cone, whose sections parallel to these planes are circles. But the tangent plane is parallel to the fixed plane **ROr**, and its intersection with

OT describes the curve of contact. Therefore the curve of contact is a circle passing (8) through the point S.

16. We have examined the two cases of circular section with reference only to the biaxal abc. If we examine the same cases with regard to the second biaxal $a'b'c'$, we shall find that their indications are reversed; the supposition which gives a node upon one biaxal, giving a circle of contact on the other: and that the node and the circle, thus corresponding, are so related, that a line drawn from $\boldsymbol{O}$ to the node passes through the circumference of the circle, cutting the plane of the circle perpendicularly; whilst every line drawn from $\boldsymbol{O}$ through the circumference of the circle is perpendicular to some nodal tangent place.

These things are evident on looking at the figure. For when $\boldsymbol{ROr}$ is a circle, it is plain that the point M is a node of the biaxal $a'b'c'$, since $\boldsymbol{OM}$ is perpendicular to the plane of the circle $\boldsymbol{ROr}$ and equal to its radius $\boldsymbol{OR}$. But we have already seen (15) that when $\boldsymbol{ROr}$ is a circle, the other biaxal abc has a circle of contact, whose plane is perpendicular to $\boldsymbol{OM}$ at the point S of its circumference. The line $\boldsymbol{OTL}$ is perpendicular, in general (11), to a tangent plane at M, and therefore perpendicular, in the present case, to a nodal tangent plane; whilst the point T, through which it passes, is on the circle of contact. It is also evident that $OT \times \boldsymbol{OL} = k^2$.

We have here an example of the general remark in the corollary of Theorem I.

17. The section made in a biaxal surface abc, by any of the principal planes of its generating ellipsoid, consists of an ellipse and a circle.

For let the plane $\boldsymbol{QOq}$ pass through one of the semiaxes a, and let it revolve round this semiaxis, while the right line OTV (9), perpendicular to the plane $\boldsymbol{QOq}$, revolves about O in the plane of the semiaxes b,c. Then the semiaxis a of the ellipsoid will always be one of the semiaxes of the ellipse $\boldsymbol{QOq}$; and if OT be equal to this semiaxis, the point T will describe a circle with the radius a about the centre O. The other semiaxis of the ellipse $\boldsymbol{QOq}$ is that semidiameter of the principal ellipse bc which lies in the intersection of the plane bc with the plane $\boldsymbol{QOq}$; and as OV is equal and perpendicular to this semidiameter, the point V describes an ellipse equal to bc, but turned round through a right angle, so that the greater axis of the ellipse described by V coincides in direction with the less axis of the ellipse bc. As the radius a of the circle is greater (4) than both the semiaxes b,c, of the ellipse, the circle will lie wholly without the ellipse.

In like manner, the section made in the biaxal surface by the plane ab consists of a circle with the radius c, and an ellipse with the semiaxes a,b; and as the radius of the circle is less than both the semiaxes of the ellipse, the circle lies wholly within the ellipse.

18. But when the section lies in the plane of the greatest and least semiaxes a,c, the circle and ellipse, of which it is composed, intersect each other. For the radius b of the circle is less than one semiaxis of the ellipse ac and greater than the other.

Leaving the ellipse *ac* in the position which it has as a section of the ellipsoid *abc*, if we describe the circle *b* with the centre **O** and radius *b*, the ellipse and the circle will cut each other in four points at the extremities of two diameters; and planes, passing through these diameters and through the semiaxis *b* of the ellipsoid, will evidently be the planes of the two circular sections of the ellipsoid. Now turning the ellipse *ac* round through a right angle (17), the circle and the ellipse in its new position will constitute the section of the biaxal surface, and will cut each other (*Fig.* 5.) in four points *n* at the extremities of two diameters *n On*, *nOn*, which are perpendicular to the two former diameters, and therefore perpendicular to the planes of the two circular sections. Consequently, the biaxal surface has four nodes at the four points *n*. These nodes, it is manifest, are alike in all their properties; and they are the only points common to the two biaxal sheets, since the points *T* and *V* (9), in the description of the biaxal surface, cannot coincide unless the section **Q** *Oq*, perpendicular to **O** *TV*, be a circle.

19. The plane of the greatest and least semiaxes *a*, *c*, of the generating ellipsoid, may be called the plane of the nodes; and the two diameters *n On*, *n On*, passing through the nodes, may be called the *nodal diameters*.

At one of the nodes *n* (*Fig.* 5) draw tangents *nf*, *nk*, to the ellipse and the circle that compose the biaxal section; and through **O** draw **O***p* perpendicular to **O***n*, cutting the circle in *p*. Then as **O***n* is perpendicular to the plane of a circular section of the ellipsoid *abc*, this circular section will have *Op* for its radius, and its circumference will cross that of the ellipse *ac* (belonging to the ellipsoid) in the point *p*. A line touching the ellipse *ac* at *p* will be parallel to every plane that touches the ellipsoid in a point of the circular section, and will therefore (6) be perpendicular to the plane which is reciprocal to the plane of the circular section. But the tangent at *p* is perpendicular to the tangent *nf*, since the two tangents would coincide if the ellipse *ac* were turned round (18) through a right angle, the point *p* then falling upon *n*. Hence the circular section and its reciprocal plane are parallel to the tangents *nk*, *nf*; and therefore two planes perpendicular to the plane of the figure and passing through these tangents, are the planes that we have called (14) the principal tangent planes at *n*.

20. Produce *Op* to meet *nf* in *v*, and conceive a parabola having its focus at *O*, its vertex at *v* (8), and its plane perpendicular to the plane of the figure. A cone, with its vertex at *n* and this parabola for its section, is (14) the nodal tangent cone.

Draw *Of* perpendicular to *nf* at *f*, and meeting *nk* in *k*. The perpendiculars let fall from *O* upon the nodal tangent planes form a cone, of which the circles described in planes perpendicular to the figure upon the diameters *nf*, *nk*, are sections (8). On the other biaxal surface *a'b'c'* there is (16) a circle of contact whose plane is perpendicular to **O***n*. This circle of contact is (16) another section of the cone last mentioned.

21. To the circle *b* and to the principal section *ac* of the ellipsoid *abc* conceive a

common tangent $d'i'$ to be drawn, in a quadrant adjacent to that which contains the node n, and let it touch the circle in d' and the ellipse ac in i'. A radius Od'', drawn through the point d' to meet the ellipsoid $a'b'c'$ in the point d'', will be reciprocal to the radius Oi', because it is perpendicular to a tangent at i', and it will be equal in length to b', because $Od'' \times Od' = k^2 = bb'$, and $Od' = b$; whence $Od'' = b'$. Therefore Od'' is in a circular section of the ellipsoid $a'b'c'$. Two planes perpendicular to the plane of the figure, and passing through the reciprocal radii Od'', Oi', are (6) reciprocal planes, and we have seen that the first of them makes a circular section in the ellipsoid $a'b'c'$. They are therefore (15) the fixed planes in the second case of Prop. V.

22. Now draw di a common tangent to the circle b and ellipse ac composing the biaxal section, and let it touch the circle in d and the ellipse in i. The lines Od, Oi, are of course perpendicular to the lines Od', Oi', and therefore perpendicular to the fixed planes just mentioned. Hence the line Od and the point d are the same as the fixed line OS and the point S in the second case of Prop. V. The plane of the circle of contact is therefore perpendicular to Od at the point d (15); and the points d and i, where its plane intersects the right lines Od, Oi, perpendicular to the fixed planes, are (8) the extremities of a diameter.

These things agree with the obvious remark, that the points of contact d and i must be points of the circle of contact; and that di must be a diameter, because the plane of the circle is perpendicular to the plane of the figure, and this latter plane divides the biaxal surface symmetrically.

As the circle and ellipse may have a common tangent opposite to each node, there are four circles of contact in planes perpendicular to the plane of the nodes.*

23. The biaxal surface belongs to a class that may be called *apsidal surfaces*, from the manner in which they are conceived to be generated.

Let G be a given surface, and O a fixed origin or pole. If a plane passing through O cut the surface G, the curve of intersection will in general have several apsides A, A', A'', &c., where the lines OA, OA', OA'', &c. are perpendicular to the curve. Through the point O conceive a right line perpendicular to the plane of the curve, and on this perpendicular take from O the distances Oa, Oa', Oa'', &c. respectively equal to the apsidal distances, OA, OA', OA'', &c. Imagine a similar construction to be made in every possible position of the intersecting plane passing through O, and the points a, a', a'', &c. will describe the different sheets of an *apsidal surface*.

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* The curves of contact on biaxal surfaces, and the conical intersections or nodes, were lately discovered by Professor Hamilton, who deduced from these properties a theory of conical refraction, which has been confirmed by the experiments of Professor Lloyd. See Transactions of the Royal Irish Academy, Vol. XVII. Part. I, pp. 132, 145; and the present paper, Art. 55—58.

The indeterminate cases of circular section—at least the case of the nodes—had occurred to me long ago; but having neglected to examine the matter attentively, I did not perceive the properties involved in it (13). *April* 2, 1834.

The apsidal surface has a centre at the point O, because the lengths Oa, Oa', Oa'', &c. may be measured on the perpendicular at either side of the intersecting plane.

Referring* to the demonstration of Theorem III. it will be seen to depend only on the supposition that the point Q is an apsis of the section made by the plane QOq; or, which is the same thing, that OQ is a position wherein the radius vector from O to the curve of section is a *maximum* or a *minimum*. Hence we have the following general theorem:—

24. PROP. VI. THEOREM. If tangent planes be applied at corresponding points A, a, on the surface G and the apsidal surface which it generates; these tangent planes will be perpendicular to each other and to the plane of the points O, A, a.

This is equivalent to saying that perpendiculars from O on the tangent planes are equal to each other, and lie in the plane of the lines OA, Oa.

25. If Q and R be reciprocal points on two reciprocal surfaces of which O is the fixed origin or pole, the tangent plane at Q will be (1) perpendicular to OR and to the plane QOR. Let a plane also perpendicular to the plane QOR pass through OQ, cutting the surface to which the point Q belongs in a certain curve, and the tangent plane at Q in a tangent to this curve. The tangent is evidently perpendicular to OQ, and therefore the point Q is an apsis of the curve.

In like manner, the point R is an apsis of the section made in the other surface by a plane passing through OR and perpendicular to the plane QOR.

26. From these observations, and from Prop. VI., it appears that if the points Q, R, in the figure of Theorem IV., be reciprocal points on *any* two reciprocal surfaces, and if the same construction be supposed to remain, the points T and M will be points on the apsidal surfaces generated by these reciprocal surfaces, and the tangent planes at T and M will be perpendicular to the lines OM and OT respectively. Also the rectangles LOT and MOS will be equal to k^2. Hence we have another general theorem:—

PROP. VII. THEOREM. The apsidal surfaces generated by two reciprocal surfaces are themselves reciprocal.

27. A very simple example of apsidal surfaces, with nodes and circles of contact, may be had by supposing the generatrix G to be a sphere, and the pole O to be within the sphere, between the surface and the centre C.

It is evident that the apsidal surface in this case will be one of revolution round the right line OC as an axis. Therefore taking for the plane of the figure (*Fig.* 6.) a plane passing through OC and cutting the sphere in a great circle of which the radius is CS, let a plane at right angles to the figure revolve about O, cutting the circle CS in the points A, A'. The section of the sphere made by the revolving plane will have only

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* Transactions of the Royal Irish Academy, Vol. XVI. Part II. p. 68.

two apsides A, A', with respect to the point O, except when the plane is perpendicular to OC. Hence if we draw the right line Oaa' perpendicular to AOA', taking Oa, Oa', always equal to OA, OA', the points a, a', will describe a section of the apsidal surface. This section will evidently consist of two circles $C'S'$; $C''S''$, equal to the circle CS, and having their centres C', C'', on the opposite sides of O in a right line $C'OC''$ perpendicular to OC; the distances OC, OC', OC'' being equal. The circles $C'S'$, $C''S''$, intersect in two points n, n', on the line OC and have two common tangents di, $d'i'$, which are bisected at right angles by OC in the points c,c'.

28. Now let the circles $C'S'$, $C''S''$, with their common tangents, or only one of the circles with the half tangents, revolve about the axis OC, and we shall have the apsidal surface with nodes at n,n', and with circles of contact described by the radii cd, $c'd'$.

The section of the sphere by a plane passing through O at right angles to On, is a circle of which O is the centre. If therefore we suppose that the point n answers to a in Prop. VI., the apsis A corresponding to n will be indeterminate, and the position of the tangent plane at n will also be indeterminate, which ought to be the case at a node.

The surface reciprocal to the sphere, the pole being at O, is evidently a surface of revolution about the axis OC (it is easily shown to be a spheroid having a focus at O); and the section of this reciprocal surface, by a plane perpendicular to the axis at O, is a circle of which O is the centre. This circumstance indicates (15) that on the apsidal surface there is a curve of contact, whose plane is parallel to the plane of circular section; which agrees with what we have already seen.

29. When the point O is without the sphere, the axis OC will pass between the circles $C'S'$, $C''S''$, without intersecting either of them. The apsidal surface, described by the revolution of one of these circles about OC, will be a circular ring. The nodes have disappeared; but the circles of contact still exist, as is evident.

PART II.—ON THE WAVE THEORY OF LIGHT.

30. Some of the foregoing propositions lead to a simple transformation of the wave theory of light.

In this theory, the *surface of waves*, or the *wave surface*, is a geometrical surface used to determine the directions and velocities of refracted or reflected rays; being the surface of a sphere in a singly refracting medium; a double surface, or a surface of two sheets, in a doubly refracting medium; a surface of three sheets on the supposition of triple refraction; and having always a centre O round which it is symmetrical. The radii of the wave surface, drawn from its centre O in different directions, represent the velocities of rays to which they are parallel.

31. We shall consider particularly the case of a doubly refracting crystal, with two plane faces parallel to each other, and surrounded by a medium of the common kind wherein the constant velocity is V; supposing, for the sake of clearness, that the crystal refracts more powerfully than the surrounding medium, so that the velocities in the crystal are less than the velocity V.

A ray $S'O$, falling on the first surface of the crystal at the point O, is partly reflected according to the common law of reflection, and partly refracted. The two refracted rays pass on to the second surface, where each of them is divided by internal reflection into a pair, the two reflected pairs being parallel to each other; while the two emergent rays—one from each refracted ray—are parallel to each other and to the incident ray $S'O$. The directions of the rays within the crystal are usually found by the following construction.

32. Describe a wave surface of the crystal, having its centre at O the point of incidence. By the nature of the wave surface, a right line OTU, drawn from the point O, will in general cut this surface in two points T, U, on the same side of O; and a ray passing through the crystal in a direction parallel to OTU will have one of the two velocities represented by the radii OT, OU, taking a line of a certain length k to represent the uniform velocity V in the external medium. With the centre O and a radius OS equal to this line k describe a sphere. As the velocities in the crystal are supposed to be less than V, the wave surface will lie wholly within this sphere. Let the plane of the figure (*Fig.* 7) be the plane of incidence, perpendicular to the parallel faces of the crystal, and intersecting the first face in the right line FA. Through the point S, where the incident ray $S'O$, produced through the crystal, cuts the surface of the sphere, draw SI at right angles to OS and meeting FA in the point I. A right line perpendicular to the plane of the figure, and passing through this point I, we shall call the right line I.

33. Through the right line I draw two planes touching the two sheets of the wave surface, on the side remote from the incident light, in the points T, T', which will lie within the sphere (32); then the incident plane wave, perpendicular to OS, will be refracted into two plane waves parallel to these two tangent planes; and the lines OT, OT', will be the directions of the refracted rays along which the refracted waves are propagated. The lengths OT, OT', represent the velocities with which the light moves along the rays; and of course the normal velocities, which are the velocities of the refracted waves, are represented by the perpendiculars OG, OH, let fall from O on the two tangent planes at T, T'. These two perpendiculars OG, OH, evidently lie in the plane of the figure; but the points T, T', in general, do not lie in this plane.

34. Again, through the right line I draw two other planes touching the wave surface, at the side of the incident light, in the points t, t'. The rays OT, OT', arriving at the second surface of the crystal, will each be divided by internal reflection into two rays parallel to Ot, Ot'; and these four reflected rays, arriving at the first surface, will each be divided,

by a new reflection, into two rays parallel to OT, OT'; and so on, for any number of reflections. Any of the rays emerging at the first surface after internal reflections, is parallel to the ray Os produced by ordinary reflection at the point of incidence; and any ray emerging at the second surface is parallel to the incident ray $S'OS$.

35. This construction may be changed into another that will be found more convenient both in theory and practice.

Through S draw SR perpendicular to OI, and meeting OG, OH, produced, in the points P, M. Then as the angles at G and R are right angles, the points I, R, G, P, are in the circumference of a circle, and therefore $OP \times OG = OI \times OR = OS^2 = k^2$; and similarly, $OM \times OH = k^2$. If then we take O for the fixed origin, or pole, and k^2 for the constant rectangle (Theorem I.), and describe the surface which is reciprocal to the wave surface, it is evident that the points P and M will be points of the surface so described, and that OT, OT', will coincide in direction with perpendiculars let fall from O on planes touching the surface at P and M, and will be inversely proportional to these perpendiculars. It follows in the very same manner, that if perpendiculars Og, Oh, let fall from O on the tangent planes at t, t', be produced to meet SR in the points p, m, these points will also be on the surface reciprocal to the wave surface.

In the present case, it is manifest that this reciprocal surface lies wholly without the sphere OS.

36. The surface reciprocal to the wave surface, the pole being at O, we shall call the *surface of refraction.* is but the index surface [illegible]

It is hardly necessary to observe that the surface of refraction has a centre at the point O, round which it is symmetrical; that it is a sphere in a singly refracting medium, a double surface in a doubly refracting medium, and a surface of three sheets if we suppose a case of triple refraction.

37. In the case that we are considering, let the figure (*Fig.* 8.) represent a section made in the double surface of refraction and its attendant sphere by the plane of incidence. Through the point S, where the incident ray $S'O$ prolonged cuts the circular section of the sphere, draw SR perpendicular to the face of the crystal, or to FA; and let SR produced cut the circle again in the point s. Then Os is the direction of the ray given by ordinary reflection at the first surface of the crystal. Produce the right line SRs both ways, to cut the surface of refraction in the points P, M, behind the crystal, and in the points p, m, before it; and conceive planes to touch the surface of refraction at the points P, M, p, m. Suppose also that perpendiculars OP', OM', Op', Om', are let fall from O upon these tangent planes, and that they intersect the planes in the points P', M', p', m', respectively.

Then from the preceding observations (33, 34, 35), it is manifest that OP', OM', are the directions of the rays into which $S'O$ is divided by refraction; that each of these refracted rays, on arriving at the second surface of the crystal, is divided by internal reflection into two rays parallel to Op',Om'; and that each of the four reflected rays, on arriving at the first surface, is again divided by reflection into two rays parallel to OP',OM'; and so on. In general, every ray going into the crystal from the first surface, whether after refraction or after any even number of internal reflections, is parallel either to OP' or to OM'; and every ray returning from the second surface of the crystal after any odd number of internal reflections, is parallel either to Op' or to Om'. Thus the direction of every ray in the interior of the crystal is the same as the direction of some one of the four lines OP',OM', Op',Om'; and the velocity of the ray is inversely as the length of this line; so that the velocity of the ray OM', for example, or of any ray parallel to OM', is to the velocity V as OS is to OM'. The little plane waves that, keeping always parallel to themselves, move along these rays, are respectively perpendicular to the lines OP,OM, Op,Om; and the lengths of these lines are inversely as the velocities of the waves estimated in directions perpendicular to their planes; so that the velocity of the wave which moves along the ray OM', or along any parallel ray, is to the velocity V as OS is to OM.

38. The ray OP' and all the rays parallel to it are perpendicular to the plane which touches at P the surface of refraction; and the waves which move along these rays are perpendicular to the right line OP. Any ray of this set may be called a *ray* P, and any of the waves a *wave* P. In like manner, the *rays* M,p,m, are rays that are perpendicular to the tangent planes at the points M,p,m, respectively; and the *waves* M,p,m, are the waves that belong to these rays, and that have their planes respectively perpendicular to the right lines OM,Op,Om. The rays P,M, all come from the first surface of the crystal; the rays p,m, from the second.

As the ordinates RP,Rp, are greater than the ordinates RM,Rm, so the rays P,p, are more refracted or more reflected than the rays M,m. The former rays may therefore be said to be *plus refracted*, or *plus reflected*, and the latter to be *minus refracted*, or *minus reflected*. Or,—for the convenience of naming,—the rays P,p, may be called *plus rays*; and the rays M,m, *minus rays*. The waves P,p, in like manner, may be termed *plus waves*, and the waves M,m, *minus waves*.

For a medium of the common kind, or a singly refracting medium, we may use the letters S and s. Thus the incident ray $S'OS$, or any ray emerging parallel to OS from the second surface of the crystal, may be marked by the letter S; while the ray Os produced by common reflection, or any ray emerging parallel to Os from the first surface, may be denoted by the letter s.

39. The course of a ray through the crystal may now be easily expressed. A ray $SMps$, for example, is a ray (S) incident on the crystal, undergoing minus refraction (M) at the first surface, plus reflection (p) at the second, and emerging (s) from the first surface in a direction parallel to Os. Of this ray the part within the crystal is Mp. A ray SPS is a ray plus refracted, and then emerging in a direction parallel to that of incidence. A ray $SPpMS$ is a ray plus refracted at the first surface, then

plus reflected at the second surface, then minus reflected at the first surface, and finally emerging from the second surface in a direction parallel to that of incidence. Its path within the crystal is ***PpM***.

These examples indicate the general method of expressing the path of a ray.

40. Suppose light to be moving in the same direction and with the same velocity along two proximate parallel rays, so that it is at the point A in one ray when it is at the point B in the other; and through the points A and B conceive two planes perpendicular to the common direction of the rays. These planes are either coincident, or maintain a constant distance. In the first case, the rays are said to be in complete accordance. In the second case, the constant distance between the planes is called the *interval* between the portions of light composing the rays, or the interval between the waves that move along the rays.

We proceed to find the lengths of these intervals in the case of rays emerging parallel to each other, at either side of the crystal that we have been hitherto considering.

41. Let the tangent planes at P, M, p, m, intersect the plane of the figure (*Fig.* 8) in the right lines $PP_{,}$, $MM_{,}$, $pp_{,}$, $mm_{,}$, which of course are tangents to the section of the surface of refraction represented in the figure; let a perpendicular at O to the face of the crystal cut these tangents in the points $P_{,}$, $M_{,}$, $p_{,}$, $m_{,}$; and let the lines OP'', OM'', Op'', Om'', respectively parallel to $PP_{,}$, $MM_{,}$, $pp_{,}$, $mm_{,}$, cut the line SRs in the points P'', M'', p'', m''.

The length of the path which a ray P describes within the crystal, is equal to the thickness Θ of the crystal divided by the cosine of the angle $P'OP_{,}$ which the path of the ray makes with a perpendicular to the faces of the crystal; and the velocity of P is equal to $V \times \frac{OS}{OP'}$ (37); dividing therefore the length of the path by the velocity, we find that the time in which a ray P crosses the crystal is equal to $\frac{\Theta \times OP'}{V \times OS \times \text{Cos } P'OP_{,}}$.

But as OP' is perpendicular to the tangent plane at P, we have $\frac{OP'}{\text{Cos } P'OP_{,}} = OP_{,} = PP''$. Therefore the time is equal to $\frac{\Theta \times PP''}{V \times OS}$. Similarly, the times in which rays M, p, m, pass from one surface of the crystal to the other, are equal to $\frac{\Theta \times MM''}{V \times OS}$, $\frac{\Theta \times pp''}{V \times OS}$, $\frac{\Theta \times mm''}{V \times OS}$, respectively.

42. Now suppose the path of a ray P to be be projected perpendicularly on a right line having any proposed direction in space. Through O conceive a right line OL parallel to the proposed direction, and meeting in L the tangent plane at P. The length of the projection is equal to the length of the path multiplied by the cosine of the angle $P'OL$ which the ray P makes with OL; that is, the projection is equal to $\Theta \frac{\text{Cos } P'OL}{\text{Cos } P'OP_{,}}$. But because OP' is perpendicular to the tangent plane at P, we have $\text{Cos } P'OL = \frac{OP'}{OL}$, and $\text{Cos } P'OP_{,} = \frac{OP'}{OP_{,}} = \frac{OP'}{PP''}$; therefore $\frac{\text{Cos } P'OL}{\text{Cos } P'OP_{,}} = \frac{PP''}{OL}$. Hence the projection is equal to $\Theta \frac{PP''}{OL}$.

If the path of a ray P be projected on the incident ray OS, then producing OS to meet $PP_{,}$ in l, we see, by what has just been proved, that the length of the projection is equal to $\Theta\,\frac{PP''}{Ol} = \Theta\,\frac{SP''}{OS}$, by similar triangles. In like manner, the projections of the paths of rays M, p, m, on the direction of the incident ray OS, are equal to $\Theta\,\frac{SM''}{OS}$, $\Theta\,\frac{Sp''}{OS}$, $\Theta\,\frac{Sm''}{OS}$, respectively.

43. Let each rectilinear path be measured in the direction in which the light moves along it; and according as the direction so measured makes an acute or an obtuse angle with the direction OS, measured from O to S, let the projection of the path on OS be reckoned positive or negative. Then if $SPmMpMS$ be any ray entering the crystal at O, and emerging from its second surface at E, and if a perpendicular EI be let fall from E upon OS, meeting OS in I; the distance OI, from O to the foot of this perpendicular, will evidently be equal to the algebraic sum of the projections of the paths P, m, M, p, M, contained within the crystal; taking each projection with its proper sign. It is obvious that the projections of the P and M rays are always positive. And as the lines Op', Om',—the directions of the rays p, m,—lie in planes which are respectively perpendicular to $pp_{,}$, $mm_{,}$, or to Op'', Om'', it is easy to see that these directions make acute or obtuse angles with OS, according as the points p'', m'', lie below the point S or above it; that is, the projections are positive or negative according as the points p'', m'', lie without the circle OS towards P, M, or within the circle. Therefore the distance OI, in the case of the figure, is equal to $\frac{\Theta}{OS}\,(SP''—Sm''+SM''—Sp''+SM'')$. Error

44. If the paths of rays P, M, p, m, be projected on the direction Os of the ordinarily reflected ray, the lengths of their projections will be $\Theta\,\frac{sP''}{OS}$, $\Theta\,\frac{sM''}{OS}$, $\Theta\,\frac{sp''}{OS}$, $\Theta\,\frac{sm''}{OS}$, respectively. The projections upon Os of the rays p, m, will be always positive; and the projections of the rays P, M, will be positive or negative according as the points P'', M'', lie above the point s or below it; that is, according as the points P'', M'', lie without the circle OS towards p and m, or within the circle. So that if $SPmMps$ be a ray entering the crystal at O and emerging from the first surface at e, and if a perpendicular ei be let fall from e upon Os, the distance Oi from the point O to the foot of this perpendicular, or the algebraic sum of the projections of the paths P, m, M, p, contained within the crystal, will be equal to $\frac{\Theta}{OS}\,(—sP''+sm''—sM''+sp'')$, in the case of the figure.

45. Let us imagine that the light in the incident ray $S'O$, instead of being interrupted at O by the crystal, had continued to move with the same velocity V in the same right line OS, leaving the point O at the moment when the refracted light enters the crystal at O. Comparing the light in this imaginary ray with that in a ray emerging parallel to it from the second surface of the crystal, after an even number of internal reflections, we shall find that the emergent is behind the imaginary ray, and

that the interval between them (40),—or the retardation of the former,—may be derived very easily from the letters that designate that ray. Let $SPmMpMS$ be any such ray. The sum of the distances of the point S from each of the points marked by the letters ($PmMpM$) that denote (39) the part of the ray contained within the crystal, is proportional to the interval of retardation; that interval being equal to $\frac{\Theta}{OS}(SP+Sm+SM+Sp+SM)$. Cura

For if from the point E, where the last internal ray M emerges from the second surface of the crystal, a perpendicular EI be let fall upon OS, meeting OS in I, the time of describing OI with the velocity V would (43) be $\frac{\Theta}{V\times OS}(SP''-Sm''+SM''-Sp''+SM'')$. But (41) the actual time of describing the broken path $PmMpM$ is $\frac{\Theta}{V\times OS}(PP''+mm''+MM''+pp''+MM'')$; and, on inspecting the figure, this time is seen to be greater than the time of describing OI, by $\frac{\Theta}{V\times OS}(SP+Sm+SM+Sp+SM)$, or by the time in which the line $\frac{\Theta}{OS}(SP+Sm+SM+Sp+SM)$ would be described with the velocity V. Consequently, at the moment when the light in the ray $SPmMpMS$ emerges at the point E from the second surface of the crystal, the light in the imaginary uninterrupted ray OS will have passed the point I by an interval equal to the line just mentioned; and as the two rays afterwards have the same velocity and parallel directions, this interval is the retardation of the emergent ray.

46. The rays emerging from the first surface after any odd number of internal reflections are to be compared with the ordinarily reflected ray Os to which they are parallel; the light in Os, which moves with the velocity V, being supposed to leave O at the moment when the refracted light enters the crystal at O. The mode of proceeding in this case is exactly similar to that in the last, and the interval is determined in the same way, using s in place of S; the retardation of the ray $SPmMps$, for example, of which the part $PmMp$ is contained within the crystal, being equal to $\frac{\Theta}{OS}(sP+sm+sM+sp)$.*

47. It is remarkable that the preceding demonstration nowise depends upon the supposition that the planes perpendicular to the rays P,M,p,m, are tangent planes to the surface of refraction at the points P,M,p,m. If we had supposed any planes—different from the plane of the figure— to pass through the points P,M,p,m, and the rays to coincide in direction with perpendiculars let fall from O upon these planes, and to have velocities inversely proportional to the lengths of the perpendiculars, the intervals of retardation would have remained unchanged. Hence the retardations are the same as if the lines OP, OM, Op, Om, were the directions of the rays in passing

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* The *change of phase*, which may take place at a surface of the crystal, is not here considered as affecting the intervals.

through the crystal; as will appear by conceiving the planes that we have spoken of to be perpendicular to these lines.

If the incident ray $S'O$ were refracted in the ordinary way with an index equal to $\frac{OP}{OS}$, it would take the direction OP; if it were refracted, in like manner, with the index $\frac{OM}{OS}$, it would take the direction OM; and if the two rays, thus ordinarily refracted, were to emerge from the second surface of the crystal in directions parallel to OS, it is evident from what has been said, that they would be in complete accordance, respectively, with the rays SPS and SMS.

If the surface of refraction should happen to have a node N, which is a point of intersection where it admits an infinite number of tangent planes (3), let the direction of the incident ray $S'OS$ be chosen, so that the right line RS perpendicular to the face of the crystal, being produced below S, may pass through N, and we shall have a cone of refracted rays formed by the perpendiculars let fall from O upon the tangent planes at N; all of which rays, on emerging parallel to OS from the second surface of the crystal, will be in complete accordance with one another. For we have just seen that if the ray $S'OS$ were supposed to emerge after being refracted in the ordinary way with an index equal to $\frac{ON}{OS}$, it would be in complete accordance with any ray of the cone.

48. The interval between any two rays emerging at the same side of the crystal is the difference of their retardations. In taking the difference, the letters that are common to the names of the two rays may be left out. Thus the ray $SPmMS$ is behind the ray SPS by the interval $\frac{\Theta}{OS}(Sm + SM) = \frac{\Theta}{OS} Mm$. The line $\frac{\Theta}{OS} Pp$ is the interval between the rays SMS and $SMpPS$, or between the reflected ray Os and the ray $SPps$; and so on.

49. The retardations of the two refracted rays SPS and SMS, emerging without internal reflection, are $\frac{\Theta}{OS} SP$ and $\frac{\Theta}{OS} SM$ respectively. The difference of these is $\Theta \frac{PM}{OS}$. Consequently, when the two refracted rays have emerged from the second surface in directions parallel to the incident ray, the light in the plus emergent ray is behind the light in the minus emergent ray by an interval equal to $\frac{\Theta \times PM}{OS}$. Or, in other words, the incident plane wave, perpendicular to OS, produces two emergent waves parallel to each other and to the incident wave, moving along the emergent rays with equal velocities V, and preserving the distance $\frac{\Theta \times PM}{OS}$ between their planes, the minus wave being foremost. If OS, the radius of the sphere, be taken for unity, PM will be a number,—generally a very small fraction,—and the interval will be the thickness of the crystal multiplied by this number.

50. Suppose the right line PMR, remaining always perpendicular to the face of the crystal, to describe a cylindrical surface, with the condition that the part PM, inter-

cepted between the two sheets of the surface of refraction, shall remain of a constant length; the point R will then describe, on the surface of the crystal, a curve whose radii OR are the sines (to the radius OS) of the angles of incidence of a cone of rays; and every ray $S'O$ of this cone, when refracted by the crystal, will afford two emergent rays, or two waves, having the same given interval between them. Lines drawn from the eye parallel to the sides of this cone are the emergent rays belonging to a ring, when rings are made to appear, in any of the usual ways, on transmitting polarised light through the plate of crystal. In nominal conformity to this, we see that the line PM describes a ring of constant breadth between the two sheets of the surface of refraction. The ring described by supposing pm to remain constant corresponds to the interval between two rays p and m reflected at the same point of the second surface of the crystal, and then emerging at the first. The other intercepts Pp, Mm, Pm, Mp, are proportional (48) to intervals like those in Newton's rings; to the intervals, namely, between the reflected ray Os and the rays $SPps$, $SMms$, $SPms$, $SMps$, emerging at the first surface after one reflection within the crystal; or to the intervals between rays that are twice reflected in the crystal and the rays transmitted without reflection.

51. The general investigation of the figure of a geometrical ring does not distinguish between the different intercepts, and will therefore include all the rings PM, pm, Pp, Mm, Pm, Mp; so that it will be sufficient to contemplate any one of them, as PM, of which the breadth PM is equal to a given line I.

The points P and M describe, in general, similar and equal curves of double curvature, which may be called *ring-edges*, as being the edges of the ring; and if we imagine the surface of refraction, carrying these curves along with it, to be shifted either way, in a direction parallel to PM, through a distance equal to I, it is clear that the new position of one of the ring-edges will exactly coincide with the first position of the other, and that therefore the curve of the latter ring-edge will be given by the intersection of the two equal surfaces in these two positions. Let $U=0$,—where U is a function of x, y, z, and given quantities—be the equation of the surface of refraction in its original position; and, the axes of coordinates being fixed, suppose that by the shifting of the surface the coordinates of a point assumed on it are diminished by the given lines f, g, h, which are the projections of the given line I on the axes of x, y, z, respectively. Then the equation of the surface in its new position will be had by substituting $x+f$, $y+g$, $z+h$, for x, y, z, in the equation $U=0$, which will thus become $U+V=0$, where V is the increment of U produced by the substitution. These two equations combined are equivalent to the equations $U=0$, $V=0$, which are therefore the equations of one of the ring-edges. If the surface had been shifted the opposite way, in a direction parallel to PM, the intersection would have been the other ring-edge, whose equations are therefore deducible from those already found, by changing the signs of f, g, h.

52. If the equation of the surface of refraction be transformed, so that the plane of xy may coincide with the face of the crystal, and the axis of z be perpendicular to it, the origin of coordinates being at the centre O, no change will be produced in x or in y by the motion of the surface, because PM, the direction of the motion, is now parallel to the axis of z; but z will be diminished or increased by I; and accordingly, if $U'=0$ be the equation of the surface in its first position, when the centre is at O, and if U' become $U'+V'$ when z becomes $z+I$,—the equation of the surface in its second position, when the centre has moved through a distance equal to I along the axis of z, will be $U'+V'=0$; and these two equations combined will give $U'=0$, $V'=0$, for the equations of one of the ring-edges. The equations of the other ring-edge are deduced from these by changing the sign of I.

The projection of each of the ring-edges on the plane of xy is the curve traced by the point R on the surface of the crystal (50). This curve may be called a *ring-trace.* Its equation is obtained by eliminating z between the equations of a ring-edge; and as the result must be the same whether I be taken positive or negative, the equation of the ring-trace, when found by this general method, will contain only even powers of I. The radii drawn from O to the points R of the ring-trace, are (50) the sines (to the radius OS,) of the angles of incidence or emergence of the rays that form an optical ring; the rays that come from this ring to the eye being parallel to the sides of the cone described by the right line $S'OS$ while the point R describes the ring-trace.

53. It is evident that tangents to the ring-edges, at the points P and M, are parallel to each other, and therefore parallel to the intersection of two planes touching the surface of refraction at P and M, because these tangent planes pass through the tangents. But the directions OP', OM', are perpendicular to the tangent planes, and therefore the plane $P'OM'$, containing the two rays, is perpendicular to the intersection of the tangent planes, and of course perpendicular to the parallel tangents. Hence the plane $P'OM'$ intersects the face of the crystal in a right line perpendicular to the projection of the parallel tangents on the face of the crystal. As this projection is a tangent to the curve described by R, it follows that the normal to the ring-trace at the point R is parallel to the line joining the points in which the two refracted rays cut the second surface of the crystal.

In like manner, taking any two consecutive rays (P and m) having a common extremity on one surface of the crystal, the line joining the points where these rays cut the other surface, is parallel to the normal at the point R of the ring-trace which is described when the intercept (Pm) between the letters that mark the rays is supposed to remain constant.

54. In all that precedes we have made no supposition about the surface of refraction except that it is a surface of two sheets; and if we supposed it to have three sheets, the conclusions would be easily extended to this hypothesis.

In the theory of Fresnel, the wave surface is* a biaxal whose generating ellipsoid has its centre at the point O, and its semiaxes parallel to the three principal directions of the crystal, the length of each semiaxis being equal to OS divided by one of the principal indices of refraction. The surface of refraction is reciprocal to the wave surface, and is (11) therefore another biaxal generated by an ellipsoid reciprocal to the former, having its centre at the same point O, and the directions of its semiaxes the same as before, the rectangle under each coincident pair of semiaxes being equal to k^2 or OS^2. Hence the semiaxes of the ellipsoid which generates the biaxal surface of refraction are equal in length to OS multiplied by each of the three principal indices. This biaxal surface is of course to be substituted for the surface of refraction in the preceding observations.

55. When the line RS, produced below S, passes through a node N of the biaxal surface of refraction, the points P, M, coincide in the point N, and the interval PM vanishes. At the point N there are an infinite number of tangent planes, and the perpendiculars from O on these tangent planes give a cone of refracted rays whose sections we have already shown how to determine (20). All the rays in this cone, on arriving at the second surface of the crystal, emerge parallel to the incident ray OS; and if the rays in the emergent cylinder be cut by a plane perpendicular to their common direction, they will all arrive at this plane at the same instant, because the interval PM vanishes. See art. 47.

56. Suppose fig. 5 to be a section of the wave surface. The right line Od will pass through N; and the circle of contact, described on the diameter di in a plane perpendicular to the right line OdN, will be a section of the refracted cone. Now it will be recollected† that, in general, the vibrations of a ray OT, which goes to any point T of the wave surface, are parallel to the line which joins the point T with the foot of the perpendicular let fall from O on the tangent plane at T. In the present case, the perpendicular is the same for all the rays of the refracted cone, and its extremity coincides with the point d : so that the line dT, drawn from d to any point T of the circle of contact, is parallel to the vibrations of the ray OT which passes through T. Conceive, therefore, a plane perpendicular to ON at the nodal point N. This plane will cut the refracted cone in a circle whose circumference will pass through N; and a line NT', drawn from the node to any other point T' of the circumference, will be the direction of the vibrations in a ray OT' which crosses the circle at this point. The plane of polarisation is perpendicular to the direction of the vibrations.

57. The transverse section of the emergent cylinder is always a very small ellipse, affording a hollow pencil of parallel rays in complete accordance (55). If the crystal be thin, this ellipse will be of evanescent magnitude. Hence the line OS will be the direction of a line drawn from the eye to the centre of the rings commonly observed

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* Trans. R. I. A. Vol. XVI. p. 76.

† Ibid.

(50) with polarised light; or it will be what is called the apparent direction of one of the *optic axes.* The diameter passing through N will be the direction of the optic axis within the crystal. There are therefore two optic axes, parallel to the two nodal diameters (19) of the surface of refraction.

As ON is equal to the mean semiaxis of the generating ellipsoid, or to the mean index of refraction, when OS is unity, it follows that the apparent direction of an optic axis is the direction of an incident ray, which, if refracted in the ordinary way, with an index equal to the mean index of refraction, would pass along a nodal diameter of the surface of refraction.

58. We have seen (15) that there is a circle of contact on the biaxal surface of refraction. If an incident ray $S'OS$ be taken, cutting the sphere in S, so that the line RS produced may pass through the circumference of this circle, it is manifest that the direction of the refracted ray will be the same through whatever point Π of the circumference the line RS may pass, because that direction is perpendicular to the tangent plane at Π, which is in fact the plane of the circle itself. If, therefore, the line RS move parallel to itself along the circumference of the circle, cutting the sphere in a series of points S, every incident ray $S\ OS$ which passes through a point S so determined, will be refracted into two rays of which one will have a fixed direction in the crystal, being perpendicular to the plane of the circle of contact, and therefore coinciding (16) with nOn, one of the nodal diameters of the wave surface. But though the direction On of the refracted ray is fixed, its polarisation changes with the incident ray from which it is derived; for if Π be the point in which the line RS, corresponding to any position of the incident ray, crosses the circle of contact, the vibrations of the refracted ray On will be contained in the plane of the lines On, $O\Pi$, and will be perpendicular to $O\Pi$. Conceive a circle described on the diameter nf in a plane perpendicular to the figure (*Fig.* 5). This circle, and the circle of contact on the surface of refraction, are (20) sections of the same cone. Let Π' therefore be the point at which $O\Pi$, in any position of the incident ray, crosses the circumference of the circle nf; and the line $\Pi'n$, drawn to the node of the wave surface, will be the corresponding direction of the vibrations in the ray On.

59. With regard to the general law of polarisation in the theory of FRESNEL, it may be observed, that if the ellipsoid abc which generates the biaxal surface of refraction be cut by a plane perpendicular to OP, the vibrations of the ray P will be parallel to the greater axis of the section, and therefore the plane of polarisation will pass through OP and the less axis; whence it is easy to show that the plane of polarisation of a ray P bisects one of the angles made by two planes intersecting in OP and passing through the nodal diameters of the surface of refraction; the bisected angle being that which contains the least semiaxis c of the generating ellipsoid. The plane of polarisation of the ray p is found in like manner. But for the rays M, m, the angle to be bisected is that which contains within it the greatest semiaxis a.

If OP' be perpendicular to a tangent plane at P, the vibrations of the ray P will be perpendicular to OP and will lie in the plane POP'. A similar remark applies to the rays M, p, m.

60. When two semiaxes a,b, of the ellipsoid abc become equal, it changes into a spheroid aac described by the revolution of the ellipse ac about the semiaxis c; and the biaxal aac, generated by this spheroid, is * composed of a sphere whose radius is a, and a concentric spheroid acc described by the revolution of the ellipse ac about the semiaxis a; so that, the diameter of the sphere being equal to the axis of revolution of the spheroid, the two surfaces touch at the extremities of the axis. This combination of a sphere and a spheroid is the surface of refraction for uniaxal crystals. In these crystals, therefore, the refracted ray whose direction is determined by the intersection of the right line RS with the surface of the sphere follows the ordinary law of a constant ratio of the sines, and is called the *ordinary* ray; whilst the other, whose variable refraction is regulated by the intersection of RS with the spheroid, is called the *extraordinary* ray. And hence uniaxal crystals are usually divided into the two classes of positive and negative, according to the character of the extraordinary ray; being called *positive* when it is the *plus* ray, and *negative* when it is the *minus* ray. The first case evidently happens when the spheroid is oblate, and therefore lies without the sphere described on its axis; the second, when the spheroid is prolate, and therefore lies within the sphere. The second case, (which is that of Iceland spar,) may be supposed to be represented in the figure (*Fig.* 8), where the elliptic section of the spheroid, made by a plane of incidence oblique to the axis, lies within the circular section of the sphere, and the minus ray is of course the extraordinary one.

61. Let PM, preserving a constant length I, move parallel to itself between the surfaces of the uniaxal sphere and spheroid, so as to form a ring (50). Then supposing the spheroid, with the ring-edge described on it by the point M, to remain fixed, imagine the sphere, carrying the ring-edge P along with it, to move parallel to PM, from P towards M, through a distance equal to I, and the two ring-edges will exactly coincide.

Hence the *uniaxal ring-edge* is the intersection of a sphere and a spheroid, the diameter of the sphere being equal to the axis of revolution of the spheroid, and the line joining their centres being perpendicular to the faces of the crystal and equal to the breadth I of the ring. And the projection of this intersection, on a plane perpendicular to the line joining the centres of the sphere and the spheroid, is the *uniaxal ring-trace.*

62. The *biaxal ring-edge* is (51) the intersection of two equal biaxal surfaces similarly posited, the line joining their centres being perpendicular to the faces of the

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* Trans. R. I. A. Vol. XVI. p. 77.

crystal and equal to the breadth of the ring. And the projection of this intersection, on a plane perpendicular to the line joining the centres of the surfaces, is the *biaxal ring-trace.**

* In applying the general theory (51, 52) to biaxal rings, it is necessary to know the equation of a biaxal surface, which may be found in the following manner. Let r, r', r'', be three rectangular radii of the generating ellipsoid abc, the two latter being the semiaxes of the section made by a plane passing through them; so that if from the centre O two distances OT, OV, equal to r', r'', be taken on the direction of r, the points T and V will belong (9) to the biaxal surface; and let a plane parallel to the plane of r', r'', and touching the ellipsoid, cut the direction of r at the distance p from the centre. Then if r make the angles α, β, γ, with the semiaxes a, b, c, we shall have, by the nature of the ellipsoid.

$$\frac{1}{r^2} = \frac{\cos^2\alpha}{a^2} + \frac{\cos^2\beta}{b^2} + \frac{\cos^2\gamma}{c^2},$$

$$p^2 = a^2\cos^2\alpha + b^2\cos^2 b + c^2\cos^2\gamma.$$

Now since the sum of the squares of the reciprocals of three rectangular radii of an ellipsoid is constant, as well as the parallelopiped described on three conjugate semidiameters, we have the equations

$$\frac{1}{r^2} + \frac{1}{r'^2} + \frac{1}{r''^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2},$$

$$p^2 r'^2 r''^2 = a^2 b^2 c^2;$$

Or,

$$\frac{1}{r'^2} + \frac{1}{r''^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \left(\frac{\cos^2\alpha}{a^2} + \frac{\cos^2\beta}{b^2} + \frac{\cos^2\gamma}{c^2}\right) = M,$$

$$\frac{1}{r'^2 r''^2} = \frac{a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma}{a^2 b^2 c^2} = N.$$

Whence it appears that r', r'', are the values of ρ in the equation

$$\frac{1}{\rho^4} - \frac{M}{\rho^2} + N = 0,$$

in which ρ denotes indifferently either semidiameter, OT or OV, of the biaxal surface. Therefore putting for M and N their values, and writing $\frac{x}{\rho}$, $\frac{y}{\rho}$, $\frac{z}{\rho}$, instead of $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and $x^2+y^2+z^2$ instead of ρ^2, we obtain, for the equation of the biaxal surface,

$$(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2) - a^2(b^2+c^2)x^2 - b^2(a^2+c^2)y^2 - c^2(a^2+b^2)z^2 + a^2b^2c^2 = 0.$$

This is the equation of the surface of refraction for a biaxal crystal in which a, b, c, are (54) the three principal indices of refraction, taking OS the radius of the sphere to be unity. The left-hand member of the equation is therefore the expression supplied by the theory of FRESNEL for the function U in art. 51.

When the faces of the crystal are parallel to any of the principal planes of the ellipsoid,—to the plane of xy for example,—the nature of the ring-trace may be found very easily. For if the difference of the two values of z, deduced from the preceding equation of the surface of refraction, be put equal to a constant quantity I, the result, when cleared of radicals, will be an equation of the fourth degree in x and y, which will be the equation of the corresponding ring-trace. This is a case that occurs frequently in practice; the crystal being often cut with its faces perpendicular to the axis of x or of z, because these lines bisect the angles made by the optic axes.

An account of a new Fulminating Silver, and its application as a Test for Chlorine, &c. By EDMUND DAVY, F. R. S., M. R. I. A., &c., *Professor of Chemistry to the Royal Dublin Society.*

Read May 23, 1831.

I LATELY had the honor of sending to the Royal Irish Academy, a paper of mine, "On a new Acid and its Combinations," published by the Royal Dublin Society. Whilst it was printing, I found that some of the compounds therein described, spontaneously exploded when brought in contact with chlorine gas. Such unexpected results appeared sufficiently interesting to merit further inquiry: accordingly, I instituted a series of experiments on the subject; and as they have led to a number of new facts which may admit of useful applications, I venture to submit some of them to the consideration of the Academy. In the present communication, I purpose to give a brief account of a new fulminating silver I have obtained, and of the principal experiments which illustrate its efficacy as a test for chlorine.

1. *Modes of making the new Fulminating Silver; its properties and composition.*

This compound may be readily prepared from Howard's well-known fulminating mercury by the following simple processes:—Put any quantity (suppose from ten to fifty grains or more,) of Howard's compound, either in a dry or moist state, into a phial, with about half an ounce of pure water, and about twenty grains of zinc filings, or granulated zinc, for every ten grains of Howard's compound used. Cork the phial and occasionally agitate its contents, for about twenty or thirty minutes. Less time will answer if the water is moderately warm. The fluid is fulminate of zinc; filter and treat it with nitrate of silver, the white precipitate which occurs is the new fulminating silver. This substance may also be easily made, by adding nitrate of silver to an aqueous solution of any of the soluble fulminates I have described in the paper already alluded to, or from Howard's fulminating silver by the means recommended above, in the case of the fulminating mercury; and after being collected on a filter, washed and dried, either in the open air or at a heat not exceeding 212° Faht. its properties appear to be the same.

The new fulminating silver, in drying on the filter, breaks into small lumps, which are easily separated from the paper, and reduced by the gentle pressure of a platina spatula, or of a card, to an impalpable powder. Its colour, when recently prepared, is white; and if dried in the dark, or in a weak light, it retains this colour for some time. On being exposed to a strong light or the sun's rays in a moist state, it soon undergoes progressive changes of colour; from greyish white to yellowish brown, brick red, blackish brown, and even black. These changes are facilitated by the presence of water, and they appear to be connected with the partial decomposition of the compound; for on exposing a little of it in water for some time to the action of the solar rays, I observed, by means of a magnifying glass, minute specks of metallic silver on the surface of the water, apparently carried there by small globules of gas, which were occasionally evolved from the compound.

The new fulminating silver, when heated to about 350° Faht. assumes a darkish tint and explodes, producing a large yellowish flame, and a loud report. It also explodes by percussion, when struck with a hammer on an anvil; and by friction, when it is rubbed between two hard surfaces; hence, it must be handled with caution.

It is insoluble, or very nearly so, in cold water. I tried in vain to dissolve a single grain of the dry compound in about eleven fluid ounces of pure water; after some days, the greater part was undissolved. The water, however, became slightly turbid, acquired a metallic taste, and on evaporating some of it to dryness, a very minute portion of the compound remained. Boiling water dissolves a certain limited quantity of the compound; but as the fluid cools, the greater part separates in minute crystals, which are principally long needles, intersecting each other.

The new fulminating silver, in a dry state and even whilst it yet retains moisture, instantly explodes when brought in contact with pure chlorine gas; and also when this gas is mixed with most other gases, as will be presently stated.

Muriatic acid decomposes the new fulminating silver, readily, converting it into chloride of silver, whilst hydrocyanic acid is evolved; and if the experiment is continued to dryness, sal-ammoniac rises in vapour, and the pure chloride of silver remains.

Strong nitric acid gradually decomposes the new fulminating silver with evolution of gas; but if diluted, fulminic acid is slowly disengaged, and nitrate of silver formed.

Strong sulphuric acid readily explodes the new compound; but when diluted, the fulminic acid is gradually evolved, and sulphate of silver produced.

The new compound is soluble, to a certain extent, in liquid ammonia, and as the alkali evaporates, minute crystals are deposited, which explode by heat, but not in chlorine gas. Solutions of the fixed alkalies dissolve a portion of the new compound, and form crystallized fulminating compounds, which I have not examined.

The new fulminating silver readily explodes by the electric spark, producing a loud report, and a reddish flame. I repeatedly succeeded in the experiment, by simply placing a little of it on the prime conductor, bringing a brass ball near it, and putting the machine in motion. I found that the very feeble charge remaining on the prime conductor, after giving the machine a few turns, was quite sufficient to explode it.

The new fulminating silver is decomposed by a number of metals, as zinc, iron, copper, &c. in cases when it is put into water, and these metals severally introduced; new fulminating compounds of each metal are in a little time produced.

As the new fulminating silver may be formed from Howard's fulminating silver, and may be readily converted into chloride of silver, (as has been stated;) little difficulty was anticipated in ascertaining its composition; and yet from the precautions to be observed, I made several experiments, (using both methods,) before I obtained any satisfactory results. In converting Howard's fulminating silver into the new compound, a given weight of the former, well dried, was very cautiously put into a small phial, (nearly filled with pure water,) with about twice its bulk of fine zinc filings. The contents of the phial were occasionally agitated gently, and after some hours the fluid, (fulminating zinc,) was filtered, and carefully treated with solution of crystallized nitrate of silver, until no farther precipitate took place. The new compound thus produced was then thrown on a filter, washed, dried at about 212° Faht. and weighed. In one experiment thus conducted, 1.41 grain of Howard's fulminating silver, afforded 0.87 grain of the new fulminating silver; but a portion could not be separated from the filter, which being dried, broken in pieces and heated, afforded successive explosions. The loss thus sustained, may be estimated at 0.03 grain, which being added to the 0.87 make 0.90. Making these results the basis of calculation, 100 grains of Howard's compound would afford about 63.8 grains of the new compound; for 1.41 : 0.90 :: 100 : 63.8 nearly. Now, according to my analysis,* 100 grains of Howard's compound, contain 26.25 grains of fulminic acid, and 63.8 : 26.25 :: 100 : 41.14 nearly. Hence, 100 grains of the new compound would consist of

$$\begin{array}{rl} 58.86 & \text{oxide of silver} \\ 41.14 & \text{fulminic acid} \\ \hline 100.00 & \end{array}$$

and taking from my experiments the proportional number of fulminic acid as 42. hydrogene being unity: the new compound would consist of two proportions of fulminic acid $2 \times 42 = 84$ and one proportion of oxide of silver 118, for

58.86 : 41.14 :: 118 : 83.7

In a second experiment, conducted like the first, 5.45 grains of Howard's compound,

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* On a new acid and its combinations. *Trans. Royal Dublin Society*, 1829.

were converted into the new compound; but only 3.02 grains of this substance could be collected from the filter, instead of about 3.48 grains: the loss on the filter, however, was nearly equivalent to the deficiency, as was ascertained by comparing the weight of the dried filter, with another, equal in every respect, and placed under similar circumstances.*

The preceding experiments afford the nearest approximations I could obtain, to the composition of the new compound, by that mode of operating. The difficulty of gaining uniform results, by converting Howard's compound into the new compound, arises partly from the facility with which the new compound, when first formed, is redissolved by an excess of nitrate of silver; and partly from its being to a certain limited extent, either dissolved by water, or partially decomposed by it.

In converting the new compound into chloride of silver, a given weight of it, (well dried,) was put into a platina crucible; a little pure water was added, and then some pure muriatic acid of moderate strength; a considerable action took place, hydrocyanic acid appeared to be evolved, and after digestion for a short time, water was put into a crucible, the chloride was thrown on a filter, washed, dried, collected and fused in a platina capsule, previously counterpoised in a very delicate balance. Operating in this manner, I found, that on fusing the chloride in the capsule it was of a dark colour, having some small specks in it like charcoal. On exposing the capsule to a full red heat, the dark coloured chloride seemed to undergo ebullition; it evolved gas, and gradually assumed a yellowish white colour. It now lost no weight on being heated to redness, and no part of it appeared to be reduced. In one experiment, thus conducted, 6 grains of the new compound afforded 4.70 grains, and in another experiment, 6 grains yielded 4.82 grains of fused chloride of silver. Now, if the mean of these experiments is taken, 6 grains will afford 4.76 grains, and 100 grains of the new compound, will yield 79.33 grains of chloride, equivalent to 54,73 grains of metallic silver, or 58.44 grains of oxide of silver. These results, so nearly correspond with those derived from a different method of examination already noticed, that I venture to regard the new fulminating silver, as a compound of one proportion of oxide of silver 118. and two proportions of fulminic acid 84. or of

58.42	oxide of silver
41.58	fulminic acid
100.00	

I made a number of comparative experiments on the two fulminating silvers, all of

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* I have since found, that in decomposing Howard's compound, by Zinc, &c. another compound of silver is formed. It is of a dull white colour, but changes on exposure to light. It does not explode by heat, but ignites and burns for an instant, leaving a brown substance, which by a strong heat is converted into silver. When decomposed over mercury, the products were oxide of silver, carbonic acid gas, and carbonate of ammonia. This substance may partly account far the loss in the above experiments; but its quantity seems too minute to affect the accuracy of the statements made.

which tended to prove that they are different compounds. Thus, I found that the new compound, exploded when heated from about 350° to 370° of Faht.; whilst Howard's compound required an increase of 100° or from about 450° to 470° Faht. to explode it. Howard's compound readily dissolves in boiling water, as M. Liebig observed, but the white silky crystals are rapidly deposited as the solution cools, and very little remains in solution at the temperature of the air. The new compound is nearly insoluble in boiling water; I could not dissolve half a grain of it in several ounces of water, kept boiling for some time in a platina crucible. Even the small portion that dissolves is found to have acquired new properties, for on being collected and dried, it will not explode in chlorine, and is probably by a loss of acid, converted into Howard's compound.

Howard's compound, however carefully prepared and dried; whether in crystals, or an impalpable powder, does not spontaneously explode in chlorine gas, or in mixtures of this gas with other gases. Whereas, the new compound, whether dried at 60° or at 212° and even before it becomes pulverulent from loss of moisture, readily explodes under such circumstances.

According to my experiments, Howard's compound contains one proportion of fulminic acid, and one proportion of oxide of silver; and it has been called fulminate of silver: as the new compound appears to contain two proportions of the acid, and one of the oxide, it is properly a bi-fulminate of silver.

5. *Experiments on the application of the new Fulminating Silver, as a test for Chlorine, &c.*

The chlorine gas employed in the following experiments, was generally made in the usual way, by adding diluted sulphuric acid to a mixture of common salt, and black oxide of manganese. Occasionally, it was procured from muriatic acid and the same oxide. Sometimes the gas was received over water, at other times it was collected in dry bottles furnished with ground stoppers. The new fulminating silver, or test, as I shall now, for convenience, call it, was made by decomposing fulminating zinc by nitrate of silver, as has been stated. It was sometimes dried on a sand bath at a temperature not exceeding 212° Faht., and sometimes in the open air. It was always, however, sufficiently dry to be pulverulent. It was commonly used in very minute quantities; a simple grain serving, on an average, for upwards of fifty, and occasionally for about one hundred separate experiments. It indicated the presence of chlorine, by instantly exploding when brought in contact with this gas, and also with other gaseous mixtures in general, which contained chlorine.

1. The test readily and repeatedly exploded when put into bottles of very impure chlorine gas, made a long time and having no vestige of colour; and also into bottles

of the gas containing about one-tenth their bulk of water, which had been exposed last summer, during several weeks, in the open air, to light and the sun's rays.

2. Water recently saturated with chlorine in one bottle, was partly transferred to another bottle. On agitating the fluids in both bottles, and introducing the test, it instantly exploded. Similar experiments were made with a saturated solution of the gas which had been excluded from light for a month, and the results were precisely similar.

3. Mixtures, both of pure and impure chlorine and hydrogene, being treated with the test, instant explosions took place, and the production of muriatic acid gas.

4. On adding the tenth of a cubic inch of nitric acid to an equal bulk of muriatic acid, chlorine gas was presently evolved; and the test, on being applied, repeatedly exploded. In cases where those acids were strong, and had acted for a short time on each other, it was only necessary to put the test on a slip of platina, and bring it to the mouth of the glass containing the acids, when an explosion took place, and the effect was produced several times. I found, indeed, that two drops of strong nitric, and one of muriatic acid, put into the same glass, produced a sufficient quantity of chlorine to explode the test several times.

5. Nitric acid being added to common salt, both in its usual state of dryness and also after being fused, the gas evolved readily, and repeatedly exploded the test. The results were precisely similar, when nitric acid was added to a number of dry chlorides, as those of potassium, lime, iron, &c. and the test applied.

6. Strong sulphuric acid was added to chloride of lime made above twelve months since; chlorine gas appeared to be evolved and the test exploded. Fluid chloride of magnesia made some time before, being treated with sulphuric acid, gas was disengaged which exploded the test.

7. A number of experiments were made to ascertain whether the power of the test would be injured or destroyed by diluting chlorine with other gases. Thus nitrogene, nitrous oxide, nitrous gas and carbonic oxide gases were mixed in different proportions with chlorine, but in every instance the test exploded with great facility. When olefiant and carburetted hydrogene gases, were separately mixed with certain proportions of chlorine, the test simply exploded when brought in contact with the mixtures; but when the same gases were severally mixed with chlorine in the proportions of about equal volumes of each, and the test applied, the mixed gases inflamed the instant the test exploded, and the interior of the tube was completely blackened from the deposition of carbon.

8. Different acid gases, as muriatic acid gas, nitrous acid gas, and carbonic acid gas, were separately mixed with chlorine gas, without any regard to proportions; but this circumstance neither prevented, or retarded in any degree, the action of the test, which instantly and repeatedly exploded on being dropped into the respective mixtures. Even gases which are known to be rapidly acted on by chlorine, as sulphu-

retted by hydrogene and sulphureous acid gas, did not hinder the action of the test provided there was the slightest excess of chlorine present.

9. Melted sulphur, and phosphorus were put into bottles of chlorine, and after the respective chlorides were formed, the test being repeatedly introduced into the bottles, in every instance, it exploded.

10. The protoxide of chlorine was collected in an open tube, and a bit of the test introduced; a double explosion took place; first of the test and then of the gas. The test being now added, it exploded as in a mixture of chlorine and oxygene. The peroxide of chlorine being treated with the test, a much louder explosion took place than in the foregoing experiment.

11. A number of experiments were made to ascertain whether the vapours of different fluids diffused through chlorine gas would prevent the usual action of the test. The chlorine gas was agitated in contact with water heated nearly to the boiling point, and whilst the hot vapour issued from the bottle, the test was several times applied, and in every trial it instantly exploded.

12. Sulphuric ether was put into a bottle of chlorine gas, and agitated; a bit of the test being now added, it exploded; inflammation took place, and carbonaceous matter was deposited on the sides of the bottle. A similar experiment being made with alcohol, the test exploded several times.

13. A few drops of oil of turpentine and of naphtha were separately put into phials of chlorine gas, a rapid action of course took place, and much heat was produced. The test, on being instantly applied, exploded in both phials; but if a short interval was suffered to elapse, and the fluids were agitated, the chlorine was all absorbed and no effect was produced on the test.

14. Strong muriatic, nitric, acetic, and hydrocyanic acids, were separately put into phials of chlorine gas and agitated; the test being repeatedly applied to all the phials, exploded in every instance. The test did not explode in any of those acids, or their respective vapours: nor in aqua regia, but when the deep orange aqua regia was agitated, a compound of chlorine was evolved from it which readily exploded the test.

15. Well stopped bottles containing Thomson's chloro-chromic acid, and chloride of sulphur, which had been made a considerable time, were opened, and the test being applied, it exploded repeatedly before it reached the fluids. In both fluids there seemed to be a partial decomposition. The chloro-chromic acid had a very strong odour of chlorine, and the gas in the bottle continued for some time to explode the test. On opening the bottle of chloride of sulphur, a quantity of the vapour and some of the fluid were forcibly expelled.

16. Being desirous of ascertaining to what extent common air might be mixed with chlorine gas, without impairing the action of the test, I made a number of experiments, using different proportions of chlorine and common air; in all of which the test readily exploded. I then collected in a cubic-inch measure divided into 100

parts, $\frac{4}{100}$ of chlorine gas, which, by absorption was reduced to about $\frac{1}{100}$, the remaining $\frac{99}{100}$ of the measure was now filled with common air, and being mixed, a bit of the test was dropped into the tube, when it readily exploded.

The foregoing experiments seem to prove that the new fulminating silver is a very delicate test of the presence of chlorine gas, nor does its delicacy appear to be impaired by exposure to the air, the light, or the sun's rays. Thus, some of the compound made last spring, was exposed on the sand bath during the summer months; part of it became of a dark brown, and part of a black colour. Some of the compound was exposed to the direct agencies of the sun's rays for some time, but the changes of colour it thus underwent, did not prevent it from instantly exploding in all cases in which it was put into chlorine gas, or into mixtures of this gas with other gases. I may also remark, that on exposing some of the test to the action of boiling water for some time, and then drying it, it exploded under the same circumstances as before.

The properties which appear to be regarded as most characteristic of chlorine, are its colour and odour. Though chlorine is easily recognised by its yellowish green colour; in cases when it is pure or nearly so, or when it forms the greater part of a gaseous mixture on which it does not act; yet it may, as is well known, be present in considerable quantity without exhibiting the least vestige of colour. Thus, in the common modes of making the gas, a considerable quantity must be generated, before any colour is apparent. And the purest chlorine, when mixed with a certain portion of common air, or other gases, on which it exerts no immediate action, is no longer distinguishable by its colour. Whereas the new test readily detects chlorine in the first bottle of air that comes over in the usual modes of making the gas; in cases when the gas is mixed with 0.98 or 0.99 of common air, and also, when even a solution of the gas in water is transferred from one bottle to another.

The *odour* of chlorine, though perhaps sufficiently characteristic, when the gas is mixed with other gases on which it exerts no action, or which have no powerful odour; yet it ceases to be so, when certain pungent gases or vapours are diffused through it. Thus, when a portion of chlorine was mingled with muriatic acid, or nitrous acid gas, or with the strongest liquid muriatic or nitric acid, the chlorine could not be satisfactorily distinguished; but in every instance of the kind the test exploded with flame an indefinite number of times. From a number of experiments I have made, I am disposed to regard what is commonly called the *odour* of chlorine, as a vague, and by no means a discriminative character, and that this odour exists in cases where we have no evidence of the presence of chlorine, or where, according to received opinions, it cannot exist. Thus, after exposing solutions of chlorine in water for several weeks or even months to the action of the sun's rays in summer, they are found still to have a strong odour which has been, I think, erroneously referred to chlorine.

I have hitherto said little concerning the specific action of chlorine gas, on the new fulminating silver. The experiments I have made throw some light on the subject, while they afford additional proofs of the extreme delicacy of the test.

I filled a long narrow-necked matrass (smaller than a florence flask), with chlorine gas; placed it upright on a table, and successively dropped into it small portions of the new fulminating silver, until the number of separate explosions exceeded six hundred; when some fragments of the test on the table exploded, and the odour of chlorine was perceived. On examining the matrass, it was found sufficiently cracked to admit the gas to escape. In the course of this experiment, the explosions were uniformly accompanied with flame, and the appearance of a small dense white cloud. These phenomena, at first, occurred in the neck of the matrass, and part of the cloud sunk into the matrass, whilst the remainder rose into the air; but they took place, lower and still lower in the glass, as the number of explosions increased. The interior of the matrass was found covered with a finely divided dark purple substance, which readily dissolved in ammonia; and the solution treated with pure nitric acid in slight excess, gave a white precipitate, which melted at a dull red heat, and was chloride of silver.

In another experiment, a dry half ounce phial, having a narrow mouth, was filled with chlorine gas; small portions of the test were introduced, until one hundred and ninety-nine explosions had taken place. A little pungent vapour arose from the phial, and a peculiar odour was emitted, resembling that of chlorine in a state of great dilution. The vapour in the phial still possessed the property of bleaching; for it soon rendered moist litmus paper, white. A dark dove coloured substance remained principally at the bottom of the phial. It was chloride of silver, and there was a very minute portion of a crystallized substance attached to the upper part of the phial, which exhibited the properties of sal-ammoniac.

In a third experiment, a long dry tube of about the capacity of two cubic inches, was filled with chlorine gas, and the test was added until it ceased to explode. There was now distinctly perceived an odour precisely similar to that of the compound which is formed when fulminate of zinc is agitated in contact with chlorine gas. This compound is a yellow, oily, volatile fluid, resembling azotane in appearance, but having none of its explosive properties. Its odour is so acrid and peculiar that it can scarcely be mistaken. Its taste is sweetish, and astringent, with a certain degree of pungency, which remains for some time on the palate. It is apparently insoluble in water, but readily forms a sort of saponaceous compound with ammonia. It does not immediately redden litmus paper, but acquires this property after a short time. It is, I presume, a compound of fulminic acid and chlorine. There appears to be another compound of the same substances, but in different proportions. I obtained it by distillation after exposing the fulminating silver, and other analogous compounds, either diffused in water or dissolved in it to the action of chlorine gas. It is a colourless, transparent, and volatile fluid, having a peculiar and disagreeable smell and a taste at first sweet, but which presently becomes sharp and enduring, somewhat resembling that of cayenne pepper. It is soluble in water, has no bleaching, but some acid properties, and may deserve farther examination.

It would seem from the foregoing results, that when the new fulminating silver is exposed to the action of chlorine gas, it is decompounded, chloride of silver is formed; one part of the fulminic acid combines with that gas to form the peculiar compound just referred to, whilst the other part is decomposed, and affords by the reunion of its elements, sal-ammoniac. It seems probable, too, that carbonic acid gas and nitrogene, are at the same time evolved.

The action of chlorine gas on the new fulminating silver, is uniformly accompanied with flame; this circumstance and the formation of ammonia above noticed, seem to favour the opinion I have advanced in the papers already referred to, that hydrogene enters into the composition of the fulminic acid.

The new fulminating silver, besides its use as a test for chlorine, might I think be employed with advantage as a substitute for Howard's fulminating mercury in the caps for percussion locks, which are now so much approved, and getting into such general use as threaten to supersede the common lock. The strong springs required in the percussion locks in which Howard's fulminating compound is used, are objectionable. The new fulminating silver requires much less percussive force to explode it than Howard's fulminating mercury; nor is the effect of the explosion of the former, accompanied with that loud, and almost deafening report, of the latter compound.

From the known analogies existing between chlorine and bromine, the vapour of the latter might be expected to explode the test, as well as the former; and this I find is the case. Thus, on putting a few drops of bromine into a small stoppered bottle, and dropping in a bit of the test; it immediately exploded in the vapour, and the experiment was repeatedly tried at different intervals in the same bottle, with the same result. The test does not explode when brought in contact with iodine, either at the common temperature of the air, or when it is raised in vapour by heat.

On the Theory of the Moist-bulb Hygrometer. *By* JAMES APJOHN, Esq. M.D. M.R.I.A., *Professor of Chemistry in the Royal College of Surgeons.*

Read November 24, 1834.

In the number of the Edinburgh Philosophical Journal just published, and which was sent me by a friend, in consequence of its containing a report of the proceedings of the late meeting of the British Association for the promotion of science, I find an anonymous paper, entitled "Observations on the Hygrometer," which induces me to take the earliest opportunity of submitting to the Academy the following remarks.

It is well known to Meteorologists that the Hygrometers of Saussure and De Luc, and all others whose indications depend upon variations in the volume or weight of the hygroscopic substance employed, have, as philosophical instruments, been in a great measure discarded. The direct determination of the Dew-point, as it is technically called, is now almost universally practised, either according to the original method of Dalton, or by means of the elegant instrument of Professor Daniell. The method of the former, however, is much too tedious for practical purposes, and the instrument of the latter, though in principle rigorously correct, requires, in order to accuracy of result, a quickness of sight, and an adroitness in observation, which few can boast of possessing. For the great purposes, therefore, of Meteorology, the condensation Hygrometer must be considered as having, as yet, but imperfectly fulfilled the high expectations which were entertained of it.

The Dew-point process was preceded by one of an analogous description, which is said to have been first practised by Hutton the Geologist. If the bulb of a thermometer be kept covered with a thin film of water, its temperature will sink beneath that of the atmosphere, (this latter being supposed unsaturated with moisture,) the maximum depression being attained when the heat received in a given time by the film from the contiguous air is exactly equal to the caloric of elasticity of the water evaporated; and it is obvious that the amount of this depression will bear some direct ratio to the degree of dryness of the air at the time of the experiment. Further than this, however, the wet-bulb Hygrometer does not go. It affords us, for example, no information as to the exact quantity of moisture present in the surrounding atmosphere at the time of observation, because it does not indicate the position of the Dew-point. That there existed, however, between this latter and its indications, some necessary connexion, so that the one might be inferred by calculation from the other, must have been very early suspected; but Sir John Leslie, after having converted his differential thermometer into an hygrometer, was the first who attempted to point it out. In this attempt, I believe I may venture to say, he was but partially successful. I have never seen any very explicit statement of the principle on which he proceeded; but the table which he published in 1820, in his " Description of Meteorological Instruments," is undoubtedly erroneous.

In a very able article on Hygrometry in Brewster's Encyclopædia, a formula for the solution of this question is elaborately investigated; but as it is in greater part tentative, assumes, contrary to the fact, that the amount of moisture which air is capable of taking up is influenced by the pressure; and, as lastly, it does not very well accord with observations—at least some which I have made—it has been adopted, I believe, but by few Meteorologists. A satisfactory solution of this problem was still viewed as a desideratum, and of this no better proof can be given than that the " theory of the moist-bulb Hygrometer" is found among the questions submitted by the first meeting of the British Association, held at York, to the renewed consideration of philosophers.

About four months since, on turning the matter in my mind, it occurred to me that the relation between the Dew-point and the temperature of a thermometer with moistened bulb might be made matter of calculation, and deduced from the theory of mixed gases and vapours—a theory which the labours of Dalton and G. Lussac have rendered as complete as any other in Physics. It was not, however, until the latter end of August that I was enabled to return to the subject, when I succeeded in arriving, by a direct rout, at a formula derived exclusively from experimental data, and which represents with unexpected accuracy, the best observations with which I have been able to compare it. To this formula (to explain, at length, the object of this hurried communication,) I am anxious to draw, without further delay, the attention of men of science, as there is a brief notice of one for accomplishing the same object in the article of Professor Jamieson's Journal already alluded to. Of this latter, which is said to belong to Mr. Ivory, I beg to say I was altogether ignorant; nor have I yet been able to refer to the number of the Philosophical Magazine in which it is stated to have been originally published.

Having disposed of these preliminary observations, I shall now proceed to explain the principle of my method of investigation; but before doing so, I wish to observe that I would have submitted my formula to the Committee of the British Association in Edinburgh, charged with the subject of Meteorology, but that, having fallen upon it during the vacation, while sojourning in the south of Ireland, I was not able, prior to the meeting, to institute test observations myself, nor had I access to books, so as to compare it with the recorded experiments of others, on the temperature of the moist-bulb Hygrometer and the corresponding Dew-point.

When in the moist-bulb Hygrometer the stationary temperature is attained, the caloric which vaporizes the water is necessarily exactly equal to that which the air imparts in descending from the temperature of the atmosphere to that of the moistened bulb; and the air which has undergone this reduction becomes saturated with moisture. Now from these facts, and the known specific heat of air, we can calculate the weight of water m' which would be converted into vapour by the heat which a given weight of air would evolve in cooling from t the temperature of the atmosphere to t' that of the moistened bulb; and we can also calculate the total quantity of moisture m which the same weight of air would contain at t' if saturated. This being accomplished, if f' be the tension of vapour at the temperature t' $(1-\frac{m'}{m})\ f'$[*] $= f''$ the tension of aqueous vapour at t'' the Dew-point. Hence, by looking in Dalton's table for f'', the Dew-point is found in the opposite column.

The value of the expression $(1-\frac{m'}{m})\ f'$ may be found in the following manner:— 1 being the specific heat of water, .267 (De la Roche and Berard) is that of air. Also 967° being the caloric of elasticity of steam at 212°, $212-50+967=1129°$ will be its caloric of elasticity at 50°, assuming, as is generally done, that the sum of the sensible and latent heats of vapour is the same at every temperature. One grain of air, therefore, in cooling through any number of degrees d, will raise the temperature of .267 grains of water through the same number, and will consequently be adequate to vaporize a quantity of water represented by $\frac{.267d}{1129}=\frac{d}{4195}$ grains; or, multiplying by the denominator, 4195 grains of air, in cooling through d degrees, give out the exact quantity of heat which constitutes the caloric of elasticity of d grains of vapour. But the volume of this weight of air, at 60°, and under a pressure of 30, is 13754 cubic inches; and, at the temperature t' and pressure p, $13754 \times \frac{448+t'}{508} \times \frac{30}{p} = 810\ \frac{448+t'}{p}$ cubic inches. Hence $(810 \times \frac{448+t'}{p}) \times (10{,}583 \times \frac{f'}{448+t'}$

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[*] This expression is obviously deduced from the fact of the tension, of vapour at a given temperature and under a given volume, being proportional to the quantity or specific gravity.

$\times\, ,305)^{*} = 2613.88 \times \frac{f'}{p}$ = the quantity of moisture which the air contains when saturated at t'. We will therefore have, on the principle already explained $(1 - \frac{d}{2613.88} + \frac{f'}{p}) \times f' = f' - \frac{pd}{2613.88} = f''$ the tension of vapour at the dew-point. If $p = 30$, $f' - \frac{d}{87} = f''$.

I shall now proceed to state, and subsequently observe upon, the objections which may be made to the method of investigation I have pursued. It may be said—

1°. That the air which is cooled by contact with the moistened bulb at the stationary temperature, is assumed, without proof, to be saturated with moisture.

2°. That the caloric of elasticity of steam is 1129 only at 50°.

3°. That the specific heat of air is .267 only under a pressure of 30.

4°. That the medium which is cooled from t to t' is not pure air, but a mixed atmosphere of air and vapour; and

5°. That the caloric, which at the temperature t' converts the water into vapour, is not derived exclusively from the air by contact, but partly also by radiation from surrounding bodies.

With respect to the first objection, I have only to observe, that air is an extremely bad conductor of heat, and that it is, therefore, very unlikely that the reduction of temperature which it experiences in the experiment in question can be effected in any other way than by actual contact with the moistened bulb. But, if such contact be established in the case of every indefinitely thin aerial shell, there can, I conceive, be no doubt but that each becomes charged with the full amount of moisture which belongs to its reduced temperature.

In reference to the second objection, it must of course be admitted that the caloric of elasticity of vapour varies with the temperature, and that it is represented by the number 1129 only at the temperature of 50°, a point chosen by me as being nearly the mean temperature of Dublin. In strictness, the number employed should be $967 + 212 - t'$, but it would be easy to shew that the uniform use of 1129 cannot give rise to any material error.

The third objection is usually considered as one of considerable weight. The specific heat of air varies with the pressure, and in order to accuracy of result, a proper correction must undoubtedly be made for this variation. But what is the law which it observes? Upon this point, different opinions would appear to be entertained. According, however, to De la Roche and Berard, (whose views, if not rigorously exact, are at least sufficiently so for my present purpose,) for small variations of

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[*] $10,583 \times \frac{f'}{448+t'} \times ,305$ = the weight of a cubic inch of vapour whose tension is f' and temperature t'.

pressure, such as occur to the natural atmosphere, the differences of specific heats under a constant volume, are proportional to the differences of pressure. And the same philosophers have shewn, that for pressures in the ratio of 1 to 1.3583, the corresponding capacities are 1 and 1.2396. Hence, as

$.3583 : .2396 :: \frac{p}{30} - 1 : \frac{x}{c} - 1$, c being the specific heat under a constant volume at 30, and x that at p,—a proportion from which we deduce

$$x = (.0223\, p + .3312)\, c.$$

But the specific heats under a constant volume, divided by the densities, give the specific heats of equal weights. And as the densities vary as the pressures directly, and as the temperatures + 448 inversely, and are, therefore, to each other in the present case as $\frac{30}{448+60}$ to $\frac{p}{448+t'}$, we will have

$$\frac{508}{30}\, c : (.0223\, p + .3312)\, c \times \frac{448+t'}{p} :: .267 : x'.$$

So that x', or the specific heat of air at temperature t' and pressure p, $= \frac{448+t'}{508} \times \frac{30}{p} \times (.0223\, p + .3312) \times .267$.

The value, therefore, of f' already given, when subjected to this correction, will become $f' - \frac{d}{87} \times \frac{448+t'}{508} \times (.0223\, p + 3312)$.

The equation for this correction, given by the writer in the Edinburgh Philosophical Journal, is exclusively a function of p; but if the method here explained be correct, and I believe it will be found so, the temperature t' of the Hygrometer has a still greater influence on its amount. They both, however, affect it in the same direction; i. e. as they rise, it increases; and as they fall, it diminishes: so that if the one should augment as the other diminishes, they will counteract, to some extent, each other's effects. When $t' = 50$, and $p = 29$,

$$f'' = f' - \frac{d}{87} \times .958,$$

i. e. the value of the subtractive quantity is diminished by its $\frac{1}{25}$th part; but if t being still 50, p be supposed 31,

$$f'' = \frac{d}{87} \times 1.002,$$

or the subtractive term is augmented only by its $\frac{1}{500}$th part,—facts from which the general conclusion may be drawn, that when $60 - t' = 30 - p$, the latter difference being measured in tenths of an inch, and that they have opposite signs, the correction may be altogether neglected.

The theoretical justness of the fourth objection must also be conceded. The medium which is in contact with the bulb of the Hygrometer is not dry air, but air charged with the amount of vapour which belongs to the existing dew-point; and as the specific heats of air and vapour are different, this mixed atmosphere, in cooling through $t^\circ - t'^\circ$, will evidently not give out the same quantity of caloric, and can therefore not convert into vapour the same quantity of water that would be cooled and vaporized by the same weight of dry air alone. In fact, for .267, the specific heat of air, we should in strictness use the specific heat of the mixture of air and vapour; or, what will answer the same purpose, multiply by the ratio of these, the value of the quantity to be deducted from f', already obtained. Now, to determine the specific heat of the mixture, the relative weights of its constituents should be multiplied by their respective capacities, and the sum of the products divided by the sum of the weights. But the weights, being obviously as the specific gravities, are to each other as $1 : .625 \cdot \frac{f''}{p}$. Also, the specific heat of air being .267, and that of vapour .847, the former is to the latter as 1 : 3.172. Hence, according to the rule given above, we will have

$$\frac{1 + .625 \cdot \frac{f'}{p} \times 3.172}{1 + .625 \cdot \frac{f''}{p}}$$

for the specific heat of the mixture of air and vapour referred to that of dry air taken as unity; and, applying the correction as already explained, we will have an equation in which f'' is the only unknown quantity, and from which, therefore, its value may be found. The equation, however, being a quadratic, and the unknown quantity in its first dimension having a coefficient of three terms, its solution would involve tedious arithmetical operations, and can therefore not be recommended as a ready means of making the correction in question. Nor is such course at all necessary, for the same object may be achieved, and with a sufficient precision, by either assigning to f'' an average value, or by deducing approximately the tension of vapour at the dew-point by the formula $f'' = f' - \frac{d}{87} \times \frac{448 + t'}{508} (.0223\, p + .3312)$, and using the value of f'' thus obtained, in order to determine that of

$$\frac{1 + .625 \, \frac{f}{p} \times 3.172}{1 + .625 \, \frac{f}{p}}$$

the specific heat of the mixture of air and vapour. The latter method is decidedly the best; and though not mathematically accurate, will not I believe exhibit a deviation from the truth until the calculation is pushed to the seventh or eighth decimal place.

I have now to notice the last circumstance which, as far as I understand the subject, can have any influence upon the accuracy of my determination of the Dew-point.

When the wet-bulb Hygrometer has attained its stationary temperature, the caloric which it loses and gains in a given time are perfectly equal. This requires no demonstration. The caloric lost also is entirely employed in converting the water into vapour; but the whole of the acquired caloric is not necessarily derived, although such is assumed to be the case, from the air cooled by contact with the bulb of the instrument. In fact, the Hygrometer is in the predicament of a cool body placed in a warm medium, and it must consequently receive from surrounding bodies, by radiation, a greater amount of caloric than it imparts to them in virtue of the same process. To the d grains, therefore, of moisture converted into vapour by the heat given out by 4195 grains of air, in cooling through d degrees, we should add the additional quantity vaporized by the heat which the bulb has in the same time received by radiation. When $t-t'$ is small, this quantity may probably be safely neglected; but it will sometimes, I make no doubt, be of sufficient magnitude to exercise an appreciable influence. I regret my inability to assign any means of determining its amount; and shall merely add, that the neglect of this correction will always tend to render the calculated Dew-point somewhat higher than the true.

Having disposed of the theory of my method, I shall now conclude by subjecting the results which it affords to the test of experiment. I shall not at present refer to my own observations, though I have of late amassed a considerable number on the Hygrometer and Dew-point. As a more unimpeachable criterion, I shall compare my formula with the observations of others, and shall select for this purpose, it being the nearest at hand, a table published in the last number of the Edinburgh Journal. The differences, it will be seen, between the corresponding numbers of the fourth and fifth* columns of this table are so small, that we may consider them as almost entirely due to errors of observation. I may add, that as in the original table there is no notice taken of the barometer, the formula, in its most complete form, could not be applied; so that a *perfect* coincidence between calculation and observation was not in this instance to be expected.

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* The numbers in the fourth column are the observed, and in the fifth the calculated dew-points.

	t	t'	t'' Observed.	t'' Calculated.
1	68.25	61.75	57.25	57.5
2	56.25	54.5	53.25	53
3	64.5	59	54.5	55
4	67.5	61.25	55.75	57
5	67.25	61.	56.75	56.75
6	63	59.	56.25	56.2
7	62.25	57.75	55.25	54.4
8	68	61.75	57.25	57.2
9	63.25	59	56.5	56
10	69.5	63	58.25	58.75
11	68.75	61	56.25	55.66
12	63.5	58	54.75	53.8
13	63.75	58	54.5	53.6
14	68	61.25	56.25	56.6
15	69	62	57.25	57.4
16	66.5	61	57.5	57.3
17	66.25	61	57.5	57.5
18	67	59.5	54.5	54
19	64.5	58.75	54.25	54.5
20	64.75	58.75	53.75	54.4
21	59	54	50	49.9
22	63.75	57.75	53.75	53.1
23	63.5	57.25	52.25	52.5
24	59.75	54.75	49.25	50.5
25	62.5	56.25	51.25	51.25
26	61.25	56.5	53.25	52.75
27	60.75	56.5	53.25	53.25
28	62.75	57.75	53.75	54
29	65 5	60.5	55.75	57.1
30	64.75	61	58.25	58.5
31	62 5	57.25	52.75	53.2
32	62.25	56.75	52.75	5 .
33	60.25	56.5	53.75	53.3
34	57.25	53.75	51.5	50.33
35	57.25	53.75	51	50.33
36	58.5	54	50.75	50.25
37	57.25	55.25	54	53.66
38	67.5	59	58.75	59
			54.40	54.74

On the Theory of the Moist-bulb Hygrometer. *By* JAMES APJOHN, Esq. M.D. M.R.I.A., *Professor of Chemistry in the Royal College of Surgeons.*

(Continued.)

Read April 27, 1335.

At the meeting of the Academy held in November last, I was permitted to read a short memoir on the subject of a formula, at which I had a considerable time previously arrived, for inferring the Dew-point from the indications of the Moist-bulb Hygrometer. This formula was deduced altogether from general considerations; and, though satisfied, from some hasty observations of my own, that it represented facts with considerable accuracy, I was not, at the time, in possession of evidence which could be considered as establishing this important point in an unequivocal manner. The table which is subjoined to my paper undoubtedly shows, that, within certain limits, my formula is in accordance with experiment; but the observed depressions in the table are, generally speaking, so small, that a formula in itself incorrect might, it must be admitted, yield results which would deviate from the observed dew-points by quantities not exceeding the possible errors of observation. Berzelius, for example, states (Traité de Chimie, tom. viii. p. 254,) that, from the experiments of August, Bohnenberger, and others, it appeared that the temperature of a thermometer with moistened bulb was an arithmetic mean between that of the air and the dew-point; and this rule, which would make $t''=2t'-t$, though utterly erroneous, would apply to the table appended to my paper, nearly as well as the formula I have deduced. The validity, therefore, of my method required to be more rigorously tested; and having been for some time engaged in experimental researches, instituted with this object, which have led to interesting, and to me most satisfactory results, I am anxious to submit them, with as little delay as possible, to the judgment of the Academy.

The equation which, as I believe, comprehends the theory of the wet-bulb hygrometer, is as follows :—

$$f'' = f' - m\,d \times \frac{p}{30}$$

in which f'' is the tension of steam at the dew-point, f' its tension at the temperature of the hygrometer, d the depression or difference between the temperature of the hygrometer and air, p the existing, and 30 the mean pressure, and m a coefficient depending upon the specific heat of air and the caloric of elasticity of its included vapour, its arithmetical value being .01149, or the equivalent vulgar fraction $\frac{1}{87}$. In the paper to which I have already referred, corrections are given for the influence on the specific heat of air, of the fluctuations of the barometer, and the moisture present in the atmosphere. These corrections are, I believe, deduced from correct principles, and should be resorted to when extreme precision is desirable. Experience, however, and a careful consideration of the subject, have satisfied me that they are, generally speaking, in their effects, much too insignificant to be objects of attention to the practical meteorologist.

The first and most obvious method of verification which presented itself to my mind, was the comparison of my formula with recorded cotemporaneous observations on the temperature of air, that shown by a moist-bulb hygrometer, and the actual dew-point. I have, however, unfortunately been able to meet but few at all suited to my purpose. Those in which $t-t'$ is small, and this is generally the case in the few registers to which I have had access, cannot, as we have already seen, serve for deciding the value of any formula. In the first report, indeed, of the British Association for the Promotion of Science, page 50, mention is made of a register of observations kept in the East Indies, which, as belonging to high temperatures, would necessarily exhibit great depressions, and would therefore be valuable as a standard of comparison; but I have in vain searched for the Calcutta Journal, "Gleanings in Science," in which they are said to be contained. In fact, the only observations I have been able to procure, adapted to my purpose, and made, apparently at least, with the necessary precision, are those adduced in the article 'Hygrometry' of Sir David Brewster's Encyclopædia, and there made, by the author of the article, the basis of a calculation for investigating the constants of a tentative formula for connecting the indications of the wet-bulb hygrometer with the dew-point. They are but two in number, and are comprehended in the following table, in which the numbers in the first column represent the temperatures of air, those in the second the corresponding indications of the hygrometer, those in the third the depressions, those in the fourth the pressures, and those in the fifth the dew-points experimentally determined by the method of Dalton :—

(1)	(2)	(3)	(4)	(5)	(6)
t	t'	d	p	t'' ob.	t'' calc.
67.2	52	15.2	29.75	35.7	35.6
56.4	49.5	6.9	30.02	39.5	42.4

The numbers in column (6) are the dew-points calculated by my formula; and while there is an almost exact correspondence between the first and the result of experiment, the second, it will be seen, is higher than the observed temperature of deposition by nearly three degrees. There is here, however, obviously some mistake. It is impossible that, with the recorded temperatures of air and hygrometer, the dew-point could have been so low; and this conclusion I do not at present draw from my theoretical views, for that would be to subject myself to the imputation of arguing in a circle; but from the following observation, made by me with great care on the 22d of March :—

t	t	d	t'' ob.
56	50	6	44

Here the temperatures t and t' differ from those taken from the Encyclopædia only by about half a degree; and nevertheless the observed dew-point 44 is higher than 39.5 by 4.5 degrees. From these observations, therefore, I am, I conceive, entitled to conclude—1st, that the series in which the depression amounts to 15°.2, being in exact accordance with my formula, lends it some degree of support; and 2dly, that my method cannot be considered as impugned by the other series, inasmuch as *this* is in some particular manifestly incorrect. But it is time to enter upon the experimental tests to which I have resorted.

If air, in reference to which t, t' and t'' have been accurately noted, be raised to any elevated temperature, and the observation be repeated in the heated air, as far as respects t and t', we will have two* separate sets of observations, from which to calculate the point of deposition; and as the amount of moisture in the air is not altered by the augmentation of temperature it has experienced, both calculations, provided our formula be correct, should give precisely the same result; *i. e.* the dew-point in the first instance determined by observation. Such is the principle of the test experiments which I first performed. The air was heated by urging it in a continued stream, by means of a double bellows, through the worm of a small still—such as are for sale in the Opticians' shops—the worm-tub being filled with water of the desired temperature; and, in order to the necessary observations, in a glass tube connected by a cork with the upper extremity of the worm, a couple of small thermometers were

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* Any number of observations, having reference to the same dew-point, may, it is obvious, be thus obtained.

placed, their bulbs being separated by about a quarter of an inch, and that of the instrument occupying the higher position being invested with a tunic of muslin kept constantly moist with water. The blast was steadily maintained until the thermometers ceased to rise, and the temperature of each was then accurately noted, the eye being assisted by a lens. Tables 1, 2, 3, and 4 exhibit the results of four distinct series of experiments thus conducted :—

Table 1.
February 8, 1835, 11 o'clock, a.m.

	t	t'	d	p	t'' ob.	t'' calc.	Diff.
1	49.6	44.7	4.9	29.6	40	39.02	—.98
2	88.5	62	26.5	29.6	40	39.18	–.82
3	80.5	59	21.5	29.6	40	39.27	–.73

Table 2.
February 9, 1835 : 11 o'clock, a.m.

	t	t'	d	p	t'' ob.	t'' calc.	Diff.
1	47.2	42.5	4.7	30.02	38	36.58	–1.42
2	76	57.5	18.5	30.02	38	40.44	+2.44

Table 3.
March 4, 1835 : 11 o'clock, a.m.

	t	t'	d	p	t'' ob.	t'' calc.	Diff.
1	48.3	43	5.3	29.76	37.5	36.41	—1.09
2	96	64	32	29.76	37.5	36.37	—1.13
3	91	62.5	28.5	29.76	37.5	37.66	+ .16
4	75	56	18	29.76	37.5	38.26	+ .76

Table 4.
March 25, 1835 : 11 o'clock, a.m.

	t	t'	d	p	t'' ob.	t'' calc.	Diff.
1	51.3	45.5	5.8	30.7	38.5	38.61	+.11
2	82	59	23	30.7	38.5	36.7	–1.8

The results exhibited in the preceding tables will, I believe, be considered by many as going far towards establishing the accuracy of my theoretical views. Although the depressions vary from 4°.7 to 28°.5, the differences between the observed dew-points and those deduced from the formula, are certainly not greater than what may fairly be ascribed to unavoidable inaccuracy of observation. But for the purpose of putting this matter in a still clearer point of view, I have calculated a number of values of m, the constant of our formula, from the preceding observations. This was easily done; for as all the observations in the same table refer to air in the same hygrometical state, each series should give the same dew-point, and the expression $f' - m\,d \times \frac{p}{30}$ must have in reference to them a constant value. $f' - m d \times \frac{p}{30}$ for one must therefore be qeual to $F' - m\,D \times \frac{p}{30}$ for any other—an equation from which we deduce $m = \frac{F' - f'}{D - d} \times \frac{30}{p}$. The application of this method gives us the following values of m :—

Table 1.

	(1 and 2)		(1 and 3)		(2 and 3)
$m =$	.01155	...	.01185	...	.01075

Table 2.

	(1 and 2)
$m =$	.01489

Table 3.

	(1 and 2)		(1 and 3)		(1 and 4)		(2 and 3)		(2 and 4)		(3 and 4)
$m =$	.01137	...	.01187	...	.01309	...	.00825	...	.00976	..	.01045

Table 4.

	(1 and 2)
$m =$	.00967

If the mean of all these values of m be taken, it will be found to be .01122, or the equivalent vulgar fraction $\frac{1}{89}$, an approximation to the coefficient $\frac{1}{87}$ employed in the formula, which, under all the circumstances, cannot but be considered as remarkably close. Indeed the difference, which is less than 3 in the fourth place of decimals, is so small, that they may be substituted indiscriminately for each other without the occurrence, at least in ordinary cases, of sensible error. Had values of m been calculated from the comparison alone of the first series of observations in each table with the subsequent ones, the mean, it is worthy of remark, would be .01156, or almost exactly $\frac{1}{87}$; and as, for such observations, $F'-f'$ and $D-d$ are necessarily greatest, they are best calculated to afford correct results, since any error of experiment would obviously, in their case, exercise the least influence.

The next test experiments performed were suggested by the formula itself. If $f''=f'-\frac{d}{87}\times\frac{p}{30}$, and f'' be supposed equal to o, a condition which can only be fulfilled in perfectly dry air, $f'=\frac{d}{87}\times\frac{p}{30}$, an equation from which we deduce $d=87f'\times\frac{30}{p}$. Hence, by determining experimentally the depression of the hygrometer in perfectly dry air, we will be able to pronounce upon the validity of the general method under discussion.

The first attempts for determining values of d experimentally, consisted in suspending a pair of thermometers, one of which had its bulb moistened, in a close corked bottle, the bottom of which was covered with a stratum of oil of vitriol; but this method was soon abandoned, as the depressions it afforded were, on an average, one-fifth less than they should be according to the formula. In fact the extreme depression could not be expected here, for it is obvious that the air, in contact with the bulb of the moist thermometer, is never perfectly dry except at the very commencement of the experiment.

