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A TREATISE

ON

ALGEBRAICAL GEOMETRY.

BY THE

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ALGEBRAICAL GEOMETRY.

PART I.

APPLICATION OF ALGEBRA TO PLANE GEOMETRY.

CHAPTER I.

INTRODUCTION.

1. The object of the present Treatise is the Investigation of Geometrical Theorems and Problems by means of Algebra.

Soon after the introduction of algebra into Europe, many problems in plane geometry were solved by putting letters for straight lines, and then working the questions algebraically; this process, although of use, did not much extend the boundaries of geometry, for each problem, as heretofore, required its own peculiar method of solution, and therefore could give but little aid towards the investigation of other questions.

It is to Descartes that we owe the first general application of algebra to geometry, and, in consequence, the first real progress in modern mathematical knowledge; in the discussion of a problem of considerable antiquity, and which admitted of an infinite number of solutions, he employed two variable quantities x and y for certain unknown lines, and then showed that the resulting equation, involving both these quantities, belonged to a series of points of which these variable quantities were the co-ordinates, that is, belonged to a curve, the assemblage of all the solutions, and hence called "the Locus of the Equation."

It is not necessary to enter into further details here, much less to point out the immense advantages of the system thus founded. However, in the course of this work we shall have many opportunities of explaining the method of Descartes; and we hope that the following pages will, in some degree, exhibit the advantages of his system.

2. In applying algebra to geometry, it is obvious that we must understand the sense in which algebraical symbols are used.

In speaking of a yard or a foot, we have only an idea of these lengths by comparing them with some known length; this known or standard length is called a unit. The unit may be any length whatever: thus, if it is an inch, a foot is considered as the sum of twelve of these units, and may therefore be represented by the number 12; if the unit is a yard, a mile may be represented by the number 1760.

But any straight line A B fig. (1) may be taken to represent the unit of length, and if another straight line C D contains the line A B an exact number (a) of times, C D is equal to (a) linear units, and omitting the words "linear units," C D is equal to (a).

In fig. (1) C D = 3 times A B, or C D = 3.

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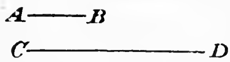

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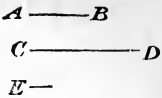


If C D does not contain A B an exact number of times, they may have a common measure E, fig. (2); let, then, C D $= m$ times E $= m$ E, and A B $= n$ E, then C D has to A B the same ratio that m E has to n E, or that m has to n, or that $\frac{m}{n}$ has to unity; hence C D $= \frac{m}{n}$ times A B $= \frac{m}{n} = b$.

In fig. (2) C D $= \frac{5}{3}$ of A B $= \frac{5}{3}$

If the lines A B and C D have no common measure, we must recur to considerations analogous to those upon which the theory of incommensurable quantities in arithmetic is founded.

We cannot express a number like $\sqrt{2}$ by integers or fractions consisting of commensurable quantities, but we have a distinct idea of the magnitude expressed by $\sqrt{2}$, since we can at once tell whether it be greater or less than any proposed magnitude expressed by common quantities; and we can use the symbol $\sqrt{2}$ in calculation, by means of reasoning founded on its being *a limit* to which we can approach, as nearly as we please, by common quantities.

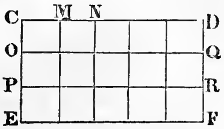
Now suppose E to be a line contained an exact number of times in A B, fig. (2), but not an exact number of times in C D, and take m a whole number, such that m E is less than C D, and $(m + 1)$ E greater than C D. Then the smaller E is, the nearer m E and $(m + 1)$ E will be to C D; because the former falls short of, and the latter exceeds, C D, by a quantity less than E. Also E may be made as small as we please; for if any line measure A B, its half, its quarter, and so on, *ad infinitum*, will measure A B. Hence we may consider C D as a quantity which, though not expressible precisely by means of any unit which is a measure of A B, may be approached as nearly as we please by such expressions. Hence C D is *a limit* between quantities commensurable with E, exactly as $\sqrt{2}$ is *a limit* between quantities commensurable with unity.

We conclude, then, that any line C D may be represented by some one of the letters a, b, c, &c., these letters themselves being the representatives of numbers either integral, fractional, or incommensurable.

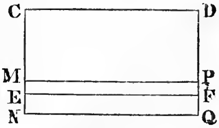
3. If upon the linear unit we describe a square, that figure is called the square unit.

Let C D F E, fig. (1), be a rectangle, having the side C D containing (a) linear units C M, M N, &c., and the side C E containing (b) linear units C O, O P, &c., divide the rectangle into square units by drawing lines parallel to C E through the points M, N, &c., and to C D through the points O, P, &c. Then in the upper row C O Q D there are (a) square units, in the second row O P R Q the same, and there are as many rows as there are units in C E, therefore altogether there are $(b \times a)$ square units in the figure, that is, C F contains $(a\, b)$ square units, or is equal in magnitude to $(a\,b)$ square units; suppressing the words "square units," the rectangle C F is equal to $a\,b$.

If C D = 5 feet and C E = 3 feet, the area C F contains 15 square feet.



(1.)



(2.)

The above proof applies only to cases where the two lines containing the rectangle can be exactly measured by a common linear unit.

Suppose C D to be measurable by any linear unit, but C E (fig. 2) not to be commensurable with C D; then, as has been shown, we may find lines C M, C N commensurable with C D approaching in magnitude as nearly as we please to C E.

Completing the rectangles C P and C Q, we see, that as C M and C N approach to C E, the rectangles C P and C Q approach to the rectangle C F, that is, the rectangle C E, C D is *the limit* of the rectangle C M, C D, just as C E is *the limit* of C M. Let therefore a and b be respectively the commensurable numbers representing C D and C M, and let c be the incommensurable number expressing C E, then the rectangle C E, C D = the limit of the rectangle C M, M P = the limit of the number $a\,b$, by the first part of this article, = the product of the respective limits of a and $b = a\,c$. *

Hence, generally, the algebraical representative of the area of a rectangle is equal to the product of those of two of its adjacent sides.

If $b = a$, the figure C F becomes the square upon C D, hence the square upon C D is equal to $(a \times a)$ times the square unit $= a^2$.

We are now able to represent all plane rectilineal figures, for such figures can be resolved into triangles, and the area of a triangle is equal to half the rectangle on the same base, and between the same parallel lines.

4. To represent a solid figure, it will be sufficient to show how a solid rectangular parallelopiped may be represented.

Let a, b, c, be, respectively, the number of linear units in the three adjacent edges of the parallelopiped; then, dividing the solid by planes parallel to its sides, we may prove, as in the last article, that the number of solid units in the figure is $a \times b \times c$, and, consequently, the parallelopiped equal to $a \times b \times c$.

The proof might be extended to the case where the edges of the parallelopiped are fractional, or incommensurable with the linear unit.

If $b = c = a$, the solid becomes a cube, and is equal to $a \times a \times a$, or a^3.

5. We proceed, conversely, to explain the sense in which algebraic expressions may be interpreted consistently with the preceding observations.

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* That "the product of the limits of two incommensurable numbers is the limit of their product," may be thus shown. Let v and w be incommensurable numbers, and let $v = m + m'$ and $w = n + n'$, m and n being commensurable numbers, and m' and n' diminishable without limit; that is, v and w are the respective limits of m and n, then $vw = mn + mn' + nm' + m'n'$, the right-hand side of this equation ultimately becomes mn, and the left-hand side of the equation is the product of the limits.

Algebraic expressions may be classed most simply under the form of homogeneous equations, as follows :—

$$x = a$$
$$x^2 + ax = bc$$
$$x^3 + ax^2 + bcx = def$$
$$x^4 + ax^3 + bcx^2 + defx = ghkl$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$x^m + ax^{m-1} + bcx^{m-2} + \&c \; . \; . \; . \; . \; . = pqrs \ldots \text{ to } m \text{ terms.}$$

In the first place, each equation may be understood as referring to linear units; thus, if L be put instead of the words 'the linear units,' the equations may be written

x times L $= a$ times L,

x^2 times L $+ ax$ times L, or $(x^2 + ax)$ times L $= bc$ times L,

$(x^4 + ax^3 + bcx^2 + defx)$ times L $= ghkl$ times L, and so on. The solution of each equation gives x times L in terms of $(a, b, c, \ldots)$ times L; and thus the letters $a, b, c, \ldots x$ are merely numbers, having reference to lines, but not to figures.

This will be equally true if L is not expressed, but understood; and it is in this sense that we shall interpret all equations beyond those of the third order.

The same reasoning would equally apply if we assumed L to represent the square or cubic unit, only it would lead to confusion in the algebraic representation of a line.

6. Again, these equations may, to a certain extent, have an additional interpretation.

For if we consider the letters in each term to be the representatives of lines drawn *perpendicular* to each other, the second equation refers to areas, and then signifies that the sum of two particular rectangles is equal to a third rectangle; the third equation refers to solid figures, and signifies, that the sum of three parallelopipeds is equal to a fourth solid.

Moreover we can pass from an equation referring to areas to another referring to lines, without any violation of principle; for, considering the second equation as referring to areas, the rectangles can be exhibited in the form of squares; and if the squares upon two lines be equal, the lines themselves are equal, or the equation is true for linear units.

7. It follows as a consequence of the additional interpretation, that every equation of the second and third order will refer to some geometrical theorem, respecting plane or solid figures; for example, the second equation, when in the form $x^2 = a\,(a - x)$ is the representation of the well-known problem of the division of a line into extreme and mean ratio.

By omitting the second and third terms of the third equation, and giving the values of $2a$, a, and a to d, e and f, respectively, we obtain the algebraic representation of the ancient problem of the duplication of the cube.

8. The solution of equations leads to various values of the unknown quantity, and there are then two methods of exhibiting these values; first, by giving to a, b, c, &c., their numerical values, and then performing any operation indicated by the algebraic symbols.

Thus, if $a = 4$, $b = 5$, and $c = 9$,

we may have $x = a + c - b = 8$ times the linear unit.

$$x = \frac{ab}{c} = \frac{20}{9} = 2\frac{2}{9} \text{ of the linear unit.}$$

$$x = \sqrt{ac} = \sqrt{36} = 6 \text{ times the linear unit.}$$

We can then draw the line corresponding to the particular value of x. This is the most practical method.

Again, we may obtain the required line from the algebraical result, by means of geometrical theorems; this method is called 'the Construction of Quantities'; it is often elegant, and is, moreover, useful to those who wish to obtain a complete knowledge of Algebraical Geometry.

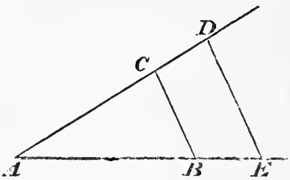
THE CONSTRUCTION OF QUANTITIES.

9. Let $x = a + b$.

In the straight line A X, let A be the point from whence the value of x is to be measured; take A B $= a$, and B C $= b$, then A C $=$ A B $+$ B C $= a + b$ is the value of x.

Let $x = a - b$, in B A take B D $= b$, then A D $=$ A B $-$ B D $= a - b$.

Let $x = \frac{ab}{c}$, then $x : a :: b : c$,



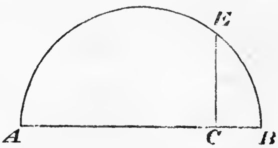
and x is a fourth proportional to the three given quantities c, b, and a; hence the line whose length is expressed by x, is a fourth proportional to three lines, whose respective lengths are c, b, and a. From A draw two lines A C D, A B E, forming any angle at A; take A B $= c$, B E $= a$, and A C $= b$, join B C, and draw D E parallel t B C; then, A B : A C :: B E : C D, or $c : b :: a :$ C D $\therefore$ C D is the required value of x.

$$\text{Let } x = \frac{abc}{de}; \text{ construct } y = \frac{bc}{e}, \text{ and then } x = \frac{ay}{d};$$

$$\text{similarly for } x = \frac{abc}{d^2}, \text{ or } \frac{ab^2}{d^2}, \text{ or } \frac{a^3}{d^2}, \text{ or } \frac{abcd}{efg}.$$

Let $x = \dfrac{abc + def}{gh} = \dfrac{abc}{gh} + \dfrac{def}{gh}$, construct each term separately, and then the sum of the terms.

10. Let $x = \sqrt{ab}$

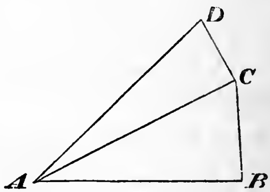


Since $x^2 = ab$, x is a mean proportional between a and b. In the straight line A B take A C $= a$, and C B $= b$; upon A B describe a semicircle, from C draw C E perpendicular to A B, and meeting the circle in E; then C E is a mean proportional to A C and C B, (Euclid, vi. 13, or Geometry, ii. 51,) and therefore C E is the required value of x.

The same property of right-angled triangles may be advantageously employed in the construction of the equation $x = \frac{a^2}{b}$; for, take AC $= b$, and draw C E perpendicular to A C and equal to a, join A E and draw E B perpendicular to A E; then C B $= \frac{a^2}{b}$.

Let $x = \sqrt{ab + cd}$, $x^2 = ab + cd = a\left(b + \frac{cd}{a}\right) = ay$ by substitution; construct y, and then $x = \sqrt{ay}$.

Again, x is a line, the square upon which is equal to the sum of the rectangles ab, cd. This sum may be reduced to a single rectangle, and the rectangle converted into a square, the base of which is the required value of x.—Euclid, i. 45, and ii. 14; or Geometry, i. 57, 58.



Let $x = \sqrt{a^2 + b^2}$; take a straight line A B $= a$, from B draw B C ($= b$) perpendicular to A B; A C is the value of x.

Let $x = \sqrt{a^2 + b^2 + c^2}$, from C draw C D ($= c$) perpendicular to A C, A D is the required value of x.

Let $x = \sqrt{a^2 - b^2} = \sqrt{(a+b)(a-b)}$; x is a mean proportional between $a + b$ and $a - b$; or by taking (in the last figure but one) A B $= a$, and A E $= b$, we have B E $= \sqrt{a^2 - b^2}$.

Let $x = \sqrt{\{a^2 + b^2 - c^2 - d^2\}}$, find $y^2 = a^2 + b^2$ and $z^2 = c^2 + d$ and then x.

Let $x = \sqrt{\frac{a^4 + bc^3}{b^2 - c^2}}$, find $y^2 = a^2 + \frac{bc^3}{a^2}$, and $z^2 = b^2 - c^2$ and then $x = \frac{ay}{z}$.

11. Of course the preceding methods will equally apply, when instead of the letters we have the original numbers, the linear unit being understood as usual.

Thus $x = \sqrt{12} = \sqrt{3.4}$ is a mean proportional between 3 and 4; hence (see last figure but one) take A C equal four times the unit, and C B equal three times the unit, C E is the value of x; or since $\sqrt{12} = \sqrt{16 - 4} = \sqrt{4^2 - 2^2}$, by constructing a right-angled triangle of which the hypothenuse is four times the linear unit, and one side twice that unit: the remaining side $= \sqrt{12}$.

Similarly $x = \sqrt{7} = \sqrt{4 + 4 - 1} = \sqrt{2^2 + 2^2 - 1^2}$, which is of the form $\sqrt{a^2 + b^2 - c^2}$.

Let $x = \sqrt{3} = \sqrt{2 + 1}$. In the last figure let A B, B C, and C D each be equal to the linear unit, then $\mathrm{A\,D} = \sqrt{3}$

Let $x = \sqrt{23} = \sqrt{5^2 - 1^2 - 1^2}$.

Let $x = \dfrac{1}{\sqrt{2}} = \sqrt{\dfrac{1}{2}} = \sqrt{\dfrac{1}{4} + \dfrac{1}{4}} = \sqrt{\left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1}{2}\right)^2}$,

then x is the hypothenuse of a right-angled triangle, each of whose sides is half the unit.

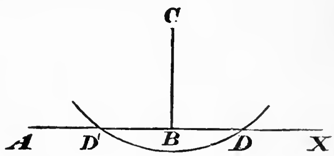
Let $x = \sqrt{\dfrac{3}{4}}$; this may be constructed as the last.

Let $x = \sqrt{\dfrac{1}{3}} = \sqrt{\dfrac{3}{9}}$; and so on for all numbers, since any finite number can be decomposed into a series of numbers representing the squares upon lines.

If the letter a be prefixed to any of the above quantities, it must be introduced under the root.

12. In constructing compound quantities, it is best to unite the several parts of the construction in one figure.

Thus if $x = a \pm \sqrt{a^2 - b^2}$, in the line A X take $\mathrm{A\,B} = a$, from B draw B C ($= b$) perpendicular to A B; with centre C and radius a describe a circle cutting A X in D and D′; A D and A D′ are the values required:



$$\text{for } \mathrm{A\,D} = \mathrm{A\,B} + \mathrm{B\,D} = a + \sqrt{a^2 - b^2}.$$

$$\mathrm{A\,D'} = \mathrm{A\,B} - \mathrm{B\,D'} = a - \sqrt{a^2 - b^2}.$$

This construction fails when b is greater than a, for then the circle never cuts the line A X; this is inferred also from the impossibility of the roots

13. Since theorems in geometry relate either to lines, areas, or solids, the corresponding equations must in each case be homogeneous, and will remain so through all the algebraic operations. If, however, one of the lines in a figure be taken as the linear unit and be therefore represented by unity, we shall find resulting expressions, such as $x = \dfrac{a}{b}$, $x = \sqrt{a}$, $x = \sqrt{a^2 + b}$,&c., in which, prior to construction, the numerical unit must be expressed; thus these quantities must be written $\dfrac{a}{b} \times 1$, $\sqrt{a \times 1}$, $\sqrt{a^2 + b \times 1}$, and then constructed as above.

CHAPTER II.

DETERMINATE PROBLEMS.

14. Geometrical Problems may be divided into two classes, Determinate and Indeterminate, according as they admit of a finite or an infinite number of solutions.

If A B be the diameter of the semicircle A E B, and it be required to find a point C in A B such, that drawing C E perpendicular to A B to meet the circumference in E, C E shall be equal to half the radius of the circle, this is a determinate problem, because there are only *two* such points in A B, each at an equal distance from the centre. Again, if it be required to find a point E out of the line A B such, that joining E A, E B, the included angle A E B shall be a right angle, this is an indeterminate problem, for there are an infinite number of such points, all lying in the circumference A E B.

The determinate class is by no means so important as the indeterminate, but the investigation of a few of the former will lead us to the easier comprehension of the latter; and therefore we proceed to the discussion of determinate problems.

15. In the consideration of a problem, the following rules are useful.

1. Draw a figure representing the conditions of the question.

2. Draw other lines, if necessary, generally parallel or perpendicular to those of the figure.

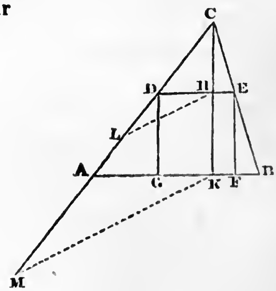
3. Call the known lines by the letters a, b, c, &c., and some of the unknown lines by the letters x, y, z, &c.

4. Consider all the lines in the figure as equally known, and from the geometrical properties of figures deduce one, two, or more equations, each containing unknown and given quantities.

5. From these equations find the value of the unknown quantities.

6. Construct these values, and endeavour to unite the construction to the original figure.

16. To describe a square in a given triangle A B C.



Let D E F G be the required squar
C H K the altitude of the triangle.

The question is resolved into finding the point H, because then the position of D E, and therefore of the square, is determined.

Let C K $= a$, A B $= b$, C H $= x$;
then by the question, D E $=$ H K,
and D E : A B :: C H : C K,
or D E : b :: x : a,

$\therefore$ D E $= \dfrac{b\,x}{a}$, and H K $= a - x$

$$\therefore \frac{bx}{a} = a - x$$

$$\therefore x = \frac{a^2}{a+b}.$$

Thus x is a third proportional to the quantities $(a+b)$ and a.

In C A take C L $= a$, produce C A to M so that L M $= b$, join M K, and draw L H parallel to M K; C H is the required value of x.

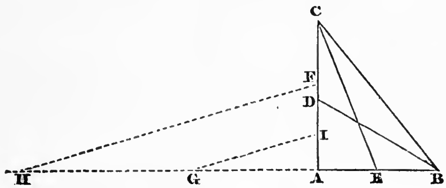
17. In a right-angled triangle the lines drawn from the acute angles to the points of bisection of the opposite sides are given, to find the triangle.

Let C E $= a$, B D $= b$, A D $=$ C D $= x$, A E $=$ E B $= y$.

Then the square upon C E $=$ square upon C A $+$ square upon A E,

$$\text{or } a^2 = 4x^2 + y^2$$

$$\text{similarly } b^2 = x^2 + 4y^2$$



whence $y = \pm \sqrt{\frac{4b^2 - a^2}{15}}$. Make any right angle A, and on one of the sides take A F $= \frac{a}{2}$, with centre F and radii b and $2a$, describe circles cutting the other side produced in G and H, respectively; draw G I parallel to F H; then 2 A I is the required value of y. Hence A D, and therefore A C and A B are found, and the triangle is determined.

18. To divide a straight line, so that the rectangle contained by the two parts may be equal to the square upon a given line b.

Let A B $= a$

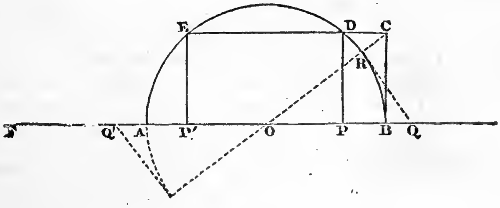
A P $= x$

Then the rectangle A P, P B $= b^2$

$$\text{or } x(a-x) = b^2$$

$$\therefore x^2 - ax = -b$$

$$\therefore x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b^2}$$



Upon A B describe a semicircle, draw B C ($= b$) perpendicular to A B, through C draw C D E parallel to A B, from D and E draw D P, E P′, perpendicular to A B; P and P′ are the required points.

If b is greater than $\frac{a}{2}$, the value of x is irrational, and therefore the problem is impossible; but then a point Q may be found in A B produced, such that, the rectangle A Q, Q B $= b^2$.

Let A Q $= x$,

$$\therefore\ x\,(x - a) = b^2$$

$$\therefore\ x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b^2}.$$

From the centre O draw the line O C cutting the circle in R, from R draw R Q perpendicular to O R, then Q is the required point; for

O Q $=$ O C $= \sqrt{\left(\frac{a^2}{4} + b^2\right)}$, and therefore A Q $= \frac{a}{2} + \sqrt{\frac{a^2}{4} + b^2}$.

Let us examine the other root $\frac{a}{2} - \sqrt{\frac{a^2}{4} + b^2}$, which is negative, and may be written in the form $-\left\{\sqrt{\frac{a^2}{4} + b^2} - \frac{a}{2}\right\}$; the magnitude of this quantity, independent of the negative sign, or its *absolute* magnitude, is evidently B Q or A Q′.

Now if the problem had been "to find a point Q in either A B produced, or B A produced, such, that the rectangle A Q, Q B $= b^2$", we might have commenced the solution by assuming the point Q to be in B A produced as at Q′; thus letting A Q′ $= x$, we should have $x\,(a + x) = b^2$, and $x = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b^2}$, of which two roots the first or $-\frac{a}{2} + \sqrt{\frac{a^2}{4} + b^2} = -\left\{\frac{a}{2} - \sqrt{\frac{a^2}{4} + b^2}\right\}$ is the absolute value of the negative root in the last question; hence the negative root of the last question is a real solution of the problem expressed in a more general form, the negative sign merely pointing out the *position* of the second point Q′. Both roots may be exhibited in a positive form by measuring x not from A, but from a point F, A F being greater than b; for letting FA $= c$, and F Q or F Q′ $= x$, we find

$$x = c + \frac{a}{2} \pm \sqrt{b^2 + \frac{a^2}{4}}.$$

The celebrated problem of dividing a given straight line in extreme and mean ratio, is solved in the same manner; letting A P $= x$ we have the rectangle A B, B P $=$ the square upon A P, or $a\,(a - x) = x^2$, whence

$x = -\frac{a}{2} \pm \sqrt{a^2 + \frac{a^2}{4}}$; here the negative root, which gives a point to the left of A, is a solution of the problem enunciated more generally *.

19. Through a point M equidistant from two straight lines A A′ and B B′ at right angles to each other, to draw a straight line P M Q, so that the part P Q intersected by A A′ and B B′ may be of a given length b.

From M draw the perpendicular lines M C, M D.

Let $\text{M D} = a$, $\text{D Q} = x$, $\text{C P} = y$,

then $\text{P Q} = \text{P M} + \text{M Q}$,

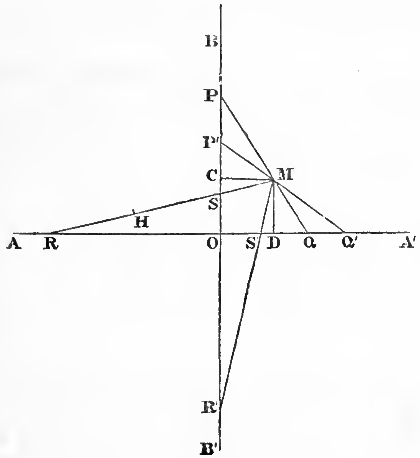
or $b = \sqrt{a^2 + y^2} + \sqrt{a^2 + x^2}$,

and $\dfrac{x}{a} = \dfrac{a}{y}$ from the similar triangles P C M, M D Q.

$$\therefore b = \sqrt{a^2 + \frac{a^4}{x^2}} + \sqrt{a^2 + x^2}$$

$$= \sqrt{a^2 + x^2}\left(1 + \frac{a}{x}\right);$$

whence $x^4 + 2ax^3 + (2a^2 - b^2)x^2 + 2a^3 x + a^4 = 0$.



We might solve this recurring equation, and then construct the four roots, as in the last problems; but since the roots of an equation of four dimensions are not easily obtained, we must, in general, endeavour to avoid such an equation, and rather retrace our steps than attempt its solution. Let us consider the problem again, and examine what kind of a result we may expect.

* Lucas de Borgo, who wrote a book on the application of this problem to architecture and polygonal figures, was so delighted with this division of a line, that he called it the Divine Proportion.

Since, in general, four lines P M Q, P′ M Q′, R S M, R′ S′ M, may be drawn fulfilling the conditions of the question, the two former, in all cases, though not always the two latter, we may conclude that there will be four solutions; but since the point M is similarly situated with respect to the two lines A A′, B B′, we may also expect that the resulting lines will be similarly situated with regard to A A′ and B B′. Thus, if there be one line P M Q, there will be another P′ M Q′ such that O Q′ = OP, and O P′ = O Q.

Again O S will be equal to O S′, and O R to O R′. Hence, if we take the perpendicular from O upon the line S R for the unknown quantity (y), we can have only two different values of this line, one referring to the lines S R and S′ R′, the other to P Q and P′ Q′; hence the resulting equation will be of *two* dimensions only. In this case the equation is

$$by^2 + 2a^2y - ba^2 = 0.$$

Again, since M R = M R′ we may take M H, H being the point of bisection of the line S R, for the unknown quantity, and then also we may expect an equation, either itself of two dimensions, or else reducible to one of that order.

$$\text{Let } MH = x; \therefore MR = x + \frac{b}{2},\ MS = x - \frac{b}{2},$$

$$\text{and } MR : MD :: RSS : O = \frac{ab}{x + \frac{b}{2}}.$$

$$MS : OD :: RS : RO = \frac{ab}{x - \frac{b}{2}};$$

but the square upon R S = square upon R O + square upon S O,

$$\therefore b^2 = \left(\frac{ab}{x - \frac{b}{2}}\right)^2 + \left(\frac{ab}{x + \frac{b}{2}}\right)^2$$

$$\therefore x^4 - \left(2a^2 + \frac{b^2}{2}\right)x^2 + \frac{b^4}{16} - \frac{a^2b^2}{2} = 0$$

$$\therefore x = \pm\sqrt{\left\{a^2 + \frac{b^2}{4} \pm a\sqrt{a^2 + b^2}\right\}}$$

an expression of easy construction; the negative value of x is useless· of the remaining two values that with the positive sign is always real, and refers to the lines M S R, M S′ R′; the other, when real, gives the lines P M Q, P′ M Q′; it is imaginary if b^2 is less than $8a^2$, that is, joining O M and drawing P M Q perpendicular to O M, if b is less than P M Q.

This question is taken from Newton's Universal Arithmetic, and is given by him to show how much the judicious selection of the unknown quantity facilitates the solution of problems. The principal point to be attended to in such questions is, to choose that line for the unknown quantity which must be liable to the least number of variations.

20. Through the point M in the last figure to draw P M Q so that the sum of the squares upon P M and M Q shall be equal to the square upon a given line b.

Making the same substitutions as in the former part of the last article, we shall obtain the equations

$$x^2 + a^2 + y^2 + a^2 = b^2,\ xy = a^2,$$

$$\therefore\ x^2 + y^2 + 2xy = b^2, \text{ and } x + y = \pm b,$$

$$\text{or } x + \frac{a^2}{x} = \pm b, \text{ whence } x = \pm\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - a^2}.$$

To construct these four values describe a circle with centre M and radius $\frac{b}{2}$, cutting A A′ in two points L, L′; with centres L, L′ and radius $\frac{b}{2}$ describe two other circles cutting A A′ again in four points: these are the required points.

21. To find a triangle A B C such that its sides A C, C B, B A, and perpendicular B D, are in continued geometrical progression.

Take any line A B $= a$ for one side, let B C $= x$,

$$AC : CB :: CB : BA :: BA : BD;$$

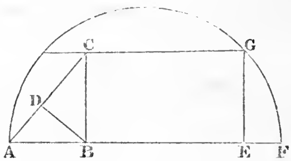
hence the triangles A C B, A D B, are equiangular, (Eucl. vi. 7, or Geometry, ii. 33,) and the angle A B C is a right angle; also A C $= \frac{x^2}{a}$, then

the square upon A C = the square upon B C + the square upon A B;

$$\therefore \frac{x^4}{a^2} = x^2 + a^2, \text{ or } x^4 - a^2x^2 - a^4 = 0,$$

$$\text{whence } x = \pm\sqrt{\frac{a^2 \pm \sqrt{5a^4}}{2}};$$

of these roots two are impossible, since $a^2\sqrt{5}$ is greater than a^2; and of the remaining two the negative one is useless.



In A B produced take B E $= a\sqrt{5}$ (11), and E F $= \frac{a}{2}$; upon AF describe a semicircle, and draw the perpendicular E G; then E G $=$

$$\sqrt{\left\{\frac{a}{2}(a + a\sqrt{5})\right\}} = \sqrt{\frac{a^2 + a^2\sqrt{5}}{2}} \text{ is the required value of } x$$

CHAPTER III.

THE POINT AND STRAIGHT LINE.

22. Determinate problems, although sometimes curious, yet, as they lead to nothing important, are unworthy of much attention. It was, however, to this branch of geometry that algebra was solely applied for some time after its introduction into Europe. Descartes, a celebrated French

philosopher, who lived in the early part of the seventeenth century, was the first to extend the connexion. He applied algebra to the consideration of curved lines, and thus, as it were, invented a new science.

Perhaps the best way of explaining his method will be by taking a simple example. Suppose that it is required to find a point P without a given line A B, so that the sum of the squares on A P and P B shall be equal to the square upon A B.

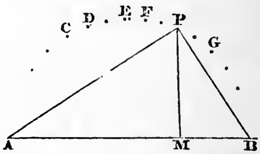
Let P be the required point, and let fall the perpendicular P M on A B.

Let A M $= x$, M P $= y$, and A B $= a$; then by the question, we have

The square on A B = the square on A P + the square on P B.

= the squares on A M, M P + the squares on P M, M B,

or $a^2 = (x^2 + y^2) + y^2 + (a - x)^2$
$= 2y^2 + 2x^2 - 2ax + a^2$
$\therefore y^2 = ax - x^2$.



Now this equation admits of an infinite number of solutions, for giving to x or A M any value, such as $\frac{a}{2}$, $\frac{a}{3}$, $\frac{a}{4}$, &c., we may, from the equation, find corresponding values of y or M P, each of them determining a separate point P which satisfies the condition of the problem.

Let C, D, E, F, &c., be the points thus determined. The number of the values of y may be increased by taking values of x between those above-mentioned and this to an infinite extent, thus we shall have an infinite number of points C, D, E, F, &c., indefinitely near to each other, so that these points ultimately form a line which geometrically represents the assemblage of all the solutions of the equation. This line A C D E F, whether curved or straight, is called the locus of the equation.

In this manner all indeterminate problems resolve themselves into investigations of loci; and it is this branch of the subject which is by far the most important, and which leads to a boundless field for research*.

23. For the better investigation of loci, equations have been divided into two classes, algebraical and transcendental.

An algebraical equation between two variables x and y is one which can be reduced to a finite number of terms involving only integral powers of x, y, and constant quantities: and it is called complete when it contains all the possible combinations of the variables together with a constant term, the sum of the indices of these variables in no term exceeding the degree of the equation; thus of the equations

$$ay + bx + c = o$$
$$ay^2 + bxy + cx^2 + dy + ex + f = o$$

the first is a complete equation of the first order, and the next is a complete equation of the second order, and so on.

Those equations which cannot be put into a finite and rational algebraical form with respect to the variables are called transcendental, for

* For the definition and examples of Loci, see Geometry, iii. § 6; and the Index, article Locus.

they can only be expanded into an infinite series of terms in which the power of the variable increases without limit, and thus the order of the equation is infinitely great, or transcends all finite orders.

$y = \sin. x$, and $y = a^x$, are transcendental equations.

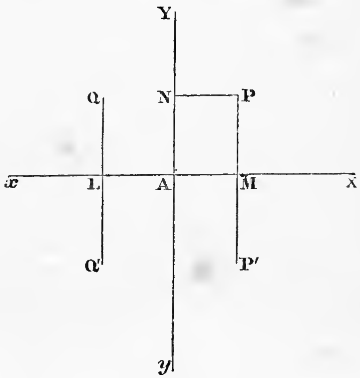
24. The loci of equations are named after their equations, thus the locus of an equation of the first order is a line of the first order; the locus of an equation of the second order is a line of the second order; the locus of a transcendental equation is a transcendental line or curve.

Algebraical equations have not corresponding loci in all cases, for the equation may be such as not to admit of any real values of both x and y; the equation $y^2 + x^2 + a^2 = 0$ is an example of this kind, where, whatever real value we give to x, we cannot have a real value of y: there is therefore no locus whatever corresponding to such an equation.

THE POSITION OF A POINT IN A PLANE.

25. The position of a point in a plane is determined by finding its situation relatively to some fixed objects in that plane; for this purpose suppose the plane of the paper to be the given plane, and let us consider as known the intersection A of two lines x X and y Y of unlimited length, and also the angle between them; from any point P, in this plane, draw P M parallel to A Y, and P N parallel to A X, then the position of the point P is evidently known if A M and A N are known. For it may be easily shown, *ex absurdo*, that there is but one point within the angle Y A X such that its distance from the lines A Y and A X is P N and P M respectively.

A M is called the abscissa of the point P; A N, or its equal M P, is called the ordinate; A M and M P are together the co-ordinates of P; X x is called the axis of abscissas, Y y the axis of ordinates. The point A where the axes meet is termed the origin.



The axes are called oblique or rectangular, according as Y A X is an oblique or a right angle. In this treatise rectangular axes as the most simple will generally be employed.

Let the abscissa A M $= x$, and the ordinate M P $= y$, then if on measuring these lengths A M and M P we find the first equal to a and the second equal to b, we have, to determine the position of this point P, the two equations

$$x = a, y = b$$

and as they are sufficient for this object, we call them, when taken together, the equations to this point.

The same point may also be defined by the equation

$$(y - b)^2 + (x - a)^2 = 0$$

for this equation can only be satisfied by the values $x = a$ and $y = b$.

And in general any equation which can only be satisfied by a single real value of each variable quantity x and y, refers to a point whose situation is determined by the co-ordinates corresponding to these values.

26. In this manner the position of any point in the angle Y A X can be determined, but in order to express the positions of points in the angle Y A x, some further considerations are necessary.

In the solution of the problem, article (18), we observed that negative quantities may be geometrically represented by lines drawn in a certain direction. An extension of this idea leads to the following reasoning.

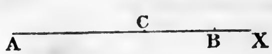
When we affix a negative sign to any quantity, we do not signify any change in its magnitude, but merely the way in which the quantity is to be used, or the operation to be performed on it. Thus the absolute magnitude of -5 is just as great as that of $+5$; but -5 means that 5 is to be subtracted, and $+5$ that it is to be added. As the sign $+$ is applied to quantities variously estimated, the sign $-$ will have in each of these various cases a corresponding meaning, necessarily following from that of the sign $+$. Whatever $+$ means, we must always have $-a+a=0$. Hence we may define $-a$ to be a quantity estimated in such a manner that the altering it by the operation indicated by $+a$ reduces the result to nothing. This is properly the meaning of the sign $-$; it depends entirely on that of the sign $+$ in every case.

The symbol of positive quantity is used in a variety of ways; but in every instance the above principle shows in what way the negative quantity must, as a necessary consequence of the meaning of the positive quantity, be used.

Thus, if we placed a mark on a pole stuck vertically into the ground, at some point in the pole which was bare at low water and covered at high water, and scored upwards the inches from that mark, we might express the height of the surface by the number of inches above the mark, positively, when the surface was above the mark; but at low water when the surface is below the mark, 11 inches for instance, we should call the height -11; because when 11 inches were added to the height, (that is, when the surface of the water was advanced 11 inches upwards, which is the direction in which the positive quantities are supposed to be reckoned,) the surface would be just at the mark, and would be *no inches* in height reckoning from the mark.

Suppose a man to advance directly from a given point p miles in the first 6 hours of a day, and to go back in the next 6 hours q miles; at the end of the 12 hours his advance from the given point would be $(p-q)$ miles. Thus, suppose $p=10$, and $q=6$, he will advance $(10-6)$ or 4 miles. But suppose he recedes 10 miles, then his advance will in the 12 hours be $(10-10)$ or 0: he will be just where he was at first. Suppose he recedes 15 miles, at the end of the 12 hours he will be 5 miles *behind* the original point. Here we say *behind*, because the movement in the direction of the original advance was considered to be *forward*. And it is clear that in this case, from an advance of 10 miles, and a recess of 15, the advance is -5; that is, it requires a further advance of 5 miles to make the man exactly as forward as he was at starting.

Now let us consider a fixed point A, and a line measured from it by positive quantities in the direction A X. Suppose the line to be described by the motion of a point from A along A X; and after the point has been carried *forward* (that is, towards X) m linear units, as to B, let it be carried



back n linear units, as to C; then altogether the advance of the point or the length of A C will be $(m - n)$ linear units.

Again, suppose n to become $= m$; that is, let the point be carried back exactly to A; then the advance of the point along A X will still be measured by $(m - n) = m - m = 0$.

Once more, let n exceed m; that is, let B C exceed A B; the advance of the point will be expressed by $(m - n)$ still; but this will now be a negative number, showing by how many linear units the point must be advanced in order to bring it forward to the original starting point A. Now any line A C may be considered to be determined by the motion of a point either simply along A C, or along first A B and then B C. We see, therefore, if we begin by reckoning distances from A in the direction A X as positive quantities, we are compelled to consider distances from A in the opposite direction as negative quantities.

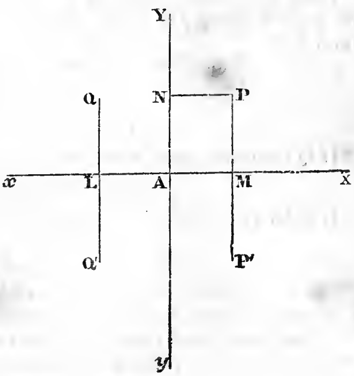
Conversely again, having designated positive quantities by lines in one direction from a given point, suppose the calculation produces a negative result, what meaning are we to assign to it? The negative result shows how much positive quantity is required to bring the whole result to nothing. Now positive quantity, by the hypothesis, is distance measured in the original direction; therefore the negative quantity shows how much distance measured in the original direction is required to bring the result to nothing. But if there be a distance from A, such that a linear units in the original direction must be subjoined to bring the result to nothing, (that is, to reduce to nothing the distance from A,) it is clear that this distance must be that of a linear units measured in a direction from A opposite to the original direction. That is, the negative quantity must be represented by lines drawn in the direction opposite to that in which the lines representing the positive quantities are drawn.

It is immaterial in which direction the line is drawn which we consider positive: but when chosen, negative quantities of the same kind must be taken in the opposite direction.

27. We are now able to express the position of points in the remaining angles formed by the axes, by considering all lines in the direction A X to be positive and those in A x to be negative: and similarly all those drawn in the direction A Y will be considered positive, and therefore those in A y will be negative.

We have then the following table of co-ordinates.

P in the angle X A Y, $+ x$, $+ y$,
Q in the angle Y A x, $- x$, $+ y$,
Q′ in the angle x A y, $- x$, $- y$,
P′ in the angle X A y, $+ x$, $- y$.



Hence the equations to a given point P are $x = a,\ y = b$

. . . . Q .. $x = -a,\ y = b$

. . . . Q′ .. $x = -a,\ y = -b$

. . . . P′ . $x = a,\ y = -b$

28. If, the abscissa AM remaining the same, the ordinate MP diminishes, the point P approaches to the axis A X; and when M P is nothing, P is situated on that axis; in this case the equations to the point P are

$$x = a,\ y = 0: \text{ or } y^2 + (x - a)^2 = 0.$$

Similarly when the point P is situated on the axis AY, its equations are

$$x = 0,\ y = b: \text{ or } (y - b)^2 + x^2 = 0.$$

If both A M and M P vanish, we have the equations to the origin A,

$$x = 0,\ y = 0: \text{ or } y^2 + x^2 = 0.$$

Ex. 1. The point whose equations are $x = 4$, $y = -2$, is situated in the angle X A y, at a distance A M = 4 times the linear unit from the axis of y, and M P′ = twice that unit from the axis of x.

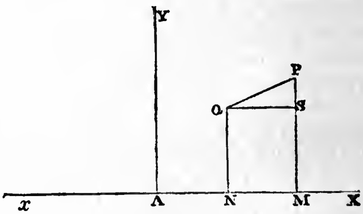
Ex. 2. The point whose equation is $(y + 3)^2 + (x + 2)^2 = 0$ is situated in the angle x A y, at distances A L = 2, L Q′ = 3, from the axes.

Ex. 3. The point whose equations are $x = 0$, $y = -3$ is in the line A y, at a distance = 3 times the linear unit.

Ex. 4. The point whose equation is $y^2 + (x + a)^2 = 0$, is in A x, at a distance a from the origin.

The preceding articles are true if the co-ordinate axes be oblique.

29. To find an expression for the distance D between two points P and Q.



Let the axes be rectangular and let the equations to

P be $x = a,\ y = b$

Q $x = a',\ y = b'$;

or in other words, let the co-ordinates of P be A M = a, M P = b, and those of Q be A N = a', N Q = b', draw Q S parallel to AX.

Then the square upon Q P = the square upon Q S + the square upon P S;

and Q S = N M = A M − A N = $a - a'$

also P S = P M − Q N = $b - b'$

$$\therefore D^2 = (a - a')^2 + (b - b')^2$$

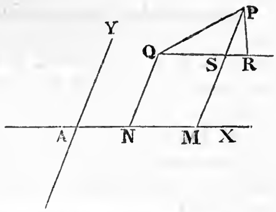
If Q be in the angle Y A x we have A N = $-a'$,

$$\therefore D^2 = (a + a')^2 + (b - b')^2.$$

If Q be at the origin we have $a' = 0$ and $b' = 0$

$$\therefore D^2 = a^2 + b^2, \text{ or } D = \sqrt{a^2 + b^2}.$$

30. If the angle between the axes be oblique and $= \omega$, draw P M and Q N parallel to A Y, and Q S parallel to A X; also let P R be drawn perpendicular to Q S; then the square upon Q P = the square on Q S + the square on P S + twice the rectangle Q S, S R;



and, $QS = a - a'$,

$PS = b - b'$,

$SR = PS \cos. PSR$

$= PS \cos. YAX$

$= (b - b') \cos. \omega$;

$$\therefore D^2 = (a - a')^2 + (b - b')^2 + 2(a - a')(b - b') \cos. \omega;$$

and when the point Q is at the origin, and therefore $a' = 0$, and $b' = 0$,

$$D^2 = a^2 + b^2 + 2ab \cos. \omega.$$

THE LOCUS OF AN EQUATION OF THE FIRST DEGREE.

31. To find the locus of an equation of the first degree between two unknown quantities.

The most general form of such an equation is,

$Ay + Bx + C = 0$, or $y = -\frac{B}{A}x - \frac{C}{A}$, or $y = \alpha x + b$ if $-\frac{B}{A} = \alpha$ and $-\frac{C}{A} = b$; we will in the first place consider the equation in its most simple form $y = \alpha x$.

Let AX, AY be the rectangular axes, then a point in the locus will be determined by giving to x a particular value as 1, 2, 3, &c.; let AM, MP and AN, NQ be the respective co-ordinates of two points P and Q thus determined;

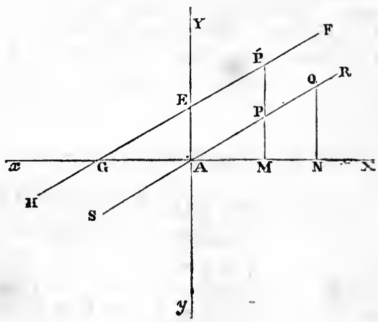
since $y = \alpha x$, we have

$$MP = \alpha . AM$$

and $NQ = \alpha . AN$

$$\therefore MP : AM :: NQ : AN;$$

therefore the triangles AMP, ANQ are similar, and the angles MAP, NAQ, equal to one another: hence the two lines AP, AQ coincide. If a third point R be taken in the locus, then, as before, AR will coincide with AP and AQ. Consequently all the lines drawn from A to the several points of the locus coincide; that is, all the points P, Q, R, &c., are in the same straight line AR, and by giving negative values to x we can determine the point S, &c., to be in the same straight line RA produced. Hence the straight line RAS produced both ways indefinitely, being the assemblage of all the points determined by the equation $y = \alpha x$, is the *locus* of that equation.



In considering the equation $y = \alpha x + b$, we observe that the new ordinate y always exceeds the former by the quantity b; hence taking A E in the axis A Y equal to b, and drawing the line H E F parallel to S R, the line H E F is the locus required.

Hence the equation of the first order belongs to the straight line.

32. To explain the nature of the equation more clearly, we will take the converse problem. To find the equation to a straight line H F, that is, to find the relation which exists between the co-ordinates, x and y, of each of its points.

Let A be the origin of co-ordinates, A X, A Y the axes; from A draw A R parallel to H F, and from any point P′ in the given line draw P′ P M perpendicular to A X and cutting A R in P.

$$\text{Let } \mathrm{AM} = x,\ \mathrm{MP'} = y, \text{ and } \mathrm{AE} = b;$$

$$\begin{aligned} \text{then } \mathrm{MP'} &= \mathrm{PM} + \mathrm{PP'} \\ &= \mathrm{AM} \tan. \mathrm{PAM} + \mathrm{AE} \\ &= x \tan. \mathrm{FGX} + b; \end{aligned}$$

$$\text{or } y = \alpha x + b, \quad \text{if } \tan. \mathrm{FGX} = \alpha.$$

If A G $= a$, we have A E $=$ A G. tan. E G A, or $b = a\,\alpha$, and therefore the equation to the straight line may be written under the form $y = \alpha x + \alpha a$.

33. In general, therefore, the equation to the straight line contains two constant quantities b and α; the former is the distance A E or is the ordinate of the point in which the line cuts the axis of y, the latter is the tangent of the angle which the line makes with the axis of x, for the angle F G A $=$ the angle P A M: hence

$$\tan. \mathrm{FGA} = \tan. \mathrm{PAM} = \frac{y - b}{x} = \alpha.$$

It is to be particularly observed that, in calling α the tangent of the angle which the line makes with the axis of x, we understand the angle F G X and not F G x.

34. In the equation $y = \alpha x + b$, the quantities α and b may be either both positive, or both negative, or one positive and the other negative; let us then examine the course of the line to which the equation belongs in each case. Now it is clear that the knowledge of two points in a straight line is sufficient to determine the position of that line; hence we shall only find the points where it cuts the axes since they are the most easily obtained.

1. α and b positive; $\therefore y = \alpha x + b$;

Let $x = 0$; $\therefore y = b$; in A Y take A D $= b$;

$y = 0$; $\therefore x = -\dfrac{b}{\alpha}$; in A x take AB $= \dfrac{b}{\alpha}$;

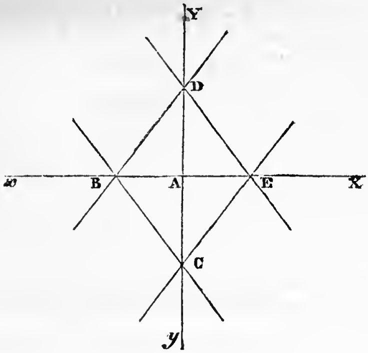
join B D; B D produced is the required locus.

2. α positive and b negative; $\therefore y = \alpha x - b$;

Let $x = 0$; $\therefore y = -b$, in A y take A C $= b$;

$y = 0$; $\therefore x = \dfrac{b}{\alpha}$; in A X take A E $= \dfrac{b}{\alpha}$

join C E; C E produced is the required locus.



3. α negative and b positive; $\therefore y = -\alpha x + b$;
Let $x = 0$; $\therefore y = b$; in A Y take AD $= b$;

$$y = 0; \therefore x = \frac{b}{\alpha}; \text{ in A X take A E} = \frac{b}{\alpha};$$

join D E; D E produced is the required locus.

4. α negative and b negative; $\therefore y = -\alpha x - b$;
Let $x = 0$; $\therefore y = -b$; in A y take A C $= b$;

$$y = 0; \therefore x = -\frac{b}{\alpha}; \text{ in A } x \text{ take AB} = \frac{b}{\alpha};$$

join B C; B C produced is the required locus.

35. The quantities α and b may also change in absolute value.

Let $b = 0$; $\therefore y = \pm \alpha x$; and the loci are two straight lines passing through the origin and drawn at angles with the axis of x whose respective tangents are $\pm \alpha$.

Let $\alpha = 0$; $\therefore y = 0 x \pm b$; $\therefore y = \pm b$ and $x = \frac{0}{0}$; the former of these results shows that every point in the locus is equidistant from the axis of x, and the latter (or $0 x = 0$) that every value of x satisfies the original equation; hence the loci are two straight lines drawn through D and C both parallel to the axis of x.

It has been stated (28), that the system of equations $y = b, x = 0$ refers to a point; we here see that the system $y = b$, $x = \frac{0}{0}$ refers to a straight line; hence, although the equation $x = \frac{0}{0}$ is generally omitted, yet it must be considered as essential to the locus.

Let $\alpha = \frac{1}{0}$; referring to article 32, the equation to the straight line may be written $y = \alpha x \pm \alpha a$ or $\frac{y}{\alpha} = x \pm a$, which when $\alpha = \frac{1}{0}$ becomes $0 y = x \pm a$; hence, as before, the system $x = \pm a$, $y = \frac{0}{0}$, or more simply the equation $x = \pm a$ denotes two straight lines parallel to the axis of y and at a distance $\pm a$ from that axis.

Again let both $\alpha = 0$, and $b = 0$; and $\therefore$ the equation $y = \alpha x + b$ becomes $y = 0\,x + 0$; and hence, $y = 0$, $x = \frac{0}{0}$, and the locus is the axis of x.

If $\alpha = \frac{1}{0}$, and $b = 0$, the equation becomes $0\,y = x + 0$; $\therefore x = 0$ and $y = \frac{0}{0}$. Hence the equation $x = 0$ denotes the axis of y.

36. By the above methods the line to which any equation of the first order belongs may be drawn.

In the following examples reference is made to parts of the last figure.

Ex. 1. $3\,y - 5\,x - 1 = 0$; let $x = 0$, $\therefore y = \frac{1}{3}$; on the axis A Y take A D one-third of the linear unit, then the line passes through D: again let $y = 0$, $\therefore x = -\frac{1}{5}$; on the axis A x take A B $= \frac{1}{5}$ of the unit, then the line passes through B; hence the line joining the points B and D is the locus required.

Ex. 2. $10\,y - 21\,x + 6 = 0$; a line situated like C E.

Ex. 3. $y - x = 0$; let $x = 0$ $\therefore y = 0$, and the line passes through the origin; also α or the tangent of the angle which the line makes with the axis of $x = 1$, therefore that angle $= 45°$; hence the straight line drawn through the origin and bisecting the angle Y A X is the required locus.

Ex. 4. $5\,y - 2\,x = 0$. The line passes through the origin as in the last example, but to find another point through which the line passes, let $x = 5$; $\therefore y = 2$: hence take A E $= 5$, and from E draw E P ($= 2$) perpendicular to A X; then the line joining the points A, P is the locus required.

Ex. 5. $ay + bx = 0$; a line drawn through A, and parallel to B C.

Ex. 6. $y^2 - 3\,x^2 = 0$; two straight lines making angles of 60° with the axis of x.

Ex. 7. $3y - 4 = 0$; take A D $= \frac{4}{3}$ of the unit, a line through D drawn parallel to A X is the locus.

Ex. 8. $x^2 + x - 2 = 0$; take A E $= 1$, and A B $= 2$, the lines drawn through E and B parallel to A Y are the required loci.

Ex. 9. $y + 2\,x = 4$. The equation to a straight line may be put under the convenient form $\frac{y}{b} + \frac{x}{a} = 1$, and since when $y = 0$, $x = a$, and when $x = 0$, $y = b$, the quantities b and a are respectively the distances of the origin from the intersection of the line with the axes of y and x. Thus Ex. 9. in this form is $\frac{y}{4} + \frac{x}{2} = 1$, take A D $= 4$, and A E $= 2$, join D E, this line produced is the required locus.

37. If the equation involve the second root of a negative quantity its locus will not be a straight line, but either a point or altogether imaginary:

thus the locus of the equation $y + 2x\sqrt{-1} - a = 0$ is a point whose co-ordinates are $x = 0$ and $y = a$, for no other real value of x can give a real value to y; but the locus of the equation $y + x + a\sqrt{-1} = 0$ is imaginary, for there are no corresponding real values of x and y. (24)

38. We have thus seen that the equation to a straight line is of the form $y = \alpha x + b$, and that its position depends entirely upon α and b.

By a given line we understand one whose position is given, that is, that α and b are given quantities; when we seek a line we require its position, so that assuming $y = \alpha x + b$ to be its equation, α and b are the two indeterminate quantities to be found by the conditions of the question: if only one can be found the conditions are insufficient to fix the position of the line.

By a given point we understand one whose co-ordinates are given; we shall generally use the letters x_1 and y_1 for the co-ordinates of a given point, and to avoid useless repetition, the point whose co-ordinates are x_1 and y_1 will be called "the point x_1, y_1." Similarly the line whose equation is $y = \alpha x + b$ will be called "the line $y = \alpha x + b$."

If in the same problem we use the equations to two straight lines as $y = \alpha x + b$ and $y = \alpha' x + b'$, it must be carefully remembered that x and y are not the same quantities in both equations; we might have used the equations $y = \alpha x + b$, and $Y = \alpha' X + b'$, X and Y being the variable coordinates of the second line, but the former notation is found to be the more convenient.

39. We regret much that in the following problems on straight lines we cannot employ an homogeneous equation as $\frac{y}{b} + \frac{x}{a} = 1$. In algebraical geometry the formulas most in use are very simple, much more so indeed than they would be if homogeneous: moreover the advantage of a uniform system of symbols and formulas is so great to mathematicians that it should not be violated without very strong reasons. To remedy in some degree this want of regularity, the student should repeatedly consider the meaning of the constants at his first introduction to the subject of straight lines.

PROBLEMS ON STRAIGHT LINES.

40. To find the equation to a straight line passing through a given point.

The point being given its co-ordinates are known; let them be $x_1\,y_1$, and let the equation to the straight line be $y = \alpha x + b$; we signify that this line passes through the point $x_1\,y_1$, by saying that when the variable abscissa x becomes x_1, then y will become y_1: hence the equation to the line becomes

$$y_1 = \alpha x_1 + b$$

$$\therefore b = y_1 - \alpha x_1$$

substituting this value for b in the first equation, we have

$$y = \alpha x + y_1 - \alpha x_1$$

$$\text{or } y - y_1 = \alpha(x - x_1)$$

The shortest method of eliminating b is by subtracting the second equation from the first, and this is the method generally adopted.

Since α, which fixes the direction of the line, is not determined, there may be an infinite number of straight lines drawn through a given point; this is also geometrically apparent.

If the given point be on the axis of x, $y_1 = 0$, and $\therefore\ y = \alpha\,(x - x_1)$; and if it be on the axis of y, $x_1 = 0\ \therefore\ y - y_1 = \alpha\, x$.

If either or both of the co-ordinates of the given point be negative, the proper substitutions must be made: thus if the point be on the axis of x and in the negative direction from A, its co-ordinates will be $-\,x_1$ and 0; therefore the equation to the line passing through that point will be

$$y = \alpha\,(x + x_1).$$

41. To find the equation to a straight line passing through two given points x_1, y_1 and x_2, y_2.

Let the required equation be $y = \alpha\, x + b$ (1)
then since the line passes through the given points, we have the equations

$$y_1 = \alpha\, x_1 + b \qquad (2)$$

$$y_2 = \alpha\, x_2 + b \qquad (3)$$

Subtracting (2) from (1)

$$y - y_1 = \alpha\,(x - x_1) \qquad (4)$$

Subtracting (3) from (2)

$$y_1 - y_2 = \alpha\,(x_1 - x_2)$$

$$\therefore\ \alpha = \frac{y_1 - y_2}{x_1 - x_2}$$

Substituting this value of α in (4), we have finally

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}\,(x - x_1)$$

The two conditions have sufficed to determine α and b, and by their elimination the position of the line is fixed, as it ought to be, since only one straight line can be drawn through the same two points.

This equation will assume different forms according to the particular situation of the given points.

Thus if the point x_2, y_2 be on the axis of x, we have $y_2 = 0$;

$$\therefore\ y - y_1 = \frac{y_1}{x_1 - x_2}\,(x - x_1);$$

if it be on the axis of y, $x_2 = 0$; $\therefore\ y - y_1 = \dfrac{y_1 - y_2}{x_1}\,(x - x_1)$;

and if it be at the origin both y_2 and $x_2 = 0$;

$$\therefore\ y - y_1 = \frac{y_1}{x_1}(x - x_1) = \frac{y_1}{x_1}x - y_1$$

$$\therefore\ y = \frac{y_1}{x_1}\,x.$$

This last equation is also thus obtained; the line passing through the origin, its equation must be of the form $y = \alpha x$ (31) where α is the tangent of the angle which the line makes with the axis of x, and this line passing through the point x_1, y_1, α must be equal to $\frac{y_1}{x_1}$ $\therefore y = \frac{y_1}{x_1} x$.

If a straight line pass through three given points, the following relation must exist between the co-ordinates of those points:

$$(y_1 x_2 - x_1 y_2) - (y_1 x_3 - x_1 y_3) + (y_2 x_3 - x_2 y_3) = 0.$$

42. To find the equation to a straight line passing through a given point x_3, y_3, and bisecting a finite portion of a given straight line.

Let the portion of the straight line be limited by the points $x_1 y_1$ and $x_2 y_2$, and therefore the co-ordinates of the bisecting point are $\frac{x_1 + x_2}{2}$, $\frac{y_1 + y_2}{2}$; hence the required equation is

$$y - y_3 = \alpha (x - x_3) = \frac{y_1 + y_2 - 2 y_3}{x_1 + x_2 - 2 x_3} (x - x_3)$$

43. To find the equation to a straight line parallel to a given straight line.

$$\text{Let } y = \alpha x + b \quad (1) \text{ be the given line}$$
$$y = \alpha' x + b' \quad (2) \ldots \text{ required line}$$

then since the lines are parallel they must make equal angles with the axis of x or $\alpha' = \alpha$ $\therefore$ the required equation is

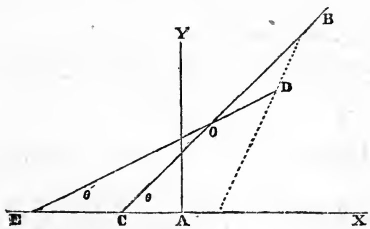
$$y = \alpha x + b' \quad (3)$$

Of course b' could not be determined by the single condition of the parallelism of the lines, since an infinite number of lines may be drawn parallel to the given line; but if another condition is added, b' will be then determined: thus if the required line passes also through a given point $x_1 y_1$, equation (2) is

$$y - y_1 = \alpha' (x - x_1)$$
$$\therefore (3) \text{ becomes } y - y_1 = \alpha (x - x_1)$$

44. To find the intersection of two given straight lines C B, E D.

This consists in finding the co-ordinates of the point O of intersection. Now it is evident that at this point they have the same abscissa and ordinate; hence if in the equations to two lines we regard x as representing the same abscissa and y the same ordinate, it is in fact saying that they are the co-ordinates X, Y of the point of intersection O.



Let $y = \alpha x + b$ be the equation to C B

and $y = \alpha' x + b'$ . . . . E D

then at O we have $Y = \alpha X + b = \alpha' X + b'$

$$\therefore X = \frac{b' - b}{\alpha - \alpha'}$$

$$\text{and } Y = \alpha X + b = \frac{\alpha b' - \alpha b}{\alpha - \alpha'} + b = \frac{\alpha b' - \alpha' b}{\alpha - \alpha'}.$$

Ex. 1. To find the intersection of the lines whose equations are

$y = 3x + 1$ and $y - 2x - 4 = 0$. $X = 3$ and $Y = 10$.

Ex. 2. To find the intersection of the lines whose equations are

$y - x = 0$ and $3y - 2x = 1$. $X = 1$ and $Y = 1$.

If a third line, whose equation is $y = \alpha'' x + b''$, passes through the point of intersection, the relation between the coefficients is

$$(\alpha b' - \alpha' b) - (\alpha b'' - \alpha'' b) + (\alpha' b'' - \alpha'' b') = 0.$$

45. To find the tangent, sine and cosine of the angle betwen two given straight lines.

Let $y = \alpha x + b$ be the equation to C B

$y = \alpha' x + b'$ . . . . E D

θ and θ' the angles which they make respectively with the axis of x; then

$$\text{tan. D O B} = \text{tan. E O C} = \text{tan. } (\theta - \theta') = \frac{\text{tan. } \theta - \text{tan. } \theta'}{1 + \text{tan. } \theta \text{ tan. } \theta'} = \frac{\alpha - \alpha'}{1 + \alpha\alpha'}$$

$$\text{also cos. D O B} = \frac{1}{\text{sec. D O B}} = \frac{1}{\sqrt{1 + (\text{tan. D O B})^2}} = \frac{1 + \alpha\alpha'}{\sqrt{(1+\alpha^2)(1+\alpha'^2)}}$$

$$\text{and sine D O B} = \text{tan. D O B} \times \text{cos. D O B} = \frac{\alpha - \alpha'}{\sqrt{1 + \alpha^2} \cdot \sqrt{1 + \alpha'^2}}.$$

46. To find the equation to a straight line making a given angle with another straight line.

Let $y = \alpha x + b$ be the given line C B,

$y = \alpha' x + b'$ required line E D,

$\beta =$ tangent of the given angle D O B.

Then $\alpha' = \text{tan. DEC} = \text{tan. (BCX} - \text{BOD)}$

$$= \frac{\text{tan. BCX} - \text{tan. BOD}}{1 + \text{tan. BCX . tan. BOD}} = \frac{\alpha - \beta}{1 + \alpha\beta}.$$

Substituting this value for α' in the second equation,

$$y = \frac{\alpha - \beta}{1 + \alpha\beta} x + b'.$$

If the required line passes also through a given point x_1, y_1, the equation is

$$y - y_1 = \frac{\alpha - \beta}{1 + \alpha\beta}(x - x_1).$$

If D be considered the given point x_1, y_1, then not only the line D O E but another (the dotted line in the figure) might be drawn, making a given angle with B C, and its equation is found, as above, to be

$$y - y_1 = \frac{\alpha + \beta}{1 - \alpha\beta}(x - x_1);$$

so that both lines are included in the equation

$$y - y_1 = \frac{\alpha \mp \beta}{1 \pm \alpha\beta}(x - x_1).$$

For example, the two straight lines which pass through the point D and cut B C at an angle of 45° are given by the equations

$$y - y_1 = \frac{\alpha - 1}{\alpha + 1}(x - x_1),$$

$$y - y_1 = \frac{1 + \alpha}{1 - \alpha}(x - x_1).$$

Also the equation to the straight line passing through D and cutting the axis of x at an angle of 135° is

$$y - y_1 = \beta(x - x_1) = \tan. 135°\,(x - x_1) = -(x - x_1),$$
or $y + x = y_1 + x_1$.

47. If the required line is to be perpendicular to the given line, β is infinitely great; therefore the fraction $\dfrac{\alpha - \beta}{1 + \alpha\beta} = \dfrac{\frac{\alpha}{\beta} - 1}{\frac{1}{\beta} + \alpha} = -\dfrac{1}{\alpha}$, or $\alpha' = -\dfrac{1}{\alpha}$; hence the equation to a straight line perpendicular to a given line $y = \alpha x + b$, is $y = -\dfrac{1}{\alpha}x + b'$.

This may be also directly proved, for drawing O E perpendicular to B C, as in the next figure, we have $\alpha' = \tan.$ O E X $= -\tan.$ O E C $= -\cot.$ O C X $= -\dfrac{1}{\alpha}$: hence in the equations to two straight lines which are perpendicular to one another we have $\alpha\alpha' + 1 = 0$; and, conversely, if in the equations to two straight lines, we find $\alpha\alpha' + 1 = 0$, these lines are perpendicular to one another.

If the perpendicular line pass also through a given point $x_1\,y_1$, its equation is

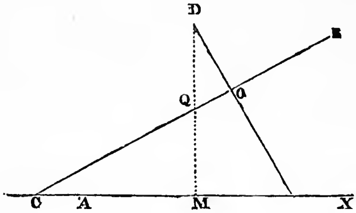
$$y - y_1 = -\frac{1}{\alpha}(x - x_1);$$

and, of course, this equation will assume various forms, agreeing with the position of the point $x_1\,y_1$; thus, for example, the line drawn through the origin perpendicular to the line $y = \alpha x + b$, is one whose equation is $y = -\dfrac{1}{\alpha}x$, for here both x_1 and $y_1 = 0$.

48. To find the length of a perpendicular from a given point D $(x_1\,y_1)$ on a given straight line C B.

Let $y = \alpha x + b$ (1) be the equation to C B,

then $y - y_1 = - \dfrac{1}{\alpha}(x - x_1)$ (2) is the equation to the perpendicular line D O E,



Let $p =$ D O; then if X and Y be the co-ordinates of O determined from (1) and (2) we have $p^2 = (X - x_1)^2 + (Y - y_1)^2$; (29)

from (2) $Y = y_1 - \dfrac{1}{\alpha}(X - x_1) = \alpha X + b$ from (1)

$$= \alpha(X - x_1) + \alpha x_1 + b,$$

$$\therefore \left(\alpha + \frac{1}{\alpha}\right)(X - x_1) = y_1 - \alpha x_1 - b,$$

$$\therefore X - x_1 = \frac{\alpha}{1+\alpha^2}(y_1 - \alpha x_1 - b), \text{ also } Y - y_1 = -\frac{1}{\alpha}(X - x_1)$$

$$\therefore p^2 = (X - x_1)^2 + (Y - y_1)^2$$

$$= (X - x_1)^2 + \frac{1}{\alpha^2}(X - x_1)^2$$

$$= \frac{1+\alpha^2}{\alpha^2}(X - x_1)^2$$

$$= \frac{1+\alpha^2}{\alpha^2}\,\frac{\alpha^2}{(1+\alpha^2)^2}\cdot(y_1 - \alpha x_1 - b)^2 = \frac{1}{1+\alpha^2}(y_1 - \alpha x_1 - b)^2$$

$$\therefore p = \pm \frac{y_1 - \alpha x_1 - b}{\sqrt{1+\alpha^2}}.$$

The superior sign is to be taken when the given point is above the given straight line, and the inferior in the contrary case.

If the given line pass through the origin $b = 0$; $\therefore p = \pm \dfrac{y_1 - \alpha x_1}{\sqrt{1+\alpha^2}}$.

If the origin be the given point, $x_1 = 0$ and $y_1 = 0$; $\therefore p = \dfrac{\pm b}{\sqrt{1+\alpha^2}}$.

There is another way of obtaining the expression for p.

Since the equation $y = \alpha x + b$ applies to all points in C O B, it must to Q, where M D or y_1 cuts C O B; $\therefore$ M Q $= \alpha x_1 + b$.

Now D O = D Q sin. D Q O,

but DQ = D M − M Q $= y_1 - \alpha x_1 - b$,

and sin. D Q O = sin. C Q M = cos. Q C M $= \dfrac{\;}{\text{sec. } QCM} =$

$$\frac{1}{\sqrt{1 + (\tan.\ QCM)^2}} = \frac{1}{\sqrt{1+\alpha^2}};$$

$\therefore$ D O or $p = \dfrac{y_1 - \alpha x_1 - b}{\sqrt{1 + \alpha^2}}$, or $\dfrac{\alpha x_1 + b - y_1}{\sqrt{1 + \alpha^2}}$, if D was below the line.

49. If the line D E had been drawn making a given angle whose tangent was β with the given line C O, the distance D O might be found; for instead of equation (2) we shall have

$$y - y_1 = \frac{\alpha - \beta}{1 + \alpha\beta}(x - x_1) \quad (46);$$

hence, following the same steps as above, we shall find

$$p = \pm \frac{y_1 - \alpha x_1 - b}{\sqrt{1 + \alpha^2}} \cdot \frac{\sqrt{1 + \beta^2}}{\beta}.$$

This expression is also very easily obtained trigonometrically.

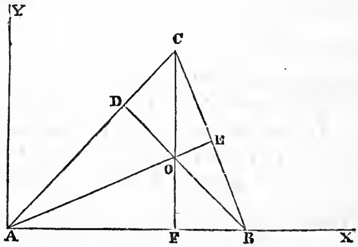
Let γ = sine of the given angle, then

$$DO = DQ \cdot \frac{\sin.\ DQO}{\sin.\ DOQ}$$

$$= \frac{y_1 - \alpha x_1 - b}{\sqrt{1 + \alpha^2}} \cdot \frac{1}{\gamma}.$$

50. The equation to the straight line may be used with advantage in the demonstration of the following theorem:—

If from the angles of a plane triangle perpendiculars be let fall on the opposite sides, these perpendiculars will meet in one point.



In the triangle A B C, let A E, B D, C F be perpendiculars from A, B and C on the opposite sides; let O be the point where A E and B D meet, then the theorem will be established by showing that the abscissa to the point O is A F.

Let A be the origin of co-ordinates,
A B the axis of x,
and A Y, perpendicular to A B, the axis of y.

Let the co-ordinates of C be x_1, y_1
.... B $x_2, 0$:

we have then the following equations,

$$\text{to AC} \quad y = \frac{y_1}{x_1} x \quad (41)$$

$$\therefore \text{ to BD, } y = \alpha(x - x_2) = -\frac{x_1}{y_1}(x - x_2) \quad (47)$$

$$\text{to BC, } y - y_2 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_2) \quad (41)$$

$$\text{or } y = \frac{y_1}{x_1 - x_2}(x - x_2) \text{ since } y_2 = 0.$$

$$\therefore \text{ to A E, } y = \alpha x = -\frac{x_1 - x_2}{y_1} x;$$

for the intersection O of B D and A E we have, by equating the values of y,

$$-\frac{x_1}{y_1}(x - x_2) = -\frac{x_1 - x_2}{y_1} x,$$

$$\therefore x_1 x - x_1 x_2 = x_1 x - x_2 x; \quad \therefore x_2 x = x_1 x_2 \text{ and } x = x_1,$$

that is, the abscissa of the point O is found to be that of C.

In the same manner it may be proved that if perpendiculars be drawn from the bisections of the sides, they will meet in one point.

Similarly we may prove that the three straight lines F C, K B, and A L, in the 47th proposition of Euclid, meet in one point within the triangle A B C.

51. We have hitherto considered the axes as rectangular, but if they be oblique, the coefficient of x, in the equation to a straight line, is not the tangent of the angle which the line makes with the axis of x.

Let ω = the angle between the axes,

θ = the angle which the line makes with the axis of x;

$$\text{then } \alpha = \frac{y-b}{x} = \frac{\sin.\ \theta}{\sin.\ (\omega - \theta)} \quad (33);$$

b remains, as before, the distance of the origin from the intersection of the line with the axis of y: hence the equation to a straight line referred to oblique axes is

$$y = \frac{\sin.\ \theta}{\sin.\ (\omega - \theta)} x + b$$

Since this equation is of the form $y = \alpha x + b$ all the results in the preceding articles which do not affect the ratio of $\frac{\sin.\ \theta}{\sin.\ (\omega - \theta)}$ will be equally true when the axes are oblique.

Thus, articles 40, 41, 42, 43, and 44, require no modification.

To find the tangent (β) of the angle between two given straight lines.

Let $\left.\begin{array}{l} y = \alpha x + b \\ y = \alpha' x + b' \end{array}\right\}$ be the equations;

from $\alpha = \frac{\sin.\ \theta}{\sin.\ (\omega - \theta)}$ we have $\tan.\ \theta = \frac{\alpha \sin.\ \omega}{1 + \alpha \cos.\ \omega}$; and, similarly,

$\tan.\ \theta' = \frac{\alpha' \sin.\ \omega}{1 + \alpha' \cos.\ \omega}$; hence $\beta = \tan.\ (\theta - \theta') = \frac{(\alpha - \alpha') \sin.\ \omega}{1 + \alpha\alpha' + (\alpha + \alpha') \cos.\ \omega}$

To find the equation to a straight line passing through a given point $x_1 y_1$, and making a given angle with a given straight line.

Let β be the tangent of the given angle,

$y = \alpha x + b$, the given line,

$y - y_1 = \alpha' (x - x_1)$, the required line.

From the last formula we have

$$\alpha' = \frac{\alpha \sin.\ \omega - \beta (1 + \alpha \cos.\ \omega)}{\sin.\ \omega + \beta (\alpha + \cos.\ \omega)}$$

and the required equation is

$$\therefore y - y_1 = \frac{\alpha \sin.\ \omega - \beta (1 + \alpha \cos.\ \omega)}{\sin.\ \omega + \beta (\alpha + \cos.\ \omega)} (x - x_1).$$

If the lines be perpendicular to each other $\beta = \frac{1}{0}$;

$$\therefore \alpha' = -\frac{1+\alpha \cos. \omega}{\alpha + \cos. \omega}.$$

and the required equation is

$$y - y_1 = -\frac{1+\alpha \cos. \omega}{\alpha + \cos. \omega}(x - x_1).$$

To find the length of the perpendicular from a given point upon a given straight line.

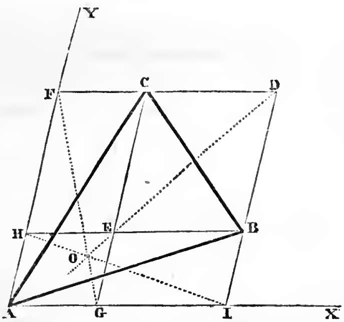
Instead of equation (2), in article 48, we must use the equation just found, and then proceeding as usual we shall find

$$p = \pm \frac{(y_1 - \alpha x_1 - b) \sin. \omega}{\sqrt{\{1 + 2 \alpha \cos. \omega + \alpha^2\}}}.$$

It will be concluded from an observation of these formulas, that oblique axes are to be avoided as much as possible; they may be used with advantage where points and lines, but not angles, are the subjects of discussion. As an instance, we shall take the following theorem.

52. If, upon the sides of a triangle as diagonals, parallelograms be described, having their sides parallel to two given lines, the other diagonals of the parallelograms will intersect each other in the same point.

Let A B C be the triangle, A X, A Y the given lines, E B D C, C F A G,



H A I B the parallelograms, the opposite diagonals D E, F G, and H I will meet in one point O.

Let A be the origin, A X, A Y the oblique axes

$x_1 y_1$ the co-ordinates of B

$x_2 y_2$. . . . . C.

To find the equation to D E;

let it be $y = \alpha x + b$

$y_2 = \alpha x_1 + b$ at D

$\therefore y - y_2 = \alpha (x - x_1)$

$y_1 - y_2 = \alpha (x_2 - x_1)$ at E

$$\therefore y - y_2 = \frac{y_1 - y_2}{x_2 - x_1}(x - x_1) \quad (1)$$

To find the equation to F G;

$$y = \alpha x + b$$
$$y_2 = 0 + b \text{ at F}$$
$$\therefore y - y_2 = \alpha x$$
$$0 - y_2 = \alpha x_2 \text{ at G}$$
$$\therefore y - y_2 = -\frac{y_2}{x_2} x \quad (2).$$

To find the equation to H I;

$$y = \alpha x + b$$
$$y_1 = 0 \quad + b \text{ at H}$$
$$\therefore y - y_1 = \alpha x$$
$$0 - y_1 = \alpha x_1 \text{ at I}$$
$$\therefore y - y_1 = -\frac{y_1}{x_1} x \quad (3).$$

Equating the values of y in (1) and (2) we find $X = \dfrac{x_1 x_2 (y_1 - y_2)}{y_1 x_2 - x_1 y_2}$;
also equating the values of y in (2) and (3) we find the same value for X; hence the abscissa for intersection being the same for any two of the lines, they must all three intersect in the same point.

Similarly we may prove that if from the angles of a plane triangle straight lines be drawn to the bisections of the opposite sides, they will meet in one point.

CHAPTER IV.

THE TRANSFORMATION OF CO-ORDINATES.

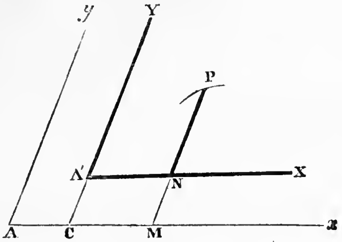
53. Before we proceed to the discussion of equations of higher orders, it is necessary to investigate a method for changing the position of the co-ordinate axes.

The object is to place the axes in such a manner that the equation to a given curve may appear in its most simple form, and conversely by the introduction of indeterminate constants into an equation to reduce the number of terms, so that the form and properties of the corresponding locus may be most easily detected.

An alteration of this nature cannot in the least change the form of the curve, but only the algebraical manner of representing it; thus the general equation to the straight line $y = \alpha x + b$ becomes $y = \alpha x$ when the origin is on the line itself. Also on examining articles 46 and 51 we see that the simplicity of an equation depends very much on the angle between the axes.

Hence in many cases not only the position of the origin but also the direction of the axes may be altered with advantage. The method of performing these operations is called the transformation of co-ordinates.

54. To transform an equation referred to an origin A, to an equation referred to another origin A′, the axes in the latter case being parallel to those in the former.



Let $A\,x$, $A\,y$ be the original axes

$A'X$, $A'Y$ the new axes

$\left.\begin{array}{l} AM = x \\ MP = y \end{array}\right\}$ original co-ordinates of P

$\left.\begin{array}{l} A'N = X \\ NP = Y \end{array}\right\}$ new co-ordinates of P

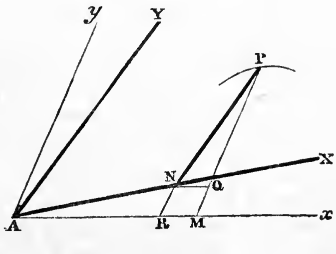
$\left.\begin{array}{l} AC = a \\ CA' = b \end{array}\right\}$ the co-ordinates of the new origin A′,

then $MP = MN + NP$, that is, $y = b + Y$,

$AM = AC + CM$, ... $x = a + X$;

substituting these values for y and x in the equation to the curve, we have the transformed equation between Y and X referred to the origin A′.

55. To transform the equation referred to oblique axes, to an equation referred to other oblique axes having the same origin.



Let $A\,x$, $A\,y$ be the original axes,

AX, AY be the new axes,

$\left.\begin{array}{l} AM = x \\ MP = y \end{array}\right\}$ original co-ordinates of P.