The next contrivance to which I resorted was as follows. A bag of India rubber cloth, furnished with a cap and stop-cock, was inflated by a bellows, and then connected, by means of a caoutchane collar, to a glass tube, traversing a cork fitted to the tubulure of the lower bottle of a Noothe's apparatus. The middle bottle of the apparatus was next filled, $\frac{2}{3}$rds, with oil of vitriol, and the pair of thermometers last described being introduced into the axis of a small tube, perforating a cork fitted to the upper opening of this bottle, a stream of air was forced by pressing on the caoutchouc bag, through the oil of vitriol, and, of course, over the thermometers; and as soon as the instrument with moistened bulb ceased to fall, the temperatures of both were noted. The following table comprehends the results of five experiments thus performed :—

	t	t'	p	d ob.	d calc.	Diff.
March 11	48.5	31.5	29.37	17	17.4	— .4
15	50.5	33.5	30.00	17	18.2	−1.2
20	54.5	35.5	30.25	19	19.4	— .4
21	57.5	37.5	30.27	20	20.8	— .8
22	54.5	36	30.35	18.5	19.7	−1.2

Now, as in all these instances, the observed depression differs from the true; this difference, though small, being always on the same side, must be ascribed either to the co-efficient m being assumed too great, or to the method of experiment employed not being calculated to afford the extreme depression. That this latter was the real cause of the discrepancy I was disposed to believe, from having observed that when the hygrometer, in the course of an experiment, became stationary, it could be made to sink a little further by pressing with great force upon the bag of air. In fact, this observation rendered it probable that the tube, between the lower and middle bottle of the Nooth, did not afford sufficient air-way; and that, therefore, there was not a sufficient current from behind to propel forward, and immediately remove, from contact with the moistened bulb, the air which had become saturated with its humidity. To bring this conjecture to the criterion of experiment, it was obviously necessary to operate so, that while the air underwent perfect desiccation, it was, at the same time, made to pass over the thermometers in a strong and continuous current; and, after some trials, I found that both objects were secured by substituting for the Nooth a series of three Wolfe's bottles, containing oil of vitriol, and connected, as in the process for preparing the water of ammonia, by glass tubes and caoutchouc collars, the bag of air being attached to a tube passing to the bottom of the first bottle, and the thermometers being placed in the axis of a tube perforating a cork inserted into one of the tubulures of the last bottle. The experiments recorded in the following table were made with this apparatus.

	t	t'	p	d ob.	d calc.	Diff.
March 26	51	33.5	30.55	17.5	17.94	+.44
27	53	34.5	30.35	18.5	17.73	—.77
28	52	34	30.21	18	17.62	—.38
29	51	33	30.05	18	17.97	—.03
30	52	33.4	29.75	18.6	18.37	—.23
31	53	34.3	29.50	18.7	19.14	+.44
April 1	56.5	35.8	29.70	20.7	20.04	—.66
2	58	37	29.72	21	20.88	—.12
3	58.2	37	29.77	21.2	20.84	—.36
4	58	37	30.03	21	20.68	—.32
5	58	37	30.15	21	20.59	—.41
6	59	37.5	30.25	21.5	20.88	—.62
7	59	38	30.26	21	21.24	+.24
8	61	38.7	30.21	22.3	21.80	—.50
10	58.3	37.7	30.35	20.6	20.96	+.36
11	58	37.5	30.45	20.5	20.75	+.35
12	56.3	36.5	30.30	19.8	20.12	+.32
13	57.5	37	30.20	20.5	20.55	+.05
14	57.5	37	30.15	20.5	20.59	+.09

Of the nineteen observations of depression in dry air registered in the preceding table, eleven are greater, and eight less than the calculated results. The mean of the plus errors of the formula is, .28, and of the minus errors, .4 of a degree; so that $.28 - .40 = -.12$ of a degree is the mean difference deducible from the whole between experiment and calculation. A closer approximation between them than this could not, I think, be anticipated, even upon the hypothesis of the strict accuracy of the formula. I may also observe that if by means of the equation $f' = m\,d \times \frac{p}{30}$, which, as we have already seen, belongs to perfectly dry air, we deduce from the preceding tables 19 values of m, the mean of all will be found almost accurately equal to $\frac{1}{87}$, a result the more entitled to confidence inasmuch as the mean pressure for the 19 experiments being but very little over 30, and the air being perfectly dry, neither of the corrections which I investigated in my former paper require to be applied.

If from the experiments already detailed I were to draw the conclusion that the equation $f'' = f' - \frac{d}{87} \times \frac{p}{30}$ will afford the dew-point with a degree of accuracy far surpassing ordinary hygrometrical observations, I would, probably, have the concurrence of most of my readers. The evidence adduced in support of the formula appears, at least to me, ample and satisfactory. For the purpose, however, of dispelling any doubts of its accuracy which may exist in the minds of others, I undertook another series of test experiments, to the description of which I shall now proceed.

The most direct method of testing our formula consists, as has been already observed, in comparing its results with dew-points experimentally determined. In order, however, that this criterion be decisive, it is not only necessary that the depressions be considerable in amount, but also, as is obvious, that the dew-points be accurately known. Now the registers to which I have had access do not perfectly satisfy either of these conditions, the depressions being generally small, and the observations made with an instrument—Daniell's hygrometer, the difficulty of observing with which is universally admitted. In reflecting on this matter it occurred to me that both difficulties might be evaded in the following simple manner. Let air, saturated with moisture, and whose temperature is, therefore, necessarily its dew-point, be heated, and let the temperature of the heated air be taken, as also that shewn by a moist bulb hygrometer, subjected to the action of a current of it. Let, then, by the application of the formula, the dew-point, belonging to the two latter observations, be calculated, and from a comparison of it with the original temperature of the air when saturated with humidity, we will be enabled to pronounce with confidence upon the value of our method.

In the experiments which I performed on this plan the air was saturated with moisture, by forcing it from a bellows through a succession of four Wolfe's bottles, connected in the usual way so as to cause the air to pass in each bottle through about two inches of water, and the air thus saturated was heated by being made to pass through a coil of copper tubing, immersed in a tub of warm water, the thermometer and hygrometer being placed with their bulbs within a quarter of an inch of each other in a narrow glass tube, attached to the farther extremity of the copper worm. The following are the results thus obtained :—

	t	t'	d	p	t'' ob.	t'' calc.	Diff.
April 17, 1835, 11 o'clock, A.M.	78	62.2	15.8	30.30	51.3	50.47	—. 83
	76	61.5	14.5	30.30	51.3	50.26	—1.04
	73	60.3	12.7	30.30	51.3	51.58	+ .28
	72	60	12	30.30	51.3	50.81	— .49
	69	58.6	10.4	30.30	51.3	50.40	— .90
April 18, 1835, 11 o'clock, A.M.	90.5	67	23.5	30.15	50.8	50.17	— .63
	82.2	64.3	17.9	30.15	50.9	51	— .10
	79	62	16.4	30.15	50.9	50.23	— .67
	71.7	60	11.7	30.15	51.2	50.66	— .54
	69	58.9	10.1	30.15	51.5	50.70	— .80
April 20, 1835, 11 o'clock, A.M.	92	69	23	30.42	54.1	54.40	+ .30
	83	65.8	17.2	30.42	54.5	54.36	— .14
	76	63.3	12.7	30.42	54.9	54.54	— .36
	68	60.3	7.7	30.42	55	54.74	— .26
April 21, 1835, 11 o'clock, A.M.	98.5	71.5	27	30.36	55.5	55.51	+ .01
	84.6	67	17.6	30.36	56	55.79	— .21
	77.5	64.5	13	30.36	56.3	55.97	— .33
	81	62.2	8.8	30.36	56.5	56.18	— .32
April 22, 1835, 11 o'clock, A.M.	83	66.5	16.5	30.51	56.8	55.87	— .93
	77	65	12	30.51	57.2	57.23	+ .03
	71.3	63	8.3	30.51	57.5	57.47	— .03
April 23, 1835, 11 o'clock, A.M.	91.8	68.6	23.2	30.51	54.1	53.70	— .40
	75.2	63.2	12	30.51	55	54.94	— .06
	72	62	10	30.51	55.1	54.98	— .12
							— .35=mean.

By a glance at the preceding table, which includes twenty-four distinct observations, we will perceive, 1st. that in the case of seven of them, the observed and calculated dew-points are almost coincident; 2d. that the difference in no instance exceeds, and in but a single instance reaches, one degree; and 3d. that the mean difference, deducible from the whole, is but .35, or about one-third of a degree Fahrenheit. It will also be noted that the difference is negative, or that the mean calculated dew-point is lower than the observed, and not vice versâ. If we were justified in considering this latter result as any thing more than accidental, it might certainly be urged as an argument against the strict accuracy either of our experiments, or our theoretical views; for the corrections for the influence of pressure and aqueous vapour on the specific heat of air being neglected in the preceding calculations, the calculated dew-points, instead of being lower, should be higher than the truth. In order, in fact, to account for the discrepancy in question, supposing it to be well established, it would be necessary to conclude either that m, the co-efficient of our hygrometric formula, is assumed somewhat too great, or that the observed depressions are a little too small. The first, I believe, to be the true solution, and am, at present disposed to consider m as more correctly represented by the fraction $\frac{1}{88}$ than $\frac{1}{87}$. This point, however, I have not as yet been able fully to satisfy myself upon, nor can the more exact determination of the value of the constant be considered a matter of much practical importance, since the formula, in its present state, conducts, as we have seen, to results which harmonize admirably with each other and with observation.

I shall conclude by subjoining a couple of tables, by the aid of which the application of my formula $f'' = f' - \frac{d}{87} \times \frac{p}{30}$, to the determination of the dew-point, is greatly facilitated. Table (A), which I have taken from the Edinburgh Encyclopædia, article hygrometry, gives the elastic force of the vapour of water for every degree Fahrenheit between 0° and 100° inclusive. Table (B) gives $\frac{d}{87 \times 30}$ for every value of d between 1 and 10. This quotient, as is obvious from a glance at the formula, is, in calculating an observation, to be multiplied by p the existing pressure, and the product, when deducted from f', as given by table (A), will afford f'', or the tension of vapour, at the dew-point. Should the depression exceed 10° the value of $\frac{d}{87 \times 30}$ may still be got from table (B) by addition. Thus if $d = 13°$, $\frac{d}{87 \times 30} =$ .00383 + .00114 = .00497.

TABLE (A.)

t	f	t	f	t	f	t	f	t	f
0	.06121	21	.13408	41	.27376	61	.54089	81	1.03350
1	.06359	22	.13906	42	.28346	62	.55913	82	1.06656
2	.06605	23	.14421	43	.29348	63	.57795	83	1.10058
3	.06861	24	.14954	44	.30384	64	.59735	84	1.13559
4	.07126	25	.15506	45	.31453	65	.61734	85	1.17161
5	.07401	26	.16076	46	.32557	66	.63795	86	1.20867
6	.07685	27	.16667	47	.33684	67	.65919	87	1.24680
7	.07980	28	.17277	48	.34875	68	.68108	88	1.28602
8	.08286	29	.17908	49	.36090	69	.70364	89	1.32636
9	.08603	30	.18561	50	.37345	70	.72688	90	1.36785
10	.08931	31	.19237	51	.38640	71	.75083	91	1.41059
11	.09270	32	.19934	52	.39977	72	.77551	92	1.45438
12	.09622	33	.20658	53	.41356	73	.80092	93	1.49948
13	.09987	34	.21404	54	.42779	74	.82710	94	1.54585
14	.10364	35	.22175	55	.44249	75	.85407	95	1.59352
15	.10755	36	.22972	56	45764	76	.88184	96	1.64251
16	.11160	37	.23796	57	.47328	77	.91042	97	1.69286
17	.11579	38	.24647	58	.48940	78	.93987	98	1.74461
18	.12013	39	.25527	59	.50604	79	.97017	99	1.79778
19	.12462	40	.26436	60	.52320	80	1.00137	100	1.85241
20	.12927								

TABLE (B.)

d	$\frac{d}{87 \times 30}$	d	$\frac{d}{87 \times 30}$	d	$\frac{d}{87 \times 30}$	d	$\frac{d}{87 \times 30}$	d	$\frac{d}{87 \times 30}$
.1	.00003	2.1	.00080	4.1	00157	6.1	0033	8.1	.00310
.2	.00007	2.2	.00084	4.2	00160	6.2	.00237	8.2	.00313
.3	.00011	2.3	.00087	4.3	00164	6.3	.00241	8 3	.00317
.4	.00015	2.4	.00091	4.4	00168	6.4	.00245	8.4	.00321
.5	.00019	2.5	.00095	4.5	00172	6.5	.00248	8.5	.00325
.6	.00022	2.6	.00099	4.6	00176	6.6	.00252	8.6	.00329
.7	.00026	2 7	.00103	4.7	00180	6.7	.00256	8.7	.00333
.8	.00030	2.8	.00107	4.8	00183	6.8	.00260	8.8	.00337
.9	.00034	2.9	.00111	4.9	00187	6.9	.00264	8.9	.00340
1	.00038	3	.00114	5	00191	7	.00268	9	.00344
1.1	.00042	3.1	.00118	5.1	00195	7.1	.00271	9.1	.00348
1.2	.00045	3.2	.00122	5.2	00199	7.2	.00275	9.2	.00352
1.3	.00049	3.3	.00126	5.3	00202	7.3	.00279	9.3	.00356
1.4	.00053	3.4	.00130	5.4	00206	7.4	.00283	9.4	.00360
1 5	.00057	3.5	.00134	5.5	00210	7.5	.00287	9.5	.00363
1 6	.00061	3.6	.00137	5.6	00214	7.6	.00291	9.6	.00367
1 7	.00065	3.7	.00141	5.7	00218	7.7	.00294	9.7	.00371
1 8	.00068	3.8	.00145	5.8	00222	7.8	.00298	9.8	.00375
1.9	.00072	3.9	.00149	5 9	00225	7.9	.00302	9.9	.00379
2	.00076	4	.00153	6	00229	8	.00306	10	00383

Theory of Conjugate Functions, or Algebraic Couples; with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time.

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Read November 4th, 1833, and June 1st, 1835.

General Introductory Remarks.

THE Study of Algebra may be pursued in three very different schools, the Practical, the Philological, or the Theoretical, according as Algebra itself is accounted an Instrument, or a Language, or a Contemplation; according as ease of operation, or symmetry of expression, or clearness of thought, (the *agere*, the *fari*, or the *sapere,)* is eminently prized and sought for. The Practical person seeks a Rule which he may apply, the Philological person seeks a Formula which he may write, the Theoretical person seeks a Theorem on which he may meditate. The felt imperfections of Algebra are of three answering kinds. The Practical Algebraist complains of imperfection when he finds his Instrument limited in power; when a rule, which he could happily apply to many cases, can be hardly or not at all applied by him to some new case; when it fails to enable him to do or to discover something else, in some other Art, or in some other Science, to which Algebra with him was but subordinate, and for the sake of which and not for its own sake, he studied Algebra. The Philological Algebraist complains of imperfection, when his Language presents him with an Anomaly; when he finds an Exception disturb the simplicity of his Notation, or the symmetrical structure of his Syntax; when a Formula must be written with precaution, and a Symbolism is not universal. The Theoretical Algebraist complains of imperfection, when the clearness of his Contemplation is obscured; when the Reasonings of his Science seem anywhere to oppose each other, or become in any part too complex or too little valid for his belief to rest firmly upon them; or when, though trial may have taught him that a rule is useful, or that a formula gives true results, he cannot prove that rule, nor understand that formula: when he cannot rise to intuition from induction, or cannot look beyond the signs to the things signified.

It is not here asserted that every or any Algebraist belongs *exclusively* to any *one* of these three schools, so as to be *only* Practical, or *only* Philological, or *only* Theoretical. Language and Thought react, and Theory and Practice help each other. No man can be so merely practical as to use frequently the rules of Algebra, and never to admire the beauty of the language which expresses those rules, nor care to know the reasoning which deduces them. No man can be so merely philological an Algebraist but that things or thoughts will at some times intrude upon signs; and occupied as he may habitually be with the logical building up of his expressions, he will feel sometimes a desire to know what they mean, or to apply them. And no man can be so merely theoretical or so exclusively devoted to thoughts, and to the contemplation of theorems in Algebra, as not to feel an interest in its notation and language, its symmetrical system of signs, and the logical forms of their combinations; or not to prize those practical aids, and especially those methods of research, which the discoveries and contemplations of Algebra have given to other sciences. But, distinguishing without dividing, it is perhaps correct to say that every Algebraical Student and every Algebraical Composition may be referred upon the whole to one or other of these three schools, according as one or other of these three views habitually actuates the man, and eminently marks the work.

These remarks have been premised, that the reader may more easily and distinctly perceive what the design of the following communication is, and what the Author hopes or at least desires to accomplish. That design is *Theoretical*, in the sense already explained, as distinguished from what is Practical on the one hand, and from what is Philological upon the other. The thing aimed at, is to improve the *Science*, not the Art nor the Language of Algebra. The imperfections sought to be removed, are confusions of thought, and obscurities or errors of reasoning; not difficulties of application of an instrument, nor failures of symmetry in expression. And that confusions of thought, and errors of reasoning, still darken the beginnings of Algebra, is the earnest and just complaint of sober and thoughtful men, who in a spirit of love and honour have studied Algebraic Science, admiring, extending, and applying what has been already brought to light, and feeling all the beauty and consistence of many a remote deduction, from principles which yet remain obscure, and doubtful.

For it has not fared with the principles of Algebra as with the principles of Geometry. No candid and intelligent person can doubt the truth of the chief properties of *Parallel Lines*, as set forth by EUCLID in his Elements, two thousand years ago; though he may well desire to see them treated in a clearer and better method. The doctrine involves no obscurity nor confusion of thought, and leaves in the mind no reasonable ground for doubt, although ingenuity may usefully be exercised in improving the plan of the argument. But it requires no peculiar scepticism to doubt, or even to disbelieve, the doctrine of Negatives and Imaginaries, when set forth (as it has commonly been) with principles like these: that a *greater magnitude may be subtracted from a less*, and that the remainder is *less than nothing;* that *two negative numbers*, or numbers denoting magnitudes each less than nothing, may be *multiplied* the one by the other, and that the product will be a *positive* number, or a number denoting a magnitude greater than nothing; and that although the *square* of a number, or the product obtained by multiplying that number by itself, is therefore *always positive*, whether the number be positive or negative, yet that numbers, called *imaginary*, can be found or conceived or determined, and operated on by all the rules of positive and negative numbers, as if they were subject to those rules, *although they have negative squares*, and must therefore be supposed to be themselves neither positive nor negative, nor yet null numbers, so that the magnitudes which they are supposed to denote can neither be greater than nothing, nor less than nothing, nor even equal to nothing. It must be hard to found a SCIENCE on such grounds as these, though the forms of logic may build up from them a symmetrical system of expressions, and a practical art may be learned of rightly applying useful rules which seem to depend upon them.

So useful are those rules, so symmetrical those expressions, and yet so unsatisfactory those principles from which they are supposed to be derived, that a growing tendency may be perceived to the rejection of that view which regarded Algebra as a SCIENCE, *in some sense analogous to Geometry*, and to the adoption of one or other of those two different views, which regard Algebra as an *Art*, or as a *Language:* as a System of Rules, or else as a System of Expressions, but not as a System of *Truths*, or Results having any other validity than what they may derive from their practical usefulness, or their logical or philological coherence. Opinions thus are tending to substitute for the Theoretical question,—"Is a Theorem of Algebra *true*?" the Practical question,—"Can it be *applied as an Instrument*, to do or to discover something else, in some research which is not Algebraical?" or else the Philological question,—"Does its *expression harmonise*, according to the Laws of Language, with other Algebraical expressions?"

Yet a natural regret might be felt, if such were the destiny of Algebra; if a study, which is continually engaging mathematicians more and more, and has almost superseded the Study of Geometrical Science, were found at last to be not, in any strict and proper sense, the Study of a Science at all: and if, in thus exchanging the ancient for the modern Mathesis, there were a gain only of Skill or Elegance, at the expense of Contemplation and Intuition. Indulgence, therefore, may be hoped for, by any one who would inquire, whether existing Algebra, in the state to which it has been already unfolded by the masters of its rules and of its language, offers indeed no rudiment which may encourage a hope of developing a SCIENCE of Algebra: a Science properly so called; strict, pure, and independent; deduced by valid reasonings from its own intuitive principles; and thus not less an object of priori contemplation than Geometry, nor less distinct, in its own essence, from the Rules which it may teach or use, and from the Signs by which it may express its meaning.

The Author of this paper has been led to the belief, that the Intuition of TIME is such a rudiment.

This belief involves the three following as components: First, that the notion of Time is connected with existing Algebra; Second, that this notion or intuition of Time may be unfolded into an independent Pure Science; and Third, that the Science of Pure Time, thus unfolded, is co-extensive and identical with Algebra, so far as Algebra itself is a Science. The first component judgment is the result of an induction; the second of a deduction; the third is a joint result of the deductive and inductive processes.

I. The argument for the conclusion that *the notion of Time is connected with existing Algebra*, is an induction of the following kind. The History of Algebraic Science shows that the most remarkable discoveries in it have been made, either expressly through the medium of that notion of *Time*, or through the closely connected (and in some sort coincident) notion of *Continuous Progression*. It is the genius of Algebra to consider what it reasons on as *flowing*, as it was the genius of Geometry to consider what it reasoned on as *fixed*. EUCLID* defined a tangent to a circle, APOLLONIUS† conceived a tangent to an ellipse, as an indefinite straight line which had only one point in common with the curve; they looked upon the line and curve not as nascent or growing, but as already constructed and existing in space; they studied them as *formed* and *fixed*, they compared the one with the other, and the proved exclusion of any second common point was to them the essential property, the constitutive character of the tangent. The Newtonian Method of Tangents rests on another principle; it regards the curve and line not as

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* *Εὐθεῖα κύκλου ἐφάπτεσθαι λέγεται, ἥτις ἁπτομένη τοῦ κύκλου καὶ ἐκβαλλομένη οὐ τέμνει τὸν κύκλον.*—EUCLID, Book III. Def. 2. Oxford Edition, 1703.

† *Ἐὰν ἐν κώνου τομῇ ἀπὸ τῆς κορυφῆς τῆς τομῆς ἀχθῇ εὐθεῖα παρὰ τεταγμένως κατηγμένην ἐκτὸς πεσεῖται τῆς τομῆς.—ἐκτὸς ἄρα πεσεῖται, διόπερ ἐφάπτεται τῆς τομῆς.*—APPOLLONIUS, Book 1. Prop. 17. Oxford Edition, 1710.

already formed and fixed, but rather as *nascent*, or in process of generation: and employs, as its primary conception, the thought of a *flowing point*. And, generally, the revolution which Newton[*] made in the higher parts of both pure and applied Algebra, was founded mainly on the notion of *fluxion*, which involves the notion of *Time*.

Before the age of Newton, another great revolution, in Algebra as well as in Arithmetic, had been made by the invention of *Logarithms;* and the "Canon Mirificus" attests that Napier[†] deduced that invention, not (as it is commonly said) from the arithmetical properties of powers of numbers, but from the contemplation of a *Continuous Progression;* in describing which, he speaks expresssly of *Fluxions*, *Velocities* and *Times*.

In a more modern age, Lagrange, in the Philological spirit, sought to reduce the Theory of Fluxions to a system of operations upon symbols, analogous to the earliest symbolic operations of Algebra, and professed to reject the notion of time as foreign to such a system; yet admitted[‡] that fluxions might be considered only as the velocities with which magnitudes vary, and that in so considering them, abstraction might be made of every mechanical idea. And in one of his own most important researches in pure Algebra, (the investigation of limits between which the sum of any number of terms in Taylor's Series is comprised,) Lagrange[||] employs the conception of *continuous progression* to show that a certain variable quantity may be made as small as can be desired. And not to dwell on the beautiful discoveries made by the same great mathematician, in the theory of singular primitives of equations, and in the algebraical dynamics of the heavens, through an extension of the conception of *variability*, (that is, in fact, of *flowingness*,) to quantities which had before been viewed as *fixed* or constant, it may suffice for the present to observe that Lagrange considered Algebra to be the *Science of Functions*[§], and that it is not easy to conceive a clearer or juster idea of a *Function* in this Science, than by regarding its essence as consisting in a *Law connecting Change with Change*. But where *Change* and *Progression* are, there is Time. The notion of Time is, therefore, inductively found to be connected with existing Algebra.[¶]

II. The argument for the conclusion that *the notion of time may be unfolded into an independent Pure Science*, or that *a Science of Pure Time is possible*, rests chiefly on the existence of certain priori

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[*] Considerando igitur quod quantitates æqualibus temporibus crescentes et crescendo genitæ, pro velocitate majori vel minori qua crescunt ac generantur evadunt majores vel minores; methodum quærebam determinandi quantitates ex velocitatibus motuum vel incrementorum quibus generantur; et has motuum vel incrementorum velocitates nominando *Fluxiones*, et quantitates genitas nominando *Fluentes*, incidi paulatim annis 1665 et 1666 in Methodum Fluxionum qua hic usus sum in Quadratura Curvarum—*Tractatus de Quad. Curv.*, Introd., published at the end of Sir I. Newton's Opticks, London 1704.

[†] Logarithmus ergò cujusque sinus, est numerus quam proximè definiens lineam, quæ æqualiter crevit intereà dum sinus totius linea proportionaliter in sinum illum decrevit, existente utroque motu synchrono, atque initio æquiveloce. Baron Napier's *Mirifici Logarithmorum Canonis Descriptio*, Def. 6, Edinburgh 1614.—Also in the explanation of Def. 1, the words *fluxu* and *fluat* occur.

[‡] Calcul des Fonctions, Leçon Premiere, page 2. Paris 1806.

[||] Donc puisque V devient nul lorsque i devient nul, il est clair qu' en faisant croître i par degrés insensibles depuis zéro, la valeur de V croîtra aussi insensiblement depuis zéro, soit en plus ou en moins, jusqu' à un certain point, après quoi elle pourra diminuer.—Calcul des Fonctions, Leçon Neuvième, page 90. Paris 1806. An instance still more strong may be found in the First Note to Lagrange's *Equations Numeriques*. Paris, 1808.

[§] On doit regarder l'algèbre comme la science des fonctions.—Calc. des Fonct., Leçon Premiere.

[¶] The word "Algebra" is used throughout this whole paper, in the sense which is commonly but improperly given by modern mathematical writers to the name "Analysis," and not with that narrow signification to which the unphilosophical use of the latter term (Analysis) has caused the former term (Algebra) to be too commonly confined. The author confesses that he has often deserved the censure which he has here so freely expressed.

intuitions, connected with that notion of time, and fitted to become the sources of a pure Science; and on the actual deduction of such a Science from those principles, which the author conceives that he has begun. Whether he has at all succeeded in *actually effecting* this deduction, will be judged after the Essay has been read; but that such a deduction is *possible*, may be concluded in an easier way, by an appeal to those intuitions to which allusion has been made. That a moment of time respecting which we inquire, as compared with a moment which we know, must either coincide with or precede or follow it, is an intuitive truth, as certain, as clear, and as unempirical as this, that no two straight lines can comprehend an area. The notion or intuition of ORDER IN TIME is not less but more deep-seated in the human mind, than the notion or intuition of ORDER IN SPACE; and a mathematical Science may be founded on the former, as pure and as demonstrative as the science founded on the latter. There is something mysterious and transcendent involved in the idea of Time; but there is also something definite and clear: and while Metaphysicians meditate on the one, Mathematicians may reason from the other.

III. That the *Mathematical Science of Time*, when sufficiently unfolded, and distinguished on the one hand from all actual Outward Chronology (or collections of recorded events and phenomenal marks and measures), and on the other hand from all Dynamical Science (or reasonings and results from the notion of cause and effect), will ultimately be found to be co-extensive and identical with Algebra, so far as Algebra itself is a Science: is a conclusion to which the author has been led by all his attempts, whether to *analyse* what is *Scientific in Algebra*, or to *construct* a *Science of Pure Time*. It is a joint result of the inductive and deductive processes, and the grounds on which it rests could not be stated in a few general remarks. The author hopes to explain them more fully in a future paper; meanwhile he refers to the present one, as removing (in his opinion) the difficulties of the usual theory of Negative and Imaginary Quantities, or rather substituting a new Theory of *Contrapositives* and *Couples*, which he considers free from those old difficulties, and which is deduced from the Intuition or Original Mental Form of Time: the opposition of the (so-called) Negatives and Positives being referred by him, *not* to the opposition of the operations of increasing and diminishing a *magnitude*, but to the simpler and more extensive contrast between the relations of *Before* and *After*,* or between the directions of *Forward* and *Backward;* and *Pairs of Moments* being used to suggest a *Theory of Conjugate Functions*,† which gives reality and meaning to conceptions that were before Imaginary,‡ Impossible, or Contradictory, because Mathematicians had derived them from that bounded notion of *Magnitude*, instead of the original and comprehensive thought of ORDER IN PROGRESSION.

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* It is, indeed, very common, in Elementary works upon Algebra, to allude to *past and future time*, as one among many *illustrations* of the doctrine of negative quantities; but this avails little for Science, so long as *magnitude* instead of PROGRESSION is attempted to be made the *basis* of the doctrine.

† The author was conducted to this Theory many years ago, in reflecting on the important symbolic results of Mr. GRAVES respecting Imaginary Logarithms, and in attempting to explain to himself the theoretical meaning of those remarkable symbolisms. The Preliminary and Elementary Essay on Algebra as the Science of Pure Time, is a much more recent developement of an Idea against which the author struggled long, and which he still longer forbore to make public, on account of its departing so far from views now commonly received. The novelty, however, is in the view and method, not in the results and details: in which the reader is warned to expect little addition, if any, to what is already known.

‡ The author acknowledges with pleasure that he agrees with M. CAUCHY, in considering every (so-called) Imaginary Equation as a symbolic representation of two separate Real Equations: but he differs from that excellent mathematician in his method generally, and especially in not introducing the sign $\sqrt{-1}$ until he has provided for it, by his Theory of Couples, a possible and real meaning, as a symbol of the couple (0,1).

CONTENTS OF THE PRELIMINARY AND ELEMENTARY ESSAY ON ALGEBRA AS THE SCIENCE OF PURE TIME.

CONTENTS OF THE THEORY OF CONJUGATE FUNCTIONS, OR ALGEBRAIC COUPLES.

PRELIMINARY AND ELEMENTARY ESSAY

ON ALGEBRA AS THE SCIENCE OF PURE TIME.

Comparison of any two moments with respect to identity or diversity, subsequence or precedence.

1. If we have formed the *thought* of any one moment of time, we may afterwards either *repeat* that thought, or else think of a *different* moment. And if any two spoken or written *names*, such as the letters A and B, be *dates*, or answers to the question *When*, denoting each a known moment of time, they must either be names of one and the *same* known moment, or else of two *different* moments. In each case, we may speak of the *pair of dates* as denoting a *pair of moments*; but in the first case, the two moments are coincident, while in the second case they are distinct from each other. To express concisely the former case of relation, that is, the case of *identity* between the moment named B and the moment named A, or of *equivalence* between the date B and the date A, it is usual to write

$$B = A ; \qquad (1.)$$

a written sentence or assertion, which is commonly called an *equation*: and to express concisely the latter case of relation, that is, the case of *diversity* between the two moments, or of *non-equivalence* between the two dates, we may write

$$B \neq A ; \qquad (2.)$$

annexing, here and afterwards, to these concise written expressions, the side-marks (1.) (2.), &c., merely to facilitate the subsequent reference in this essay to any such assertion or result, whenever such reference may become necessary or convenient. The latter case of relation, namely, the case (2.) of diversity between two moments, or of non-equivalence between two dates, subdivides itself into the two cases of *subsequence* and of *precedence*, according as the moment B is later or earlier than A. To express concisely the former sort of diversity, in which the moment B is *later* than A, we may write

$$B > A\,; \qquad (3.)$$

and the latter sort of diversity, in which the moment B is *earlier* than A, may be expressed concisely in this other way,

$$B < A. \qquad (4.)$$

It is evident that

$$\text{if } B = A, \text{ then } A = B\,; \qquad (5.)$$

$$\text{if } B \neq A, \text{ then } A \neq B\,; \qquad (6.)$$

$$\text{if } B > A, \text{ then } A < B\,; \qquad (7.)$$

$$\text{if } B < A, \text{ then } A > B. \qquad (8.)$$

Comparison of two pairs of moments, with respect to their analogy or non-analogy.

2. Considering now any two other dates C and D, we perceive that they may and must represent either the *same* pair of moments as that denoted by the former pair of dates A and B, or else a *different* pair, according as the two conditions,

$$C = A, \text{ and } D = B, \qquad (9.)$$

are, or are not, both satisfied. If the new pair of moments be the *same* with the old, then the connecting relation of identity or diversity between the moments of the one pair is necessarily the same with the relation which connects in like manner the moments of the other pair, because the pairs themselves are the same. But even if the *pairs* be *different*, the *relations* may still be the same; that is, the moments C and D, even if not both respectively coincident with the moments A and B, may still be related to each other exactly as those moments, (D to C as B to A;) and thus the two pairs, A, B and C, D may be *analogous*, even if they be *not coincident* with each other. An *analogy* of this sort (whether between coincident or different pairs) may be expressed in writing as follows,

$$\mathrm{D}-\mathrm{C}=\mathrm{B}-\mathrm{A}, \text{ or, } \mathrm{B}-\mathrm{A}=\mathrm{D}-\mathrm{C}; \qquad (10.)$$

the interposed mark =, which before denoted identity of moments, denoting now identity of relations : and the written assertion of this identity being called (as before) an *equation*. The conditions of this exact identity between the relation of the moment D to C, and that of B to A, may be stated more fully as follows: that if the moment B be identical with A, then D must be identical with C ; if B be later than A, then D must be later than C, and exactly so much later; and if B be earlier than A, then D must be earlier than C, and exactly so much earlier. It is evident, that whatever the moments A B and C may be, there is always one, and only one, connected moment D, which is thus related to C, *exactly* as B is to A ; and it is not difficult to perceive that the same moment D is also related to B, exactly as C is to A : since, in the case of coincident pairs, D is identical with B, and C with A ; while, in the case of pairs analogous but not coincident, the moment D is later or earlier than B, according as C is later or earlier than A, and exactly so much later or so much earlier. If then the pairs A, B, and C, D, be analogous, the pairs A C and B D, which may be said to be *alternate* to the former, are also analogous pairs ; that is,

$$\text{if } \mathrm{D}-\mathrm{C}=\mathrm{B}-\mathrm{A}, \text{ then } \mathrm{D}-\mathrm{B}=\mathrm{C}-\mathrm{A}; \qquad (11.)$$

a change of statement of the relation between these four moments A B C D, which may be called *alternation* of an analogy. It is still more easy to perceive, that if any two pairs AB and CD be analogous, then the *inverse* pairs BA and DC are analogous also, and therefore that

$$\text{if } \mathrm{D}-\mathrm{C}=\mathrm{B}-\mathrm{A}, \text{ then } \mathrm{C}-\mathrm{D}=\mathrm{A}-\mathrm{B}, \qquad (12.)$$

a change in the manner of expressing the relation between the four moments A B C D, which may be called *inversion* of an analogy. Combining inversion with alternation, we find that

$$\text{if } \mathrm{D}-\mathrm{C}=\mathrm{B}-\mathrm{A}, \text{ then } \mathrm{B}-\mathrm{D}=\mathrm{A}-\mathrm{C}; \qquad (13.)$$

and thus that all the eight following written sentences express only one and the same relation between the four moments A B C D :

$$\left.\begin{array}{ll} \mathrm{D}-\mathrm{C}=\mathrm{B}-\mathrm{A}, & \mathrm{B}-\mathrm{A}=\mathrm{D}-\mathrm{C}, \\ \mathrm{D}-\mathrm{B}=\mathrm{C}-\mathrm{A}, & \mathrm{C}-\mathrm{A}=\mathrm{D}-\mathrm{B}, \\ \mathrm{C}-\mathrm{D}=\mathrm{A}-\mathrm{B}, & \mathrm{A}-\mathrm{B}=\mathrm{C}-\mathrm{D}, \\ \mathrm{B}-\mathrm{D}=\mathrm{A}-\mathrm{C}, & \mathrm{A}-\mathrm{C}=\mathrm{B}-\mathrm{D}; \end{array}\right\} \qquad (14.)$$

any one of these eight written sentences or *equations* being equivalent to any other.

3. When the foregoing relation between four moments A B C D does not exist, that is, when the pairs AB and CD are not analogous pairs, we may mark this *non-analogy* by writing

$$\mathrm{D} - \mathrm{C} \ddagger \mathrm{B} - \mathrm{A} ; \qquad (15.)$$

and the two possible cases into which this general conception of non-analogy or diversity of relation subdivides itself, namely, the case when the analogy fails on account of the moment D being *too late*, and the case when it fails because that moment D is *too early*, may be denoted, respectively, by writing in the first case,

$$\mathrm{D} - \mathrm{C} > \mathrm{B} - \mathrm{A}, \qquad (16.)$$

and in the second case,

$$\mathrm{D} - \mathrm{C} < \mathrm{B} - \mathrm{A} ; \qquad (17.)$$

while the two cases themselves may be called, respectively, a *non-analogy of subsequence*, and a *non-analogy of precedence*. We may also say that the relation of D to C, as compared with that of B to A, is in the first case a relation of *comparative lateness*, and in the second case a relation of *comparative earliness*.

Alternations and inversions may be applied to these expressions of non-analogy, and the case of D *too late* may be expressed in any one of the eight following ways, which are all equivalent to each other,

$$\left.\begin{array}{ll} \mathrm{D} - \mathrm{C} > \mathrm{B} - \mathrm{A}, & \mathrm{B} - \mathrm{A} < \mathrm{D} - \mathrm{C}, \\ \mathrm{D} - \mathrm{B} > \mathrm{C} - \mathrm{A}, & \mathrm{C} - \mathrm{A} < \mathrm{D} - \mathrm{B}, \\ \mathrm{C} - \mathrm{D} < \mathrm{A} - \mathrm{B}, & \mathrm{A} - \mathrm{B} > \mathrm{C} - \mathrm{D}, \\ \mathrm{B} - \mathrm{D} < \mathrm{A} - \mathrm{C}, & \mathrm{A} - \mathrm{C} > \mathrm{B} - \mathrm{D}; \end{array}\right\} \qquad (18.)$$

while the other case, when the analogy fails because the moment D is *too early*, may be expressed at pleasure in any of the eight ways following,

$$\left.\begin{array}{ll} \mathrm{D} - \mathrm{C} < \mathrm{B} - \mathrm{A}, & \mathrm{B} - \mathrm{A} > \mathrm{D} - \mathrm{C}, \\ \mathrm{D} - \mathrm{B} < \mathrm{C} - \mathrm{A}, & \mathrm{C} - \mathrm{A} > \mathrm{D} - \mathrm{B}, \\ \mathrm{C} - \mathrm{D} > \mathrm{A} - \mathrm{B}, & \mathrm{A} - \mathrm{B} < \mathrm{C} - \mathrm{D}, \\ \mathrm{B} - \mathrm{D} > \mathrm{A} - \mathrm{C}, & \mathrm{A} - \mathrm{C} < \mathrm{B} - \mathrm{D}. \end{array}\right\} \qquad (19.)$$

In general, if we have any analogy or non-analogy between two pairs of moments, A B and C D, of which we may call the first and fourth mentioned moments, A and D, the *extremes*, and the second and third mentioned moments, namely, B and C, the *means*, and may call A and C the *antecedents*, and B and D the *consequents;* we do not disturb this analogy or non-analogy by interchanging the means among themselves, or the extremes among themselves; or by altering equally, in direction and in degree, the two consequents, or the two antecedents, of the analogy or of the non-analogy, or the two moments of either pair; or, finally, by altering oppositely in direction, and equally in degree, the two extremes, or the two means. In an analogy, we may also put, by inversion, extremes for means, and means for extremes; but if a non-analogy be thus inverted, it must afterwards be changed in kind, from subsequence to precedence, or from precedence to subsequence.

Combinations of two different analogies, or non-analogies, of pairs of moments, with each other.

4. From the remarks last made, it is manifest that

$$\left.\begin{array}{ll}\text{if} & D-C=B-A,\\ \text{and} & D'-D=B'-B,\\ \text{then} & D'-C=B'-A;\end{array}\right\}\qquad(20.)$$

because the second of these three analogies shews, that in passing from the first to the third, we have either made no change, or only altered equally in direction and in degree the two consequent moments B and D of the first analogy. In like manner,

$$\left.\begin{array}{ll}\text{if} & D-C=B-A,\\ \text{and} & C'-C=A'-A,\\ \text{then} & D-C'=B-A';\end{array}\right\}\qquad(21.)$$

because now, in passing from the first to the third analogy, the second analogy shews that we have either made no change, or else have only altered equally, in direction and degree, the antecedents A and C. Again,

$$\left.\begin{array}{ll}\text{if} & D-C=B-A,\\ \text{and} & D'-D=C'-C,\\ \text{then} & D'-C'=B-A;\end{array}\right\}\qquad(22.)$$

because here we have only altered equally, if at all, the two moments C and D of one common pair, in passing from the first analogy to the third. Again,

$$\left.\begin{array}{ll}\text{if} & D-C=B-A,\\ \text{and} & C-C'=B'-B,\\ \text{then} & D-C'=B'-A;\end{array}\right\}\qquad(23.)$$

because now we either do not alter the means B and C at all, or else alter them oppositely in direction and equally in degree. And similarly,

$$\left.\begin{aligned} &\text{if} && D - C = B - A, \\ &\text{and} && D' - D = A - A', \\ &\text{then} && D' - C = B - A', \end{aligned}\right\} \qquad (24.)$$

because here we only alter equally, if at all, in degree, and oppositely in direction, the extremes, A and D, of the first analogy. It is still more evident that if two pairs be analogous to the same third pair, they are analogous to each other; that is

$$\left.\begin{aligned} &\text{if} && D - C = B - A, \\ &\text{and} && B - A = D' - C', \\ &\text{then} && D - C = D' - C'. \end{aligned}\right\} \qquad (25.)$$

And each of the foregoing conclusions will still be true, if we change the first supposed analogy $D - C = B - A$, to a non-analogy of subsequence $D - C > B - A$, or to a non-analogy of precedence $D - C < B - A$, provided that we change, in like manner, the last or concluded analogy to a non-analogy of subsequence in the one case, or of precedence in the other.

It is easy also to see, that if we still suppose the first analogy $D - C = B - A$ to remain, we cannot conclude the third analogy, and are not even at liberty to suppose that it exists, in any one of the foregoing combinations, unless we suppose the second also to remain: that is, if two analogies have the same antecedents, they must have analogous consequents; if the consequents be the same in two analogies, the antecedents must themselves form two analogous pairs; if the extremes of one analogy be the same with the extremes of another, the means of either may be combined as extremes with the means of the other as means, to form a new analogy; if the means of one analogy be the same with the means of another, then the extremes of either may be combined as means with the extremes of the other as extremes, and the resulting analogy will be true; from which the principle of inversion enables us farther to infer, that if the extremes of one analogy be the same with the means of another, then the means of the former may be combined as means with the extremes of the latter as extremes, and will thus generate another true analogy.

On continued Analogies, or Equidistant Series of Moments.

5. It is clear from the foregoing remarks, that in any analogy

$$B' - A' = B - A, \qquad (26.)$$

the two moments of either pair $A\ B$ or $A'\ B'$ cannot coincide, and so reduce themselves to one single moment, without the two moments of the other pair A′ B′ or A B being also identical with each other; nor can the two antecedents A A′ coincide, without the two consequents B B′ coinciding also, nor can the consequents without the antecedents. The only way, therefore, in which two of the four moments A B A′ B′ of an analogy can coincide, without the two others coinciding also, that is, the only way in which an analogy can be constructed with three distinct moments of time, is either by the two extremes A B′ coinciding, or else by the two means B A′ coinciding; and the principle of inversion permits us to reduce the former of these two cases to the latter. We may then take as a sufficient type of every analogy which can be constructed with three distinct moments, the following:

$$\mathrm{B}' - \mathrm{B} = \mathrm{B} - \mathrm{A}; \qquad (27.)$$

that is, the case when an extreme moment B′ is related to a mean moment B, as that mean moment B is related to another extreme moment A; in which case we shall say that the three moments A B B′ compose a *continued analogy*. In such an analogy, it is manifest that the three moments A B B′ compose also an *equidistant series*, B′ being exactly so much later or so much earlier than B, as B is later or earlier than A. The moment B is evidently, in this case, exactly intermediate between the two other moments A and B′, and may be therefore called *the middle moment*, or the *bisector*, of the interval of time between them. It is clear that whatever two distinct moments A and B′ may be, there is always one and only one such bisector moment B; and that thus a continued analogy between three moments can always be constructed in one but in only one way, by inserting a mean, when the extremes are given. And it is still more evident, from what was shewn before, that the middle moment B, along with either of the extremes, determines the other extreme, so that it is always possible to complete the analogy in one but in only one way, when an extreme and the middle are given.

6. If, besides the continued analogy (27.) between the three moments A B B′, we have also a continued analogy between the two last B B′ of these three and a fourth moment B″, then the *four* moments A B B′ B″ may themselves also be said to form another *continued analogy*, and an *equidistant series*, and we may express their relations as follows:

$$\mathrm{B}'' - \mathrm{B}' = \mathrm{B}' - \mathrm{B} = \mathrm{B} - \mathrm{A}. \qquad (28.)$$

In this case, the interval between the two extreme moments A and B″ is *trisected* by the two intermediate moments B and B′, and we may call B the *first trisector*, and B′ the *second trisector* of that interval. If the first extreme moment A and the first trisector moment B be given, it is evidently possible to complete the continued analogy or equidistant series in one and in only one way, by supplying the second trisector B' and the second extreme B''; and it is not much less easy to perceive that any two of the four moments being given, (together with their names of position in the series, as such particular extremes, or such particular trisectors,) the two other moments can be determined, as necessarily connected with the given ones. Thus, if the extremes be given, we must conceive their interval as capable of being trisected by two means, in one and in only one way; if the first extreme and second trisector be given, we can bisect the interval between them, and so determine (in thought) the first trisector, and afterwards the second extreme; if the two trisectors be given, we can continue their interval equally in opposite directions, and thus determine (in thought) the two extremes; and if either of these two trisectors along with the last extreme be given, we can determine, by processes of the same kind, the two other moments of the series.

7. In general, we can imagine a continued analogy and an equidistant series, comprising any number of moments, and having the interval between the extreme moments of the series divided into the next lesser number of portions equal to each other, by a number of intermediate moments which is itself the next less number to the number of those equal portions of the whole interval. For example, we may imagine an equidistant series of five moments, with the interval between the two extremes divided into four partial and mutually equal intervals, by three intermediate moments, which may be called the first, second, and third *quadrisectors* or quarterers of the total interval. And it is easy to perceive, that when any two moments of an equidistant series are given, (as such or such known moments of time,) together with their places in that series, (as such particular extremes, or such particular intermediate moments,) the other moments of the series can then be all determined; and farther, that the series itself may be continued forward and backward, so as to include an unlimited number of new moments, without losing its character of equidistance. Thus, if we know the first extreme moment A, and the third quadrisector B'' of the total interval (from A to B''') in any equidistant series of five moments, $A\ B\ B'\ B''\ B'''$, we can determine by trisection the two first quadrisectors B and B', and afterwards the last extreme moment B'''; and may then continue the series, forward and backward, so as to embrace other moments B^{iv}, B^{v}, &c., beyond the fifth of those originally conceived, and others also such as E, E', E'', &c., behind the first of the original five moments, that is, preceding it in the order of progression of the series; these new moments forming with the old an equidistant series of moments, (which comprehends as a part of itself the original series of five,) namely, the following unlimited series,

$$\ldots.\mathrm{E}''\,\mathrm{E}'\,\mathrm{E}\,\mathrm{A}\,\mathrm{B}\,\mathrm{B}'\,\mathrm{B}''\,\mathrm{B}'''\,\mathrm{B}^{\mathrm{IV}}\,\mathrm{B}^{\mathrm{V}}\ldots, \qquad (29.)$$

constructed so as to satisfy the conditions of a continued analogy,

$$\ldots\mathrm{B}^{\mathrm{V}}-\mathrm{B}^{\mathrm{IV}}=\mathrm{B}^{\mathrm{IV}}-\mathrm{B}'''=\mathrm{B}'''-\mathrm{B}''=\mathrm{B}''-\mathrm{B}'=\mathrm{B}'-\mathrm{B}=\mathrm{B}-\mathrm{A}=\mathrm{A}-\mathrm{E}=\mathrm{E}-\mathrm{E}'=\mathrm{E}'-\mathrm{E}''\ldots \qquad (30.)$$

8. By thus constructing and continuing an equidistant series, of which any two moments are given, we can arrive at other moments, as far from those two, and as near to each other, as we desire. For no moment B can be so distant from a given moment A, (on either side of it, whether as later or as earlier,) that we cannot find others still more distant, (and on the same side of A, still later or still earlier,) by continuing (in both directions) any given analogy, or given equidistant series; and, therefore, no two given moments, C and D, if not entirely coincident, can possibly be so near to each other, that we cannot find two moments still more near by treating any two given distinct moments (A and B), whatever, as extremes of an equidistant series of moments sufficiently many, and by inserting the appropriate means, or intermediate moments, between those two given extremes. Since, however far it may be necessary to continue the equidistant series C D...D$''$, with C and D for its two first moments, in order to arrive at a moment D$''$ more distant from C than B is from A, it is only necessary to insert as many intermediate moments between A and B as between C and D$''$, in order to generate a new equidistant series of moments, each nearer to the one next it than D to C. Three or more moments A B C &c. may be said to be *uniserial* with each other, when they all belong to one common continued analogy, or equidistant series; and though we have not proved (and shall find it not to be true) that *any* three moments whatever are thus uniserial moments, yet we see that if any two moments be given, such as A and B, we can always find a third moment B^{n} uniserial with these two, and differing (in either given direction) by less than any interval proposed from any given third moment C, whatever that may be. This possibility of indefinitely appproaching (on either side) to any given moment C, by moments uniserial with any two given ones A and B, increases greatly the importance which would otherwise belong to the theory of continued analogies, or equidistant series of moments. Thus if any two given dates, C and D, denote two distinct moments of time, ($\mathrm{C} \neq \mathrm{D}$,) however near to each other they may be, we can always conceive their diversity detected by inserting means sufficiently numerous between any two other given distinct moments A and B, as the extremes of an equidistant series, and then, if necessary, extending this series in both directions beyond those given extremes, until some one of the moments B'' of the equidistant series thus generated is found to fall between the two near moments C and D, being later than the earlier, and earlier than

the later of those two. And, therefore, reciprocally, if in any case of two given dates C and D, we can prove that no moment B″ *whatever*, of all that can be imagined as uniserial with two given distinct moments A and B, falls thus between the moments C and D, we shall then have a sufficient proof that those two moments C and D are identical, or, in other words, that the two dates C and D represent only one common moment of time, ($C=D$,) and not two different moments, however little asunder.

And even in those cases in which we have not yet succeeded in discovering a rigorous proof of this sort, identifying a sought moment with a known one, or distinguishing the former from the latter, the conception of continued analogies offers always a method of research, and of nomenclature, for investigating and expressing, or, at least, conceiving as investigated and expressed, with any proposed degree of approximation if not with perfect accuracy, the situation of the sought moment in the general progression of time, by its relation to a known equidistant series of moments sufficiently close. This might, perhaps, be a proper place, in a complete treatise on the *Science of Pure Time,* to introduce a regular system of *integer ordinals,* such as the words *first, second, third,* &c., with the written marks 1, 2, 3, &c., which answer both to them and to the *cardinal* or quotitative numbers, *one, two, three,* &c.; but it is permitted and required, by the plan of the present essay, that we should treat these spoken and written names of the integer ordinals and cardinals, together with the elementary laws of their combinations, as already known and familiar. It is the more admissible in point of method to suppose this previous acquaintance with the chief properties of integer numbers, as set forth in elementary arithmetic, because these properties, although belonging to the Science of Pure Time, as involving the conception of succession, may all be deduced from the unfolding of that mere conception of *succession,* (among things or thoughts as *counted,*) without requiring any notion of *measurable intervals,* equal or unequal, between successive moments of time. Arithmetic, or the *science of counting,* is, therefore, a part, indeed, of the *Science of Pure Time,* but a part so simple and familiar that it may be presumed to have been previously and separately studied, to some extent, by any one who is entering on Algebra.

On steps in the progression of time; their application (direct or inverse) to moments, so as to generate other moments; and their combination with other steps, in the way of composition or decomposition.

9. The foregoing remarks may have sufficiently shewn the importance, in the general study of pure time, of the conception of a continued analogy or equidistant

series of moments. This conception involves and depends on the conception of the repeated transference of one common ordinal relation, or the continued application of one common mental step, by which we pass, in thought, from any moment of such a series to the moment immediately following. For this, and for other reasons, it is desirable to study, generally, the properties and laws of the transference, or application, direct or inverse, and of the composition or decomposition, of ordinal relations between moments, or of steps in the progression of time; and to form a convenient system of written signs, for concisely expressing and reasoning on such applications and such combinations of steps.

In the foregoing articles, we have denoted, by the complex symbol $\mathrm{B} - \mathrm{A}$, the ordinal relation of the moment B to the moment A, whether that relation were one of identity or of diversity; and if of diversity, then whether it were one of subsequence or of precedence, and in whatever degree. Thus, having previously interposed the mark $=$ between two equivalent signs for one common moment of time, we came to interpose the same sign of equivalence between any two marks of one ordinal relation, and to write

$$\mathrm{D} - \mathrm{C} = \mathrm{B} - \mathrm{A},$$

when we designed to express that the relations of D to C and of B to A were coincident, being both relations of identity, or both relations of diversity; and if the latter, then both relations of subsequence, or both relations of precedence, and both in the same degree. In like manner, having agreed to interpose the mark $\neq$ between the two signs of two moments essentially different from each other, we wrote

$$\mathrm{D} - \mathrm{C} \neq \mathrm{B} - \mathrm{A},$$

when we wished to express that the ordinal relation of D to C (as identical, or subsequent, or precedent) did *not* coincide with the ordinal relation of the moment B to A; and, more particularly, when we desired to distinguish between the two principal cases of this non-coincidence of relations, namely the case when the relation of D to C (as compared with that of B to A) was comparatively a relation of lateness, and the case when the same relation (of D to C) was comparatively a relation of earliness, we wrote, in the first case,

$$\mathrm{D} - \mathrm{C} > \mathrm{B} - \mathrm{A},$$

and in the second case,

$$\mathrm{D} - \mathrm{C} < \mathrm{B} - \mathrm{A},$$

having previously agreed to write

$$\mathrm{B} > \mathrm{A}$$

if the moment B were later than the moment A, or

$$B < A$$

if B were earlier than A.

Now, without yet altering at all the foregoing conception of B − A, as the symbol of an *ordinal relation* discovered by the comparison of two moments, we may in some degree abridge and so far simplify all these foregoing expressions, by using a simpler symbol of relation, such as a single letter a or b &c. or in some cases the character 0, or other simple signs, instead of a complex symbol such as B − A, or D − C, &c. Thus, if we agree to use the symbol 0 to denote the relation of identity between two moments, writing

$$A - A = 0, \qquad (31.)$$

we may express the equivalence of any two dates B and A, by writing

$$B - A = 0, \qquad (32.)$$

and may express the non-equivalence of two dates by writing

$$B - A \mp 0\,; \qquad (33.)$$

distinguishing the two cases when the moment B is later and when it is earlier than A, by writing, in the first case,

$$B - A > 0, \qquad (34.)$$

and in the second case,

$$B - A < 0, \qquad (35.)$$

to express, that as compared with the relation of identity 0, the relation B − A is in the one case a relation of comparative lateness, and in the other case a relation of comparative earliness: or, more concisely, by writing, in these four last cases respectively, which were the cases before marked (1.) (2.) (3.) and (4.)

$$a = 0, \qquad (36.)$$

$$a \mp 0, \qquad (37.)$$

$$a > 0, \qquad (38.)$$

$$a < 0, \qquad (39.)$$

if we put, for abridgement,

$$B - A = a\,. \qquad (40.)$$

Again, if we put, in like manner, for abridgement,

$$D - C = b, \qquad (41.)$$

the analogy (10.) namely,

$$D - C = B - A,$$

may be concisely expressed as follows,

$$b = a\,; \qquad (42.)$$

while the general non-analogy (15.),

$$D - C \neq B - A,$$

may be expressed thus,

$$b \neq a\,, \qquad (43.)$$

and the written expressions of its two cases (16.) and (17.), namely,

$$D - C > B - A$$

$$\text{and} \quad D - C < B - A,$$

may be abridged in the following manner,

$$b > a\,, \qquad (44.)$$

$$\text{and} \quad b < a\,. \qquad (45.)$$

Again, to denote a relation which shall be exactly the inverse or opposite of any proposed ordinal relation a or b, we may agree to employ a complex symbol such as $\Theta\, a$ or $\Theta\, b$, formed by prefixing the mark Θ, (namely, the initial letter O of the Latin word Oppositio, distinguished by a bar across it, from the same letter used for other purposes,) to the mark a or b of the proposed ordinal relation; that is, we may agree to use $\Theta\, a$ to denote the ordinal relation of the moment A to B, or $\Theta\, b$ to denote the ordinal relation of C to D, when the symbol a has been already chosen to denote the relation of B to A, or b to denote that of D to C: considering the two assertions

$$B - A = a\,, \text{ and } A - B = \Theta\, a, \qquad (46.)$$

as equivalent each to the other, and in like manner the two assertions

$$D - C = b, \text{ and } C - D = \Theta\, b, \qquad (47.)$$

and similarly in other cases. In this notation, the theorems (5.) (6.) (7.) (8.) may be thus respectively written:

$$\Theta\, a = 0, \text{ if } a = 0\,; \qquad (48.)$$

$$\Theta\, a \neq 0, \text{ if } a \neq 0\,; \qquad (49.)$$

$$\Theta\, a < 0, \text{ if } a > 0\,; \qquad (50.)$$

$$\Theta\, a > 0, \text{ if } a < 0\,; \qquad (51.)$$

and the theorem of inversion (12.) may be written thus:

$$\Theta\,\mathrm{b} = \Theta\,\mathrm{a}, \text{ if } \mathrm{b} = \mathrm{a}. \qquad (52.)$$

The corresponding rules for inverting a non-analogy shew that, in general,

$$\Theta\,\mathrm{b} \not= \Theta\,\mathrm{a}, \text{ if } \mathrm{b} \not= \mathrm{a}; \qquad (53.)$$

and more particularly, that

$$\Theta\,\mathrm{b} < \Theta\,\mathrm{a}, \text{ if } \mathrm{b} > \mathrm{a}, \qquad (54.)$$

$$\text{and} \quad \Theta\,\mathrm{b} > \Theta\,\mathrm{a}, \text{ if } \mathrm{b} < \mathrm{a}. \qquad (55.)$$

It is evident also that

$$\text{if } \mathrm{a}' = \Theta\,\mathrm{a}, \text{ then } \mathrm{a} = \Theta\,\mathrm{a}'; \qquad (56.)$$

that is, the opposite of the opposite of any proposed relation a is that proposed relation itself; a theorem which may be concisely expressed as follows:

$$\Theta\,(\Theta\,\mathrm{a}) = \mathrm{a}; \qquad (57.)$$

for, as a general rule of notation, when a complex symbol (as here $\Theta\,\mathrm{a}$) is substituted in any written sentence (such as here the sentence $\mathrm{a} = \Theta\,\mathrm{a}'$) instead of a simple symbol (which the symbol a', notwithstanding its accent, is here considered to be), it is expedient, and in most cases necessary, for distinctness, to record and mark this using of a complex as a simple symbol, by some such written warning as the enclosing of the complex symbol in parentheses, or in brackets, or the drawing of a bar across it. However, in the present case, no confusion would be likely to ensue from the omission of such a warning; and we might write at pleasure

$$\Theta\,(\Theta\,\mathrm{a}) = \mathrm{a},\ \Theta\,\{\Theta\,\mathrm{a}\} = \mathrm{a},\ \Theta\,[\Theta\,\mathrm{a}] = \mathrm{a},\ \Theta\,\overline{\Theta\,\mathrm{a}} = \mathrm{a}, \text{ or simply } \Theta\,\Theta\,\mathrm{a} = \mathrm{a}. \qquad (58.)$$

10. For the purpose of expressing, in a somewhat similar notation, the properties of alternations and combinations of analogies, set forth in the foregoing articles, with some other connected results, and generally for the illustration and development of the conception of ordinal *relations* between moments, it is advantageous to introduce that other connected conception, already alluded to, of *steps* in the progression of time; and to establish this other symbolic definition, or conventional manner of writing, namely,

$$\mathrm{B} = (\mathrm{B} - \mathrm{A}) + \mathrm{A}, \text{ or } \mathrm{B} = \mathrm{a} + \mathrm{A} \text{ if } \mathrm{B} - \mathrm{A} = \mathrm{a}; \qquad (59.)$$

this notation $\mathrm{a} + \mathrm{A}$, or $(\mathrm{B} - \mathrm{A}) + \mathrm{A}$, corresponding to the above-mentioned conception of a certain *mental step* or *act of transition*, which is determined in direction and degree by the ordinal relation a or $\mathrm{B} - \mathrm{A}$, and may, therefore, be called "the step a," or the step $B - A$, and which is such that by making this mental step, or performing this act of transition, we pass, in thought, from the moment A to the moment B, and thus suggest or generate (in thought) the latter from the former, as a mental product or *result* B of the *act* a and of the *object* A. We may also express the same relation between B and A by writing

$$A = (\Theta\, a) + B, \text{ or more simply } A = \Theta\, a + B, \qquad (60.)$$

if we agree to write the sign $\Theta\, a$ without parentheses, as if it were a simple or single symbol, because there is no danger of causing confusion thereby; and if we observe that the notation $A = \Theta\, a + B$ corresponds to the conception of another step, or mental act of transition, $\Theta\, a$, exactly opposite to the former step a, and such that by it we may *return* (in thought) from the moment B to the moment A, and thus may generate A as a result of the act $\Theta\, a$ and of the object B. The mark $+$, in this sort of notation, is interposed, as a *mark of combination,* between the signs of the *act* and the *object,* so as to form a complex sign of the *result;* or, in other words, between the sign of the transition (a or $\Theta\, a$) and the sign of the moment (A or B) *from* which that transition is made, so as to express, by a complex sign, (recording the suggestion or generation of the thought,) that other moment (B or A) *to* which this mental transition conducts. And in any transition of this sort, such as that expressed by the *equation* $B = a + A$, we may call (as before) the moment A, *from* which we pass, the *antecedent,* and the moment B, *to* which we pass, the *consequent,* of the ordinal *relation* a, or $B - A$, which suggests and determines the transition. In the particular case when this ordinal relation is one of *identity,* ($a = 0$,) the mental transition or *act* (a or 0) makes no change in the *object* of that act, namely in the *moment* A, but only leads us to *repeat* the thought of that antecedent moment A, perhaps with a new name B; in this case, therefore, the transition may be said to be *null,* or a *null step,* as producing no real alteration in the moment from which it is made. A step *not null,* ($a \neq 0$,) corresponds to a relation of *diversity,* and may be called, by contrast, an *effective* step, because it is an act of thought which really alters its object, namely the moment to which it is applied. An effective step a must be either a *late-making* or an *early-making* step, according as the resultant moment $a + A$ is later or earlier than A; but even a *null* step 0 may be regarded as *relatively late-making,* when compared with an early-making step a, ($0 + A > a + A$, if $a < 0$,) or as *relatively early-making* if compared with a late-making step b; ($0 + A < b + A$, if $b > 0$;) and, in like manner, of two unequal early-making steps, the lesser may be regarded as relatively late-making, while of two unequal late-making steps the lesser step may be considered as relatively early-making. With these conceptions of the *relative effects* of any two steps a and b, we may enunciate in words the non-analogy (44.), ($b > a$, that is, $b + A > a + A$,) by saying that the step b as compared with the step a is *relatively late-making ;* and the opposite non-analogy (45.), ($b < a$, that is, $b + A < a + A$,) by saying that the step b as compared with a is *relatively early-making.*

11. After having made any one step a from a proposed moment A to a resulting moment represented (as before) by $a + A$, we may conceive that we next make from this new moment $a + A$ a new step b, and may denote the new result by the new complex symbol $b + (a + A)$; enclosing in parentheses the sign $a + A$ of the *object* of this new *act* of mental transition, or (in other words) the sign of the new antecedent moment, to mark that it is a complex used as a simple symbol ; so that, in this notation,

$$\text{if } B - A = a, \text{ and } C - B = b, \text{ then } C = b + (a + A). \qquad (61.)$$

It is evident that the *total change* or *total step,* effective or null, from the first moment A to the last moment C, in this successive transition from A to B and from B to C, may be considered as *compounded* of the two successive or *partial steps* a and b, namely the step a from A to B, and the step b from B to C ; and that the *ultimate ordinal relation* of C to A may likewise be considered as *compounded* of the two *intermediate* (or suggesting) ordinal relations b and a, namely, the relation b of C to B, and the relation a of B to A ; a composition of steps or of relations which may conveniently be denoted, by interposing, as a mark of combination, between the signs of the component steps or of the component ordinal relations, the same mark + which was before employed to combine an act of transition with its object, or an ordinal relation with its antecedent. We shall therefore denote the compound transition from A to C, or the compound relation of C to A, by the complex symbol $b + a$, writing,

$$C - A = b + a, \text{ if } B - A = a, \text{ and } C - B = b, \qquad (62.)$$

that is,

$$c = b + a, \text{ if } B = a + A, \; C = b + B, \; C = c + A. \qquad (63.)$$

For example, the case of coincidence between the moments A and C, that is, the case when the resulting relation of C to A is the relation of identity, and when therefore the total or compound transition from A to C is null, because the two component or successive steps a and b have been exactly opposite to each other, conducts to the relations,

$$\Theta a + a = 0 ; \; b + \Theta b = 0. \qquad (64.)$$

In general, the establishment of this new complex mark $b + a$, for the compound mental transition from A through B to C, permits us to regard the two written assertions or equations

$$C = (b + a) + A \text{ and } C = b + (a + A), \qquad (65.)$$

as expressing the same thing, or as each involving the other; for which reason we are at liberty to omit the parentheses, and may write, more simply, without fear of causing confusion,

$$C = b + a + A, \text{ if } C = b + B, \text{ and } B = a + A: \qquad (66.)$$

because the complex symbol $b + a + A$ denotes only the one determined moment C, whether it be interpreted by first applying the step a to the moment A, so as to generate another moment denoted by the complex mark $a + A$, and afterwards applying to this moment the step denoted by b, or by first combining the steps a and b into one compound step $b + a$, and afterwards applying this compound step to the original moment A.

In like manner, if three successive steps $a\ b\ c$ have conducted successively (in thought) from A to B, from B to C, and from C to D, and therefore ultimately and upon the whole from A to D, we may consider this total transition from A to D as compounded of the three steps $a\ b\ c$; we may also regard the resulting ordinal relation of D to A as compounded of the three relations c, b, a, namely of the relation c of D to C, the relation b of C to B, and the relation a of B to A; and may denote this compound step or compound relation by the complex symbol $c + b + a$, and the last resulting moment D by the connected symbol $c + b + a + A$; in such a manner that

$$\left.\begin{array}{c} D - A = c + b + a, \text{ and } D = c + b + a + A, \\ \text{if } B - A = a,\ C - B = b, \text{ and } D - C = c. \end{array}\right\} \qquad (67.)$$

For example,

$$\left.\begin{array}{c} c + \Theta a + a = c,\ \ c + b + \Theta b = c, \\ \Theta b + b + a = a,\ \ c + \Theta c + a = a. \end{array}\right\} \qquad (68.)$$

Remarks of the same kind apply to the composition of more successive steps than three. And we see that in any complex symbol suggested by this sort of composition, such as $c + b + a + A$, we are at liberty to enclose any two or more successive component symbols, such as c or b or a or A, in parentheses, with their proper combining marks $+$, and to treat the enclosed set as if they formed only one single symbol; thus,

$$\left.\begin{aligned} c + b + a + A = c + b + (a + A) &= c + (b + a) + A \\ &= (c + b + a) + A, \text{ \&c.,} \end{aligned}\right\} \qquad (69.)$$

the notation $c + (b + a) + A$, for example, directing us to begin by combining (in thought) the two steps a and b into one compound step $b + a$, and then to apply successively this compound step and the remaining step c to the original moment A; while the notation $(c + b + a) + A$ suggests a previous composition (in thought) of all the three proposed steps a, b, c, into one compound step $c + b + a$, and then the application of this one step to the same original moment. It is clear that all these different processes must conduct to one common result; and generally, that as, by the very meaning and conception of a *compound step,* it may be *applied to any moment* by applying in their proper order its component steps successively, so also may these components be *compounded* successively *with any other step,* as a mode of compounding with that other step the whole original compound.

We may also consider *decomposition* as well as composition of steps, and may propose to deduce either of two components a and b from the other component and from the compound $b + a$. For this purpose, it appears from (68.) that we have the relations

$$a = \Theta b + c, \text{ and } b = c + \Theta a, \text{ if } c = b + a; \qquad (70.)$$

observing that a problem of decomposition is plainly a determinate problem, in the sense that if any one component step, such as here the step denoted by $\Theta b + c$, or that denoted by $c + \Theta a$, has been found to conduct to a given compound c, when combined in a given order with a given component b or a, then no other component a or b, essentially different from the one thus found, can conduct by the same process of composition to the same given compound step. We see then that each of the two components a and b may be deduced from the other, and from the compound c, by compounding with that given compound the opposite of the given component, in a suitable order of composition, which order itself we shall shortly find to be indifferent.

Meanwhile it is important to observe, that though we have agreed, for the sake of conciseness, to omit the parentheses about a complex symbol of the kind Θa, when combined with other written signs by the interposed mark $+$, yet it is in general necessary, if we would avoid confusion, to retain the parentheses, or some such connecting mark or marks, for any complex symbol of a step, when we wish to form, by prefixing the mark of opposition Θ, a symbol for the opposite of that step: for example, the opposite of a compound step $b + a$ must be denoted in some such manner as $\Theta\,(b + a)$, and not merely by writing $\Theta b + a$. Attending to this remark, we may write

$$\Theta\,(b + a) = \Theta a + \Theta b, \qquad (71.)$$

because, in order to destroy or undo the effect of the compound step $b + a$, it is sufficient first to apply the step Θb which destroys the effect of the last component step b, and afterwards to destroy the effect of the first component step a by applying its opposite Θa, whatever the two steps denoted by a and b may be. In like manner,

$$\Theta\,(c + b + a) = \Theta a + \Theta b + \Theta c; \qquad (72.)$$

and similarly for more steps than three.

12. We can now express, in the language of *steps*, several other general theorems, for the most part contained under a different form in the early articles of this Essay.

Thus, the propositions (20.) and (21.), with their reciprocals, may be expressed by saying that if equivalent steps be similarly combined with equivalent steps, whether in the way of composition or of decomposition, they generate equivalent steps; an assertion which may be written thus:

$$\text{if } a' = a, \text{ then } \left.\begin{array}{l} b + a' = b + a,\ \ a' + b = a + b, \\ b + \Theta a' = b + \Theta a,\ \ \Theta a' + b = \Theta a + b, \\ \Theta b + a' = \Theta b + a,\ \ a' + \Theta b = a + \Theta b, \ \&c. \end{array}\right\} \qquad (73.)$$

The proposition (25.) may be considered as expressing, that if two steps be equivalent to the same third step, they are also equivalent to each other; or, that

$$\text{if } a'' = a' \text{ and } a' = a, \text{ then } a'' = a. \qquad (74.)$$

The theorem of alternation of an analogy (11.) may be included in the assertion that in the composition of any two steps, the order of those two components may be changed, without altering the compound step; or that

$$a + b = b + a. \qquad (75.)$$

For, whatever the four moments A B C D may be, which construct any proposed analogy or non-analogy, we may denote the step from A to B by a symbol such as a, and the step from B to D by another symbol b, denoting also the step from A to C by b', and that from C to D by a'; in such a manner that

$$B - A = a,\ D - B = b,\ C - A = b',\ D - C = a'; \qquad (76.)$$

and then the total step from A to D may be denoted either by $b + a$ or by $a' + b'$, according as we conceive the transition performed by passing through B or through C; we have therefore the relation

$$a' + b' = b + a, \qquad (77.)$$

which becomes

$$a + b' = b + a, \qquad (78.)$$

when we establish the analogy

$$D - C = B - A, \text{ that is, } a' = a; \qquad (79.)$$

we see then that if the theorem (75.) be true, we cannot have the analogy (79.) without having also its alternate analogy, namely

$$b = b', \text{ or } D - B = C - A: \qquad (80.)$$

because the compound steps $a + b'$ and $a + b$, with the common second component a, could not be equivalent, if the first components b' and b were not also equivalent to each other. The theorem (75.) includes, therefore, the theorem of alternation.

Reciprocally, from the theorem of alternation considered as known, we can infer the theorem (75.), namely, the indifference of the order of any two successive components a, b, of a compound step: for, whatever those two component steps a and b may be, we can always apply them successively to any one moment A, so as to generate two other moments B and C, and may again apply the step a to C so as to generate a fourth moment D, the moments thus suggested having the properties

$$B = a + A, \; C = b + A, \; D = a + C, \qquad (81.)$$

and being therefore such that

$$D - A = a + b, \; D - C = a = B - A; \qquad (82.)$$

by alternation of which last analogy, between the two pairs of moments A B and C D, we find this other analogy,

$$D - B = C - A = b, \; D = b + B = b + a + A, \qquad (83.)$$

and finally,

$$b + a = D - A = a + b. \qquad (84.)$$

The propositions (22.) (23.) (24.), respecting certain combinations of analogies, are included in the same assertion (75.) ; which may also, by (71.), be thus expressed,

$$\mathrm{a} + \mathrm{b} = \Theta\,(\Theta\,\mathrm{a} + \Theta\,\mathrm{b}),\ \text{or,}\ \ \mathrm{b} + \mathrm{a} = \Theta\,(\Theta\,\mathrm{b} + \Theta\,\mathrm{a})\,; \qquad (85.)$$

that is, by saying that it comes to the same thing, whether we compound any two steps a and b themselves, or first compound their opposites $\Theta\,\mathrm{a}$, $\Theta\,\mathrm{b}$, into one compound step $\Theta\,\mathrm{b} + \Theta\,\mathrm{a}$, and then take the opposite of this. Under this form, the theorem of the possibility of reversing the order of composition may be regarded as evident, whatever the number of the component steps may be ; for example, in the case of any three component steps a, b, c, we may regard it as evident that by applying these three steps successively to any moment A, and generating thus three moments B, C, D, we generate moments related to A as A itself is related to those three other moments B', C', D', which are generated from it by applying successively, in the same order, the three respectively opposite steps, $\Theta\,\mathrm{a}$, $\Theta\,\mathrm{b}$, $\Theta\,\mathrm{c}$; that is, if

$$\left.\begin{array}{ll} \mathrm{B} = \mathrm{a} + \mathrm{A}, & \mathrm{B}' = \Theta\,\mathrm{a} + \mathrm{A}, \\ \mathrm{C} = \mathrm{b} + \mathrm{B}, & \mathrm{C}' = \Theta\,\mathrm{b} + \mathrm{B}', \\ \mathrm{D} = \mathrm{c} + \mathrm{C}, & \mathrm{D}' = \Theta\,\mathrm{c} + \mathrm{C}', \end{array}\right\} \qquad (86.)$$

then the sets $\mathrm{B}'\,\mathrm{A}\,\mathrm{B}$, $\mathrm{C}'\,\mathrm{A}\,\mathrm{C}$, $\mathrm{D}'\,\mathrm{A}\,\mathrm{D}$, containing each three moments, form so many continued analogies or equidistant series, such that

$$\left.\begin{array}{l} \mathrm{B} - \mathrm{A} = \mathrm{A} - \mathrm{B}' \\ \mathrm{C} - \mathrm{A} = \mathrm{A} - \mathrm{C}' \\ \mathrm{D} - \mathrm{A} = \mathrm{A} - \mathrm{D}' \end{array}\right\} \qquad (87.)$$

and therefore not only $\mathrm{b} + \mathrm{a} = \Theta\,(\Theta\,\mathrm{b} + \Theta\,\mathrm{a})$, as before, but also

$$\mathrm{c} + \mathrm{b} + \mathrm{a} = \Theta\,(\Theta\,\mathrm{c} + \Theta\,\mathrm{b} + \Theta\,\mathrm{a}), \qquad (88.)$$

that is, by (72.) and (57.),

$$\mathrm{c} + \mathrm{b} + \mathrm{a} = \mathrm{a} + \mathrm{b} + \mathrm{c}\,; \qquad (89.)$$

and similarly for more steps than three.

The theorem (89.) was contained, indeed, in the reciprocal of the proposition (24.), namely, in the assertion that

$$\left.\begin{array}{rl} \text{if} & \mathrm{D} - \mathrm{C} = \mathrm{B} - \mathrm{A}, \\ \text{and} & \mathrm{D}' - \mathrm{C} = \mathrm{B} - \mathrm{A}', \\ \text{then} & \mathrm{D}' - \mathrm{D} = \mathrm{A} - \mathrm{A}', \end{array}\right\} \qquad (90.)$$

and, therefore, by alternation,

$$\mathrm{D}' - \mathrm{A} = \mathrm{D} - \mathrm{A}' ; \qquad (91.)$$

for, whatever the three steps $a\,b\,c$ may be, we may always conceive them applied successively to any moment A, so as to generate three other moments B, C, D', such that

$$\mathrm{B} = a + \mathrm{A}, \quad \mathrm{C} = b + \mathrm{B}, \quad \mathrm{D}' = c + \mathrm{C}, \qquad (92.)$$

and may also conceive two other moments A' and D such that $\mathrm{B}\ \mathrm{C}\ \mathrm{D}$ may be successively generated from A' by applying the same three steps in the order c, b, a, so that

$$\mathrm{B} = c + \mathrm{A}', \quad \mathrm{C} = b + \mathrm{B}, \quad \mathrm{D} = a + \mathrm{C} ; \qquad (93.)$$

and then the two first analogies of the combination (90.) will hold, and, therefore, also the last, together with its alternate (91.); that is, the step from A to D', compounded of the three steps $a\,b\,c$, is equivalent to the step from A' to D, compounded of the same three steps in the reverse order $c\,b\,a$.

Since we may thus reverse the order of any three successive steps, and also the order of any two which immediately follow each other, it is easy to see that we may interchange in any manner the order of three successive steps; thus

$$\left.\begin{array}{r} c + b + a = c + a + b = b + c + a \\ = a + b + c = a + c + b = b + a + c. \end{array}\right\} \qquad (94.)$$

We might also have proved this theorem (94.), without previously establishing the less general proposition (89.), and in a manner which would extend to any number of component steps; namely, by observing that when any arrangement of component steps is proposed, we may always reserve the first (and by still stronger reason any other) of those steps to be applied the last, and leave the order of the remaining steps unchanged, without altering the whole compound step; because the components which followed, in the proposed arrangement, that one which we now reserve for the last, may be conceived as themselves previously combined into one compound step, and this then interchanged in place with the reserved one, by the theorem respecting the arbitrary order of any two successive steps. In like manner, we might reserve any other step to be the last but one, and any other to be the last but two, and so on; by pursuing which reasoning it becomes manifest that when any number of component steps are applied to any original moment, or compounded with any primary step, their order may be altered at pleasure, without altering the resultant moment, or the whole compounded step: which is, perhaps, the most important and extensive property of the composition of ordinal relations, or steps in the progression of time.

On the Multiples of a given base, or unit-step; and on the Algebraic Addition, Subtraction, Multiplication, and Division, of their determining or multipling Whole Numbers, whether positive, or contra-positive, or null.

13. Let us now apply this general theory of successive and compound steps, from any one moment to any others, or of component and compound ordinal relations between the moments of any arbitrary set, to the case of an equidistant series of moments,

$$\ldots \mathrm{E}'' \; \mathrm{E}' \; \mathrm{E} \; \mathrm{A} \; \mathrm{B} \; \mathrm{B}' \; \mathrm{B}'' \ldots \qquad (29.)$$

constructed so as to satisfy the conditions of a continued analogy,

$$\ldots \mathrm{B}'' - \mathrm{B}' = \mathrm{B}' - \mathrm{B} = \mathrm{B} - \mathrm{A} = \mathrm{A} - \mathrm{E} = \mathrm{E} - \mathrm{E}' = \mathrm{E}' - \mathrm{E}'', \text{ \&c.}; \qquad (30.)$$

and first, for distinctness of conception and of language, let some one moment A of this series be selected as a standard with which all the others are to be compared, and let it be called the *zero-moment;* while the moments B, B′, &c. which *follow* it, in the order of progression of the series, may be distinguished from those other moments E, E′, &c., which *precede* it in that order of progression, by some two contrasted epithets, such as the words *positive* and *contra-positive:* the moment B being called the *positive first*, or the first moment of the series on the positive side of the zero; while in the same plan of nomenclature the moment B′ is the *positive second,* B″ the *positive third,* E the *contra-positive first,* E′ the *contra-positive second,* and so forth. By the nature of the series, as composed of equi-distant moments, or by the conditions (30.), all the positive or *succeeding* moments B B′ &c. may be conceived as *generated* from the zero-moment A, by the continual and successive application of one common step a, and all the contra-positive or *preceding* moments E E′ &c. may be conceived as generated from the same zero-moment A, by the continual and successive application of the opposite step Θ a, so that we may write

$$\mathrm{B} = \mathrm{a} + \mathrm{A}, \;\; \mathrm{B}' = \mathrm{a} + \mathrm{B}, \;\; \mathrm{B}'' = \mathrm{a} + \mathrm{B}', \text{ \&c.}, \qquad (95.)$$

and

$$\mathrm{E} = \Theta\, \mathrm{a} + \mathrm{A}, \;\; \mathrm{E}' = \Theta\, \mathrm{a} + \mathrm{E}, \;\; \mathrm{E}'' = \Theta\, \mathrm{a} + \mathrm{E}', \text{ \&c.}; \qquad (96.)$$

while the standard or zero-moment A itself may be denoted by the complex symbol $\Theta + \mathrm{A}$, because it may be conceived as generated from itself by applying the null step

0. Hence, by the theory of compound steps, we have expressions of the following sort for all the several moments of the equi-distant series (29.):

$$\left.\begin{array}{l}
\cdots\cdots\cdots\cdots \\
E'' = \Theta\, a + \Theta\, a + \Theta\, a + A, \\
E' = \Theta\, a + \Theta\, a + A, \\
E = \Theta\, a + A, \\
A = 0 + A, \\
B = a + A, \\
B' = a + a + A, \\
B'' = a + a + a + A, \\
\cdots\cdots\cdots\cdots
\end{array}\right\} \qquad (97.)$$

with corresponding expressions for their several ordinal relations to the one standard moment A, or for the acts of transition which are made in passing from A to them, namely:

$$\left.\begin{array}{l}
\cdots\cdots\cdots\cdots\cdots \\
E'' - A = \Theta\, a + \Theta\, a + \Theta\, a, \\
E' - A = \Theta\, a + \Theta\, a, \\
E - A = \Theta\, a, \\
A - A = 0, \\
B - A = a, \\
B' - A = a + a, \\
B'' - A = a + a + a, \\
\quad \&c.
\end{array}\right\} \qquad (98.)$$

The simple or compound step, a, or a + a, &c., from the zero-moment A to any positive moment B or B′ &c. of the series, may be called a *positive step;* and the opposite simple or compound step, Θ a, or Θ a + Θ a, &c., from the same zero-moment A to any contra-positive moment E or E′, &c., of the series, may be called a *contra-positive step;* while the null step 0, from the zero-moment A to itself, may be called, by analogy of language, the *zero-step*. The original step a is supposed to be an effective step, and not a null one, since otherwise the whole series of moments (97.) would reduce themselves to the one original moment A; but it may be either a late-making or an early-making step, according as the (mental) order of progression of that series is from earlier to later, or from later to earlier moments. And the whole series or system of steps (98.), simple or compound, positive or contra-positive, effective or null, which serve to generate the several moments of the equi-distant series (29.) or (97.) from the original or standard moment A, may be regarded as a *system of steps generated from the original step* a, by a *system of acts* of generation which are all of one common kind; each step having therefore a certain *relation* of its own to that original step, and these relations having all a general resemblance to each other, so that they may be conceived as composing a certain *system of relations*, having all one common character. To mark this *common generation* of the system of steps (98.) from the one original step a, and their *common relation* thereto, we may call them all by the common name of *multiples* of that original step, and may say that they are or may be (mentally) formed by *multipling* that common *base*, or *unit-step*, a; distinguishing, however, these several multiples among themselves by peculiar or special names, which shall serve to mark the peculiar relation of any one multiple to the base, or the special act of multipling by which it may be conceived to be generated therefrom.

Thus, the null step, or zero-step, 0, which conducts to the zero-moment A, may be called, according to this way of conceiving it, the *zero multiple* of the original step a; and the positive (effective) steps, simple or compound, a, $a + a$, $a + a + a$, &c., may be called by the general name of *positive multiples* of a, and may be distinguished by the special ordinal names of *first*, *second*, *third*, &c., so that the original step a is, in this view, its own first positive multiple; and finally, the contra-positive (but effective) steps, simple or compound, namely, Θa, $\Theta a + \Theta$, $\Theta a + \Theta a + \Theta a$, &c., may be called the *first contra-positive multiple* of a, the *second contra-positive multiple* of the same original step a, and so forth. Some particular multiples have particular and familiar names; for example, the second positive multiple of a step may also be called the *double* of that step, and the third positive multiple may be called familiarly the *triple*. In general, the original step a may be called (as we just now agreed) the common *base (*or *unit)* of all these several multiples; and the ordinal name or number, (such as zero, or positive first, or contra-positive second,) which serves as a special mark to distinguish some one of these multiples from every other, in the general series of such multiples (98.), may be called the *determining ordinal:* so that any one multiple step is sufficiently described, when we mention its base and its determining ordinal. In conformity with this conception of the series of steps (98.,) as a *series of multiples of the base* a, we may denote them by the following series of written symbols,

$$\ldots\ldots\ 3\,\Theta a,\ 2\,\Theta a,\ 1\,\Theta a,\ 0a,\ 1a,\ 2a,\ 3a,\ \ldots \qquad (99.)$$

and may denote the moments themselves of the equi-distant series (29.) or (97.) by the symbols,

$$\left.\begin{aligned} &\ldots\ldots \\ \mathrm{E}'' &= 3\,\Theta\,\mathrm{a} + \mathrm{A}, \\ \mathrm{E}' &= 2\,\Theta\,\mathrm{a} + \mathrm{A}, \\ \mathrm{E} &= 1\,\Theta\,\mathrm{a} + \mathrm{A}, \\ \mathrm{A} &= \ \ 0\,\mathrm{a} + \mathrm{A}, \\ \mathrm{B} &= \ \ 1\,\mathrm{a} + \mathrm{A}, \\ \mathrm{B}' &= \ \ 2\,\mathrm{a} + \mathrm{A}, \\ \mathrm{B}'' &= \ \ 3\,\mathrm{a} + \mathrm{A}, \\ &\ \&\mathrm{c.}; \end{aligned}\right\} \qquad (100.)$$

in which

$$0\,\mathrm{a} = 0, \qquad (101.)$$

and

$$\left.\begin{aligned} 1\,\mathrm{a} &= \mathrm{a}, & 1\,\Theta\,\mathrm{a} &= \Theta\,\mathrm{a}, \\ 2\,\mathrm{a} &= \mathrm{a} + \mathrm{a}, & 2\,\Theta\,\mathrm{a} &= \Theta\,\mathrm{a} + \Theta\,\mathrm{a}, \\ 3\,\mathrm{a} &= \mathrm{a} + \mathrm{a} + \mathrm{a}, & 3\,\Theta\,\mathrm{a} &= \Theta\,\mathrm{a} + \Theta\,\mathrm{a} + \Theta\,\mathrm{a}, \\ &\&\mathrm{c.}, & &\&\mathrm{c.} \end{aligned}\right\}. \qquad (102.)$$

The written sign 0 in $0\,\mathrm{a}$ is here equivalent to the spoken name *zero*, as the determining ordinal of the null step from A to A, which step was itself also denoted before by the same character 0, and is here considered as the *zero-multiple* of the base a; while the written signs 1, 2, 3, &c., in the symbols of the positive multiples $1\,\mathrm{a}$, $2\,\mathrm{a}$, $3\,\mathrm{a}$, &c., correspond to and denote the determining positive ordinals, or the spoken names *first positive, second positive, third positive*, &c.; and, finally, the remaining written signs $1\,\Theta$, $2\,\Theta$, $3\,\Theta$, &c., which are combined with the written sign of the base a, in the symbols of the contra-positive multiples $1\,\Theta\,\mathrm{a}$, $2\,\Theta\,\mathrm{a}$, $3\,\Theta\,\mathrm{a}$, &c., correspond to and denote the determining ordinal names of those contra-positive multiples, that is, they correspond to the spoken names, *first contra-positive, second contra-positive, third contra-positive*, &c.: so that the series of signs of multiple steps (99.), is formed by combining the symbol of the base a with the following series of ordinal symbols,

$$\ldots 3\,\Theta,\ 2\,\Theta,\ 1\,\Theta,\ 0,\ 1,\ 2,\ 3,\ \&\mathrm{c.} \qquad (103.)$$

We may also conceive this last series of signs as equivalent, not to *ordinal* names, such as the numeral word *first*, but to *cardinal* names, such as the numeral word *one;* or more fully, *positive cardinals, contra-positive cardinals*, and the *null cardinal* (or number *none*); namely, the system of all possible answers to the following complex question: "*Have any* effective steps (equivalent or opposite to the given base a) been made (from the standard moment A), and if any, then *How many*, and *In which direction?*" In this view, $3\,\Theta$ is a written sign of the *cardinal* name or number *contra-positive three*, as a possible answer to the foregoing general question; and it implies, when prefixed to the sign of the base a, in the complex written sign $3\,\Theta\,\mathrm{a}$ of the corresponding multiple step, that this multiple step has been formed, (as already shown in the equations (102.),) by making three steps equal to the base a in length, but in the direction opposite thereto. Again, the mark 1 may be regarded as a written sign of the cardinal number *positive one*, and $1\,\mathrm{a}$ denotes (in this view) the step formed by making one such step as a, and in the same direction, that is, (as before,) the original step a itself; and 0 denotes the cardinal number *none*, so that $0\,\mathrm{a}$ is (as before) a symbol for the null step from A to A, which step we have also marked before by the simple symbol 0, and which is here considered as formed by making *no* effective step like a. In general, this view of the numeral signs (103.), as denoting *cardinal* numbers, conducts to the same ultimate interpretations of the symbols (99.), for the steps of the series (98.), as the former view, which regarded those signs (103.) as denoting *ordinal* numbers.

If we adopt the latter view of those numeral signs (103.), which we shall call by the common name of *whole* (or *integer*) *numbers*, (as distinguished from certain broken or fractional numbers to be considered afterwards,) we may conveniently continue to use the word *multiple* (occasionally) as a verb active, and may speak of the several multiple steps of the series (98.), or (99.), as formed from the base a, by *multipling that base by the several whole* (cardinal) *numbers:* because every multiple step may be conceived as generated (in thought) from the base, by a certain mental act, of which the cardinal number is the mark. Thus we may describe the multiple step $3\,\Theta\,\mathrm{a}$, (which is, in the ordinal view, the third contra-positive multiple of a,) as formed from the base a by *multipling it by contra-positive three.* Some particular acts of multipling have familiar and special names, and we may speak (for instance) of *doubling* or *tripling* a step, instead of describing that step as being multipled by positive two, or by positive three. In general, to distinguish more clearly, in the written symbol of a multiple step, between the base and the determining number (ordinal or cardinal), and to indicate more fully the performance of that mental act (directed by the number) which generates the multiple from the base, the mark × may be inserted between the sign of the base, and the sign of the number; and thus we may denote the series of multiple steps (99.) by the following fuller symbols,

$$\ldots\ 3\,\Theta \times \mathrm{a},\ 2\,\Theta \times \mathrm{a},\ 1\,\Theta \times \mathrm{a},\ 0 \times \mathrm{a},\ 1 \times \mathrm{a},\ 2 \times \mathrm{a},\ 3 \times \mathrm{a},\ \&\text{c.}, \qquad (104.)$$

and which $1 \times \mathrm{a}$ (for example) denotes the original step a itself, and $2 \times \mathrm{a}$ represents the double of that original step.

It is manifest that in this notation

$$\left.\begin{aligned} n\,\Theta \times \mathrm{a} &= n \times \Theta\,\mathrm{a} = \Theta\,(n \times \mathrm{a}) = \Theta\,(n\,\Theta \times \Theta\,\mathrm{a}), \\ \text{and } n \times \mathrm{a} &= n\,\Theta \times \Theta\,\mathrm{a} = \Theta\,(n\,\Theta \times \mathrm{a}) = \Theta\,(n \times \Theta\,\mathrm{a}), \end{aligned}\right\} \qquad (105.)$$

if n denote any one of the positive numbers 1, 2, 3, &c. and if $n\,\Theta$ denote the corresponding contra-positive number, $1\,\Theta$, $2\,\Theta$, $3\,\Theta$, &c.; for example, the equation $2\,\Theta \times \mathrm{a} = 2 \times \Theta\,\mathrm{a}$ is true, because it expresses that the second contra-positive multiple of the base a is the same step as the second positive multiple of the opposite base or step $\Theta\,\mathrm{a}$, the latter multiple being derived from this opposite base by merely doubling its length without reversing its direction, while the former is derived from the original base a itself by both reversing it in direction and doubling it in length, so that both processes conduct to the one common compound step, $\Theta\,\mathrm{a} + \Theta\,\mathrm{a}$. In like manner the equation $2 \times \mathrm{a} = 2\,\Theta \times \Theta\,\mathrm{a}$ is true, because by first reversing the direction of the original step a, and then taking the reversed step $\Theta\,\mathrm{a}$ as a new base, and forming the second contra-positive multiple of it, which is done by reversing and doubling, and which is the process of generation expressed by the symbol $2\,\Theta \times \Theta\,\mathrm{a}$, we form in the end the same compound step, $\mathrm{a} + \mathrm{a}$, as if we had merely doubled a. We may also conveniently annex the mark of opposition Θ, at the left hand, to the symbol of any whole number, n or $n\,\Theta$ or 0, in order to form a symbol of its opposite number, $n\,\Theta$, n, or 0; and thus may write

$$\Theta\,n = n\,\Theta, \;\; \mathrm{O}\,(n\,\mathrm{O}) = n, \;\; \Theta\,0 = 0\,; \qquad (106.)$$

if we still denote by n any positive whole number, and if we call two whole numbers *opposites* of each other, when they are the determining or multipling numbers of two opposite multiple steps.

14. Two or more multiples such as $\mu \times \mathrm{a}$, $\nu \times \mathrm{a}$, $\xi \times \mathrm{a}$, of the same base a, may be compounded as *successive steps* with each other, and the resulting or compound step will manifestly be itself some multiple, such as $\omega \times \mathrm{a}$, of the same common base a; the signs μ, ν, ξ, denoting here any arbitrary whole numbers, whether positive, or contra-positive, or null, and ω denoting another whole number, namely the determining number of the compound multiple step, which must evidently depend on the determining numbers $\mu\ \nu\ \xi$ of the component multiple steps, and on those alone, according to some general law of dependence. This law may conveniently be denoted, in writing, by the same mark of combination + which has been employed already to form the complex symbol of the compound step itself, considered as depending on the component steps; that is, we may agree to write

$$\omega = \nu + \mu, \text{ when } \omega \times \mathrm{a} = (\nu \times \mathrm{a}) + (\mu \times \mathrm{a}), \qquad (107.)$$

and

$$\omega = \xi + \nu + \mu, \text{ when } \omega \times \mathrm{a} = (\xi \times \mathrm{a}) + (\nu \times \mathrm{a}) + (\mu \times \mathrm{a}), \qquad (108.)$$

together with other similar expressions for the case of more component steps than three. In this notation,

$$\left.\begin{array}{c}(\nu \times \mathrm{a}) + (\mu \times \mathrm{a}) = (\nu + \mu) \times \mathrm{a}, \\ (\xi \times \mathrm{a}) + (\nu \times \mathrm{a}) + (\mu \times \mathrm{a}) = (\xi + \nu + \mu) \times \mathrm{a}, \\ \&\text{c.}\end{array}\right\} \qquad (109.)$$

whatever the whole numbers $\mu\ \nu\ \xi$ may be; equations which are to be regarded here as true by definition, and as only serving to explain the meaning attributed to such complex signs as $\nu + \mu$, or $\xi + \nu + \mu$, when $\mu\ \nu\ \xi$ are any symbols of whole numbers: although when we farther assert that the equations (109.) are true independently of the base or unit-step a, so that symbols of the form $\nu + \mu$ or $\xi + \nu + \mu$ denote whole numbers independent of that base, we express in a new way a theorem which we had before assumed to be evidently true, as an axiom and not a definition, respecting the composition of multiple steps.

In the particular case when the whole numbers denoted by $\mu\ \nu\ \xi$ are positive, the law of composition of those numbers expressed by the notation $\nu + \mu$ or $\xi + \nu + \mu$, as explained by the equations (109.), is easily seen to be the law called *addition* of numbers (that is of quotities) in elementary arithmetic; and the quotity of the compound or resulting whole number is the arithmetical *sum* of the quotities of the component numbers, this arithmetical *sum* being the answer to the question, *How many* things or thoughts does a total group contain, if it be composed of *partial groups* of which the quotities are given, namely the numbers to be arithmetically *added*. For example, since $(3 \times \mathrm{a}) + (2 \times \mathrm{a})$ is the symbol for the total or compound multiple step composed of the double and the triple of the base a, it must denote the quintuple or fifth positive multiple of that base, namely $5 \times \mathrm{a}$; and since we have agreed to write

$$(3 \times \mathrm{a}) + (2 \times \mathrm{a}) = (3 + 2) \times \mathrm{a},$$

we must interpret the complex symbol $3 + 2$ as equivalent to the simple symbol 5; in seeking for which latter number *five*, we *added*, in the arithmetical sense, the given component numbers *two* and *three* together, that is, we formed their arithmetical *sum*, by considering how many steps are contained in a total group of steps, if the component or partial groups contain two steps and three steps respectively. In like manner, if we admit in arithmetic the idea of the cardinal number *none*, as one of the possible answers to the fundamental question *How many*, the rules of the arithmetical addition of this number to others, and of others to it, and the properties of the arithmetical sums thus composed, agree with the rules and properties of such combinations as $0 + \mu$, $\xi + \nu + 0$, explained by the equations (109.), when the whole numbers, μ, ν, ξ, are positive; we shall, therefore, not clash in our enlarged phraseology with the language of elementary arithmetic, respecting the addition of numbers regarded as answers to the question *How many*, if we now establish, as a definition, in the more extensive *Science of Pure Time*, that any combination of whole numbers $\mu\ \nu\ \xi$, of the form $\nu + \mu$, or $\xi + \nu + \mu$, interpreted so as to satisfy the equations (109.), is the *sum* of those whole numbers, and is composed by *adding* them together, whether they be positive, or contra-positive, or null. But as a mark that these words *sum* and *adding* are used in ALGEBRA (as the general Science of Pure Time), in a more extensive sense than that in which *Arithmetic* (as the science of counting) employs them, we may, more fully, call $\nu + \mu$ the *algebraic sum* of the whole numbers μ and ν, and say that it is formed by the operation of *algebraically adding* them together, ν to μ.

In general, we may extend the arithmetical names of *sum* and *addition* to every algebraical combination of the class marked by the sign +, and may give to that combining sign the arithmetical name of *Plus;* although in Algebra the idea of *more*, (originally implied by *plus*,) is only occasionally and accidentally involved in the conception of such combinations. For example, the written symbol $\mathrm{b} + \mathrm{a}$, by which we have already denoted the compound step formed by *compounding* the step b as a successive step with the step a, may be expressed in words by the phrase "a plus b," (such written algebraic expressions as these being read from right to left,) or "the algebraic sum of the steps a and b;" and this algebraic sum or compound step $\mathrm{b} + \mathrm{a}$ may be said to be formed by "algebraically adding b to a:" although this compound step is only occasionally and accidentally greater in length than its components, being necessarily shorter than one of them, when they are both effective steps with directions opposite to each other. Even the *application* of a step a to a moment A, so as to generate another moment $\mathrm{a} + \mathrm{A}$, may not improperly be called (by the same analogy of language) the *algebraic addition* of the step to the moment, and the moment generated thereby may be called their *algebraic sum*, or "the original moment *plus* the step;" though in this sort of combination the moment and the step to be combined are not even homogeneous with each other.

With respect to the process of calculation of an algebraic sum of whole numbers, the following rules are evident consequences of what has been already shown respecting the composition of steps. In the first place, the numbers to be added may be added in any arbitrary order; that is,

$$\left.\begin{array}{l} \nu+\mu=\mu+\nu, \\ \xi+\nu+\mu=\mu+\xi+\nu= \&c., \\ \quad\&c.\,; \end{array}\right\} \qquad (110.)$$

we may therefore collect the positive numbers into one algebraical sum, and the contra-positive into another, and then add these two partial sums to find the total sum, omitting (if it anywhere occur) the number None or Zero, as not capable of altering the result. In the next place, positive numbers are algebraically added to each other, by arithmetically adding the corresponding arithmetical numbers or quotities, and considering the result as a positive number; thus positive two and positive three, when added, give positive five: and contra-positive numbers, in like manner, are algebraically added to each other, by arithmetically adding their quotities, and considering the result as a contra-positive number; thus,contra-positive two and contra-positive three have contra-positive five for their algebraic sum. In the third place, a positive number and a contra-positive, when the quotity of the positive exceeds that of the contra-positive, give a positive algebraic sum, in which the quotity is equal to that excess; thus positive five added to contra-positive three, gives positive two for the algebraic sum: and similarly, a positive number and a contra-positive number, if the quotity of the contra-positive exceed that of the positive, give a contra-positive algebraic sum, with a quotity equal to the excess; for example, if we add positive three to contra-positive five, we get contra-positive two for the result. Finally, a positive number and a contra-positive, with equal quotities, (such as positive three and contra-positive three,) destroy each other by addition; that is, they generate as their algebraic sum the number None or Zero.

It is unnecessary to dwell on the algebraical operation of *decomposition of multiple steps*, and consequently of whole or *multipling numbers*, which corresponds to and includes the operation of arithmetical *subtraction;* since it follows manifestly from the foregoing articles of this Essay, that the decomposition of numbers (like that of steps) can always be performed by *compounding* with the given compound number (that is, by algebraically *adding* thereto) the *opposite* or opposites of the given component or components: the number or numbers proposed to be subtracted are therefore either to be neglected if they be null, since in that case they have no effect, or else to be changed from positive to contra-positive, or from contra-positive to positive, (their quotities being preserved,) and then added (algebraically) in this altered state. Thus, positive five is subtracted algebraically from positive two by adding contra-positive five, and the result is contra-positive three; that is, the given step $2 \times \mathrm{a}$ or $2\,\mathrm{a}$ may be decomposed into two others, of which the given component step $5 \times \mathrm{a}$ is one, and the sought component step $3\,\Theta\,\mathrm{a}$ is the other.

15. Any multiple step $\mu \times \mathrm{a}$ may be treated as a new base, or new unit-step; and thus we may generate from it a new system of multiple steps. It is evident that these multiples of a multiple of a step are themselves also multiples of that step; that is, if we first multiple a given base or unit-step a by any whole number μ, and then again multiple the result $\mu \times \mathrm{a}$ by any other whole number ν, the final result $\nu \times (\mu \times \mathrm{a})$ will necessary be of the form $\omega \times \mathrm{a}$, ω being another whole number. It is easy also to see that the new multipling number, such as ω, of the new or derived multiple, must pend on the old or given multipling numbers, such as μ and ν, and on those alone; and the law of its dependence on them may be conveniently expressed by the same mark of combination $\times$ which we have already used to combine any multipling number with its base; so that we may agree to write

$$\omega = \nu \times \mu, \text{ when } \omega \times \mathrm{a} = \nu \times (\mu \times \mathrm{a}). \qquad (111.)$$

With this definition of the effect of the combining sign $\times$, when interposed between the signs of two whole numbers, we may write

$$\nu \times (\mu \times \mathrm{a}) = (\nu \times \mu) \times \mathrm{a} = \nu \times \mu \times \mathrm{a}, \qquad (112.)$$

omitting the parentheses as unnecessary; because, although their absence permits us to interpret the complex symbol $\nu \times \mu \times \mathrm{a}$ either as $\nu \times (\mu \times \mathrm{a})$ or as $(\nu \times \mu) \times \mathrm{a}$, yet both the processes of combination thus denoted conduct to one common result, or ultimate multiple step. (Compare article 11.)

When μ and ν are positive numbers, the law of combination expressed by the notation $\nu \times \mu$, as above explained, is easily seen to be that which is called *Multiplication* in elementary Arithmetic, namely, the arithmetical addition of a given number ν of equal quotities μ; and the resulting quotity $\nu \times \mu$ is the arithmetical *product* of the numbers to be combined, or the product of μ multiplied by ν: thus we must, by the definition (112.), interpret 3×2 as denoting the positive number 6, because $3 \times (2 \times \mathrm{a}) = 6 \times \mathrm{a}$, the triple of the double of any step a being the sextuple of that step; and the quotity 6 is, for the same reason, the arithmetical product of 2 multiplied by 3, in the sense of being the answer to the question, How many things or thoughts (in this case, steps) are contained in a total group, if that total group be composed of 3 partial groups, and if 2 such things or thoughts be contained in each of these? From this analogy to arithmetic, we may in general call $\nu \times \mu$ the *product*, or (more fully) the *algebraic product*, of the whole numbers μ and ν, whether these, which we may call the *factors* of the product, be positive, or contra-positive, or null; and may speak of the process of combination of those numbers, as the *multipling*, or (more fully) the *algebraic multipling* of μ by ν: reserving still the more familiar arithmetical word "multiplying" to be used in algebra in a more general sense, which includes the operation of multipling, and which there will soon be occasion to explain.

In like manner, three or more whole numbers, μ, ν, ξ, may be used successively to multiple a given step or one another, and so to generate a new derived multiple of the original step or number; thus, we may write

$$\xi \times \{\nu \times (\mu \times \mathrm{a})\} = \xi \times \{(\nu \times \mu) \times \mathrm{a}\} = (\xi \times \nu \times \mu) \times \mathrm{a}, \qquad (113.)$$

the symbol $\xi \times \nu \times \mu$ denoting here a new whole number, which may be called the algebraic *product* of the *three* whole numbers μ, ν, ξ, those numbers themselves being called the *factors* of this product. With respect to the actual processes of such *multipling*, or the rules for forming such algebraic *products* of whole numbers, (whether positive, or contra-positive, or null,) it is sufficient to observe that the product is evidently null if any one of the factors be null, but that otherwise the product is contra-positive or positive, according as there is or is not an odd number (such as one, or three, or five, &c.) of contra-positive factors, because the direction of a step is not changed, or is restored, when it is either not reversed at all, or reversed an even number of times; and that, in every case, the quotity of the algebraic product is the arithmetical product of the quotities of the factors. Hence, by the properties of arithmetical products, or by the principles of the present essay, we see that in forming an algebraical product the order of the factors may be altered in any manner without altering the result, so that

$$\nu \times \mu = \mu \times \nu, \;\; \xi \times \nu \times \mu = \mu \times \xi \times \nu = \&c., \&c.; \qquad (114.)$$

and that any one of the factors may be decomposed in any manner into algebraical parts or component whole numbers, according to the rules of algebraic addition and subtraction of whole numbers, and each part separately combined as a factor with the other factors to form a partial product, and then these partial products algebraically added together, and that the result will be the total product; that is,

$$\left.\begin{array}{l} (\nu' + \nu) \times \mu = (\nu' \times \mu) + (\nu \times \mu), \\ \nu \times (\mu' + \mu) = (\nu \times \mu') + (\nu \times \mu), \;\; \&c. \end{array}\right\} \qquad (115.)$$

Again, we saw that if a factor μ be null, the product is then null also,

$$\nu \times 0 = 0 ; \qquad (116.)$$

because the multiples of a null multiple step are all themselves null steps. But if, in a product of two whole numbers, $\nu \times \mu$, the first factor μ (with which by (114.) the second factor ν may be interchanged) be given, and effective, that is, if it be any given positive or contra-positive whole number, $(\mu \neq 0,)$ then its several multiples, or the products of the form $\nu \times \mu$, form an indefinite series of whole numbers,

$$\ldots\ 3\,\Theta \times \mu,\ 2\,\Theta \times \mu,\ 1\,\Theta \times \mu,\ 0 \times \mu,\ 1 \times \mu,\ 2 \times \mu,\ 3 \times \mu,\ \ldots \qquad (117.)$$

such that any proposed whole number ω, whatever, must be either a number of this series, or else included between two successive numbers of it, such as $\nu \times \mu$ and $(1+\nu) \times \mu$, being on the positive side of one of them, and on the contra-positive side of the other, in the complete series of whole numbers (103.). In the one case, we can satisfy the equation

$$\omega = \nu \times \mu, \text{ or, } \Theta\,(\nu \times \mu) + \omega = 0, \qquad (118.)$$

by a suitable choice of the whole number ν; in the other case, we cannot indeed do this, but we can choose a whole number ν, such that

$$\omega = \rho + (\nu \times \mu), \text{ or, } \Theta(\nu \times \mu) + \omega = \rho, \qquad (119.)$$

ρ being a whole number which lies between 0 and μ in the general series of whole numbers (103.), and which therefore has a quotity less than the quotity of that given first factor μ, and is positive or contra-positive according as μ is positive or contra-positive. In each case, we may be said (by analogy to arithmetical division) to have *algebraically divided* (or rather *measured*), accurately or approximately, the whole number ω by the whole number μ, and to have found a whole number ν which is either the *accurate quotient* (or *measure*), as in the case (118.), or else the *next preceding integer*, as in the other case (119.); in which last case the whole number ρ may be called the *remainder* of the division (or of the *measuring*). In this second case, namely, when it is impossible to perform the division, or the *measuring*, exactly, in whole numbers, because the proposed *dividend*, or *mensurand*, ω, is not contained among the series (117.) of multiples of the proposed *divisor*, or *measurer*, μ, we may choose to consider as the approximate integer *quotient*, or *measure*, the *next succeeding* whole number $1 + \nu$, instead of the next preceding whole number ν; and then we shall have a different *remainder*, $\Theta\,\mu + \rho$, such that

$$\omega = (\Theta\mu + \rho) + (\overline{1+\nu} \times \mu), \qquad (120.)$$

which new remainder $\Theta \mu + \rho$ has still a quotity less than that of μ, but lies between 0 and $\Theta \mu$, instead of lying (like ρ) between 0 and μ, in the general series of whole numbers (103.), and is therefore contra-positive if μ be positive, or positive if μ be contra-positive. With respect to the actual process of calculation, for discovering whether a proposed algebraical division (or measuring), of one whole number by another, conducts to an accurate integer quotient, or only to two approximate integer quotients, a next preceding and a next succeeding, with positive and contra-positive remainders; and for actually finding the names of these several quotients and remainders, or their several special places in the general series of whole numbers: this algebraical process differs only by some slight and obvious modifications (on which it is unnecessary here to dwell,) from the elementary arithmetical operation of dividing one quotity by another; that is, the operation of determining what multiple the one is of the other, or between what two successive multiples it is contained. Thus, having decomposed by arithmetical division the quotity 8 into the arithmetical sum of 1×5 and 3, and having found that it falls short by 2 of the arithmetical product 2×5, we may easily infer from hence that the algebraic whole number *contra-positive eight* can be only approximately measured (in whole numbers), as a mensurand, by the measurer *positive five;* the next succeeding integer quotient or measure being *contra-positive one*, with *contra-positive three* for remainder, and the next preceding integer quotient or measure being *contra-positive two*, with *positive two* as the remainder. It is easy also to see that this algebraic measuring of one whole number by another, corresponds to the accurate or approximate measuring of one step by another. And in like manner may all other arithmetical operations and reasonings upon quotities be generalised in Algebra, by the consideration of multiple steps, and of their connected positive and contra-positive and null whole numbers.

On the Sub-multiples and Fractions of any given Step in the Progression of Time; on the Algebraic Addition, Subtraction, Multiplication, and Division, of Reciprocal and Fractional Numbers, positive and contra-positive; and on the impossible or indeterminate act of sub-multipling or dividing by zero.

16. We have seen that from the thought of any one step a, as a base or unit-step, we can pass to the thought of a series or system of multiples of that base, namely, the series (98.) or (99.) or (104.), having each a certain relation of its own

to the base, as such or such a particular multiple thereof, or as mentally generated from that base by such or such a particular act of multipling; and that every such particular relation, and every such particular act of multipling, may be distinguished from all such other relations, and from all such other acts, in the entire series or system of these relations, and in the entire system of these acts of multipling, by its own special or determining whole number, whether ordinal or cardinal, and whether positive, or contra-positive, or null. Now every such relation or act must be conceived to have a certain inverse or reciprocal, by which we may, in thought, connect the base with the multiple, and return to the former from the latter: and, generally, the conception of passing (in thought) from a base or unit-step to any one of its multiples, or of returning from the multiple to the base, is included in the more comprehensive conception of passing from any one such multiple to any other; that is, from any one step to any other step *commensurable* therewith, two steps being said to be *commensurable* with each other when they are multiples of one common base or unit-step, because they have then that common base or unit for their *common measurer*. The base, when thus compared with one of its own multiples, may be called a *sub-multiple* thereof; and, more particularly, we may call it the "second positive sub-multiple" of its own second positive multiple, the "first contra-positive sub-multiple" of its own first contra-positive multiple, and so forth; retaining always, to distinguish any one sub-multiple, the determining ordinal of the multiple to which it corresponds: and the act of returning from a multiple to the base, may be called an act of *sub-multipling* or (more fully) of sub-multipling *by* the same determining cardinal number by which the base had been multipled before; for example, we may return to the base from its second contra-positive multiple, by an act of thought which may be called sub-multipling by contra-positive two. Some particular sub-multiples, and acts of sub-multipling, have particular and familiar names; thus, the second positive sub-multiple of any given step, and the act of sub-multipling a given step by positive two, may be more familiarly described as the *half* of that given step, and as the act of *halving* it. And the more comprehensive conception above mentioned, of the act of passing from any one step b to any other step c commensurable therewith, or from any one to any other multiple of one common measure, or base, or unit-step a, may evidently be resolved into the foregoing conceptions of the acts of multipling and sub-multipling; since we can always pass first by an act of sub-multipling from the given step b, considered as a multiple of the base a, to that base a itself, as an auxiliary or intermediate thought, and then proceed, by an act of multipling, from this auxiliary thought or step, to its other multiple c. Any one step c may therefore be considered as a multiple of a sub-multiple of any other step b, if those two steps be commensurable; and the act of passing from the one to the other is an act compounded of sub-multipling and multipling.

Now, all acts thus compounded, besides the acts of multipling and sub-multipling themselves, (and other acts, to be considered afterwards, which may be regarded as of the same kind with these, being connected with them by certain intimate relations, and by one common character,) may be classed in algebra under the general name of *multiplying acts*, or acts of *algebraic multiplication;* the *object* on which any such *act* operates being called the *multiplicand*, and the *result* being called the *product;* while the *distinctive thought or sign* of such an act is called the *algebraic multiplier*, or *multiplying number:* whatever this distinctive thought or sign may be, that is, whatever conceived, or spoken, or written *specific rule* it may involve, for specifying one particular act of multiplication, and for distinguishing it from every other. The relation of an algebraic product to its algebraic multiplicand may be called, in general. *ratio*, or *algebraic ratio;* but the particular ratio of any one particular product to its own particular multiplicand, depends on the particular act of multiplication by which the one may be generated from the other: the *number* which specifies the *act* of multiplication, serves therefore also to specify the resulting *ratio*, and every number may be viewed either as the *mark of a ratio*, or as the *mark of a multiplication*, according as we conceive ourselves to be *analytically examining* a product already formed, or *synthetically generating* that product.

We have already considered that series or system of *algebraic integers*, or *whole numbers*, (positive, contra-positive, or null,) which mark the several possible ratios of all multiple steps to their base, and the several acts of multiplication by which the former may be generated from the latter; namely all those several acts which we have included under the common head of *multipling*. The inverse or reciprocal acts of *sub-multipling*, which we must now consider, and which we have agreed to regard as comprehended under the more general head of *multiplication*, conduct to a new class of multiplying numbers, which we may call *reciprocals of whole numbers*, or, more concisely, *reciprocal numbers;* and to a corresponding class of ratios, which we may call *reciprocals of integer ratios*. And the more comprehensive conception of the act of passing from one to another of any two commensurable steps, conducts to a correspondingly extensive class of multiplying acts, and therefore also of multiplying numbers, and of ratios, which we may call *acts of fractioning*, and *fractional numbers*, or *fractional ratios;* while the *product* of any such act of fractioning, or of multiplying by any such fractional number, that is, the *generated step* which is any multiple of any sub-multiple of any proposed step or *multiplicand*, may be called a *fraction* of that step, or of that multiplicand. A fractional number may therefore always be determined, in thought and in expression, by *two whole numbers,* namely the sub-multipling number, called also the *denominator,* and the multipling number, called also the *numerator,* (of the fraction or fractional number,) which mark the two successive or component acts that make up the complex act of fractioning. Hence also the reciprocal number, or reciprocal of any proposed whole number, which marks the act of multiplication conceived to be equivalent to the act of sub-multipling by that whole number, coincides with the fractional number which has the same whole number for its denominator, and the number 1 for its numerator, because a step is not altered when it is multipled by positive one. And any whole number itself, considered as the mark of any special act of multipling, may be changed to a fractional number with positive one for its denominator, and with the proposed whole number for its numerator; since such a fractional number, considered as the mark of a special act of multiplication, is only the complex mark of a complex act of thought equivalent to the simpler act of multipling by the numerator of the fraction; because the other component act, of sub-multipling by positive one, produces no real alteration. Thus, the conceptions of whole numbers, and of reciprocal numbers, are included in the more comprehensive conception of fractional numbers; and a complete theory of the latter would contain all the properties of the former.

17. To form now a notation of fractions, we may agree to denote a fractional number by writing the numerator over the denominator, with a bar between; that is, we may write

$$c = \frac{\nu}{\mu} a, \text{ or more fully, } c = \frac{\nu}{\mu} \times a, \qquad (121.)$$

when we wish to express that two commensurable steps, b and c, (which we shall, for the present, suppose to be both effective steps,) may be severally formed from some one common base or unit-step a, by multipling that base by the two (positive or contra-positive) whole numbers μ and ν, so that

$$b = \mu \times a, \quad c = \nu \times a. \qquad (122.)$$

[We shall suppose throughout the whole of this and of the two next following articles, that all the steps are effective, and that all the numerators and denominators are positive or contra-positive, excluding for the present the consideration of null steps, and of null numerators or null denominators.]

Under these conditions, the step c is a fraction of b, and bears to that step b the fractional ratio $\frac{\nu}{\mu}$, called also "the ratio of ν to μ;" and c may be deduced or generated as a product from b by a corresponding act of fractioning, namely, by the act of multiplying b as a multiplicand by the fractional number $\frac{\nu}{\mu}$ as a multiplier, or finally by the complex act of first submultipling b by the denominator μ, and then multipling the result a by the numerator ν. Under the same conditions, it is evident that we may return from c to b by an inverse or reciprocal act of fractioning, namely, by that new complex act which is composed of submultipling instead of multipling by ν, and then multipling instead of submultipling by μ; so that

$$b = \frac{\mu}{\nu} \times c, \quad \text{when } c = \frac{\nu}{\mu} \times b: \qquad (123.)$$

on which account we may write

$$b = \frac{\mu}{\nu} \times \left(\frac{\nu}{\mu} \times b\right), \quad \text{and } c = \frac{\nu}{\mu} \times \left(\frac{\mu}{\nu} \times c\right), \qquad (124.)$$

whatever (effective) steps may be denoted by b and c, and whatever (positive or contra-positive) whole numbers may be denoted by μ and ν. The two acts of fractioning, marked by the two fractional numbers $\frac{\nu}{\mu}$ and $\frac{\mu}{\nu}$, are therefore opposite or *reciprocal acts*, of which each destroys or undoes the effect of the other; and the fractional numbers themselves may be called *reciprocal fractional numbers*, or, for shortness, *reciprocal fractions:* to mark which reciprocity we may use a new symbol ᴚ, (namely, the initial letter of the word Reciprocatio, distinguished from the other uses of the same letter by being written in an inverted position,) that is, we may write

$$\frac{\nu}{\mu} = \text{ᴚ}\,\frac{\mu}{\nu}, \quad \frac{\mu}{\nu} = \text{ᴚ}\,\frac{\nu}{\mu}, \qquad (125.)$$

whatever positive or contra-positive whole numbers may be marked by μ and ν. In this notation,

$$\text{ᴚ}\,\text{ᴚ}\,\frac{\nu}{\mu} = \text{ᴚ}\left(\text{ᴚ}\,\frac{\nu}{\mu}\right) = \text{ᴚ}\,\frac{\mu}{\nu} = \frac{\nu}{\mu}; \qquad (126.)$$

or, to express the same thing in words, the reciprocal of the reciprocal of any fractional number is that fractional number itself. (Compare equation (57.)).

It is evident also, that

$$a = \frac{1}{\mu} \times b, \quad \text{and } b = \frac{\mu}{1} \times a, \text{ if } b = \mu \times a; \qquad (127.)$$

that is, the whole number μ, regarded as a multiplier, or as a ratio, may be put under the fractional form $\frac{\mu}{1}$, so that we may write

$$\frac{\mu}{1} = \mu; \qquad (128.)$$

and the reciprocal of this whole number, or the connected reciprocal number $\text{u}\,\mu$ to multiply by which is equivalent to submultipling by μ, coincides with the reciprocal fractional number $\frac{1}{\mu}$, so that

$$\frac{1}{\mu} = \text{u}\,\frac{\mu}{1} = \text{u}\,\mu: \qquad (129.)$$

results which were indeed anticipated in the remarks made at the close of the foregoing article, respecting the extent of the conception of fractional numbers, as including whole numbers and their reciprocals. As an example of these results, the double of any step a may be denoted by the symbol $\frac{2}{1} \times \mathrm{a}$ as well as by $2 \times \mathrm{a}$, and the half of that step a may be denoted either by the symbol $\frac{1}{2} \times \mathrm{a}$, or by $\text{u}\,2 \times \mathrm{a}$. The symbol $\text{u}\,1$ is evidently equivalent to 1, the number positive one being its own reciprocal; and the opposite number, contra-positive one, has the same property, because to reverse the direction of a step is an act which destroys itself by repetition, leaving the last resulting step the same as the original; we have therefore the equations,

$$\text{u}\,1 = 1, \quad \text{u}\,\Theta\,1 = \Theta\,1. \qquad (130.)$$

By the definition of a fraction, as a multiple of a submultiple, we may now express it as follows:

$$\frac{\nu}{\mu} \times \mathrm{b} = \nu \times \left(\frac{1}{\mu} \times \mathrm{b}\right) = \nu \times (\text{u}\,\mu \times \mathrm{b}). \qquad (131.)$$

Besides, under the conditions (122.), we have, by (112.) and (114.), that is, by the principle of the indifference of the order in which any two successive multiplings are performed,

$$\mu \times \mathrm{c} = \mu \times (\nu \times \mathrm{a}) = (\mu \times \nu) \times \mathrm{a} = (\nu \times \mu) \times \mathrm{a} = \nu \times (\mu \times \mathrm{a}) = \nu \times \mathrm{b}; \quad (132.)$$

so that a fractional product $\mathrm{c} = \frac{\nu}{\mu} \times \mathrm{b}$ may be derived from the multiplicand b, by first multipling by the numerator ν and then submultipling by the denominator μ, instead of first submultipling by the latter and afterwards multipling by the former; that is, in any act of fractioning, we may change the order of the two successive and component acts of submultipling and multipling, without altering the final result, and may write

$$\frac{\nu}{\mu} \times \mathrm{b} = \frac{1}{\mu} \times (\nu \times \mathrm{b}) = \mathfrak{U}\,\mu \times (\nu \times \mathrm{b}). \qquad (133.)$$

In general it may easily be shown, by pursuing a reasoning of the same sort, that in any set of acts of multipling and submultipling, to be performed successively on any one original step, the order of succession of those acts may be altered in any arbitrary manner, without altering the final result. We may therefore compound any proposed set of successive acts of fractioning, by compounding first the several acts of submultipling by the several denominators into the one act of submultipling by the product of those denominators, and then the several acts of multipling by the several numerators into the one act of multipling by the product of those numerators, and finally the two acts thus derived into one last resultant act of fractioning; that is, we have the relations,

$$\left.\begin{aligned} \frac{\nu'}{\mu'} \times \left(\frac{\nu}{\mu} \times \mathrm{b}\right) &= \frac{\nu' \times \nu}{\mu' \times \mu} \times \mathrm{b}, \\ \frac{\nu''}{\mu''} \times \left\{\frac{\nu'}{\mu'} \times \left(\frac{\nu}{\mu} \times \mathrm{b}\right)\right\} &= \frac{\nu'' \times \nu' \times \nu}{\mu'' \times \mu' \times \mu} \times \mathrm{b}, \\ \&\text{c.} \qquad & \end{aligned}\right\} \qquad (134.)$$

We may also introduce or remove any positive or contra-positive whole number as a factor in both the numerator and the denominator of any fraction, without making any real alteration; that is, the following relation holds good:

$$\frac{\nu}{\mu} = \frac{\omega \times \nu}{\omega \times \mu}, \qquad (135.)$$

whatever positive or contra-positive whole numbers may be denoted by $\mu\ \nu\ \omega$; a theorem which may often enable us to put a proposed fraction under a form more simple in itself, or more convenient for comparison with others. As particular cases of this theorem, corresponding to the case when the common factor ω is contra-positive one, we have

$$\frac{\nu}{\mu} = \frac{\Theta\,\nu}{\Theta\,\mu}, \quad \frac{\Theta\,\nu}{\mu} = \frac{\nu}{\Theta\,\mu}; \qquad (136.)$$

that is, the denominator of any fraction may be changed from contra-positive to positive, or from positive to contra-positive, without making any real change, provided that the numerator is also changed to its own opposite whole number. Two fractional numbers, such as $\frac{\Theta\,\nu}{\mu}$ and $\frac{\nu}{\mu}$, may be said to be *opposites*, (though *not recipro-*

cals), when (though *not* themselves the marks of *opposite acts*), they *generate opposite steps*, such as the steps $\frac{\Theta\nu}{\mu} \times \mathrm{b}$ and $\frac{\nu}{\mu} \times \mathrm{b}$; and to mark this opposition we may write

$$\frac{\Theta\nu}{\mu} = \Theta\frac{\nu}{\mu}. \qquad (137.)$$

Hence every fractional number, with any positive or contra-positive whole numbers μ and ν for its denominator and numerator, may be put under one or other of the two following forms:

$$\text{Ist. } \frac{n}{m}, \quad \text{or} \quad \text{IInd. } \Theta\frac{n}{m}, \qquad (138.)$$

(m and n denoting positive whole numbers,) according as the proposed whole numbers μ and ν agree or differ in respect of being positive or contra-positive; and in the Ist case we may say that the fractional number itself is *positive*, but in the IInd case that it is *contra-positive:* definitions which agree with and include the former conceptions of positive and contra-positive whole numbers, when we consider these as equivalent to fractional numbers in which the numerator is a multiple of the denominator; and lead us to regard the reciprocal of any positive or contra-positive whole number (and more generally the reciprocal of any positive or contra-positive fractional number) as positive or contra-positive like it; a fractional number being equivalent to the reciprocal of a whole number, when the denominator is a multiple of the numerator. A fraction of a late-making step b is itself a late-making or an early-making step, according as the multiplying fractional number is positive or contra-positive; and as we have agreed to write $\mathrm{b} > 0$ when b is a late-making step, so we may now agree to write

$$\frac{\nu}{\mu} > 0, \quad \text{when} \quad \frac{\nu}{\mu} \times \mathrm{b} > 0 \quad \text{and} \quad \mathrm{b} > 0, \qquad (139.)$$

that is, when $\frac{\nu}{\mu}$ is a *positive fractional number*, and to write, on the contrary,

$$\frac{\nu}{\mu} < 0, \quad \text{when} \quad \frac{\nu}{\mu} \times \mathrm{b} < 0 \quad \text{and} \quad \mathrm{b} > 0, \qquad (140.)$$

that is, when $\frac{\nu}{\mu}$ is a *contra-positive fractional number.* More generally, we shall write

$$\frac{\nu'}{\mu'} > \frac{\nu}{\mu}, \quad \text{if} \quad \frac{\nu'}{\mu'} \times \mathrm{b} > \frac{\nu}{\mu} \times \mathrm{b}, \quad \mathrm{b} > 0, \qquad (141.)$$

and

$$\frac{\nu'}{\mu'} < \frac{\nu}{\mu}, \quad \text{if} \quad \frac{\nu'}{\mu'} \times \mathrm{b} < \frac{\nu}{\mu} \times \mathrm{b}, \quad \mathrm{b} > 0; \qquad (142.)$$

and shall enunciate these two cases respectively, by saying that in the first case the fractional number $\frac{\nu'}{\mu'}$ is *on the positive side*, and that in the second case it is *on the contra-positive side*, of the other fractional number $\frac{\nu}{\mu}$; or that in the first case $\frac{\nu'}{\mu'}$ *follows* and that in the second it *precedes* $\frac{\nu}{\mu}$, in the general progression of numbers, from contra-positive to positive: definitions which may easily be shown to be consistent with each other, and which extend to whole numbers and their reciprocals, as included in fractional numbers, and to the number zero itself as compared with any of these. Thus, every positive number is on the positive side of zero and of every contra-positive number; while zero is on the positive side of all contra-positive numbers, but on the contra-positive side of all positive numbers: for example,

$$2 > 0,\ 2 > \Theta\, 3,\ \Theta\, 3 < 0,\ \Theta\, 3 < 2,\ 0 > \Theta\, 3,\ 0 < 2. \qquad (143.)$$

Of two unequal positive whole numbers, the one which has the greater quotity is on the positive side, but among contra-positive numbers the reverse is the case; for example,

$$3 > 2,\ \Theta\, 3 < \Theta\, 2: \qquad (144.)$$

and in general a relation of subsequence or precedence between any two whole or fractional numbers is changed to the opposite relation of precedence or subsequence, by altering those numbers to their opposites, though a relation of equality or coincidence remains unaltered after such a change. Among reciprocals of positive whole numbers, the reciprocal of that which has the lesser quotity is on the positive side of the other, while reciprocals of contra-positive numbers are related by the opposite rule; thus

$$\frac{1}{2} > \frac{1}{3},\quad \frac{1}{\Theta\, 2} < \frac{1}{\Theta\, 3},\ \text{that is, } \mathrm{u}\, 2 > \mathrm{u}\, 3,\ \mathrm{u}\, \Theta\, 2 < \mathrm{u}\, \Theta\, 3. \qquad (145.)$$

In general, to determine the ordinal relation of any one fractional number $\frac{\nu}{\mu'}$ to another $\frac{\nu}{\mu}$, as subsequent, or coincident, or precedent, in the general progression of numbers, it is sufficient to prepare them by the principle (135.) so that their denominators may be equal and positive, and then to compare their numerators; for which reason it is always sufficient to compare the two whole numbers $\mu \times \mu \times \mu' \times \nu'$ and $\mu' \times \mu' \times \mu \times \nu$, and we have

$$\frac{\nu'}{\mu'} \overset{>}{\underset{<}{=}} \frac{\nu}{\mu},\ \text{according as } \mu \times \mu \times \mu' \times \nu' \overset{>}{\underset{<}{=}} \mu' \times \mu' \times \mu \times \nu: \qquad (146.)$$

the abridged notation $\overset{>}{\underset{<}{=}}$ implying the same thing as if we had written more fully

"$>$ or $=$ or $<$." If it had been merely required to prepare two fractional numbers so as to make them have a common denominator, without obliging that denominator to be positive, we might have done so in a simpler manner by the formula (135.), namely by multipling the numerator and denominator of each fraction by the denominator of the other fraction, that is, by employing the following expressions,

$$\frac{\nu'}{\mu'} = \frac{\mu \times \nu'}{\mu \times \mu'}, \quad \frac{\nu}{\mu} = \frac{\nu \times \mu'}{\mu \times \mu'}; \qquad (147.)$$

a process which may be still farther simplified when the original denominators have any whole number (other than positive or contra-positive one) for a common factor, since it is sufficient then to multiple by the factors which are not thus common, that is, to employ the expressions,

$$\frac{\nu'}{\omega \times \mu'} = \frac{\mu \times \nu'}{\omega \times \mu \times \mu'}, \quad \frac{\nu}{\omega \times \mu} = \frac{\nu \times \mu'}{\omega \times \mu \times \mu'}. \qquad (148.)$$

A similar process of preparation applies to more fractions than two.

18. This reduction of different fractional numbers to a common denominator is chiefly useful in combining them by certain operations which may be called *algebraical addition and subtraction of fractions,* (from their analogy to the algebraical addition and subtraction of whole numbers, considered in the 14th article, and to the arithmetical operations of addition and subtraction of quotities,) and which present themselves in considering the composition and decomposition of fractional steps. For we compound, as successive steps, any two or more fractions $\frac{\nu}{\mu} \times \text{b}$, $\frac{\nu'}{\mu'} \times \text{b}$, &c., of any one effective step b, and generate thereby a new effective step, this resultant step will evidently be itself a fraction of the step b, which we may agree to denote as follows:

$$\left.\begin{array}{r} \left(\frac{\nu'}{\mu'} \times \text{b}\right) + \left(\frac{\nu}{\mu} \times \text{b}\right) = \left(\frac{\nu'}{\mu'} + \frac{\nu}{\mu}\right) \times \text{b}, \qquad\qquad \\ \left(\frac{\nu''}{\mu''} \times \text{b}\right) + \left(\frac{\nu'}{\mu'} \times \text{b}\right) + \left(\frac{\nu}{\mu} \times \text{b}\right) = \left(\frac{\nu''}{\mu''} + \frac{\nu'}{\mu'} + \frac{\nu}{\mu}\right) \times \text{b}, \quad \&\text{c.}; \end{array}\right\} \qquad (149.)$$

and the resultant fractional number $\frac{\nu'}{\mu'} + \frac{\nu}{\mu}$ or $\frac{\nu''}{\mu''} + \frac{\nu'}{\mu'} + \frac{\nu}{\mu}$ &c. may be called the algebraical *sum* of the proposed fractional numbers $\frac{\nu}{\mu}$, $\frac{\nu'}{\mu'}$, $\frac{\nu''}{\mu''}$, &c. and may be said to be formed by algebraically *adding* them together; definitions which agree with those established in the 14th article, when the fractional numbers reduce themselves to whole numbers. If the denominators of the proposed fractions be the same, it is sufficient to add the numerators, because then the proposed fractional steps are all multiples of one common sub-multiple of the common unit-step b, namely of that sub-multiple which is determined by the common denominator ; it is therefore sufficient, in other cases, to prepare the fractions so as to satisfy this condition of having a common denominator, and afterwards to add the numerators so prepared, and to combine their sum as the new or resulting numerator of the resulting fractional sum, with the common denominator of the added fractions as the denominator of the same fractional sum ; which may, however, be sometimes simplified by the omission of common factors, according to the principle (135.). Thus

$$\frac{\nu'}{\mu'} + \frac{\nu}{\mu} = \frac{(\nu' \times \mu) + (\mu' \times \nu)}{\mu' \times \mu}, \text{ or more concisely } \frac{\nu'}{\mu'} + \frac{\nu}{\mu} = \frac{\nu'\mu + \mu'\nu}{\mu'\mu}, \text{ \&c.; } \quad (150.)$$

for, as a general rule of algebraic notation, we may omit at pleasure the mark of multiplication between any two simple symbols of factors, (except the arithmetical signs 1, 2, 3, &c.,) without causing any confusion ; and when a product thus denoted, by the mere juxta-position of its factors, (without the mark ×,) is to be combined with other symbols in the way of addition, by the mark +, it is not necessary to enclose that symbol of a product in parentheses : although in this Elementary Essay we have often used, and shall often use again, these combining and enclosing marks, for greater clearness and fulness. It is evident that the addition of fractions may be performed in any arbitrary order, because the order of composition of the fractional steps is arbitrary.

The algebraical *subtraction* of one given fractional number $\frac{\nu'}{\mu'}$ from another unequal fractional number $\frac{\nu}{\mu}$, is an operation suggested by the decomposition of a given compound fractional step $\frac{\nu}{\mu} \times \mathrm{b}$ into a given component fractional step $\frac{\nu'}{\mu'} \times \mathrm{b}$ and a sought component fractional step $\frac{\nu''}{\mu''} \times \mathrm{b}$, (these three steps being here supposed to be all effective :) and it may be performed by compounding the opposite of the given component step with the given compound step, or by algebraically adding the opposite $\Theta \frac{\nu'}{\mu'}$ of the given fractional number $\frac{\nu'}{\mu'}$ to the other given fractional number $\frac{\nu}{\mu}$, according to the rule (150.). When we thus subtract one fractional number from another with which it does not coincide, the result is positive or contra-positive according as the fraction from which we subtract is on the positive or contra-positive side of the other ; and thus we have another general method, besides the rule (146.), for examining the ordinal relation of any two unequal fractions, in the general progression of numbers. This ordinal relation between any two fractional (or whole) numbers α and β, is not altered by adding any fractional (or whole) number γ to both, nor by subtracting it from both; so that

$$\gamma+\beta \gtreqless \gamma+\alpha, \text{ and } \Theta\,\gamma+\beta \gtreqless \Theta\,\gamma+\alpha, \text{according as } \beta \gtreqless \alpha. \qquad (151.)$$

19. Again, the composition and decomposition of *successive acts of fractioning* (instead of successive fractional *steps*) conduct to algebraical operations of *multiplication* and *division* of fractional numbers, which are analogous to the arithmetical operations of multiplication and division of quotities. For if we first multiply a given step b by a given fractional number $\frac{\nu}{\mu}$, that is, if we first perform on b the act of fractioning denoted by this number, and so form the fractional step $\frac{\nu}{\mu} \times \mathrm{b}$, we may then perform on the result another act of fractioning denoted by another fractional number $\frac{\nu'}{\mu'}$, and so deduce another fractional step $\frac{\nu'}{\mu'} \times \left(\frac{\nu}{\mu} \times \mathrm{b}\right)$, which will evidently be itself a fraction of the original step b, and might therefore have been deduced from b by one compound act of fractioning; and thus we may proceed to other and other fractions of that step, and to other compound acts of fractioning, which may be thus denoted,

$$\left.\begin{aligned} \frac{\nu'}{\mu'} \times \left(\frac{\nu}{\mu} \times \mathrm{b}\right) &= \left(\frac{\nu'}{\mu'} \times \frac{\nu}{\mu}\right) \times \mathrm{b}, \\ \frac{\nu''}{\mu''} \times \left\{\frac{\nu'}{\mu'} \times \left(\frac{\nu}{\mu} \times \mathrm{b}\right)\right\} &= \left(\frac{\nu''}{\mu''} \times \frac{\nu'}{\mu'} \times \frac{\nu}{\mu}\right) \times \mathrm{b}, \quad \&\text{c.}; \end{aligned}\right\} \qquad (152.)$$

and the resultant fractional numbers $\frac{\nu'}{\mu'} \times \frac{\nu}{\mu}$, $\frac{\nu''}{\mu''} \times \frac{\nu'}{\mu'} \times \frac{\nu}{\mu}$, &c., which thus express the resultant acts of fractioning, derived from the proposed component acts marked by the fractional numbers $\frac{\nu}{\mu}$, $\frac{\nu'}{\mu'}$, $\frac{\nu''}{\mu''}$, &c., may be called the *algebraic products* of those proposed fractional numbers, and may be said to be formed by *algebraically multiplying* them as *fractional factors* together; definitions which agree with the definitions of product and multiplication already established for whole numbers. The same definitions shew that every fraction may be regarded as the product of the numerator (as one factor) and the reciprocal of the denominator (as another); and give, in general, by (134.), the following rule for the calculation of a fractional product

$$\frac{\nu'}{\mu'} \times \frac{\nu}{\mu} = \frac{\nu' \times \nu}{\mu' \times \mu}, \quad \frac{\nu''}{\mu''} \times \frac{\nu'}{\mu'} \times \frac{\nu}{\mu} = \frac{\nu'' \times \nu' \times \nu}{\mu'' \times \mu' \times \mu}, \ \&\text{c.} \qquad (153.)$$

The properties (114.) and (115.) of algebraic products of whole numbers extend to products of fractional numbers also; that is, we may change in any manner the order of the fractional factors; and if we resolve any one of those factors into two or more algebraic parts by the rules of algebraic addition and subtraction, we may combine each part separately as a partial factor with the other factors proposed, so as to form by algebraic multiplication a partial fractional product, and then add together those partial products algebraically to obtain the total product: or, in written symbols,

$$\frac{\nu'}{\mu'} \times \frac{\nu}{\mu} = \frac{\nu}{\mu} \times \frac{\nu'}{\mu'}, \text{ \&c.,} \qquad (154.)$$

and

$$\frac{\nu}{\mu} \times \left(\frac{\nu''}{\mu''} + \frac{\nu'}{\mu'}\right) = \left(\frac{\nu}{\mu} \times \frac{\nu''}{\mu''}\right) + \left(\frac{\nu}{\mu} \times \frac{\nu'}{\mu'}\right), \text{ \&c.,} \qquad (155.)$$

because

$$\frac{\nu}{\mu} \times \left(\mathrm{b}'' + \mathrm{b}'\right) = \left(\frac{\nu}{\mu} \times \mathrm{b}''\right) + \left(\frac{\nu}{\mu} \times \mathrm{b}'\right), \qquad (156.)$$

whatever steps may be denoted by b' and b'' and whatever fractional (or whole) number by $\frac{\nu}{\mu}$. We may also remark that

$$\gamma \times \beta \overset{>}{\underset{<}{=}} \gamma \times \alpha, \text{ according as } \beta \overset{>}{\underset{<}{=}} \alpha, \text{ if } \gamma > 0, \qquad (157.)$$

but that

$$\gamma \times \beta \overset{<}{\underset{>}{=}} \gamma \times \alpha, \text{ according as } \beta \overset{>}{\underset{<}{=}} \alpha, \text{ if } \gamma < 0, \qquad (158.)$$

$\alpha\, \beta\, \gamma$ denoting any three fractional (or whole) numbers.

The deduction of one of two fractional factors from the other and from the product, may be called (by analogy to arithmetic) the *algebraic division* of the given fractional product as a *dividend*, by the given fractional factor as a *divisor;* and the result, which may be called the *quotient,* may always be found by algebraically multiplying the proposed dividend by the reciprocal of the proposed divisor. This more general conception of quotient, agrees with the process of the 15th article, for the division of one whole number by another, when that process gives an accurate quotient in whole numbers; and when no such integral and accurate quotient can be found, we may still, by our present extended definitions, conceive the numerator of any fraction to be divided by the denominator, and the quotient of this division will be the fractional number itself. In this last case, the fractional number is not exactly equal to any whole number, but lies between two successive whole numbers, a next preceding and a next succeeding, in the general progression of numbers; and these may be discovered by the process of approximate division above mentioned, while each of the two remainders of that approximate division is the numerator of a new fraction, which retains the proposed denominator, and must be added algebraically as a *correction* to the corresponding *approximate integer quotient,* in order to express, by the help of it, the quotient of the accurate division. For example,

$$\frac{8}{5} = \frac{3}{5} + 1 = \frac{\Theta\,2}{5} + 2, \text{ and } \frac{\Theta\,8}{5} = \frac{2}{5} + \Theta\,2 = \frac{\Theta\,3}{5} + \Theta\,1.$$

In general, a fractional number may be called a *mixed number,* when it is thus expressed as the algebraic sum of a whole number and a *proper fraction,* this last name being given to a fractional number which lies between zero and positive or contra-positive one. We may remark that an ordinal relation between two fractional numbers is not altered by dividing them both by one common positive divisor; but if the divisor be contra-positive, it changes a relation of subsequence to one of precedence, and conversely, without disturbing a relation of coincidence.

20. In all the formulæ of the three last articles, we have supposed that all the numerators and all the denominators of those formulæ are positive or contra-positive whole numbers, excluding the number zero. However, the general conception of a fraction as *a multiple of a sub-multiple,* permits us to suppose that the multipling number or numerator is zero, and shows us that then the fractional step itself is null, if the denominator be different from zero; that is,

$$\frac{0}{\mu} \times \mathrm{b} = 0 \text{ if } \mu \neq 0. \qquad (159.)$$

Thus, although we supposed, in the composition (149.) of successive fractional steps, (with positive or contra-positive numerators and denominators,) that the resultant step was effective, yet we might have removed this limitation, and have presented the formulæ (150.) for fractional sums as extending even to the case when the resultant step is null, if we had observed that in every such case the resultant numerator of the formula is zero, while the resultant denominator is different from zero, and therefore that the formula rightly expresses that the resultant fraction or sum is null. For example, the addition of any two opposite fractional numbers, such as $\frac{\nu}{\mu}$ and $\frac{\Theta\,\nu}{\mu}$, in which μ and ν are different from zero, conducts to a null sum, under the form $\frac{\Theta\,\nu + \nu}{\mu}$, in which the numerator $\Theta\,\nu + \nu$ is zero, while the denominator is different from zero.

But it is not so immediately clear what ought to be regarded as the meaning of a fractional sign, in the case when the denominator is null, and when therefore the act of fractioning prescribed by the notation involves a sub-multipling by zero. To discuss this case, we must remember that to sub-multiple a step b by a whole number μ, is, by its definition, to find another step a, which, when multipled by that whole number μ, shall produce the proposed step b; but, whatever step a may be, the theory of multiple steps (explained in the 13th article) shows that it necessarily produces the null step 0, when it is multipled by the null number zero; that is, the equation

$$0 \times \mathrm{a} = 0 \qquad (160.)$$

is true independently of a, and consequently we have always

$$0 \times \mathrm{a} \neq \mathrm{b}, \text{ if } \mathrm{b} \neq 0. \qquad (161.)$$

It is, therefore, impossible to find any step a, in the whole progression of time, which shall satisfy the equation

$$\frac{1}{0} \times \mathrm{b} = \mathrm{a}, \text{ or } 0 \times \mathrm{a} = \mathrm{b}, \qquad (162.)$$

if the given step b be effective; or, in other words, it is impossible to sub-multiple an effective step by zero. The fractional sign $\frac{1}{0}$ denotes therefore an *impossible act*, if it be applied to an effective step: and *the zero-submultiple of an effective step* is a phrase which involves a contradiction. On the other hand, if the given step b be null, it is not only possible to choose some one step a which shall satisfy the equations (162.), but every conceivable step possesses the same proposed property; in this case, therefore, the proposed conditions lay no restriction on the result, but at the same time, and for the same reason, they fail to give any information respecting it: and the act of sub-multipling a null step by zero, is indeed a possible, but it is also an *indeterminate act*, or an act with an indeterminate result; so that the *zero-submultiple of a null step*, and the written symbol $\frac{1}{0} \times 0$, are spoken or written signs which do not specify any thing, although they do not involve a contradiction. We see then that while a fractional number is in general the sign of a possible and determinate act of fractioning, it loses one or other of those two essential characters whenever its denominator is zero; for which reason it becomes comparatively unfit, or at least inconvenient, in this case, for the purposes of mathematical reasoning. And to prevent the confusion which might arise from the mixture of such cases with others, it is convenient to lay down this *general rule*, to which we shall henceforth adhere: that *all* *denominators and divisors are to be supposed different from zero* unless the contrary be mentioned expressly; or that we shall *never sub-multiple nor divide by a null number* without expressly recording that we do so.

On the Comparison of any one effective Step with any other, in the way of Ratio, and the Generation of any one such step from any other, in the way of Multiplication; and on the Addition, Subtraction, Multiplication, and Division of Algebraic Numbers in general, considered thus as Ratios or as Multipliers of Steps.

21. The foregoing remarks upon fractions lead naturally to the more general conception of *algebraic ratio,* as a complex relation of any one effective step to any other, determined by their *relative largeness* and *relative direction;* and to a similarly extended conception of algebraic *multiplication,* as an *act* (of thought) which enlarges, or preserves, or diminishes the magnitude, while it preserves or reverses the direction, of any effective step proposed. In conformity with these conceptions, and by analogy to our former notations, if we denote by a and b any two effective steps, of which a may be called the *antecedent* or the *multiplicand,* and b the *consequent* or the *product,* we may employ the symbol $\frac{\mathrm{b}}{\mathrm{a}}$ to denote the *ratio* of the consequent b to the antecedent a, or the algebraic *number* or *multiplier* by which we are to multiply a as a *multiplicand* in order to generate b as a product: and if we still employ the mark of multiplication $\times$, we may now write, in general,

$$\mathrm{b} = \frac{\mathrm{b}}{\mathrm{a}} \times \mathrm{a}: \qquad (163.)$$

or, more concisely,

$$\mathrm{b} = a \times \mathrm{a}, \text{ if } \frac{\mathrm{b}}{\mathrm{a}} = a, \qquad (164.)$$

that is, if we employ, for abridgement, a simple symbol, such as the italic letter a, to denote the same ratio or multiplier which is more fully denoted by the complex symbol $\frac{\mathrm{b}}{\mathrm{a}}$.

It is an immediate consequence of these conceptions and definitions, that the following relation holds good,

$$\frac{\nu \times \mathrm{a}}{\mu \times \mathrm{a}} = \frac{\nu}{\mu}, \qquad (165.)$$

a denoting any effective step, and μ and ν denoting any positive or contra-positive whole numbers; since the fractional ratio denoted by the symbol $\frac{\nu}{\mu}$ is the ratio of the multiple step $\nu \times a$ to the multiple step $\mu \times a$. In like manner it follows, from the same conceptions and definitions, that

$$\frac{\frac{\nu}{\mu} \times b}{b} = \frac{\nu}{\mu}, \text{ and reciprocally } b' = \frac{\nu}{\mu} \times b \text{ if } \frac{b'}{b} = \frac{\nu}{\mu}; \tag{166.}$$

and more generally, that

$$\frac{\frac{b}{a} \times c}{c} = \frac{b}{a}, \tag{167.}$$

and reciprocally,

$$d = \frac{b}{a} \times c \text{ if } \frac{d}{c} = \frac{b}{a}; \tag{168.}$$

whatever effective steps may be denoted by a, b, c, d, and whatever fraction by $\frac{\nu}{\mu}$.

We may also conceive combinations of ratios with each other, by operations which we may call Addition, Subtraction, Multiplication, and Division of Ratios, or of *general algebraic numbers,* from the analogy of these operations to those which we have already called by the same names, in the theories of whole numbers and of fractions. And as we wrote, in treating of whole numbers,

$$\omega = \nu + \mu \text{ when } \omega \times a = (\nu \times a) + (\mu \times a), \tag{107.}$$

and

$$\omega = \nu \times \mu \text{ when } \omega \times a = \nu \times (\mu \times a); \tag{111.}$$

and, in the theory of fractions,

$$\frac{\nu''}{\mu''} = \frac{\nu'}{\mu'} + \frac{\nu}{\mu} \text{ when } \frac{\nu''}{\mu''} \times b = \left(\frac{\nu'}{\mu'} \times b\right) + \left(\frac{\nu}{\mu} \times b\right), \tag{149.}$$

and

$$\frac{\nu''}{\mu''} = \frac{\nu'}{\mu'} \times \frac{\nu}{\mu} \text{ when } \frac{\nu''}{\mu''} \times b = \frac{\nu'}{\mu'} \times \left(\frac{\nu}{\mu} \times b\right), \tag{152.}$$

with other similar expressions; so we shall now write, in the more general theory of ratios,

$$\frac{b''}{a''} = \frac{b'}{a'} + \frac{b}{a} \text{ when } \frac{b''}{a''} \times c = \left(\frac{b'}{a'} \times c\right) + \left(\frac{b}{a} \times c\right), \tag{169.}$$

and

$$\frac{b''}{a''} = \frac{b'}{a'} \times \frac{b}{a}, \text{ when } \frac{b''}{a''} \times c = \frac{b'}{a'} \times \left(\frac{b}{a} \times c\right): \tag{170.}$$

and shall suppose that similar definitions are established for the algebraical sums and products of more than two ratios, or general algebraic numbers. It follows that

$$\left.\begin{aligned}\frac{b'}{a}+\frac{b}{a}&=\frac{b'+b}{a}\\ \frac{b''}{a}+\frac{b'}{a}+\frac{b}{a}&=\frac{b''+b'+b}{a},\\ \&c.&\end{aligned}\right\}\qquad(171.)$$

and that

$$\left.\begin{aligned}\frac{b'}{b}\times\frac{b}{a}&=\frac{b'}{a},\\ \frac{b''}{b'}\times\frac{b'}{b}\times\frac{b}{a}&=\frac{b''}{a},\quad\&c.\end{aligned}\right\}\qquad(172.)$$

A ratio between any two effective steps may be said to be *positive* or *contra-positive*, according as those two steps are *co-directional* or *contra-directional*, that is, according as their directions agree or differ; and then the product of any two or more positive or contra-positive ratios will evidently be contra-positive or positive according as there are or are not an odd number of contra-positive ratios, as factors of this product; because the direction of a step is not altered or is restored, if it either be not reversed at all, or be reversed an even number of times.

Again, we may say, as in the case of fractions, that we *subtract* a ratio when we add its *opposite*, and that we *divide* by a ratio when we multiply by its *reciprocal*, if we agree to say that two ratios or numbers are *opposites* when they generate *opposite steps* by multiplication from one common step as a multiplicand, and if we call them *reciprocals* when their corresponding acts of multiplication are *opposite acts*, which destroy, each, the effect of the other; and we may mark such opposites and reciprocals, by writing, as in the notation of fractions,

$$\frac{b'}{a'}=\Theta\,\frac{b}{a}\text{ when }\frac{b'}{a'}\times c=\Theta\left(\frac{b}{a}\times c\right),\qquad(173.)$$

and

$$\frac{b'}{a'}=\text{ꓤ}\,\frac{b}{a},\text{ when }\frac{b'}{a'}\times\left(\frac{b}{a}\times c\right)=c:\qquad(174.)$$

definitions from which it follows that

$$\frac{\Theta\,b}{a}=\Theta\,\frac{b}{a},\qquad(175.)$$

and that

$$\frac{a}{b}=\text{ꓤ}\,\frac{b}{a}.\qquad(176.)$$

And as, by our conceptions and notations respecting the ordinal relation of one fractional number to another, (as subsequent, or coincident, or precedent, in the general progression of such numbers from contra-positive to positive,) we had the relations,

$$\frac{\nu'}{\mu'} \overset{>}{\underset{<}{=}} \frac{\nu}{\mu}, \text{ when } \frac{\nu'}{\mu'} \times \mathrm{a} \overset{>}{\underset{<}{=}} \frac{\nu}{\mu} \times \mathrm{a}, \quad \mathrm{a} > 0;$$

so we may now establish, by analogous conceptions and notations respecting ratios, the relations,

$$\frac{\mathrm{b}''}{\mathrm{a}''} \overset{>}{\underset{<}{=}} \frac{\mathrm{b}'}{\mathrm{a}'}, \text{ when } \frac{\mathrm{b}''}{\mathrm{a}''} \times \mathrm{a} \overset{>}{\underset{<}{=}} \frac{\mathrm{b}'}{\mathrm{a}'} \times \mathrm{a}, \quad \mathrm{a} > 0: \qquad (177.)$$

that is, more fully,

$$\frac{\mathrm{b}''}{\mathrm{a}''} > \frac{\mathrm{b}}{\mathrm{a}}, \text{ when } \left(\frac{\mathrm{b}''}{\mathrm{a}''} \times \mathrm{a}\right) + \mathrm{A} > \left(\frac{\mathrm{b}'}{\mathrm{a}'} \times \mathrm{a}\right) + \mathrm{A}, \qquad (178.)$$

$$\frac{\mathrm{b}''}{\mathrm{a}''} = \frac{\mathrm{b}'}{\mathrm{a}'}, \text{ when } \left(\frac{\mathrm{b}''}{\mathrm{a}''} \times \mathrm{a}\right) + \mathrm{A} = \left(\frac{\mathrm{b}'}{\mathrm{a}'} \times \mathrm{a}\right) + \mathrm{A}, \qquad (179.)$$

and

$$\frac{\mathrm{b}''}{\mathrm{a}''} < \frac{\mathrm{b}'}{\mathrm{a}'}, \text{ when } \left(\frac{\mathrm{b}''}{\mathrm{a}''} \times \mathrm{a}\right) + \mathrm{A} < \left(\frac{\mathrm{b}'}{\mathrm{a}'} \times \mathrm{a}\right) + \mathrm{A}; \qquad (180.)$$

the symbol A denoting any moment of time, and a any late-making step. The relation (179.) is indeed an immediate consequence of the first conceptions of steps and ratios; but it is inserted here along with the relations (178.) and (180.), to show more distinctly in what manner the comparison and arrangement of the moments

$$\left(\frac{\mathrm{b}'}{\mathrm{a}'} \times \mathrm{a}\right) + \mathrm{A}, \left(\frac{\mathrm{b}''}{\mathrm{a}''} \times \mathrm{a}\right) + \mathrm{A}, \text{ \&c.} \qquad (181.)$$

which are suggested and determined by the ratios or numbers $\frac{\mathrm{b}'}{\mathrm{a}'}$, $\frac{\mathrm{b}''}{\mathrm{a}''}$, &c., (in combination with a standard moment A and with a late-making step a,) enable us to compare and arrange those ratios or numbers themselves, and to conceive an indefinite progression of ratio from contra-positive to positive, including the indefinite progression of whole numbers (103.), and the more comprehensive progression of fractional numbers considered in the 17th article: for it will soon be shown, that though every fractional number is a ratio, yet there are many ratios which cannot be expressed under the form of fractional numbers. Meanwhile we may observe, that the theorems (151.) (157.) (158.) respecting the ordinal relations of fractions in the general progression of number, are true, even when the symbols $\alpha\ \beta\ \gamma$ denote ratios which are not reducible to the fractional form; and that this indefinite progression of number, or of ratio, from contra-positive to positive, corresponds in all respects to the thought from which it was deduced, of the progression of time itself, from moments indefinitely early to moments indefinitely late.

22. It is manifest, on a little attention, that the ratio of one effective step b to another a, is a relation which is entirely determined when those steps are given, but which is not altered by multiplying both those steps by any common multiplier, whether positive or contra-positive; for the *relative largeness* of the two steps is not altered by doubling or halving both, or by enlarging or diminishing the magnitudes of both in any other common ratio of magnitude, that is, by multiplying both by any common positive multiplier: nor is their *relative direction* altered, by reversing the directions of both. We have then, generally,

$$\frac{\frac{b'}{a'} \times b}{\frac{b'}{a'} \times a} = \frac{b}{a}; \qquad (182.)$$

and in particular, by changing a' to a, and b' to c,

$$\frac{\frac{c}{a} \times b}{c} = \frac{b}{a}. \qquad (183.)$$

Hence, by (167.), the two steps $\frac{c}{a} \times b$ and $\frac{b}{a} \times c$ are related in one common ratio, namely the ratio $\frac{b}{a}$, to the common step c, and therefore are equivalent to each other; that is, we have the equation,

$$\frac{c}{a} \times b = \frac{b}{a} \times c, \qquad (184.)$$

whatever three effective steps may be denoted by a b c.

In general, when any four effective steps a b c d are connected by the relation

$$\frac{d}{c} = \frac{b}{a}, \qquad (185.)$$

that is, when the ratio of the step d to c is the same as the ratio of the step b to a, these two pairs of steps a, b and c, d may be said to be *analogous* or *proportional pairs*; the steps a and c being called the *antecedents* of the analogy, (or of the proportion) and the steps b and d being called the *consequents*, while a and d are the *extremes* and b and c the *means*. And since the last of these four steps, or the second consequent d, may, by (168.), be expressed by the symbol $\frac{b}{a} \times c$, we see, by (184.), that it bears to the first consequent b the ratio $\frac{c}{a}$ of the second antecedent c to the first antecedent a; that is,

$$\frac{\mathrm{d}}{\mathrm{b}} = \frac{\mathrm{c}}{\mathrm{a}} \text{ if } \frac{\mathrm{d}}{\mathrm{c}} = \frac{\mathrm{b}}{\mathrm{a}} : \qquad (186.)$$

a theorem which shows that we may transform the expression of an *analogy* (or *proportion*) between two pairs of effective steps in a manner which may be called *alternation.* (Compare the theorem (11.).)

It is still more easy to perceive that we may *invert* an analogy or proportion between any two pairs of effective steps; or that the following theorem is true,

$$\frac{\mathrm{c}}{\mathrm{d}} = \frac{\mathrm{a}}{\mathrm{b}}, \text{ if } \frac{\mathrm{d}}{\mathrm{c}} = \frac{\mathrm{b}}{\mathrm{a}}. \qquad (187.)$$

Combining inversion with alternation, we see that

$$\frac{\mathrm{b}}{\mathrm{d}} = \frac{\mathrm{a}}{\mathrm{c}}, \text{ if } \frac{\mathrm{d}}{\mathrm{c}} = \frac{\mathrm{b}}{\mathrm{a}}. \qquad (188.)$$

(Compare the theorems (12.) and (13.).)

In general, if any two pairs of effective steps a, b and c, d be analogous, we do not disturb this analogy by interchanging the extremes among themselves, or the means among themselves, or by substituting extremes for means and means for extremes; or by altering *proportionally,* that is, altering in one common ratio, or multiplying by one common multiplier, whether positive or contra-positive, the two consequents, or the two antecedents, or the two steps of either pair: or, finally, by altering *in inverse proportion,* that is, multiplying respectively by any two reciprocal multipliers, the two extremes, or the two means. The analogy (185.) may therefore be expressed, not only in the ways (186.), (187.), (188.), but also in the following:

$$\frac{a \times \mathrm{d}}{\mathrm{c}} = \frac{a \times \mathrm{b}}{\mathrm{a}}, \quad \frac{\mathrm{d}}{a \times \mathrm{c}} = \frac{\mathrm{b}}{a \times \mathrm{a}}, \quad \frac{a \times \mathrm{d}}{a \times \mathrm{c}} = \frac{\mathrm{b}}{\mathrm{a}}, \qquad (189.)$$

$$\frac{\text{ᴚ}\, a \times \mathrm{d}}{\mathrm{c}} = \frac{\mathrm{b}}{a \times \mathrm{a}}, \quad \frac{\mathrm{d}}{\text{ᴚ}\, a \times \mathrm{c}} = \frac{a \times \mathrm{b}}{\mathrm{a}}, \qquad (190.)$$

a denoting any ratio of one effective step to another, and ᴚ a denoting the reciprocal ratio, of the latter step to the former.

23. We may also consider it as evident that if any effective step c be compounded of any others a and b, this relation of compound and components will not be disturbed by altering the magnitudes of all in any common ratio of magnitude, that is by doubling or halving it, or multiplying all by any common positive multiplier; and we saw, in the 12th article, that the same relation of compound and components is not disturbed by reversing the directions of all: we may therefore mul-

tiply all by any common multiplier a, whether positive or contra-positive, and may establish the theorem,

$$a \times \mathrm{c} = (a \times \mathrm{b}) + (a \times \mathrm{a}), \text{ if } \mathrm{c} = \mathrm{b} + \mathrm{a}; \qquad (191.)$$

which gives, by the definitions (169.) (170.) for the sum and product of two ratios, this other important relation,

$$a \times (b' + b) = (a \times b') + (a \times b), \qquad (192.)$$

if b, b', and $b' + b$, denote any three positive or contra-positive numbers, connected with each other by the definition (169.), or by the following condition,

$$(b' + b) \times \mathrm{d} = (b' \times \mathrm{d}) + (b \times \mathrm{d}), \qquad (193.)$$

in which d denotes any arbitrary effective step. The definitions of the sum and product of two ratios, or algebraic numbers, give still more simply the theorem,

$$(b' + b) \times a = (b' \times a) + (b \times a). \qquad (194.)$$

The definition (169.) of a sum of two ratios, when combined with the theorem (75.) respecting the arbitrary order of composition of two successive steps, gives the following similar theorem respecting the addition of two ratios,

$$b + a = a + b. \qquad (195.)$$

And if the definition (170.) of a product of two ratios or multipliers be combined with the theorem (186.) of alternation of an analogy between two pairs of steps, in the same way as the definition of a compound step was combined in the 12th article with the theorem of alternation of an analogy between two pairs of moments, it shows that as any two steps a, b, may be applied to any moment, or compounded with each other, either in one or in the opposite order, ($\mathrm{b} + \mathrm{a} = \mathrm{a} + \mathrm{b}$,) so any two ratios a and b may be applied as multipliers to any step, or combined as factors of a product with each other, in an equally arbitrary order; that is, we have the relation,

$$b \times a = a \times b. \qquad (196.)$$

It is easy to infer, from the thorems (195.) (196.), that the opposite of a sum of two ratios is the sum of the opposites of those ratios, and that the reciprocal of the product of two ratios is the product of their two reciprocals; that is,

$$\Theta\,(b + a) = \Theta\,b + \Theta\,a, \qquad (197.)$$

and

$$\text{ꓤ}\,(b \times a) = \text{ꓤ}\,b \times \text{ꓤ}\,a. \qquad (198.)$$

And all the theorems of this article, respecting pairs of ratios or of steps, may easily be extended to the comparison and combination of more ratios or steps than two. In particular, when any number of ratios are to be added or multiplied together, we may arrange them in any arbitrary order; and in any multiplication of ratios, we may treat any one factor as the algebraic sum of any number of other ratios, or partial factors, and substitute each of these separately and successively for it, and the sum of the partial products thus obtained will be the total product sought. As an example of the multiplication of ratios, considered thus as sums, it is plain from the foregoing principles that

$$\begin{aligned}(d+c)\times(b+a) &= \{d\times(b+a)\}+\{c\times(b+a)\}\\ &= (d\times b)+(d\times a)+(c\times b)+(c\times a)\\ &= d\,b+d\,a+c\,b+c\,a, \qquad (199.)\end{aligned}$$

and that

$$\begin{aligned}(b+a)\times(b+a) &= (b\times b)+(2\times b\times a)+(a\times a)\\ &= b\,b+2\,b\,a+a\,a, \qquad (200.)\end{aligned}$$

whatever positive or contra-positive ratios may be denoted by $a\ b\ c\ d$.

And though we have only considered effective steps, and positive or contra-positive ratios, (or algebraic numbers,) in the few last articles of this Essay, yet the results extend to null steps, and to null ratios, also; provided that for the reasons given in the 20th article we treat all such null steps as consequents only and not as antecedents of ratios, admitting null ratios themselves but not their reciprocals into our formulæ, or employing null numbers as multipliers only but not as divisors, in order to avoid the introduction of symbols which suggest either impossible or indeterminate operations.

On the insertion of a Mean Proportional between two steps; and on Impossible, Ambiguous, and Incommensurable Square-Roots of Ratios.

24. Three effective steps $a\ b\ b'$ may be said to form a *continued analogy* or *continued proportion*, when the ratio of b' to b is the same as that of b to a, that is, when

$$\frac{b'}{b} = \frac{b}{a}; \qquad (201.)$$

a and b' being then the *extremes*, and b the *mean*, or the *mean proportional* between a and b', in this continued analogy; in which b' is also the *third proportional* to a and b, and a is at the same time the third proportional to b' and b, because the analogy may be inverted thus,

$$\frac{\mathrm{a}}{\mathrm{b}} = \frac{\mathrm{b}}{\mathrm{b}'}. \qquad (202.)$$

When the condition (201.) is satisfied, we may express b' as follows,

$$\mathrm{b}' = \frac{\mathrm{b}}{\mathrm{a}} \times \mathrm{a}; \qquad (203.)$$

that is, if we denote by a the ratio of b to a, we shall have the relations

$$\mathrm{b} = a \times \mathrm{a}, \; \mathrm{b}' = a \times \mathrm{b} = a \times a \times \mathrm{a}; \qquad (204.)$$

and reciprocally when these relations exist, we can conclude the existence of the continued analogy (201.). It is clear that whatever effective steps may be denoted by a and b, we can always determine, (or conceive determined,) in this manner, one third proportional b' and only one; that is, we can complete the continued analogy (201.) in one, but in only one way, when an extreme a and the mean b are given: and it is important to observe that whether the ratio a of the given mean b to the given extreme a be positive or contra-positive, that is, whether the two given steps a and b be co-directional or contra-directional steps, the product $a \times a$ will necessarily be a positive ratio, and therefore the deduced extreme step b' will necessarily be co-directional with the given extreme step a. In fact, without recurring to the theorem of the 21st article respecting the cases in which a product of contra-positive factors is positive, it is plain that the continued analogy requires, by its conception, that the step b' should be co-directional to b, if b be co-directional to a, and that b' should be contra-directional to b if b be contra-directional to a; so that in every possible case the extremes themselves are co-directional, as both agreeing with the mean or both differing from the mean in direction. *It is, therefore, impossible to insert a mean proportional between two contra-directional steps;* but for the same reason *we may insert either of two opposite steps as a mean proportional between two given co-directional steps;* namely, either a step which agrees with each, or a step which differs from each in direction, while the common magnitude of these two opposite steps is exactly intermediate in the way of ratio between the magnitudes of the two given extremes. (We here assume, as it seems reasonable to do, the conception of the general existence of such an exactly intermediate magnitude, although the nature and necessity of this conception will soon be more fully considered.) For example, it is impossible to insert a mean proportional between the two contra-directional (effective) steps a and $\Theta\, 9\, \mathrm{a}$, that is, it is impossible to find any step b which shall satisfy the conditions of the continued analogy

$$\frac{\Theta\, 9\, \mathrm{a}}{\mathrm{b}} \doteq \frac{\mathrm{b}}{\mathrm{a}}, \qquad (205.)$$

or any number or ratio a which shall satisfy the equation

$$a \times a \doteq \Theta\, 9 : \qquad (206.)$$

whereas it is possible to insert in two different ways a mean proportional b between the two co-directional (effective) steps a and $9\, \mathrm{a}$, or to satisfy by two different steps b (namely, by the step $3\, \mathrm{a}$, and also by the opposite step $\Theta\, 3\, \mathrm{a}$) the conditions of the continued analogy

$$\frac{9\, \mathrm{a}}{\mathrm{b}} \doteq \frac{\mathrm{b}}{\mathrm{a}}, \qquad (207.)$$

and it is possible to satisfy by two different ratios a the equation

$$a \times a \doteq 9, \qquad (208.)$$

namely, either by the ratio 3 or by the opposite ratio $\Theta\, 3$. In general, we may agree to express the two opposite ratios a which satisfy the equation

$$a \times a \doteq b \ (> 0), \qquad (209.)$$

by the two symbols

$$\sqrt{b} \ (> 0) \text{ and } \Theta \sqrt{b} \ (< 0), \qquad (210.)$$

b and $\sqrt{b}$ being positive ratios, but $\Theta \sqrt{b}$ being contra-positive; for example,

$$\sqrt{9} \doteq 3, \ \Theta \sqrt{9} \doteq \Theta\, 3. \qquad (211.)$$

With this notation we may represent the two opposite steps of which each is a mean proportional between two given co-directional (effective) steps a and b', by the symbols

$$\sqrt{\frac{\mathrm{b}'}{\mathrm{a}}} \times \mathrm{a}, \text{ and } \Theta \sqrt{\frac{\mathrm{b}'}{\mathrm{a}}} \times \mathrm{a}; \qquad (212.)$$

and shall have for each the equation of a continued analogy,

$$\frac{\mathrm{b}'}{\sqrt{\frac{\mathrm{b}'}{\mathrm{a}}} \times \mathrm{a}} \doteq \frac{\sqrt{\frac{\mathrm{b}'}{\mathrm{a}}} \times \mathrm{a}}{\mathrm{a}}, \quad \frac{\mathrm{b}'}{\Theta \sqrt{\frac{\mathrm{b}'}{\mathrm{a}}} \times \mathrm{a}} \doteq \frac{\Theta \sqrt{\frac{\mathrm{b}'}{\mathrm{a}}} \times \mathrm{a}}{\mathrm{a}}. \qquad (213.)$$

We may also call the numbers $\surd b$ and $\ominus \surd b$ by the common name of *roots*, or (more fully) *square-roots* of the positive number b; distinguishing them from each other by the separate names of the *positive square-root* and the *contra-positive square-root* of that number b, which may be called their common *square:* though we may sometimes speak simply of *the square-root* of a (positive) number, meaning then the positive root, which is simpler and more important than the other.

25. The idea of the *continuity of the progression from moment to moment in time* involves the idea of a similarly *continuous progression in magnitude* from any one effective step or interval between two different moments, to any other unequal effective step or other unequal interval; and also the idea of a *continuous progression in ratio,* from any one degree of inequality, in the way of relative largeness or smallness, as a relation between two steps, to any other degree. Pursuing this train of thought, we find ourselves compelled to conceive the existence (assumed in the last article) of a determined magnitude b, exactly intermediate in the way of ratio between any two given unequal magnitudes a and b′, that is, larger or smaller than the one, in exactly the same proportion in which it is smaller or larger than the other; and therefore also the existence of a determined number or ratio a which is the exact square-root of any proposed (positive) number or ratio b. To show this more fully, let A B D be any three given distinct moments, connected by the relations

$$\frac{\mathrm{D} - \mathrm{A}}{\mathrm{B} - \mathrm{A}} = b, \ b > 1, \qquad (214.)$$

which require that the moment B should be situated between A and D; and let C be any fourth moment, lying between B and D, but capable of being chosen as near to B or as near to D as we may desire, in the continuous progression of time. Then the two ratios

$$\frac{\mathrm{C} - \mathrm{A}}{\mathrm{B} - \mathrm{A}} \text{ and } \frac{\mathrm{D} - \mathrm{A}}{\mathrm{C} - \mathrm{A}}$$

will both be positive ratios, and both will be *ratios of largeness,* (that is, each will be a relation of a larger to a smaller step,) which we may denote for abridgement as follows,

$$\frac{\mathrm{C} - \mathrm{A}}{\mathrm{B} - \mathrm{A}} = x, \ \frac{\mathrm{D} - \mathrm{A}}{\mathrm{C} - \mathrm{A}} = y = \amalg x \times b; \qquad (215.)$$

but by choosing the moment C sufficiently near to B we may make the ratio x approach as near as we desire to the ratio of equality denoted by 1, while the ratio y

will tend to the given ratio of largeness denoted by b; results which we may express by the following written sentence,

$$\text{if } \underline{\mathrm{L}}\,\mathrm{C} = \mathrm{B}, \text{ then } \underline{\mathrm{L}}\,x = 1 \text{ and } \underline{\mathrm{L}}\,y = b, \qquad (216.)$$

prefixing the symbol $\underline{\mathrm{L}}$, (namely the initial letter L of the Latin word Limes, distinguished by a bar drawn under it,) to the respective marks of the variable moment C and variable ratios x, y, in order to denote the respective *limits* to which those variables tend, while we vary the selection of one of them, and therefore also of the rest. Again, we may choose the moment C nearer and nearer to D, and then the ratio x will tend to the given ratio of largeness denoted by b, while the ratio y will tend to the ratio of equality; that is,

$$\text{if } \underline{\mathrm{L}}\,\mathrm{C} = \mathrm{D}, \text{ then } \underline{\mathrm{L}}\,x = b,\ \underline{\mathrm{L}}\,y = 1; \qquad (217.)$$

and if we conceive a continuous progression of moments C from B to D, we shall also have a continuous progression of ratios x, determining higher and higher degrees of relative largeness (of the increasing step $\mathrm{C} - \mathrm{A}$ as compared with the fixed step $\mathrm{B} - \mathrm{A}$) from the ratio of equality 1 to the given ratio of largeness b, together with another continuous but opposite progression of ratios y, determining lower and lower degrees of relative largeness (of the fixed step $\mathrm{D} - \mathrm{A}$ as compared with the increasing step $\mathrm{C} - \mathrm{A}$) from the same given ratio of largeness b down to the ratio of equality 1; so that we cannot avoid conceiving the existence of some one determined state of the progression of the moment C, for which the two progressions of ratio *meet*, and for which they give

$$\text{ꓤ}\, x \times b = y = x, \text{ that is } \frac{\mathrm{D} - \mathrm{A}}{\mathrm{C} - \mathrm{A}} = \frac{\mathrm{C} - \mathrm{A}}{\mathrm{B} - \mathrm{A}}, \qquad (218.)$$

having given at first $y > x$, and giving afterwards $y < x$. And since, in general,

$$\frac{\mathrm{D} - \mathrm{A}}{\mathrm{C} - \mathrm{A}} \times \frac{\mathrm{C} - \mathrm{A}}{\mathrm{B} - \mathrm{A}} = \frac{\mathrm{D} - \mathrm{A}}{\mathrm{B} - \mathrm{A}}, \text{ that is, } (\text{ꓤ}\, x \times b) \times x = b, \qquad (219.)$$

we can and must by (218.) and (214.), conceive the existence of a positive ratio a which shall satisfy the condition (209.), $a \times a = b$, if $b > 1$, that is, we must conceive the existence of a positive square-root of b, if b denote any positive ratio of largeness. A reasoning of an entirely similar kind would prove that we must conceive the existence of a positive square-root of b, when b denotes any positive ratio of smallness, ($b < 1$;) and if b denote the positive ratio of equality, ($b = 1$,) then it evidently has that ratio of equality itself for a positive square-root. We see then by this more full examination what we before assumed to be true, that every positive number or ratio b has a positive (and therefore also a contra-positive) square-root.

And hence we can easily prove another important property of ratios, which has been already mentioned without proof; namely that several ratios can and must be conceived to exist, which are incapable of being expressed under the form of whole or fractional numbers; or, in other words, that every effective step a has other steps *incommensurable* with it; and therefore that when any two distinct moments A and B are given, it is possible to assign (in various ways) a third moment C which shall not be *uniserial* with these two, in the sense of the 8th article, that is, shall not belong in common with them to any one equi-distant series of moments, comprising all the three. For example, the positive square-root of 2, which is evidently intermediate between 1 and 2 in the general progression of numbers, and which therefore is not a whole number, cannot be expressed as a fractional number either; since if it could be put under the fractional form $\frac{n}{m}$, so that

$$\sqrt{2} = \frac{n}{m}, \qquad (220.)$$

we should then have

$$2 = \frac{n}{m} \times \frac{n}{m} = \frac{n \times n}{m \times m}, \qquad (221.)$$

that is,

$$n \times n = 2 \times m \times m; \qquad (222.)$$

but the arithmetical properties of quotities are sufficient to prove that this last equation is impossible, whatever positive whole numbers may be denoted by m and n. And hence, if we imagine that

$$\mathrm{b} = \sqrt{2} \times \mathrm{a}, \ \mathrm{a} > 0, \qquad (223.)$$

the step b which is a mean proportional between the two effective and co-directional steps a and $2\,\mathrm{a}$ (of which the latter is double the former) will be *incommensurable* with the step a (and therefore also with the double step $2\,\mathrm{a}$); that is, we cannot find nor conceive any other step c which shall be a *common measurer* of the steps a and b, so as to satisfy the conditions

$$\mathrm{a} = m\,\mathrm{c}, \ \mathrm{b} = n\,\mathrm{c}, \qquad (224.)$$

whatever positive or contra-positive whole numbers we may denote m and n; because, if we could do this, we should then have the relations,

$$\mathrm{b} = \frac{n}{m}\,\mathrm{a}, \quad \sqrt{2} = \frac{n}{m}, \qquad (225.)$$

of which the latter has been shown to be impossible. Hence finally, if A and B be any two distinct moments, and if we choose a third moment C such that

$$\frac{\mathrm{C} - \mathrm{A}}{\mathrm{B} - \mathrm{A}} = \sqrt{2}, \qquad (226.)$$

the moment C will not be uniserial with A and B, that is, no one equi-distant series of moments can be imagined, comprising all the three. And all that has here been shown respecting the square-root of two, extends to the square-root of three, and may be illustrated and applied in an infinite variety of other examples. We must then admit the existence of pairs of steps which have no common measurer; and may call the ratio between any two such steps an *incommensurable ratio*, or *incommensurable number*.

More formal proof of the general existence of a determined positive square-root, commensurable or incommensurable, for every determined positive ratio: continuity of progression of the square, and principles connected with this continuity.

26. The existence of these incommensurables, (or the necessity of conceiving them to exist,) is so curious and remarkable a result, that it may be usefully confirmed by an additional proof of the general existence of square-roots of positive ratios, which will also offer an opportunity of considering some other important principles.

The existence of a positive square-root $a = \sqrt{b}$, of any proposed ratio of largeness $b > 1$, was proved in the foregoing article, by the comparison of the two opposite progressions of the two ratios x and $\text{ᴜ}\, x \times b$, from the states $x = 1$, $\text{ᴜ}\, x \times b = b$, for which $\text{ᴜ}\, x \times b > x$, to the states $x = b$, $\text{ᴜ}\, x \times b = 1$, for which $\text{ᴜ}\, x \times b < x$; for this comparison obliged us to conceive the existence of an intermediate state or ratio a between the limits 1 and b, as a *common state* or *state of meeting* of these two opposite progressions, corresponding to the conception of a *moment* at which the decreasing ratio $\text{ᴜ}\, x \times b$ *becomes exactly equal* to the increasing ratio x, having been *previously a greater ratio* (or a ratio of greater relative largeness between steps), and becoming *afterwards a lesser ratio* (or a ratio of less relative largeness). And it was remarked that an exactly similar comparison of two other inverse progressions would prove the existence of a positive square-root $\sqrt{b}$ of any proposed positive ratio b of smallness, $b < 1$, $b > 0$. But instead of thus comparing, with the progression of the positive ratio x, the connected but opposite progression of the connected positive ratio ʁ $x \times b$, and showing that these progressions meet each other in a certain intermediate state or positive ratio a, we might have compared the two connected and not opposite progressions of the two connected positive ratios x and $x \times x$, of which the latter is the square of the former; and might have shown that the square $(= x \times x = x\,x)$ increases *constantly and continuously* with the root $(= x)$, from the state zero, so as to *pass successively through every state* of positive ratio b. To develope this last conception, and to draw from it a more formal (if not a more convincing) proof than that already given, of the necessary existence of a conceivable positive square-root for every conceivable positive number, we shall here lay down a few *Lemmas,* or preliminary and auxiliary propositions.

$$\textit{Lemma I.}\quad \text{If } x' \overset{>}{\underset{<}{=}} x, \text{ and } x > 0,\ x' > 0, \text{ then } x'x' \overset{>}{\underset{<}{=}} x\,x; \qquad (227.)$$

that is, the square $x'x'$ of any one positive number or ratio x', is greater than, or equal to, or less than the square xx of any other positive number or ratio x, according as the number x' itself is greater than, or equal to, or less than the number x; one number x' being said to be *greater* or *less* than another number x, when it is on the positive or on the contra-positive side of that other, in the general progression of numbers considered in the 21st article. This Lemma may be easily proved from the conceptions of ratios and of squares; it follows also without difficulty from the theorem of multiplication (200.). And hence we may obviously deduce as a *corollary* of the foregoing Lemma, this converse proposition:

$$\text{if } x'x' \overset{>}{\underset{<}{=}} x\,x, \text{ and } x > 0,\ x' > 0, \text{ then } x' \overset{>}{\underset{<}{=}} x; \qquad (228.)$$

that is, if any two proposed positive numbers have positive square-roots, the root of the one number is greater than, or equal to, or less than the root of the other number, according as the former proposed number itself is greater than, or equal to, or less than the latter proposed number.

The foregoing Lemma shows that the square *constantly* increases with the root, from zero up to states indefinitely greater and greater. But to show that this increase is *continuous* as well as constant, and to make more distinct the conception of such continuous increase, these other Lemmas may be added.

Lemma II. If a' and a'' be any two unequal ratios, we can and must conceive the existence of some intermediate ratio a; that is, we can always choose a or conceive it chosen so that

$$a > a', \quad a < a'', \quad \text{if} \quad a'' > a'. \qquad (229.)$$

For then we have the following relation of subsequence between moments,

$$a'' (\mathrm{B} - \mathrm{A}) + \mathrm{A} > a' (\mathrm{B} - \mathrm{A}) + \mathrm{A}, \text{ if } \mathrm{B} > \mathrm{A}, \qquad (230.)$$

by the very meaning of the relation of subsequence between ratios, $a'' > a'$, as defined in article 21.; and between any two distinct moments it is manifestly possible to insert an intermediate moment, indeed as many such as we may desire: it is, therefore, possible to insert a moment C between the two non-coincident moments

$$a' (\mathrm{B} - \mathrm{A}) + \mathrm{A} \text{ and } a'' (\mathrm{B} - \mathrm{A}) + \mathrm{A},$$

such that

$$\mathrm{C} > a' (\mathrm{B} - \mathrm{A}) + \mathrm{A}, \quad \mathrm{C} < a'' (\mathrm{B} - \mathrm{A}) + \mathrm{A}, \text{ if } \mathrm{B} > \mathrm{A}, \ a'' > a'; \qquad (231.)$$

and then if we put, for abridgement,

$$a = \frac{\mathrm{C} - \mathrm{A}}{\mathrm{B} - \mathrm{A}}, \qquad (232.)$$

denoting by a the ratio of the step or interval $\mathrm{C} - \mathrm{A}$ to the step or interval $\mathrm{B} - \mathrm{A}$, we shall have

$$\left.\begin{array}{r} \mathrm{C} = a (\mathrm{B} - \mathrm{A}) + \mathrm{A}, \ \mathrm{B} > \mathrm{A}, \\ a (\mathrm{B} - \mathrm{A}) + \mathrm{A} > a' (\mathrm{B} - \mathrm{A}) + \mathrm{A}, \\ a (\mathrm{B} - \mathrm{A}) + \mathrm{A} < a'' (\mathrm{B} - \mathrm{A}) + \mathrm{A}, \end{array}\right\} \qquad (233.)$$

and therefore finally,

$$a > a', \ a < a'',$$

as was asserted in the Lemma. We see, too, that the ratio a is not determined by the conditions of that Lemma, but that an indefinite variety of ratios may be chosen, which shall all satisfy those conditions.

Corollary. It is possible to choose, or conceive chosen, a ratio a, which shall satisfy all the following conditions,

$$\left.\begin{array}{l} a > a', \ a > b', \ a > c', \ldots \\ a < a'', \ a < b'', \ a < c'', \ldots \end{array}\right\} \qquad (234.)$$

if the least (or hindmost) of the ratios a'', b'', c'', ... be greater (or farther advanced in the general progression of ratio from contra-positive to positive) than the greatest (or foremost in that general progression) of the ratios a', b', c', &c.

For if c'' (for example) be the least or hindmost of the ratios $a'', b'', c'', \ldots$ so that

$$c'' \leqq a'', \; c'' \leqq b'', \; c'' \leqq d'', \ldots \qquad (235.)$$

and if b (for example) be the greatest or foremost of the ratios $a', b', c', \ldots$ so that

$$b' \geqq a', \; b' \geqq c', \; b' \geqq d', \ldots \qquad (236.)$$

(the abridged sign $\leqq$ denoting what might be more fully written thus, "$<$ or $=$", and the other abridged sign $\geqq$ denoting in like manner "$>$ or $=$",) then the conditions (234.) of the Corollary will all be satisfied, if we can satisfy these two conditions,

$$a > b', \; a < c''; \qquad (237.)$$

and this, by the Lemma, it is possible to do, if we have the relation

$$c'' > b', \qquad (238.)$$

which relation the enunciation of the Corollary supposes to exist.

Remark.—If the ratios $a' \; b' \; c' \ldots a'' \; b'' \; c'' \ldots$ be all actually given, and therefore limited in number; or if, more generally, the least of the ratios $a'' \; b'' \; c'' \ldots$ and the greatest of the ratios $a' \; b' \; c' \ldots$ be actually given and determined, so that we have only to choose a ratio a intermediate between two given unequal ratios; we can then make this choice in an indefinite variety of ways, even if it should be farther required that a should be a fractional number $\frac{\nu}{\mu}$, since we saw, in the 8th article, that between any two distinct moments, such as a' (B − A) + A and a'' (B − A) + A, it is possible to insert an indefinite variety of others, such as $\frac{\nu}{\mu}$ (B − A) + A, *uniserial* with the two moments A and B, and giving therefore fractions such as $\frac{\nu}{\mu}$, intermediate (by the 21st article) between the ratios a' and a''. But if, instead of actually knowing the ratios $a' \; b' \; c' \ldots a'' \; b' \; c'' \ldots$ themselves, in (234.), we only know a *law* by which we may assign such ratios without end, this law may lead us to conceive new conditions of the form (234.), incompatible with some (and perhaps ultimately with all) of these selections of fractional ratios $\frac{\nu}{\mu}$, although they can never exclude *all ratios a whatever*, unless they be incompatible with each other, that is, unless they fail to possess the relation mentioned in the Corollary. The force of this remark will soon be felt more fully.

Lemma III. If b denote any given positive ratio, whether it be or be not the square of any whole or of any fractional number, it is possible to find, or to conceive as found, one positive ratio a, and only one, which shall satisfy all the conditions of the following forms:

$$a > \frac{n'}{m'}, \quad a < \frac{n''}{m''}, \qquad (239.)$$

m' n' m'' n'' denoting here any positive whole numbers whatever, which can be chosen so as to satisfy these relations,

$$\frac{n' \, n'}{m' \, m'} < b, \quad \frac{n'' \, n''}{m'' \, m''} > b. \qquad (240.)$$

For if the proposed ratio b be not the square of any whole or fractional number, then the existence of such a ratio a may be proved from the two preceding Lemmas, or from their Corollaries, by observing that the relations (240.) give

$$\frac{n'' \, n''}{m'' \, m''} > \frac{n' \, n'}{m' \, m'}, \text{ and therefore } \frac{n''}{m''} > \frac{n'}{m'}; \qquad (241.)$$

so that no two conditions of the forms (239.) are incompatible with each other, and there must be *at least one* positive ratio a which satisfies them all. And to prove in the same case that there is *only one* such ratio, or that if any one positive ratio a satisfy all the conditions (239.), no greater ratio c $(> a)$ can possibly satisfy all those conditions, we may observe that however little may be the excess $\Theta \, a + c$ of the ratio c over a, this excess may be multiplied by a positive whole number m' so large that the product shall be greater than unity, in such a manner that

$$m' \, (\Theta \, a + c) > 1, \qquad (242.)$$

and therefore

$$\Theta \, a + c > \frac{1}{m'}, \quad \text{and} \quad c > \frac{1}{m'} + a; \qquad (243.)$$

and that then another positive (or null) whole number n' can be so chosen that

$$\frac{n' \, n'}{m' \, m'} < b, \quad \frac{1 + n'}{m'} \times \frac{1 + n'}{m'} > b, \qquad (244.)$$

with which selection we shall have, by (239.) (240.) (243.),

$$a > \frac{n'}{m'}, \quad c > \frac{1 + n'}{m'}: \qquad (245.)$$

whereas, if c satisfied the conditions (239.) it ought to be less than this fraction $\frac{1 + n'}{m'}$, because the square of this positive fraction is greater by (244.) than the proposed ratio b. In like manner it may be proved that in the other case, when b is the square of a positive fractional or positive whole number $\frac{n}{m}$, one positive ratio a and only one, namely the number $\frac{n}{m}$ itself, will satisfy all the conditions (239.); in both cases, therefore, the Lemma is true: and the consideration of the latter case shows, that, under the conditions (239.),

$$a = \frac{n}{m} \text{ if } b = \frac{n\,n}{m\,m}, \quad \frac{n}{m} > 0. \qquad (246.)$$

In no case do the conditions (239.) exclude *all* ratios a whatever; but except in the case (246.) they *exclude all fractional ratios:* for it will soon be shown that the one ratio a which they do not exclude has its square always $=b$, and must, therefore, be an incommensurable number when b is not the square of any integer or fraction. (Compare the *Remark* annexed to the Corollary of the IInd Lemma.)

Lemma IV. If b' and b'' be any two unequal positive ratios, it is always possible to insert between them an intermediate fractional ratio which shall be itself the square of another fractional ratio $\frac{n}{m}$; that is, we can always find, or conceive found, two positive whole numbers m and n which shall satisfy the two conditions,

$$\frac{n\,n}{m\,m} > b', \quad \frac{n\,n}{m\,m} < b'', \text{ if } b'' > b', \; b' > 0. \qquad (247.)$$

For, by the theorem of multiplication (200.), the square of the fraction $\frac{1+n'}{m}$ may be expressed as follows,

$$\frac{1+n'}{m} \times \frac{1+n'}{m} = \frac{1}{m\,m} + \frac{2\,n'}{m\,m} + \frac{n'\,n'}{m\,m}; \qquad (248.)$$

that is, its excess over the square of the fraction $\frac{n'}{m}$ is $\frac{1}{m\,m} + \frac{2\,n'}{m\,m}$, which is less than $\frac{2}{m} \times \frac{1+n'}{m}$, and constantly increases with the positive whole number n' when the positive whole number m remains unaltered; so that the $1+n'$ squares of fractions with the common denominator m, in the following series,

$$\frac{1}{m} \times \frac{1}{m}, \; \frac{2}{m} \times \frac{2}{m}, \; \frac{3}{m} \times \frac{3}{m}, \dots \frac{n'}{m} \times \frac{n'}{m}, \; \frac{1+n'}{m} \times \frac{1 \times n'}{m}, \qquad (249.)$$

increase by increasing differences which are each less than $\frac{2}{m} \times \frac{1+n'}{m}$, and therefore than $\frac{1}{k}$, if we choose m and n' so as to satisfy the conditions

$$m = 2\,i\,k, \quad 1+n' = i\,m, \qquad (250.)$$

i and k being any two positive whole numbers assumed at pleasure: with this choice, therefore, of the numbers m and n', some one (at least) such as $\frac{n\,n}{m}$ among the squares of fractions (249.), that is, some one at least among the following squares of fractions,

$$\frac{1}{2\,i\,k} \times \frac{1}{2\,i\,k}, \; \frac{2}{2\,i\,k} \times \frac{2}{2\,i\,k}, \; \frac{3}{2\,i\,k} \times \frac{3}{2\,i\,k}, \ldots \frac{2\,i\,i\,k}{2\,i\,k} \times \frac{2\,i\,i\,k}{2\,i\,k}, \qquad (251.)$$

of which the last is $= i\,i$, must lie between any two proposed unequal positive ratios b' and b'', of which the greater b'' does not exceed that last square $i\,i$, and of which the difference $\Theta\, b' + b''$ is not less than $\frac{1}{k}$; and positive whole numbers i and k can always be so chosen as to satisfy these last conditions, however great the proposed ratio b'' may be, and however little may be its excess $\Theta\, b' + b'$ over the other proposed ratio b'.

27. With these preparations it is easy to prove, in a new and formal way, the existence of *one determined positive square root* $\sqrt{b}$ for every proposed positive ratio b, whether that ratio b be or be not the square of any whole or of any fractional number; for we can now prove this *Theorem:*

The square $a\,a$ of the determined positive ratio a, of which ratio the existence was shown in the IIId. Lemma, is equal to the proposed positive ratio b in the same Lemma; that is,

$$\left.\begin{array}{c} \text{if } \; a > \frac{n'}{m'} \; \text{ whenever } \; \frac{n'\,n'}{m'\,m'} < b, \\ \text{and } a < \frac{n''}{m''} \; \text{ whenever } \; \frac{n''\,n''}{m''\,m''} > b, \\ \text{then } \; a\,a = b, \; a = \sqrt{b}, \end{array}\right\} \qquad (252.)$$

$m'\;n'\;m''\;n''$ being any positive whole numbers which satisfy the conditions here mentioned, and b being any determined positive ratio.

For if the square $a\,a$ of the positive ratio a, determined by these conditions, could be greater than the proposed positive ratio b, it would be possible, by the IVth Lemma, to insert between them some positive fraction which would be the square of another positive fraction $\frac{n}{m}$; that is, we could choose m and n so that

$$\frac{nn}{mm} > b, \; \frac{nn}{mm} < aa: \qquad (253.)$$

and then, by the Corollary to the Ist Lemma, and by the conditions (252.), we should be conducted to the two following incompatible relations,

$$\frac{n}{m} < a, \; a < \frac{n}{m}. \qquad (254.)$$

A similar absurdity would result, if we were to suppose $a\,a$ less than b; $a\,a$ must therefore be equal to b, that is, the theorem is true. It has, indeed, been here assumed as evident, that every determined positive ratio a has a determined positive square $a\,a$; which is included in this more general but equally evident principle, that any two determined positive ratios or numbers have a determined positive product.

We find it, therefore, proved, by the most minute and rigorous examination, that if we conceive any positive ratio x or a to increase constantly and continuously from 0, we must conceive its square $x\,x$ or aa to increase constantly and continuously with it, so as to pass successively but only once through every state of positive ratio b: and therefore that every determined positive ratio b has one determined positive square root $\surd b$, which will be commensurable or incommensurable, according as b can or cannot be expressed as the square of a fraction. When b cannot be so expressed, it is still possible to *approximate in fractions* to the incommensurable square root $\surd b$, by choosing successively larger and larger positive denominators, and then seeking for every such denominator m' the corresponding positive numerator n' which satisfies the two conditions (244.); for although every fraction thus found will be less than the sought root $\surd b$, yet the error, or the positive correction which must be added to it in order to produce the accurate root $\surd b$, is less than the reciprocal of the denominator m', and therefore may be made as little different as we please from 0, (though it can never be made exactly $= 0$,) by choosing that denominator large enough. This process of approximation to an incommensurable root $\surd b$ is capable, therefore, of an indefinitely great, though never of a perfect accuracy; and using the notation already given for *limits*, we may write

$$\surd b = \underline{\text{L}}\, \frac{n'}{m'}, \text{ if } \frac{n'\,n'}{m'\,m'} < b, \; \frac{1+n'}{m'} \times \frac{1+n'}{m'} > b, \qquad (255.)$$

and may think of the incommensurable root as the *limit* of the varying fractional number.