$\left.\begin{array}{l} AN = X \\ NP = Y \end{array}\right\}$ new co-ordinates of P

Let the angle $x\,A\,y = \omega$, $x\,A\,X = \theta$, $x\,A\,Y = \theta'$;

Draw N R parallel to P M, and N Q parallel to A M,

then $y = MP = MQ + QP = NR + QP$

$$= AN\,\frac{\sin.\ NAR}{\sin.\ NRA} + PN\,\frac{\sin.\ PNQ}{\sin.\ PQN}$$

$$= X\,\frac{\sin.\ \theta}{\sin.\ \omega} + Y\,\frac{\sin.\ \theta'}{\sin.\ \omega};$$

and $x = AM = AR + RM = AR + NQ$

$$= AN\,\frac{\sin.\ ANR}{\sin.\ ARN} + PN\,\frac{\sin\ NPQ}{\sin.\ NQP}$$

$$= X\,\frac{\sin.\ (\omega - \theta)}{\sin.\ \omega} + Y\,\frac{\sin.\ (\omega - \theta')}{\sin.\ \omega}$$

$$\therefore\ y = \frac{X \sin.\ \theta + Y \sin.\ \theta'}{\sin.\ \omega}$$

$$x = \frac{X \sin.\ (\omega - \theta) + Y \sin.\ (\omega - \theta')}{\sin.\ \omega}.$$

56. Let the original axes be oblique, and the new rectangular, or $\theta' - \theta = 90°$.

$$\therefore\ y = \frac{X\ \sin.\ \theta + Y\ \cos.\ \theta}{\sin.\ \omega}$$

$$x = \frac{X \sin.\ (\omega - \theta) - Y \cos.\ (\omega - \theta)}{\sin.\ \omega}.$$

57. Let the original axes be rectangular, or $\omega = 90°$.

$$\therefore\ y = X \sin.\ \theta + Y \sin.\ \theta',$$
$$x = X \cos.\ \theta + Y \cos.\ \theta'$$

58. Let both systems be rectangular, or $\omega = 90°$ and $\theta' - \theta = 90°$

$$\therefore\ y = X \sin.\ \theta + Y \cos.\ \theta,$$
$$x = X \cos.\ \theta - Y \sin.\ \theta$$

59. These forms have been deduced from the first, but each of them may be found by a separate process. The first and last pairs are the most useful. Perhaps they may be best remembered if expressed in the following manner.

Both systems oblique, the formulas (55) become

$$y = \{X \sin.\ X A x + Y \sin.\ Y A x\}\,\frac{1}{\sin.\ x A y}$$

$$x = \{X \sin.\ X A y + Y \sin.\ Y A y\}\,\frac{1}{\sin.\ x A y}.$$

Both systems rectangular, the formulas (58) become

$$y = X \cos.\ X A y + Y \cos.\ Y A y$$
$$x = X \cos.\ X A x + Y \cos.\ Y A x.$$

If the situation of the origin be changed as well as the direction of the axes, we have only to add the quantities a and b to the values of x and y respectively; however, in such a case, it is most convenient to perform each transformation separately.

If the new axis of X falls below the original axis of x, the angle θ must be considered as negative, therefore its sine will be negative and its cosine positive. Hence the formulas of transformation will require a slight alteration before applied to this particular case.

Since the values of x and y are in all cases expressed by equations of the first order, the degree of an equation is never changed by the transformation of co-ordinates.

60. Hitherto we have determined the situation of a point in a plane by its distance from two axes, but there is also another method of much use. Let S be a fixed point, and S B a fixed straight line; then the point P is also evidently determined if we know the length S P and the angle P S B.

If S P $= r$ and P S B $= \theta$, r and θ are called the polar co-ordinates of P. S is called the pole, and S P the radius vector, because a curve may be supposed to be described by the extremity of the line S P revolving round S, the length of S P being variable. The fixed straight line S B is also called the axis.

To transform an equation between co-ordinates x and y into another between polar co-ordinates r and θ,

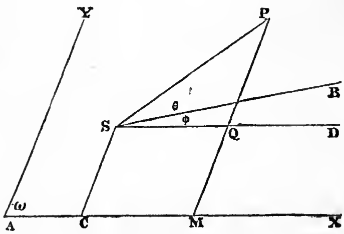
Draw S D parallel to A X, and let the angle B S D $= \phi$, and the angle Y A X $= \omega$.

Let A M $= x$, M P $= y$, A C $= a$, C S $= b$,

$$\left.\begin{aligned} \text{then } y = \text{M P} = \text{M Q} + \text{Q P} = b + r\frac{\sin.\ (\theta+\phi)}{\sin.\ \omega} \\ x = \text{A M} = \text{A C} + \text{S Q} = a + r\frac{\sin.\{\omega-(\theta+\phi)\}}{\sin.\ \omega.} \end{aligned}\right\} \quad (1)$$

Let S B coincide with S D, or $\phi = 0$;

$$\left.\begin{aligned} \therefore\ y = b + r\frac{\sin.\ \theta}{\sin.\ \omega} \\ x = a + r\frac{\sin.\ (\omega-\theta)}{\sin.\ \omega} \end{aligned}\right\} \quad (2)$$



61. Let the original axes be also rectangular, or $\omega = \frac{\pi}{2}$;

$$\left.\begin{aligned} \therefore\ y = b + r\sin.\ \theta \\ x = a + r\cos.\ \theta \end{aligned}\right\} \quad (3)$$

and if the origin A be the pole, we have $a = 0$ and $b = 0$.

$$\left.\begin{aligned} \therefore\ y = r\sin.\ \theta \\ x = r\cos.\ \theta \end{aligned}\right\} (4).$$

Of these formulas (3) and (4) are the most useful.

62. Conversely, to find r and θ in terms of x and y: from (1) we have

$$\frac{x-a}{y-b} = \frac{\sin.\{\omega-(\theta+\phi)\}}{\sin.\ (\theta+\phi)} = \sin.\ \omega\cot.\ (\theta+\phi) - \cos.\ \omega;$$

$$\therefore\ \tan.\ (\theta+\phi) = \frac{(y-b)\sin.\ \omega}{x-a+(y-b)\cos.\ \omega}$$

$$\therefore\ \theta+\phi = \tan.^{-1}\left\{\frac{(y-b)\sin.\ \omega}{x-a+(y-b)\cos.\ \omega}\right\},$$

where the symbol $\tan.^{-1}a$ is equivalent to the words "a circular arc whose radius is unity, and tangent a."

also $r^2 = (x-a)^2 + (y-b)^2 + 2(x-a)(y-b)\cos.\ \omega \ldots (30)$

63. If the axes are rectangular, or $\omega = \frac{\pi}{2}$, the pole at the origin, and therefore $a = 0$ and $b = 0$, and also $\phi = 0$, we have $\frac{y}{x} = \tan.\theta$, and therefore

$$\theta = \tan.^{-1}\frac{y}{x}$$

$$\text{and } r^2 = x^2 + y^2 \ldots . (29)$$

and these are the formulas generally used.

From $\tan.\theta = \frac{y}{x}$ we have $\cos.\theta = \frac{1}{\sqrt{1+(\tan.\theta)^2}} = \frac{1}{\sqrt{1+\frac{y^2}{x^2}}} = \frac{x}{\sqrt{y^2+x^2}}$ and $\sin.\theta = \cos.\theta \times \tan.\theta = \frac{y}{\sqrt{y^2+x^2}}$; hence the value of θ may also be expressed by the equations

$$\theta = \sin.^{-1}\frac{y}{\sqrt{y^2+x^2}} \text{ or } \theta = \cos.^{-1}\frac{x}{\sqrt{y^2+x^2}}.$$

CHAPTER V.

ON THE CIRCLE.

64. Following the order of this treatise, our next subject of discussion would be the loci of the general equation of the second degree; but there is one curve among these loci, remarkable for the facility of its description and the simplicity of its equation: this curve, we need scarcely say, is the circle; and as the discussion of the circle is admirably fitted to prepare the reader for other investigations, we proceed to examine its analytical character.

The common definition of the circle states, that the distance of any point on the circumference of the figure from the centre is equal to a given line called the radius.

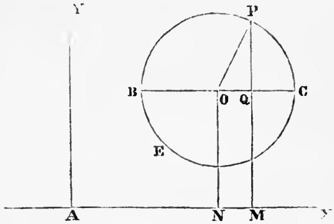
If a and b be the co-ordinates of the centre, x and y those of any point on the circumference, and r the radius, the distance between those points is $\sqrt{\{(y-b)^2 + (x-a)^2\}}$ (29): hence the equation to the circle is

$$(y-b)^2 + (x-a)^2 = r^2$$

65. To obtain this equation directly from the figure,
let A be the origin,
A X, A Y the rectangular axes,
$\left.\begin{array}{l} AN = a \\ NO = b \end{array}\right\}$ the co-ordinates of the centre,

$\left.\begin{array}{l} AM = x \\ MP = y \end{array}\right\}$ those of any point P on the circumference;

B O Q C a line parallel to the axis of x.



Then the square upon O P = the square upon P Q + the square upon O Q,

and $PQ = PM - QM = y - b$

also $OQ = AM - AN = x - a$;

$\therefore r^2 = (y - b)^2 + (x - a)^2$

or $(y - b)^2 + (x - a)^2 = r^2$ (1)

If the axis of x or that of y passes through the centre, the equation (1) becomes respectively

$$\left.\begin{array}{r} (y - b)^2 + x^2 = r^2 \\ \text{or } y^2 + (x - a)^2 = r^2 \end{array}\right\} (2)$$

If the origin be at any point of the circumference as E, we have then the equation of condition $a^2 + b^2 = r^2$; expanding (1) and reducing it by means of this condition, we have

$$y^2 - 2by + x^2 - 2ax = 0 \text{ (3).}$$

If the origin is at B, B O being the axis of x, we have $b = 0$ and $a = r$

$$\therefore y^2 + x^2 - 2rx = 0$$

$$\text{or } y^2 = 2rx - x^2 \text{ (4).}$$

Again, placing the origin at the centre O, we have $b = 0$ and $a = 0$;

$$\therefore y^2 + x^2 = r^2 \text{ (5).}$$

The above equations are all useful, but those most required are (1), (4), and (5).

66. Equation (1), if expanded, is

$$y^2 + x^2 - 2by - 2ax + a^2 + b^2 - r^2 = 0.$$

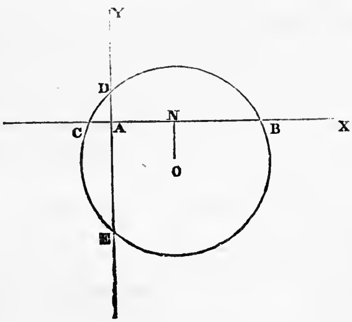
This differs from the complete equation of the second order (23) in having the coefficients of x^2 and y^2 unity, and by having no term containing the product xy.

Any equation of this form being given, we can, by comparing it with the above equation, determine the situation of its locus, that is, find the position of the centre, and the magnitude of the corresponding circle.

Ex. 1. $y^2 + x^2 + 4y - 8x - 5 = 0$.

here $b = -2$, $a = 4$, and $a^2 + b^2 - r^2 = -5$;

$$\therefore r^2 = a^2 + b^2 + 5 = 25.$$



Let A be the origin of co-ordinates, A X, A Y the axes.

In AX take AN=4 times the linear unit, from N draw NO perpendicular to A X, but downwards, and equal to 2; then O is the centre of the circle. With centre O and radius 5 describe a circle; this is the locus required. The points where it cuts the axis of x are determined by putting $y=0$;

$$\therefore x^2 - 8x - 5 = 0;$$
$$\therefore x = 4 \pm \sqrt{21};$$
$$\text{hence } AB = 4 + \sqrt{21} \text{ and } AC = 4 - \sqrt{21}.$$

Similarly putting $x = 0$, we find A D = 1 and A E = 5.

67. The shortest way of describing the locus is to put the equation into the form $(y-b)^2 + (x-a)^2 = r^2$.

For example, the equation

$$y^2 + x^2 + cy + dx + e = 0,$$

becomes, by the addition and subtraction of $\frac{c^2}{4}$ and $\frac{d^2}{4}$,

$$y^2 + cy + \frac{c^2}{4} + x^2 + dx + \frac{d^2}{4} + e - \frac{c^2}{4} - \frac{d^2}{4} = 0,$$

$$\text{or } (y + \frac{c}{2})^2 + (x + \frac{d}{2})^2 = \frac{c^2 + d^2}{4} - e,$$

where we observe directly that $-\frac{d}{2}$ and $-\frac{c}{2}$ are the co-ordinates of the centre, and that $\sqrt{\left\{\frac{c^2+d^2}{4} - e\right\}}$ is the radius of the required locus.

Ex. 2. $y^2 + x^2 + 4y - 4x - 8 = 0$

add and subtract 8, and the equation becomes

$$y^2 + 4y + 4 + x^2 - 4x + 4 - 16 = 0$$
$$\text{or } (y+2)^2 + (x-2)^2 = 16$$

hence the co-ordinates of the centre are $a = 2$ and $b = -2$, and the radius is 4.

Ex. 3. $2y^2 + 2x^2 - 4y - 4x + 1 = 0$; $a = 1$, $b = 1$, $r = \sqrt{\frac{3}{2}}$.

4. $y^2 + x^2 - 6y + 4x - 3 = 0$; $a = -2$, $b = 3$, $r = 4$.

5. $6y^2 + 6x^2 - 21y - 8x + 14 = 0$; $a = -\frac{2}{3}$, $b = \frac{7}{4}$, $r = \frac{13}{12}$.

6. $y^2 + x^2 + 4y - 3x = 0$; $a = \frac{3}{2}$, $b = -2$, $r = \frac{5}{2}$.

7. $y^2 + x^2 - 4y + 2x = 0$; $a = -1$, $b = 2$, $r = \sqrt{5}$.

In these last two examples there is no occasion to calculate the length of the radius, for the circumference of the circle passes through the origin of co-ordinates, as do the loci of all equations which want the last or constant term.

8. $y^2 + x^2 - 4y = 0$; $a = 0$, $b = 2$, $r = 2$.

9. $y^2 + x^2 + 6x = 0$; $a = -3$, $b = 0$, $r = 3$.

10. $y^2 + x^2 - 6x + 8 = 0$; $a = 3$, $b = 0$, $r = 1$.

In the last three examples the centre of the circle is on the axes.

68. We have seen that the equation to the circle referred to rectangular axes does not contain the product $x y$, and also that the coefficients of y^2 and x^2 are each unity; we have, moreover, seen that generally an equation of the second degree of this form has a circle for its locus, but there are some exceptions to this last rule.

For example, the equation $y^2 + x^2 - 8y - 12x + 52 = 0$ is apparently of the circular form; its locus, however, is not a circle, but a point whose co-ordinates are $x = 6$ and $y = 4$, for it may be put under the form $(y - 4)^2 + (x - 6)^2 = 0$, the only real solution of which is $x = 6$ and $y = 4$; and this will always be the case when $r^2 = 0$, hence a point may be considered as a circle whose radius is indefinitely small.

Again, the equation $y^2 + x^2 - 4y + 2x + 9 = 0$, may be put under the form $(y-2)^2 + (x+1)^2 = -4$; but there are no possible values of x and y that can satisfy this equation, therefore the locus is imaginary. (24).

69. To find the equation to the tangent to a circle.

Let the origin of co-ordinates be at the centre, and x', y' any point on the circumference.

Then the equation to the straight line through x', y' is

$$y - y' = \alpha (x - x');$$

the equation to the radius through x', y' is $y = \dfrac{y'}{x'} x$;

but the tangent being perpendicular to the radius, we have $\alpha = -\dfrac{x'}{y'}$ (47.)

$$\therefore y - y' = -\frac{x'}{y'}(x - x'),$$

$$\text{or } y y' - y'^2 = - x x' + x'^2;$$

$$\therefore y y' + x x' = y'^2 + x'^2$$

$$= r^2.$$

The equation $y y' + x x' = r^2$, thus found, may be easily remembered, from the similarity of its form to that of the equation to the circle, it being obtained at once from $y^2 + x^2 = r^2$ by changing y^2 or $y y$ into $y y'$, and x^2 or $x x$ into $x x'$.

If we take the general equation to the circle, $(y - b)^2 + (x - a)^2 = r^2$, the equation to the radius is

$$y - b = \frac{y' - b}{x' - a}(x - a) \ldots . (41)$$

$$\therefore \alpha = -\frac{x' - a}{y' - b}, \text{ and the equation to the tangent is}$$

$$y - y' = \alpha (x - x') = -\frac{x' - a}{y' - b}(x - x')$$

The equation $(y' - b)^2 + (x' - a)^2 = r^2$ enables us finally to reduce the equation to the tangent to the form

$$(y - b)(y' - b) + (x - a)(x' - a) = r^2$$

70. To find the equation to the tangent of a circle parallel to a given straight line.

Let $y = \alpha x + b$ be the given line,
and $y y' + x x' = r^2$ the required tangent, in which x', y' are unknown.

Since these lines are parallel, $-\dfrac{x'}{y'} = \alpha$, (43) or $-\dfrac{\sqrt{r^2 - y'^2}}{y'} = \alpha$,

$\therefore\ y' = \pm \dfrac{r}{\sqrt{1 + \alpha^2}}$.

Hence by substitution in the equation $y = -\dfrac{x'}{y'}x + \dfrac{r^2}{y'}$, we have

$$y = \alpha x \pm r\sqrt{1 + \alpha^2};$$

consequently two tangents can be drawn parallel to the given line.

71. To find the intersection of a straight line and circle:

Let the centre of the circle be the common origin, and let the equations be $y = \alpha x + b$, and $y^2 + x^2 = r^2$; at the point of intersection, y and x must be the same for both. $r^2 - x^2 = (\alpha x + b)^2$,

$$\text{whence } x = \frac{-\alpha b \pm \sqrt{\{r^2(1+\alpha^2) - b^2\}}}{1 + \alpha^2};$$

there being two values of x, we have two intersections; these values may be constructed, and the points of intersection found.

If $r^2(1+\alpha^2) = b^2$ the two values of x are equal, and the line will touch the circle. If $r^2(1+\alpha^2)$ is less than b^2 the line will not meet the circle.

Ex. 1. $y^2 + x^2 = 25$, $y + x = 1$, $x = 4$ and -3, $y = -3$ and 4

Ex. 2. $y^2 + x^2 = 25$, $y + x = 5$; $x = 5$ and 0, $y = 0$ and 5

Ex. 3. $y^2 + x^2 = 25$ $4y + 3x = 25$: The line touches the circle.

We may observe that the combination of an equation of the first order with any equation of two dimensions will, as above, give an equation of the second order for solution; and hence there can be only two intersections of their loci.

72. If the axes be oblique and inclined to each other at an angle ω, the equation to the circle is

$$(y - b)^2 + (x - a)^2 + 2(y - b)(x - a)\cos.\ \omega = r^2, \qquad (30)$$

and $y^2 + x^2 + 2xy\cos.\ \omega = r^2$, if the origin be at the centre; hence the equation $y^2 + cxy + x^2 + dy + ex + f = 0$, belongs to the circle in the particular case where the co-ordinate angle is one whose cosine $= \dfrac{c}{2}$.

Comparing it with the general equation to the circle, we find

$$2\cos.\ \omega = c, \qquad -2b - 2a\cos.\ \omega = d,$$

$$-2a - 2b\cos.\ \omega = e, \qquad a^2 + b^2 + 2ab\cos.\ \omega - r^2 = f;$$

whence, by elimination, we obtain $a = \dfrac{2e - cd}{c^2 - 4}$, $b = \dfrac{2d - ce}{c^2 - 4}$,

$$\text{and } r^2 = \frac{ced - e^2 - d^2}{c^2 - 4} - f;$$

nce the co-ordinates of the centre and the radius being known, the s can be drawn.

Ex. 1. $y^2 + xy + x^2 + y + x - 1 = 0$;

$2\cos.\ \omega = 1$; $\therefore \omega = 60°$; hence this equation will give a circle if the axes be inclined at an angle of $60°$; the co-ordinates of the centre are $a = -\frac{1}{3}$, $b = -\frac{1}{3}$; and the radius $= \frac{2}{\sqrt{3}}$

The equation to this circle, when referred to the centre as origin, and to rectangular axes, is obviously $y^2 + x^2 = r^2 = \frac{4}{3}$.

Ex. 2. $y^2 + \sqrt{2} \,.\, xy + x^2 - 9 = 0$.

This will give a circle if the axes be inclined at an angle of $45°$, the centre is at the origin of co-ordinates, and the radius $= 3$.

Of course c must never be equal to, or greater than, ± 2, for cos. ω must be less than unity.

If the circle be referred to oblique co-ordinates, the equation to the radius is $y - b = \frac{y' - b}{x' - a}(x - a) \ldots . (41)$

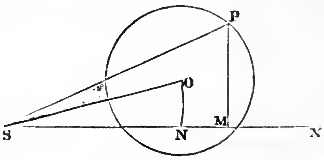
and the equation to the tangent is

$$y - y' = -\frac{(x' - a) + (y' - b)\cos.\omega}{(y' - b) + (x' - a)\cos.\omega}(x - x') \ldots . (51)$$

and reducing as in article 69 we have the equation to the tangent

$$(y - b)(y' - b) + (x - a)(x' - a) + (x - a)(y' - b)\cos.\omega + (x' - a)(y - b)\cos.\omega = r^2.$$

73. To find the polar equation to the circle.



Let the pole be at the origin S, and the angle P S M ($= \theta$) be measured from the axis of x.

Let $\left.\begin{array}{l} SM = x \\ MP = y \end{array}\right\}$ be rectangular co-ordinates of P

and $\left.\begin{array}{l} SN = a \\ NO = b \end{array}\right\}$ .... O

Let $SP = u$, $SO = c$, and angle $OSX = \alpha$; then by the formulas (61) or by the figure $y = u \sin.\theta$, $x = u \cos.\theta$, $a = c \cos.\alpha$, and $b = c \sin.\alpha$.

Substituting these values of x and y in the equation to the circle,

$$y^2 + x^2 - 2by - 2ax + a^2 + b^2 - r^2 = 0,$$

we have

$$u^2(\sin.\theta)^2 + u^2(\cos.\theta)^2 - 2cu\sin.\alpha\sin.\theta - 2cu\cos.\alpha\cos.\theta + c^2(\cos.\alpha)^2 + c^2(\sin.\alpha)^2 - r^2 = 0,$$

$$\text{or} \quad u^2 - 2cu\{\sin.\theta\sin.\alpha + \cos\theta\cos.\alpha\} + c^2 - r^2 = 0,$$

$$\text{or} \quad u^2 - 2cu\cos.(\theta - \alpha) + c^2 - r^2 = 0.$$

74. If a and b are not expressed in terms of the polar co-ordinates c and α, the polar equation is then of the form

$$u^2 - 2\{b\sin.\theta + a\cos.\theta\}u + a^2 + b^2 - r^2 = 0.$$

If the origin be on the circumference we have $a^2 + b^2 = r^2$, and therefore the equation to the circle becomes

$$u = 2\,(b \sin.\,\theta + a \cos.\,\theta)$$

If the axis of x passes through the centre, $b = 0$, and the equation is

$$u^2 - 2\,a\,u \cos.\,\theta + a^2 - r^2 = 0.$$

$$\text{Whence } u = a \cos.\,\theta \pm \sqrt{r^2 - a^2\,(\sin.\,\theta)^2}\,;$$

which equation may also be directly obtained from the triangle S P O.

CHAPTER VI.

DISCUSSION OF THE GENERAL EQUATION OF THE SECOND ORDER.

75. The most general form in which this equation appears is

$$a\,y^2 + b\,x\,y + c\,x^2 + d\,y + e\,x + f = 0\,;$$

where a, b, c, &c., are constant coefficients.

Let the equation be solved with respect to y and x separately, then

$$y = -\frac{b\,x + d}{2\,a} \pm \frac{1}{2\,a}\sqrt{\{(b^2 - 4\,a\,c)\,x^2 + 2\,(b\,d - 2\,a\,e)\,x + d^2 - 4\,a\,f\}}; \quad (1)$$

$$x = -\frac{by + e}{2\,c} \pm \frac{1}{2\,c}\sqrt{\{(b^2 - 4\,ac)\,y^2 + 2\,(b\,e - 2\,c\,d) + - \ 4\,c\,f\}} \quad (2).$$

On account of the double sign of the root in (1), there are, in general, two values of y; hence there are two ordinates corresponding to the same abscissa: these ordinates may be constructed whenever the values of x render the radical quantity real; but if these values render it nothing, there is only one ordinate, and if they make it imaginary, no corresponding ordinate can be drawn, and therefore there is no point of the curve corresponding to such a value of x. Hence, to know the extent and limits of the curve, we must examine when the quantity under the root is real, nothing, or imaginary.

This will depend on the algebraical sign of the quantity

$$(b^2 - 4\,a\,c)\,x^2 + 2\,(b\,d - 2\,a\,e)\,x + d^2 - 4\,a\,f.$$

In an expression of this form, a value may be given to x, so large that the sign of the whole quantity depends only upon that of its first term, or upon that of its coefficient $b^2 - 4\,a\,c$, since x^2 is always positive for any real value of x.

For, writing the expression in the form $m\,(x^2 + \frac{n}{m}\,x + \frac{p}{m})$ let q be the absolute value of the greater of the two quantities $\frac{n}{m}$ and $\frac{p}{m}$; then substituting $r = \pm\,(q + 1)$ for x, the expression becomes

$$m\left\{q^2 + 2q + 1 \pm \frac{n}{m} q \pm \frac{n}{m} + \frac{p}{m}\right\},$$

which, whatever be the values of $\frac{n}{m}$ and $\frac{p}{m}$, is positive, and the same is true for any magnitude greater than $\pm r$; hence the sign of the expression depends upon that of m.

When $b^2 - 4ac$ is negative, real values may be given to x, either positive or negative, greater than $\pm r$, which will render y imaginary. The curve will then be limited in both the positive and negative directions of x.

When $b^2 - 4ac$ is positive, all values of x not less than $\pm r$ will render y real, and therefore the curve is of infinite extent in both directions of x.

Lastly, when $b^2 - 4ac$ is nothing, the quantity under the root becomes

$$2(bd - 2ae)x + d^2 - 4af.$$

If $bd - 2ae$ be positive, real positive values may be given to x, which shall render y real; but if a negative value be given to x greater than $\frac{d^2 - 4af}{2(bd - 2ae)}$, y is imaginary; therefore the curve will be of indefinite extent in the direction of x positive and limited in the opposite direction.

But if $bd - 2ae$ be negative, exactly opposite results will follow, that is, the curve will be of indefinite extent in the direction of x negative and limited in the opposite direction.

Taking equation (2) we should find similar results.

The curves corresponding to the equation of the second degree, may therefore be divided into three distinct classes.

1. $b^2 - 4ac$ negative, curves limited in every direction.
2. $b^2 - 4ac$ positive, curves unlimited in every direction.
3. $b^2 - 4ac$ nothing, curves limited in one direction, but unlimited in the opposite direction.

76. *First class $b^2 - 4ac$ negative.*

Let $-\frac{b}{2a} = \alpha$, $-\frac{d}{2a} = l$, $\frac{b^2 - 4ac}{4a^2} = -\mu$,

and let x_1 and x_2 be the roots of the equation

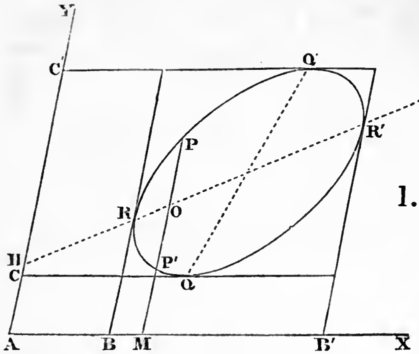
$$(b^2 - 4ac)x^2 + 2(bd - 2ae)x + d^2 - 4af = 0.$$

Then equation (1) or

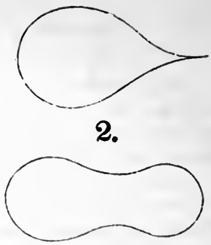
$$y = -\frac{bx + d}{2a} \pm \sqrt{\left\{\frac{b^2 - 4ac}{4a^2}\left(x^2 + 2\frac{bd - 2ae}{b^2 - 4ac}x + \frac{d^2 - 4af}{b^2 - 4ac}\right)\right\}}$$

becomes by substitution

$$y = \alpha x + l \pm \sqrt{\{-\mu(x - x_1)(x - x_2)\}}$$



1.



2.

Let A be the origin of co-ordinates, A X, A Y the oblique axes.

Let H H$'$ be the line represented by the equation $y = \alpha x + l$, M O one of its ordinates corresponding to any value of x between x_1 and x_2.

Along the line M O take O P and O P$'$ each equal to $\sqrt{\{-\mu(x-x_1)(x-x_2)\}}$, then P and P$'$ are two points in the curve, for

$$\text{M P} = \text{M O} + \text{O P} = \alpha x + l + \sqrt{\{-\mu(x-x_1)(x-x_2)\}}$$
$$\text{M P}' = \text{M O} - \text{O P}' = \alpha x + l - \sqrt{\{-\mu(x-x_1)(x-x_2)\}}$$

If we repeat this construction for all the real values of x which render the root real we obtain the different points of the curve.

The line H H$'$ is called a diameter of the curve, for it bisects all the chords P P$'$ which are parallel to the axis of y.

The reality of y depends on the reality of the radical quantity, which last depends on the form of the factors $(x - x_1)$ and $(x - x_2)$, that is, on the roots x_1 and x_2. Now these may enter the equation in three forms—real and unequal—real and equal—or both imaginary.

Case 1. Let x_1 and x_2 be real and unequal, take A B $= x_1$, A B$' = x_2$, then if $x = x_1$ or x_2 the quantity $-\mu(x-x_1)(x-x_2)$ vanishes, and the ordinate to the curve coincides with the ordinate to the diameter, therefore drawing through B and B$'$ two lines B R and B$'$ R$'$ parallel to A Y the curve cuts the diameter in R and R$'$.

For all values of x between x_1 and x_2 there are two real values of y, for $x - x_1$ is positive and $x - x_2$ is negative, and therefore $-\mu(x-x_1)(x-x_2)$ is positive.

For all values of $x > x_2$ or $< x_1$, $-\mu(x-x_1)(x-x_2)$ is negative, the root being impossible cannot be constructed, hence there is no real value of y corresponding to such values of x, and therefore the curve is entirely confined between the two lines B R and B$'$ R$'$.

Similarly by taking equation (2) in (75), we shall find that a straight line Q Q$'$ is a diameter; that the curve cuts it in two points Q, Q$'$: that drawing lines parallel to A X through Q and Q$'$ the curve is confined between those parallels.

We have thus determined that the curve exists and only exists between certain parallel lines: its form is not yet ascertained. We might by giving a variety of values to x between x_1 and x_2 determine a variety of points P, Q, &c., and thus arrive at a tolerably exact idea of its course, but independently of this method, its form cannot much differ from that in the figure, for supposing it to be such as in fig. (2) a straight line could be drawn cutting it in more points than two which is impossible (71).

This oval curve is called the Ellipse.

If we require the points where the curve cuts A X, put $y = 0$, then the roots of the equation $cx^2 + ex + f = 0$ are the abscissas of the points of intersection, and the curve will cut the axis in two points, touch it in one, or never meet it, according as these roots are real and unequal, real and equal, or imaginary. Similarly putting $x = 0$ we find the points, if any, where the curve meets the axis of y.

Case 2. Let the roots x_1 and x_2 be real and equal,

$$\therefore y = \alpha x + l \pm (x - x_1)\sqrt{-\mu}$$

which is imaginary except when $x = x_1$, therefore the locus is the poin whose co-ordinates are x_1 and $\alpha x_1 + l$, or $\dfrac{2ae - bd}{b^2 - 4ac}$ and $\dfrac{2cd - be}{b^2 - 4ac}$.

Case 3. Let x_1 and x_2 be impossible, then no real value can be given to x to make $(x - x_1)(x - x_2)$ negative, for the roots are of the form $\pm p + q\sqrt{-1}$ and $\pm p - q\sqrt{-1}$ $\therefore (x - x_1)(x - x_2) = x^2 \pm 2px + p^2 + q^2 = (x \pm p)^2 + q^2$ which quantity is always positive for a real value of x. Hence in this case the radical quantity being impossible there is no locus.

We have not examined the equation of x in terms of y at length, for the results of the latter are dependent on those of the former. By comparing equations (1) and (2) in (75), we see that c stands in one equation where a stands in the other, and therefore that the radical quantities are contemporaneously possible, equal, or impossible, provided that a and c have the same sign, which is the case when $b^2 - 4ac$ is negative.

In discussing a particular example reduce it to the forms

$$y = \alpha x + l \pm \sqrt{\{-\mu(x - x_1)(x - x_2)\}}$$
$$x = \alpha' y + l' \pm \sqrt{\{-\mu'(y - y_1)(y - y_2)\}}$$

there are then three cases.

Case 1. x_1 and x_2 real and unequal. The locus is called an ellipse, its boundaries are determined from x_1, x_2, y_1 and y_2, its diameters are drawn from $y = \alpha x + l$ and $x = \alpha' y + l'$, and its intersections with the axes found by putting x and y successively $= 0$ in the original equation.

Case 2. x_1 and x_2 real and equal: the locus is a point.

Case 3. x_1 and x_2 impossible: the locus is imaginary.

Ex. 1. $y^2 - 2xy + 2x^2 - 2y - 4x + 9 = 0$. Case 1. Fig 1.

$AB = 2$, $AB' = 4$, $AC = 4 - \sqrt{2}$, $AC' = 4 + \sqrt{2}$, $AH = 1$

Ex. 2. $y^2 + xy + x^2 + y + x - 5 = 0$. Case 1.

The curve cuts the diameters when $AB = 2\frac{1}{3}$, $AB' = -3$, $AC = 2\frac{1}{3}$, $AC' = -3$, and it cuts the axes at distances $1 \cdot 7$ and $-2 \cdot 7$ nearly;

These six points are sufficient to determine its course.

Ex. 3. $y^2 + 2xy + 3x^2 - 4x = 0$. Case 1.

Ex. 4. $y^2 - 2xy + 2x^2 - 2y + 2x = 0$. Case 1.

Ex. 5. $y^2 + 2x^2 - 10x + 12 = 0$. Case 1.

Ex. 6. $y^2 - 2xy + 3x^2 - 2y - 10x + 19 = 0$. Case 2. The intersection of the diameters in Ex. 1.

Ex. 7. $y^2 - 4xy + 5x^2 + 2y - 4x + 2 = 0$. Case 3.

It is to be observed that no accurate form of the curve is here found, that will be hereafter ascertained, all that we can at present do, is to obtain an idea of the situation of the locus.

77. *Second class, $b^2 - 4ac$ positive.*

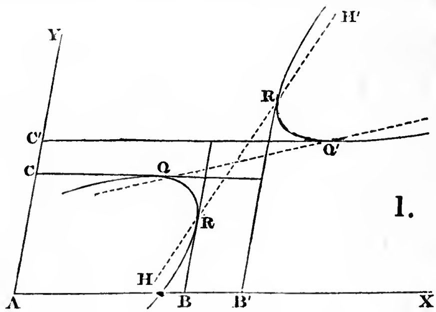
Arranging and substituting as in (76) the equation becomes

$$y = \alpha x + l \pm \sqrt{\{\mu (x - x_1)(x - x_2)\}}$$

Let H H′ be the diameter whose equation is $y = \alpha x + l$.

Then as before there are three forms of the roots x_1 and x_2.

Case 1. Let x_1 and x_2 be real and unequal, let A B $= x_1$ and A B′ $= x_2$, draw B R, B′ R′ parallel to A Y, the curve meets the diameter in R and



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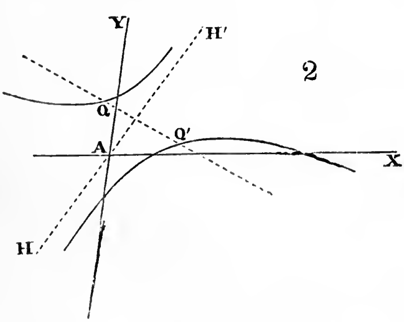

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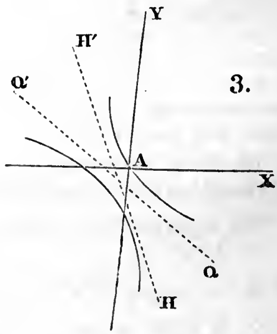


3.

R′. The radical quantity is imaginary for all values of x between x_1 and x_2 but real beyond these limits, hence no part of the curve is between the parallels B R, B′ R′, but it extends to infinity beyond them.

Taking the equation for x in terms of y, we may draw the diameter QQ' and determine the lines CQ, $C'Q'$ parallel to AX between which no part of the curve is found, and beyond which x is always possible.

From this examination it results that the form of the locus must be something like that in fig. 1, consisting of two opposite arcs with branches proceeding to infinity.

This curve is called the Hyperbola.

We must observe that the second diameter does not necessarily meet the curve, for the contemporaneous possibility or impossibility of the radical quantities depends on the signs of a and c, and these may be different in the hyperbola; so that one radical quantity may have possible and the other impossible roots.

Case 2. x_1 and x_2 real and equal.

$$y = \alpha x + l \pm (x - x_1)\sqrt{\mu}$$

this is the equation to two straight lines.

Case 3. x_1 and x_2 imaginary; whatever real values be given to x the radical quantity is real, and therefore there must be four infinite branches. Also since $\mu(x-x_1)(x-x_2)$ can never vanish, (76, Case 3.) the diameter HH' never meets the curve, but we may draw the other diameter as in the first case.

If neither diameter meets the curve, yet they will at least determine where the curve does not pass, we must then find the intersections with the axes. If these will not give a number of points sufficient to determine the locality of the curve we must have recourse to other methods to be explained hereafter.

In discussing a particular example reduce it to the forms

$$y = \alpha x + l \pm \sqrt{\{\mu(x-x_1)(x-x_2)\}}$$
$$x = \alpha' y + l' \pm \sqrt{\{\mu(y-y_1)(y-y_2)\}}$$

there are then three cases.

Case 1. x_1 and x_2 real and unequal. The locus is an hyperbola, its boundaries determined from x_1, x_2, y_1 and y_2, the diameters are drawn from $y = \alpha x + l$ and $x = \alpha' y + l'$, and its intersections with the axes found by putting x and y separately equal nothing.

Case 2. x_1 and x_2, real and equal. The locus consists of two straight lines which intersect each other.

Case 3. x_1 and x_2 impossible. The locus is an hyperbola, draw the diameters, and find the intersection, if any, of the curve with either diameter and with the axes.

Ex. 1. $y^2 - 3xy + x^2 + 1 = 0$. Case 1. Fig. 1. The origin being at the intersection of the dotted lines.

The equations to the diameters are $y = \frac{3x}{2}$ and $y = \frac{2x}{3}$, $AB' = \frac{2}{\sqrt{5}}$.

$AB = -\frac{2}{\sqrt{5}}$, $AC' = \frac{2}{\sqrt{5}}$, $AC = -\frac{2}{\sqrt{5}}$.

Ex. 2. $y^2 - 2xy - x^2 + 2 = 0$. Case 1. The two diameters pass through the origin and make an angle of 45° with the axes, the second QQ' never meets the curve, $AB' = 1$ and $AB = -1$; the curve intersects the axis of x at distances $\pm\sqrt{2}$.

Ex. 3. $4y^2 - 4xy - 3x^2 + 8y + 4x + 16 = 0.$ Case 1.
Ex. 4. $y^2 - 4xy - 5x^2 - 2y + 40x - 26 = 0.$ Case 1.
Ex. 5. $y^2 - 6xy + 8x^2 + 2x - 1 = 0.$ Case 2. The equations to the two straight lines are $y - 4x + 1 = 0$, and $y - 2x - 1 = 0.$
Ex. 6. $y^2 + 3xy + 2x^2 + 2y + 3x + 1 = 0.$ Case 2.
Ex. 7. $y^2 - 4xy - x^2 + 10x - 10 = 0.$ Case 3. Fig. 2.
Ex. 8. $y^2 + 3xy + x^2 + y + x = 0.$ Case 3. Fig. 3.

Here neither diameter meets the curve; but the curve passes through the origin and cuts the axis of x at a distance -1, and that of y also at a distance -1.

Ex. 9. $y^2 - x^2 - 2y + 5x - 3 = 0.$ Case 3.
The diameters are parallel to the axes, but the curve never meets that diameter whose equation is $x = \frac{5}{2}$.

Ex. 10. $y^2 - x^2 - y = 0.$ Case 3.

78. *Third Class.* $b^2 - 4ac = 0.$

In this case the general equation becomes

$$y = -\frac{bx+d}{2a} \pm \frac{1}{2a}\sqrt{\{2(bd - 2ae)x + d^2 - 4af\}} \quad (1)$$

$$\text{Let } -\frac{b}{2a} = \alpha, \; -\frac{d}{2a} = l, \; \frac{bd - 2ae}{2a^2} = \nu$$

And let x_1 be the root of the equation

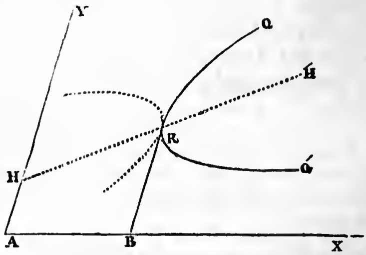
$$2(bd - 2ae)x + d^2 - 4af = 0.$$

Substituting equation (1) becomes

$$y = \alpha x + l \pm \sqrt{\{\nu(x - x_1)\}}$$

The locus of $y = \alpha x + l$ is a diameter H H′ as before.

Let ν be positive, then if $x = x_1$ the root vanishes; or if $AB = x_1$ and BR be drawn parallel to AY, the curve cuts the diameter in R. As x increases from x_1 to ∞, y increases to ∞, hence there are two arcs RQ, RQ′ extending to infinity. If x be less than x_1, y is impossible, or no part of the curve extends to the negative side of B.



Let ν be negative, then the results are contrary, and the curve only extends on the negative side of B; this case is represented by the dotted curve.

This curve is called the Parabola.

$$\text{If } bd - 2ae = 0, \quad y = \alpha x + l \pm \sqrt{\left\{\frac{d^2 - 4af}{4a^2}\right\}},$$

and the locus consists of two parallel straight lines; and, according as $d^2 - 4af$ is positive, nothing, or negative, these lines are both real, or unite into one, or are both imaginary.

In discussing a particular example, reduce it to the form

$$y = \alpha x + l \pm \sqrt{\{\nu(x - x_1)\}}$$

Case 1. ν positive or negative. The locus is called a parabola; draw the diameter and find the points where the curve cuts the axes and diameter.

Case 2. $\nu = 0$. The locus consists of two parallel straight lines, or one straight line, or is imaginary.

Ex. 1. $y^2 - 2xy + x^2 - 2y - 1 = 0$. Case 1.

Ex. 2. $y^2 - 2xy + x^2 - 2y - 2x = 0$. Case 1.

Ex. 3. $y^2 + 2xy + x^2 + 2y + x + 3 = 0$. Case 1.

Ex. 4. $y^2 - 2xy + x^2 - 1 = 0$. Case 2. Two parallel straight lines.

Ex. 5. $y^2 - 2xy + x^2 + 2y - 2x + 1 = 0$. Case 2. One straight line.

Ex. 6. $y^2 + 2xy + x^2 + 1 = 0$. Case 2. Imaginary locus.

79. Before we leave this subject, it may be useful to recapitulate the results obtained from the investigation of the general equation

$$ay^2 + bxy + cx^2 + dy + ex + f = 0$$

If $b^2 - 4ac$ be negative, the locus is an ellipse admitting of the following varieties :—

1. $c = a$, and $\frac{b}{2a}$ = cosine of the angle between the axes, locus a circle. (72.)

2. $(bd - 2ae)^2 = (b^2 - 4ac)(d^2 - 4af)$. Locus a point.

3. $(bd - 2ae)^2$ less than $(b^2 - 4ac)(d^2 - 4af)$. Locus imaginary.

If $b^2 - 4ac$ be positive, the locus is an hyperbola admitting of one variety.

1. $(bd - 2ae)^2 = (b^2 - 4ac)(d^2 - 4af)$. Locus two straight lines.

Lastly, if $b^2 - 4ac = 0$, the locus is a parabola admitting of the following varieties,—

1. $bd - 2ae = 0$. Locus two parallel straignt lines.

2. $bd - 2ae = 0$, and $d^2 - 4af = 0$. Locus one straight line.

3. $bd - 2ae = 0$, and d^2 less than $4af$. Locus imaginary.

Apparently another relation between the coefficients would be obtained in each variety, by taking the equation of x in terms of y; but on examination, it will be found that in each case the last relation is involved in the former.

CHAPTER VII

REDUCTION OF THE GENERAL EQUATION OF THE SECOND ORDER

80. In order to investigate the properties of lines of the second order more conveniently, we proceed to reduce the general equation to a more simple form, which will be effected by the transformation of co-ordinates. Taking the formulas in (54.)

$$y = y' + n, \text{ and } x = x' + m$$

and substituting in the general equation, we have

$a(y' + n)^2 + b(x' + m)(y' + n) + c(x' + m)^2 + d(y' + n) + e(x' + m) + f = 0;$

or arranging

$a y'^2 + b x' y' + c x'^2 + (2an + bm + d) y' + (2cm + bn + e) x' + an^2 + bmn + cm^2 + dn + emf = 0.$

As we have introduced two indeterminate quantities, m and n, we are at liberty to make two hypotheses respecting the new co-efficients in the last equation; let, therefore, the co-efficients of x' and y' each $= 0$.

$$\therefore 2an + bm + d = 0, \text{ and } 2cm + bn + e = 0;$$

whence we find by elimination, $m = \dfrac{2ae - bd}{b^2 - 4ac}$, and $n = \dfrac{2cd - be}{b^2 - 4ac}$.

The value of the constant term, or f', may be obtained from the equation,

$$f' = an^2 + bmn + cm^2 + dn + em + f = 0;$$

or, since $2an + bm + d = 0$, and $2cm + bn + e = 0$,

Multiply the first of these two equations by n, and the second by m; and, adding the results, we have

$$2an^2 + 2bmn + 2cm^2 + dn + em = 0;$$

$$\therefore an^2 + bmn + cm^2 = -\frac{dn + em}{2};$$

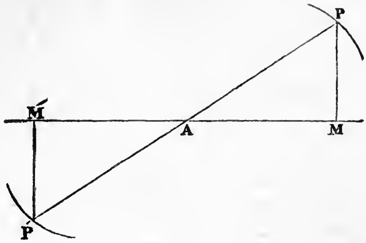
hence $f' = -\dfrac{dn + em}{2} + dn + em + f = \dfrac{dn + em}{2} + f$, which, by the substitution of the values of m and n, becomes

$$f' = \frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f.$$

The reduced equation is now of the form

$$ay'^2 + bx'y' + cx'^2 + f' = 0.$$

81. The point A, which is the new origin of co-ordinates, is called the centre of the curve, because every chord passing through it is bisected in that point. For the last equation remains the same when $-x$ and $-y$ are substituted for $+x$ and $+y$: hence, for every point P in the curve, whose co-ordinates are x and y, there is another point P′, whose co-ordinates are $-x$ and $-y$, or A M′ and M′ P′; hence, by comparing the right angled triangles, A M P, A M′ P′, we see that the vertical angles at A are equal, and therefore, the line P A P′ is a straight line bisected in A.



Whenever, therefore, the equation remains the same on the substitution of $-x$ and $-y$ for $+x$ and $+y$ respectively, it belongs to a locus referred to its centre.

If the equation be of an even order, this condition will be satisfied if the sum of the exponents of the variables in every term be even; thus, in the general equation of the second order, $a y^2 + b x y + c x^2 + d y + e x + f = 0$, the sum of the exponents in each of the three first terms is 2, and in the two next terms is 1; changing the signs of x and y, the equation does not remain the same; or for one point P, there is not another point P′ opposite and similarly situated with respect to the origin; hence that origin is not the centre of the curve. But the equation $a y^2 + b x y + c x^2 + f' = 0$, refers just as much to the point P′ as to P, and thus the origin is here the centre of the curve.

If the equation be of an odd order, the sum of the exponents in each term must be odd, and the constant term also must vanish; for if both these conditions are not fulfilled, the equation would be totally altered by putting $-x$ and $-y$ for $+x$ and $+y$ respectively. Therefore a locus may be referred to a centre if it be expressed by an equation which, by transformation, can be brought under either of the two following conditions:—

(1) Where the sum of the indices of every term is even, whether there be a constant or not, as $a y^2 + b x y + c x^2 + f = 0$.

(2) Where the sum of the indices in every term is odd, and there is no constant term, as $a y^3 + b x y^2 + c x^2 y + d x^3 + e y + f x = 0$.

Now it has been stated (59) that no equation can be so transformed that the new equation shall be of a lower or higher degree than the original one. Hence, if the original equation be of an even degree, the transformed equation will be so too, and the locus can be transferred to a centre only where the equation can be brought under the first condition; but if the original equation be of an odd degree, the transformed equation also will be of an odd degree, and the locus can only be transferred to a centre when the equation can be brought under the second condition. Hence we have a test, whether a locus with a given equation can be referred to a centre or not. If the axes can be transferred so that (1) The original equation being of an even degree, the co-efficients of all the terms, the sum of whose exponents is odd, vanish. (2) The original equation being of an odd degree, the co-efficients of all the terms, the sum of whose exponents is even, and also the constant term, vanish, then the locus may be referred to a centre, and not otherwise.

Now in the transformation which we effect by making $y = y' + n$, and $x = x' + m$, we can destroy only two terms; we cannot therefore bring, by any substitution, an equation of higher dimensions than the second under the necessary conditions, unless from some accidental relation of the original co-efficients of that equation. But in the case of equations of the second degree, we can always bring them under the first condition, unless the values of the indeterminate quantities, m and n, are found to be impossible or infinite.

In curves of the second order, we see that the values of m and n are real and finite, unless $b^2 - 4\,a\,c = 0$; consequently the ellipse and hyperbola have a centre and the parabola has not; hence arises the division of these curves into two classes, central and non-central.

In the case where $b^2 - 4\,a\,c = 0$, and at the same time $2\,a\,e - b\,d$ or $2\,c\,d - b\,e$ vanish, the equation becomes that to a straight line, as appears on inspecting the equations (1) and (2) in (75).

If by the transformation the term f' should vanish, the equation becomes of the form $a\,y^2 + b\,x\,y + c\,x^2 = 0$; whence

$y = \{-b \pm \sqrt{b^2 - 4\,a\,c}\}\dfrac{x}{2\,a}$; and the curve is reduced to two straight lines which pass through the centre; or if $b^2 - 4\,a\,c$ is negative, the locus is the centre itself (25).

82. The central class may have their general equation still further reduced by causing the term containing the product of the variables to vanish, which is done by another transformation of co-ordinates. Taking the formulas in (58) let

$$\text{Let } y' = x'' \sin.\theta + y'' \cos.\theta,$$

$$x' = x'' \cos.\theta - y'' \sin.\theta,$$

substituting in the equation $a\,y'^2 + b\,x'\,y' + c\,x'^2 + f' = 0$, we have

$$a(x''\sin.\theta + y''\cos\theta)^2 + b(x''\sin\theta + y''\cos.\theta)(x''\cos\theta - y''\sin.\theta) + c(x''\cos.\theta - y''\sin.\theta)^2 + f' = 0$$

$$\therefore y''^2\{a(\cos.\theta)^2 - b\sin.\theta\cos.\theta + c(\sin\theta)^2\} + x''^2\{a(\sin.\theta)^2 + b\sin.\theta\cos.\theta + c(\cos.\theta)^2\}$$

$$+ x''\,y''\{2\,a\sin.\theta\cos.\theta + b(\cos.\theta)^2 - b(\sin.\theta)^2 - 2\,c\sin.\theta\cos.\theta\} + f' = 0.$$

Let the co-efficient of $x''\,y'' = 0$,

$$\therefore 2\,a\sin.\theta\cos.\theta + b(\cos.\theta)^2 - b(\sin.\theta)^2 - 2\,c\sin.\theta\cos.\theta = 0,$$

$$\text{or } (a - c)\,2\sin.\theta\cos.\theta + b\{(\cos.\theta)^2 - (\sin.\theta)^2\} = 0,$$

$$\therefore (a - c)\sin.2\theta + b\cos.2\theta = 0;$$

and dividing by $\cos.2\theta$, we have

$$\tan.2\theta = \frac{-b}{a - c}.$$

Here θ is the angle which the new axis of x makes with the original one (58); hence, if the original rectangular axes be transferred through an angle θ, such that $\tan.2\theta = \dfrac{-b}{a - c}$, the transformed equation will have no term containing the product $x'' y''$, that is, the equation, when referred to its new rectangular axes, will be reduced to the simple form

$$a' y''^2 + c' x''^2 + f' = 0.$$

83. As a tangent is capable of expressing all values from 0 to ∞, positive or negative, it follows that the angle θ has always a real value, whatever be the values of a, b, and c, and thus it is always possible to destroy the term containing $x y$.

The values of sin. 2θ and cos. 2θ are thus obtained from that of tan. 2θ;

$$\cos. 2\theta = \frac{1}{\pm\sqrt{1 + (\tan. 2\theta)^2}} = \frac{1}{\pm\sqrt{1 + \left(\frac{b}{a - c}\right)^2}} = \frac{a - c}{\pm\sqrt{(a - c)^2 + b^2}}$$

$$\text{And } \sin. 2\theta = \cos. 2\theta . \tan. 2\theta = \frac{-b}{\pm\sqrt{(a - c)^2 + b^2}}.$$

Since θ must be less than 90°, and therefore sin. 2θ positive, the sign of the radical quantity must be taken positive or negative, according as b is itself negative or positive.

84. To express the co-efficients a' and c' of the transformed equation in terms of the co-efficients in the original equation.

Taking the expressions for the co-efficients in article (82) we have

$$a' = a (\cos. \theta)^2 - b \sin. \theta \cos. \theta + c (\sin. \theta)^2$$

$$c' = a (\sin. \theta)^2 + b \sin. \theta \cos. \theta + c (\cos. \theta)^2,$$

$$\therefore a' - c' = a\{(\cos.\theta)^2 - (\sin.\theta)^2\} - 2b \sin.\theta\cos.\theta + c\{(\sin.\theta)^2 - (\cos\theta)^2\}$$

$$= a \cos. 2\theta - b \sin. 2\theta - c \cos. 2\theta$$

$$= (a - c) \cos. 2\theta - b \sin. 2\theta;$$

$$\text{but } \cos. 2\theta = \frac{a - c}{\pm\sqrt{(a - c)^2 + b^2}}, \text{ and } \sin. 2\theta = \frac{-b}{\pm\sqrt{(a - c)^2 + b^2}};$$

hence substituting, we have

$$a' - c' = \frac{(a - c)^2}{\pm\sqrt{(a - c)^2 + b^2}} + \frac{b^2}{\pm\sqrt{(a - c)^2 + b^2}}$$

$$= \frac{(a - c)^2 + b^2}{\pm\sqrt{(a - c)^2 + b^2}},$$

$$\text{or } a - c' = \pm\sqrt{(a - c)^2 + b^2}$$

$$\text{Also } a' + c' = a + c,$$

$$\therefore a' = \tfrac{1}{2}\{a + c \pm \sqrt{(a - c)^2 + b^2}.\}$$

$$c' = \tfrac{1}{2}\{a + c \mp \sqrt{(a - c)^2 + b^2}.\}$$

Hence the final equation is

$$\tfrac{1}{2}\{a + c \pm \sqrt{(a - c)^2 + b^2}\} y''^2 + \tfrac{1}{2}\{a + c \mp \sqrt{(a - c)^2 + b^2}\} x''^2 + \frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f = 0.$$

The upper or lower sign to be taken all through this article, according as the sign of b in the original equation is negative or positive

85. Hitherto in this chapter we have been making a number of alterations in the *form* of the original equation: the following figures will show the corresponding alterations which have been made in the *position* of the curve. The ellipse is used in the figure, in preference to the hyperbola solely on account of its easier description.

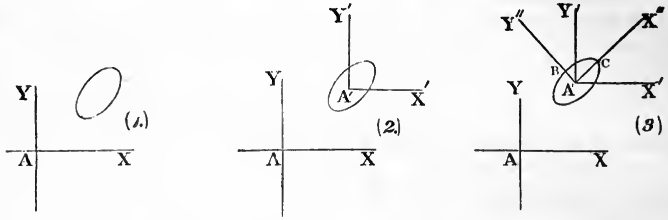


Fig. 1. We have here the original position of the curve referred to rectangular axes A X and A Y, and the corresponding equation is

$$a\,y^2 + b\,x\,y + c\,x^2 + d\,y + e\,x + f = 0.$$

Fig. 2. The origin is here transferred from A to the centre of the curve A′, the co-ordinates of which are $m = \dfrac{2\,a\,e - b\,d}{b^2 - 4\,a\,c}$, and $n = \dfrac{2\,c\,d - b\,e}{b^2 - 4\,a\,c}$.

The new axes A′ X′ and A′ Y′ are parallel to the former axes, and the equation to the curve is

$$a\,y'^2 + b\,x'\,y' + c\,x'^2 + f' = 0.$$

Fig. 3. The origin remains at A′, but the curve is referred to the new rectangular axes A′ X″ and A′ Y″, instead of the former ones A′ X′ and A′ Y′. The axis A′ X′ has been transferred through an angle X′ A′ X″ into the position A′ X″, the angle X′ A′ X″, or θ, being determined by the equation $\tan.\ 2\,\theta = \dfrac{-\,b}{a - c}$, and the equation to the curve is now

$$a'\,y''^2 + c'\,x''^2 + f' = 0.$$

86. In the ellipse and hyperbola the word "axis" is used in a limited sense to signify that portion of the central rectangular axis which is bounded by the curve.

To find the lengths of the axes, put x'' and y'' successively $= 0$, we then obtain the points where the curve cuts the axes, or, in other words, we have the lengths of the semi-axes.

In the equation

$$a'\,y''^2 + c'\,x''^2 + f' = 0,$$

$$\text{Let } y'' = 0, \ \therefore\ c'\,x''^2 + f' = 0, \text{ and } x'' = \sqrt{\frac{-f'}{c'}}.$$

$$\text{Let } x'' = 0, \ \therefore\ a'\,y''^2 + f' = 0, \text{ and } y'' = \sqrt{\frac{-f'}{a'}}.$$

In fig. 3, the semi-axes are A′ C and A′ B, so that A′ C $= \sqrt{\dfrac{-f'}{c'}}$, and A′ B $= \sqrt{\dfrac{-f'}{a'}}$; putting for a', c', and f', their values in terms of the original co-efficients (80, 84) we have the squares upon the semi-axes both comprehended in the formula,

$$-\frac{2}{a+c\pm\sqrt{(a-c)^2+b^2}}\left(\frac{a\,e^2+c\,d^2-b\,d\,e}{b^2-4\,a\,c}+f\right).$$

Let the equation $a'\,y''^2+c'\,x''^2+f'=0$, be written in the form $\left(-\frac{a'}{f'}\right)y''^2+\left(-\frac{c'}{f'}\right)x''^2=1.$

Then, if the curve is an ellipse, we must have $b^2-4\,a\,c$ negative, or, since $b=0$ in the present case, we must have $-4\left(-\frac{a'}{f'}\right)\left(-\frac{c'}{f'}\right)$ negative, and therefore $-\frac{a'}{f'}$ and $-\frac{c'}{f'}$ both positive; thus both axes meet the curve, (the case where both $-\frac{a'}{f'}$ and $-\frac{c'}{f'}$ are negative, would give an imaginary locus). If the curve is an hyperbola, $b^2-4\,a\,c$ is positive, and therefore $-4\left(-\frac{a'}{f'}\right)\left(-\frac{c'}{f'}\right)$ must be positive, or one of the values, $\left(-\frac{a'}{f'}\right)$ positive, and the other $\left(-\frac{c'}{f'}\right)$ negative; hence, one of the axes in the hyperbola has an impossible value, and therefore does not meet the curve.

The relative lengths of the axes will depend entirely on the magnitude o. $\frac{a'}{f'}$ and $\frac{c'}{f'}$.

87. Hitherto the original co-ordinates have been rectangular, but if they were oblique, considerable alterations must be made in some of the formulas.

Articles 80 and 81 are applicable in all cases, but 82, 83, and 84, must be entirely changed; the method pursued will be nearly the same as in the more simple case; but on account of the great length of some of the operations, we cannot do more than indicate a few steps, and give the results *.

To destroy the co-efficient of the term containing the product of the variables, take the formulas in (55)

$$y'=\frac{x''\sin.\theta+y''\sin.\theta'}{\sin.\omega}$$

$$x'=\frac{x''\sin.(\omega-\theta)+y''\sin.(\omega-\theta')}{\sin.\omega.}$$

Substituting in the equation $a\,y'^2+b\,x'\,y'+c\,x'^2+f'=0$, we have

$$y''^2\left\{a\,(\sin.\theta')^2+b\sin.\theta'\sin.\overline{\omega-\theta'}+c\,(\sin.\overline{\omega-\theta'})^2\right\}\frac{1}{(\sin.\omega)^2}$$

$$+\,x''^2\left\{a\,(\sin.\theta)^2+b\sin.\theta\sin.\overline{\omega-\theta}+c\,(\sin.\overline{\omega-\theta})^2\right\}\frac{1}{(\sin.\omega)^2}$$

* This article, and the following ones marked with an asterisk, had better be omitted at the first reading of the subject.

$$+ x'' y'' \left\{ 2\, a \sin.\, \theta.\, \sin.\, \theta' + b \sin.\, \theta' \sin.\, \overline{\omega - \theta} + b \sin.\, \theta \sin.\, \overline{\omega - \theta'} + \right.$$

$$\left. 2\, c \sin.\, \overline{\omega - \theta}.\, \sin.\, \overline{\omega - \theta'} \right\} \frac{1}{(\sin.\, \omega)^2} + f' = 0.$$

Let the co-efficient of $x'' y'' = 0$; expanding $\sin.\, \overline{\omega - \theta}$, $\sin.\, \overline{\omega - \theta'}$ and dividing by $\cos.\, \theta.\, \cos\, \theta'$ we shall obtain the equation

$\left\{ a - b \cos.\, \omega + c\, (\cos.\, \omega)^2 \right\} 2 \tan.\, \theta.\, \tan.\, \theta' + \left\{ b - 2\, c \cos.\, \omega \right\} \sin.\, \omega.$
$\left\{ \tan.\, \theta + \tan.\, \theta' \right\} + 2\, c\, (\sin.\, \omega)^2 = 0.$

Whence for any given value of θ a value of θ' and consequently of $(\theta' - \theta)$ may be found, so that there are an infinite number of pairs of axes to which if the curve be referred, its equation may assume the form

$$a' y^2 + c' x^2 + f' = 0.$$

Let us now examine these pairs of axes, to find what systems can be rectangular:

For this purpose we must have $\theta' - \theta = \frac{\pi}{2}$ and therefore $\tan.\, \theta' = - \frac{1}{\tan.\, \theta}$.

By substituting this value of $\tan.\, \theta'$ in the equation containing $\tan.\, \theta$ and $\tan.\, \theta'$, we have $- 2 \left\{ a - b \cos.\, \omega + c\, (\cos.\, \omega)^2 \right\} + \left\{ 2\, c \cos.\, \omega - b \right\}$

$$\sin.\, \omega.\, \frac{2}{\tan.\, 2\, \theta} + 2\, c\, (\sin.\, \omega)^2 = 0.$$

$$\therefore \tan.\, 2\, \theta = \frac{c \sin.\, 2\, \omega - b \sin.\, \omega}{a - b \cos.\, \omega + c \cos.\, 2\, \omega}.$$

There are two angles which have got the same $\tan.\, 2\, \theta$ separated from each other by 180°, therefore there are two angles θ, which would satisfy the above equation; however, as they are separated by an angle of 90°, the second value only applies to the new axis of y''.

Hence there is only one *system* of rectangular axes, and their position is fully determined by the last formula.

*88. To find the co-efficients a' and c' in terms of the co-efficients of the original equation, the new axes being supposed rectangular. Taking the co-efficients in the general transformed equation given above, putting $\theta' = \frac{\pi}{2} + \theta$, and multiplying by $(\sin.\, \omega)^2$, we have,

$$a' (\sin.\, \omega)^2 = a\, (\cos.\, \theta)^2 - b \cos.\, \theta \cos.\, \overline{\omega - \theta} + c\, (\cos.\, \overline{\omega - \theta})^2$$
$$c' (\sin.\, \omega)^2 = a\, (\sin.\, \theta)^2 + b \sin.\, \theta \sin.\, \overline{\omega - \theta} + c\, (\sin.\, \overline{\omega - \theta})^2$$

$$\therefore (a' - c')\, (\sin.\, \omega)^2 = a \left\{ (\cos.\, \theta)^2 - (\sin.\, \theta)^2 \right\} - b \left\{ \cos.\, \theta \cos.\, \overline{\omega - \theta} + \right.$$
$$\left. \sin.\, \theta \sin.\, \overline{\omega - \theta'} \right\} + c \left\{ (\cos.\, \overline{\omega - \theta})^2 - (\sin.\, \overline{\omega - \theta})^2 \right\}$$
$$= a \cos.\, 2\, \theta - b \cos.\, (\omega - 2\, \theta) + c \cos.\, (2\, \omega - 2\, \theta)$$
$$= \left\{ a - b \cos.\, \omega + c \cos.\, 2\, \omega \right\} \cos.\, 2\, \theta + (c \sin.\, 2\, \omega - b \sin.\, \omega) \sin.\, 2\, \theta.$$

Also, following the method in (83) we find from $\tan.\, 2\, \theta$ that

$$\cos.\, 2\, \theta = \frac{a - b \cos.\, \omega + c \cos.\, 2\, \omega}{\pm \mathrm{M}}, \text{ and } \sin.\, 2\, \theta = \frac{c \sin.\, 2\, \omega - b \sin.\, \omega}{\pm \mathrm{M}}$$

Where $M = \pm \sqrt{\{a^2 + b \ + c^2 - 2b(a + c)\cos.\omega + 2ac\cos.2\omega\}}$
or $= \pm \sqrt{\{(a + c - b\cos.\omega)^2 + (b^2 - 4ac)(\sin.\omega)^2\}}$.
Hence $(a' - c')(\sin.\omega)^2 = \pm M$
and $(a' + c')(\sin.\omega)^2 = a - b\cos.\omega + c$

$$\therefore a' = \{a - b\cos.\omega + c \pm M\}\frac{1}{2(\sin.\omega)^2}$$

$$\text{and } c' = \{a - b\cos.\omega + c \mp M\}\frac{1}{2(\sin.\omega)^2}$$

Hence the final equation is

$$\{a - b\cos.\omega + c \pm M\}\frac{y''^2}{2(\sin.\omega)^2} + \{a - b\cos.\omega + c \mp M\}\frac{}{2(\sin.\omega)^2}$$
$$+ \frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f = 0$$

And the $\pm$ sign is to be used according as $c\sin.2\omega - b\sin.\omega$ is positive or negative, since 2θ is assumed to be positive.

These analytical transformations may be geometrically represented as in (85). In figures (1), (2), and (3) we must suppose the axes A X, A Y, and also the axes A′X′ and A′ Y′, to contain the angle ω.

The article (86) will equally apply when the original axes are oblique; the value of the square on the semi-axis is,

$$\frac{-2(\sin.\omega)^2}{a - b\cos.\omega + c \pm M} \cdot \left(\frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f\right)$$

89. We shall conclude the discussion of the central class by the application of the results already obtained to a few examples.

The original axes rectangular.

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \quad (1)$$

$$\left.\begin{array}{l} y = y' + n \\ x = x' + m \end{array}\right\} \text{formulas to be used.}$$

$$m = \frac{2ae - bd}{b^2 - 4ac}\ (2),\quad n = \frac{2cd - be}{b^2 - 4ac}\ (3)$$

$$f' = \frac{dn + em}{2} + f = \frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f\ (4)$$
$$ay'^2 + bx'y' + cx' + f' = 0\ \ (5)$$

$$\left.\begin{array}{l} y' = x''\sin.\theta + y''\cos.\theta \\ x = x''\cos.\theta - y''\sin.\theta \end{array}\right\} \text{formulas to be used.}$$

$$\tan.2\theta = \frac{-b}{a - c}\quad (6)$$

$$a' = \frac{1}{2}\left\{a + c \pm \text{M}\right\} \quad (7) \qquad c' = \frac{1}{2}\left\{a + c \mp \text{M}\right\} \quad (8)$$

$$\text{M} = \pm \sqrt{(a-c)^2 + b^2}, \ \pm \text{ according as } b \text{ is } \mp$$

$$\left(-\frac{a'}{f'}\right) y''^2 + \left(-\frac{c'}{f'}\right) x''^2 = 1 \qquad (9);$$

(2) and (3) determine the situation of the centre, and together with (4) reduce the equation to the form (5); (6) determines the position of the rectangular axes passing through the centre, (7) and (8) enable us to reduce the equation to its most simple form (9): and the co-efficients of y''^2 and x''^2 inverted are respectively the squares upon the semi-axes measured along the axes of y'' and x''.

Ex. 1. $y^2 - xy + x^2 + y + x - 1 = 0$; locus an ellipse.

$$m = -1;\ n = -1;\ f' = -2;$$

$$y'^2 - x'y' + x'^2 - 2 = 0$$

tan. $2\theta = \frac{1}{0}$ $\therefore 2\theta = 90^\circ$ and $\theta = 45^\circ$; b is negative, and $\therefore$ M $= +1$

$$a' = \frac{3}{2} \text{ and } c' = \frac{1}{2}$$

$$\therefore \frac{3}{2} y''^2 + \frac{1}{2} x''^2 - 2 = 0$$

$$\text{or } \frac{3}{4} y''^2 + \frac{1}{4} x''^2 = 1$$

The squares on the semi-axes are $\frac{4}{3}$ and 4; hence the semi-axes themselves are $\frac{2}{\sqrt{3}}$ and 2, and therefore the lengths of the axes are 4 and $\frac{4}{\sqrt{3}}$.

Ex. 2. $3y^2 - 4xy + 3x^2 + y - x - \frac{9}{10} = 0$; locus an ellipse.

The reduced equation is $5y''^2 + x''^2 = 1$. The axes are 2 and $\frac{2}{\sqrt{5}}$

Ex. 3. $2y^2 + xy + x^2 - 2y - 4x + 3 = 0$; locus an ellipse.

The reduced equation is $\frac{3 - \sqrt{2}}{2} y''^2 + \frac{3 + \sqrt{2}}{2} x''^2 = 1.$

Ex. 4. $5y^2 + 6xy + 5x^2 - 22y - 26x + 29 = 0$; locus an ellipse.

$$m = 2,\ n = 1,\ f' = \frac{dn + em}{2} + f = -8$$

$$\therefore 5y'^2 + 6x'y' + 5x'^2 - 8 = 0$$

tan. $2\theta = \infty$ $\therefore \theta = 45^\circ$; hence the formulas of transformation are

$$y' = \frac{x'' + y''}{\sqrt{2}} \text{ and } x' = \frac{x'' - y''}{\sqrt{2}};$$

$$\therefore \frac{5}{2}(x''+y'')^2 + 6\frac{x''^2-y''^2}{2} + \frac{5}{2}(x''-y'')^2 - 8 = 0$$

$$\text{or } 4y''^2 + x''^2 = 4$$

Ex. 5. $5y^2 + 2xy + 5x^2 - 12y - 12x = 0$; locus an ellipse.

$$2y''^2 + 3x''^2 = 6$$

Ex. 6. $2y^2+x^2+4y-2x-6=0$; locus an ellipse.

Let $y = y'+n$ and $x = x'+m$, hence the transformed equation is

$$2(y'+n)^2 + (x'+m)^2 + 4(y'+n) - 2(x'+m) - 6 = 0$$

or $2y'^2 + x'^2 + 4(n+1)y' + 2(m-1)x' + 2n^2 + m^2 + 4n - 2m - 6 = 0$.

Let $n+1=0$ and $m-1=0$ $\therefore m=1$ and $n=-1$ and $f'=-9$; hence the transformed equation is

$$2y'^2 + x'^2 = 9$$

and no further transformation is requisite. The axes are 6 and $3\sqrt{2}$.

Ex. 7. $y^2-10xy+x^2+y+x+1=0$; locus an hyperbola.

$$6y''^2 - 4x''^2 + \frac{9}{8} = 0.$$

Ex. 8. $4y^2-8xy-4x^2-4y+28x-15=0$; locus an hyperbola.

$$y''^2 - x''^2 = \frac{-1}{2\sqrt{2}}$$

Here the axes are each $= \sqrt[4]{2}$, that which is measured along the new axis of x'' alone meeting the curve.

Ex. 9. $y^2 - 2xy - x^2 - 2 = 0$; locus an hyperbola.

The origin is already at the centre, and thus only one transformation is necessary.

tan. $2\theta=1$ $\therefore 2\theta=45°$; $M=\sqrt{8}$, $a'=\sqrt{2}$, $c'=-\sqrt{2}$, $y'^2-x'^2=\sqrt{2}$.

*90. The axes oblique.

The values of m, n, and f' remain as for rectangular axes.

$$\tan. 2\theta = \frac{c \sin. 2\omega - b \sin. \omega}{a - b \cos. \omega + c \cos. 2\omega}$$

$$a' = \{a - b\cos.\omega + c \pm M\}\frac{1}{2(\sin\omega)^2}.$$

$$c' = \{a - b\cos.\omega + c \mp M\}\frac{1}{2(\sin.\omega)^2}.$$

$$M = \pm\sqrt{\{a^2+b^2+c^2-2b(a+c)\cos.\omega + 2ac\cos.2\omega\}}.$$

$$\pm \text{ as } c \sin. 2\omega - b\sin\omega \text{ is } \pm.$$

Ex. 1. $y^2 + xy + x^2 + y + x - \frac{1}{6} = 0$; the angle between the axes being 45°.

$m = -\frac{1}{3}$, $n = -\frac{\ }{3}$; tan $2\theta=1$ $\therefore 2\ \ =45°$; $M=+(1-\sqrt{2})$.

$a' = 3 - \frac{3}{\sqrt{2}}$, $c' = 1 + \frac{1}{\sqrt{2}}$ and $f' = -\frac{1}{2}$.

The reduced equation is

$$3\,(2 - \sqrt{2}\,)\,y''^2 + (2 + \sqrt{2}\,)\,x''^2 = 1.$$

The curve is an ellipse, and the squares upon the semi-axes are

$$\frac{1}{3\,(2 - \sqrt{2}\,)} \text{ and } \frac{1}{2 + \sqrt{2}}$$

Ex. 2. $7\,y^2 + 16\,x\,y + 16\,x^2 + 32\,y + 64\,x + 28 = 0$. The angle $\omega = 60°$.

$$m = -2,\ n = 0,\ f' = \frac{d\,n + e\,m}{2} + f = -36.$$

The form of the equation is now

$$7\,y'^2 + 16\,x'\,y' + 16\,x'^2 - 36 = 0\,;$$

since tan. $2\,\theta = 0$, the reduction to rectangular axes is effected by merely transferring the axis of y' through $30°$; hence, putting $\theta = 0$, and $\omega = 60$, the formulas (56) of transformation become

$$y' = \frac{2\,y''}{\sqrt{3}}, \text{ and } x' = x'' - \frac{y''}{\sqrt{3}}.$$

Substituting these values in the last equation, it becomes

$$4\,y''^2 + 16\,x''^2 - 36 = 0\,;$$

Hence the axes of the ellipse are 3 and 6.

Ex. 3. $y^2 - 3\,x\,y + x^2 + 1 = 0$; the angle $\omega = 60°$.

$m = 0$, $n = 0$, tan. $2\,\theta = \sqrt{3}$, $\therefore\ \theta = 30°$; $M = +4$.

$a' = 5$, $c' = \frac{-1}{3}$, $f' = 1$, and the reduced equation is

$$5\,y''^2 - \frac{1}{3}\,x''^2 = -1.$$

The curve is an hyperbola, of which the axes are $2\sqrt{3}$ and $\frac{2}{\sqrt{5}}$; the first of these, which is the greatest, is measured along the new axis of x'' The second axis never meets the curve.

91. It was observed, at the end of art. 81, that the curves corresponding to the general equation of the second order were divided into two classes, one class having a centre or point such that every chord passing through it is bisected in that point, and another class having no such peculiar point. This fact was ascertained from the inspection of the values of the two indeterminate quantities m and n introduced into the equation by means of the transformation of co-ordinates, and for the purpose of destroying certain terms in the general equation. The values of m and n were found to be infinite, that is, there was no centre when the relation among the three first terms of the co-efficients of the general equation was such that $b^2 - 4\,a\,c = 0$.

This relation $b^2 - 4\,a\,c = 0$ being characteristic of the parabola, it follows that the general equation of the second order belonging to a parabola is not capable of the reduction performed in art. (80); that is, we cannot destroy the co-efficients of both x and y, or reduce the equation to the form $a\,y^2 + b\,x\,y + c\,x^2 + f = 0$, or, finally, to the form $a\,y^2 + c\,x^2 + f = 0$.

Although, however, we cannot thus reduce the parabolic equation, we are yet able to reduce it to a very simple form, in fact to a much more simple form than that of either of the above equations. This will be effected by a process similar to that already used for the general equation, only in a different order. We shall commence by transferring the axes through an angle θ, and thus destroy two terms in the equation, so that it will be reduced to the form $a\,y^2 + d\,y + e\,x + f = 0$; we shall then transfer the axes parallel to themselves, and by that means destroy two other terms, so that the final equation will be of the form

$$a\,y^2 + e\,x = 0$$

92. Taking the formulas in (58), let

$$y = x' \sin.\theta + y' \cos.\theta$$
$$x = x' \cos.\theta - y' \sin.\theta$$

substituting these values in the general equation

$$a\,y^2 + b\,x\,y + c\,x^2 + d\,y + e\,x + f = 0,$$

nd arranging, we obtain the equation

$$\begin{array}{l|l|l|l|l|l}
a(\cos.\theta)^2 & y'^2 + 2a\sin.\theta\cos.\theta & x'y' + a(\sin.\theta)^2 & x'^2 + d\cos.\theta & y' + d\sin.\theta & x' + f = 0 \\
-b\sin.\theta\cos.\theta & \quad + b(\cos.\theta)^2 & + b\sin.\theta\cos.\theta & \quad - e\sin.\theta & \quad + e\cos.\theta & \\
+c(\sin.\theta)^2 & \quad - b(\sin.\theta)^2 & \quad + c(\cos.\theta)^2 & & & \\
 & - 2c\sin.\theta\cos.\theta & & & &
\end{array}$$

Let the co-efficient of $x'\,y' = 0$

$$\therefore 2(a - c)\sin\theta\cos.\theta + b\left\{(\cos.\theta)^2 - (\sin.\theta)^2\right\} = 0,$$
$$\text{or } (a - c)\sin.2\theta + b\cos.2\theta = 0,$$

$$\text{and } \tan.2\theta = \frac{-b}{a-c}, \text{ as in (82.)}$$

Hence, if the axes be transfered through an angle θ such that $\tan.2\theta = \frac{-b}{a-c}$ the transformed equation will have no term containing the product of the variables; that is, it will be of the form

$$a'\,y'^2 + c'\,x'^2 + d'\,y' + e'\,x' + f = 0.$$

But, since this last equation belongs to a parabola, the relation among the co-efficients of the three first terms must be such that the general condition $b^2 - 4\,a\,c = 0$ holds good. In this case, since $b' = 0$, we must have $-4\,a'\,c' = 0$; hence either a' or c' must $= 0$, that is, the transformation which has enabled us to destroy the co-efficient of the term containing $x'\,y'$ will of necessity destroy the co-efficient of either x'^2 or y'^2. And this will soon be observed upon examining the values of the co-efficients of x'^2 and y'^2.