The only additional remark which need be made, at present, on the subject of the progression of the square $x\,x$, or $a\,a$, as depending on the progression of the root x, or a, is that since (by the 24th article) the square remains positive and unchanged when the root is changed from positive to contra-positive, in such a manner that

$$\Theta\, a \times \Theta\, a = a \times a, \qquad (256.)$$

the square $a a$ must be conceived as *first* constantly and continuously *decreasing* or *retrograding* towards 0, and *afterwards* constantly and continuously *increasing* or *advancing* from 0, if the root a be conceived as constantly and continuously increasing or advancing, in the general progression of ratio, from states indefinitely far from 0 on the contra-positive side, to other states indefinitely far from 0, but on the positive side in the progression.

On Continued Analogies, or Series of Proportional Steps; and on Powers, and Roots, and Logarithms of Ratios.

28. Four effective steps $\mathrm{a}\ \mathrm{b}\ \mathrm{b}'\ \mathrm{b}''$ may be said to form a continued analogy or continued proportion, a and b'' being the extremes, and b and b' the means, when they are connected by one common ratio in the following manner:

$$\frac{\mathrm{b}''}{\mathrm{b}'} = \frac{\mathrm{b}'}{\mathrm{b}} = \frac{\mathrm{b}}{\mathrm{a}}; \qquad (257.)$$

and if we denote for abridgement this common ratio by a, we may write

$$\mathrm{b} = a \times \mathrm{a},\ \mathrm{b}' = a \times a \times \mathrm{a},\ \mathrm{b}'' = a \times a \times a \times \mathrm{a}. \qquad (258.)$$

Reciprocally, when $\mathrm{b}\ \mathrm{b}'\ \mathrm{b}''$ can be thus expressed, the four steps $\mathrm{a}\ \mathrm{b}\ \mathrm{b}'\ \mathrm{b}''$ compose a continued *analogy;* and it is clear that if the first extreme step a and the common ratio a be given, the other steps can be deduced by the multiplications (258.) It is easy also to perceive, that if the two extremes a and b'' be given, the two means b and b' may be conceived to be determined (as necessarily connected with these) in one and in only one way; and thus that the insertion of *two mean proportionals* between two given effective steps, is never impossible nor ambiguous, like the insertion of a single mean proportional. In fact, it follows from the theorems of multiplication that the product $a \times a \times a$, which may be called the *cube* of the number or ratio a, is not obliged (like the square $a \times a$) to be always a positive ratio, but is positive or contra-positive according as a itself (which may be called the *cube-root* of this product

$a \times a \times a$) is positive or contra-positive; and on examining the law of its progression, (as we lately examined the law of the progression of the square,) we find that the cube $a \times a \times a$ increases constantly and continuously with its cube-root a from states indefinitely far from zero, on the contra-positive side, to states indefinitely far advanced on the positive side of zero, in the general progression of ratio, so as to pass successively but only once through every state of contra-positive or positive ratio, instead of first decreasing or retrograding, and afterwards increasing or advancing, like the square. Thus every ratio has one and only one cube-root, (commensurable or incommensurable,) although a ratio has sometimes two square-roots and sometimes none, according as it is positive or contra-positive; and when the two extreme effective steps a and b'' of the continued analogy (257.) are given, we can always conceive the cube-root a of their ratio $\frac{\mathrm{b}''}{\mathrm{a}}$ determined, and hence the two mean steps or mean proportionals of the analogy, b and b'.

29. In general, as we conceived a continued analogy or *series of equi-distant moments*, generated from a single standard moment A, by the *repetition* of a forward *step* a and of a backward step $\Theta\,\mathrm{a}$; so we may now conceive, as another sort of continued analogy, a *series of proportional steps*, generated from a single standard (effective) step a, by the *repetition* of the *act of multiplication* which corresponds to and is determined by some one multiplier or ratio $a\,(\neq 0)$, and of the inverse or reciprocal act of multiplication determined by the reciprocal multiplier or ratio $\text{ꓤ}\,a$: namely, the following series of proportional steps,

$$\ldots \text{ꓤ}\,a \times \text{ꓤ}\,a \times \text{ꓤ}\,a \times \mathrm{a},\ \text{ꓤ}\,a \times \text{ꓤ}\,a \times \mathrm{a},\ \text{ꓤ}\,a \times \mathrm{a},\ \mathrm{a},\ a \times \mathrm{a},\ a \times a \times \mathrm{a},\ a \times a \times a \times \mathrm{a}, \ldots \tag{259.}$$

which may also be thus denoted,

$$\ldots \text{ꓤ}\,(a\,a\,a) \times \mathrm{a},\ \text{ꓤ}\,(a\,a) \times \mathrm{a},\ \text{ꓤ}\,a \times \mathrm{a},\ 1 \times \mathrm{a},\ a \times \mathrm{a},\ a a \times \mathrm{a},\ a a a \times \mathrm{a}, \ldots \tag{260.}$$

and in which we may consider the system or series of ratios or multipliers,

$$\ldots \text{ꓤ}\,(a\,a\,a),\ \text{ꓤ}\,(a\,a),\ \text{ꓤ}\,a,\ 1,\ a,\ a\,a,\ a\,a\,a, \ldots \tag{261.}$$

to be a *system generated* from the original ratio or multiplier a, by a *system of acts* of generation having all one common character: as we before considered the system of multiple steps (98.),

$$\ldots \Theta\,\mathrm{a} + \Theta\,\mathrm{a} + \Theta\,\mathrm{a},\ \Theta\,\mathrm{a} + \Theta\,\mathrm{a},\ \Theta\,\mathrm{a},\ 0,\ \mathrm{a},\ \mathrm{a} + \mathrm{a},\ \mathrm{a} + \mathrm{a} + \mathrm{a}, \ldots$$

to be a system of steps generated from the original step a by a system of acts of generation to which we gave the common name of acts of multiplying.

In conformity with this conception, we may call the original ratio a the *base* of the system of ratios (261.) and may call those ratios by the common name of *powers* of that common base, and say that they are (or may be) formed by acts of *powering* it. And to distinguish any one such power, or one such act of powering, from all the other powers in the system, or from all the other acts of powering, we may employ the aid of *determining numbers,* ordinal or cardinal, in a manner analogous to that explained in the 13th article for a system of multiple steps. Thus, we may call the ratios a, $a\,a$, $a\,a\,a$, ... by the common name of *positive powers* of the base a, and may distinguish them by the special ordinal names *first, second, third,* &c.; so that the ratio a is, in this view, its own first positive power; the second positive power is the square $a\,a$, and the third positive power is the cube. Again, we may call the ratio 1, which immediately precedes these positive powers in the series, the *zero-power* of the base a, by analogy to the zero-multiple in the series of multiple steps, which immediately preceded in that series the system of positive multiples; and the ratios $\text{Я}\,a$, $\text{Я}\,(a\,a)$, $\text{Я}\,(a\,a\,a)$, ... which precede this zero-power 1 in the series of powers (261.), may be called, by the same analogy, from their order of position, *contra-positive powers* of a, so that the reciprocal $\text{Я}\,a$ of any ratio a is the *first contra-positive power* of that ratio, the reciprocal $\text{Я}\,(a\,a)$ of its square is its second contra-positive power, and so on. We may also distinguish the several corresponding acts of powering by the corresponding cardinal numbers, positive, or contra-positive, or null, and may say (for example) that the third positive power $a\,a\,a$ is formed from the base a by the act of *powering by positive three;* that the second contra-positive power $\text{Я}\,(a\,a)$ is formed from the same base a by *powering by contra-positive two;* and that the zero-power 1 is (or may be) formed from a by powering that base by the null cardinal or number *none.* In written symbols, answering to these thoughts and names, we may denote the *series of powers* (261.), and the *series of proportional steps* (260.), as follows,

$$\ldots\, a^{\ominus 3},\ a^{\ominus 2},\ a^{\ominus 1},\ a^{0},\ a^{1},\ a^{2},\ a^{3},\ \ldots \qquad (262.)$$

and

$$\ldots\, a^{\ominus 3}\times \mathrm{a},\ a^{\ominus 2}\times \mathrm{a},\ a^{\ominus 1}\times \mathrm{a},\ a^{0}\times \mathrm{a},\ a^{1}\times \mathrm{a},\ a^{2}\times \mathrm{a},\ a^{3}\times \mathrm{a},\ \ldots \qquad (263.)$$

in which

$$a^{0}=1, \qquad (264.)$$

and

$$\left.\begin{array}{ll} a^{1}=a, & a^{\ominus 1}=\text{Я}\ a, \\ a^{2}=a\,a, & a^{\ominus 2}=\text{Я}\ (a\,a), \\ a^{3}=a\,a\,a, & a^{\ominus 3}=\text{Я}\ (a\,a\,a), \\ \quad\&\text{c.} & \quad\&\text{c.} \end{array}\right\} \qquad (265.)$$

And we may give the name of *exponents* or *logarithms* to the determining numbers, ordinal or cardinal,

$$\ldots \Theta\, 3,\ \Theta\, 2,\ \Theta\, 1,\ 0,\ 1,\ 2,\ 3, \ldots\ldots \qquad (266.)$$

which answer the question "*which in order* is the Power?" or this other question "*Have any* (effective) acts of multiplication, equivalent or reciprocal to the original act of multiplying by the given ratio a, been combined to produce the act of multiplying by the Power; and if any, then *How many*, and *In which direction*, that is, whether are they similar or opposite in effect, (as enlarging or diminishing the step on which they are performed,) to that original act?" Thus 2 is the logarithm of the square or second power $a\,a$, when compared with the base a; 3 is the logarithm of the cube $a\,a\,a$, 1 is the logarithm of the base a itself, $\Theta\, 1$ is the logarithm of the reciprocal u a, and 0 is the logarithm of the ratio 1 considered as the zero-power of a.

With these conceptions and notations of powers and logarithms, we can easily prove the relation

$$a^{\nu} \times a^{\mu} = a^{\nu + \mu}, \qquad (267.)$$

for any integer logarithms μ and ν, whether positive, or contra-positive, or null; and this other connected relation

$$b^{\nu} = a^{\nu \times \mu} \text{ if } b = a^{\mu}; \qquad (268.)$$

which may be thus expressed in words: "Any two powers of any common base may be multiplied together by adding their logarithms," and "Any proposed power may be powered by any proposed whole number, by multiplying its logarithm by that number," if the sum of the two proposed logarithms in the first case, or the multiple of the proposed logarithm in the second case, be employed as a new logarithm, to form a new power of the original base or ratio; the logarithms here considered being all whole numbers.

30. The act of passing from a base to a power, is connected with an inverse or reciprocal act of returning from the power to the base; and the conceptions of both these acts are included in the more comprehensive conception of the act of passing from any one to any other of the ratios of the series (261.) or (262.). This act of passing from any one power a^{μ} to any other power a^{ν} of a common base a, may be still called in general an act of *powering*; and more particularly, (keeping up the analogy to the language already employed in the theory of multiple steps,) it may be called the act of *powering by the fractional number* $\frac{\nu}{\mu}$. By the same analogy of

definition, this fractional number may be called the *logarithm* of the resulting power, and the power itself may be denoted in written symbols as follows,

$$(a^{\mu})^{\frac{\nu}{\mu}} = a^{\nu}, \qquad (269.)$$

or thus,

$$c = b^{\frac{\nu}{\mu}}, \text{ if } b = a^{\mu}, \ c = a^{\nu}. \qquad (270.)$$

(The analogous formula (121.) ought to have been printed $\mathrm{c} = \frac{\nu}{\mu}\,\mathrm{b}$, and not $\mathrm{c} = \frac{\nu}{\mu}\,\mathrm{a}$, when $\mathrm{b} = \mu \times \mathrm{a}$, $\mathrm{c} = \nu \times \mathrm{a}$.)

In the particular case when the numerator ν is 1, and when, therefore, we have to power by the reciprocal of a whole number, we may call the result $(a^{\mu})^{\frac{1}{\mu}}$, that is a^{1}, $= a$, a *root* or more fully *the μ'th root* of the power or ratio a^{μ}; and we may call the corresponding act of powering, an *extraction of the μ'th root*, or a *rooting by the (whole) number μ*. Thus, to power any proposed ratio b by the reciprocal number $\frac{1}{2}$ or $\frac{1}{3}$, is to extract the second or the third root, that is, (by what has been already shown,) the square-root or the cube-root, of b, or to root the proposed ratio b by the number 2 or 3; and in conformity with this last mode of expression, the following notation may be employed,

$$a = \sqrt[\mu]{b} \text{ when } b = a^{\mu}, \ a = b^{\frac{1}{\mu}}; \qquad (271.)$$

so that a square-root $\sqrt{b}$ may also be denoted by the symbol $\sqrt[2]{b}$, and the cube-root of b may be denoted by $\sqrt[3]{b}$. And whereas we saw, in considering square-roots that a contra-positive ratio $b < 0$ has no square-root, and that a positive ratio $b > 0$ has two square-roots, one positive $= \sqrt{b}$ and the other contra-positive $= \Theta \sqrt{b}$, of which each has its square $= b$; we may consider the new sign $b^{\frac{1}{2}}$ or $\sqrt[2]{b}$ as denoting indifferently either of these two roots, reserving the old sign $\sqrt{b}$ to denote specially that one of them which is positive, and the other old sign $\Theta \sqrt{b}$ to denote specially that one of them which is contra-positive. Thus $\sqrt{b}$ and $\Theta \sqrt{b}$ shall still remain determinate signs, implying each a determinate ratio, (when $b > 0$,) while $\sqrt[2]{b}$ and $b^{\frac{1}{2}}$ shall be used as ambiguous signs, susceptible each of two different meanings. But $\sqrt[3]{b}$ is a determinate sign, because a ratio has only one cube-root. In general, an *even* root, such as the second, fourth, or sixth, of a proposed ratio b, is ambiguous if that ratio be positive, and impossible if b be contra-positive; because an even power, or a power with an even integer for its logarithm, is always a positive ratio, whether the base be positive or contra-positive: but an *odd* root, such as the third or fifth, is always possible and determinate.

31. It may, however, be useful to show more distinctly, by a method analogous to that

of the 26th and 27th articles, that for any proposed positive ratio b whatever, and for any positive whole number m, it is possible to determine, or conceive determined, one positive ratio a, and only one, which shall have its m'th power $= b$; and for this purpose to show that the power a^m increases constantly and continuously from zero with a, so as to pass successively, but only once, through every state of positive ratio b. On examining the proof already given of this property, in the particular case of the power a^2, we see that in order to extend that proof to the more general case of the power a^m, we have only to generalise, as follows, the Ist, IIId, and IVth Lemmas, and the Corollary of the Ist, with the Theorem resulting from all four, retaining the IInd Lemma.

Vth *Lemma*: (generalised from Ist.)

$$\text{If } y \overset{>}{\underset{>}{=}} x, \text{ and } x > 0,\ y > 0, \text{ then } y^m \overset{>}{\underset{<}{=}} x^m. \tag{272.}$$

When $m = 1$, this Lemma is evident, because the first powers y^1 and x^1 coincide with the ratios y and x. When $m > 1$, the Lemma may be easily deduced from the conceptions of ratios, and of powers with positive integer exponents; it may also be proved by observing that the difference $\Theta\, x^m + y^m$, between the powers x^m and y^m, in which the symbol $\Theta\, x^m$ denotes the same thing as if we had written more fully $\Theta\,(x^m)$, and which may be obtained in one way by the subtraction of x^m from y^m, may also be obtained in another way by multiplication from the difference $\Theta\, x + y$ as follows:

$$\Theta\, x^m + y^m = (\Theta\, x + y) \times (x^{\Theta 1 + m} y^{0} + x^{\Theta 2 + m} y^{1} + \ldots + x^{1} y^{\Theta 2 + m} + x^{0} y^{\Theta 1 + m}), \tag{273.}$$

and is, therefore, positive, or contra-positive, or null, according as the difference $\Theta\, x + y$ of the positive ratios x and y themselves is positive, or contra-positive, or null, because the other factor of the product (273.) is positive. For example,

$$\Theta\, x^3 + y^3 = (\Theta\, x + y) \times (x^2 + x\,y + y^2); \tag{274.}$$

and, therefore, when x and y and consequently $x^2 + x\,y + y^2$ are positive, the difference $\Theta\, x^3 + y^3$ and the difference $\Theta\, x + y$ are positive, or contra-positive, or null together.

As a *Corollary* of this Lemma, we see that, conversely,

$$\text{if } y^m \overset{>}{\underset{<}{=}} x^m, \text{ and } x > 0,\ y > 0, \text{ then } y \overset{>}{\underset{<}{=}} x. \tag{275.}$$

Thus, the power x^m and the root x increase *constantly* together, when both are positive ratios.

The *logic* of this last deduction, of the Corollary (275.) from the Lemma (272.), must not be confounded with that erroneous form of argument which infers the truth of the antecedent of a true hypothetical proposition from the truth of the consequent; that is, with the too common *sophism:* If A be true then B is true; but B is true, therefore A is true. The Lemma (272.) asserts three hypothetical propositions, which are tacitly supposed to be each transformed, or logically converted, according to this *valid* principle, that the falsehood of the consequent of a true hypothetical proposition infers the falsehood of the antecedent; or according to this just formula: If A were true then B would be true; but B is false, therefore A is not true. Applying this just principle to each of the three hypothetical propositions of the Lemma, we are entitled to infer, by the general principles of Logic, these three converse hypothetical propositions:

$$\left.\begin{array}{l} \text{if } y^m \ngtr x^m, \text{ then } y \ngtr x; \\ \text{if } y^m \neq x^m, \text{ then } y \neq x; \\ \text{if } y^m \nless x^m, \text{ then } y \nless x; \end{array}\right\} \qquad (276.)$$

x and y being here any positive ratios, and m any positive whole number, and the signs $\ngtr$ $\nless$ denoting respectively "not $>$" and "not $<$" as the sign $\neq$ denotes "not $=$". And if, to the propositions (276.), we join this principle of intuition in Algebra, as the Science of Pure Time, that a variable moment B must either follow, or coincide with, or precede a given or variable moment A, but cannot do two of these three things at once, and therefore (by the 21st article) that a variable ratio y must also bear one but only one of these three ordinal relations to a given or variable ratio x, which shows that

$$\left.\begin{array}{l} \text{when } y^m > x^m, \text{ then } y^m \neq x^m \text{ and } y^m \nless x^m, \\ \text{when } y^m = x^m, \text{ then } y^m \nless x^m \text{ and } y^m \ngtr x^m, \\ \text{when } y^m < x^m, \text{ then } y^m \ngtr x^m \text{ and } y^m \neq x^m, \end{array}\right\} \qquad (277.)$$

and that

$$\left.\begin{array}{l} \text{when } y \neq x \text{ and } y \nless x, \text{ then } y > x, \\ \text{when } y \nless x \text{ and } y \ngtr x, \text{ then } y = x, \\ \text{when } y \ngtr x \text{ and } y \neq x, \text{ then } y < x, \end{array}\right\} \qquad (278.)$$

we find finally that the Corollary (275.) is true. The same logic was tacitly employed in deducing the Corollary of the Ist Lemma, in the hope that it would be mentally supplied by the attentive reader. It has now been stated expressly, lest any should confound it with that dangerous and common fallacy, of inferring, in Pure Science, the necessary truth of a premiss in an argument, from the known truth of the conclusion.

Resuming the more mathematical part of the research, we may next establish this

VIth *Lemma* (generalised from IIId): There exists one positive ratio a, and only one, which satisfies all the following conditions,

$$\left.\begin{array}{l} a > \frac{n'}{m'}, \text{ whenever } \left(\frac{n'}{m'}\right)^m < b, \\ a < \frac{n''}{m''} \text{ whenever } \left(\frac{n''}{m''}\right)^m > b\,; \end{array}\right\} \qquad (279.)$$

b being any given positive ratio, and m any given positive whole number, while m' n' m'' n' are also positive but variable whole numbers. The proof of this Lemma is so like that of the IIId, that it need not be written here; and it shows that in the particular case when the given ratio b is the m^{th} power of a positive fraction $\frac{n_{,}}{m_{,}}$, then a is that fraction itself. In general, it will soon be shown that under the conditions of this Lemma the m^{th} power of a is b.

VIIth *Lemma* (generalised from IVth). It is always possible to find, or to conceive as found, two positive whole numbers $m_{,}$ and $n_{,}$, which shall satisfy the two conditions

$$\left(\frac{n_{,}}{m_{,}}\right)^m > b', \ \left(\frac{n_{,}}{m_{,}}\right)^m < b'', \text{ if } b'' > b', \ b' > 0, \qquad (280.)$$

m being any given positive whole number; that is, we can insert between any two unequal positive ratios b' and b'' an intermediate fractional ratio which is itself the m^{th} power of a fraction.

For, when $m = 1$, this Lemma reduces itself to the IInd; and when $m > 1$, the theorem (273.) shows that the excess of $\left(\frac{1+n}{m_{,}}\right)^m$ over $\left(\frac{n}{m_{,}}\right)^m$ may be expressed as follows:

$$\Theta \left(\frac{n}{m_{,}}\right)^m + \left(\frac{1+n}{m_{,}}\right)^m = \frac{1}{m_{,}} \times p, \qquad (281.)$$

in which

$$p = \left(\frac{n}{m_{,}}\right)^{\Theta 1 + m} + \left(\frac{n}{m_{,}}\right)^{\Theta 2 + m} \left(\frac{1+n}{m_{,}}\right) + \left(\frac{n}{m_{,}}\right)^{\Theta 3 + m} \left(\frac{1+n}{m_{,}}\right)^2$$

$$+ \ldots + \left(\frac{n}{m_{,}}\right)^2 \left(\frac{1+n}{m_{,}}\right)^{\Theta 3 + m} + \frac{n}{m_{,}} \left(\frac{1+n}{m_{,}}\right)^{\Theta 2 + m} + \left(\frac{1+n}{m_{,}}\right)^{\Theta 1 + m}; \qquad (282.)$$

for example, when $m=3$, the excess of the cube $\left(\frac{1+n}{m_{,}}\right)^3$ over the cube $\left(\frac{n}{m_{,}}\right)^3$, is

$$\Theta\left(\frac{n}{m_{,}}\right)^3+\left(\frac{1+n}{m_{,}}\right)^3=\frac{1}{m_{,}}\times\left\{\left(\frac{n}{m_{,}}\right)^2+\frac{n}{m_{,}}\,\frac{1+n}{m_{,}}+\left(\frac{1+n}{m_{,}}\right)^2\right\}. \qquad (283.)$$

In general, the number of the *terms* (or added parts) in the expression (282.), is m, and they are all unequal, the least being $\left(\frac{n}{m_{,}}\right)^{\Theta 1+m}$, and the greatest being $\left(\frac{1+n}{m_{,}}\right)^{\Theta 1+m}$; their sum, therefore, is less than the m^{th} multiple of this greatest term, that is,

$$p<m\times\left(\frac{1+n}{m_{,}}\right)^{\Theta 1+m}, \qquad (284.)$$

and therefore the excess (281.) is subject to the corresponding condition

$$\Theta\left(\frac{n}{m_{,}}\right)^m+\left(\frac{1+n}{m_{,}}\right)^m<\frac{m}{m_{,}}\left(\frac{1+n}{m_{,}}\right)^{\Theta 1+m}; \qquad (285.)$$

for example,

$$\Theta\left(\frac{n}{m_{,}}\right)^3+\left(\frac{1+n}{m_{,}}\right)^3<\frac{3}{m_{,}}\left(\frac{1+n}{m_{,}}\right)^2. \qquad (286.)$$

However this excess (281.) increases constantly with n, when $m_{,}$ remains unaltered, because p so increases; so that the $1+n$ fractions of the series

$$\left(\frac{1}{m_{,}}\right)^m,\ \left(\frac{2}{m_{,}}\right)^m,\ \left(\frac{3}{m_{,}}\right)^m,\ \ldots\left(\frac{1+n}{m_{,}}\right)^m, \qquad (287.)$$

increase by increasing differences, (or advance by increasing intervals,) which are each less than $\frac{m}{m_{,}}\left(\frac{1+n}{m_{,}}\right)^{\Theta 1+m}$, and therefore than $\frac{1}{k}$, if we choose $m_{,}$ and n so as to satisfy the conditions

$$1+n=i\,m_{,},\ \ m_{,}=k\,m\times i^{\Theta 1+m}=\frac{k\,m\,i^m}{i}, \qquad (288.)$$

i and k being any two positive whole numbers assumed at pleasure; with this choice, therefore, of the numbers $m_{,}$ and n, some one (at least), such as $\left(\frac{n_{,}}{m_{,}}\right)^m$, of the series of powers of fractions (287.), of which the last is $=i^m$, will fall between any two proposed unequal positive ratios b' and b'', if the greater b'' does not exceed that last power i^m, and if the difference $\Theta\, b' + b''$ is not less than $\frac{1}{k}$; and these conditions can be always satisfied by a suitable choice of the whole numbers i and k, however large may be the given greater positive ratio b'', and however little may be its given excess over the lesser positive ratio b'.

Hence, finally, this Theorem :

$$\left.\begin{aligned} &\text{If } a > \frac{n'}{m'},\ \text{and } a < \frac{n''}{m''}, \\ &\text{whenever } \left(\frac{n'}{m'}\right)^m < b,\ \left(\frac{n''}{m''}\right)^m > b, \\ &\text{then } a^m = b,\ a = \sqrt[m]{b} = b^{\frac{1}{m}}; \end{aligned}\right\} \qquad (289.)$$

b denoting any given positive ratio, and m any given positive whole number, while m' n' m'' n'' are any arbitrary positive whole numbers which satisfy these conditions, and a is another positive ratio which the VIth Lemma shows to be determined.

For if a^m could be $> b$, we could, by the VIIth Lemma, insert between them a positive fraction of the form $\left(\frac{n_{,}}{m_{,}}\right)^m$, such that

$$\left(\frac{n_{,}}{m_{,}}\right)^m > b,\ \left(\frac{n_{,}}{m_{,}}\right)^m < a^m; \qquad (290.)$$

and then by the Corollary of the Vth Lemma, and by the conditions (289.), we should deduce the two incompatible relations

$$\frac{n_{,}}{m_{,}} < a,\ \ a < \frac{n_{,}}{m_{,}}, \qquad (291.)$$

which would be absurd. A similar absurdity would follow from supposing that a^m could be less than b; a^m must therefore be $= b$, that is, the Theorem is true. It has, indeed, been all along assumed as evident that every determined positive ratio a has a determined positive m^{th} power a^m, when m is a positive whole number; which is included in this more general but also evident principle, that any m determined positive ratios or numbers have a determined positive product.

Every positive ratio b has therefore one, and only one, positive ratio a for its m^{th} root, which is commensurable or incommensurable, according as b can or cannot be put under the form $\left(\frac{n_{,}}{m_{,}}\right)^m$; but which, when incommensurable, may be theoretically conceived as the accurate limit of a variable fraction,

$$a = \sqrt[m]{b} = \underline{\text{L}}\,\frac{n'}{m'},\ \text{if } \left(\frac{n'}{m'}\right)^m < b,\ \left(\frac{1+n'}{m'}\right)^m > b, \qquad (292.)$$

and may be practically approached to, by determining such fractions $\frac{n'}{m'}$, with larger and larger whole numbers m' and n' for their denominators and numerators. And whether m be odd or even, we see that the power a^m increases *continuously* (as well as constantly) with its positive root or base a, from zero up to states indefinitely greater and greater. But if this root, or base, or ratio a be conceived to advance constantly and continuously from states indefinitely far from zero on the contra-positive side to states indefinitely far upon the positive side, then the power a^m will either advance constantly and continuously likewise, though not with the same quickness, from contra-positive to positive states, or else will first constantly and continuously retrograde to zero, and afterwards advance from zero, remaining always positive, according as the positive exponent or logarithm m is an odd or an even integer. It is understood that for any such positive exponent m,

$$0^m = 0, \qquad (293.)$$

the powers of 0 with positive integer exponents being considered as all themselves equal to 0, because the repeated multiplication by this null ratio generates from any one effective step a the series of proportional steps,

$$\mathrm{a},\ 0 \times \mathrm{a} = 0,\ 0 \times 0 \times \mathrm{a} = 0,\ \ldots, \qquad (294.)$$

which may be continued indefinitely *in one direction*, and in which all steps after the first are null; although we were obliged to exclude the consideration of such null ratios in forming the series (259.) because we wished to continue that series of steps indefinitely in two opposite directions.

32. We are now prepared to discuss completely the meaning, or meanings, if any, which ought to be assigned to any proposed symbol of the class $b^{\frac{\nu}{\mu}}$, b denoting any proposed ratio, and μ and ν any proposed whole numbers. By the 30th article, the symbol $b^{\frac{\nu}{\mu}}$ denotes generally the ν'th power of a ratio a of which b is the μ'th power; or, in other words, the νth power of a μth root of b; so that the mental operation of passing from the ratio b to the ratio $b^{\frac{\nu}{\mu}}$, is compounded, (when it can be performed at all,) of the two operations of first rooting by the one whole number μ, and then powering by the other whole number ν: and we may write,

$$b^{\frac{\nu}{\mu}} = (\sqrt[\mu]{b})^{\nu} = (b^{\frac{1}{\mu}})^{\nu}. \qquad (295.)$$

The ratio b, and the whole numbers μ and ν, may each be either positive, or contra-positive, or null; and thus there arise many cases, which may be still farther sub-divided, by distinguishing between odd and even values of the positive or contra-positive whole numbers. For, if we suppose that B denotes a positive ratio, and that m and n denote positive whole numbers, we may then suppose

$$\left.\begin{array}{l} b=\mathrm{B}, \text{ or } b=0, \text{ or } b=\Theta\ \mathrm{B}, \\ \mu=m, \text{ or } \mu=0, \text{ or } \mu=\Theta\ m, \\ \nu=n, \text{ or } \nu=0, \text{ or } \nu=\Theta\ n, \end{array}\right\} \quad (296.)$$

and thus shall obtain the twenty-seven cases following,

$$\left.\begin{array}{lll} \mathrm{B}^{\frac{n}{m}}, & \mathrm{B}^{\frac{0}{m}}, & \mathrm{B}^{\frac{\Theta\ n}{m}} \\ \mathrm{B}^{\frac{n}{0}}, & \mathrm{B}^{\frac{0}{0}}, & \mathrm{B}^{\frac{\Theta\ n}{0}} \\ \mathrm{B}^{\frac{n}{\Theta\ m}}, & \mathrm{B}^{\frac{0}{\Theta\ m}}, & \mathrm{B}^{\frac{\Theta\ n}{\Theta\ m}} \end{array}\right\} \quad (297.)$$

$$\left.\begin{array}{lll} 0^{\frac{n}{m}}, & 0^{\frac{0}{m}}, & 0^{\frac{\Theta\ n}{m}}, \\ 0^{\frac{n}{0}}, & 0^{\frac{0}{0}}, & 0^{\frac{\Theta\ n}{0}}, \\ 0^{\frac{n}{\Theta\ m}}, & 0^{\frac{0}{\Theta\ m}}, & 0^{\frac{\Theta\ n}{\Theta\ m}}, \end{array}\right\} \quad (298.)$$

$$\left.\begin{array}{lll} (\Theta\ \mathrm{B})^{\frac{n}{m}}, & (\Theta\ \mathrm{B})^{\frac{0}{m}}, & (\Theta\ \mathrm{B})^{\frac{\Theta\ n}{m}}, \\ (\Theta\ \mathrm{B})^{\frac{n}{0}}, & (\Theta\ \mathrm{B})^{\frac{0}{0}}, & (\Theta\ \mathrm{B})^{\frac{\Theta\ n}{0}}, \\ (\Theta\ \mathrm{B})^{\frac{n}{\Theta\ m}}, & (\Theta\ \mathrm{B})^{\frac{0}{\Theta\ m}}, & (\Theta\ \mathrm{B})^{\frac{\Theta\ n}{\Theta\ m}}, \end{array}\right\} \quad (299.)$$

which we may still farther sub-divide by putting m and n under the forms

$$\left.\begin{array}{l} m=2\ i, \text{ or } m=\Theta\ 1+2\ i, \\ n=2\ k, \text{ or } n=\Theta\ 1+2\ k, \end{array}\right\} \quad (300.)$$

in which i and k themselves denote positive whole numbers. But, various as these cases are, the only difficulty in discussing them arises from the occurrence, in some, of the ratio or number 0; and to remove this difficulty, we may lay down the following rules, deduced from the foregoing principles.

To power the ratio 0 by any positive whole number m, gives, by (293.), the ratio 0 as the result. This ratio 0 is, therefore, at least *one* m'th root of 0; and since no positive or contra-positive ratio can thus give 0 when powered by any positive whole number, we see that the *only* m'th root of 0 is 0 itself. Thus,

$$0^{\frac{1}{m}}=0, \quad (301.)$$

and generally,

$$0^{\frac{n}{m}} = 0. \qquad (302.)$$

To power any positive ratio a, whether positive, or contra-positive, or null, by the number or logarithm 0, may be considered to give 1 as the result; because we can always construct at least this series of proportional steps, beginning with any one effective step a, and proceeding indefinitely in one direction:

$$1 \times \mathrm{a},\ a \times \mathrm{a},\ a \times a \times \mathrm{a},\ \ldots; \qquad (303.)$$

and we may still call the ratio 1 the *zero-power*, and the ratios a, $a \times a$, ... the *positive powers* of the ratio a, even when we cannot continue this series of proportional steps (303.) backward, like the series (259.), so as to determine any contra-positive powers of a; namely, in that particular case when $a = 0$. We may, therefore, consider the equation (264.), $a^{0} = 1$, as including even this particular case $a = 0$; and may write

$$0^{0} = 1, \qquad (304.)$$

and, therefore, by (301.) and (295.)

$$0^{\frac{0}{m}} = 1: \qquad (305.)$$

we are also conducted to consider the symbols

$$0^{\ominus m}, \quad 0^{\frac{\ominus n}{m}}, \qquad (306.)$$

as absurd, the ratio 0 having no contra-positive powers.

From the generality which we have been led to attribute to the equation $a^{0} = 1$, it follows that the symbol

$$1^{\frac{1}{0}}, \text{ and more generally } 1^{\frac{\nu}{0}}, \qquad (307.)$$

is indeterminate, or that it is equally fit to denote all ratios whatever; but that the symbol

$$b^{\frac{1}{0}}, \text{ or } b^{\frac{\nu}{0}}, \text{ if } b \neq 1, \qquad (308.)$$

is absurd, or that it cannot properly denote any ratio. In particular, the symbols

$$0^{\frac{1}{0}}, \quad 0^{\frac{\nu}{a}}, \qquad (309.)$$

are absurd, or denote no ratios whatever. In like manner the symbol

$$0^{\frac{1}{\Theta\, m}}, \text{ and more generally } 0^{\frac{\nu}{\Theta\, m}}, \qquad (310.)$$

is absurd, or denotes no ratio, because no ratio a can satisfy the equation

$$a^{\Theta\, m}=0. \qquad (311.)$$

We have thus discussed all the nine cases (298.), of powers in which the base is 0, and have found them all to be impossible, except the two first, in which the exponents are $\frac{n}{m}$, and $\frac{0}{m}$, and in which the resulting powers are respectively 0 and 1. We have also obtained sufficient elements for discussing all the other cases (297.) and (299.), with their sub-divisions (300.), as follows.

1st. $\mathrm{B}^{\frac{n}{m}}$ is determined and positive, unless m is even, and n odd; in which case it becomes of the form $\mathrm{B}^{\frac{\Theta\, 1+2\, k}{2\, i}}$, and is ambiguous, being capable of denoting either of two opposite ratios, a positive or a contra-positive. To distinguish these among themselves, we may denote the positive one by the symbol

$$\mathrm{B}_{\circ}^{\frac{\Theta\, 1+2\, k}{2\, i}}, \qquad (312.)$$

and the contra-positive one by the symbol

$$\Theta\, \mathrm{B}_{\circ}^{\frac{\Theta\, 1+2\, k}{2\, i}}; \qquad (313.)$$

for example, the two values of the square-root $\sqrt[2]{\mathrm{B}}$ or $\mathrm{B}^{\frac{1}{2}}$, may be denoted for distinction by the two separate symbols

$$\mathrm{B}_{\circ}^{\frac{1}{2}}=\sqrt{\mathrm{B}}, \quad \Theta\, \mathrm{B}_{\circ}^{\frac{1}{2}}=\Theta \sqrt{\mathrm{B}}. \qquad (314.)$$

The other three cases of the notation $\mathrm{B}^{\frac{n}{m}}$, namely, the symbols

$$\mathrm{B}^{\frac{\Theta\, 1+2\, k}{\Theta\, 1+2\, i}}, \quad \mathrm{B}^{\frac{2\, k}{\Theta\, 1+2\, i}}, \quad \mathrm{B}^{\frac{2\, k}{2\, i}}, \qquad (315.)$$

denote determined positive ratios.

2d. The three cases

$$\mathrm{B}^{\frac{1+\Theta\, (2\, k)}{\Theta\, 1+2\, i}}, \quad \mathrm{B}^{\frac{\Theta\, (2\, k)}{\Theta\, 1+2\, i}}, \quad \mathrm{B}^{\frac{\Theta\, (2\, k)}{2\, i}}, \qquad (316.)$$

of the notation $\mathrm{B}^{\frac{\Theta\, n}{m}}$, are symbols of determined positive ratios; but the case

$B^{\frac{1+\Theta(2k)}{2i}}$ is ambiguous, this symbol denoting either a determined positive ratio or a determined contra-positive ratio, which may be thus respectively marked, when we wish to distinguish them from each other,

$$\underset{\circ}{B}^{\frac{1+\Theta(2k)}{2i}}, \quad \Theta\,\underset{\circ}{B}^{\frac{1+\Theta(2k)}{2i}}. \qquad (317.)$$

In general, we may write,

$$B^{\frac{\Theta\,n}{m}} = \mathfrak{A}\,(B^{\frac{n}{m}}), \qquad (318.)$$

the latter of these two symbols having the same meaning or meanings as the former.

3d. The symbols

$$B^{\frac{\Theta\,1+2k}{1+\Theta(2i)}}, \quad B^{\frac{2k}{1+\Theta(2i)}}, \quad B^{\frac{2k}{\Theta(2i)}}, \qquad (319.)$$

included in the form $B^{\frac{n}{\Theta\,m}}$, denote determined positive ratios; but the other symbol $B^{\frac{\Theta\,1+2k}{\Theta(2i)}}$, included in the same form $B^{\frac{n'}{\Theta\,m}}$, is ambiguous, denoting either a determined positive or a determined contra-positive ratio,

$$\underset{\circ}{B}^{\frac{\Theta\,1+2k}{\Theta(2i)}}, \text{ or } \Theta\,\underset{\circ}{B}^{\frac{\Theta\,1+2k}{\Theta(2i)}}. \qquad (320.)$$

In general, we may write

$$B^{\frac{n}{\Theta\,m}} = \mathfrak{A}\,(B^{\frac{n}{m}}). \qquad (321.)$$

4th. In like manner, we may write,

$$B^{\frac{\Theta\,n}{\Theta\,m}} = B^{\frac{n}{m}}, \qquad (322.)$$

the former symbol having always the same meaning or meanings as the latter. The cases

$$B^{\frac{1+\Theta(2k)}{1+\Theta(2i)}}, \quad B^{\frac{\Theta(2k)}{1+\Theta(2i)}}, \quad B^{\frac{\Theta(2k)}{\Theta(2i)}}, \qquad (323.)$$

are symbols of determined positive ratios; but the case $B^{\frac{1+\Theta 2k)}{\Theta(2i)}}$ is ambiguous, and includes two opposite ratios, which may be thus respectively denoted,

$$B^{\frac{1+\Theta(2k)}{\Theta(2i)}}, \quad \Theta\,\underset{\circ}{B}^{\frac{1+\Theta(2k)}{\Theta(2i)}}. \qquad (324.)$$

In general, we shall denote by the symbol

$$\underset{\circ}{B}^{\frac{\nu}{\mu}}, \text{ or } \underset{\circ}{b}^{\frac{\nu}{\mu}}, \text{ if } b > 0,\ \mu \neq 0,\ \nu \neq 0, \tag{325.}$$

that positive ratio which is either the only value, or at least one of the values of the symbol $B^{\frac{\nu}{\mu}}$ or $b^{\frac{\nu}{\mu}}$; and it is important to observe that this positive ratio is not changed, when the form of the fractional logarithm $\frac{\nu}{\mu}$ is changed, as if it were a fractional multiplier, by the rule (135.), to the form $\frac{\omega \times \nu}{\omega \times \mu}$, or (as it may be more concisely written) $\frac{\omega\,\nu}{\omega\,\mu}$; that is,

$$\underset{\circ}{B}^{\frac{\omega\,\nu}{\omega\,\mu}} = \underset{\circ}{B}^{\frac{\nu}{\mu}} : \tag{326.}$$

a theorem which is easily proved by means of the relation (268.), combined with the determinateness (already proved) of that positive ratio which results from powering or rooting any proposed positive ratio by any positive or contra-positive whole number.

5th. With respect to the five remaining notations of the group (297.), namely, those in which 0 occurs, we have

$$B^{\frac{0}{m}} = 1\,; \quad B^{\frac{0}{\Theta\,m}} = 1\,; \tag{327.}$$

also the symbols

$$B^{\frac{n}{0}}, \quad B^{\frac{\Theta\,n}{0}}, \tag{328.}$$

are each indeterminate when $B = 1$, and absurd in the contrary case; and, finally, the symbol

$$B^{\frac{0}{0}} \tag{329.}$$

is absurd when $B \neq 1$, but determined and $= 1$, when $B = 1$.

6th. Proceeding to the group (299.), the symbols

$$(\Theta\,B)^{\frac{n}{0}}, \quad (\Theta\,B)^{\frac{0}{0}}, \quad (\Theta\,B)^{\frac{\Theta\,n}{0}}, \tag{330.}$$

are absurd; the symbols

$$(\Theta\,B)^{\frac{0}{m}}, \quad (\Theta\,B)^{\frac{0}{\Theta\,m}}, \tag{331.}$$

are determined and each $= 1$, if m be odd, but otherwise they are absurd; and the four remaining symbols

$$(\Theta\,B)^{\frac{n}{m}}, \quad (\Theta\,B)^{\frac{\Theta\,n}{m}}, \quad (\Theta\,B)^{\frac{n}{\Theta\,m}}, \quad (\Theta\,B)^{\frac{\Theta\,n}{\Theta\,m}}, \tag{332.}$$

are absurd if m be even, but denote determined ratios when m is odd, which ratios are positive if n be even, but contra-positive if n be odd.

It must be remembered that all the foregoing discussion of the cases of the general notation $b^{\frac{\nu}{\mu}}$, for powers with *fractional* logarithms, is founded on the definition laid down in the 30th article, that $b^{\frac{\nu}{\mu}}$ denotes the ν'th power of a μ'th root of b, or in other words, the ν'th power of a ratio a of which b is the μ'th power. When no such ratio a can be found, consistently with the previous conception of powers with *integer* logarithms, the symbol $b^{\frac{\nu}{\mu}}$ is pronounced to be *absurd*, or to be incapable of denoting any ratio consistently with its general definition; and when two or more such ratios a can be found, each having its μ'th power $= b$, we have pronounced that the fractional power $b^{\frac{\nu}{\mu}}$ is *ambiguous* or *indeterminate*, except in those cases in which the second component act of powering by the numerator ν has happened to destroy the indeterminateness. And with respect to powers with *integer* exponents, it is to be remembered that we always define them by a reference to a series of proportional steps, of which at least the original step (corresponding to the zero-power) is supposed to be an *effective* step, and which can always be continued indefinitely, at least in the positive direction, that is, in the way of *repeated multiplication* by the ratio proposed as the base, although in the particular case of a null ratio, we cannot continue the series backward by *division*, so as to find any contra-positive powers. These definitions appear to be the most natural; but others might have been assumed, and then other results would have followed. In general, the definitions of mathematical science are not altogether arbitrary, but a certain discretion is allowed in the selection of them, although when once selected, they must then be consistently reasoned from.

33. The foregoing article enables us to assign one determined positive ratio, and only one, as denoted by the symbol $\underset{\circ}{b}^{\alpha}$, when b is any determined positive ratio, and α any fractional number with a numerator and a denominator each different from 0: it shows also that this ratio $\underset{\circ}{b}^{\alpha}$ does not change when we transform the expression of the fractional logarithm α by introducing or suppressing any whole number ω as a factor common to both numerator and denominator; and permits us to write

$$\underset{\circ}{b}^{\Theta\alpha} \dot{=} \mathrm{u}\,(\underset{\circ}{b}^{\alpha}), \qquad (333.)$$

$\Theta\,\alpha$ being the opposite of the fraction α in the sense of the 17th article. More generally, by the meaning of the notation $\underset{\circ}{b}^{\alpha}$, and by the determinateness of those positive ratios which result from the powering or rooting of determined positive ratios by determined integer numbers, (setting aside the impossible or indeterminate case of rooting by the number 0,) we have the relation

$$\underset{\circ}{b}^{\beta} \times \underset{\circ}{b}^{\alpha} = \underset{\circ}{b}^{\beta+\alpha}, \qquad (334.)$$

which is analogous to (267.); and the relation

$$\underset{\circ}{c}^{\beta} = \underset{\circ}{b}^{\beta\times\alpha} \text{ if } c = \underset{\circ}{b}^{\alpha}, \qquad (335.)$$

analogous to (268.): α and β denoting here any two commensurable numbers. And it is easy to see that while the fractional exponent or logarithm α increases, advancing successively through all fractional states in the progression from contra-positive to positive, the positive ratio $\underset{\circ}{b}^{\alpha}$ increases constantly if $b > 1$, or else decreases constantly if $b < 1$, $(b > 0,)$ or remains constantly $= 1$ if $b = 1$. But to show that this increase or decrease of the power with the exponent is *continuous* as well as constant, we must establish principles for the interpretation of the symbol $\underset{\circ}{b}^{\alpha}$ when α is not a fraction.

When α is incommensurable, but b still positive, it may be proved that we shall still have these last relations (334.) and (335.), if we interpret the symbol $\underset{\circ}{b}^{\alpha}$ to denote that determined positive ratio c which satisfies the following conditions:

$$\left.\begin{array}{l} c = \underset{\circ}{b}^{\alpha} > \underset{\circ}{b}^{\frac{n'}{m'}} \text{ whenever } \alpha > \frac{n'}{m'}, \\ c = \underset{\circ}{b}^{\alpha} < \underset{\circ}{b}^{\frac{n''}{m''}} \text{ whenever } \alpha < \frac{n''}{m''}, \\ \qquad\qquad \text{if } b > 1; \end{array}\right\} \qquad (336.)$$

or else these other conditions,

$$\left.\begin{array}{l} c = \underset{\circ}{b}^{\alpha} < \underset{\circ}{b}^{\frac{n'}{m'}} \text{ whenever } \alpha > \frac{n'}{m'}, \\ c = \underset{\circ}{b}^{\alpha} > \underset{\circ}{b}^{\frac{n''}{m''}} \text{ whenever } \alpha < \frac{n''}{m''}, \\ \qquad\qquad \text{if } b < 1,\ b > 0; \end{array}\right\} \qquad (337.)$$

or finally this equation,

$$c = \underset{\circ}{b}^{\alpha} = 1, \text{ if } b = 1. \qquad (338.)$$

The reader will soon perceive the reasonableness of these interpretations; but he may desire to see it proved that the conditions (336.) or (337.) can always be satisfied by one positive ratio c, and only one, whatever determined ratio may be denoted by α, and whatever positive ratio (different from 1) by b. That *at least one* such positive ratio $c = \underset{\circ}{b}^{a}$ can be found, whatever incommensurable number the exponent a may be, is easily proved from the circumstance that none of the conditions (336.) are incompatible with one another if $b > 1$, and that none of the conditions (337.) are incompatible with each other in the contrary case, by reason of the constant increase or constant decrease of the fractional power $\underset{\circ}{b}^{\frac{n}{m}}$ for constantly increasing values of the fractional exponent $\frac{n}{m}$. And that *only one* such positive ratio $c = \underset{\circ}{b}^{a}$ can be found, or that no two different positive ratios c, c', can *both* satisfy *all* these conditions may be proved for the case $b > 1$ by the following process, which can without difficulty be adapted to the other case.

The fractional powers of b comprised in the series

$$\underset{\circ}{b}^{\frac{1}{m}}, \underset{\circ}{b}^{\frac{2}{m}}, \underset{\circ}{b}^{\frac{3}{m}}, \ldots \underset{\circ}{b}^{\frac{im}{m}}, \underset{\circ}{b}^{\frac{1+im}{m}}, \qquad (339.)$$

increase (when $b > 1$) by increasing differences, of which the last is

$$\Theta \, \underset{\circ}{b}^{\frac{im}{m}} + \underset{\circ}{b}^{\frac{1+m}{m}} = b^{i} (\Theta \, 1 + \underset{\circ}{b}^{\frac{1}{m}}); \qquad (340.)$$

this last difference, therefore, and by still stronger reason each of the others which precede it, will be less than $\frac{1}{k}$, if

$$l > k \, b^{i} \qquad (341.)$$

and

$$\Theta \, 1 + \underset{\circ}{b}^{\frac{1}{m}} < \frac{1}{l}: \qquad (342.)$$

and this last condition will be satisfied, if

$$m > l \, (\Theta \, 1 + b), \qquad (343.)$$

l and m (like i and k) denoting any positive whole numbers; for then we shall have

$$1 + \frac{m}{l} > b, \qquad (344.)$$

and by still stronger reason

$$\left(1 \times \frac{1}{l}\right)^{m} > b, \; 1 + \frac{1}{l} > \underset{\circ}{b}^{\frac{1}{m}}, \qquad (345.)$$

observing that

$$\left(1 + \frac{1}{l}\right)^{m} > 1 + \frac{m}{l}, \text{ if } m > 1, \qquad (346.)$$

because, by the theorem of multiplication (273.), or (281.),

$$\Theta 1 + (1 + \tfrac{1}{l})^{m} = \tfrac{1}{l} \left\{ 1 + (1 + \tfrac{1}{l}) + (1 + \tfrac{1}{l})^{2} + \ldots + (1 + \tfrac{1}{l})^{\Theta 1 + m} \right\}. \quad (347.)$$

If then c c' be any two proposed unequal positive ratios, of which we may suppose that c' is the greater,

$$c' > c, \; c > 0, \qquad (348.)$$

we may choose two positive whole numbers i, k, so large that

$$b^{i} > c', \; \frac{1}{k} < \Theta\, c + c', \qquad (34\,9.)$$

and two other positive whole numbers l, m, large enough to satisfy the conditions (341.) (343.) ; and then we shall be sure that some one at least, such as $\underset{\circ}{b}^{\frac{n}{m}}$, of the fractional powers of b comprised in the series (339.) will fall between the two proposed unequal ratios c c', so that

$$c < \underset{\circ}{b}^{\frac{n}{m}}, \; c' > \underset{\circ}{b}^{\frac{n}{m}}. \qquad (350.)$$

If then the one ratio c satisfy all the conditions (336.), the incommensurable number a must be $< \frac{n}{m}$, and therefore, by the 2nd relation (350.), the other ratio c' cannot also satisfy all the conditions of the same form, since it is $> \underset{\circ}{b}^{\frac{n}{m}}$, although $a < \frac{n}{m}$. In like manner, if the greater ratio c' satisfy all the conditions of the form (336.) the lesser ratio c cannot also satisfy them all, because in this case the incommensurable number a will be $> \frac{n}{m}$. No two unequal positive ratios, therefore, can satisfy all those conditions : they are therefore satisfied by one positive ratio and only one, and the symbol $\underset{\circ}{b}^{a}$ (interpreted by them) denotes a determined positive ratio when $b > 1$. For a similar reason the same symbol $\underset{\circ}{b}^{a}$, interpreted by the conditions (337.), denotes a determined positive ratio when $b < 1$, $b > 0$; and for the remaining case of a positive base, $b = 1$, the symbol $\underset{\circ}{b}^{a}$ denotes still, by (338.) a determined positive ratio, namely, the ratio 1. The exponent or logarithm a has, in these late investigations, been supposed to be incommensurable ; when that exponent a is commensurable, the base b being still positive, we saw that the symbol $\underset{\circ}{b}^{a}$ can be interpreted more easily, as a power of a root, and that it always denotes a determined positive ratio.

Reciprocally, in the equation

$$c = \underset{\circ}{b}^{a}, \qquad (351.)$$

if the power c be any determined positive ratio, and if the exponent a be any determined ratio, positive or contra-positive, we can deduce the positive base or ratio b, by calculating the inverse or reciprocal power

$$b = \underset{\circ}{c}^{1/a}; \qquad (352.)$$

as appears from the relation (335.) which extends, as was above announced, together with the relation (334.), even to the case of incommensurable exponents. The proof of the important extension last alluded to, will easily suggest itself to those who have studied the foregoing demonstrations; and they will perceive that with the foregoing rules for the interpretation of the symbol $\underset{\circ}{b}^{x}$, for the case of an incommensurable exponent, the power $\underset{\circ}{b}^{a}$ increases (as was said above) *continuously* as well as *constantly* with the exponent a if the base b be > 1, or else decreases continuously and constantly if that positive base be < 1, but remains constantly $= 1$ if $b = 1$. It is therefore possible to find one determined exponent or logarithm a, and only one, which shall satisfy the equation (351.), when the power c and the base b are any given positive ratios, except in the impossible or indeterminate case when this base b is the particular ratio 1; and the number a thus determined, whether positive or contra-positive or null, may be called "the logarithm of c to the base b," and may be denoted by the symbol

$$a = \log_{b}. c. \qquad (353.)$$

It is still more easy to perceive, finally, that when this logarithm a is given, (even if it be incommensurable,) the power c increases constantly and continuously from zero with the base b, if $a > 0$, or else decreases constantly and continuously towards zero if $a < 0$, or remains constant and $= 1$, if $a = 0$.

Remarks on the Notation of this Essay, and on some modifications by which it may be made more like the Notation commonly employed.

34. In the foregoing articles we have constantly denoted *moments*, or indivisible points of time, by small capital letters, A, B, A′, B′, &c.; and *steps*, or transitions from one such moment to others, by small Roman letters, a, b, a′, b′, &c. The mark − has been interposed between the marks of two moments, to express the ordinal relation of those two moments, or the step which must be made in order to pass from one to the other; and the mark + has been inserted between the marks of a step and a moment, or between the marks of two steps, to denote the application of the

step to the moment, or the composition of the two steps with each other. For the decomposition of a step into others, we have used no special mark; but employed the theorem that such decomposition can be performed by compounding with the given compound step the opposites of the given component steps, and a special notation for such opposite steps, namely, the mark* Θ prefixed; so that we have written $\Theta\,\mathrm{a}$ to denote the step opposite to the step a, and consequently $\Theta\,\mathrm{a}+\mathrm{b}$ to denote the algebraical excess of the step b over the step a, because this excess may be conceived as a step compounded of b and $\Theta\,\mathrm{a}$. However, we might have agreed to write

$$(\mathrm{b}+\mathrm{A})-(\mathrm{a}+\mathrm{A})=\mathrm{b}-\mathrm{a}, \qquad (354.)$$

denoting the step from the moment $\mathrm{a}+\mathrm{A}$ to the moment $\mathrm{b}+\mathrm{A}$, for conciseness by $\mathrm{b}-\mathrm{a}$; and then $\mathrm{b}-\mathrm{a}$ would have been another symbol for the algebraical excess of the step b over the step a, and we should have had the equation

$$\mathrm{b}-\mathrm{a}=\Theta\,\mathrm{a}+\mathrm{b}. \qquad (355.)$$

We might thus have been led to interpose the mark $-$ between the marks of a compound step b and a component step a, in order to denote the other component step, or the algebraical *remainder* which results from the algebraical *subtraction* of the component from the compound.

Again, we have used the Greek letters $\mu\ \nu\ \xi\ \rho\ \omega$, with or without accents, to denote *integer numbers* in general, and the italic letters $i\ k\ l\ m\ n$ to denote positive whole numbers in particular; using also the earlier letters $\alpha\ \beta\ \gamma\ a\ b\ c\ d$ to denote any ratios whatever, commensurable or incommensurable, and in one recent investigation the capital letter B to denote any positive ratio: and employed, in the combination of these symbols of numbers, or of ratios, the same marks of *addition* and of *opposition*, $+$ and Θ, which had been already employed for steps, and the mark of multiplication $\times$, without any special mark for *subtraction*. We might, however, have agreed to write, in general,

$$(b\times\mathrm{a})-(a\times\mathrm{a})=(b-a)\times\mathrm{a}, \qquad (356.)$$

as we wrote

$$(b\times\mathrm{a})+(a\times\mathrm{a})=(b+a)\times\mathrm{a};$$

and then the symbol $b-a$ would have denoted the algebraical excess of the number

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* This mark has been printed, for want of the proper type, like a Greek Theta in this Essay: it was designed to be printed thus $\ominus$.

b over the number a, or the remainder obtained by the algebraical *subtraction* of the latter number from the former; and we should have had the equation,

$$b - a = \Theta\, a + b, \qquad (357.)$$

which is, with respect to *numbers*, or ratios, what the equation (355.) is, with respect to steps. And when such symbols of remainders, $\mathrm{b} - \mathrm{a}$ or $b - a$, are to be combined with other symbols in the way of algebraical *addition*, it results, from principles already explained, that they need not be enclosed in parentheses; for example, we may write simply $\mathrm{c} + \mathrm{b} - \mathrm{a}$ instead of $\mathrm{c} + (\mathrm{b} - \mathrm{a})$, because the sum denoted by this last notation is equivalent to the remainder $(\mathrm{c} + \mathrm{b}) - \mathrm{a}$. But the parentheses (or some other combining mark) must be used, when a remainder is to be *subtracted*; thus the symbol $\mathrm{c} - \mathrm{b} - \mathrm{a}$ is to be interpreted as $(\mathrm{c} - \mathrm{b}) - \mathrm{a}$, and not as $\mathrm{c} - (\mathrm{b} - \mathrm{a})$, which latter symbol is equivalent to $(\mathrm{c} - \mathrm{b}) + \mathrm{a}$, or $\mathrm{c} - \mathrm{b} + \mathrm{a}$.

35. With this way of denoting the algebraical subtraction of steps, and that of numbers, we have the formula,

$$0 - \mathrm{a} = \Theta\, \mathrm{a}, \quad 0 - a = \Theta\, a, \qquad (358.)$$

0 denoting in the one a null step, and in the other a null number. And if we farther agree to suppress (for abridgement) this symbol 0 when it occurs in such combinations as the following, $0 + \mathrm{a}$, $0 - \mathrm{a}$, $0 + a$, $0 - a$, writing, in the case of steps,

$$0 + \mathrm{a} = + \mathrm{a}, \quad 0 - \mathrm{a} = - \mathrm{a}, \qquad (359.)$$

and similarly, in the case of numbers,

$$0 + a = + a, \quad 0 - a = - a, \qquad (360.)$$

and, in like manner,

$$\left.\begin{array}{l} 0 + \mathrm{a} \pm \mathrm{b} = + \mathrm{a} \pm \mathrm{b}, \quad 0 - \mathrm{a} \pm \mathrm{b} = - \mathrm{a} \pm \mathrm{b}, \\ 0 + a \pm b = + a \pm b, \quad 0 - a \pm b = - a \pm b, \end{array}\right\} \qquad (361.)$$

we shall then have the formula

$$+ \mathrm{a} = \mathrm{a}, \quad - \mathrm{a} = \Theta\, \mathrm{a}, \qquad (362.)$$

and

$$+ a = a, \quad - a = \Theta\, a, \qquad (363.)$$

of which the one refers to steps and the other to numbers. With these conventions, the prefixing of the mark + to an isolated symbol of a step or of a number, does not change the meaning of the symbol; but the prefixing of the mark − converts that symbol into another, which denotes the opposite of the original step, or the opposite of the original number; so that the series of whole numbers (103.) or (266.) may be written as follows:

$$\ldots\ -3,\ -2,\ -1,\ 0,\ +1,\ +2,\ +3,\ \ldots \qquad (364.)$$

Also, in this notation,

$$\left.\begin{array}{l} \mathrm{b} \pm (+\mathrm{a}) = \mathrm{b} \pm \mathrm{a},\ \ \mathrm{b} \pm (-\mathrm{a}) = \mathrm{b} \mp \mathrm{a}, \\ b \pm (+a) = b \pm a,\ \ b \pm (-a) = b \mp a. \end{array}\right\} \qquad (365.)$$

36. Finally, as we wrote, for the case of commensurable steps,

$$\frac{\nu \times \mathrm{a}}{\mu \times \mathrm{a}} = \frac{\nu}{\mu},$$

μ and ν being here whole numbers, so we may agree to write, in general,

$$\frac{b \times \mathrm{a}}{a \times \mathrm{a}} = \frac{b}{a}, \qquad (366.)$$

whatever ratios a and b may be; and then this symbol $\frac{b}{a}$ will denote, in general, the algebraic quotient obtained by dividing the number or ratio b by the number or ratio a; whereas we had before no general way of denoting such a quotient, except by the mark u prefixed to the symbol of the divisor a, so as to form a symbol of the reciprocal number $\mathrm{u}\, a$, to multiply by which latter number is equivalent to dividing by the former. Comparing the two notations, we have the formula,

$$\frac{1}{a} = \mathrm{u}\, a, \qquad (367.)$$

and generally

$$\frac{b}{a} = \mathrm{u}\, a \times b = b \times \mathrm{u}\, a. \qquad (368.)$$

These two marks Θ and u have been the only *new* marks introduced in this Elementary Essay; although the notation employed for powers differs a little from the common notation: especially the symbol $\underset{\circ}{b}^{a}$, suggested by those researches of Mr. Graves respecting the general expression of powers and logarithms, which were the first occasion of the conception of that Theory of Conjugate Functions to which we now proceed.

END OF THE PRELIMINARY AND ELEMENTARY ESSAY.

THEORY OF CONJUGATE FUNCTIONS,

OR ALGEBRAIC COUPLES.

On Couples of Moments, and of Steps, in Time.

1. When we have imagined any one moment of time A_1, which we may call a *primary moment*, we may again imagine a moment of time A_2, and may call this a *secondary moment*, without regarding whether it follows, or coincides with, or precedes the primary, in the common progression of time; we may also speak of this primary and this secondary moment as forming a *couple of moments*, or a *moment-couple*, which may be denoted thus, (A_1, A_2). Again, we may imagine any other two moments, a primary and a secondary, B_1 and B_2, distinct from or coincident with each other, and forming another *moment-couple*, (B_1, B_2); and we may compare the latter couple of moments with the former, moment with moment, primary with primary, and secondary with secondary, examining how B_1 is ordinally related to A_1, and how B_2 is ordinally related to A_2, in the progression of time, as coincident, or subsequent, or precedent; and thus may obtain a *couple of ordinal relations*, which may be thus separately denoted $B_1 - A_1$, $B_2 - A_2$, or thus collectively, as a *relation-couple*,

$$(B_1 - A_1,\ B_2 - A_2).$$

This couple of ordinal relations between moments may also be conceived as constituting a complex *relation of one moment-couple to another*; and in conformity with this conception it may be thus denoted,

$$(B_1,\ B_2) - (A_1,\ A_2),$$

so that, comparing this with the former way of representing it, we may establish the written equation,

$$(\mathrm{B}_1,\ \mathrm{B}_2) - (\mathrm{A}_1,\ \mathrm{A}_2) = (\mathrm{B}_1 - \mathrm{A}_1,\ \mathrm{B}_2 - \mathrm{A}_2). \qquad (1.)$$

Instead of conceiving thus a couple of ordinal relations between moments, or a relation between two couples of moments, discovered by the (analytic) *comparison* of one such couple of moments with another, we may conceive a *couple of steps* in the progression of time, from moment to moment respectively, or a single complex step which we may call a *step-couple* from one moment-couple to another, serving to *generate* (synthetically) one of these moment-couples from the other; and if we denote the two separate steps by a_1, a_2, (a_1 being the step from A_1 to B_2, and a_2 being the step from A_2 to B_2,) so that in the notation of the Preliminary Essay,

$$\mathrm{B}_1 = \mathrm{a}_1 + \mathrm{A}_1,\ \ \mathrm{B}_2 = \mathrm{a}_2 + \mathrm{A}_2,$$
$$\mathrm{B}_1 = (\mathrm{B}_1 - \mathrm{A}_1) + \mathrm{A}_1,\ \ \mathrm{B}_2 = (\mathrm{B}_2 - \mathrm{A}_2) + \mathrm{A}_2,$$

we may now establish this analogous notation for couples,

$$\left.\begin{aligned}(\mathrm{B}_1,\ \mathrm{B}_2) &= (\mathrm{a}_1 + \mathrm{A}_1,\ \ \mathrm{a}_2 + \mathrm{A}_2)\\ &= (\mathrm{a}_1,\ \mathrm{a}_2) + (\mathrm{A}_1,\ \mathrm{A}_2)\\ &= \{(\mathrm{B}_1,\ \mathrm{B}_2) - (\mathrm{A}_1,\ \mathrm{A}_2)\} + (\mathrm{A}_1,\ \mathrm{A}_2),\end{aligned}\right\} \qquad (2.)$$

the symbol $(\mathrm{B}_1,\ \mathrm{B}_2) - (\mathrm{A}_1,\ \mathrm{A}_2)$ corresponding now to the conception of the *step-couple* by which we may pass from the *moment-couple* $(\mathrm{A}_1,\ \mathrm{A}_2)$ to the *moment-couple* $(\mathrm{B}_1,\ \mathrm{B}_2)$, and the equivalent symbol $(\mathrm{a}_1,\ \mathrm{a}_2)$ or $(\mathrm{B}_1 - \mathrm{A}_1,\ \mathrm{B}_2 - \mathrm{A}_2)$ corresponding now to the conception of the *couple of steps* a_1, a_2, from the two moments A_1, A_2, to the two moments B_1, B_2, respectively. The step a_1, or $\mathrm{B}_1 - \mathrm{A}_1$ may be called the *primary step* of the couple $(\mathrm{a}_1,\ \mathrm{a}_2)$, and the step a_2 or $\mathrm{B}_2 - \mathrm{A}_2$ may be called the *secondary step.*

A step-couple may be said to be *effective* when it actually changes the moment-couple to which it is applied; that is, when one at least of its two coupled steps is effective: and in the contrary case, that is, when both those coupled steps are separately null, the step-couple itself may be said to be *null* also. And effective step-couples may be distinguished into *singly effective* and *doubly effective* step-couples, according as they alter *one* or *both* of the two moments of the moment-couples to which they are applied. Finally, a singly effective step-couple may be called a *pure primary* or *pure secondary* step-couple, according as only its primary or only its secondary step is effective, that is, according as it alters only the primary or only the secondary moment. Thus (0, 0) is a null step-couple, $(\mathrm{a}_1,\ \mathrm{a}_2)$ is a doubly effective step-couple, and $(a_1, 0)$ $(0, a_2)$ are singly effective step-couples, the former $(a_1, 0)$ being a pure primary, and the latter $(0, a_2)$ being a pure secondary, if 0 denote a null step, and a_1 a_2 effective steps.

On the Composition and Decomposition of Step-Couples.

2. Having stepped from one couple of moments (A_1, A_2) to another couple of moments (B_1, B_2) by one step-couple (a_1, a_2), we may afterwards step to a third couple of moments (C_1, C_2) by a second step-couple (b_1, b_2), so as to have

$$\left.\begin{array}{l}(C_1, C_2) = (b_1, b_2) + (B_1, B_2), \\ (B_1, B_2) = (a_1, a_2) + (A_1, A_2);\end{array}\right\} \qquad (3.)$$

and then we may consider ourselves as having made upon the whole a compound couple of steps, or a *compound step-couple*, from the first moment-couple (A_1, A_2) to the third moment-couple (C_1, C_2), and may agree to call this compound step-couple the *sum* of the two component step-couples (a_1, a_2), (b_1, b_2), or to say that is formed by *adding* them, and to denote as follows,

$$(C_1, C_2) - (A_1, A_2) = (b_1, b_2) + (a_1, a_2); \qquad (4.)$$

as, in the language of the Preliminary Essay, the two separate compound steps, from A_1 to C_1 and from A_2 to C_2 are the *sums* of the component steps, and are denoted by the symbols $b_1 + a_1$ and $b_2 + a_2$ respectively. With these notations, we have evidently the equation

$$(b_1, b_2) + (a_1, a_2) = (b_1 + a_1, b_2 + a_2); \qquad (5.)$$

that is, the *sum of two step-couples may be formed by coupling the two sum-steps.* Hence, also,

$$(b_1, b_2) + (a_1, a_2) = (a_1, a_2) + (b_1, b_2), \qquad (6.)$$

that is, *the order of any two component step-couples may be changed without altering the result;* and

$$(a_1, a_2) = (a_1, 0) + (0, a_2), \qquad (7.)$$

that is, *every doubly effective step-couple is the sum of a pure primary and a pure* *secondary.* In like manner, we can conceive sums of more than two step-couples, and may establish, for such sums, relations analogous to those marked (5.) and (6.); thus,

$$\left.\begin{aligned}(c_1, c_2)+(b_1, b_2)+(a_1, a_2)&=(c_1+b_1+a_1, c_2+b_2+a_2),\\ &=(a_1, a_2)+(b_1, b_2)+(c_1, c_2) \text{ \&c.}\end{aligned}\right\} \quad (8.)$$

We may also consider the *decomposition* of a step-couple, or the *subtraction* of one such step-couple from another, and may write,

$$(b_1, b_2)-(a_1, a_2)=(b_1-a_1, b_2-a_2), \quad (9.)$$

$(b_1, b_2)-(a_1, a_2)$ being that sought step-couple which must be compounded with or added to the given component step-couple (a_1, a_2) in order to produce the given compound step-couple (b_1, b_2). And if we agree to suppress the symbol of a null step-couple, when it is combined with others or others with it in the way of addition or subtraction, we may write

$$\left.\begin{aligned}(a_1, a_2)&=(0, 0)+(a_1, a_2)=+(a_1, a_2),\\ (-a_1, -a_2)&=(0, 0)-(a_1, a_2)=-(a_1, a_2),\end{aligned}\right\} \quad (10.)$$

employing a notation analogous to that explained for single steps in the 35th article of the Preliminary Essay. Thus $+(a_1, a_2)$ is another way of denoting the step-couple (a_1, a_2) itself; and $-(a_1, a_2)$ is a way of denoting the *opposite* step-couple $(-a_1, -a_2)$.

On the Multiplication of a Step-Couple by a Number.

3. From any proposed moment-couple (A_1, A_2), and any proposed step-couple (a_1, a_2), we may generate a series of other moment-couples

$$\ldots (E'_1, E'_2), (E_1, E_2), (A_1, A_2), (B_1, B_2), (B'_1, B'_2) \ldots \quad (11.)$$

by repeatedly applying this step-couple (a_1, a_2), itself, and the opposite step-couple $-(a_1, a_2)$, or $(-a_1, -a_2)$, in a way analogous to the process of the 13th article of the Preliminary Essay, as follows:

$$\left.\begin{array}{l}
\qquad\qquad\cdots\cdots\cdots\cdots \\
(E'_1, E'_2) = (-a_1, -a_2) + (-a_1, -a_2) + (A_1, A_2), \\
(E_1, E_2) = (-a_1, -a_2) + (A_1, A_1), \\
(A_1, A_2) = (A_1, A_2), \\
(B_1, B_2) = (a_1, a_2) + (A_1, A_2), \\
(B'_1, B'_2) = (a_1, a_2) + (a\ , a_2) + (A_1, A_2), \\
\qquad\qquad\cdots\cdots\cdots\cdots
\end{array}\right\} \qquad (12.)$$

and a series of *multiple step-couples*, namely

$$\left.\begin{array}{l}
\qquad\qquad\cdots\cdots\cdots\cdots \\
(E'_1, E'_2) - (A_1, A_2) = (-a_1, -a_2) + (-a_1, -a_2), \\
(E_1, E_2) - (A_1, A_2) = (-a_1, -a_2) \\
(A_1, A_2) - (A_1, A_2) = (0, 0), \\
(B_1, B_2) - (A_1, A_2) = (a_1, a_2), \\
(B'_1, B'_2) - (A_1, A_2) = (a_1, a_2) + (a_1, a_2), \\
\qquad\qquad\cdots\cdots\cdots\cdots
\end{array}\right\} \qquad (13.)$$

which may be thus more concisely denoted,

$$\left.\begin{array}{l}
\qquad\qquad\cdots\cdots\cdots\cdots \\
(E'_1, E'_2) = -2\,(a_1, a_2) + (A_1, A_2), \\
(E_1, E_2) = -1\,(a_1, a_2) + (A_1, A_2), \\
(A_1, A_2) = 0\,(a_1, a_2) + (A_1, A_2), \\
(B_1, B_2) = +1\,(a_1, a_2) + (A_1, A_2), \\
(B'\ , B'_2) = +2\,(a_1, a_2) + (A_1, A_2), \\
\qquad\qquad\cdots\cdots\cdots\cdots
\end{array}\right\} \qquad (14.)$$

and

$$\left.\begin{array}{l}
\qquad\qquad\qquad\cdots\cdots\cdots\cdots \\
(E'_1, E'_2) - (A_1, A_2) = -2\,(a_1, a_2) = -2 \times (a_1, a_2), \\
(E_1, E_2) - (A_1, A_2) = -1\,(a_1, a_2) = -1 \times (a_1, a_2), \\
(A_1, A_2) - (A_1, A_2) = 0\,(a_1, a_2) = 0 \times (a_1, a_2), \\
(B_1, B_2) - (A_1, A_2) = +1\,(a_1, a_2) = +1 \times (a_1, a_2), \\
(B'_1, B'_2) - (A_1, A_2) = +2\,(a_1, a_2) = +2 \times (a_1, a_2), \\
\qquad\qquad\qquad \&c.
\end{array}\right\} \qquad (15.)$$

We may also conceive step-couples which shall be *sub-multiples* and *fractions* of a given step-couple, and may write

$$(c_1, c_2) = \frac{\nu}{\mu} \times (b_1, b_2) = \frac{\nu}{\mu}\,(b_1, b_2), \qquad (16.)$$

when the two step-couples $(\mathrm{b}_1, \mathrm{b}_2)$ $(\mathrm{c}_1, \mathrm{c}_2)$ are related as multiples to one common step-couple $(\mathrm{a}_1, \mathrm{a}_2)$ as follows:

$$(\mathrm{b}_1, \mathrm{b}_2) = \mu \times (\mathrm{a}_1, \mathrm{a}_1), \ (\mathrm{c}_1, \mathrm{c}_2) = \nu \times (\mathrm{a}_1, \mathrm{a}_2), \qquad (17.)$$

μ and ν being any two proposed whole numbers. And if we suppose the *fractional multiplier* $\frac{\nu}{\mu}$ in (16.) to tend to any *incommensurable limit* a, we may denote by $a \times (\mathrm{b}_1, \mathrm{b}_2)$ the corresponding limit of the fractional product, and may consider this latter limit as the *product* obtained by multiplying the step-couple $(\mathrm{b}_1, \mathrm{b}_2)$ by the *incommensurable multiplier* or number a; so that we may write,

$$\left.\begin{array}{c}(\mathrm{c}_1, \mathrm{c}_2) = a \times (\mathrm{b}_1, \mathrm{b}_2) = a\,(\mathrm{b}_1, \mathrm{b}_2), \\ \text{if } (\mathrm{c}_1, \mathrm{c}_2) = \underline{\mathrm{L}}\,(\frac{\nu}{\mu}\,(\mathrm{b}_1, \mathrm{b}_2)) \text{ and } a = \underline{\mathrm{L}}\,\frac{\nu}{\mu},\end{array}\right\} \qquad (18.)$$

using $\underline{\mathrm{L}}$ as the mark of a limit, as in the notation of the Preliminary Essay. It follows from these conceptions of the multiplication of a step-couple by a number, that generally

$$a \times (\mathrm{a}_1, \mathrm{a}_2) = (a\,\mathrm{a}_1, a\,\mathrm{a}_2), \qquad (19.)$$

whatever steps may be denoted by a_1, a_2, and whatever number (commensurable or incommensurable, and positive or contra-positive or null) may be denoted by a. Hence also we may write

$$\frac{(a\,\mathrm{a}_1, a\,\mathrm{a}_2)}{(\mathrm{a}_1, \mathrm{a}_2)} = a, \qquad (20.)$$

and may consider the number a as expressing the *ratio* of the step-couple $(a\,\mathrm{a}_1, a\,\mathrm{a}_2)$ to the step-couple $(\mathrm{a}_1, \mathrm{a}_2)$.

On the Multiplication of a Step-Couple by a Number-Couple; and on the Ratio of one Step-Couple to another.

4. The formula (20.) enables us, in an infinite variety of cases, to assign a single number a as the ratio of one proposed step-couple $(\mathrm{b}_1, \mathrm{b}_2)$ to another $(\mathrm{a}_1, \mathrm{a}_2)$; namely, in all those cases in which the primary and secondary steps of the one couple are proportional to those of the other: but it fails to assign such a ratio, in all those

other cases in which this condition is not satisfied. The spirit of the present Theory of Couples leads us, however, to conceive that the ratio of any one effective step-couple to any other may perhaps be expressed in general by a *number-couple*, or couple of numbers, a primary and a secondary ; and that with reference to this more general view of such ratio, the relation (20.) might be more fully written thus,

$$\frac{(a_1\,\mathrm{a}_1,\ a_1\,\mathrm{a}_1)}{(\mathrm{a}_1,\ \mathrm{a}_2)} = (a_1,\ 0), \qquad (21.)$$

and the relation (19.) as follows,

$$(a_1,\ 0) \times (\mathrm{a}_1,\ \mathrm{a}_2) = (a_1,\ 0)\,(\mathrm{a}_1,\ \mathrm{a}_2) = (a_1\,\mathrm{a}_1,\ a_1\,\mathrm{a}_2), \qquad (22.)$$

the single number a_1 being changed to the couple $(a_1, 0)$, which may be called a *pure primary number-couple*. The spirit of this theory of primaries and secondaries leads us also to conceive that the ratio of any step-couple $(\mathrm{b}_1, \mathrm{b}_2)$ to any pure primary step-couple $(\mathrm{a}_1, 0)$, may be expressed by coupling the two ratios $\frac{\mathrm{b}_1}{\mathrm{a}_1}$, $\frac{\mathrm{b}_2}{\mathrm{a}_1}$, which the two steps b_1, b_2 bear to the effective primary step a_1; so that we may write

$$\frac{(\mathrm{b}_1,\ \mathrm{b}_2)}{(\mathrm{a}_1,\ 0)} = \left(\frac{\mathrm{b}_1}{\mathrm{a}_2},\ \frac{\mathrm{b}_2}{\mathrm{a}_2}\right),\quad \frac{(a_1\,\mathrm{a}_1,\ a_2\,\mathrm{a}_1)}{(\mathrm{a}_1,\ 0)} = (a_1,\ a_2), \qquad (23.)$$

and in like manner, by the general connexion of multiplication with ratio,

$$(a_1,\ a_2) \times (\mathrm{a}_1,\ 0) = (a_1,\ a_2)\,(\mathrm{a}_1,\ 0) = (a_1\,\mathrm{a}_1,\ a_2\,\mathrm{a}_1). \qquad (24.)$$

From the relations (22.) (24.), it follows by (5.) that

$$(b_1 + a_1,\ 0)\,(\mathrm{a}_1,\ \mathrm{a}_2) = (b_1,\ 0)\,(\mathrm{a}_1,\mathrm{a}_2) + (a_1,\ 0)\,(\mathrm{a}_1,\ \mathrm{a}_2), \qquad (25.)$$

and that

$$(a_1,\ a_2)\,(\mathrm{b}_1 + \mathrm{a}_1,\ 0) = (a_1, a_2)\,(\mathrm{b}_1,\ 0) + (a_1,\ a_2)\,(\mathrm{a}_1,\ 0)\,; \qquad (26.)$$

and the spirit of the present extension of reasonings and operations on single moments, steps, and numbers, to moment-couples, step-couples, and number-couples, leads us to determine (if we can) what remains yet undetermined in the conception of a number-couple, as a multiplier or as a ratio, so as to satisfy the two following more general conditions,

$$(b_1 + a_1,\ b_2 + a_2)\,(\mathrm{a}_1,\ \mathrm{a}_2) = (b_1,\ b_2)\,(\mathrm{a}_1,\ \mathrm{a}_2) + (a_1,\ a_2)\,(\mathrm{a}_1,\ \mathrm{a}_2), \qquad (27.)$$

and

$$(a_1,\ a_2)\,(\mathrm{b}_1 + \mathrm{a}_1,\ \mathrm{b}_2 + \mathrm{a}_2) = (a_1,\ a_2)\,(\mathrm{b}_1,\ \mathrm{b}_2) + (a_1,\ a_2)\,(\mathrm{a}_1,\ \mathrm{a}_2), \qquad (28.)$$

whatever numbers may be denoted by $a_1\, a_2\, b_1\, b_2$, and whatever steps by $\mathrm{a}_1\, \mathrm{a}_2\, \mathrm{b}_1\, \mathrm{b}_2$. With these conditions we have

$$(a_1, a_2)\ (\mathrm{a}_1, \mathrm{a}_2) = (a_1, 0)\ (\mathrm{a}_1, \mathrm{a}_2) + (0, a_2)\ (\mathrm{a}_1, \mathrm{a}_2), \qquad (29.)$$

$$(0, a_2)\ (\mathrm{a}_1, \mathrm{a}_2) = (0, a_2)\ (\mathrm{a}_1, 0) + (0, a_2)\ (0, \mathrm{a}_2), \qquad (30.)$$

and, therefore, by (22.) and (24.), and by the formula for sums,

$$\begin{aligned}(a_1, a_2)\ (\mathrm{a}_1, \mathrm{a}_2) &= (a_1\,\mathrm{a}_2,\ a_1\,\mathrm{a}_1) + (0,\ a_2\,\mathrm{a}_1) + (0, a_2)\ (0, \mathrm{a}_2)\\ &= (a_1\,\mathrm{a}_1,\ a_1\,\mathrm{a}_2 + a_2\,\mathrm{a}_1) + (0, a_2)\ (0, \mathrm{a}_2), \qquad (31.)\end{aligned}$$

in which the product $(0, a_2)\ (0, \mathrm{a}_2)$ remains still undetermined. It must, however, by the spirit of the present theory, be supposed to be some step-couple,

$$(0, a_2)\ (0, \mathrm{a}_2) = (\mathrm{c}_1, \mathrm{c}_2)\,; \qquad (32.)$$

and these two steps $\mathrm{c}_1\ \mathrm{c}_2$ must each vary proportionally to the product $a_2\,\mathrm{a}_2$, since otherwise it could be proved that the foregoing conditions, (27.) and (28.), would not be satisfied; we are, therefore, to suppose

$$\mathrm{c}_1 = \gamma_1\, a_2\,\mathrm{a}_2, \quad \mathrm{c}_2 = \gamma_2\, a_2\,\mathrm{a}_2, \qquad (33.)$$

that is,

$$(0, a_2)\ (0, \mathrm{a}_2) = (\gamma_1\, a_2\,\mathrm{a}_2,\ \gamma_2\, a_2\,\mathrm{a}_2), \qquad (34.)$$

γ_1, γ_2, being some two constant numbers, independent of a_2 and a_2, but otherwise capable of being chosen at pleasure. Thus, the general formula for the product of a step-couple $(\mathrm{a}_1, \mathrm{a}_2)$ multiplied by a number-couple (a_1, a_2), is, by (31.) (34.) and by the theorem for sums,

$$\begin{aligned}(a_1, a_2)\ (\mathrm{a}_1, \mathrm{a}_2) &= (a_1\,\mathrm{a}_1,\ a_1\,\mathrm{a}_2 + a_2\,\mathrm{a}_1) + (\gamma_1\, a_2\,\mathrm{a}_2,\ \gamma_2\, a_2\,\mathrm{a}_2)\\ &= (a_1\,\mathrm{a}_1 + \gamma_1\, a_2\,\mathrm{a}_2,\ a_1\,\mathrm{a}_2 + a_2\,\mathrm{a}_1 + \gamma_2\, a_2\,\mathrm{a}_2): \qquad (35.)\end{aligned}$$

and accordingly this formula satisfies the conditions (27.) and (28.), and includes the relations (22.) and (24.), whatever arbitrary numbers we choose for γ_1, and γ_2; provided that after once choosing these numbers, which we may call the *constants of multiplication*, we retain thenceforth unaltered, and treat them as independent of both the multiplier and the multiplicand. It is clear, however, that the simplicity and elegance of our future operations and results must mainly depend on our making a simple and suitable choice of these two constants of multiplication; and that in making this choice, we ought to take care to satisfy, if possible, the essential condition that there shall be always *one determined number-couple to express the ratio of any one determined step-couple to any other*, at least when the latter is not null : since this was the very object, to accomplish which we were led to introduce the conception of these number-couples. It is easy to show that no choice simpler than the following,

$$\gamma_1 = -1, \ \gamma_2 = 0, \qquad (36.)$$

would satisfy this essential condition : and for that reason we shall now select these two numbers, contra-positive one and zero, for the two constants of multiplication, and shall establish finally this formula for the multiplication of any step-couple $(\mathrm{a}_1, \mathrm{a}_2)$ by any number-couple $(a_1, 0)$,

$$(a_1, a_2)\ (\mathrm{a}_1, \mathrm{a}_2) = (a_1\ \mathrm{a}_1 - a_2\ \mathrm{a}_2,\ a_2\ \mathrm{a}_1 + a_1\ \mathrm{a}_2). \qquad (37.)$$

5. In fact, whatever constants of multiplication $\gamma_1\ \gamma_2$ we may select, if we denote by $(\mathrm{b}_1, \mathrm{b}_2)$ the product of the step-couple $(\mathrm{a}_1, \mathrm{a}_2)$ by the number-couple (a_1, a_2), so that

$$(\mathrm{b}_1, \mathrm{b}_2) = (a_1, a_2) \times (\mathrm{a}_1, \mathrm{a}_2), \qquad (38.)$$

we have by (35.) the following expressions for the two coupled steps b_1, b_2, of the product,

$$\left.\begin{array}{l} \mathrm{b}_1 = a_1\ \mathrm{a}_1 + \gamma_1\ a_2\ \mathrm{a}_2, \\ \mathrm{b}_1 = a_1\ \mathrm{a}_2 + a_2\ \mathrm{a}_1 + \gamma_2\ a_2\ \mathrm{a}_2, \end{array}\right\} \qquad (39.)$$

and therefore

$$\left.\begin{array}{l} \beta_1 = a_1\ \alpha_1 + \gamma_1\ a_2\ \alpha_2\ , \\ \beta_2 = a_1\ \alpha_2 + a_2\ \alpha_1 + \gamma_2\ a_2\ \alpha_2\ , \end{array}\right\} \qquad (40.)$$

if $\alpha_1\ \alpha_2\ \beta_1\ \beta_2$ denote respectively the ratios of the four steps $\mathrm{a}_1\ \mathrm{a}_2\ \mathrm{b}_1\ \mathrm{b}_2$ to one effective step c, so that

$$\mathrm{a}_1 = \alpha_1\ \mathrm{c}, \ \mathrm{a}_2 = \alpha_2\ \mathrm{c}, \qquad (41.)$$

and

$$\mathrm{b}_1 = \beta_1\ \mathrm{c}, \ \mathrm{b}_2 = \beta_2 \quad ; \qquad (42.)$$

from which it follows that

$$\left.\begin{array}{l} a_1\ \{\alpha_1\ (\alpha_1 + \gamma_2\ \alpha_2) - \gamma_1\ \alpha_2^2\} = \beta_1\ (\alpha_1 + \gamma_2\ \alpha_2) - \beta_2\ \gamma_1\ \alpha_2\ , \\ a_2\ \{\alpha_1\ (\alpha^1 + \gamma_2\ \alpha_2) - \gamma_1\ \alpha_2^2\} = \beta_2\ \alpha^1 - \beta_1\ \alpha_2\ ; \end{array}\right\} \qquad (43.)$$

in order therefore that the numbers $a_1\ a_2$ should always be determined by the equation (38.), when $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are not both null steps, it is necessary and sufficient that the factor

$$a_1\ (a_1 + \gamma_2\ a_2) - \gamma_1\ a_2^{\ 2} = (a_1 + \tfrac{1}{2}\gamma_2\ a_2)^2 - (\gamma_1 + \tfrac{1}{4}\gamma_2^{\ 2})\ a^{\ 2} \qquad (44.)$$

should never become null, when a_1 and a_2 are not both null numbers; and this condition will be satisfied if we so choose the constants of multiplication $\gamma_1\ \gamma_2$ as to make

$$\gamma_1 + \tfrac{1}{4}\gamma_2^{\ 2} < 0, \qquad (45.)$$

but not otherwise. Whatever constants $\gamma_1\ \gamma_2$ we choose, we have, by the foregoing principles,

$$\frac{(\mathfrak{c},\ 0)}{(\mathfrak{c},\ 0)} = (1,\ 0)\ ; \quad \frac{(0,\ \mathfrak{c})}{(\mathfrak{c},\ 0)} = (0,\ 1)\ ; \quad \frac{(0,\ \mathfrak{c})}{(0,\ \mathfrak{c})} = (1,\ 0)\ ; \qquad (46.)$$

and finally

$$\frac{(\mathfrak{c},\ 0)}{(0,\ \mathfrak{c})} = \left(-\frac{\gamma_2}{\gamma_1},\ \frac{1}{\gamma_1}\right), \qquad (47.)$$

because, when we make, in (43.),

$$a_1 = 0,\ a_2 = 1,\ \beta_1 = 1,\ \beta^2 = 0, \qquad (48.)$$

we find

$$a_1 = \frac{-\gamma_2}{\gamma_1},\ a_2 = \frac{1}{\gamma_1}\ ; \qquad (49.)$$

so that although the ratio of the pure primary step-couple $(\mathfrak{c},\ 0)$ to the pure secondary step-couple $(0,\ \mathfrak{c})$ can never be expressed as a *pure primary number-couple*, it may be expressed as a *pure secondary number-couple*, namely $\left(0,\ \frac{1}{\gamma_1}\right)$, if we choose 0, as in (36.), for the value of the secondary constant γ_2, but not otherwise: this choice $\gamma_2 = 0$ is therefore required by simplicity. And since by the condition (45.), the primary constant γ_1 must be contrapositive, the simplest way of determining it is to make it contrapositive one, $\gamma_1 = -1$, as announced in (36.). We have therefore justified that selection (36.) of the two constants of multiplication; and find, with that selection,

$$\frac{(\mathfrak{c},\ 0)}{(0,\ \mathfrak{c})} = (0,\ -1), \qquad (50.)$$

and generally, for the ratio of any one step-couple to any other, the formula

$$\frac{(\mathfrak{b}_1,\ \mathfrak{b}_2)}{(\mathfrak{a}_1,\ \mathfrak{a}_2)} = \frac{(\beta_1\ \mathfrak{c},\ \beta_2\ \mathfrak{c})}{(a_1\ \mathfrak{c},\ a_2\ \mathfrak{c})} = \left(\frac{\beta_1\, a_1 + \beta_2\, a_2}{a_1^{\ 2} + a_2^{\ 2}},\ \frac{\beta_2\, a_1 - \beta_1\, a_2}{a_1^{\ 2} + a_2^{\ 2}}\right). \qquad (51.)$$

On the Addition, Subtraction, Multiplication, and Division, of Number-Couples, as combined with each other.

6. Proceeding to operations upon number-couples, considered in combination with each other, it is easy now to see the reasonableness of the following definitions, and even their necessity, if we would preserve in the simplest way, the analogy of the theory of couples to the theory of singles:

$$(b_1, b_2) + (a_1, a_2) = (b_1 + a_1, b_2 + a_2); \qquad (52.)$$

$$(b_1, b_2) - (a_1, a_2) = (b_1 - a_1, b_2 - a_2); \qquad (53.)$$

$$(b_1, b_2)(a_1, a_2) = (b_1, b_2) \times (a_1, a_2) = (b_1 a_1 - b_2 a_2, b_2 a_1 + b_1 a_2); \qquad (54.)$$

$$\frac{(b_1, b_2)}{(a_1, a_2)} = \left(\frac{b_1 a_1 + b_2 a_2}{a_1^2 + a_2^2}, \frac{b_2 a_1 - b_1 a_2}{a_1^2 + a_2^2}\right). \qquad (55.)$$

Were these definitions even altogether arbitrary, they would at least not contradict each other, nor the earlier principles of Algebra, and it would be possible to draw legitimate conclusions, by rigorous mathematical reasoning, from premises thus arbitrarily assumed: but the persons who have read with attention the foregoing remarks of this theory, and have compared them with the Preliminary Essay, will see that these definitions are really *not arbitrarily chosen*, and that though others might have been assumed, no others would be equally proper.

With these definitions, addition and subtraction of number-couples are mutually inverse operations, and so are multiplication and division; and we have the relations,

$$(b_1, b_2) + (a_1, a_2) = (a_1, a_2) + (b_1, b_2), \qquad (56.)$$

$$(b_1, b_2) \times (a_1, a_2) = (a_1, a_2) \times (b_1, b_2), \qquad (57.)$$

$$(b_1, b_2)\{(a'_1\ a'_2) + (a_1, a_2)\} = (b_1, b_2)\ (a'_1, a'_2) + (b_1, b_2)\ (a_1, a_2): \qquad (58.)$$

we may, therefore, extend to number-couples all those results respecting numbers, which have been deduced from principles corresponding to these last relations. For example,

$$\{(b_1, b_2) + (a_1, a_2)\} \times \{(b_1, b_2) + (a_1, a_2)\} =$$

$$(b_1, b_2)\ (b_1, b_2) + 2\,(b_1, b_2)\ (a_1, a_2) + (a_1, a_2)\ (a_1, a_2), \qquad (59.)$$

in which

$$2\,(b_1,\ b_2)\ (a_1,\ a_2)=(2,\ 0)\ \ (b_1,\ b_2)\ \ (a_1,\ a_2)=(b_1,\ b_2)\ (a_1,\ a_2)+(b_1,\ b_2)\ (a_1,\ a_2)\,; \qquad (60.)$$

for, in general, we may *mix the signs of numbers with those of number-couples,* if we consider every single number a as equivalent to a pure primary number-couple,

$$a=(a,\ 0). \qquad (61.)$$

When the pure primary couple (1, 0) is thus considered as equivalent to the number 1, it may be called, for shortness, the *primary unit;* and the pure secondary couple (0, 1) may be called in like manner the *secondary unit.*

We may also agree to write, by analogy to notations already explained,

$$\left.\begin{array}{l}(0,\ 0)+(a_1,\ a_2)=+(a_1,\ a_2),\\ (0,\ 0)-(a_1,\ a_2)=-(a_1,\ a^2)\,;\end{array}\right\} \qquad (62.)$$

and then $+(a_1,\ a_2)$ will be another symbol for the number-couple $(a_1,\ a_2)$ itself, and $-(a_1,\ a_2)$ will be a symbol for the *opposite number-couple* $(-a_1,\ -a_2)$. The *reciprocal* of a number-couple $(a_1,\ a_2)$ is this other number-couple,

$$\frac{1}{(a_1,\ a_2)}=\frac{(1,\ 0)}{(a_1,\ a_2)}=\left(\frac{a_1}{{a_1}^2+a_2^2},\ \ \frac{-a_2}{{a_1}^2+{a_2}^2}\right)=\frac{(a_1\ ,\ -a_2\ )}{{a_1}^2+{a_2}^2}. \qquad (63.)$$

It need scarcely be mentioned that the insertion of the sign of coincidence = between any two number-couples implies that those two couples coincide, number with number, primary with primary, and secondary with secondary; so that *an equation between number-couples is equivalent to a couple of equations between numbers.*

On the Powering of a Number-couple by Single Whole Number.

7. Any number-couple $(a_1,\ a_2)$ may be used as a *base* to generate a series of *powers,* with integer *exponents,* or *logarithms,* namely, the series

$$\ldots\ (a_1,\ a_2)^{-2},\ (a_1,\ a_2)^{-1},\ (a_1,\ a_2)^0,\ (a_1,\ a_2)^1,\ (a_1,\ a_2)^2,\ \ldots \qquad (64.)$$

in which the *first positive power* $(a_1,\ a_2)^1$ is the base itself, and all the others are formed from it by repeated multiplication or division by that base, according as they follow or precede it in the series; thus,

$$(a_1,\ a_2)^0=(1,\ 0), \qquad (65.)$$

and

$$\left.\begin{array}{ll}(a_1, a_2)^1=(a_1, a_2), & (a_1, a_2)^{-1}=\dfrac{(1, 0)}{(a_1, a_2)}, \\ (a_1, a_2)^2=(a_1, a_2)\,(a_1, a_2), & (a_1, a_2)^{-2}=\dfrac{(1, 0)}{(a_1, a_2)\,(a_1, a_2)}, \\ \&c. & \&c.\end{array}\right\} \qquad (66.)$$

To *power* the couple (a_1, a_2) by any *positive* whole number m, is, therefore, to *multiply*, m times successively, the *primary unit*, or the couple $(1, 0)$, by the proposed couple (a_1, a_2); and to power (a_1, a_2) by any *contra-positive* whole number $-m$, is to *divide* $(1, 0)$ by the same couple (a_1, a_2), m times successively: but to power by 0 produces always $(1, 0.)$. Hence, generally, for any whole numbers μ, ν,

$$\left.\begin{array}{c}(a_1, a_2)^{\mu}\ (a_1, a_2)^{\nu}=(a_1, a_2)^{\mu+\nu}, \\ ((a_1, a_2)^{\mu})^{\nu}=(a_1, a_2)^{\mu\,\nu}.\end{array}\right\} \qquad (67.)$$

8. In the theory of single numbers,

$$\frac{(a+b)^m}{1\times 2\times 3\ldots\times m}=\frac{a^m}{1\times 2\times 3\ldots\times m}+\frac{a^{m-1}}{1\times 2\times 3\times\ldots(m-1)}\,\frac{b^1}{1}+\frac{a^{m-2}}{1\times 2\times 3\times\ldots(m-2)}\,\frac{b^2}{1\times 2}+\ldots$$

$$+\frac{a^1}{1}\,\frac{b^{m-1}}{1\times 2\times 3\times\ldots(m-1)}+\frac{b^m}{1\times 2\times 3\times\ldots m}; \qquad (68.)$$

and similarly in the theory of number-couples,

$$\frac{\{(a_1, a_2)+(b_1, b_2)\}^m}{1\times 2\times 3\times\ldots m}=\frac{(a_1, a_2)^m}{1\times 2\times 3\times\ldots m}+\frac{(a_1, a_2)^{m-1}}{1\times 2\times 3\times\ldots(m-1)}\,\frac{(b_1, b_2)^1}{1}$$

$$+\frac{(a_1, a_2)^{m-2}}{1\times 2\times 3\times\ldots(m-2)}\,\frac{(b_1, b_2)^2}{1\times 2}+\ldots$$

$$+\frac{(a_1, a_2)^1}{1}\,\frac{(b_1, b_2)^{m-1}}{1\times 2\times 3\times\ldots(m-1)}+\frac{(b_1, b_2)^m}{1\times 2\times 3\times\ldots m}; \qquad (69.)$$

m being in both these formula a positive whole number, but $a\ b\ a_1\ a_2\ b_1\ b_2$ being any numbers whatever. The latter formula, which includes the former, may easily be proved by considering the product of m unequal factor sums,

$$(a_1, a_2)+(b_1^{(1)}, b_2^{(1)}),\ (a_1, a_2)+(b_1^{(2)}, b_2^{(2)}),\ldots\ (a_1, a_2)+(b_1^{(m)}, b_2^{(m)}); \qquad (70.)$$

for, in this product, when developed by the rules of multiplication, the power $(a_1; a_2)^{m-n}$ is multiplied by the sum of all the products of n factor couples such as $(b_1^{(1)}, b_2^{(1)})$ $(b_1^{(2)}, b_2^{(2)})\dots(b_1^{(n)}, b_2^{(n)})$; and the number of such products is the number of combinations of m things, taken n by n, that is,

$$\frac{1 \times 2 \times 3 \times \dots \times m}{1 \times 2 \times 3 \times \dots (m-n) \times 1 \times 2 \times 3 \times \dots n}, \qquad (71.)$$

while these products themselves become each $=(b_1, b_2)^n$, when we return to the case of equal factors.

The formula (69.) enables us to determine separately the primary and secondary numbers of the power or couple $(a_1, a_2)^m$, by treating the base (a_1, a_2) as the sum of a pure primary couple $(a_1, 0)$ and a pure secondary $(0, a_2)$, and by observing that the powering of these latter number-couples may be performed by multiplying the powers of the numbers a_1 a_2 by the powers of the primary and secondary units, (1, 0) and (0, 1) ; for, whatever whole number i may be,

$$\left.\begin{array}{l} (a_1, 0)^i = a_1^i\,(1, 0)^i, \\ (0, a_2)^i = a_2^i\,(0, 1)^i. \end{array}\right\} \qquad (72.)$$

We have also the following expressions for the powers of these two units,

$$\left.\begin{array}{l} (1,\ 0)^i = (1,\ 0), \\ (0,\ 1)^{4k-3} = (0,\ 1), \\ (0,\ 1)^{4k-2} = (-1,\ 0), \\ (0,\ 1)^{4k-1} = (0,\ -1), \\ (0,\ 1)^{4k} = (1,\ 0)\,; \end{array}\right\} \qquad (73.)$$

that is, the powers of the primary unit are all themselves equal to that primary unit ; but the first, second, third, and fourth powers of the secondary unit are respectively

$$(0, 1)\ (-1, 0),\ (0, -1),\ (1, 0),$$

and the higher powers are formed by merely repeating this period. In like manner we find that the equation

$$(a_1, a_2)^m = (b_1, b_2), \qquad (74.)$$

is equivalent to the two following,

$$\left.\begin{array}{l} b_1 = a_1^m - \frac{m\,(m-1)}{1 \times 2}\,a_1^{m-2}\,a_2^2 + \frac{m\,(m-1)\,(m-2)\,(m-3)}{1 \times 2 \times 3 \times 4}\,a_1^{m-4}a_2^4 - \&c. \\ b_2 = m\,a_1^{m-1}\,a_2 - \frac{m\,(m-1)\,(m-2)}{1 \times 2 \times 3}\,a_1^{m-3}a_2^3 + \&c. \end{array}\right\} \qquad (75.)$$

For example, the square and cube of a couple, that is, the second and third positive powers of it, may be developed thus,

$$(a_1, a_2)^2 = \{(a_1, 0) + (0, a_2)\}^2 = (a_1^2 - a_2^2, 2\,a_1\,a_2), \qquad (76.)$$

and

$$(a_1, a_2)^3 = \{(a_1, 0) + (0, a_2)\}^3 = (a_1^3 - 3\, a_1\, a^2_2,\ 3\, a_1^{\,2}\, a_2 - a_2^3). \qquad (77.)$$

9. In general, if

$$(a_1,\ a_2)\ (a'_1,\ a'_2) = (a''_1,\ a''_2), \qquad (78.)$$

then, by the theorem or rule of multiplication (54.)

$$a''_1 = a_1\, a'_2 - a_2\, a'_2,\ \ a''_2 = a_2\, a'_1 + a_1\, a'_2, \qquad (79.)$$

and therefore

$$a''^2_1 + a''^2_2 = (a_1^2 + a_2^2)\ (a'^2_1 + a'^2_2)\,; \qquad (80.)$$

and in like manner it may be proved that

$$\left.\begin{array}{l} \text{if}\ \ (a_1,\ a_2)\ (a'_1,\ a'_2)\ (a''_1,\ a''_2) = (a'''_1,\ a'''_2), \\ \text{then}\ (a'''^2_1 + a'''^2_2) = (a_1^2 + a_2^2)\ (a'^2_1 + a'^2_2)\ (a''^2_1 + a''^2_2), \end{array}\right\} \qquad (81.)$$

and so on, for any number m of factors. Hence, in particular, when all these m factors are equal, so that the product becomes a power, and the equation (74.) is satisfied, the two numbers $b_1\ b_2$ of the *power-couple* must be connected with the two numbers $a_1\ a_2$ of the *base-couple* by the relation

$$b_1^2 + b_2^2 = (a_1^2 + a_2^2)^m. \qquad (82.)$$

For example, in the cases of the square and cube, this relation holds good under the forms

$$(a_1^2 - a^2)^2 + (2\, a_1\, a_2)^2 = (a_1^2 + a_2^2)^2, \qquad (83.)$$

and

$$(a_1^3 - 3\, a_1\, a_2^2)^2 + (3\, a_1^2\, a_2 - a_2^3)^2 = (a_1^2 + a_2^2)^3. \qquad (84.)$$

The relation (82.) is true even for powers with contra-positive exponents $-m$, that is,

$$b_1^2 + b_2^2 = (a_1^2 + a_2^2)^{-m}\ \text{if}\ (b_1,\ b_2) = (a_1,\ a_2)^{-m}\,; \qquad (85.)$$

for in general

$$\left.\begin{array}{l} \text{if}\ \ (b_1,\ b_2) = \dfrac{(a_1, a_2)\ (a'_1, a_2)\ (a''_1, a''_2)\ \ldots}{(c_1,\ c_2)\ (c'_1,\ c'_2)\ (c''_1,\ c''_2)\ \ldots} \\ \text{then}\ \ (b_1^2 + b_2^2) = \dfrac{(a_1^2 + a_2^2)\ (a'^2_1 + a'^2_2)\ (a''^2_1 + a''^2_2)\ldots}{(c_1^2 + c_1^2)\ (c'^2_1 + c'^2_2)\ (c''^2_1 + c''^2_2)\ldots} \end{array}\right\} \qquad (86.)$$

On a particular Class of Exponential and Logarithmic Function-Couples, connected with a particular Series of Integer Powers of Number-Couples.

10. The theorem (69.) shows, that if we employ the symbols $\mathrm{F}_m(a_1, a_2)$ and $\mathrm{F}_m(b_1, b_2)$ to denote concisely two number-couples, which depend in the following way on the couples (a_1, a_2) and (b_1, b_2),

$$\mathrm{F}_m(a_1, a_2) = (1, 0) + \frac{(a_1, a_2)^1}{1} + \frac{(a_1, a_2)^2}{1 \times 2} + \ldots + \frac{(a_1, a_2)^m}{1 \times 2 \times 3 \times \ldots m}, \qquad (87.)$$

$$\mathrm{F}_m(b_1, b_2) = (1, 0) + \frac{(b_1, b_2)^1}{1} + \frac{(b_1, b_2)^2}{1 \times 2} + \ldots + \frac{(b_1, b_2)^m}{1 \times 2 \times 3 \times \ldots m}, \qquad (88.)$$

and if we denote in like manner by the symbol

$$\mathrm{F}_m\left((a_1, a_2) + (b_1, b_2)\right) = \mathrm{F}_m(a_1 + b_1,\ a_2 + b_2) \qquad (89.)$$

the couple which depends in the same way on the sum $(a_1, a_1) + (b_1, b_2)$, or on the couple $(a_1 + b_1,\ a_2 + b_2)$, and develope by the rule (69.) the powers of this latter sum, we shall have the relation

$$\begin{aligned}
&\{\mathrm{F}_m(a_1, a_2) \times \mathrm{F}_m(b_1, b_2)\} - \mathrm{F}_m\left((a_1, a_2) + (b_1, b_2)\right) = \\
&\frac{(a_1, a_2)^m}{1 \times 2 \times 3 \times \ldots m}\left\{\frac{(b_1, b_2)^1}{1} + \frac{(b_1, b_2)^2}{1 \times 2} + \ldots + \frac{(b_1, b_2)^m}{1 \times 2 \times 3 \times \ldots m}\right\} \\
&+ \frac{(a_2, a_2)^{m-1}}{1 \times 2 \times 3 \times \ldots (m-1)}\left\{\frac{(b_1, b_2)^2}{1 \times 2} + \ldots + \frac{(b_1, b_2)^m}{1 \times 2 \times 3 \times \ldots m}\right\} \\
&+ \ldots\ldots \\
&+ \frac{(a_1, a_2)^1}{1}\ \frac{(b_1, b_2)^m}{1 \times 2 \times 3 \times \ldots m}. \qquad (90.)
\end{aligned}$$

This expression may be farther developed, by the rule for the multiplication of a sum, into the sum of several terms or couples, (c_1, c_2), of which the number is

$$1 + 2 + 3 + \ldots + m = \frac{m\,(m+1)}{2}, \qquad (91.)$$

and which are of the form

$$(c_1, c_2) = \frac{(a_1, a_2)^i}{1 \times 2 \times 3 \times \ldots i} \times \frac{(b_1, b_2)^k}{1 \times 2 \times 3 \times \ldots k}, \qquad (92.)$$

i and k being positive integers, such that

$$i \not> m, \ k \not> m, \ i + k > m; \qquad (93.)$$

and if we put for abridgment

$$\sqrt{a_1{}^2 + a_2{}^2} = \alpha, \ \ \sqrt{b_1{}^2 + b_2{}^2} = \beta, \qquad (94.)$$

and

$$\gamma = \frac{\alpha^i \, \beta^k}{1 \times 2 \times 3 \times \ldots i \times 1 \times 2 \times 3 \times \ldots k}, \qquad (95.)$$

we shall have, by principles lately explained,

$$\sqrt{c_1{}^2 + c_2{}^2} = \gamma, \qquad (96.)$$

and therefore

$$c_1 \not> +\gamma, \ c_1 \not< -\gamma, \ c_2 \not> +\gamma, \ c_2 \not< -\gamma: \qquad (97.)$$

if then the entire sum (90.) of all these couples (c_1, c_2) be put under the form

$$\Sigma\,(c_1, c_2) = (\Sigma\, c_1, \Sigma\, c_2), \qquad (98.)$$

the letter Σ being used as a mark of summation, we shall have the corresponding limitations

$$\left.\begin{array}{l} \Sigma\, c_1 \not> \Sigma\,\gamma, \ \Sigma\, c_1 \not< -\Sigma\,\gamma, \\ \Sigma\, c_2 \not> \Sigma\,\gamma, \ \Sigma\, c_2 \not< -\Sigma\,\gamma, \end{array}\right\} \qquad (99.)$$

$\Sigma\,\gamma$ being the positive sum of the $\frac{m\,(m+1)}{2}$ such terms as that marked (95.). This latter sum depends on the positive whole number m, and on the positive numbers α, β; but whatever these two latter numbers may be, it is easy to show that by taking the former number sufficiently great, we can make the positive sum $\Sigma\,\gamma$ become smaller, that is nearer to 0, than any positive number δ previously assigned, however small that number δ may be. For if we use the symbols $\mathrm{F}_m\,(\alpha)$, $\mathrm{F}_m\,(\beta)$, $\mathrm{F}_m\,(\alpha + \beta)$, to denote positive numbers connected with the positive numbers α, β, $\alpha + \beta$, by relations analogous to those marked (87.) and (88.), so that

$$\mathrm{F}_m\,(\alpha) = 1 + \frac{\alpha}{1} + \frac{\alpha^2}{1 \times 2} + \ldots + \frac{\alpha^m}{1 \times 2 \times 3 \times \ldots m}, \qquad (100.)$$

it is easy to prove, by (68.), that the product $\mathrm{F}_m\,(\alpha) \times \mathrm{F}_m\,(\beta)$ exceeds the number

$\mathrm{F}_m(\alpha+\beta)$ by $\Sigma\gamma$, but falls short of the number $\mathrm{F}_{2m}(\alpha+\beta)$, that is of the following number

$$\mathrm{F}_{2m}(\alpha+\beta)=1+\frac{(\alpha+\beta)^1}{1}+\frac{(\alpha+\beta)^2}{1\times 2}+\ldots+\frac{(\alpha+\beta)^{2m}}{1\times 2\times 3\times\ldots\times 2m}; \qquad (101.)$$

so that

$$\Sigma\gamma=\left(\mathrm{F}_m(\alpha)\times\mathrm{F}_m(\beta)\right)-\mathrm{F}_m(\alpha+\beta), \qquad (102.)$$

and

$$\Sigma\gamma<\mathrm{F}_{2m}(\alpha+\beta)-\mathrm{F}_m(\alpha+\beta): \qquad (103.)$$

if then we choose a positive integer n, so as to satisfy the condition

$$n+1>2(\alpha+\beta), \text{ that is } \frac{\alpha+\beta}{n\times 1}<\tfrac{1}{2}, \qquad (104.)$$

and take $m>n$, we shall have

$$\frac{(\alpha+\beta)^m}{1\times 2\times 3\times\ldots m}<\frac{1}{2^{m-n}}\,\frac{(\alpha+\beta)^n}{1\times 2\times 3\times\ldots n}, \text{ and therefore } <\delta, \qquad (105.)$$

however small the positive number δ may be, and however large $\alpha+\beta$ may be, if we take m large enough; but also

$$\mathrm{F}_{2m}(\alpha+\beta)-\mathrm{F}_m(\alpha+\beta)=\frac{(\alpha+\beta)^m\eta}{1\times 2\times 3\times\ldots m} \text{ and therefore } <\delta\times\eta, \qquad (106.)$$

in which

$$\eta=\frac{\alpha+\beta}{m\times 1}+\frac{(\alpha+\beta)^2}{(m+1)(m+2)}+\ldots+\frac{(\alpha+\beta)^m}{(m+1)(m+2)\times\ldots(2m)}, \qquad (107.)$$

and, therefore,

$$\eta<1, \qquad (108.)$$

because

$$\frac{\alpha+\beta}{m+1}<\tfrac{1}{2}, \quad \frac{(\alpha+\beta)^2}{(m+1)\ (m+2)}<\frac{1}{2^2}, \ldots \frac{(\alpha+\beta)^m}{(m+1)\ (m+2)\times\ldots(2\,m)}<\frac{1}{2^m}; \qquad (109.)$$

therefore, combining the inequalities (103.) (106.) (108.), we find finally

$$\Sigma\gamma<\delta. \qquad (110.)$$

And hence, by (99.), the two sums $\Sigma\,c_1$, $\Sigma\,c_2$, may both be made to approach as near as we desire to 0, by taking m sufficiently large; so that, in the notation of limits already employed,

$$\underline{\mathrm{L}}\ \Sigma\gamma=0, \quad \underline{\mathrm{L}}\ \Sigma\,c_1=0, \quad \underline{\mathrm{L}}\ \Sigma\,c_2=0, \qquad (111.)$$

and, therefore,

$$\mathrm{L}\left\{ \mathrm{F}_m(a)\ \mathrm{F}_m(\beta) - \mathrm{F}_m(a+\beta) \right\} \doteq 0, \qquad (112.)$$

$$\underline{\mathrm{L}}\left\{ \mathrm{F}_m(a_1, a_2)\ \mathrm{F}_m(b_1, b_2) - \mathrm{F}_m\big((a_1, a_2) + (b_1, b_2)\big) \right\} = (0, 0). \qquad (113.)$$

In the foregoing investigation, a and β denoted positive numbers; but the theorem (113.) shows that the formula (112.) holds good, whatever numbers may be denoted by a and β, if we still interpret the symbol $\mathrm{F}_m(a)$ by the rule (100.).

11. If a still retain the signification (94.), it results, from the foregoing reasonings, that the primary and secondary numbers of the couple

$$\mathrm{F}_{m+m'}(a_1, a_2) - \mathrm{F}_m(a_1, a_2) \qquad (114.)$$

are each

$$\ngtr \mathrm{F}_{m+m'}(a) - \mathrm{F}_m(a), \text{ and } \nless \mathrm{F}_m(a) - \mathrm{F}_{m+m'}(a); \qquad (115.)$$

and, therefore, may each be made nearer to 0 (on the positive or on the contra-positive side) than any proposed positive number δ by choosing m large enough, however large m' and a may be, and however small δ may be: because in the expression

$$\mathrm{F}_{m+m'}(a) - \mathrm{F}_m(a) \doteq \frac{a^m}{1 \times 2 \times 3 \times \dots m} \left\{ \frac{a}{m+1} + \frac{a^2}{(m+1)\ (m+2)} + \dots + \frac{a^{m'}}{(m+1)\dots(m+m')} \right\} \qquad (116.)$$

the positive factor $\frac{a^m}{1 \times 2 \times 3 \times \dots m}$ may be made $< \delta$, that is, as near as we please to 0, and also the other factor, as being $< \frac{1}{n} + \frac{1}{n^2} + \dots + \frac{1}{n^{m'}}$, and therefore $< \frac{1}{n-1}$, if $m + 1 > n\, a$. Pursuing this train of reasoning, we find that as m becomes greater and greater without end, the couple $\mathrm{F}_m(a_1, a_2)$ tends to a determinate *limit-couple*, which depends on the couple (a_1, a_2), and may be denoted by the symbol $\mathrm{F}_\infty(a_1, a_2)$, or simply $\mathrm{F}(a_1, a_2)$,

$$\mathrm{F}(a_1, a_2) = \mathrm{F}_\infty(a_1, a_2) = \underline{\mathrm{L}}\ \mathrm{F}_m(a_1, a_2); \qquad (117.)$$

and similarly, that for any determinate number a, whether positive or not, the number $\mathrm{F}_m(a)$ tends to a determinate *limit-number*, which depends on the number a, and may be denoted thus,

$$\mathrm{F}(a) = \mathrm{F}_\infty(a) = \underline{\mathrm{L}}\ \mathrm{F}_m(a). \qquad (118.)$$

It is easy also to prove, by (112.), that this *function*, or *dependent number*, $\mathrm{F}(a)$, must always satisfy the conditions

$$\mathrm{F}(a) \times \mathrm{F}(\beta) = \mathrm{F}(a+\beta), \qquad (119.)$$

and that it increases constantly and continuously from positive states indefinitely near to 0 to positive states indefinitely far from 0, while α increases or advances constantly, and continuously, and indefinitely in the progression from contra-positive to positive; so that, for every positive number β, there is some determined number α which satisfies the condition

$$\beta = \mathrm{F}\ (\alpha), \qquad (120.)$$

and which may be thus denoted,

$$\alpha = \mathrm{F}^{-1}\ (\beta). \qquad (121.)$$

It may also be easily proved that we have always the relations,

$$\mathrm{F}\ (\alpha) = \underset{\circ}{e}^{\alpha},\ \ \mathrm{F}^{-1}(\beta) = \log_{e}.\ \beta, \qquad (122.)$$

if we put, for abridgement,

$$\mathrm{F}\ (1) = e, \qquad (123.)$$

and employ the notation of powers and logarithms explained in the Preliminary Essay. A power $\underset{\circ}{b}^{\alpha}$, when considered as depending on its exponent, is called an *exponential function* thereof; its most general and essential properties are those expressed by the formulæ,

$$\underset{\circ}{b}^{\alpha} \times \underset{\circ}{b}^{\beta} = \underset{\circ}{b}^{\alpha+\beta},\ \ \underset{\circ}{b}^{1} = b, \qquad (124.)$$

of which the first is independent of the base b, while the second specifies that base; and since, by (113.), the function-couple $\mathrm{F}\ (a_1, a_2)$ satisfies the analogous condition,

$$\mathrm{F}\ (a_1, a_2) \times \mathrm{F}\ (b_1, b_2) = \mathrm{F}\,\big((a_1, a_2) + (b_1, b_2)\big) = \mathrm{F}\ (a_1 + b_1,\ a_2 + b_2), \qquad (125.)$$

(whatever numbers $a_1\ a_2\ b_1\ b_2$ may be,) we may say by analogy that this function-couple $\mathrm{F}\ (a_1, a_2)$ is an *exponential function-couple*, and that its *base-couple* is

$$\mathrm{F}\ (1, 0) = (e, 0): \qquad (126.)$$

and because the exponent α of a power $\underset{\circ}{b}^{\alpha}$, when considered as depending on that power, is called a *logarithmic function* thereof, we may say by analogy that the couple (a_1, a_2) is a *logarithmic function*, or *function-couple*, of the couple $\mathrm{F}\ (a_1, a_2)$, and may denote it thus,

$$(a_1, a_2) = \mathrm{F}^{-1}(b_1, b_2), \ \text{ if } \ (b_1, b_2) = \mathrm{F}\ (a_1, a_2). \qquad (127.)$$

In general, if we can discover any law of dependence of one couple $\Phi\,(a_1, a_2)$, upon another (a_1, a_2), such that for all values of the numbers $a_1\ a_2\ b_1\ b_2$ the condition

$$\Phi\,(a_1, a_2)\ \Phi\,(b_1, b_2) = \Phi\,(a_1 + a_2,\ b_1 + b_2) \qquad (128.)$$

is satisfied, then, whether this function-couple $\Phi\,(a_1, a_2)$ be or be not coincident with the particular function-couple $\mathrm{F}\,(a_1, a_2)$, we may call it (by the same analogy of definition) an *exponential function-couple*, calling the particular couple $\Phi\,(1, 0)$ its *base*, or *base-couple;* and may call the couple (a_1, a_2), when considered as depending inversely on $\Phi\,(a_1, a_2)$, a *logarithmic function*, or *function-couple*, which we may thus denote,

$$(a_1, a_2) = \Phi^{-1}(b_1, b_2), \ \text{ if } \ (b_1, b_2) = \Phi\,(a_1, a_2). \qquad (129.)$$

12. We have shown that the particular exponential function-couple $(b_1, b_2) = \mathrm{F}\,(a_1, a_2)$ is always possible and determinate, whatever determinate couple (a_1, a_2) may be; let us now consider whether, inversely, the particular logarithmic function-couple $(a_1, a_2) = \mathrm{F}^{-1}\,(b_1, b_2)$ is always possible and determinate, for every determined couple (b_1, b_2). By the exponential properties of the function F, we have

$$\begin{aligned}(b_1, b_2) &= \mathrm{F}\,(a_1, a_2) = \mathrm{F}\,(a_1, 0)\ \mathrm{F}\,(0, a_2) = \mathrm{F}\,(a_1)\ \mathrm{F}\,(0, a_2)\\ &= (\underset{\circ}{e}^{a_1} \cos a_2,\ \underset{\circ}{e}^{a_1}\ \sin a_2),\end{aligned} \qquad (130.)$$

if we define the functions $\cos a$ and $\sin a$, or more fully the *cosine* and *sine* of any number a, to be the primary and secondary numbers of the couple $\mathrm{F}\,(0, a)$, or the numbers which satisfy the *couple-equation*,

$$\mathrm{F}\,(0, a) = (\cos a,\ \sin a). \qquad (131.)$$

From this definition, or from these two others which it includes, namely from the following expressions of the functions *cosine* and *sine* as limits of the sums of series, which are already familiar to mathematicians,

$$\left.\begin{aligned}\cos a &= 1 - \frac{a^2}{1\times2} + \frac{a^4}{1\times2\times3\times4} - \&c.\\ \sin a &= a - \frac{a^3}{1\times2\times3} + \frac{a^5}{1\times2\times3\times4\times5} - \&c.\end{aligned}\right\} \qquad (132.)$$

it is possible to deduce all the other known properties of these two functions; and especially that they are *periodical functions*, in such a manner that while the variable number a increases constantly and continuously from 0 to a certain constant positive number $\frac{\pi}{2}$, (π being a certain number between 3 and 4,) the function $\sin a$ increases with it (constantly and continuously) from 0 to 1, but $\cos a$ decreases (constantly and continuously) from 1 to 0; while a continues to increase from $\frac{\pi}{2}$ to π, $\sin a$ decreases from 1 to 0, and $\cos a$ from 0 to -1; while a increases from π to $\frac{3\pi}{2}$, $\sin a$ decreases from 0 to -1, but $\cos a$ increases from -1 to 0; while a still increases from $\frac{3\pi}{2}$ to 2π, $\sin a$ increases from -1 to 0, and $\cos a$ from 0 to 1, the sum of the squares of the cosine and sine remaining always $= 1$; and that then the same changes recur in the same order, having also occurred before for contra-positive values of a, according to this *law of periodicity*, that

$$\cos(a \pm 2\,i\,\pi) = \cos a, \quad \sin(a \pm 2\,i\,\pi) = \sin a, \qquad (133.)$$

i denoting here (as elsewhere in the present paper) any positive whole number. But because the proof of these well known properties may be deduced from the equations (132.), without any special reference to the theory of couples, it is not necessary, and it might not be proper, to dwell upon it here.

It is, however, important to observe here, that by these properties we can always find (or conceive found) an indefinite variety of numbers a, differing from each other by multiples of the constant number 2π, and yet each having its cosine equal to any one proposed number β_1, and its sine equal to any other proposed number β_2, provided that the sum of the squares of these two proposed numbers β_1, β_2, is $= 1$; and reciprocally, that if two different numbers a both satisfy the conditions

$$\cos a = \beta_1, \quad \sin a = \beta_2, \qquad (134.)$$

β_1 and β_2 being two given numbers, such that $\beta_1^2 + \beta_2^2 = 1$, then the difference of these two numbers a is necessarily a multiple of 2π. Among all these numbers a, there will always be one which will satisfy these other conditions

$$a > -\pi, \quad a \ngtr \pi, \qquad (135.)$$

and this particular number a may be called the *principal solution* of the equations (134.), because it is always nearer to 0 than any other number a which satisfies the same equations, except in the particular case when $\beta_1 = -1$, $\beta_2 = 0$; and because, in this particular case, though the two numbers π and $-\pi$ are equally near to 0, and both satisfy the equations (134.), yet still the principal solution π, assigned by the conditions (135.), is simpler than the other solution $-\pi$, which is rejected by those last conditions. It is therefore always possible to find not only one, but infinitely many number-couples (a_1, a_2), differing from each other by multiples of the constant couple (0, 2π), but satisfying each the equation (130.), and therefore each entitled to be represented by, or included in the meaning of, the general symbol $\mathrm{F}^{-1}(b_1, b_2)$, whatever proposed effective couple (b_1, b_2) may be. For we have only to satisfy, by (130.), the two separate equations

$$\underset{\circ}{e}^{\,a_1} \cos a_2 = b_1,\ \underset{\circ}{e}^{\,a_1} \sin a_2 = b_2; \qquad (136.)$$

which are equivalent to the three following,

$$\underset{\circ}{e}^{\,a_1} = \sqrt{b_1^2 + b_2^2}, \qquad (137.)$$

and

$$\cos a_2 = \frac{b_1}{\sqrt{b_1^2 + b_2^2}},\ \sin a_2 = \frac{b_2}{\sqrt{b_1^2 + b_2^2}}; \qquad (138.)$$

and if α be the *principal solution* of these two last equations, we shall have as their most general solution

$$a_2 = \alpha + 2\,\omega\,\pi, \qquad (139.)$$

while the formula (137.) gives

$$a_1 = \log_e . \sqrt{b_1^2 + b_2^2} : \qquad (140.)$$

the couple (a_1, a_2) admits therefore of all the following values, consistently with the conditions (130.) or (136.),

$$(a_1, a_2) = \mathrm{F}^{-1}(b_1, b_2) = (\log_e . \sqrt{b_1^2 + b_2^2},\ \alpha + 2\,\omega\,\pi), \qquad (141.)$$

in which ω is any whole number, and α is a number $> -\pi$, but not $> \pi$, which has its cosine and sine respectively equal to the proposed numbers b_1, b_2, divided each by the square-root of the sum of their squares. To specify any one value of (a_1, a_2), or $\mathrm{F}^{-1}(b_1, b_2)$, corresponding to any one particular whole number ω, we may use the symbol $\underset{\omega}{\mathrm{F}}^{-1}(b_1, b_2)$; and then the symbol $\underset{\circ}{\mathrm{F}}^{-1}(b_1, b_2)$ will denote what may be called the *principal value* of the inverse or logarithmic function-couple $\mathrm{F}^{-1}(b_1, b_2)$, because it corresponds to the principal value of the number a_2, as determined by the conditions (138.).

On the Powering of any Number-Couple by any Single Number or Number-Couple.

13. Resuming now the problem of powering a number-couple by a number, we may employ this property of the exponential function F,

$$(\text{F}(a_1, a_2))^{\mu} = \text{F}(\mu a_1, \mu a_2), \qquad (142.)$$

μ being any whole number whether positive or contrapositive or null; which easily follows from (125.), and gives this expression for the μ'th power, or *power-couple*, of any effective number-couple,

$$(b_1, b_2)^{\mu} = \text{F}(\mu \, \text{F}^{-1}(b_1, b_2)). \qquad (143.)$$

Reciprocally if (a_1, a_2) be an mth root, or *root-couple*, of a proposed couple (b_1, b_2), so that the equation (74.) is satisfied, then

$$(a_1, a_2) = (b_1, b_2)^{\frac{1}{m}} = \text{F}\left(\frac{1}{m}\text{F}^{-1}(b_1, b_2)\right). \qquad (144.)$$

This last expression admits of many values, when the positive whole number m is > 1, on account of the indeterminateness of the inverse or logarithmic function F^{-1}; and to specify any one of these values of the root-couple, corresponding to any one value $\underset{\omega}{\text{F}}^{-1}$ of that inverse function, which value of the root we may call *the ωth value* of that root, we may employ the notation

$$(b_1 \underset{\omega}{,} b_2)^{\frac{1}{m}} = \text{F}\left(\frac{1}{m}\underset{\omega}{\text{F}}^{-1}(b_1, b_2)\right); \qquad (145.)$$

we may also call the particular value

$$(b_1 \underset{0}{,} b_2)^{\frac{1}{m}} = \text{F}\left(\frac{1}{m}\underset{0}{\text{F}}^{-1}(b_1, b_2)\right), \qquad (146.)$$

the *principal value* of the root-couple, or the *principal m'th root* of the couple (b_1, b_2). In this notation,

$$(1 \underset{\omega}{,} 0)^{\frac{1}{m}} = \text{F}\left(0, \frac{2\omega\pi}{m}\right), \qquad (147.)$$

$$(b_1 \underset{\omega}{,} b_2)^{\frac{1}{m}} = (b_1 \underset{0}{,} b_2)^{\frac{1}{m}} (1 \underset{\omega}{,} 0)^{\frac{1}{m}}; \qquad (148.)$$

so that generally, *the ω'th value of the m th root of any number-couple is equal to the principal value of that root multiplied by the ωth value of the m th root of the* *primary unit* (1, 0). The mth root of any couple has therefore *m distinct values,* and no more, because the mth root of the primary unit (1, 0) has m distinct values, and no more, since it may be thus expressed, by (147.) and (131.),

$$(1 \underset{\omega}{,} 0)^{\frac{1}{m}} = \left(\cos. \frac{2\,\omega\,\pi}{m} , \ \sin. \frac{2\,\omega\,\pi}{m} \right), \qquad (149.)$$

so that, by the law of periodicity (133.), for any different whole number ω',

$$(1 \underset{\omega'}{,} 0)^{\frac{1}{m}} = (1 \underset{\omega}{,} 0)^{\frac{1}{m}}, \qquad (150.)$$

and therefore generally,

$$(b_1 \underset{\omega'}{,} b_2) = (b_1 \underset{\omega}{,} b_2)^{\frac{1}{m}}, \qquad (151.)$$

if

$$\omega' = \omega \pm i\,m, \qquad (152.)$$

but not otherwise. For example, the cube-root of the primary unit (1, 0) has three distinct values, and no more, namely

$$(1 \underset{0}{,} 0)^{\frac{1}{3}} = (1, 0)\,;\ (1 \underset{1}{,} 0)^{\frac{1}{3}} = \left(-\frac{1}{2}, \ \frac{\surd 3}{2} \right); \ (1 \underset{2}{,} 0)^{\frac{1}{3}} = \left(-\frac{1}{2}, \ -\frac{\surd 3}{2} \right)\,; \qquad (153.)$$

so that each of these three couples, but no other, has its cube = (1, 0). Again the couple $(-1, 0)$ has two distinct square-roots, and no more, namely

$$(-1 \underset{0}{,} 0)^{\frac{1}{2}} = (0, 1)\,;\ (-1 \underset{1}{,} 0)^{\frac{1}{2}} = (0, \ -1). \qquad (154.)$$

In general we may agree to denote the *principal square-root* of a couple (b_1, b_2) by the symbol

$$\sqrt{(b_1, b_2)} = (b_1 \underset{0}{,} b_2)^{\frac{1}{2}}; \qquad (155.)$$

and then we shall have the particular equation

$$\sqrt{(-1, 0)} = (0, 1)\,; \qquad (156.)$$

which may, by the principle (61.), be concisely denoted as follows,

$$\sqrt{-1} = (0, 1). \qquad (157.)$$

In the THEORY OF SINGLE NUMBERS, the symbol $\sqrt{-1}$ is *absurd,* and denotes an IMPOSSIBLE EXTRACTION, or a merely IMAGINARY NUMBER; but in the THEORY OF COUPLES, the same symbol $\sqrt{-1}$ is *significant,* and denotes a POSSIBLE EXTRACTION, or a REAL COUPLE, namely (as we have just now seen) the *principal square-root of the couple* $(-1, 0)$. In the latter theory, therefore, though not in the former, this sign $\sqrt{-1}$ may properly be employed; and we may write, if we choose, for any couple (a_1, a_2) whatever,

$$(a_1, a_2) = a_1 + a_2 \sqrt{-1}, \qquad (158.)$$

interpreting the symbols a_1 and a_2, in the expression $a_1 + a_2 \sqrt{-1}$, as denoting the pure primary couples $(a_1, 0)$ $(a_2, 0)$, according to the law of mixture (61.) of numbers with number-couples, and interpreting the symbol $\sqrt{-1}$, in the same expression, as denoting the secondary unit or pure secondary couple $(0, 1)$, according to the formula (157.). However, the notation (a_1, a_2) appears to be sufficiently simple.

14. In like manner, if we write, by analogy to the notation of fractional powers of numbers,

$$(c_1, c_2) = (b_1, b_2)^{\frac{\nu}{\mu}}, \qquad (159.)$$

whenever the two couples (b_1, b_2) and (c_1, c_2) are both related as integer powers to one common base couple (a_1, a_2) as follows,

$$(b_1, b_2) = (a_1, a_2)^{\mu}, \quad (c_1, c_2) = (a_1, a_2)^{\nu}, \qquad (160.)$$

(μ and ν being any two whole numbers, of which μ at least is different from 0,) we can easily prove that this *fractional power-couple* (c_1, c_2), or this result of powering the couple (b_1, b_2) by the fractional number $\frac{\nu}{\mu}$, has in general many values, which are all expressed by the formula

$$(c_1, c_2) = (b_1, b_2)^{\frac{\nu}{\mu}} = \mathrm{F}\left(\frac{\nu}{\mu}\mathrm{F}^{-1}(b_1, b_2)\right), \qquad (161.)$$

and of which any one may be distinguished from the others by the notation

$$(b_1 \underset{\omega}{,} b_2)^{\frac{\nu}{\mu}} = \mathrm{F}\left(\frac{\nu}{\mu}\underset{\omega}{\mathrm{F}}{}^{-1}(b_1, b_2)\right). \qquad (162.)$$

We may call the couple thus denoted *the ω'th value of the fractional power,* and in particular we may call

$$(b_1 \underset{0}{,} b_2)^{\frac{\nu}{\mu}} = \mathrm{F}\left(\frac{\nu}{\mu}\underset{0}{\mathrm{F}}{}^{-1}(b_1, b_2)\right) \qquad (163.)$$

the *principal value.* The ω'th value may be formed from the principal value, by multiplying it by the ω'th value of the corresponding fractional power of the primary unit, that is, by the following couple,

$$\underset{\omega}{(1,\,0)}^{\frac{\nu}{\mu}} = \left(\cos\frac{2\,\omega\,\nu\,\pi}{\mu},\ \sin\frac{2\,\omega\,\nu\,\pi}{\mu}\right); \qquad (164.)$$

and therefore the number of distinct values of any fractional power of a couple, is equal to the number m of units which remain in the denominator, when the fraction $\frac{\nu}{\mu}$ has been reduced to its simplest possible expression, by the rejection of common factors.

15. Thus, the *powering of any couple* (b_1, b_2) *by any commensurable number* x may be effected by the formula,

$$(b_1,\,b_2)^x = \mathrm{F}\,(x\,\mathrm{F}^{-1}\,(b_1,\,b_2)); \qquad (165.)$$

or by these more specific expressions,

$$\begin{aligned}\underset{\omega}{(b_1,\,b_2)}^x &= \mathrm{F}\,(x\,\underset{\omega}{\mathrm{F}}^{-1}\,(b_1,\,b_2))\\ &= \underset{0}{(b_1,\,b_2)}^x\,\underset{\omega}{(1,\,0)}^x, \qquad (166.)\end{aligned}$$

in which

$$\underset{\omega}{(1,\,0)}^x = (\cos\overline{2\,\omega\,x\,\pi},\ \sin\overline{2\,\omega\,x\,\pi}): \qquad (167.)$$

and it is natural to extend the same formulæ by definition, for reasons of analogy and continuity, even to the case when the exponent or number x is *incommensurable,* in which latter case *the variety of values of the power is infinite, though no confusion can arise, if each be distinguished from the others by its specific ordinal number,* or *determining integer* ω.

And since the spirit of the present theory leads us to extend all operations with single numbers to operations with number-couples, we shall further define (being authorised by this analogy to do so) that *the powering of any one number-couple* (b_1, b_2) *by any other number-couple* (x_1, x_2) is the calculation of a third number-couple (c, c_2), such that

$$(c_1,\,c_2) = (b_1,\,b_2)^{(x_1,\,x_2)} = \mathrm{F}\,((x_1,\,x_2)\times\mathrm{F}^{-1}\,(b_1,\,b_2)); \qquad (168.)$$

or more specifically of any one of the infinitely many couples corresponding to the infinite variety of *specific ordinals* or *determining integers* ω, according to this formula,

$$\begin{aligned}\underset{\omega}{(b_1,\,b_2)}^{(x_1,\,x_2)} &= \mathrm{F}\,((x_1,\,x_2)\times\underset{\omega}{\mathrm{F}}^{-1}\,(b_1,\,b_2))\\ &= \underset{0}{(b_1,\,b_2)}^{(x_1,\,x_2)}\,\underset{\omega}{(1,\,0)}^{(x_1,\,x_2)}, \qquad (169.)\end{aligned}$$

in which the factor $\underset{0}{(b_1,\,b_2)}^{(x_1,\,x_2)}$ may be called the *principal value* of the general power-couple, and in which the other factor may be calculated by the following expression,

$$
\begin{aligned}
(1 \underset{\omega}{,} 0)^{(x_1,\, x_2)} &= \mathrm{F}\,((x_1, x_2) \times (0, 2\,\omega\,\pi)) \\
&= \mathrm{F}\,(-2\,\omega\,\pi\,x_2,\ 2\,\omega\,\pi\,x_1) \\
&= \underset{0}{e}^{\,-2\,\omega\,\pi\,x_2}(\cos 2\,\omega\,\pi\,x_1,\ \sin 2\,\omega\,\pi\,x_1). \quad (170.)
\end{aligned}
$$

For example,

$$(1 \underset{0}{,} 0)^{(x_1,\, x_2)} = (1, 0), \qquad (171.)$$

and

$$(e \underset{0}{,} 0)^{(x_1,\, x_2)} = \mathrm{F}\,(x_1, x_2)\,; \qquad (172.)$$

also

$$(e \underset{\omega}{,} 0)^{(x_1,\, x_2)} = \mathrm{F}\,((x_1, x_2) \times (1, 2\,\omega\,\pi)). \qquad (173.)$$

On Exponential and Logarithmic Function-Couples in general.

16. It is easy now to discover this general expression for an exponential function-couple:

$$\Phi\,(x_1, x_2) = \mathrm{F}\,((x_1, x_2) \times (a_1, a_2))\,; \qquad (174.)$$

in which (a_1, a_2) is any constant couple, independent of (x_1, x_2). This *general exponential function* Φ includes the particular function F, and satisfies (as it ought) the condition of the form (128.),

$$\Phi\,(x_1, x_2)\ \Phi\,(y_1, y_2) = \Phi\,(x_1 + y_1, x_2 + y_2)\,; \qquad (175.)$$

its *base,* or *base-couple,* which may be denoted for conciseness by (b_1, b_2), is, by the 11th article, the couple

$$(b_1, b_2) = \Phi\,(1, 0) = \mathrm{F}\,(a_1, a_2)\,; \qquad (176.)$$

and if we determine that integer number ω which satisfies the conditions

$$a_2 - 2\,\omega\,\pi > -\pi,\quad a_2 - 2\,\omega\,\pi \not> \pi, \qquad (177.)$$

we shall have the general transformation

$$\Phi\,(x_1, x_2) = (b_1 \underset{\omega}{,} b_2)^{(x_1,\, x_2)}. \qquad (178.)$$

And the general *inverse exponential* or *logarithmic function-couple,* which may, by (129.), be thus denoted,

$$(x_1, x_2) = \Phi^{-1}(y_1, y_2), \quad \text{if} \quad (y_1, y_2) = \Phi\,(x_1, x_2), \qquad (179.)$$

may also, by (174.) and (176.), be thus expressed:

$$\Phi^{-1}(y_1, y_2) = \frac{\mathrm{F}^{-1}(y_1, y_2)}{\mathrm{F}^{-1}(b_1, b_2)}; \qquad (180.)$$

it involves, therefore, *two arbitrary integer numbers*, when only the couple (y_1, y_2) and the base (b_1, b_2) are given, and it may be thus more fully written,

$$\underset{\omega}{\overset{\omega'}{\Phi}}{}^{-1}(y_1, y_2) = \underset{\omega}{\overset{\omega'}{\log}}{}_{(b_1, b_2)} \cdot (y_1, y_2) = \frac{\underset{\omega'}{\mathrm{F}}{}^{-1}(y_1, y_2)}{\underset{\omega}{\mathrm{F}}{}^{-1}(b_1, b_2)}. \qquad (181.)$$

For example, the general expression for the logarithms of the primary unit (1, 0) to the base (*e*, 0), is

$$\underset{\omega}{\overset{\omega'}{\log}}{}_{(e, 0)} \cdot (1, 0) = \frac{(0, 2\,\omega'\pi)}{(1, 2\,\omega\pi)} = \frac{(2\,\omega'\pi, 0)}{(2\,\omega\pi, -1)}, \qquad (182.)$$

or, if we choose to introduce the symbol $\sqrt{-1}$, *as explained in the 13th article,* that is, as denoting the couple (0, 1) according to the law of mixture of numbers with number-couples, then

$$\underset{\omega}{\overset{\omega'}{\log}}{}_{e} \cdot 1 = \frac{2\,\omega'\pi\sqrt{-1}}{1 + 2\,\omega\pi\sqrt{-1}} = \frac{2\,\omega'\pi}{2\,\omega\pi - \sqrt{-1}}. \qquad (183.)$$

In general,

$$\underset{\omega}{\overset{\omega'}{\log}}{}_{(b_1, b_2)} \cdot (y_1, y_2) = \frac{\underset{0}{\mathrm{F}}{}^{-1}(y_1, y_2) + (0, 2\,\omega'\pi)}{\underset{0}{\mathrm{F}}{}^{-1}(b_1, b_2) + (0, 2\,\omega\pi)}. \qquad (184.)$$

The integer number ω may be called the *first specific ordinal,* or simply the ORDER, and the other integer number ω' may be called the *second specific ordinal,* or simply the RANK, of the particular logarithmic function, or *logarithm-couple,* which is determined by these two integer numbers. This existence of *two arbitrary and independent integers in the general expression of a logarithm,* was discovered in the year 1826, by Mr. GRAVES, who published a Memoir upon the subject in the Philosophical Transactions for 1829, and has since made another communication upon the same subject to the British Association for the Advancement of Science, during the meeting of that Association at Edinburgh, in 1834: and it was he who proposed these names of *Orders and Ranks of Logarithms.* But because Mr. GRAVES employed, in his reasoning, the usual principles respecting *Imaginary Quantities*, and was content to prove the symbolical necessity without showing the interpretation, or inner meaning, of his formulæ, the present *Theory of Couples* is published to make manifest that hidden meaning: and to show, by this remarkable instance, that expressions which seem according to common views to be merely symbolical, and quite incapable of being interpreted, may pass into the world of thoughts, and acquire reality and significance, if Algebra be viewed as not a mere Art or Language, but as the Science of Pure Time. The author hopes to publish hereafter many other applications of this view; especially to Equations and Integrals, and to a Theory of Triplets and Sets of Moments, Steps, and Numbers, which includes this Theory of Couples.

THE END.

Researches on the Action of Ammonia upon the Chlorides and Oxides of Mercury.

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Read November 30th and December 28th 1835.

SECTION I.—*Of the Action of Ammonia upon the Bichloride of Mercury.*

THE action of ammonia upon corrosive sublimate is of so remarkable a nature, as to have attracted attention from a very early period. There have been, therefore, very many facts elicited; but the discordance in the quantitative results of the most eminent chemists who have applied themselves to its elucidation, renders an attentive and fundamental re-examination of the subject necessary before any conclusion can be justly drawn. In the elementary works on chemistry, a theory is given as an ascertained truth, which we shall find to be totally devoid of foundation; and almost every analyst that has written, has brought forward a different hypothesis of his own.

Ammonia may be made to act upon corrosive sublimate in either of two ways—first, by passing the gas directly over dry deuto-chloride; or, secondly, by adding solution of ammonia to that of corrosive sublimate. The reaction of the first case has only been examined by Grouvelle and Rose,*—that of the second has obtained an amount of study bestowed on few bodies. The memoirs of Fourcroy,† of Hennell,‡ Guibourt,§ Soubeiran,|| and Mitcherlich,¶ have shown the interest attached by chemists to the investigation of the nature of these bodies; but, by the absolute discordance of the results, and by the accidental misinterpretation of some of them, have demonstrated the existence of some latent error, and the necessity for new investigations, in which it should be sought for, detected, and avoided.

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* Poggendorff, Annalen der Physik, vol. xci.

† Journal de l'Ecole Polytechnique, vol. vi. p. 312.

‡ Quarterly Journal of Science, vol. xviii. p. 291.

§ Journal de Pharmacie, vol. vi.

|| Soubeiran, Journal de Pharmacie, vol. xii. 24; Annales de Chimie, vol. xxxvi. p. 220.

¶ Mitcherlich, Poggendorff, vol. lxxxv. p. 410.

The *white precipitate*, produced by adding water of ammonia to a solution of corrosive sublimate, is the substance with which we shall commence, as a knowledge of its history will be found to facilitate very much the study of those remaining.

§ 1.—*Of the White Precipitate of Mercury.*

It is well known that on adding water of ammonia to a solution of corrosive sublimate, there is obtained a milk-white precipitate, insoluble in water, and possessing properties which, as they are made the foundation of many experiments hereafter to be described, I shall briefly mention. This precipitate, when first produced, is milk-white, very bulky, depositing but slowly, and rather aluminous-looking. If very hot water be used in preparing it, or if it be washed very much, it loses its pure white colour, and acquires a yellow tinge; if boiled for a few minutes in the liquor, it is completely decomposed, and then results a canary-yellow powder, very heavy and granular. This white precipitate is perfectly insoluble, as such, in water. An occasional appearance of solution results from its being decomposed, and one or other of its elements entering into new combinations. When heated in a glass tube, sealed at one end, it is completely decomposed below a red heat; there is disengaged a mixture of ammonia and azote, some water, and calomel sublimes, generally darkened by some ammonia, from which, however, it can readily be freed.

White precipitate dissolves readily in nitric or in muriatic acids; when mixed with an alcali, as potash or soda, or with lime or baryta, there is ammonia disengaged, and the mass becomes yellowish, but the decomposition is never complete. The real nature of the products we shall hereafter study; but it may be remarked, that no excess of alcali can expel all the ammonia of the body.

If we add to white precipitate a solution of iodide of potassium, there is deposited a red powder, and much ammonia is disengaged. The powder is biniodide of mercury, and the liquor contains free potash; all the ammonia is liberated. Sulphuret of barium in solution produces a similar reaction—all the ammonia being disengaged, and all the quicksilver deposited as bisulphuret.

In order to obtain white precipitate of sufficient purity for examination, some precautions must be observed, the neglect of which has produced much of the confusion in results; different chemists having analyzed heterogeneous products. To a cold solution of bichloride of mercury, there is to be added water of ammonia in very slight excess; the whole is to be thrown on a filter, and as much as possible of the liquor allowed to come away before any washing is attempted. It may then be cautiously washed with as much distilled water as may suffice to remove the original liquor from the mass, but over excess avoided, as, even by cold water, some portions are decomposed, and the milk-white colour of the powder lost.

Having, by careful repetition of the above method, obtained a product such as I might consider pure, I entered upon its analysis. I think it advisable, however, before commencing the details of my own experiments, or noticing the results to which they have led, to bring forward a tabular statement of the analytical results of other writers, in order to show in what obscurity the subject was involved. The results are taken, divested of the theories to which they have respectively conducted their authors, and are exhibited in the real quantities of chlorine, quicksilver, and ammonia, which were obtained as the products of the experimental trials.

There were obtained from 100 parts :

Authors.	Mercury.	Chlorine.	Ammonia.
Fourcroy,*	74,1	13,2	6,03
Hennell,†	74,24	13,14	6,31
Mitcherlich,‡	76,37	13,82	7,1
Guibourt,§	78,31	13,31	4,45
Soubeiran,‖	82,1	7,8	5,3

A glance is sufficient to show the impossibility of deducing from such a chaos any general expression for the composition of white precipitate. In attempting to find some principle by which it could be set in order, I rested my probabilities of success on the number of trials I would make ; for, conscious that the chance of incorrectness in any one analysis should be still greater with me than with any one of those chemists whose ill success I have exhibited above, I trusted to diminishing the errors by taking averages of numerous results, and obtaining the results themselves by processes differing in principle, so as to render it very unlikely that any one error could pervade all. I shall therefore describe in order the results of each different method of analysis.

A.—When a solution of corrosive sublimate is precipitated by ammonia, the liquor contains no mercury, but much chlorine. All the mercury of the sublimate is contained in the white precipitate, and a portion of the chlorine is removed, and remains in the solution as sal-ammoniac. It is evident that on this principle may be founded a method of determining the amount of mercury and chlorine in white precipitate; and this was accordingly that first made use of.

One hundred grains of corrosive sublimate were dissolved in cold water, the solution decomposed by a slight excess of ammonia, and the precipitate thrown on a weighed filter, and washed with cold water. The precipitate was carefully dried and weighed ; the liquor added to the washings was acidulated by nitric acid, and precipitated by solu-

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* By result given by Leopold Gmelin, Handbuch.

† Quarterly Journal of Science.

‡ Poggendorff, vol. lxxxv.

§ Result given by Thenard, Traité Elementaire.

‖ Journal de Pharmacie, vol. xii.

tion of nitrate of silver; the chloride produced was collected on a weighed filter, and its quantity determined. In this way were obtained the results given in the following table:

No. of Experiments.	Precipitate.	Chlorine in liquor.
1	91,3	12,9
2	92,4	13,3
3	92,9	13,15
4	95,4	12,7
5	93,4	12,95
Mean	93,1	13,00

But 100 grains of sublimate contain,

Mercury	74,09
Chlorine	25,91

The quantity of chlorine contained in the liquor was, therefore, evidently one-half of that in the corrosive sublimate, and we have by this method, in 93,1 of white precipitate,

Mercury	74,09
Chlorine	12,91

Or in one hundred parts,

Mercury	79,57
Chlorine	13,87

B.—When white precipitate is heated, there is obtained, besides gaseous matter and watery vapour, the whole of the mercury and chlorine united as calomel. This fact, which confirms the relation between the quantities of mercury and chlorine found in the preceding results, affords a means of determining the actual amount, which was next put in practice. The operation was thus performed. A small tube retort was taken, and sometimes a straight tube having a strong bulb blown at one extremity, and it was accurately tared; there was next introduced a amount of white precipitate sufficient to about half-fill the bulb, and the whole weighed; the increase indicated the quantity of white precipitate employed. The bulb was now heated, and the tube itself so warmed, as perfectly to get rid of any water that might tend to deposit itself in the throat of the apparatus, but guarding against the loss of any calomel. When the latter has been completely sublimated, it is generally dark-coloured, owing to the contact of free ammonia; but, by allowing the tube to cool, and the atmospheric air to gain admission, and then again heating the calomel, it is finally obtained quite white. The apparatus was then again weighed; the loss from the gross weight gave the amount of volatile ingredients; the excess above tare gave the weight of the calomel, from which the quantities of quicksilver and chlorine may be calculated. The following table contains the results of this method.

No. of Ex.	Matter used.	Calomel.	Calomel per cent.
1	20,42	18,95	92,80
2	19,42	18,07	92,53
3	12,14	11,28	92,91
4	14,71	13,79	93,68

These results, which agree very closely with each other, give us as a mean from one hundred of white precipitate 92,98 of calomel, containing

Mercury	79,14.
Chlorine	13,84.

C.—To obtain a value for the quicksilver by reduction, the usual method was pursued. The white precipitate was dissolved in muriatic acid, and precipitated by solution of protochloride of tin. The quicksilver was dried in the cold, and there was obtained from one hundred grains of white precipitate, in one experiment, 77,3, and in a second 78,1 of mercury, giving a mean of 77,7 per cent.

D.—105,40 of white precipitate were dissolved in muriatic acid, and the liquor diluted with four times its volume of water. Sulphuretted hydrogen was then passed through until the resulting precipitate became perfectly black, and deposited readily. The whole was then thrown on a weighed filter, and washed with distilled water.

The washings and liquor were cautiously evaporated to dryness, and the residue weighed. There were thus obtained:

Filter and sulphuret	143,58 grs.	
Filter	48,35	
Bisulphuret of mercury		95,23
Consisting of Mercury	82,17	
Sulphur	13,06	
And Sal-ammoniac		23,59 grs.
Consisting of Muriatic acid	16,04	
Ammonia	7,55	
Or from 100 of white precipitate:		
Mercury	77,96	
Ammonia	7,16	

E.—For the determination of the ammonia of white precipitate, potash or lime cannot be employed, as only one-half of the ammonia is thus separated. In addition to the mode applied in (D), which is perhaps the best, the following means were had recourse to:

a. 100 grains of white precipitate were introduced into a flask, and a solution of sulphuret of barium poured on it; from the flask a bent tube conducted to a tall jar containing dilute muriatic acid. By the application of heat, the ammonia and much water were driven over; the liquor in the jar was then evaporated to dryness. There remained sal-ammoniac 21,57 grs. consisting of—

Muriatic acid	14,85
Ammonia	6,72

β. 100 grains of white precipitate, treated with iodide of potassium by the same method, gave sal-ammoniac 19,83, consisting of—

Muriatic acid	13,50
Ammonia	6,33

F.—In all theories of the composition of white precipitate hitherto advanced, oxygen is enumerated as a constituent to a very considerable amount, generally so much as to peroxidize the whole of the quicksilver. The results hitherto obtained in my experiments would not appear to leave room for so much oxygen, and I therefore endeavoured to obtain an estimate of the amount of that element by direct experiment. The principle of which I made use, is the following. When white precipitate is heated there are obtained ammonia and azote, water and calomel, but no free oxygen; therefore all the oxygen of the substance has formed water at the expense of the ammonia. I resolved upon obtaining this water, and determining from its weight how much oxygen the white precipitate contained.

A small retort was blown of strong glass, and of a capacity of from 0,2 to 0,3 of a cubic inch, the neck being about two inches long. To this was tightly connected a small tube, containing sometimes dry lime, and at others fused potash, and communicating by a narrow tube with the mercurial pneumatic trough. The retort was carefully weighed, and the white precipitate introduced; the whole then weighed, and thus the quantity of materiel determined; the desiccating tube was also carefully weighed; and the apparatus having been connected, heat was applied to the retort until the chlorine and mercury had all sublimed as calomel. The water was driven completely out of the neck of the retort, and passed into the tube with lime or potash, where it remained. The azote and ammonia were collected in a jar over the mercury.

After the operation, the retort was weighed: the residue was calomel; the loss, water, azote and ammonia: the desiccating tube was weighed; the increase gave the quantity of water formed. The mixture of gases was corrected for pressure and temperature, and analyzed by water.

Expt. 1st.—Retort and material	99,93
Retort	77,72
White precipitate	22,21
Tube with lime before	269,00
Tube with lime after	269,22
Water condensed	0,22

In this experiment, owing to an accident, the estimates for the calomel and gaseous matters were rendered unavailable.

Expt. 2d.—Retort and material	85,25
Retort	64,78
White precipitate	20,47

Retort and sublimate residue	-		84,22
Retort	-	-	64,78
Sublimate residue		19,44	
Tube with lime before		-	268,72
Tube with lime after		-	268,86.
Water condensed 0,14.			

The gaseous mixture, reduced to mean temperature and pressure, amounted to 4,24 cubic inches, of which 2,67 were absorbed by water. Of the remainder, 0,23 were the atmospheric air of the apparatus; there were obtained therefore—

	Cub. Inch.		
Ammonia	2,67	weighing	0,488 grs.
Azote	1,34		0,404

But the sublimed residue of calomel was dark-coloured from having absorbed ammonia; it must, therefore, be corrected by subtracting from it that quantity, and adding it to the ammonia actually obtained. Now it has been shown that 100 of white precipitate give 92,28 of calomel; therefore 20,47 should have given 19,135, which had absorbed 0,305 of ammonia.

Summing up these results, we obtain—

$$\left.\begin{array}{ll} \text{Calomel} & 19{,}135 \\ \text{Azote} & 0{,}404 \\ \text{Ammonia} & 0{,}793 \\ \text{Water} & 0{,}140 \end{array}\right\} 20{,}472$$

It is unnecessary to give all the details of other experiments similarly made; the results will suffice.

Expt. 3d gave for 12,14 white precipitate—

$$\left.\begin{array}{ll} \text{Calomel} & 11{,}280 \\ \text{Azote} & 0{,}310 \\ \text{Ammonia} & 0{,}470 \\ \text{Water} & 0{,}080 \end{array}\right\} 12{,}14$$

Experiment 4th furnished the calomel result, No. 2 of the table, page 426. But as some gas was lost in shifting a jar, the azote or ammonia could not be determined; the desiccating tube had not sensibly increased in weight.

It is not necessary to use the above results for the chlorine or mercury constituents, as there have been already gotten numbers by simpler processes, and consequently less exposed to error. The difference is, however, very trivial. The product of water gives the the following:—

Expt.	Material.	Water.	Water per cent.
1	22,21	0,22	0,990
2	20,47	0,14	0,684
3	12,14	0,08	0,658
4	19,42	0,00	0,000

Giving as the average of water per cent. 0,583; but of this a portion probably consists of moisture, hygrometric, or arising from imperfect desiccation of the precipitate.

To obtain the ammonia, we must convert the azote into ammonia, and add it to that actually gotten; we thus find—

Expt. 2d gives 1,282, or 6,26 per cent.
3d 0,845, or 6,96.
Giving an average of 6,61 per cent of ammonia.

Summing up all the analytical results for white precipitate, we have—

Process.	Mercury.	Chlorine.	Ammonia.	Water.
A	79,57	13,87		
B	79,14	13,84		
C	77,70			
D	77,96		7,16	
E			6,53	
F			6,61	0,583

And we obtain the final mean of

Mercury	78,60
Chlorine	13,85
Ammonia	6,77
Water	0,58
Loss	0,20
	100,00

The theory of white precipitate given by the majority of chemists supposes it to consist of an atom of peroxide of mercury, united to an atom of sal-ammoniac. This view, which is founded on the experiments of Fourcroy and Hennell, gives the following numerical results:—

1 atom	mercury	202,8	or	74,46
2	oxygen	16,0		5,88
1	muriatic acid	36,42		6,29
1	Ammonia	17,15		13,37
		272,37		100,00

The fallacy of this theoryis completely proved by the great difference in the determination of the mercury. The result of Fourcroy was obtained so long since, that it necessarily participates in the imperfections of the analytic methods of that time; and Hennell's results are evidently of a very inaccurate description. He states that he got a quantity of mercury very near the atomic proportion, but does not give the number actually gotten; hence there cannot be placed implicit confidence in the theory he supports. In addition, the existence of a quantity of oxygen, so large per cent. (5,88) in white precipitate, is completely impossible. My researches, directed expressly to that object, sufficiently disprove such an idea.

Another circumstance which demonstrates the incorrectness of the popular theory of this body, is the fact that 100 of sublimate yield only 93,1 of product; on Hennell's view, there should be 99,5 of white precipitate for

1 atom	of mercury	202,80	give	1 atom	mercury	202,80
2	of chlorine	70,84		2	oxygen	16,00
				1	muriatic acid	36,42
		273,64		1	ammonia	17,15
						272,37

The source of error in such analyses, evidently consisted in not having dried the white precipitate until it ceased to lose weight.

In the Memoir already quoted, George Mitcherlich adopts the hypothesis just described, and his doing so has created great confusion. He gives the name Chlor-Wasserstoff Säure to the hypothetic dry muriatic acid; and his formula $(\dot{N}H^3 + \ddot{M}) + \ddot{H}g$ is so constructed. This is shown by the numbers for muriatic acid and ammonia, 10,7 and 7,1, which, he states, form sal-ammoniac, (dry). In order to get his value for chlorine, given in the table of results, p. 425, I had to add to his muriatic acid, half the oxygen which he gives to the oxide of mercury. In fact, his analysis, correctly interpreted, overturns the very hypothesis which it has been supposed to be in accordance with, for—

		Mercury	76,37
Dry muriatic acid	10,7	Chlorine	13,82
Oxygen -	3,12		
		Ammonia	7,10
			97,29

leaves only a vacancy of 2,71 per cent for the oxygen to oxidize the whole of the quicksilver.

Not having access to Guibourt's Memoir, I can only speak of his opinions and results, by the references made to him by Thenard,* and other writers. He considers white precipitate to be composed of corrosive sublimate, peroxide of mercury and ammonia in proportions, giving the formula—

$$\left\{3\,(2\,Ch + Hg) + 2\,NH^3\right\} + \left\{3\,\ddot{H}g + 2\,NH^3\right\}$$

which gives as the per centage result—

Mercury	78,71	100,00
Chlorine	13,78	
Oxygen	3,10	
Ammonia	4,41	

---

* Traité Elémentaire de Chimie.

Here the mercury and chlorine agree pretty nearly with my determination, but the quantity of ammonia is only about one-half. I think it probable, that Guibourt determined the ammonia by means of potash, thinking the decomposition perfect, as it is generally described, and as I myself at first considered it to be. In order to show the probability of this being the source of Guibourt's error, I shall describe the following, one of the earliest experiments I made in the matter.

100 grains of white precipitate were mixed with a very strong solution of potash, in a flask from which a bent tube passed to a jar containing dilute muriatic acid. The flask was heated until most of the water in it had distilled over. The liquor in the jar was then evaporated to dryness, and gave 11,5 grs. of sal-ammoniac, consisting of

Muriatic acid	7,84
Ammonia	3,66

Now, supposing the loss to be oxygen, which was the method generally pursued in analyzing this substance hitherto, we should have the result—

$$\left.\begin{array}{lr} \text{Mercury} & 78{,}60 \\ \text{Chlorine} & 13{,}85 \\ \text{Ammonia} & 3{,}66 \\ \text{Oxygen} & 3{,}89 \end{array}\right\} 100{,}00$$

These results agree so closely as to point out the manner in which Guibourt's analysis, so correct in the chlorine and quicksilver constituents, became erroneous by reducing the ammonia to nearly one-half the actual amount.

I shall return to the causes of error in Soubeiran's analysis; at present it is not necessary to advert to them, as he evidently analyzed a body quite different from white precipitate.

The simplest view to take of the existence of the chlorine in this substance, is to suppose it united with half the mercury as corrosive sublimate; it is almost the only view possible. Then, in what state is the remainder of the mercury? We may suppose it peroxidized, and the oxide united with the ammonia giving the formula $(2\,Ch + Hg) + (\ddot{H}g + 2\,NH^3)$ and the following numerical arrangements :—

2	atoms mercury	405,60	or	77,00
2	oxygen	16,00		3,04
2	chlorine	70,84		13,45
2	ammonia	34,30		6,51
		526,74		100,00

This agrees closely with Mitcherlich's, and also with some of my own analyses. It differs, nevertheless, from the mean of my results in the quantities of mercury and chlorine, and particularly in the quantity of oxygen. This hypothesis supposes the existence of 3,04 per cent. of oxygen, a body of which I could not determine the existence as a constituent at all, and which the relative quantities of the other elements would appear to exclude. It is therefore necessary to examine whether any other method of arrangement is more suitable.

From Dumas' researches on oxamide, benzamide, &c. it follows, that by the action of ammonia on an oxide, there may be formed water and a compound of the body NH^2, with the basis of the oxide. If we consider this to have taken place in white precipitate, we should have the formula $(2\,Ch + Hg) + (2\,NH^2 + Hg)$, giving—

Hg	202,80	or	79,73
Ch	35,42		13,93
NH^2	16,15		6,34
	254,37		100,00

And this compound should give by decomposition 6,73 per cent. of ammonia.

I should not wish to adopt too positively the opinion that white precipitate is a compound of deuto-chloride and deutamide of mercury, although the non-existence by experiment of oxygen as a constituent, renders it extremely probable. In addition, the decomposition of white precipitate by iodide of potassium appears to afford a presumption that the mercury is not oxydized, as red oxide of mercury does not decompose iodide of potassium; on the other theory, the reaction is at once explained. Thus—

$$4\left\{I + K\right\} + \left\{(2\,Ch + Hg) + (2\,NH^2 + Hg)\right\} =$$

$$2(2\,I + Hg) + 2\,(Ch + K) + 2\,(NH^2 + K).$$

and $$2(NH^2 + K) + 2\,\dot{H} = 2\,\dot{K} + 2\,NH^3.$$

The question whether ammonia, in acting on metallic oxides, forms water and metallic amides, is one of the most interesting now beginning to be examined; but, notwithstanding the bearing which the results just described, have on the question, I do not wish to come to any positive conclusion, until a more extensive basis for induction is obtained. The atomic weight of mercury is so large, and preponderates so much on the other constituents, as to make small differences in their quantities fall within the limits of necessary error; and hence, I shall leave the two explanations until, by experiments on the compounds of a metal of a smaller combining number, I shall have an opportunity of seeing them diverge at a greater angle.

To conclude, the white precipitate yields, as the mean results of my analyses,

Mercury	78,60	100,00
Chlorine	13,85	
Ammonia	6,77	
Hygrometric water, loss, and oxygen	0,78	

and there are two formulæ which give results approaching closely to those, viz.—

1		2	
$(2Ch + Hg) + (2NH^2 + Hg)$		$(2Ch + Hg) + (2NH^3 + \ddot{Hg})$	
which gives		which gives	
Hg	79,73	Hg	77,00
Ch	13,93	Ch	13,45
NH^2	6,34	NH^3	6,51
		Ox	3,04
	100,00		

It is evident, that although neither is so far removed as to be beyond admissibility, the balance of probability is inclined much to the side of the former.

§ 2.—*Of the Powder formed by the Action of Water on White Precipitate.*

It is generally stated by chemical writers, that by the action of much boiling water, white precipitate is decomposed completely, red oxide of mercury being left behind. I never could succeed in effecting this; but the reaction that did take place appearing to me perfectly definite, and identical in its results at different times, I was induced to examine it in detail.

When white precipitate is boiled in water, it is changed into a heavy canary-yellow powder, subsiding rapidly, and very easily dried, when it appears granular. This powder is not quite insoluble in water; when heated, it gives out ammonia, azote, water, and there sublimes a mixture of calomel and metallic mercury: it dissolves readily in muriatic or nitric acids. Alcalies appear to have scarcely any action upon it, except slightly altering its colour; when digested with iodide of potassium, there is ammonia disengaged, and a brown powder formed. To this reaction I shall hereafter recur. In order to determine the composition of this yellow powder, the following experiments were made:—

A.—100 parts of corrosive sublimate were dissolved in water, and ammonia added in excess. The mass, in place of being filtered cold, was boiled until the light-white precipitate was changed into the clear yellow heavy powder; it was then filtered, and the quantity of product determined. The liquor and washings were acidulated by nitric acid, and precipitated by nitrate of silver, and the chloride abstracted from the sublimate thus determined; the liquor contained a very small trace of mercury. Several experiments were made on this plan, the result of which are exhibited in the following table:

100 parts of sublimate gave—

Expt.	Yellow powder.	Chlorine in liquor.
1	83,5	19,25
2	83,3	18,50
3	84,7	18,90
mean	83,83	18,89

Now 100 of sublimate, contain

Mercury	74,09
Chlorine	25,91

Hence we see that there have been abstracted from the sublimate three-fourths of its chlorine; the remaining fourth, and all the mercury, existing in the yellow powder. We have therefore in 83,83 parts of it—

Mercury	74,09
Chlorine	71,02

Or in one hundred parts,

Mercury	88,381
Chlorine	8,374

B.—When white precipitate already prepared is boiled with water, there is obtained a similar yellow powder, and the supernatant liquor is found to contain only sal-ammoniac. As we know, within very strict limits, the composition of the white powder, we can make use of this reaction to illustrate the nature of the yellow product:

100 parts of white precipitate were boiled with water until completely converted into the yellow powder; the liquor, which was quite neutral, was acidulated; and the chlorine dissolved precipitated as chloride of silver, from which its quantity was obtained by calculation. The following table gives the results of experiments conducted in this manner:—

100 of white precipitate gave—

Expt.	Yellow powder.	Chlorine in liquor.
1	90,00	5,93
2	88,50	6,50
3	90,30	6,40
mean	89,60	6,29

But 100 of white precicipate contain

Mercury	78,60
Chlorine	13,85

Therefore 89,60 of the yellow powder contain

Mercury	78,60
Chlorine	7,56

and 100 contain

Mercury	87,95
Chlorine	8,44

C.—100 grains of white precipitate were boiled with water until completely decomposed; the resulting yellow powder weighed 91,15 grains. The liquor was cautiously evaporated to dryness, and gave 10,23 of sal-ammoniac, consisting of

$$\left.\begin{array}{lr}\text{Chlorine} & 6{,}76 \\ \text{Hydrogen} & {,}19 \\ \text{Ammonia} & 3{,}28\end{array}\right\} 10{,}23$$

Therefore there are obtained, by this experiment, for the constituents of yellow powder,

Mercury	86,23
Chlorine	7,77
Ammonia	3,83

D.—It has been already stated, that when this powder is heated, it is resolved into ammonia, azote, water, calomel, and quicksilver. Having found that, by performing this operation in a very small retort, the water and gases could be dissipated without any remarkable loss of the other constituents, I made some trials in this way to determine the amount of the chlorine and quicksilver. For this purpose a higher temperature is required than for the corresponding analysis of white precipitate, and the condensation of the mercurial vapour must be very carefully effected. In other respects, the manipulation was the same, and the following table contains the results:

Expt.	Quantity of Material.	Subliemd Residue.	Sublimed Residue from 100 parts.
1	14,30	13,37	93,50
2	19,65	18,53	94,30
3	23,72	22,35	94,22
	Per cent. mean		94,01

From this result we can easily calculate the quantities of chlorine and mercury the residue contains, for—

Let m = the residue = 94,01

x = the quantity of chlorine

y = the quantity of mercury

a = atomic weight of chlorine = 35,42

b = atomic weight of mercury = 202,8

This is (1) $x = m - y$

and (2) $\frac{x}{y} = \frac{a}{2b.}$ by other processes.

Then $\frac{m - y}{y} = \frac{a}{2b} \therefore 2bm = (a+2b)\, y$

and $y = \frac{2bm}{a+2b}$

We thus find 100 of yellow powder, to contain

Mercury	86,46
Chlorine	7,55

E.—105,28 grains of yellow powder were dissolved in muriatic acid, and the solution having been somewhat diluted, was decomposed by a current of sulphuretted hydrogen gas. The perfectly black sulphuret was collected on a weighed filter, and the liquor evaporated to dryness, and the residual sal-ammoniac, weighed:

The filter and sulphuret	126,71
Filter - -	23,00

Sulphuret of mercury 103,71, consisting of

Sulphur - -	14,22
Mercury - -	89,49

The sal-ammoniac weighed 12,86 grs. consisting of

Chlorine - -	8,50
Hydrogen -	,24
Ammonia -	4,12

Therefore, the yellow powder consisted of

in 105,28 parts—		in 100 parts—	
Mercury	89,49	Mercury	85,00
Ammonia	4,12	Ammonia	3,91

Summing up these different results, we have—

Process.	Mercury.	Chlorine.	Ammonia.
A	88,381	8,374	
B	87,95	8,44	
C	86,23	7,77	3,83
D	86,46	7,55	
E	85,00		3,91

And taking the mean result of all, we get, for the composition of this yellow powder,

Mercury	86,80	100,00
Chlorine	8,03	
Ammonia	3,87	
Oxygen and loss	1,30	

In the processes, A and B, a small quantity of the yellow powder was lost, in consequence of its being not perfectly insoluble in water. This quantity, I have reason to believe, varied from one to two per cent. and by the means of calculation we employed, the mercury and chlorine constituents are given above what is correct in that proportion. As far as I can judge, by considering the circumstances of the experiments, I conceive the mean to be consequently too high; and I believe the analysis C. by itself, to approach closer to the truth. By it we have in 100 of the yellow powder—

Mercury	86,23	100,00
Chlorine	7,77	
Ammonia	3,83	
Oxygen and loss	2,17	

This yellow powder is generated evidently by the reaction of water on white precipitate, in which one-half of the chlorine and ammonia are converted into sal-ammoniac, a corresponding portion of the mercury being oxydized. I shall compare the results of this reaction with each of the formulæ for white precipitate that I have previously given.

$$2\left\{(2\,\mathrm{Ch} + \mathrm{Hg}) + (2\,\mathrm{NH}^3 + \ddot{\mathrm{H}}\mathrm{g})\right\} + 2\dot{\mathrm{H}} =$$

$$\left\{(2\,\mathrm{Ch} + \mathrm{Hg}) + (3\,\ddot{\mathrm{H}}\mathrm{g} + 2\,\mathrm{NH}^3)\right\} + 2\,\mathrm{ChNH}^4$$

Here two atoms of white precipitate and two of water, mutually reacting, give two atoms of sal-ammoniac and one of the powder. On this idea, the yellow powder should consist of

Mercury	84,12	100,00
Chlorine	7,36	
Ammonia	3,56	
Oxygen	4,96	

These numbers, with the exception of oxygen, fall below the lowest experimental result, and probabilities are therefore rather against this formula being true. Let us next try the result of reaction with that formula for white precipitate which does not include oxygen, and supposes the ammonia to exist as amidogene.

$$2\left\{(2\,Ch+Hg)+(2\,NH^{2}+Hg)\right\}+4\,\dot{H}=$$

$$\left\{(2\,Ch+Hg)+2\ddot{H}g+(2NH^{2}+Hg)\right\}+2\,Ch\,NH^{4}$$

There are here equally produced by the two atoms of white powder and two of water, two of sal-ammoniac, and one of the yellow powder, of which the composition should be,

Mercury	85,72	100,00
Chlorine	7,48	
Amidogene	3,42	
Oxygen	3,38	

and it should yield in analysis 3,63 per cent. of ammonia.

The definite composition of this yellow powder is thus evident, and the decomposition by which it is formed perfectly explained. We see that all the results tend to show that in these bodies the ammonia is not united with oxide of mercury, but rather the metal with amidogene. The perfect demonstration of this principle, however, must be sought for in the other metals.

§ 3.—*Of the products of the Action of Alcalies in Excess on White Precipitate.*

Grouvelle and other chemists have stated that by the action of an excess of alcali on a sublimate solution, there is produced the ammoniuret of mercury which was discovered by Fourcroy and examined by Guibourt, and to which I shall hereafter speedily recur; and even Dumas states, that "the same compound (the ammoniuret) is obtained by pouring ammonia into a solution of corrosive sublimate, and then adding caustic-potash in excess." My anxiety to obtain pure ammoniuret of mercury, joined to the interest of the preceding investigations, led me to examine the nature of the products thus obtained; and the results, as correcting an error very generally fallen into, are worthy of being described.

When corrosive sublimate is decomposed by ammonia, the quantity of alcali in excess does not appear to interfere much with the reaction before described. If the liquors be cold, there is obtained white precipitate; and if it be boiled, the heavy yellowish powder is produced. The liquor retaining in the former, one-half, in the latter, three-fourths of the chlorine of the sublimate. Again if white precipitate be boiled in water, rendered strongly alcaline by ammonia, we obtain the yellowish powder, and half the chlorine and half the ammonia of the precipitate are disengaged. Thus, water of ammonia acts on white precipitate only as water itself does, the nature of the reaction being the same in both instances. Again when white precipitate was treated with potash for analysis, as in p. 432. it has been seen that the ammonia disengaged was but one-half what it contained, the formation of the yellowish powder being the limit at which the decomposition stops. In these cases, however, the powder product is not so bright in colour as that produced by the action of mere water. It does not appear to be quite so pure, but in its properties it manifests complete identity.

Nevertheless, in order to leave no room for doubt upon the matter, I decomposed corrosive sublimate by a great excess of ammonia, added a strong solution of potash, and boiled for a considerable time. The yellowish white powder produced, was separated by the filter and washed until the liquors ceased to effect turmeric paper: dried carefully. It weighed from 100 of sublimate, 85 grains. When heated it gave out water, ammonia, and azote, and calomel with metallic mercury sublimed. When suddenly heated, it puffed up, more so than the pure yellow powders, which was probably the reason of its having been confounded with the ammoniuret which possesses a very slight detonating property.

To analyze it, 66,83 grains were dissolved in muriatic acid, and having been diluted were decomposed by a current of sulphuretted hydrogen gas. The black sulphuret was collected on a weighed filter, and having been carefully dried, weighed 67,70 grains, consisting of

Mercury	58,42
Sulphur	9,28

The liquor evaporated, gave sal-ammoniac 6,58 grains, consisting of

Chlorine	4,35
Hydrogen	,12
Ammonia	2,11

There were thus obtained from 66,83 of this powder—

$$\left.\begin{array}{lr} \text{Mercury} & 58{,}42 \\ \text{Chlorine} & 4{,}35 \\ \text{Ammonia} & 2{,}11 \\ \text{Oxygen and loss} & 1{,}95 \end{array}\right\} 66{,}83$$

But the yellow powder by water, should have given from the formula:

$$(2\,\mathrm{Ch}+\mathrm{Hg})+2\,\ddot{\mathrm{H}}\mathrm{g}+(2\,\mathrm{NH}^2+\mathrm{Hg})$$

$$\left.\begin{array}{lr} \text{Mercury} & 57{,}30 \\ \text{Chlorine} & 5{,}00 \\ \text{Amidogene} & 2{,}28 \\ \text{Oxygen} & 2{,}25 \end{array}\right\} 66{,}83$$

This result proves the identity of the effect in the two cases of the action of water and of an alkali.

Rose and Grouvelle have already shown that when dry ammonia is passed over melted sublimate, an atom of the former is absorbed by one of the latter, and a white mass formed. The history of its properties has been so well given in Rose's Memoir, that it is unnecessary for me to do more than mention my results as having been confirmatory of his. This compound is decomposed by water giving white precipitate and sal-alembroth, as may be at once seen :

$$4\,(2\,\mathrm{Cl}+\mathrm{Hg})+4\,\mathrm{NH}^3 =$$

$$\left\{(2\mathrm{Cl}+\mathrm{Hg})+(2\,\mathrm{NH}^2+\mathrm{Hg})\right\} + 2\left\{(2\,\mathrm{Cl}+\mathrm{Hg})+(\mathrm{Cl}+\mathrm{NH}^4)\right\}$$

SECTION II.—*Of the Action of Ammonia on the Proto-Chloride of Mercury.*

§ 1.—*Action of Liquid Ammonia upon Calomel.*

The decomposition resulting from the action of water of ammonia upon the protochloride of mercury, does not appear to have attracted particular attention, as all writers who speak at all upon the subject, mention ammonia, along with potash and soda, as decomposing calomel into black oxide of mercury. Hennell in particular, states expressly, that calomel decomposed by excess of ammonia, yields a black powder containing in 100 parts, 96 of mercury and four of oxygen. I was therefore rather surprised when experiment showed me that a reaction of a totally different nature takes place, giving rise to a compound possessed of very remarkable properties.

When water of ammonia is poured on calomel, whether sublimed or precipited, the the mass immediately becomes black, and the appearance is not altered by boiling the mixture for a long time. While yet wet the powder remains almost black, but it becomes much lighter on drying, so that when quite dry it is of a dark-grey. This powder is not altered by exposure to air, or to a moderate heat ; a portion of it was exposed in a platinum crucible on a sand bath for several hours to a temperature of 180° *Fahrenheit*, without being altered in weight or colour. When moistened it becomes nearly as dark as when first generated, but it again loses its black colour on being dried : boiled with water it does not appear altered in its composition. When this powder is heated in a tube sealed at one end, it first gives a trace of water, with much azote and ammonia ; then there sublimes calomel mixed with metallic mercury, the decomposition being accompanied with that sort of effervescence which appears in the heating of so many of the substances under examination.

For the examination of this body, an order of analysis similar to that adopted for white precipitate was pursued.

A.—148,15 grains of precipitated calomel were boiled for some minutes with a great excess of water of ammonia, and the whole thrown on a filter. The black powder thus obtained weighed 141,92 grains corresponding to 95,79 grains from 100 of calomel.

The liquor that had been filtered off was acidulated by nitric acid and nitrate of silver added in excess; the chloride of silver precipitated was collected and dried: it weighed 44,44 grains corresponding to 30,0 from 100 of calomel; and the 30,0 grains of chloride of silver containing 7,401. But calomel consists in 100 parts of

Mercury	85,117
Chlorine	14,883

Therefore we have by this experiment, the black powder composed of

Mercury	85,117	and	88,85
Chlorine	7,482		7,76
Other matters	3,191		3,39
	95,790		100,00

No. 2.—153,36 grains of calomel were boiled with water of ammonia for a few minutes, and filtered. The dry dark-grey powder weighed 146,71 grains, corresponding to 95,66 per cent.

The liquor treated with nitrate of silver gave 44,03 of chloride of silver, corresponding to 28,71 of chloride per cent. and which contains 7,08 of chlorine.

Thus we obtain,

Mercury	85,117	or	88,98
Chlorine	7,803		8,15
Other matters	2,740		2,87
	95,660		100,00

The mean of these experiments gives,

Mercury	88,91
Chlorine	7,95
Other matters	3,14
	100,00

B.—As the above method necessarily throws the chlorine and mercury estimate rather too high, the following experiment was made, in which the necessary loss produces an opposite effect:

101,37 of the powder were boiled with strong muriatic acid, and an acid solution of proto-chloride of tin added. The reduction of the quicksilver took place readily, and large well-formed globules appeared; the metal collected and carefully dried, weighed 89,39 grains, or 100 of the powder had given 88,18.

C.—51,42 of the grey powder were dissolved in dilute aqua regia, and a current of sulphuretted hydrogen in excess passed through the liquor. It was found, that owing to free chlorine, the sulphur precipitated invalidated the result. The whole was therefore mixed with nitric acid, and boiled until the sulphuret of mercury was completely decomposed; the liquor was then freed from the particles of pure sulphur and evaporated until all free nitric acid and chlorine were completely dissipated. Being then treated by sulphuretted hydrogen, it yielded a sulphuret, pure and jet black, which collected and dried, weighed 52,39 grains, consisting of

$$\left.\begin{array}{ll}\text{Sulphur} & 7{,}19 \\ \text{Mercury} & 45{,}20\end{array}\right\} 52{,}39$$

The 51,42 grains therefore contained 45,20 of mercury
or 100,00 - - - 87,90

In this experiment so much ammonia was lost by the treatment with nitric acid, that its quantity could not be determined.

D.—As in none of these former analyses had the ammonia constituent been determined, the following experiments were made for the purpose of ascertaining its precise quantity:

1st. 66,43 grains were boiled with an excess of solution of iodide of potassium, and the flask being connected with a bent tube dipping into dilute muriatic acid, the heat was kept up until all the ammonia and about half the water had passed over. The liquor was then evaporated to dryness, and yielded a residue of 6,96 grs. of sal-ammoniac, consisting of

$$\left.\begin{array}{ll}\text{Muriatic acid} & 4{,}73 \\ \text{Ammonia} & 2{,}33\end{array}\right\}$$

or 100 of powder gives 3,36 of ammonia.

The action of potash on the grey powder liberates ammonia likewise; but it was found so difficult to obtain complete decomposition that the method was abandoned. Another process tried, consisted in repeatedly distilling strong muriatic acid off the powder, in order to convert it into metallic mercury, corrosive sublimate, and sal-ammoniac, and thus obtain a quantative result; but this method also was found of so imperfect action, that it could not be well applied.

Summing up the results of the analyses above recorded, we have for 100 parts of the powder:

Process.	Mercury.	Chlorine.	Ammonia.
A	88,91	7,95	
B	88,18		
C	87,90		
D			3,36

Or the mean result is—

Mercury	88,33
Chlorine	7,95
Ammonia	3,36
Loss, &c.	0,36
	100,00

It is evident that we have here a body precisely corresponding to white precipitate; the mercury, however, being in proto-combination. Water of ammonia acting on calomel, abstracts half the chlorine, which is replaced by a corresponding quantity of ammonia in some form of combination. We can accordingly construct two formulæ corresponding to white precipitate; in the first half, the mercury being conserved as protoxidized and combined with an atom of ammonia: in the second, that half of the mercury being directly united to amidogene. The former theory gives from the formula $(Ch + Hg) + (\dot{H}g + NH^3)$.

$$\left.\begin{array}{lr} \text{Mercury} & 87,00 \\ \text{Chlorine} & 7,59 \\ \text{Oxygen} & 1,73 \\ \text{Ammonia} & 3,68 \end{array}\right\} 100,00$$

and 100 of calomel should yield 97,84 of product; whilst the second, from the formula $(Ch + Hg) + (NH^2 + Hg)$ gives,

$$\left.\begin{array}{lr} \text{Mercury} & 88,72 \\ \text{Chlorine} & 7,74 \\ \text{Amidogene} & 3,54 \end{array}\right\} 100,00$$

and 100 of calomel should yield 95,95 of product, which is almost precisely the quantity obtained in experiment.

We here find the evidence in favour of the existence of amidogene in combination to be almost insuperable. I shall nevertheless retain all through this paper the two methods of expression, until by examining the compounds of the other metals, the differences may become so much larger, as to completely prevent their falling within possible limits of error of observation.

SECTION III.—*Of the Action of Ammonia upon Peroxide of Mercury.*

The accurate examination of the action of ammonia upon peroxide of mercury is of very great importance, as the compound resulting; the ammoniuret of mercury is one of a very remarkable class of bodies, viz. the fulminating compounds containing ammonia; and in addition, the experiments of Guibourt, the only chemist I believe who has made analyses of it, would appear to demonstrate in it, a relation between the number of atoms of ammonia and oxygen, which must influence the ammoniacal theories to a very great extent. These circumstances made me trace out the properties of this body with more exactness than should have been otherwise required.

I have not been able to prepare a substance possessing the external characters of the ammoniuret of mercury described by Fourcroy and Thenard. I have varied in every manner I could imagine, the method of obtaining it; but, although I got a substance constantly the same in its properties and composition, it differed much in appearance from that described by the French chemists. They state, that by digesting liquid ammonia on red oxide of mercury during eight or ten days, the oxide gradually covers itself with a yellowish-white powder which generally passes to a very fine white. I have never obtained it of a pure white, but always with a tinge of yellow, possessing an appearance and affording on analysis, results always the same. The constancy of its properties justifies me, I should think, in considering it as pure, notwithstanding its not exactly agreeing with their result. Unfortunately they did not publish any quantative analysis of their product; the only one known to me is that in Guibourt's thesis.

In order to prepare ammoniuret of mercury, I precipitated a solution of sublimate by potash, and the precipitate having been well washed from all excess of alcali, was put into a bottle of water of ammonia and left for some days; its colour became much lighter, but never completely white. Other portions of recently precipitated peroxide were boiled in water of ammonia for a few minutes, until the colour ceased to undergo any change: the reaction was very much accelerated by heat. These different portions of product had all the same colour, and were indifferently, but without mixture, used in the following examination without any difference of properties becoming observable.

When this ammonuret is heated it gives off much ammonia, and azote; a considerable quantity of water collects in the tube, and the matter remaining becomes dark-red, like peroxide; but if it be allowed to cool, it reassumes its whitish colour, and is evidently still unaltered ammoniuret. The reaction evidently does not consist in a separation of the ammoniuret into ammonia and peroxide; but, from the commencement to the termination, there are disengaged water, ammonia, azote, oxygen, and metallic mercury. The ammoniuret, like many other mercurial compounds, is dark-red when hot, but of a whitish colour when cold. When a quantity of the ammonuret is suddenly thrown on ignited coals it explodes very feebly, and far inferiorly to fulminating gold with which its discoverers have compared it: it dissolves readily in nitric or muriatic acid.

To analyse this compound, processes of a simple nature were sufficient.

A.—72,07 grains of ammoniuret were dissolved in muriatic acid, and the liquor having been diluted was decomposed by sulphuretted hydrogen. The resulting sulphuret dried and weighed, amounted to 70,08 grains, consisting of

Sulphur	9,61
Mercury	60,47

The liquor and washings evaporated to dryness, gave sal-ammoniac, 9,21 grains, consisting of

Muriatic acid	6,28
Ammonia	2,93

Hence, supposing the mercury to exist as peroxide, we have as the result of the analysis:

Mercury	60,47	72,07
Oxygen	4,78	
Ammonia	2,93	
Water and loss	3,89	

or in one hundred parts—

Mercury	83,90
Oxygen	6,63
Ammonia	4,07
Water and loss	5,40

2.—The following analysis was made on a portion of ammoniuret prepared at a different time and in another manner than that used in the former experiment.

67,57 grains were dissolved in muriatic acid and decomposed by a stream of sulphuretted hydrogen. The precipitated sulphuret weighed 65,37 grains, consisting of

Sulphur	8,96	65,37
Mercury	56,41	

The liquor evaporated to dryness gave 8,15 grains of sal-ammoniac, consisting of

Muriatic acid	5,54
Ammonia	2,61

we have therefore the result—

Mercury	56,41	67,57
Oxygen	4,46	
Ammonia	2,61	
Water and loss	4,09	

or in 100 parts—

Mercury	83,48
Oxygen	6,59
Ammonia	3,86
Water and loss	6,07

a result almost identical with the former.

B.—52,22 grains were dissolved in muriatic acid and decomposed by chloride of tin. There were obtained 43,74 of mercury corresponding to 83,76 per cent.

C.—As the constancy of the amount of mercury and ammonia in the preceding results, proved completely that the loss did not arise from error, but probably from water present, the following experiment was made to ascertain whether water existed in such quantity: A small green-glass retort was blown, with a pretty long neck; to it was attached a tube containing potash; and the ammoniuret in the retort having been decomposed by a red heat, its gaseous elements were allowed to escape, the mercury condensed in the neck of the retort and the water in the potash-tube; the result, though not absolutely true, is sufficiently accurate for the determination of the point required.

Weighth of retort and material		75,38
Weight of retort		63,00
Ammoniuret used	12,38 grains	
Weight of retort and mercury-residue		73,35
Weight of retort		63,00
Mercury remaining	10,35	
Weight of potash-tube before		278,28
Weight of potash-tube after		278,95
Water absorbed	0,67	

We thus obtain as results—

Mercury	10,35	83,62
Water	,67	5,39
Gases and loss	1,36	10,99

But the gases consist of oxygen and ammonia, the former being such as to peroxidize the mercury; and assuming the remainder to be ammonia without loss, we have,

Mercury	83,62	100,00
Oxygen	6,60	
Ammonia	4,39	
Water	5,39	

These results summed up, give—

Process.	Mercury.	Oxygen.	Ammonia.	Water.
A, No. 1.	83,90	6,63	4,07	5,40
— No. 2.	83,48	6,59	3,86	6,07
B	83,76	6,60		
C	83,62	6,60	4,39	5,39

Giving a mean result of

$$\left.\begin{array}{lr} \text{Mercury} & 83{,}68 \\ \text{Oxygen} & 6{,}60 \\ \text{Ammonia} & 4{,}10 \\ \text{Water} & 5{,}62 \end{array}\right\} 100{,}00$$

on abstracting the water, we have—

$$\left.\begin{array}{lr} \text{Mercury} & 88{,}67 \\ \text{Oxygen} & 6{,}99 \\ \text{Ammonia} & 4{,}34 \end{array}\right\} 100{,}00$$

The only analysis of this substance that I am aware of having been published, is that of Guibourt, already quoted, and he considers it to be a compound of oxide of mercury and ammonia in such proportion that the hydrogen of the ammonia could convert the oxygen of the oxide of mercury into water, consequently his formula is the following $(3\,\ddot{Hg} + 2NH^3)$ and the per centage result:

$$\left.\begin{array}{lr} \text{Mercury} & 88{,}08 \\ \text{Oxygen} & 6{,}95 \\ \text{Ammonia} & 4{,}97 \end{array}\right\} 100{,}00$$

with which my analyses may be considered as completely agreeing. In the abstracts of Guibourt's paper that I have seen, there is not any notice taken of the water present; but yet its constant value shows it to be a chemical ingredient, and we have its atomic proportion, thus—

$$\frac{2\,Hg}{3\,H} = \frac{405{,}6}{27} = \frac{83{,}68}{5{,}57} \text{ or nearly } \frac{83{,}68}{5{,}62}$$

The compound $(3\,\ddot{Hg} + 2\,NH\ \ + 4\dot{H})$ gives us in per cent. composition, the following:

$$\left.\begin{array}{lr} \text{Mercury} & 83{,}72 \\ \text{Oxygen} & 6{,}60 \\ \text{Ammonia} & 4{,}72 \\ \text{Water} & 4{,}96 \end{array}\right\} 100$$

a result agreeing very closely with that of experiment.

Admitting that the azotic element is engaged in the combination as amidogene, and not as ammonia, the above formula converts itself into

$$(2\,\ddot{Hg} + (2\,NH^2 + Hg) + 6\,\dot{H})$$

a method of arrangement which we have already met with as an element of the yellow powder, formed by water on white precipitate.

Further development of a method of observing the Dip and the Magnetic Intensity at the same time, and with the same Instrument. *By the* Rev. Humphrey Lloyd, M.A., F.R.S., M.R.I.A. *Fellow of Trinity College, and Professor of Natural and Experimental Philosophy in the University of Dublin.*

Read December 28. 1835.

On a former occasion I had the honour of submitting to the Academy a new method of observing, as applied to Terrestrial Magnetism, in which the dip and the intensity of the magnetic force were determined with the same instrument, and by one observation. As this method has fully realized the expectations which I ventured at that time to entertain respecting it, I feel it my duty to enter somewhat more minutely into its details, and to explain the modifications which experience has led me to adopt in the practice of it. The ordinary dipping needle is supported on an axle which is supposed to pass precisely through its centre of gravity; and, consequently, the position which it assumes, when placed in the magnetic meridian, is the direction of the magnetic force. But if one of the arms of the needle be loaded with a weight, the needle will no longer rest in the line of the dip, but will assume a new position of equilibrium under the combined influence of magnetism and gravity;—the inclination of the needle to the horizon being connected with the dip, the magnetic force, and the moment of the added weight, by a very simple relation. This is the simple principle of the method which has been already laid before the Academy. In order to apply it, let us supose two small weights to be attached in succession to the southern arm of the needle, at fixed distances from its centre; and let the statical moments of these weights be μ and ν, and the corresponding inclinations of the needle to the horizon ζ and θ; then, it has been shown[*] that

$$\mu \cos \zeta = \phi \sigma \sin (\delta - \zeta), \qquad (1)$$
$$\nu \cos \theta = \phi \sigma \sin (\delta - \theta); \qquad (2)$$

in which δ denotes the dip, ϕ the earth's magnetic force, and σ a constant depending on the distribution of magnetism in the needle itself. If therefore the angles ζ and θ

[*] Page 161.

be observed in the usual manner, and if the ratio of the moments μ and ν be previously ascertained, these equations will give the dip, and the relative force, at the several places of observation.

The degree of accuracy with which these elements are thus determined is, however, not independent of the moments of the added weights; and, for a given amount of friction of the axle on its supports, the errors of the final results will vary with the position of equilibrium of the needle. It has been already shown* from theoretical considerations that the probable error in the determination of the *dip*, arising from the friction of the axle, will be least when the needle is entirely unloaded, and of course in the line of the dip;—while the probable error in the determination of the *force* is least, when the needle is at right angles to the same line. Hence the most advantageous mode of applying the preceding method consists in observing the position of the needle,—first, when unloaded,—and, secondly, when loaded with a weight sufficient to bring it into a position nearly perpendicular to the line of the dip.

It is obvious that if $\mu = 0, \zeta = \delta$; or the first of the observed inclinations becomes equal to the dip, when there is no weight whatever acting with or against the directive force. This condition, however, is never perfectly attained in practice. Owing to the want of perfect coincidence of the centre of gravity of the needle with the axle, the weight of the needle itself has a certain moment, which must deflect it from the true line of the dip. But, as this deflexion is, in all cases, small, it will be convenient to consider the angle ζ as the approximate value of the dip, and to seek the correction necessary to reduce it to its exact value. For this purpose, let equation (1) be divided by (2), and let the ratio of the moments, $\frac{\mu}{\nu}$, be denoted by ρ; then

$$\rho \frac{\cos \zeta}{\cos \theta} = \frac{\sin (\delta - \zeta)}{\sin (\delta - \theta)}.$$

Now making

$$\delta = \zeta + \varepsilon, \qquad (3)$$

the 2d member of the preceding equation becomes $\frac{\sin \varepsilon}{\sin (\zeta - \theta)}$ *q. p.* since ε is a very small quantity, and we have

$$\sin \varepsilon = \rho \frac{\cos \zeta}{\cos \theta} \sin (\zeta - \theta). \qquad (4)$$

The dip is therefore determined by means of the two equations (3) and (4); and the correction due to the want of perfect balance of the needle is inferred from the two observed angles, without the reversal of the poles.