93. Let the co-efficient b in the original equation be negative, that is, let $b = -2\sqrt{a\,c}$.

From tan. $2\,\theta$ we have $\cos.2\theta = \dfrac{1}{\sqrt{1 + (\tan.2\theta)^2}} = \dfrac{1}{\sqrt{1+\left(\frac{b}{a-c}\right)^2}} =$

$$= \frac{a-c}{\pm\sqrt{b^2+(a-c)^2}} = \frac{a-c}{\pm(a+c)} \text{ and } \sin 2\,\theta = \frac{-b}{\pm(a+c)} = \frac{-b}{(a+c)},$$

since sin. 2 θ must be positive, and b is itself negative;

$$\text{hence } \cos.\,\theta = \sqrt{\frac{1+\cos.\,2\,\theta}{2}} = \sqrt{\tfrac{1}{2}\left(1+\frac{a-c}{a+c}\right)} = \sqrt{\frac{a}{a+c}},$$

$$\text{and } \sin.\,\theta = \sqrt{\frac{1-\cos.\,2\,\theta}{2}} = \sqrt{\frac{c}{a+c}}.$$

Substituting these values of sin. θ and cos. θ in the general transformed equation, we have

$$a' = \frac{a\,a}{a+c} - \frac{b\sqrt{a\,c}}{a+c} + \frac{c\,c}{a+c} = \frac{a^2+2\,a\,c+c^2}{a+c} = a+c.$$

$$c' = \frac{a\,c}{a+c} + \frac{b\sqrt{a\,c}}{a+c} + \frac{c\,a}{a+c} = \frac{a\,c-2\,a\,c+a\,c}{a+c} = 0.$$

$$d' = \frac{d\sqrt{a}-e\sqrt{c}}{\sqrt{a+c}}$$

$$e' = \frac{d\sqrt{c}+e\sqrt{a}}{\sqrt{a+c}}$$

And the transformed equation is now

$$(a+c)\,y'^2 + \frac{d\sqrt{a}-e\sqrt{c}}{\sqrt{a+c}}\,y' + \frac{d\sqrt{c}+e\sqrt{a}}{\sqrt{a+c}}\,x' + f = 0.$$

And it is manifest that if b had been positive all the way through this article, the reduced equation would have been

$$(a+c)\,x'^2 + \frac{d\sqrt{c}-e\sqrt{a}}{\sqrt{a+c}}\,y' + \frac{d\sqrt{a}+e\sqrt{c}}{\sqrt{a+c}}\,x' + f = 0.$$

94. In order to reduce the equation still lower, let us transfer the axes parallel to themselves by means of the formulas $y' = y'' + n$ and $x' = x'' + m$ (54.)

then the equation $a'\,y'^2 + d'\,y' + e'\,x' + f = 0$ becomes

$$a'\,(y''+n)^2 + d'\,(y''+n) + e'\,(x''+m) + f = 0,$$

$$\text{or } a'\,y''^2 + (2\,a'\,n + d')\,y'' + e'\,x'' + a'\,n^2 + d'\,n + e'\,m + f = 0.$$

And since we have two independent quantities, m and n, we can make two hypotheses respecting them; let, therefore, their values be such that the co-efficient of y' and the constant term in the equation each $= 0$· that is, let

$$2\,a'\,n + d' = 0, \text{ and } a'\,n^2 + d'\,n + e'\,m + f = 0;$$

$$\text{whence } n = \frac{-d'}{2\,a'} \text{ and } m = \frac{d'^2 - 4\,a'\,f}{4\,a'\,e'},$$

and the reduced equation is now of the form

$$a'\,y''^2 + e'\,x'' = 0;$$

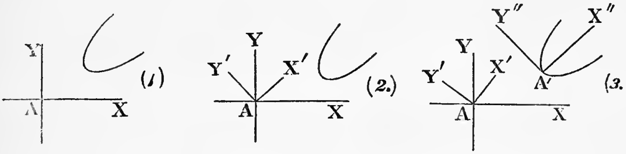
and it is manifest that if b had been positive, the equation $c' x'^2 + d' y' + e' x' + f = 0$ would have been reduced to the form

$$c' x''^2 + d' y'' = 0,$$

where the values of m and n would be found from the equations

$$m = -\frac{e'}{2 c'}, \text{ and } n = \frac{e^2 - 4 c' f}{4 c' d'}.$$

95. The following figures will exhibit the changes which have taken place in regard to the *position* of the locus corresponding to each analytical change in the *form* of the equation :—



In fig. 1, the curve is referred to rectangular axes A X and A Y, and the equation is

$$a y^2 + b x y + c x^2 + d y + e x + f = 0.$$

In fig. 2, the axes are transferred into the position A X′, A Y′, the angle X A X′ or θ being determined by the equation $\tan. 2 \theta = \frac{-b}{a - c}$, the corresponding equation is, for b negative,

$$a' y'^2 + d' y' + e' x' + f = 0.$$

If b is positive, the curve would originally have been situated at right angles to its present position, and the reduced equation would be

$$c' x'^2 + d' y' + e' x' + f = 0.$$

In fig. 3, the position of the origin is changed from A to A′, the co-ordinates of A′ being measured along A X′ and A Y′, and their values determined by the equations

for b negative, $n = \frac{-d'}{2 a'}$ and $m = \frac{d'^2 - 4 a' f}{4 a' e'}$

for b positive, $m = \frac{-e'}{2 c'}$ and $n = \frac{e'^2 - 4 c' f}{4 c' d'}$.

The reduced equation is

for b negative, $a' y''^2 + e' x'' = 0$.
for b positive, $c' x''^2 + d' y'' = 0$.

96 *. If the original axes are oblique, the transformation of the general equation must be effected by means of the formulas in (55). The values of a', b', and c' will be exactly the same as in (87).

We may then let $b' = 0$, and also find $\tan. 2 \theta$ when the axes are rect-

* See Note, Art. 87.

angular, whence, as in (87), we shall find that there is but one such system of axes.

The same value of θ which destroys the term in $x'\,y'$ will, as in (93), also destroy the term in x'^2 or y'^2; hence the reduced equation will be

for c sin. $2\,\omega - b$ sin. ω positive, $a'\,y'^2 + d'\,y' + e'\,x' + f = 0$.
for c sin. $2\,\omega - b$ sin. ω negative, $c'\,x'^2 + d'\,y' + e'\,x' + f = 0$.

97. To find the values of a', c', d', and e'.

The values of a' and c' are best deduced from those in art. (88), Since $b^2 - 4\,a\,c = 0$, we have for c sin. $2\,\omega - b$ sin. ω positive

$$\text{M} = a - b \cos.\,\omega + c$$

$$a' = \{a - b \cos.\,\omega + c\} \frac{1}{(\sin.\,\omega)^2}$$

$$c' = 0$$

$$\cos.\,2\,\theta = \frac{a - b \cos.\,\omega + c \cos.\,2\,\omega}{a - b \cos.\,\omega + c};$$

$$\sin.\,\theta = \frac{\sin.\,\omega \sqrt{c}}{\sqrt{a - b \cos.\,\omega + c}}, \text{ and } \cos.\,\theta = \sqrt{\frac{a - b \cos.\,\omega + c\,(\cos.\,\omega)}{a - b \cos.\,\omega + c}}$$

Also $d' = \dfrac{d \cos.\,\theta - e \cos.\,(\omega - \theta)}{\sin\,\omega}$ from the transformed equation

$$= \frac{(d - e \cos.\,\omega) \sqrt{\{a - b \cos.\,\omega + c\,(\cos.\,\omega)^2\}} - e \sqrt{c}\,(\sin.\,\omega)^2}{\sin.\,\omega \sqrt{\{a - b \cos.\,\omega + c\}}}$$

and $e' = \dfrac{d \sin.\,\theta + e \sin.\,(\omega - \theta)}{\sin.\,\omega}$

$$\frac{(d - e \cos.\,\omega) \sqrt{c} + e \sqrt{\{a - b \cos.\,\omega + c\,(\cos.\,\omega)^2\}}}{\sqrt{\{a - b \cos.\,\omega + c\}}}$$

and the reduced equation is now of the form

$$a'\,y'^2 + d'\,y' + e'\,x' + f = 0;$$

For c sin. $2\,\omega - b$ sin. ω negative, the corresponding values of a', c', M, d', and e' are

$$\text{M} = -(a - b \cos.\,\omega + c)$$

$$a' = 0$$

$$c' = (a - b \cos.\,\omega + c) \frac{1}{(\sin.\,\omega)^2}$$

sin. θ and cos. θ merely change values,

hence $d' = \dfrac{(d - e \cos.\,\omega) \sqrt{c} - e \sqrt{\{a - b \cos.\,\omega + c\,(\cos.\,\omega)^2\}}}{\sqrt{\{a - b \cos.\,\omega + c\}}}$

and $e' = \dfrac{(d - e \cos.\,\omega) \sqrt{\{a - b \cos.\,\omega + c\,(\cos.\,\omega)^2\}} + e \sqrt{c}\,(\sin.\,\omega)^2}{\sin.\,\omega \sqrt{\{a - b \cos.\,\omega + c\}}}$

and the reduced equation is now of the form

$$c'\,x'^2 + d'\,y' + e'\,x' + f = 0.$$

The transformation required to reduce the equations still lower is performed exactly as in (94); and, by making the angle between the original axes oblique, the figures in (95) will exhibit the changes in the *position* of the curve.

98. We shall conclude the discussion of this class of curves by the application of the results already obtained to a few examples.

Ex. 1. $y^2 - 6xy + 9x^2 + 10y + 1 = 0$; locus a parabola.

$$\tan. 2\theta = \frac{-b}{a-c} = -\frac{3}{4}; \text{ hence } \theta \text{ may be found by the tables.}$$

b is negative;

$$\therefore \text{by (93) } a' = a + c = 10, c' = 0, d' = \sqrt{10} \text{ and } e' = 3\sqrt{10},$$

$$\therefore 10y'^2 + \sqrt{10}\,y' + 3\sqrt{10}\,x' + 1 = 0.$$

Also by (94) $n = \dfrac{-d'}{2a'} = \dfrac{-1}{2\sqrt{10}}$, and $m = \dfrac{-1}{4\sqrt{10}}$;

and the final equation is

$$y''^2 + \frac{3}{\sqrt{10}}x = 0.$$

Ex. 2. $y + 2xy + x^2 + y - 3x + 1 = 0$; locus a parabola.

$$x''^2 + \sqrt{2}\,y = 0.$$

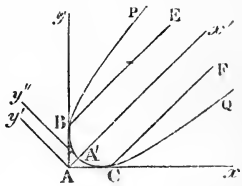
Ex. 3. $\sqrt{y} + \sqrt{x} = \sqrt{d}$. This equation may be put under the form $y + x - d = 2\sqrt{xy}$; or

$$y^2 - 2xy + x^2 - 2dy - 2dx + d^2 = 0;$$

and the locus is a parabola because it satisfies the condition

$$b^2 - 4ac = 0.$$

By tracing the curve as in (78) we shall find its position to be that of



PBCQ in the figure; and $y = x \pm d$ are the equations to the diameters BE and CF.

A x', A y', are the new axes, θ or x A x being 45°.

$$a' = 2,\ d' = 0,\ e' = -2d\sqrt{2},\ n = o,\ m = \frac{d}{2\sqrt{2}};$$

the last two quantities are to be measured along the new axes, therefore take A A'$= \dfrac{d}{2\sqrt{2}}$, and A' is the new origin.

The final equation is

$$y^2 = dx\sqrt{2}.$$

Ex. 4. $y = d + ex + fx^2$. The locus is a parabola, since $b^2 - 4ac$ or $0 - 4.0.f = 0$.

Let $y = y' + n$, and $x = x' + m$;

$$\therefore y' + n = d + e(x' + m) + f(x' + m)^2;$$

$$\therefore f x'^2 + (2 m f + e) x' - y' + f m^2 + e m + d - n = 0.$$

Let $2 m f + e = 0$, and $f m^2 + e m + d - n = 0$

$$\therefore m = \frac{-e}{2 f} \text{ and } n = \frac{4 d f - e^2}{4 f},$$

and the equation is reduced at once to the form

$$f x'^2 - y' = 0.$$

99. The axes oblique.

$y^2 - 2 x y + x^2 - 6 x = 0$; the angle between the axes being 60°.

Here, $c \sin. 2 \omega - b \sin. \omega$ is positive.

$$\sin. \theta = \frac{\sin. 60}{\sqrt{3}} = \frac{1}{2} \quad \therefore \theta = 30^\circ$$

$$M = 3,\ a' = 4,\ c' = 0,\ d' = 6,\ e' = -2\sqrt{3},\ m = -\frac{3\sqrt{3}}{8},$$

$$n = -\frac{3}{4};$$

$$\therefore 4 y^2 - 2\sqrt{3}\ x = 0.$$

CHAPTER VIII

THE ELLIPSE.

100. In the discussion of the general equation of the second order, we have seen that, supposing the origin of co-ordinates in the centre, there is but one system of rectangular axes to which, if the corresponding ellipse be referred, its equation is of the simple form

$$\left(\frac{-a'}{f'}\right) y^2 + \left(\frac{-c'}{f'}\right) x^2 = 1$$

$$\text{or, } P y^2 + Q x^2 = 1$$

where the coefficients P and Q are both positive. (86, 87.)

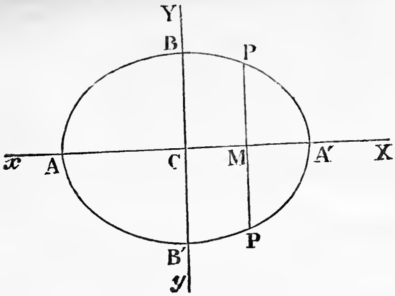
We now proceed to deduce from this equation the various properties of the ellipse.

To exhibit the coefficients in a better form; let C be the centre of the curve; X x, Y y, the rectangular axes meeting in C; C M $= x$, M P $= y$.

Then at the points where the curve cuts the axes, we have

$$y = 0, \quad Q x^2 = 1, \quad \therefore x = \pm \frac{1}{\sqrt{Q}}.$$

$$x = 0, \quad P y^2 = 1, \quad \therefore y = \pm \frac{1}{\sqrt{P}}.$$



In the axis of x take $C\,A' = \dfrac{1}{\sqrt{Q}}$ and $C\,A = -\dfrac{1}{\sqrt{Q}}$,

also in the axis of y take $C\,B = \dfrac{1}{\sqrt{P}}$ and $C\,B' = -\dfrac{1}{\sqrt{P}}$,

then the curve cuts the axes at the points A, A′, B, and B′.

Also if $C\,A = a$ and $C\,B = b$, and a be greater than b, we have Q $= \dfrac{1}{a^2}$ and $P = \dfrac{1}{b^2}$, therefore the equation to the curve becomes

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$$

$$\text{or} \qquad a^2 y^2 + b^2 x^2 = a^2 b^2$$

$$\text{or} \qquad y^2 = \frac{b^2}{a^2}(a^2 - x^2).$$

101. We have already seen (76) that the curve is limited in every direction.

The points A, A′, B, and B′ determine those limits. From the last equation we have

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2} \quad (1), \quad \text{and } x = \pm \frac{a}{b}\sqrt{b^2 - y^2} \quad (2),$$

from (1) if x is greater than $\pm\, a$, y is impossible, and from (2) if y is greater than $\pm\, b$, x is also impossible; hence straight lines drawn through the points A, A′, B and B′ parallel to the axes, completely enclose the curve.

Again from (1) for every value of x less than a we have two real and equal values of y, that is, for any abscissa C M less than C A′ we have two equal ordinates M P, M P′, the $\pm$ sign determining their opposite directions.

Also as x increases from 0 to $+\, a$ these values of y decrease from $\pm\, b$ to 0, hence we have two equal arcs B P A′, B′ P′ A′ exactly similar and opposite to one another.

If x be negative, and decrease from 0 to $-\, a$, x^2 is positive, and the same values of y must recur, hence there are two equal and opposite arcs B A, B′A. Therefore the whole curve is divided into two equal parts by the axis of x.

From (2) the curve appears in the same way to be divided into two equal parts by the axis of y: hence it is said to be symmetrical with respect to those axes.

Its concavity must also be turned towards the centre, otherwise it might be cut by a straight line in more points than two, which is impossible (71).

102. From the equation $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ we have

$$CP = \sqrt{x^2+y^2} = \sqrt{x^2 + \frac{b^2}{a^2}(a^2-x^2)} = \sqrt{b^2 + \frac{a^2-b^2}{a^2}x^2}$$

hence C P is greatest when x is greatest, that is, when $x = a$, in which case C P becomes also equal to a, hence C A or C A′ is the greatest line that can be drawn from the centre to the curve. Again C P is least when $x = 0$, in which case C P becomes equal to b, hence C B is the least line that can be drawn from C to the curve. The axes A A′ and B B′ are thus shown to be the greatest and least lines that can be drawn through the centre. The greater A A′ is called the axis major, or greater axis, or transverse axis, and B B′ the axis minor or lesser axis.

103. The points A, B, A′ and B′ are called the vertices or summits of the curve. Any of these points may be taken for the origin, thus let A be the origin, A C the axis of x, and let the axis of y be parallel to C B, and A M $= x'$.

Then $x = \text{C M} = \text{A M} - \text{A C} = x' - a$

$$\therefore\ y^2 = \frac{b^2}{a^2}(a^2 - x^2) = \frac{b^2}{a^2}\{a^2 - (x'-a)^2\} = \frac{b^2}{a^2}(2\,a\,x' - x'^2)$$

or suppressing the accents, $y^2 = \frac{b^2}{a^2}(2\,a\,x - x^2) = \frac{b^2}{a^2}\,x\,(2\,a - x)$.

This last equation is geometrically expressed by the following proportion. The square upon M P : the rectangle A M, M A′ :: the square upon B C : the square upon A C.

Hence the square upon the ordinate varies as the rectangle contained by the segments of the axis major.

If the origin be at C, C A′ the axis of y and C B the axis of x, we have, putting x for y and y for x, the equation $y^2 = \frac{a^2}{b^2}(b^2 - x^2)$, and if the origin be at B, $y^2 = \frac{a^2}{b^2}(2\,b\,x - x^2)$.

104. If the axes major and minor were equal to one another, the equation to the ellipse would become $y^2 = a^2 - x^2$, which is that to a circle whose diameter is $2\,a$, hence we see as in (79) that the circle is a species of ellipse. As we advance we shall have frequent occasion to remark the analogy existing between these two curves.

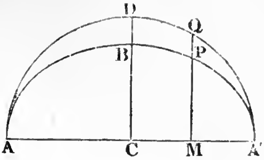
Let A D Q A′ be the circle described upon A A′ as diameter, and M Q or Y be an ordinate corresponding to the abscissa C M or x, let M P ($=y$) be the corresponding ordinate to the ellipse, then we have

$$Y^2 = a^2 - x^2$$

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\therefore y^2 = \frac{b^2}{a^2} Y^2 \text{ and } y = \frac{b}{a} Y$$

$$\therefore y : Y :: b : a$$



thus the ordinate to the ellipse has to the corresponding ordinate of the circle the constant ratio of the axis minor to the axis major.

Since b is less than a the circle is wholly without the ellipse, except at A and A′ where they meet. Similarly if a circle be described on the axis minor, it is wholly within the ellipse except at B and B′. Thus the elliptic curve lies between the two circumferences.

THE FOCUS.

105. The equation $y^2 = \frac{b^2}{a^2}(2\,a\,x - x^2)$ may be put under the form $y^2 = l\,x - \frac{l}{2\,a}x^2$, in which case the quantity $l = \frac{2\,b^2}{a}$ is called the principal Parameter or Latus Rectum.

Since $l = \frac{2\,b^2}{a} = \frac{4\,b^2}{2\,a}$ the Latus Rectum is a third proportional to the axis major and minor.

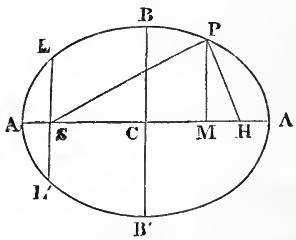
106. To find from what point in the axis major a double ordinate and be drawn equal to the Latus Rectum.

Here $4\,y^2 = l^2$ or $\frac{4\,b^2}{a^2}(a^2 - x^2) = \frac{4\,b^4}{a^2}$

$$\therefore a^2 - x^2 = b^2$$

$$\text{or } x^2 = a^2 - b^2$$

$$\text{and } x = \pm\sqrt{a^2 - b^2}.$$



With centre B and radius a describe a circle cutting the axis major in the points S and H, then we have $CH = +\sqrt{a^2 - b^2}$ and $CS = -\sqrt{a^2 - b^2}$, thus S and H are the points through either of which if an ordinate at L S L′ be drawn, it is equal to the Latus Rectum; henceforward then we shall consider this line as the Latus Rectum or principal parameter of the ellipse.

The two points S and H thus determined are called the Foci, for a reason to be hereafter explained.

107. The fraction $\frac{\sqrt{a^2 - b^2}}{a}$ which represents the ratio of C S to C A

is called the excentricity, because the deviation of this curve from the circular form, that is, its ex-centric course, depends upon the magnitude of this ratio.

If the excentricity, which is evidently less than unity, be represented by the letter e, we have $\frac{\sqrt{a^2 - b^2}}{a} = $: whence $e^2 = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2} \therefore \frac{b^2}{a^2} = 1 - e^2$ and the equation to the ellipse may be put under the form

$$y^2 = (1 - e^2)(a^2 - x^2).$$

108. The line S C is sometimes called the ellipticity; its value, as above, is $a\,e$; but it is also expressed by the letter c. Also since $a^2 - b^2 = a^2 e^2$ we have $b^2 = a^2 - a^2 e^2 = (a - ae)(a + a\,e)$; hence

The rectangle A S, S A′ = The square upon B C.

109. To find the distance from the focus to any point P in the curve.

Let $SP = r$, $HP = r'$,

$$\therefore r^2 = (y - y')^2 + (x - x')^2 \ldots (29)$$

also y', x' being the co-ordinates of S, we have $y' = 0$ and $x' = -a\,e$,

$$\begin{aligned}\therefore r^2 &= y^2 + (x + a\,e)^2 \\ &= (1 - e^2)(a^2 - x^2) + (x + a\,e)^2 \\ &= a^2 - x^2 - e^2 a^2 + e^2 x^2 + x^2 + 2\,a\,e\,x + a^2 e^2 \\ &= a^2 + 2\,a\,e\,x + e^2 x^2 \\ &= (a + e\,x)^2\end{aligned}$$

$$\therefore SP = a + e\,x; \text{ similarly } HP = a - e\,x.$$

In all questions referring to the absolute magnitude of S P or H P we must give to x its proper sign; thus if P is between B and A, the absolute magnitude of S P is $a - e\,x$, because x is itself negative.

By the addition of S P and H P, we have $SP + HP = 2\,a = AA'$; that is, the sum of the distances of any point on the curve from the foci is equal to the axis major

This property is analogous to that of the circle, where the distance of any point from the centre is constant.

110. This property of the ellipse is so useful, that we shall prove the converse. To find the locus of a point P, the sum of whose distances from two fixed points S and H is constant or equal $2\,a$.

Let $SH = 2c$, bisect S H in C, which point assume to be the origin of rectangular axes C A′, C B; let $CM = x$, and $MP = y$,

$$\text{then } SP = \sqrt{(c + x)^2 + y^2}$$

$$HP = \sqrt{(c - x)^2 + y^2}$$

$$\text{but } SP + HP = 2\,a, \quad \text{or } SP = 2\,a - HP$$

$$\therefore \sqrt{(c + x)^2 + y^2} = 2\,a - \sqrt{(c - x)^2 + y^2}$$

$$\therefore (c + x)^2 + y^2 = 4\,a^2 - 4\,a\sqrt{(c - x)^2 + y^2} + (c - x)^2 + y^2;$$

hence, transposing and dividing by 4, we have

$$a\sqrt{(c - x)^2 + y^2} = a^2 \cdot c\,x$$

$$\therefore\ a^2 y^2 = a^4 - a^2 c^2 + c^2 x^2 \quad a^2 x^2$$
$$= (a^2 - c^2)(a^2 - x^2)$$
$$\text{and } y^2 = \frac{a^2 - c^2}{a^2}(a^2 - x^2)$$

Hence the locus is an ellipse whose axes are $2a$ and $2\sqrt{a^2 - c^2}$, and whose foci are S and H.

THE TANGENT.

111. To find the equation to the tangent to the ellipse at any point.

Let $x'\ y'$ be the point P

. . . $x''\ y''$ be any other point Q

the equation to the line P Q through these two points is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'), \quad (41)$$

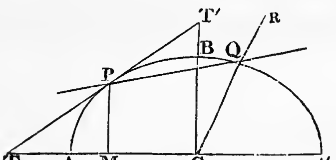
Now this cutting line or secant PQ will come to the position T P T′ or just touch the curve when Q comes to P, and the equation P Q will become the equation to the tangent P T when $x'' = x'$ and $y'' = y'$.

In this case the term $\frac{y' - y''}{x' - x''}$ becomes $\frac{0}{0}$, but its value may yet be found, for since the points $x'\ y'$, $x''\ y''$ are on the curve, we have

$$a^2 y'^2 + b^2 x'^2 = a^2 b^2$$
$$a^2 y''^2 + b^2 x''^2 = a^2 b^2$$
$$\therefore\ a^2 (y'^2 - y''^2) + b^2 (x'^2 - x''^2) = 0\,;$$
$$\text{or } a^2 (y' - y'')(y' + y'') + b^2 (x' - x'')(x' + x'') = 0,$$
$$\therefore\ \frac{y' - y''}{x' - x''} = -\frac{b^2}{a^2}\,\frac{x' + x''}{y' + x''}$$
$$= -\frac{b^2}{a^2}\frac{x'}{y'} \text{ when } x'' = x' \text{ and } y'' = y.$$

∴ The equation to the tangent is

$$y - y' = -\frac{b^2}{a^2}\frac{x'}{y'}(x - x')$$



By multiplication $a^2 y y' - a^2 y'^2 = -b^2 x x' + b^2 x'^2$

$$\therefore\ a^2 y y' + b^2 x x' = a^2 y'^2 + b^2 x'^2$$
$$= a^2 b^2.$$

In the figure C M is x' and M P is y', and x and y are the co-ordinates of any point in T P T'.

The equation to the tangent is easily recollected, since it may be obtained from that to the curve $a^2 y^2 + b^2 x^2 = a^2 b^2$ by putting $y y'$ for y^2 and $x x'$ for x^2.

112. That P T is a tangent is evident, since a straight line cannot cut the curve in more points than two, and here those two have gradually coalesced; it may, however, be satisfactory to show that every point in P T except P is without the curve.

Let x_1 and y_1 be the co-ordinates of any point R; then if $a^2 y_1^2 + b^2 x_1^2$ is greater than $a^2 b^2$, the point R is without the curve. For, join the point R with the centre of the ellipse by a line cutting the curve in Q, and let x and y be the co-ordinates of Q, then if $a^2 y_1^2 + b^2 x_1^2$ is greater than $a^2 b^2$, or than $a^2 y^2 + b^2 x^2$, we have $b^2 (x_1^2 - x^2)$ greater than $a^2 (y^2 - y_1^2)$; but b is less than a, therefore $x_1^2 - x^2$ must be greater than $y^2 - y_1^2$, or $x_1^2 + y_1^2$ greater than $x^2 + y^2$, and therefore C R greater than C Q (29), or R is without the curve.

In the present case we have the two equations.

$$a^2 y y' + b^2 x x' = a^2 b^2$$

$$a^2 y'^2 + b^2 x'^2 = a^2 b^2$$

$$\therefore\ a^2 y'^2 - 2 a^2 y y' + b^2 x'^2 - 2 b^2 x x' = - a^2 b^2$$

$$\text{or } a^2 (y' - y)^2 + b^2 (x' - x)^2 = a^2 y^2 + b^2 x^2 - a^2 b^2$$

$$\therefore\ a^2 y^2 + b^2 x^2 = a^2 b^2 + a^2 (y' - y)^2 + b^2 (x' - x)^2$$

which is greater than $a^2 b^2$.

But y and x are the co-ordinates of any point in the tangent; therefore generally any point on the tangent is without the curve; in the particular case where $y = y'$, and $x = x'$, that is at P, we have the equation $a^2 y^2 + b^2 x^2 = a^2 b^2$, therefore at that point the tangent coincides with the curve.

113. If the vertex A be the origin, the equation to the curve is

$$y^2 = \frac{b^2}{a^2} (2 a x - x^2) \text{ or } a^2 y^2 + b^2 x^2 - 2 a b^2 x = 0,$$

and the equation to the tangent, found exactly as above, is

$$a^2 y y' + b^2 x x' - a b^2 (x + x') = 0;$$

If the equation to the ellipse be $y^2 = m x - n x^2$, the equation to the tangent is $y y' = \dfrac{m}{2} (x + x') - n x x'$.

Generally, if the equation to the curve be

$$a y^2 + b x y + c x^2 + d y + e x + f = 0,$$

the equation to the tangent is

$$y - y' = - \frac{2 c x' + b y' + e}{2 a y' + b x' + d} (x - x'),$$

$$\text{or } (2 a y' + b x' + d) y + (2 c x' + b y' + e) x + d y + e x + 2 f = 0.$$

Again let $y = \alpha x + d$ be the equation to a tangent to the ellipse; then, comparing this with the equation $a^2 y y' + b^2 x x' = a^2 b^2$, and eliminating x' and y' by means of the equation $a^2 y'^2 + b^2 x'^2 = a^2 b^2$, we have

$$a^2 \alpha^2 + b^2 = d^2,$$

and this is the necessary relation among the co-efficients of the equation $y = \alpha x + d$ when it is a tangent to the curve.

114. To find the point where the tangent cuts the axes.

In the equation $a^2 y y' + b^2 x x' = a^2 b^2$ put $y = 0 \therefore b^2 x x' = a^2 b^2$, and $x = \frac{a^2}{x'} = \text{C T}$; similarly $y = \text{C T}' = \frac{b^2}{y'}$; hence we have

The rectangle C T, C M = The square upon A C,

and The rectangle C T′, M P = The square upon B C

Since C T$\left(= \frac{a^2}{x'}\right)$ does not involve y', it is the same for all ellipses which have the same axis major, and same abscissa for the point of contact; and, as the circle on the axis major may be considered as one of these ellipses, the distance C T is the same for an ellipse and its circumscribing circle.

Again, since C T $= \frac{a^2}{x'}$ is independent of the sign of y', the tangents, at the two extremities of an ordinate, meet in the same point on the axis. The equation to the lower tangent is found by putting $-y'$ for y' in the general equation to the tangent (111).

115. The distance M T from the foot of the ordinate to the point where the tangent meets the axis of x, is called the subtangent.

In the ellipse, M T = C T − C M $= \frac{a^2}{x'} - x' = \frac{a^2 - x'^2}{x'}$;

Hence, The rectangle C M, M T = The rectangle A M, M A′.

116. The equation to the tangent being $a^2 y y' + b^2 x x' = a^2 b^2$, let $x' = a$; and $\therefore y' = 0$, $\therefore b^2 a x = a^2 b^2$ and $x = a$; hence the tangent, at the extremity of the axis major, is perpendicular to that axis. At B, the equation to the tangent is $y = b$; hence the tangent at B is perpendicular to the axis minor.

The equation to the tangent being $a^2 y y' + b^2 x x' = a^2 b^2$, or

$$y = -\frac{b^2 x'}{a^2 y'} x + \frac{b^2}{y'}.$$

If P C be produced to meet the curve again in P′, the signs of the co-ordinates of P′ are both contrary to those of P; hence the co-efficient $-\frac{b^2 x'}{a^2 y'}$ remains the same for the tangent at P′, or the tangents at P and P′ are parallel (43).

117. To find the equation to the tangent at the extremity of the Latus Rectum.

The equation to the tangent is generally

$$a^2 y y' + b^2 x x' = a^2 b^2$$

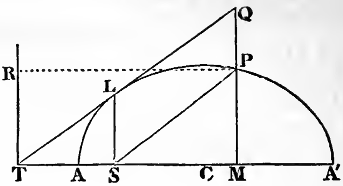
At L, $x' = -ae$ and $y' = \frac{b^2}{a}$,

$$\therefore\ a^2 y \frac{b^2}{a} - b^2 x\, a e = a^2 b^2$$

$$y = a + e x.$$

If the ordinate y, or M Q, cut the ellipse in P, we have S P $= a + e x$ (109)

$$\therefore\ \text{M Q} = \text{S P}.$$



118. To find the point where this particular tangent cuts the axis, let $y = 0$; $\therefore\ x = \text{CT} = -\frac{a}{e}$

From T draw T R perpendicular to A C, and from P draw P R parallel to A C; then, taking the absolute values of C M and C T, we have

$$\text{PR} = \text{MT} = \text{CT} + \text{CM} = \frac{a}{e} + x = \frac{a + e x}{e} = \frac{1}{e}.\text{SP};$$

Consequently, the distances of any point P from S, and from the line T R, are in the constant ratio of $e\ :\ 1$.

This line T R is called the directrix; for, knowing the position of this line and of the focus, an ellipse of any excentricity may be described, as will hereafter be shown.

If $x = 0$, we have $y = a$. Thus the tangent, at the extremity of the Latus Rectum, cuts the axis of y where that axis meets the circumscribing circle.

By producing Q M to meet the ellipse again in P′, it may be proved that

The rectangle Q P, Q P′ = The square on S M.

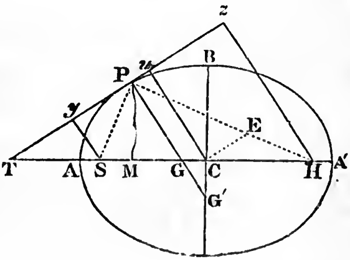
119. To find the length of the perpendicular from the focus on the tangent.

Let S y, H z, be the perpendiculars on the tangent P T.

Taking the expression in (48.) we have

$$p = -\frac{y_1 - \alpha x_1 - d}{\sqrt{1 + \alpha^2}}.$$

Where $y_1 = 0$ and $x_1 = -ae$ are co-ordinates of the point S, and $y = \alpha x + d$ is the equation to the line P T. But the equation to P T is also



$$y = -\frac{b^2 x'}{a^2 y'} x + \frac{b^2}{y'};$$

$$\therefore \alpha = -\frac{b^2 x'}{a^2 y'} \text{ and } d = \frac{b^2}{y'};$$

$$\therefore p = -\frac{-\frac{b^2 x'}{a^2 y'} a e - \frac{b^2}{y'}}{\sqrt{\left\{1 + \frac{b^4 x'^2}{a^4 y'^2}\right\}}}$$

$$= \frac{a b^2. (a + e x')}{\sqrt{a^4 y'^2 + b^4 x'^2}}$$

And $a^4 y'^2 + b^4 x'^2 = a^2 (a^2 b^2 - b^2 x'^2) + b^4 x'^2 = a^2 b^2 \left\{a^2 - \frac{a^2 - b^2}{a^2} x'^2\right\}$

$$= a^2 b^2 (a^2 - e^2 x'^2);$$

$$\therefore p = \frac{a b^2 (a + e x')}{a b \sqrt{(a^2 - e^2 x'^2)}} = b \sqrt{\frac{a + e x'}{a - e x'}}.$$

Let S P or $a + e x' = r$, and H P or $a - e x' = 2a - r = r'$,

$$\therefore p^2 = \frac{b^2 r}{2a - r} = \frac{b^2 r}{r'}.$$

Similarly, if H $z = p'$, we have $p'^2 = b^2 \frac{r'}{r}$.

By multiplication we have $p p' = b^2$; Hence,

The rectangle S y, H z = The square upon B C.

120. To find the locus of y or z in the last article.

The equation to the curve at P is $a^2 y'^2 + b^2 x'^2 = a^2 b^2$ (1)

The equation to the tangent at P is $a^2 y y' + b^2 x x' = a^2 b^2$ (2).

The equation to the perpendicular S y (the co-ordinates of S being $-c, 0$) is $y = \alpha (x + c)$ and this line being perpendicular to the tangent (2), we have $\alpha = \frac{a^2 y'}{b^2 x'}$; and therefore the equation to S y is

$$y = \frac{a^2 y'}{b^2 x'} (x + c) \quad (3).$$

If we eliminate y' and x' from (1) (2) and (3), we shall have an equation involving x and y; but this elimination supposes x and y to be the same for both (2) and (3), and therefore can only refer to their intersection. Hence, the resulting equation is the locus of their intersection.

From (3) $\frac{y'}{x'} = \frac{b^2}{a^2} \frac{y}{x + c} = \frac{b^2}{x' y} - \frac{b^2 x}{a^2 y}$ from (2);

$$\therefore \frac{1}{x'} = \frac{y^2 + x (x + c)}{a^2 (x + c)}, \therefore x' = \frac{a^2 (x + c)}{y^2 + x (x + c)},$$

$$\text{and } y' = \frac{b^2 y x'}{a^2 (x + c)} = \frac{b^2 y}{y^2 + x (x + c)}.$$

Substituting these values of x' and y', in (1), we have

$$a^2 b^4 y^2 + b^2 a^4 (x+c)^2 = a^2 b^2 \{y^2 + x(x+c)\}^2$$

$$\therefore\ b^2 y^2 + a^2 (x+c)^2 = \{y^2 + x(x+c)\}^2;$$

Or, $a^2 y^2 - c^2 y^2 + a^2 (x^2 + c)^2 = y^4 + 2x(x+c)y^2 + x^2 (x+c)^2;$

$$\begin{aligned}\therefore\ a^2 \{y^2 + (x+c)^2\} &= y^4 + y^2 \{2x(x+c) + c^2\} + x^2 (x+c)^2\\ &= y^4 + y^2 x^2 + y^2 (x+c)^2 + x^2 (x+c)^2\\ &= y^2 (y^2 + x^2) + (y^2 + x^2)(x+c)^2\\ &= (y^2 + x^2)\{y^2 + (x+c)^2\};\end{aligned}$$

$$\therefore\ a^2 = y^2 + x^2.$$

This is the equation to a circle whose radius is a. Hence, the locus of y is the circle described on the axis major as diameter.

From the equation to S y, combined with that to C P $\left(y = \frac{y}{x'}x\right)$, we may prove that C P and S y meet in the directrix.

121. To find the angle which the focal distance S P makes with the tangent P T.

The equation to the tangent is $y = -\frac{b^2 x'}{a^2 y'}x + \frac{b^2}{y'}$.

The equation to S P passing through S $(-c, 0)$ and P (x', y') is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x') = \frac{y'}{x' + c}(x - x');$$

And tan. S P T = tan. (P S C − P T C) $= \dfrac{\dfrac{y'}{x'+c} + \dfrac{b^2 x}{a^2 y'}}{1 - \dfrac{y'}{x'+c}\dfrac{b^2 x'}{a^2 y'}}$

$$= \frac{a^2 y'^2 + b^2 x'^2 + b^2 c x'}{y'\{(x'+c)a^2 - b^2 x'\}} = \frac{a^2 b^2 + b^2 c x'}{y'\{(a^2 - b^2)x' + a^2 c\}}$$

$$= \frac{b^2 (a^2 + c x')}{c y' (a^2 + c x')} = \frac{b^2}{c y'}.$$

To pass from tan. S P T to tan. H P T we must put $-c$ for c in the preceding investigation; this would evidently lead us to the equation tan. H P T $= -\frac{b^2}{c y'}$; hence, tan. H P z = tan. (180 − H P T) = − tan. H P T $= \frac{b^2}{c y'}$, or the two angles S P T, H P z are equal; thus the tangent makes equal angles with the focal distances.

It is a property of light that, if a ray proceeding from H in the direction H P be reflected by the line z P y, the angle S P y of the reflected ray will equal the angle H P z. Now, in the ellipse, these angles are equal;

hence, if a light be placed at H, all rays which are reflected by the ellipse will proceed to S. Hence, these points, S and H, are called foci.

This very important property is also thus proved from article 119.

$$S\,y = p = b\sqrt{\frac{r}{r'}};\ \text{and } H\,z = p' = b\sqrt{\frac{r'}{r}};$$

$$\therefore\ S\,y : H\,z :: r : r' :: S\,P : H\,P;$$

hence the triangles $S\,P\,y$ and $H\,P\,z$ are similar, and the angle $S\,P\,y$ equal to the angle $H\,P\,z$.*

122. To find the length of the perpendicular $C\,u$, from the centre, on the tangent:

$$p = -\frac{y_1 - \alpha\,x_1 - d}{\sqrt{1+\alpha^2}};$$

here $y_1 = 0,\ x_1 = 0,\ \alpha = -\dfrac{b^2\,x'}{a^2\,y'}$, and $d = \dfrac{b^2}{y'}$;

$$\therefore C\,u = \frac{\dfrac{b^2}{y'}}{\sqrt{\left\{1 + \dfrac{b^4\,x'^2}{a^4\,y'^2}\right\}}} = \frac{a^2\,b^2}{\sqrt{\{a^4\,y'^2 + b^4\,x'^2\}}} = \frac{a^2\,b^2}{a\,b\,\sqrt{a^2 - e^2\,x'^2}}\quad (119)$$

$$= \frac{a\,b}{\sqrt{(a + e\,x')\,(a - e\,x')}} = \frac{a\,b}{\sqrt{r\,r'}}$$

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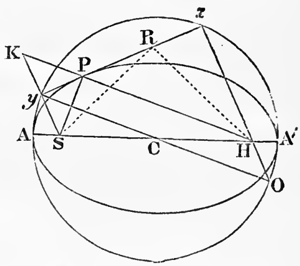
* The following geometrical method of drawing a tangent to the ellipse, and proving that the locus of the perpendicular from the focus on the tangent is the circumscribing circle, will be found useful.



Let $A\,P\,A'$ be the ellipse, P any point on it. Join $S\,P$ and $H\,P$, and produce $H\,P$ to K, making $P\,K = P\,S$; bisect the angle $K\,P\,S$ by the line $y\,P\,z$, and join $S\,K$, cutting $P\,y$ in y.

1. $P\,y$ is a tangent to the ellipse; for if R be any other point in the line $P\,y$, we have $S\,R + R\,H = K\,R + R\,H$, greater than $K\,H$, greater than $2\,a$; hence, R and every other point in $z\,P\,y$ except P is without the ellipse.

2. The locus of y is the circumscribing circle. Draw $H\,z$ parallel to $S\,y$, and join $C\,y$; then, because the triangles $S\,P\,y$, $K\,P\,y$ are equal, we have the angle $S\,y\,P$ a right angle, or $S\,y$ and $H\,z$ are perpendicular to the tangent. Also, since $S\,y = K\,y$, and $S\,C = C\,H$, we have $C\,y$ parallel to $K\,H$, and $C\,y = \frac{1}{2}\,K\,H = \frac{1}{2}\,(S\,P + P\,H) = C\,A$.

3. The rectangle $S\,y,\ H\,z$ = the square on $B\,C$. Let $Z\,H$ meet the circle again in O and join $C\,O$; then, because the angle $y\,z\,O$ is a right angle, and that the points y and O are in the circumference of the circle, the line $y\,C\,O$ must be a straight line, and a diameter. Hence, the triangles $C\,S\,y$, $C\,H\,O$ are equal; and the rectangle $S\,y,\ H\,z$ = the rectangle $Z\,H,\ H\,O$ = the rectangle $A\,H,\ H\,A'$ = the square on $B\,C$ (108).

4. Let $S\,P = r$, $H\,P = 2\,a - S\,P = 2\,a - r$, $S\,y = p$, and $H\,z = p'$, then $p^2 = \dfrac{b^2\,r}{2\,a - r}$;

For by similar triangles, $S\,y : S\,P :: H\,z : H\,P\ \therefore\ p = \dfrac{r}{2\,a - r}\,p'$; and, as above,

$$p\,p' = b^2\ \therefore\ p^2 = \frac{b^2\,r}{2\,a - r}.$$

123. To find the locus of u:

$$a^2 y'^2 + b^2 x'^2 = a^2 b^2 \quad (1).$$

$$a^2 y y' + b^2 x x' = a^2 b^2 \quad (2).$$

$$y = \frac{a^2 y'}{b^2 x'} x \; . \; . \; . \; (3), \text{ the equation to C } u.$$

Proceeding, as in (120.), to eliminate x' and y', we arrive at the final equation $b^2 y^2 + a^2 x^2 = (y^2 + x^2)^2$; the locus is an oval meeting the ellipse at the extremities of the axes, and bulging out beyond the curve, something like the lowest of figures 2 in page 44. We shall have occasion to trace this curve hereafter.

124. To find the angle which the distance C P makes with the tangent, we have the equation

$$\text{to C P, } y = \frac{y'}{x'} x; \text{ and to P T, } y = -\frac{b^2 x'}{a^2 y'} x + \frac{b^2}{y'};$$

hence tan. C P T is found $= \dfrac{a^2 b^2}{c^2 x' y'}$.

125. From C u = C y sin. C y u, we have

$$\frac{a b}{\sqrt{r r'}} = a \text{ sin. C } y u \therefore \text{ sin. C } y u = \frac{b}{\sqrt{r r'}};$$

Also from H z = H P sin. H P z, we have

$$b \sqrt{\frac{r'}{r}} = r' \text{ sin. H P } z \therefore \text{ sin. H P } z, = \frac{b}{\sqrt{r r'}};$$

$\therefore$ angle C y u = angle H P z, and C y is parallel to H P.

Hence, if C E be drawn parallel to the tangent P T, and meeting H P in E, we have P E = C y = A C.

THE NORMAL.

126. The normal to any point of a curve is a straight line drawn through that point, and perpendicular to the tangent at that point.

To find the equation to the normal P G.

The equation to a straight line through the point P (x' y') is

$$y - y' = \alpha (x - x')$$

This line must be perpendicular to the tangent whose equation is

$$y = -\frac{b^2 x'}{a^2 y'} x + \frac{b^2}{y'}, \quad \therefore \alpha = \frac{a^2 y'}{b^2 x'};$$

and the equation to the normal is

$$y - y' = \frac{a^2 y'}{b^2 x'} (x - x')$$

127. To find the points where the normal cuts the axes :

Let $y = 0 \therefore - y' = \dfrac{a^2 y'}{b^2 x'}(x - x') \therefore x = x' - \dfrac{b^2 x'}{a^2} = \dfrac{a^2 - b^2}{a^2} x'$ $= e^2 x' = \text{C G}$.

Let $x = 0 \therefore y = y' - \dfrac{a^2 y'}{b^2} = - \dfrac{a^2 - b^2}{b^2} y' = - \dfrac{a^2 e^2}{b^2} y' = \text{C G}'$.

Hence $\text{S G} = \text{S C} - \text{C G} = a e - e^2 x' = e(a - e x') = e.\ \text{S P}$.

The distance M G, from the foot of the ordinate to the foot of the normal, is called the subnormal:

Its value is $x - x' = - \dfrac{b^2}{a^2} x'$.

128. From the above values of M G, C G and C G′ we

have $\text{P G} = \sqrt{\left\{ y'^2 + \dfrac{b^4}{a^4} x'^2 \right\}} = \sqrt{\left\{ \dfrac{b^2}{a^2}(a^2 - x'^2) + \dfrac{b^4 x'^2}{a^4} \right\}}$

$$= \frac{b}{a} \sqrt{\left\{ a^2 - x'^2 + \frac{b^2 x'^2}{a^2} \right\}} = \frac{b}{a} \sqrt{\left\{ a^2 - \frac{a^2 - b^2}{a^2} x'^2 \right\}}$$

$$= \frac{b}{a} \sqrt{\left\{ a^2 - e^2 x'^2 \right\}} = \frac{b}{a} \sqrt{r r'};$$

and similarly $\text{P G}' = \dfrac{a}{b} \sqrt{r r'}$, consequently,

The rectangle, P G, P G′ $= r r'$ = the rectangle S P, H P.

The greatest value of the normal is when $x' = 0$; hence, at the extremity of the axis minor, we have the greatest value of the normal $= b$. Similarly, the least value of the normal is at the extremity of the axis major, the value being then $= \dfrac{b^2}{a}$, or half the Latus Rectum (105.).

Also, $\text{S G}' = \dfrac{a e}{b} \sqrt{r r'}$, and $\text{G G}' = \dfrac{a e^2}{b} \sqrt{r r'} \therefore \text{G G}' = e.\text{S G}'$.

If a perpendicular G L be drawn from G upon S P or H P, the triangles P G L, S P y, and H P z, are similar; hence

$$\text{P L} = \text{P G}.\frac{p}{r}, \text{ or } = \text{P G}\frac{p'}{r'} = \frac{b^2}{a} = \tfrac{1}{2} \text{ the Latus Rectum.}$$

129. Since the tangent makes equal angles with the focal distances, the normal, which is perpendicular to the tangent, also makes equal angles with the focal distances. This theorem may be directly proved from the above value of C G; for S G : H G :: S C − C G : H C + C G :: $a e - e^2 x' : a e + e^2 x' :: a - e x' : a + e x'$ :: S P : H P; hence, the angle S P H is bisected by the line P G.—Euclid, VI. 3, or Geometry, ii. 50 *.

THE DIAMETERS.

130. A diameter was defined in (76.) to be a line bisecting a system of parallel chords. We shall now prove that all the diameters of the ellipse

* The absolute values of S P and H P are here taken.—See 109.

are straight lines, and that they pass through the centre, which last circumstance is evident, since no line could bisect every one of a system of parallel chords without itself passing through the centre.

Let $y = \alpha x + c$ be the equation to any chord ;

$$a^2 y^2 + b^2 x^2 = a^2 b^2, \text{ the equation to the curve.}$$

Transfer the origin to the bisecting point x' y' of the chord, by putting $y + y'$ for y and $x + x'$ for x, then the equation to the chord becomes $y + y' = \alpha (x + x') + c$ or $y = \alpha x$, since $y' = \alpha x' + c$; also the equation to the curve becomes $a^2 (y + y')^2 + b^2 (x + x')^2 = a^2 b^2$.

To find where the chord intersects the curve, put αx for y in the second equation :

$$\therefore a^2 (\alpha x + y')^2 + b^2 (x + x')^2 = a^2 b^2;$$

$$\text{or, } (a^2 \alpha^2 + b^2) x^2 + 2 (a^2 \alpha y' + b^2 x') x + a^2 y'^2 + b^2 x'^2 = a^2 b^2.$$

But since the origin is at the bisection of the chord, the two values of x must be equal to one another, and have opposite signs, or the second term of the last equation must $= 0$.

$$\therefore a^2 \alpha y' + b^2 x' = 0.$$

This equation gives the relation between x' and y' ; and, since it is independent of c, it will be the same for any chord parallel to $y = \alpha x + c$; hence, considering x' and y' as variable, it is the equation to the assemblage of all the middle points, or to their locus.

This equation is evidently that to a straight line passing through the centre. Conversely, any straight line passing through the centre is a diameter.

131. A pair of diameters are called conjugate when each bisects all the chords parallel to the other.

Hence, the axes major and minor are conjugate diameters, and the equation $a^2 y^2 + b^2 x^2 = a^2 b^2$, which we have generally employed, is that to the ellipse referred to its centre and rectangular conjugate diameters.

If the curve be referred to oblique co-ordinates, and its equation remains of the same form, that is, containing only x^2, y^2, and constant quantities, the new axes will also be conjugate diameters ; for each value of one coordinate will give two equal and opposite values to the other. We shall, therefore, pass from the above equation to another referred to oblique conjugate diameters, by determining, through the transformation of co-ordinates, all the systems of axes, for which the equation to the ellipse preserves this same form.

Let the equation be $a^2 y^2 + b^2 x^2 = a^2 b^2$; the formulas for transformation are (57),

$$y = x' \sin. \theta + y' \sin. \theta',$$

$$x = x' \cos. \theta + y' \cos. \theta',$$

$$\therefore a^2 (x' \sin. \theta + y' \sin. \theta')^2 + b^2 (x' \cos. \theta + y' \cos. \theta')^2 = a^2 b^2,$$

$$\text{or } \{a^2 (\sin. \theta')^2 + b^2 (\cos. \theta')^2\} y'^2 + \{a^2 (\sin. \theta)^2 + b^2 (\cos. \theta)^2\} x'^2$$
$$+ 2 \{a^2 \sin. \theta \sin. \theta' + b^2 \cos. \theta \cos. \theta'\} x' y' = a^2 b^2.$$

In order that this equation may be of the conjugate form, it must not contain the term $x' y'$; but since we have introduced two indeterminate quantities, θ and θ', we are enabled to put the co-efficient of $x' y' = 0$; hence we have the condition

$$a^2 \sin. \theta \sin. \theta' + b^2 \cos. \theta \cos. \theta' = 0,$$

or dividing by $a^2 \cos.\theta \cos.\theta'$,

$$\tan.\theta.\tan.\theta' = -\frac{b^2}{a^2}.$$

Now this condition will not determine both the angles θ and θ', but for any value of the one angle it gives a real value for the other; and hence there is an infinite number of pairs of axes to which, if the curve be referred, its equation is of the required conjugate form.

If, in the next figure, we draw C P making any angle θ with C A', and C D making an angle θ' (whose tangent is $-\frac{b^2}{a^2}\cot.\theta$) with C A', then C P and C D are conjugate diameters. Also since the product of the tangents is negative, if C P be drawn in the angle A' C B, C D must be drawn in the angle B C A.

132. There is no occasion to examine the above equation of condition in the case where θ or $\theta' = 0$, for then we have the original axes; but let us examine whether there are any other systems of rectangular axes.

Let $\theta' = 90° + \theta$, $\therefore \sin.\theta' = \cos.\theta$, and $\cos.\theta' = -\sin.\theta$,

hence the equation of condition becomes

$$(a^2 - b^2)\sin.\theta\cos.\theta = 0,$$

and since, by the nature of the ellipse, a^2 cannot $= b^2$, we must have $\theta = 0$, or $\theta = 90°$, both which values give the original axes again; hence the only system of rectangular diameters is that of the axes. This remark agrees with article 87.

We may observe in the above transformation that, although we have introduced *two* indeterminate quantities θ and θ', it does not follow that we can destroy *two* terms in the transformed equation, unless the values of these quantities are real: for example, if we attempt to destroy any other term as the second, we find $\tan.\theta = \frac{b}{a}\sqrt{-1}$, a value to which there is no corresponding angle θ; hence, in putting the co-efficient of $x'y' = 0$, we adopted the only possible hypothesis.

133. The equation to the curve is now

$$\{a^2(\sin.\theta')^2 + b^2(\cos.\theta')^2\}y'^2 + \{a^2(\sin.\theta)^2 + b^2(\cos.\theta)^2\}x'^2 = a^2b^2.$$

If we successively make $y' = 0$, and $x' = 0$, we have the distances from the origin to the points in which the curve cuts the new axes; let these distances be represented by a_1 and b_1, the former being measured along the axis of x', and the latter along the axis of y'; then we have

$$y' = 0, \ \therefore \{a^2(\sin.\theta)^2 + b^2(\cos.\theta)^2\}a_1^2 = a^2b^2,$$
$$x' = 0, \ \therefore \{a^2(\sin.\theta')^2 + b^2(\cos.\theta')^2\}b_1^2 = a^2b^2.$$

And the transformed equation becomes

$$\frac{a^2b^2}{b_1^2}y'^2 + \frac{a^2b^2}{a_1^2}x'^2 = a^2b^2,$$

$$\text{or,}\quad \frac{y'^2}{b_1^2} + \frac{x'^2}{a_1^2} = 1,$$

$$\text{or,}\quad a_1^2y'^2 + b_1^2x'^2 = a_1^2b_1^2,$$

Where the lengths of the new conjugate diameters are $2a_1$ and $2b_1$.

134. From the transformation we obtain the three following equations:

$$a_1^2 \{a^2 (\sin. \theta)^2 + b^2 (\cos. \theta)^2\} = a^2 b^2 \quad (1),$$

$$b_1^2 \{a^2 (\sin. \theta')^2 + b^2 (\cos. \theta')^2\} = a^2 b^2 \quad (2),$$

$$\left.\begin{array}{l} a^2 \sin. \theta \sin. \theta' + b^2 \cos. \theta \cos. \theta' = 0, \\ \text{or,} \quad \tan. \theta \tan. \theta' = -\dfrac{b^2}{a^2} \end{array}\right\} \quad (3)$$

Putting $1 - (\sin. \theta)^2$ for $(\cos. \theta)^2$ in (1), we have

$$a_1^2 (a^2 - b^2) (\sin. \theta)^2 = a^2 b^2 - a_1^2 b^2,$$

$$\text{and } a_1^2 (a^2 - b^2) (\cos. \theta)^2 = a_1^2 a^2 - a^2 b^2,$$

$$\therefore (\tan. \theta)^2 = \frac{b^2}{a^2} \frac{a^2 - a_1^2}{a_1^2 - b^2}.$$

Putting b_1 for a_1 in this expression, we have the value of $(\tan. \theta')^2$, as found from (2)

$$(\tan. \theta')^2 = \frac{b^2}{a^2} \frac{a^2 - b_1^2}{b_1^2 - b^2};$$

hence by multiplication,

$$(\tan. \theta)^2 (\tan. \theta')^2 = \frac{b^4}{a^4} \frac{a^2 - a_1^2}{a_1^2 - b^2} \frac{a^2 - b_1^2}{b_1^2 - b^2} = \frac{b^4}{a^4} \text{ from (3}$$

$$\therefore (a^2 - a_1^2) (a^2 - b_1^2) = (a_1^2 - b^2) (b_1^2 - b^2),$$

$$\text{or, } a^4 - a^2 b_1^2 - a_1^2 a^2 + a_1^2 b_1^2 = a_1^2 b_1^2 - a_1^2 b^2 - b^2 b_1^2 + b^4;$$

$$\therefore a^4 - b^4 = a^2 b_1^2 + a_1^2 a^2 - a_1^2 b^2 - b^2 b_1^2,$$

$$= a^2 (a_1^2 + b_1^2) - b^2 (a_1^2 + b_1^2),$$

$$= (a^2 - b^2) (a_1^2 + b_1^2),$$

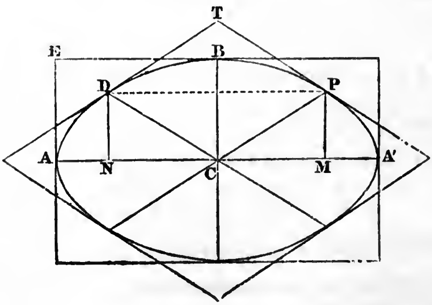
$$\therefore a^2 + b^2 = a_1^2 + b_1^2,$$

that is, the sum of the squares upon the conjugate diameters is equal to the sum of the squares upon the axes.

135. Again, multiplying (1) and (2) together, and (3) by itself, and then subtracting the results, we have

$$a_1^2 b_1^2 \{a^4 (\sin. \theta)^2 (\sin. \theta')^2 + b^4 (\cos. \theta)^2 (\cos. \theta')^2 + a^2 b^2 (\sin. \theta')^2 (\cos. \theta)^2 + a^2 b^2 (\sin. \theta)^2 (\cos. \theta')^2\} = a^4 b^4,$$

$$a^4 (\sin \theta)^2 (\sin. \theta')^2 + b^4 (\cos. \theta)^2 (\cos. \theta')^2 + 2 a^2 b^2 \sin. \theta \sin. \theta' \cos. \theta \cos. \theta' = 0;$$



$$\therefore\ a_1^2 b_1^2 a^2 b^2 \{ (\sin.\theta')^2 (\cos.\theta)^2 - 2\sin.\theta\sin.\theta'\cos.\theta\cos.\theta' + (\sin.\theta)^2(\cos.\theta')^2 \} = a^4 b^4,$$

$$\text{or, } a_1^2 b_1^2 \{ \sin.\theta'\cos.\theta - \sin.\theta\cos.\theta' \}^2 = a^2 b^2$$

$$\text{or, } a_1^2 b_1^2 \{ \sin.(\theta' - \theta) \}^2 = a^2 b^2;$$

$$\therefore\ a_1 b_1 \sin.(\theta' - \theta) = a\, b.$$

Now $\theta' - \theta$ is the angle P C D, between the conjugate diameters C P and C D; hence drawing straight lines at the extremities of the conjugate diameters, parallel to those diameters, we have, from the above equation, the parallelogram P C D T = the rectangle A C B E, and therefore the whole parallelogram thus circumscribing the ellipse is equal to the rectangle contained by the axes *.

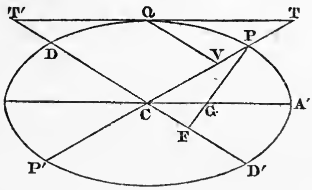
If the extremities of the conjugate diameters be joined, it is readily seen that the inscribed figure is a parallelogram, and that its area is equal to half that of the above circumscribed parallelogram.

We may remark, in passing, that the circumscribed parallelogram, having its sides parallel to a pair of conjugate diameters, is the least of all parallelograms circumscribing the ellipse; and that the inscribed parallelogram, having conjugate lines for its diameters, is the greatest of all inscribed parallelograms.

136. Returning to article (133.), the equation to the curve, suppressing the accents on x' and y', as no longer necessary, is

$$a_1^2 y^2 + b_1^2 x^2 = a_1^2 b_1^2.$$

In the figure, C P $= a_1$, C D $= b_1$ C V $= x$, and V Q $= y$.



* The theorems in articles 134 and 135 may be proved also in the following manner: Referring the curve to its rectangular axes, as in article (138.), let the co-ordinates of P be x' and y'; then the equation to C D is $a^2 y y' + b^2 x x' = 0$, and eliminating x and y between this equation and that to the curve $(a^2 y^2 + b^2 x^2 = a^2 b^2)$, we have the co-ordinates C N and D N, fig. 135, of the intersection of C D with the curve, C N $= x = \frac{a y'}{b}$ and D N $= y = \frac{b x'}{a}$; hence we have

$$a_1^2 + b_1^2 = x'^2 + y'^2 + x^2 + y^2 = x'^2 + y'^2 + \frac{a^2 y'^2}{b^2} + \frac{b^2 x'^2}{a^2} = \frac{b^2 x'^2 + a^2 y'^2}{b^2} + \frac{a^2 y'^2 + b^2 x'^2}{a^2} = \frac{a^2 b^2}{b^2} + \frac{a^2 b^2}{a^2} = a^2 + b^2.$$

Also the triangle P C D = the trapezium P M N D − the triangles P C M and D C N

$$= (x + x')\frac{y + y'}{2} - \frac{x y + x' y'}{2} = \frac{x' y + y' x}{2} = \tfrac{1}{2}\left\{ x'\frac{b x'}{a} + y'\frac{a y'}{b} \right\} = \frac{b^2 x'^2 + a^2 y'^2}{2 a b}$$

$= \frac{a^2 b^2}{2 a b} = \frac{a b}{2}$, therefore the parallelogram P C D T $= a\, b$.

No notice has been taken of the positional value of the abscissa C N, since this is entirely a question of absolute values.

Putting the equation into the form

$$y^2 = \frac{b_1^2}{a_1^2}(a_1^2 - x^2) = \frac{b_1^2}{a_1^2}(a_1 - x)(a_1 + x),$$

we have the square upon the ordinate Q V : the rectangle P V, V P′ :: the square upon C D : the square upon C P.

137. The equation to the tangent at any point Q (x' y') found exactly as in (111.) is $a_1^2\, y\, y' + b_1^2\, x\, x' = a_1^2\, b_1^2$.

The points T and T′, where it cuts the new axes, are determined as in (114.); whence $C\,T = \frac{a_1^2}{x'}$, $C\,T' = \frac{b_1^2}{y'}$; and the tangents drawn at the two extremities of a chord meet in the diameter to that chord (114.).

138. Let the ellipse be now referred to its rectangular axes, and let the co-ordinates of P be $x'\, y'$, then the equation to C P is $y = \frac{y'}{x'}\, x$, and the equation to C D is

$$y = x \tan.\theta' = -\frac{b^2}{a^2}\cot.\theta = -\frac{b^2\, x'}{a^2\, y'}\, x,$$

$$\text{or,}\quad a^2\, y\, y' + b^2\, x\, x' = 0.$$

But the equation to the tangent at P is

$$a^2\, y\, y' + b^2\, x\, x' = a^2\, b^2;$$

hence CD or the diameter conjugate to C P is parallel to the tangent at P.

From this circumstance the conjugate to any diameter is often defined to be the line drawn through the centre, and parallel to the tangent at the extremity of the diameter.

The equation to the conjugate diameter is readily remembered, since it is the same as that to the tangent without the last term, and therefore may be deduced from the equation to the curve, as at the end of article 111. The three equations are

$$a^2\, y^2 + b^2\, x^2 = a^2\, b^2,\ \text{to the curve,}$$

$$a^2\, y\, y' + b^2\, x\, x' = a^2\, b^2,\ \text{to the tangent,}$$

$$a^2\, y\, y' + b^2\, x\, x' = 0,\ \text{to the conjugate.}$$

The equation to the tangent D T passing through the point D, whose co-ordinates are $\frac{b\, x'}{a}$ and $-\frac{a\, y'}{b}$ (*note* 135), and parallel to C P, is

$$y - \frac{b\, x'}{a} = \frac{y'}{x'}\left(x + \frac{a\, y'}{b}\right)$$

or reducing

$$y\, x' - x\, y' = a\, b.$$

And the equation to C P is

$$y\, x' - x\, y' = 0.$$

These equations to the tangents and conjugate diameters, combined with the equation to the curve, will be found useful in the solution of problems relating to tangents.

139. Let x' and y' be rectangular co-ordinates of P; then, from the equation $a_1^2 + b_1^2 = a^2 + b^2$, we have $b_1^2 = a^2 + b^2 - a_1^2 = a^2 + b^2 - x'^2 - y'^2 = a^2 + b^2 - x'^2 - b^2 + \frac{b^2}{a^2} x'^2 = a^2 - \frac{a^2 - b^2}{a^2} x'^2 = a^2 - e^2 x'^2$
$= (a - ex')\,(a + ex') = r\,r'$.
That is, the square upon the conjugate diameter C D = the rectangle under the focal distances S P and H P.

140. Draw P F perpendicular upon the conjugate diameter C D, then by (135.) the rectangle P F, C D $= a\,b$,

$$\therefore \text{PF} = \frac{a\,b}{b_1} = \frac{a\,b}{\sqrt{a^2 + b^2 - a_1^2}} = \frac{a\,b}{\sqrt{r\,r'}}.\ *$$

It was shown in (128.) that $\text{PG} = \frac{b}{a}\sqrt{r\,r'}$, and $\text{PG}' = \frac{a}{b}\sqrt{r\,r'}$;

hence, The rectangle P G, P F = The square on B C,
and The rectangle P G′, P F = The square on A C,
and The rectangle P G, P G′ = The square on C D

SUPPLEMENTAL CHORDS.

141. Two straight lines drawn from a point on the curve to the extremities of a diameter are called supplemental chords. They are called principal supplemental chords if that diameter be the axis major.

Referring the ellipse to its axes, let P P′ be a diameter, Q P, Q P′ two supplemental chords; then, if $x'\,y'$ be the co-ordinates of P, $-x'$, $-y'$ are those of P′; hence, the equation to Q P is $y - y' = \alpha\,(x - x')$,

and the equation to Q P′ is $y + y' = \alpha'\,(x + x')$.

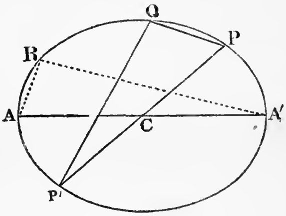
At the point of intersection, y and x are the same for both equations, being the co-ordinates of Q; hence, $y^2 - y'^2 = \alpha\,\alpha'\,(x^2 - x'^2)$;

but $a^2 y^2 + b^2 x^2 = a^2 b^2$ at Q,

and $a^2 y'^2 + b^2 x'^2 = a^2 b^2$ at P;

$$\therefore y^2 - y'^2 = -\frac{b^2}{a^2}\,(x^2 - x'^2);$$

$$\therefore \alpha\,\alpha' = -\frac{b^2}{a^2};$$



that is, The product of the tangents of the angles, which a pair of supplemental chords makes with the axis major, is constant.

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* If the distance C P $= u$, and $p =$ the perpendicular from the centre on the tangent at P, this equation is

$$p^2 = \frac{a^2 b^2}{a^2 + \quad - u^2}.$$

If the curve was referred to any conjugate diameters, $2\,a_1$ and $2\,b_1$, we should find exactly in the same manner that the product of the tangents of the angles, which a pair of supplemental chords makes with any axis $2\,a_1$, is constant, and equal to $-\dfrac{b_1^2}{a_1^2}$.

The equation to a chord Q P being $y - y' = \alpha\,(x - x')$, the equation to its supplemental chord Q P′ is $y + y' = -\dfrac{b^2}{a^2\,\alpha}(x + x')$.

In the circle $b = a \therefore \alpha\,\alpha' = -1$, which proves that in the circle the supplemental chords are at right angles to each other, a well-known property of that figure.

The converse of the proposition is thus proved.

Let A C A′ be any diameter, C the origin, and $\alpha\,\alpha' = -\dfrac{b_1^2}{a_1^2}$, then the equation to A R is $y = \alpha\,(x + a_1)$ (1), and the equation to A′ R is $y = \alpha'\,(x - a_1) = -\dfrac{b^2}{a_1^2\,\alpha}(x - a_1)$ (2). To find the intersection of the lines A R and A′ R, let y and x be the same for (1) and (2), and eliminate α by multiplication; hence,

$y^2 = -\dfrac{b_1^2}{a_1^2}(x^2 - a_1^2)$; or $a_1^2\,y^2 + b_1^2\,x^2 = a_1^2\,b_1^2$, and the locus of R is an ellipse whose axes are $2\,a_1$ and $2\,b_1$.

142. The equation $\alpha\,\alpha' = -\dfrac{b^2}{a^2}$ is remarkable, as showing that $\alpha\,\alpha'$ is the same, not only for different pairs of chords drawn to the extremities of the same diameter, but also for pairs of chords drawn to the extremities of any diameter; hence, if from the extremity of the axis major we can draw one chord A R parallel to Q P′, the supplemental chord R A′ will be parallel to Q P: this is possible in all cases, except when one chord is parallel or perpendicular to the axis.

143. To find the angle between two supplemental chords.

Let x, y be the co-ordinates of Q, and x' y' those of P,

$$\text{Then tan. } P\,Q\,P' = \frac{\alpha - \alpha'}{1 + \alpha\,\alpha'} = \frac{\dfrac{y - y'}{x - x'} - \dfrac{y + y'}{x + x'}}{1 - \dfrac{b^2}{a^2}} = \frac{2 a^2}{a^2 - b^2}\,\frac{x'\,y - y'\,x}{x^2 - x'^2};$$

$$\text{or, } = -\frac{2\,b^2}{a^2 - b^2}\,\frac{x'\,y - y'\,x}{y^2 - y'^2}.$$

For the principal supplemental chords, we have $x' = a$, $y' = 0$;

$$\therefore \text{ tan. } A\,R\,A' = -\frac{2\,b^2}{a^2 - b^2}\,\frac{a}{y}.$$

This value of the tangent being negative, the angle A R A′ is always obtuse, which is also evident, since all the points on the ellipse are within the circumscribing circle.

As y increases, the numerical value of the tangent decreases, or the angle increases (since the greater the obtuse angle, the less is its tangent); hence, the angle is a maximum when y is, that is, when $y = b$. This shows that the angle A B A′ is the greatest angle contained by the principal supplemental chords, and therefore by any supplemental chords. Also, its supplement B A B′ is the least angle contained by any supplemental chords. The angle between the chords being thus limited by the angles A′B A, B A B′, of which the former is greater, and the latter less, than a right angle, chords may be drawn containing any angle between these limits. This is done by describing a segment of a circle, containing the given angle, upon any diameter, except the axis, and joining the extremities of the diameter with the points of intersection of the ellipse and circle. Also, from the value of tan. P Q P′, it appears that, if the angle be a right angle, the two chords are perpendicular to the axes.

144. It was shown in (131.) that if θ and θ' were the angles which conjugate diameters make with the axis major, $\tan. \theta. \tan. \theta' = -\frac{b^2}{a^2}$, but α, α' being tangents of the angles which two supplemental chords make with the same axis, we have $\alpha\alpha' = -\frac{b^2}{a^2}$; $\therefore \tan. \theta. \tan. \theta' = \alpha\alpha'$; hence, if $\tan. \theta = \alpha$, we have $\tan. \theta' = \alpha'$; or if one diameter be parallel to any chord, the conjugate diameter is parallel to the supplemental chord.

145. Since supplemental chords can be drawn containing any angle within certain limits, conjugate diameters parallel to these chords may be drawn containing any given angle within the same limits.

Also, since the angle between the principal supplemental chords is always obtuse, the angle P C D between the conjugate diameters is also obtuse, and is the greatest when they are parallel to A B and A B′. In this case, being symmetrically situated with respect to the axes, they are equal to one another.

The magnitude of the equal conjugate diameters is found from the equation $a_1^2 + b_1^2 = a^2 + b^2$, $\therefore a_1^2 = \frac{a^2 + b^2}{2}$.

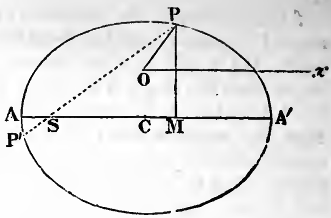
The equation to the ellipse referred to its equal conjugate diameters is $y^2 + x^2 = a_1^2$; however, this must not be confounded with the equation to the circle, which only assumes this form when referred to *rectangular* axes.

THE POLAR EQUATION.

146. Instead of an equation between rectangular co-ordinates x and y, we may obtain one between polar co-ordinates u and θ.

Let the curve be referred to the centre C, and to rectangular axes, and let the co-ordinates of the pole O be x' and y', θ the angle which the radius vector O P, or u, makes with a line O x parallel to the axis of x; then, by (61.), or by inspection of the figure, we have

$$y = y' + u \sin. \theta$$
$$x = x' + u \cos. \theta \cdot$$



also $a^2 y^2 + b^2 x^2 = a^2 b^2$;

$\therefore$ by substitution, $a^2 (y' + u \sin.\theta)^2 + b^2 (x' + u \cos.\theta)^2 = a^2 b^2$;

Whence u may be found in terms of θ and constant quantities.

147. Let the centre be the pole:

$$\therefore x' = 0 \text{ and } y' = 0,$$

$$a^2 u^2 (\sin.\theta)^2 + b^2 u^2 (\cos.\theta)^2 = a^2 b^2;$$

$$u^2 = \frac{a^2 b^2}{a^2 (\sin.\theta)^2 + b^2 (\cos.\theta)^2} = \frac{a^2 b^2}{a^2 (\sin.\theta)^2 + (a^2 - a^2 e^2)(\cos.\theta)^2}$$

$$= \frac{a^2 b^2}{a^2 - a^2 e^2 (\cos.\theta)^2} = \frac{a^2 (1 - e^2)}{1 - e^2 (\cos.\theta)^2}.$$

148. Let the focus S be the pole:

$\therefore y' = 0,\ x' = -ae = -c$, and u becomes r;

hence the transformed equation (146.) becomes

$$a^2 (r \sin.\theta)^2 + b^2 (-c + r \cos.\theta)^2 = a^2 b^2;$$

$$\therefore a^2 r^2 (\sin.\theta)^2 + b^2 r^2 (\cos.\theta)^2 - 2 b^2 r c \cos.\theta + b^2 c^2 = a^2 b^2;$$

or, $a^2 r^2 (\sin.\theta)^2 + a^2 r^2 (\cos.\theta)^2 - c^2 r^2 (\cos.\theta)^2 - 2 b^2 r c \cos.\theta = a^2 b^2 - b^2 c^2 = b^4$ since $a^2 - b^2 = c^2$.

$$\text{or, } a^2 r^2 = c^2 r^2 (\cos.\theta)^2 + 2 b^2 r c \cos.\theta + b^4$$

$$= (c r \cos.\theta + b^2)^2;$$

$$\therefore a r = c r \cos.\theta + b^2$$

$$r = \frac{b^2}{a - c \cos.\theta} = \frac{a^2 (1 - e^2)}{a - a e \cos.\theta} = \frac{a (1 - e^2)}{1 - e \cos.\theta}.$$

149. Let any point on the curve be the pole:
Expanding the terms of the polar equation in (146.), and reducing by means of the equation $a^2 y'^2 + b^2 x'^2 = a^2 b^2$, we have

$$u = -2 \frac{a^2 y' \sin.\theta + b^2 x' \cos.\theta}{a^2 (\sin.\theta)^2 + b^2 (\cos.\theta)^2}.$$

If the pole is at A, we have $y' = 0$, and $x' = -a$,

$$\therefore u = \frac{2 b^2 a \cos.\theta}{a^2 (\sin.\theta)^2 + b^2 (\cos.\theta)^2} = \frac{2 a (1 - e^2) \cos.\theta}{1 - e^2 (\cos.\theta)^2}.$$

150. When the focus is the pole, the equation is often obtained directly from some known property of the curve.

Let $SP = r$, $CM = x$, and $ASP = \theta$,

$$\text{then } SP = a + e x \ (109.)$$

$$= a + e (SM - SC)$$

$$= a + e (-r \cos.\theta - a e)$$

$$\therefore r + e r \cos.\theta = a - a e^2 \text{ and } r = \frac{a (1 - e^2)}{1 + e \cos.\theta}.$$

This is the equation generally used in astronomy, the focus S being the place of the sun, and the ellipse the approximate path of the planet.

$$\text{Let } a\,(1 - e^2) = \frac{b^2}{a} = p, \text{ where } p \text{ is the parameter. (105.)}$$

Then the last equation may be written under the following forms:

$$r = \frac{p}{2} \cdot \frac{1}{1 + e\cos.\theta} = \frac{p}{2} \cdot \frac{1}{1 - e + 2e\left(\cos.\frac{\theta}{2}\right)^2}$$

$$= \frac{p}{2} \cdot \frac{1}{(1+e)\left(\cos.\frac{\theta}{2}\right)^2 + (1-e)\left(\sin.\frac{\theta}{2}\right)^2}$$

If θ be measured, not from S A, but from a line passing through S, and making an angle α with S A, the polar equation is

$$r = \frac{p}{2} \cdot \frac{1}{1 + e\cos.(\theta - \alpha)}.$$

151. If P S meet the curve again P′, let S P′ $= r'$,

$$\text{then } r = \frac{p}{2} \cdot \frac{1}{1 + e\cos.\theta}$$

$$\text{and } r' = \frac{p}{2} \cdot \frac{1}{1 + e\cos.(\pi - \theta)} = \frac{p}{2} \cdot \frac{1}{1 - e\cos.\theta};$$

$$\therefore r + r' = \text{P P}' = \frac{p}{1 - e^2(\cos.\theta)^2}$$

$$\text{and } r\,r' = \frac{p^2}{4} \cdot \frac{1}{1 - (e^2\cos.\theta)^2} = \frac{p}{4}(r + r')$$

or the rectangle S P, S P′ $= \frac{1}{4}$ of the rectangle under the principal parameter and focal chord.

152. Let C D, or b_1, be the semi-diameter parallel to S P, then (147.)

$$b_1^2 = \frac{a^2(1 - e^2)}{1 - e^2(\cos.\theta)^2} = \frac{a}{2}\,\frac{p}{1 - e^2(\cos.\theta)^2} = \frac{a}{2}(r + r'),$$

$$\therefore r + r = \frac{2\,b_1^2}{a};$$

that is, a focal chord at any point P, is a third proportional to the axis major and diameter to that chord.

CHAPTER IX.

THE HYPERBOLA.

153. In the discussion of the general equation of the second order, we observed that, referring the curve to the centre and rectangular axes, the equation to the hyperbola assumed the form

$$\left(\frac{a'}{-f'}\right)y^2 + \left(\frac{c'}{-f'}\right)x^2 = 1$$

where the co-efficients have different signs, 85. 86.

Let $\left(\frac{a'}{-f'}\right)$ be negative, then the equation becomes

$$-\left(\frac{a'}{-f'}\right)y^2 + \left(\frac{c'}{-f'}\right)x^2 = 1;$$

$$\text{or } P\,y^2 - Q\,x^2 = -1.$$

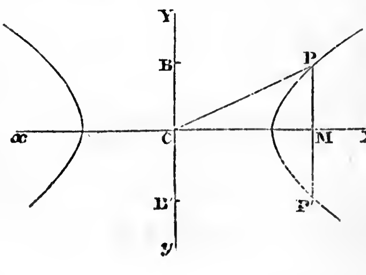
We now proceed to investigate this equation, and to deduce from it all the properties of the hyperbola.

154. Let the curve be referred to its centre C, and rectangular axes X x, Y y, meeting in C; C M $= x$, and M P $= y$; then, at the point where the curve cuts the axes, we have

$$y = 0,\ Q\,x^2 = 1,\ \therefore x = \pm\frac{1}{\sqrt{Q}},$$

$$x = 0,\ P\,y^2 = -1,\ \therefore y = \pm\sqrt{\frac{-1}{P}}$$

In the axis of x take C A $= \frac{1}{\sqrt{Q}}$, and CA$' = -\frac{1}{\sqrt{Q}}$, and the curve cuts the axis X x in A and A: Since the value of y is impossible, the other axis never meets the curve; nevertheless we mark off two points, B and B$'$, in that axis, whose distances from C are C B $=$ $+\frac{1}{\sqrt{P}}$ and C B$' = -\frac{1}{\sqrt{P}}$.



Also if C A $= a$, and C B $= b$, we have Q $= \frac{1}{a^2}$, P $= \frac{1}{b^2}$; therefore the equation to the curve becomes

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = -1;$$

$$\text{or } a^2 y^2 - b^2 x^2 = -a^2 b^2;$$

$$\text{or } y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

155. From the last equation we have

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2} \ (1) \text{ and } x = \pm \frac{a}{b}\sqrt{y^2 + b^2} \ (2).$$

From (1) if x be less than $\pm a$, y is impossible; if, therefore, lines be drawn through A and A′, parallel to C Y, no part of the curve is found between these lines.

Again, for every value of x, greater than a, we have two real and equal values of y; that is, for any abscissa C M, greater than C A, we have two equal and opposite ordinates, M P, M P′.

Also as x increases from a to ∞, these values of y increase from 0 to $\pm \infty$; hence, we have two arcs A P, A P′, exactly equal and opposite to each other, and extending themselves indefinitely.

If x be negative, x^2 being positive, the same values of y must recur; hence, there are again two equal and opposite arcs which form another branch extending from A′ to ∞; thus the whole curve is divided into two equal parts by the axis of x.

From (2) it appears to be divided into two equal parts by the axis of y; hence it is symmetrical with respect to the axes; and its concavity is turned towards the axis of x, otherwise it might be cut by a straight line in more points than two, (71.)

156. If P be any point on the curve, we have

$$C P = \sqrt{x^2 + y^2} = \sqrt{x^2 + \frac{b^2}{a^2}(x^2 - a^2)} = \sqrt{\frac{a^2 + b^2}{a^2}x^2 - b^2};$$

hence C P is least when x is least, that is, when $x = a$, in which case C P becomes also equal to a; hence C A, or C A′, is the least line that can be drawn from the centre to the curve: thus, the axis A A′ is the least line that can be drawn through the centre to meet the curve. The other axis, B B′, never meets the curve.

In the equation Py^2 − Q$x^2 = -1$, the imaginary axis may be greater or less than the real one, according as Q is greater or less than P; hence the appellation of axis major cannot be generally applied to the real axis of the curve. In this treatise we shall call A A′ the transverse axis, and B B′ the conjugate axis.

157. The points A, A′ are called the vertices, or summits of the curve· either of these points may be taken for the origin by making proper substitutions.

Let A be the origin, A M $= x'$;

Then $x =$ C M $=$ C A + A M $= a + x'$;

$$\therefore y^2 = \frac{b^2}{a^2}(x^2 - a^2) = \frac{b^2}{a^2}\{(a + x')^2 - a^2\} = \frac{b^2}{a^2}\{2\,a\,x' + x'^2\},$$

or, suppressing accents, $y^2 = \dfrac{b^2}{a^2}(2\,a\,x + x^2) = \dfrac{b^2}{a^2}\,x\,(2\,a + x)$.

This last equation is geometrically expressed by the following proportion:

The square upon M P : rectangle A M, M A′ :: the square upon B C : the square upon A C.

If the origin be at A′, the equation is $y^2 = \dfrac{b^2}{a^2}(x^2 - 2\,a\,x)$.

158. If $a = b$, the equation to the hyperbola becomes $y^2 - x^2 = -a^2$; this curve is called the equilateral hyperbola, and has, to the common hyperbola, the same relation that the circle has to the ellipse.

159. The analogy between the ellipse and hyperbola will be found to be very remarkable; the equations to the two curves differ only in the sign of b^2; for if, in the equation to the ellipse $a^2\,y^2 + b^2\,x = a^2\,b^2$, we put $-b^2$ for b^2, we have the equation to the hyperbola: hence we might conclude that many of the algebraical results found in the one curve will be true for the other, upon changing b^2 into $-b^2$ in those results; and in fact this is the case, the same theorems are generally true for both, and may be proved in the same manner: for this reason we shall not enter at length into the demonstration of all the properties of the hyperbola, but merely put down the enunciations and results, with a reference at the end of each article to the corresponding one in the ellipse, except in those cases where there may be any modification required in the working. To prevent any doubt about the form of the figure, we shall insert figures in those places where they may be wanted; and, with this assistance, we trust that the present plan will offer no difficulty.

THE FOCUS.

160. The equation $y^2 = \dfrac{b^2}{a^2}(2\,a\,x + x^2)$ may be put under the form $y^2 = l\,x + \dfrac{l}{2\,a}\,x^2$, in which case the quantity $l = \dfrac{2\,b^2}{a}$ is called the principal parameter, or the Latus Rectum

Since $l = \dfrac{2\,b^2}{a} = \dfrac{4\,b^2}{2\,a}$, the Latus Rectum is a third proportional to the transverse and conjugate axes.

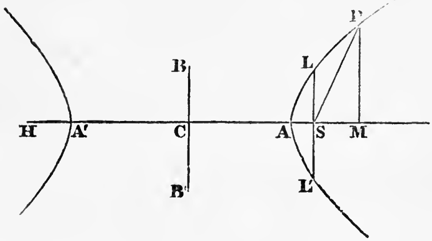
161. To find from what point in the transverse axis a double ordinate can be drawn equal to the Latus Rectum,

$$\text{Here } 4\,y^2 = l^2, \text{ or } \frac{4\,b^2}{a^2}(x^2 - a^2) = \frac{4\,b^4}{a^2};$$

$$\therefore x^2 - a^2 = b^2.$$

$$\text{or, } x^2 = a^2 + b^2;$$

$$\therefore x = \pm \sqrt{a^2 + b^2}$$



Join A B, then A B $= \sqrt{a^2 + b^2}$; with centre C and radius A B describe a circle cutting the transverse axis in the points S and H, we have then C S $= \sqrt{a^2 + b^2}$, and C H $= -\sqrt{a^2 + b^2}$; thus S and H are the points through either of which, if an ordinate as L S L′ be drawn, it is equal to the Latus Rectum.

The two points S and H, thus determined, are called the foci.

162. The fraction $\dfrac{\sqrt{a^2 + b^2}}{a}$, which represents the ratio of C S to C A, is called the eccentricity: if this quantity, which is evidently greater than unity, be represented by the letter e, we have $\sqrt{a^2 + b^2} = a\,e$, whence $e^2 = \dfrac{a^2 + b^2}{a^2} = 1 + \dfrac{b^2}{a^2}$; $\therefore \dfrac{b^2}{a^2} = e^2 - 1$, and the equation to the hyperbola may be put under the form

$$y^2 = (e^2 - 1)(x^2 - a^2).$$

163. Since $a^2 + b^2 = a^2 e^2$, we have $b^2 = a^2 e^2 - a^2 = (a\,e - a)(a\,e + a)$;

Or the rectangle A S, S A′ = the square upon B C.

164. To find the distance from the focus to any point P in the curve, proceeding exactly as in (109.) we find

$$\text{S P} = e\,x - a,\ \text{H P} = e\,x + a;$$

Hence H P − S P $= 2\,a =$ A A′, that is the difference of the distances of any point in the curve from the foci is equal to the transverse axis.

165. Conversely, To find the locus of a point, the difference of whose distances from two fixed points S and H is constant or equal $2\,a$.

If S H $= 2\,c$, the locus is an hyperbola, whose axes are $2\,a$ and $2\sqrt{a^2 + c^2}$, and whose foci are S and H. (110.)

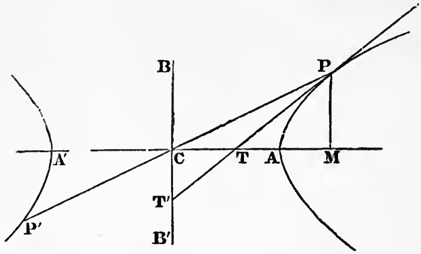
THE TANGENT.

166. To find the equation to the tangent at any point P $(x'\,y')$,

The required equation obtained as in (111.) is

$$a^2\,y\,y' - b^2\,x\,x' = -\,a^2\,b^2.$$

This form is easily recollected, since it may be obtained from the equation to the curve $a^2 y^2 - b^2 x^2 = - a^2 b^2$, by putting $y\,y'$ for y^2, and $x\,x$ for x^2.



167. To find the points where the tangent cuts the axes;

Let $y = 0$, $\therefore x = \frac{a^2}{x'} = \mathrm{C\,T}$; similarly $y = \mathrm{C\,T'} = - \frac{b^2}{y'}$; hence we have

The rectangle C T, C M = the square upon A C;
and The rectangle C T′, M P = the square upon B C.

Since C T $\left(= \frac{a^2}{x'}\right)$ is always less than C A, the tangent to any point of the branch P A cuts the transverse axis between C and A.

The subtangent M T $= x' - \frac{a^2}{x'} = \frac{x'^2 - a^2}{x'}$. (115.)

The tangent at the extremity A of the transverse axis is perpendicular to that axis (116.).

If P C be produced to meet the curve again in P′, the tangents at P and P′ will be found to be parallel (116.).

168. To find the equation to the tangent at the extremity of the Latus Rectum,

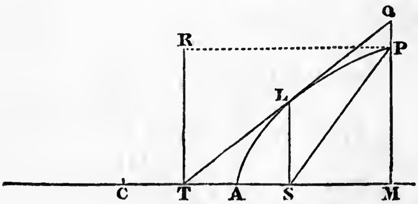
Generally the equation to the tangent is

$$a^2 y\, y' - b^2 x\, x' = - a^2 b^2;$$

at L, $x' = a\,e$, $y' = \frac{b^2}{a}$;

$$\therefore a^2 y \frac{b^2}{a} - b^2 x\, a\, e = - a^2 b^2,$$

$$y = e\,x - a.$$



Let the ordinate y, or M Q, cut the curve in P, then we have S P $= e\,x - a$ (164.).

$$\therefore \text{M Q} = \text{S P}.$$

Also C T $= \dfrac{a}{e}$, hence from T draw T R perpendicular to A C, and from P draw P R parallel to A C, then we have

$$\text{P R} = \text{M T} = \text{M C} - \text{C T} = x - \frac{a}{e} = \frac{e\,x - a}{e} = \frac{1}{e}.\text{S P}.$$

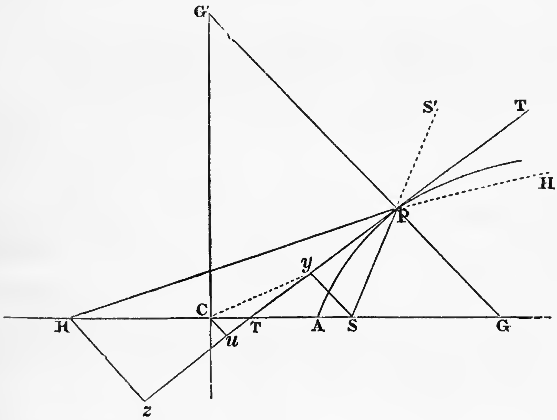
Consequently, the distances of any point P from S, and from the line T R, are in the constant ratio of $e : 1$.

The line T R is called the directrix.

If $x = 0$, we have $y = -a$; hence the tangent at the extremity of the Latus Rectum cuts the axis of y at the point where the circle on the transverse axis cuts the axis of y.

169. To find the length of the perpendicular from the focus on the tangent.

Let S y, H z be the perpendiculars on the tangent P T.



Taking the expression in (48.) we have

$$p = -\frac{y_1 - \alpha\,x_1 - d}{\sqrt{1 + \alpha^2}}$$

here $y_1 = 0$ and $x_1 = a\,e$ are co-ordinates of the point S, and $y = \alpha\,x + d$ is the equation to P y; but the equation to P y (166.) is also

$$y = \frac{b^2\,x'}{a^2\,y'}\,x - \frac{b^2}{y'} \;\therefore\; \alpha = \frac{b^2\,x'}{a^2\,y'} \text{ and } d = -\frac{b^2}{y'};$$

$$p = -\frac{-\dfrac{b^2\,x'}{a^2\,y'}\,a\,e + \dfrac{b^2}{y'}}{\sqrt{\left\{1 + \dfrac{b^4\,x'^2}{a^4\,y'^2}\right\}}} = +\frac{a\,b^2\,(e\,x' - a)}{\sqrt{\{a^4\,y'^2 + b^4\,x'^2\}}}$$

$$= \frac{a\,b^2\,(e\,x' - a)}{a\,b\,\sqrt{\{e^2\,x'^2 - a^2\}}} = b\sqrt{\frac{e\,x' - a}{e\,x' + a}}.$$

Let S P $= r$, and H P $= 2a + r = r'$ $\therefore p = b\sqrt{\frac{r}{r'}}$, or $p^2 = b^2 \frac{r}{2a+r}$

Similarly if H $z = p'$, we have $p'^2 = b^2 \frac{r'}{r}$.

By multiplication we have $p\,p' = b^2$; $\;$: hence

The rectangle S y, H z = the square upon B C.

170. To find the locus of y or z in the last article.

The equation to the curve at P is $a^2 y'^2 - b^2 x'^2 = - a^2 b^2$

The equation to the tangent at P is $a^2 y\, y' - b^2 x\, x' = - a^2 b^2$.

The equation to S y is $y = \frac{-a^2 y'}{b^2 x'}(x - c)$

By eliminating x' and y', exactly as in (120.), we arrive at the equation

$$a^2 = y^2 + x^2;$$

Hence the locus of y is a circle described on the transverse axis as diameter.

171. To find the angle which the focal distance S P makes with the tangent P T.

The equation to the tangent is $y = \frac{b^2 x'}{a^2 y'} x - \frac{b^2}{y'}$, and the equation to S P is, $y - y' = \frac{y'}{x' - c}(x - x')$,

hence tan. S P T = tan. (P S X − P T X)

$$= \frac{\frac{y'}{x'-c} - \frac{b^2 x'}{a^2 y'}}{1 + \frac{b^2 x'}{a^2 y'}\frac{y'}{x'-c}} = \frac{a^2 y'^2 - b^2 x'^2 + b^2 c\, x'}{a^2 y' x' - a^2 c\, y' + b^2 x' y'}$$

$$= \frac{b^2 (c\, x' - a^2)}{y' c (c\, x' - a^2)} = \frac{b^2}{c\, y'}.$$

Similarly tan. H P T $= \frac{b^2}{c\,y'}$, $\therefore$ the angles S P T, H P T are equal; thus the tangent makes equal angles with the focal distances.

Produce S P to S′, then it is a property of light, that if a ray proceeding from H be reflected by the line T P T′, the angle S′ P T′ of the reflected ray will equal the angle H P T. Now, in the hyperbola, these angles are equal; hence if a light be placed at H, all rays which are incident on the curve will be reflected as if diverging from S; or if a body of rays proceeding to S be incident on the curve, they will converge to H. Hence these points S and H are called foci.

This important property of the curve is also thus proved from article (169.),

$$S\,y = p = b\sqrt{\frac{r}{r'}}, \text{ and } H\,z = p' = b\sqrt{\frac{r'}{r}};$$

$$\therefore S y : H z :: r : r' :: S P : H P;$$

$\therefore$ angle $S P y = H P z$, and the tangent makes equal angles with the focal distances *.

172. To find the length of the perpendicular $C u$ from the centre on the tangent.

$$p = - \frac{y_1 - \alpha x_1 - d}{\sqrt{1 + \alpha^2}},$$

here $y_1 = 0$, $x_1 = 0$, $\alpha = \frac{b^2 x'}{a^2 y'}$, and $d = - \frac{b^2}{y'}$, $\therefore C u = \frac{a b}{\sqrt{r r}}$

173. To find the locus of u.

The equation to $C u$ is $y = - \frac{a^2 y'}{b^2 x'} x$, eliminating x' y' from this equation, and the equation to the tangent, we find as in (123.), the resulting equation to be $a^2 x^2 - b^2 y^2 = (x^2 + y^2)^2$, which cannot be discussed at present.

174. From the equation to the tangent, and that to $C P$, we find, as in (124.),

$$\tan. C P T = \frac{a^2 b^2}{c^2 x' y'}.$$

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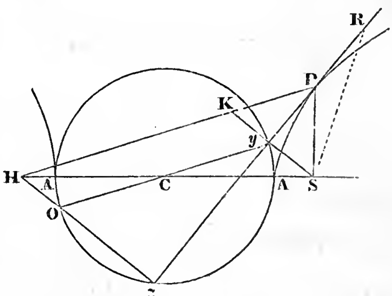
* The following geometrical method of drawing a tangent to the hyperbola, and proving that the locus of the perpendicular from the focus on the tangent is the circle on the transverse axis, will be found useful.



Let $A P$ be the hyperbola, P any point on it; join $S P$ and $H P$, and in $H P$ take $P K = P S$; bisect the angle $S P K$ by the line $P y z$, and join $S K$, cutting $P y$ in y.

1. $P y$ is a tangent to the hyperbola; for if R be any other point in the line $P y$, we have $H R - S R = H R - K R$ is less than $H K$ (Geom. i. 10) less than $2 a$, hence R, and every other point in $P y$, is without the curve.

2. The locus of y is the circle on the transverse axis: draw $H z$ parallel to $S y$, and join $C y$; then, because the triangles $S P y$, $K P y$ are equal, we have the angle $S y P$ a right angle, or $S y$ and $H z$ are perpendicular to the tangent. Also since $S y = K y$, and $S C = C H$, we have $C y$ parallel to $H K$, and $C y = \frac{1}{2} H K = \frac{1}{2} (H P - S P) = C A$.

3. The rectangle $S y$, $H z =$ the square on $B C$. Let $z H$ meet the circle again in O, and join $C O$; then the line $O C y$ is a straight line and a diameter, hence the triangles $C S y$, $C H O$ are equal, and the rectangle $S y$, $H z =$ the rectangle $H O$, $H z =$ the rectangle $H A'$, $H A =$ the square upon $B C$.

4. Let $S P = r$, $H P = 2 a + r$, $S y = p$ and $H z = p'$, then $p^2 = \frac{b^2 r}{2 a + r}$; for by similar triangles, $S y : S P :: H z : H P$, $\therefore p = \frac{r}{2 a + r} p'$, and, as above, $p p' = b^2$, $\therefore p^2 = \frac{b^2 r}{2 a + r}$.

From C u = C y sin. C y u, we have

$$\frac{a\,b}{\sqrt{r\,r'}} = a \text{ sin. C } y\,u \therefore \text{ sin. C } y\,u = \frac{b}{\sqrt{r\,r'}}.$$

Also from H z = H P sin. H P z, we have

$$b\sqrt{\frac{r'}{r}} = r' \text{ sin. H P } z, \therefore \text{ sin. H P } z = \frac{b}{\sqrt{r\,r'}},$$

$\therefore$ angle C y u = angle H P z and C y is parallel to H P.

And if C E be drawn parallel to the tangent P T, and meeting H P in E, we have P E = C y = A C.

THE NORMAL.

175. The equation to the line passing through the point P $(x'\,y')$, and perpendicular to the tangent $\left(y = \frac{b^2\,x'}{a^2\,y'}\,x - \frac{b^2}{y'}\right)$ is

$$y - y' = -\frac{a^2\,y'}{b^2\,x'}(x - x').$$

To find where the normal P G cuts the axes.

$$\text{Let } y = 0 \therefore -y' = -\frac{a^2\,y'}{b^2\,x'}(x - x') \therefore x = x' + \frac{b^2\,x'}{a^2}$$

$$= \frac{a^2 + b^2}{a^2}\,x' = e^2\,x' = \text{C G}.$$

$$\text{Let } x = 0 \therefore y = y' + \frac{a^2\,y'}{b^2} = \frac{a^2 + b^2}{b^2}\,y' = \frac{a^2\,e^2}{b^2}\,y' = \text{C G}'.$$

Also the subnormal M G = $x - x' = \frac{b^2\,x'}{a^2}$; and S G = e. S P.

176. From the above values of C G, C G′, and M G′, we may demonstrate that P G = $\frac{b}{a}\sqrt{r\,r'}$, P G′ = $\frac{a}{b}\sqrt{r\,r'}$, and consequently that

The rectangle P G, P G′ = $r\,r'$ = the rectangle S P, H P.

Also S G′ = $\frac{a\,e}{b}\sqrt{r\,r'}$, G G′ = $\frac{a\,e^2}{b}\sqrt{r\,r'}$, and $\therefore$ G G′ = e.SG′.

177. Since the tangent makes equal angles with the focal distances, the normal, which is perpendicular to the tangent, also makes equal angles with the focal distances, one of them being first produced as to H′. This theorem may be directly proved from the above value of C G; for S G : H G :: $e^2\,x' - a\,e : e^2\,x' + a\,e :: e\,x' - a : e\,x' + a$ :: S P : H P, hence the angle S P H′ is bisected by the line P G.

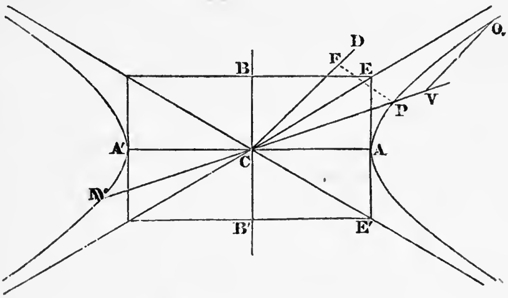
THE DIAMETERS.

178. It may be proved as for the ellipse (130.), that all the diameters of the hyperbola pass through the centre, and that any line through the

centre is a diameter. If $y = \alpha x + c$ be the equation to any chord, $a^2 x y - b^2 x = 0$ is the equation to the diameter bisecting all chords parallel to $y = \alpha x + c$.

179. In the ellipse all the diameters must necessarily meet the curve; but this is not the case in the hyperbola, as will appear by finding the co-ordinates of intersection of the diameter and the curve.

Let $y = \beta x$ be the equation to a diameter C P, and substitute this value of y in the equation to the curve.



$$a^2 y^2 - b^2 x^2 = - a^2 b^2,$$
$$\therefore a^2 \beta^2 x^2 - b^2 x^2 = - a^2 b^2,$$
$$x^2 = \frac{a^2 b^2}{b^2 - a^2 \beta^2},$$
$$\therefore x = \pm \frac{a b}{\sqrt{b^2 - a^2 \beta^2}}.$$

These values are impossible, if $a^2 \beta^2$ is greater than b^2, that is, if β is greater than $\frac{b}{a}$; and if $\beta = \pm \frac{b}{a}$, the diameter meets the curve only at an infinite distance. The limits of the intersecting diameters are thus determined; through A, B and B′ draw lines parallel to the axes meeting in E and E′, then tan. E C A $= \frac{b}{a}$, and tan. E′ C A $= - \frac{b}{a}$, hence C E and C E′ produced are the lines required. Hence, in order that a diameter meet the curve, it must be drawn within the angle E C E′; thus the line C D never meets the curve.

The curve is symmetrical with respect to these lines C E, C E′, since the axis bisects the angle E C E′.

180. The hyperbola has an infinite number of pairs of conjugate diameters. This is proved by referring the equation to other axes by means of the formulas of transformation (57.)

$$y = x' \sin. \theta + y' \sin. \theta',$$
$$x = x' \cos. \theta + y' \cos. \theta';$$

hence the equation $a^2 y^2 - b^2 x^2 = - a^2 b^2$ becomes

$$\{a^2 (\sin. \theta')^2 - b^2 (\cos. \theta')^2\} y'^2 + \{a^2 (\sin. \theta)^2 - b^2 (\cos. \theta)^2\} x'^2$$
$$+ 2 \{a^2 \sin. \theta \sin. \theta' - b^2 \cos. \theta \cos. \theta'\} x' y' = - a^2 b^2.$$

In order that this equation be of the conjugate form, let the co-efficient of $y' x' = 0$,

$$\therefore\ a^2 \sin.\theta \sin.\theta' - b^2 \cos.\theta \cos.\theta' = 0,$$

$$\text{or,}\quad \tan.\theta \tan.\theta' = \frac{b^2}{a^2}.$$

Hence for any value of θ, we have a real value of θ', that is, there is an infinite number of pairs of axes to which, if the curve be referred, its equation is of the required conjugate form.

If $\tan.\theta$ be less than $\frac{b}{a}$, $\tan.\theta'$ must be greater than $\frac{b}{a}$, that is, if one diameter C P, in the last figure, meets the curve, the conjugate diameter C D does not; therefore in each system of conjugate diameters one is imaginary. Also, since the product of the tangents is positive, both angles are acute, or both obtuse; in the figure they are both acute, but for the opposite branch they must be both obtuse.

181. As in article (132.), it appears that there can be only one system of rectangular conjugate diameters.

182. The equation to the curve is now

$$\{a^2 (\sin.\theta')^2 - b^2 (\cos.\theta')^2\}\, y'^2 + \{a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2\}\, x'^2 = -a^2 b^2.$$

If we successively make $y' = 0$, and $x' = 0$, we have the distances from the origin to the points in which the curve cuts the new axes; but as we already know (180.) that one of these new axes never meets the curve, we must represent one of these distances by an imaginary quantity.

Let the axis of x' meet the curve at a distance a_1 from the centre, and let the length of the other semi-axis be b_1 connected with the symbol $\sqrt{-1}$, that is, let the new conjugate diameters be $2\,a_1$ and $2\,b_1 \sqrt{-1}$, then we have

$$y = 0 \ \therefore\ \{a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2\}\, a_1^2 = -a^2 b^2,$$

$$x = 0 \ \therefore\ \{a^2 (\sin.\theta')^2 - b^2 (\cos.\theta')^2\}\, (-b_1^2) = -a^2 b^2,$$

And the transformed equation becomes

$$\frac{a^2 b^2}{b_1^2}\, y'^2 - \frac{a^2 b^2}{a_1^2}\, x'^2 = -a^2 b^2.$$

$$\text{or,}\quad \frac{y'^2}{b_1^2} - \frac{x'^2}{a_1^2} = -1,$$

$$\text{or,}\quad a_1^2 y'^2 - b_1^2 x'^2 = -a_1^2 b_1^2.$$

183. From the transformation we obtain the three following equations:

$$a_1^2 \{a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2\} = -a^2 b^2 \qquad (1),$$

$$b_1^2 \{a^2 (\sin.\theta')^2 - b^2 (\cos.\theta')^2\} = +a^2 b^2 \qquad (2),$$

$$\left.\begin{aligned} &a^2 \sin.\theta \sin\ \theta' - b^2 \cos.\theta \cos.\theta' = 0,\\ &\text{or,}\quad \tan.\theta \tan.\theta' = \frac{b^2}{a^2}\end{aligned}\right\}\qquad (3).$$

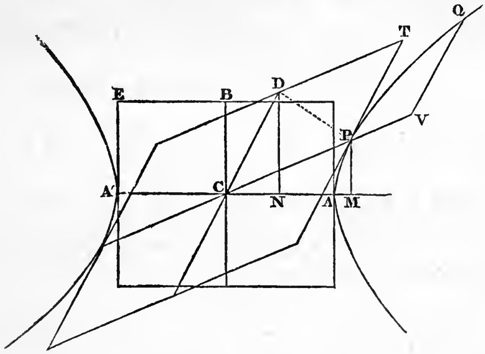
Following the steps exactly as in article (134.), or, which amounts to the same thing, putting $-b^2$ for b^2, and $-b_1^2$ for b_1^2 all through that article, we arrive at the result

$$a_1^2 - b_1^2 = a^2 - b^2,$$

or, the difference of the squares upon the conjugate diameters is equal to the difference of the squares upon the axes.

184. Again, multiplying (1) and (2) together, and (3) by itself, then subtracting the results, and reducing, as in the article (135.), we have

$$a_1\, b_1 \sin.\,(\theta' - \theta) = a\, b.$$



Now $\theta' - \theta$ is the angle P C D between the conjugate diameters C P and C D; hence, drawing straight lines at the extremities of the conjugate diameters, parallel to those diameters, we have, from the above equation, the parallelogram P C D T = the rectangle A′ C B E, and hence the whole parallelogram thus inscribed in the figure is equal to the rectangle contained by the axes *.

185. Returning to article (182.), the equation to the curve, suppressing the accents on x' and y', as no longer necessary, is

$$a_1^2\, y_1^2 - b_1^2\, x^2 = -\, a_1^2\, b_1^2;$$

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* The theorems in articles 183 and 184 may be proved also in the following manner:—

Referring the curve to its rectangular axes, as in art. (187.), let the co-ordinates of P be x' and y'; then the equation to C D is $a^2\, y\, y' - b^2\, x\, x' = 0$, and eliminating x and y between this equation and that to the curve $(a^2\, y^2 - b^2\, x^2 = -\, a^2\, b^2)$ we have the co-ordinates C N and D N, independent of the sign $\sqrt{-1}$, with which they are both affected,

$$\text{C N} = x = \frac{a\, y'}{b}, \text{ and D N} = y = \frac{b\, x'}{a};$$

Hence we have

$$a_1^2 - b_1^2 = x'^2 + y'^2 - x^2 - y^2 = x'^2 + y'^2 - \frac{a^2\, y'^2}{b^2} - \frac{b^2\, x'^2}{a^2} = \frac{b^2\, x'^2 - a^2\, y'^2}{b^2}$$

$$+ \frac{a^2\, y'^2 - b^2\, x'^2}{a^2} = \frac{a^2\, b^2}{b^2} + \frac{-\, a^2\, b^2}{a^2} = a^2 - b^2.$$

Also the triangle P C D = the trapezium P M N D + the triangle D C N − the triangle P C M

$$= (x' - x)\,\frac{y + y'}{2} + \frac{x\, y - x'\, y'}{2} = \frac{x'\, y - y'\, x}{2} = \frac{1}{2}\left\{x'\frac{b\, x'}{a} - y'\frac{a\, y'}{b}\right\}$$

$$= \frac{b^2\, x'^2 - a^2\, y'^2}{2\, a\, b} = \frac{a^2\, b^2}{2\, a\, b} = \frac{a\, b}{2},$$

therefore the parallelogram P C D T = $a\, b$.

In the last figure, C P $= a_1$, C D $= b_1$, C V $= x$ and Q V $= y$. Putting the equation into the form

$$y^2 = \frac{b_1^2}{a_1^2}(x^2 - a_1^2) = \frac{b_1^2}{a_1^2}(x - a_1)(x + a_1),$$

we have the square upon Q V : the rectangle P V, V P' :: the square upon C D : the square upon C P.

186. The equation to the tangent at any point Q $(x' \, y')$ is

$$a_1^2 \, y \, y' - b_1^2 \, x \, x' = - a_1^2 \, b_1^2.$$

187. Let the curve be referred to its axes C A, C B, and let the co-ordinates of P be $x' \, y'$, then the equation to C P being $y = \frac{y'}{x'} x$, the equation to C D is $y = x \tan. \theta = x \frac{b^2}{a^2} \cot. \theta = \frac{b^2}{a^2} \frac{x'}{y'} x$, or,

$$a^2 \, y \, y' - b^2 \, x \, x' = 0.$$

But the equation to the tangent at P is

$$a^2 \, y \, y' - b^2 \, x \, x' = - a^2 \, b^2;$$

hence C D, or the diameter conjugate to C P, is parallel to the tangent at P.

The equation to the conjugate diameter is the same as that to the tangent, omitting the last term $- a^2 \, b^2$.

188. Let x' and y' be the rectangular co-ordinates of P; then from the equation $a_1^2 - b_1^2 = a^2 - b^2$, we have

$$b_1^2 = a_1^2 - a^2 + b^2 = x'^2 + y'^2 - a^2 + b^2 = x'^2 + \left(\frac{b^2}{a^2} x'^2 - b^2\right) - a^2 + b^2$$

$$= \frac{a^2 + b^2}{a^2} x'^2 - a^2 = e^2 \, x'^2 - a^2 = (e \, x' - a)(e \, x' + a) = r \, r';$$

That is, the square upon the conjugate diameter C D = the rectangle under the focal distances S P and H P.

189. If P F be drawn perpendicular from P upon the conjugate C D, (see the last figure but one,) we have the rectangle P F, C D $= a \, b$, (184.).

$$\therefore \text{P F} = \frac{a \, b}{b_1} = \frac{a \, b}{\sqrt{a_1^2 - a^2 + b^2}} = \frac{a \, b}{\sqrt{r \, r'}};\text{*}$$

Also P G $= \frac{b}{a} \sqrt{r \, r'}$, and P G' $= \frac{a}{b} \sqrt{r \, r'}$,

Hence the rectangle P G, P F = the square on B C;

And the rectangle P G', P F = the square on A C;

And the rectangle P G, P G' = the square on C D.

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* If the distance C P $= u$, and $p =$ the perpendicular from the centre on the tangent, this equation is

$$p^2 = \frac{a^2 \, b^2}{u^2 - a^2 + b^2}$$

SUPPLEMENTAL CHORDS.

190. Two straight lines drawn from a point on the curve to the extremities of a diameter are called supplemental chords; they are called principal supplemental chords if that diameter be the transverse axis.

The equations to a pair of chords are

$$y - y' = \alpha\,(x - x')$$
$$y + y' = \alpha'\,(x + x');$$

Whence $\alpha\,\alpha' = \frac{b^2}{a^2}$ as in (141.); hence the product of the tangents of the angles which a pair of supplemental chords makes with the transverse axis is constant; the converse is proved as in 141.

191. The angle between two supplemental chords is found from the expression

$$\tan.\ P\,Q\,P' = \frac{2\,b^2}{a^2 + b^2}\,\frac{x'\,y - y'\,x}{y^2 - y'^2}.$$

And, if A R, A′ R be principal supplemental chords drawn to any point R on the curve,

$$\tan.\ A\ R\ A' = \frac{2\,a\,b^2}{(a^2 + b^2)\,y}.$$

The angle A R A′ is always acute, and diminishes from a right angle to 0; the supplemental angle A A′ R′ increases at the same time from a right angle to 180°; hence, the angle between the supplemental chords may be any angle between 0 and 180°.

Chords may be drawn containing any angle between these limits, by describing on any diameter, except the axes, a segment of a circle containing the given angle, and then joining the extremities of the diameter with the point where the circle intersects the hyperbola. And therefore principal supplemental chords parallel to these may be drawn.

192. Conjugate diameters are parallel to supplemental chords (144.); and therefore they may be drawn containing any angle between 0 and 90°.

193. There are no equal conjugate diameters in the hyperbola, but in that particular curve where $b = a$, we have the equation

$$a_1^2 - b_1^2 = a^2 - b^2 = 0;$$

hence the conjugate diameters a_1 and b_1 are always equal to each other.

The equation to this curve, called the equilateral hyperbola, is

$$y^2 - x^2 = -a^2.$$

THE ASYMPTOTES.

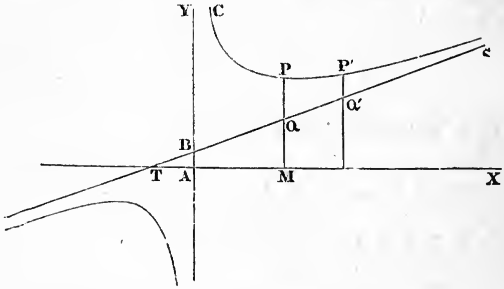
194. We have now shown that most of the properties of the ellipse apply to the hyperbola with a very slight variation: there is, however, a whole class of theorems quite peculiar to the latter curve, and these arise from the curious form of the branches extending to an infinite distance; it appears from the equation $\tan.\ \theta\ .\ \tan\ \theta' = \frac{b^2}{a^2}$ in (180.), that as $\tan.\ \theta$

approaches to $\frac{b}{a}$, tan. θ' approaches also to $\frac{b}{a}$, and thus, as a point P recedes along the curve from the origin, the conjugate diameters for that point approach towards a certain line C E, fig. (179.), and finally at an infinite distance come indefinitely near to that line.

We now proceed to show that the curve itself continually approaches to the same line C E, without ever actually coinciding with it. But as this species of line is not confined to the hyperbola, we shall state the theory generally.

195. Let C P P′ be a curve whose equation has been reduced to the form

$$y = a\,x + b + \frac{c}{x};$$



And let T B S be the line whose equation is

$$y = a\,x + b.$$

For any value of x we can find from this last equation a corresponding ordinate M Q, and by adding $\frac{c}{x}$ to M Q, we determine a point P in the curve: similarly we can determine any number of corresponding points (P′, Q′, &c.) in the curve and straight line.

Since $\frac{c}{x}$ decreases as x increases, the line P′ Q′ will be less than P Q, and the greater x becomes, the smaller does the corresponding P′ Q′ become; so that when x is infinitely great, P′ Q′ is infinitely small, or the curve approaches indefinitely near to the line T B S, but yet never actually meets it: hence T B S is called an asymptote to the curve, from three Greek words signifying "never coinciding."

The equation to the asymptote T B S is $y = a\,x + b$, or is the equation to the curve, with the exception of the term involving the inverse power of x.

196. The reasoning would have been as conclusive if there had been more inverse powers of x; and in general if the equation to a curve can be put into the form

$$y = \&c. + m\,x^3 + n\,x^2 + a\,x + b + \frac{c}{x} + \frac{d}{x^2} + \&c.$$

Then the equation to the curvilinear asymptote is

$$y = \&c. + m\,x^3 + n\,x^2 + a\,x + b$$

Also the equation $y = \&c. + m\,x^3 + n\,x^2 + a\,x + b + \frac{c}{x}$ gives a curve much more asymptotic than the preceding equation, and hence arises a series of curves, each " more nearly coinciding" with the original curve.

197. Let us apply this method to lines of the second order, whose general equation is (75.)

$$y = -\frac{b\,x + d}{2\,a} \pm \frac{1}{2\,a}\sqrt{\{(b^2 - 4\,a\,c)\,x^2 + 2\,(b\,d - 2\,a\,e)\,x + d^2 - 4\,a\,f\}}$$

$$= -\frac{b\,x + d}{2\,a} \pm \sqrt{\{m\,x^2 + n\,x + p\}}\text{, by substitution,}$$

$$= -\frac{b\,x + d}{2\,a} \pm x\sqrt{m}\left\{1 + \frac{n\,x + p}{m\,x^2}\right\}^{\frac{1}{2}}.$$

$$= -\frac{b\,x + d}{2\,a} \pm x\sqrt{m}\left\{1 + \tfrac{1}{2}\left(\frac{n\,x + p}{m\,x^2}\right) - \tfrac{1}{8}\left(\frac{n\,x + p}{m\,x^2}\right)^2 + \&c.\right\}$$

$$= -\frac{b\,x + d}{2\,a} \pm \sqrt{m}\left\{x + \tfrac{1}{2}\,\frac{n}{m}\right\} \pm \frac{\text{constant terms}}{\text{powers of } x}.$$

Hence the equation to the asymptote is

$$y = -\frac{b\,x + d}{2\,a} \pm \sqrt{m}\left\{x + \tfrac{1}{2}\,\frac{n}{m}\right\}$$

$$= -\frac{b\,x + d}{2\,a} \pm \frac{\sqrt{b^2 - 4\,a\,c}}{2\,a}\left\{x + \frac{b\,d - 2\,a\,c}{b^2 - 4\,a\,c}\right\}$$

Now $b^2 - 4\,a\,c$ is negative in the ellipse, and therefore there is no locus to the above equation in this case; also if $b^2 - 4\,a\,c = 0$, the equation to the asymptote, found as above, will contain the term $\sqrt{x}$, and therefore will belong to a curvilinear asymptote; hence the hyperbola is the only one of the three curves which admits of a rectilinear asymptote.

It appears from the $\pm$ sign, that there are two asymptotes, and that the diameter $y = -\frac{b\,x + d}{2\,a}$ bisects them. Also these asymptotes pass through the centre; for giving to x the value $\frac{2\,a\,e - b\,d}{b^2 - 4\,a\,c}$, we have

$$y = -\frac{b\,x + d}{2\,a} = \frac{2\,c\,d - b\,e}{b^2 - 4\,a\,c},$$

and these values of x and y are the co-ordinates of the centre (80.).

198. If the equation want either of the terms x^2 or y^2, a slight operation will enable us to express the equation in a series of inverse powers of or x; thus if the equation be

$$b\,x\,y + c\,x^2 + d\,y + e\,x + f = 0,$$

we have $y = -\dfrac{c x^2 + e x + f}{b x + d} = -\dfrac{c x^2 + e x + f}{b x\left(1 + \dfrac{d}{b x}\right)}$

$$= -\frac{c x^2 + e x + f}{b x}\left(1 + \frac{d}{b x}\right)^{-1}$$

$$= -\left(\frac{c x}{b} + \frac{e}{b} + \frac{f}{b x}\right)\left(1 - \frac{d}{b x} + \left(\frac{d}{b x}\right)^2 -, \&c.\right)$$

Hence the equation to the asymptote, found by multiplying and neglecting inverse powers of x, is

$$y = -\frac{c x}{b} - \frac{e}{b} + \frac{c d}{b^2},$$

$$\text{or,}\quad b y + c x = \frac{c d - b e}{b}.$$

The other asymptote is determined by the consideration that if, for any finite value of x, we obtain a real infinite value of y, that value of x determines the position of an asymptote.

Here when $b x + d = 0$, we have $y = \infty$; hence a line drawn parallel to the axis of y, and through the point $x = -\dfrac{d}{b}$, is the required asymptote.

If the equation be

$$a y^2 + b x y + d y + e x + f = 0,$$

the equations to the asymptotes are

$$a y + b x = \frac{a e - b d}{b}, \text{ and } b y + e = 0;$$

and the second asymptote is parallel to the axis of y.

If the equation be

$$b x y + d y + e x + f = 0,$$

the equations to the asymptotes are

$$b x + d = 0, \text{ and } b y + e = 0;$$

the former asymptote being parallel to the axis of y, and the latter parallel to that of x.

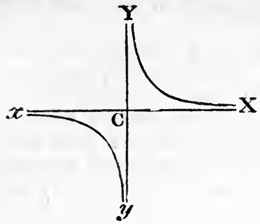
199. Lastly, if the equation be

$$b x y + f = 0,$$

the asymptotes are then the axes themselves, and the curve is referred to its centre and asymptotes as axes.

The position of the curve in this case is directly obtained from the equation $y = -\dfrac{f}{b x} = \dfrac{m}{x}$, by substitution.

Let C X and C Y be the axes, then for $x = 0$, $y = \infty$; as x increases y decreases, and when $x = \infty$, $y = 0$; hence we have the branch Y X.



For x negative, y is negative; and as x increases from 0 to ∞, y decreases from ∞ to 0; hence another branch $y\,x$, equal and similar to the former.

200. To find the equation to the asymptotes from the equation to the hyperbola referred to its centre and axes,

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2} = \pm \frac{b}{a} x \sqrt{1 - \frac{a^2}{x^2}} = \pm \frac{b}{a} x \left\{1 - \tfrac{1}{2}\frac{a^2}{x^2} +, \&c.\right\}$$

Hence the equation to the asymptotes is

$$y = \pm \frac{b}{a} x.$$

To draw these lines, complete the parallelogram on the principal axes (see the figure, art. 179.); the diagonals of this parallelogram are the loci of the last equation, and therefore are the asymptotes required: thus C E and C E′, when produced, are the asymptotes.

The equation to the asymptotes, referred to the centre and rectangular axes, is readily remembered, since it is the same as the equation to the curve without the last term; the two equations are

$$a^2 y^2 - b^2 x^2 = -a^2 b^2, \text{ to the curve,}$$
$$a^2 y^2 - b^2 x^2 = 0 \quad , \text{ to the asymptotes.}$$

If the curve be referred to conjugate axes, the equations are

$$a_1^2 y^2 - b_1^2 x^2 = -a_1^2 b_1^2, \text{ to the curve,}$$
$$a_1^2 y^2 - b_1^2 x^2 = 0 \quad , \text{ to the asymptotes.}$$

201. If $b = a$, the equation to the hyperbola referred to its centre and rectangular axes is $y^2 - x^2 = -a^2$, therefore the equation to the asymptotes is $y^2 - x^2 = 0$, or $y = \pm x$; hence these asymptotes cut the axes at an angle of 45°, or the angle between them is 90°; hence the equilateral hyperbola is also called the rectangular hyperbola.

202. If the curve be referred to its vertex A and rectangular axes, the equation to the curve is

$$y = \pm \frac{b}{a}(x^2 - 2\,a\,x)^{\frac{1}{2}} = \pm \frac{b}{a} x \left(1 + \frac{2\,a}{x}\right)^{\frac{1}{2}},$$

and, expanding and neglecting inverse powers of x, the equation to the asymptotes is

$$y = \pm\left(\frac{b}{a} x + l\right).$$

203. If we take the equation to any line $\left(y = \frac{b}{a} x + c\right)$ parallel to the asymptote, and eliminate y between this equation and the equation to the curve, we find only one value of x; and thus a straight line parallel to the asymptote cuts the hyperbola only in one point.

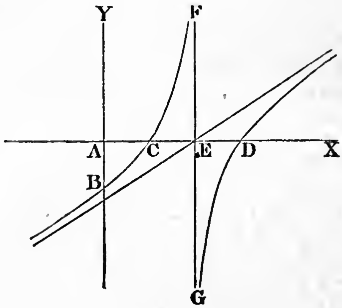
204. In article (77.) it was stated that, in some cases, the form of the curve could not be readily ascertained: thus, when the curve cuts neither diameter, there might be some difficulty in ascertaining its correct position: the asymptotes will, however, be found very useful in this respect: for example, if the equation is $x y = x^2 + b x + c^2$, or $y = x + b + \frac{c^2}{x}$, we have for $x = 0$, $y = \infty$; and when x becomes very great, y approximates to $x + b$; hence the lines A Y and T B S, in figure (194), will represent the asymptotes of the curve; and since the curve never cuts the axes, its course is entirely confined within the angle Y B S and the opposite angle T B A; hence the position of the curve is at once determined, as in figure (194).

Ex. 2. $y(x - 2) = (x - 1)(x - 3)$, or $y = \frac{(x-1)(x-3)}{x-2}$

In the first place we ascertain that the curve is an hyperbola by the test $b^2 - 4ac$ being positive; then draw the rectangular axes A X, A Y: to find the points where the curve cuts the axes,

$$\text{Let } x = 0, \quad \therefore y = -\tfrac{3}{2} = \text{A B},$$
$$\text{Let } y = 0, \quad \therefore x = 1 = \text{A C},$$
$$\text{also } x = 3 = \text{A D},$$

thus the curve passes through the points B, C, and D.



Again, to find the asymptotes, we have $y = \infty$ for $x = 2$; hence, if A E $= 2$, the line F E G, drawn perpendicular to A X, is one asymptote.

To find the other, we have

$$y = \frac{(x-1)(x-3)}{x-2} = \frac{(x-1)(x-3)}{x\left(1 - \frac{2}{x}\right)} = \frac{(x-1)(x-3)}{x}\left(1 - \frac{2}{x}\right)^{-1}$$

$$= \frac{x^2 - 4x + 3}{x} \left\{ 1 + \frac{2}{x} +, \text{\&c.} \right\} = \left(x - 4 + \frac{3}{x} \right)$$

$$\left\{ 1 + \frac{2}{x} +, \text{\&c.} \right\} = x - 4 + 2 + \frac{a}{x} +, \text{\&c.};$$

hence the equation to the asymptote is $y = x - 2$, and therefore this line must be drawn through the point E, making an angle of 45° with A X.

We can now trace the course of the curve completely; for all values of x less than 1, y is negative, hence the branch B C; for x greater than 1, but less than 2, y is positive and increases from 0 to ∞, hence the branch C F; for x greater than 2, but less than 3, y is negative, hence the branch G D; and for x greater than 3, y is positive and approximating to $x - 2$, hence the branch from D extending to the second asymptote.

For negative values of x, y is negative, and increases from $-\frac{3}{2}$ to ∞, approximating also to the value $-x-2$; hence the curve extends downwards from B towards the asymptote.

Ex. 3. $y(x-a) = x(x-2a)$. Here $x = a$ and $y = x - a$, are the equations to the asymptotes. The figure is like the last, supposing that A and C coincide.

Ex. 4 $y^{\frac{1}{2}} = a\,x^{-\frac{1}{2}} + x^{\frac{1}{2}}$. The axis of y is one asymptote, since $x = 0$ gives $y = \infty$: Also

$$y = x\left(1 + \frac{a}{x}\right)^2 = x + 2a + \frac{a^2}{x};$$

hence $y = x + 2a$ gives the other asymptote.

205. In order to discuss an equation of the second order completely, we have given, in Chapter VII., a general method of reducing that equation to its more simple forms.

In that chapter we showed that the equation, when belonging to an hyperbola, could be reduced to the form $a y^2 + c x^2 + f = 0$. (84.)

Now the same equation can be reduced also to the form $x y = k^2$; and as this form is of use in all discussions about asymptotes, we shall proceed to its investigation.

206. Let the general equation be referred to rectangular axes, and let it be

$$a y^2 + b x y + c x^2 + d y + e x + f = 0.$$

Let $x = x' + m$, and $y = y' + n$, and then, as in article (80.), put the co-efficients of x' and y' each $= 0$; by this means the curve is referred to its centre, and its equation is reduced to the form

$$a y'^2 + b x' y' + c x'^2 + f' = 0.$$

Again, to destroy the co-efficients of x'^2 and y'^2, take the formulas of transformation from rectangular to oblique co-ordinates (57.).

$$y' = \quad x'' \sin.\theta + y'' \sin.\theta',$$

$$x' = \quad x'' \cos.\theta + y'' \cos.\theta';$$

then, by substituting and arranging, the central equation becomes

$$y''^2 \{a(\sin.\theta')^2 + b \sin.\theta' \cos.\theta' + c(\cos.\theta')^2\}$$

$$+ x''^2 \{a(\sin.\theta)^2 + b \sin.\theta \cos.\theta + c(\cos.\theta)^2\}$$

$$+ x'' y'' \{2 a \sin. \theta' \sin. \theta + b (\sin. \theta \cos. \theta' + \sin. \theta' \cos. \theta) + 2 c \cos. \theta' \cos. \theta\} + f' = 0.$$

There are two new indeterminate quantities θ and θ' introduced; therefore we may make two suppositions respecting the co-efficients in the transformed equation; hence, letting the co-efficients of x''^2 and $y''^2 = 0$, we have

$$a (\sin. \theta)^2 + b \sin. \theta \cos. \theta + c (\cos. \theta)^2 = 0 \qquad (1),$$
$$a (\sin. \theta')^2 + b (\sin. \theta') (\cos. \theta') + c (\cos. \theta')^2 = 0 \quad (2).$$

Dividing the first of these two equations by $(\cos. \theta)^2$, we have

$$a (\tan. \theta)^2 + b \tan. \theta + c = 0;$$

$$\text{hence} \quad \tan \theta = \frac{b \pm \sqrt{b^2 - 4 a c}}{2 a}.$$

From the similarity of the equation (1) and (2), it is evident that we shall arrive at the same value for $\tan. \theta'$; hence, letting one of the above values refer to θ, the other will refer to θ; or both the new axes are de termined in position from the above values of $\tan. \theta$.

The equation is now reduced to the form

$$b' x'' y'' + f' = 0.$$

207. To find the value of b', we have

$$b' = 2 a \sin. \theta' \sin \theta + b (\sin. \theta \cos. \theta' + \sin. \theta' \cos. \theta) + 2 c \cos. \theta' \cos. \theta,$$
$$= \cos. \theta' \cos. \theta \{2 a \tan. \theta' \tan. \theta + b (\tan. \theta' + \tan. \theta) + 2 c\}.$$

From the equation involving $\tan. \theta$, we have

$$\tan. \theta \,.\, \tan. \theta' = \frac{c}{a}, \quad \tan. \theta + \tan. \theta' = - \frac{b}{a},$$

and therefore $\cos. \theta \cos. \theta' = \dfrac{a}{\sqrt{(a - c)^2 + b^2}}$;

$$\therefore\ b' = \frac{a}{\sqrt{(a - c)^2 + b^2}} \left\{2 c - \frac{b^2}{a} + 2 c\right\} = - \frac{b^2 - 4 a c}{\sqrt{(a - c)^2 + b^2}}.$$

$$\text{Also} \quad f' = \frac{a e^2 - c d^2 - b d e}{b^2 - 4 a c} + f \quad (80.).$$

Hence the final equation is

$$- \frac{b^2 - 4 a c}{\sqrt{(a - c)^2 + b^2}} x'' y'' + \frac{a e^2 + c d^2 - b d e}{b^2 - 4 a c} + f = 0.$$

208. If the original axes are oblique we must take the formulas in (56.), and then, following the above process, we find

$$\tan. \theta = \frac{\pm \sqrt{b^2 - 4 a c} - b + 2 c \cos. \omega}{2 (a + c (\cos. \omega)^2 - b \cos. \omega)},$$

$$b' = \frac{- (b^2 - 4 a c)}{\sqrt{\{(a + c - b \cos. \omega)^2 + (b^2 - 4 a c) (\sin. \omega)^2\}}}.$$

209. The following examples relate to the reduction of the general equation referred to rectangular axes, to another equation referred to the asymptotes.

Ex 1. $y^2 - 10\,x\,y + x^2 + y + x + 1 = 0$,

$$m = \frac{1}{8},\ n = \frac{1}{8},\ f' = \frac{9}{8};\ \tan.\theta = 5 \pm 2\sqrt{6},\ b' = -\frac{48}{5};$$

$$\therefore\ -\frac{48}{5}x''\,y'' + \frac{9}{8} = 0,$$

$$\text{or,}\quad x''\,y'' = \frac{15}{128}.$$

Ex. 2. $4\,y^2 - 8\,x\,y - 4\,x^2 - 4\,y + 28\,x - 15 = 0$,

$$b' = -8\sqrt{2},\ f' = 2,$$

$$\therefore\ -8\sqrt{2}\,x''\,y'' + 2 = 0,$$

$$\text{or,}\quad x''\,y'' = \frac{1}{4\sqrt{2}}.$$

Ex. 3. $\dfrac{a}{x} + \dfrac{b}{y} = 1$, or $x\,y = a\,y + b\,x$.

The axes are here parallel to the asymptotes (198.): in order to transfer the origin to the centre, let $y = y' + n$ and $x = x' + m$, hence we have $m = a$, $n = b$, and the reduced equation is

$$x'\,y' = a\,b.$$

210. If θ and θ' be the angles which the asymptotes make with the original rectangular axes, we have from the equation (206.),

$$a\,(\tan.\theta)^2 + b\tan.\theta + c = 0,$$

$$\therefore\ \tan.\theta.\ \tan.\theta' = \frac{c}{a}.$$

Now when $c = -a$, this equation becomes $\tan.\theta.\tan.\theta' = -1$, or, $\tan.\theta.\tan.\theta' + 1 = 0$; hence by (47.), the angle between the asymptotes is in this case $= 90°$; and thus whenever, in the general hyperbolic equation, we have $c = -a$, the curve is a rectangular hyperbola.

Ex. 4. $y^2 - x^2 = \sqrt{2}$.

The curve is a rectangular hyperbola, and is referred to its centre and rectangular axes; also taking the two values of $\tan.\theta$ in (206.), we have $\tan.\theta = 1$, and $\tan.\theta' = -1$; hence $\theta = 45°$ and $\theta' = -45°$, and the formulas of transformation become

$$y = \frac{x' - y'}{\sqrt{2}},\ x = \frac{x' + y'}{\sqrt{2}};$$

$$\therefore\ \left(\frac{x' - y'}{\sqrt{2}}\right)^2 - \left(\frac{x' + y'}{\sqrt{2}}\right)^2 = \sqrt{2},$$

$$\text{or}\quad -2\,x'\,y' = \sqrt{2},$$

$$\text{and}\quad x'\,y' = -\frac{1}{\sqrt{2}}.$$

In this example the curve is placed as in the next figure, and at first was referred to the axes C X and C Y, but now is referred to the asymptotes C x and C y, supposing C y and C x to change places, and the angle x C y = 90°.

211. Conversely given the equation $xy = k^2$, to find the equation referred to the rectangular axes, and thence to deduce the lengths of the axes

For this purpose we use the formulas of transformation from oblique to rectangular axes (56.).

$$y = \frac{x' \sin. \theta + y' \cos. \theta}{\sin. \omega},$$

$$x = \frac{x' \sin. (\omega - \theta) - y' \cos. (\omega - \theta)}{\sin. \omega};$$

substituting these values in the equation $xy = k^2$, we have

$$x'^2 \sin. \theta \sin (\omega - \theta) - y'^2 \cos. \theta \cos. (\omega - \theta)$$
$$+ x' y' \{\cos. \theta \sin. (\omega - \theta) - \sin. \theta \cos. (\omega - \theta)\} = k^2 (\sin. \omega)^2.$$

Let the co-efficient of $x' y' = 0$,

$\therefore \cos. \theta \sin. (\omega - \theta) - \sin. \theta \cos. (\omega - \theta)$, or $\sin. (\omega - 2\theta) = 0$;

$$\therefore \omega = 2\theta, \text{ and } \theta = \frac{\omega}{2};$$

hence the new rectangular axis of x, determined by the angle θ, bisects the angle ω between the asymptotes; this agrees with the remark at the end of (179.).

The transformed equation, putting $\theta = \frac{\omega}{2}$, is

$$x'^2 \left(\sin. \frac{\omega}{2}\right)^2 - y'^2 \left(\cos. \frac{\omega}{2}\right)^2 = k^2 (\sin. \omega)^2,$$

or, putting $2 \sin. \frac{\omega}{2} \cos. \frac{\omega}{2}$ for $\sin. \omega$, and dividing

$$\frac{y'^2}{4k^2 \left(\sin. \frac{\omega}{2}\right)^2} - \frac{x'^2}{4k^2 \left(\cos. \frac{\omega}{2}\right)^2} = -1;$$

Comparing this with the equation $\frac{y^2}{b^2} - \frac{x^2}{a^2} = -1$, we have

$$a = 2k \cos. \frac{\omega}{2}, \text{ and } b = 2k \sin. \frac{\omega}{2};$$

hence the lengths of the semi-axes are determined.

If the equation had been $xy + ax + by + c = 0$, first refer the curve to its centre, and then proceed as above.

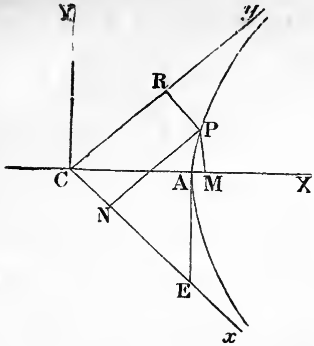
212. To deduce the equation $xy = k^2$ from the equation to the curve referred to the centre and rectangular axes.

Let C X, C Y be the rectangular axes,

C x, C y the asymptotes, or the new axes,

$\left.\begin{array}{l} \text{CM} = x \\ \text{MP} = y \end{array}\right\}$ the original co-ordinates of P,

$\left.\begin{array}{l} \text{CN} = x' \\ \text{NP} = y' \end{array}\right\}$ the new co-ordinates of P.



Then taking the formulas of transformation from rectangular to oblique axes (57),

$$y = x' \sin.\theta + y' \sin.\theta',$$
$$x = x' \cos.\theta + y' \cos.\theta',$$

and substituting in the equation $a^2 y^2 - b^2 x^2 = -a^2 b^2$, we have

$$a^2 (x' \sin.\theta + y' \sin.\theta')^2 - b^2 (x' \cos.\theta + y' \cos.\theta')^2 = -a^2 b^2,$$

$$\text{or, } \{a^2 (\sin.\theta')^2 - b^2 (\cos.\theta')^2\} y'^2 + \{a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2\} x'^2,$$
$$+ 2 \{ a^2 \sin\theta \sin.\theta' - b^2 \cos.\theta \cos.\theta' \} x' y' = -a^2 b^2.$$

In order that this equation may be of the required form, it must not contain the terms in x'^2 and y'^2; but since we have introduced two indeterminate quantities, we can make the two suppositions that the co-efficients of these terms shall $= 0$;

$$\therefore\ a^2 (\sin.\theta')^2 - b^2 (\cos.\theta')^2 = 0,$$
$$a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2 = 0,$$

From the last of these equations we have $\tan.\theta = \pm \frac{b}{a}$, and as we obtain from the other equation the same value of $\tan.\theta'$, it follows that the values of θ and θ' are both contained in the equation $\tan.\theta = \pm \frac{b}{a}$, that is, if $\tan.\theta \left(= -\frac{b}{a}\right)$ refers to the axis of x, then $\tan.\theta \left(= +\frac{b}{a}\right)$ refers to the axis of y, (we have chosen $\tan.\theta = -\frac{b}{a}$ for the axis of x, in order to agree with the figure).

The equation to the curve referred to its asymptotes is now

$$2 \{a^2 \sin.\theta \sin.\theta' - b^2 \cos.\theta \cos.\theta'\} x' y' = -a^2 b^2,$$
$$\text{or, } \quad 2 \cos.\theta \cos.\theta' \{a^2 \tan.\theta \tan.\theta' - b^2\} x' y' = -a^2 b^2;$$

but since $\tan.\theta = \pm \frac{b}{a}$, we have

$$\cos.\theta = \frac{1}{\sqrt{1+(\tan.\theta)^2}} = \frac{a}{\sqrt{a^2+b^2}} = \cos.\theta';$$

$$\therefore 2\frac{a^2}{a^2+b^2}\left\{-a^2\frac{b^2}{a^2}-b^2\right\}x\,y' = -a^2b^2.$$

$$\text{or,}\quad -\frac{4\,a^2b^2}{a^2+b^2}x'\,y' = -a^2b^2;$$

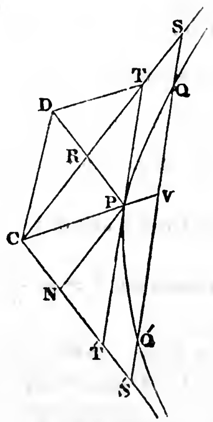
$$\therefore x'\,y' = \frac{a^2+b^2}{4}.$$

If $b = a$, or the curve be the rectangular hyperbola, the equation referred to the asymptotes is $x\,y = \frac{a^2}{2}$.

213. The angle between the asymptotes is 2θ; if therefore P R be drawn parallel to C N, the area P N C R $= x$ sin. $2\theta = x\,y$. 2 sin. θ cos. $\theta = \frac{a^2+b^2}{4} \cdot 2 \cdot \frac{b}{\sqrt{a^2+b^2}} \cdot \frac{a}{\sqrt{a^2+b^2}} = \frac{a\,b}{2}$.

Thus all the parallelograms constructed upon co-ordinates parallel to the asymptotes are equal to each other, and to half the rectangle in the semi-axes.

214. Let C S, C S′ be the asymptotes to the curve referred to conjugate diameters C P, C D $(a_1\,b_1)$, then if P T be parallel to C D, it is a tangent at P (187.); T P T′ is also a double ordinate to the asymptote, for the equation to C S is $y = \pm\frac{b_1}{a_1}x$, and when $x = a_1, y = \pm b_1$. Hence P T $=$ P T′, or the parts of the tangent contained between the point of contact and the asymptotes are equal to each other, and to the semi-conjugate diameter.



215. Join D T, then D P is a parallelogram; also because C D is equal and parallel to P T′, we have the line D P parallel to the asymptote

C S′. Hence, if the conjugate diameters be given, the asymptotes may always be found by completing the parallelogram upon the conjugate diameters, and then drawing the diagonals. Also, if the asymptotes be given, a conjugate diameter to C P may be found by drawing P R parallel to C S′ and taking P D double of P R.

If the asymptotes be given, a tangent may be drawn by taking C T double of C R, and joining P T.

If the position of the focus is known, the length of the conjugate axis is equal to the perpendicular, from the focus on the asymptote.

216. To find the equation to the tangent P T, when referred to the asymptotes as axes,

Let x', y' be the co-ordinates of P, and $x''\ y''$ co-ordinates of another point on the curve.

$$\therefore\ y' = \frac{k^2}{x'},\ \text{and}\ y'' = \frac{k^2}{x''}$$

$$\therefore\ \frac{y' - y''}{x' - x''} = -\frac{k^2}{x'\,x''} = -\frac{y'}{x''}$$

$$\therefore\ y - y' = -\frac{y'}{x''}(x - x')\ \text{is the equation to a secant.}$$

When $x'' = x'$ we have the equation to the tangent

$$y - y' = -\frac{y'}{x'}(x - x') = -\frac{y'}{x'}x + y'$$

$$\therefore\ x'y + y'x = 2\,x'y' = 2\,k^2.$$

This equation to the tangent is readily obtained from the equation to the curve ($xy = k^2$ or $xy + xy = 2k^2$) by putting $x'y$ and xy' successively for xy, and then adding the results.

Let $y = 0\ \therefore$ C T′ $= 2\,x' = 2$ C N ; and C T $= 2\,y' = 2$ N P ;

The triangle C T T′ $= \frac{1}{2}.\ 2\,x'.\ 2\,y'$ sin. T C T $= 2\,x'\,y'$ sin. $2\,\theta = a\,b$, (213.)

217. The two parts S Q, S′Q′ of any secant S Q Q′S comprised between the curve and its asymptote are equal; for if the diameter C P V and its conjugate C D be drawn, we have V Q = V Q′ from the equation to the curve $\left(y = \pm\frac{b_1}{a_1}\sqrt{x^2 - a_1^2}\right)$, and from the equation to the asymptotes $\left(y = \pm\frac{b_1}{a_1}x,\right)$ we have V S = V S $\therefore$ S Q = S′Q′.

218. If Y and y are the ordinates V S, V Q respectively, we have

$$Y^2 - y^2 = \frac{b_1^2}{a_1^2}x^2 - \frac{b_1^2}{a_1^2}(x^2 - a_1^2)$$

$$\text{or } (Y - y)(Y + y) = b_1^2.$$

Thus the rectangle S Q, Q S′ = the square upon C D.

THE POLAR EQUATION.

219. Let the curve be referred to the centre C, and to rectangular axes C A, C B, and let the co-ordinates of the pole O be x' and y', O being situated anywhere in the plane of the curve and P any point on the curve, as in (146.), θ the angle which the radius vector O P or u makes with a line parallel to the axis of x. Then we have by (61.)

$$y = y' + u \sin.\theta$$
$$x = x' + u \cos.\theta$$
$$\text{also,} \quad a^2 y^2 - b^2 x^2 = -a^2 b^2$$
$$\therefore \; a^2 (y' + u \sin.\theta)^2 - b^2 (x' + u \cos.\theta)^2 = -a^2 b^2$$

220. Let the centre be the pole, $\therefore$ $x' = 0$, and $y' = 0$,

$$\therefore \; u^2 = \frac{-a^2 b^2}{a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2} = \frac{a^2 (e^2 - 1)}{e^2 (\cos.\theta)^2 - 1}.$$

221. Let the focus S be the pole,

$$\therefore \; y' = 0, x' = a e \text{ and } u \text{ becomes } r,$$

Substituting these values, and following the steps in (148.), we find

$$r = \frac{b^2}{a - c\cos.\theta} = \frac{a(e^2 - 1)}{1 - e\cos\theta}$$

If the angle A S P $= \theta$, we have

$$r = \frac{a(e^2 - 1)}{1 + e\cos.\theta}$$

This is the equation generally used. It may easily be obtained from the equation $r = e x - a$, fig. (161) $= e (a e - r \cos.\theta) - a$,

$$\therefore \; r = \frac{a(e^2 - 1)}{1 + e\cos.\theta}.$$

222. If $\frac{p}{2} = a(e^2 - 1)$ we have $r = \frac{p}{2} \cdot \frac{1}{1 + e\cos.\theta} =$ S P, and if P S meet the curve again in P′, we have the rectangle S P, S P′ $= \frac{p}{4}$ (P S + S P′) $= \frac{p}{4}$ P P′.

The length of the chord through the focus $= 2\frac{b_{,}^2}{a}$ where $b_{,}$ is the diameter to that chord.

THE CONJUGATE HYPERBOLA.

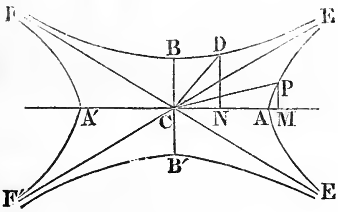
223. There is another equation to the hyperbola, not yet investigated. If $\left(\frac{c'}{-f'}\right)$ be negative in article 153, the equation is $P y^2 - Q x^2 = 1$

or $a^2 y^2 - b^2 x^2 = a^2 b^2$ if $P = \frac{1}{b^2}$, and $Q = \frac{1}{a^2}$. If we examine the course of this curve, we shall find that $B B' = 2 b$ is the real or transverse axis, and $A A'$, or $2 a$, is the conjugate axis, and that the curve extends indefinitely from B to B′, so that it is, in form, like the hyperbola already investigated, but only placed in a different manner.

Both curves are represented in the next figure; the real axis of the one being the conjugate or imaginary axis of the other.

It is evident from the form of the equations that both curves have got common asymptotes E C E′, F C F′.

224. Let C P and C D be two conjugate diameters to the original hyperbola A P E, it is required to find the locus of D.



Let $C M = x'$, $M P = y'$, $C N = x$, $N D = y$,

$$\text{then } a_1^2 - b_1^2 = a^2 - b^2;$$

$$\therefore x'^2 + y'^2 = x^2 + y^2 + a^2 - b^2$$

but the equation to C D is

$$a^2 y y' - b^2 x x' = 0 \qquad \therefore x' = \frac{a^2 y y}{b^2 x},$$

$$\therefore x'^2 + y'^2 = \frac{a^4 y^2 + b^4 x^2}{b^4 x^2} y'^2 = x^2 + y^2 + a^2 - b^2,$$

$$\therefore y'^2 = \frac{b^4 x^2}{a^4 y^2 + b^4 x^2} (x^2 + y^2 + a^2 - b^2),$$

$$\text{and } x'^2 = \frac{a^4 y^2}{a^4 y^2 + b^4 x^2} (x^2 + y^2 + a^2 - b^2).$$

Substituting these values in the equation $a^2 y'^2 - b^2 x'^2 = - a^2 b^2$, and reducing, we have $a^2 y^2 \quad b^2 x^2 = a^2 b^2$, hence the locus of D is the conjugate hyperbola, and hence arises its name.

By changing the sign of the constant term in the equation to any hyperbola, referred to its centre, we directly obtain the equation to its conjugate, referred to the same axes of x and y. Both curves are comprised in the form

$$(a^2 y^2 - b^2 x^2)^2 = a^4 b^4, \text{ or } x^2 y^2 = k^4.$$

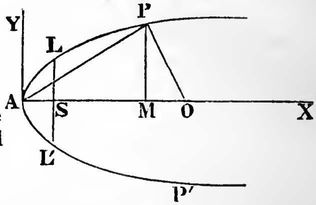
CHAPTER X.

THE PARABOLA.

225. **The** equation to the parabola referred to rectangular axes, has been reduced to the form $a' y^2 + e' x = 0$ (94.).

From this equation we proceed now to deduce all the important properties of the parabola.

$$\text{Let } \frac{e'}{-a'} = p, \quad \therefore y^2 = px$$



Let A be the origin; A X, A Y the axes; then for $x = 0$ we have $y = 0$, and the curve passes through the origin A.

For each positive value of x there are two equal and opposite values of y, which increase from 0 to ∞, according as x increases from 0 to ∞; hence there are two equal arcs, A P and A P′, proceeding from A, without any limit. This curve is symmetrical with respect to its axis A X, and its concavity is turned towards that axis, otherwise it could be cut by a straight line in more points than one.

For every negative value of x, y is imaginary.

226. The point A is called the vertex of the parabola; A X, A Y the principal axes; but, generally speaking, A X alone is called the Axis of the parabola. Thus the equation to the curve referred to its axis and vertex is $y^2 = px$.

From this equation we have The square upon the ordinate = The rectangle under the abscissa and a constant quantity; or the square upon the ordinate varies as the abscissa.

227. The last property of this curve points out the difference between the figures of the hyperbola and parabola; both have branches extending to infinity, but of a very different nature; for the equation to the hyperbola is $y^2 = \frac{b^2}{a^2}(x^2 - a^2) = \frac{b^2}{a^2}x^2\left(1 - \frac{a^2}{x^2}\right)$, and therefore, for large values of x, the values of y^2 increase nearly as the corresponding values of x^2 or y varies nearly as x; hence the hyperbolic branch rises much more rapidly than that of the parabola, whose ordinate varies only as $\sqrt{x}$. When x is very great, the former takes nearly the course of the line $y = \frac{b}{a}x$, but in the parabola, y is not much increased by an increase of x, and therefore the curve tends rather towards parallelism with the axis of x.

228. The equation to the parabola may be derived from that of the ellipse by considering the axis major of the ellipse to be infinite.

Let C be the centre, and S the focus of an ellipse whose equation is

$$y^2 = \frac{2\,b^2}{a}\,x - \frac{b^2}{a^2}\,x^2. \quad (105.)$$

Let $m = \text{A S} = \text{A C} - \text{S C} = a - \sqrt{a^2 - b^2}$, (fig. 106.)

$$\therefore\ b^2 = 2\,a\,m - m^2;$$

$$\therefore\ y^2 = \left(4\,m - \frac{2\,m^2}{a}\right)x - \left(\frac{2\,m}{a} - \frac{m^2}{a^2}\right)x^2.$$

Now if a be considered to vary, this will be the equation to a series of ellipses, in which the distance A S, or m, is the same for all, but the axis major different for each; thus giving to a any particular value, we have a corresponding ellipse. Let now a be infinite, then, since all the other terms vanish, the equation becomes $y^2 = 4\,m\,x$; hence the ellipse has gradually approached to the parabolic form, as its axes enlarged, and finally coincided with it when the axis major was infinite *.

In the same manner the equation to the parabola may be derived from that to the hyperbola.

THE FOCUS.

229. The quantity p, which is the co-efficient of x in the equation to the parabola, is called the principal parameter, or Latus Rectum of the parabola.

Since $p = \frac{y^2}{x}$, the principal parameter is a third proportional to any abscissa and its corresponding ordinate.

In article (228.) we have used the equation $y^2 = 4\,m\,x$ for the parabola, merely to avoid fractions with numerical denominators; it appears that many of the operations in this chapter are similarly shortened, without losing any generality, by merely putting $4\,m$ for p; hence we shall use the equation $y^2 = 4\,m\,x$ in most of the following articles, recollecting that all the results can be expressed in terms of the principal parameter, by putting $\frac{p}{4}$ for m wherever m occurs.

230. To find the position of the double ordinate which is equal to the Latus Rectum.

Let $2\,y = 4\,m$, $\quad\therefore\ 4\,y^2 = 16\,m^2$, or $16\,m\,x = 16\,m^2$, and $x = m$.

In A X take A S $= m$, then the ordinate L S L′ drawn through S, is the Latus Rectum.

The point S is called the focus.

The situation of the focus S may be also thus determined:

Let A M $= x$, M P $= y$, join A P, and draw P O perpendicular to A P,

Then A M : M,P :: M P : M O $= \frac{y^2}{x} = 4\,m$, $\therefore$ A S $= m = \frac{1}{4}$ M O

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* If x is very small when compared with a, the equation to the ellipse is very nearly that to a parabola; and this is the reason that the path of a comet near its perihelion appears to be a portion of a parabola.

231. To find the distance of any point P in the curve from the focus:

Let $SP = r$, $AM = x$, $MP = y$; also at s, $y' = 0$, and $x' = m$,

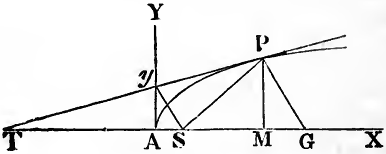
$$\therefore r^2 = (y - y')^2 + (x - x')^2 = y^2 + (x - m)^2 = 4mx + (x - m)^2$$
$$= (x + m)^2;$$
$$\therefore r = SP = x + m.$$

THE TANGENT.

232. To find the equation to the tangent at any point P (x', y') of the parabola.

The equation to a secant through two points on the curve (x', y') $(x'' y'')$ is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'),$$



Also $y'^2 = 4mx'$, and $y''^2 = 4mx''$;

$$\therefore y'^2 - y''^2 = 4m(x' - x''),$$

$$\text{and} \quad \frac{y' - y''}{x' - x''} = \frac{4m}{y' + y''}$$

Thus the equation to the secant becomes

$$y - y' = \frac{4m}{y' + y''}(x - x');$$

but when the two points coincide $y'' = y'$, and the secant becomes a tangent,

$$\therefore y - y' = \frac{4m}{2y'}(x - x'),$$

$$\text{or} \quad yy' - y'^2 = 2m(x - x'),$$

$$\therefore yy' = y'^2 + 2m(x - x') = 4mx' + 2m(x - x'$$

$$\therefore yy' = 2m(x + x').$$

This equation is immediately deduced from that to the curve

$$(y^2 = 4mx = 2m(x + x))$$

by writing yy' for y^2, and $x + x'$ for $x + x$.

233. To find the points where the tangent cuts the axes.

Let $y = 0$, $\therefore x + x' = 0$, $\therefore x = -x'$, or $AT = -AM$;

Hence the absolute value of the sub-tangent MT is 2 AM.

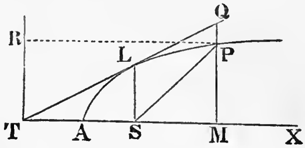
Let $x = 0$, $\therefore\ y = 2\,m\,\dfrac{x'}{y'} = \dfrac{y'^2}{2\,y'} = \dfrac{y'}{2}$, $\quad\therefore$ A $y = \frac{1}{2}$ M P.

234. The equation to the tangent being $y\,y' = 2\,m\,(x + x')$, we have at the vertex A, x' and y' each $= 0$, therefore the equation to the tangent becomes $2\,m\,x = 0$, or $x = 0$;

But $x = 0$ is the equation to the axis A Y;

Hence the tangent at the vertex of the parabola coincides with the axis of y.

235. To find the equation to the tangent at the extremity of the principal parameter.



$$y\,y' = 2\,m\,(x + x')$$

At L we have $x' = m$, and $y' = 2\,m$,

$$\therefore\ 2\,m\,y = 2\,m\,(x + m),$$

$$\therefore\ y = x + m,$$

If the ordinate y or M Q cut the parabola in P, we have S P $= x + m$ (231.), $\therefore$ M Q $=$ S P.

236. To find the point where this particular tangent cuts the axis of x.

Let $y = 0$, $\therefore\ x =$ A T $= -\,m = -$ A S.

From T draw T R perpendicular to A X, and from P draw P R parallel to A X, then taking the absolute value of A T, we have

P R $=$ A T $+$ A M $= m + x =$ S P.

Consequently the distances of any point P from S, and from the line T R, are equal to one another.

This line, T R, is called the directrix; for knowing the position of this line and of the focus, a parabola may be described.

This tangent cuts the axis at an angle of 45°. (35. Ex. 3:)

237. To find the length of the perpendicular S y from the focus on the tangent.

Taking the expression in (48.) we have

$$\text{S}\,y = -\,\frac{y_1 - \alpha\,x_1 - b}{\sqrt{1 + \alpha^2}}.$$

But from the figure 232, we have $y_1 = 0$, and $x_1 = m$ for the coordinates of the point S, and $y = \alpha\,x + b$ is the equation to the line P T; also the equation to P T is

$$y = \frac{2\,m}{y'}\,(x + x'),$$

$$\therefore\ \alpha = \frac{2\,m}{y'}, \text{ and } b = \frac{2\,m\,x'}{y'},$$

$$\therefore\ \text{S}\,y = \frac{\dfrac{2\,m}{y'}\,m + \dfrac{2\,m\,x'}{y'}}{\sqrt{\left\{1 + \dfrac{4\,m^2}{y'^2}\right\}}} = \frac{2\,m\,(m + x')}{\sqrt{\{y'^2 + 4\,m^2\}}} = \frac{2\,m\,(m + x')}{\sqrt{\{4\,m\,x' + 4\,m^2\}}}$$

$$= \sqrt{m\,(m + x')} = \sqrt{m\,r}, \text{ if S P} = r;$$

Hence the square on $S\,y$ = the rectangle $S\,P$, $S\,A$;

$$\text{or,}\quad S\,P : S\,y :: S\,y : S\,A.$$

238. To find the locus of y in the last article.

The equation to the tangent $P\,T$, fig. 232, is $y = \frac{2\,m}{y'}\,(x + x')$;

Hence the equation to $S\,y$ passing through the point $(m, 0)$, and perpendicular to $P\,T$, is

$$y = -\frac{y'}{2\,m}\,(x - m).$$

To find where this line cuts the axis of y, put $x = 0$, $\therefore\ y = \frac{y'}{2}$, but this is the point where the tangent at P cuts the same axis (233.); hence the tangent and the perpendicular on it from the focus meet in the axis $A\,Y$, or the locus of y is the axis $A\,Y$.

239. Again, to find where the perpendicular $S\,y$ cuts the directrix, put

$$x = -m,\ \therefore\ y = -\frac{y'}{2\,m}\,(x - m) = -\frac{y'}{2\,m}\,(-m - m) = y',$$

but this is the ordinate $M\,P$; hence a tangent being drawn at any point P, the perpendicular on it from the focus cuts the directrix in the point where the perpendicular from P on the directrix meets that directrix.

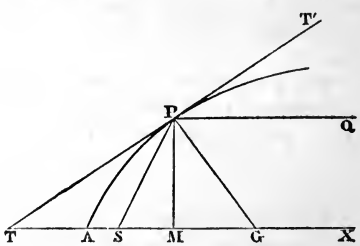
240. To find the angle which the tangent makes with the focal distance

The equation to the tangent $P\,T$ is $y = \frac{2\,m}{y'}\,(x + x')$.

The equation to the focal distance $S\,P$ through the points S $(= 0, m)$ and P $(= x', y')$ is

$$y = \frac{y'}{x' - m}\,(x \quad m),$$

And $\tan.\, S\,P\,T = \tan.\,(P\,S\,X - P\,T\,X)$



$$= \frac{\dfrac{y'}{x' - m} - \dfrac{2\,m}{y'}}{1 + \dfrac{y'}{x' - m}\,\dfrac{2\,m}{y'}}$$

$$= \frac{y'^2 - 2\,m\,(x' - m)}{y'\,(x' - m) + 2\,m\,y'} = \frac{4\,m\,x' - 2\,m\,x' + 2\,m^2}{y'\,(x' + m)} = \frac{2\,m\,(x' + m)}{y'\,(x' + m)}$$

$$= \frac{2\,m}{y'} = \frac{y'}{2\,x'},\quad \text{since } 2\,m = \frac{y'^2}{2\,x'}:$$

But $M\,P = M\,T\ \tan.\,P\,T\,M$, $\therefore\ \tan.\,P\,T\,M = \frac{y'}{2\,x'}$, $\therefore\ \tan.\,S\,P\,T$

$=$ tan. S T P $=$ tan. T′P Q, if P Q be drawn parallel to the axis of x. Thus the tangent at P makes equal angles with the focal distance, and with a parallel to the axis through P.