The value of the constant coefficient, ρ, in equation (4), will be given by the formula

$$\rho = \frac{\cos \theta \sin (\delta - \zeta)}{\cos \zeta \sin (\delta - \theta)}$$

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* Page 167.

when the corresponding values of the angles δ, ζ and θ, are accurately known at some one station. A series of cotemporaneous observations were made for this purpose, in the Philosophy School of Trinity College, with a dipping needle of the ordinary form, and with two needles which have been constructed for the observation of the dip and the force by the the present method. The following are the results:

	Need. I.	*Need. III.*		*Need. IV.*	
Date	(δ)	(ζ)	(θ)	(ζ)	(θ)
April 18	70° 51′.2	71° 1′.9	− 6° 37′.5	70° 48′.4	− 8° 8′.7
—— 19	70 49.0	71 6.6	− 7 41.9	70 48.1	− 7 48.8
—— 21	70 51.4	71 1.6	− 7 31.3	70 48.2	− 8 10.6
Mean	70 50.5	71 3.4	− 7 16.9	70 48.2	− 8 2.7

Substituting the mean results in the formula, we find

Needle III. $\log \rho = \bar{2}.\ 06670$, $\rho = -.01166$.

Needle IV. $\log \rho = \bar{3}.\ 31175$, $\rho = +.00205$.

When the value of ρ is small, as in the case of Needle IV, the variations in the value of ε arising from moderate changes in the angles on which it depends, are inconsiderable; and, accordingly, where the district over which the observations extend is limited, the correction may be regarded as *constant*. Thus the computed difference in the value of this correction at London and Dublin is only 0.′2 for Needle IV. But as a very slight abrasion, or oxidation of the surface, will affect in a very sensible manner the position of equilibrium of the needle, it is probable that the correction will undergo some change in the course of time. The present value of the correction of Needle IV in Dublin appears to be + 1′.5.*

The dip being known, the relative values of the magnetic intensity at different stations will be given by equation (2). But before we can apply this equation, it is necessary to examine, a little more particularly, the coefficients σ and ν which enter it.

The former of these quantities is the statical moment of the free magnetism of the needle, or the value of the integral $\int qrdm$, taken throughout its extent,—dm being the element of the mass, q the quantity of free magnetism which belongs to it, and r

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* It has been here assumed that all constant errors are removed by reversal in the mean of the various readings taken with the ordinary needle, or that the deviations of the results from the absolute dip are equally probable on the positive and on the negative side. This however, it has been elsewhere shown, is not the case, and a correction seems to be required even in needles whose poles are reversed. This correction in the case of Needle I appears to be +10′, so that the true correction of Needle IV in Dublin is + 11′.5

its distance from the centre. Now this quantity (which we may denominate the *magnetic moment)* varies with the temperature, and this variation must be taken into account before we can make any accurate inference from the formula. Let τ denote the temperature of observation, τ' a certain *standard* temperature, and σ' the corresponding value of σ. Then, assuming the changes of the magnetic moment to be proportional to the changes of temperature, we have

$$\sigma=\sigma'\left[\,1-\alpha\,(\tau-\tau')\,\right]; \qquad (5)$$

in which α is a constant whose value is to be determined by observation.

In order to obtain the value of this constant, the Needles III and IV were suspended horizontally by a few filaments of silkworm's thread, and vibrated in a large glass bell, the air of which was heated from beneath by means of a spirit lamp. The time of 100 vibrations was observed at the artificial temperature, and at the ordinary temperature of the room before and after. The following are the results:

Needle III.			*Needle IV.*		
Hour	Time	Temp.	Hour	Time	Temp.
10^h 28′	222″.64	52°.2	12^h 28′	238″.00	58°.2
10 46	222.50	58.5	12 49	238.00	58.5
1 42	222.80	63.5	3 39	238.45	58.9
1 52	222.84	63.8	3 50	238.56	58.6
Mean	222.70	61.0	Mean	238.25	58.5
12 5	222.97	90.3	2 4	238.80	78.8
12 19	222.84	91.1	2 15	238.76	79.5
12 29	223.28	92.3	Mean	238.78	79.2
Mean	223.03	91.2			

Hence we have for Needle III

$$T'=222''.70\,,\quad T-T=0''.33\,,\quad \tau-\tau'=30^\circ.2\,;$$

and substituting in the formula

$$\alpha=\frac{-(\sigma-\sigma')}{\sigma'(\tau-\tau')}=\frac{2\,(T-T')}{T'(\tau-\tau')}$$

there is $\alpha=00010$. For Needle IV

$$T'=238.''25\,,\quad T-T'=0''.53\,,\quad \tau-\tau'=20^\circ.7\,;$$

and $\alpha=.00021$. But as these two needles were made at the same time by the same artist, and are therefore probably similar in temper, as they are in material and size, it is natural to suppose that the effects of temperature will be the same on both, and that the difference here observed is due to the uncertainties attending observations of this nature. Taking then the mean of the preceding results as the most probable value of the coefficient for both needles, we have

$$\alpha=.00016.$$

The quantity ν in formula (2) is the sum (or difference) of the moments of the weight of the needle and of the added weight. Accordingly, if m denote the *mass* of the added weight, and a the distance of its point of application from the axle,

$$\nu - \mu = m a g ;$$

or, since $\mu = \rho \nu$,

$$\nu = \frac{m a g}{1-\rho}.$$

Now the force of gravity, g, varies with the latitude ; and the variation is expressed by the formula

$$g = g' (1 - e \cos 2\lambda),$$

in which g' is the force at the latitude of 45°, λ the latitude of the place of observation, and e a constant whose numerical value is .002588. Accordingly, substituting this value of g in the preceding expression, and employing the symbol ν' to denote the value of ν corresponding to the latitude of 45°, or the quantity $\frac{m a g'}{1-\rho}$, we have

$$\nu = \nu' \ 1 - e \cos 2\lambda). \qquad (6)$$

The equations (2) (5) (6) contain all that is requisite for the comparison of the magnetic force at different places of the earth's surface, and under different circumstances as to temperatnre. The expression for the force, obtained from them by substitution, is

$$\phi = \frac{\nu' \cos \theta}{\sigma' \sin (\delta - \theta)} \times \frac{1 - e \cos 2\lambda}{1 - a (\tau - \tau')}. \qquad (7)$$

This expression is peculiarly adapted to logarithmic computation : for, since $e \cos 2\lambda$ and $a\ (\tau - \tau')$ are very small fractions whose squares and higher powers may be neglected,

$$\log (1 - e \cos 2\lambda) = - M e \cos 2\lambda ,$$
$$\log [1 - a (\tau - \tau')] = - M a \ (\tau - \tau') ;$$

M being the modulus of the common system, whose numerical value is .43429. Hence, if we take

$$\Delta = \log \cos \theta - \log \sin (\delta - \theta) + M a \ (\tau - \tau') - M e \cos 2 \lambda ,$$
$$\Delta_{,} = \log \cos \theta_{,} - \log \sin (\delta_{,} - \theta_{,}) + M a \ (\tau_{,} - \tau') - M e \cos 2 \lambda_{,} , \qquad (8)$$

in which $\delta_{,}$, $\theta_{,}$, &c. denote the values of δ, θ, &c. at the place for which the force is taken as unit, we have

$$\log \phi = \log \nu' - \log \sigma' + \Delta ,$$
$$0 = \log \nu' - \log \sigma' + \Delta_{,} ;$$

and, subtracting,

$$\text{og}\ \phi = \Delta - \Delta_{,}, \qquad (9)$$

The last term in equations (8), or the correction for the variation of gravity, may be omitted when the places at which the magnetic force is compared do not differ considerably in latitude; for in that case the quantity $Me\ (\cos 2\lambda_{,} - \cos 2\lambda)$, which enters the logarithmic formula for the force, may from its smallness be disregarded.

In applying this method of observation to the determination of the changes which the intensity of the magnetic force undergoes at the same place, it will be convenient to substitute an approximate formula for the preceding. To obtain this, we have only to differentiate the equations (8) and (9), and we find

$$\frac{d\phi}{\phi} = -\tan\theta\, d\theta - \cot(\delta - \theta)\,(d\delta - d\theta) + \alpha d\tau.$$

in which we are to substitute for $\tan\theta$, $\cot(\delta - \theta)$, and α, their values belonging to the particular place and needle. If the needle be loaded so that $\delta - \theta$ is very nearly equal to 90°, the term multiplied by $\cot(\delta - \theta)$ may be neglected, and we have

$$\frac{d\phi}{\phi} = -\tan\theta\, d\theta + \alpha d\tau.$$

This seems to offer a very simple means of observing the diurnal and menstrual variations of the total intensity, and of ascertaining the law of a phenomenon of which nothing certain is as yet known.

I shall now adduce, in exemplification of this method, the results which I have obtained on the direction and intensity of the magnetic force in Dublin; and shall compare these results with those of the received method, the observations being made, for the most part, at the same time. The needle employed (Needle IV) is one of those made expressly for the practice of this method; and care has been therefore taken to preserve its magnetic state undisturbed. In each arm are drilled three small holes, close to one another, coinciding, as accurately as could be effected, with the axis of form of the needle, and distant from its centre by about two-thirds of the length of the arm. The weight is a small piece of brass wire, which is introduced into one of the holes on the southern arm,—the diameter of the wire corresponding accurately to that of the hole; the magnitude of the weight is such as to bring the needle into a position nearly perpendicular to the line of the dip. The length of the needle is $4\frac{1}{2}$ inches.

The following are the observations, and the results computed by the formulæ (3) (8) and (9); they constitute three distinct comparisons of Dublin and London, the first of which was made in the year 1834, and the other two in 1835.

I. August and September 1834.

Place	Date	Hour	Temp.	(ζ)	(θ)
London	Aug. 28	2^{h} $35'$	70°.0	69° 8′.0	− 11° 53′.2
	29	12 42	68.5	69 8.5	− 12 7.4
	—	1 18	67.7	69 5.6	− 12 26.1
	Mean,		68.7	69 7.4	− 12 8.9
Dublin	Sept. 22	2 15	61.0	71 2.2	− 8 5.6
	23	2 45	62.5	70 53.8	− 7 37.0
	26	2 40	66.2		− 7 59.6
	29	2 40	62.5	70 44.8	− 8 9.5
	Mean,		63.0	70 53.6	− 7 57.9

London.

$\zeta = 69° \; 7'.4$, $\epsilon = +1'.5$
$\delta = 69 \; 8.9$.
$\theta = -12 \; 8.9$, log cos $= 9.99016$
$\delta - \theta = 81 \; 17.8$, log sin $= 9.99497$
diff. $= -.00481$
$\tau - \tau' = +8°.7$, corr. $= +.00060$
$\Delta_{,} = -.00421$

Dublin.

$\zeta = 70° \; 53'.6$, $\epsilon = +1'.5$
$\delta = 70 \; 55.1$.
$\theta = -7 \; 57.9$, log cos $= 9.99579$
$\delta - \theta = 78 \; 53.0$, log sin $= 9.99177$
diff. $= +.00402$
$\tau - \tau' = +3°.0$, corr. $= +.00021$
$\Delta = +.00423$

log $\phi = \Delta - \Delta_{,} = +.00844$, $\phi = 1.0196$.

II. September 1835.

Place	Date	Hour	Temp.	(ζ)	(θ)
Dublin	Sept. 4	1^{h} $56'$	71·8	70° 43′.6	− 12° 29′.4
	5	2 19	65.5	70 52.8	− 13 15.4
	7	4 8	70.0	70 52.2	− 13 20.8
	15	12 25	62.0	70 55.0	− 13 10.5
	Mean		67.3	70 50.9	− 13 4.0
London	Sept. 19	1 10	68.0	69 7.4	− 16 54.2
	22	11 32	70.8	69 12.4	− 17 16.3
	—	1 45	70.0	69 13.6	− 16 49.4
	Mean		69.6	69 11.1	− 17 0.0

London.

$\zeta = 69° \; 11'1$, $\epsilon = +1'.5$
$\delta = 69 \; 12.6$.
$\theta = -17 \; 0.0$, log cos $= 9.98060$
$\delta - \theta = 86 \; 12.6$, log sin $= 9.99905$
diff. $= -.01845$
$\tau - \tau' = + \; 9°.6$, corr. $= +.00066$
$\Delta_{,} = -.01779$

Dublin.

$\zeta = 70° \; 50'.9$, $\epsilon = +1'.5$
$\delta = 70 \; 52.4$.
$\theta = -13 \; 4.0$, log cos $= 9.98861$
$\delta - \theta = 83 \; 56.4$, log sin $= 9.99756$
diff. $= -.00895$
$\tau - \tau' = +7.° \; 3$, corr. $= +.00050$
$\Delta = -.00845$

log $\phi = \Delta - \Delta_{,} = +.00934$, $\phi = 1.0217$.

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* The coefficient of $\tau - \tau'$ in the correction, or the value of Ma, is .000069.

III. October and November 1835.

Place	Date	Hour	Temp.	(ζ)	(θ)
London	Oct. 23	11^h 33′	50°.5	69° 10′.6	− 16° 32′.0
	—	2 25	51.6	69 2.2	− 16 35.4
	24	1 16	53.8	69 6.0	− 16 44.4
	Mean		52.0	69 6.3	− 16 37.3
Dublin	Nov. 5	12 22	56.2	70 49.6	− 12 54.6
	—	2 32	52.8	70 45.8	− 12 54.5
	6	1 15	49.0	70 53.9	− 12 36.6
	Mean		52.7	70 49.8	− 12 48.6

London,

$\zeta = 69° \; 6'.3, \quad \varepsilon = +1'.5$

$\delta = 69 \; 7.8.$

$\theta = -16 \; 37.3, \text{ log cos } = 9.98146$

$\delta - \theta = 85 \; 45.1, \text{ log sin } = 9.99881$

$\text{diff.} = -.01735$

$\tau - \tau' = -8° \; 0, \quad \text{corr.} = -.00055$

$\Delta_{,} = -.01790$

Dublin.

$\zeta = 70° \; 49'.8, \quad \varepsilon = +1'.5$

$\delta = 70 \; 51.3.$

$\theta = -12 \; 48.6, \text{ log cos } = 9.98905$

$\delta - \theta = 83 \; 39.9, \text{ log sin } = 9.99734$

$\text{diff.} = -.00829$

$\tau - \tau' = - 7.°3, \quad \text{corr.} = -.00050$

$\Delta = -.00879$

$\log \phi = \Delta - \Delta_{,} = +.00911, \qquad \phi = 1.0212.$

In order to reduce these results to the same epoch, the values of the dip obtained in the autumn of 1834 must be diminished by 3′, that being nearly the amount of the annual decrease of dip in this part of the globe at the present time. We have then, on summing up,

	($\delta_{,}$)	(δ)	(ϕ)
I.	69° 5′.9	70° 52′.1	1.0196
II.	69 12.6	70 52.4	1.0217
III.	69 7.8	70 51.3	1.0212
Mean	69 8.8	70 51.9	1.0208

We may now compare the preceding determinations of the Intensity with those obtained by the received method. We have for this purpose three comparisons of the horizontal part of the magnetic force at Dublin and London, made in the summer and autumn of last year; the latter two having been cotemporaneous with (II) and (III) of the preceding table. The results of these comparisons are given in the annexed table, which contains 1. the place, and 2. the date of the observation; 3. the name of needle employed; 4. the number of observations; 5. the mean time of 100 vibrations, corrected for temperature and for the rate of the chronometer; and 6. the computed ratio of the horizontal intensities. The observations were made in the manner adopted by Professor Hansteen, viz. by allowing the needle to make 360 vibrations commencing with the arc of 20°, and noting the time of completion of every 10th vibration during the interval by a chronometer. The needles are short cylinders, 2½ inches long, and .13 of an inch in diameter; such being the form and size recommended by the same skilful and indefatigable observer.

	Place	Date	Cyl.	No.	Time	Intensity
(I)*	London	July 8–19	R (c)	19	441.″59	
		Aug. 30,31	——	6	441.46	
		Mean	——	25	441.53	1.0000
	Dublin	Aug. 16	——	3	454.06	.9456
	London	July 19,20	R (d)	8	438.57	
		Aug. 28,29	——	6	439.07	
		Mean	——	14	438.82	1.0000
	Dublin	Aug. 14.	——	3	452.11	.9421
(II)	London	Sept. 19–22	L (1)	3	235.98	1.0000
	Dublin	—— 12–15	——	4	243.52	.9390
	London	Sept. 19–22	L (2)	3	284.01	1.0000
	Dublin	—— 12–15	——	3	293.31	.9376
(III)	London	Oct. 23,24	L (1)	3	235.42	1.0000
	Dublin	Nov. 5,6	——	3	243.87	.9319
	London	Oct. 23,24	L (2)	4	283.11	1.0000
	Dublin	Nov. 5,6	——	3	293.27	.9319

The mean of these three determinations gives the horizontal component of the magnetic force in Dublin equal to .9380, that in London being unity. To deduce from this the ratio of the total force at the two places, we must know the amount of the dip at each. The following are the results of the observations made with the needles (I) and (IV), in the autumn of 1835. In taking the mean, only half the weight has been allowed to each observation with the latter,—the number of readings from which the dip is deduced in the statical method being half of that taken in the ordinary process.

Place	Date	Needle	No.	Dip
Dublin	Sept. 4–15	I	6	70° 53.′5
		IV	6	70 52.0
		Mean	12	70 53.0
London	Sept. 19–25	I	9	69 6.3
	Sept.—Oct.	IV	7	69 9.8
		Mean	16	69 7.3

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* The observations of comparison (I) were made by Captain James Ross.

Hence, if ϕ denote, as before, the total intensity of the magnetic force at Dublin, that at London being unity, we have

$$\phi = .9380 \times \frac{\cos\ (69^\circ\ 7.'3)}{\cos\ (70^\circ 53.'0)} = 1.0208.*$$

The mean results of the two methods, then, agree in a very remarkable manner; the agreement extending to the fourth place of decimals inclusive. But the differences between the partial results and the mean (by which we are accustomed to judge of the value of observations) are very different in the two cases. The greatest of these differences, in the method which forms the subject of this paper, is only .0012; while the greatest difference, in the three comparisons of the horizontal intensity, amounts to .0061, and the corresponding difference in the value of the total force is .0066.

The difference .006, though it does not appear to be greater than that commonly met with in different comparisons of the horizontal force at two places, is yet much beyond the limits of the errors of observation; and, to account for it, we must suppose the horizontal force to have varied at one or both of the places of observation. The existence of such variations seems to be well established. Besides the regular periodical changes dependent on the hour and on the season, the horizontal force appears to be liable also to accidental fluctuations, or irregular oscillations round its mean state; and the variations of the latter kind (like those of the barometer in our climates) are probably more considerable than those that are periodic and regular. These variations are, in all probability, the effects of changes both in the intensity and direction of the magnetic force; but the latter appear to be (in these high magnetic latitudes) the predominating cause. The relation which subsists among these changes is obtained by differentiating, the equation $h = \phi \cos \delta$, considering h, ϕ, and δ, as all variable; dividing the result by the equation itself, we find

$$\frac{dh}{h} = \frac{d\phi}{\phi} - \tan \delta \sin 1'. \, d\delta;$$

the change of dip, $d\delta$, being expressed in minutes. When the dip is 71°, the last term of this equation becomes $-.00084 \times d\delta$; so that considering the change of dip as the sole cause of the effect observed, a variation of .006 in the amount of the horizontal force will be produced by a variation of 7′ in the dip; and this is, probably, within the limits of the irregular changes to which that element is subject.

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[*] The values of the dip employed in the preceding calculation are the apparent values, reduced to Needle I, as the standard. When the correction due to the latter needle (see note, p. 451) is applied, there will be a small alteration, amounting to +.0007, in the computed value of the relative intensity.

If the preceding views be correct, it will follow that, where the dip is considerable, we cannot hope to determine with any accuracy the relative values of the total intensity, by the observation of its horizontal component, unless the dip be observed *at the same time* at each place. Still, however, the final values of the intensity will be affected, to a large amount, by the errors of observation to which the results obtained with the dipping needle are liable. The chief of these errors is that due to the friction of the axle on its supports; and it has been already shown* that if the amount of that error in the natural position of the needle be denoted (in parts of radius) by e, the induced error in the determination of the force by the received method, will be

$$e \tan \delta;$$

while the error, arising from the same cause, in the needle, is

$$\frac{e}{\sin(\delta - \theta)}.$$

Accordingly, when $\delta - \theta = 90°$, or when the position of the needle when loaded is perpendicular to the line of the dip, the error in the determination of the force is reduced to e; and is less than the corresponding error in the common method in the ratio of unity to the tangent of the dip.

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* p. 167—8.

On the Laws of the Double Refraction of Quartz. *By* JAMES MACCULLAGH, *Fellow of Trinity College, Dublin.*

Read February 22, 1836.

The singular laws of the double refraction of quartz, which have been discovered by the successive researches of Arago, Biot, Fresnel, and Airy, are known merely as so many independent facts; they have not been connected by a theory of any kind. I propose, therefore, to show how these laws may be explained hypothetically, by introducing differential coefficients of the third order into the equations of vibratory motion.

Suppose a plane wave of light to be propagated within a crystal of quartz. Let the coordinates x, y, z, of a vibrating molecule be rectangular, and take the axis of z perpendicular to the plane of the wave, and the axis of y perpendicular to the axis of the crystal. Let us admit that the vibrations are accurately in the plane of the wave, and of course parallel to the plane of xy. Then, using ξ and η to denote, at any time t, the displacements parallel to the axes of x and y respectively, we shall assume the two following equations for explaining the laws of quartz:—

$$\frac{d^2\xi}{dt^2} = \mathrm{A}\frac{d^2\xi}{dz^2} + \mathrm{C}\frac{d^3\eta}{dz^3}. \qquad (1.)$$

$$\frac{d^2\eta}{dt^2} = \mathrm{B}\frac{d^2\eta}{dz^2} - \mathrm{C}\frac{d^3\xi}{dz^3}. \qquad (2.)$$

The peculiar properties of this crystal depend on the constant C. When $\mathrm{C}=0$, the third differentials disappear, and the equations are reduced to the ordinary form, in which state they ought to agree with the common equations for uniaxal crystals. Hence, putting a for the reciprocal of the ordinary index, b for the reciprocal of the extraordinary, and ϕ for the angle made by the axis of z with the axis of the crystal, we must have

$$\mathrm{A}=a^2, \qquad \mathrm{B}=a^2-(a^2-b^2)\sin^2\phi, \qquad (3.)$$

supposing the velocity of propagation in air to be unity.

Now, from the nature of equations (1.) and (2.), the vibrations must be elliptical. In fact, if we put

$$\xi = p\cos\left\{\frac{2\pi}{l}(st-z)\right\}, \qquad \eta = q\sin\left\{\frac{2\pi}{l}(st-z)\right\}, \qquad (4.)$$

where p, q, s, l, are constant quantities, the differential equations will be satisfied by assigning proper values to s and to the ratio $\frac{q}{p}$. For, after substituting in equations (1.) and (2.) the values of the partial differential coefficients obtained by differentiating formulæ (4.), we shall find that every term of each equation will have the same sine or cosine for a factor; omitting, therefore, the common factors, and making $\frac{q}{p}=k$, we shall get the two following equations of condition:

$$s^2 = \mathrm{A} - \frac{2\pi}{l}\mathrm{C}k, \qquad (5.)$$

$$s^2 = \mathrm{B} - \frac{2\pi}{l}\times\frac{\mathrm{C}}{k}. \qquad (6.)$$

Subtracting these, we have

$$\mathrm{A} - \mathrm{B} + \frac{2\pi\mathrm{C}}{l}\left(\frac{1}{k} - k\right) = 0, \qquad (7.)$$

which, by formulæ (3.), becomes

$$k^2 - \frac{l}{2\pi\mathrm{C}}\left(a^2 - b^2\right)\sin^2\phi .\ k = 1. \qquad (8.)$$

Let us now interpret these results. It is obvious, from formulæ (4.), that s is the velocity of propagation for a wave whose length is l, and that each vibrating molecule describes a little ellipse whose semiaxes p and q are parallel to the directions of x and y. But the number k, expressing the ratio of the semiaxes, has two values, one of which is the negative reciprocal of the other, as appears by equation (8.); and each value of k has a corresponding value of s determined by equation (5.) or (6.) Hence there will be two waves elliptically polarized, and moving with different velocities, the ratio of the axes being the same in both ellipses; but the greater axis of the one will coincide with the less axis of the other. The difference of sign in the two values of k, shows that if the vibration be from left to right in one wave, it will be from right to left in the other. These laws were discovered by Mr. Airy.

The law by which the ellipticity of the vibrations depends on the inclination ϕ, and on the colour of the light, is contained in equation (8.). The value of the constant C will be determined presently. In the mean time we may observe, that C denotes a line, whose length is very small, compared with the length of a wave.

When $\phi=0$, the light passes along the axis of the crystal. In this case we have $k^2=1$, and $k=\pm 1$; which shows that there are two rays, circularly polarized in opposite directions. The value of s for each ray may be had from equation (5.) or (6.), by putting $+1$ and -1 successively for k. Calling these values s' and s'', we find

$$s'^2=a^2-2\pi\frac{C}{l}, \qquad s'=a\left(1-\frac{\pi C}{a^2 l}\right); \qquad (9.)$$

$$s''^2=a^2+2\pi\frac{C}{l}, \qquad s''=a\left(1+\frac{\pi C}{a^2 l}\right). \qquad (10.)$$

Suppose a plate of quartz to have two parallel faces perpendicular to the axis, and conceive a ray of light, polarized in a given plane, to fall perpendicularly on it. The incident rectilinear vibration may be resolved into two opposite circular vibrations, which will pass through the crystal with different velocities; and which, after their emergence into air, will again compound a rectilinear vibration, whose direction will make a certain angle ρ with that of the incident vibration: so that the plane of polarization will appear to have been turned round through an angle equal to ρ, called the angle of rotation. This angle may be determined by means of the preceding formulæ. Putting θ for the thickness of the crystalline plate, the circularly polarized wave whose velocity is s', will pass through it in the time

$$\frac{\theta}{s'}=\frac{\theta}{a}\left(1+\frac{\pi C}{a^2 l}\right);$$

and the wave whose velocity is s'', in the time

$$\frac{\theta}{s''}=\frac{\theta}{a}\left(1-\frac{\pi C}{a^2 l}\right).$$

Therefore, if δ be the difference of the times, we have

$$\delta=\frac{2\pi C\theta}{a^3 l}. \qquad (11.)$$

But, since the velocity of propagation in air is supposed to be unity, the time and the space described are represented by the same quantity; and therefore δ, which is evidently a line, denotes the distance between the fronts of the two circularly polarized waves, when they emerge into air. The waves being at this distance from each other, if we conceive, at the same depth in each of them, a molecule performing its circular vibration, and carrying a radius of its circle along with it, the two radii will revolve in contrary directions, and will always cross each other in a position parallel to the incident rectilinear vibration. Now let two series of such waves be superposed, so as to agitate every molecule by their compound effect, and it is evident, that, when the radius vector of one component vibration attains the position just mentioned, the radius vector of the other will be separated from it by an angle equal to $\frac{2\pi\delta}{\lambda}$, where

λ is the length of a wave in air. The resultant rectilinear vibration will bisect this angle; and therefore ρ, the angle of rotation, will be equal to $\frac{\pi\delta}{\lambda}$. Hence, substituting for δ its value, and observing that l, the length of a wave in quartz, is equal to $a\lambda$, we find

$$\rho = \frac{2\pi^2 C\theta}{a^4 \lambda^2}\,; \qquad (12.)$$

which gives the experimental law of M. Biot, that the angle of rotation is directly as the thickness of the crystal, and inversely as the square of the length of a wave for any particular colour. By changing the sign of C, we should have an equal rotation in the opposite direction. And here we may remark, that C may be made negative in all the preceding equations, its magnitude remaining. There are two kinds of quartz, the right-handed and left-handed, distinguished by the sign of C.

The angle of rotation, for a given colour and thickness, is known from M. Biot's experiments. We can therefore find the value of C by means of the last formula; and substituting this value in equation (8.), we shall be able to compute k when ϕ and l are given. Now it happens that Mr. Airy *, by a very ingenious method of observation, has determined the values of k in red light for two different values of ϕ; and of course we must compare these observed values of k with the independent results of theory. As Mr. Airy's experiments were made upon red light, we shall select, for the object of our calculations, the red ray which is marked by the letter C in the spectrum of Fraunhofer. For this ray, Fraunhofer has given the length λ, which, expressed in parts of an English inch, is equal to .0000258; and M. Rudberg has found $a = .64859$, $b = .64481$. Moreover, from the experiments of M. Biot, we may collect, that the arc of rotation, produced by the thickness of a millimetre, is something more than 19 degrees for the ray we have chosen; so that the fraction $\frac{1}{3}$ may be taken to express nearly the length of that arc in a circle whose radius is unity. We have, therefore, $\theta = .03937$ inch, and $\rho = .333$. Substituting these values in the formula

$$\frac{l}{2\pi C} = \frac{\pi\theta}{a^3\lambda\rho}$$

derived from (12.), we find

$$\frac{l}{2\pi C} = 52710\,;$$

from which it appears that C is about twenty thousand times less than the millionth part of an inch.

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* Transactions of the Cambridge Philosophical Society, vol. iv. p. 205.

Again, since $a^2-b^2=.00489$, we have

$$\frac{l}{2\pi C}(a^2-b^2)=258,$$

so that equation (8.) becomes

$$k^2-258\sin^2\phi.\ k=1. \qquad (13.)$$

The results of this formula are compared with Mr. Airy's experiments in the following table, in which the less root is taken for k, and its sign is neglected.

Values of ϕ.	Values of k.	
	Observed.	Calculated.
6° 15′	tan 16° 38′ = .2987	.2980
8° 54′	tan 8° 56′ = .1572	.1579

The angles ϕ, in the first column, are deduced from the observed inclinations of the rays in air to the axis of the crystal; and as k was observed to be somewhat different for the ordinary and extraordinary rays, its mean values are given in the second column. The exact coincidence between these and the calculated values is, perhaps, in some degree accidental; but a less perfect agreement would be sufficient to confirm the theory.

The magnitude of k varies considerably with the colour of the light, increasing from the red to the violet, while the coefficient of $\sin^2\phi.\,k$, in formula (13.) diminishes. If we take the violet ray H, for example, this coefficient will be about 159. But it would be useless to make any more calculations, as we have no experiments with which they might be compared.

The figure of the wave surface yet remains to be examined.

Eliminating k between formulæ (5.) and (6.), we obtain the equation

$$(s^2-A)(s^2-B)=4\pi^2\frac{C^2}{l^2}, \qquad (14.)$$

which expresses the nature of the surface, s being a perpendicular from the origin on a tangent plane. From this equation it follows that the two values of s can never become equal in quartz, as they do in other crystals; for if we solve the equation for s^2, and put the radical equal to zero, we shall get the condition

$$(A-B)^2+16\pi^2\frac{C^2}{l^2}=0,$$

which cannot be fulfilled, since the quantity which ought to vanish is the sum of two squares. The two sheets, or *nappes*, of the wave surface, are therefore absolutely separated.

It is commonly assumed that one of the rays is refracted according to the ordinary law; but this is not the case, since neither of the values of s is constant. However, the ray which has the greater velocity, (a being greater than b,) may still, for convenience, be called the ordinary ray. Of the two roots of equation (8), the one k_o, whose numerical value (supposing ϕ not to vanish) is less than unity, corresponds to this ray. When C is positive, k_o is negative; and when C is negative, k_o is positive: therefore in both kinds of quartz, by formulæ (5) and (6), we have $s_o^2 > \mathrm{A}$, and $s_e^2 < \mathrm{B}$; denoting by s_o and s_e the respective velocities of propagation of the ordinary and extraordinary waves. Hence, if we conceive a sphere of the radius a, with its centre at the origin, and a concentric prolate spheroid, whose semiaxis of revolution is also equal to a, and parallel to the axis of the crystal, while the radius of its equator is equal to b, the ordinary *nappe* of the wave surface will fall entirely without the sphere, and the extraordinary *nappe* entirely within the spheroid, whether the crystal be right-handed or left-handed. With respect to the little ellipse in which the vibrations are performed, and of which the semiaxes parallel to x and y are represented by p and q respectively, it is evident that $p > q$ for the ordinary wave, since $k_o < 1$; and that $p < q$ for the extraordinary wave. When C vanishes, the minor axis of each ellipse also vanishes, and the rays become plane-polarized, the ordinary vibrations being then parallel to the direction of x, and the extraordinary parallel to that of y. This is exactly what ought to happen on the supposition that the vibrations of a plane-polarized ray* are parallel to its plane of polarization; a supposition which was kept in view in framing the fundamental equations (1.) and (2.).

To show, with precision, how the two kinds of quartz are to be distinguished by the sign of C, we must give definite directions to the axes of coordinates. To this end, let us imagine the plane of xy to be horizontal, and a circle to be described in it with the origin O for its centre; and let the north, east, and south points of this circle be marked respectively with the letters N, E, S. Let the direction of $+x$ be eastward, from O to E; that of $+y$ northward, from O to N; and that of $+z$ vertically downwards; the progress of the light through the crystal being also downwards, and the plane of the wave moving parallel, as before, to the plane of xy. Then the crystal will be right handed or left handed, according as C is positive or

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* On this point there are two very different opinions. Fresnel supposed, as is well known, that the vibrations of a plane-polarized ray are perpendicular to its plane of polarization; whereas, according to M. Cauchy, whom I have followed, they are parallel to that plane. I am induced to adopt the latter supposition, because I have succeeded, by means of hypotheses which are grounded on it, in discovering the laws of reflexion from crystallized surfaces; laws which include, as a particular case, those discovered by Fresnel for ordinary media. The hypotheses alluded to, along with some of their results, are published in the *London and Edinburgh Philosophical Magazine*, vol. viii. p. 103, in a letter to Sir David Brewster. See also vol. vii. p. 295, of the same Journal. I hope soon to offer the Academy a detailed account of my researches on this subject.

negative. For, if C be positive, k_o will be negative, and formulæ (4) will become, by exhibiting the sign of k_o,

$$\xi = p \cos\left\{\frac{2\pi}{l}(st-z)\right\}, \qquad \eta = -k_o\, p \sin\left\{\frac{2\pi}{l}(st-z)\right\}, \qquad (15.)$$

for the ordinary vibration ; and

$$\xi = k_o\, q \cos\left\{\frac{2\pi}{l}(st-z)\right\}, \qquad \eta = q \sin\left\{\frac{2\pi}{l}(st-z)\right\}, \qquad (16.)$$

for the extraordinary vibration. Now if we suppose the arc $\frac{2\pi}{l}(st-z)$ either to vanish, or to be a multiple of the circumference, the molecule will be at the east point of its vibration; and upon increasing the time a little, the value of η will become negative in (15.), and positive in (16.),so that the movement will be towards the south in the first case, and towards the north in the second. Therefore, when C is positive, the ordinary vibration takes place in the direction NES, or from left to right, and the extraordinary in the direction SEN, or from right to left, supposing a spectator to look in the direction of the progress of the light. It may be shown, in like manner, that, when C is negative, the ordinary and extraordinary vibrations are in the directions SEN and NES, or from right to left and from left to right respectively. Now if a plane-polarized ray be transmitted along the axis of the crystal, the plane of polarization will be turned in the direction of the ordinary vibration, because this vibration, being propagated more quickly, will be in advance of the other, upon emerging from the crystal. Hence, the rotation is from left to right when C is positive, and from right to left when C is negative; and the crystal is called right-handed in the first case, and left-handed in the second.

We have all along supposed that C is a constant quantity, and the agreement of our results with experiment proves that this supposition is at least very nearly true in the neighbourhood of the axis. It is probable, however, not only that C varies with ϕ, but that it becomes different in equations (1.) and (2.) ; that is to say, it is probable that the following equations

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} &= \mathrm{A}\frac{d^2\xi}{dz^2} + \mathrm{C}\frac{d^3\eta}{dz^3}, \\ \frac{d^2\eta}{dt^2} &= \mathrm{B}\frac{d^2\eta}{dz^2} - \mathrm{C}'\frac{d^3\xi}{dz^3}, \end{aligned}\right\} \qquad (17.)$$

in which C′ is a little different from C, would be more correct than those which we have assumed. Indeed Mr. Airy's experiments seem to indicate that C′ is greater than C; for he found, as we have already said, that the ratio of the axes of the little ellipse described by a vibrating molecule is somewhat different for the two rays, being more nearly a ratio of equality for the ordinary than for the extraordinary ray. Now if we set out from equations (17.), instead of (1.) and (2.), and proceed in all respects as before, we shall arrive at the formula

$$k^2 - \frac{l}{2\pi C}\left(a^2 - b^2 \right)\sin^2\phi . \ k = \frac{C'}{C} \qquad (18.)$$

instead of formula (8.). The quantity $\frac{C'}{C}$ will be greater than unity, if C' be greater than C, and the value of k_0 will be greater than before. This seems to be the explanation of the difference between the ratios observed by Mr. Airy.

It may be proper to state briefly the the considerations which led to the foregoing theory. Beginning with the simple case of a ray passing along the axis, the first thing to be explained was the law of M. Biot, that the angle of rotation varies inversely as the square of l or of λ. Now it was remarked by Fresnel, who first resolved the phenomena of rotation into the interference of two circularly polarized waves, that the interval δ between these waves, at their emergence from the crystal, must be inversely as l, if the angle of rotation be inversely as the square of l. This remark suggested* to me the idea of adding, to the equations of the common theory, terms containing the third differential coefficients of the displacements; for it was evident that such additional terms would give, in the value of s^2, a part inversely proportional to l. It was also evident, that the third differential coefficient of ξ should be combined with the second differential coefficients of η, and the third of η with the second of ξ, in order that, after substitutions such as we have indicated in deducing formulæ (5.) and (6.), the sines or cosines might disappear by division, and that thus the value of s^2 might be independent of the time, as it ought to be. This kind of reasoning led me to assume the equations

$$\frac{d^2\xi}{dt^2} = a^2\frac{d^2\xi}{dz^2} + C\frac{d^3\eta}{dz^3}, \qquad (19.)$$

$$\frac{d^2\eta}{dt^2} = a^2\frac{d^2\eta}{dz^2} + D\frac{d^3\xi}{dz^3}, \qquad (20.)$$

for the case of a ray passing along the axis of quartz; and then, substituting in these equations the values of the differential coefficients obtained by differentiating the formulæ

$$\xi = p\cos\left\{\frac{2\pi}{l}(st - z)\right\}, \qquad \eta = \pm\, p\sin\left\{\frac{2\pi}{l}(st - z)\right\},$$

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* "The singular relation between the interval of retardation [δ] and the length of the wave [l] seems to afford the only clue to the unravelling of this difficulty."—*Report on Physical Optics*, by Professor Lloyd; (*Fourth Report of the British Association*, p. 409). It was in reading this Report, that Fresnel's remark, about the relation between δ and l, first came to my knowledge.

which express a circular vibration, (from right to left, or from left to right, according to the sign of the second p,) the result was

$$s^2 = a^2 \mp \frac{2\pi}{l} C$$

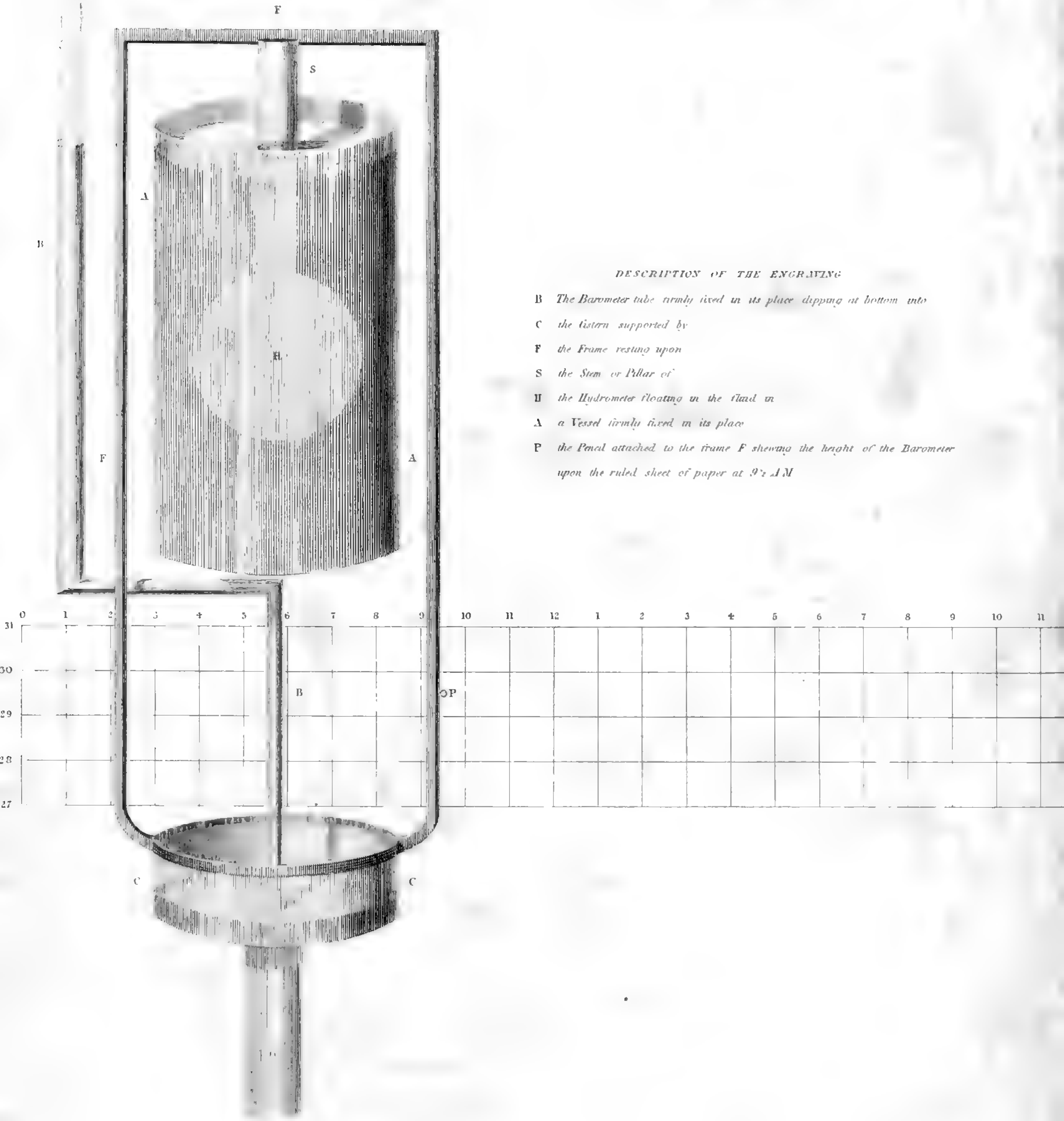
from (19.), and

$$s^2 = a^2 \pm \frac{2\pi}{l} D$$

from (20.); which showed that $D = -C$, since the values of s, corresponding to the same circular vibration, ought to be equal. The transition from this simple case to that of a ray inclined at a given angle ϕ to the axis, was easily made, by taking into account the doubly refracting structure of the crystal. This was done by supposing ξ and η parallel to the principal directions in the plane of the wave, and by changing a^2, in equation (20.), into $a^2 - (a^2 - b^2) \sin^2\phi$; and thus the fundamental equations (1.) and (2.) were obtained.

PROFESSOR STEVELLY'S

Self Registering Barometer



DESCRIPTION OF THE ENGRAVING

B *The Barometer tube firmly fixed in its place dipping at bottom into*

C *the Cistern supported by*

F *the Frame resting upon*

S *the Stem or Pillar of*

H *the Hydrometer floating in the fluid in*

A *a Vessel firmly fixed in its place*

P *the Pencil attached to the frame F shewing the height of the Barometer upon the ruled sheet of paper at 9½ A M*

An Investigation of the Principles upon which a new Self-Registering Barometer may be constructed. By JOHN STEVELLY, Esq. *Professor of Natural Philosophy in the Belfast Institution.*

Read Nov. 30, 1835.

My attention having been of late much directed to subjects connected with Meteorology, I could scarcely fail to remark how much time was consumed in making and recording observations, and how very limited, as to their extent, were our best observations, when considered in relation to the multiplicity of changes which are taking place at times when no person is present to observe or record. It is, therefore, an object of much interest to endeavour to construct instruments of every description, as far as is practicable, in such a way as that they may register their own indications, not only at stated hours of the day, but if possible at all times, and may place as clearly as possible the general results before the mind.

The barometer and the rain gauge are unquestionably two very important instruments in this science. The following observations will, I trust, clearly show how a self-registering barometer may be constructed; and a method of constructing a rain gauge capable of registering its own indications, may be so readily deduced from this, that it would be a waste of time to allude to it more particularly.

It is very obvious that if a pencil can be caused to rise and fall through equal distances, in such a way as to correspond with equal elevations and depressions of the barometer, this pencil may be made to press against a sheet of paper divided by twenty-four equidistant vertical lines, which would represent the twenty-four hours of the day; while the sheet may be carried by clock-work across the pencil laterally or in a direction at right angles to the vertical lines. The pencil would thus trace out on the sheet of paper, a curve which would present to the eye a correct view of the actual oscillations of the barometer during the several parts of that day; the height at each hour, or intervening portion of time, being readily and distinctly traceable. A second portion of this same sheet, or a second sheet, may then have the curve for the next day traced upon it similarly; and so on from day to day; these sheets being then dated, and arranged in consecutive order, would afford a correct register of the barometer, and thus we should have within our reach a means of ascertaining the height at which it had ranged at any instant of past time that our researches may render desirable.

But farther, the same sheet of paper being used on many successive days, for a great length of time, it is obvious that the points at which the point of the pencil arrived oftenest would become darker than those points at which it was only occasionally found at the same hour, and therefore the succession of these darker points would at length trace out the true curve of mean diurnal oscillation. In this case, and perhaps indeed always, it would be best to have the sheet of paper strained upon the surface of a cylinder, which the clock-work should cause to turn round once in twenty-four hours. In this way it is obvious that nearly all the labour and tedium of observing, recording, and afterwards reducing actual observations to mean results, may be avoided.

Most, if not all of these advantages, I conceive, may be attained by the use of an instrument, one modification of which I had the honour of hastily describing to the subsection of useful arts at the late meeting of the British Association in Dublin. I shall now give a more full and a more general description of the instrument, together with the formulæ, which will in all ordinary circumstances of variety of construction, give the connection of its scale with the scale of the common barometer. I shall also briefly point out the manner in which its indications are affected by changes of temperature, at least so far as may be required for making the necessary corrections.

During the oscillations of the common barometer, when it falls, a certain quantity of mercury is added to that already in the cistern, which of course increases its weight by so much. On the contrary, when the barometer rises, mercury retires from the cistern into the tube; the cistern thus becoming by so much lighter than it was before. If then the tube of the barometer be firmly sustained in its place, but the cistern be suspended by any mechanical means, so as to descend by arithmetical distances for equal additions to its weight, and to rise similarly when its weight has been similarly diminished; an index carried by the cistern may be made either to point to a fixed scale placed beside the instrument, as in the common barometer, or to mark on a sheet of paper a variety of positions corresponding with the synchronous variations of the height of the barometer, while, as will appear just now, the range of the scale of this new barometer may be made to bear any proportion, that may be desired, to the three-inch scale of the common barometer.

Although various mechanical means of suspending the cistern may be readily devised, either on the principle of the lever, or by certain curved surfaces made to turn freely on an axis, or even by simple counterpoise, the alteration of the buoyant force of a metallic cylinder as it is drawn more or less out of a fluid being made the means of restoring the equilibrium, yet for reasons which I shall not now stop to detail, I should prefer to any of these, the method of suspending it, derived from the buoyant force of a hydrometer; and I also conceive that a hydrometer floating in mercury is to be preferred, although almost any other fluid may be made to answer. This method of suspending, or, to speak more correctly, of weighing the cistern, was suggested to me by having long since observed that a most simple, cheap, and sensible instrument for weighing, may be constructed on the hydrometrical principle, by using an index, or the wire of a microscope attached to the vessel in which the hydrometer floats, for noting the position of the mark upon the stem of the hydrometer, instead of using the surface of the fluid for that purpose, as is usually done.

The following description of a barometer constructed upon these principles will, I trust, be readily understood.

Let B be a barometer tube, which may be of iron, turned accurately cylindrical, internally, where the upper surface of the mercury rises and falls , externally, where it dips into the cistern; its shape and dimensions, in other respects, is of no consequence. This tube must be firmly fixed in its place. Let C be the cistern which should be cylindrical near the surface of the mercury, suspended by the frame F from the cylindrical pillar or stem S of a hydrometer H, which floats in a vessel A cylindrical near its upper part, firmly fixed in its place, and filled to a proper height with mercury, oil, water, or any other proper fluid; it is now obvious, that if the barometer should fall, mercury would descend from B into C, add to the weight of C, and cause it to sink until the stem S should have displaced as much additional fluid as would be equal in weight to the weight added to C; on the contrary, if the barometer should rise, mercury would ascend from C into B, and the hydrometer would rise as far as was necessary for re-establishing the equilibrium by the emerging of part of S. A scale, it is also obvious, may be placed beside an index attached to any moveable part of instrument, which scale shall bear a certain proportion to the scale of the common barometer. Let us now endeavour to discover what that proportion is.

Let h denote the height of the common barometer at any instant (say when it stands at 30 inches).

Let δh denote a change in that height caused by a change of atmospheric pressure.

Let $\delta h'$ denote the vertical alteration of level of the surface of the mercury in the tube B, caused by that change.

Let $\delta h''$ denote the vertical descent or ascent of the cistern and hydrometer in consequence of that change; it is then obvious that $\delta h''$ is the part of the scale of the new barometer which corresponds with δh upon the scale of the common barometer.

Let $\delta h'''$ denote the rise or fall of the surface of the mercury in the cistern.

Let δh^{v} denote the rise or fall of the surface of the mercury in the vessel A, in consequence of the sinking or rising of the stem of the hydrometer.

Let s denote the internal cross section of the tube B at its upper part; s' the external cross section of B where it dips into the cistern ; s'' the excess of the cross section of the cistern, at the surface of the mercury, above s' ; s''' the cross section of the pillar or stem of the hydrometer; s^{v} the excess of the cross section of the vessel *A*, at the surface of the fluid it contains, above s'''. All these quantities are supposed to have one inch for their linear unit. Let w denote the weight of one cubic inch of mercury, and if the hydrometer float in any other fluid than mercury, let w' denote the weight of one cubic inch of that other fluid.

Then $\delta h''-\delta h'''$ is the part of the lower portion of the tube *B* which emerges from the mercury in the cistern, or becomes more immersed in it, in consequence of the oscillations of the instrument. And $\delta h''+\delta h^{v}$ is the portion of the stem or pillar of the hydrometer which becomes immersed in the fluid in the vessel *A*, or emerges from it in consequence of the same oscillation.

Again $\delta h.'\ s.\ w-(\delta h''-\delta h''')\ s.'\ w$ is the alteration of weight of the cistern, and $(\delta h'+\delta h^{v})\ s.'''\ w'$ is the alteration of the buoyant force of the hydrometer, and that the equilibrium may continue these must be equal; hence

$$\delta h.'s-(\delta h''-\delta h''')\,s'=(\delta h''+\delta h\ )\,s.'''\ \frac{w'}{w} \qquad \text{(D)}$$

Also because the perpendicular distance of the surface of the mercury in the tube ***B*** from the surface of that in the cistern *C* is at all times equal to the height of the common barometer, therefore

$$\delta h=\delta h'-\delta h''+\delta h''' \qquad (1)$$

And from considering the cause of the rising or falling of the surface of the mercury in the cistern, it will appear that

$$\delta h'.s-(\delta h''-\delta h''')\,s'=\delta h'''\ s'',\ \text{hence}$$

$$\delta h'''=\frac{\delta h's-\delta h''s'}{s''-s'} \qquad (2)$$

and by elimination from (1) and (2) we get

$$\delta h=\frac{\delta h'\,(s''-s')-\delta h''\,(s''-s')+\delta h's-\delta h''s'}{s''-s'};\ \text{hence}$$

$$\delta h'=\frac{\delta h\,(s''-s')+\delta h''s''}{s+s''-s'} \qquad (3)$$

and by substitution of this in (2) we get

$$\delta h'''=\frac{\delta h\ s+\delta h''s-\delta h''s'}{s+s''-s'} \qquad (4)$$

Also by considering the cause of the rising or falling of the surface of the fluid in the vessel *A* it will be seen that

$$(\delta h''+\delta h^{v})\ s'''=\delta h.^{v}\ s^{v}.)\quad \text{Hence—}$$

$$\delta h^{v}=\frac{\delta h''\ s''}{s^{v}\div s'''} \qquad (5)$$

Then by substituting in *D* the respective values for $dh,'$ $\delta h'''$ and δh^{v} found in (3) (4) and (5) we obtain

$$\frac{\delta h\ (s''-s')\,s+\delta h''.\,s''.s-\delta h''.s'\ (s+s''-s')+\delta h.s.s'+\delta h''.s.s'-\delta h''.s'.s'}{s+s''-s'} =$$

$$\left(\frac{\delta h''\,(s^{v}-s''')+\delta h''s'''}{s^{v}-s'''}\right)\ s'''\ \times\frac{w'}{w}$$

Hence concinnating both sides, we have

$$\frac{\delta h.s''.s+\delta h''.s''\ (s-s')}{s+s''-s'} = \frac{\delta h''.s'''.s^{v}}{s^{v}-s'''}\times\frac{w'}{w}$$

Hence we deduce the general equation

$$\delta h=\delta h''\left[\frac{s^{v}.s'''(s+s''-s')}{s''.s\ (s^{v}-s''')}\times\frac{w'}{w}-\frac{s-s'}{s}\right] \qquad \text{(E)}$$

In which equation it is obvious that the multiplier of $\delta h''$ is a constant quantity; and since that multiplier is capable of receiving any value by altering the dimensions of the cross sections, of the upper and lower part of the tube *B*, of the cistern, of the pillar of the hydrometer, and of the vessel *A*, we have it in our power to establish any relation we may desire between δh and $\delta h''$ that is between the scale of the common barometer and of this new barometer.

When the fluid used in the vessel *A* is mercury, which for many reasons is to be preferred, $w'=w$ and equation *E* becomes

$$\delta h=\delta h''\left[\frac{s^{v}.s'''(s+s''-s')}{s.s''\ (s^{v}-s''')}-\frac{s-s'}{s}\right] \qquad \text{(G)}$$

Equation *(E)* is more general; but when *(G)* is used, a much less ball will be required for the hydrometer, the other dimensions being the same. In these equations the effects of diversified atmospheric pressure alone, is considered; no account being as yet taken of a variation of temperature.

In the modification of this instrument, which I described popularly to the subsection of useful arts of the British Association, I supposed the cross sections of the cistern and of the vessel *A* to be so large as that the level of the surfaces of the mercury in them was not materially altered by the oscillations of the instrument; also, in order to diminish or remove its oscillations, I supposed the cross section of the part of the tube *B* which dips into the cistern, to be very small. Now, if in the equation *(G)* we suppose s^{v} and s'' to be very large and s' to be very small, it assumes the form

$$\delta h=\delta h''\ \left(\frac{s'''-s}{s}\right)$$

in which case if s''' the cross section of the stem of the hydrometer, were equal to s the cross section of the upper part of the tube *B*, the multiplier of $\delta h''$ would be cipher and $\delta h''$ would be infinite for any finite δh, or the new instrument would require an infinitely long scale, and would therefore be useless; but if s''' were double of s, the multiplier of $\delta h''$ would be unity and the scale of the instrument would be precisely the same as that of the common barometer; between these, any desired length of scale may be obtained by proportioning the pillar of the hydrometer to the cross section of the upper part of the tube *B*. But of course the formula affords the more correct rule for constructing the instrument, so as to suit the use for which it is intended. I shall, however, give one other example of a particular construction of it, before I proceed to examine the effects of changes of temperature upon the indications of this barometer.

If the internal cross section of the upper part of the tube *B* be made equal to the external cross section of that part of it which dips into the cistern, that is if $s=s'$; and if the cross section of the vessel *A* near the surface of the fluid it contains be so large that s^{v} shall be nearly equal to $s^{v}-s'''$; then will (G) become

$$\delta h=\delta h''\frac{s'''}{s} \quad \text{or,} \quad \delta h'' : \delta h :: s : s'''$$

that is, the scale of this instrument will bear to the scale of the common barometer, the ratio of the internal cross section of the upper part of the tube *B* to the cross section of the stem or pillar of the hydrometer. Now, these conditions may be obtained while the passage for the mercury through the lower part of the tube *B*, shall be as small as is necessary for getting rid of the irregular oscillations, before alluded to; and since the cross section of the vessel *A*, must be always pretty large, this rule will be sufficiently exact to enable a workman to guess at the size that the instrument will be when constructed, as also to know before hand, pretty exactly what length of scale an instrument of which he had only made the tube *B* and the hydrometer stem would require; or, vice versa, what relative magnitudes he should give these, in order to produce an instrument with a required length of scale; all these considerations also will enable an intelligent person before hand to calculate with sufficient exactness the cost of a rerequired instrument constructed on this principle.

Let us now suppose the pressure of the atmosphere to remain unvaried, but the temperature to change, and let us endeavour briefly to investigate the manner in which the indications of the instrument are affected, and the allowances to be made, corrections to be applied, or the precautions to be adopted in consequence of that change.

Any person who has attended to the subject of compensating pendulums, will readily perceive, that there is a certain position in the frame which supports the cistern, depending upon the materials of which the hydrometer is constructed, and upon its dimensions, as also upon the frame, and scale, or whatever supports the sheet of paper on which the curve of barometric oscillation is to be delineated; at which position, if the index or tracing pencil be fixed, any alteration of temperature will move it as much upwards, in consequence of the expansion or contraction of one part, as downwards, in consequence of the corresponding expansion or contraction of other parts, so that the position of the index will remain unchanged, notwithstanding any change of temperature experienced by the frame work of the instrument. I shall accordingly suppose the index or pencil to be so placed, and then we need not take into account the effect of any expansion or contraction of the mere frame work of the instrument.

But there are three other causes, in a barometer of this particular construction, affecting its indications by changes of temperature, which I shall now notice, and endeavour to trace their effects. The first is, that when the mercury in the tube expands, by an increase of temperature, part of it will be required to descend into the cistern, namely, the pillar which would have been otherwise raised upon the expansion of its cross section; but since the material of the tube (suppose it to be iron) expands likewise, the quantity of mercury that will truly have to descend into the cistern, is only the column corresponding to the excess of the expansion of the mercury above the expansion of the tube; the effect is, of course, reversed when contraction takes place by a fall of temperature. It is then, upon the excess of the expansion of the bulk of mercury in the tube above the expansion of the capacity of the tube, and *e contra*, that this effect depends. It is likewise to be observed, that a mere vertical lengthening or shortening of the barometric column, by changes of temperature, since it leaves the weight of the column unchanged, has therefore, no effect upon the weight of the cistern. The second cause is, that the effect of the capacity of the part of the tube which is plunged into the mercury in the cistern upon its weight, alters on three accounts—first, because the capacity of that part of the tube itself alters; secondly, because the level of the surface of the mercury in the cistern is changed; also, thirdly, the specific gravity of the mercury which it displaces, is changed; and on these accounts, the weight of mercury in the cistern, is virtually changed, and the indication of the instrument affected. The third cause is, that the buoyant force of the hydrometer is affected: first, by its own change of bulk; secondly, by the change of the weight of the bulk of mercury, whose place it occupies; thirdly, by the alteration of the portion of its pillar or stem which is immersed in the fluid, partly by the sinking or rising of the hydrometer itself; partly by the alteration of the level of the surface of the mercury in which it floats. Upon these principles we can, in the following way investigate the effect of changes of temperature upon the indications of the instrument.

Let us suppose the materials of which the parts of the instrument are constructed, to be cast iron and mercury.

Let c = capacity of the mercury in the barometer tube B at 30 inches pressure and 60° temperature.

Let c' = bulk of same tube B which is immersed in fluid in cistern.

c'' = bulk of fluid in the cistern.

c''' = bulk of fluid in vessel A.

c^{v} = capacity of part of hydrometer immersed in fluid in vessel A.

Let δm, denote the variation of cubic unit of mercury $\Big\}$ for a change of temperature of 1° Fahrenheit.
δi cast iron

$dt =$ the total change of temperature from 60°.
$dh =$ the sinking of the index of the instrument caused by that change.
$dh' =$ the elevation of the surface of the fluid in the cistern, so caused.
$dh'' =$ the elevation of the surface of the fluid in vessel A.

Then will $c\,(\delta m - \delta i)\,dt =$ the column of mercury which descends from B.
$c''\,(\delta m - \delta i)\,dt =$ change of volume of mercury in cistern.
$c'\,\delta i\,dt =$ change of bulk of the part of B immersed in cistern.
$dh' - dh =$ total alteration of immersion of B in fluid in cistern.

Then by considering the causes of alteration of level of surface in the cistern, it will be seen that

$$(dh' - dh)\,\frac{s'}{s''} + \frac{c'\,di\,dt}{s''} + \frac{c''\,(\delta m - \delta i)\,dt}{s''} + \frac{c\,(\delta m - \delta i)\,dt}{s''} = dh'. \quad \text{Hence}$$

$$dh' = \frac{(c+c'')\,(\delta m - \delta i)\,dt + c'\,\delta i\,dt - s'\,dh}{s'' - s'}; \qquad \text{and}$$

$$dh' - dh) = \frac{(c+c'')\,(\delta m - \delta i)\,dt + c'\,\delta i\,dt - s''\,dh}{s'' - s'}. \qquad \text{hence}$$

$$\frac{(c+c'')\,(\delta m - \delta i)\,dt + c'\,\delta i\,dt - s''\,dh}{s'' - s'}\,.\;s'\,w = \left\{\text{weight virtually added to the cistern, in consequence of change of part of tube } B \text{ immersed in cistern.}\right.$$

$$c'\,\delta m\,dt.\,w = \left\{\text{weight lost by cistern in consequence of the decrease of specific gravity of fluid displaced by part of } B \text{ immersed in cistern.}\right.$$

$$c\,(\delta m - \delta i)\,dt.\,w = \left\{\text{weight of mercury which descends from tube } B \text{ into the cistern.}\right.$$

$$\frac{(cs'' + c''s')\,(\delta m - \delta i)\,dt + c's'\,(\delta m + \delta i)\,dt - c's''\,\delta m\,dt - s's''\,dh}{s'' - s'}\,.w = \left\{\text{total weight gained by the cistern.}\right.$$

$$\text{Again, } (dh + dh'')\,\frac{s'''}{s^{v}} = \left\{\text{rising of surface in vessel } A\text{, in consequence of total sinking of the pillar of the hydrometer.}\right.$$

$$\frac{(c''' - c^{v})\,(\delta m - \delta i)\,dt}{s^{v}} = \left\{\text{rising of surface in vessel } A\text{, in consequence of change of volume of mercury, vessel } A\text{, and hydrometer.}\right.$$

hence,

$$\partial h'' = \frac{(dh + dh'')\,s'''}{s^{v}} + \frac{(c''' - c^{v})\,(\delta m - \delta i)\,dt}{s^{v}}, \quad \text{from whence we deduce}$$

$$dh'' = \frac{dh.\,s''' + (c''' - c^{v})\,(\delta m - \delta i)\,dt}{s^{v} - s'''}; \quad \text{and from hence}$$

$$dh + dh'' = \frac{s^{v}dh + (c''' - c^{v})\,(\delta m - \delta i)\,dt}{s^{v} - s'''} = \text{total sinking of pillar of hydrometer; from}$$

whence we infer

$$\left(\frac{s^{v}.\,dh + (c''' - c^{v})\,(\delta m - \delta i)\,dt}{s^{v} - s'''}\right) s''.\,w = \left\{\text{increase of buoyant force caused by the sinking of the hydrometer.}\right.$$

also

$$c^{v}\,(\delta m-\delta i)\,dt.\,w=\left\{\text{diminution of buoyant force caused by the increased volume of the mercury and the hydrometer.}\right.$$

hence by subtraction we deduce

$$\frac{s^{v}\,s^{\prime\prime\prime}\,dh+(c^{\prime\prime\prime}\,s^{\prime\prime\prime}-c^{v}v^{v})\,(\delta m-\delta i)\,dt}{s^{v}-s^{\prime\prime\prime}}\,.w=\left\{\text{total increase of the buoyant force of the hydrometer.}\right.$$

hence by equating the total gain of weight of the cistern to the total gain of buoyant force of the hydrometer, and dividing both sides by *w*, we deduce the equation

$$\frac{(cs^{\prime\prime}+c^{\prime\prime}s^{\prime})\,(\delta m-\delta i)\,dt+c^{\prime}s^{\prime}\,(\delta m+\delta i)\,dt-c^{\prime}s^{\prime\prime}\,\delta m\,dt-s^{\prime}s^{\prime\prime}\,dh}{s^{\prime\prime}-s^{\prime}}=$$

$$\frac{s^{v}s^{\prime\prime\prime}\,dh+(c^{\prime\prime\prime}s^{\prime\prime\prime}-c^{v}s^{v})\,(\delta m-\delta i)\,dt}{s^{v}-s^{\prime\prime\prime}}$$

from whence by the ordinary processes of concinnation we deduce

$$dh=\frac{[(cs^{\prime\prime}+c^{\prime\prime}s^{\prime})\,(s^{v}-s^{\prime\prime\prime})-(c^{\prime\prime\prime}s^{\prime\prime\prime}-c^{v}s^{v})\,(s^{\prime\prime}-s_{\prime})]\,(\delta m-\delta i)+c^{\prime}s^{\prime}\,(s^{v}-s^{\prime\prime\prime})\,(\delta m+\delta i)-c^{\prime}s^{\prime\prime}\,(s^{v}-s^{\prime\prime\prime})\,\delta m}{s^{v}s^{\prime\prime\prime}\,(s^{\prime\prime}-s^{\prime})+s^{\prime\prime}s^{\prime}\,(s^{v}-s^{\prime\prime\prime})}\times dt \qquad \text{(K)}$$

being the value, for the entire effect of change of temperature equal to the number of degrees of Fahrenheit, *dt.* upon the indication of the instrument.

From equation *(K)*, when the dimensions of the several parts of a barometer on this principle are known, we might, were it desirable, calculate the rising or falling of the instrument, consequent upon a change of temperature, for we know from the mean of very accurate observations by several eminent philosophers, that $\delta m=.0183345\ qp$, and that $\delta i=.003367.\ qp$. But any dependance upon such a calculation would in practice be by no means desirable, as the sources of error would be too many to admit of much confidence in the result. The better practical method, however, is readily deduced from a careful consideration of the foregoing equation, as follows.

In equation *(K)* it is obvious that $s, s^{\prime}, s^{\prime\prime}, s^{\prime\prime\prime}, s^{v}\ dm$, and δi are all independant of the amount of atmospheric pressure, but that $c, c^{\prime}, c^{\prime\prime}, c^{\prime\prime\prime}$ and c^{v} are all dependant upon that pressure, and are each functions of $\delta h^{\prime\prime}$ of the form $\gamma+\gamma^{\prime}\,\delta h^{\prime\prime}$; γ and $\gamma^{\prime}$ being certain constant quantities. The entire equation *(K)* therefore, if such values were substituted for each of these equations, would assume the form

$$dh=(\Gamma+\Gamma^{\prime}\,\delta h^{\prime\prime})\,dt \qquad \text{(L)}$$

In which Γ and $\Gamma^{\prime}$ are constants, the values of which are very easily discoverable by two simple experiments, in which the effects of two known changes of temperature upon a given instrument, while the pressure remained unvaried, should be accurately noted, or more correctly should be deduced from means of many pairs of observations, the pressure being maintained unchanged during each pair. Should the fluid in which the hydrometer is sustained, be any other than mercury, the division of both sides of the equation from which we deduce (K) by w, cannot be resorted to, as the w which multiplies the second member, is the weight of a cubic inch of that other fluid; but practically this would be of little consequence, as the *form* of the equation (L) would not be affected by this difference.

As I stated in the beginning of this paper, it is obvious that a self-registering rain gauge may be constructed on a similar principle, by conducting the rain from a proper funnel into a vessel like the cistern sustained by a hydrometer, the descent of which would become an index of the quantity of rain received by the funnel, and a pencil might be made to trace at the several portions of the entire day, the height at which the gauge stood at each instant by means of a sheet of paper carried across it by clock work.

ANTIQUITIES.

On an Astronomical Instrument of the Ancient Irish. *By* SIR WILLIAM BETHAM, *Ulster King of Arms*, M.R.I.A., F.A.S., *&c. &c.*

Read 23d of May, 1836.

We learn from Cæsar that the Druids of Gaul were the philosophers and men of science, as well as the priests, of the Celtæ.

First—As priests, they were occupied in expounding religion, and ordering and directing sacrifices.

Secondly—The education of youth was solely committed to them.

Thirdly—As judges they settled all disputes and controversies.

Fourthly—That their learning and science was derived from the British Islands, where all persons who wished to obtain a perfect education, went to study, and to become adepts required attention for twenty years.

Fifthly—That they were a literate people, and in their common concerns of life, both public and private, they used, in writing, a character similar to the Greek.

Sixthly—They taught the doctrine of the metempsychosis.

Lastly—"They also taught the youth many points touching the motion of the heavenly bodies, the magnitude of the earth, the nature of the world and of things, and the dignity and power of the gods."

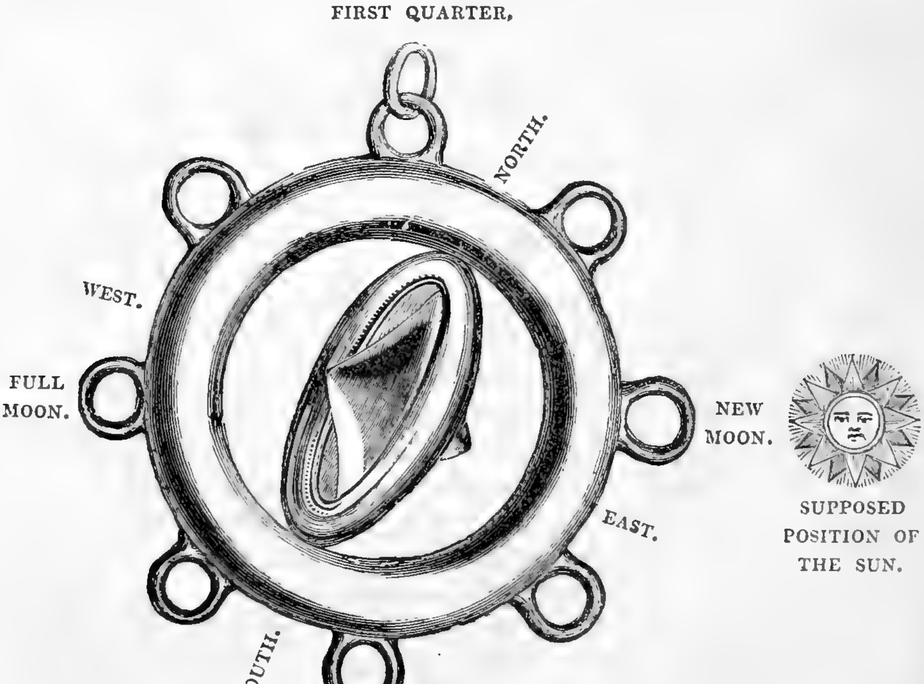
It thus appears that the Celtæ were learned in the sciences of their day. Their youth were taught astronomy and other abstruse and difficult subjects, the doctrines of Pythagoras, and the learning of the Egyptians. In fact, they were highly instructed, and knew as much as was then known by any people.

Hibernia being inhabited by the same race of people as Gaul, was, in fact, as one of the British Islands, the chief seat of Druidic learning, for Cæsar distinctly states, that all who wished to perfect themselves in the sciences and learning of the Druids, went to the British Islands for instruction. I trust, the time has arrived when the subject of Antient Irish History can be logically discussed, with patient candour, free, as well from the sneers of those who condemn, because they do not understand, as from the injudicious zeal of those who, by crediting every wild notion, have injured the cause they have endeavoured to promote.

I beg to call the attention of the Academy to a very singular instrument lately found in Ireland, which has fortunately been preserved from the melting pot by our excellent friend and zealous antiquary, the Dean of St. Patrick's. If I am not greatly mistaken, it is a Celtic astronomical instrument, invented to exhibit to the pupil a diagram of the earth's polar inclination, and the phenomena of the phases of the moon. It is certainly a rude effort, but it displays considerable ingenuity, and a progress, or, at least an attempt, towards demonstration. It is very nearly the diagram now made use of, in elementary works on astronomy, for the same purpose.

People possessing so correct a notion of the heavenly bodies, and skill to form an instrument so nearly representing their motions and real state, were, at all events, a discriminating race. It required a long period of study, a succession of ages of learning, with a state of quietude and undisturbed repose, to have arrived at such a point of scientific knowledge.

It is of Celtic brass, and evidently was in common use; for though parts of many have been found, no other, as far as I have seen, was perfect. It consists of a circle, the outside edge of which represents the moon's orbit, having on it eight rings representing the different phases of the planet. In the inside of this circle, is another fixed on an axis, in the line of the inclination of the poles, on which this, which represents the earth, traverses.



It is not possible, I think, to imagine any other use for this instrument, or to deny that assigned to it.

On the upper ring was another loose one, apparently for the purpose of hanging it up, which was broken in the pocket of the Dean as he brought it to me; but it is preserved, and exactly fits the broken part, so as to indicate its true position, as represented in the drawings.

The outside edge is the equator or ecliptic, with the phases of the moon. The interior moveable circle exhibits the inclination of the earth's axis to the equator.

It will be observed, that the learning and doctrines of the Druids, as related by Cæsar, exactly tally with those of the Phenician philosopher Pythagoras, which, as far as astronomy is concerned, is shortly summed up in the following passage in a work published by Mr. Mason Good, and Doctor Olynthus Gregory:

"In astronomy his (Pythagoras) inventions were many and great. It is reported, that he discovered and maintained the true system of the world, which places the sun in the centre, and makes all the planets revolve about him; from him it is to this day called the Old or Pythagorean System, and is the same as that revived by Copernicus. He discovered that Lucifer and Hesperus were but one and the same, being the planet Venus, though formerly thought to be two different stars. The *invention of the obliquity of the Zodiac*, is also ascribed to him. He first gave the world the name of *Kosmos*, from the order and beauty of all things comprehended in it, asserting that it was made according to musical proportion; for he held that the sun (by him and his followers termed the fiery globe of unity) was seated in the midst of the universe, and the earth and planets moving around him; so he held that the seven planets had an harmonious motion, and their distances from the sun corresponded to the musical intervals or divisions of the monochord.

"Pythagoras and his followers held the *transmigration of souls*, making them successively occupy one body after another."

Let us see how exactly this agrees with what Cæsar says of the Druids:—

"Imprimis hoc volunt persuadere *non interire animas, sed ab aliis post mortem transire ad alios:* atque hos maxime ad virtutem exercitari putant metu mortis neglecto. Multa præterea *de sideribus atque eorum motu, de mundi, ac terrarum magnitudine,* de rerum natura de deorum immortalium vi ac potestate disputant, et juventuti transdunt."—*Lib.* VI. 13.

In this short passage is condensed the precise philosophy of Pythagoras which is declared to have been taught to the youth by the Druid philosophers of the Celtæ.

Pythagoras was born at Sidon, in Phenicia, about the 47th Olympiad, or 590 years before our era; but the Greeks say his father was a Greek merchant of Samos, where he was brought when young; but his thirst for knowledge not being satisfied with the ignorance which prevailed at that place, at eighteen years of age he travelled, first to

Syros, to see Pherecydes the philosopher, then to Miletus, to Thales, then to Sidon, in Phenicia, where he remained some time, and then to Egypt, where Solon and Thales had been before him, and stayed there twenty-five years. He also went to Chaldea, to visit Babylon.

I am aware that Pythagoras is claimed by the Greeks, but they admit his birth and education to have been Phenician. The Greeks were furtive in this respect. According to them, Hercules was a Greek, and at the same time, they admit the Tyrian Hercules was the most ancient who bore that name. There is little doubt, that had the Tyrian Hercules not existed, we should never have heard of the Greek hero. The fact is, they borrowed their science, learning, gods, heroes, and philosophers, all from the Phenicians; and there are strong grounds for doubting every Greek relation which has reference to their national vanity.

Pythagoras taught what he learned from his masters at Sidon—the science, learning, and philosophy, long known and taught in Phenicia; which, however, being new to the Greeks when promulgated by Pythagoras, were by them attributed to him as his own discoveries, and the works of his great mind.

Here we have a glimpse of the reality on which is grounded the tradition of the ancient learning of the ancestors of the Irish people—that they were a colony of Phenicians. If not clearly established, every advance in the inquiry seems to confirm that opinion.

On the Ring Money of the Celtæ, and their System of Weights, which appears to have been what is now called Troy Weight. By SIR WILLIAM BETHAM, M.R.I.A., *Ulster King of Arms, &c. &c.*

Read 23d of May, and 27th of June, 1836.

That the Britons at Cæsar's invasion, were considerably advanced in civilization, fully appears from his statements. They had large ships, and much trade and commercial intercourse with Gaul, Spain, and Germany. The island was populous ; they had good houses ; great abundance of cattle ; and, among other proofs of a polished and cultivated people, they possessed a well regulated and graduated metallic circulating medium. " Hominum est infinita multitudo, creberrimaque edificia, ferè Gallicis consimilia, pecoris magnus numerus, utuntur autem nummo aureo aut annulis ferreis ad certum pondus examinatus pro nummo. Nascitur ibi plumbum album in mediterraneis regionibus, in maritimis ferrum, sed ejus exigua est copia, ære utuntur importato." Many circumstances are stated in this short passage which only could refer to a people somewhat advanced in civilized life. These observations are equally applicable to the Celtæ of Hibernia, which the remains daily found under her surface fully substantiate.

Suetonius says, (in Cæsare, cap. 54,) " In Gallia fana templaque Deum donis referta expilavit, urbes diruit, sæpius ob prædam quam ob delictum, unde factum est ut auro abundaret ternisque millibus nummum in libras promercale, per Italiam provinciasque divenderet."

The great abundance of gold found by Cæsar in the cities and temples of Gaul, absolutely diminished its value in Italy.

It appears also, from Cæsar and Diodorus Siculus, that there was a great abundance of gold among the Celtæ ; the people wore the richest and most ponderous ornaments of that metal, as torques, chains, breastplates, and even had the frontlets of their helmets covered with plates of gold. Many specimens of all these have been found of great weight and value in Gaul and Britain ; but, in recent times, those found in Ireland greatly exceed them, both in number and value. Several are still in the Dean of St. Patrick's and other museums.

Many reasons may be given for the more frequent occurrence of these remains in Ireland than in either Britain or Gaul The former has ever been more a grazing than an agricultural country, and therefore its surface has been much less disturbed than the other parts of Celtica; the plough and the spade have been less active, therefore the remains of the magnificence of former ages have there remained undiscovered.

The cultivation of the potato has been, however, of late years, greatly instrumental to the discovery of these antiquities: most of those found without the bogs, have been brought to light by the potato-planter's spade.

The existence of bogs, or peat moss, of such immense extent in Ireland, has also greatly operated in preserving the remains of antiquity which have been therein deposited, either by accident or design—I say design, for no doubt many valuable articles have been purposely hidden on sudden alarms.

They were secure places of secretion of the precious metals from the incursions of an enemy or a sudden emergency; this description of property could be deposited without any external indication to lead the plunderer to the spot; and a token or clue could easily and safely be fixed to guide the owner, when the danger had passed, to the certain recovery of his property. In many instances, no doubt, the possessors of the secret fell by the swords of their enemies, and thus the treasure remained, until accident, in recent times, brought it to light.

The bog is also of such a nature as sometimes to overgrow the level of the ground adjoining, and overflow it, and thus cover even the habitations of men, and with them all their valuables. An instance of this having taken place has lately been discovered by Captain Mudge, R. N., while employed in making a survey of the coast of Ireland. In cutting out a bog he discovered a series of wooden buildings of a very rude character; and the stone axes with which the timbers had been hewed into shape were found on the spot, which were, no doubt, covered by the moving bog, while in the very act of construction, and so remained until discovered by Captain Mudge. These must have been built of a period previous to the Celtic invasion—as stone axes were the implements of the *Tuath de daonæ*, or Northern People.

The aggregate amount of the articles of manufactured *gold* found in the course of twelve months in the bogs and fields of Ireland, is truly surprising—most of them of exquisite and elaborate workmanship, particularly torques, helmets, breastplates, bracelets, rings, and ring-money, with many implements, the use of which it is difficult even to suggest. Besides those which come under the notice of the antiquary or the curious investigator, immense quantities are silently broken up and sold by the finders as old gold, lest the owners of the soil should make their claim, and deprive the finders of the fruits of their good fortune.

Ancient silver articles are, however, of much rarer occurrence, at least those which may be considered Celtic. It may safely be said that there are found a thousand articles in gold to one of silver. This may possibly be attributed to the ease with which gold was collected, compared with the exertion necessary to obtain silver—the latter requiring all the labour and skill of mining and refining operations, while gold is found frequently, if not generally, in a pure state in the soil washed down by the mountain streams.

Cæsar tells us, that the Gauls "*use for money, gold and iron rings, by certain weight.*" The latter have perished by oxidation, but the two former are found, in great abundance, in the fields and bogs in every part of Ireland. These curious remains are so perfectly anolagous to the accounts given of the Britons and Gauls, by Greek and Roman writers, that they of themselves afford the most powerful testimony of the identity of the origin of the ancient Irish and that people. To gold and iron, may be added silver and brass rings of a graduated weight.

There are also great quantities of rings of jet, coal, or ebony, found in our bogs, which may possibly have passed as a circulating medium : but there is not, as far as I have discovered, any authority beyond conjecture that they were so used ; although it is well known that such substances have, in other countries, been used as a circulating medium.

It has often been objected against the Irish pretensions to early civilization, that no very ancient coins, or medals, of the early Irish monarchs have been found ; and, certainly, the absence of any indication of a metallic circulating medium, would supply a fair inference of a low state of commercial intercourse ; but, on the other hand, the appearance of a well-regulated, convenient, and graduated circulating medium of the precious metals, demonstrates an advanced progress in civilization. There have, however, been found in Ireland many specimens of very ancient silver coins, the legends on which have never been deciphered, or appropriated to any monarchs or people, and yet remain unexplained.

These were, certainly, very early attempts at coinage ; and as exactly similar coins are often found in Britain, it is a fair conclusion, that they were the production of the ancient Celtic inhabitants of each country.

But it is not necessary to rest the pretensions of the Celtæ, to the possession of a metallic circulating medium, on these rude specimens of early coinage ; the period of their fabrication is recent, when compared with the circulating medium which, we have now irrefragible testimony, existed for ages previous to our era, in all parts of Celtica.

It is very remarkable, that *rings of gold and iron being used as money* among the Britons, appears to have been an idea new to Cæsar ; there is no remark that any such medium was known at the time to have existed among any nation with whom he was acquainted.

The kings of Lydia, we are told by Herodotus, were the first who coined metallic money, about 600 years A.C. and the practice soon after obtained universally among the nations with whom they had intercourse. The discontinuance, and a lapse of 560 years, in Cæsar's day, had obliterated the recollection that the ancient currency of rings had ever existed. Recent investigations have exhibited evidence, that the most ancient of nations used rings of gold and silver for money, or a metallic circulating medium.

So extensively a commercial people as the Phenicians, of whom the Celtæ were unquestionably colonists, could not long carry on their affairs of trade by the means of barter and exchange. They would soon feel the necessity for something defined to represent property, and the precious metals would be naturally suggested as the readiest mean, and weight would be adopted as the measure. They were, in all probability, the inventors of ring-money, for they were certainly the first people who carried on an extensive commerce.

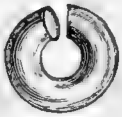
Gold and silver wire, cut into equal lengths, were most likely the first attempt at money, because the pieces could more easily be made of the required weight and value. Straight pieces are inconvenient of carriage, would wear a purse, or bag, and escape from small apertures; this inconvenience naturally suggested twisting the wire into the form of a ring, such were the gold ring-money of the Gael of Gaul, Britain, and Ireland. The brass rings were cast, first, like those of gold, afterwards in a perfect ring; and both are every day found in Ireland in great numbers.

Vast quantities of articles in gold and brass are also found, the use of which have not a little puzzled the learned antiquaries. Vallancey calls them pateræ, but as *pateræ* they are of the most inconvenient shape; they will not stand, so as to hold a liquid; and, having two cups, one would discharge itself while a person was drinking from the other. Vallancey supposes them sacrificial cups, and that they were used to pour forth oblations to the gods; others have fancied them to be used to cover the breasts of the *dea mater*. All these speculations, I conceive, are untenable, as some of them have flat surfaces, instead of cups, consequently could not be used as pateræ, as they would contain no liquid whatever.

The objections against their being fibulæ, are equally cogent; a buckle, or fastening of gold, of *fifty-six ounces weight*, appears absurd; besides, undoubted fibulæ of the precious metals and brass are found in Ireland, in great quantities, of convenient and palpable shapes.

Their peculiar form appear to render them incapable of application to any active operative purpose; and the only conclusion which appears satisfactory as to their use is, that they were ingots of gold, or the larger species of the circulating medium, and but a variety of the ring-money.

The following specimens will illustrate my notice of the transition from the straight wire to the ring, and from them to the larger ingots or gold money.

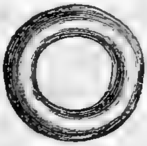


This is the most common form of the smaller gold ring-money found in Ireland. They are made of pieces of gold wire formed into the required thickness, cut into lengths of equal weights, and then bent round into the shape above represented. I have seen counterfeits which have been occasionally found, made of cast brass, exactly of the same shape and size, and so neatly covered over with a coating of gold plate, as to defy detection unless weighed. There can be no doubt of the fraud being ancient, for the brass is of copper and tin, the same as the brazen spear-heads and other Celtic utensils of that metal.

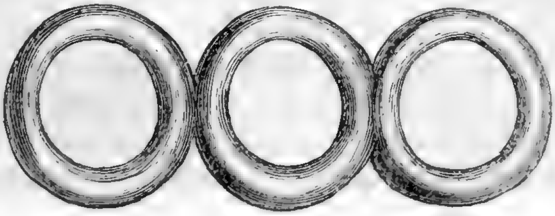
The smallest I have seen, weighed 12 grains, or half a pennyweight, which seems to have been the *unit* by which the larger were graduated, for all have been found to bear a relative value to it; taking this for the *unit*, the rest are its multiples. Among those in the possession of the Dean of St. Patrick's, and G. Petrie, Esq. are the following:

			oz.	*dwt.*	*grs.*			
No. 1	*1.	weighing	0	0	12			
2	1.	*Dean of St. Patrick's,*	0	1	12	equal to	3	of No. 1*
3	3.	*do.*	0	2	12	=	5	of do.
4	2.	*do.*	0	3	16	=	$7\frac{2}{3}$	
5	1.	*do.*	0	5	0	=	10	
6	1.	*do.*	0	11	0	=	22	
7	1.	*do.*	0	11	8	=	$22\frac{1}{3}$	
8	1.	*George Petrie, Esq.*	0	8	0	=	16	
9	1.	*Mr. Stewart, Goldsmith,*	12	0	0	=	480	
10	1.	*Alderman West,*	13	7	0	=	534	

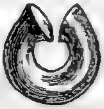
It will be observed, that there are but Nos. 4 and 7 in all these which do not contain in weight the exact value of their multiple No. 1*: thus No. 2 is equal to 3 of No. 1—No. 5 equal to 10—No. 6 to 22—Nos. 4 and 7, these may have been wasted by use or fraud. They, however, contain each two fractional thirds of No. 1.; and it is possible they may have so graduated for convenience of exchange, as our half-crown is contrived to represent 2s. 6d., or as the old quarter of a guinea represented 5s. 3d.



This is a specimen of the simple cast brass ring, and is found in immense quantities, from the weight of one pennyweight to that of several ounces. Many are found double and treble, of which the following wood-cut will give a correct idea:

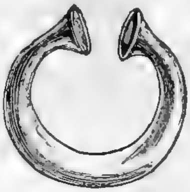


In the Memoir lately published by the Ordnance Surveyors, mention is made, from the Irish Annals, of many rings having been presented by the princes of Ireland to the corb or successor of St. Columbkill. In 1151, Cooly O'Flynn presented one weighing two ounces, and another in 1153, of one ounce. In 1004, Brian Borroihme presented to the altar at Armagh, a ring of gold weighing 20 ounces. It is observed that all these are described of equal weight in *ounces*. Such passages are of very frequent occurrence in the Irish Annals.

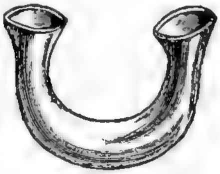


This is of brass, and is the first variation or change from the perfect circle ; it is larger than the largest of the perfect brass rings : it was cast, and they are found in very great quantities together in bogs ; the points are flat.

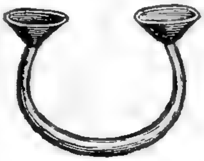
This is also of brass, but larger than the last specimen which it much resembles, but the points are larger and at a greater distance, with flat surfaces. It is obviously a progressive and farther deviation from the perfect circle, as the weight and value increased. Its diameter is somewhat more than two and a-half inches. An immense number of these, many thousands, as many as loaded a large cart, were found in a kind of tumulus, in the county of Monaghan, a few years since. They were sold for old brass, which is the general fate of the brazen articles found in Ireland. One specimen, of which the above is a representation, is preserved in the museum of George Petrie, Esq.

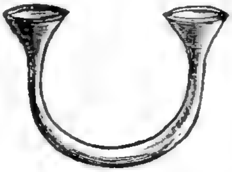


This specimen is of gold, and weighed 4 oz. 16 dwts. ; it was found, about two years since, with ten others, under a large stone in the County of Mayo. They were sold to Alderman West of Dublin, who sent them to me for inspection and to have drawings made. This has more of the circle than the last specimen, but varies in having the points of a concave or cup-like shape.

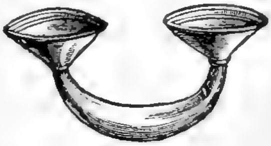


This was found with the last, which it resembles, but its points are more distant; it is of gold, and weighed 9 oz.; it has deep cups at the points, which originally, perhaps, were made to regulate its weight.

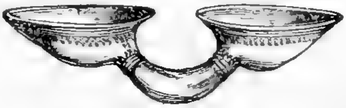




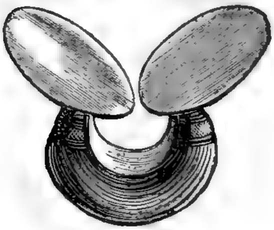
These are very attenuated specimens. The first only weighed 16 dwts. 12 grs., and is in the Museum of the Royal Dublin Society; the other two ounces. The cups are large and very thin; the transition is gradual from the last.



This is also of gold, and weighed twelve ounces; the cups are very large. Another weighed 3 oz. 12 dwts. I have seen many other specimens of greater weight; all, however, of equal penny-weights.



This is also of gold, and weighed 19 oz.: it was found in a stone chest in the church-yard of Ballymoney, in the County of Antrim. I had one nearly, if not precisely, of the same figure, which weighed 36 oz.; and Vallancey mentions one which weighed 56 oz. The cups of this are five inches in diameter. Two others are mentioned by Vallancey, one weighing 15 oz., the other 1 oz. 12 grs. It should be observed, that one or two have been found of a weight two or three grains less than the even twelve grains, which may have been lost by attrition, but they are very few— I have met with but two or three, and they may be considered exceptions to the rule.



This specimen is singular in its shape, from its very broad, thin, and flat points, which are of equal thickness, except just at the junction with the stem. They are found of various sizes and weights, some of the plates are two inches in diameter, but all very thin. One specimen, in the collection of Alderman West, has, on the back of one of the plates, a small loop, through which a cord of the size of a pack-thread might pass.

It is very difficult to decide on this specimen; it bears a strong resemblance in some respects to its predecessors, and yet it differs so much that it might be considered a different article altogether.

To look at the last five specimens, unconnected with the others, no one would imagine them *ring-money;* but, seeing the gradual variations, I think they may fairly be considered as the same. The necessity of stringing small money is obvious to preserve them from loss, but for the larger and more valuable this precaution is not required.

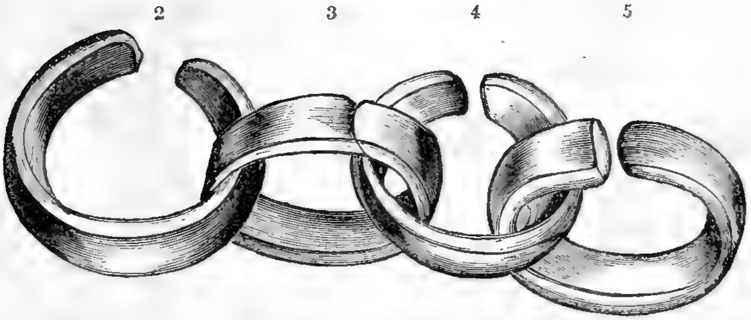
These ring coins of the Gael suggest a very early period of civilization, before the Phenicians struck medallic coins on flat plates, with the effigies of a sovereign, or emblem of a people, and a legend or inscription. The Phenicians were, at a very early period, acquainted with the art of coining money; and as there are very few, if any, instances of Phenician coins found in Ireland, the period of their intercourse must have been of very remote antiquity. I have seen but one brass coin which was thought to be Phenician, and that of doubtful character—their intercourse most likely ceased with Ireland before they coined money on plates.

SILVER RINGS.

Since the foregoing was written, the Dean of Saint Patrick's has placed in my hands six rings *of silver*, of very rude workmanship. I have never seen any similar, and, at first sight, could scarcely believe they could be money; but further investigation, and their graduated weight, led me to the affirmative conclusion. I have weighed them, and find they also, like gold rings, are all multiples of the half penny-weights, as follows:

		oz.	*dwts.*	*grs.*
No. 1	weighs	2	10	12
2	——	1	14	12
3	——	1	7	12
4	——	1	2	0
5	——	0	19	12
*6	——	0	19	0

These also are so nicely adjusted to their weight that they nearly balance the scales.



BRASS RINGS.

Mr. Petrie has also placed in my hands three cast brass rings, which fit most cu-

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* No. 6 is exactly like the others in shape. Mr. Alderman Brady has lately purchased several other silver rings exactly like these, and of the same graduated weight.

riously one within another; they have been considered money, and are perforated laterally through in such a manner as to admit a thong of leather or cord to pass through them. They are of the old Celtic brass. Their weight also is of equal penny-weights.

	oz.	*dwts.*
No. 1, the outside, weighs	7	9
2, the second, weighs	3	18
3, the centre, weighs	1	13

Having been perforated, they are not so exactly balanced as the gold and silver, though they are very nearly of equal penny-weights, but just turning the scale.

The smaller specimens, however, of the brass ring-money which are perfect, are quite as accurately balanced as the gold and silver; of these I lay before the Academy eight specimens, and it is singular that, with one exception, they exhihit odd multiples of the unit of twelve grains.

	dwts.	*grs.*
No. 1, or the largest is	19	12
2, ...	15	12
3, ...	8	12
4, ...	3	12
5, ...	2	0
6, treble rings	8	12
7, double rings......	7	12
8, ditto...........	3	12

No. 5 alone consists of even penny-weights. I have weighed a great many of these rings, and found them, without a single exception, multiples of half a penny-weight. It would, indeed, be difficult to persuade ourselves that this circumstance could be accidental. The ring money, gold, silver, and brass, as Cæsar tells us, was "*ad certum pondus*;" and that weight, all our specimens show, was formed on the same scale, or, perhaps, was derived from the same original as the Troy weight. A pound Troy of gold thus formed 480 rings, weighing each half a penny-weight, 40 of which were equal to an ounce. It is not easy to say whether their system was duodecimal or decimal—from these specimens it might have been either. If we could to a certainty say their penny-weight consisted of 24 grains, we might conclude it was duodecimal; and the coincidence of all the other fractional parts being so accurately Troy weight, and several of the specimens being of duodecimal multiples of the unit, and all consistent with with it, seem to lead to the conclusion, that it was the same ancient system from which the Troy weight was derived.

To what remote period of antiquity do these singular facts carry us back! To many ages before the time of Cæsar, or even Herodotus. The latter speaks, as I have observed, of the Lydians as the first who coined medallic money, at least six centuries before our era. These are no visionary speculations: we have here the remains and imperishable reliques of these early times to verify the whole; and recent investigations and discoveries, in a most singularly convincing manner, come to our aid by showing that the fresco paintings in the tombs of Egypt exhibit people bringing, as tribute, to the foot of the throne of Pharoah, bags of *gold and silver rings*, at a period before the Exodus of the Israelites.

It is right to observe that many of these weights were taken by me before I had an idea that these things were money.

The Troy weight is said to have been brought to Europe from Palestine and Egypt by the Crusaders, and obtained its present name from the town of Troyes, in France, where it was first used at the great fair held there. There is reason to believe that it is the old Phenician mercantile standard weight which once prevailed all over the East, and that, like most other commercial improvements, originated with that great commercial people, as they must have first felt the necessity for such a means of adjustment of those commodities which were disposed of by weight.

The old Celtic (rnꞅe) *unsha* was the exact ounce Troy—it is a compound word aon, *one*, and ꞅe, *sixth*, or the one-sixth part of a given weight, containing the quantity of our half pound Troy; the name of this weight I have not yet been able to ascertain. The weight of twelve ounces, now called a pound, having eventually prevailed in general computation of larger quantities, the word *unsha*, as the *twelfth part* of a pound, became the Celtic, and from them the Latin, word signifying the *twelfth part of anything*, even of time.

I submit the foregoing remarks, if not with reluctance, yet with diffidence; for although I feel satisfied that the evidence sustains all the conclusions I have drawn, I am more anxious for truth than for any hypothesis; and shall be glad if the discussion produces a correct result, even if it should be contrary to what I myself have fondly fancied to be irrefragibly established.

NOTES.

In the interesting work published by Mr. Wilkinson on the Thebaid and Egypt, in which it appears, that the most ancient money of that country, even before the Exodus of the Israelites, was gold and silver rings of a graduated weight, as the following extracts will show:

"In the second line black chiefs of Cush or Ethiopia bring presents of *gold rings*, copper, skins, fans, or umbrellas, of feather-work, and an ox bearing on its horns an artificial garden and a lake of fish."—*Wilkinson's Thebaid, p.* 136.