This important theorem may also be deduced from the property in article 233. It is there proved that the absolute value of A T is A M, hence we have S T $=$ S A $+$ A T $= m + x =$ S P, and therefore the angle S P T $=$ angle S T P $=$ angle Q P T′.

If a ray of light, proceeding in the direction Q P, be incident on the parabola at P, it will be reflected to S, on account of the equal angles Q P T′, S P T: similarly all rays coming in a direction parallel to the axis, and incident on the curve, would converge to S; and if a portion of the curve revolve round its axis, so as to form a hollow concave mirror, all rays from a distant luminous point in the direction of the axis would be concentrated in S. Thus, if a parabolic mirror be held with its axis pointing to the sun, a very powerful heat will be found at the focus.

Again, if a brilliant light be placed in the focus of such a mirror, all the rays, instead of being lost in every direction, will proceed in a mass parallel to the axis, and thus illuminate a very distant point in the direction of that axis. This property of the curve has led to the adoption of parabolic mirrors in many light-houses.

THE NORMAL.

241. To find the equation to the normal P G, at a point P $(x'y')$.

The equation to a straight line, through P, is $y - y' = \alpha\,(x - x')$, and as this line must be perpendicular to the tangent whose equation is $y = \frac{2\,m}{y'}\,(x + x')$, we have $\alpha = -\frac{y'}{2\,m}$, hence the equation to the normal is $y - y' = -\frac{y'}{2\,m}\,(x - x')$.

242. To find the point where the normal cuts the axis of x.

Let $y = 0 \quad \therefore\ x - x' = m$, or the subnormal M G is constant and equal to half the principal parameter.

Hence S G $=$ S M $+$ M G $= x' - m + 2\,m = x' + m =$ S P

And P G $= \sqrt{y'^2 + 4\,m^2} = \sqrt{4\,m\,x' + 4\,m^2} = \sqrt{4\,m\,(x' + m)} = \sqrt{4\,m\,r}$.

Hence the normal P G is a mean proportional between the principal parameter and the distance S P.

THE DIAMETERS.

243. It was shown in article 81, that the parabola has no centre.

Since for every positive value of x there are two equal and opposite values of y, the axis of x is a diameter, but that of y is not; hence the axes cannot be called conjugate axes. The parabola has an infinite number of diameters, all parallel to the axis; to prove this,

Let $y = \alpha\,x + b$ be the equation to any chord,

$y^2 = p\,x$ the equation to the curve.

Transfer the origin to the bisecting point $x'y'$ of the chord, then the equations become $y = \alpha\,x$, and $(y + y')^2 = p\,(x + x')$.

To find where the chord intersects the curve, put αx for y in the second equation.

$$\therefore\ (\alpha x + y')^2 = p(x + x')$$
$$\text{or } \alpha^2 x^2 + (2\alpha y' - p)x + y'^2 - p x' = 0$$

But since the origin is at the bisection of the chord, the two values of x must be equal to one another, and have opposite signs; hence the second term of the last equation must $= 0$, $\therefore\ 2\alpha y' - p = 0$.

This equation gives the value of y', and since it is independent of b, it will be the same for any chord parallel to $y = \alpha x + b$; hence $y = \dfrac{p}{2\alpha}$ is the equation to the locus of all the middle points of a system of parallel chords, and this equation is evidently that to a straight line parallel to the axis; and conversely.

244. To transform the equation into another referred to a new origin and to new axes, and so that it shall preserve the same form,

$$\text{Let } x = a + x' \cos.\theta + y' \cos.\theta',$$
$$\text{and} \quad y = b + x' \sin.\theta + y' \sin.\theta', \quad (57.)$$

Substituting these values in the equation $y^2 = p x$ and arranging, we have

$$y'^2(\sin.\theta')^2 + x'^2(\sin.\theta)^2 + 2x'y'\sin.\theta\sin.\theta' + y'(2b\sin.\theta' - p\cos.\theta')$$
$$+ x'(2b\sin.\theta - p\cos.\theta) + b^2 - ap = 0.$$

And as this equation must be of the form $y^2 = p x$, we must have

$$(\sin.\theta)^2 = 0 \quad \ldots \quad (1),$$
$$2\sin.\theta.\sin.\theta' = 0 \quad \ldots \quad (2),$$
$$2b\sin.\theta' - p\cos.\theta' = 0 \quad (3),$$
$$b^2 - ap = 0 \quad \ldots \quad (4).$$

Hence the equation becomes

$$y'^2(\sin.\theta')^2 + (2b\sin.\theta - p\cos.\theta)x' = 0;$$
$$\text{or since } \theta = 0, \quad y'^2(\sin.\theta')^2 - p x' = 0.$$

245. On the examination of the equations (1) (2) (3) and (4), it appears from (1) that the new axis of x' is parallel to the original axis of x; and θ being 0 from (1), of course (2) is destroyed, and thus the equations of condition are reduced to three: but there are four unknown quantities, hence there are an infinite number of points to which, if the origin be transferred, the equation may be reduced to the same simple form.

We may take the remaining three quantities a, b and θ', in any order, and arrive at the same results. Suppose a is known, then from (4), $b^2 = p a$, this equation shows that a must be taken in a positive direction from A, and also that the new origin must be taken on the curve itself, or the new origin is at some point P on the curve, as in the next figure.

From (3) we have $\tan.\theta = \dfrac{p}{2b} = \dfrac{b}{2a}$;

but this is exactly the value of the tangent of the angle which a tangent P T to the curve makes with the Axis (240.): hence the new axis of y is a tangent to the curve at the new origin P.

The results are therefore these,—the new origin is at any point P on the curve (see the next figure). The axes are one (P X′) parallel to the axis A X, and the other (P Y′) is a tangent at the new origin P. Lastly, from the form of the equation, the new axis of x is a diameter.

246. The equation is $y'^2 = \frac{p}{(\sin.\ \theta')^2} x' = p'x$ where $p' = \frac{p}{(\sin \theta')^2}$

$= p\,(\text{cosec.}\ \theta')^2 = p\,(1 + \cot.\ \theta')^2 = p\left(1 + \frac{4\,a^2}{b^2}\right) = p + 4\,a$

$= 4\left(\frac{p}{4} + a\right) = 4\ \text{S P}$ (231.)

Hence the new parameter at P is four times the focal distance S P.

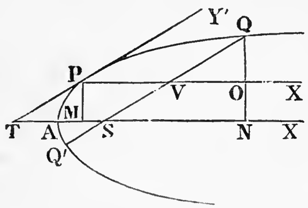
247. The equation to the parabola, when we know the position and direction of the new axes, is readily obtained from the original equation referred to rectangular co-ordinates.

Let the point P be the new origin, P X′, P Y′ the new axes, angle Y′ P X′ $= \theta$.

Also, let A N $= x$, N Q $= y$ be the rectangular co-ordinates of Q.

And A M $= a$, M P $= b$ . . . . . . P.

P V $= x'$, V Q $= y'$ be the new co-ordinates of Q.



Then $y = \text{Q N} = \text{M P} + \text{O Q} = b + y' \sin.\ \theta$,

$x = \text{A N} = \text{A M} + \text{P V} + \text{V O} = a + x' + y' \cos.\ \theta.$

Substituting these values in the equation $y^2 = p\,x$,

we have $(b + y' \sin.\ \theta)^2 = p\,(a + x' + y' \cos \theta)$;

$\therefore\ y'^2 (\sin.\ \theta)^2 + (2\,b \sin.\ \theta - p \cos.\ \theta)\,y' + b^2 = p\,a + p\,x$;

but $b^2 = p\,a$, and $\tan.\ \theta = \tan.\ \text{P T M} = \frac{b}{2\,a} = \frac{b}{2\frac{b^2}{p}} = \frac{p}{2\,b}$,

$$\therefore\ 2\,b \sin.\theta - p \cos.\theta = 0,$$

and the equation is reduced to the form

$$y'^2 (\sin.\ \theta)^2 = p\,x'.$$

Also from $(\tan.\ \theta)^2 = \frac{b^2}{4\,a^2}$ we have $(\cos.\ \theta)^2 = \frac{4\,a^2}{4\,a^2 + b^2}$

and $(\sin.\ \theta)^2 = \frac{b^2}{4\,a^2 + b^2} = \frac{p}{4\,a + p}$;

$$\therefore y'^2 \frac{p}{4a+p} = p x'$$

$$y'^2 = (4a+p)\,x' = 4\left(a + \frac{p}{4}\right) x' = p'\,x', \text{ where } p' = 4\,SP.$$

Hence the square upon the ordinate = the rectangle under the abscissa and parameter.

248. To find the length of the ordinate which passes through the focus:

Here, $x = PV = ST = SP = r \;\therefore\; y^2 = px = 4r \,.\, r = 4r^2$
$$\therefore y = 2r$$

Hence, $QQ' = 4\,SP$.

Thus the ordinate through the focus is equal to four times the focal distance S P, is equal to the parameter at the point P.

Hence, generally, if the origin of co-ordinates be at any point P on the parabola, and if the axes be a diameter and a tangent at P, the parameter to the point P is that chord which passes through the focus.

249. The equation to a tangent at any point Q $(x'\ y')$, referred to the new axes PX', PY', is

$$y y' = \frac{p'}{2}\left(x + x'\right)$$

Let $y = 0 \therefore x = -x'$, hence the sub-tangent = twice the abscissa.

Let $x = 0 \therefore y = \dfrac{p'x'}{2y} = \dfrac{y'}{2} = \frac{1}{2}$ the ordinate.

For y put $-y$, then we have the equation to the tangent at the other extremity Q' of the ordinate QVQ'; hence it may be proved that tangents at the two extremities of a chord meet in a diameter to that chord.

250. If the chord QVQ' pass through the focus, as in the figure, the co-ordinates of Q are $y' = 2\,SP = 2r$, and $x' = PV = SP = r$, also

$p' = 4r$; hence the equation to the tangent at Q, or $y y' = \dfrac{p'}{2}(x + x')$

becomes $y = x + r$, and similarly the equation to the tangent at Q' is $-y = x + r$, and these lines meet the axis PX' at a distance $-r$ from P, that is, tangents at the extremity of any parameter meet in the directrix.

Also, the angle between these tangents is determined from the equation

$$\tan.\theta = \frac{(\alpha - \alpha')\sin.\omega}{1 + \alpha\alpha' + (\alpha + \alpha')\cos.\omega} \qquad (51.)$$

$$= \frac{(1 + 1)\sin.\omega}{1 - 1 + 0} \text{ (since } \alpha = 1 \text{ and } \alpha = -1)$$

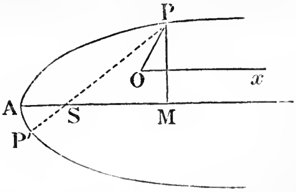
$$= \frac{1}{0} = \tan.\,90^\circ.$$

Hence, pairs of tangents drawn at the extremities of any parameter meet in the directrix at right-angles.

THE POLAR EQUATION

251. To find the polar equation to the curve.

Let the co-ordinates of any point O be x' and y', and let θ be measured from a line O x, which is parallel to the axis of the curve:



Then by (61.), or by inspection of the figure, we have

$$y = y' + u \sin.\theta$$

$$x = x' + u \cos.\theta$$

Substituting these values of x and y in the equation $y^2 = p\,x$, we have

$$(y' + u \sin.\theta)^2 = p\,(x' + u \cos.\theta)$$

252. Let the pole be at any point on the curve,

$$\therefore\ y'^2 + 2\,u\,y' \sin.\theta + u^2 (\sin.\theta)^2 = p\,x' + p\,u \cos.\theta\,;$$

$$\text{or, } u\,(\sin.\theta)^2 = p \cos.\theta - 2\,y' \sin.\theta, \quad \text{since } y'^2 = p\,x'\,;$$

$$\therefore\ u = \frac{p \cos.\theta - 2\,y' \sin.\theta}{(\sin.\theta)^2},$$

And if the vertex be the pole, we have $y' = 0$;

$$\therefore\ u = \frac{p \cos.\theta}{(\sin.\theta)^2}.$$

253. Let the focus S be the pole, $\therefore\ y' = 0,\ x' = \dfrac{p}{4}$ and u becomes r; hence the general equation $(y' + u \sin.\theta)^2 = p\,(x' + u \cos.\theta)$ becomes

$$r^2 (\sin.\theta)^2 = \frac{p^2}{4} + p\,r \cos.\theta,$$

$$\text{or } r^2 (\sin.\theta)^2 + r^2 (\cos.\theta)^2 = \frac{p^2}{4} + p\,r \cos.\theta + r^2 (\cos.\theta)^2$$

$$\therefore\ r^2 = \left(\frac{p}{2} + r \cos.\theta\right)^2$$

$$r = \frac{p}{2} + r \cos.\theta\,; \quad \text{or } r = \frac{p}{2}\cdot\frac{1}{1 - \cos.\theta}.$$

The polar equation in this case is also easily deduced from article (231). Let angle A S P $= \theta$,

then $r = \text{S P} = \text{A M} + \text{A S} = 2\,\text{A S} + \text{S M} = \dfrac{p}{2} - r \cos.\theta$;

$$\therefore\ r = \frac{p}{2}\,\frac{1}{1 + \cos.\theta} = \frac{p}{\left(\cos.\dfrac{\theta}{2}\right)^2}$$

254. If P S meet the curve again in P′, we have S P′

$$\frac{p}{2} \cdot \frac{1}{1 - \cos.(\pi - \theta)} = \frac{p}{2} \quad \frac{1}{1 + \cos.\theta};$$

hence the rectangle

$$\mathrm{P\,S, S\,P'} = \frac{p}{4} \quad \frac{p}{1 - (\cos.\theta)^2} = \frac{p}{4}(\mathrm{S\,P + S\,P'}) = \frac{p}{4}, \mathrm{P\,P'}.$$

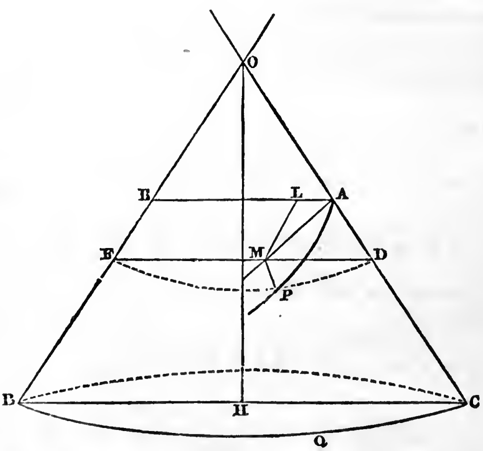
CHAPTER XI

THE SECTIONS OF A CONE.

255. It is well known that the three curves, the ellipse, the hyperbola, and parabola, were originally obtained from the section of a cone, and that hence they were called the conic sections. We shall now show the manner in which a cone must be cut by a plane, in order that the section may be one of these curves.

A right cone is the solid generated by the revolution of a right-angled triangle about one of its perpendicular sides.

The fixed side, O H, about which the triangle revolves, is called the axis; and the point O, where the hypothenuse of the triangle meets the axis, is called the vertex of the cone. If the revolving hypothenuse be produced above the vertex, it will describe another cone, having the same axis and vertex. Any point in the hypothenuse of the triangle describes a circle; hence, the base of the triangle describes a circular area called the base of the cone.



Section made by planes which pass through the vertex and along the axis are called vertical sections; these are, evidently, triangles.

If a plane pass through the cone in any direction, the intersection of it with the surface of the cone is called a conic section. The nature of the ine thus traced will be found to be different, according to the various positions of the cutting plane. It is our purpose to show, generally, to what class of curves a section must necessarily belong; and, afterwards, to point out the particular species of curve due to a given position of the cutting plane.

256. Let O B Q C be a right cone, O the vertex, O H the axis, B C Q the circular base, P A the line in which the cutting plane meets the surface of the cone; A being the point in the curve nearest to the vertex O. Let O B H C A be a vertical plane passing through the axis O H and perpendicular to the cutting plane P A M.

A M, the intersection of these planes, is a straight line, and is called the axis of the conic section, the curve being symmetrically placed with regard to it.

Let F P D be a section parallel to the base, it is therefore a circle, and F M D, its intersection with the vertical plane O B H C A, is a diameter.

Since both this last plane F P D and the cutting plane P A M are perpendicular to the vertical plane O B H C, M P the intersection of the two former is perpendicular to the vertical plane, (Euc. xi. 19, or Geometry iv. 18,) and, therefore, to all lines meeting it in that plane. Hence M P is perpendicular to F D and to A M.

Let the angle O A M, which is the inclination of the cutting plane to the side of the cone, $= \alpha$, and let the $\angle$ A O B $= \beta$, draw A E parallel to B H and M L parallel to O B.

Let A M $= x$, M P $= y$, and A O $= a$.

Then by the property of the circle

The square on M P = the rectangle F M, M D;

$$\text{and } MD = \frac{MA \sin. MAD}{\sin. MDA} = x \frac{\sin. \alpha}{\cos. \frac{\beta}{2}};$$

$$\text{Also, } FM = EA - AL = \frac{AO \sin. AOE}{\sin. OEA} - \frac{AM \sin. AML}{\sin. ALM};$$

But angle O E A $= 90° - \frac{\beta}{2}$, angle A L M $= 90° + \frac{\beta}{2}$, and if we produce M L to meet O A, we shall find that the angle A M L $= 180° - (\alpha + \beta)$;

$$\text{hence } FM = a \frac{\sin \beta}{\cos. \frac{\beta}{2}} - x \frac{\sin. (\alpha + \beta)}{\cos. \frac{\beta}{2}};$$

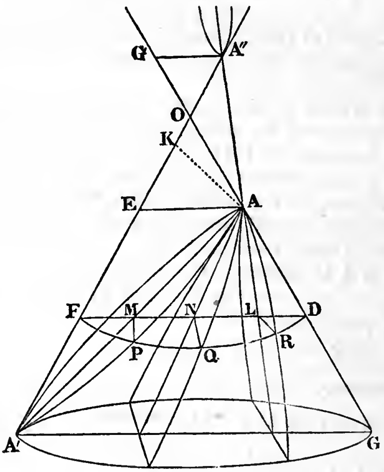
$$\therefore\ y^2 = x \frac{\sin. \alpha}{\cos. \frac{\beta}{2}} \left\{ a \frac{\sin. \beta}{\cos. \frac{\beta}{2}} - x \frac{\sin. (\alpha + \beta)}{\cos. \frac{\beta}{2}} \right\}$$

$$\text{or } y^2 = \frac{\sin. \alpha}{\left(\cos. \frac{\beta}{2}\right)^2} \{a \sin. \beta . x - \sin. (\alpha + \beta)\, x^2\};$$

which equation being of the second degree, it follows that the sections of the cone are curves of the second degree.

Comparing this with the equation $y^2 = p\,x + q\,x^2$, which represents an ellipse, a parabola, or an hyperbola, according as q is negative, nothing, or positive; we observe that the section is an ellipse, a parabola, or an hyperbola, according as sin. $(\alpha + \beta)$ is positive, nothing, or negative. To investigate these various cases, we shall suppose the cutting plane to move about A, so that α may take all values from 0 to 180°

257. Let $\alpha = 0$, $\therefore y^2 = 0$, and $y = 0$; this is the equation to the straight line which is the axis of x.



And this appears, also, from the figure; for when $\alpha = 0$, the cutting plane just touches the cone, and hence the line of intersection A M is in the position A O.

258. Let $\alpha + \beta$ be less than 180°. The curve is an ellipse. In the figure the angles A O E and O A M being together less than 180°, the lines O E and A M meet in A′, or the sectional plane cuts both sides of the cone.

259. Let M be the centre of the ellipse, then F M = ½ A E and M D = ½ A′ G;

$\therefore$ The square on the axis minor = The rectangle A E, A′ G.

Also by drawing perpendiculars from A and E upon A′ G, it may be proved that

The square on the axis major = The square on A G + The rectangle A E, A′ G.

And $\therefore$ The distance between the foci = A G.

If the straight line A K be drawn making the angle E A K = the angle E A A′, then A K is the latus rectum of the section.

And if a circle be inscribed in the triangle A′ A O, it will touch the line A A′ in the focus of the section. (Geometry, Appendix, prop. 21.)

260. Let $\alpha = 90° - \frac{\beta}{2}$, then $\frac{\sin. \alpha \sin. (\alpha + \beta)}{\left(\cos. \frac{\beta}{2}\right)^2} = 1$,

and the equation is that to a circle, the cutting plane being parallel to the base.

261. Let $\alpha + \beta = 180°$, $\therefore \sin. (\alpha + \beta) = 0$, and the curve is a parabola. The plane, continuing to turn, has now come into the position A N Q, the axis A N being parallel to O F, or the cutting plane parallel to a side of the cone.

The equation to the parabola is $y^2 = 4\ a \left(\sin. \frac{\beta}{2}\right)^2 x$.

If A K be drawn making the angle E A K = the angle A O K, then A K is the latus rectum of the section, and the circle which touches A O, A N and O F, will touch A N in the focus of the parabola.

262. Let $\alpha + \beta$ be greater than 180° $\therefore$ sin. $(\alpha + \beta)$ is negative, and the curve is an hyperbola; The cutting plane is now in the position A L R; in this case the lines A L, E O must meet if produced backwards, or the plane cuts both cones, and the curve consists of two branches, one on the surface of each cone.

As in the ellipse, it may be proved that the square on the conjugate axis = the rectangle A E, A″ G′; that A G′ is the distance between the foci, that A K is the latus rectum, and that the circle touching A′ O, O A and A L touches A L at the focus.

263. We may also suppose a to have different values, or the cutting plane to meet the cone in some other point than A, for example:

Let $a = 0 \quad \therefore\ y^2 = -\frac{\sin. \alpha \sin. (\alpha + \beta)}{\left(\cos. \frac{\beta}{2}\right)^2} x^2$;

Since sin. α and $\left(\cos. \frac{\beta}{2}\right)^2$ are positive, the rationality of this equation will depend upon sin. $(\alpha + \beta)$.

If $\alpha + \beta$ is less than 180° the radical quantity is impossible, and the only solution of the equation is $x = 0$ and $y = 0$, or the section is a point; this is the case when the cutting plane passes through the vertex O, and is parallel to any elliptic section A P A′.

If $\alpha + \beta$ is greater than 180° we have two straight lines which cut each other at the origin. In this case the cutting plane is drawn through O, parallel to A L R, and the intersection with the cone is two straight lines meeting in O.

264. We may conclude from this discussion, that the conic sections are seven: a point, a straight line, two straight lines which intersect, a circle, an ellipse, an hyperbola, and a parabola or all the curves of the second degree and their varieties, with the exception of two parallel lines, which is a variety of the parabola.

The three latter sections, the ellipse, hyperbola, and parabola, are those which are usually termed "conic sections," and which have been the study and delight of mathematicians since the time of Plato. In his school they were first discovered; and his disciples, excited, no doubt, by the many beautiful properties of these curves, examined them with such industry, that in a very short time several complete treatises on the conic sections were published. Of these, the best still extant is that of Apollonius of Perga. It is in eight books, four of which are elementary; and four on the abstruser properties of these curves. The whole work is well worth attention, as showing how much could be done by the ancient analysis, and as giving a very high opinion of the geometrical genius of the age.

Apollonius gave the names of ellipse and hyperbola to those curves—Hyperbola, because the square on the ordinate is equal to a figure "exceeding" ("ὑπερβαλλον") the rectangle under the abscissa and latus rectum by another rectangle.—B. i. p. 13.

Ellipse, because the square on the ordinate is "defective" ("ελλειπον") with regard to the same rectangle.—p. 14.

It is not known who gave the name of parabola to that curve—probably Archimedes, because the square of the ordinate is equal ("παραβαλλον") to the rectangle of the abscissa and latus rectum.

Thus, the ancients viewed these curves geometrically, in the same manner as we are accustomed to express them by the equations:

$$y^2 = p\,x + \frac{p}{2\,a}\,x^2,$$

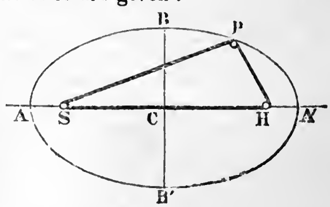
$$y^2 = p\,x - \frac{p}{2\,a}\,x^2,$$

$$y^2 = p\,x.$$

DESCRIPTION OF THE CONIC SECTIONS BY CONTINUED MOTION.

265. The conic sections being curves of great importance, not only from their mathematical properties, but also from their usefulness in the arts and sciences, it becomes necessary that we should be able to describe these curves with accuracy. Now, a curve may be drawn in two ways, either by "mechanical description" or by "points." As an instance of the first method we may mention the circle, described by the compasses, or by means of a string fastened at one end to the centre, and the other carried round by the hand, the hand tracing the curve. This mechanical method, or, as it is sometimes called, "that by continued motion," is not always practicable: no curve is so simple, in this respect, as the circle; hence we are often obliged to have recourse to the second method, or that by points: this is done by taking the equation to the curve and from some property expressed geometrically, finding a number of points, all of which belong to the curve, and then neatly joining these points with a pen or other instrument. We shall commence with the mechanical description of these curves.

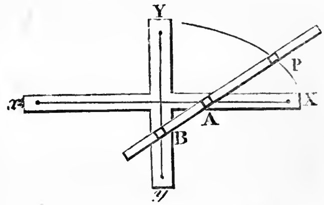
266. To trace an ellipse of which the axes are given:



Let A A′, B B′ be the axes: with centre B and radius A C describe a circle cutting A A′ in S and H, these points are the foci. Place pegs at S and H. Let one extremity of a string be held at A, and pass the string round H back again to A, and there join its two ends by a knot, so that its length shall be just double of A H; place a pen or other pointed instrument within this string, and move it round the points S and H, so that the string be always stretched; the pen will trace out the required ellipse. For if P be one of its positions, we have

$$SP + PH + HS = 2AH = AA' + HS;$$
$$\therefore SP + PH = AA'.$$

267. Another method is by means of an instrument called the elliptic compasses, or the trammel.



Let Xx and Yy be two rulers with grooves in them, and fastened at right angles to each other. Let B P be a third ruler, on which take B P equal to the semi-axis major, and P A the semi-axis minor. At B a peg is so fixed that the point B with the peg can move along Yy; a similar peg is fixed at A. By turning the ruler B P round, a pen placed at P will trace out the curve. Suppose C to be the point where the axes meet, $CM = x$ and $MP = y$, the rectangular co-ordinates of P, and suppose that B N is drawn parallel to C M and meeting P M in N, then $AM = \frac{b}{a}$ B N, and

The square on A P = the square on P M + the square on A M,

$$\text{or } b^2 = y^2 + \frac{b^2}{a^2}x^2,$$

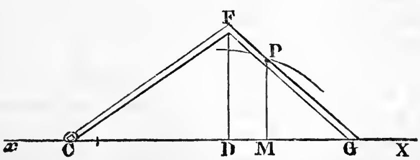
$$\therefore a^2y^2 + b^2x^2 = a^2b^2.$$

268. The following is also a very simple method of describing the ellipse. Xx is a ruler of any length, C F, F G are two rulers, each equal to half the sum of the semi-axes. These rulers are fastened together by a moveable joint at F, and F C turns round a pivot at C; F P is taken equal to half the difference of the semi-axes. Let the point G slide along the line Xx, then the point P will trace out the curve. Draw F D and P M perpendicular to C X, and let $CM = x$, and $MP = y$, then

The square on F G = the square on F D + the square on D G;

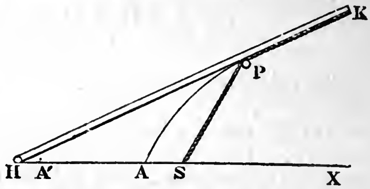
$$\text{or } \left(\frac{a+b}{2}\right)^2 = \left(\frac{a+b}{2}\frac{y}{b}\right)^2 + \left(\frac{a+b}{2}\frac{x}{a}\right)^2$$

$$\therefore \frac{y^2}{b^2} + \frac{x^2}{a^2} = 1.$$



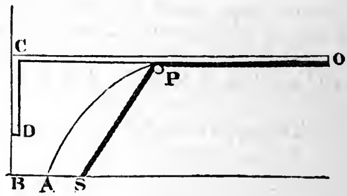
For a description of the Elliptograph, and other instruments for describing ellipses, we must refer our readers to the treatise on Practical Geometry, where an extremely good account is given of all the instruments, and also the advantages and disadvantages of each are well exhibited

269. To trace the hyperbola by continued motion, let A A′ be the transverse axis, S H the distance between the foci, H P K a ruler movable about H. A string, whose length is less than H K by A A′ is fastened to K and S; when the ruler is moved round H, keep the string stretched, and in part attached to the ruler by a pencil as at P; then, since the difference of H P and P S is constantly the same, the point P will trace out the curve.



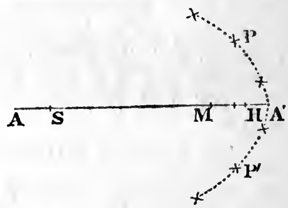
If the length of the string be H K, a straight line perpendicular to H S will be traced out; and if the string be greater than H K, the opposite branch, or that round H, will be described.

270. To trace the parabola by continued motion. Let S be the focus, and B C the directrix. Apply a carpenter's square O C D to the ruler B C, fasten one end of a thread whose length is C O to O, and the other end to the focus S; slide the square D C O along B C, keeping the thread tight by means of a pencil P, and in part attached to the square. Then since S P = P C, the point P will describe a parabola.

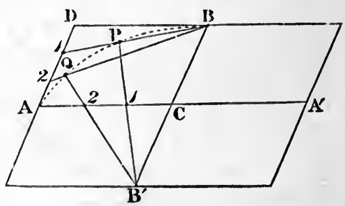


Description of the Conic Sections by Points.

271. Given the axes of an ellipse to describe the curve. Let A A′ be the axis major, S and H the foci. With centre S, and any radius A M less than A A′, describe a circle, and with centre H and radius A′ M describe a second circle, cutting the former in two points P and P′; then since S P + P H = A M + M A′ = A A′, P is a point in the required curve; and thus any number of points may be found, and the curve described.



272. Given a pair of conjugate diameters to describe the curve. Let A A′, B B′, be the conjugate diameters. Through B draw B D parallel to A C, and through A draw A D parallel to B C. Divide A D and A C into the same number of equal parts as three. From B draw lines to the dividing points in A D, and from B′ draw lines to the dividing points in A C; the intersections P, Q, of these lines are points in the ellipse.



For let C be the origin; $CA = a_1$, $CB = b_1$,

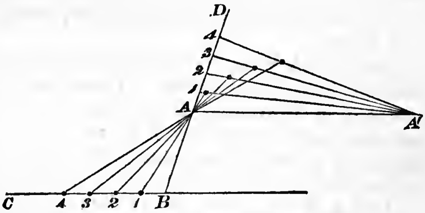
Then the equation to B P is $y - b_1 = \dfrac{b_1}{3a_1}x$;

and the equation to B′P is $y + b_1 = -\dfrac{3b_1}{a_1}x$.

Hence the product of the tangents of the angles which these lines B P, B′P make with the axis of $x = -\dfrac{b_1}{3a_1}\dfrac{3b_1}{a_1} = -\dfrac{b_1^2}{a_1^2}$ and is constant; therefore P is a point in the curve (141).

Innumerable points may be thus found in the four quadrants of the figure.

273. The following is perhaps the best method of tracing the ellipse by points:



Let A A′ be a diameter and A B equal and parallel to the conjugate diameter. Through B draw B C parallel to A A′ and equal to any multiple of A A′. In B A produced, take A D the same multiple of A B. Divide B C into any number of equal parts, and A D into the same number of equal parts. Through A draw lines to the points of division in B C, and through A′ draw lines to the points of division in A D; the intersections of corresponding lines will give points in an ellipse whose conjugate diameters are A A′ and A B. The proof is the same as in the last case.

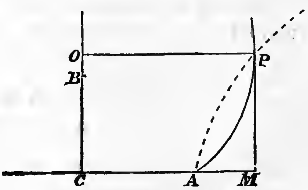
274. Given the axes of an hyperbola to trace the curve.

Let A A′ be the transverse axis, S and H the foci, which are given points; with centre S and any radius A M greater than A A′, describe a circle, and with centre H and radius A′ M describe a second circle, cutting the former in two points P and P, these are points in the required curve.

The proof is much the same as that for the ellipse (271.)

Again, if, in article 273, B C was taken to the right of B instead of the left, as in the figure, the intersections of the corresponding lines will give an hyperbola.

275. To describe the rectangular hyperbola by points.

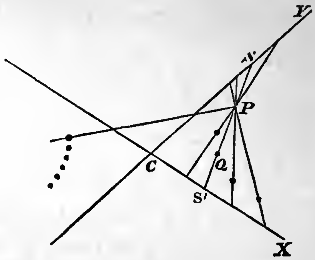


Let C A, C B be the equal semi-axes with any centre O in C B produced and with radius O A, describe a circle; draw O P perpendicular to C O meeting the circle in P, then P is a point in the curve; Let $CM = x$, $MP = y$; then the square on CO = the square on OA − the square on CA;

$$\text{or } y^2 = x^2 - a^2.$$

276. Given the asymptotes C X, C Y of an hyperbola, and one point P in the curve, to describe the curve by points.

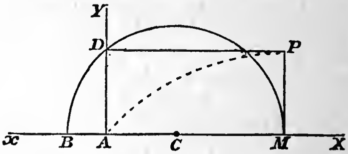
Through P draw any line S P S′ terminated by the asymptotes; in it take S′Q = S P; then Q is a point in the curve (217), and similarly any number of points may be found.



Together with the asymptotes, another condition must always be given to enable us to trace the curve, for the position of the asymptotes only gives us the ratio of the axes, and not the axes themselves.

277. To describe the parabola by points, when the principal parameter p is given.

Let A X, A Y be the rectangular axes; in A x take A B = p; with any centre C in A X and radius C B describe a circle B D M, cutting A Y in D and A X in M, draw D P and M P perpendicular to A Y and AX respectively; then P is a point in the curve.



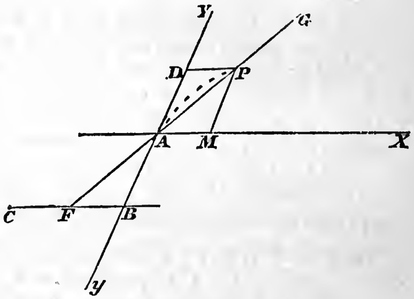
Let A M = x, M P = y; then the square on A D = the rectangle B A, A M,

$$\text{or } y^2 = p\,x.$$

278. Given the angle between the axes and any parameter p' to describe the curve.

Let A X, Y A y be the axes, A B the parameter. Through B draw C B parallel to A X. Through A draw any line F A G, meeting B C in F; in A Y take A D = B F, and draw D P parallel to A X, cutting A G in P, then P is a point in the curve.

Draw M P parallel to A Y, and let

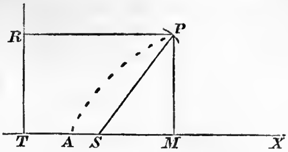


$$AM = x, \text{ and } MP = y,$$

$$\text{then } MP : MA :: AB : FB,$$

$$\text{or } y : x :: p' : y; \quad \therefore y^2 = p'\,x.$$

279. Given the position of the directrix T R and the focus S, to trace any of the conic sections by points.



Draw S T perpendicular to T R, then T S produced will be the axis of the curve.

Let $e : 1$ be the ratio of the distance of any point P in the curve from the focus and from the directrix; hence if A S : A T :: $e : 1$; A is a point in the curve. Take any point M in A X, and with centre S and radius equal to e times T M, describe a circle; draw M P perpendicular to A X, and meeting the circle in P, then P is a point in the curve.

Let A be the origin of rectangular co-ordinates, $AM = x$, $MP = y$, $AS = m$, and $\therefore AT = \frac{m}{e}$;

$$\text{then } SP = e \,.\, TM = e \,.\, PR$$

$$\therefore \quad y^2 + (x - m)^2 = e^2\left(x + \frac{m}{e}\right)^2$$

$$\text{or } y^2 + x^2 - 2\,m\,x + m^2 = e^2\,x^2 + 2\,e\,m\,x + m^2;$$

$$\therefore \quad y^2 + (1 - e^2)\,x^2 - 2\,m\,x\,(1 + e) = 0;$$

which is the equation to the curves of the second order.

Let e be less than unity, $\therefore y^2 = (1 - e^2)\left\{\frac{2\,m}{1-e}\,x - x^2\right\}$.

Comparing this equation with that to the ellipse $y^2 = \frac{b^2}{a^2}\,(2\,a\,x - x^2)$, we have

$$2\,a = \frac{2\,m}{1 - e} \text{ and } \frac{b^2}{a^2} = 1 - e^2,$$

$$\therefore \quad b^2 = \frac{m^2}{(1 - e)^2}\,(1 - e^2) = m^2\frac{1 + e}{1 - e};$$

hence the curve is an ellipse whose axes are $\frac{2\,m}{1 - e}$ and $2\,m\sqrt{\frac{1+e}{1-e}}$.

Let e be greater than unity, $\therefore y^2 = (e^2 - 1)\left\{\frac{2\,m}{e-1}\,x + x^2\right\}$;

and the curve is an hyperbola, whose axes are $\frac{2\,m}{e - 1}$ and $2\,m\sqrt{\frac{e+1}{e-1}}$.

Let e be equal to unity, $\therefore y^2 = 4\,m\,x$;

the curve is a parabola, whose principal parameter is $4\,m$.

280. The general equation to all the conic sections being

$$y^2 + (1 - e^2)\,x^2 - 2\,m\,x\,(1 + e) = 0,$$

it follows that if we find any property of the ellipse from this equation, it will be true for the hyperbola and parabola, making the necessary changes in the value of e:

Thus the equation to the tangent is

$$y\,y' + (1 - e^2)\,x\,x' - m\,(1 + e)\,(x + x') = 0, \text{ for the ellipse,}$$

$$y\,y' - (e^2 - 1)\,x\,x' - m\,(1 + e)\,(x + x') = 0, \text{ for the hyperbola,}$$

$$\text{and } y\,y' - 2\,m\,(x + x') = 0, \text{ for the parabola.}$$

Also most of the results found in Chapter VIII. for the ellipse will be true for the hyperbola, by putting $-b^2$ for b^2; and will be true for the parabola by transferring the origin to the vertex of the ellipse, by then putting $\frac{m}{1-e}$ for a, and $m^2\,\frac{1+e}{1-e}$ for b^2; and then making $e = 1$. Thus the equation to the tangent at the extremity of the Latus Rectum in the ellipse, when the origin is at the vertex, is

$$y = a + e\,(x - a)\ (117),$$

$$\text{or,}\quad y = a\,(1 - e) + e\,x,$$

$$\text{for } a \text{ put } \frac{m}{1-e}, \text{ and then let } e = 1;$$

$$\therefore\ y = m + x, \text{ as in (235).}$$

281. If $SP = r$, and $ASP = \theta$, the polar equation to the curve thus traced is easily found:

$$SP = e.\,PR = e\,(TS + SM),$$

$$\text{or}\quad r = e\left(m\,\frac{1+e}{e} - r\cos.\theta\right);$$

$$\therefore\ r = \frac{m\,(1+e)}{1 + e\cos.\theta}.$$

Or since $m\,(1 + e) = \frac{b^2}{a}$ for the ellipse and hyperbola, and $= 2\,m$ for the parabola, we have (putting p for the principal parameter) the genera polar equation to the three curves,

$$r = \frac{p}{2}\,\frac{1}{1 + e\cos.\theta}\ (150).$$

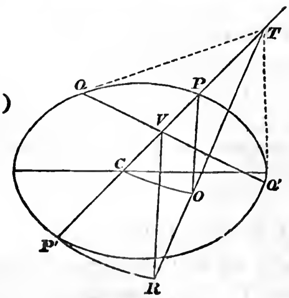
282. To draw a tangent at a given point P on the ellipse.

Draw the ordinate M P, and produce it to meet the circumscribing circle in Q, from Q draw a tangent to the circle meeting the axis major produced in T, join P T; this line is a tangent to the ellipse (114).

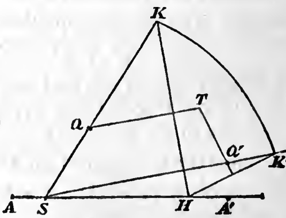
Again, taking the figure in the note appended to Art. 121, join S P, H P, and produce H P to K, so that $PK = PH$; join S K; the line $P\,y$ bisecting S K is a tangent.

283. To draw a tangent to the ellipse from a point T without the curve.

(1)



(2)



Draw the line T P C P′ through the centre, fig. 1.; draw a conjugate diameter to C P: then the question is reduced to finding a point V in C P, through which a chord Q V Q′ is to be drawn, so that T Q and T Q′ may be tangents.

Take C V a third proportional to C T and C P, then V is the required point (136).

Again, with centre T and radii T P′, T C describe circles C O, P′ R, draw any line T O R, cutting these circles in O and R; join P O, and draw R V parallel to P O: then it may be proved by similar triangles that C V is a third proportional to C T and C P, and therefore V is the required point.

284. If the axes, and not the ellipse, are given, take S and H the foci, fig. 2, with centre S and radius A A′ describe a circle, and with centre T and radius T H describe another circle, cutting the former in the points K and K′; join S K and SK′, H K and H K′; from T draw the lines T Q and TQ perpendicular to H K and H K′, these lines meet S K and SK′ in the required points Q and Q′. The proof will readily appear upon joining H Q and H Q′, and referring to the note, page 77.

285. To draw a tangent to the hyperbola at a given point P on the curve.

Join S P and H P, note, page 97; in H P take PK = S P, and join S K; the line P Y bisecting S K is the required tangent.

286. To draw a tangent from a given point T without the curve.

The two methods given (283) for the ellipse will apply, with the necessary alteration of figure, to the hyperbola.

287. To draw a tangent to the parabola at a given point P on the curve.

Draw an ordinate P M to the axis, fig. 232, and in the axis produced take A T = A M, join P T; this line is a tangent (233); or take S T = S P, and join P T.

288. To draw a tangent to a parabola from a given point T without the curve.

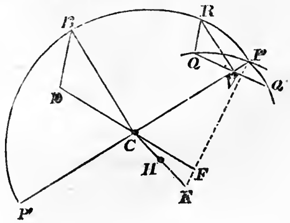
Draw a diameter T P V parallel to the axis, and cutting the curve in P, take P V = P T, and draw an ordinate Q V Q′ to the abscissa P V, then T Q and T Q′ are the required tangents (249).

If the directrix and focus be given, but not the curve; with centre T and radius T S describe a circle, cutting the directrix in the points R and R′, join R S and R′ S; draw R Q and R′ Q′ parallel to the axis, and then TQ and T Q′ perpendicular to R S and R′ S (239).

289. An arc Q P Q′ of a conic section, being traced on a plane to find to which of the curves it belongs; and also the axes and focus of the section.

Draw a line L through the middle of two parallel chords, and another line L′ through the middle of other two parallel chords, if the lines L, L are parallel, the curve is a parabola, if they meet on the concave side of the curve it is an ellipse, if on the convex side it is an hyperbola. (130. 243.)

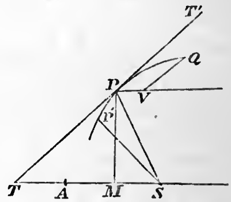
290. Let the curve be an ellipse, the point where the lines L L′ meet is the centre C; let P P′ be a diameter, its conjugate C D is thus found; describe a circle on P P′ as diameter, and draw V R, C B perpendicular to P P′; join R Q, and draw B D parallel to R Q, meeting a line parallel to Q V, passing through C; then C D is the conjugate diameter (136).



To find the length and position of the axes; draw P F perpendicular on C D, and produce it to E, making P E = C D, join C E, and bisect C E in H; join P H; then from the triangle C P E we have the side C E in terms of C P and C D $= \sqrt{\{a_1^2 + b_1^2 - 2\, a_1 b_1 \sin.\ (\theta' - \theta)\}} =$ $\sqrt{\{a^2 + b^2 - 2\, a\, b\}} = a - b$ ∴; C H $= \dfrac{a - b}{2}$; also from the same triangle we have P H $= \dfrac{a + b}{2}$; hence P H + H E is the small-axis major, and P H − H E is the semi-axis minor.

In H P take H K = H E, then C K is the direction of the axis-major.

291. If the arc Q P Q′ be an hyperbola, the conjugate diameter may be found by a process somewhat similar to that for the ellipse; the asymptotes may then be drawn by Art. 215. The direction of the axes bisects the angle of the asymptote, and their length is determined by drawing a tangent P T, and perpendicular P M, to the axis, and taking C A a mean proportional between C M and C T (167).



292. If the arc be a portion of a parabola, draw T P T′ parallel to Q V, and then draw P S, making the angle S P T = the angle T′ P V; repeat this construction for another point P′, then the junction of PS and P′S determines the focus (240); the axis is parallel to P V, and the vertex is found by drawing a perpendicular on the axis, and then bisecting T M (233).

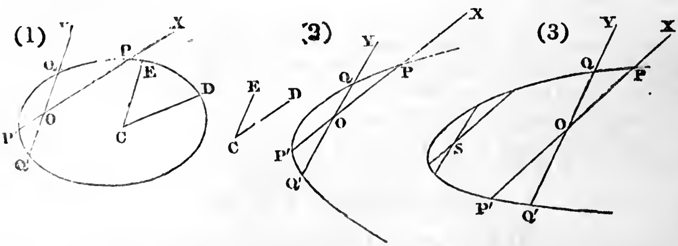
293. We shall conclude the subject of conic sections with the following theorem.

If through any point within or without a conic section two straight lines making a given angle with each other, be drawn to meet the curve, the rectangle contained by the segments of the one will be in a constant ratio to the rectangle contained by the segments of the other.

Case 1. The ellipse and hyperbola.

Let C D, C E be two semi-diameters parallel to the chords P O P′, Q O Q′; then, wherever chords parallel to these be drawn, we shall always have the following proportion:

The rectangle P O, O P′ : the rectangle Q O, O Q :: the square on C D : the square on C E.



Let O be the origin of oblique axes O X, O Y: then the equation to the curve will be of the form

$$a y^2 + b x y + c x^2 + d y + e x + f = 0.$$

Let $x = 0$; $\therefore$ $a y^2 + d y + f = 0$, and the product of the roots being $\frac{f}{a}$, we have

$$\text{The rectangle } Q O, O Q' = \frac{f}{a};$$

$$\text{Similarly the rectangle } P O, O P' = \frac{f}{c};$$

$\therefore$ the rectangle $Q O$, $O Q'$: the rectangle $P O$, $O P'$ :: $\frac{f}{a} : \frac{f}{c}$ :: $c : a$

Now, let the origin be transferred to the centre without changing the direction of the axes, then the form of the equation is

$$a y^2 + b x y + c x^2 + f' = 0 \quad (81).$$

Let $x = 0$; $\therefore$ the square on $C E = \frac{-f'}{a}$; and the square on $C D = \frac{-f'}{c}$; $\therefore$ the square on $C E$: the square on $C D$; :: $c : a$;

$\therefore$ the rectangle $Q O$, $O Q'$: the rectangle $P O$, $O P'$ :: the square on $C E$: the square on $C D$.

In the hyperbola fig. (2), $C E$ and $C D$ do not meet the curve; but in order to show that these lines are semi-diameters, let the axis of y be carried round till it becomes conjugate to $C D$, then the formulas for transformation in (55) become for $\theta = 0$,

$$y = y' \frac{\sin.\, \theta'}{\sin.\, \omega}, \; x = x' + y' \frac{\sin.\, (\omega - \theta')}{\sin.\, \omega}.$$

If these values of x and y be substituted in the general central equation above, and it be reduced to the conjugate form by putting $b' = 0$, the transformed equation is of the form $a' y^2 + c x^2 + f' = 0$, where c and f' are not changed, and $-\frac{f'}{c}$ is the square on the semi-diameter along the axis of x (86); hence the theorem is true for the hyperbola.

Case 2. The Parabola fig. (3.)

As before, we have the rectangle $P O$, $O P$: the rectangle $Q O$, $O Q'$ :: $c : a$.

Let P and Q be the parameters to the chords $P O P'$ and $Q O Q'$; transfer the origin to the focus, the axes remaining parallel to $P O$, and $Q O$, by which transformation c and a are not altered.

Now in this case, the chords passing through the focus, we have the rectangle $P S$, $S P'$: the rectangle $Q S$, $S Q'$; :: $\frac{p}{4} P : \frac{p}{4} Q$ (254) and also as $c : a$; hence the rectangle $P O$, $O P'$: the rectangle $Q O$, $O Q'$ :: $c : a$ :: $P : Q$.

294. If the point O be without the curves, and the points P P' coincide as well as Q and Q', or the lines become tangents, we have for the ellipse and hyperbola,

The square on O P : the square on O Q :: the square on C D : the square on C E;

or O P : O Q :: C D : CE.

For the parabola;

The square on O P : the square on O Q :: S P : S Q;

hence it may be proved that, if a polygon circumscribe an ellipse, the algebraical product of its alternate segments are equal. And the same theorem will apply to tangents about an hyperbola; the tangents commencing from any point in the asymptotes.

CHAPTER XII.

ON CURVES OF THE HIGHER ORDERS.

295. Having completed the discussion of lines of the second order, we should naturally proceed to the investigation of the higher orders; but the bare mention of the number of those in the next or third order (for they amount to eighty) is quite sufficient to show that their complete investigation would far exceed the limits of an elementary treatise like the present. Nor is it requisite: we have examined the sections of the cone at great length, because, from their connexion with the system of the world, every property of these curves may be useful; but it is not so with the higher orders; generally speaking they possess but few important qualities, and may be considered more as objects of mathematical curiosity than of practical utility.

The third order is chiefly remarkable from its investigation having been first undertaken by Newton. Of the eighty species now known, seventy-two were examined by him; eight others, which escaped his searching eye, have since been discovered.

Those who wish to study these curves, may refer to Newton's "Enumeratio Linearum tertii Ordinis;" or to the work of Stirling upon the same subject.

Of the fourth order there are above five thousand species, and the number in the higher orders is so enormous as to preclude the possibility of their general investigation in the present state of analysis.

A systematic examination of curves being thus impossible, all that we can do is to give a selection, taking care that amongst them shall be found all the algebraical or transcendental curves which are most remarkable either for their utility or history.

We shall generally introduce them as examples of indeterminate problems, that is, of problems leading to final equations, containing two variables. We shall then trace the loci of those equations, and explain, when necessary, anything relating to the construction or properties of the curves.

It would be useless to give any general rules for the working of these questions; those given for determinate problems will here serve equally well; but, in both cases, experience is the only sure guide. In the solution of these problems we shall not always follow the same, nor even the easiest, method; but we shall endeavour to vary the manner, so that an attentive observer may learn how to act in any particular case.

We commence with problems leading to loci of the second order.

296. Given the straight line A B ($= a$) to find the point P without A B, so that A P : P B :: m : 1.

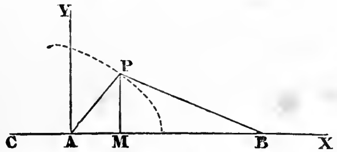
Let A be the origin of rectangular co-ordinates, A X and A Y the axes, $AM = x$, $MP = y$, and $\therefore MB = a - x$,

$$\text{then } AP : PB :: m : 1$$

$$\text{or } \sqrt{x^2 + y^2} : \sqrt{(a-x)^2 + y^2} :: m : 1;$$

$$\therefore x^2 + y^2 = m^2 (a - x)^2 + m^2 y^2,$$

$$\text{or } (1 - m^2) y^2 + (1 - m^2) x^2 + 2 m^2 a x - m^2 a^2 = 0,$$



$$\text{or } y^2 + \left(x + \frac{m^2 a}{1 - m^2}\right)^2 = \frac{m^2 a^2}{(1 - m^2)^2}.$$

This equation shows that there are an infinite number of points satisfying the conditions of the problem, all situated on the circumference of a circle (66).

To draw this circle; in A x take $AC = \frac{m^2 a}{1 - m^2}$, and with centre C and radius $\frac{m a}{1 - m^2}$ describe a circle; this is the required locus.

If $m = 1$, reverting to the original equation we have $x = \frac{a}{2}$, which is the equation to a straight line drawn through the bisection of A B, and parallel to A Y.

297. If perpendiculars be drawn to two lines given in position from a point P, and the distance between the feet of the perpendiculars be a constant quantity a, required the locus of P.

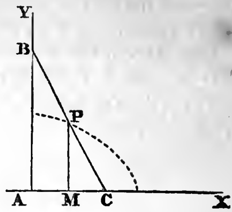
Let the intersection of the given lines be the origin of rectangular axes, take one of the lines for the axis of x, and let $y = \alpha x$ be the equation to the other; then the equation to the line passing through P (x y'), and perpendicular to the line $y = \alpha x$, is $y - y' = -\frac{1}{\alpha}(x - x')$; then from these two equations the co-ordinates of the point where their loci meet, that is, the co-ordinates of the foot of the perpendicular are readily obtained; and then the final equation found, by art. 29, is $y'^2 + x'^2 = a^2 \frac{(1 + \alpha^2)}{\alpha^2}$, which belongs to a circle whose centre is at the intersection of the lines.

298. A given straight line B C moves between two straight lines, A B, A C, so that its extremities B C are constantly on those lines; to find the curve traced out by any given point P in B C.

Let the lines A B, A C be the axes of y and x,

$AM = x$, $BP = a$,

$MP = y$, $PC = b$,



and let B A C be a right angle;

then $AM : BP :: MC : PC$,

$$x : a :: \sqrt{b^2 - y^2} : b$$

$$\therefore b^2 x^2 = a^2 b^2 - a^2 y^2,$$

$$\text{or } a^2 y^2 + b^2 x^2 = a^2 b^2,$$

which is the equation to an ellipse whose centre is A and axes $2a$, $2b$. If a ladder be placed against a wall, and its foot drawn along the ground at right angles to the wall, any step will trace out a quarter of an ellipse, and the middle step will trace out a quadrant of a circle.

If the co-ordinate axes be inclined at an angle θ, we have

$$AB = \frac{a+b}{b}\, y, \text{ and } AC = \frac{a+b}{a}\, x,$$

$$\text{Whence } a^2 y^2 + b^2 x^2 - 2\, a\, b \cos.\theta \,.\, x\, y - a^2 b^2 = 0,$$

which is the equation to an ellipse (76).

It is easy to see that from this problem arises a very simple mechanical method of describing the ellipse.

If a straight line B C of variable length move between two straight lines A B, A C, so that the triangle A B C is constant, the curve traced out by a point P which divides B C in a given ratio is an hyperbola.

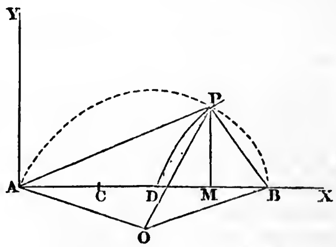
299. Given the line A B ($= c$) to find a point P without A B, such that drawing P A and P B, the angle P B A may be double of P A B.

Let A be the origin; A X, A Y the rectangular axes:

The equation to A P is $y = \alpha x$, (1)

and that to B P is $y = \alpha' (x - c)$;

but $\alpha' = \tan. PBX = -\tan. PBA = -\tan. 2PAB = -\dfrac{2\alpha}{1-\alpha^2}$;



$$\therefore y = \frac{-2\alpha}{1-\alpha^2}(x - c) \quad (2)$$

Eliminating α between the equations (1) and (2), we have $y^2 = 3x^2 - 2cx$, hence the locus of P is an hyperbola; comparing its equation with the equation $y^2 = \dfrac{b^2}{a^2}(x^2 - 2ax)$, we find the axes to be $\dfrac{2c}{3}$ and

$\frac{2c}{\sqrt{3}}$, and the centre at C where $AC = \frac{c}{3}$.

By this hyperbola, a circular arc may be trisected; for if A P B be the arc to be trisected, describe the hyperbola D P as above, and let the curves intersect in P; then if O be the centre of the circle, the angle $AOP = 2ABP = 4PAB = 2POB$, or the arc P B is one-third of B P A.

This problem may also be thus solved:

$$\text{Let } AM = x,\ MP = y, \text{ and angle } PAB = \theta;$$

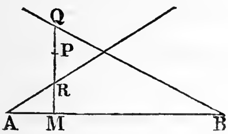
Then $\tan.\theta = \frac{y}{x}$, and $\tan.2\theta = \frac{y}{c - x}$ but $\tan.2\theta = \frac{2\tan.\theta}{1 - (\tan.\theta)^2}$,

$$\therefore \frac{y}{c - x} = \frac{\frac{2y}{x}}{1 - \frac{y^2}{x^2}}, \text{ or } y^2 = 3x^2 - 2cx.$$

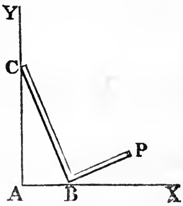
On examination it will be seen that the above two methods of solution are identical.

300. The following problems give loci of the second order.

1. From the given points A and B, (fig. 1,) two straight lines given in position are drawn, M R Q is a common ordinate to these lines, and M P is taken in M R Q a mean proportional to M Q and M R; required the locus of P.



(1)



(2)

2. A common carpenter's square C B P, (fig. 2,) moves in the right angle X A Y, so that the point C is always in A Y, and the right angle B in the line A X; required the locus of P.

3. If the base and difference of the angles at the base of a triangle be given, the locus of the vertex is an equilateral hyperbola

4. To find a point P, from which, drawing perpendiculars on two given straight lines, the enclosed quadrilateral shall be equal to a given square.

301. Let A Q A′ be an ellipse, A A′ the axis major, Q Q′ any ordinate, join A Q and A′ Q′; required the locus of their intersection P.

Let C be the origin of rectangular co-ordinates.

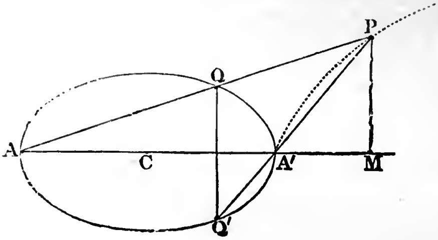
$$CM = x,\ MP = y,\ CN = x', \text{ and } NQ = y'.$$

Then the equation to A Q is $y = \alpha x + c$

which at A is $0 = -\alpha a + c$;

$$\therefore\ y = \alpha(x + a),$$

At Q it becomes $y = \alpha\,(x' + a)$ $\therefore\ \alpha = \dfrac{y'}{x' + a}$;



Hence the equation to A Q is $y = \dfrac{y'}{x' + a}\,(x + a)$, (1)

And similarly that to A′Q′ is $y = \dfrac{-\,y'}{x' - a}\,(x - a)$, (2)

Also $a^2\,y'^2 + b^2\,x'^2 = a^2\,b^2$; (3)

Eliminating x' and y' between (1) and (2), we have

$$x' = \frac{a^2}{x} \text{ and } y' = \frac{a\,y}{x}.$$

Substituting in (3) we obtain the final equation

$$a^2\,\frac{a^2\,y^2}{x^2} + \frac{b^2\,a^4}{x^2} = a^2\,b^2,$$

$$\text{or } a^2\,y^2 - b^2\,c^2 = -\,a^2\,b^2,$$

which is the equation to an hyperbola, whose centre is C, and transverse axis 2 a.

The method of elimination used in this problem is of great use; the principle admits of a clear explanation. We have the equations to A Q and A′ Q′; putting x and y the same for both equations intimates that x and y are the co-ordinates C M and M P in one particular case of intersection; but the elimination of x' and y' intimates that x and y are also always the co-ordinates of intersection, and therefore that the resulting equation belongs to the locus of their intersection.

302. To find the locus of the centres of all the circles drawn tangential to a given line A X, and whose circumferences pass through a given point Q $(a\ b)$.

Let S Q M be one of these circles, referred to rectangular axes A x, A y.

x, y the co-ordinates of its centre P,

x', y' . . . . . . any point on its circumference.

Then the equation to S Q M is

$$(y' - y)^2 + (x' - x)^2 = r^2; \quad (65)$$

but passing through Q, it becomes

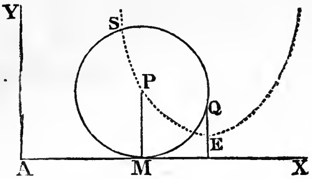
$$(b - y)^2 + (a - x)^2 = r^2,$$

and, being tangential to A X, we have $r = y$,

$$\therefore\ (b - y)^2 + (a - x)^2 = y^2,$$

$$\text{or } x^2 - 2\,a\,x - 2\,b\,y + a^2 + b^2 = 0.$$

This is the equation to a parabola (78).



It may be put in the form $(x - a)^2 = 2\,b\left(y - \frac{b}{2}\right)$. Hence if we transfer the origin to the point E $\left(a,\ \frac{b}{2}\right)$, we have the equation $x^2 = 2\,b\,y$, and the curve is referred to its vertex E, which is the centre of the least circle.

If, instead of the circle passing through a given point, it touch a given circle, a parabola is again the locus of P.

303. Let A B, B C, C D, and D A (fig. 1, p. 148) be four straight lines given in position, to find the locus of a point P, such, that drawing the lines P E, P F, P G, and P H making given angles with A B, B C, C D, and D A, we may have the rectangle P E, P F = the rectangle P G, P H.

Let O be the origin of rectangular axes O X, O Y; x and y the co-ordinates of P; β, β', β'' and β''' the cosecants of the angles which the lines P E, P F, &c., make with A B, B C, &c. Then the equation

to A B being $y' = \alpha\,x' + b$ we have $\mathrm{P\,E} = \dfrac{y - \alpha\,x - b}{\sqrt{1 + \alpha^2}}\,\beta$ (49)

to B C . . $y' = \alpha'\,x' + b'$. . . $\mathrm{P\,F} = \dfrac{y - \alpha'\,x - b'}{\sqrt{1 + \alpha'^2}}\,\beta'$

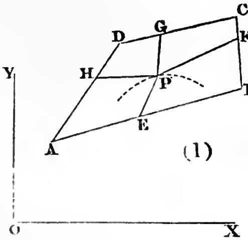
to D C . . $y' = \alpha''\,x' + b''$ . . $\mathrm{P\,G} = \dfrac{y - \alpha''\,x - b''}{\sqrt{1 + \alpha''^2}}\,\beta''$

to A D . . $y = \alpha'''\,x + b'''$ . . $\mathrm{P\,H} = \dfrac{y - \alpha'''\,x - b'''}{\sqrt{1 + \alpha'''^2}}\,\beta'''$;

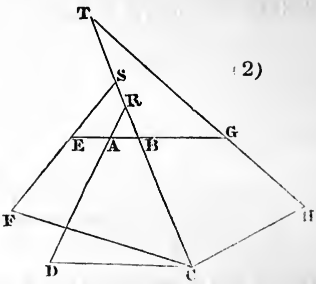
$$\therefore \text{ by the question } \frac{y - \alpha\,x - b}{\sqrt{1 + \alpha^2}}\ \frac{y - \alpha'\,x - b'}{\sqrt{1 + \alpha'^2}}\,\beta\,\beta'$$

$$= \frac{y - \alpha''\,x - b''}{\sqrt{1 + \alpha''^2}}\ \frac{y - \alpha'''\,x - b'''}{\sqrt{1 + \alpha'''^2}}\,\beta''\,\beta''.$$

This equation being evidently of two dimensions, the locus of P is a conic section, the particular species of which depends on the situation of the given lines.



(1)



(2)

This problem may be expressed much more generally. Suppose 3, 4, 5 or a greater number of lines to be given in position, required a point from which, drawing lines to the given lines, each making a given angle with them, the rectangle of two lines thus drawn from the given point may have a given ratio to the square on the third, if there are three; or to the rectangle of the two others, if there are four: or again, if there are five lines, that the parallelopiped composed of three lines may have a given ratio to the parallelopiped of the two remaining lines, together with a third given line, or to the parallelopiped composed of the three others, if there are six: or again, if there are seven, that the algebraical product of four may have a given ratio to the algebraical product of the three others and a given line, or to the four others, if there are eight, and so on.

This was a problem which very much perplexed the ancient geometricians. Pappus says, that neither Euclid nor Apollonius could give a solution. He himself knew that when there are only three or four lines the locus was a conic section, but he could not describe it, much less could he tell what the curve would be when the number of lines were more than four. When the number of lines were seven or eight, the ancients could scarcely enunciate the problem, for there are no figures beyond solids, and without the aid of algebra, it is impossible to conceive what the product of four lines can mean.

It was this problem which Descartes successfully attacked, and which, most probably, led him to apply algebra generally to geometry. The following solution is that given by Descartes, with a few abbreviations:

A B, A D, E F and G H (fig. 2) are the given lines, C the required point from which are drawn the lines C B, C D, C F and C H making given angles C B A, C D A, C F E, and C H G. A B ($= x$) and B C ($= y$) are the principal lines to which all the others will be referred. Suppose the given lines to meet C B in the points R, S, T, and A B in the points A, E and G. Let A E $= c$ and A G $= d$.

Then since all the angles of the triangle A B R are known, we have $BR = \alpha . AB = \alpha x$; $\therefore$ $CR = \alpha x + y$ and $CD = \beta (\alpha x + y)$; also $BS = \alpha' . BE = \alpha' (c + x)$; $\therefore$ $CS = y + \alpha' (c + x)$ and $CF = \beta' \{y + \alpha' (c + x)\}$; also $BT = \alpha'' . BG = \alpha'' (d - x)$; $\therefore$ $CT = y + \alpha'' (d - x)$ and $CH = \beta'' \{ y + \alpha'' (d - x) \}$; then since the rectangle C B, C F = the rectangle C D, C H, we have the equation

$$y\,\beta'\,\{y + \alpha'\,(c + x)\} = \beta\,(\alpha\,x + y)\,\beta''\,\{y + \alpha''\,(d - x)\}.$$

This equation Descartes showed to belong to a conic section which he described. He also gave the following numerical example:

Let E A = 3, A G = 5, A B = B R, B S = $\frac{1}{2}$ B E, G B = B T, C D = $\frac{3}{2}$ C R, C F = 2 C S, C H = $\frac{2}{3}$ C T, the angle A B R = 60°, and the rectangle C B, C F = the rectangle C D, C H. By the above method he found the equation to be

$$y^2 + x\,y + x^2 - 2\,y - 5\,x = 0;$$

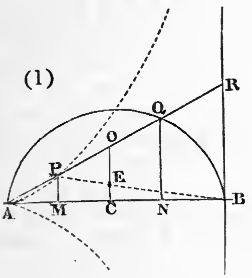
which he showed belonged to a circle. Taking the expressions in art. (72) we have the co-ordinates of the centre $\frac{8}{3}$ and $-\frac{1}{3}$, and the radius $= \frac{\sqrt{19}}{3}$.

304. Let A Q B be a semi-circle of which A B is the diameter, B R an indefinite straight line perpendicular to A B, A Q R a straight line meeting the circle in Q and B R in R; take A P = Q R; required the locus of P.

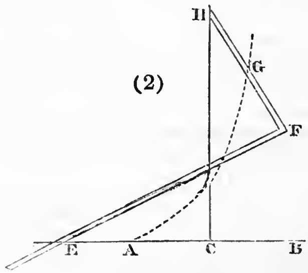
Let A be the origin of rectangular axes, and A B the axis of x.

A B = $2\,a$, A M = x, M P = y, and draw Q N parallel to M P;

(1)



(2)



then since A P = Q R, we have A M = B N,

and A M : M P :: A N : N Q;

that is, $x : y :: (2\,a - x) : \sqrt{(2\,a - x)\,x}$; (65)

$$\therefore\ y^2 = \frac{x^3}{2\,a - x} \text{ and } y = \pm\sqrt{\frac{x^3}{2\,a - x}}.$$

The following table gives the corresponding values of x and y:

	1	2	3	4	5	6
Values of x	0	a	$< 2a$	$2a$	$> 2a$	$-$
Values of y	0	a	possible	∞	impos.	impos.

From (1) the curve passes through the origin, from (2) it bisects the semi-circular arc A Q B, from (3) there are possible values of y for all values of x less than $2a$, from (4) there is an infinite ordinate at B, or B R is an asymptote to the curve: from these values we thus obtain an infinite arc proceeding from A to meet the asymptote B R. Again, from (5) for any value of x greater than $2a$, y is impossible, or no part of the curve is found to the right of the asymptote; and from (6) no part of the curve is on the left of A. Also, for every value of x there are two of y equal and opposite; hence there is a branch below A B similar to the one above it.

Diocles, a mathematician of the sixth century, invented this curve, which he called the Cissoid, from a Greek word signifying "ivy," because this curve climbs up its asymptote like ivy up a tree. He employed it in solving the celebrated problem of the insertion of two mean proportionals between given extremes.

Before his time, Pappus had reduced the problem to this case:

Let B C, C E be the two extremes, and A Q B a circle whose centre is C and radius C B; draw an indefinite straight line B E P through E and then draw the straight line A P O Q meeting B E and C E produced, and also meeting the circle at Q in such a manner that O Q = O P, then C O will be the first of the two mean proportionals. But the point P could not be directly found: hence, Diocles invented this curve to determine a series of points which will solve the problem for any length of C E: for example, suppose that B C, C E and the cissoid be drawn, join B E meeting the curve in P, then since O R = O A and Q R = A P we have O Q = O P.

From the definition of the curve it can be readily described by points; but as this is only a tentative process at best, and therefore not geometrically correct, Newton invented a very simple instrument for describing the curve by continued motion:

Let C H (fig. 2, p. 149) be a straight line parallel to B R; take A E = A C and let E F H be a common carpenter's square, the side F E being of indefinite length, and F H = A B; move this square so that the longer leg F E always passing through E, the extremity H of the other slides along C H, the middle point G of F H traces out the cissoid.

To obtain the polar equation to this curve:

$$\text{Let } y = r \sin.\theta \text{ and } x = r \cos.\theta;$$

Substitute these values in the equation $y^2 = \dfrac{x^3}{2a-x}$

$$\therefore\ r^2 (\sin.\theta)^2 = \frac{r^3 (\cos.\theta)^3}{2a - r\cos.\theta};\quad \text{whence } r = 2a \sin.\theta\,.\tan\theta\,.$$

Ex. If a perpendicular be drawn from the vertex of a parabola to a tangent, the locus of their intersection is the cissoid.

305. If C be a point in the diameter A B of the circle A Q B, and M Q any ordinate, join B Q, and draw C P parallel to B Q, meeting M Q in P required the locus of P.

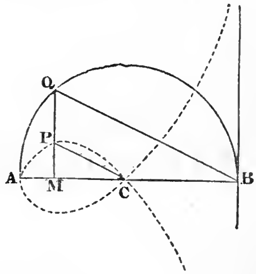
Let $AM = x$,

$MP = y$,

$AB = a$,

$AC = b$;

then $BM : MQ :: CM : MP$,



$$\text{or } (a - x) : \sqrt{ax - x^2} :: (b - x) : y,$$

$$\therefore\ y = \pm (b - x) \sqrt{\frac{x}{a - x}}.$$

Hence the following table of values:

	1	2	3	4	5	6
Values of x	0	b	a	$< a$	$> a$	$-$
Values of y	0	0	$\pm \infty$	pos.	imp.	imp

From (1) and (2) the curve passes through A and C; from (3) the ordinate at B is an asymptote to the curve; from (4) there are two arcs between A and C, also two between C and B; from (5) and (6) no part of the curve extends to the right of B or the left of A.

If $b = 0$, the oval between A and C disappears, and the curve is the cissoid of Diocles.

If b is negative, or the point C on the left of A, the curve consists of two branches proceeding from A to the asymptote through B, and the point C, though not on the curve, yet essentially belongs to it. This insulated point is called a conjugate point. The theory of such points will be fully explained in the treatise on the Differential Calculus.

Ex. A point Q is taken in the ordinate M P of the parabola, always equidistant from P, and from the vertex of the parabola; required the locus of Q.

306. M Q is an ordinate to the semicircle A Q B, and M Q is produced to P, so that M P : M Q :: A B : A M to find the locus of P.

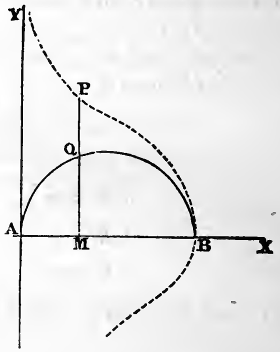
Let A B X and A Y be the rectangular axes.

$A M = x$,

$M P = y$,

$A B = 2a$;

Then M P : M Q :: A B : A M,



$$\text{or } y : \sqrt{2ax - x^2} :: 2a : x,$$

$$yx = \pm 2a\sqrt{2ax - x^2},$$

$$\therefore y = \pm 2a\sqrt{\frac{2a - x}{x}};$$

	1	2	3	4	5
Values of x	0	$2a$	$< 2a$	$> 2a$	neg.
Values of y	$\pm\infty$	0	pos.	imp.	imp.

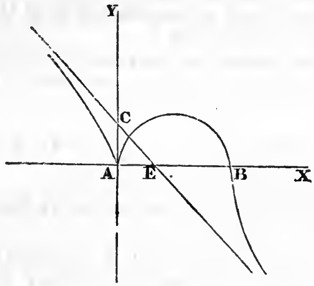
From (1) we have the ordinate at the origin infinite, and therefore an asymptote to the curve; from (2) the curve cuts the axis at B; from (3) the curve extends between A and B; from (4) no part of the curve is beyond B; from (5) no part is to the left of A.

This curve is called the Witch, and is the invention of an Italian lady, Maria Gaetana Agnesi, Professor of Mathematics in the University of Bologna, A.D. 1748.

307. In the circle the square on the ordinate is equal to the rectangle under the segments of the diameter; required the form of the curve on which the curve upon the ordinate is equal to the parallelopiped, of which the base is the square on one segment, and the altitude is the other segment, or $y^3 = x^2 (2a - x)$.

Let A be the origin Y, A X, Athe rectangular axes, and $A B = 2a$.

Let $x = 0$ or $= 2a$, $\therefore\ y = 0$; hence the curve passes through A and B; for $x < 2a$, y is positive; but when x is $> 2a$, y increases negatively to infinity, since the third root of a negative quantity is negative and possible. Again, y is positive for all negative values of x, and increases to ∞; also for each value of x, there is only one real value of y, the other two roots of an equation $y^3 \pm 1 = 0$, being always impossible.



Expanding the equation we have

$$y = -x\sqrt[3]{1 - \frac{2a}{x}} = -x\left\{1 - \frac{1}{3}\frac{2a}{x} - \frac{4a}{9x^2} + \text{\&c.}\right\};$$

$\therefore$ the equation to the asymptote is $y = -x + \frac{2a}{3}$ (195).

In A Y take A C $= \frac{2a}{3}$, and in A X take A E $= \frac{2a}{3}$, join C E, this line produced is an asymptote to the curve.

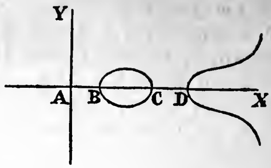
Ex. Find the locus of the equation, $y^3 + x^3 = a^3$; and of the equation $y^3 = a^2x - x^3$.

308. To trace the curve whose equation is $ay^2 = x^3 + mx^2 + nx + p$.

Case (1). Suppose the roots of this equation to be real and unequal, and to be represented by the letters a, b, and c, of which a is less than b and b less than c, then the equation is of the form

$$y = \pm\sqrt{\left\{\left(\frac{x-a}{a}\right)(x-b)\ (x-c)\right\}}.$$

	1	2	3	4	5	6	7	8	9	10
Values of x	0	a	$< a$	$>a<b$	b	$>b<c$	c	$> c$	∞	$-$
Values of y	imp.	0	imp.	pos.	0	imp.	0	pos.	$\pm\infty$	imp.



Let A be the origin, A X, A Y, the axes; A B $= a$, A C $= b$, and A D $= c$;

From (2) (5) and (7) the curve passes through B, C, and D; from (3) and (6) no part of the curve is found between A and B, or C and D; from (4) there are two branches between B and C; from (8) and (9) the curve proceeds from D to ∞, and from (10) no part of the curve is on the left of A.

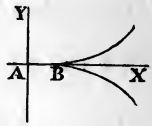
If the roots had been negative, the curve would have the same form, but would be rather differently situated with regard to the origin.

Case (2). If two roots be equal, the equation is $y = (x-c)\sqrt{\frac{x-a}{a}}$,

or $y = \pm (x-a)\sqrt{\frac{x-c}{a}}$; in the former case the figure is nearly the same as above, when the points C and D coincide; in the latter, supposing the points B and C to coincide, or the oval to become a conjugate point.

Case (3). If two of the roots be impossible, we have only the bell-shaped part of the curve from D.

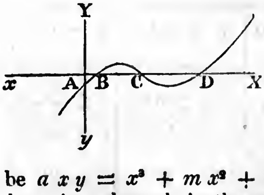
Case (4). If the three roots be equal, the equation is $a\,y^2 = (x-a)^3$.



The figure now consists of two branches proceeding from B with their convexity towards the axis. This curve is called the semi-cubical parabola; its equation is the most simple when the origin is at the vertex B; that is, putting x instead of $x - a$, when $a\,y^2 = x^3$.

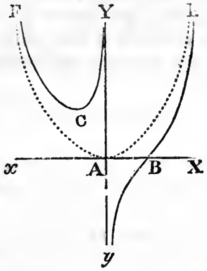
This curve is remarkable as being the first curve which was rectified, that is, the length of any portion of it was shown to be equal to a number of the common rectilinear unit.

309. The equation $a^2\,y = x^3 + m\,x^2 + n\,x + p$, can be traced exactly as in the last article: the accompanying figure applies to the case when the three roots are positive, real, and unequal. If two of them be equal, one of the semi-ovals disappears; if three are equal, both disappear: in this case the equation is of the form $a^2\,y = (x-a)^3$, or $a^2\,y = x^3$, if the origin be transferred to B; the curve is then called the cubical parabola.



310. If the equation be $a\,x\,y = x^3 + m\,x^2 + n\,x + p$, the axis of y is an asymptote, and there is a branch in the angle Y A x; the rest

of the curve is like that in the last figure, supposing the lower branch from B to come to A y as the asymptote, the form will vary as the roots vary. We shall take the case where $y = \frac{x^3 - a^3}{a x}$.



	1	2	3	4	5	6	7
Values of x	0	a	$< a$	$> a$	∞	$-$	$-\infty$
Values of y	∞	0	$-$	$+$	∞	$+$	$+\infty$

From (1) Ay is an asymptote; from (2) the curve cuts the axis at B (A B $= a$); from (3, 4, 5) it is below the axis of x from A to B, and above from B to ∞; from (6) and (7) we have the branch F C Y.

This curve is called the trident, from its form. This curve enables us to point out the difference between what are called parabolic and hyperbolic branches of a curve: B y and C Y are hyperbolic, because they admit of a straight line Y y for the asymptote; but B E and C F are parabolic, because they admit of a parabolic asymptote, represented by the dotted curve F A E,

whose equation is $y = \frac{x^2}{a}$ (196).

Ex. Find the locus of the equation $x^2 y - y - x^3 - x = 0$.

If the equation be $x y^2 + a^2 y = x^3 + m x^2 + n x + p$, the form of the curve will depend on the nature of the roots of the equation $x^4 + m x^3 + n x^2 + p x + \frac{a^4}{4} = 0$; there will be no difficulty in any particular case. Generally the equation to the asymptotes is $y = \pm (x + \frac{1}{2} m)$; and the axis of y is an asymptote.

311. If the terms x^3 and $m x^2$ are wanting, the equation is

$$x y^2 + a^2 y = n x + p,$$

$$\therefore y = \frac{-a^2 \pm \sqrt{\{a^4 + 4 p x + 4 n x^2\}}}{2 x}$$

If the denominator of this expression had been constant, the equation would have belonged to an ellipse, hyperbola, or parabola, according as n was negative, positive or nothing; hence if such constant quantity be replaced by the variable quantity $2x$, the conic section becomes "hyperbolized" by having an infinite branch proceeding to the axis of y as an asymptote.

For the nine figures corresponding to the values of p, see Newton, Enum. Lin. Tert. Ord.

From the last article it appears that all curves of the third order have infinite branches; and this must necessarily be the case, for every equation of an odd degree has at least one real root, so that there is always one real value of y corresponding to any real value of x.

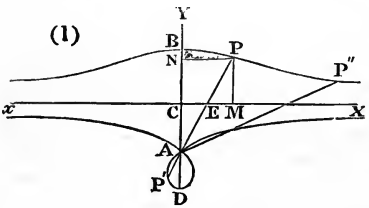
312. The conchoid of Nicomedes.

Let X x (fig. 1) be an indefinite straight line, A a given point, from which draw the straight line A C B perpendicular to X x, and also any number of straight lines A E P, A E P″, &c.; take E P always equal to C B, then the locus of P is the conchoid.

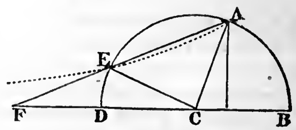
If in E A we take E P′ = E P the locus of P′ is called the inferior conchoid; both conchoids form but one curve, that is, both are expressed by the same equation.

C B is called the modulus, and X x the base or rule;

(1)



(2)



$$\text{Let } AC = a, \quad CM = x,$$

$$CB = b, \quad MP = y,$$

$$\text{then} \quad EP : PM :: AP : AN,$$

$$\text{or} \quad b : y :: \sqrt{x^2 + (a+y)^2} : a + y,$$

$$\therefore y^2 x^2 + y^2 (a+y)^2 = b^2 (a+y)^2,$$

$$\therefore x^2 = (b^2 - y^2)\left(\frac{a+y}{y}\right)^2$$

$$\therefore x = \pm \frac{a+y}{y} \sqrt{b^2 - y^2}.$$

We have three cases according as b is $> a$, $= a$, or $< a$.

Case 1. $b > a$.

		2	3	4	5	6	7	8
Values of y	0	b	$< b$	$> b$	$-a$	$-b$	$< -a$	$> -a, < -b$
Values of x	∞	0	pos.	imp.	0	0	pos.	pos.

From (1) X x is an asymptote; from (2) the curve passes through B; from (3) and (4) the curve extends from the asymptote upwards to B and no higher; hence the branch B P P$''$. Again from (5) and (6) the curve passes through A and D if C D $= b$; from (7) there is a branch A $\times$ extending from A to the asymptote; and from (8) the curve exists between A and D; the double value of x gives the same results along Cx.

Case 2. $b = a$; in the table of values put $b = a$, and omit (8); thus the figure will be the same as the preceding, with the exception of the oval A P$'$ D, which vanishes by the coincidence of A and D.

Case 3. $b < a$; in the table of values put b for a in (7), and for (8) write "if y is $> -b$, x is impossible;" the upper part of the curve is not altered, but the point D falls between A and C; from (8) no part of the curve is between D and A; but from (5) A is a point not on the curve, but belonging to it, and called a conjugate point. In this case the lower curve is similar to the upper one.

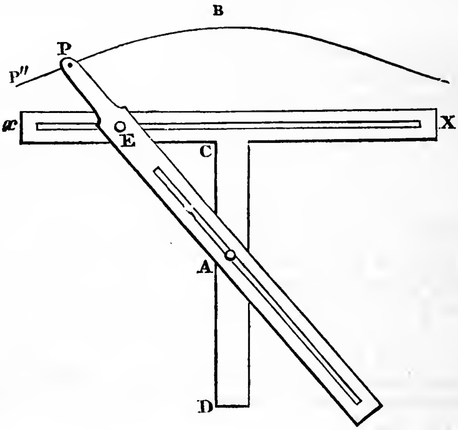
The generation of the conchoid gives a good idea of the nature of an asymptote, for the line E P must always be equal to C B, and this condition manifestly brings the curve continually nearer to C X, as at P$''$, so that the curve, though never actually coinciding with C X, approaches nearer to it than by any finite distance.

This curve was invented by Nicomedes, a Greek geometrician, who flourished about 200 years B.C. He called it the Conchoid, from a Greek word signifying "a shell;" it was employed by him in solving the problems of the duplication of the cube, and the trisection of an angle.

To show how the curve may be applied to the latter problem, let B C A (fig. 2) be the angle to be trisected; draw A E F meeting the circle in E, and the diameter produced in F, and so that the part E F equal the radius C A, then it is directly seen that the arc D E is one-third of B A.

Now it is not possible by the common geometry, that is, with the straight line and circle alone, to draw the line A E F, so that E F shall be equal to C A (the tentative process, though easy, being never considered geometrically correct), and for a long time the ancient geometricians would not hear of any other mathematical instruments than the ruler and compasses; hence the problem was quite insuperable: finding at last that this was the case, they began to invent some curves to assist in the solution of this and other problems: of these curves, the most celebrated is the conchoid of Nicomedes. It may be thus applied to the present problem. Let A be the pole of the inferior conchoid, B F the asymptote or base, and A C the modulus, the intersection of the curve with the circle evidently gives the required point E. The superior conchoid may also be used for the same purpose.

Unless the curve could be described by continued motion, the solution would be incomplete. Nicomedes therefore invented the following simple machine for describing it. Let x X be a straight ruler with a groove cut in it; C D is another ruler fixed at right angles to x X; at A there is a fixed pin, which is inserted in the groove of a third ruler A E P; in A P is a fixed pin at E, which is inserted into the groove of x X; P E is any given length; then, by the constrained motion of the ruler P E A, a pencil at P will trace out a conchoid, and another pencil fixed in E A would trace out the inferior curve.



This curve was formerly used by architects; the contour of the shaft of a column being the portion B P P'' of a conchoid.

The polar equation to the conchoid is thus found:

Let A (fig. 1, page 156) be the pole, A P $= r$, P A B $= \theta$;

$$\therefore\ y + a = r \cos.\ \theta, \text{ and } x = r \sin.\ \theta.$$

Substituting these values in the equation, and reducing, we arrive at the polar equation $r = a \sec.\ \theta + b$.

The polar equation may, however, be much more easily obtained from the definition of the curve. We have

$$r = \text{A P} = \text{A E} + \text{E P} = \text{A C} \sec.\ \text{C A E} + \text{C B} = a \sec.\ \theta + b.$$

313. The following method of obtaining the equation to the conchoid will be found applicable to many similar problems.

Let any number of lines, A E P, fig. 1, be drawn cutting C X in different points E, &c.; from each of these points E as centre, and with radius b describe a circle cutting the line A E P in P and P'; the locus of the point P is the conchoid.

Let A be the origin of the rectangular co-ordinates.

A B the axis of y, and A X parallel to C X in the figure.

Let the general equation to the line A E P be $y = \alpha x$, where α is in determinate:

Then $y' = a$, and $x' = \frac{a}{\alpha}$ are the equations to the point E ;

The equation to the circle which has the point E for its centre and radius b, is

$$(y - y')^2 + (x - x')^2 = b^2,$$

$$\text{or} \quad (y - a)^2 + \left(x - \frac{a}{\alpha}\right)^2 = b^2.$$

And eliminating α between this equation, and that to the line A E P, we have the final equation to the curve,

$$(y - a)^2 + \left(x - \frac{a\,x}{y}\right)^2 = b^2,$$

$$\text{or } (y - a)^2 \frac{x^2 + y^2}{y^2} = b^2.$$

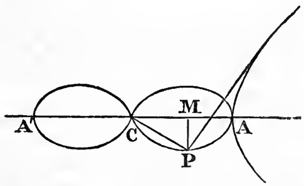
In general if the line C X be a curve whose equation is $y = f(x)$, the co-ordinates of the point E are found by eliminating x and y from the equations $y = \alpha\ x$, and $y = f\ (x)$; hence we find $x = f'\ (\alpha)$, and $y = \alpha f'\ (\alpha)$, and the equation to the circle is

$$\{y - \alpha f'(\alpha)\}^2 + \{x - f'(\alpha)\}^2 = b^2,$$

And the general equation to the curve is

$$\left\{y - \frac{y}{x} f'\left(\frac{y}{x}\right)\right\}^2 + \left\{x - f'\left(\frac{y}{x}\right)\right\}^2 = b^2.$$

314. A perpendicular is drawn from the centre of an hyperbola upon a tangent, find the locus of their intersection.



The equation to the tangent is

$$a^2\,y\,y' - b^2\,x\,x' = -\,a^2\,b^2. \qquad (1)$$

The equation to the perpendicular on it from the centre is

$$y = -\frac{a^2}{b^2}\frac{y'}{x'}\,x. \qquad (2)$$

In order to get the equation to their intersection, we must eliminate x' and y' from these two equations and that to the hyperbola; from (1) and (2) we find

$$x' = \frac{a^2\,x}{x^2 + y^2},\ y' = \frac{-\,b^2\,y}{x^2 + y^2}.$$

Substituting in the equation $a^2\,y'^2 - b^2\,x'^2 = -\,a^2\,b^2$, we have

$$(x^2 + y^2)^2 + b^2\,y^2 - a^2\,x^2 = 0,$$

which is an equation of the fourth degree.

We shall only investigate the figure in the case when $b = a$, that is when the hyperbola is equilateral, in which case the equation is $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$,

$$\therefore y^4 + (2x^2 + a^2) y^2 + x^4 - a^2 x^2 = 0,$$

$$\text{and } y = \pm \sqrt{\left\{ - \left(x^2 + \frac{a^2}{2} \right) \pm a \sqrt{2x^2 + \frac{a^2}{4}} \right\}}.$$

If the sign of the interior root be negative, y is impossible; hence we shall only examine the equation

$$y = \pm \sqrt{\left\{ - \left(x^2 + \frac{a^2}{2} \right) + a \sqrt{2x^2 + \frac{a^2}{4}} \right\}};$$

here y is impossible, if $x^2 + \frac{a^2}{2}$ is $> a \sqrt{2x^2 + \frac{a^2}{4}}$,

$$\text{if } x^4 + a^2 x^2 + \frac{a^4}{4} \text{ is } > 2a^2 x^2 + \frac{a^4}{4}$$

$$\text{if } x^4 \text{ is } > a^2 x^2,$$

$$\text{if } x \text{ is } > \pm a;$$

hence we have the following table :

	1	2	3	4
Values of x	0	$\pm a$	$< \pm a$	$> \pm a$
Values of y	0	0	pos.	imp.

From (1) the curve passes through C; from (2) it passes through A and A′; from (3) it has two branches from C to A and from C to A′; from (4) it does not extend beyond A and A′.

We may judge yet more nearly of the form of these ovals, for the tangent at the vertex of the hyperbola being perpendicular to the axis, the oval will cut the axis at A at a right angle; and again at C in an angle of 45°, because the tangent nearly coinciding with the asymptote, the perpendicular on it makes an angle of 45° with the axis ultimately.

This curve was invented by James Bernouilli; it is called the Lemniscata, and forms one of a series of curves corresponding to different values of b.

To find the polar equation to the lemniscata,

$$\text{Let } y = r \sin. \theta, \text{ and } x = r \cos. \theta;$$

hence the equation $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$ becomes $r^2 = a^2 \cos. 2\theta$.

Any curve that is of the form of this figure is called a lemniscata.

315. In the following example the curve may be easily traced by points.

Let a circle be described with centre C and any radius C Q; draw the ordinate Q M, and in Q C take Q P = Q M; the locus of P is a lemniscata.

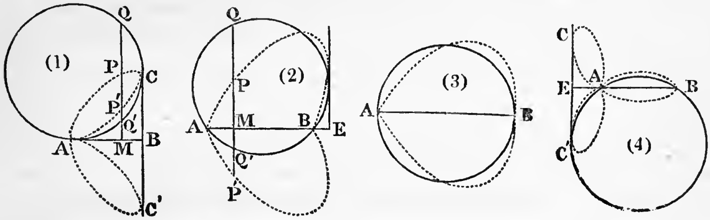
Again, if in M Q we take M R = a third proportional to M Q and C M, the locus of R is another lemniscata whose equation is

$$x^4 - a^2 x^2 + a^2 y^2 = 0.$$

The equation $a^2 (y - a)^2 = (x - a)^2 (2ax - x^2)$ belongs to the same curve referred to a different origin.

Ex. Trace the locus of the equation $y^2 = x^2 \cdot \dfrac{a^2 + x^2}{a^2 - x^2}$.

316. A M, fig. 1, is a tangent to a circle A C Q, M Q an ordinate to the abscissa A M; M P is taken a mean proportional between A M and M Q; required the locus of P.



Let A M $= x$, and M P $= y$, be the rectangular co-ordinates of P, and let the radius of the circle $= b$,

then the square on M P = the rectangle A M, M Q.

To find M Q, we have the equation to the circle

$$(y - y')^2 + (x - x')^2 = r^2,$$

$$\text{or } y^2 - 2by + x^2 = 0, \text{ since } x' = 0, \text{ and } y' = r = b,$$

$$\therefore \text{MQ} = b \pm \sqrt{b^2 - x^2},$$

$$\therefore \text{MP or } y = \pm \sqrt{\{bx \pm x\sqrt{b^2 - x^2}\}}.$$

Since $b^2 - x^2$ is $< b^2$, there are four values of y to each positive value of $x < b$, and no value of y to x negative; hence if A B $= b$, fig. 1, the straight line C B C′ perpendicular to A B is a limit to the curve, and when $x = b$, the ordinate to the curve is equal to the extreme ordinate of the circle, that is, to the tangent B C.

Between $x = 0$, and $x = b$, we have four values of y, which give the two dotted ovals of fig. (1).

To make the question more general we shall suppose the line A B to be a chord of the circle, figs. (2) (3) (4).

Then if b and a are the co-ordinates to the centre of the circle, and A the origin, the equation to the curve will be

$$y = \pm \sqrt{\{bx \pm x\sqrt{b^2 + 2ax - x^2}\}},$$

and we have four cases depending on the values of b and a; hence we have four curves of different forms, yet partaking of the same character and generation.

Case (1). $a = 0$, fig. (1) already discussed.

Case (2). a and b positive, fig. (2). A E $= a + \sqrt{a^2 + b^2}$.

Case (3). $b = 0$, fig. (3).

Case (4). b negative, fig. (4), the equation is

$$y = \pm \sqrt{\{-bx \pm x\sqrt{b^2 + 2ax - x^2}\}}.$$

There are two values of y for x positive, and $< 2a$; but four values for x negative, and $< \sqrt{a^2 + b^2} - a$, that is, $<$ A E.

The gradual transition of one curve to another is apparent, but that the same problem should produce such very different curves as (2) and (4) requires some explanation.

In fig. (1) P and P′ are determined by mean proportionals between A M and M Q, and also between A M and M Q′. Moreover P may be in Q M produced as well as in M Q, thus we have the double oval, fig. (1.) On the left of A the abscissas A M will be negative, and the ordinates M Q positive; hence no possible mean proportional can exist, or no part of the curve can be on the left of A.

In fig. (2) A M and M Q determine the points P and P′; but A M and M Q′ give only an imaginary locus.

Fig. (3) requires no comment.

In fig. (4) the reasoning on fig. (2) will explain the positive side of A; on the left of A the abscissa and both ordinates are negative; therefore two mean proportionals can be found, or four points in the curve for each abscissa.

Such curves may be invented at pleasure, by taking the parabola or other curves for the base instead of the circle.

Ex. To find the locus of the equation $y^4 + 2axy^2 - ax^3 = 0$.

317. To find a point P′, such that drawing straight lines to two given points S and H, we may have the rectangle S P, H P constant.

Join the points S and H, and bisect S H in C; let C be the origin of rectangular axes, S H $= 2a$, C M $= x$, M P $= y$ and let the rectangle S P, H P, $= ab$.

Then since S M $= a + x$, and H M $= a - x$, we have

$$\{y^2 + (a + x)^2\}\{y^2 + (a - x)^2\} = a^2 b^2,$$

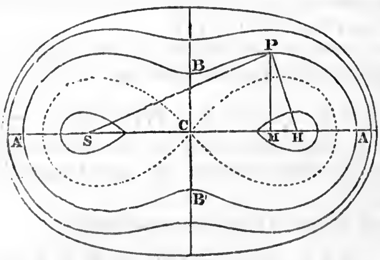
$$\text{or } (y^2 + x^2 + a^2 + 2ax)(y^2 + x^2 + a^2 - 2ax) = a^2 b^2,$$

$$\text{or, } \{y^2 + x^2 + a^2\}^2 - 4a^2x^2 = a^2 b^2;$$

$$\text{hence } y = \pm\sqrt{\{-(a^2 + x^2) + a\sqrt{b^2 + 4x^2}\}}.$$

$$\text{Let } y = 0, \therefore x = \pm\sqrt{a(a \pm b)} \qquad (1).$$

$$\text{Let } x = 0, \therefore y = \pm\sqrt{a(b - a)} \qquad (2).$$



1. Let a be less than b.

Then from (1) we have the points A and A′, and from (2) we have the points B and B′.

Also by comparing the values of y in the original equation and in equation (2) we shall find that M P is greater than C B as long as x is greater than $\sqrt{2a(2a-b)}$; thus the form of the curve must be like that of the figure A P B A′ B′ A.

As b increases, the oval becomes flatter at the top, and takes the form of the outer curves.

2. Let $a = b$, then we have the dotted curve passing through C; also since the equation becomes $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$ the locus is in this case the lemniscata of Bernouilli.

3. Let a be greater than b.

Then from (1) we have two values of x, and from (2) an impossible value of y; hence the curve must consist of the two small oval figures round S and H.

As b decreases, the little ovals decrease; and when $b = 0$, we have the points S and H themselves for the locus.

These curves are called the ovals of Cassini, that celebrated astronomer having imagined that the path of a planet was a curve like the exterior one in the above figure.

The equation $(y^2 + x^2)^2 = b^2 y^2 + a^2 x^2$, found in art. (123), gives a figure like that in case 1.

318. There are some cases in which it is useful to introduce a third variable; for example, if the equation be $y^4 + x^2 y^2 + 2y^3 + x^3 = 0$, it requires the solution of an equation of three or four dimensions, in order to find corresponding values of x and y; to avoid this difficulty, assume $x = uy$,

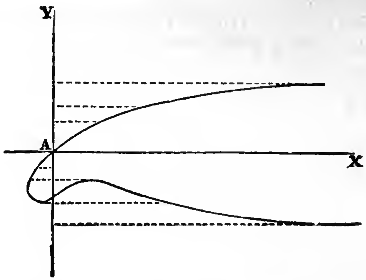
$$\therefore\ y^4 + u^2 y^4 + 2y^3 \quad u^3 y^3 = 0,$$

$$\text{or,}\qquad y + u^2 y + 2 - u^3 = 0,$$

$$\therefore\ y = \frac{u^3 - 2}{u^2 + 1}, \text{ and } x = u . \frac{u^3 - 2}{u^2 + 1};$$

from these equations we can find a series of corresponding values for x and y.

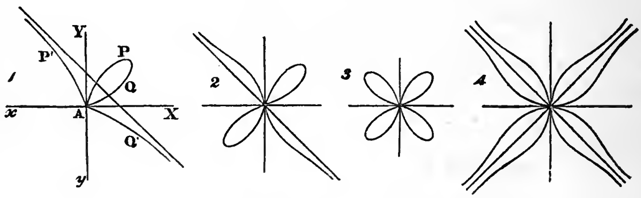
$u = -3$	$y = -2\frac{9}{10}$,	$x = 8\frac{7}{10}$
$u = -2$	$= -2$	$= 4$
$= -1$	$= -1\frac{1}{2}$	$= 1\frac{1}{2}$
$= -\frac{1}{2}$	$= -1\frac{7}{10}$	$= \frac{17}{20}$
$= 0$	$= -2$	$= 0$
$= \frac{1}{2}$	$= -1\frac{1}{2}$	$= -\frac{3}{4}$
$= 1$	$= -\frac{1}{2}$	$= -\frac{1}{2}$
$= 1\frac{1}{2}$	$= \frac{11}{26}$	$= \frac{33}{52}$
$= 2$	$= 1\frac{1}{5}$	$= 2\frac{2}{5}$
$= 3$	$= 2\frac{1}{2}$	$= 7\frac{1}{2}$
$= 4$	$= 3\frac{11}{17}$	$= 14\frac{10}{17}$
&c.	&c.	&c.



Also when $y = 0$, $x = 0$, hence the curve passes through A. Let A X, A Y be the axes; along the axis of y take values equal to those in the table for y; and from the points thus determined draw lines equal to the corresponding values in the table for x (these are the dotted lines in the figure); by this method we obtain a number of points in the curve sufficient to determine its course.

This example is taken from the "Analyse des Lignes Courbes, by G. Cramer. Geneva. 1750," a work which will be found extremely useful in the study of algebraical curves.

319. To trace the curve whose equation is $y^5 - 5ax^2y^2 + x^5 = 0$.



Let x be very small $\therefore$ x^5 being exceedingly small may be omitted, and the equation becomes $y^5 = 5ax^2y^2$, or $y^3 = 5ax^2$, which is the equation to a semi-cubical parabola P A P′ fig. (1.); and if y be very small, we have $x^3 = 5ay^2$, which gives the parabola Q A Q′; hence near the origin the curve assumes the forms of the two parabolic branches. Again when x is infinitely great, x^2 may be neglected in comparison with x^5 and the equation becomes $y^5 = -x^5$, $\therefore y = -x$; hence for x positive, we have an infinite branch in the angle X A y, and for x negative an infinite branch in the angle Y A x.

To find the asymptote:

$$y^5 = -x^5 + 5ax^2y^2,$$

$$= -x^5\left(1 - 5\frac{ay^2}{x^3}\right).$$

$$\therefore y = -x\left(1 - 5a\frac{y^2}{x^3}\right)^{\frac{1}{5}},$$

$$= -x\left\{1 - a\frac{y^2}{x^3} - 2a^2\frac{y^4}{x^6} -, \&c.\right\}$$

$$= -x + a\left(\frac{y}{x}\right)^2 + 2a^2\frac{y^2}{x^5} +, \&c.$$

$$= -x + a + 2\frac{a^2}{x^3} +, \&c. \text{ when } y = -x:$$

Therefore the equation to the asymptote is $y + x = a$; this being drawn and the branches A P′, A Q′ produced towards it, we have nearly a correct idea of the curve.

If the equation be $y^5 - 5a^2x^2y + x^5 = 0$, the curve will be traced in the same manner, fig. (2).

If the equation be $y^6 - a^2x^2y^2 + x^6 = 0$, we have fig. (3);

And the equation $y^6 - a^2x^2y^2 - x^6 = 0$ will give fig. (4).

Ex. Find the locus of the equation $y^4 - 4a^2xy - x^4 = 0$.

For the above method of tracing curves of this species, see a treatise on the Differential Calculus, by Professor Miller. Cambridge, 1832.

320. B C is a straight line of given length $(2b)$, having its extremities always in the circumferences of two equal circles, to find the locus of the middle point P of the line B C.

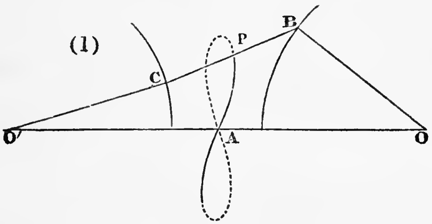
Let the line joining the centres O, O′ of the circles be the axis of x, and let the origin of rectangular axes be at A, the bisecting point of O O′.

Let $x\,y$ be the co-ordinates of B.

$x'\,y'$ . . . . . . C.

X Y . . . . . . P.

(1)



$$AO = AO' = a,$$
$$OB = O'C = c,$$

$$\left.\begin{array}{lr} \text{the equation to B is } y^2 + (x-a)^2 = c^2 & (1) \\ \text{to C is } y'^2 + (x'+a^2)^2 = c^2 & (2) \\ \text{also } (y-y')^2 + (x-x')^2 = 4b^2 & (3) \\ 2Y = y + y' & (4) \\ 2X = x + x' & (5) \end{array}\right\}$$

From these five equations we must eliminate the four quantities y, x, y' and x'; from (1) and (2)

$$y^2 - y'^2 + x^2 - x'^2 - 2a(x + x') = 0,$$

$$\text{or } (y - y')Y + (x - x')X - 2aX = 0 \qquad (6),$$

from (4) and (5) $y^2 + y'^2 + x^2 + x'^2 + 2yy' + 2xx' = 4Y^2 + 4X$

from (3) $y^2 + y'^2 + x^2 + x'^2 - 2yy' - 2xx' = 4b^2$,

from (1) and (2) $2y^2 + 2y'^2 + 2x^2 + 2x'^2 - 4a(x-x') = 4c^2 - 4a^2$,

$\therefore$ by substitution $4a(x - x') = 4(Y^2 + X^2 + b^2 + a^2 - c^2)$,

$$\text{or } (x - x') = \frac{Y^2 + X^2 + m^2}{a}, \text{ if } m^2 = a^2 + b^2 - c^2,$$

and from (6)

$$y - y' = \{2a - (x - x')\} \frac{X}{Y} = \left(2a - \frac{Y^2 + X^2 + m^2}{a}\right) \frac{X}{Y}$$

Substituting these values of $x - x'$ and $y - y'$ in (3), we have

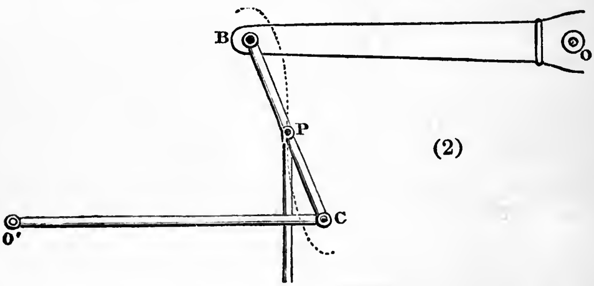
$$\left\{2a - \frac{Y^2 + X^2 + m^2}{a}\right\}^2 \frac{X^2}{Y^2} + \left\{\frac{Y^2 + X^2 + m^2}{a}\right\}^2 = 4b^2,$$

$$\text{or } 4a^2 X^2 - 4(Y^2 + X^2 + m^2) X^2 + \left(\frac{Y^2 + X^2 + m^2}{a}\right)^2 (X^2 + Y^2) = 4b^2 Y^2$$

$$\therefore\ a^2 X - b^2 Y^2 - X^2 (X^2 + Y^2 + m^2) + (X^2 + Y^2)\left(\frac{X^2 + Y^2 + m^2}{2a}\right)^2 = 0.$$

This equation, being of the sixth dimension, and the highest terms being both positive, the curve must be limited in every direction : when X is very small, there are four values of Y; also when $X = 0$, we have $Y = 0$; hence the curve is a species of double oval, or lemniscata.

If the circles be unequal, and P be any point in the line B C, the curve will be of the same nature, but the investigation is much longer.



(2)

The very beautiful contrivance of Watt to reduce a circular to a rectilinear motion is well known to every one. Suppose the point B to be the extremity of an engine-beam, moveable about its centre O, this beam is required to moved a piston-rod always in the same vertical position; it is plain that this motion cannot be obtained by fixing the piston-rod to B, or to any point in O B. Suppose now, a beam O′ C, called the radius-rod, to move about a centre O′, and join the extremities B, C, by a bar B C; the extremity of the piston-rod is fixed to the middle of the beam B C, and its motion, according to the above demonstration, is in a portion of the curve, such as the dark part of the lemniscate in the first figure, and consequently the rod itself continues much more in the same vertical line than if attached to B. The comparative lengths of the rods necessary to render the motion as nearly vertical as possible are stated in most works on the steam-engine, and in the Mechanics' Magazine. For a more complete but very different method of finding the equation to the above curve, see " Prony, Hydraulique.'

321. We have no space for the discussion of any higher algebraic curves, if it were necessary; but in fact we have not the means: it must

have been already seen that many of the preceding curves have not been drawn with mathematical exactness; for unless we took the trouble of tracing them by points, we could not easily determine their curvature; we shall therefore pass to the consideration of the general equation of the nth dimension, and then proceed to the intersection of algebraic curves*.

322. The general equation of the nth degree, with all its terms complete, is

$$y^n + (a\,x + b)\,y^{n-1} + (c\,x^2 + d\,x + e)\,y^{n-2} + \;.\;\;.\;.\; + f\,x^n + g\,x^{n-1} + h\,x^{n-2} + \;.\;.\;.\;.\;.\; + k\,x + l = 0;$$

it contains all the possible combinations of x and y, so that the sum of the exponents in no one term exceeds n.

The number of terms is $1 + 2 + 3 + \;.\;.\;.\;. + (n\;+1)$, or is the sum of an arithmetic progression, whose first term and common difference is unity, and the number of terms is $n + 1$; therefore the sum of this series is $\dfrac{(n+2)\quad(n+1)}{2}$.

The number of independent constants is (dividing by the co-efficient of y^n if necessary) one less than the number of terms in the equation, that is,

$$= \frac{(n+2)\quad(n+1)}{2} - 1 = \frac{n\quad(n+3)}{2}$$

323. An algebraic curve of the n th degree may pass through as many given points as it has arbitrary constants, that is, through $\dfrac{n\;\;(n+3)}{2}$ points, for giving to x and y their values at each one of the given points, we have $\dfrac{n\;\;(n+3)}{2}$ different equations, by which the values of the constants may be determined. For example,

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* We must refer our readers to our treatise on the Differential Calculus for information on the curvature of lines. It must not, however, be imagined that algebraic geometry is incapable of exhibiting the form of curves; the following method of determining the curvature is an instance to the contrary.

Let y_1, y_2, and y_3 be three consecutive ordinates, at equal distances from each other; then drawing a corresponding figure, it will be seen that the curve is concave or convex to the axis, according as y_2 is $>$ or $<$ $\dfrac{y_1 + y_3}{2}$; as an example, take the cubical parabola, whose equation is $a^2\,y = x^3$, then the curve is convex, if $2\,x^3$ is $< (x-1)_3 + (x\;+1)^3$ is $> 2\,x^3 + 6\,x$, which it is, and therefore the curve is convex. The distances at which the ordinates are drawn from each other must depend on the constants in the equation.

Again, to determine the angle at which a curve cuts the axis of x, transfer the origin to that point; then the tangent to the curve at that point and the curve itself make the same angle with the axis; but the value of the tangent of the angle which the tangent to the curve makes with the axis is then $\dfrac{y}{x} = \dfrac{0}{0}$, which may be any value whatever: for example, let $a^2\;y = x \;\;\therefore \dfrac{y}{x} = \dfrac{x^2}{a^2} = 0$ when $x = 0$, therefore the curve coincides with the axis at the origin. Again, take the example in art. 307 $y^3 = x^2\;(2\,a - x)$; at A we have $\dfrac{y^3}{x^3} = \dfrac{2\,a - x}{x} = \dfrac{1}{0}$, and at B we have $\dfrac{y}{2\,a - x} = \dfrac{x^2}{y^2} = \dfrac{4\;a^2}{0}$; hence the curve cuts the axis of x in both cases at an angle of 90°.

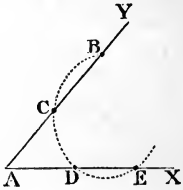
The general equation to the conic sections, dividing by the co-efficient of y^2, is

$$y^2 + b\,x\,y + c\,x^2 + d\,y + e\,x + f = 0,$$

in which there are five co-efficients, and therefore a conic section may pass through five given points; substituting the co-ordinates of the given points separately for x and y we obtain five equations from which the constants can be determined, and thence we have the particular curve required; it will be an ellipse, hyperbola, or parabola, according as $b^2 - 4c$ is negative, positive, or nothing. (79.)

324. The elimination is long, but the trouble may be much lessened by assuming one of the given points for the origin, and two lines drawn from the origin to other two given points for the axes.

For example, if it be required to pass a conic section through four given points B C D E, join B C and D E, and let them meet in A; let A B be the axis of y and A D the axis of x,



Let $\mathrm{AC} = y_1$, $\mathrm{AB} = y_2$,

$\mathrm{AD} = x_1$, $\mathrm{AE} = x_2$;

Assume the equation to be

$$y^2 + b\,x\,y + c\,x^2 + d\,y + e\,x + f = 0;$$

we have for $x = 0$, $y_1^2 + d\,y_1 + f = 0$

and $y_2^2 + d\,y_2 + f = 0$;

$$\therefore \quad d = -(y_1 + y_2), \text{ and } f = y_1\,y_2.$$

Similarly for $y = 0$, $e = -c\,(x_1 + x_2)$, and $f = c\,x_1\,x_2$;

equating the values of f, we have $c = \dfrac{y_1\,y_2}{x_1\,x_2}$.

Substituting and dividing by $y_1\,y_2$, we have

$$\frac{y^2}{y_1\,y_2} + \frac{b}{y_1\,y^2}\,x\,y + \frac{x^2}{x_1\,x_2} - \frac{y_1 + y_2}{y_1\,y_2}\,y + \frac{x_1 + x_2}{x_1\,x_2}\,x + 1 = 0,$$

an equation involving only one unknown co-efficient b.

There are some restrictions depending on the situation of the given points; thus no more than two can be in the same straight line, or else the conic section degenerates into two straight lines.

The five given points are the same as five conditions expressed analytically; four are sufficient if the curve is to be a parabola; for $b^2 - 4c = 0$, is equivalent to one. If the curve has a centre, whose position is given, three other conditions suffice, because we may assume the equation to be $y^2 + b\,x\,y + c\,x^2 + f = 0$. If the position of two conjugate diameters be given, only two more conditions are requisite.

Newton, in his Universal Arithmetic, gives excellent methods for describing, by continued motion, a conic section passing through five given points.

325. If it be required to pass a curve, whose species is not given, through a number of given points, we may with advantage assume the equation to be of the form

$$y = a + b x + c x^2 + d x^3 +, \text{ \&c.}$$

The elimination of the constants is more regular, and therefore easier in this equation than in any other: such curves are called parabolic (the three first terms giving the common parabola) and consist of a series of sinuosities, such as in (309), which are easily traced. For the elimination of the constants, see Lagrange, or Lardner's Algebraic Geometry, art. 617.

326. We saw in article (79) that the general equation of the second order sometimes gave straight lines for the loci; such will be the case whenever any equation is reducible into rational factors of the first degree; so that we must not always conclude that an equation of the nth order has a curve of the nth order for its locus. If the equation be reducible into factors of lower degrees, there will be a series of lines corresponding to those factors; thus if an equation of the 4th degree be composed of one factor of two degrees, and two factors of the 1st degree, the loci are a conic section and two straight lines; and hence a general equation of any order embraces under it all curves of inferior orders: if any of the factors be impossible, their loci are either points, or imaginary.

If the sum of the indices of x and y be the same in every term, the loci are either straight lines or points; for an equation of this species will have the form

$$y^n + a y^{n-1} x + b y^{n-2} x^2 \quad \dots + l x^n = 0,$$

$$\text{or} \quad \left(\frac{y}{x}\right)^n + a\left(\frac{y}{x}\right)^{n-1} + b\left(\frac{y}{x}\right)^{n-2} \dots + l = 0,$$

let the roots of this equation be α, β, γ, &c., then the equation will be

$$\left(\frac{y}{x} - \alpha\right)\left(\frac{y}{x} - \beta\right)\left(\frac{y}{x} - \gamma\right) \dots = 0,$$

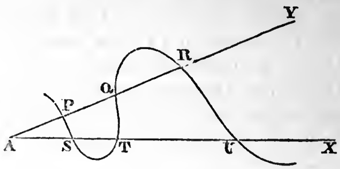
each factor of which being $= 0$, its corresponding locus is evidently a straight line; if the roots of the equation be impossible, the corresponding loci are points.

Ex. $y^2 - 2xy \sec. \alpha + x^2 = 0$. The locus consists of two straight lines whose equations are $y = x \dfrac{1 \pm \sin. \alpha}{\cos. \alpha} = x \tan. \left(45° \pm \dfrac{\alpha}{2}\right)$ and therefore the lines pass through the origin, and are inclined to the axis of x at angles of $45° \pm \dfrac{\alpha}{2}$.

327. Since the general equation includes all equations below it, the properties of the curve of n dimensions will generally be true for the lower orders, and also for certain combinations of the lower orders; thus, a property of a line of the third degree will be generally true for a conic section, or for a figure consisting of a conic section and a straight line, or for three straight lines. Moreover the lower orders of curves have generally some analogy to the higher curves, and hence the properties of inferior orders often lead to the discovery of those of the superior.

328. From the application of the theory of equations to curves, an immense number of curious theorems arise, which may be seen in the

works of Waring and Maclaurin: we have only room for two or three of the most important.



If two straight lines, A X, A Y cut a curve of n dimensions, in the points P Q R, &c., S T U, &c., so that A P, A Q, A R, &c. $= y_1, y_2, y_3$, &c. respectively, and A S, A T, A U, &c. $= x_1, x_2, x_3$, &c. respectively, then if A X and A Y move parallel to themselves, we shall always have $y_1 . y_2 . y_3 .$ &c. : $x_1 . x_2 . x_3 .$ &c., in a constant ratio.

Let the equation to the curve be referred to the origin A, and to axes A X, A Y, by means of the transformation of co-ordinates, and suppose the equation to be

$$y^n + (a x + b) y^{n-1} + \ldots c x^n + d x^{n-1} + \ldots k x + l = 0.$$

$$\text{Let } y = 0 \quad \therefore c x^n + d x^{n-1} + \ldots . k x + l = 0. \qquad (1)$$

$$x = 0 \quad \therefore y^n + b y^{n-1} + \ldots . k y + l = 0. \qquad (2)$$

The roots of (1) are A S, A T, A U, &c.; $\therefore x_1 . x_2 . x_3 .$ &c. $= \frac{l}{c}$.

The roots of (2) are A P, A Q, A R, &c.; $\therefore y_1 . y_2 . y_3 .$ &c. $= l$

$$\therefore y . y_2 . y_3, \&c. : x_1 . x_2 . x_3, \&c. :: c : 1.$$

Now the transformation of the axes, parallel to themselves, never alters the co-efficients of y^n and x^n; hence the above ratio is constant for any parallel position of A X and A Y.

Article 293 is an example of this theorem.

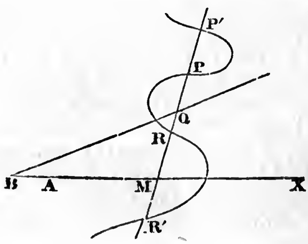
329. A diameter was defined in (76) to be a straight line, bisecting a system of parallel chords; more generally it is a line, such that if any one of its parallel chords be drawn, meeting the curve in various points, the sum of the ordinates on one side shall equal the sum on the other; thus, in the figure, if P Q + P′ Q + &c. = R Q + R′ Q = &c., and the same be true for all lines parallel to P R, then B Q is a diameter.

To find the equation to the diameter B Q let the equation to the curve, referred to A X and a parallel to P Q, &c.

$$y^n + (a x + b) y^{n-1} + (c x^2 + d x + e) y^{n-2} +, \&c. = 0.$$

Let M Q $= u$, and P Q $= y'$, $\therefore y = y' + u$,
by substitution we have

$$y'^n + (a x + b + n u) y'^{n-1} + \{c x^2 + d x + e + \overline{n-1}\, u . \overline{a x + b} + n . \frac{n-1}{2} u^2\} y'^{n-2} +, \&c. = 0$$



By the definition the sum of the values of y' must equal nothing, and that sum is the co-efficient of the second term in the last equation with its sign changed,

$$\therefore\ a x + b + n u = 0,$$

$$\text{or}\quad u = -\frac{a x + b}{n},$$

and this is the equation to the diameter B Q.

Again, by the same reasoning, the equation

$$c x^2 + d x + e + \overline{n-1}\, u \,.\, \overline{a x + b} + n \,.\, \frac{n-1}{2} u^2 = 0$$

is that to a conic section drawn so that the sum of the products of the values of y, taken two and two together, shall equal nothing.

We might proceed on with the co-efficient of the fourth term.

These curves are sometimes called curvilinear diameters.

330. The method of finding the centre, if any, of a curve, is given in (81); the operation is too long to apply it to a general equation of high dimensions, and therefore we shall take an example among the lines of the third order as fully illustrating the subject.

Let the equation be $x y^2 + e y = a x^3 + b x^2 + c x + d$, under which form are comprehended most of the curves of the third order.

Let $x = x + m$, $y = y + n$; the transformed equation is

$$\begin{gathered} x y^2 + 2 n x y + m y^2 + (2 n m + e) y - a x^3 - (3 a m + b) x^2 \\ + (n^2 - 3 a m^2 - 2 b m - c)\, x + m n^2 + e n - a m^3 - b m^2 \\ - c m - d = 0; \end{gathered}$$

in order that the curve may have a centre, the 2nd, 3rd, 6th, and last or constant term must each $= 0$; $\therefore n = 0$, $m = 0$, $b = 0$, $d = 0$, so that the corresponding curve has a centre, which is the origin, only when the co-efficients b and d are wanting.

CHAPTER XIII.

ON THE INTERSECTION OF ALGEBRAIC CURVES.

331. The intersection of a straight line with a line of the nth order is found by eliminating y from the two equations; hence the resulting equation in terms of x will be of the nth order, and therefore may have n real roots; thus there may be n intersections: there may be less, since some of the roots of the resulting equation may be equal to one another, or some impossible.

Generally speaking, a curve of n dimensions may be cut by a straight line drawn in some direction in n points; but the curve, in its most general form, must be taken; otherwise certain points as conjugate and multiple points, must be considered as evanescent ovals or evanescent branches of the curve, and thus a line passing through such points is equivalent to two or more intersections.

332. The intersections of two lines of the mth and nth orders are found also by eliminating y from both; hence the resulting equation may be of the mnth order, or there may be $m\,n$ intersections; there are often less, for not all the real roots of the equation $X = 0$ will give points of intersection: for example, if we eliminate y from the equations

$$y^2 = 2\,a\,x - x^2 \text{ and } y^2 = 2\,a\,(x - b) \text{ we find } x = \sqrt{2\,a\,b};$$

hence, apparently, there is always an intersection corresponding to the abscissa $\sqrt{2\,a\,b}$; but this is not the case; for then $y^2 = 2\,a\,(\sqrt{2ab} - b)$, and therefore y is impossible, if b is $> 2\,a$, which is evident on drawing the two curves; hence after the abscissa is found, we must examine the corresponding ordinates in each curve; if they are not real, there can be no intersection corresponding to such abscissa.

If we have the two equations $y^2 + 2\,x = 0$, $y^2 + 4\,x^2 - 10\,x - 16 = 0$, the elimination y gives the abscissas of intersection $x = 4$ and $x = -1$, the second of which alone determines a point of intersection.

333. In finding the intersections of lines, we often fall upon a final equation of an order higher than the second, or arrive at an equation whose roots are of a form not readily constructed; to avoid this difficulty a method is often used which consists in drawing a line which shall pass through all the required points of intersection, and thus determine their situation.

Let $y = f\,(x)$ * (1), and $y = \phi\,(x)$ (2), be the equations to two lines, then at the point of intersection they have the same ordinates and abscissas; or calling X and Y the co-ordinates of the point of intersection, we have simultaneously $Y = f(X)$ and $Y = \phi\,(X)$; hence $f(X) = \phi\,(X)$, from which equation X and Y might be obtained, and their values constructed

But since $$Y = f\,(X) \quad \ldots\ldots \quad (3)$$

and $$Y = \phi\,(X) \quad \ldots\ldots \quad (4)$$

we have by addition $$2\,Y = f\,(X) + \phi\,(X) \quad \ldots\ldots \quad (5)$$

or by multiplication $$Y^2 = f\,(X)\,.\,\phi\,(X) \quad \ldots\ldots \quad (6)$$

or generally $$Y = F\,\{f(X), \phi\,(X)\} \quad \ldots\ldots \quad (7)$$

F implying any function arising from the addition, subtraction, multiplication, &c. of (3) and (4).

Now any one of these equations gives a true relation between the co-ordinates X and Y of the point of intersection of (1) and (2); but by supposing X and Y to vary, it will give a relation between a series of points, of which the required point of intersection is certainly one; that is, drawing the locus of (5) or (6) or (7), it must pass through the required point of intersection of (1) and (2).

It is manifest that if one of the equations (5), (6), or (7), be a circle

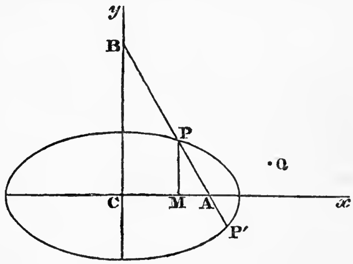
* The symbols $F\,(x)$, $f\,(x)$, $\phi\,(x)$, serve to denote different functions of x, that is, indicate expressions into which the same quantity x enters, but combined in different ways with given quantities. But $f\,(x)$, $f\,(y)$, indicate similar formulæ for both x and y; thus, if $f\,(x) = 2\,a\,x - x^2$, then $f\,(y) = 2\,a\,y - y^2$, or $2\,b\,y - y^2$.

or straight line, it will be much easier to draw this circle or straight line than to find the intersection by means of elimination.

Also we may often find the intersection of (1) and (2), when one of them is a given curve, by drawing the locus of the other, and this method is the simplest when that other is a straight line.

We shall give a few examples to illustrate the subject.

334. From a given point Q without an ellipse, to draw a tangent to it.



Let the co-ordinates of Q be m and n, and let X and Y be the co-ordinates of the point P, where the required tangent meets the curve.

Then by (111) the equation to the tangent through P is

$$a^2 y \mathrm{Y} + b^2 x \mathrm{X} = a^2 b^2,$$

and since this passes through Q we have

$$a^2 n \mathrm{Y} + b^2 m \mathrm{X} = a^2 l^2 \qquad (1)$$

$$\text{and} \quad a^2 \mathrm{Y}^2 + b^2 \mathrm{X}^2 = a^2 b^2. \qquad (2)$$

From (1) and (2) we might, by elimination, find X and Y, and their constructed values would be the co-ordinates C M, M P of the required point.

Now (1) is not the equation to any straight line, but only gives the relation between C M and M P; but if we suppose X and Y to vary, it will give the relation between a series of points, of which P is certainly one; and therefore, if the line whose equation is (1) be drawn, it must pass through P, and consequently, with the ellipse (2), will completely fix the situation of P.

To draw the line (1),

Let $\mathrm{X} = 0$; $\therefore \mathrm{Y} = \dfrac{b^2}{n}$; Let $\mathrm{Y} = 0$; $\therefore \mathrm{X} = \dfrac{a^2}{m}$;

in C y take C B $= \dfrac{b^2}{n}$, and in C x take C A $= \dfrac{a^2}{m}$; join B A; B A produced is the locus of (1), and it cuts the ellipse in two points P and P ; hence if Q P and Q P′ be joined, they are the tangents required.

The same method may be employed in drawing tangents to the parabola and hyperbola.

To take the more general case, let $a y^2 + c x^2 + d y + e x = 0$ (1) be the equation to the curves of the second order referred to axes parallel to conjugate diameters.

Then the equation to a tangent at a point $x' y'$ is

$$a y y' + c x x' + \frac{d}{2}(y + y') + \frac{e}{2}(x + x') = 0,$$

or $(2ay' + d)y + (2cx' + e)x + dy' + ex' = 0$ (2).

Let this tangent pass through a point $m\,n$, then (2) becomes

$$(2ay' + d)n + (2cx' + e)m + dy' + ex' = 0 \quad (3),$$

or, $(2an + d)y' + (2cm + e)x' + dn + em = 0$ (4).

Now let x' and y' in (4) be considered variable, and construct the straight line, which is the locus of (4); this with the curve itself, determines the position of the secant line which joins the two points on the curve, whence tangents are drawn to the point $m\,n$.

335. Again, suppose the secant line (4) to pass through a given point $m'\,n'$; Then the equation (4) becomes

$$(2an + d)n' + (2cm + e)m' + dn + em = 0 \quad (5),$$

and of course the point $m\ n$, whence tangents were originally drawn, must have a particular position corresponding to each secant line passing through $m'\,n'$; if therefore we make m and n variable in (5) we shall have the equation to the locus of the point $m\,n$

$$(2an' + d)n + (2cm' + e)m + dn' + em' = 0$$

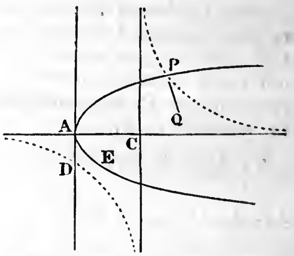
where m and n are the variable co-ordinates.

Hence we have the following theorem: if from any point secants be drawn to a line of the second order, and from the two points where each of these secants intersect the curve, tangents be drawn meeting each other, the locus of all such points of concourse is a straight line.

336. To draw a normal to a parabola from a point Q $(a, b,)$ not on the curve.

Let $y^2 = 4mx$, be the equation to the curve, and let X and Y be the co-ordinates of the required point, then the equation to the tangent at the point X Y, is by (232)

$$Yy = 2m(X + x),$$



and therefore that to the normal at X Y is

$$y - Y = -\frac{Y}{2m}(x - X),$$

and since it passes through $(a\,b)$ we have

$$b - Y = -\frac{Y}{2m}(a - X),$$

or, $XY - (a - 2m)Y - 2mb = 0$, (1)

also, $Y^2 = 4mX$ . . . . . . (2)

The elimination of X gives $Y^3 - 4m(a - 2m)Y - 8m^2 b = 0$ (3), an equation whose roots would give the three required ordinates.

To avoid this equation we shall construct the locus of (1), which is the equation to an equilateral hyperbola. The axis of x is one asymptote (198), and the other is parallel to the axis of y, and at a distance A C $= a - 2m$ from A; the equation to the hyperbola referred to its centre C and asymptotes is $XY = 2mb$; moreover the hyperbola cuts the axis of y in the point D, where $AD = \frac{2mb}{2m - a}$; hence this hyperbola (the dotted curve in the figure) may be constructed.

We have drawn the figure, so that there shall be only one intersection of the curves, and hence only one normal is drawn from Q. If the curves touched, as at E, there would be two normals; and if the hyperbola cut the parabola in the lower branch, there would be three normals drawn from Q. These cases correspond respectively to the equation (3), having one real root; three real roots of which two are equal; and, lastly, three real and unequal roots.

337. We must particularly observe that, in the construction of loci, those are to be selected which admit of the easiest description, and of all curves the circle is to be preferred; hence, in the present case, we must look carefully to see if it is possible, by any combination of (1) and (2), to obtain the equation to the circle; for by **333** this will pass through the required normal points.

Multiply (1) by Y, then

$$XY^2 - (a - 2m)\,Y^2 - 2mb\,Y = 0,$$

$$\text{or,}\quad X \,.\, 4mX - (a - 2m)\,4mX - 2mb\,Y = 0\,;$$

$$\therefore\ X^2 - (a - 2m)\,X - \frac{b}{2}\,Y = 0,$$

$$\text{and } Y^2 - 4mX = 0, \text{ from (2)}$$

$$\therefore \text{ by addition}\quad Y^2 + X^2 - (a + 2m)\,X - \frac{b}{2}\,Y = 0$$

which is the equation to a circle, the co-ordinates of whose centre are $\frac{a}{2} + m$ and $\frac{b}{4}$, and whose radius is $\sqrt{\left\{\left(\frac{a}{2} + m\right)^2 + \frac{b^2}{16}\right\}}$. Although this circle passes through the vertex of the parabola, yet that point is not one of the required intersections, but merely arises from the multiplication of (1) by Y.

If the parabola and circle be drawn, the latter in various situations according to the position of Q, we shall see, as before, that there will be one, two or three intersections: such practice will be found very useful.

The problem of drawing a normal to an ellipse is of the same nature, only in this case there may be four intersections.

338. The intersection of curves has been employed in the last articles to avoid the resolution of equations resulting from elimination, but the principle may be extended, so as to render curves generally subservient to the solution of equations; for as two equations combined produce one whose roots give the intersection of their loci, so that one may in its turn be separated into two, whose loci can be drawn, and their intersection will determine the roots of the one.

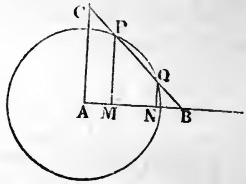
This method, known by the name of "the construction of equations," was much used by mathematicians before the present methods of approximation were invented; it is even now useful to a certain extent, and therefore we proceed to explain it.

Let there be two equations: $y + x = a$ (1), $y^2 + x^2 = b^2$ (2), by elimination we find

$$x^2 - a\,x + \frac{a^2 - b^2}{2} = 0 \quad (3).$$

We already know that the roots of (3) are the abscissas to the points of intersection of the loci (1) (2); but, conversely, it is manifest that the roots of (3) can be determined by drawing the loci of (1) and (2), and measuring the abscissas of intersection.

Hence if it be required to exhibit geometrically the roots of (3), let it be decomposed into the two equations (1) and (2), and let C P Q B be the locus of (1), and the circle E P Q of (2), having the same origin and axes: draw the ordinates M P, N Q, then A M and A N are the roots of (3).



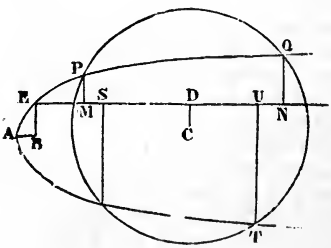
The method consists in parting any given equation into two others, and then drawing the loci of those two; and as it is obvious that there are a great many equations which, when combined together, may produce the given equation, so we may construct a great many loci, whose intersections will give the required roots: thus, in the above case, the equation (3) may be resolved into the two $x^2 = a\,y$, and $a\,y - a\,x + \dfrac{a^2 - b^2}{2} = 0$, and the corresponding parabola and straight line being drawn, their intersections will give the roots of (3).

In general the roots of an equation can be found by the intersection of any two species of curves whose indices, multiplied together, are equal to the index of the equation: thus, a straight line and a curve of the third order will give the solution of an equation of the third order; and any two conic sections, except two circles, will give the roots of an equation of the fourth order.

339. As equations of the third and fourth order are of frequent recurrence in mathematical researches, we proceed to the solution of the complete equation of the fourth order,

$$y^4 + p\,y^3 + q\,y^2 + r\,y + s = 0.$$

Here the circle and parabola, as curves of easy description, ought to be chosen, and assuming the equation to the parabola a slight artifice will give us that to the circle.



Let $y + \dfrac{p}{2}\,y = x$ (1);

$$\therefore\ y^4 + p\,y^3 + \frac{p^2}{4}\,y^2 = x^2$$

$$\text{but}\quad y^4 + p\,y^3 + q\,y^2 + r\,y + s = 0,$$

$$\therefore \text{ by subtraction } \left(q - \frac{p^2}{4}\right) y^2 + x^2 + r\,y + s = 0,$$

$$\text{or from (1), } \left(q - \frac{p^2}{4}\right)\left(x - \frac{p}{2}\,y\right) + x^2 + r\,y + s = 0;$$

$$\text{or } x^2 + \left(q - \frac{p^2}{4}\right) x + \left(r - \frac{p\,q}{2} + \frac{p^3}{8}\right) y + s = 0,$$

$$\text{and from (1), } y^2 + \frac{p}{2}\,y - x \qquad = 0,$$

$$\therefore\ y^2 + x^2 + \left(r + \frac{p^3}{8} + \frac{p}{2} - \frac{p\,q}{2}\right) y + \left(q - 1 - \frac{p^2}{4}\right) x + s = 0 \quad (2).$$

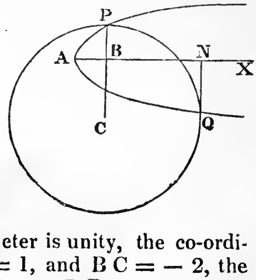
The locus of (1) is the parabola A E Q, the origin being at E $\left(\text{BE} = \frac{p}{4}\right)$, and the co-ordinates rectangular. The locus of (2) is the circle Q P R; the co-ordinates E D, D C of the centre, and the radius are readily determined from (2). The roots of the equation are drawn as if two, P M, Q N were positive, and other two R S, T U were negative. If the circle touch the parabola, two roots are equal; the cases of three or four equal roots can only be discussed by the principles of osculation, but as two roots are sufficient to depress the equation to one of the second order, we need not here consider those cases. If there be only two intersections, two roots are impossible; and if there be no intersection, all four roots are impossible.

340. In practice the operation is shortened by first taking away the second term of the equation; for example, to construct the roots of the equation

$$x^4 + 8\,x^3 + 23\,x^2 + 32\,x + 16 = 0. \qquad (1).$$

Let $x = y - 2$, and the reduced equation is

$$y^4 - y^2 + 4\,y - 4 = 0. \qquad (2).$$



Let $y^2 = x$ (3);

$\therefore$ by substitution $x^2 - x + 4\,y - 4 = 0$,

$\therefore$ by addition $y^2 + x^2 + 4\,y - 2\,x - 4 = 0$,

or, $(y + 2)^2 + (x - 1)^2 = 9$ (4).

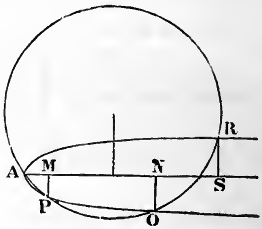
Let P A Q be the parabola (3), whose parameter is unity, the co-ordinates of the centre C of the circle (4) are A B = 1, and B C = − 2, the radius = 3. Describing this circle, the ordinates B P and Q N are the possible roots of (2); measuring these values we shall find P B = 1, and Q N = − 2; hence the possible roots of (2) are 1 and − 2, and therefore those of (1) are − 1 and − 4.

341. The construction of equations of the third order is involved in that

of the fourth order. Take away the second term, if necessary, multiply the resulting equation $Y = 0$ by y, and then proceed precisely as in the last article. The circle will always pass through the vertex of the parabola, bu this intersection gives the root $y = 0$, introduced by multiplication, and has therefore nothing to do with the roots of the given equation. This circumstance of the circle passing through the vertex of the parabola, is singularly convenient, as it entirely saves the trouble of calculating the radius to decimal places, which is often necessary in the preceding cases.

Ex. 1. $x^3 - 6x^2 - x + 6 = 0$. Let $x = y + 2$;

$$\therefore y^3 - 13y - 12 = 0,$$
$$\text{or, } y^4 - 13y^2 - 12y = 0.$$
$$\text{Let } y^2 - x = 0 \quad - \ (1)$$
$$\therefore x^2 - 13x - 12y = 0,$$
$$\therefore y^2 - 12y + x^2 - 12x = 0,$$
$$\text{or, } (y-6)^2 + (x-6)^2 = 72 \quad (2).$$



The three roots of y, as given by the figure, are 4, -1 and -3; hence the values of x are 6, -1 and -1.

Ex. 2. $4y^3 + 6y - 5 = 0$. There is one possible root nearly $= \dfrac{1}{\sqrt{2}}$.

Ex. 3. $4y^3 - 3y + 1 = 0$.

There never can be any difficulty in constructing the loci of these equations; having once drawn a parabola, whose parameter is unity, with tolerable exactness, it will serve for the construction of any number of such equations.

As another example, we take the following question.

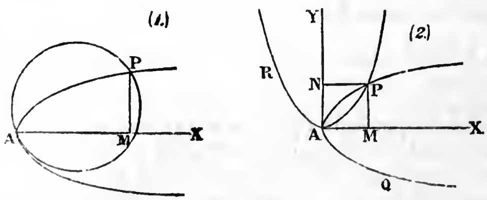
342. To find two mean proportionals between two given lines a and b,

Let y and x be the required lines;

$$\text{then } a : y :: y : x, \quad \therefore y^2 = a x \qquad (1),$$
$$y : x :: x : b, \quad \therefore x^2 = b y \qquad (2),$$
$$\therefore y^4 = a^2 x^2 = a^2 b y, \text{ or } y^3 - a^2 b = 0;$$

but by addition of (1) and (2), $\quad y^2 - by + x^2 - ax = 0,$

$$\text{or, } \left(y - \frac{b}{2}\right)^2 + \left(x - \frac{a}{2}\right)^2 = \frac{a^2 + b^2}{4} \qquad (3).$$



Let P A Q be the parabola (1), then the intersection of the circle (3) will give M P and A M, the two mean proportionals required.

The other roots of the equation $y^3 - a^2 b = 0$ are impossible.

This problem was one of those so much celebrated by the ancient mathematicians. Menechme, of the school of Plato, was the first who gave a solution of it: his method being particularly ingenious, as well as being the first instance known of the application of geometrical loci to plain problems, is well worth insertion.

With a parameter a, draw the parabola P A Q (fig. 2), and on A Y perpendicular to A X describe the parabola P A R with parameter b.

Then the rectangle a, A M or a, N P is equal to the square on M P;

$\therefore$ a, M P and N P are in continued proportion.

Again, the rectangle b, A N or b, M P is equal to the square on N P;

$\therefore$ M P, N P, and b, are in continued proportion;

hence we have at the same time the two proportions

$$a : \text{M P} :: \text{M P} : \text{N P and M P} : \text{N P} :: \text{N P} : b;$$

$\therefore$ a, M P, N P, and b, are in continued proportion.

Menechme also gave a second solution depending on the intersection of a parabola and hyperbola.

343. To find a cube which shall be double of a given cube.

Let a be a side of the given cube, then the equation to be solved is

$$y^3 = 2\,a^3, \text{ or } y^4 - 2\,a^3\,y = 0,$$

Let $y^2 = a\,x$ (1), $\therefore$ $a^2\,x^2 - 2\,a^3\,y = 0$, or, $x^2 - 2\,a\,y = 0$;

$\therefore$ by addition, $y^2 - 2\,a\,y + x^2 - a\,x = 0$ (2);

The loci of (1) and (2) being drawn, the ordinate P M of their intersection is the side of the required cube.

This problem, like the former, occupied the attention of the early geometricians; they soon discovered that its solution is involved in the preceding one; for if $b = 2\,a$, the resulting equations are the same.

In this manner a cube may be found which shall be m times greater than a given cube.

344. We may thus find any number of mean proportionals between two given quantities a and b.

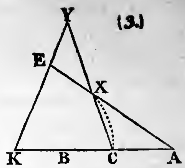
For if y be the first of the mean proportionals, they will form the following progression:

$$a,\ y,\ \frac{y^2}{a},\ \frac{y^3}{a^2},\ \frac{y^4}{a^3},\ \text{\&c.}$$

Let there be four mean proportionals, then the sixth term of the progression being b we have $\frac{y^5}{a^4} = b$, or $y^5 - a^4\,b = 0$.

Describe the parabola whose equation is $y^2 = a\,x$, and then draw the locus of the equation $y\,x^2 - a^3\,b = 0$. The last curve consists of an hyperbolic branch in each of the angles Y A X, Y A x, and therefore the ordinate corresponding to the real root is readily found.

345. Newton constructed equations by means of the conchoid of Nicomedes: he justly observes that those curves are to be preferred whose mechanical description is the easiest; and he adds, that of all curves, the conchoid next to the circle is, in this respect, the most simple. See the instrument in (312). The following is one of the many examples given in the Universal Arithmetic.



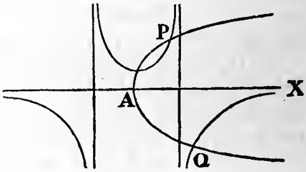
Let the equation be $x^3 + q\,x + r = 0$, draw a straight line K A, of any length n. In K A take K B $= \frac{q}{n}$, and bisect B A in C; with centre k and radius K C describe a circle, in which inscribe the straight line C X $= \frac{r}{n^2}$; join A X, and between the lines C X and A X produced, inscribe E Y equal to C A, so that, when produced, it passes through the point K.

A geometrical proof follows to show that, from this construction, the equation for the length of X Y is $x^3 + q\,x + r = 0$, so that X Y is a root of the equation.

The conchoid is employed to insert the line E Y between C X and C A.

Let K be the pole, A X E the base, and C A the modulus; then the common description of the curve determines the point Y on the line C X Y, such that E Y = C A.

346. With regard to the higher equations, there is not much advantage in constructions, since it is extremely difficult to draw the curves with sufficient exactness. The method, however, is so far useful as enabling us to detect the number of impossible roots in any equation, as we can generally trace the curves with sufficient accuracy to determine the number of intersections, though not the exact points of intersection.



Ex. $y^5 - 3\,y + 1 = 0$.

Let $y^2 = x$ . . . . . (1),

$\therefore\ y\,x^2 - 3\,y + 1 = 0$, (2)

or $y = \dfrac{1}{3 - x^2}$;

the locus of (1) is a parabola P A Q, that of (2) is a curve of the third order, and there are three intersections; and, therefore, three possible roots, two positive, and one negative.

347. There is some uncertainty in the employment of curves in finding roots; we stated in (332), that real roots may correspond to imaginary intersections; so, on the contrary, imaginary intersections, or what is the same, the absence of intersections, does not always prove the absence of real roots; for example, if to prove the equation $x^4 + 15\,x + 14 = 0$ we assume $y^2 = x^3$ (1), and therefore $x\,y^2 + 15\,x + 14 = 0$ (2), the loci of (1) and (2) will not intersect, but yet two roots are possible. The error was in choosing a curve (1), which proceeds only in the positive direction, when from the form of the equation it is apparent that there are negative roots. Taking the circle and common parabola for the loci, as in (340), we shall find the roots to be $-$ 1 and $-$ 2. Hence, in general, to ascertain real roots it will be advisable to try more than two curves.

CHAPTER XIV.

TRANSCENDENTAL CURVES.

348. It was stated in art. (23), that those equations which cannot be put into a finite and rational algebraical form with respect to the variables, are called Transcendental; of this nature are the equations $y = \sin. x$ and $y = a^x$. In Chapter XII. we have obtained the equations to curves, generally from some distinct Geometrical property of those curves; but there are many curves whose equations thus obtained cannot be expressed in the ordinary language of algebra; that is, the equation resulting from the description or generation of the curve is dependent upon Trigonometrical or Logarithmical quantities; these curves, from the nature of their equations, are called Transcendental.

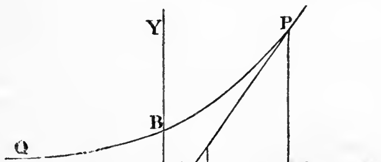
We shall here investigate the equations and the forms of the most celebrated of these curves, and mention a few of the remarkable properties belonging to them, although they can be only fully investigated by the higher calculus.

349. In this class will be found some curves, as the Cardioide, whose equations may be expressed in finite algebraic terms; but these examples are only particular cases of a species of curves decidedly Transcendental, and which cannot be separated from the rest without injury to the general arrangement.

Some of the Transcendental class have been called Mechanical curves, because they can be described by the continued motion of a point; but this name as a distinction is erroneous, for it is very probable that all curves may be thus described by a proper adjustment of machinery.

THE LOGARITHMIC CURVE.

350. The curve Q B P, of which the abscissa A M is the logarithm of the corresponding ordinate M P, is called the Logarithmic curve.



Let A M $= x$, M P $= y$, then $x = \log. y$, that is, if a be the base of the system of logarithms, $y = a^x$.

To examine the course of the curve we find when $x = 0$, $y = a^0 = 1$; as x increases from 0 to ∞, y increases from 1 to ∞; as $-x$ increases to ∞, y decreases from 1 to 0. In A Y take A B $=$ the linear unit, then the curve proceeding from B to the right of A B, recedes from the axis of x, and on the left continually approaches that axis, which is therefore an asymptote.

This curve was invented by James Gregory; Huyghens discovered that if P T be a tangent meeting A X in T, M T is constant and equal to the modulus $\left(\frac{1}{\log. a}\right)$ of the system of logarithms. Also that the whole area M P Q x extending infinitely towards x is finite, and equal to twice the triangle P M T, and that the solid described by the revolution of the same area about X x is equal to $1\frac{1}{2}$ times the cone, by the revolution of P T M about X x.

That such areas and solids are finite is curious, but not unintelligible to those who are accustomed to the summation of decreasing infinite series.

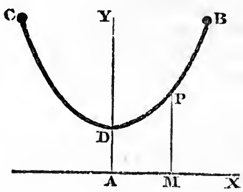
If the equation be $y = a^{-x}$, the curve is the same, but placed in the opposite direction with regard to the axis of y.

351. The equation to the curve called the Catenary, formed by suspending a chain, or string, between two points B and C, is

$$y = \tfrac{1}{2}(e^x + e^{-x})$$

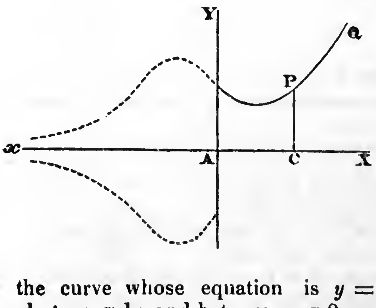
where A M $= x$, M P $= y$,

and A D $= 1$.



This equation cannot be obtained by the ordinary algebraical analysis; but it is evident that the curve may be traced from this equation, by adding together the ordinates of two logarithmic curves corresponding to the equations $y = e^x$ and $y = e^{-x}$.

352. Trace the locus of the equation $y = a^{\frac{1}{x}}$. (Fig. 1.)



353. To trace the curve whose equation is $y = x^x$. Let $x = 0$ $\therefore$ $y = 1$; let $x = 1$ $\therefore$ $y = 1$; and between $x = 0$ and $x = 1$, we have y less than 1; also x increases from 1 to ∞, y increases to infinity.

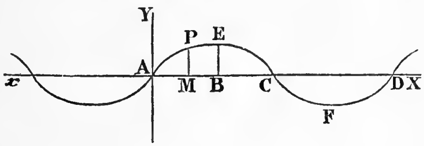
hence if A B = 1 (fig. 2,) and A C = 1, we have the branch B P Q corresponding to positive values of x.

Let x be negative $\therefore y = (-x)^{-x} = \dfrac{1}{(-x)^x}$; now if we take for x three consecutive values, as 2, $2\frac{1}{2}$, 3, it is evident that y will be positive, impossible, or negative; hence the curve must consist of a series of isolated points above and below the axis A x.

For further information on this subject see a very interesting memoir by M. Vincent, in the fifteenth volume of the "Annales des Math." M. Vincent calls such discontinuous branches by the name "Branches Ponctuées;" and he also shows, that in the common logarithmic curve there must be a similar branch below the axis of x, corresponding to fractional values of x with even denominators.

THE CURVE OF SINES.

354. The curve A P E C, of which the ordinates M P, B E are the sines of the corresponding abscissas A M, A B, is called the Curve of Sines.



Let A M $= x$, M P $= y$, then the equation is $y = \sin. x$,

$$y = r \sin. \frac{x}{r}.$$

	1	2	3	4	5
Values of x	0	$\frac{\pi r}{2}$	πr	$\frac{3 \pi r}{2}$	$2 \pi r$
Values of y	0	r	0	$- r$	0

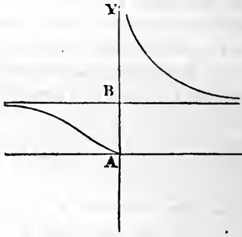
Take A B $= \frac{\pi r}{2}$, A C $= \pi r$, A D $= 2 \pi r$; then from (1) the curve cuts the axis at A; from (2), if B E $= r$, the curve passes through E, and this is the highest point of its course, because between (1) and (2) y increases, and between (2) and (3) y decreases; the curve cuts the axis again in C; from C, y increases negatively until it equals $- r$, and then decreases to 0, so that we have a second branch C F D equal and similar to the first. Beyond D the values of y recur, and the curve continues the same course *ad infinitum;* also since $\sin. (- x) = - \sin. x$ there is a similar branch to the left of A.

This curve may be supposed to arise from the development of circular arcs into a straight line $X x$, ordinates being drawn corresponding to the sines of these circular arcs.

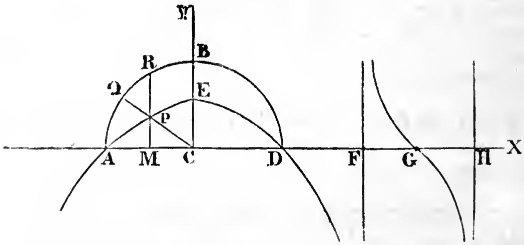
In a similar manner the curve of cosines, of versed sines, of tangents, &c., may readily be investigated.

If the ordinates of the curve of sines be increased or diminished in a given ratio, the resulting curve ($y = m$ sin. x) is the curve formed by the simple vibration of a musical chord: hence this curve is called the Harmonic Curve.

355. The accompanying figure belongs to the curve whose equation is $y = x$ tan. x. Such curves are useful in finding the roots of an equation as x tan. $x = a$; for, supposing the curve to be described, in A Y take A B $= a$, and from B draw a line parallel to A X; then the ordinates corresponding to the points of intersection of this straight line with the curve are the values of y, that is, of x tan. x.



THE QUADRATRIX.



356. Let C be the centre of a circle A Q B D; let the ordinate M R move uniformly from A to B C, and in the same time let the radius C Q, turning round C, move from C A to C B; then the intersection P of C Q and R M traces out a curve called the Quadratrix.

Let A be the origin, A M $= x$, M P $= y$, A C $= r$, angle A C Q $= \theta$,

Then A M : A C :: A Q : A B,

$$x : r :: r\theta : \frac{\pi r}{2} \therefore \theta = \frac{\pi x}{2r}.$$

But M P $=$ M C tan. θ,

$$\therefore y = (r - x) \tan. \frac{\pi x}{2r}, \text{ which is the equation to the curve.}$$

When $x = 0$, $y = 0$; $\therefore$ the curve passes through A; as x increases from 0 to r, y increases, because the tangent increases faster than the angle; when $x = r = \text{A C}, y = \frac{0}{0}$, the real value of which found by the Differential Calculus is $\frac{2r}{\pi}$; hence if $\text{C E} = \frac{2r}{\pi}$, the curve passes through E; as x increases beyond r the tangent diminishes but is negative, and so is $r - x$; $\therefore y$ is positive and diminishes until it finally becomes 0, when $x = 2r = \text{A D}$; when x is greater than $2r$ the tangent is positive, therefore y is negative and increases; when $x = 3r$, the tangent $= \infty$; $\therefore y = -\infty$; this gives an asymptote through F. As x increases beyond $3r$ the tangent decreases but is negative; hence y is positive; when $x = 4r, y = 0$, when $x = 5r, y = -\infty$, and between $x = 4r$ and $5r$, y is negative: therefore we have the branch between the asymptotes at F and H, and proceeding onwards we should find a series of branches like the last. The part of the curve to the left of A is the same as that to the right of D.

This curve was invented most probably by a Greek mathematician of the name of Hippias, a cotemporary of Socrates; his object was to trisect an angle, or generally to divide an angle into any number of equal parts, and this would be done if the curve could be accurately drawn; thus to trisect an angle A C Q, draw the quadratrix and the ordinate M P, trisect the line A M in the points N and O, draw the ordinates N S, O T to the quadratrix. Then from the equation $\theta = \frac{\pi x}{2r}$, we shall see that C S and C T trisect the angle A C Q.

This curve was afterwards employed by Dinostratus to find the area or quadrature of the circle, and hence its name: supposing the point E to be determined by mechanical description we have the value of π given by the equation $\text{C E} = \frac{2r}{\pi}$, and therefore the ratio of the circumference to the diameter of the circle would be known.

There is another quadratrix, that of Tschirnhausen, which is generated by drawing two lines through Q and M parallel respectively to A C and B C, and finding the locus of their intersection; its equation will be

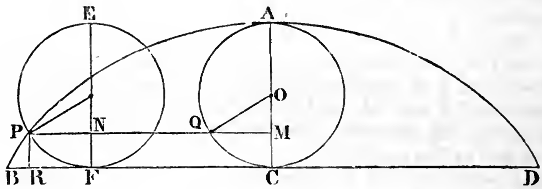
$$y = r \cos. \left(\frac{\pi}{2} - \theta\right) = r \sin. \theta = r \sin. \frac{\pi x}{2r}.$$

THE CYCLOID

357. If a circle E P F be made to roll in a given plane upon a straight line B C D, the point in the circumference which was in contact with B at the commencement of the motion, will, in a revolution of the circle, describe a curve B P A D, which is called the cycloid.

This is the curve which a nail in the rim of a carriage-wheel describes in the air during the motion of the carriage on a level road; hence the generating circle E P F is called the wheel. The curve derives its name from two Greek words signifying "circle formed."

The line B D which the circle passes over in one revolution is called the base of the cycloid; if A Q C be the position of the generating circle in the middle of its course, A is called the vertex and A C the axis of the curve. The description of the curve shows that the line B D is equal to the circumference of the circle, and that B C is equal to half that circumference. Hence also if E P F be the position of the generating circle, and P the generating point, then every point in the circular arc P F having coincided with B F, we have the line B F $=$ the arc P F, and F C $=$ the arc E P or A Q ;



Draw P N Q M parallel to the base B D.

Let A be the origin of rectangular axes,

A C the axis of x, and O the centre of the circle A Q C.

Let A M $= x$, A O $= a$,

M P $= y$, angle A O Q $= \theta$;

then by the similarity of position of the two circles, we have

$$\mathrm{PN} = \mathrm{QM}, \text{ and } \mathrm{PQ} = \mathrm{NM};$$

$$\therefore \mathrm{MP} = \mathrm{PQ} + \mathrm{QM} = \mathrm{NM} + \mathrm{QM} = \mathrm{FC} + \mathrm{QM} = \text{arc } \mathrm{AQ} + \mathrm{QM}$$

$$\text{that is, } y = a\theta + a \sin.\theta = a(\theta + \sin.\theta) \quad (1)$$

$$x = a - a\cos.\theta = a \text{ vers.}\theta \quad (2)$$

The equation between y and x is found by eliminating θ between (1) and (2)

$$\cos.\theta = \frac{a - x}{a} \quad \therefore \sin.\theta = \frac{\sqrt{2ax - x^2}}{a}$$

$$\text{and } y = a\theta + a\sin.\theta$$

$$= a \cos.^{-1}\left(\frac{a - x}{a}\right) + \sqrt{2ax - x^2}$$

But we can obtain an equation between x and y from (1) alone; that is, from the equation, M P $=$ arc A Q $+$ Q M.

For arc A Q $=$ a circular arc whose radius is a and versed sine x

$$= a\left\{\text{a circular arc whose radius is unity and vers. sin. } \frac{x}{a}\right\}$$

$$= a \text{ vers.}^{-1}\frac{x}{a}$$

$$\therefore y = a \text{ vers.}^{-1}\frac{x}{a} + \sqrt{2ax - x^2}$$

If the origin is at B, B R $= x$ and R P $= y$, the equations are

$$x = a\theta - a \sin. \theta$$

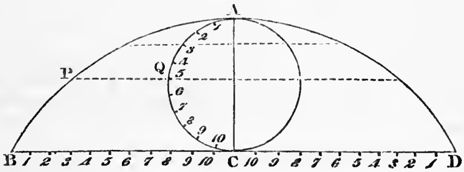
$$y = a - a \cos. \theta.$$

We shall not stop to discuss these equations, as the mechanical description of the curve sufficiently indicates its form.

The cycloid, if not first imagined by Galileo, was first examined by him; and it is remarkable for having occupied the attention of the most eminent mathematicians of the seventeenth century.

Of the many properties of this curve the most curious are that the whole area is three times that of the generating circle, that the arc A P is double of the chord of A Q, and that the tangent at P is parallel to the same chord. Also that if the figure be inverted, a body will fall from any point P on the curve to the lowest point A in the same time; and if a body falls from one point to another point, not in the same vertical line, its path of quickest descent is not the straight line joining the two points, but the arc of a cycloid, the concavity or hollow side being placed upwards.

358. Given the base of a cycloid to trace the curve.

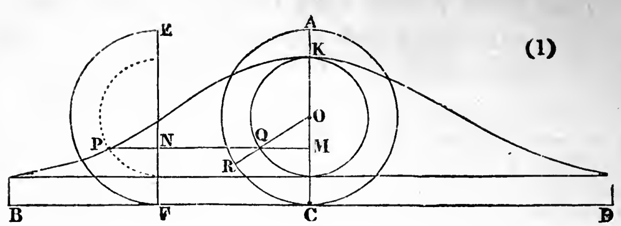


Let the base B D be divided into twenty-two equal parts, and let them be numbered from B and D towards the middle point C; from C draw the perpendicular line C A equal to 7 of these parts; and on A C describe a circle A Q C. Along the circumference mark off the same number of equal parts, either by measurement or by applying the line B C to the circle C A. In the figure the point Q is supposed to coincide with the end of the fifth division from the top.

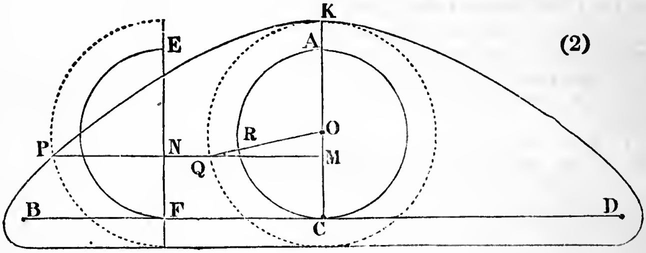
Then the arc C Q being equal to the length C 5 measured on the base, if P Q be drawn parallel to the base, and equal to the remainder of the base, that is, to B 5 or A Q, it is evident that P is a point in the cycloid, and thus any number of points may be found.

The ratio of the circumference to the diameter of a circle is generally taken as in this case to be as 22 to 7.

359. Instead of the point P being on the circumference of the circle, it may be anywhere in the plane of that circle, either within or without the circumference. In the former case the curve is called the prolate cycloid or trochoid, (fig. 1,) in the latter case the curtate or shortened cycloid, (fig. 2.)



(1)



(2)

B D is the base on which the generating circle A R C rolls, O the centre of the generating circle, P the describing point when that circle is at F. Draw P N Q M parallel to the base.

Let A be the origin of rectangular axes,

A M $= x$, M P $= y$, A O $= a$, K O $= m\,a$, angle A O R $= \theta$, then

M P = M N + P N = M N + Q M = F C + Q M = arc A R + Q M

and A M = A O + O M ;

$$\therefore\ y = a\,\theta + m\,a \sin.\,\theta$$

$$\text{and}\quad x = a \text{ vers. } \theta.$$

These are the equations to the prôlate, curtate, or common cycloid, according as m is less than, greater than, or equal to, unity.