"A continuation of these presents follows in the third line, where, besides *rings** of gold, and bags of precious stones, are the cameleopard, &c."

Thothmes III. (1495 B.C. p. 154.) "The money used at that epoch was, as I have already observed, of *gold and silver rings*."

"On the right hand are some very elegant vases of what has been called the *Greek style*, but common in the oldest tombs of Thebes. They are ornamented as usual with arabasques and other devices. Indeed, all these forms of vases, the *Tuscan* border, and the greater part of the painted ornaments, which exist on Greek remainss, are found on Egyptian monuments of the earliest epoch, even before the Exodus of the Israelites, and plainly removes all doubts as to their original invention. Above these are, chariot makers and other artisans. Others are employed *in weighing gold and silver rings*, the property of the deceased; their weights are an entire calf—the head of an ox (the half weight), and small oval balls (the quarter weights.) They have a very ingenious mode of preventing the scale from sinking when the object they have weighed is taken out by means of a ring upon the beam," p. 151—vide Genesis xliii. 21, "*our money in full weight*."

Pomponius Mela thus speaks of the Phenicians: "Phœnicen illustravere Phœnices solers hominum genus, et ad belli pacisque munia eximium literas et literarum operas aliasque etiam artes maria navibus adire classe configere, imperitare gentibus regnum prælianque commenti."—*Vixit A.D. I. under Claudius, De Situ orbis.*

They were the shrewdest and most acute of mankind—skilled eminently in the arts of peace and war, and by their skill and valour, kept the empire of the seas, and governed nations—skilled in science, literature, and the arts—addicted to navigation and commerce.

When Pliny and other ancient authors declares the *Pœni* to have been the inventors of navigation and astronomy, he intended the Phenicians and not the Carthagenians. It was under the conduct of the Phenicians, the fleets of Solomon sailed to Ophir and Tarshish, from the ports of Eilath and Esiongeber, in the Red Sea. Ophir was the general name of the eastern coasts of Africa, which was always a great market for gold-dust. Tharshish was also a general name for *all distant countries.*

The Greeks, in their lists of the nations who have been masters of the Mediterranean, give the seventh place to the Phenicians, and the eighth to the Egyptians; but they were always reproached by the Egyptians, as novices in antiquities, as they really were. The Egyptians do not appear to have ever been a naval power, except by their allies, the Phenicians.

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* The *money* of the Ethiopians and Egyptians was in *rings of gold and silver*, like those still in use about Sennaar. I had interpreted the hieroglyphic signifying silver, "*wrought-gold*," but the white colour of the *rings* placed opposite to others painted yellow, (in another tomb at Thebes) decides the question in favour of the word silver.

Arbuthnot on Weights and Measures. 4to. *London*, 1754.

"The Romans used the *libra*, which they divided into twelve uncia; and the later Greeks had their litra, which they divided into the same measure."—p. 47.

"The ratel, or litra, used all over Egypt, is of different quantities in several places, and in the same place for several goods, but always divided into twelve parts, which are their ounces."—p. 55.

According to Appian—"As to Gaul, Cæsar exacted from it yearly *quadringenties*, £322,916. 13s. 4d." —p. 194.

"Lipsius is of opinion that *quatermillies* should be read for *quadringenties*, which would make the sum ten times bigger, viz. £3,229,166. 13s. 4d. But it is not probable that Gaul would be able to pay such a sum yearly. However Velleius Paterculus affirms that Gaul was reckoned on the same footing with Egypt as to taxes.—p. 194.

"The commerce of the Phœnicians lying more towards the west than that of the Egyptians, was the occasion of their being celebrated as the inventors of astronomy and navigation. When Pliny names the Pæni, as inventors of navigation, it must not be understood of the Carthagenians, but the Phœnicians, from whom they were descended. They navigated into the ocean by the straights of Gibraltar—established many colonies—Thebes in Beotia, Cadiz, and Carthage itself, which was built fifty years before the destruction of Troy. It was under the conduct of the Phœnicians that Solomon's fleet sailed to Ophir and Tharsis, from the ports of Ailath and Esiongeber on the Red Sea. Ophir was the general name of the eastern coast of Africa—and Tharsis, that of the western coast, both of Africa and Spain. This commerce, Jehosaphat, king of Judah, endeavoured to restore; but his enterprise was blasted by the destruction of his vessels in the harbour."—p. 218.

"It is past all doubt that the Cape of Good Hope was doubled in those early times, and that the Portuguese were not the first discoverers of that navigation."—p. 219.

"The Phœnicians were much older sailers than the Greeks; the naval expedition of their Hercules mentioned by Sanchoniathan, under the name of *Malcanthus*, being 300 years before that of Jason."

"The Corinthians were said to (have been) be the inventors of weights and measures, though both their sea craft and arithmetic came originally from the Phœnicians."—p. 248.

"Strabo relates, that the commodities of Britain were corn, cattle, gold, silver, iron, skins, leather, and hunting dogs; and speaking of the Cassiterides, tin and lead. Tacitus adds pearls."—p. 253.

"Tin and lead were used in the time of the Trojan war."—p. 254.

"Arabia was a country of great commerce in the time of the Romans. *Aden* had in its harbour, ships from all parts of the world. The Gerrheans and the Mineans, ancient inhabitants of Arabia, carried their spices, by land, to the frontiers of Palestine."—p. 364.

Aden was afterwards called *Portus Romanus*.

The Affinity of the Phenician and Celtic Languages, illustrated by the Geographical names in Ptolemy and the Periplous of Arrian. By SIR WILLIAM BETHAM, S.R.I.A., *Ulster King of Arms, &c. &c.*

Read 23d of May, and 27th of June, 1836.

Many years since, I ventured to suggest, that the inhabitants of Celtic Gaul, Britain and Ireland, at Cæsar's invasion, spoke the language now called Irish or Gaelic, and that they were the only true Celtæ. I have since, being satisfied that the ancient Caledonians were an exception, they being of Cimbric or Gothic origin, and the ancestors of the Picts and Welsh.

With this exception, farther investigation have convinced me of the accuracy of my opinions; and I have had the satisfaction of the approval of many most competent individuals who have given the subject consideration, and flatter myself, that the time is not far distant when there will be no doubts on the subject.

The more difficult question of—from what branch of the great family of mankind the Celtæ proceeded—I have endeavoured to answer by showing the cognate character of the languages of the Celtæ and the Phenicians, if they were not precisely the same, and that the former being a colony of the latter, the Celtic language demonstrated the signification of the names of Greco-Phenician geography, as well as those of the deities of the Greek and Roman theology.

" Language, (says the writer in the *Quarterly Review**) under the guidance of the extensive research and philosophic spirit of modern philology, has been the safest clue to the application of remote races."

" The grounds from which we may infer the affiliation or the relationship of the different races of mankind and similarity—

1st. Of languages, including written characters.

2d. Religion.

3d. Civil Institutions.

4th. Manners.

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* Sept. 1835, p. 433. Micali on the ancient peoples of Italy.

5th. Of Arts, and perhaps

6th. Physical Form.

But these points of similarity must be so marked and peculiar, as not to be resolved into the common habits and usages of mankind in a similar period of civilization."

The surest test, indeed, of the affinity of nations, is the community of language· The Phenicians, no doubt, were the earliest civilized maritime visitors of Western Europe, and conferred names on countries and places when they discovered and visited them for the first time; and though their language may be said to have been lost, or hitherto unknown, yet it remains, in a degree, in the names they conferred. The names given to places by maritime discoverers, are usually, if not universally, significant; and, as it is well known that Ptolemy borrowed the names for his geography from the Phenician mariners, if we can discover an existing tongue in which the names to be found in Ptolemy *are significant*, and accurately describe the peculiar features of the places and things which bear them, we may fairly conclude that language to be a branch of the Phenician, and the people speaking it, the descendants of their colonists.

That language is found in the Gaelic or Irish and Scottish Celtic.

By these names we are able to trace the proceedings and voyages of these primitive mariners in every portion of the world, known to the ancients, and their wanderings and maritime enterprise, at a period, compared with which, the Greek and Roman was, are occurrences of yesterday. It may, without any great stretch of imagination, be surmised that even Egypt was greatly indebted to the illustrious *Homeritæ*, or *Arabian Phenicians*, as well for the first dawn of her commerce and consequent civilization, as for her knowledge of alphabetic writing.

In my volume, on the Gael and Cymbri, I have treated largely on the names of that part of Europe, inhabited by Phenician Celtæ, which appear clearly significant in the Gaelic tongue, and fully bear me out in presuming the fact of the Celtæ being Phenician colonies.

A test has lately presented itself to my mind, which at once appeared likely to settle the question satisfactorily and conclusively.

I felt strongly, if my hypothesis be true, that the Gaelic and Phenician were originally the same tongue, the names of places in the Eastern or Indian Seas, also *avowedly borrowed* from the Phenicians by Ptolemy, ought also to be as significant as those of Europe, which he had from the same source. I now propose to give the result of my investigation into this proposition.

It was my intention to have made a few brief remarks on the Etrurians, and on Italy generally, but on dipping into that question, so much did it interest me, and led me so far, that I found that I should do injustice to a subject which deserves a more critical and serious investigation. I have discovered enough to induce me to think that the solving the mystery of the origin of the Pelasgi and the Etruscans is within my reach: I shall, therefore, reserve that subject for a future paper.

Heeren's Researches has rendered services to historic literature, surpassing all other writers, and too much praise can scarcely be given, or any eulogium, perhaps, be too laudatory, for the soundness of his reasoning and the general accuracy of his deductions; it is, therefore, with great diffidence and reluctance I venture to promulgate opinions differing, or even slightly varying from his, and but for the irresistible force of the evidence I have to adduce, as it appears to my mind, I would not venture on so great a responsibility.

Heeren supposes, and argues on the supposition, that Sidon, Tyre and Aradus, were the first seats of Phenician greatness—that they were a Syrian people, *a Canaanitish tribe*, which settled on the coast, the descendants of Ham. "*The oldest of these, the first-born of Canaan, according to the Mosaic record, was Sidon*," the foundress *of the trade and navigation of the Phenicians*." It is true, as far as the western greatness of the Phenicians was concerned, that Sidon was their first seat, but erroneous as to the eastern world and their origin; they were no originally Syrians, or descendants of Ham, but from Shem; they were Arabians, from Yemen, or Arabia Felix, and originally Chaldeans.

It is singular that Heeren should have considered so lightly, and worthy of so little weight, what Herodotus states so distinctly and clearly, that the Phenicians were a colony of the *Homeritæ* (which name he says, means the same as Phenician *in their language)* a great commercial people, who inhabited the southern coast of Arabia Felix; and Dyonisius declares they came from Chaldea, to the Persian Gulph, and eventually settled on the southern coast of Arabia." There, from their ports of Aden, Hargia, Cana, &c. they carried on their commerce with all parts of the Erythrean ocean for ages before they knew of the existence of the Mediterranean. They first possessed the island of Tyrus (now Tylos) and Aradus, in the Persian Gulph, from which it is said, and truly, I think, the Tyrian cities borrowed their names. The inhabitants of those islands, claimed to be the parent country of their Syrian namesakes.

Heeren observes—"The principal direction in which the Phenician race extended itself by colonization, was towards the west, *because, from their situation, their sea trade could take no other*. But notwithstanding this, so soon as their land trade through Asia had reached the coasts of the Indian ocean, the want of settlements there must naturally have been felt. Traces of them, though certainly only doubtful traces, are found both on the Persian and Arabian Gulphs. The names of two islands in the midst of the Persian Gulph, named *Tyrus*, or *Tylos*, and *Aradus*, bear striking marks of Phenician origin; and in these have lately been discovered vestiges of Phenician workmanship and buildings."

Again, in the chapter on Babylonish commerce, he says—"First, however, there remains an investigation, as obscure as it is important, concerning some islands situated near this coast, which, as they are said to have been eminent trading places, must not be passed over in silence. In the Greek geographers, for instance, we read of two islands named Tyrus, or Tylos, and Aradus, *which boasted that they were the mother country of the Phenicians, and exhibited relics of Phenician temples.*"

Pliny and Strabo are the principal authorities, yet they are both indebted to more ancient authors. "On sailing farther south from Gerrha, (says Strabo) we came to two islands, where there are to be seen Phenician temples; and the inhabitants assure us that the cities of Phenicia *are colonies from them.*"

It appears very extraordinary, that with this testimony before him, Heeren should still speak of the Phenicians *as a Syrian tribe,* and of their commerce, as originating from the Syrian cities, and that we should meet with such passages as the following:

"Having thus shown the direction and extent of the trade and navigation of the Phenicians, towards the west, let us now bend our course eastwards, and trace their progress on the two great south-western gulphs of Asia, the Arabian and Persian. In these, it has already been stated, they had *partially settled ;* and thus gained secure harbours, from which to set forth on their still more distant enterprises.

"It must, however, be at once perceived, that their navigation *could not have a like undisturbed continuance with that of the Mediterranean : as the proper dominions of the Phenicians never stretched so far as either of these gulphs.* It depended upon their political relations, how far they could make use of the harbours they possessed there. For even though the way might be open to caravans, the dominant nations of inner Asia might not be always willing to allow foreign colonies on their coasts.

Their navigation upon the Arabian gulph, arose out of their connexion with the Jews, and the extension of dominions of the latter under David." So far Heeren.

Had the learned Professor allowed due weight to the evidence of Herodotus, Strabo, and Pliny, one would conceive it impossible he could not have observed, that the operations in the east must have been carried on by a people possessing territory, cities, and harbours, on those seas. The whole tenor of the evidence is so strong, that it appears really astonishing, so acute and able a reasoner as Heeren, should not have seen that the Syrian cities were, and must have been, but colonies of the Homeritæ or Asiatic Phenicians, as they are stated to be by Herodotus, Strabo, and Pliny.

We must, therefore, look to a much more remote period for the origin and history of the Phenicians, to a time when the sites of their Syrian cities of Tyre, Sidon, and Aradus, were to be marked out. Herodotus says, Tyre was 2400 years before his time; and the prophet Ezekiel speaks of it about the same time, as being of the "*antiquity of ancient days.*"

Arabia Felix, or the kingdom of Yemen, may safely be considered as the previous country of the Phenicians, where, under the names of Homeritæ, and Sabeans, they established and carried on an extensive commerce with India and all the coasts of the Erythrean or Indian ocean. Their ships visited the eastern coasts of Africa, and supplied Nubia and the countries about the upper regions of the Nile, with the rich manufactures and products of India, and obtained gold, ivory, and other products of that country, in return. Their ports of *Cana, Aden, Saba*, and *Macala*, situated in Arabia Felix, at the entrance of the Arabian Gulph, were admirably circumstanced as to locality for the purposes of commerce, both with Asia and Africa.

The direct testimony of Herodotus would be sufficient, in my mind, to settle the question, that the Sabeans or Homeritæ were the same people as the Phenicians; but the evidence I shall soon lay before the Academy, comes with irresistible force, in corroboration of the statements of the father of history, and shows forth his fidelity and accuracy in a most extraordinary, if not an unexpected manner.

We are now contemplating a period of remote antiquity—a time long before any written history, previous to the foundation of those most ancient cities, Sidon and Tyre, of whom the prophet Ezekiel says, "*whose antiquity is of ancient days;*" and the former is said to have been the first seat of Phenician commerce on the Mediterranean. It is not, perhaps, venturing too much, if we suggest, that through the African ports of *Sabe*, *Avalites Emporium*, and *Malao*, commerce and consequent civilization laid the foundation of Egypt's glory and greatness; and that the Sabean and Homeritæ were the first to "lighten up the flame of commerce and consequent humanity" among that most ancient and magnificent people.

With respect to the extent of the commercial and geographical discovery of the Homeritæ, or Arabian Phenicians, history is silent, or nearly so: it has not come down to us but in imperfect glimpses, confused and unsatisfactory: the queen of the Sabeans, who visited Solomon, appears as sovereign of a highly cultivated nation, and supplies a few hints. Dyonisius (Periegetes) declares, they were originally from Chaldea; but little, indeed, do we know of their acts or the extent of their power and commerce. The extent of their maritime discovery, however, for the first time, if I have not deceived myself, are within our reach; but before we enter into the detail, that the subject may be better understood, it may be as well to glance briefly at the origin and early history of the Arabians.

The old patriarchal government and history of the Arabians, as detailed in the sacred writings, prove them to have been a very ancient people, and trace them back to ages near the deluge. They are divided into classes—the primitive Arabians, and the descendants of Ishmael, from whom the present Arabians are descended.

The primitive Arabians are generally derived in descent, from *Jokten* the son of *Eber*, or *Heber*, of the line of *Shem*, whose son *Jarab*, or *Yarab*, is said (after the confusion of Babel) to have founded the kingdom of *Yemen*, and his brother *Jorham*, that of *Hejaz*.

The kingdom *Yemen* was governed by princes of the tribe *Hamyar*, son of *Saber*, great grandson of Joktan, but at length passed to the descendants of his brother Cohlan, who retained the title of *king of the Harngarites,* (by the Greeks called *Homerites).* It is said to have continued in existence 2020 years, when the inundation of Aram, soon after the time of Alexander the Great, dislodged many of the tribes who emigrated to other countries.

Such is briefly the early received history of Arabia. It rests mainly on tradition, and, as was usual, especially with the Greeks, a personage is constructed to give name to a people. *Hamyar* is made the patriarch and ancestor of the Homeritæ. Herodotus, however, tells us *the name of Homeritæ was significant,* and had *the same meaning* as *Phenicians,* being evidently conferred, in consequence of the profession and habits of the people, that is, a *mariner* or *navigator of the sea.* This statement of the father of history is more rational and satisfactory than any tradition, especially when dependant on so slender a foundation.

The *Homeritæ* may, therefore, be considered the most ancient and primitive inhabitants of Arabia Felix, who flourished for ages, as the greatest, and perhaps, the only maritime commercial people of antiquity. They were conquered, and probably exterminated long after the foundation of Tyre, by the warlike descendants of Ishmael; and their commerce and mercantile settlements, or colonies, if they formed any, were transferred to their Syrian colonies who, in their turn, possessed of the empire of the seas, and after a long possession of the commerce of the world, and great glory, they eventually fell under the sword of a conqueror, whose sagacity and enlarged mind discovered the cause of their greatness, and ever assiduously promoted the commerce of his own subjects. Alexander acquired thereby, the name *of Great,* more deservedly, than from the deeds of arms which made his name terrible, as a scourge of the human race.

After the fall of Tyre, Carthage, her most illustrious western colony, succeeded to her commerce and consequent wealth, power, and dignity, in the western world. In her turn, she fell a victim to the jealous rivalry of Rome; and with Carthage, the Phenician race, *as rulers,* ceased altogether. The savage decree of *delenda est Carthago,* was extended to her records, muniments and monuments, which were so sedulously destroyed, that scarcely a vestige remains of this once great and illustrious people. The Greeks, indeed, acknowledge that they owe to the Phenician Cadmus their alphabet; but the Romans would not tolerate the idea, and did all they could to efface the recollection and remembrance of a greater and more illustrious nation than themselves—a people who did so much to promote the cause of humanity and the civilization of mankind; who conquered but the bad passions of mankind, and by commerce, taught man that it was the interest of all that each should be prosperous and happy.

A dark and almost impenetrable cloud has since obscured the Phenician story; even the language was apparently annihilated; and thus was removed the only certain means, and key, or clue for ascertaining the extent of their commerce, geographical discoveries, and extended empire, in the names given by them to the countries, rivers, places, and people, they visited for commercial purposes. The course of events placed most of their European colonies under the power of the Romans, whose policy led them to make their newly conquered provinces essentially *Roman,* in customs and language; and thus the Punic tongue soon ceased to be colloquial in the countries conquered by the Romans.

In one solitary, separated corner of the remote west, a colony of Phenicians escaped the overwhelming influence of the Roman sword, and kept their language and traditions pure and unmixed. Ireland was never visited by a Roman, at least we have no historical notice: the Romanised Britons probably visited the island for commercial purposes, but never with a view to conquest.

The Irish language, although losing ground every day, but before it was entirely extinguished, the zeal of modern investigation has discovered its identity with the Phenician, demonstrating the fact almost beyond a question; and the discovery has been productive of the most important results to history, geography, and philology, and opens a view into events which occurred long before the existence of written history.

Whenever the names of places, as they appear in Ptolemy's Geography, are *significant,* and explain their peculiarity of character in the Gaelic or Celtic tongue, we may conclude they were conferred by the Phenicians. By these names we may now define and mark out the extent of the voyages and discoveries of those adventurous and bold mariners, with almost unerring accuracy. They have, in fact, put a label upon each which tells its story, if not so fully and clearly as certainly as if we possessed it written by Sanconiathon, or in the records of Tyre and Sidon, or their more ancient sisters of Saba, Aden, Avalites, Semaa and Corana.

The geography of the ancient known world, which has hitherto been a jargon of unmeaning names, without sense or import, becomes intelligible, clear, and significant. The character of many nations, and tribes who inhabited regions, of which no written or even traditional history is extant, is strikingly exhibited, and the reasons for the names they bear at this day, made clear, explicit, and satisfactory.

The deadly and unwholesome malaria of the Deltas of the Ganges, Surawady, and other great rivers of the east, is pointed out with alarming epithets, and the savage cruelty of the people, and the dangerous rocks and unhospitable coasts, are made known to voyagers, by warnings contained in their names; while places of secure retreat and supply, are expressed by such inviting demonstrations, as would not fail to induce the mariners to visit a port, whose name indicated secure and refreshing repose after his labours, sufferings, and deprivations.

The extent and importance of the early commerce of the Homeritæ, Sabeans, or Phenicians of Arabia, may be traced by their names along the coasts of the eastern seas, where they established, *chiefly upon islands*, at convenient points, *fortified depots* or *entrepots* for their goods and merchandize, which answered, as in modern times, the double purpose of protecting warehouses and supply to their vessels in long voyages.

Herodotus, as before stated, tells us that the Homeritæ, who inhabited the southern coast of Arabia, were also called by the Greeks, Phenicians, and that the name meant in Arabic, the same thing as Phenician. By Arabic, must be meant the ancient language of Arabia Felix—*that of the Homeritæ*, not what is now spoken by the descendants of Ishmael.

Herodotus in this, as in all other cases, is faithfully correct—***Phenicia*** is the country of the *ploughers of the sea*—Homeritæ, that *of mariners*—ꝼeın, *a ploughman*—oıcc, *of the sea*—ȷıa, *country*;—ua, *the country*—maꞃaıꝺe, *of mariners*. The old Arabian and the Phenician were the same language.

The Arabian Gulph was called the Red Sea, by the Romans and Greeks, from Erythorus, the son of Perseus and Andromeda. This is one of those ingenious fictions, by which, as Sammes observes, "the Greeks endeavoured to bring down the origin of every thing to their own pitiful era," and was invented to disguise and obscure its real origin, or ignorantly for want of a better. The name of the *Erythrean* ocean, probably arose from its position—oıꞃ, *east*—ıꞇꞃoꞅ, *headland*. The Phenicians of Tyre and Sidon called the sea which washed the shores of Arabia Felix the country of their ancestors; muıꞃ oıꞃı ꞇꞃoꞅ—the sea beyond the eastern headland, alluding to what is now called Cape Gardefan, the north-eastern point of Africa. The Greeks copying the sound, made the sea, *Erythrean*, and the people, *Erythreans*.

Erythrus was but a personification of the *Erythreans*, *Homeritæ*, or ***Phenicians***, the same people, and first mariners and ship-builders of the human race, who first brought ship-building and navigation, for commercial purposes, to any perfection, in the same manner as the Tyrians were personified by Hercules.

From their ports of Cana, Aden, Saba, Sanaa, and Corana, they visited, for the first time, the coasts of the whole Indian ocean, as far as the Straits of Malacca; and as they discovered new countries, islands, rivers, promontories, estuaries, coasts, or people, they gave them appropriate names, expressive of their respective products, relative positions, appearance, qualities, or other palpable and obviously striking circumstances.

This conclusion is so natural, that it would be insulting to the understanding, to use arguments to establish so evident a proposition. Thus we find in Ptolemy and other other ancient geographers, as among modern discoveries, places distinguished by names, which, in the Celtic, indicate—the *round hill*, the *good market*, the *swampy marshy inlet*, the *happy tribe*, *the welcome*, the *island of gentle showers*, the *fruitful hill*, the *pleasant town on the sea*, the *promontory of turtles*, the *brilliant principality on the sea*, the *farthest torrent* or *great river*, the *eastern island fruitful in corn*, the

the *bounteous islands*, the *island of rich earth, or fruitful soil*, the *land of love*, the *good harbour for ships*, the *harbour of refuge*, the *fortified depot for goods*, the *highland tribe*, the *narrow district*. Mariners are warned against other places by fearful denominations, as—the *coast of death and evil*, the *gulph of the power of death*, the *gulph of cruel pirates*, the *deceitful invitation*, or *false bay*, the *land of robbers*, the *unhealthy country*, the *weedy river*, the *muddy stream*, the *quarrelsome people*, the *shipwreck rocks*, the *inhospitable coast;* and many others, all of which are appropriate and descriptive of the places, too palpably to be mistaken, and too obviously Gaelic, to admit of question.

It is now proposed to take a coasting investigation of the ancient Ptolemaic names of M. D'Anville's Map of the world known to the ancients, commencing at the north-eastern point of the Arabian Gulph, or Red Sea, at Elana and Ezion Geber, a port mentioned in the Sacred Writings; thence down to the Erythrean, or Indian Ocean by the Straits of Babelmandel, along the coasts of Arabia to the Persian Gulph, the Gulphs of Cutch and Cambay, and the Malabar coast to Cape Cormorin and the island of Ceylon; then up the Coromandel to the Ganges; and again, southward, on the coast of the Birman empire, to the Straits of Malacca, and passing Sincapore to northward, up to the Gulph of Siam, which appears to have been the farthest limit of Phenician navigation in that direction. We shall then proceed down the eastern coast of Africa, from Cape Gardefan to Zanguebar, an island a few degrees south of the equator, beyond which the names do not, as far as I have investigated, indicate Phenician origin:—

Elana, a ruinous town on the north-eastern branch of the Red Sea, in Arabia Petrea; aeꝺ, *the eye;* leana, *a swampy plain.* The eye or inlet of the swampy plain.

Ezion Geber—aꞅon, *bad;* ᵹabaċ, *dangerous.* The bad or dangerous harbour. "The ships of Tarshish were destroyed by a storm at Ezion Geber."

Sina Mountain—ꞅin, *round;* a, *hill, or mount.* The round hill, or mount.

Madion, a town in Arabia Petrea, on the Red Sea; maꝺ, *a field, or plain;* aoine, *fasting.* The unfruitful plain, or hungry plain.

Pharan—promontory. The cape or head land of Arabia which divides the two bays or gulphs—one ending at Elana, the other at Suez; ꝼaꞃan, *a turtle.* The promontory of turtles.

Phenicon, a town in the Gulph of Suez; ꝼein, *a plougher;* oice, *of the sea.* Mariner's town.

Raunat, a town on the Red Sea in Arabia, in about lat. 26° N.; ꞃanaꝺ, *a market.* Place of commerce.

Leucecome, a town in Arabia, lat. 25° N. on the Red Sea; leoᵹ, *a marsh or swamp;* coim, *a cover.* The covered marsh.

Latrippa, now *Medina*—la, *the place;* ꞇꞃeaba, *of the tribes.* The place of meeting of the tribes.

Betuis, river; beit, *double,* probably a double river like the Guadalquiver, or Betis, in Spain. This river discharges itself about lat. 23° north.

Chersonesus.—This name occurs about lat. 24° N. on the coast of Arabia, in the Red Sea; but I have sought in vain for a promontory or peninsula answering to the meaning of the Greek. I am, therefore, inclined to think, that it is a corruption of a Phenician term having a similar sound: probably coir-sonas, *deceitful happiness,* like our false bay, deceitful promise, treacherous port, of which we have numerous instances in our own geography.

Badeo, a town in Arabia, about lat. 22° N.; baide, *gratitude, friendship, kindness,*

Sacacia, a town about lat. 21° N.; saic, *plenty, abundance;* aice, *a leading to.* The town in a fruitful neighbourhood.

Æli, now *Heli,* a town in Arabia Felix, lat. 18° N., on the coast: perhaps aile, *perfume, odour, smell.*

Gasandi, now *Ghezan,* a town in lat. 14° N.; geis, *promise;* aon, *country.* The town or land of promise, or invitation.

Orine, an island near the coast of Ethiopia, lat. 16° N. This was probably a market for gold, and the Ophir of the Scriptures. Its contiguity to the African gold country, and its name indicates such a circumstance; oir, *golden;* inis, *island.* The golden island, or market for gold. Ophir is indicative of the same idea; oir, *golden;* fear, *man.* Men inhabiting the country of gold. Men who dealt in gold—as we call them goldsmiths, refiners, or workers.

Sava, a town in Arabia, lat. 14° N. This town may have acquired the appellation from quarrel or fight having occurred there; saḃad, *a skirmish;* or from saḃaid, *sorrel,* which may be had in abundance there.

Musa, a town in Arabia Felix, in lat. 12° N.; meas, *fruit;* a, *hill, or eminence.* The fruitful hill.

Ocelis, a town or place in Arabia Felix, near the Straits of Babelmandel; oic, *the sea;* aleas, *a pleasant place.* A pleasant or agreeable residence on the sea.

Having now examined most of the names mentioned in Ptolemy's Geography in the Arabian Gulph, I now proceed to the south and south-east coast of Arabia, on the Erythrean, or Indian Ocean, a country, as before stated, which was the habitation of Sabeans and Homeritæ, or Arabian Phenicians, before their discovery of the Mediterranean sea, and the building of Sidon and Tyre.

Tamala, the wet or oozy hills; taoṁ, *swampy;* al, *rocks.* This name indicates an unwholesome position.

Minæa, the hilly country on the western shore of the Hargiah river: this is undoubtedly mianaċ, *abounding in ore—the mine country.* The Phenicians were the first great miners.

Abisama, now called Cape Aden, a peninsula in Arabia Felix—*the abode of delight;* aḃaḋ, *abode;* saiṁ, *pleasure or delight.*

Cana Emporium—the Hargiah river—*the chief market;* caan̄, *the head,* or chief market—the metropolis of the Homeritæ, or Arabian Phenicians, or mariners.

Corana, a town on the west branch of the Hargiah river, situated where the ruins of Dhafen now appear: the site or city of a fair, or annual market; cor, *a place, district, or neighbourhood;* aonaic, *of a fair or market.*

Sanaa, the capital of Arabia Felix—the river or stream of ships; sa, *a stream;* nai, *ships.*

Mariaba, now called *Mareb,* a city in the mountainous country to the east of the Hargiah river—perhaps *the boast or glory of seamen;* maraide, *of the seamen;* aibeis, *boast or glory*—pronounced Maraba.

Saphor, a place on the east bank of the Hargiah; sa, *the river;* for, *protection, a defence, or fortress, on a stream.*

Catabanum, a place in the mountains to the east of Mareb—*the worship of the woman goddess,* or female deity; cata, *the worship;* bean, *the woman.* This was, no doubt, a temple to Onvana, the same deity who was worshipped by the Celtæ—the Diana, or Ardurena, whose temple was erected in the ards, or high mountains.

Chatramotidæ, a people living on this fruitful coast to the east of the Hargiah—a name which well expresses their position; caṫ, *a tribe;* troġ, *the shore;* motaċ, *fruitful.*

Sochor, a country about one hundred miles east of the Hargiah, on the coast now called Doan, or Dofar; soicear, *bountiful, plentiful, liberal.*

Prion river.—If the water of this river be foul, dirty, unwholesome, or fetid, it is from the Celtic word brean, which means all that. Thus the Severn, or Sabrina, was so called, as *the dirty stream.*

Savalitæ, a people residing to the east of the mountains, now called Lous Kebir. The *people rich, or wealthy, in precious stones;* saiḋḃir, *rich, opulent, wealthy;* lic, *precious stones.*

Corte, island—now called *Maziera;* perhaps caot, *showery.*

Syagros, promontory—now called *Ras al Had,* or *Rosalgate,* the eastern point of Arabia. The farther promontory; sia, *farther;* gros, *nose, or promontory.*

Moscha, now *Muscat;* mos, *drought, exhaustion;* caota, *shower.* The place of few showers, or little rain.

PERSIAN GULPH.

Heeren tells us that there are indications of the Phenicians in the Persian Gulph; and Herodotus distinctly says, they came originally from Chaldea through it, before they settled on the southern coast of Arabia. I have before mentioned the islands of Aradus and Tylos, or Tyre, and therefore need not again advert to them.

Indus, or *Sind*—sind is also the old name for the Shannon river—the *old river;* sind, *old;* aḃan, *river.*

MALABAR COAST, FROM THE GULPH OF CAMBAY, TO CAPE CORMORIN AND THE ISLAND OF CEYLON.

Dachanabades, the country of misfortune and plunderers, or robbers; daiġ, *plunderer ;* ana, ill-luck, or *misfortune ;* bad, *district.*

Pandionis regio, from 10 to 12° N. lat.—*The mountaineers ;* pannal, a *crew or body of men, a tribe ;* dion, *surface, top, summit.*

Baragazenus, Sinus. The Gulph of Cambay—*the gulph of the small fruitful stream ;* baraċ, *fruitful ;* gaisin, *smaller torrent, or stream.*

Baragaza, now the town of *Baroach.* The town on that stream.

Calliana, now called *Bombay.* The small or narrow district ; caoilaing, *the narrow district,* the country between the Ghaut Mountains and the sea.

Tyndis, now called *Choule,* or in that neighbourhood, neat and clean ; dein, *neat ;* deis, *clean.*

Suppora, a place about lat. 18° N. The pleasant town ; suḃa, *pleasure, delight ;* ara, *country.*

Musiris, in 18° N. lat. ; mus, *pleasant ;* iris, *variegated, pretty place,* near Severndroog.

Limyrica—Rampara, in lat. 17° N. The place of trade ; laim, *a hand, or adroitness ;* a, *for ;* reic, *trade.* Limeric in Ireland, may be from the same root ; though it is generally considered to be derived from laimrig, a ford, or pier. The latter may be also the origin of this name.

Barace, now *Goa.*—The lowland swampy district, with aquatic plants growing in it, from barrog, *aquatic plants.*

Nelcynda. The *first cloud,* probably the monsoon first makes its appearance here ; neal, *a cloud ;* cionda, *first.*

Sisecreinæ island. The protection of trade or merchandize ; seasda, *defence, protection* ; creana, *the merchant's business or merchandize.* The coast to the north is marked PIRATÆ, by D'Anville. This island was a fortified depot for merchandize. I had written thus far, when referring to a modern map, I found that it now goes by the name of the *Fortified Island,* from the remains of ancient earth-works which cover the whole extent of its shores.*

Cotonara, now called *Canara. The shore of the country on the sea ;* gaot, *the sea ;* an, *of the ;* ara, *country.*

Elancin Port—eilain, *an island or peninsula ;* ceañ, *a head.* The port of the headland.

Aii Cottiara. The tribe living on the country on the sea ; ai, *tribe ;* gaot, *the sea ;* ara, *country ;* now called Cranganore.

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* Captain Eatwell of the Indian Navy, who knows it well, assured me that the island is nearly covered by antient earth-works.

NAMES ON THE ISLAND OF CEYLON.

Taprobana, the old name of Ceylon. *The bright sparkling principality of the ocean ;* tab, *the ocean ;* ruad, *lordship or principality ;* ban, *white or bright.* Ceylon produces greater variety of precious stones than any other country.

Comaria, now called Cape Cormorin, the south point of Hindostan ; comar, *the point : nose, promontory ;* ia, *a country.*

Dagana, a place in the south point of Ceylon, now called Matura ; dag, *good ;* ana, *fair weather, or climate.*

Sindocan, now *Columbo*—sion, *stormy weather ;* docan̄, *harm, injury. The place of storms.*

Anuro Grannan, a place on the Arippo river ; anuair, *once, at one time, formerly ;* grianan, *the royal residence.*

Ganges, now called the Montiganga river, discharges itself at Trincomalee—*the further torrent*, or most northern in the island.

NAMES ON THE COAST OF COROMANDEL, BEGINNING AT CAPE CORMORIN.

Sosicure, a town in lat. 8° N. The mountain town ; so, *this ;* siogaire, *mountaineer.* A place built on, or near, the mountain.

Calligicum—cala, *a port ;* geocam, *of wanderers, or voyagers.* One of the mouths of the river.

Nigama, now *Negapatam*, negam, *an indenture* in the coast ; *a bay, an inlet.*

Chaberris, now *Caverypatnam ;* cabar, *a conjunction or confluence with ;* ris, *a river that joins others.*

Maliarphia, now *Meliapour*, the land of merchants ; malia, *a merchant ;* fia, *land or country.*

Malanga, a town on the river, about 13° N. lat. The good harbour for ships ; ma, *good ;* longa, *for ships.* Here we have the Celtic *longa*, for ships, again.

Tyna, river in lat. 14° N. The same as the Tyne, in England ; teign, *hasty, rapid river.*

Palura, a town in lat. 16° N. The town of strife or contention ; bal, *a town ;* ura, *strife or contention.*

Mesolia, now *Masulipatam.* The cliff of fishing ; meas, *fish ;* ail, *cliff.*

Calingon (Portus). The harbour for ships ; ca, *house, refuge ;* longa, *of ships.* Here is *longa* again.

Tyndis, river—now *Guadaveri.* The rapid but shallow river ; teign, *rapid ;* diosc, *shallow or dry.*

Calinga, now *Calingapatan ;* ca, *harbour ;* longa, *of ships ; longa, again.*

Cosamba, now *Ganjam.* The river which runs through a ravine, or high bank ; cos, *a ravine ;* aṁan, *a river.*

Cambysum, now called the *Subunreeka* river. The crooked silk river; cam, crooked or winding; biosar, *silk.*

Tilogrannum.—The Delta land at the mouth of the Ganges, now called the Sunderbunds, a place of jungle and disease. The land of deformity, ugliness, or danger; talaṁ, *land, earth, district;* graneaḋ, or graneam, *ugliness, abomination, baseness:* a most appropriate name.

NAMES ON THE EASTERN COAST OF THE BAY OF BENGAL BEGINNING AT THE MOUTH OF THE CHITTAGONG RIVER.

Catabeda river, now called the *Chittagong*. The river of friendly boatmen; cata, *friendship;* badaċ, *having boats.*

Baracura river, now called the *Sunkar*. The weedy river, or the marshy river with weeds; barrog, *plants that grow in the water;* currac, *a marsh, or fen.*

Triglyphon, the river of Aracan—*the dirty river with three branches;* tri, *three;* glaib, *dirty water;* aḃan, *river.*

Tecosanna—the house of dissolution or death; teac, *house;* do, *of;* sanad, *dissolution, or dying.* The mouth of the river of Aracan: perhaps one of the most unhealthy and pestiferous places on the globe.

Saboa, a town on the coast, a little to the south of the above; saḃ, *death;* ua, *country.* The unhealthy country, or district of death.

Mareura, one of the mouths of the river Ava, the Surawaddy: the sea of strife; mara, *sea;* ura, *contention, strife.*

Barabenna, a place at one of the mouths of the Ava river: the head or promontory of strife; bara, *strife, anger;* a, *of;* beñ, *head.*

Tamala, The S. W. promontory of the Delta, formed by the mouths of the Ava river, now called Cape Negrais: the place or habitation of the plague, or disease from Malaria; taom, *the plague or pestilence* arising from unwholesome exhalations; aill, *a place* or locality; taom, also means ooze, or swampy low land: it also means death.

Sabaricus sinus—the Gulph of Martaban or Pegu: the gulph into which the Surawaddy and other rivers empty themselves—the gulph of the power of death; sab, *death;* aracas, *power.* In this bay is the entrance to the port of Rangoon.

Lestæ Daonæ—The people of that part of the coast of the Burman Empire which lies on the Bay of Bengal, between 9 and 17° N. lat. This appellation is very remarkable; its literal meaning in Gaelic is, the contentious or quarrelsome people; liosda, *contentious* or *quarrelsome;* daoine, *people*: the pronunciation is precisely Lestæ Daonæ, as given by D'Anville.

Berobe, now called *Mergi*, a place on the coast, in lat. 14° N.; be, *visage;* reab, *crafty—the people of dissimulation, or fraud.*

Ligor, a place on the peninsula of Malacca, about 8° N. lat; liaġor, *a tongue;* which exactly answers its character.

Malencolen; maol, *bald, naked;* an, *of the;* colam, *body.* The land of naked people.

Zabe, a town on the point of the peninsula of Malacca; ꞅab, death.

ISLANDS ON THE SAME COAST.

Bazacata, in about 18° N., now called *Cheduba;* baꞅaꝺ, *slaying;* ceaꝺaċ, *striking:* the island which strikes with disease and death.

Andaman, in lat. 14° N.—the evil island; anꝺaċ, *evil;* man, *sin:* the very abominable islands: they are called the islands of the cannibals, or anthropophagi.

Maniolæ, the little Andaman island, about 12° N.; man, *sin, wickedness;* ıola, *loss, destruction:* the island of wickedness and destruction; the people here are also called anthropophagi.

Barussæ, now *Nicobar;* baꞃꞃ, *death or disease;* uꞅacꝺ, *power:* the islands of death or disease.

Sabadibæ, in lat. 6° N.; ꞅabaꝺ, *quarrell;* ıbe, *people:* the quarrelsome people.

Jabaidii, now *Sumatra;* ıa, *country;* baıꝺ, *of love:* the rich and lovely country.

Perimulicus sinus—the Straits of Malacca. The gulph of cruel or savage pirates; pıoꞃaıꝺ, *a pirate, or sea robber;* mıolac, *cruel, savage, beastly:* pronounced Perimulac. This name is appropriate, even to this day.

Malacca is probably derived from this word mıolac, *cruel, savage;* ıa, *country.* The cruel country.

NAMES ON THE EASTERN COAST OF THE PENINSULA OF MALACCA.

Thagora, in lat. 4° north; ꞇaᵹaꞃ, *fight, battle:* the place of skirmish.

Balonga, now called Patara, in lat. 9° north: this is the same name as that of Barlonga, on the coast of Portugal; baꞃꞃ, *death, destruction*; lonᵹa, *of ships.* No doubt given for the same reason—the dangerous rocky islands on this coast, where many ships have been wrecked.

Cotiaris river, the river of Cambojia—the river of boats; coꞇeoıꞃ, *a boat builder.* It is a singular fact, that the houses are built in this river, upon stakes, or piles, driven into the bottom of the shores of the river. The inhabitants communicate by boats.

Cattigara, a little to the north of Camboja river. Whether this place had its name from caꞇaᵹaꝺ, *temptation,* it is impossible to say; but it is probable, from the pronunciation, it is the same.

D'Anville traces the coast very little farther northward, and gives no names but Sinhou, which he calls Sinarum Metropolis, or the chief city of the Chinese.

SOUTHERN COAST OF ARABIA AND ISLANDS.

Dioscoridis insula, now called Socotora, an island in lat. 13° north, about five leagues from the north-east cape of Africa or Gardefan. This name indicates its fruitfulness; ꝺıaꞃꞃac, *abounding in fruit, fertile;* oıꞃ, *eastern;* ıꝺ, *high land:* the eastern island of high land, abounding in fruits and corn.

Sacalitæ sinus et insulæ, islands on the south-east of Arabia; ꞅaıc, *plenty, abundance;* lıꞇeaꝺ, *overflowing, fruitful, bounteous;* islands of abundance.

Avalites sinus et emporium, on the coast of Africa, near Babelmandel—*the welcome;* a, *the;* ꝼaılꞇe, *welcome:* the desired port.

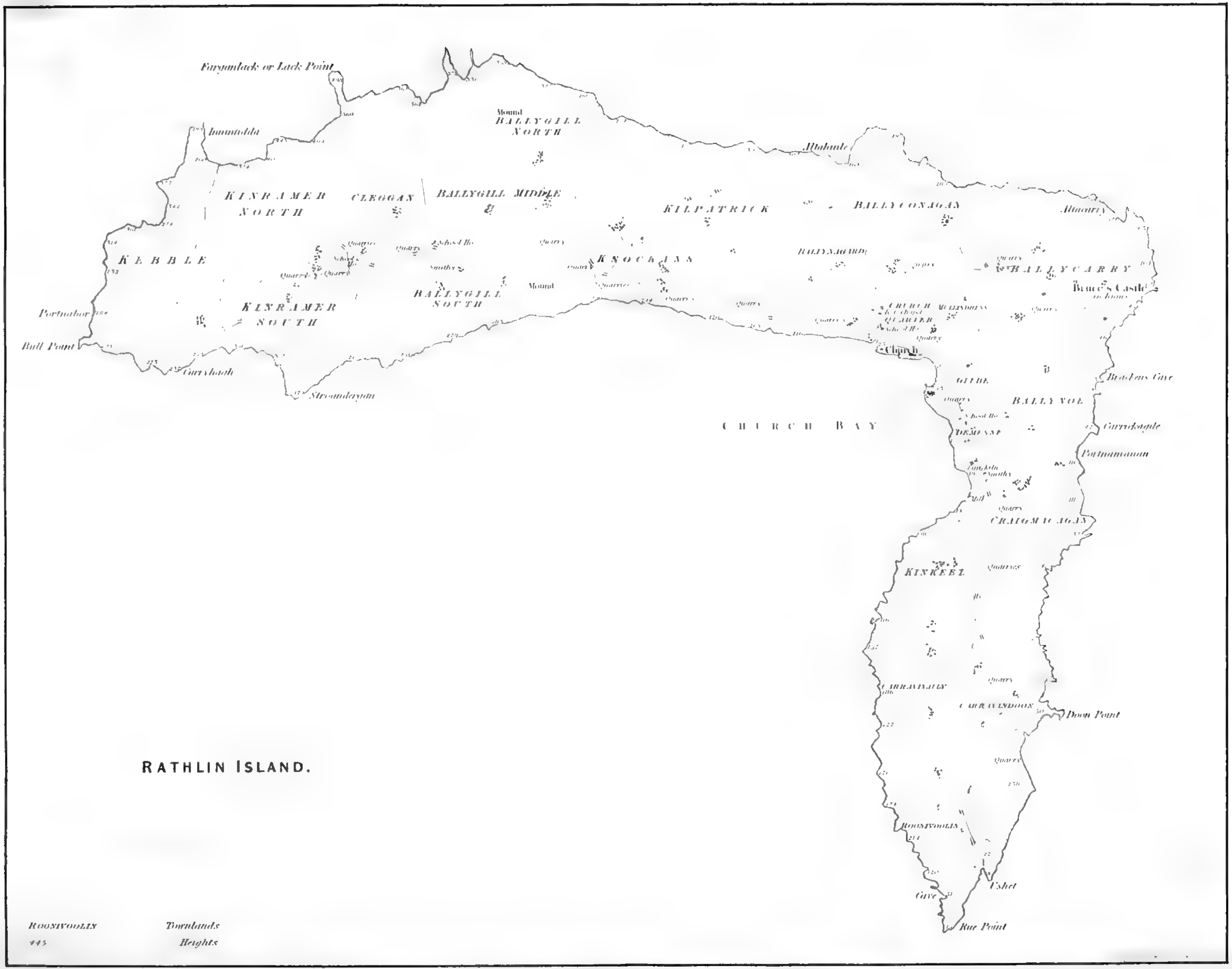
Aronata, Promontorium, the north-east Cape of Africa, now called Cape Gardefan, or Gardefoy. The commencement or beginning of the broad sea; aꞃeanac.

Barbaria or *Asania*—the country from Cape Gardefan to the river of Patta. Both these names very significantly describe this district, which is most wretched, unhealthy, unhospitable, and barren; there is not a river, or creek, for several hundred miles; baꞃbaꞃ, *deadly;* ıa, *country;* aꞅan, *evil;* ıa, *country.*

Rapta, the capital of Barbaria, about lat. 3° south, on the banks of the river: perhaps ꞃaobꞇa, *torn:* separated by convulsion.

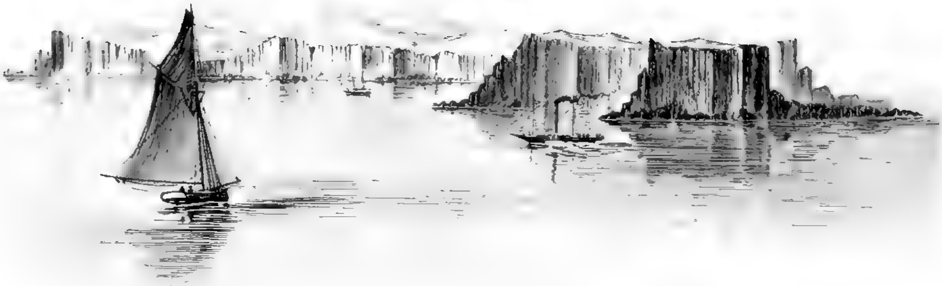
Menuthias Insulæ, now *Zanguebar*, an island situated in about lat. 6½° south; mnın, *riches, abundance, goods, wealth;* uaꞇ̇, *earth or mould.* The rich island of fertile soil. This is a very accurate description of Zanguebar. Captain Owen, who lately surveyed this coast, says this island is eminently fruitful, and produces sugar, grain, and fruit, in the greatest abundance.

I now leave this etymological examination to the scrutinizing criticism of the most incredulous and sceptical, as well as to the candid. I feel convinced of the utter impossibility of my conclusions being otherwise than correct on the whole. I may have fallen into some inaccuracies, but that so many coincidences of accurate description, and precise terms of meaning and import, should be accidental, appear to me quite impossible. A few names and sounds might happen perchance; but that *all* the names on lines of coasts, of some thousand miles extent, should be significant, and in correct and applicable terms, of the nature and properties of the places described in any language but that of the people who conferred these names, cannot I think be seriously asserted. It may also be observed, that where two names are given by Ptolemy, each are found equally expressive of the nature and qualities of the place.



RATHLIN ISLAND.

Reduced from the Ordnance Survey



Notes on the Statistics and Natural History of the Island of Rathlin, off the Northern Coast of Ireland. By JAMES DRUMMOND MARSHALL, M.D., *Secretary to the Natural History Society of Belfast, &c.*

Rathlin forms one of that vast multitude of islands which are every where scattered round the shores of Ireland. Upwards of six hundred of these have been enumerated, of which one hundred and forty are inhabited—the others, consisting chiefly of islets and holms, are tenanted only by cattle or birds.

This island is situated off the northern coast of the county of Antrim, in north latitude 55° 15′. It is a part of the county, a rectory of the diocese of Connor, and a distinct parish, having been separated from that of Ballintoy in 1723.

Doctor Hamilton justly observes, that "the name of this island has suffered so many variations in its orthography, as to render it now very difficult to determine what may be the most proper. It is called *Ricnia*, by Pliny; *Ricina*, by Ptolemy; *Riduna*, by Antoninus; *Raclinda*, by Buchanan, the Scotch historian, (who classes it among the Ebudæ or Western Isles of Scotland); *Raghlin*, by Sir James Ware; *Rathlin*, by Sir William Petty, and most of the modern map-makers."* Mr. J. O'Donovan, to whom the student of Irish antiquities is largely indebted, farther informs us, that "it was called Reċraind and Raclaind by Tigernach; *Rachra*, by John Lesley; *Rachrine*, by John Fordun, in his Scoti-chronicon; and Reacrainn, by Dudley M'Firbis."†

The nearest point of Rathlin lies about three miles from the promontory on the mainland, called Fairhead, but from Ballycastle it is nearly 5½. The usual point of disembarkation in Rathlin, is Church bay, which lies at the distance of 7½ miles from

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* In a note he subjoins, that "*Raghery*, as pronounced in Ireland, corresponds with the spelling and sound of the name in use at this day. If one were inclined to speculate in the dangerous field of etymology, perhaps *Ragh Erin*, or the Fort of Erin, might appear to be somewhat in the midst of these various sounds; and the command of the Irish coast, which must have attended the possessors of this island in early ages, might make it not unaptly be styled the Fortress of Ireland."

† See note to a translation "from the Autograph of the Four Irish Masters, in the Library of the Royal Irish Academy," by J. O'D. in the Dublin Penny Journal, vol. I. p. 315. In the "Ancient Topography of Ireland," contained in the 11th Number of the Collectanea, (p. 411), it is stated, that "all these words are derived from *Rach*, *Ridh*, *Rudh*, *Riada*, and *Reuda*, a tribe or habitation; and *ean* or *lean*, water; whence the habitation in the water—the present isle of Rathlin."

Ballycastle; this, therefore, may be considered the mean distance of the island from the mainland.

The *form* of the island has been compared, like Italy, to that of a boot,[*] the toe pointing to the coal-works of Ballycastle—the heel, where Bruce's Castle is situated, to Cantire—and the top to the great western ocean. Towards the middle, which lies opposite Ballycastle, it is bent in an angle, and thus is formed Church bay, almost the only good harbour in Rathlin.

The *length* of the island from the Bull or western point, to Bruce's Castle on the extreme east, is 5½ English miles. From Rue-point, the most southerly, to Altacarry, at the north-east extremity, the distance is upwards of four miles. The greatest *breadth* of the island at any part is 1¼ miles, and the narrowest, half a mile.

The *highest* point of Rathlin is 447 feet above the level of the sea, it is in North Kenramer,[†] at the north-western extremity of the island. The other highest points are:

Kenramer Head,	445	feet.	Sliebh-an-all, -	347	feet.
Farganlack Point,	347	do.	Bull Point, -	295	do.
Lough Cleggan,	350	do.	Slack-na-Calye -	240	do.
Altahuile -	300	do.	Broagh-mor-na-Hoosid	237	do.
Kintroan Head	318	do.			

So precipitous are the cliffs, that from the vicinity of Bruce's Castle, round the whole northern shore, by the Bull-point to the church in Church bay, the lowest point is 180 feet above the level of the sea, and the mean height may be said to be 300 feet.

From the striking similitude existing between the Island of Rathlin and the adjoining continent, it is the general opinion that this island had, at one period, formed a part of the county of Antrim, from which it has been separated by some violent convulsion of nature. All geologists who have made this the subject of inquiry, have stated, that in geological structure the island and adjacent continent are accurately the same, the principal strata in both being limestone[‡] and basalt. Along the north-eastern coast of Ireland, for a space of at least sixty miles, these strata every where present themselves—in one place the limestone rises to a considerable height above the level of the sea, and in another gives place to the basalt. On the range of cliffs running westward, and forming the northern boundary of Church bay in Rathlin, we see the limestone rising abruptly from the ocean, and forming a line of coast fantastically beautiful. At Kenbaan-head, on the mainland, corresponding in situation to these cliffs, we have the limestone appearing in a similar manner. In both, the lime-

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[*] "Sir William Petty says it resembles *an Irish stockin, the toe of which pointeth to the main lande.*"—J. O'D.

[†] "Ceañ Raṁaꞃ, the thick head, or promontory. The south end was anciently called Ceañ Caol, or slender head, being a small point of land, pointing to the north coast of Antrim."—J. O'D.

[‡] The term *limestone* must be considered synonymous with the English *chalk.*

stone is overlaid by the basalt. To the east of Bruce's Castle, the limestone disappears; and at Doon Point, which is situated exactly opposite to Fair Head, that singularly beautiful arrangement of basaltic pillars, which is very imperfectly delineated in the sketch, may be observed.

Sandstone, coals, iron, ore, &c. the substances which form the eastern side of Ballycastle-bay, and which appear different from the common mineral productions of the country, may also be traced directly opposite, running under Rathlin, which, in connection with other circumstances, would tend materially to confirm the opinion of their being a continuation of the same general strata.

Doctor Hamilton entertained the idea, that this island, standing as it were in the midst between this and the Scottish coast, may be the surviving fragment of a large tract of country, which at some period of time has been buried in the deep, and may have formerly united Staffa and the Giant's Causeway.

I have already mentioned, that the limestone traverses the island from west to east. It is stated, that this chalk or white limestone, when crossed by a basaltic dyke, often undergoes a remarkable alteration near the point of contact, the limestone becoming granular marble, highly phosphorescent when subjected to heat. On the west side of Church bay, within a distance of 100 feet, the chalk is thus intersected by three basaltic dykes, and converted in each instance into granular marble. Doctor Hamilton supposes, from the sandy texture of this marble, called by him phosphorescent calcareous sandstone, that it might be considered to have been originally formed of waterworn grains cemented together; but the gradual change of the white limestone into this substance, together with the fact of marine exuviæ being sometimes contained in it, seems to preclude this opinion, and renders it more probable that some local circumstances may have converted the original limestone into this state.

"This sandstone," Doctor Hamilton remarks, "occurs near Larne, and in Island Magee; but in point of phosphorescent qualities, that found in Raghery is much superior."

I was some time ago favoured by Mrs. Mant, of Down and Connor House, with some specimens of minerals and fossils, which were collected in Rathlin by the late Rev. Andrew O'Beirne, while curate of the island. In consequence of no catalogue having been furnished with these, it is uncertain whether they are indigenous to the island, or procured from boulders that may have been washed across from the opposite shore. They all appeared, however, to be similar to those found on the Antrim shore, containing some beautiful specimens of the zeolite family, with calcareous spar, hornblende, &c.

My friend, Mr. M'Adam, M.G.S. examined the fossils, which seemed all to belong to the lias formation: the only species he could at the time determine, were the *lima antiquata* and the *avicula inequivalvis;* a number of obscure *terebratulæ* could also be discerned. The specimen of fossil coal was similar to that found at the Giant's Causeway and elsewhere.

When strolling along the base of the white cliffs in Church bay, I was frequently struck with the appearance of nodules of dark flint which occurred amid the substance of the limestone, and which have been before noticed by Dr. Hamilton. When broken, they were of a dark grey colour, and imperfect transparency.

Zeolite and calcareous spar I found in great abundance in detached pieces in different parts of the island, but more particularly on the southern side of Church bay, where the basalt is strewn about in fragments of every shape and size. On breaking some of these, beautiful crystals of quartz and zeolite appeared in the cells and cavities.

A mineral somewhat resembling the *puozzolana* of Italy has been frequently found along the shores of Raghery. As described by Dr. Hamilton, it is "very cellular, sharp and cutting in its feel, of a specific gravity little superior to that of water, and of the character of a basaltic cinder broken down." The specimens which I procured were not exactly analogous in appearance to that just mentioned; but as it is said to exist in the state of a gritty powder before its contact with water, and those pieces which I obtained had, I should think, been submitted to the action of that fluid, it is scarcely correct to institute a comparison.

Mr. Gage mentioned to me, that specimens of this puozzolana had been forwarded for experiment, and that it was supposed that it might be serviceable for the same important purposes as those volcanic products found at Naples and in the Canary Islands.

In the island of Rathlin there are several systems of *basaltic pillars*. At Kenramer, or the western extremity, many may be observed, the pillars standing in a vertical position, although none of them are very regular. At one of these headlands, called Croch-an-Teriv, I noticed a peculiar arrangement, which I do not believe has been mentioned by any writer on Rathlin. The pillars were arranged like the spokes of a wheel, each running out from a common centre. The headland is between three and four hundred feet high, and the columns near its summit, against the face of the cliff, being in some places fractured and interspersed with grass, present an aspect more interesting than if they had been arranged in unbroken regularity.

At *Thivigh*, or the side point, there is a headland which slopes down into the sea and is covered with grass; but the section, sideways, exhibits two assemblages of square pillars not unlike those of Fairhead.

Near Ushet, on the south-east side of a hill, named in Irish Broagh-mor-na-Hoosid, there is an elegant causeway, which runs to the extent of between four and five hundred yards. The pillars are pentagonal and hexagonal, a few only having seven sides.

But the most remarkable disposition of columns, I believe, at present known, occurs at Doon Point, on the south-eastern side of the island. In Dr. Hamilton's words, "the base of this little promontory is a natural pier or mole; above this is a collection of columns, many of which are curved, apparently assumed in conformity with the surface on which they rest, and inducing a belief that they were so moulded when in a state of softness; and above this arrangement there is a variety of differently disposed columns, partaking of every position and form in which basalt has yet been discovered."

I took the sketch of Doon Point, from which I made the drawing, from a headland, a short distance to the south; and on examining the seat which I occupied, I found it to be a basaltic column, one of a series which projected quite horizontally through the face of the headland, thus exhibiting their ends to the view of any one looking from below. The arrangement of the basalt on this spot was so varied that it would have required hours to note the different positions of the columns.

Caves are very numerous in Raghery, particularly on the northern shore of the island, where the heavy sea which almost constantly beats against its base, has formed many excavations in both the limestone and basalt. In sailing round the shore, these caves can, for the most part, in calm weather, be entered by a boat; they vary much both in extent and form, some retiring only a few feet from the edge of the cliff, and others extending as far as the eye can reach.

The finest of the caves which I had an opportunity of seeing, is *Bruce's Cave,* situated a short distance north-west of the castle. It rises at its entrance, as well as I could conjecture, sixty or seventy feet, and appeared from forty to fifty in depth, formed of noble arches of dark basalt, resting in layers behind one another. Bruce's Cave faces the Northern Ocean; and, as I have remarked, the sea which sets in on this part of the coast is tremendous. Although the day I obtained a view of it was almost perfectly calm, yet the swell was too great to permit us to approach its entrance, and I was reluctantly compelled to pass it after a very imperfect inspection. The other caves on this part of the island were all smaller than that just mentioned. Some of these we entered, but others were so situated as to prevent the boat approaching them. The echoes excited by the firing of our guns in some of these were awfully grand, and the rocks rung for many seconds. We never failed to disturb the corvorant from his apparently dismal seat in the most retired part of the cave, and his hard croak sounded in unison with the appearance of his habitation.

The scenery on this shore far exceeded in grandeur any I had before witnessed. The rocks are not only of very considerable height, but from their rising abruptly from the surface of the water, appear much more so than they really are. The line of the coast is not straight, but intersected into different amphitheatres, the bases of which are composed of huge masses of limestone, converted by the incessant rolling of the waves into the most fantastical shapes. Above the limestone, the grass and earth appear, and still higher the basalt, arranged in many places in regular and beautiful columns, similar to those at Fairhead, or the Causeway, and exhibiting, on one of the cliffs, the singular appearance of which I have already spoken.

On the southern side of Church-bay, on the headland called Cloch-a-doos, there are *four caves* situated a short distance from the water's edge, but considerably above that elevation. In one of those which I entered, the mouth of the cave was at least thirty or forty feet above the level of the sea; but the floor of the cave gradually descended towards its extremity, which was too retired for inspection except by torch-light. Doctor Berger notices an interesting geological fact about some of these caves: "Although excavated in the basaltic rock, and at a point remote from any calcareous formation, they are nevertheless invested with calcareous stalactites depending from their roofs, and by their droppings on the floor, depositing a crust of about an inch in thickness." Doctor Berger thinks this circumstance worthy of attention, since "calcareous matter seems evidently, from the situation of the caverns, to have been derived from that which enters as a chemical ingredient into the composition of the basaltic rock, separated from the mass, and deposited in its present situation by the percolation of water, which the rains or springs must have furnished."

In those which I visited, I did not observe any calcareous depositions; but I obtained from Mr. M'Donnell a stalactite of very considerable size, which he had procured from one of these caves some time before the period of my visit. I may add, that these are the caves in which the bones of different animals were found by Doctor Andrews during the past summer, a notice of which was laid before the Meeting of the British Association in Edinburgh; and from the deposition of marine exuviæ, it was conjectured that the respective levels of the sea and land had there undergone some material change.

Rathlin is, generally speaking, fertile, and in those parts where the land is well cultivated, good crops are produced. It has been already mentioned that a bed of limestone traverses the island from east to west, and lime is in consequence abundantly procured for manure.

The *crops* usually grown, are barley, potatoes, and oats, with occasionally some wheat and flax. The wheat would grow well in many parts of the island, were the attempt made to procure its introduction. At present it grows only on the farms belonging to Mr Gage and Mr. M'Donnell; but on these, the crop was fully equal to what I had seen on the mainland; and the opinion universally expressed was, that wheat might be grown as abundantly as any of the other grains.

Of barley, besides the quantity necessary for home consumption, the inhabitants are enabled annually to export upwards of 90 tons to Scotland.

The oats, wheat, and flax which are grown, are also made use of on the island; and potatoes are exported to Ballycastle and other places, whence formerly they were imported into Rathlin.

The reason for this increased attention to agricultural pursuits, appears obvious, when we consider that *kelp* now obtains in the market scarcely half the price it did twenty years ago, and hence the inhabitants have of late years bestowed their labour on the improvement of their farms; and although these now occupy the greater portion of their time, they find leisure to make a small quantity of kelp.

The most fertile parts of the island are the valleys; the hills are generally rocky and sterile—some so entirely covered with broken masses of rock, as to render it a matter of much difficulty to force a passage among them. The stone is made use of for building houses and fences, and so extensive has been its use, that no less than thirty quarries are to be found on the island; but many of these are not now wrought.

Rathlin is liberally supplied with fresh water, both by *springs* and *lakes*. Of the former there are above thirty, the most remarkable of which is one situated about a quarter of a mile north-west of Bruce's Castle. The water in this well rises and falls with the tide, although it is about one hundred feet above the level of the sea. Its rise and fall vary, it is said, from four to six inches, and at spring and neap tides there is a proportional variation. There are two lakes: one, called Lough Cleggan, situated on the north-western side of the island, is three hundred and fifty feet above the level of the sea. It extends over the space of ten or twelve acres. The other, Lough Runaolin, covers upwards of thirty English acres, and is situated at the Ushet end. The streams of water which flow into these are very small, so insignificant, indeed, as not to deserve a name.

Besides these lakes, there are fifty or sixty sheets of water of different sizes, but the principal of these are in marshy districts. The largest marsh is towards Doon Point, in a very wild and uninhabited district. On its surface I found, in great luxuriance, the white and yellow water-lilies, *(nymphæa alba* and *nuphar lutea)* and also the common reed.

The leaves and flowers of the lilies covered the greater part of the water's surface, and the beautiful variety of their white and yellow petals gave an air of beauty and interest to the marsh which it would not otherwise have possessed. This was the only part on the island where these plants were to be found.

The *soil* in Rathlin is in most places light and dry, but in parts the clay is firm. and in one of the valleys about the centre of the island, it is well adapted for making bricks. As I have just mentioned, marshes are very abundant, and in most of the valleys, peat is dug and used by the natives as fuel; but the supply thus furnished is not adequate to the demand, and it consequently forms one of their imports from Ballycastle. It is, however, principally imported by the inhabitants of the southern end of the island; for towards the Kenramer, or western end, fuel is much more abundant.

The quality of the soil is, on the whole, good; for every where I saw the appearance of flourishing crops. The potatoes were planted principally in ridges, and each house had its potato garden attached. This vegetable appeared to thrive well throughout the island; and I was very much surprised, when sailing along the base of the white cliffs in Church bay, to see little plats of potatoes appearing here and there among the enormous masses of rock which had at one time been dislodged from the adjoining precipices. The only way of getting access to these potato-fields is, by descending the steep declivities by which they are on every side surrounded; and I conceived that the value of all the vegetables grown on these sequestered spots, would not repay the trouble necessary for their culture. I afterwards learned, that the seed was planted every second year only—that is, that as many potatoes were allowed to remain in the ground during the winter, as would produce a sufficiently large crop the following summer, and by this means the labour was materially diminished.

The pasturage ground in Rathlin is very extensive, as might be supposed in so rocky an island. On all the headlands large flocks of sheep may be seen picking out the tufts of grass from amid the rocky enclosures; and no place furnishes a finer breed than Raghery. Each family has a flock, however few in number, and they are enabled to kill a sheep occasionally for home consumption, unless an unusually unproductive season should force them to sell these useful animals. The sheep and goat will thrive well in situations where scarcely any other domestic animal could gain subsistence, and hence, to the inhabitants of islands, they are invaluable. The trouble attendant on rearing them is very trifling—they are turned out to graze in common, each with its owner's mark upon it, and they roam over the rocky eminences of Raghery, free and undisturbed.

The *climate* of Rathlin is but little different from that of the mainland, the variations of temperature being nearly the same. In winter there is no snow of any consequence, the weather usually continuing very mild. *Fogs* are, however, very prevalent, particularly in spring and autumn, and they are frequently so dense as to prevent the island being seen even at a short distance. Hence many vessels are exposed to danger in rough weather in approaching this rock-bound isle, and shipwrecks frequently take place on its shores, from which no one survives to tell the tale. Not a winter elapses, during which at least one vessel is not wrecked; and at the Bull Point, fragments are frequently driven on shore, as notifications to the islanders that some unfortunate crew have sunk to rise no more.

In March 1833, two vessels were wrecked on the eastern side of the island, the remains of which were washed on the rocks near Doon Point. The night was dark and stormy, the crew, in all probability, ignorant of the coast, and a strong easterly wind setting in on the shore. On such a shore, no hope of rescue could for a moment be entertained: the islanders were doomed to hear the shrieks of their expiring fellow-creatures, without having it in their power to render the slightest assistance. The vessels beat to pieces near Doon, and in the morning their timbers lay here and there on the rocky points of the shore.

The mariner, in coasting by Rathlin, has every thing to contend with. Even if the weather be calm, the tides are so irregular, and set down the channel with such a force, that considerable risk is incurred. If a storm arise and the wind should be blowing on the shore of the island nearest the vessel, but little chance of escape can be offered. The shores consist of one entire range of rocks; and these so steep and rugged, as to prevent any aid being afforded by those on shore, should even a pause in the fury of the elements enable her cries to be heard by the islanders.

Surrounded as it is by a wild and turbulent sea—from the precipitous nature of its shores, and the number of caves every where to be found, Rathlin has been in former days a favourite resort of smugglers. In Queen's Anne's reign, a French privateer had made Church bay her head quarters; and there is not the smallest doubt that were there any persons on the island favourable to illicit trade, the system might be pursued to a great extent. Any one who would take the trouble of walking round Raghery, must be struck with the impossibility of a body of men preventing boats or vessels from having intercourse with the island, unless by keeping a sentinel and guard on each of the headlands. Although many vessels can be boarded coming through the channel in fine weather, it would be out of the question attempting it during the gales of wind which, for many months in the year, sweep round its shores.

The channel between Rathlin and the mainland has, it is said, a strong resemblance to the Straits of Reggio between Sicily and the coast of Calabria, particularly in the indenting of its shores, the velocity of its tides, and the vortices produced by counter currents. Like it, the water is frequently agitated and thrown into ridges and whirlings by the violence of the current, the particular direction of certain winds, and the irregular conformation of the coasts. At times it likewise happens, as I have observed, that a very dense vapour is accumulated over the waters of the channel. If the atmosphere be highly impregnated with these vapours and dense exhalations not previously dispersed by the action of the winds or waves, or rarefied by the sun, it then happens that in this vapour, as in a curtain extended along the channel for some height above the sea, the extraordinary phenomenon called the *Fata Morgana* may be observed.

In one instance, many years ago, a gentleman of undoubted veracity, the commander of a corps of yeomen, being at some distance from the shore, with a party in his pleasure-boat, distinctly saw a body of armed men going through their exercise on the beach; and so complete was the deception, that he supposed it had been a field-day which he had forgotten. A woman also, at a time when an alarm of French invasion prevailed, very early on a summer's morning, saw a numerous fleet of French vessels advancing in full sail up the channel. She withdrew in amazement to call her friends to witness the spectacle, but on her return the whole had vanished!

On the evening I first crossed the channel to Raghery, the boatmen were pointing out the most interesting objects to my notice, and enlivening the time by the relation of any little occurrences of more than ordinary note which they had met with. One of them spoke of the dense fog which they frequently encountered in crossing to and fro; and after mentioning the deceptions they occasionally experienced, he stated, that one morning very lately they had been crossing to Ballycastle during a fog, when suddenly they perceived a brig in full sail bearing down on them. The illusion was so great, that the crew used every exertion to escape being run down, as they momentarily anticipated; and they had just accomplished their object, when the vessel totally disappeared.

In connexion with this subject, I may mention, that a belief was formerly prevalent among the inhabitants, that a green island rises, every seventh year, out of the sea between Bengore and Rathlin. Many individuals, they say, have distinctly seen it adorned with woods and lawns, and crowded with people selling yarn, and engaged in the common occupations of a fair. Could this have been the *Fata Morgana?*

Here oft, 'tis said, Morgana's fairy train
Sport with the senses of the wondering swain:
Spread on the eastern haze a rainbow light,
And charm with visions fair th' enchanted sight.
At first a beauteous island scene behold,
Like that Hy Brasail found, by swains of old,
In ocean's depths;—and then a rustic throng,
With booths and tents the forest glades among;
Next, warrior bands in scarlet files arise,
Chariots and steeds, and towers that reach the skies:
But soon they flit, and bounding in the breeze,
Embattled navies plough the azure seas;
Sail crowds on sail, the boiling wake grows hoar,
And whitening surges climb each sculptured prore.
Thus shifts each pageant, like the scenes that fall
Through lens, or lantern, pictured on the wall
Of chamber dark—till all dissolves away,
As filmy vapour in the noon-tide ray.

Drummond's Poem on the Giant's Causeway.

The *tides* of Rathlin are most remarkable; and to navigate the channel between the island and the opposite coast, requires more than ordinary skill and caution. Against the north-west point, the Bull, the great body of water which flows from the ocean during flood-tide to supply the northern part of the Irish channel, is first interrupted and broken in its course, and counter-tides are here created. Thus, along part of the coasts of Antrim, Derry, and Donegal, the flood-tide appears to flow *nine* hours, and the ebb only *three.* In Church bay, in Rathlin, the same is the case; but at Archill bay, south of Bruce's Castle, the ebb runs nine hours, and the flood only three. So prevalent are these currents round the island, that I have frequently observed that one tide or current will be setting round part of the coast for a short distance from the shore, while another will be running in the very opposite direction at the distance of some perches. A person navigating the channel, must not leave either shore without reference to the state of the tides, otherwise, the passage which might be both safe and expeditious, may become the very reverse. In leaving Ballycastle for Church bay, the time usually chosen is, at the last of ebb-water, when the current is setting down towards the north, and when the boat will be carried to the north-western side of Church bay ; and at the time this is reached, the flood-tide sets down the channel towards Fairhead, and carries the boat in nearly the opposite direction to that in which she started, viz. into the heart of the bay. If, however, there be a smart breeze, and the wind fair, a good boat will make the passage in a direct line across the channel from Church bay to Ballycastle, without paying that strict attention to the tides, which, under other circumstances, is absolutely necessary.

Lying at so considerable a distance from the mainland, the depth of water every where round the shores of Rathlin is considerable. There are but few adjacent banks—one, a cod bank called *Skirnaw*, lies between it and Isla, in Scotland. While I remained on the island, I had opportunities of ascertaining the depth of water at different parts along the coast, and although it varied much, it generally exceeded ten fathoms within a very short distance of the shore. In many places a fifty-fathom line did not reach the bottom ; and I was much struck with the accuracy of the fishermen in pointing out most minutely the particularly deep or shallow parts, and calculating correctly the depth of water at those places where we found it desirable to gain the information.

The greatest depth of the channel between Rathlin and Ballycastle has been ascertained to be fifty-three fathoms, and between the north-east of Ireland and the west of Scotland, ninety fathoms

There is good anchorage in Church and Archill bays, in ten to twenty fathom water ; and also at Ushet, where a small rude pier or quay is formed for the accommodation of the boats in that part of the island, and where an occasional collier or merchantman may be seen at anchor.

The *traditions* of the island of Rathlin do not reach beyond the commencement of the fifth century, when we learn that it was well inhabited, and was garrisoned by a small army. At the period alluded to, St. Comgall, a religious individual, landed here, but was instantly seized and driven out of the island. After him came St. Columba, the celebrated missionary of the north, who founded in Raghery a religious establishment, and placed over it Colman, the son of Roi. Here it flourished for the space of nearly 300 years, in peace and quietness, until the latter end of the eighth century, when (to use the words of Doctor Hamilton) " the northern storm filling at once the whole horizon, and bursting impetuously from the ocean, overwhelmed the island, burying in blind and brutal destruction the inoffensive ministers of the Christian religion, in the very moment when they were cultivating the olive branch, and preaching peace and good will among men."

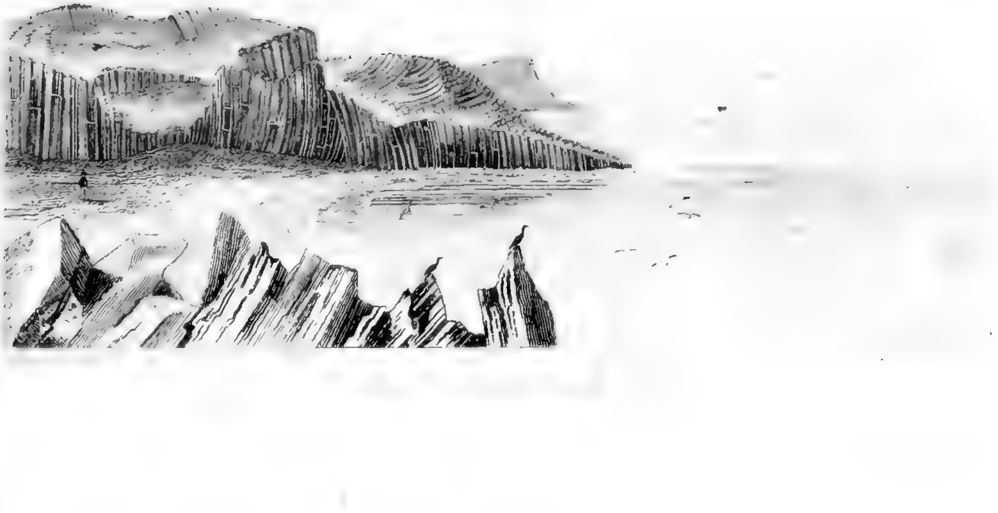
In 790, the monastery established by St. Columba was ravaged and destroyed by the Danes; and in 973 by a second visitation of these freebooters, who put the abbot of the island to death.

Its vicinity to Ireland rendering it an object of importance to an invading enemy, it became a scene of contention between the inhabitants of the opposite coasts of Scotland and Ireland. The memory of a dreadful massacre perpetrated by the Campbells, a Highland clan, is still preserved, and a place called Sloc-na-Calleach, perpetuates a tradition of the destruction, by precipitation over the rocks, of all the women in advanced life, then resident on the island. Doctor Hamilton remarks, "that the remembrance of this horrid deed remains so strongly impressed on the minds of the present inhabitants, that no person of the name of Campbell is allowed to settle on the island." This feeling, however, seems now to have subsided; for during my visit to Raghery in the summer of 1834, no such enmity to the name of Campbell was manifested.

During the civil wars which devastated Scotland after the appointment of Baliol to the throne of that kingdom, Robert Bruce was driven out and obliged to seek shelter in the isle of Raghery, in a fortress whose ruined walls still retain the name of the illustrious fugitive. His enemies, however, pursued him even to this remote spot, and forced him to embark in a little skiff and seek refuge on the ocean. The ruins of Bruce's Castle are situated on a bold headland at the extreme eastern part of the island, immediately fronting Scotland. Although apparently very lofty, the height of the rock on which the castle stood, is marked, according to the late survey, between seventy and eighty feet only above the level of the sea.

It rises perpendicularly from the water's edge; and about forty or fifty feet from the eastern extremity, a deep chasm traverses the ground, insulating, as it were, the huge mass on which the outer part of the fortress has been situated. On this, the ruins now standing, consist only of part of a wall fronting the west, entirely destitute of all ornament and style of architecture. About eighty or one hundred feet on the western side of the chasm, the remains of another part of the building are still visible, from which we may fairly infer, that the castle had originally been of very considerable extent. In the face of the rock fronting the south, and immediately under the wall, there is the appearance of a small cave, in which, it is said by some, that Bruce concealed himself, the castle not having been built at the time of his residence there.

Considerable difference of opinion has, however, been manifested concerning the exact date of the erection of this building. Its antiquity is dated at least five hundred years back; and it is supposed to be considerably older, as the time which Bruce spent on the island was not sufficient to erect it. One of the strongest proofs alleged of its antiquity, is, that the lime with which it is built has been burned with sea-coal, the cinders of which are still visible, and bear so strong a resemblance to the cinder of the Ballycastle coal, as makes it extremely probable that at some period so



early as the year 1500, sea-coal had been used as fuel in Raghery. On this subject Doctor Hamilton remarks—" It might be imagined that the coals were brought from " Britain, but a little reflection will show that supposition to be extremely improbable " even so late as the time of Robert Bruce. It was but just then that the English " themselves had discovered the use of sea-coal as fuel; and we find that in the time " of Edward I. after being tried in London, they were immediately prohibited on " a hasty opinion that the vapour was noxious to the health of the inhabitants. It is " not, therefore, to be readily believed, that at this early period, England could have " had any extensive export trade in coals; or, if so, it must have been to some populous and civilized country—to some safe harbour—to a great and commercial " town; but, at the time we speak of, the British charts do not lay down a single " village on all this line of coast. Further reflection on the subject," continues he, " might lead me to suppose, that the building of this castle was of much more ancient " date, because, in the time of Edward I. the kingdom of Ireland was an almost uninterrupted forest, so that the abundance of more convenient fuel would then have " anticipated all necessity for searching for fossil coal; indeed, for several ages subsequent to the year 1171, at which time the English invaders found Ireland to be a " country overrun with wood, instructions may frequently be found among the annual " orders of government, to have successive portions of forest cleared away, for the " purpose of rendering the country more accessible to the English forces; and it was " not until four hundred years after, about the latter end of Elizabeth's reign, that " any considerable progress was made in this work of devastation."

In 1550, Rathlin was invaded by the English, who were repulsed with the loss of a vessel and several men. We are told in the Annals of the Four Masters, as translated by Mr. J. O'Donovan, that in 1551, "the Lord Chief Justice (Anthony St. Leger) marched at the head of an army into Ulster, and dispatched the crew of four ships to the island of Reac ꞃınn, to plunder it. James and Colla, the two sons of Mac Donnell of Scotland, were on the island to defend it. A battle ensued, which ended in the total defeat of the English; not one of whom survived the battle, excepting the lieutenant who commanded them on this excursion, whom the Albanians *(Scots)* kept as a prisoner, until they got in his stead their own brother, Sorley Boy M'Donnell, who had been imprisoned in Dublin a year before that time, besides other ransoms."

In 1558, the Scots took possession of the island; but were soon expelled with dreadful slaughter by the Lord Deputy Sussex, who seized upon it for the English crown, and left in it a garrison for its defence. In 1575, General Morris landed here with a body of men from Carrickfergus, and having killed two hundred and forty of the islanders, seized upon the castle. In consequence of successive barbarities committed on the peaceable inhabitants of Raghery by various savage invaders during the unsettled ages of Ireland, this little island became at length totally uninhabited, in which state it is represented in a manuscript of the country, so late as 1580, now in the hands of the Macdonalds; and it is further mentioned, that some Highlanders who fled to it for safety, were forced to feed on colt's flesh for want of other provisions.

Subsequent to this date, but little is related concerning the island. Doctor Francis Hutchinson, Bishop of Down and Connor, who published an Irish Almanac and other works, procured for the inhabitants of Raghery a translation of the Church Catechism into Irish, with the English annexed. It was published in Belfast, in the Roman letter; but so scarce has it now become, that it is a matter of uncertainty whether or not a copy of the Raghery catechism is at present extant.

About the year 1740, the island became the property of Mr Gage, in whose family it still continues. Mr. Gage is both the rector and magistrate of Rathlin; and from the interest he feels, and the attention he bestows on the inhabitants, is much respected by all. His residence is situated in Church bay, within a short distance of the church; and in no house is the virtue of hospitality more freely exercised. To Mrs. Gage I am particularly indebted for many facts noticed in my paper, and for opportunities afforded me of observing whatever was rare or interesting on the island. To Mr. and Mrs. Macdonnell I must also stand indebted for many acts of kindness.

Dr. Hamilton, in his "Letters on the County of Antrim," and Doctor Drummond in his "Giant's Causeway," are almost the only writers who have noticed any of the particulars of the island. Doctor Hamilton's remarks have now been written many years; and although comprising much valuable information, are, nevertheless, interspersed with many observations, which, after the lapse of fifty years of civilization and improvement, would now admit of considerable modification. These, where differing from those made during my visit, I have noticed under their respective heads, paying, at the same time, all due deference to an authority so highly respected.

Rathlin contains about 2000 Irish plantation acres, or 3239 English. The population is stated by Dr. Hamilton at 1200, or 130 families; but there are now not more than 1050 inhabitants. Of this number, upwards of 900 are Roman Catholics, and the remainder belonging to the Established Church. From this statement, it will appear that the population has latterly much decreased; a great number of the young men go to Glasgow and Greenock, or to some of the ports in Ireland, to learn different trades, particularly that of ship-carpenters; and emigration to America has, of late years, taken away considerable numbers. In one of the years immediately preceding my visit, upwards of forty had left the island for America; and during the summer of 1834, sixteen had emigrated.

The inhabitants are simple, laborious, honest, and exceedingly attached to their native soil. Doctor Hamilton says, "that in conversation they always speak of Ireland as a foreign kingdom—and a common and heavy curse among them is, 'may Ireland be your latter end.'" In my intercourse with them, however, I did not observe that repugnance to visit the mainland, experienced by Doctor Hamilton. This, no doubt, has arisen from the more frequent intercourse of late years, kept up between the islanders and their brethren across the water; and the distinction made by him between the inhabitants of the Kenramer or western end, and the Ushet or southern extremity of the island, however correct in his time, has now in a great degree ceased to exist. In consequence of the Kenramer end being of a much more precipitous character, and not being well situated for intercourse with Ballycastle, its inhabitants retain more of their primitive manners than those residing further to the south, who, from having more frequent communication with the mainland, have acquired a more intimate acquaintance with English manners and customs.

The *houses* in Rathlin, with the exception of those of Mr Gage, Mrs. M'Donnell, and a very few others, are of the very poorest description. I was particularly struck with this circumstance during my excursion through the island, and one house might almost be taken as a fair sample of all the others: they are chiefly of one story, and built of stone—many without a window or chimney, the inhabitants being obliged to be content with the light admitted by the door, and the hole made in the roof, by which part only, and that a small portion of the smoke is carried off. To each dwelling-house there is commonly attached a small shed for cattle.

The land is divided into small farms; each house has a plot for vegetables, while the remainder is occupied by potatoes and barley. The only *trees* on the island, are those planted in Mr. Gage's garden, and in the immediate vicinity of his house; the hedges number but two or three, the fields being divided by stone walls and fences. That trees once abounded in Raghery, is beyond a doubt. One of its ancient names is, as has been observed, employed in consequence of the abundance of wood with which it was furnished; and although now scarcely a tree is to be seen, we must not infer that this has always been the case. In fact, we have indisputable evidence in the wood which is occasionally dug up in some of the bogs. Hazel is frequently found in considerable quantities—oak is also found, but of small dimensions. Were sufficient care bestowed in the selection and planting of trees in situations similar to Raghery, they would frequently succeed, where, at the present day, they fail.

The average number of a family in Rathlin does not exceed six or seven, a few only contain so many as ten. This is generally attributed to the want of early medical assistance and other causes. Their food consists of potatoes, oaten and barley-bread, and fish; and their own mutton, pork, and beef, in winter, unless when the price of meat is high, or the season unusually unproductive, when they prefer exporting their cattle.

The inhabitants employ themselves in tilling the ground, in fishing, and in making kelp. Towards the latter end of the last century, when kelp gave a good price, it furnished to the industrious people of Rathlin a rich source of wealth. At the period I allude to, one hundred tons were annually exported, and the kelp alone frequently paid the entire rent of the island. Now, however, the demand and price have both declined, and thirty tons constitute the annual exportation. The kelp is usually exported to Scotland in a vessel which pays an annual visit to the island ; and as it would be very inconvenient to the islanders to bring their kelp to Church bay, just at the time when the vessel would be anchored there, Mr. Gage has kindly fitted up a small store-house, where the kelp is allowed to remain till the time of its exportation.

In Rathlin all the larger *fuci* are indiscriminately used in the manufacture of kelp; but the *fucus nodosus*, or bladder-wrack, and the *fucus serratus*, are the species most employed. Besides these, they use the *chorda filum*, or long tangle which they call *rrog ;* this, and the *laminaria digitata*, as well as the other species, are furnished in great abundance round all the shores, and many a kelp-kiln bears testimony to their profusion. At low water the women and children walk out on the ledges of the rocks, and with old knives or reaping hooks, cut off the sea-weed, which is then borne to the shore, and spread out before the sun to dry : in the evening it is made into little heaps or hillocks, and in the morning again shaken out, just in the manner in which they make hay. This process is continued till the weeds are dry enough to be burnt. An excavation of about five feet long, two or three broad, and two deep, is then made in the ground, and lined with large stones ; and in this, which is called the kelp-kiln, the dried weeds are burned, the fire being kept up by constantly throwing them on the flame. During this process, the alkali, and every thing not capable of being dissipated by the heat, accumulate in the bottom of the kiln ; and when liquid, like molten metal, are stirred about with an iron rod till they form the hard bluish mass called kelp. I know nothing which produces a more beautiful and picturesque effect, than when sauntering along the sea-shore on a calm summer's evening, we see the numerous kelp-kilns sending forth their dense wreaths of smoke. The effect is more striking when the kiln is viewed from the sea, or from a point of land where the yellow smoke can be contrasted with the dark-green of a verdant back-ground. At night, if the kiln be kept burning, the figures of the persons engaged in the occupation are thrown finely forward by the fire ; and this, aided by the reflection from their faces, gives such an air of wildness to the scene, as to lead the spectator to imagine that he is witnessing a nocturnal sacrifice, or the infernal cauldron of Macbeth's witches :

"What clouds of smoke in azure curls aspire
From many an altar's dark and smouldering fire ?
What shadowy forms dim gleam upon the sight,
Now hid in fume—now clear with sudden light?
Do stern Crom-cruach's priests to life return,
And here, once more, their fires unholy burn.?
Or Albyn's friends their beacon lights relume,
To guide the spoiler through the midnight gloom ?
Or those dread sisters who, on blasted plain,
Are wont to meet in thunder, lightning, rain;
Their cauldron boil, and round it as they go,
Their cursed enchantments in the hell-broth throw ?

Ah! no, a race inured to toil severe,
Of manners simple, and of heart sincere,
Sons of the rock and nurslings of the surge,
Around the kiln their daily labours urge;
O'er the dried weed the smoky volume coils,
And deep beneath, the precious kali boils."

Drummond.

The *customs* and *manners* of the islanders differ in some points from those of their fellow-men on this side of the channel. The island is divided into townlands; and when one of the natives dies, even an infant, the inhabitants of his townland will do no work that can possibly be avoided: whether they deem it irreligious, or merely a fit opportunity for self-indulgence, I cannot determine. They observe all their holidays and fast-days very strictly; and hence, between these and the funeral holidays, they idle no inconsiderable portion of time. They possess, in an eminent degree, the virtue of hospitality. I found them remarkably kind and obliging—ever ready to point out to my attention any object particularly worthy of notice, or to give me any information I required.

The only *tradesmen* in Rathlin are, a smith, two or three tailors and shoemakers, and a few boat-builders. The natives manufacture the chief part of their woollen and linen clothing from their own wool and flax, grown and spun on the island; they also plait straw hats, and knit their own stockings. A public-house has been established within these few years, and a shop for groceries and a few medicines.

There are three *public schools*—one male and two female, to which competent teachers are appointed, and which are under the superintendence of Mr. Gage's family, who devote much attention to the children. A mixture of Irish and Gaelic forms the native dialect; and though the English has made some progress, by many it is neither spoken nor understood. On entering some of the cottages, I not unfrequently found the only house-keepers to be two or three children under seven or eight years of age. On putting a question to them, I was answered by a wild stare, evincing their total ignorance of the language in which I addressed them. In different parts of the island, the women particularly, knew nothing of the English tongue.

To the taste which some at least of the islanders possess for *music*, I can bear witness; and I shall make an extract from the journal which I kept, as it contains a short account of the first evening I spent in Raghery:

"I reached Ballycastle on Friday the 27th June, 1834, at 7 P. M. My first inquiry was for a boat to Rathlin, but to my disappointment I was informed that one had started about an hour before my arrival. I found one, however, which had just reached Ballycastle quay, laden with potatoes from the island; and although the crew did not purpose returning till morning, I induced them to leave for the island this evening, which they did at 8 o'clock. Although already sufficiently impatient to visit

Raghery, an additional stimulus was afforded me in the mildness of the evening. I could not have chosen a more delightful one—the sun was sinking in the waves, and he had already dipped so low as to throw the island into deep shade; the water was clear and calm—not a ripple disturbed its surface; and every thing around me wore a pleasing aspect.

"After a most agreeable sail, I landed on the island, and was escorted to the public-house in Church bay, where I was informed by my boatmen I should get good accommodation. Although not exactly corresponding with the character bestowed on it, I endeavoured to make myself comfortable. The boatmen, after their pull across the channel, had no objection to drink my health, and they accordingly retired to the large room above that in which I was seated. The joke and laugh now circulated, and being musically inclined, a song was commenced. The peculiarity of the tune attracted my notice, and one of the men having kindly requested my presence, if I wished to hear a Raghery song, I gladly joined the party. A new song was now begun—it was a duet, to which a chorus was attached, sung by the whole party. The two principal performers took hold of each other by the right hand, and kept time with the tune by striking their hands thus entwined, on the table. The song lasted at least fifteen minutes, and was sung in their native language, with greater spirit and warmth of feeling than is usually displayed by more fashionable vocalists. I retired to my room, highly pleased with this my first introduction to the inhabitants of Rathlin."

The *diseases* most prevalent in Raghery are, asthma, rheumatism, and diseases of the eye. The asthma and ophthalmic diseases are principally occasioned by residing in ill-constructed and ill-ventilated houses, in an atmosphere of smoke, by which the lungs and eyes must inevitably suffer. In the turf of the island, sulphur abounds, as I experienced by inhaling its smoke for some time; and the lungs of the inhabitants being thus daily exposed for many hours to the fumes of sulphur, may lay the foundation of asthma. I not unfrequently found the junior members of families afflicted with this disease, although under other circumstances it affects adults only.

One of the greatest desiderata of the islanders, is a physician, the want of which is severely felt. In summer and in calm weather, they can, in cases of emergency, command one at Ballycastle; but in winter, when the waters of the channel are fretted by the fury of the winds, and the wide expanse of water exhibits but one sheet of foam, it were in vain to seek for aid from the mainland. One, two, or three weeks, not unfrequently elapse without any communication with the opposite shore; and a physician has been detained for many days on the island, waiting the return of milder weather. Should, therefore, any accident occur, or symptoms arise requiring immediate surgical or medical aid, the poor inhabitants are doomed, perhaps, to see their nearest and dearest friends cut off for want of timely assistance.

On my first passage across the channel, the boatmen, my Kenramer friends, were very anxious to discover the purport of my visit to Raghery. Though they all conversed in their native dialect, I saw from their looks, and manners, the anxiety they were unable to suppress. At length, one of them ventured to inquire, if I was "one of the gentlemen who looked after the schools?" I answered I was not. "Maybe you would be one of the government gentlemen going to take observations?" said another. Here again they were unsuccessful, and I was at length obliged to inform them of my profession. A physician so seldom visited their island, that they looked upon their present voyager as a kind of godsend; and I observed, that during the remainder of the passage, I was treated with much increased deference.

Questions on the treatment of different diseases now flowed on me; and we had become uncommonly good friends when we reached Raghery. They invited me to visit them when in their part of the island, which I did about a week afterwards. I was proceeding along the banks of Lough Cleggan, in pursuit of a ring-plover, and in the course of the chase, I was led over a small hill forming the southern boundary of the lake. From the summit of this, I overheard my Kenramer friends busied in cutting turf; they immediately recognised me, and they and their companions forming a party of about twenty in number, gathered round me, and, for at least half an hour, did all in their power to aid my search for the ring-plover's nest. They pointed out to me their flock of sheep and herd of cattle grazing in the valley in which their cottages were situated; the animals, though belonging to eight or ten families, were all kept together, each having its owner's mark. The inhabitants of this and other townlands gain a livelihood by cultivating their land and carrying the overplus of their crops to Ballycastle; and from their residing three hundred feet above the level of the sea, fishing and kelp-making are not pursued, as along the southern shores.

One day, while sauntering on the road at Church bay, an old man overtook and accosted me. "I understand, sir," said he, "that you are a doctor." "You are right," I replied. "If it was not taking too great a liberty, then, might I ask what you would recommend for my leg which has been hurt." I examined the wound, and prescribed what I thought necessary. He now wished to pay me my fee for advice, although his dress bespoke any thing but wealth in its wearer. Of course, I expressed my obligation for his good intention, but declined the proffered kindness. What then could he do to oblige me? he would carry my gun—my shot-belt—my botanical box, or any thing else I had; something he must do to evince his gratitude for my attention, and I was at length obliged to send him on some errand to one of the adjoining houses. I met him frequently afterwards in different parts of the island, and he never failed to inform all the invalids in the townland of my approach; and when I had reached the houses, he would present himself with a train of followers, all desirous of medical advice. In many places he acted as interpreter; and some days after I had first met him, I was greatly amused by the dexterity he displayed in discovering the invalids' complaints, and making me acquainted with them. Had I wanted practice in my profession, Raghery offered me every facility.

ZOOLOGY.

The Zoology of Rathlin does not, so far as I am acquainted, offer anything novel, or afford any species not hitherto known as natives of Ireland. Placed at so short a distance from the mainland, it cannot be expected that the animals frequenting Rathlin should be found differing in any considerable degree from those found in the other parts of the country; and the few remarks I have to offer on this part of the subject, will rather serve to mark the *habitats* of some of the species, than throw light on that interesting, though, hitherto, too neglected subject—the Natural History of Ireland.

The only quadrupeds on the island, with the exception of those which are domesticated, are the

COMMON HARE. *(Lepus timidus.)*
NORWAY RAT. *(Mus decumanus.)*
COMMON MOUSE. *(Mus musculus.)*
SHREW MOUSE. *(Sorex araneus.)*

The HARE (*Lepus timidus*) is very rarely seen in Rathlin; and I was unable to procure a specimen, although anxious to examine whether it belonged to the British species, or to the variety lately discovered to be peculiar to Ireland. The total want of underwood, such as whins, broom, briar, or heath, must account for the scarcity of this animal. Dr. Barry states that the same reason, he is satisfied, operates in the Orkney Islands where the hare is now literally extinct, although in former times it must have been abundant, as it formed an object of the chase to the ancient Earls of Orkney.

The BROWN OR NORWEGIAN RAT. As Dr. Hamilton remarks, we might have expected that the native black rat would have found an asylum in Raghery, although driven from the other parts of Ireland; but here the brown rat only is found, having entirely usurped the place of its predecessor. It infests all the dwelling-houses, barns, and store-houses, on the island, and is here, as elsewhere, universally detested.

The MOUSE (*Mus musculus*) is found in and near all the houses on the island. I was unable to learn whether the field mouse is seen here, but

The SHREW MOUSE (*Sorex araneus*) is found occasionally in the fields.

LAND BIRDS.

CINEREOUS EAGLE. (*Falco albicilla.*) This eagle generally frequents the island during summer, and on some of the lofty headlands makes its eyrie. It is, however, but rarely seen. It feeds on young lambs, sickly sheep, fish, and, when pressed by hunger, on carrion. A friend on the island informed me that he one day saw a sea eagle strike down a raven, and afterwards soar away apparently regardless of his fallen victim.

PEREGRINE FALCON. (*Falco peregrinus.*) This falcon, which is found on many of the headlands in the County of Antrim, may be also seen on the precipices on the northern shore of Rathlin, where it annually rears its young. During my visit to the island the situation of one of their nests was pointed out, and a man descended the cliff and procured two of the young, which were brought to me for sale.

SPARROW HAWK (*Falco nisus*) is occasionally seen hovering over the corn fields in search of mice and small birds, on which it subsists.

KESTREL. (*Falco tinnunculus.*) The kestrel is seen in different parts of the island; and on the cliffs on the southern shore I had frequent opportunities of observing it. The young *starlings,* which were, at the time I allude to, just fledged and frequenting the island in large flocks, offered to this and the other birds of prey ample means of subsistence.

WHITE OWL. (*Strix flammea.*) This is the only species of owl found in Rathlin, and is but very rarely seen.

RAVEN. (*Corvus corax.*) A few pairs annually breed on some of the precipitous cliffs on the island, but it is a bird only occasionally seen. It is destructive here as elsewhere to young lambs and sickly sheep, which it attacks and commonly destroys. Chickens and ducks not unfrequently fall victims to its voracity.

HOODED CROW. (*Corvus cornix.*) This wary bird is found in Rathlin throughout the year. They associate mostly in pairs, frequenting the inland parts of the island, and occasionally resorting to the shore in search of shell-fish, and other marine substances, on which they sometimes feed. Like the raven, they breed in the most inaccessible parts of the rocks.

MAGPIE. (*Corvus pica.*) From the total want of wood on the island, the magpie is very rare, one or two pairs only being seen.

CORNISH-CHOUGH. (*Pyrrhocorax graculus.*) This is called by the islanders the *Jackdaw,* and is by far the most numerous species on the island. In the month of July I found them everywhere, associated in large flocks, at one place frequenting inland situations, and at another congregated on the sea-shore. They had just collected together their different families, now fully fledged, and were picking up their food (which consisted chiefly of insects) either on the shore, in the crevices of rocks, or in the pasture fields. Mr. Selby mentions that the chough will not alight on the turf if it can possibly avoid it, always preferring gravel, stones, or walls. On Rathlin its choice of situation seemed to be but sparingly exhibited, as I found it frequenting the corn and pasture fields in even greater numbers than along the shores. This may have arisen from the shores being in general high and rocky, not affording them the same opportunities of seeking their food as a low, sandy, or gravelly beach. They breed on the lofty cliffs overhanging the sea; the eggs are of a whitish colour, speckled at the larger end with brown. The chough is of a restless active disposition, hopping or flying about from place to place; it is also very shy, and can be with difficulty approached. Temminck says that the legs of this bird, before the first moult, are of a dark colour, while Montagu affirms that they are orange-coloured from the first. The young which I examined were about six weeks old, and in them the bills were of a brownish orange; not of that brilliant colour which marks the adult plumage, but certainly exhibiting enough of the orange to lead us to conjecture that they would become completely of that colour after the moult. The legs could not be called "orange-coloured," for although there was a tinge of that colour, yet the brown predominated. I should, therefore, agree with Temminck in stating the legs and feet to be "dark-coloured" in the young birds.

STARLING. (*Sturnus vulgaris.*) This is one of the most common birds in Rathlin. It is found over the greater part of the island, but principally about Church bay, where the houses are more numerous, and where there are a few trees and shrubs. In July they were assembled in flocks of one or two hundred, dispersing themselves over the fields and along the sea-shores. They frequented the more rocky parts of the pasture fields, and seemed busy picking among the loose stones, both here and on the shore, for insects of various kinds. Mr. Low, in his "Fauna Orcadensis," says, that in the Orkney Islands they feed, during the severity of winter, on the *sea-louse*, (oniscus marinus,) which they obtain by turning over the small stones on the beach with their bills. It is more than probable that this insect constitutes part of their food in Rathlin, when the small and inefficient supply of berries and grain has been consumed; but I regret that I did not make the matter certain, by examining the contents of the stomach in my specimens. The cry uttered by this bird was a kind of chatter, which, when coming from a number assembled together, was rather harsh and grating to the ear. They build among the rocks; the young are of a brown colour, lighter on the belly and throat.

CUCKOO. (*Cuculus canorus.*) This well-known bird annually visits Rathlin for two or three weeks in May, when her pleasing note is occasionally heard. Her eggs must be deposited here, as in other places, in the nest of the tit-lark, or some other of the smaller birds, but, so far as I could learn, it has not yet been discovered.

CHIMNEY SWALLOW. (*Hirundo rustica.*) This bird is not very common on the island, being only occasionally seen in the vicinity of the houses in Church bay.

MARTIN. (*Hirundo urbica.*) This swallow is very generally distributed, being found in all parts of the island, as well inland as along the cliffs which overhang the sea. It is the most numerous of the genus in Rathlin. The situations it selects for the purposes of incubation are the out-houses in different parts of the island, and the lofty precipices near the sea. The latter it does not select without regard to eligibility of situation, for I invariably found it frequenting the range of white cliffs running along the north-western side of Church bay. These have a southerly aspect; and being formed almost entirely of limestone, become in summer good reflectors of heat, and, consequently, render the situation well adapted to the nests of the martin. These are placed against projecting masses of the rock, and built of mud and the usual materials. One of these birds which I shot had its mouth completely filled with insects, among which were a large dragon-fly, and one of the tipulæ. The former was alive and apparently unhurt. How it had procured this insect, struck me at the time as singular, for the bird was killed on one of the rocks at least a mile distant from any marsh or pond, the usual residence of the dragon-fly.

SWIFT. (*Cypselus murarius.*) This bird I found in different parts of the island, in populous districts, and near barren and rocky cliffs. Although I saw it so often flying about the headlands on the sea-shore, and conjectured that it had its nest formed either in the earth at the summits of the cliffs, or in holes ready made in the rocks, I was unable to ascertain the fact, from the inaccessible height at which it usually remained. Mr. Selby thinks, that the cry made by swifts, when flying in small parties near houses, steeples, &c. to be the consequence of irritability, excited by the highly electrical state of the atmosphere; and not, as Mr. White supposed, a serenade to their respective families. From what Mr. Selby says, it would be imagined that the cry was only uttered in very sultry close weather with approaching thunder-storms, as these are the periods at which he mentions it as being heard. From the opinions of these justly celebrated naturalists, I should be inclined to differ, and suppose the scream to be uttered by the swift when in the enjoyment of the fine weather in which it delights, and when an abundant store of insects is afforded it, for at these times is its cry most audible. I have very frequently heard a flock screaming when no thunderstorm portended, and when the weather was particularly mild, though cool; and in some places where they abound, I have never seen them without hearing their scream. Mr. Selby's remark may be, generally speaking, correct; but in the north of Ireland, so far as my observation has extended, the opinion I have formed appears correct.

THRUSH. (*Turdus musicus.*) One or two pairs are found on the island.

BLACKBIRD. (*Turdus merula.*) About Mr. Gage's house one or two are occasionally seen; they breed in the garden.

REDBREAST. (*Sylvia rubecula.*) Very rarely seen.

WREN. (*Troglodytes Europæus.*) Very rare.

WHEATEAR. (*Saxicola œnanthe.*) This bird is very numerous on the island, and may be seen in almost every direction, flitting from stone to stone, or fence to fence. All situations seemed equally well adapted to it—the barren moor, the gravelly beach, or the rocky headland. It arrives on the island about April, and after performing the process of incubation, departs for a warmer region. Its nest is placed under a large stone, or in the crevices and interstices of rocks and fences. In July, the young birds are very numerous, and were readily distinguished from the adult by the brown on the upper parts of the plumage, which in the old birds are bluish-grey. It is called by the islanders, the *stonechat*, or *stonecheck*.

Stonechat. (*Saxicola rubicola.*) This lively little bird is common in Rathlin, although not so frequently met with as the foregoing.

Pied Wagtail. (*Motacilla alba.*) This well-known bird is as numerous in Rathlin as on the mainland. It prefers the neighbourhood of the houses, and resorts to the moist watery places which abound on the island, and where a constant supply of insects may be procured.

Rock Pipit. (*Anthus aquaticus.*) This is a constant resident, and is found round all the coasts, the situation being very rocky, and consequently well adapted to its habits. It flits from stone to stone, along the shore, and may always be seen, generally in search of the smaller marine insects, its usual food.

Sky-Lark. (*Alauda arvensis.*) The island is enlivened by the sweet notes of this delightful songster ; it is a constant resident. It builds in the corn and pasture-fields ; and in winter, assembles in large flocks in the same way as on the mainland.

Yellow Bunting. (*Emberiza citrinella.*) Not very numerous.

Reed Bunting. (*Emberiza schœniculus.*) I saw a pair of these birds on a swamp in the southern part of the island where the common reed (*arundo phragmites*) grew in considerable quantities, and where they had formed their nest. This was the only situation in which I found them.

House Sparrow. (*Fringilla domestica.*) Very common.

Common Linnet. (*Fringilla cannabina.*) This little bird was common throughout the island; and at the time of my visit was in its full summer dress, having the bright red markings on the head and breast.

Rock Dove. (*Columba livia.*) This bird, better known by the name of wild pigeon, is not unfrequently seen in Rathlin. It frequents the rocky precipices overhanging the sea, and is found in all districts in the island. I found it in flocks of ten to twenty in the corn fields and open moors. They were very wild, and could not be approached within gun-shot. They breed in the caves along the shore ; but from the inaccessible situations they occupy, their nests are seldom disturbed.

Partridge. (*Perdix coturnix.*) But very rarely met with.

Quail. (*Perdix coturnix.*) I heard this little bird occasionally in the corn fields.

WATER BIRDS.

Common Heron. (*Ardea cinerea.*) There are but few places in the British isles, where this common, though not uninteresting, bird is not met with ; and in the island of Rathlin a few pairs have taken up their abode. The numerous ponds and marshes with which Rathlin is furnished, render it a desirable situation for a bird like the heron, whose food consists of small fish; for eels abound on the island, and afford it an easily obtained and plentiful supply. The ponds and marshes are for the most part totally devoid of planting or other shelter, and surrounded with bleak barren hills, on which scarcely a trace of vegetation is to be seen—yet here, on these dreary solitudes, the heron may be seen standing in its apparently listless attitude, on a stone at the edge of the water, watching for its prey. The heron is gifted with extreme wariness; and the situation it selects here, precludes the possibility of approaching within gun-shot, for it can see and be seen from a great distance. On one of the marshes at the southern extremity of the island, the water covers the ground to any depth, only after heavy rains. Here the marsh is almost entirely covered by the *nymphæa alba*, *nuphar lutea*, and *arundo phragmites*, the latter of which grows to a considerable height. This is the only spot on the island where their nests could be placed, there being no trees, their usual resort, during the season of incubation. This affords us another instance of the facility with which birds can adapt themselves to situations different from those usually selected; for it is a well known fact, that herons almost invariably place their nests at the summit of some lofty trees; yet in such a spot as Rathlin, where no trees are to be had, they choose such a situation as I have described. Here the reeds are sufficiently tall to enable them successfully to conceal their lurking places; and the ground is so soft as to prevent any disturbance from persons inclined to annoy them. A boy told me, that in the summer of 1833, he ran a heron down; it had, in all probability, gorged itself with food, and was unable to make its escape from among the reeds and grass.

Common Curlew. (*Numenius arquata.*) The curlew is occasionally seen during the summer season, although by no means numerous. In winter, however, they assemble in large flocks; and in severe weather, approach the houses and cultivated grounds, in search of food.

Redshank Sandpiper. (*Totanus calidris.*) I saw only one pair of these birds on the island, although I expected to find them much more numerous, the marshes and bleak hills being favourite situations.

Common Sandpiper. (*Totanus hypoleucos.*) This bird, like the preceding, I was surprised at finding so rarely, having seen only two or three round the shores, and never having met with it about the ponds or marshes.

Common Snipe. (*Scolopax gallinago.*) On inquiry, I was told that this bird was seen only very rarely in winter, and was, therefore, quite unprepared for finding it in July; which, however, I did on the marsh at the southern end of the island. I was watching the motions of another bird, when I suddenly heard what I thought was the bleating of a goat in the neighbourhood. I looked round—the sound was now before me, at one time very loud, and, at the next moment, apparently at the further extremity of the marsh. After looking frequently around me in vain, I at last heard it so distinctly above me, that I looked up, and there the snipe was hovering, sweeping down repeatedly towards the marsh, and again wheeling aloft, all the time uttering its peculiar bleating cry. Its young, I conjectured, were at the time in the marsh ; and its anxiety was severely tried, while my companion and I remained in the vicinity.

MEADOW OR CORN CRAKE. (*Crex pratensis.*) The rail is plenty in the corn fields in all parts of the island, and forms her nest in similar situations to those on the opposite coast of Antrim.

COMMON GALLINULE. (*Gallinula chloropus.*) This bird inhabits the marsh which I have before alluded to in the southern extremity of the island, and makes its nest among the reeds and water-lilies covering its surface.

CRESTED OR GREEN LAPWING. (*Vanellus cristatus.*) I found this plover on some of the high grounds on the island ; but at this season, they are by no means so numerous as in winter.

RINGED PLOVER. (*Charadrius hiaticula.*) This handsome little plover is not unfrequently seen in summer, on the high, retired, and stony parts of Rathlin, where it forms its nest. I found it also on the gravelly beach of Church bay, where, near high-water mark, its eggs have been frequently found. In the month of July they were running about, accompanied by their young, for whom they evinced the greatest attachment.

BEAN OR WILD GOOSE. (*Anser ferus.*) Seen in small flocks in winter.

BRENT GOOSE. (*Anser brenta.*) Occasionally met with in Church bay.

WHISTLING SWAN. (*Cygnus ferus.*) Seen sometimes in hard winters.

COMMON WILD DUCK. (*Anas boschas.*) Occasionally shot in winter.

COMMON TEAL. (*Querquedula crecca.*) In small flocks on the marshes and ponds in winter.

COMMON WIGEON. (*Mareca penelope.*) Rarely seen in winter.

NORTHERN DIVER. (*Colymbus glacialis.*) Frequents the bays and shores of Raghery in winter and spring, and has been seen both in adult and immature plumage.

FOOLISH GUILLEMOT. (*Uria troile.*) These birds were congregated in very considerable numbers on the north-western extremity of the island, where the high and precipitous rocks afford them facilities for incubation. They were not, however, so plentiful as either the razor-bill auks or puffins, but they frequented the same rocks indiscriminately. This guillemot lays one large egg on the bare rock, to which it is secured as it were by its peculiarly conical shape, being very large at one end, and diminishing rapidly towards the other. It is thus prevented rolling off the rock ; but it was in former times supposed to be retained on the rock by some glutinous substance applied to one side by the bird. The young guillemots I had frequent opportunities of examining ; they were, when excluded from the shell, covered with a dark grey down, of a whitish colour underneath.

Black Guillemot. (*Uria grylle.*) This bird frequents the southern or Ushet extremity of the island—a place totally devoid of any other sea-fowl—and the shores which immediately front Ballycastle, where I found them in number about thirty, flying backwards and forwards among the rocks, where they had established themselves. I saw only one pair on the northern shores, and could not ascertain whether they bred there or not. At their breeding haunts on the southern shore they were very wary, and could scarcely be approached; but the day I visited the immediate vicinity of the spot I allude to, was so stormy, and the sea ran so high, that I dared not keep the boat closer to the rocks, in order to examine their breeding places more particularly. The black guillemots were easily distinguished from all the others, by the dark plumage and the white spots on the wings.

Razor-bill Auk. (*Alca torda.*) This auk was found associated with the foolish guillemot in countless numbers on the northern shores of Rathlin. It was, however, much more plentiful than the guillemot, but so much resembling it in general appearance, that by the boatmen they were invariably confounded, and, while sitting on the rocks, regarded as belonging to the same species. The cry of the razor-bill auk is a kind of croak, harsh and disagreeable; and by an imitation of it, the birds, securing themselves behind the ledges of rock, are drawn out from their lurking places by the fowlers. The egg is similar in size and markings to that of the guillemot; the young were covered with dark grey down, the bill slightly hooked at the tip, but not presenting the peculiar marks which characterize that of the adult.

Common Puffin. (*Fratercula arctica.*) These little birds breed in great numbers at the Bull Point, and the headlands adjoining, where the rocks are based with mould, and intersected and covered here and there with patches of grass; thus affording them facilities for scooping out their nests. These we found wherever the earth appeared among the rocks. Here they excavate or burrow in the mould to the depth of two or three feet; and, at the extremity of the excavation, the egg, which is white and about the size of a hen's, is deposited on the bare earth. From being surrounded by the damp mould, it appears, when taken from the hole, of a dirty brown, but, on being washed, it acquires its natural colour. The puffins seemed equally numerous as the razor-bill auks; they took possession of the earthy parts, while the latter sat close beside them on all those bare ledges of rock not otherwise occupied. These birds, with a few guillemots, were met with in considerable numbers along the range of white cliffs facing the south, and forming the northern boundary of Church bay; they were not, however, by any means so numerous as on the northern side of the Bull Point. The opinion prevails here, as well as elsewhere, that the puffins feed their young with sorrel, when they become, as it is stated, too fat to allow them to make their escape from their burrowed nests. This idea I conceived might have originated in consequence of the quantity of the plant not unfrequently found growing, as in Rathlin, in the vicinity of their nests.

COMMON CORVORANT. (*Phalacrocorax carbo.*) This bird breeds in the caves round the northern coast.

CRESTED CORVORANT. (*Phalacrocorax cristatus.*) We found this corvorant in pairs, frequenting the numerous caves with which the northern and western shores of Rathlin are indented. They formed their nests on the high ledges of rock, almost touching the summit of the caves; the nest was composed of fuci of various kinds, matted and plastered together; the eggs were of a bluish-green colour. We sometimes, by good management, entered the caves ere the corvorants had left, and at such times we found them sitting, with the neck and head thrust over the ledge of rock, looking down on the boat as it made its way to the inner extremity of the cave. On firing our guns, they would drop into the water as if they had been shot, and, with great expertness, dive under the boat, and make their way out to sea. This species seemed much more numerous than the preceding.

SOLAN GANNET. (*Sula bassana.*) Occasionally seen fishing in the channel. The first time I crossed to Rathlin, two or three pairs were very busily engaged at the fry, then passing down along the coast.

COMMON GULL. (*Larus canus.*) This species occupied one of the large natural amphitheatres formed on the north-western side of the island, and which seemed to be occupied by no other species. Their nests were placed towards the summits of the cliffs in situations equally inaccessible from above or below; and, when disturbed, the birds would soar away at such a distance as to leave them free and undisturbed by any intruder.

KITTIWAKE. (*Larus rissa.*) This is by far the most common species of gull in Rathlin. On all the precipitous headlands north of the Bull, with the exception of a few, these birds take up their summer residence; and they were, during my visit, in such countless multitudes, as to darken the air above our heads. I have never witnessed so great a congregation of birds as along the headlands of Raghery. Every pinnacle and ledge of rock was tenanted by the razor-bill, puffin, or kittiwake gull; and, numerous as the others were, the latter far outstripped them in number. The nests were formed of dried grass, sea-weed, &c.; and the eggs, usually two in number, are of a grey colour, blotched and dotted with brown and purple. When I looked down from a height on these nests, it appeared wonderful how the birds found room to sit and hatch their eggs or tend their young, for five or six nests were placed on a shelf of rock so close to each other that the birds sat in contact, and, if not peaceably inclined, would have thrown the whole into confusion, and prevented each other from fulfilling the process of incubation. Yet they all seemed to live in harmony; and, except when one unintentionally occupied a nest not its own, (which very rarely happened,) they never attempted to disturb one another. The young, when first excluded from the shell, are covered with a greyish down, intermixed with white. Their food consisted chiefly of fry. For two or three miles along the base of these cliffs the rocks were covered with eggs, from which the young had been liberated—young birds which had been precipitated from the rocks, and with the excrement and feathers of the adult birds.

HERRING GULL. (*Larus argentatus.*) This gull occupied the summits of the cliffs, tenanted below by the foregoing species; but their nests, like those of the common gull, were placed far beyond reach, except by lowering a man by a rope. Besides being found on the northern side of the island, these birds occupied the range of white cliffs on the northern side of Church bay; here they remained quite secure, for they scarcely ever ventured lower than the middle of the precipices, and could in this manner effectually escape the gun of the fowler, either from the summit or base. This species I also found in pairs on the eastern coast, although on this part of the island it was rare.

The cry of the herring gull is very similar to that of the common gull, and the two were not unfrequently confounded with each other, when soaring towards the summits of their respective cliffs.

Among the AMPHIBIA, I may briefly mention the COMMON SEAL. (*Phoca vitulina.*) This animal frequents the numerous caves in Rathlin, particularly during the winter season, when it is seen in very considerable numbers in Church bay and other parts of the island. It varies in size from three to five feet. It is seldom or never seen nearer to the shore than high-water mark, and it generally prefers the rocks at the mouth of its cave, where it will lie for hours, basking in the sun. It is generally taken by the natives with the gun, but may also be secured by throwing a net across the mouth of the cave, and driving the seal out from the interior extremity, to which, when alarmed, it retires. The skin and oil are made use of; the former in the manufacture of shoes, caps, &c. and the latter, for burning.

FISHES.

Although classed among the Mammalia, I shall in the present instance, place the CETACEA at the head of the fishes.

COMMON WHALE. (*Balæna mysticetus.*) This monster of the deep is occasionally seen in the channel between Rathlin and Ballycastle, though of late years very rarely. Part of the skeleton of one may be seen on the shore of Church bay; but whether belonging to one of these animals which had been killed by the natives, or one accidentally thrown on shore, I could not ascertain.

PORPOISE. (*Delphinus phocæna*) is frequently seen in large shoals or herds in the channel and round the coast during summer; but in the more inclement season, they desert these shores.

GRAMPUS. (*Delphinus orca*). This voracious fish is also met with in great numbers, during the summer months; it is said to be very mischievous, and not unfrequently to endanger boats.

FROG-FISH, ANGLER OR SEA-DEVIL. (*Lophius piscatorius.*) This fish has been taken on the coast of Raghery.

SKATE. (*Raia batis.*) The skate is taken in deep water, in considerable numbers round the shores; and were it not for the prejudice entertained towards it by the islanders, might afford them, as in Scotland, Shetland, and other places, a nutritive article of food; but here they will eat the skate, only when nothing better can be had. When salted and well dried, it will keep for upwards of a twelvemonth.

LESSER DOG-FISH (*Squalus catulus*) is often caught on the long line, during summer, and when captured, is valued by the natives for the oil which it affords.

CONGER-EEL. (*Muræna conger.*) Conger eels are very abundant round all the coasts, and often take the bait on the long line. They furnish a small quantity of oil, but are never used as food in Raghery.

LAUNCE. (*Ammodytes tobianus.*) This little fish, commonly known by the name of sand-eel, is very abundant round the island, and furnishes a favourite food to the different sea-fowl frequenting Raghery. Almost every sea-fowl I had an opportunity of examining, had the mouth and stomach filled with the fry of this fish; and from the innumerable flocks of birds which reside here during summer, the quantity of fry devoured at this period must be quite incalculable.

COD-FISH. (*Gadus morhua.*) This valuable fish was formerly very abundant round the shores of Rathlin, but of late years it has been but occasionally caught.

The only cod bank near the island is called Skirnaw, and lies between Rathlin and Isla. The red cod-fish is much esteemed.

COAL-FISH. (*Gadus carbonarius.*) This fish, known through all its stages, by the names of *pickoc*, *blockan*, *glashan*, and *grey lord*, was in former times a most abundant species in Rathlin, and furnished a cheap, wholesome, and nutritious food. In latter years, however, they have become scarce; and on an average, one fish may now be caught, where at least twenty were captured before. So numerous were they formerly, that they could be taken by a common boat hook, or a pole armed with iron; and in Church bay, they not unfrequently loaded a boat in this manner, and in a very short space of time. In Orkney, they were equally abundant, as Mr. Barry mentions that they were caught in myriads, and valued not only as articles of food, but for the quantity of oil furnished by the livers.

Round the coast of the county of Antrim, they were taken in the following manner. At the ebb and flow of tide, two men rowed against the current, so that the boat continued nearly stationary, the impulse of the oars counteracting the force of the stream. The hook was coarsely dressed with a goose feather thrown on the water, and greedily caught by the fishes, which were often so plentiful as literally to cover the surface of the water.

From Rathlin, the coal-fish after being salted, was exported in very considerable numbers; and the quantity of oil collected during the summer season, served the inhabitants for lighting their lamps during winter.

At Drainsbay, near Larne, in 1810, 456 fishes of this species, supposed to weigh upwards of five tons, were captured by a single boat in one night. The coal-fish, in Scotland is, according to my friend Dr. Neill, called the *sillock*, till it attains the length of five inches; and the *pittock*, when it measures twelve or fourteen inches.

LYTHE OR POLLACK. (*Gadus pollachus.*) The lythe is caught in summer in the deep pools, which, from the craggy nature of the coasts, are very numerous. They are sometimes caught so large, as to weigh seven pounds. The bait used in Raghery is crab.

LING (*Gadus molva*) is not an uncommon visitor of Rathlin; but few are captured in comparison to the number frequenting the coasts.

FATHER-LASHER. (*Cottus scorpius.*) This fish is found in the small pools regularly left by the ebbing tide.

PLAISE (*Pleuronectes platessa*) is generally caught on the long line, particularly on the eastern coast towards the great cod-bank already alluded to.

SOLE. (*Pleuronectes solea.*) Occasionally caught with the plaise.

TURBOT. (*Pleuronectes maximus.*) This delicate fish is not unfrequently taken near the island; and specimens have been obtained, weighing twenty pounds.

WRASSE. (*Labrus tinca,*) and the BALLAN-WRASSE. (*Labrus ballanus.*) Both indiscriminately called by the islanders, *murrans*, are caught in considerable numbers in Raghery. The capture of these fish occupies a great proportion of the boys on the island—as on fine days, almost every projecting point of rock in some parts of the island contains one or two fishers. The bait principally used is crab. I could scarcely persuade some of them that the hooks were too large in comparison with the size of the fishes' mouths; and their bait was nibbled away almost as fast as it was put on, in consequence of the disproportion of the hook to the fish's mouth.

The wrasse is easily caught, but not much esteemed, their flesh being soft and watery, and the bones small and numerous. They are found on all the rocky parts of the coast in the county of Antrim, frequenting deep wracky holes, where, from the brilliancy of their colour, on a clear day they may be seen to a considerable depth.

SHORT-SPINED STICKLEBACK. (*Gasterosteus brachycentrus.*) This species of stickleback, first added to the British fauna by William Thompson, Esq. Vice-President of the Belfast Natural History Society, and described and figured in "Yarrell's British Fishes," I found the only species of three-spined stickleback inhabiting Rathlin. It must be noticed as a remarkable fact, that, of all the species of three-spined stickleback, common to England and Scotland, none should be found in Ireland, while in their place we have a truly continental species, the only inhabitant of our Irish lakes and rivers. The specimens which I obtained in Raghery were all smaller than those frequenting the mainland; and although nearly agreeing with the "*brachycentrus*," had the appearance of the lateral plates extending towards the tail. Mr. Thompson examined those specimens which I procured on the island, and came to the conclusion, that although exhibiting an apparent difference, they belonged to the same species. I found them in all the ponds and small streams in Rathlin.

FIFTEEN-SPINED STICKLEBACK. (*Gasterosteus spinachia.*) This species does not, like the other, frequent lakes or rivers, but prefers the rocky pools of salt water which occur round the coasts. I found it in such situations in Rathlin.

GREY GURNARD. (*Trigla gurnardus.*) During the months of June, July, and August, this fish is very abundant round the shores, and is caught in great numbers by the natives, who hang and dry what they do not want for immediate use. The gurnard is easily captured; by a bit of fish-skin, or similar substance, tied firmly on the hook, many hundreds may be caught without ever changing the bait. In Larne, and other places on the coast of the County of Antrim, this fish occasionally furnishes a rich harvest to the industrious fisherman. When the shoals of gurnard make their appearance on the coast, the gulls congregate in innumerable flocks near the fishing stations, by which means the boatmen are directed to their prey. They leave the shore for these fishing grounds about four in the morning, so as to reach their destination about six. From this till nine or ten they continue fishing; and a boat will frequently take, in a morning's fishing, from four hundred to seven hundred gurnards. It is affirmed by some, that, when brought into the boat, the gurnard utters a croaking noise; but the accuracy of this I have never had attested.

HERRING. (*Clupea heringus.*) This valuable fish is seldom met with between Lough Swilly and the Point of Tor, on the Coast of Antrim; and in the vicinity of Rathlin it rarely appears. This may, perhaps, be occasioned by the very powerful currents which sweep round the shores of the island; and to avoid contending with these streams, the herring may keep more towards the middle of the channel, between Rathlin and Scotland.

SALMON. (*Salmo salar.*) In the immediate vicinity of the island it is seldom seen, but, on the opposite coast, at Ballycastle and Carrick-a-rede, it occurs in great numbers.

WHITE-TROUT OR SEA-TROUT. (*Salmo trutta.*) Occasionally caught.

Among the INSECTA, I may briefly allude to the following:

COMMON CRAB OR PARTIN. (*Cancer pagurus.*) These are very abundant round all the shores, and are sought after by the boys for bait to the wrasse or murran.

LOBSTER. (*Cancer gammarus.*) Around many parts of the coast, these are caught in considerable numbers in summer. A large fishing smack anchors at some favourable situation, as off Stroanadergan point, on the western side of the island. The men are provided with lobster baskets, made of a conical shape, with a hole at the top, sufficiently large to allow the animals to crawl in, but prevent their leaving their prison when once within its walls. Each basket is furnished with a piece of fish, flesh, or other similar bait to attract the lobsters; and left down for some hours. They are then drawn up, the lobsters taken out and again sunk, until they have thus taken the requisite number. When caught, their claws are tied together to prevent them injuring each other; and they are put into other baskets which are suspended in the water over the vessel's side, that by this means they may be kept alive till enough have been caught for the market.

One of these fishing smacks will be furnished with twenty or thirty baskets, or lobster-pots, as they are termed, and will take in a morning, not unfrequently, several lobsters from each. To prevent the pots sinking too far, or going astray, each is provided with a large piece of cork-wood, which floats on the surface, and points out the situation. When the vessel has remained here for two or three days, the pots are lifted, the anchor weighed, and the lobsters carried from the coast of Rathlin into Liverpool, Dublin, or some other port, where they bring a handsome price.

BOTANY.

The following are the names of a few plants which were observed on the island; but as the writer's attention was not directed to the *Botany* of Rathlin, they must be considered merely as a list of those accidentally noticed in his excursions through the Island.

The species marked thus (*), were found on the island by the late J. Templeton, Esq.

Ranunculus *acris*, Linn. Upright meadow crowfoot. Frequent.

Ranunculus *sceleratus*, L. Celery-leaved crowfoot. In pools, &c.

Ranunculus *repens*, L. Creeping crowfoot. Common.

Caltha *palustris*, L. Common marsh-marigold. Marshes and ditches.

Nymphæa *alba*, L. Great white water-lily. Covering one of the marshes in the southern extremity of the island.

Nuphar *lutea*, L. Common yellow water-lily. Growing with the white water-lily.

Fumaria *officinalis*, L. Common fumitory. Frequent.

Cochlearia *officinalis*, L. Common scurvy-grass. Rocks on the shore.

* Cochlearia *coronopus*, L. Common wart-cress. On the waste grounds, common.

* Crambe *maritima*, L. Sea-kale. Gravelly shore of Church bay.

Viola *tricolor*, L. Pansy or heart's-ease. Fields, rather frequent.

Polygala *vulgaris*, L. Common milk-wort. Hills, common.

Malva *sylvestris*, L. Common mallow. On one of the hills on the northern shore of the island.

Silene *maritima*, With. Sea campion. Not common.

Silene *armeria*, L. Common catch-fly. Not frequent.

Geranium *robertianum*, L. Stinking crane's-bill. Common on waste grounds.

* Rhodiola *rosea*, L. Rose-root. On rocks at the north end in great abundance.

* Ulex *nanus*, Forst. Dwarf-furze. Sown in Rathlin by Mr. Gage, in 1790.

Anthyllis *vulneraria*, L. Common kidney vetch. Common.

Trifolium *pratense*, L. Common purple trefoil or red clover. In great abundance in front of Mr. Gage's house.

* Trifolium *agrarium?*

Potentilla *anserina*, L. Silver-weed. Road-side and meadows, frequent.

Rosa *spinosissima*, L. Burnet-leaved rose. Frequent in many parts of the island.

Cratægus *oxyacantha*, L. Hawthorn. Very rare.

Daucus *carota*, L. Wild carrot. Borders of fields.

Hydrocotyle *vulgaris*, L. Common white-rot. Marshy grounds.

Galium *verum*, L. Yellow bed-straw. Common.

Galium *palustre*, L. White water bed-straw. Ditches and marshes.

Jasione *montana*, L. Annual sheep's-bit. On the pasture grounds.

Senecio *vulgaris*, L. Common groundsel. Not common.

Bellis *perennis*, L. Common daisy. Frequent.

Arctium *lappa*, L. Common burdock. Road-sides, common.

Carduus *acanthoides*, L. Welted thistle. Common.

Carduus *tenuiflorus*, Curt. Slender-flowered thistle. Road-sides.

Leontodon *taraxacum*, L. Dandelion. Not common.

Myosotis *arvensis*, Hoffm. Field scorpion grass. Common.

Myosotis *palustris*, Roth. Forget me not. Ditches and marshy ground.

Plantago *coronopus*, L. Buck's-horn plantain. Waste grounds.

Statice *armeria*, L. Sea gilliflower. Rocks on sea-side, and on the hills.

Erythræa *centaurium*, Pers. Common centaury. Not frequent.

Anagallis *arvensis*, L. Common pimpernel. Corn fields and hills, common.

* Anagallis *cærulea*, Schreb. Blue pimpernel.

* Anagallis *tenella*, L. Bog pimpernel.

Veronica *beccabunga*, L. Brooklime. Ditches and marshy ground.

Veronica *chamædrys*, L. Germander speedwell. Waste grounds, common.

Euphrasia *officinalis*, L. Eye-bright. Hills, abundant.

Rumex *crispus*, L. Curled dock. Common.

Rumex *acetosa*, L. Common sorrel. Pasture grounds, and by the sides of the marshes and ponds.

Polygonum *amphibium*, L. Amphibious persicaria. Marshy grounds.

Urtica *dioica*, L. Great nettle. Common.

* Ceratophyllum *demersum*, L. Common horn-wort. Large lake (Lough Runaolin) in southern end of Rathlin.

Lemna *minor*, L. Lesser duck-weed. Marshes, common.

Poa *pratensis*, L. Smooth-stalked meadow-grass. Frequent.

Poa *maritima*, Huds. Creeping sea meadow-grass. Sea shore.

Poa *annua*, L. Annual meadow-grass. Road-side.

Festuca *ovina*, L. Sheep's fescue-grass. Abundant.

Arundo *phragmites*, L. Common reed. Covering great part of the marsh in the southern part of the island.

* Triticum *loliaceum*, L. Dwarf sea wheat-grass. On dry ground.

* Lolium *arvense*, L. Short-awned annual darnel. In dry places, as on hills above Church bay.

Aspidium *filix mas*, L. Abundant at the entrance to one of the caves, south of Church bay; growing also on borders of the marshes.

Chara *vulgaris*, L. Common chara. Ditches and rivulets.

On the Affinity of the Hiberno-Celtic and Phenician Languages. By SIR WILLIAM BETHAM, F.S.A. M.R.I.A. *Secretary of Foreign Correspondence, Member of the Royal Academy of Sciences of Lisbon, &c. &c.*

Read 28th of November, 1836.

IN my former paper, read the 23d May and 27th June, I have stated that the names of places in Ptolemy's Geography are significant of their local position, circumstances, or peculiar qualities, in the Hiberno-Celtic language; from which it may fairly be inferred, that the Celtæ must have been an early colony of Phenicians, as all those names were avowedly borrowed from the Phenician mariners.

Before I proceed to lay before the Academy the results of my more recent investigations, I wish to say something in answer to the observation—which, as it has been made by many, requires, perhaps, some preliminary remarks—viz. *That my theory derives the Celtæ, and all their early learning and science, from Ireland and the Irish.*

This is altogether an erroneous notion. I claim for Ireland itself no pre-eminence in science, learning, or the arts, above the other branches of the Celtæ; all I demand for Ireland is—*That her people, being a branch of the great colonizing people of antiquity, enjoyed an equal portion of civilization with the mother country, in the ratio which colonies usually possess.*

It is no part of my theory that the other colonies of the Phenicians, or branches of the Celtæ, derived anything from Ireland, or the British islands, further than what Cæsar asserts, that the chief seat of Druidic science and learning was from thence. Were I to assert that the early Greeks and Romans borrowed their learning, science, and civilization, from the Irish, I should receive and deserve the ridicule due to such an assertion. I shall not, however, fear it, when I assert that they derived those blessings from the Phenician ancestors of the Irish Celtæ. Were I to assert that the Etruscans and Pelasgi were descended from the Irish, I should receive the derision such a declaration would justly call for; but I do not fear it when I assert that they were colonies of the same great people.

Fortunately, the Phenician language has been preserved by their Irish descendants, and by it we are enabled to unravel difficulties, solve problems, and elucidate facts which, without an acquaintance with that tongue, must have remained for ever unexplained and inexplicable mysteries.

ON ITALY AND ITS ANTIENT INHABITANTS.

Previously to the building of Rome, the history of the various antient people of Italy is involved in the deepest obscurity, and of the inhabitants of the more northern and western portions of Europe we absolutely know nothing whatever. We learn, indeed, from Pliny, (3 c. 5) Strabo, (5) Plutarch, (in Romulo) and Mela, (2 c. 4) that the Etruscans, or Tuscans, occupied the countries west of the Tiber, between that river and the Tyrhenian sea; and that they were divided into twelve tribes or districts, having each a chief, monarch, or leader, called a Lucoman.

I have long been satisfied that the Etruscans were an early Phenician colony, and of the same race as the Pelasgi. They are both represented as civilized polished people; and the remains of the former, lately brought to light by the excavations of the Prince of Canino, as well as those which have been long known, exhibit a progress and perfection in the arts which moderns are happy to copy, seldom equal, and never excel.

It may be asked, what other civilized people of that period, except the Phenicians, possessed *a local habitation or a certain country?* The Pelasgi are said to have spread abroad and settled colonies, but no historian has ever given them an *original country*. Were they not the Phenicians, disguised under another specific name? as the Phenicians who inhabited the southern coasts of Arabia were called *Homeritæ*, which, Herodotus tells us, *meant the same* as Phenician, each indicating a seaman or mariner.

The researches of Micali on the antient peoples of Italy, has thrown considerable light on this most interesting subject. He says, " In the religion of the Etruscans there is rather a general resemblance to the great oriental systems than to that which is purely and exclusively Egyptian—monuments of Phenician and other eastern superstitions appear mingled with those of an Egyptian character."

A writer in the Quarterly Review for September, 1835, says, in the critique on Micali's work, "The Etruscan language stands alone a problem and a mystery, not merely allied to none of the older dialects of Italy, but bearing no resemblance to *any language* to which it has been compared.

The means of explaining and unravelling this difficulty has, hitherto, been wanting, but I shall endeavour to show that at least some of the Etruscan words and names are significant in an existing tongue, and indicate a resemblance so striking and palpable, that it will be difficult, if not impossible, to avoid recognizing them as cognate tongues.

Exclusive of proper names, but few Etruscan words have come down to us through the Latin writers. One most remarkable word, however, is mentioned by Suetonius, in the life of Augustus, which is *Æsar*. He relates, that the electric fluid having struck the statue of Augustus at Rome, melted the *C* from the name Cæsar on its pedestal. The augurs declared that C denoting one hundred, and God being called *Æsar* in the Etruscan language, the emperor had but one hundred days to remain on earth, when he would have his apotheosis, and be taken to the gods.* The death of Augustus took place shortly after, and apparently verified the prediction.

Aosar is the (Irish) Celtic word for God; and not only is the word itself to be found in the Irish Dictionary, but its roots, or the two words of which it is a compound—aos, *age*, and ar, *ruling, guiding, dividing, judging, controlling*, i. e. *God*, as *the guider, ruler, and controller of ages, the eternal ruler.*

I have collated the names of the Etrurian *nations* with the Celto-Phenician, and the following is the result:—

Veientes. The people living in an undulating or sloping country, from faoin, undulating or sloping.

Clusini. Cluasin, *a pot, or porringer*, of brass or other metal; probably from the manufacture of such articles being carried on in the place.

Cortonensis. Caore, *sheep; dun, a hill.* A hilly country, favourable to the feeding of sheep.

Arretani. The arable or agricultural country: aoireaḋ, *a ploughed field or land;* tana, *country.*

Vetuloni. Featal, *a cup or vase* of earthenware; ana, *rich or productive of.* This name is very interesting on account of the recent discoveries on the site of the old city or town of Vetulonia by the Prince of Canino. *Featalon* means the manufactory of earthen vessels, vases, cups, &c.

Volaterrani. Folada, *cattle;* tir, *land;* ana, *rich. The country rich in cattle.*

Russellani. Rus, *a wood;* alaineacṫ, *beautiful. The well-wooded or beautiful country of groves.*

Volcinii. Fáol, *wild, fierce, bold;* scian, *a knife or dagger.*

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* "Mors quoque ejus, de qua de hinc dicam, divinitasque post mortem, evidentissimis ostentis præcognita est. Cum lustrum in campo Martio magna populi frequentia conderet aquila eum sæpius circumvolavit; transgressaque in vicinam ædem, super nomen Agrippæ ad primam literam sedit: quo animadverso, vota, quæ in proximum lustrum suscipi mos est, collegam suum Tiberium nuncupare jussit: nam se quanquam conscriptis paratisque jam tabulis negavit suscepturum quæ non esset soliturus. *Sub idem tempus ictu fulminis ex inscriptione statuæ ejus prima nominis litera effluxit; Responsum est, centum solos dies post hac victurum, quem numerum C litera notaret; futurumque, ut inter Deos referretur quod* ÆSAR, *id est, reliqua pars e Cæsaris nomine, Etrusca linqua* DEUS *vocaretur.*" *(Suet. in Aug. 97.)*

Tarquinii. Tarconas, *a ferry or passage over a river.* These people resided in a town situated on the river Arone.

Falisci. Fall, *a precipice or cliff;* uisge, *water.* The people residing on or near a cataract or fall of water.

Coretani. The pastoral or sheep feeding tribe or people. Caor, *sheep ;* taṅá, *country.*

Lucamo, or *Lucamon.* The title of the hereditary chiefs of the Etrurian nation, derived from leic, *a precious stone*, and caoman, *a noble, chief, or prince.* One ornamented with or wearing a diadem, as a mark of dignity or sovereignty.

The name of the country *Tuscia* is probably derived from the circumstance of its being the first settlement made by the Phenicians on the coasts of Italy. From tosac, *a beginning or commencement.*

The Irish have many figurative refinements of language, by which they denominate the Deity: as aosar, *the eternal ruler ;* tosac gan tosac, *the beginning without a beginning*, or *the first cause.* Their common name is dia, which also means *a day ;* so dia aois, *the antient of days*, may be the true origin of *deus.* How little foundation is there for the assertion that the Irish language is without terms of art or refined ideas; it would be made only in total ignorance or relying on the ignorant declaration of others. The language is rich in such refinements as could only have originated among a thinking and reflecting people.

The Greeks called Etruria τυῤῥηνια, and the sea the Tyrhenian sea. This is probably from tir, *land ;* eigean, of *necessity*, *force, compulsion, violence*, or *conquest*, from being driven upon it first by stress of weather or other necessity.

THE GREEKS.

The first civilized people who are said to have visited Greece were the Pelasgi, a most mysterious nation. We know little of them but their name, and their character as a civilized people ; but from whence they came, or of their original country, not a word. They are said to have settled colonies in Argolis, in Peloponesus, Thessaly, and Epīrus, while others extend them over all Greece, and even into Thrace ; and, as usual, their name, for want of correct knowledge, is derived by the Greeks from an individual called *Pelasgus*, who is said to have been their first king. But this derivation is unworthy of a moment's consideration.

Two theories have been promulgated respecting their origin. One bringing them from the barbarous hordes residing on the Caspian and Euxine sea ; the other makes them *autochthones*, or *aborigines ;* both unsupported by evidence. An admirable German writer, Conrad Mannert, of Nuremberg, (in his *Geografie der Griechen*, 1792–5) gives a more probable hypothesis, while he rejects both the former. He says, " *they every where met, on their arrival, with races of men less civilized than them-*

selves ; some living in forests, and others just formed into civil societies. The name Pelasgi *was never assumed by them,* but was given them by the Greeks. The name was more antiently written Πελαργοὶ, and was applied to them in familiar language by the Greeks, from the resemblance they bore to *storks* and other *birds of passage,* when they first became known to the Greeks; for it seems before they fixed themselves permanently in Greece they would *appear and disappear from the coasts at almost stated and regular intervals.*" How exactly, I may say, this notion agrees with the habits of mariners in their trading voyages. And it may be asked, who were the commercial mariners of antiquity? *The Phenicians alone.*

Again, Mannert says, "All the Pelasgic colonies which established themselves among the early Greeks, brought with them the elements of civilization and the arts. *Whence did they obtain them?*" Certainly not among the ignorant barbarians of the Caspian and Euxine.

Again, "The Pelasgi are acknowledged, by the concurrent voice of antiquity, to have brought with them into Greece a peculiar and distinct system of religion. They are acknowledged, moreover, *to have been the founders of Grecian theology.* They established an oracle at Dodona, instituted the mysteries of the Cabiri, and there is reason to believe that those of Eleusis were of similar origin."

Few will venture, I think, to question the truth of these observations. Now if we can discover an antient language, in which all the names of the Greek divinities and heroes are significant of their peculiar attributes, we may justly conclude it to be a cognate language with that of the Pelasgi; and if that language also be proved to be the same as the Phenicians and Etrurians, it follows that they were a branch of the same people as the Phenicians and Etrurians.

"Profound night (says Mannert) rests on this portion of history; a single gleam of light pierces the darkness which involves it. On the one side of the Pelasgi many tribes of Illyrians practised navigation; as, for example, the Phœacians of the island of Scheria, afterwards Corcyra. At the head of the Adriatic existed long established commercial cities, and artificial canals were seen at an early period. Every thing seems to indicate that at an early period the shores of the Adriatic were inhabited by civilized communities." Thus far Mannert. We may add that the name of the city of *Venice* itself would indicate a Phenician origin; and their position, and the character of the people, strongly corroborates the same idea. Phenice and Venice are very similar in sound, and both suggest a common origin or meaning; that is, *of a mariner or plougher of the seas.* Mannert, indeed, supposes the name to be derived from the Sclavonic *wenden, to rove about,* and that they were a northern race; but he was not acquainted with the Celtic language, or he would not have made so improbable a conjecture.

Herodotus states that letters were introduced into Greece by Cadmus, but Diodorus claims a previous possession of written characters; and Pausanius mentions an inscription on a monument earlier than the time of Cadmus. The learned editor of the last edition of Lempriere's Classical Dictionary meets this with a very conclusive answer. "How came," says he, "the alphabet used by the Greek nation to bear so close a resemblance in the names, order, and very form of the letters, to the alphabets of the nations which belonged to the Shemetic race; namely, to those of the Phenicians, Samaritans, and Jews; or, to speak more correctly, to that of the Phenicians, for those and the Jews, until the time of Cyrus, used the same characters? One of two suppositions must be the answer to this question. Either the Phenicians introduced an alphabet into Greece so far superior to the old Pelasgic as to be adopted in its stead, or the *alphabet of Cadmus and that of the Pelasgi were the same.*

"The first supposition will be found extremely difficult to support. It takes for granted what few, if any, will be willing to allow, that there existed in those early ages a sufficient degree of mental activity and refinement, on the part of the rude inhabitants of Greece, to induce them to discriminate between the comparative advantages of two rival systems of alphabetic writing." The most rational conclusion is, that the Pelasgic and Cadmean alphabets were the same, and both were Phenician.

The history of Greece, previous to the period when Cadmus taught them the use of an alphabet, is nearly a blank, and involved in dark fable for near 800 years after. Rome was founded about the year 704, A. C. But both these periods are, however, recent when compared with the glorious era of Tyre and Sidon; and it will not be denied that the Greeks, when first visited by the Pelasgi, were nearly as ignorant and illiterate barbarians as the South-Sea islanders were, on their first discovery, to the English.

Their learning, science, arts, and the whole of their mythology, with its appendages, were all borrowed from their schoolmasters. They really have nothing antient of their own. It must have appeared to every scholar how absurd, far-fetched, and puerile, are most of the attempts made to derive Greek names from their own language.

The Greek mythology seems to have been a disfigured and corrupted paraphrase of the Phenician system: each mythos appears a confused representation of something they had learned without being acquainted with its precise or defined meaning; and every story being involved in a mist, is exaggerated and distorted by being viewed through it.

Most of the names of their divinities and heroes have no meaning in the Greek language, but appear mere barbarous and unmeaning epithets. This is equally true with the Romans, who, indeed, invented some new divinities, and gave them names indicating their supposed attributes, but the names of their old deities are equally without meaning in the Latin.

It is, however, very remarkable, that although many of the Greek names for the same divinity differ from the Roman, yet both being derived from the same Phenician source, by different channels, each are significant in that language, and express the same, or common, attributes of the given deity, so as to mark their identity with extraordinary precision. Thus VULCAN means the *adroit, or able and intelligent, smith.* HEPHAISTOS, *one skilled in the effects of heat or fire.* VENUS, *the woman of the community,* or the courtesan. MERETRIX, *violent lust or lechery.* Αφροδιτε, *the produce of froth.* MERCURY, *swift cap,* or the man with wings on his cap. HERMES, *the ray of the sun.* BACCHUS, *the lame or staggering drunkard.* Διονυσος, *the drunken man,* &c.

It would be inexpedient, even if time would permit, to fatigue the Academy with a collation with the Celto-Phenician of every name to be found in the Greek and Roman mythology; but it is necessary to state that they are found, almost without an exception, significant in too remarkable a manner to be mistaken.

I shall now proceed to state the meaning, in the Celtic, of some of the most remarkable of the divinities and heroes of the Greeks and Romans, and commence with the *most antient.*

Uranus. The most antient of the gods. This name is derived from the sun's supposed orbit round the earth. Uꞃ, *the sun ;* ain, *circle.* The idea of the motion of the sun round the earth, his benificent warmth, vivifying and generating powers, naturally became the object of early and grateful devotion ; and the Greeks worshipped the sun, (the Phenician or Tyrian Hercules,) under the name of Ουρανος, which is nothing more than the Phenician uꞃain Hellenised.

Cœlus. The same deity as Uranus, under another name. From Ceil, *a husband, generator, father, progenitor, creator ;* aoiꞅ, *of ages.*

Thea, or *Tithea*—she is represented as the wife of Cœlus—is merely ꝺia, *plenty, abundance, riches.* Thus the generator or creator produced all things from his abundant power or capability of production. This goddess is also called *Titea,* which is a compound of ꞇi, *power, government, rule, design, intention, contemplation,* and ꝺia, *divinity* or *god.* Thus Ceil, the creator, by his power, wisdom, prudence, and fore-knowledge, produced *Saturn, Chronos, or Time,* heaven, earth, and all things. Her name of Ops may be from ob, or op, *force,* or *violence, of parturition. Rhea,* from ꞃe, *the moon.*

Saturn, or *Chronos.* The former of these names is palpably Phenician—it literally means *the Lord,* ꞅe ꞇiaꞃna ; the latter equally so, *Time,* Cꞃon.

The *Titans,* brothers of Saturn, meant nothing more than *the Princes.* ꞇi, *the ;* ꞇañ, *princes.*

INFERNAL DEITIES, RIVERS, &c.

Pluto. The god of hell, or the infernal regions. Blotac, is one who dwells in a cave, or under the earth ; from blot, *a cave* or *mine.* From wealth being found in a *mine* Plutus, the god of wealth, has his name, as also the word Πλουτος, *wealth.*

Acheron. A river in hell. *The river of bitterness, severity.* Acar, *bitter, severe ;* aḃan, *river.*

Abastor. One of Pluto's horses, so called because he barked like a dog. Abostrac, *barking like a dog.*

Ades or *Hades.* Pluto ; the infernal regions. Aid, *cold ;* ais, *death.*

Hecate. The same as Proserpine, Luna, and Diana. *The eye of death.* Ecc, *death ;* aed, *eye.*

Cerberus. The dog of hell. Cearboire, *the worryer, slaughterer ;* or cear, *death, burial ;* berac, *barking, the incessant barker.*

Charon. The ferryman over the Styx. *The river of interment.* Cear, *burial, interment ;* aḃan, *river.*

Barathrum. The infernal regions. *The change of death.* Barr, *death ;* atruiġim, *I change.*

Gorgones. Frightful women. Gorg, *cruel, fierce, frightful ;* gean, *women.*

Nemesis. The goddess of revenge. Nemaise, *terrible, cruel, revengeful.*

Tartarus. The region of hell, where the most impious and guilty were punished. *The place of scorn, reproach, and contempt.* Tar, *contempt, reproach, scorn ;* tarrac, *horrid, terrible, fearful, appalling.*

Tisiphone. One of the furies—represented with *a whip in her hand.* Teas, *hot ;* siṫbe, *a whip or scourge ;* feanaim, *to flay, scourge, excoriate.* These words, compounded, give the exact sound of the Greek, or Latin name, with its equally correct meaning.

Typhon. "A famous giant, *son of Tartarus and Terra,* who had a hundred heads, like those of a serpent or dragon. Flames of fire were darted from his eyes and mouth, and he uttered horrid yells, like the dissonant shrieks of different animals. He was no sooner born than he made war against heaven and frightened the gods." Such is the description of this portentous being, and very accurately does the personification of the monster describe the subterranean fires and volcanoes of Etna, which its name indicate. Ti, *burning, or fire ;* f'on, *under, or below, the earth.* The hundred heads are the peaks of the mountain, the mouth and eyes are the craters which vomit forth fire, and the horrid yells are the thunder-like voices and hissings. The making war against heaven are the missiles projected from the craters. "Jupiter is said to have put Typhon to flight and crushed him below Etna."

The Egyptians reckoned *Typhon* to be the cause of all evil. Ti also signifies *judg-*

ment; f'on, *below,* in which sense it meant *the infernal regions,* and without doubt was the origin of *Tartarus,* or *Hell.* He is represented as the father of Geryon, Cerberus, and Orthos. *Geryon—the rugged, dreadful river, with the rocky bed. Cerberus*—the dog of hell: cear, *death;* berac, incessant *talking, barking, shouting.* The noises of Etna were supposed to proceed from the barking of Cerberus: our word *bark* was probably from the Phenician *berac. Orthos*—or, *noise, sound;* tos, *first. The vigilant dog who gave the first bark.*

Bacchus. The god of revels and drinking. Bac, *loving;* as, *drink, liquor, wine;* or bacaċ, *lame, staggering,* from drink.

Dyonusus (Διονυσος). The Greek name of Bacchus. Daoneas, *manliness;* as, *drink. The drunken manly hero.*

Eleleus. A name of Bacchus, from the word ελελευ, which the Bacchanals loudly repeated during the festivals. This is the cry of the Irish, of *ululu,* or *pillulu,* at funerals and when melancholy drunk.

Euhyus. A name of Bacchus. Eug, *dying,* or *dead;* as, *drink*—i. e. *dead drunk.*

Iacchus. A surname of Bacchus. Iac, *a yell, scream, shout;* as, *drink.* From the noise the Bacchanals made at his feasts. Ιαχειν, *shout;* ιακχάζω, *to shout, revel, to be drunk.* This word is also, no doubt, from the Phenician root.

Mænades. The Bacchantes, priestesses of Bacchus. This word is generally derived from μαινομαι, *to be furious;* but I would rather say from the Phenician meanad, *gaping, yawning, vomiting.*

Orgia. Festivals in honour of Bacchus. Orgios, *cheer, entertainment, revelry.*

Orphica. A name by which the orgies of Bacchus were called—*because they were introduced by Orpheus?* Or, *sound, voice, music;* fioc, *concert, combined, united, a choir.* Thus singing in a choir is so called, and not from Orpheus.

Diana. Dia, *divinity;* na, *the*—na, is the feminine of the article, an, the. *The goddess.* Diana was called *the goddess,* in consideration of her eminence. "*Great is Diana of the Ephesians.*"

Echo. A daughter of *Air and Tellus.* Eccosg, *a model, shape, likeness, repetition, simulance.*

Eolus. Eolas, *knowledge, science, philosophy, art;* king of the winds, a great astronomer and inventer of sails. How much more palpable is this derivation than the Greek αιολος, *varius.*

Fortuna. The goddess of fortune. This was a very antient deity. She has been represented standing on the prow of a ship and holding the rudder in her hands, which is probably one of her most antient representations. The import of the name Fortuna is, for, *protection, defence;* tuin, *from the waves,* which is well expressed in the above figure.

Janus. His temples were closed in time of peace and open in war; ian, *the blade of a sword.*

Iris. One of the Oceanides, messenger of Juno and the gods. Ur, *the sun ;* aer, *a shower. The rainbow.* The production of the sun and a shower.

Lares. Gods who presided over houses, families, and grounds, or estates. Lar, *the ground, floor, hearth.*

Lucina. The goddess who presided over the birth of children. Her mother brought her forth without pain, from whence her name. lu, *little ;* ci, lamentation ; na, *of the.*

Luna. Diana, the moon. luan, *moon ;* na, the feminine article *the.* Literally, *the moon.*

Napæ. Certain divinities who presided over hills and woods. It is said from ναπη, *a grove*, rather from, naob, a *holy person*, a *nymph.* The Irish call the fairies the *good people ;* naoiṁ, the *nymphs*—see *nymphs.*

Naiades. Inferior deities who presided over rivers, springs, wells, and fountains. They are said to resort to the woods near the stream or fountains over which they presided, and hence their names, (says Lempriere) ναιειν, *to flow.* The true derivation is from naoṁ, *sacred or holy*, or a holy person ; dia, *divinity*, or a person made a divinity—*a demi-deity* pronounced *Naidia.*

Nemesis. The goddess of revenge. Neamaise, *terrible, cruel, revengeful.* She is also called *Adrastia*, from (says Lempriere) the temple of *Adrastus*, king of Argus, but rather from andras ; the Phenician and Celtic name for a fury or infernal deity.

Neptune. The god of the sea or waves. Naob, a *holy person*, a *saint* or *divinity ;* ton̄, *the waves.*

Nereus. A deity of the sea. Niarḋeas, *the south-west ;* pronounced *Nereus.*

Orthia. A name of Diana at Sparta—whence boys were whipped at her sacrifices. Or, *voice, shouting, exclamation, crying ;* dia, *goddess.* The goddess of *crying, shouting*, or *exclamation.*

Orthus. The dog of Geryon. Or, *noise ;* tos, *first,*

Orus. The sun, skilled in medicine, the benefactor of the human race. An Egyptian deity. Ur, *the sun ;* ais, *a shower, or water.* Thus the united influence of these two agents acquired for this god the name of benefactor of man and divine honours.

Osiris. A deity of the Egyptians. Os, *above, superior ;* ur, *the sun ;* ais, *water* or *showers.* The deity who commanded and directed the influence of the sun and the rain. *The supreme director of all things.* God above, the sun, the rain, and genial influence.

Pan. The god of shepherds. Péan, *a reed ;* so called from playing on a musical instrument made of reeds, which shepherds played on.

Parcæ. The fates. Barr, *death ;* cai, *a road ; the road or course to death.*

Atropos. One of the *Parcæ* or *Fates.* She cuts the thread of life. A, *the ;* droċ, *dark ;* bar, *death.* The common derivation of *a non* (τρεπω,) *muto ;* is not so palpable, although she be inexorable.

Clotho. The youngest of the *Parcæ.* She presided over the moment of birth, held the distaff, and spun the thread of life; whence her name κλωθειν, *to spin ;* dloċtad, *separating, loosing, spinning.*

Lachesis. One of the Parcæ, whose name is said to be derived from λαχειν, *to measure out by lot ;* rather from lac, *reckoning ;* aois, of ages. She is represented spinning the thread of life; aois, is repeated to signify the *plural ;* the Phenician word would stand thus, lac-aois-aois—which is pronounced exactly as the Greek or Latin name.

Priapus. *The phallus.* The god of generation. Priobáid, *privacy, secrecy.*

Sancus. A deity of the Sabines. San, *holy ;* cos, *foot.* The holy foot.

Sibillæ. The sybyls or fortune-telling women. Sia, *a fairy or witch ;* beala, *mouth.* *The women who foretold events.*

Sichæus, Shicharbus, or Acerbas. The husband of Dido, put to death by Pygmalion. Siogaċ, *indolent, inactive.* The latter name from his death. Sicaire, *a motive, occasion, reason.* Bas, *for death ;* acar, *sour, bitter,* or acair, *poison ;* bas, *death.*

Taautes. A Phenician deity, the same as the Saturn of the Latins, and probably the Thaut, or Thoth, the Mercury of the Egyptians. (*Cic. de Nat. Deorum,* 3 c. 22. *Varro.*) Teutates. Dia, *the god, Taat* or *Tait,* the Celtic or Phenician god of trade, one of the deified heroes or patriarchs of the Phenician Gael. (See Gael and Cymbræ, 225, &c.) Ta, pronounced *tha,* or *thor,* is *being, God.* Literally, *am, I am, existence,* "*I am, has sent me unto you ;*" tais, is *merciful, clement, beneficent, compassionate.* Thus we have *Thortais,* or *Taautes, the merciful God.*

Triton. A sea deity, the son of Neptune ; powerful among the sea deities. Triat, *lord, sovereign,* or *king ;* toñ, *of the waves or billows.* King or lord or prince of the billows.

Venus. The goddess of beauty, the mother of love, the queen of laughter, the mistress of the graces, and patroness of courtezans. Some mention more than one. Plato mentions two : Venus Urania, the daughter of Uranus ; and Venus Popularia, the daughter of Jupiter and Dione. Cicero speaks of four, a daughter of Cœlus and Light ; one sprung from the froth of the sea, the third daughter of Jupiter and Dione ; and the fourth *born at Tyre,* the Astarte of the Phenicians. Of these the Venus sprung from the froth of the sea, after the mutilated parts of the body of Uranus had been thrown there by Saturn is most known. The name of this goddess is very apposite ; ḃean, *the woman ;* aos, *of the people or community ;* pronounced *Vanus, the prostitute, courtezan, or woman of the town.*

Vertumnus. The god who presided over spring, orchards, fruits, and village pursuits. Fear, *a man ;* tuaim, *a village or farmers house ;* neas, *active, vigorous, attentive. The good husbandman.*

Vesta. The goddess of chastity and female virtue. Uas, *gentleness ;* ta, *existence, gentleness personified.*

Vulcan. The god of the antients who presided over fire, and was patron of all artists who worked in iron and metals. Folg, *clever, active, intelligent, adroit ;* gaḃan̄, *smith,* pronounced *folgaun,* or *vulgaun.* The clever smith. *Tubal Cain?*

Alecto. (A, ληγω, *non desino,* not laid aside,) one of the furies. Ala, *a wound ;* ecc, *death ;* to, *feminine.* The woman who inflicts wounds and death.

Harpylæ or Harpies. Winged monsters with the faces of women and bodies of vultures ; three in number. There names—*Aello, Ocypete,* and *Celeno.* Ar, *plague ;* boileac, *altogether,* or *entirely.* They emitted an infectious smell. *Aello,* aile, *smell.*

Occipete. Occ, *with ;* cī, *lamentation ;* peit, *music.*

Celeno. Ci, *lamentation ;* lion, a *snare* or *net.*

Medusa. One of the Gorgons, whose hair was composed of serpents, and who changed into stones, or killed, whoever they looked upon. Maiteas, *necromancy, sorcery, magic.*

Narcissus. A beautiful youth, who, seeing his own image reflected in a stream, became enamoured of it, thinking it the *nymph* of the water. Naoḃ, a *nymph ;* ceas, *sight, vision ;* as, a *stream,* or *waterfall*—Naocea sas, *the sight of a nymph in the stream.*

Orpheus. Or, *sound, voice, music ;* fios, *skill, knowledge, science.*

Pandora. Bean, *the woman ;* dorrac or dorrḋa, *harsh, rough, fierce, cruel, austere, unpleasant. The woman of mischief.*

Pythia. The priestess of Apollo at Delphos. She always delivered the oracles in hexameter verses, and with musical intonation. Piteaḋ, *music ;* from whence the name.

Titii. The priests of Apollo at Rome. Titin, *the sun.*

Tityus. The giant son of Terra or Jupiter. Tit, *the earth ;* tiu, *bulky, large, gigantic.*

Parthenope. One of the syrens. Bairr, *death ;* tein, *approaching ;* aoibe, *civility, politeness, deceitful invitation.*

Pasiphae. The wife of Minos enamoured of a bull. Ba, *a cow ;* sabas, *taste, relish. The propensity, fancy, or disposition of a cow.*

Pygmalion. King of Tyre, son of *Belus,* or *Baal.* Beag, *little ;* maoilinn, *mule.* The *little mule* or person of a low stature and obstinate disposition.

Pythagoras. The philosopher. He was most probably a Phenician nobleman, or—Peit, *musician ;* tiagur, *nobleman. The noble musician.* He was a great poet and musician.

Sanconiathon. The Phenician historian born at Berytus. San, *holy ;* con, *understanding, sense,* or *wise man ;* niod, *real ;* tain, *of the country. The sacred writer, or wise recorder of the events of his country.*

Sesostris. The great king of Egypt, and conqueror of Asia, &c. His father ordered all the male children born on the same day with him, to be educated at the public expense. Seis, *pleasure, delight, happiness, fortune, success ;* sos, *knowledge, science, learning ;* treise, *force, strength, power.* His Egyptian name was *Ramesis the Great.*

Tages. A son of Genius, who taught the Etrurians *augury* and *divination.* He is said to have been found under a clod by a Tuscan ploughman, and assumed a human shape to instruct this nation, so celebrated for their knowledge of incantations. Tagḃais, *chance, fortune, hope ;* pronounced *Tages.*

Talus. The inventor of the saw, compasses, and other instruments. Tailm, *instruments,* or *tools ;* tal, *an adze.*

Tantalus. King of Lydia, son of Jupiter or *Pluto.* Represented as punished in hell with insatiable thirst, in the midst of a pool of water which recedes as he brings his mouth to it ; a bough hangs over his head, loaded with delicious fruit, which, as he attempts to take it, is removed from his grasp by a gust of wind. Others say, a large rock or stone was suspended over his head, ready to crush him to pieces; all the same idea of a state if constant excitement, fear, and trembling disappointment. Tain, *water, dropping,* or *falling ;* tal, *receding ;* ais, *back, water receding backwards.*

Tlepolemus. Son of Hercules. Tlat, *soft ;* peollaid, *a skin without hair ;* mas, *comely.* The handsome man with a fine soft skin, without hair on it. The union of the *Tl*, a peculiarity of the Irish language is singularly exhibited in this word.

Triptolemus. Son of Oceanus and Terra. Treaḃ, *tilling, ploughing ;* talam, *the ground ;* aois, *to a people, or community.* Ceres gave him her chariot, and he travelled in it over the world, distributing corn, and teaching people agriculture. This name is a very singular confirmation of the identity of the Phenician and Celtic tongues. The three words express the correct idea, and is pronounced exactly like the Greek name.

Haruspex. A soothsayer who drew omens from the resistance, throes, pangs in dying, and the inspection of the entrails of beasts sacrificed. The usual derivation of this word is *ab aris inspiciendis*, which is but a guess, although not so improbable as most of the classical derivations. The true derivation is, ar, *judging, conducting, deciding, guiding ;* uspoig, *from a throe, pang, gasp, heave, agony.* This superstition had its origin with the Chaldeans, and their Phenician descendants communicated it to the Greeks and the Romans.

ANTIENT GEOGRAPHIC NAMES.

Abyssinia. Literally, *the country of rain.* A, *the ;* baos, *rain ;* sineaḋ, *extension, progress, dilatation.* The country or district where, heavy, or intense, rain or showers fall. The water which supplies the Nile, falls in a great measure in Abyssinia, and it had its name from the Phenicians, who were the first civilized people who penetrated into it.

Ægyptus. The civilized valley. Ai, *learned, cizilized, cultivated ;* gibis, *valley, or glyn.*

Æthiopia. The civilized country of springs, or wells. Ai, *civilized ;* tiobar, *of wells ;* ia, *country.*

Ætna. The holy hill. Ait, *a hill, or mountain ;* naoṁ, *holy.*

Africa. The desert country. A, *the ;* frac, *barren, desert, bleak ;* ia, *country.* From its barren plains and districts, (the name given it by ourselves is *the Desert,)* which eventually became the denomination of the whole continent.

Adriaticum Mare. The Phoceans first made the Greeks acquainted with sea; but it had its name from the Phenicians, who, as was their custom, endeavoured to prevent other nations from trading to those places they found a source of wealth; they, therefore, called this *the sea of enchantment.* A, *the;* draoideacḋ, *enchanted ;* oice, *sea.* The Greeks and Romans added, Πελαγος, Pontus, and *Mare*, because they knew not the intent of its name, the last syllable of which means *sea.* Therefore, the additions were surplusage.

Euxinus Pontus. Here a similar unnecessary addition has been made. The first word means the *little sea.* Oice, *sea ;* in, *little.*

Parnassus. One of the highest mountains in Greece. Bar, *chief, principal, highest ;* neas, *hill ;* as, *the sign of the comparative. The highest hill in its neighbourhood.*

Samothrace. An island in the Egean sea, where the Eleusinian mysteries were carried to the highest perfection. It was an early settlement of the Phenicians. The name is from the antient god Saṁain, *heaven, or Cœlus ;* and treise, *power, strength.* The residence of the gods.

Scylla. The rocks in the straits of Messina in Sicily, many rocks in Ireland are called Sceligs, from scallac, *bald, bare, naked, without verdure.*

Tamyras. A river in Phenicia, between Tyre and Sidon. Ta, *the ;* mear, *quick, rapid ;* as, *water, or cataract.*

Tanas. A river in Numidia. Tanas, *a spirit, apparition, ghost.* The antients always affixed a genius or spirit to a river. Or teiṁ, *swift, hasty ;* as, *stream.*

Tentyra. A place in Thrace opposite Samothrace. Tean, *rugged ;* tir, *land.*

Thule. The island which the antients considered the farthest point of the inhabited world, called *Ultima thule*, supposed to be *Iceland.* Tuil, pronounced *Thule*, is *rest, sleep, repose.* The last place of rest.

Vatican. A hill at Rome, covered with stagnated waters, long disregarded on that account by the Romans. Feat, *a fen or bog ;* a, *the ;* cean, *head, or top ; the hill covered with bog or swamp.* Heliogabalus first drained and cleared it.

NYMPHS.

Nymph. The word Nymph (νύμφη) is derived from the Phenician. Naoṁ, *holy, sacred, sanctified, set apart,* and was used by the Greeks to designate a bride, as well as a female divinity ; naoṁ, which in pronunciation sounds nearly the same, is used by the Irish at this day in the same sense, it means *a holy person, or saint,* and *a bride.* It is spelled by the Highland Scottish Gael, *Naomh.*

The derivation from the word Lympha, and that from τοῦ ἀεὶ νέας φαίνεσθαι, *their perpetual youthful appearance* it is unnecessary to refute. They were the imaginary divinities, or spirits, who were supposed to inhabit the seas, rivers, fountains, springs, woods, trees, mountains, &c. &c. from which our modern fairy elves are derived, as well as the genii of the east.

The *Dryades*, are probably derived from draoideac, *witchcraft, sorcery, magic, elfish craft,* which has ever been supposed to be performed by the agency of the *genii, fairies, demons, or evil spirits* ; or under the Greek and Roman mythology, the demi-deity, who presided over fountains, woods, &c. &c.

The *Hamadryades* were formed by the addition of the word omna, *a large tree, or timber ;* the supposed residence of the spiritual being, for the sake of euphony the Celto-Phenicians often dropped a consonant which interfered with the agreeable sound of a word.

The number of nymphs according to Hesiod, amounted to 3000; it would, therefore, be a laborious task to collate their names, so as to ascertain their respective meanings. I shall, therefore, merely take the Neriedes in the order their names stand in Lempriere's Classical Dictionary, which will afford ample testimony, that they were all derived from Phenician roots: their signification is not only striking, but highly poetical.

Sao. The light, flitting, aerial nymph. Saoḃ, *light, &c.*

Amphitrite. The nymph of the sea-weed or fuci. Am, *the deep sea, or ocean ;* fitreac, *sea-weed, fuci.*

Proto. The arrogant, conceited nymph. Prod, or brod, *arrogance, conceit.*

Galatea. The bright, fair, or beautiful nymph of the wave or froth. galla, *fairness, brightness ;* teaġ, *froth, delusive vision or appearance.*

Thoe. The silent nymph. Toa, *silence, quietude.*

Eucrate. Ea, *the ;* creat, *swan.*

Eudora. The leading, early, first, or prominent nymph. Ea, *the ;* dora, *leader.*

Gallena. The nymph of the rock. Gallan, *a rock.*

Glauce. The sea-green nymph. Glas, *sea-green.*

Thetis. The nymph of the deep water. Tet, *smooth,* pronounced *Thet ;* is, *under.*

Spio. The reproachful nymph. Sṗiod, *reproach.*

Cymothoe. The nymph of the gentle quiet wave. Cima, *a small wave ;* toa, *silent.*

Melita. The nymph who sports on the wave. Mealad, *sporting, enjoying.*

Thalia. The nymph of the calm. Talaḋ, *rocking to sleep, composing.*

Agave. The nymph of the boisterous wave. A, *the ;* gaḃac, *dangerous, perilous, boisterous.*

Eulimine. The nymph who leaps or dances on the waves, or from wave to wave. Ea, *the ;* liminaċ, *leaper, or dancer.*

Erato. The nymph who runs on the waves. Ea, *the ;* rata, *running.*

Pasithea. The gentle nymph who deviously sports in the waves. Ba, *good ;* sitead, *bending, twisting, revolving.*

Doto. The female sea-snail or nautilus. Dao, *the sea-snail ;* to, *female.*

Eunice. The nimble nymph. Ea, *the ;* nes, *nimble.*

Neasa. The generous, friendly nymph. Neasa, *generous, friendly.*

Dynamene. The nymph who appears and disappears rapidly. Deinmne, *swift, active, nimble, supple, hasty, rapid.*

Ferusa. The nymph who sports in devious contortions. Fiaras, *crooked, twisted, bent.*

Protomelia. The nymph who sports in the phosphoric spark of the wave. Brot, *fire ;* to, *feminine, or female ;* mealad, *enjoying, sporting.*

Actea. The nymph of despair. Eig, *a cry ;* taoġ, *frenzy, passion.*

Panope. The nymph of the estuary. Bean, *the woman ;* aba, *an estuary.*

Doris. The invisible nymph. Do-aireac, *invisible.*

Cymatolege. The nymph of the wet rock or breaker. Cim, *a drop, water, or wave ;* atoleic, *on a rock or stone.*

Hippothoe. The silent, happy nymph. Aoib, *joyous, pleasant ;* o, *from* toi, *silent,*

Cymadoce. The nymph of the white foamy wave. Cim, *a wave ;* o, *from ;* dos, *scum, froth of the wave.*

Neso. The nimble nymph. Nes, *quick, nimble, agile.*

Eupompe. The nymph who sounds the conc or musical shell. Euḋ, *music ;* beim, *to sound, strike.*

Pronoe. The complaining or bewailing nymph. Broineag, *complaining.*

Themiste. The nymph of the flowing tide. Taom, *to pour out, overflow, throw water out of a vessel;* airte, *out of her.*

Glauconome. The nymph with long waving hair. Gleacḋac, *with waving hair;* naom, *a nymph.*

Halimede. The walking nymph of the rocks. Al, *a rock;* imeaċd, *walking or pacing.*

Pontoporia. The nymph of the wells, or spring, district. Bean, *a woman;* tobar, *a spring, or well;* ia, *a district, or country.*

Evagora. The kind and smiling nymph. Aoib, *civil, courteous, pleasant;* a, *the;* gor, *laughter, or smiling.*

Liagora. The nymph of the pleasant countenance. lia, *the face, or countenance;* gor, *laughter, cheerfulness.*

Polynome. The nymph of the cavern. Poll, *a hole, pit, or cavern;* naom, *a nymph.*

Laomedia. The mid-day. Laomeadon, *the mid-day or noon.*

Lysianassa. The nymph of the light of the promontory. lear, *light;* na, *of the;* near, *promontory, headland.*

Antonoe. The nymph of the spark of the wave. An, *a spark of fire;* ton, *of the wave;* naom, *nymph.*

Menippe. The sipping nymph. Mion, *little;* ib, *drinker.*

Evarne. The nymph of the marble rocks. Ibear, *marble;* na, *of the.*

Psamathe. The nymph of the summer breeze. Saim, *the summer;* ata, *breeze.*

Nemertes. The nymph of the boat. Naom, *nymph;* artaċ, *boat.*

The most incredulous must admit, that all those names are extremely apposite; and some, if not most of them, palpably derived from the same language as the Hiberno-Celtic. It is scarcely possible so great, so universal a coincidence in name and import can have been accidental. If they stood alone, they would amount almost to demonstration, but supported as they are by other evidence and analogies, they appear to me to be irresistible.

The further prosecution of this investigation is likely to dispel much of the mist, and unravel many of the difficulties in which antient history is involved. Every step we advance opens a new vista and field of view, and, at the same time, points out the means of further progress.

It was my intention to have said something of the Cabiri, but I think it a subject worthy separate consideration, and a future paper.

On the Ring Money of the Celtæ. *By* SIR WILLIAM BETHAM, M.R.I.A. F.S.A. *Secretary for Foreign Correspondence, Member of the Royal Academy of Sciences of Lisbon, &c.*

Read 28th of November, 1836, and 9th of January, 1837.

IN a letter from my friend, Richard Sainthill of Cork, dated the 5th (Nov. 1836) instant, is the following paragraph :—

"Cork, 5th November, 1836.

"I am anxiously expecting your paper on the Ring Money of Ireland, which Mr. Orpen will at any time send me. A vessel going to Africa, to trade with the natives, was wrecked last summer on the coast here ; among the articles on board for barter, were some boxes of cast iron pieces, extremely like one species of gold articles found in Ireland, annexed is a rude outline, one of which was saved from the iron foundry, where they were sold and melted. I understand they pass in barter at about the value of one halfpenny, but the similarity in shape to the old Irish articles is curious."

Another letter from the same gentleman, dated 12th November, instant, he says—

"I had the pleasure to receive your letters and paper on the Ring Money of Ireland last night, and the present is merely to request you to accept the enclosed specimen of the Anglo-African Iron Ring Money. I will procure the information you require, and send it to you as soon as possible."

I have now the pleasure of placing before the Academy specimens of the Irish bronze, the iron found on board the wrecked vessel, and two other specimens, both found in Italy, one said to be found in Herculaneum, but this I should doubt. This latter is much corroded, and is also very singularly incrusted with what appears to me to be crystals of carbonate of lime.

I yesterday received a letter from my friend T. C. Croker, dated 25th November, (1836,) of which the following is an extract :—

"I have got some curious information for you respecting the ring money now current in Africa, which goes completely to establish your theory. Not a shadow of doubt can now exist on the subject. My informants are our friend Sainthill and an Egyptian traveller, Mr. Bonomi. Their letters to me are not immediately at hand, or I should send them over to you."

These are most unexpected confirmations of the theory I had ventured to propound, respecting the larger description of ring money; with respect to the smaller, no one could have any doubt, I mean those of mere rings.

9th of January, 1837.

On the 28th of November, 1836, I had the honour to place before you specimens of certain articles of cast iron, found on board a vessel wrecked on the coast of Cork, so exactly similar in size and shape to those found of Celtic brass in Ireland, that it is quite impossible to refuse assent to the notion of their being of the same use and intent. I have since obtained the most satisfactory information on the subject, which I have now the pleasure of placing before the Academy.

Although very unwilling to intrude on the time of the Academy more than is absolutely necessary to convey the information, yet, rather than not be sufficiently explicit, I think it best to give the information in the order I received it, and then proceed to make a few observations.

In a letter from my friend Mr. Sainthill of Cork, dated 7th of December last, he says—

"The gentleman who collected the African trading bracelet money, is named Mr. Abraham Abell, a Quaker, to whom I applied for particulars which he promised, and I have no doubt intends to give. I am weary with waiting. I can only inform you, that the schooner Magnificent, the property (I believe) of Sir John Tobin of Liverpool, on her voyage to Africa, was wrecked at Ballycotten Bay, about a year ago. A nephew of Sir John Tobin's is working the gunpowder mills at Ballincollig, near Cork, for his uncle, but since your letter I have not met him."

In another letter, dated the 10th of December, 1836, Mr. Sainthill says—"I this day met Mr. Tobin of Ballincollig gunpowder mills, I spoke to him respecting the African money, he promised most kindly to procure me every information. He informed me that the articles are manufactured at Birmingham, and are a composition of brass and copper, they are called *manillas,* and are worn as ornaments, and pass as the representative of money in Africa. They send out about forty chests annually, and, according to the number of vessels on the coast, a greater or lesser number represent *a bar*, which is a certain ideal standard of value, like our pound; and as our pound, according to the rate of exchange, say with France, for instance, would to-day represent twenty-four franks, to-morrow twenty-six, next day twenty-two; so *the bar* one day represents so many manillas, and to-morrow, if a dozen vessels arrived, so many less."

Having a slight acquaintance with Sir John Tobin, and knowing his gentlemanly courtesy, I wrote to request all the information he could give me on the subject, and I had the pleasure of receiving a most satisfactory letter in answer to my queries, of which the following is an extract of so much as refers to the subject:—

"On the subject of the schooner Magnificent, which was lost some where near Cork, some time since; she was bound to the river Bonney, or New Calabar, which is not far from the kingdom of Benin. The trade to these rivers for palm oil and ivory, is cotton goods of a great variety, gunpowder, muskets, and an extensive number of other articles and *manillas,* both of iron, and copper mixed, which is the money that the people of the Eboe and Brass country, and all the natives in that neighbourhood, go to market with. On Wednesday next, I will send you a manilla of each kind. This vessel did not belong to me, but the master of her was named Tobin, and had been a master of one of my vessels.

"If at any time I can give you any information of the West Coast of Africa and its inhabitants, and their customs and habits, I shall have great pleasure, having in early life seen a good deal of these people.

"I am, dear Sir, yours, &c.

"J. Tobin."

On the 28th of December, 1836, the day after I received Sir John Tobin's letter, which I have just read, I received the parcel containing the books which I had the pleasure of presenting just now, from my learned and valued friend Dr. Hibbert, of York, (late of Edinburgh,) in which, among other things, he was kind enough to send me the following :—

"Extract of a letter from Edward Jones, Esq. Captain of the 1st Regiment of Royal Lancashire Militia, to Dr. Hibbert, dated 9, Bridge-street, Manchester, 16th November, 1836—

"'The annexed two sketches are taken from a cast of the species of money now, at the present day, passing current among the Africans. It so strongly resembles what we saw in Ireland, that I thought you might be interested in a copy of it. A Mr. Dyson, who was for some years surgeon on board an African merchantman, brought it with him; and the first opportunity I shall make inquiries respecting this coin, and other sorts in use among the natives.

"'I am told that in the country they are made of solid gold as in Ireland, but that they are now counterfeited in England, and sent out to Africa in large chests, especially from Liverpool.

"'Do you think this will throw any light on the antient Irish relics, &c. &c.'

"I wrote to Captain Jones, in reply, that I believed Sir William Betham had published some remarks on the gold relics found in Ireland, and that I would communicate to him the information given me.

"S. Hibbert."

Thus has corroborating information and evidence crowded upon me, in a very remarkable manner, from different quarters and in appropriate season.

Mr. J. Bonomi, who travelled with Lord Prudhoe in Egypt, Nubia, and Sennaar, writes to my friend T. C. Croker as follows :

"You ask me for a note on the ring money of Africa—here it is. So little has the interior of that country changed, in that particular, since the days of the Pharaohs, that to this day, among the inhabitants of Sennaar, pieces of gold, in the form of rings, pass current. The rings of gold have a cut in them,

for the convenience of keeping them together, the gold being so pure you easily bind them and unite them in the manner of a chain. The money is weighed, as in the days of Joseph.

"I shall soon be able to show you a cast from an Egyptian basso-relievo, where occurs an example of the money in the form of a chain; and you may see, in Heskyn's work, plenty of rings of gold.

"Yours, very truly,

"J. Bonomi."

These gold rings are identically the same, as to shape and character, with those found in Ireland; the sketch of one most accurately represents the other.

The *manillas* are still more interesting, in as much as they give *the name* by which they are known among the Africans; a name, no doubt, the same which they bore when they were first introduced among these people so many centuries, aye, it may be said, thousands of years ago, by the Phenicians, the same commercial people who introduced them into Ireland. *So little has the customs of the negro race of Western Africa, like those of Sennaar, changed since the period of their intercourse with the Phenicians*, that these *manillas* pass current as money among them as they did two or three thousand years ago. Africa seems to have stood still, while the rest of the world progressed in civilization; her deadly unwholesome climate forbade intercourse beyond the exchange of manufactured goods for her raw materials.

This name *manilla* is in itself a powerful testimony; it is no doubt the name the articles bore in Phenicia, and by which they were known when first introduced to the knowledge of the African negro nations, who have preserved it to our day. In the Celto-Phenician it literally means *the value or representation of property.* Main, *riches, patrimony, goods, value,* and eallaċ, *cattle*, or any description of property; the word *chattles* rightly expresses it. Thus it appears that as *pecunia* had its name from *pecus, cattle,* because flocks and herds were the first riches, and a number of cattle were the standard of value before money existed, and where it was not to be had; so *manilla* means literally the *value of cattle or goods,* or the representative of the value of cattle, or any *chattle* property.

Money was so scarce in Ireland, in the fourteenth and fifteenth century, that the fines and amerciaments, mentioned on the rolls as imposed by the courts, were pigs, sheep, and cattle; but it is not necessary to use arguments in support of so self-evident a proposition.

Benin and Calabar are situated on the Gulph of Guinea, in latitude from 7 to 10 N.; longitude, 5 to 10 east of London. It would appear from those facts that the Phenicians had penetrated to the Gulph of Guinea, and were acquainted with the whole of this coast, probably beyond the line. We know they circumnavigated Africa, by order of Pharaoh Necho, king of Egypt; but it now appears that they traded regularly to the coasts of Guinea, and there introduced a money which still bears a Phenician name, and is still as much in estimation as it was when the merchant princes of Tyre supplied them with manillas in exchange for their gold, ivory, and palm oil. The Tobins, and other English merchants, equally eminent and illustrious with those of Tyre, now occupy the Tyrians' position, while the negroe's is not a whit more elevated than it was two or three thousand years since.

The Romans knew nothing on the west coast of Africa beyond the port of Sala, now Sallee, in latitude 34 N. and a very narrow slip on the north coast, not even so far as the Great Desert; except, perhaps, they may be said to have been acquainted with the existence of the Desert, but had no intercourse with any people beyond it. The coast of Guinea, within ten degrees of the equator, was far beyond their ken; consequently, during their sway, the people of that district having no intercourse with any great commercial people, their customs and habits of commerce received no impetus likely to produce any change of their antient mode of traffic, and the metallic currency they learned from the Phenicians remains unchanged to the present day. The English finding the manillas current, naturally availed themselves of the facilities which they possessed of *fabricating* them, it can scarcely be justly called *counterfeiting*, because they bear no impress or mark of authority.

The Carthaginians may have carried on the trade with these coasts after the destruction of Tyre, but there is no evidence that they or the Romans ever visited them. The intercourse of the English, French, Spaniards, Portuguese, and Dutch, is of a very recent date. .

When I first ventured to assert that these things were money, the smile of something very like pity, if not of contempt, met me on all sides; it was considered so wild a proposition, that sage individuals almost considered it a proof of a disordered imagination, and some laughed outright, it appeared so preposterous and improbable. I have never doubted the truth of my opinion from the moment I first saw the specimen of the Monaghan manilla in the museum of my friend Mr. George Petrie; and it is no small gratification to me to have been able to collect such irrefragable evidence in its support, and to conduct the inquiry to so triumphant and satisfactory a conclusion.

This matter of the ring money has arisen out of the investigation into the question of *who were the Scoti?* and the tracing them to be the Celtæ, led to the consequent question of *who were the Celtæ?* which being answered that they were Phenicians, have led to further questions and inquiries respecting that great people, as well as to *who were the Pelasgi and Etruscans?* and in short, into the investigation of the history of the proceedings of the illustrious nation, who so abundantly sowed the seed of civilization in the world. As we proceed new lights break in upon us, the investigation and elucidation of one question opens a vista to our sight into farther mysteries, and eventually enables us to force a passage into the remotest penetralia of antiquity. If I have not deceived myself, I think I see my way still farther, and that I shall be able on a future occasion to produce evidence to illustrate the progress of the nations of the most remote antiquity, and to throw some light on the channel by which civilization poured to the west.

I have now the pleasure of laying before you the specimens of the two descriptions of manillas alluded to in Sir John Tobin's letter, which I have since received from that gentleman. The price at first cost is as under:

The copper manilla is £105. per ton.
The cast iron manilla £22. per ton.

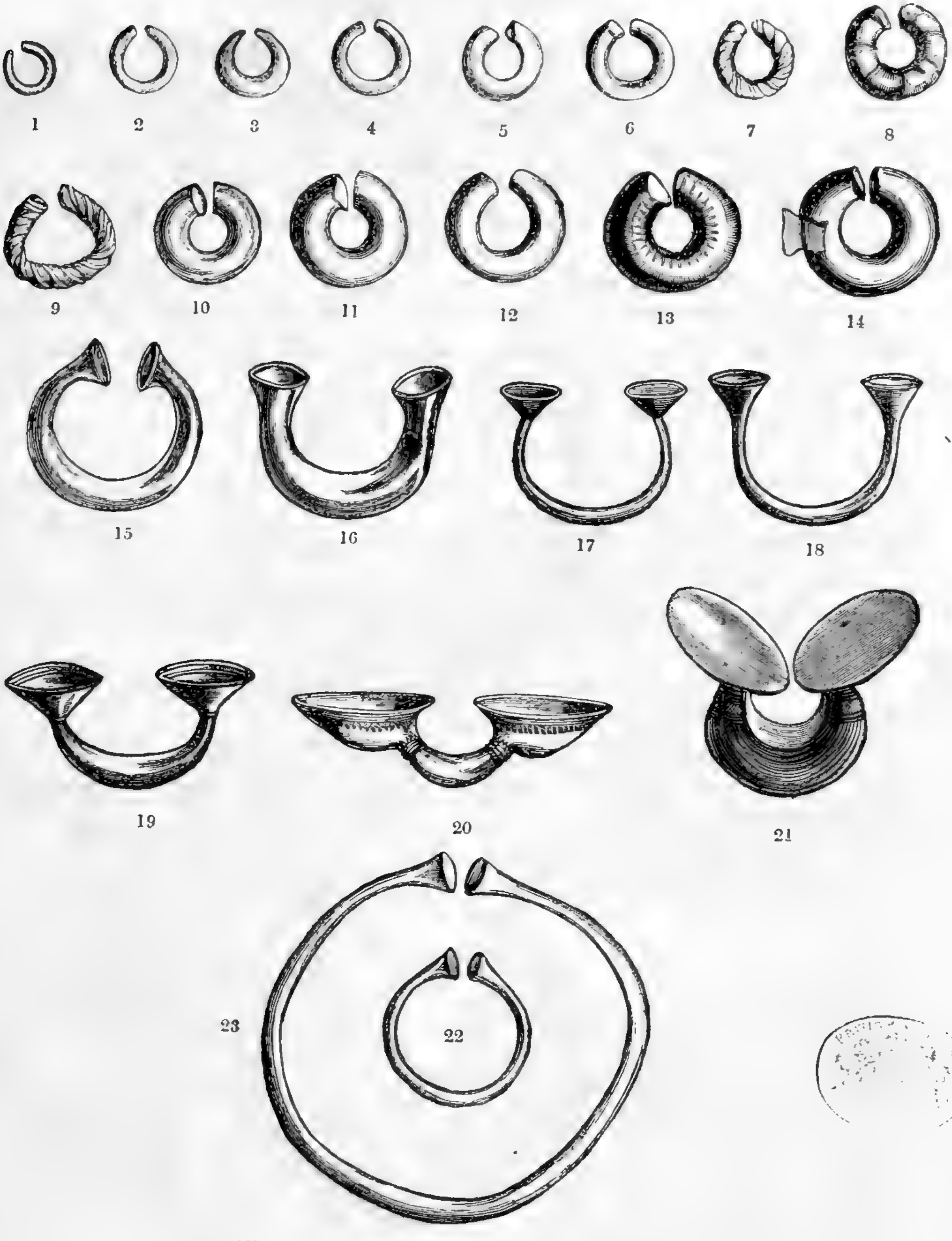
The copper manilla weighs somewhat more than two and a half ounces avoirdupois, or about six to the pound, and would be about twopence each; but they, no doubt, pass for much more in Africa, I believe about fourpence.

The iron manilla weighs rather more than an ounce and a half, and gives ten to the pound, or about one farthing each; I believe they pass in Africa for about one halfpenny.

The weight of the ancient Celtic brass manilla found in Monaghan, is somewhat more than three ounces avoirdupois.

As nothing tends so much to the right understanding of the subject as figures of the articles in juxta-position, I have placed the wood-cuts of all the different specimens in one point of view.

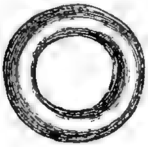
GOLD RING MONEY, OR MANILLAS, FOUND IN IRELAND.



WEIGHTS OF THE ABOVE SPECIMENS.

No. 1,	…… 12 grs.
2,	1 dwt. 12 grs.
3,	2 dwt. 12 grs.
4,	2 dwt. 12 grs.
5,	2 dwt. 12 grs.
6,	3 dwt. 16 grs.
7,	3 dwt. 16 grs.
8,	6 dwt.
No. 9,	5 dwt.
10,	10 dwt.
11,	11 dwt.
12,	8 dwt.
13,	11 dwt. 12 grs.
14,	Counterfeit, plated with gold over brass.
15,	4 oz. 16 dwt.
No. 16,	9 oz.
17,	16 dwt. 12 grs.
18,	2 oz.
19,	3 oz. 12 dwt.
20,	19 oz. to 56 oz.
21,	1 oz.
22,	…… 12 dwt.
23,	…… 2 dwt.

BRASS MANILLAS.



24

Antient Irish Brass Rings of various graduated weight and sizes.

25

Antient Brass or Bronze Manilla found in Monaghan.

26

Manilla in Copper and Cast Iron, fabricated in England, and now passing current as money in Africa.

ERRATA.

Page
321, last line, *for* $\Theta +$ *read* $\theta +$
323, line 17, *for* $+ \Theta$ *read* $+ \Theta \mathrm{a}$,
330, line 11, *for* pend *read* depend
336, equations (121.) *for* a *read* b
342, line 18, *for* For *read* For if
347, 6th line from foot, *for* thorems *read* theorems
360, last line but 2, *for* denote *read* denote by
383, last line, *for* B *read* $\underset{\circ}{\mathrm{B}}$
384, line 7, *for* $\genfrac{}{}{0pt}{}{\omega\nu}{\omega\mu}$ *read* $\frac{\omega\nu}{\omega\mu}$
387, equation (340.) *for* $1 + m$ *read* $1 + i\,m$
391, before (358.) *for* formula *read* formulæ
391, before (362.) *for* formula *read* formulæ
391, last line, *for* one *read* one set
394, line 10, *for* A_1 to B_2 *read* A_1 to B_1
395, before (4.) *for* denote *read* denote it
399, in (21.) *for* $(a_1 \mathrm{a}_1 , a_1 \mathrm{a}_1)$ *read* $(a_1 \mathrm{a}_1 , a_1 \mathrm{a}_2)$
399, in (23.) *for* $\left(\frac{\mathrm{b}_1}{\mathrm{a}_2}, \frac{\mathrm{b}_2}{\mathrm{a}_2}\right)$ *read* $\left(\frac{\mathrm{b}_1}{\mathrm{a}_1}, \frac{\mathrm{b}_2}{\mathrm{a}_1}\right)$
400, line 6, *for* $(a_1 \mathrm{a}_2 , a_1 \mathrm{a}_1)$ *read* $(a_1 \mathrm{a}_1 , a_1 \mathrm{a}_2)$
400, last but 3, *for* retain *read* retain them
401, before (37.) *for* $(a_1 , 0)$ *read* (a_1 , a_2)
401, 2d formula (39.) *for* b_1 *read* b_2
401, in (42.) *for* β_2 *read* $\beta_2 \mathrm{c}$
402, end of (44.) *for* a_2 *read* $a_2{}^2$
402, in (48.) *for* β^2 *read* β_2
404, in (62.) *for* a^2 *read* a_2
404, in (63.) *for* $a_1{}^2 + a_2$ *read* $a_1{}^2 + a_2{}^2$
405, after (69.) *for* formula *read* formulæ
407, in (78.) *for* (a''_1 , a'_2) *read* (a''_1 , a''_2)
407, in (79.) *for* $a_1 a'_2$ *read* $a_1 a'_1$
407, in (83.) *for* $a_1{}^2 - a^2$ *read* $a_1{}^2 - a_2{}^2$
408, in (90.) *for* $(a_2 , a_2)^{m-1}$ *read* $(a_1 , a_2)^{m-1}$
410, in (104.) *for* $n \times 1$ *read* $n+1$
410, in (107.) *for* $m \times 1$ *read* $m+1$
417, in (151.) *for* $(b_1 \underset{\omega'}{,} b_2)$ *read* $(b_1 \underset{\omega'}{,} b_2)^{\frac{1}{m}}$
421, in (182.) *for* $(2\,\omega'\,\pi . 0)$ *read* $(2\,\omega'\,\pi, 0)$