If the vertex K of the curve be the origin of co-ordinates in figs. (1) and (2,) we have K O $= a$, and A O $= m a$: also M P = F C + Q M = arc A R + Q M

$$\therefore\ y = m\,a\,\theta + a \sin.\,\theta$$

$$= m \text{ vers.}^{-1}\frac{x}{a} + \sqrt{2\,a\,x - x^2}$$

The curve whose equations are $y = a\,\theta$, and $x = a$ vers. θ is called the companion to the cycloid.

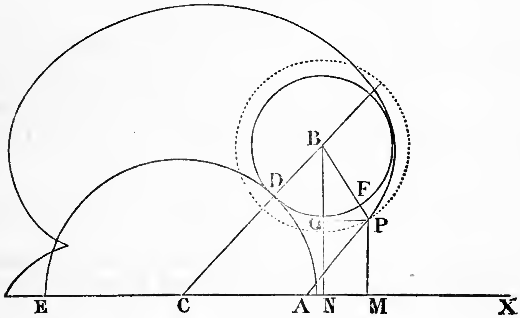
360. The class of cycloids may be much extended by supposing the base on which the circle rolls to be no longer a straight line, but itself a curve: thus let the base be a circle, and let another circle roll on the circumference of the former; then a point either within or without the

circumference of the rolling circle will describe a curve called the epitrochoid; but if the describing point is on the circumference, it is called the epicycloid.

If the revolving circle roll on the inner or concave side of the base, the curve described by a point within or without the revolving circle is called the hypotrochoid; and when the point is on the circumference it is called the hypocycloid.

To find the equation to the epitrochoid.

Let C be the centre of its base E D, and B the centre of the revolving circle D F in one of its positions: C A M the straight line passing through the centres of both circles at the commencement of the motion; that is, when the generating point P is nearest to C or at A.



Let C A be the axis of x,

$C M = x$, $M P = y$,

$C D = a$, $D B = b$,

$B P = m b$, angle $A C B = \theta$

Draw B N parallel to M P, and P Q parallel to E M. Then, since every point in D F has coincided with the base A D, we have $D F = a\theta$, and

angle $D B F = \dfrac{a\theta}{b}$; also angle $F B Q =$ angle $F B D -$ angle $Q B D$

$$= \frac{a\theta}{b} - \left(\frac{\pi}{2} - \theta\right) = \frac{a+b}{b}\theta - \frac{\pi}{2}$$

Now $C M = C N + N M = C B \cos. B C N + P B \sin. P B Q$

$$= (a+b)\cos.\theta + m b \sin.\left(\frac{a+b}{b}\theta - \frac{\pi}{2}\right)$$

And $M P = B N - B Q = (a+b)\sin.\theta - m b \cos.\left(\frac{a+b}{b}\theta - \frac{\pi}{2}\right)$

$$\left.\begin{aligned} \text{or } x &= (a+b)\cos.\theta - m b \cos.\frac{a+b}{b} \\ \text{and } y &= (a+b)\sin.\theta - m b \sin.\frac{a+b}{b}\theta \end{aligned}\right\} \quad (1)$$

The equations to the epicycloid are found by putting b for $m\,b$ in (1.)

$$\left.\begin{aligned} \therefore\ x &= (a+b)\cos.\theta - b\cos.\frac{a+b}{b}\theta \\ \text{and } y &= (a+b)\sin.\theta - b\sin.\frac{a+b}{b}\theta \end{aligned}\right\}\ (2)$$

The equations to the hypotrochoid may be obtained in the same manner as the system (1), or more simply by putting $-b$ for b in the equations (1.)

$$\left.\begin{aligned} \therefore\ x &= (a-b)\cos.\theta + m\,b\cos.\frac{a-b}{b}\theta \\ \text{and } y &= (a-b)\sin.\theta - m\,b\sin.\frac{a-b}{b}\theta \end{aligned}\right\}\ (3)$$

The equations to the hypocycloid are found by putting $-b$ for both b and $m\,b$ in system (1.)

$$\left.\begin{aligned} \therefore\ x &= (a-b)\cos.\theta + b\cos.\frac{a-b}{b}\theta \\ \text{and } y &= (a-b)\sin.\theta - b\sin.\frac{a-b}{b}\theta \end{aligned}\right\}\ (4)$$

We have comprehended all the systems in (1), but each of them might be obtained from their respective figures.

361. The elimination of the trigonometrical quantities is possible, and gives finite algebraic equations whenever a and b are in the proportion of two integral numbers. For then $\cos.\theta$, $\cos.\frac{a+b}{b}\theta$, $\sin.\theta$, &c., can be expressed by trigonometrical formulas, in terms of $\cos.\phi$ and $\sin.\phi$, where ϕ is a common submultiple of θ and $\frac{a+b}{b}\theta$; and then $\cos.\phi$ and $\sin.\phi$ may be expressed in terms of x and y. Also since the resulting equation in $x\,y$ is finite, the curve does not make an infinite series of convolutions, but the wheel or revolving circle, after a certain number of revolutions, is found, having the generating point exactly in the same position as at first, and thence describing the same curve line over again.

For example, let $a = b$, the equations to the epicycloid become

$$x = a\,(2\cos.\theta - \cos.2\,\theta)$$
$$y = a\,(2\sin.\theta - \sin.2\,\theta)$$
$$\therefore\ x = a\,(2\cos.\theta - 2\,(\cos.\theta)^2 + 1)$$
$$\text{and } y = 2\,a\sin.\theta\,(1 - \cos.\theta).$$

From the first of these equations we find $\cos.\theta$, and then from the second we have $\sin.\theta$, adding the values of $(\cos.\theta)^2$ and $(\sin.\theta)^2$ together and reducing, we have

$$(y^2 + x^2 - 3\,a)^2 = 4\,a^4\left(3 - \frac{2\,x}{a}\right)$$
$$\text{or } \{x^2 + y^2 - a^2\}^2 - 4\,a^2\,\{(x-a)^2 + y^2\} = 0.$$

This curve, from its heart-like shape, is called the cardioide.

Let A be the origin; that is, for x put $x + a$ in the last algebraical equation, and then by transformation into polar co-ordinates, the equation to the cardioide becomes

$$r = 2a(1 - \cos.\phi).$$

362. If $b = \frac{a}{2}$ the equations (4) to the hypocycloid become

$$x = a \cos.\theta$$
$$\text{and } y = 0;$$

and the hypocycloid has degenerated into the diameter of the circle A C E.

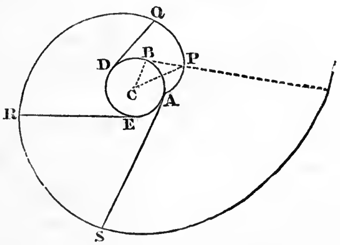
In the same case the equations to the hypotrochoid become

$$x = \frac{a}{2}(1 + m)\cos.\theta$$

$$y = \frac{a}{2}(1 - m)\sin.\theta;$$

which by the elimination of θ give the equation to an ellipse, whose axes are $a(1 + m)$ and $a(1 - m)$.

363. If a thread coinciding with a circular axis be unwound from the circle, the extremity of the thread will trace out a curve called the involute of the circle.



Thus suppose a thread fixed round the circle A B C D; then if it be unwound from A, the extremity in the hand will trace out the curve A P Q R S; the lines B P, D Q, C R, A S, which are particular positions of the thread, are also tangents to the circle, and each of them is equal to the length of the corresponding circular arc measured from A.

The curve makes an infinite number of revolutions, the successive branches being separated by a distance equal to the circumference of the circle.

To find the equation to the involute.

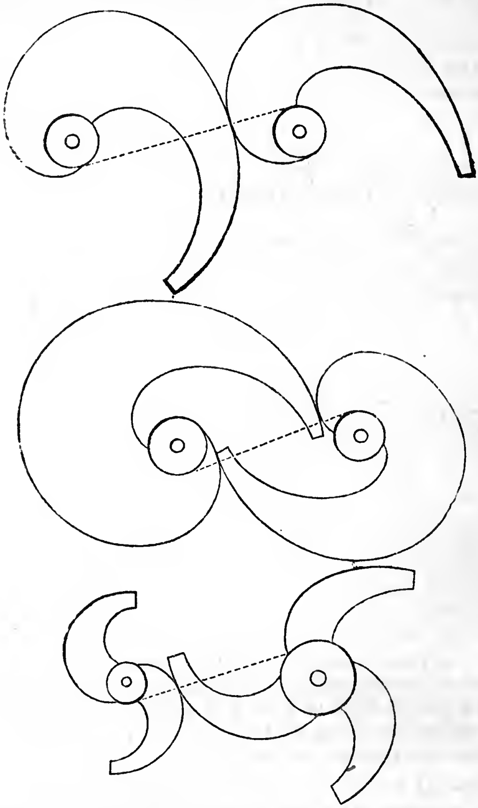
Let C A $= a$, C P $= r$, and angle A C P $= \theta$; then from the triangle B C P, we have B C $=$ P C cos. P C B, or angle P C B $= \cos.^{-1}\frac{a}{r}$;

$$\therefore \text{B P} = \text{B A} = a\left(\cos.^{-1}\frac{a}{r} + \theta\right)$$

$$\text{or } \sqrt{(r^2 - a^2)} = a\left(\cos.^{-1}\frac{a}{r} + \theta\right)$$

$$\therefore \theta = \frac{\sqrt{\{r^2 - a^2\}}}{a} - \cos.^{-1}\frac{a}{r}.$$

The involute of the circle is usefully employed in toothed wheels ; for there is less waste of power in passing from one tooth to another when they are of this form than in any other case.



In the figures (2) and (3) we have examples of two equal wheels which have each two teeth ; and by turning one wheel the other wheel will be kept in motion by means of the continual contact of the teeth. The dotted line of contact is, by the property of the involute, a common tangent to the two wheels ; this dotted line is the constant line of contact, and the force is the same in every part of a revolution.

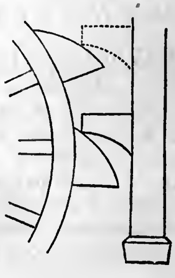


Fig. (3) is another example ; and by making the teeth smaller and more numerous we shall have toothed wheels always in contact, and therefore giving no jar or shake to the machinery.

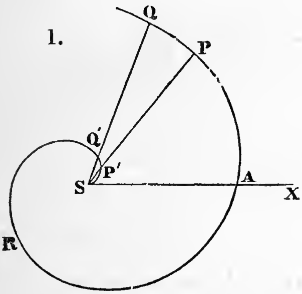
Again, in raising a piston or hammer, the involute of the circle is the best form for the teeth of the turning-wheel, as the force acts on the piston entirely in a vertical direction.

ON SPIRALS.

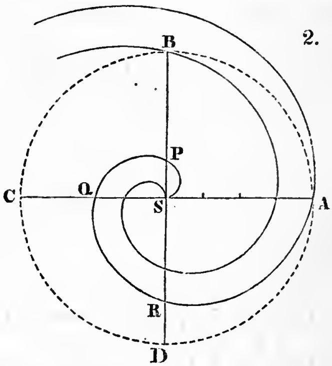
364. There is one class of transcendental curves which are called spirals, from their peculiar twisting form. They were invented by the ancient geometricians, and were much used in architectural ornaments. Of these curves, the most important as well as the most simple, is the spiral invented by the celebrated Archimedes.

This spiral is thus generated: Let a straight line S P of indefinite length move uniformly round a fixed point S, and from a fixed line S X, and let a point P move uniformly also along the line S P, starting from S, at the same time that the line S P commences its motion from S X, then the

1.



2.



point will evidently trace out a curve line S P Q R A, commencing at S, and gradually extending further from S. When the line S P has made one revolution, P will have got to a certain point A, and S P still continuing to turn as before, we shall have the curve proceeding on regularly through a series of turnings, and extending further from S.

To examine the form and properties of this curve, we must express this method of generation by means of an equation between polar co-ordinates.

$$\text{Let } SP = r,\ SA = b,\ ASP = \theta;$$

then since the increase of r and θ is uniform, we have

$$SP : SA :: \text{angle } ASP : \text{four right angles} :: \theta : 2\pi$$

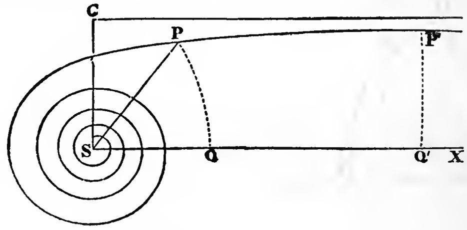
$$\therefore r = \frac{b\theta}{2\pi} = a\theta, \text{ if } a = \frac{b}{2\pi}.$$

From this equation it appears that when S P has made two revolutions or $\theta = 4\pi$, we have $r = 2b$, or the curve cuts the axis S X again at a distance 2 S A; and similarly after 3, 4, n revolutions it meets the axis S X at distances 3; 4, n times S A. Archimedes discovered that the area S P Q R A is equal to one-third of the area of the circle described with centre S and radius S A.

365. The spiral of Archimedes is sometimes used for the volutes of the capitals of columns, and in that case the following description by points is useful.

Let a circle A B C D, fig. (2), be described on the diameter C S A, and draw the diameter B D at right angles to C A; divide the radius S A into four equal parts, and in S B take S P $= \frac{1}{4}$ S A, in S C take S Q $= \frac{1}{2}$ S A, and in S D take S R $= \frac{3}{4}$ S A; then from the equation to the curve these points belong to the spiral; by subdividing the radius S A and the angles in each quadrant we may obtain other points as in the figure. In order to complete the raised part in the volute, another spiral commences from S B.

366. The spiral of Archimedes is one of a class of spirals comprehended in the general equation $r = a\theta^n$. Of this class we shall consider the cases where $n = -1$, and $n = -\frac{1}{2}$.



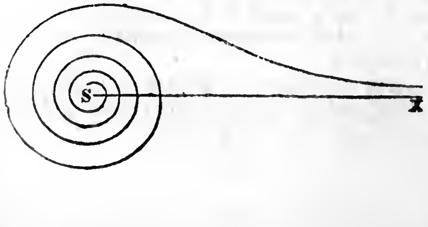
Let $n = -1 \quad \therefore r = a\theta^{-1}$

Let S be the pole, S X the axis from which the angle θ is measured, S P $= r$.

When $\theta = 0$, $r = \infty$; as θ increases, r decreases very rapidly at first and more uniformly afterwards; as θ may go on increasing ad infinitum r also may go on diminishing ad infinitum without ever actually becoming nothing: hence we have an infinite series of convolutions round S: Describe a circular arc P Q with centre S and radius S P, then P Q $= r\theta$ $= a$; and since this value of a is the same for all positions of P, we must have P Q $=$ P′ Q′ $=$ the straight line S C at an infinite distance, and therefore the curve must approach to an asymptote drawn through C parallel to S X.

This curve is called the reciprocal spiral from the form of its equation, since the variables are inversely as each other, or the hyperbolic spiral, from the similarity of the equation to that of the hyperbola referred to its asymptotes ($xy = k^2$).

367. Let $n = -\frac{1}{2}$; $\therefore r = a\,\theta^{-\frac{1}{2}}$ or $r^2\theta = a^2$. This curve, called the lituus or trumpet, is described as in the figure; proceeding from the asymptote S X, it makes an infinite series of convolutions round S.



368. If in the equation $r\theta = a$, we always deduct the constant quantity, b, we have the equation $(r - b)\theta = a$; this curve commences its course like the reciprocal spiral; but as θ increases we have $r - b$ approximating to nothing, or r approximating to b; hence the spiral, after an infinite number of convolutions, approaches to an asymptotic circle, whose centre is S, and radius b.

369. Trace the spiral whose equation is $\theta\sqrt{ar - r^2} = b$; this curve has an infinite number of small revolutions round the pole, and gradually extends outwards to meet an asymptotic circle whose radius is a.

370. The spiral whose equation is $(r - a)^2 = b^2\theta$ commences its course from a point in the circumference of the circle whose radius is a, and extends outwards round S in an infinite series of convolutions. This curve is formed by twisting the axis of the common parabola round the circumference of a circle, the curve line of the parabola forming the spiral.

371. The curve whose equation is $r = a^\theta$ is called the logarithmic spiral, for the logarithm of the radius vector is proportional to the angle θ. Examining all the values of θ from 0 to $\pm\infty$ we find that there are an infinite series of convolutions round the pole S. This curve is also called the equiangular spiral, for it is found by the principles of the higher analysis that this curve cuts the radius vector in a constant angle.

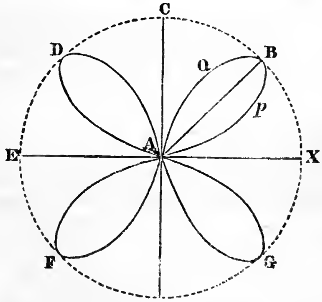
Descartes, who first imagined this curve, found also that the whole length of the curve from any point P to the pole was proportional to the radius vector at P.

372. It will often happen that the algebraical equation of a curve is much more complicated than the polar equation; the conchoid art. 312 is an example. In these cases it is advisable to transform the equation from algebraical to polar co-ordinates, and then trace the curve from the polar equation.

For example, if the equation be $(x^2 + y^2)^{\frac{3}{2}} = 2axy$, there would be much difficulty in ascertaining the form of the curve from this equation; but let $x = r\cos.\theta$ and $y = r\sin.\theta$ (61)

$$\therefore r^3 = 2ar^2\cos.\theta\sin.\theta,$$

$$\text{or } r = a\sin.2\theta.$$



Let A be the origin of polar co-ordinates; A X the axis whence θ is measured; with centre A and radius a describe a circle B C D. Then for $\theta = 0$ we have $r = 0$, as θ increases from 0 to 45°, r increases from

0 to a; hence the branch A P B. Again, as θ increases from 45° to 90°, sin. 2θ diminishes from 1 to 0; $\therefore$ r diminishes, and we trace the branch B Q A. As θ increases from 90° to 180°, sin. 2θ increases and decreases as before; hence the similar oval in the second quadrant. By following θ from 180° to 360°, we shall have the ovals in the third and fourth quadrant: and since the sine of an arc advances similarly in each quadrant of the circle, we have the four ovals similar and equal.

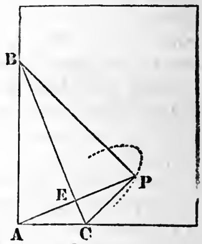
In this case we have paid no regard to the algebraical sign of r; we have considered θ to vary from 0 to 360°, which method we prefer to that of giving θ all values from 0 to 180°, and then making the sign of r to vary.

If the equation had been $(x^2 + y^2)^2 = 2a^2 x y$, we should have found two equal and similar ovals in the first and third quadrant.

The locus of the equation $r = a$ (cos. θ — sin. θ) is the same kind of figure differently situated with respect to the lines A X and A C.

The equation to the lemniscata $r^2 = a^2$ cos. 2θ art. (314), may be similarly traced.

373. In many indeterminate problems we shall find that polar co-ordinates may be very usefully employed. For example,



Let the corner of the page of a book be turned over into the position B C P, and in such a manner that the triangle B C P be constant, to find the locus of P.

Let A P $= r$, angle P A C $= \theta$, and let the area A B C $= a^2$; then since the triangles A B E, P B E are equal, we have A E $= \frac{r}{2}$, and the angle A E B a right angle $\therefore$ A E = A C cos. θ, and A E = A B cos $\left(\frac{\pi}{2} - \theta\right)$ = A B sin. θ $\therefore$ $\frac{r^2}{4} = \frac{a^2}{2}$ sin. θ cos. θ, or $r^2 = a^2$ sin. 2θ. Hence the locus is an oval A P B Q as in the last figure.

If a point be taken in the radius vector S P of a parabola so that its distance from the focus is equal to the perpendicular from the focus on the tangent, the locus of the point is the curve whose equation is $r = a$ sec. $\frac{\theta}{2}$.

PART II.

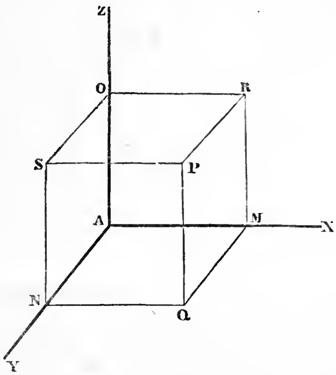
APPLICATION OF ALGEBRA TO SOLID GEOMETRY.

CHAPTER I.

INTRODUCTION.

374. In the preceding part of this Treatise lines and points have always been considered as situated in one plane, and have been referred to two lines called axes situated in that plane. Now we may readily imagine a curve line, the parts of which are not situated in one plane; also, if we consider a surface, as that of a sphere, for example, we observe immediately that all the points in such a surface cannot be in the same plane; hence the method of considering figures which has been hitherto adopted cannot be applied to such cases, and therefore we must have recourse to some more general method for investigating the properties of figures.

375. We begin by showing how the position of a point in space may be determined



Let three planes Z A X, Z A Y, and X A Y, be drawn perpendicular to each other, and let the three straight lines A X, A Y, A Z be the intersections of these planes, and A the common point of concourse.

From any point P in space draw the lines P Q, P R, and P S respectively perpendicular to the planes X A Y, Z A X, and Z A Y; then the position of the point P is completely determined when these three perpendicular lines are known.

Complete the rectangular parallelopiped A P, then P Q, P R, and P S are respectively equal to A O, A N, and A M.

These three lines A M, A N, and A O, or more commonly their equals A M, M Q, and Q P, are called the co-ordinates of P, and are denoted by the letters x, y, and z respectively.

The point A is called the origin.

The line A X is called the axis of x, the line A Y is called the axis of y, and the line A Z is called the axis of z.

The plane X A Y is called the plane of $x\,y$, the plane Z A X is called the plane of $z\,x$, and the plane Z A Y is called the plane of $z\,y$ *.

From P we have drawn three perpendicular lines, P Q, P R, and P S, on the three co-ordinate planes. The three points, Q, R, and S are called the projections of the point P on the planes of $x\,y$, $x\,z$, and $z\,y$ respectively.

The method of projections is so useful in the investigation and description of surfaces, that we proceed to give a few of the principal theorems on the subject so far as may be required in this work.

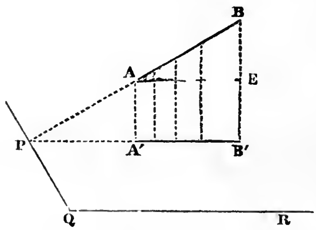
PROJECTIONS.

376. If several points be situated in a straight line, their projections on any one of the co-ordinate planes are also in a straight line.

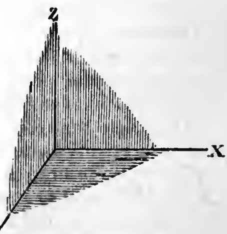
For they are all comprised in the plane passing through the given straight line, and drawn perpendicular to the co-ordinate plane; and as the intersection of any two planes is a straight line, the projections of the points must be all in one straight line.

This plane, which contains all the perpendiculars drawn from different points of the straight line, is called the projecting plane; and its intersection with the co-ordinate plane is called the projection of the straight line.

377. To find the length of the projection of a straight line upon a plane.



Let A B be the line to be projected on the plane P Q R; produce A B to meet this plane in P; draw A A′ and B B′ perpendicular to the plane, and meeting it in A′ and B′. Join A′ B′; then A′ B′ is the projection of A B.



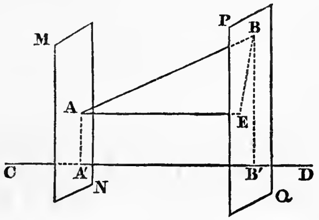
* This system of co-ordinate planes may be represented by the sides and floor of a room, the corner being the origin of the axes, the plane X Y is then represented by the floor of the room, and the two remaining planes by the two adjacent sides of the room.

Since A B and A′ B′ are in the same plane, they will meet in P. Let the angle B P B′ or the angle of the inclination of A B to the plane $= \theta$, and in the projecting plane A B′ draw A E parallel to A′ B′, then

$$A'B' = AE = AB \cos. BAE = AB \cos. \theta$$

The same proof will apply to the projection of a straight line upon another straight line, both being in the same plane.

378. To find the length of the projection of a straight line upon another straight line not in the same plane.



Let A B be the line to be projected; C D the line upon which it is to be projected. From A and B draw lines A A′ and B B′ perpendicular to C D, then A′ B′ is the projection of A B.

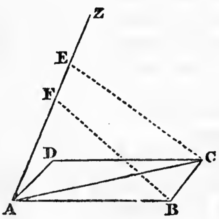
Through A and B draw planes M N and P Q perpendicular to C D. These planes contain the perpendicular lines A A′ and B B′.

From A draw A E perpendicular to the plane P Q, and therefore equal and parallel to A′ B′; join B E; then the triangle A B E having a right angle at E, we have $A'B' = AE = AB \cos. BAE$, and angle B A E is equal to the angle θ of inclination between A B and C D; hence

$$A'B' = AB \cos. \theta.$$

Also any line equal and parallel to A B has an equal projection A′ B′ on C D, and the projection of A B on any line parallel to C D is of the same length as A′ B′.

379. The projection of the diagonal of a parallelogram on any straight line is equal to the sum of the projections of the two sides upon the same straight line.



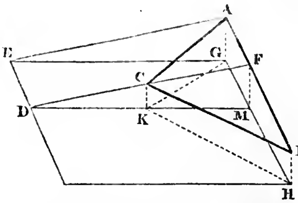
Let A B C D be a parallelogram, A Z any straight line through A inclined to the plane of the parallelogram. From C and B draw perpendiculars C E and B F upon A Z, then A E is the projection of A C upon A Z or $AE = AC \cos. CAZ$; and A F is the projection of A B upon

A Z or A F = A B cos. B A Z. Also F E is the projection of B C or A D upon A Z or F E = B C cos. D A Z

$$\text{and } AE = AF + FE;$$

hence the projection of A C = the sum of projections of A B and B C.

380. To find the projection of the area of any plane figure on a given plane E D G H.



Let A B C be a triangle inclined to the given plane E D G H at an angle θ; draw A E, C D, perpendicular to the intersection E D of these planes; then the triangle A B C and its projection G K H have equal bases A B, G H, but unequal altitudes C F, K M;

$$\therefore \text{ area } ABC : GKH :: CF : KM :: DF : DM :: 1 : \cos.\theta$$

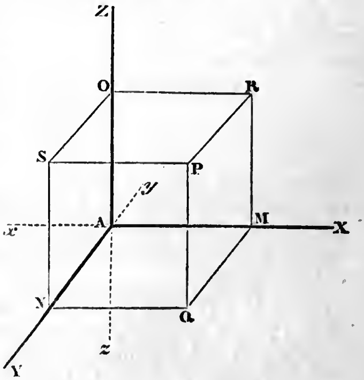
$$\text{or area } GKH = ABC \cos.\theta;$$

and this being true for any triangle, is true for any polygon, and therefore ultimately for any plane area.

CHAPTER II.

THE POINT AND STRAIGHT LINE.

381. We have already explained how the position of a point in space is determined by drawing perpendicular lines from it upon three fixed planes called the co-ordinate planes. If, then, on measuring the lengths of these three perpendicular lines or co-ordinates of P we find A M$=a$, A N$=b$, and A O$=c$, we have the position of a point P completely determined by the three equations $x=a$, $y=b$ and $z=c$; and as these are sufficient for that object, they are called the equations to the point P.



This point may also be defined as in Art. (25) by the equation $(x-a)^2+(y-b)^2+(z-c)^2=0$, since the only values that render this expression real are $x=a$, $y=b$, and $z=c$.

382. The algebraical signs of the co-ordinates x, y, and z, are determined as in Plane Geometry, by the directions of the co-ordinate lines: thus A O is positive or negative according as it is drawn from A along A Z or A z, that is, according as it is above or below the plane of $x y$: and so on for the other lines: hence we have the following values of co-ordinates for a point in each of the eight compartments into which space is divided by the co-ordinate planes.

$+x+y+z$	a point P situated in the angle	X A Y Z
$+x-y+z$	. . .	X A y Z
$-x-y+z$	. . .	x A y Z
$-x+y+z$	. . .	x A Y Z
$+x+y-z$	. . .	X A Y z
$+x-y-z$	. . .	X A y z
$-x-y-z$	. . .	x A y z
$-x+y-z$	. . .	x A Y z

383. A point also may be situated in one of the co-ordinate planes, in which case the co-ordinate perpendicular to that plane must $= 0$; thus, if the point be in the plane of $x y$, its distance z from this plane must $= 0$: hence the equations to the point in the plane of $x y$ are

$$x = a, y = b, z = 0$$
$$\text{or } (x-a)^2 + (y-b)^2 + z^2 = 0.$$

If the point be in the plane of $x z$, the equations are

$$x = a, y = 0, z = c$$

And if the point be in the plane of $y z$

$$x = 0, y = b, z = c.$$

Also, if the point be on the axis of x, its distance from the planes $x y$ and $y z = 0$, therefore the equations to such a point are

$$x = a, y = 0, z = 0;$$

and so on for points situated on the other axes.

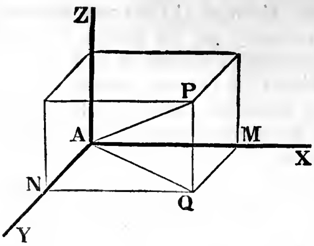
384. The points Q, R, and S, in the last figure, are the projections of the point P on the co-ordinate planes; on referring each of these points to the axes in its own plane, we have

The equations to Q on $x y$ are $x = a$, $y = b$
R on $x z$ are $x = a$, $z = c$
S on $y z$ are $y = b$, $z = c$

Hence we see that the projections of the point P on two of the co-ordinate planes being known, the projection on the third plane is necessarily given: thus, if S and R are given, draw S N and R M parallel to A Z, also N Q and M Q respectively parallel to A X and A Y, and the position of Q is known.

385. To find the distance A P of a point from the origin of co-ordinates A.

Let A X, A Y, and A Z be the rectangular axes; A M $= x$, M Q $= y$, and P Q $= z$, the co-ordinates of P.



The square on A P = the square on A Q + the square on P Q

= the squares on A M, M Q + the square on P Q

or $d^2 = x^2 + y^2 + z^2$.

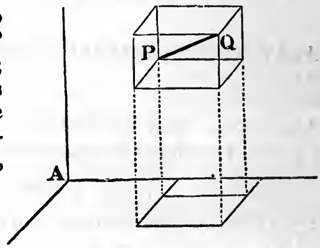
386. Let α, β, γ, be the angles which A P makes with the axis of x, y, and z, respectively;

$$\text{then } x = AM = AP \cos PAM = d \cos \alpha$$
$$y = MQ = AN = AP \cos. PAN = d \cos. \beta$$
$$z = PQ = AP \sin. PAQ = d \cos. \gamma$$
$$\therefore d^2 = x^2 + y^2 + z^2 = d^2 (\cos. \alpha)^2 + d^2 (\cos. \beta)^2 + d^2 (\cos. \gamma)^2$$
$$\therefore (\cos. \alpha)^2 + (\cos. \beta)^2 + (\cos. \gamma)^2 = 1.$$

387. Again $d^2 = x^2 + y^2 + z^2 = x\, d \cos. \alpha + y\, d \cos. \beta + z\, d \cos. \gamma$

$$\therefore d = x \cos. \alpha + y \cos. \beta + z \cos. \gamma.$$

388. To find the distance between two points, let the co-ordinates of the points P and Q be respectively $x\ y\ z$ and $x_1\ y_1\ z_1$; then the distance between these points is the diagonal of a parallelopiped, the three contiguous sides of which are the differences of the parallel co-ordinates; hence, by the last article we have



$$d^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$$

If d_1 and d_2 be the distances of the points $x_1\ y_1\ z_1$ and $x_2\ y_2\ z_2$ respectively from the origin, the above expression may be put in the form

$$d^2 = d_1^2 + d_2^2 - 2\,(x_1\, x_2 + y_1\, y_2 + z_1\, z_2).$$

THE STRAIGHT LINE.

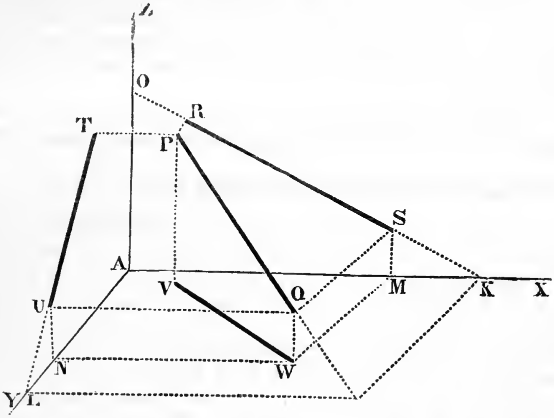
389. A straight line may be considered as the intersection of two planes, and therefore its position will be known if the situation of these planes is known; hence it may be determined by the projecting planes, and the situation of these last is fixed by their intersections with the co-ordinate planes, that is, by the projections of the line upon the co-ordinate planes; hence, the position of a straight line is geometrically fixed by knowing its projections; and it is also algebraically determined by the equations to those projections taken conjointly. Taking the axis of z as the axis of abscissas the equation to the projection on the plane $x\ z$ is of the form

$x = \alpha z + a$ (31), and the equation to the projection on the plane of $y z$ is $y = \beta z + b$.

As these two equations fix the position of the straight line in space, they are, taken together, called the equations to a straight line.

390. To illustrate this subject we shall let P Q be a portion of the straight line, R S its projection on $x z$, T U its projection on $y z$, V W its projection on $x y$;

And let $x = \alpha z + a$, be the equation to R S, and $y = \beta z + b$, be the equation to T U:



then any point Q in the projecting plane P Q R S has the same values of z and x that its projection S has, that is, the co-ordinates A M and M S are the same as N W and W Q; hence there is the same relation between them in each case; and therefore, the equation $x = \alpha z + a$ expresses not only the relation between the x and z of all the points in R S, but also of all the points in the plane P Q R S.

Similarly the equation $y = \beta z + b$ not only relates to T U, but also to all the points in the plane T U Q P.

Therefore, the system of the two equations exists for all the points in the straight line P Q, the intersection of the two projecting planes, and for this line only; hence, the equations to the straight line P Q are

$$\left.\begin{array}{l} x = \alpha z + a \\ y = \beta z + b \end{array}\right\}.$$

The elimination of z between these two equations gives

$$\frac{1}{\alpha}(x - a) = \frac{1}{\beta}(y - b)$$

$$y - b = \frac{\beta}{\alpha}(x - a)$$

and this is the relation between the co-ordinates A M and M W of the projection W of any point Q in the line P Q; and therefore, this last equation is that to the projection V W on the plane $x y$.

391. In the equations $x = \alpha z + a$ and $y = \beta z + b$, a is the distance

of the origin from the intersection of R S with A X, or $a =$ A K; similarly $b =$ A L.

Let $x = 0 \therefore z = -\frac{a}{\alpha} =$ A O $\therefore$ A K $= -\alpha$ A O, but A K = A O tan. A O K $= -$ A O tan. Z O R $\therefore$ α is tangent of the angle which R S makes with A Z, and similarly β is tangent of the angle which U T makes with A Z.

392. The straight line will assume various positions according to the algebraical signs of a, b, α and β: however, it would be of very little use to go through all the cases arising from these changes of sign, especially as they offer nothing of consequence, and no one case presents any difficulty. We shall only consider the cases where the absolute value of a, b, α and β is changed.

Let $a = 0$ and $b = 0$, then $x = \alpha z$ and $y = \beta z$, and the two projections pass through the origin, and therefore the line itself passes through the origin; the equation to the third projection is $y = \frac{\beta}{\alpha} x$.

Let $a = 0$ then $x = \alpha z$ and $y = \beta z + b$, the projection on $x z$ passing through the origin, the line itself must pass through the axis A Y perpendicular to $x z$: similarly, if $b = 0$, the equations $x = \alpha z + a$, $y = \beta z$ belong to a line passing through the axis of x, and if the equations are $y = \alpha x$, $y = \beta z + b$, the straight line passes through the axis of z: this last case may be represented by supposing (in the last figure) W V to pass through A, then the equation to V W is of the form $y = \alpha x$, and the equation to O T U is $y = \beta z + b$; now, if two planes be drawn, one through T U perpendicular to $y z$, and the other through V W perpendicular to $x y$, both planes pass through the point O, and therefore the line itself must pass through O.

393. Let $\beta = 0 \therefore x = \alpha z + a$, $y = b$, the line is in a plane parallel to $x z$ and distant from it by the quantity b. If the last figure be adapted to this case we should have U T perpendicular to A Y, and therefore P Q equal and parallel to R S situated in the plane W N U Q perpendicular to $x y$.

Let $\alpha = 0 \therefore x = a$, $y = \beta z + b$, the line is in a plane parallel to to $y z$.

Similarly $z = c$, $y = \alpha' x + a'$ belong to a line in a plane parallel to $x y$.

394. A straight line may also be situated in one of the co-ordinate planes as in the plane of $y z$; for example, the equations to such a line are $y = \beta z + b$, $x = 0$. If the line be in the plane of $x z$ the equations are $x = \alpha z + a$, $y = 0$; and if the line be in the plane of $x y$ the equations become $y = \alpha' x + a'$ $z = 0$.

395. If the straight line be perpendicular to one of the co-ordinate planes, as $x y$ for example; α and β must each equal 0, and therefore the equations to this line are

$$x = a, y = b, z = \frac{0}{0}.$$

Similarly the equations to a line perpendicular to $x z$ are

$$x = a, y = \frac{0}{0}, z = c$$

and the equations to a line perpendicular to $y\ z$ are

$$x = \frac{0}{0},\ y = b,\ z = c.$$

396. To find the point where a straight line meets the co-ordinate planes:

Let $x = \alpha\, z + a$ and $y = \beta\, z + b$ be the equations to the line; when it meets the plane of $x y$ we have $z = 0\ \therefore\ x = a,\ y = b$ are the equations to the required point.

Similarly $z = -\dfrac{b}{\beta},\ x = -\dfrac{\alpha}{\beta}\,b + a$ are the equations to the point where the line meets the plane of $x\ z$, and $z = -\dfrac{a}{\alpha},\ y = -\dfrac{\beta}{\alpha}\,a + b$ are the equations to the point where the line pierces the plane of $z\ y$.

397. There are four constant quantities in the general equations to a straight line, and if they are all given, the position of the line is completely determined; for we have only to give to one of the variables as z a value z', and we have

$$x = \alpha\, z + a = \alpha\, z' + a = x' \text{ and } y = \beta\, z + b = \beta\, z' + b = y';$$

or, x' and y' are also necessarily determined; hence, taking A M $= x'$, (see the last figure,) and drawing M W ($= y'$) parallel to A Y, and lastly, drawing from W a perpendicular W Q $= z'$, the point Q thus determined belongs to the line; and similarly, any number of points in the line are determined, or the position of the line is completely ascertained. Again, the straight line may be subject to certain conditions, as passing through a given point, or being parallel to a given line; or, in other words, conditions may be given which will enable us to determine the quantities α, β, a and b, supposing them first to be unknown; in this manner arises a series of Problems on straight lines similar to those already worked for straight lines situated in one plane (40, 50).

PROBLEMS ON STRAIGHT LINES.

398. To find the equations to a straight line passing through a given point:

Let the co-ordinates of the given point be x_1, y_1 and z_1, and let the equations to the straight line be $x = \alpha\, z + a$, $y = \beta\, z + b$.

Now since this line passes through the given point, the projections of the line must also pass through the projections of the point; hence the projection $x = \alpha\, z + a$ passing through x_1 and z_1, we have $x_1 = \alpha\, z_1 + a$,

$$\therefore\ x - x_1 = \alpha\,(z - z_1)$$

$$\text{and similarly } y - y_1 = \beta\,(z - z_1)$$

hence these are the equations required: α and β being indeterminate, there may be an infinite number of straight lines passing through the given point.

If the given point be in the plane of $x\,y$, we have $z_1 = 0$,

$$\therefore\ \left.\begin{aligned} x - x_1 &= \alpha\, z \\ y - y_1 &= \beta\, z \end{aligned}\right\}$$

If the given point be on the axis of x, we have $z_1 = 0$ and $y_1 = 0$,

$$\left.\begin{array}{l} x - x_1 = \alpha z \\ y = \beta z \end{array}\right\}$$

And the equation would assume various other forms according to the position of the given point.

399. To find the equations to a straight line passing through two given points, $x_1\, y_1\, z_1$ and $x_2\, y_2\, z_2$.

Since the line passes through the point $x_1\, y_1\, z_1$ its equations are

$$x - x_1 = \alpha\,(z - z_1)$$
$$y - y_1 = \beta\,(z - z_1).$$

And since the line also passes through $x_2\, y_2\, z_2$ the last equations become

$$x_2 - x_1 = \alpha\,(z_2 - z_1)$$
$$y_2 - y_1 = \beta\,(z_2 - z_1)$$

$$\therefore \alpha = \frac{x_2 - x_1}{z_2 - z_1} \text{ and } \beta = \frac{y_2 - y_1}{z_2 - z_1};$$

hence the equations to the required line are

$$x - x_1 = \frac{x_2 - x_1}{z_2 - z_1}\,(z - z_1)$$

$$y - y_1 = \frac{y_2 - y_1}{z_2 - z_1}\,(z - z_1).$$

These equations will assume many various forms dependent on the position of the given point, for example: If the first point be in the plane of $y\,z$, and the second in the axis of x, we have $x_1 = 0$; $y_2 = 0$, $z_2 = 0$

$$\therefore x = \frac{x_2}{-z_1}\,(z - z_1)$$

$$y - y_1 = \frac{y_1}{z_1}\,(z - z_1).$$

If the second point be the origin, we have $x_2\, y_2\, z_2$ each $= 0$,

$$\therefore x - x_1 = \frac{x_1}{z_1}\,(z - z_1) = \frac{x_1}{z_1}\,z - x_1$$

$$y - y_1 = \frac{y_1}{z_1}\,(z - z_1) = \frac{y_1}{z_1}\,z - y_1;$$

hence the equations to a point passing through the origin are

$$x = \frac{x_1}{z_1}\,z, \text{ and } y = \frac{y_1}{z_1}\,z.$$

And these equations may be also obtained by considering that the projections pass through the origin, and therefore their equations are of the form $x = \alpha\,z$, $y = \beta\,z$, and the first passing through $x_1\,z_1$ we have

$$\alpha = \frac{x_1}{z_1}, \text{ and similarly } \beta = \frac{y_1}{z_1}.$$

400. To find the equation to a straight line parallel to a given straight line.

Since the lines are parallel their projecting planes on any one of the co-ordinate planes are also parallel, and therefore the projections themselves parallel; hence, if the equations to the given line are

$$x = \alpha z + a, \; y = \beta z + b,$$

the equations to the required line are

$$x = \alpha z + a', \; y = \beta z + b'.$$

If the straight line pass also through a given point $x_1\, y_1\, z_1$ its equations are

$$x - x_1 = \alpha (z - z_1), \; y - y_1 = \beta (z - z_1).$$

401. To find the intersection of two given straight lines.

Two straight lines situated in one plane must meet in general, but this is not necessarily the case if the lines be situated anywhere in space; hence there must be a particular relation among the constant quantities in the equation in order that the lines may meet: to find this relation, let the equations to the lines be

$$\left.\begin{array}{l} x = \alpha z + a \\ y = \beta z + b \end{array}\right\} \qquad \left.\begin{array}{l} x = \alpha' z + a' \\ y = \beta' z + b' \end{array}\right\}$$

For the point of intersection the projected values of x, y and z must be the same in all the equations; hence

$$\alpha z + a = \alpha' z + a' \text{ and } z = \frac{a' - a}{\alpha - \alpha'}$$

$$\text{and } \beta z + b = \beta' z + b' \text{ and } z = \frac{b' - b}{\beta - \beta'},$$

$$\therefore \frac{a' - a}{\alpha - \alpha'} = \frac{b' - b}{\beta - \beta'};$$

$$\text{or, } (a' - a)\,(\beta' - \beta) = (b' - b)\,(\alpha' - \alpha).$$

And this is the relation which must exist amongst the constants in order that the two lines may meet.

Having thus determined the necessary relation among the constants, the co-ordinates of intersection are given by the equations

$$z = \frac{a' - a}{\alpha - \alpha'} \text{ or } = \frac{b' - b}{\beta - \beta'}$$

$$y = \beta z + b = \beta \frac{b' - b}{\beta - \beta'} + b = \frac{\beta b' - \beta' b}{\beta - \beta'}$$

$$x = \alpha z + a = \alpha \frac{a' - a}{\alpha - \alpha'} + a = \frac{\alpha a' - \alpha' a}{\alpha - \alpha'}.$$

402. To find the angles which a straight line (l) makes with the co-ordinate axes; and thence with the co-ordinate planes:

Let the equations to the given line be

$$x = \alpha z + a$$
$$y = \beta z + b;$$

the equations to the parallel line through the origin are

$$x = \alpha z, \; y = \beta z;$$

also let r be the distance of any point (x, y, z) in this last line from the origin:

$$\therefore r^2 = x^2 + y^2 + z^2$$
$$= (\alpha^2 + \beta^2 + 1)\, z^2$$
$$\text{or, } z^2 = \frac{r^2}{1 + \alpha^2 + \beta^2}$$

But $l\,x$, $l\,y$, and $l\,z$ being the angles which either line makes with the axes of x, y and z respectively, we have from the second line

$$\cos.\, l\,x = \frac{x}{r} = \frac{\alpha z}{r} = \frac{\alpha}{\sqrt{1 + \alpha^2 + \beta^2}}$$
$$\cos.\, l\,y = \frac{y}{r} = \frac{\beta z}{r} = \frac{\beta}{\sqrt{1 + \alpha^2 + \beta^2}}$$
$$\cos.\, l\,z = \frac{z}{r} = \frac{1}{\sqrt{1 + \alpha^2 + \beta^2}}$$

Also $(\cos.\, l\,x)^2 + (\cos.\, l\,y)^2 + (\cos.\, l\,z)^2 = 1$;

and this is the equation connecting the three angles which any straight line makes with the rectangular axes.

Since the system is rectangular, the angle which a line makes with any axis is the complement of the angle which it makes with the plane perpendicular to that axis: hence the angles which a line makes with the coordinate planes are given.

403. To find the cosine, sine, and tangent of the angle between two given straight lines.

Let the equations to the two straight lines be

$$\left.\begin{array}{l} x = \alpha\, z + a \\ y = \beta\, z + b \end{array}\right\} \qquad \left.\begin{array}{l} x = \alpha'\, z + a' \\ y = \beta'\, z + b' \end{array}\right\}.$$

These two lines may meet, or they may not meet; but in either case their mutual inclination is the same as that of two straight lines parallel to them and passing through the origin; hence the problem is reduced to find the angle between the lines represented by the equations

$$\left.\begin{array}{l} x = \alpha\, z \\ y = \beta\, z \end{array}\right\} (1) \qquad \left.\begin{array}{l} x = \alpha'\, z \\ y = \beta'\, z \end{array}\right\} (2)$$

Let $r =$ the distance of a point $x\ y\ z$ in (1) from the origin,
$r_1 =$ . . . . $x_1\, y_1\, z_1$ in (2) . . . .
$d =$ the distance between these points,
$\theta =$ the angle between the given lines,

then
$$d^2 = r^2 + r_1^2 - 2\, r\, r_1 \cos.\, \theta$$
$$= (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \quad (388)$$
$$= x^2 + y^2 + z^2 + x_1^2 + y_1^2 + z_1^2 - 2\,(x\,x_1 + y\,y_1 + z\,z_1)$$
$$= r^2 + r_1^2 - 2\,(x\,x_1 + y\,y_1 + z\,z_1)$$
$$\therefore r\, r_1 \cos.\, \theta = x\,x_1 + y\,y_1 + z\,z_1$$

Now $x x_1 + y y_1 + z z_1 = \alpha z\, \alpha' z_1 + \beta z\, \beta' z_1 + z\, z_1 = (\alpha\alpha' + \beta\beta' + 1)\, z\, z_1$

And $r\, r_1 = \sqrt{(x^2 + y^2 + z^2)}\, \sqrt{(x_1^2 + y_1^2 + z_1^2)}$
$= z\, z' \sqrt{(\alpha^2 + \beta^2 + 1)} \sqrt{(\alpha'^2 + \beta'^2 + 1)}$.

$$\therefore \cos.\, \theta = \frac{x\,x_1 + y\,y_1 + z\,z_1}{r\, r_1}$$

$$= \frac{\alpha\alpha' + \beta\beta' + 1}{\sqrt{(\alpha^2 + \beta^2 + 1)}\sqrt{(\alpha'^2 + \beta'^2 + 1)}}$$

Hence $\sin.\ \theta = \dfrac{\sqrt{\{(\alpha\beta' - \alpha'\beta)^2 + (\alpha - \alpha')^2 + (\beta - \beta')^2\}}}{\sqrt{(\alpha^2 + \beta^2 + 1)}\sqrt{(\alpha'^2 + \beta'^2 + 1)}}$.

And $\tan.\ \theta = \dfrac{\sin.\ \theta}{\cos.\ \theta} = \dfrac{\sqrt{\{(\alpha\beta' - \alpha'\beta)^2 + (\alpha - \alpha')^2 + (\beta - \beta')^2\}}}{\alpha\alpha' + \beta\beta' + 1.}$

The value of the cosine of the angle between two straight lines may also be expressed in terms of the angles which the two straight lines l and l_1 make with the co-ordinate axes.

For $x = r \cos.\ l\,x$, $\quad y = r \cos.\ l\,y$, $\quad z = r \cos.\ l\,z$,

and $x_1 = r_1 \cos.\ l_1\,x$, $\quad y_1 = r_1 \cos.\ l_1\,y$, $\quad z_1 = r_1 \cos.\ l_1\,z$,

$$\therefore \cos.\ \theta = \frac{x\,x_1}{x\,x_1} + \frac{y\,y_1}{r\,r_1} + \frac{z\,z_1}{r\,r_1}$$

$$= \cos.\ l\,x \cos.\ l_1 x + \cos.\ l\,y \cos.\ l_1 y + \cos.\ l\,z \cos.\ l_1\,z.$$

404. If the lines are parallel, we must have $\sin.\ \theta = 0$.

$$\therefore (\alpha\beta' - \alpha'\beta)^2 + (\alpha - \alpha')^2 + (\beta - \beta')^2 = 0,$$

an equation which cannot be satisfied unless by supposing $\alpha = \alpha'$, $\beta = \beta'$, and $\alpha\beta' = \alpha'\beta$, the first two of these conditions are the same as those already shown to determine the parallelism of two lines (400), and the third condition is only a necessary consequence of the other two, and therefore implies nothing further.

405. If the lines are perpendicular to each other, we must have $\cos.\ \theta = 0$.

$$\therefore \alpha\alpha' + \beta\beta' + 1 = 0.$$

or, $\cos.\ l\,x \cos.\ l_1\,x + \cos.\ l\,y \cos.\ l_1\,y + \cos.\ l\,z \cos.\ l_1\,z = 0$;

Now, one line in space is considered as perpendicular to a second straight line, whenever it is in a plane perpendicular to this second line; hence an infinite number of lines can be drawn perpendicular to a given line; and this appears from the above equation, for there are four constants involved in the equation to the perpendicular line, and only one equation between them.

406. If the lines also meet, we have then the additional equation,

$$(a' - a)\,\beta' - \beta = (b' - b)\,(\alpha' - \alpha) \quad (401).$$

However, even yet an infinite number of straight lines can be drawn, meeting the given line at right angles, for an infinite number of planes can be drawn perpendicular to the given line, and in each plane an infinite number of straight lines can be drawn passing through the given line.

407. To find the equation to a straight line passing through a given point $x_1\,y_1\,z_1$, and meeting a given line (1) at right angles.

Let the equations to the lines be,

$$\left.\begin{aligned} x &= \alpha z + a \\ y &= \beta z + b \end{aligned}\right\} (1) \qquad \left.\begin{aligned} x - x_1 &= \alpha'(z - z_1) \\ y - y_1 &= \beta'(z - z_1) \end{aligned}\right\} (2)$$

hence the two equations of condition are,

$$\alpha\alpha' + \beta\beta' + 1 = 0 \quad (3)$$

$$(a' - a)(\beta - \beta') - (b' - b)(\alpha - \alpha') = 0$$

or since $a' = x_1 - \alpha' z_1$, and $b' = y_1 - \beta' z_1$

$$(x_1 - \alpha' z_1 - a)(\beta - \beta') - (y_1 - \beta' z_1 - b)(\alpha - \alpha') = 0 \quad (4).$$

The elimination of α' and β' from (3) and (4) give the equations

$$\beta' = \frac{\{(x_1 - a)\,\alpha + z_1\}\,\beta - (y_1 - b)\,(1 + \alpha^2)}{(y_1 - b)\,\beta + (x_1 - a)\,\alpha - (\alpha^2 + \beta^2)\,z_1}$$

$$\alpha' = \frac{\{(y_1 - b)\,\beta + z_1\}\,\alpha - (x_1 - a)\,(1 + \beta^2)}{(y_1 - b)\,\beta + (x_1 - a)\,\alpha - (\alpha^2 + \beta^2)\,z_1}.$$

These values of α' and β' substituted in (2) give the final equation to the straight line, passing through a given point, and meeting a given straight line at right angles.

In particular cases other methods may be adopted, for example, to find the equations to a straight line passing through the axis of y at right angles to that axis:
here $x_1 = 0$, and $z_1 = 0$, therefore the equations to the line are

$$x = \alpha z$$
$$y - y_1 = \beta z$$

but because the line is perpendicular to the axis of y we have $\beta = 0$, hence the required equations are $x = a\,z$, $y = y_1$. By assuming the axes of co-ordinates to be conveniently situated, this and many other problems may be worked in a shorter manner. This will be shown hereafter.

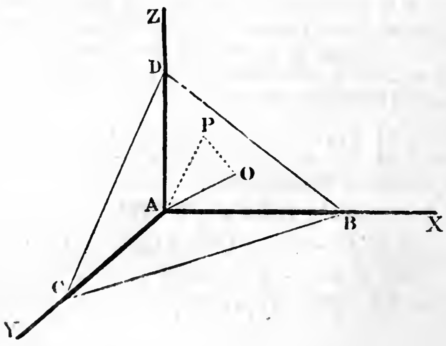
CHAPTER III.

THE PLANE.

408. A Plane may be supposed to be generated by the motion of a straight line about another straight line perpendicular to it.

Let A be the origin, A X, A Y, A Z the axes, B C D a portion of a plane, A O the perpendicular from the origin upon this plane, P any point in this plane; then, according to the above definition, we suppose the plane to be formed by the revolution of a line like O P round A O, the angle A O P being a right angle.

To find the equation to the plane.
Let x, y, z, be the co-ordinates of P, and x_1, y_1, z_1, those of O, and let the fixed distance AO $= d$.



Then the square on A P = the square on A O + the square on O P;
or, $x^2 + y^2 + z^2 = d^2 + (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$

$$= d^2 + x^2 + y^2 + z^2 + x_1^2 + y_1^2 + z^2 - 2\,x x_1 - 2\,y y_1 - 2\,z z_1.$$

$$\therefore\ 2\,(x x_1 + y y_1 + z z_1) = d^2 + d^2 = 2\,d^2$$

$$\text{or } x x_1 + y y_1 + z z_1 = d^2.$$

409. Let $\frac{x_1}{d^2} = m$, $\frac{y_1}{d^2} = n$, and $\frac{z_1}{d^2} = p$, then the above equation becomes

$$m x + n y + p z = 1.$$

And it is under this form that we shall generally consider the equation to the plane.

Let $\frac{d^2}{x_1} = a$, $\frac{d^2}{y_1} = b$, and $\frac{d^2}{z_1} = c$, then the equation to the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

And this is perhaps the most intelligible form in which the equation to the plane can be put, the constants a, b and c being equal to A B, A C and A D the respective distances of the origin from the intersection of the plane with the co-ordinate axes; this is found by putting y and z both $= 0$, hence $\frac{x}{a} = 1$, or A B $= a$, and similarly for the other lines.

410. Let the word "plane" be represented by the letter P, and let the angles which A O or d makes with the co-ordinate axes be represented by $d\,x$; $d\,y$; $d\,z$; and let the angles which the plane makes with the same axes be denoted by Px; Py; Pz; then, since A O B is a right angle, and A B O is the angle which the plane makes with A X, we have

$$d = a \cos.\, d\,x = a \sin.\, \mathrm{P}\,x$$

$$d = b \cos.\, d\,y = b \sin.\, \mathrm{P}\,y$$

$$d = c \cos.\, d\,z = c \sin.\, \mathrm{P}\,z;$$

therefore the last equation to the plane may be put in either of the forms

$$x \cos.\, d\,x + y \cos.\, d\,y + z \cos.\, d\,z = d$$

$$\text{or } x \sin.\, \mathrm{P}\,x + y \sin.\, \mathrm{P}\,y + z \sin.\, \mathrm{P}\,z = d.$$

411. Let P, $y\,z$ represent the angle which the plane makes with the co-ordinate plane $y\,z$, then since angle O A B. is equal to the angle of inclination of the plane to $y\,z$, we have cos. $d\,x$ = cos. P, $y\,z$, hence the equation of the plane becomes

$$x \cos.\, \mathrm{P}, y\,z + y \cos.\, \mathrm{P}, x\,z + z \cos.\, \mathrm{P}, x\,y = d.$$

412. Since by (386) $(\cos.\, d\,x)^2 + (\cos.\, d\,y)^2 + (\cos.\, d\,z)^2 = 1$
we have $(\cos.\, \mathrm{P}, y\,z)^2 + (\cos.\, \mathrm{P}\,x\,z)^2 + (\cos.\, \mathrm{P}\,x\,y)^2 = 1$ *.

* If A be the area of a plane P, the projections of this area on the co-ordinate planes are represented by A cos. P, $x\,y$; A cos. P, $x\,z$; A cos. P, $y\,z$; hence $(\mathrm{A} \cos.\, \mathrm{P}, x\,y)^2 + (\mathrm{A} \cos.\, \mathrm{P}, x\,z)^2 + (\mathrm{A} \cos.\, \mathrm{P}, y\,z)^2 = \mathrm{A}^2 \left\{(\cos.\, \mathrm{P}, x\,y)^2 + (\cos.\, \mathrm{P}, x\,z)^2 + (\cos.\, \mathrm{P}, y\,z)^2\right\} = \mathrm{A}^2$ by (412). This theorem, referring to the numerical values of the projected areas, is of use in finding the area of a plane between the three co-ordinate planes. Thus, if the equation to a plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, we have by the last figure the area A B C $= \frac{a\,b}{2}$; area A D C $= \frac{b\,c}{2}$, and area A B D $= \frac{a\,c}{2}$; hence the area B C D $= \sqrt{\frac{1}{4}\,(a^2\,b^2 + a^2\,c^2 + b^2\,c^2)}$ by the above theorem. The volume of the pyramid A C D B $= \frac{c}{3}\,\frac{a\,b}{2} = \frac{a\,b\,c}{6}$.

413. To find the angles which a plane makes with the co-ordinate planes in terms of the co-efficients of the equation to the plane.

Let the equation to the plane be

$$m x + n y + p z = 1.$$

Now the equation to a plane expressed in terms of the angles which it makes with the co-ordinate planes is given by (411.)

$$x \cos. \text{P}, y z + y \cos. \text{P}, x z + z \cos. \text{P}, x y = d,$$

hence equating co-efficients, we have

$$m = \frac{\cos. \text{P}, y z}{d}, n = \frac{\cos. \text{P}, x z}{d}, p = \frac{\cos. \text{P}, x y}{d}$$

$$\therefore m^2 + n^2 + p^2 = \frac{1}{d^2} \text{ and } d = \frac{1}{\sqrt{m^2 + n^2 + p^2}};$$

$$\text{and } \cos. \text{P}, y z = m d = \frac{m}{\sqrt{m^2 + n^2 + p^2}}$$

$$\text{Cos. P}, x z = n d = \frac{n}{\sqrt{m^2 + n^2 + p^2}}$$

$$\text{Cos. P}, x y = p d = \frac{p}{\sqrt{m^2 + n^2 + p^2}}.$$

414. The equation to the plane will assume various forms according to the various positions of the plane.

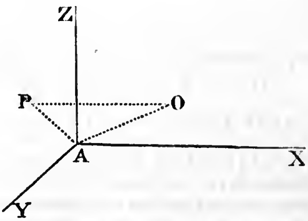
Let the plane pass through the origin, then $d = 0$; therefore, putting $d = 0$ in the equation, art. (408), we have the equation to the plane passing through the origin; but as the equation to the plane has been obtained on the supposition of d being finite, it becomes necessary to give an independent proof for this particular case.

Let A O ($= d$) be the length of a perpendicular from the plane to a given point O; whose co-ordinates are x_1, y_1, z_1; x, y, z, as before, the co-ordinates of any point P in the plane, then the square on O P = the square on A O + the square on A P;

or $(x - x_1)^2 + (y - y_1)^2 + (z - z_1) = d^2 + x^2 + y^2 + z^2$.

$$\therefore \quad - 2 (x x_1 + y y_1 + z z_1) + d^2 = d^2,$$

$$\text{or } x x_1 + y y_1 + z z_1 = 0.$$



So that the equation to the plane in this case is the same as the original equation without the constant term.

415. Let the plane be parallel to any of the co-ordinate planes, as $x y$ for example, then $a = \infty$ and $b = \infty$; therefore the equation $\frac{x}{a} + \frac{y}{b} +$

$\frac{z}{c} = 1$ becomes $0\,x + 0\,y + \frac{z}{c} = 1$; $\therefore\ z = c$, $x = \frac{0}{0}$ and $y = \frac{0}{0}$; of these three equations the first signifies that every point in the plane is equidistant from the plane $x\,y$, and the other two signify that for this single value of z, every possible value of x and y will give points in the plane. The two latter equations are generally omitted; and we say that for a plane parallel to $x\,y$ the equation is $z = c$; similarly for the plane parallel to $x\,z$ it is $x = a$, and for a plane parallel to $y\,z$ the equation is $y = b$.

The equations to a co-ordinate plane, as $x\,y$ for example, are $z = 0$, $x = \frac{0}{0}$; $y = \frac{0}{0}$; or, more simply, $z = 0$.

416. The lines B C, B D, and D C, where the plane intersects the co-ordinate planes, are called the traces of the plane. The equations to these traces are found, from the equation to the plane, by giving to x, y, or z the particular values which they have when the plane intersects the co-ordinate planes.

Let the equation to the plane be $m\,x + n\,y + p\,z = 1$; then for the intersection B C we have the equations

$$z = 0,\ m\,x + n\,y = 1.$$

Similarly the equations to the traces B D and C D are respectively

$$y = 0,\ m\,x + p\,z = 1$$
$$x = 0,\ n\,y + p\,z = 1.$$

PROBLEMS ON THE PLANE.

417. To find the equation to a plane parallel to a given plane.

Let the given plane be $m\,x + n\,y + p\,z = 1$,
and the required plane be $m'\,x + n'\,y + p'\,z = 1$.
Then the planes being parallel, their traces on the co-ordinate planes must be parallel; now their traces on $x\,z$ are

$$m\,x + p\,z = 1,\ m'\,x + p'\,z = 1;$$

$$\therefore\ \frac{m}{p} = \frac{m'}{p'},\ \text{or}\ m' = \frac{m}{p}\,p';\ \text{similarly}\ n' = \frac{n}{p}\,p'.$$

Hence the required equation becomes

$$\frac{m}{p}\,p'\,x + \frac{n}{p}\,p'\,y + p'\,z = 1,$$

$$\text{or}\ m\,x + n\,y + p\,z = \frac{p}{p'}$$

In this case the resulting equation contains one indeterminate constant p', and therefore shows that an infinite number of planes can be drawn parallel to a given plane, which is also geometrically evident. Three conditions are apparently given, since the three traces of one plane are parallel to the three traces of the other plane; but if the traces on two of the co-ordinate planes be parallel, the traces on the third co-ordinate plane are necessarily parallel; for if $\frac{m}{p} = \frac{m}{p'}$, and $\frac{n}{p} = \frac{n'}{p'}$, we have $\frac{m}{n} =$

$\frac{m'}{n'}$, or $m x + n y = 1$ parallel to $m' x + n' y = 1$. Thus, in reality, only two conditions are given to determine the three constants.

418. To find the equation to a plane parallel to a given plane, and passing through a given point x_1, y_1, z_1.

Let $m' x + n' y + p' z = 1$ be the required plane, then since the plane passes through $x_1\, y_1\, z_1$ we have

$$m' x_1 + n' y_1 + p' z_1 = 1$$

$$\therefore\ m' (x - x_1) + n' (y - y_1) + p' (z - z_1) = 0.$$

$$\text{Also } \frac{m'}{p'} = \frac{m}{p}, \text{ and } \frac{n'}{p'} = \frac{n}{p};$$

$$\therefore\ \frac{m}{p} p' (x - x_1) + \frac{n}{p} p' (y - y_1) + p' (z - z_1) = 0;$$

$$\text{or } m (x - x_1) + n (y - y_1) + p (z - z_1) = 0.$$

419. To find the intersection of a straight line and plane.

Let $m x + n y + p z = 1$ be the equation to the plane,

$$\left.\begin{array}{l} x = \alpha z + a \\ y = \beta z + b \end{array}\right\} \text{the equations to the line;}$$

then, since the co-ordinates of the point of intersection are common, we have

$$m (\alpha z + a) + n (\beta z + b) + p z = 1,$$

$$\therefore\ z = \frac{1 - m a - n b}{m \alpha + n \beta + p},$$

$$\text{and } x = \alpha z + a = \frac{\alpha - n b a + n \beta a + p a}{m \alpha + n \beta + p}$$

$$y = \beta z + b = \frac{\beta - m a \beta + m a b + p b}{m \alpha + n \beta + p}.$$

Thus the required point of intersection is found.

420. To find the conditions that the straight line and plane be parallel or coincide.

If they are parallel, the values of x, y, and z must be infinite

$$\therefore\ m \alpha + n \beta + p = 0.$$

If they coincide, the values of x, y, and z must be indeterminate, or each $= \frac{0}{0}$.

$$\therefore\ m \alpha + n \beta + p = 0, \text{ and } 1 - m a - n b = 0;$$

and these are the two conditions for coincidence, the numerators of x and y being both given $= 0$ by combining the last two equations.

Hence, to find the equation to a plane coinciding with a given straight line, we have the two conditions

$$m a + n b = 1,$$
$$m \alpha + n \beta + p = 0;$$

whence, by elimination, we have

$$m = \frac{\beta + p b}{a \beta - b \alpha} \text{ and } n = - \frac{\alpha + p a}{a \beta - b \alpha};$$

therefore the equation to the plane is

$$(\beta + p\,b)\,x - (\alpha + p\,a)\,y + p\,(a\,\beta - b\,\alpha)\,z = a\,\beta \quad b\,\alpha,$$

where p remains indeterminate.

421. To find the equation to a plane coinciding with two given lines.

$$m\,x + n\,y + p\,z = 1.$$

$$\left.\begin{array}{l} x = \alpha\,z + a \\ y = \beta\,z + b \end{array}\right\} \qquad \left.\begin{array}{l} x = \alpha'\,z + a' \\ y = \beta'\,z + b' \end{array}\right\}$$

the plane coinciding with the given lines, we have

$$m\,a + n\,b = 1 \quad (1) \qquad m\,\alpha + n\,\beta + p = 0 \quad (3)$$
$$m\,a' + n\,b' = 1 \quad (2) \qquad m\,\alpha' + n\,\beta' + p = 0 \quad (4)$$

From (1) and (2) we have m and n, and these values being substituted in (3) and (4), give two values of p, hence we have the equation of condition

$$(\beta' - \beta)\,(a - a') + (\alpha' - \alpha)\,(b - b') = 0.$$

This equation is verified either if the lines are parallel (in which case $\alpha' = \alpha$ and $\beta' = \beta$), or if they meet; hence in either of these cases a plane may be drawn coinciding with the two lines; the equation to this plane is found, from the values of m, n, and p, to be

$$(b' - b)\,x - (a' - a)\,y + \{\,(a' - a)\,\beta - (b' - b)\,\alpha\,\}\,z = a\,b' - a'\,b.$$

422. If it be required to find the equation to a plane which coincides with one given straight line, and is parallel to another given straight line, we have the three equations

$$\left.\begin{array}{r} m\,a + n\,b = 1 \\ m\,\alpha + n\,\beta + p = 0 \end{array}\right\} \text{ for coincidence with one line,}$$

$$m\,\alpha' + n\,\beta' + p = 0 \text{ for parallelism with the other;}$$

and from these three equations we may determine m, n, and p, and then substitute these values in the general equation to the plane.

423. To find the intersection of two given planes.

Let the equations to the two planes be

$$m\,x + n\,y + p\,z = 1$$
$$m'\,x + n'\,y + p'\,z = 1.$$

By the elimination of z we obtain an equation between x and y, which belongs to the projection of the intersection of the planes on $x\,y$,

hence $(m\,p' - m'\,p)\,x + (n\,p' - n'\,p)\,y = p' - p$

is the projection on $x\,y$ of the required intersection.

Similarly

$$(m\,n' - m'\,n)\,x + (p\,n' - p'\,n)\,z = n' - n$$

is the equation to the projection on $x\,z$.

But the equations to the projections of a line on two co-ordinate planes are called the equations to the line itself; hence the above two equations are the required equations to the intersection.

The third projection is given by the other two, or it may be found separately

$$(n\,m' - n'\,m)\,y + (p\,m' - m\,p')\,z = m' - m.$$

424. To find the intersection of three planes.

Let the intersection of the first and second, found as in the last article, be expressed by the equations

$$x = \alpha z + a$$
$$y = \beta z + b,$$

and let the intersection of the first and third planes be denoted by the equations

$$x = \alpha' z + a'$$
$$y = \beta' z + b'.$$

Then, finding the intersection of these two lines from their four equations, we have the values of x, y, and z, corresponding to the point of intersection of the two lines, and therefore to the point of intersection of the three planes.

In this manner we may find the relation among the co-efficients of any number of planes meeting in one point.

425. To find the relation among the coefficients of the equations to four planes so that they may meet in the same straight line.

Let the equations be

$$m x + n y + p z = 1$$
$$m_1 x + n_1 y + p_1 z = 1$$
$$m_2 x + n_2 y + p_2 z = 1$$
$$m_3 x + n_3 y + p_3 z = 1.$$

Then the first and second plane intersect in a line whose equations are

$$x = \alpha z + a$$
$$y = \beta z + b$$

The first and third intersect in the line

$$x = \alpha_1 z + a_1$$
$$y = \beta_1 z + b_1$$

And the first and fourth in the line

$$x = \alpha_2 z + a_2$$
$$y = \beta_2 z + b_2$$

Now, in order that these intersections all coincide, we must have

$$\alpha = \alpha_1 = \alpha_2\,;\ \beta = \beta_1 = \beta_2\,;\ a = a_1 = a_2\,;\ \text{and } b = b_1 = b_2.$$

And the values of α, β, a and b are given in terms of m, n, p, &c., by article (423), hence the relation among the co-efficients is found.

The same relation exists among the co-efficients of any number o planes meeting in one point.

426. To find the relation among the co-efficients of a straight line and plane, so that they may be perpendicular to one another.

Let $(x_1\, y_1\, z_1)$ be the point in which the plane and line meet, then the equation to the plane is

$$m\,(x - x_1) + n\,(y - y_1) + p\,(z - z_1) = 0 \quad (1)$$

And the equations to the line are

$$\left.\begin{array}{l} x = \alpha z + a \\ y = \beta z + b \end{array}\right\} (2)$$

Also let the equations to a line perpendicular to (2) and passing through the point (a, y, z_1) in (2) be

$$\left.\begin{aligned} x - x_1 &= \alpha' (z - z_1) \\ y - y_1 &= \beta' (z - z_1) \end{aligned}\right\} \quad (3)$$

But since these two lines are perpendicular to one another, we have the cosine of the angle between them $= 0$,

$$\therefore \ \alpha \alpha' + \beta \beta' + 1 = 0 \quad (402)$$

Now, this equation combined with that to the last line (3), will give the relation among the co-ordinates of x, y, and z, so that the point to which they refer is always in a locus perpendicular to the first given line; hence substituting for α' and β', we have the equation to the plane which is the locus of all the lines perpendicular to (2), this equation is

$$\alpha \frac{x - x_1}{z - z_1} + \beta \frac{y - y_1}{z - z_1} + 1 = 0$$

$$\text{or } \alpha (x - x_1) + \beta (y - y_1) + z - z_1 = 0 \ (4)$$

and as this equation (4) must coincide with (1) we have, by equating the co-efficients,

$$\alpha = \frac{m}{p}, \text{ and } \beta = \frac{n}{p},$$

and these are the conditions required.

427. Hence, if the line be given, the equation to the plane perpendicular to it is

$$\alpha x + \beta y + z = \frac{1}{p}.$$

Or if the plane be given, the equations to the straight line perpendicular to it are

$$x = \frac{m}{p} z + a,$$

$$y = \frac{n}{p} z + b.$$

From the form of these equations to the plane and perpendicular straight line, it appears that the trace of the plane is perpendicular to the projection of the line upon the same co-ordinate plane.

428. If the plane pass through a given point $x_1 \, y_1 \, z_1$, and be perpendicular to a given straight line, ($x = \alpha z + a$, $y = \beta z + b$) its equation is

$$\alpha (x - x_1) + \beta (y - y_1) + z - z_1 = 0.$$

429. If the straight line pass through a given point, and be perpendicular to a given plane ($m x + n y + p z = 1$) its equations are

$$x - x_1 = \frac{m}{p} (z - z_1),$$

$$y - y_1 = \frac{n}{p} (z - z_1).$$

430. To find the length of a perpendicular from a given point on a given plane.

Let $x_1 \, y_1 \, z_1$ be the co-ordinates of the given point,

$m x + n y + p z = 1$ the equation to the given plane.

It was shown in Art. 413, that if d be the perpendicular distance of the origin from a plane, whose equation is

$$mx + ny + pz = 1$$

we have $d = \dfrac{1}{\sqrt{m^2 + n^2 + p^2}}.$

Now, the equation to the plane, parallel to the given plane, and passing through the given point, is

$$m(x - x_1) + n(y - y_1) + p(z - z_1) = 0 \text{ (418).}$$

or $\dfrac{mx + ny + pz}{mx_1 + ny_1 + pz_1} = 1.$

Hence the distance d_1 of the origin from this plane is

$$d_1 = \frac{mx_1 + ny_1 + pz_1}{\sqrt{m^2 + n^2 + p^2}}.$$

But the distance of the given point from the given plane is evidently the distance between the two planes, that is, $= d_1 - d$

$$= \frac{mx_1 + ny_1 + pz_1 - 1}{\sqrt{m^2 + n^2 + p^2}}.$$

431. To find the distance of a point from a straight line.

Let the equations to the given line be $x = \alpha z + a$, $y = \beta z + b$, then the equation to the plane passing through the given point $x_1 y_1 z_1$, and perpendicular to the given line, is

$$\alpha(x - x_1) + \beta(y - y_1) + z - z_1 = 0.$$

Eliminating x, y, and z by means of the above equations to the straight line, we find

$$z = \frac{\alpha(x_1 - a) + \beta(y_1 - b) + z_1}{1 + \alpha^2 + \beta^2};$$

or, if this fraction $= \dfrac{M}{N}$, we have

$$z = \frac{M}{N}, \quad x = \alpha\frac{M}{N} + a, \quad y = \beta\frac{M}{N} + b.$$

These are the co-ordinates of the intersection of the given line, with the perpendicular plane passing through the given point; and the required perpendicular line (P) is the distance of the given point from this intersection.

Hence $P^2 = (x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2$

$$= \left(x_1 - a - \alpha\frac{M}{N}\right)^2 + \left(y_1 - b - \beta\frac{M}{N}\right)^2 + \left(z_1 - \frac{M}{N}\right)^2$$

which, after expansion and reduction, becomes

$$= (x_1 - a)^2 + (y_1 - b)^2 + z_1^2 - \frac{M^2}{N}.$$

432. If the given point be the origin, we have $x_1\ y_1\ z_1$, each equal $= 0$

$$\therefore P^2 = a^2 + b^2 - \frac{(\alpha a + \beta b)^2}{1 + \alpha^2 + \beta^2}.$$

433. To find the angle θ between two given planes.

Let the equations to the planes be

$$mx + ny + pz = 1 \quad (1)$$
$$m_1 x + n_1 y + p_1 z = 1 \quad (2).$$

Then, if from the origin we draw perpendiculars on each of these planes, the angle between these perpendiculars is equal to the angle between the planes: let the equations to the two lines be

$$\left.\begin{array}{l} x = \alpha z \\ y = \beta z \end{array}\right\} (3) \qquad \left.\begin{array}{l} x = \alpha' z \\ y = \beta' z \end{array}\right\} (4).$$

In order that (3) may be perpendicular to (1), we must have

$$\alpha = \frac{m}{p}, \beta = \frac{n}{p} \text{ (426), and similarly } \alpha' = \frac{m_1}{p_1}, \beta' = \frac{n_1}{p_1}.$$

Then the angle between the two lines is found from the expression

$$\cos. \theta = \frac{\alpha\alpha' + \beta\beta' + 1}{\sqrt{(1 + \alpha^2 + \beta^2)}\sqrt{(1 + \alpha'^2 + \beta'^2)}} \text{ (403).}$$

$$\therefore \cos. \theta = \frac{m m_1 + n n_1 + p p_1}{\sqrt{m^2 + n^2 + p^2}\sqrt{m_1^2 + n_1^2 + p_1^2}}.$$

434. This value of cos. θ may also be expressed in another form by means of Art. (413.)

$$\cos. \theta = \cos. P, x \cos. P', x + \cos. P, y \cos. P', y + \cos. P, z \cos. P', z.$$

or $\cos. \theta = \cos. P, yz \cos. P', yz + \cos. P, xz \cos. P', xz + \cos. P, xy \cos. P', xy.$

435. If the planes be perpendicular to each other, we have cos. $\theta = 0$.

$$\therefore m m_1 + n n_1 + p p_1 = 0.$$

Hence, if the equation to any plane be $mx + ny + pz = 1$, the equation to the plane perpendicular to it is

$$m_1 x + n_1 y - \frac{m m_1 + n n_1}{p} z = 1,$$

where two constants remain indeterminate.

436. If the planes be parallel, we have cos. $\theta = 1$; and putting therefore the expression for cos. θ equal to unity, we shall arrive at the results,

$$\frac{m}{n} = \frac{m_1}{n_1} \text{ and } \frac{m}{p} = \frac{m_1}{p_1};$$

the same as already obtained when two planes are parallel.

437. To find the angle between a straight line and a plane.

This angle is the angle which the line makes with its projection on the plane; and therefore, drawing a perpendicular from any point in the line to the plane, is the complement of the angle which this perpendicular makes with the given line.

Let the equations to the plane and the line be

$$mx + ny + pz = 1$$
$$x = \alpha z + a, y = \beta z + b,$$

then the equations to the perpendicular from any point $x_1 y_1 z_1$ in the line to the plane are $x = \frac{m}{p}(z - z_1), y = \frac{n}{p}(z - z_1)$. (429)

$$\therefore \cos. (\pi - \theta) = \sin. \theta = \frac{\alpha\frac{m}{p} + \beta\frac{n}{p} + 1}{\sqrt{1 + \alpha^2 + \beta^2}\sqrt{1 + \frac{m^2}{p^2} + \frac{n^2}{p^2}}}$$

$$= \frac{m\alpha + n\beta + p}{\sqrt{1 + \alpha^2 + \beta^2}\sqrt{m^2 + n^2 + p^2}}.$$

CHAPTER IV.

THE POINT, STRAIGHT LINE, AND PLANE REFERRED TO OBLIQUE AXES.

438. If the co-ordinate axes are not rectangular but inclined to each other at any given angles, they are then called oblique axes. The equations to the point, Art. (381.) remain exactly the same as before, but the quantities a, b, and c, are no longer the representatives of lines drawn perpendicular to the co-ordinate planes, but of lines respectively parallel to the oblique axes.

439. To find the distance of a point from the origin referred to oblique axes.

Let A X, A Y, A Z, be the oblique axes; and let x, y, z, be the co-ordinates of P, draw P N perpendicular on A Q produced,

then the sq. on AP=the sqs. on AQ and PQ+twice the rectangle AQ,QN.

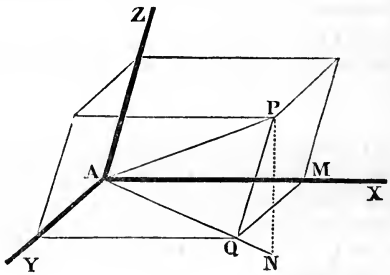
$$\text{Now, Q N} = \text{P Q cos. P Q N} = z \text{ cos. Z A Q}$$

$$\text{and A Q cos. Z A Q} = \text{A M cos. M A Z} + \text{M Q cos. Y A Z (379)}$$

$$= x \text{ cos. X A Z} + y \text{ cos. Y A Z}$$

$$\therefore \text{ the rectangle A Q, Q N} = z\,(x \text{ cos. X Z} + y \text{ cos. Y Z})$$

$$\text{also the square on A Q} = x^2 + y^2 + 2\,x\,y \text{ cos. Y X,}$$



$$\therefore\ d^2 = x^2 + y^2 + z^2 + 2\,x\,y \text{ cos. X Y} + 2\,x\,z \text{ cos. X Z} + 2\,y\,z \text{ cos. Y Z.}$$

440. To find the distance between two points when the axes are oblique

Let $x\ y\ z$ be the co-ordinates of one point,

and $x_1\ y_1\ z_1$ . . the other point,

then the distance between these points is the diagonal of a parallelopiped, of which the sides are the differences of parallel co-ordinates (388); hence,

$$d^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 + 2\,(x - x_1)\,(y - y_1) \text{ cos. X Y}$$
$$+ 2\,(x - x_1)\,(z - z_1) \text{ cos. X Z} + 2\,(y - y_1)\,(z - z_1) \text{ cos. Y Z.}$$

441. To find the equation to a straight line referred to oblique co-ordinates. The straight line must be considered to be the intersection of two planes formed by drawing straight lines through the several points of the given straight line parallel respectively to the planes of $x\,z$, $y\,z$; the traces of these planes on the co-ordinate planes are of the same *form* as for rect-

angular axes; that is, the equation to the traces, and therefore to the line itself are of the *form*

$$x = \alpha z + a$$
$$y = \beta z + b$$

but the values of α and β are not the tangents of any angles, but the ratio of the sines of the angles which each trace makes with the axes in its plane (51).

The quantities a and b remain the same as when the straight line is referred to rectangular co-ordinates, and since the equations are of the same form as before, those problems which do not affect the inclination of lines will remain the same as before.

442. To find the angle between two straight lines referred to oblique co-ordinates we shall follow the plan adopted in Art. 402.

Let the equations to the parallel lines through the origin be

$$\left.\begin{array}{l} x = \alpha z \\ y = \beta z \end{array}\right\} (1) \qquad \left.\begin{array}{l} x = \alpha' z \\ y = \beta' z \end{array}\right\} (2)$$

And let r be the distance of a point $x\,y\,z$ in (1) from the origin, and r_1 the distance of a point $x_1\,y_1\,z_1$ in (2) from the origin.

Then if d be the distance between these points, we have

$$d^2 = r^2 + r_1^2 - 2\,r\,r'\cos.\theta$$
$$= (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 + 2\,(x - x_1)\,(y - y_1)\cos.\,XY$$
$$+ 2(x - x_1)\,(z - z_1)\cos.\,XZ + 2\,(y - y_1)\,(z - z_1)\cos.\,YZ,$$
$$= r^2 + r_1^2 - 2\,(x\,x_1 + y\,y_1 + z\,z_1)$$
$$- 2\,\{(x_1\,y + x\,y_1)\cos.\,XY + (x_1\,z + x\,z_1)\cos.\,XZ + (y_1\,z + y\,z_1)\cos.\,YZ\}$$
$$\therefore r\,r_1\cos.\theta = x\,x_1 + y\,y_1 + z\,z_1$$
$$+ \{(x_1\,y + x\,y_1)\cos.\,XY + (x_1\,z + x\,z_1)\cos.\,XZ + (y_1\,z + y\,z_1)\cos.\,YZ\}$$
$$\therefore \cos.\theta = \frac{x\,x_1 + y\,y_1 + z\,z_1 + \&c.}{r\,r_1} =$$
$$\frac{\alpha\,\alpha' + \beta\,\beta' + 1 + (\alpha'\,\beta + \alpha\,\beta')\cos.\,XY + (\alpha' + \alpha)\cos.\,XZ + (\beta' + \beta)\cos.\,YZ}{\sqrt{\{1 + \alpha^2 + \beta^2 + 2\alpha\,\beta\cos.XY + 2\,\alpha\cos.\,XZ + 2\beta\cos.\,YZ\}}\ \sqrt{\{1 + \alpha'^2 + \beta'^2\,\&c.\}}}$$

443. To find the equation to a plane referred to oblique axes.

We consider a plane as the locus of all the straight lines which can be drawn perpendicular to a given straight line, and passing through a given point in that given straight line.

Let the equations to the given line be

$$x = \alpha z + a$$
$$y = \beta z + b$$

Also the equations to the straight line passing through a point x_1, y_1, z_1, in the above line, are

$$x - x_1 = \alpha'\,(z - z_1)$$
$$y - y_1 = \beta'\,(z - z_1)$$

But these two last lines being perpendicular to each other, we have the angle θ between them $= 90°$, or $\cos.\theta = 0$; hence by the last article:

$$\alpha\,\alpha' + \beta\,\beta' + 1 + (\alpha'\,\beta + \alpha\,\beta')\cos.XY + (\alpha' + \alpha)\cos.XZ + (\beta' + \beta)\cos.YZ = 0$$

and eliminating α' and β'

$$\alpha \frac{x - x_1}{z - z_1} + \beta \frac{y - y_1}{z - z_1} + 1 + \left(\frac{x - x_1}{z - z_1} \beta + \frac{y - y_1}{z - z_1} \alpha\right) \cos. \mathrm{X\,Y}$$

$$+ \left(\frac{x - x_1}{z - z_1} + \alpha\right) \cos. \mathrm{X\,Z} + \left(\frac{y - y_1}{z - z_1} + \beta\right) \cos. \mathrm{Y\,Z} = 0$$

or, $(\alpha + \beta \cos. \mathrm{X\,Y} + \cos. \mathrm{X\,Z})(x - x_1) + (\beta + \alpha \cos. \mathrm{X\,Y} + \cos. \mathrm{Y\,Z})(y - y_1) + (1 + \alpha \cos. \mathrm{X\,Z} + \beta \cos. \mathrm{Y\,Z})(z - z_1) = 0$

and this equation, which is the locus of all the straight lines meeting the given straight line at a given point and at right angles, is called the equation to the plane.

444. To find the conditions that a straight line be perpendicular to a given plane;

The method is the same as that in article 426.

The equation to a plane passing through a point $x_1\,y_1\,z_1$ in the given line is

$$m(x - x_1) + n(y - y_1) + p(z - z_1) = 0.$$

But the equations to the given line being

$$x = \alpha z + a, \; y = \beta z + b$$

the equation to the plane perpendicular to it is given at the end of the last article; hence, equating co-efficients we have

$$m = \alpha + \beta \cos. \mathrm{X\,Y} + \cos. \mathrm{X\,Z},$$
$$n = \beta + \alpha \cos. \mathrm{X\,Y} + \cos. \mathrm{Y\,Z},$$
$$p = 1 + \alpha \cos. \mathrm{X\,Z} + \beta \cos. \mathrm{Y\,Z}.$$

From these equations we have the values of m, n, p; or the values of α and β in terms of m, n, p.

445. To find the angle between a plane and straight line.

Let the given equations be

$$m x + n y + p z = 1 \quad (1)$$
$$\left.\begin{aligned} x &= \alpha z + a \\ y &= \beta z + b \end{aligned}\right\} \quad (2)$$

And let the equations to a straight line perpendicular to the given plane be

$$\left.\begin{aligned} x &= \alpha' z + a' \\ y &= \beta' z + b' \end{aligned}\right\} (3)$$

where α' and β' have the values of α and β in the last article.

Also the angle between the lines (2) and (3) is given in article (442.), and the angle between the plane and the line (1) being the complement of the angle between the two lines (2) and (3) may be obtained.

446. To find the angle between two planes.

The equations to the lines perpendicular to the given planes, and passing through the origin are given by Article (444.); and the angle between these lines, which is the angle between the given planes, is given by Article (442.)

CHAPTER V.

THE TRANSFORMATION OF CO-ORDINATES.

447. To transform an equation referred to an origin A to an equation referred to another origin A′, the axes in the latter case being parallel to those in the former.

The co-ordinates of the new origin being a, b, and c, it is evident that if a point be referred to this new origin and to the new axes, that each original ordinate is equivalent to the new ordinate together with the corresponding ordinate to the new origin; hence if x, y, z be the original co-ordinate of a point P, and X, Y, Z the new co-ordinates, we have

$$x = a + \mathrm{X},$$
$$y = b + \mathrm{Y},$$
$$z = c + \mathrm{Z};$$

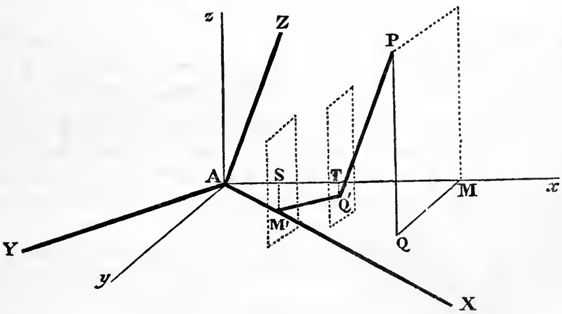
Substituting these values for x, y and z in the equation to the surface, we have the transformed equation between X, Y, and Z referred to the origin A′.

448. To transform the equation referred to rectangular axes to an equation referred to oblique axes having the same origin.

Let A x, A y, A z be the original axes,

A X, A Y, A Z the new axes,

$$\left.\begin{aligned}\mathrm{AM} &= x\\ \mathrm{MQ} &= y\\ \mathrm{QP} &= z\end{aligned}\right\}\quad \left.\begin{aligned}\mathrm{AM'} &= \mathrm{X}\\ \mathrm{M'Q'} &= \mathrm{Y}\\ \mathrm{Q'P} &= \mathrm{Z}\end{aligned}\right\}$$



Through the points M′, Q′, P draw planes parallel to $y\,z$, or, which is the same thing, perpendicular to A x and meeting A x in S, T and M (these planes are represented by the dotted lines in the figure). Then A S, S T and T M are the respective projections of A M′, M′ Q′ and Q′ P on A x, also

$$AM = AS + ST + TM,$$

$$= AM' \cos. XAx + M'Q' \cos. YAx + Q'P \cos. ZAx \quad (378)$$

$$\therefore \left.\begin{array}{l} x = X\cos. Xx + Y\cos. Yx + Z\cos. Zx \\ y = X\cos. Xy + Y\cos. Yy + Z\cos. Zy \\ z = X\cos. Xz + Y\cos. Yz + Z\cos. Zz \end{array}\right\}$$

$$\text{Or,} \quad \left.\begin{array}{l} x = mX + m_1 Y + m_2 Z \\ y = nX + n_1 Y + n_2 Z \\ z = pX + p_1 Y + p_2 Z \end{array}\right\}$$

where m is put for cos. X x, &c.

We have also, by art. 402, the following equation between the angles which one straight line, as A X, makes with the axes of x, y, z.

$$(\cos. Xx)^2 + (\cos. Xy)^2 + (\cos. Xz)^2 = 1,$$

Hence the following system,

$$\left.\begin{array}{l} m^2 + n^2 + p^2 = 1 \\ m_1^2 + n_1^2 + p_1^2 = 1 \\ m_2^2 + n_2^2 + p_2^2 = 1 \end{array}\right\} \quad 2.$$

449. If the new system be rectangular, we have also the equations in art. (405), which signify that the new axes are perpendicular to each other; hence the system

$$\left.\begin{array}{l} m m_1 + n n_1 + p p_1 = 0 \\ m m_2 + n n_2 + p p_2 = 0 \\ m_1 m_2 + n_1 n_2 + p_1 p_2 = 0 \end{array}\right\} \quad 3.$$

Hence we observe that of the nine cosines involved in the system (1) three are determined by the system (2), and other three by the system (3); and therefore that there are only three arbitrary angles remaining.

450. In the place of these three systems the following three may also be used:

$$\left.\begin{array}{l} X = mx + ny + pz \\ Y = m_1 x + n_1 y + p_1 z \\ Z = m_2 x + n_2 y + p_2 z \end{array}\right\} \quad 4.$$

$$\left.\begin{array}{l} m^2 + m_1^2 + m_2^2 = 1 \\ n^2 + n_1^2 + n_2^2 = 1 \\ p^2 + p_1^2 + p_2^2 = 1 \end{array}\right\} \quad 5. \qquad \left.\begin{array}{l} mn + mp + np = 0 \\ m_1 n_1 + m_1 p_1 + n_1 p_1 = 0 \\ m_2 n_2 + m_2 p_2 + n_2 p_2 = 0 \end{array}\right\} \quad 6.$$

For, multiplying the values of x, y and z in (1) by m, n and p respectively; then adding the results together, and reducing by means of (2) and (3), we have $X = mx + ny + pz$; and repeating this operation with the other multipliers $m_1 n_1 p_1$ and $m_2 n_2 p_2$, we have the system (4). Also, since the distance of P from the origin is the same for both systems, we have $x^2 + y^2 + z^2 = X^2 + Y^2 + Z^2$; putting here, for X, Y and Z, their values in (4), and then equating co-efficients on both sides, we have the two systems (5) and (6).

Whenever we see the systems (2) and (3), we may replace them by (5) and (6); this may be proved independently of any transformation of co-ordinates, by assuming the quantities $m\,n\,p$, &c. to be connected as in (1).

451. The transformation from oblique axes to others oblique, is effected by drawing a perpendicular from M in the last figure upon the

plane of $y\,z$, and by projecting x, X, Y, and Z on this perpendicular, we shall have

$$x \sin. x, y\,z = \mathrm{X} \sin. \mathrm{X}, y\,z + \mathrm{Y} \sin. \mathrm{Y}, y\,z + \mathrm{Z} \sin. \mathrm{Z}, y\,z;$$

and similarly for the other two, x and y,

$$y \sin. y, x\,z = \mathrm{X} \sin. \mathrm{X}, x\,z + \mathrm{Y} \sin. \mathrm{Y}, x\,z + \mathrm{Z} \sin. \mathrm{Z}, x\,z,$$

$$z \sin. z, x\,y = \mathrm{X} \sin. \mathrm{X}, x\,y + \mathrm{Y} \sin. \mathrm{Y}, x\,y + \mathrm{Z} \sin. \mathrm{Z}, x\,y.$$

452. Another useful method of transformation from rectangular axes to others also rectangular, is the following:

Let the equations to the axes of X, Y and Z be respectively

$$\left.\begin{array}{l} x = \alpha\,z \\ y = \beta\,z \end{array}\right\} \qquad \left.\begin{array}{l} x = \alpha_1\,z \\ y = \beta_1\,z \end{array}\right\} \qquad \left.\begin{array}{l} x = \alpha_2\,z \\ y = \beta_2\,z \end{array}\right\}$$

and let

$$m = \frac{1}{\sqrt{1 + \alpha^2 + \beta^2}},\ m_1 = \frac{1}{\sqrt{1 + \alpha_1^2 + \beta_1^2}}, m_2 = \frac{1}{\sqrt{1 + \alpha_2^2 + \beta_2^2}}$$

then by art. (402.) we have

$$\cos. \mathrm{X}\,x = m\,\alpha, \quad \cos. \mathrm{X}\,y = m\,\beta, \quad \cos. \mathrm{X}\,z = m\,; \text{ \&c.}$$

Hence by substitution, the first formulas for transformation in art. (448.) become

$$x = m\,\alpha\,\mathrm{X} + m_1\,\alpha_1\,\mathrm{Y} + m_2\,\alpha_2\,\mathrm{Z}$$

$$y = m\,\beta\,\mathrm{X} + m_1\,\beta_1\,\mathrm{Y} + m_2\,\beta_2\,\mathrm{Z}$$

$$z = m\,\mathrm{X} + m_1\,\mathrm{Y} + m_2\,\mathrm{Z}.$$

And the nine angles in (1) are replaced by the six unknown terms $\alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2$.

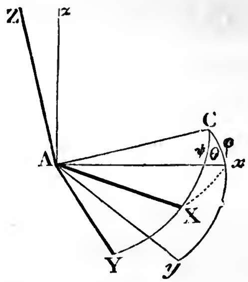
Instead of these systems, we may obtain a system involving only five arbitrary constants by supposing the solid trihedral angle formed by the original co-ordinate planes to turn about the origin into a new position: such a system has been ably discussed by M. Gergonne in the "Annales de Maths.," tome vii. p. 56.

453. It appears throughout these articles that only three arbitrary quantities are absolutely necessary; and therefore it might be supposed that formulas for transformation would be obtained involving only three angles: such formulas have been discovered by Euler, and as they are generally useful in various branches of analysis, we proceed to their investigation.

Let A C be the intersection of the original plane of $x\,y$ with the new plane of X Y, and suppose the plane C X Y A to lie above the plane C $x\,y$ A, which last we may assume to be the plane of the paper.

Let a sphere be described with centre A and radius unity, cutting all the axes in the points indicated by their respective letters.

Let $C\,x = \phi$, $C\,X = \psi$, and let the angle $X\,C\,x$ between the planes $x\,y$ and $X\,Y$ be called θ.



Then the object is to substitute in formula (1) art. (448.) the values of the cosines in terms of the new variables ϕ, ψ, and θ.

This is effected by means of the elementary theorem in spherical trigonometry for finding one side of a triangle in terms of the other two and the included angle. In the triangles $C\,X\,x$ and $C\,Y\,x$, we have

$$\cos.\, X\,x = \cos.\, \theta \sin \psi \sin.\, \phi + \cos.\, \psi \cos.\, \phi$$

$$\cos.\, Y\,x = \cos.\, \theta \sin.\, (90^\circ + \psi) \sin.\, \phi + \cos.\, (90^\circ + \psi) \cos.\, \phi$$

$$= \cos.\, \theta \cos.\, \psi \sin.\, \phi - \sin.\, \psi \cos.\, \phi.$$

Similarly $\cos.\, X\,y$ and $\cos.\, Y\,y$ may be found.

Also, supposing $Z\,x$ and $Z\,C$ to be joined by arcs of the sphere, we have from the triangle $Z\,C\,x$

$$\cos.\, Z\,x = \cos.\, Z\,C\,x \sin.\, Z\,C \sin.\, C\,x + \cos.\, Z\,C \cos.\, C\,x$$

$$= \cos.\, (90^\circ + \theta) \sin.\, 90^\circ \sin.\, \phi + \cos.\, 90^\circ \cos \phi$$

$$= - \sin.\, \theta \sin.\, \phi.$$

Similarly $\cos.\, Z\,y$, $\cos.\, X\,z$, and $\cos.\, Y\,z$ may be determined.

And $\cos.\, Z\,z = \cos.\, \theta$; hence the system (1) becomes

$$\begin{aligned} x = {} & X\,(\cos.\, \theta \sin.\, \psi \sin.\, \phi + \cos.\, \psi \cos.\, \phi) \\ & + Y\,(\cos.\, \theta \cos.\, \psi \sin.\, \phi - \sin.\, \psi \cos.\, \phi) \\ & - Z \sin.\, \theta \sin.\, \phi \end{aligned}$$

$$\begin{aligned} y = {} & X\,(\cos.\, \theta \sin.\, \psi \cos.\, \phi - \cos.\, \psi \sin.\, \phi) \\ & + Y\,(\cos.\, \theta \cos.\, \psi \cos.\, \phi + \sin.\, \psi \sin.\, \phi) \\ & - Z \sin.\, \theta \cos.\, \phi \end{aligned}$$

$$z = X \sin.\, \theta \sin.\, \psi + Y \sin.\, \theta \cos.\, \psi + Z \cos.\, \theta.$$

These are the formulas investigated, but in a different manner, by Laplace, " Méc. Cél." i. p. 58. They will be found in most works on this subject, but often with some slight alteration in the algebraic signs of the terms, arising from the various positions of A C.

THE INTERSECTION OF A SURFACE BY A PLANE.

454. The last system may be advantageously employed in finding the nature of the intersection of curve surfaces made by planes. If we propose to cut a surface, as a cone for example, by a plane, we should eliminate z from the equations to the surface and plane; but this gives us the equation to the projection of their intersection on $x\,y$, not the equation to the intersection itself; and as the projection will not always suffice to determine the nature of a curve, it is requisite to find the equation to that curve traced on the cutting plane.

This may be done by a transformation of co-ordinates.

Let the cutting plane be that of X Y, and the trace A C the axis of X, the surface will then be referred to new axes X, Y, Z, of which X and Y are in the cutting plane. By putting $Z = 0$ in the equation thus transformed, we shall have the intersection of the surface with the plane X Y, which is the intersection required.

Now, as the present object is only to obtain the curve of intersection, we may at first put $Z = 0$, and then transform the equation.

Let therefore $Z = 0$, and the angle C A X or $\psi = 0$, then the last formulas become

$$\begin{aligned} x &= \mathrm{X} \cos.\phi + \mathrm{Y} \sin.\phi \cos.\theta \\ y &= -\mathrm{X} \sin.\phi + \mathrm{Y} \cos.\phi \cos.\theta \\ z &= \mathrm{Y} \sin.\theta. \end{aligned}$$

These formulas may be separately investigated, with great ease, without deduction from the general case.—See "Francœur," vol. ii. art. 369, or "Puissant, Géométrie," art. 134.

455. In applying these formulas to a particular case, a little consideration will greatly alleviate the labour of transformation: thus, in many cases, we may suppose the cutting plane to be perpendicular to $x\,z$, without at all diminishing the generality of the result, but only adding much to its simplicity; for in this case the trace A C either coincides with A y or y A produced, and therefore $\phi = 90°$; hence the ast formulas become

$$\begin{aligned} x &= +\mathrm{Y} \cos.\theta \\ y &= -\mathrm{X} \\ z &= \mathrm{Y} \sin.\theta. \end{aligned}$$

These formulas may be readily investigated by drawing a figure like the last, but letting A C, A X and y A produced coincide, $\phi = 90°$ and C Y $=$ 90°, and then taking the original formulas (1) in art. 448.

456. If in the above cases the origin is also changed, we must intro duce the quantities a, b, c into the left side of the above equations.

CHAPTER VI.

THE SPHERE AND SURFACES OF REVOLUTION.

457. A CURVE surface as a sphere being given for discussion, we proceed as in plane geometry to find its equation from some known property of the surface; and generally we arrive at a relation between three unknown quantities x, y, and z, which relation is expressed by the symbol $f(x, y, z) = 0$, or $z = f(x, y)$. This equation is called the equation to the surface, and it corresponds to all points of the surface, and to it alone.

458. Conversely, an equation of the form $f(x, y, z) = 0$, where x, y, and z represent the co-ordinates of a point, refers to some surface. That it cannot belong to all the points in a solid may be thus shown.

Let there be two equations $f(x, y, z) = 0$, and $f'(x, y, z) = 0$; giving to x, y, and z the same values in both these equations, and then eliminating z, we have the equation to the intersection of the above loci projected on the plane of xy: this equation is of the form $\phi(x\ y) = 0$, and therefore it belongs to a *line*. Similarly the projections of the intersection on the other co-ordinate planes are *lines*; but if the projections of a locus on three different planes are lines, the locus itself must be a line, that is, it cannot be a surface. Hence the intersection of the two loci of $f(x, y, z) = 0$, and $f'(x, y, z) = 0$ being a line, each of these equations must belong to a surface.

459. Surfaces as well as lines are divided into orders, and for the same object, to avoid the confusion of ideas and to allow us to unite the important properties of generality and simplicity in our investigations as far as possible. Hence a plane which is the locus of a simple equation between three unknown quantities is called a surface of the first order; the locus of an equation of two dimensions between three unknown quantities is called a surface of the second order, and so on. The length, rather than the difficulty of the mathematical operations, renders this part of the subject tedious. Hence we shall omit many of the investigations which merely require manual labour, and rather dwell upon what we consider the important steps.

A much more serious difficulty arises from the state of the figures: we cannot give complete graphical illustrations of this part of geometry, and a mind unaccustomed to the conception of solid figures cannot always comprehend the meaning of the corresponding analytical results. We have endeavoured to obviate this difficulty as much as possible by descriptions of what the figures intend to represent, and to these descriptions we beg the particular attention of our readers, for we are convinced that this part of geometry is by no means difficult, if attention be paid to the form of the body; but without this care it is quite unintelligible.

We commence with the discussion of the Sphere.

THE SPHERE.

460. To find the equation to the surface of a sphere.

Let the surface be referred to rectangular axes, and let x, y, z be the co-ordinates of any point on the surface, and a, b, c the corresponding co-ordinates of the centre. Then since the surface is such that the distance of any point in it from the centre of the sphere is constant or equal to a line r, called the radius, we have by art. (388.)

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

461. This equation will assume various forms corresponding to the position of the centre.

Let the centre be in the plane of xy $\therefore$ $c = 0$,

$$\therefore\ (x-a)^2 + (y-b)^2 + z^2 = r^2.$$

Let the centre be on the axis of z $\therefore$ $a = 0$, and $b = 0$,

$$\therefore\ x^2 + y^2 + (z-c)^2 = r^2.$$

462. Let the centre be the origin $\therefore$ $a = b = c = 0$, and the equation is

$$x^2 + y^2 + z^2 = r^2.$$

And this is the equation to the surface of the sphere most generally used.

463. The general equation upon expansion becomes

$$x^2 + y^2 + z^2 - 2ax - 2by - 2bz + a^2 + b^2 + c^2 - r^2 = 0.$$

And hence the sphere corresponding to any equation of this form may be described as for the circle, art. 67.

464. The sections of a surface made by the co-ordinate planes are called the principal sections of the surface, and the boundaries of the principal sections are called the traces of the surface on the co-ordinate planes.

The equation to a trace is determined by putting the ordinate perpendicular to the plane of the trace $= 0$ in the general equation. Thus, to find the curve in which the sphere cuts the plane of xy, put $z = 0$, and then we have the equation to the points where the plane and sphere meet, which in this case is

$$x-a)^2 + (y-b)^2 + c^2 = r^2.$$

Hence the section on xy is a circle as long as x and y have real values And, similarly, the other traces are circles.

The theorem that the intersection of any plane with a sphere is a circle. is best proved geometrically, as in Geometry, *b*. v. 19.

465. To find the equation to the tangent plane to a sphere.

Let $x_1\, y_1\, z_1$ be the co-ordinates of the point on the surface through which the tangent plane passes, and let the equation to the spherical surface be

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2;$$

then the equation to the plane passing through the point $x_1\, y_1\, z_1$ is

$$m(x-x_1) + n(y-y_1) + p(z-z_1) = 0.$$

Also, the equations to the radius passing through the points $(a\ b\ c$ $(x_1\, y_1\, z_1)$ are

$$x - x_1 = \frac{x_1 - a}{z_1 - c}(z - z_1), \quad y - y_1 = \frac{y_1 - b}{z_1 - c}(x - x_1).$$

And since every line in the tangent plane, and therefore the plane itself, is perpendicular to the radius at the point of tangence, we have from the equations to the plane and line

$$\frac{m}{p} = \frac{x_1 - a}{z_1 - c}, \quad \frac{n}{p} = \frac{y_1 - b}{z_1 - c} \quad (426.)$$

Hence the equation to the tangent plane becomes

$$\frac{x_1 - a}{z_1 - c}(x - x_1) + \frac{y_1 - b}{z_1 - c}(y - y_1) + z - z_1 = 0,$$

or, $(x_1 - a)(x - x_1) + (y_1 - b)(y - y_1) + (z_1 - c)(z - z_1) = 0$

This equation may be modified by means of the condition

$$(x_1 - a)^2 + (y_1 - b)^2 + (z_1 - c)^2 = r^2,$$

or, $(x_1 - a)(x_1 - a) + (y_1 - b)(y_1 - b) + (z_1 - c)(z_1 - c) = r^2$.

Adding this equation, term by term, to the above one for the tangent plane, we have

$$(x_1 - a)(x - a) + (y_1 - b)(y - b) + (z_1 - c)(z - c) = r^2.$$

466. If the origin is in the centre of the sphere, the equation to the tangent plane is

$$x x' + y y' + z z' = r^2,$$

which equation is at once obtained from that to the sphere $x^2 + y^2 + z^2 = r^2$, or, $x x + y y + z z = r^2$, by putting $x x'$, $y y'$, and $z z'$ for $x x$, $y y$, and $z z$ respectively.

The line in which the tangent plane cuts any co-ordinate plane is found by putting the ordinate perpendicular to that plane $= 0$; and the point in which the tangent plane cuts any axis is found by putting the two variables measured along the other axes each 0.

467. The equation to the spherical surface referred to oblique co-ordinates by (440.) is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 + 2(x - a)(y - b)\cos. XY + 2(x - a)(z - c)\cos. XZ + 2(y - b)(z - c)\cos. YZ = r^2.$$

ON COMMON SURFACES OF REVOLUTION.

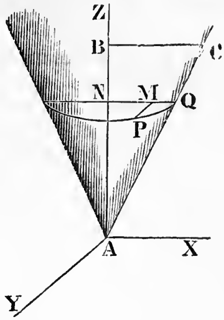
468. A right cone is formed by the revolution of the hypothenuse of a right-angled triangle about one of its sides.

Let A C be the side which revolves about A B as an axis, so that any section Q P perpendicular to the axis is a circle.

Let A X, A Y, A Z be the rectangular axes to which the cone is referred, having the origin at the vertex of the cone, and the axis of Z coincident with the axis of the cone.

Let $\left.\begin{array}{l} AN = z \\ NM = x \\ MP = y \end{array}\right\}$ be the co-ordinates of any point on the surface.

Then the squares on N M and M P = the square on N P



and N P = N Q = A N tan. C A B,

therefore the equation to the surface is

$$x^2 + y^2 = e^2 z^2,$$

where e = tangent of the semiangle of the cone.

469. Let the line A C be a curve, as a parabola, for example, in which case the surface is called the common paraboloid.

Let the equation to the generating parabola A Q C be N Q = $\sqrt{p z}$ · Then the squares on N M, M P = the square on N P = the square on N Q,

$$\therefore\ x^2 + y^2 = p z.$$

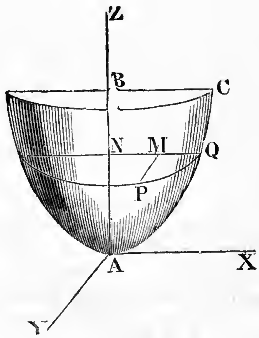
470. Let A C be an ellipse, centre and origin at B.

Let B N = z, N M = x, and M P = y, C B = b and B A = a.

Then the squares on N M and M P = the square on N Q; and N Q being an ordinate to the ellipse A Q C, whose semiaxes are a and b, we have

$$\text{N Q} = \frac{b}{a}\sqrt{a^2 - z^2},$$

and therefore the equation to the surface is



$$x^2 + y^2 = \frac{b^2}{a^2}(a^2 - z^2);$$

$$\text{or, } x^2 + y^2 + \frac{b^2}{a^2} z^2 = b^2$$

Let a and b change places in the equation, we have then for the surface of revolution round the axis minor the equation

$$x^2 + y^2 + \frac{a^2}{b^2} z^2 = a^2.$$

The former surface is called the prolate spheroid, the latter the oblate spheroid.

471. The equation to the hyperboloid round the transverse axis is

$$x^2 + y^2 - \frac{b^2}{a^2} z^2 = - b^2.$$

And putting a for b and b for a, we have the surface by revolution round the conjugate axis.

472. In general the equation to all these surfaces may be comprehended under the form $x^2 + y^2 = f(z)$ if A Z be the axis of revolution; or, $z^2 + y^2 = f(x)$ if A X be the axis of revolution.

To find the curve of intersection of a plane and a surface of revolution.

473. Let the section be made by a plane perpendicular to $x z$, and as the nature of the curve is the same in whatever part of the cutting plane we place the origin, we shall let the origin be in the plane $x z$.

Then the formulas for transformation are

$$x = a + y \cos. \theta$$
$$y = - x$$
$$z = c + y \sin. \theta.$$

Hence by substitution in the equation to a surface, we shall have the required curve of intersection.

474. Let the surface be a paraboloid

$$x^2 + y^2 = p z$$
$$\therefore (a + y \cos. \theta)^2 + x^2 = p (c + y \sin. \theta)$$

or, $y^2 (\cos. \theta)^2 + x^2 + (2 a \cos. \theta - p \sin. \theta) y = 0$, since $a^2 = p c$;

hence the curve of intersection is a line of the second order.

It is an ellipse generally (76); a circle if $\theta = 0$; and a parabola similar to the generating one, if $\theta = 90°$.

475. Let the surface be the spheroid formed by the revolution of an ellipse round its axis major

$$x^2 + y^2 + \frac{b^2}{a^2} z^2 = b^2;$$

by substitution this equation becomes

$$y^2 \{ (\cos. \theta)^2 + \frac{b^2}{a^2} (\sin. \theta)^2 \} + x^2 + 2 y \{ c \frac{b^2}{a^2} \sin. \theta - a_1 \cos. \theta \} = 0.$$

This is the equation to an ellipse generally, and to a circle when $\theta = 0$.

476. Let the surface be the hyperboloid, whose equation is

$$x^2 + y^2 - \frac{b^2}{a^2} z^2 = - b^2,$$

the sections will be found to depend on the value of $\tan. \theta$: if $\tan. \theta$ is less than $\frac{b}{a}$, the curve is an ellipse; if it is equal to $\frac{b}{a}$, the curve is a parabola; and if $\tan. \theta$ is greater than $\frac{b}{a}$, it is an hyperbola; and lastly, a circle if $\theta = 0$

CHAPTER VII.

SURFACES OF THE SECOND ORDER.

477. The general equation to surfaces of the second order is

$$a x^2 + b y^2 + c z^2 + 2 d x y + 2 e x z + 2 f y z + 2 g x + 2 h y + 2 i z + k = 0$$

the number 2 being prefixed to some of the terms merely for convenience. In order to discuss this equation, that is, to examine the nature and position of the surfaces which it represents, we shall render it more simple by means of the transformation of co-ordinates.

Let the origin be transferred by putting

$$x = x' + m,\ y = y' + n,\ z = z' + p,$$

substituting these values in the general equation, and then putting the terms containing the first powers of the variables each $= 0$, we have the equation

$$a x'^2 + b y'^2 + c z'^2 + 2 d x' y' + 2 e x' z' + 2 f y' z' + k' = 0.$$

This equation remains the same if we change x', y', z', into $-x', -y', -z'$ respectively; thence we conclude that any straight line drawn through the origin, and intercepted by the surface, will be divided into two equal parts at the origin; this new origin therefore will be the centre of the surface, attributing to this expression the same signification as we did in treating of curves of the second order (81.)

478. The values of m, n, and p, are to be determined from the three equations

$$\begin{aligned} a m + d n + e p + g &= 0, \text{ co-efficient of } x', \\ b n + d m + f p + h &= 0, \quad \ldots \quad y', \\ c p + e m + f n + i &= 0, \quad \ldots \quad z'. \end{aligned}$$

Eliminate p from the first and second of these equations, and also from the first and third, then from the two resulting equations eliminate n, and we shall arrive at an equation of the first order involving m, whence we have the value of m, and therefore of n and p.

The denominator of the values of m, n and p is

$$a b c + 2 d e f - a f^2 - b e^2 - c d^2$$

hence, if this quantity $= 0$, the values of m and p are infinite, or the surface has no centre when there is this relation among the co-efficients of the original equation. This circumstance corresponds to the case of the parabola in lines of the second order (81.)

479. To destroy the co-efficients of the terms involving $x' y'$, $x' z'$, and $y' z'$, we must have recourse to another transformation of co-ordinates.

Taking the formulas in (452) we have

$$x' = m\,\alpha\,x'' + m_1\,\alpha_1\,y'' + m_2\,\alpha_2\,z''$$
$$y' = m\,\beta\,x'' + m_1\,\beta_1\,y'' + m_2\,\beta_2\,z''$$
$$z' = m\,x'' \quad + m_1\,y'' \quad + m_2\,z''$$

Substituting in the general equation, and then putting the co-efficients of $x''\,y''$, $x''\,z''$, and $y''\,z''$, each $= 0$, we have the three equations

$$(a\,\alpha + d\,\beta + e)\,\alpha_1 + (d\,\alpha + b\,\beta + f)\,\beta_1 + e\,\alpha + f\beta + c = 0 \ldots\ldots x''\,y''$$
$$(a\,\alpha + d\,\beta + e)\,\alpha_2 + (d\,\alpha + b\,\beta + f)\,\beta_2 + e\,\alpha + f\beta + c = 0 \ldots\ldots x''\,z''$$
$$(a\,\alpha_2 + d\,\beta_2 + e)\,\alpha_1 + (d\,\alpha_2 + b\,\beta_2 + f)\,\beta_1 + e\,\alpha_2 + f\beta_2 + c = 0 \ldots y''\,z''$$

Our object is now to ascertain if this transformation can always be effected, that is, to determine the possibility of the values of the six unknown quantities in the last three equations.

480. The equations to the new axis of y'' are $x = \alpha_1\,z$, $y = \beta_1\,z$ (452.); hence, by substitution, the first of the above three equations becomes

$$(a\,\alpha + d\,\beta + e)\,x + (d\,\alpha + b\,\beta + f)\,y + (e\,\alpha + f\,\beta + c)\,z = 0,$$

which is the equation to a plane passing through the origin.

Now the co-ordinates of every point in this plane satisfy the condition that the co-efficient of $x''\,y'' = 0$, that is, give the necessary relation between α_1 and β_1; hence, if the new axis of y'' be drawn in this plane the condition is still satisfied. Thus, the direction of the axis of x'' being quite arbitrary, that of y'' is determined to be in the particular plane given above; and the term $x''\,y''$ is gone.

Again, by a similar elimination of α_2 and β_2 from the co-efficient of $x''\,z''$, and from the equations of z'' ($x = \alpha_2\,z$, $y = \beta_2\,z$), we have, from the similarity of the equations, the same plane as before; hence, if the axis of z'' be also drawn in this plane, the term $x''\,z''$ will disappear.

Also, α_2 and β_2 being thus obtained, the relation between α_1 and β_1 may be found from the co-efficient of $y''\,z'' = 0$.

Thus, fixing upon any position of the axis of x'', that is, giving any values to α and β, we have determined a plane passing through the origin, in which plane any two straight lines whatever drawn from the origin may be the axes of y'' and z'', and one of them as z'' being so drawn, α_2 and β_2 are given, and then the relation between α_1 and β_1 is determined from the co-efficient of $x''\,y'' = 0$.

But since the relation between these quantities α_1 and β_1, and not the quantities themselves, is given by the last equation, it appears that there are an infinite number of systems to which, if the axes be transferred, the products of the variables may be destroyed.

481. Let the new axes be rectangular.

In this case the axis of x'' must be perpendicular to the plane of $y''\,x''$, or the line whose equations are $x = \alpha\,z$, $y = \beta\,z$ is perpendicular to the plane

$$(a\,\alpha + d\,\beta + e)\,x + (d\,\alpha + b\,\beta + f)\,y + (e\,\alpha + f\,\beta + c)\,z = 0$$
$$\therefore\ a\,\alpha + d\beta + e = (e\,\alpha + f\beta + c)\,\alpha \ (426)$$
$$d\,\alpha + b\,\beta + f = (e\,\alpha + f\beta + c)\,\beta$$

Substituting in the first of these equations the value of α obtained from the second, we have the following equation for β:

$$\begin{aligned}&\{ (a-b) f e + (f^2 - e^2) d \} \beta^3 \\ &+ \{ (a-b)(c-b) e + (2 d^2 - f^2 - e^2) e + (2c - a - b) f d \} \beta^2 \\ &+ \{ (c-a)(c-b) d + (2 c^2 - f^2 - d^2) d + (2b - a - c) f e \} \beta \\ &+ \{ (a-c) f d + (f^2 - d^2) e \} = 0.\end{aligned}$$

This equation of the third degree has at least one real value for β, and hence a real value of α; thus the position of the axis of x'' is found, and also the position of the perpendicular plane in which y'' and z'' are situated.

Again, we might find a plane $x''\,z''$ perpendicular to y'', and such that the terms in $x''\,y''$, $y''\,z''$ should disappear, and the necessary conditions will, as appears from the similarity of the equations, lead to the same equation of the third degree in β_1, and the same is true for the axis of z''.

Hence the three roots of the above equation of the third degree are the three real values of β, β_1 and β_2.

These three quantities give the three corresponding values of α, α_1 and α_2, and since there are only one value of each quantity, it appears that there is only one system of rectangular axes to which the curve surface can be referred so as not to contain the products of the variables. For further information on this subject, see "Annales Math." ii. p. 144.

482. By the last transformation, the equation when the locus has a centre is reduced to the form

$$a_1\,x''^2 + b_1\,y''^2 + c_1\,z''^2 + k_1 = 0$$
$$\text{or, } \mathrm{L}\,x^2 + \mathrm{M}\,y^2 + \mathrm{N}\,z^2 = 1$$

by substitution and the suppression of accents, which are no longer necessary.

The order of transformation might have been inverted, by first destroying the products of the variables exactly in every respect as in the last article, and then the resulting equation must be deprived of three terms by a simple change of the origin; the result, after both transformations, is

$$\mathrm{L}\,x^2 + \mathrm{M}\,y^2 + \mathrm{N}\,z^2 + \mathrm{P}\,x = 0.$$

483. The central equation involves three distinct cases, which depend on the signs of the quantities L, M, and N.

(1) They may be all positive.

(2) Two may be positive, and the third negative.

(3) One may be positive, and the other two negative.

They cannot be all negative.

Substituting for L, M and N, the constants $\frac{1}{a^2}$ $\frac{1}{b^2}$ $\frac{1}{c^2}$ respectively, where a is $> b$ and $b > c$, the three cases are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

The readiest way of obtaining the form of these surfaces is by sections either in planes parallel to the co-ordinate planes, or on the co-ordinate planes. We remark again, that in the latter case they are called the principal sections or traces.

THE ELLIPSOID.

484. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

For the trace on $x\,y$, $z = 0$, $\therefore \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

. . . . $x\,z$, $y = 0$, $\therefore \dfrac{x^2}{a^2} + \dfrac{z^2}{c^2} = 1$

. . . . $y\,z$, $x = 0$, $\therefore \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

Therefore the principal sections are ellipses.

Let $z = m$ $\therefore$ the section parallel to $x\,y$ is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 - \dfrac{m^2}{c^2}$ *

$y = n$ . . . . $x\,z$ is $\dfrac{x^2}{a^2} + \dfrac{z^2}{c^2} = 1 - \dfrac{n^2}{c^2}$

$x = p$ . . . . $y\,z$ is $\dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1 - \dfrac{p^2}{c^2}$.

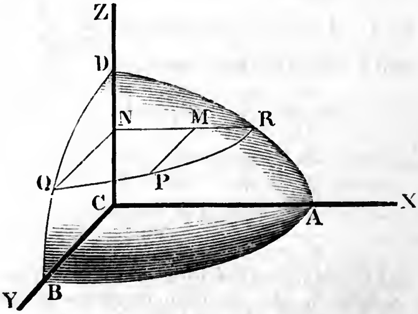
The first of these equations is an ellipse from m, or $z = 0$ to $z = c$; when $z = c$ the curve becomes a point, and when z is greater than c the ellipse is imaginary, therefore the surface is limited in the direction of z. Similarly it may be proved, that the other sections are ellipses, and the surface is limited in the directions of x and y. From the circumstance of this surface being thus limited in every direction, and also from the above sections being all ellipses, this surface is called the ellipsoid.

The diameters $2\,a$, $2\,b$, $2\,c$ of the principal sections are called the diameters of the ellipsoid, and their extremities are the vertices of the surface.

If $b = a$, the equation becomes $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} + \dfrac{z^2}{c^2} = 1$, which is the equation to a spheroid by revolution round the axis of z.

If any other two co-efficients are equal, we have spheroids round the other axes; and if $a = b = c$, the surface becomes a sphere.

485. To render the conception of this surface clear we subjoin a figure representing the eighth part of an ellipsoid.



* This equation belongs to the projection on $x\,y$, but since the plane of $x\,y$ is parallel to that of $z = m$, the projection is exactly the same in form as the curve of section itself.

A B is part of the ellipse on $x\,y$
A D . . . . $x\,z$
B D . . . . $y\,z$,

and the section Q P R parallel to $x\,y$ is also an ellipse.

The surface may be conceived to be generated by a variable ellipse C A B moving upwards parallel to itself with its centre in C Z. Let N Q R be one position of this variable ellipse; and let

C N $= z$,	C A $= a$,	N R $= x_1$;
N M $= x$,	C B $= b$,	N Q $= y_1$
M P $= y$,	C D $= c$,	

Then from the ellipse Q P R we have

$$\frac{x^2}{x_1^2}+\frac{y^2}{y_1^2}=1$$

Also from the ellipses D R A and D Q B we have

$$\frac{x_1^2}{a^2}+\frac{z^2}{c^2}=1,\text{ and }\frac{y_1^2}{b^2}+\frac{z^2}{c^2}=1$$

Therefore $\frac{x_1^2}{a^2}=\frac{y_1^2}{b^2}$; and multiplying the first equation by $\frac{x_1^2}{a^2}$ or its equal $\frac{y_1^2}{b^2}$, we have $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{x_1^2}{a^2}=1-\frac{z^2}{c^2}$.

$$\therefore\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

THE HYPERBOLOID.

486. Case 2. $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

The principal sections are

$$\text{on } x\,y,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (1)$$

$$\text{on } x\,z,\ \frac{x^2}{a^2}-\frac{z^2}{c^2}=1 \quad (2)$$

$$\text{on } y\,z,\ \frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \quad (3).$$

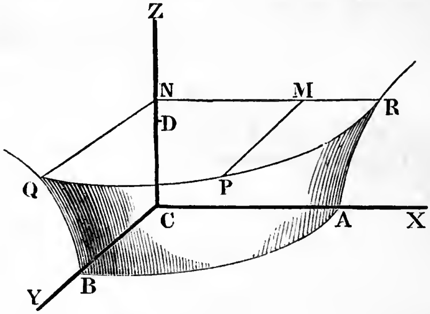
(1) is the equation to an ellipse whose axes are $2\,a$ and $2\,b$; (2) and (3) are hyperbolas with the same imaginary conjugate axis $2c\sqrt{-1}$; if x is less than a, or y less than b, z is imaginary.

Giving to z, y, and x the values m, n, and p, respectively, we have the section parallel to $x\,y$ an ellipse, to $y\,z$ and $x\,z$ hyperbolas.

487. The accompanying figure represents a portion of the eighth part of this surface. A B is the ellipse on $x\,y$, A R the hyperbola on $x\,z$, and B Q is the other hyperbola on $y\,z$. This surface may also be conceived

to be generated by a variable ellipse C A B moving parallel to itself with it centre in C Z. Let N Q R be one position of this variable ellipse ; and le

$$\text{C N} = z, \qquad \text{C A} = a, \qquad \text{N R} = x_1\,;$$
$$\text{N M} = x, \qquad \text{C B} = b, \qquad \text{N Q} = y_1\,;$$
$$\text{M P} = y, \qquad \text{C D} = c,$$



Then from the ellipse P Q R, we have

$$\frac{x^2}{x_1^2} + \frac{y^2}{y_1^2} = 1$$

Also from the hyperbolas A R and B Q we have

$$\frac{x_1^2}{a^2} - \frac{z^2}{c^2} = 1, \text{ and } \frac{y_1^2}{b^2} - \frac{z^2}{c^2} = 1,$$

therefore $\frac{x_1^2}{a^2} = \frac{y_1^2}{b^2}$; and multiplying the first equation by $\frac{x_1^2}{a^2}$ or its equal $\frac{y_1^2}{b^2}$, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x_1^2}{a^2} = 1 + \frac{z^2}{c^2}$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

This surface is called the hyperboloid of one sheet because it forms one continuous surface or sheet.

If $a = b$ the surface becomes the common hyperboloid of revolution round the *conjugate* axis.

488. Through the origin draw a line, whose equations are $x = \alpha z$, $y = \beta z$, and substituting in the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, we have

$$\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - \frac{1}{c^2}\right) z^2 = 1$$

$$\therefore z = \pm \frac{a\,b\,c}{\sqrt{b^2 c^2 \alpha^2 + a^2 c^2 \beta^2 - a\,b^2}};$$

hence this line meets the surface as long as the denominator of the fraction is real and finite; let $b^2 c^2 \alpha^2 + a^2 c^2 \beta^2 = a^2 b^2$, then the line only meets the surface at an infinite distance, or is an asymptote to the surface. The last equation gives the relation between α and β, when the corresponding line is an asymptote; and if for α and β we substitute their general values $\frac{x}{z}$ and $\frac{y}{z}$, we obtain an equation between x, y, z, whose locus will consist of all the asymptotes to the surface, because the co-ordinates of any point in it have the required relation above.

The equation to this surface is

$$b^2 c^2 x^2 + a^2 c^2 y^2 = a^2 b^2 z^2$$

$$\text{or, } \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}:$$

We shall hereafter show (art. 514.) that this is the equation to a cone whose vertex is the origin, and whose base, or section parallel to the axis, is an ellipse.

489. Case 3.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

The principal sections are

$$\text{on } xy, \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (1)$$

$$\text{on } xz, \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \qquad (2)$$

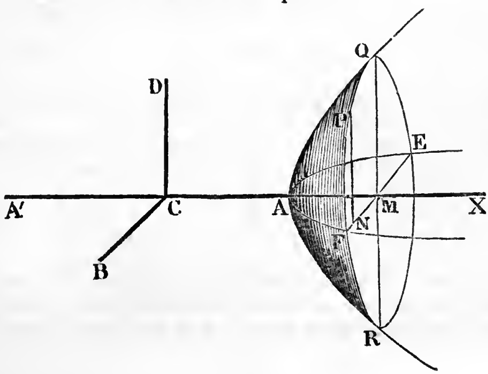
$$\text{on } yz, \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1 \qquad (3).$$

(1) is an hyperbola whose axes are $2a$ and $2b\sqrt{-1}$; (2) is an hyperbola whose axes are $2a$ and $2c\sqrt{-1}$; (3) is imaginary, therefore the plane of yz never meets the surface.

Of the sections parallel to the co-ordinate planes, those parallel to xy and xz are hyperbolas, and that parallel to yz is an ellipse, whose equation is

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{p^2}{a^2} - 1;$$

hence this ellipse is imaginary, if p or x is less than $\pm a$; therefore, if two planes are drawn parallel to yz, and at distances $\pm a$ from the centre, no part of the surface can be between these planes.



In the figure E A F represents the hyperbolic section on $x y$, and Q A R that on $x z$; E Q F R is an elliptic section parallel to $y z$. There is an equal and opposite sheet with its vertex at A′; hence the surface is called the hyperboloid of two sheets.

490. The equation to the surface is deduced from the figure; let $A M = x$, $M N = y$, $N P = z$; $Q M = z_1$, $M F = y_1$;

Then from the elliptic section Q P F R we have

$$\frac{z^2}{z_1^2} + \frac{y^2}{y_1^2} = 1.$$

Also from the hyperbolas E A F and Q A R we have

$$\frac{y_1^2}{b^2} - \frac{x^2}{a^2} = -1, \text{ and } \frac{z_1^2}{c^2} - \frac{x^2}{a^2} = -1$$

therefore $\frac{y_1^2}{b^2} = \frac{z_1^2}{c^2}$; and multiplying the first equation by $\frac{z_1^2}{c^2}$ or its equal $\frac{y_1^2}{b^2}$, we have

$$\therefore \frac{z^2}{c^2} + \frac{y^2}{b^2} = \frac{z_1^2}{c^2} = \frac{x^2}{a^2} - 1$$

$$\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

491. $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$, is the equation to the conical asymptote; hence both in case (2) and (3) we have the conical asymptote by omitting the constant term in the equations.

ON SURFACES WHICH HAVE NO CENTRE.

492. In this case the general equation can be deprived of the products of the variables, as in (479); it will then be of the form

$$a x^2 + b y^2 + c z^2 + 2 g x + 2 h y + 2 i z + k = 0.$$

In order to deprive this equation of three more terms, let

$$x = m + x', \; y = n + y', \; z = p + z',$$

$$\therefore a x'^2 + b y'^2 + c z'^2 + 2 (a m + g) x' + 2 (b n + h) y' + 2 (c p + i) z' + k' = 0:$$

Let the co-efficients of x', y' and $z' = 0$;

$$\therefore m = -\frac{g}{a}, \; n = -\frac{h}{b}, \; p = -\frac{i}{c}:$$

But since this class has no centre, the values of some, or all the quantities m, n, p, must be infinite; therefore, either one, two, or three of the co-efficients a, b, c, must $= 0$. Thus the original transformation which deprived the equation of the terms $x y$, $x z$, and $y z$, has of itself destroyed one or two of the co efficients of x^2, y^2, or z^2; this corresponds to the case in art. 92. Now, all three co-efficients cannot $= 0$, for then we fall upon the equation to a plane: hence we have only two cases left, when a vanishes, or when a and b both vanish.

493. Let $a = 0$, then, as we have three quantities, m, n and p to determine, we may let $k' = 0$ as well as the co-efficients of y' and z'; hence the equation is reduced to the form

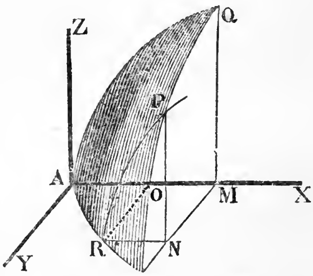
$$b y'^2 + c z'^2 + 2 g x' = 0.$$

$$\text{or} \left(-\frac{b}{2g}\right) y'^2 + \left(-\frac{c}{2g}\right) z'^2 = x'.$$

This equation has two varieties depending upon the signs of the quantities $-\frac{b}{2g}$ and $-\frac{c}{2g}$.

494. Case 1. Let the signs of y'^2 and z'^2 be both alike and positive, (if they were negative we should change the sign of x' to reduce the equation to the same form) substituting $\frac{1}{l}$ for $-\frac{b}{2g}$, and $\frac{1}{l'}$ for $-\frac{c}{2g}$, and suppressing the accents on x, y and z as no longer necessary, the equation is of the form

$$\frac{y^2}{l} + \frac{z^2}{l'} = x.$$



For the principal sections we have

on $x y$, $y^2 = l x$. (1)

on $x z$, $z^2 = l x'$. (2)

on $y z$, $l' y^2 + l z^2 = 0$ (3)

(1) and (2) are parabolas extending on the side of x positive; (3) is a point, which is the origin itself.

For the sections parallel to

$$x y, \text{ put } z = p, \therefore \frac{y^2}{l} = x - \frac{p^2}{l'} \qquad (1)$$

$$x z, \text{ put } y = n, \therefore \frac{z^2}{l'} = x - \frac{n^2}{l} \qquad (2)$$

$$y z, \text{ put } x = m, \therefore \frac{y^2}{l} + \frac{z^2}{l'} = m \qquad (3)$$

(1) and (2) are parabolas, equal to those of the principal sections respectively, (the equation differing by a constant term, implies that the origin is differently situated with regard to the curve): (3) is an ellipse.

495. In the figure A Q and A R are parts of the parabolas on $x z$ and $x y$, and the surface is described by the motion of the parabola A Q, parallel to itself, its vertex moving along the parabola A R. Let P R N be one position of the generating parabola, and let A M $= x$, M N $= y$, N P $= z$, and draw R O parallel to A Y or M N; then from the parabola R P we have

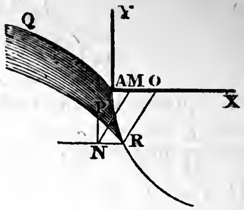
$$z^2 = l' \,\mathrm{RN} = l' (\mathrm{AM} - \mathrm{AO}) = l' \left(x - \frac{y^2}{l} \right)$$

$$\therefore \frac{z^2}{l'} + \frac{y^2}{l} = x.$$

This surface is called the elliptic paraboloid, and is composed of one entire sheet, like the paraboloid of revolution.

496 Case 2. Let the signs of y'^2 and z'^2 be different.

$$\therefore \frac{y^2}{l} - \frac{z^2}{l'} = x.$$



For the principal sections we have

on $x y$, $y^2 = l x$ (1)
on $x z$, $z^2 = - l'x$ (2)
on $y z$, $l'y^2 - l z^2 = 0$ (3)

(1) and (2) are parabolas, the first corresponding to x positive, and the second to x negative; (3) belongs to two straight lines through the origin.

The sections in planes parallel to $x y$ and $x z$ are parabolas, and those parallel to $y z$ are hyperbolas.

497. A Q is the parabola on $x z$, and A R is that on $x y$; and the surface is described by the motion of the parabola A Q parallel to itself, its vertex moving along the parabola A R. Let R P N be one position of the generating parabola, and let A M $= x$, M N $= y$, and N P $= z$, and draw R O parallel to M N; then from the parabola R Q we have

$$z^2 = l' \mathrm{RN} = l' (\mathrm{AO} - \mathrm{AM}) = l' \left(\frac{y^2}{l} - x,\right)$$

$$\therefore \frac{y^2}{l} - \frac{z^2}{l'} = x.$$

This surface is called the hyperbolic paraboloid.

498. The equations to the elliptic and hyperbolic paraboloids may be deduced from those of the ellipsoid and hyperboloid of one sheet, as the equation to the parabola was deduced from that to the ellipse (228) by supposing the centre to be infinitely distant.

Let the origin be transferred to a vertex of the surface, by putting $x - a$ for x, then the equation to the ellipsoid and hyperboloid is

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1.$$

Let m and m' be the distances of the vertex from the foci or the sections on $x y$ and $x z$;

$$\therefore b^2 = a^2 - (a - m)^2 = 2 a m - m^2$$

$$\text{and } c^2 = 2 am' - m'^2;$$

therefore, by substitution, the equation

$$\frac{x^2}{a^2} - \frac{2 x}{a} + 1 + \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1$$

$$\text{becomes } \frac{x^2}{a^2} - \frac{2 x}{a} + \frac{y^2}{2 a m - m^2} \pm \frac{z^2}{2 a m' - m'^2} = 0$$

$$\text{or } \frac{x^2}{a} - 2 x + \frac{y^2}{2 m - \frac{m^2}{a}} \pm \frac{z^2}{2 m' - \frac{m'^2}{a}} = 0,$$

$$\text{or } \frac{y^2}{2 m} \pm \frac{z^2}{2 m'} - 2 x = 0, \text{ when } a \text{ is infinite.}$$

And hence results obtained for the ellipsoid and hyperboloid will be true for the paraboloids, after making the above substitutions.

499. We stated in article 492, that both a and b might vanish; in this case the equation will be

$$c z^2 + 2 g x + 2 h y + 2 i z + k = 0.$$

And by the transformation in art. 492, we cannot destroy the co-efficients of x and y, but we may destroy that of z, and also the constant term k; hence the transformed equation is reduced to the form

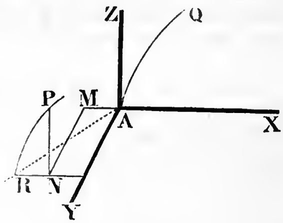
$$c z^2 + 2 g x + 2 h y = 0;$$

$$\text{or } z^2 = l x + l' y \text{ if } -\frac{2g}{c} = l, \text{ and } -\frac{2h}{c} = l'.$$

500. There are two cases depending on the signs of l and l', which may be both positive, or one positive and the other negative.

Case 1. l and l' both positive.

The section on $x y$ is $l x + l' y = o$ (1)
on $x z$ is $z^2 = l x$ (2)
on $y z$ is $z^2 = l' y$ (3)



(1) is a straight line A B; (2) is a parabola A Q; (3) is also a parabola, not in the figure; the sections on the planes parallel to the above are similar in each case. The surface is formed by the motion of the parabola A Q parallel to itself, its vertex describing the straight line A R; let R P N be one position of the generating parabola; let A M $= x$, M N $= y$, N P $= z$,

$$\text{then } z^2 = l.\, \text{R N} = l\left(\frac{l'}{l} y + x\right) = l' y + l x.$$

Since this surface is a cylinder with a parabolic base, it is not usually classed among the surfaces of the second order.

Case 2. If the signs of l and l' be different, the surface will be the same, but situated in a different manner.

CHAPTER VIII.

CYLINDRICAL AND CONICAL SURFACES.

501. Our notion of surfaces will be very much enlarged, if we take into consideration the general character of classes of surfaces, defining them by their peculiar method of generation, and then expressing that definition in a general algebraical form. For example, we have been accustomed, in common geometry, to consider a cylinder as a surface generated by a straight line, which is carried round the circumference of a given circle, and always parallel to a given straight line. (Geom. b. v

def. 1.) But it is evident that if the base be not a circle, but any other curve, as a parabola, for instance, we shall have a surface partaking of the essential cylindrical character, and which, with others of the same kind, come under a more extended definition; and similarly for conical and many other surfaces.

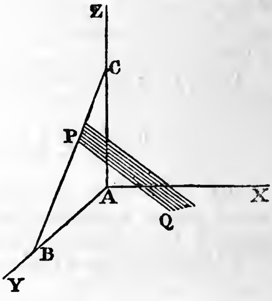
Having seized upon this general character, method of generation, or law by which the lines are compelled to move, the next step is to express this fact in algebraical language; that is, to obtain an equation between co-ordinates x, y, and z, of any point on the surface, which equation shall belong to the class of surfaces in the first instance, and then can be adapted to any particular surface in that class.

THE PLANE.

502. In order to prepare the reader for this subject, we shall take a simple case: to find the surface generated by the motion of a straight line, parallel to itself, and constrained to pass through a given straight line.

Let A X, A Y, A Z be rectangular axes, and let the equations to the given straight line B C (supposed for the sake of simplicity to be in the plane of y z) be

$$\left.\begin{array}{r} nY + pZ = 1 \\ X \quad\quad = 0 \end{array}\right\} 1$$



Also, let the equations to the generating line P Q, in any one of its positions, be

$$\left.\begin{array}{l} x = \alpha z + a \\ y = \beta z + b \end{array}\right\} 2.$$

Now, α and β are the tangents of the angles which the projections of P Q make with the axes A X and A Y respectively; and in the motion of P Q, parallel to itself, the projections also remain parallel to themselves respectively; and hence α and β are always constant, and therefore are known or given quantities. But a and b being the co-ordinates of the point where the line P Q meets the plane of x y, they change with every change of position of P Q; and therefore, being variable, must not appear in the final equation to the surface. Now, these variable quantities, a and b, can be expressed in terms of the other variable quantities x, y, z; and hence we can thus estimate them from the two given systems above.

At the point P, where P Q meets B C, we have, by comparison of (1) and (2),

$$X = x = 0$$
$$Z = z = -\frac{a}{\alpha}$$
$$Y = y = -\frac{a\beta}{\alpha} + b$$

But the system (1) is true for any values of X, Y, Z; therefore, by substitution in (1), we have

$$n\left(-\frac{a\beta}{\alpha} + b\right) + p\left(-\frac{a}{\alpha}\right) = 1$$

and this is the equation connecting a and b together, or expressing the relation which the variable quantities a and b have to each other, or the relation which any quantities equal to a and b have to each other; that is, substituting for a and b the quantities $x - \alpha z$, and $y - \beta z$ from (2), we shall have the relation between the quantities x, y, and z, which is called the equation to a surface.

$$\therefore -n\frac{\beta}{\alpha}\left(x - \alpha z\right) + n\,(y - \beta z) - \frac{p}{\alpha}\left(x - \alpha z\right) = 1;$$

$$\text{or, } -\frac{n\beta + p}{\alpha}\,x + n\,y + p\,z = 1,$$

which is the equation to a plane; and this is the most general method of determining the equation to a plane; for it can be thus found for any system of co-ordinate axes, and it is determined from the most obvious character of the plane.

We now proceed to the discussion of surfaces formed by the motion of a straight line constrained to move after some given law or condition.

ON CYLINDRICAL SURFACES.

503. Definition. A cylindrical surface is generated by a straight line, which moves parallel to itself in space, and describes, with its extremity, a given curve.

The straight line which moves is called the Generatrix; and the given curve is called the Directrix.

To find the equation to the surface,

Let the equation to the generatrix, in any one of its positions, be

$$x = \alpha z + a$$
$$y = \beta z + b$$

Now, the generatrix, in its movement, always moving parallel to itself; the quantities α and β remain the same for every position of the generatrix; but the quantities a and b, which are the co-ordinates of the point where the generatrix meets the plane of $x\,y$, are constant for the same position of the generatrix, but vary when the generatrix passes from one position to another. Thus, when any point on the surface changes its position without quitting the generatrix, a and b are both constant; and when the point moves from one position of the generatrix to another, a and b are both variable; hence these two quantities, being constant together, and variable together, must be dependent on each other in some way or another; which general dependence is expressed by saying that one of them is a function of the other

$$\therefore b = \phi(a);$$

or, putting for b and a their values as above, we have

$$y - \beta z = \phi(x - \alpha z),$$

which is the general equation to cylindrical surfaces.

504. The form of the function ϕ will depend upon the nature of the directrix in any particular case.

Let the equations to the directrix be

$$\left.\begin{array}{r} F(X, Y, Z,) = 0 \\ f(X, Y, Z,) = 0 \end{array}\right\}$$

Then as the generatrix must in all its positions meet the directrix, the equations to this curve and to the generatrix must exist simultaneously for the points of intersection; thus having four equations we may eliminate x, y, z, and arrive at an equation between a, b, and constant quantities, which will determine the form of the function ϕ.

Substituting in this equation for a and b their values $x - \alpha z$, $y - \beta z$, we have the actual equation to the particular cylinder required.

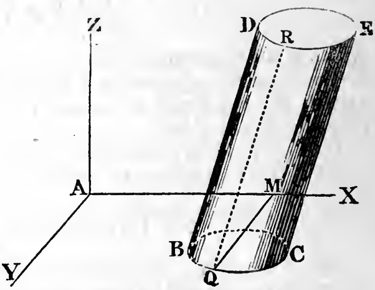
505. Ex. 1. Let the directrix be the circle B Q C, in the plane of $x\,y$, and let x_1 and y_1 be the co-ordinates of its centre; then the equations to the directrix are

$$\left.\begin{array}{r} (X - x_1)^2 + (Y - y_1)^2 = r^2 \\ Z = 0 \end{array}\right\} (1)$$

Let B D, Q R, C E, be various positions of the generatrix whose general equation is

$$\left.\begin{array}{r} x = \alpha z + a \\ y = \beta z + b \end{array}\right\} (2)$$

to express that the generatrix meets



the circle as at Q, the equations (1) and (2) must exist together

$$\therefore Z = z = 0$$
$$X = x = a$$
$$Y = y = b$$

substituting these values in (1), we have

$$(a - x_1)^2 + (b - y_1)^2 = r^2 \qquad (3)$$

hence the form of the function ϕ is determined.

Substituting in (3) the values of a and b from (2), we have

$$(x - \alpha z - x_1)^2 + (y - \beta z - y_1)^2 = r^2$$

This is the equation to an oblique cylinder, with circular base, situated in the plane of $x\,y$.

506. Let the centre of the circle be at the origin,

$$\therefore x_1 = 0 \text{ and } y_1 = 0$$
$$\therefore (x - \alpha z)^2 + (y - \beta z)^2 = r^2$$

And if the origin be at the extremity of a diameter parallel to the axis of x,

$$(x - \alpha z)^2 + (y - \beta z)^2 = 2r(x - \alpha z)$$

507. Let the axis of the cylinder be parallel to the axis of z; then α and β

each $= 0$, since they are the tangents of the angles which the projection of the generatrix on $x\,z$ and $y\,z$ make with A Z;

$$\therefore\ (x - x_1)^2 + (y - y_1)^2 = r^2;$$

and if the axis coincide with A Z, $x^2 + y^2 = r^2$, $z = 0$;

in these cases the cylinder is called a right cylinder, and its equation is the same as that of the directrix.

If the directrix be a circle on $x\,z$, the equation to the right cylinder will be

$$x^2 + z^2 = r^2.$$

508. Let the directrix be a parabola on $x\,y$, vertex at the origin, and axis coincident with the axis of x.

Then the equations to the directrix and generatrix are

$$\left.\begin{array}{l} Y^2 = p\,X \\ Z = 0 \end{array}\right\} 1 \qquad \left.\begin{array}{l} x = \alpha\,z + a \\ y = \beta\,z + b \end{array}\right\} 2;$$

therefore at the points of junction we have

$$\begin{array}{l} Z = z = 0 \\ X = x = a \\ Y = y = b; \end{array}$$

then by substituting in (1) we have

$$b^2 = p\,a$$

$$\therefore\ (y - \beta\,z)^2 = p\,(x - \alpha\,z)$$

which is the equation to an oblique parabolic cylinder, whose base is on $x\,y$.

509. Let the directrix be a parabola on $x\,z$, axis A X, and vertex at A; and let the generatrix be parallel to the plane $x\,y$.

The equations are

$$\left.\begin{array}{l} Z^2 = p\,X \\ Y = 0 \end{array}\right\} 1 \qquad \left.\begin{array}{l} y + \alpha\,x = a \\ z = b \end{array}\right\} 2$$

Then the equation to the surface is

$$z^2 = \frac{p}{\alpha}\,y + p\,x. \qquad \text{See article (499).}$$

ON CONICAL SURFACES.

510. Definition. A conical surface is generated by the movement of a straight line, which passes constantly through a given point, and also describes a given curve.

The given point is called the centre of the surface, the straight line which moves is called the generatrix, and the given curve is called the directrix.

Let a, b, c, be the co-ordinates of the centre; then the equations to the generatrix are

$$\begin{array}{l} x - a = \alpha\,(z - c) \\ y - b = \beta\,(z - c). \end{array}$$

Now when a point on the surface changes its position without quitting the generatrix, the quantities α, β are constant, but when the point passes from one generatrix to another, they are both variable; hence being constant together, and variable together, they are functions of one another;

$$\therefore\ \beta = \phi\,(\alpha), \text{ or substituting their equals,}$$

$$\frac{y - b}{z - c} = \phi\left(\frac{x - a}{z - c}\right) \text{ which is the general equation}$$

to conical surfaces

511. The form of the function ϕ will depend upon the nature of the directrix in any particular case.

By combining the equations to the generatrix and directrix we may, as for cylindrical surfaces, eliminate x, y, z, in a particular case, and thus arrive at an equation between α and β, which will determine the form of the function ϕ.

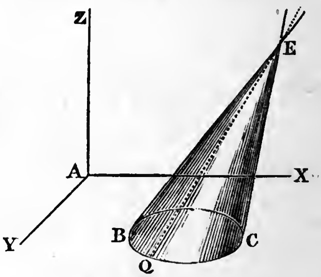
Substituting in this equation for α and β their values $\frac{x-a}{z-c}$ and $\frac{y-b}{z-c}$ we obtain the actual equation to the particular conical surface.

512. Ex. Let the directrix be a circle B Q C in the plane of $x\,y$.

The equations to this directrix are

$$\left.\begin{array}{c}(X-x_1)^2+(Y-y_1)^2=r^2\\ Z=0\end{array}\right\}\quad(1)$$

And the equations to the generatrix B E, or Q E passing through the point E (a, b, c), are



$$\left.\begin{array}{l}x-a=\alpha\,(z-c)\\ y-b=\beta\,(z-c)\end{array}\right\}\quad(2)$$

To express that the generatrix meets the circle, the equations (1) and (2) must coexist.

$$\therefore\ Z=z=0$$
$$X=x=a-\alpha\,c$$
$$Y=y=b-\beta\,c$$

hence by substitution in (1) we have

$$(a-\alpha\,c-x_1)^2+(b-\beta\,c-y_1)^2=r^2\quad(3)$$

Putting for α and β their values from (2) and reducing

$$\left(\frac{a\,z-c\,x}{z-c}-x_1\right)^2+\left(\frac{b\,z-c\,y}{z-c}-y_1\right)^2=r^2$$

This is the equation to an oblique cone with a circular base situated in the plane of $x\,y$.

Let the centre of the circle be at the origin $\therefore x_1=o$ and $y_1=o$;

$$\therefore\ (a\,z-c\,x)^2+(b\,z-c\,y)^2=r^2\,(z-c)^2$$

513. Let the axis of the cone be parallel to the axis of z $\therefore$ $a=x_1$ and $b=y_1$, and the general equation becomes

$$\left(\frac{x-a}{z-c}\right)^2+\left(\frac{y-b}{z-c}\right)^2=\frac{r^2}{c^2}$$

In this case the cone is called a right cone.

Also, if in this case the origin be at the centre of the circle, we have $a=o$ and $b=o$,

$$\therefore\ x^2+y^2=\frac{r^2}{c^2}\,(z-c)^2.$$

514. Directrix an ellipse on $x\,y$, whose centre is the origin, and the centre of the cone in the axis of z; then the equation to the cone is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{z-c}{c}\right)^2$$

or putting z, for $z-c$ that is, measuring from the centre of the cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}.$$

In this simple case, the equation to the surface is easily found by the method in article (468). Taking the figure in that article, and supposing every section, like P Q, to be an ellipse, whose axes x_1 and y_1 are always proportional to the axes a and b of an ellipse whose centre is in A Z, and at a distance c from A, we have the equation to P Q

$$\frac{x^2}{x_1^2}+\frac{y^2}{y_1^2}=1,$$

$$\text{but } y_1=\frac{b}{a}x_1, \text{ and } x_1=\frac{a}{c}z$$

$$\therefore \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}.$$

515. Let the directrix be a parabola parallel to $x\,y$, and vertex in the axis of z. The equations to the directrix and generatrix are

$$\left.\begin{array}{l} Y^2=p\,X \\ Z\ =d \end{array}\right\} 1 \qquad \left.\begin{array}{l} x-a=\alpha\,(z-c) \\ y-b=\beta\,(z-c) \end{array}\right\}$$

at the points of junction we have

$$\begin{array}{l} Z=z=d \\ X=x=a+\alpha\,(d-c) \\ Y=y=b+\beta\,(d-c) \end{array}$$

hence the final equation is

$$\left\{b+\frac{y-b}{z-c}(d-c)\right\}^2=p\left\{a+\frac{x-a}{z-c}(d-c)\right\}$$

516. Let the vertex or centre of the cone be at the origin $\therefore a=b=c=o$, and the equation to a cone whose directrix is $\{y^2=p\,x,\ z=d\}$ and whose vertex is at the origin, is

$$d\,y^2=p\,x\,z.$$

517. The following method of finding the equation to a right cone whose vertex is at the origin, is sometimes useful.

Let the length of the axis of the cone be k, and suppose this axis to pass through the origin, and be perpendicular to a given plane or base whose equation therefore will be of the form

$$\alpha\,x+\beta\,y+\gamma\,z=k$$

where α, β, γ are the co-sines of the angles which k makes with the axis of x, y, and z (410).

Also suppose x, y, and z to be the co-ordinates of a point on the circumference of this base, and let θ be the angle which the generatrix of the cone makes with its axis, then by the property of the right-angle triangle we have the equation

$$k=\sqrt{(x^2+y^2+z^2)}\cos.\theta$$

Hence by equating the values of k we have the equation,

$$(\alpha\,x+\beta\,y+\gamma\,z)^2=(x^2+y^2+z^2)\,(\cos.\theta)^2$$

And this is the equation to any point in the surface, since α, β, γ remain the same for a plane parallel to the base and passing through any point $(x\ y\ z)$ of the surface.

If the axis of the cone coincides with the axis of z, we have $\alpha = \beta = o$ and $\gamma = 1$; $\quad z^2 = (x^2 + y^2 + z^2)(\cos.\ \theta)^2$

518. To find the curve of intersection of a plane and an oblique cone, we may suppose the cutting plane to pass through the origin of co-ordinates without detracting from the generality of the result. Substituting for x, y, z, in the equation, their values in 455, we readily find that the sections are lines of the second order and their varieties.

ON CONOIDAL SURFACES.

519. Definition. A conoidal surface is generated by the movement of a straight line constantly parallel to a plane, one extremity of the line moving along a given straight line, the other describing a given curve.

We shall commence with a simple case. Let the axis of z be one directrix, and let the generatrix be parallel to the plane of $x\ y$: then the equations to the generatrix in any one position are

$$\left.\begin{array}{l} y = \alpha\, x \\ z = b \end{array}\right\}$$

Now it is evident that when a point moves on the surface without quitting the generatrix, α and b are both constant, but when it passes from one position of the generatrix to another α and b are both variable; hence these quantities, being constant together and variable together, are functions of one another.

$$\therefore b = \phi\,(\alpha) \text{ or substituting their values.}$$

$$z = \phi\left(\frac{y}{x}\right)$$

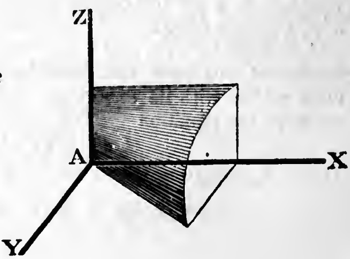
which is the general equation to all conoidal surfaces.

520. The form of the function ϕ will depend upon the nature of the second directrix.

By combining the equations to the generatrix and this directrix, we may, as before, eliminate x, y, z, and arrive at an equation between b and a, we must then substitute the values of b and a, their general values z and $\frac{y}{x}$, and we shall obtain the equation to the particular conoidal surface.

521. Let the second directrix be a circle parallel to $y\ z$, and the centre in the axis of x, therefore the equations to this directrix are

$$\left.\begin{array}{r} Z^2 + Y^2 = r^2 \\ X = a \end{array}\right\} (1)$$



Then where this directrix meets the generatrix we have

$$Z = z = b$$
$$X = x = a$$
$$Y = y = \alpha\, a$$
$$\therefore b^2 + \alpha^2 a^2 = r^2$$

Hence the required equation is

$$z^2 + a^2 \frac{y^2}{x^2} = r^2.$$

This surface partaking of the form and generation of both the cone and the wedge, was called the cono-cuneus by Wallis, who investigated many of its properties.

If the axis of x be one directrix, and the other be a circle parallel to $x\,z$, and the generatrix be parallel to $y\,z$, the equation is

$$x^2 + \frac{a^2 z^2}{y^2} = r^2.$$

522. Let the axis of z be one directrix, any straight line the other, and let the generatrix move parallel to $x\,y$.

Then the equations to the second directrix are

$$\begin{aligned} X &= \mu Z + m \\ Y &= \nu Z + n \end{aligned}$$

Also the equations to the generatrix being $y = \alpha x$, $z = b$, we have at the points of junction

$$\begin{aligned} Z &= z = b \\ Y &= y = \nu b + n \\ X &= x = \frac{\nu b + n}{\alpha} \end{aligned}$$

$$\therefore \frac{\nu b + n}{\alpha} = \mu b + m$$

$$\therefore (\nu z + n) \frac{x}{y} = \mu z + m$$

$$\text{or} \quad \nu z x - \mu z y + n x - m y = o.$$

523. Let the axis of z be one directrix, and let the second directrix be the thread of a screw whose axis is coincident with the axis of z.

The thread of a screw, or the curve called the helix, is formed by a thread wrapped round the surface of a right cylinder, so as always to make the same angle with the axis; or if the base of a right-angled triangle coincide with the base of the cylinder, and the triangle be wrapped round the cylinder, the hypothenuse will form the helix A P.

To find the equations to the helix,

Let the centre of the cylindrical base be the origin of rectangular axes. C M $= x$, M Q $= y$, P Q $= z$ and the radius of the cylinder $= a$.

Then P Q bears a constant ratio to A Q; namely, that of the altitude to the base of the describing triangle

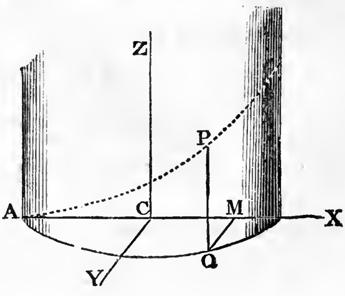
$\therefore$ P Q $= e$ A Q

and A Q is a circular arc whose sine is y and radius a:

$$\therefore z = e\,a \sin.^{-1} \frac{y}{a},$$

$$\text{or } z = e\,a \cos.^{-1} \frac{x}{a};$$

$$\text{also } x^2 + y^2 = a^2$$



And these are the equations to the projections of the helix.

To return to the problem, which is to find the surface described by a line subject to the conditions that it be parallel to the base of the cylinder, that it passes through the axis, and that it follows the course of the helix.

The equations to the directrix (if c be the interval between two threads) are

$$z = e\,a \sin.^{-1}\frac{y}{a} + c$$
$$x^2 + y^2 = a^2$$

And the equations to the generatrix being $y = \alpha\, x$, $z = b$; we have

$$z = b\,;\ x = \frac{y}{\alpha} = \sqrt{a^2 - y^2}\ \therefore\ y = \frac{\alpha\, a}{\sqrt{1+\alpha^2}}$$

$$\therefore\ b = e\,a \sin.^{-1}\frac{\alpha}{\sqrt{1+\alpha^2}} + c$$

hence the equation to the surface is

$$z = e\,a \sin.^{-1}\frac{y}{\sqrt{y^2 + x^2}} + c$$

This surface is the under side of many spiral staircases.

524. A straight line passes through two straight lines whose equations are $x = a$, $y = b$; and $x = a_1$, $z = b_1$; and also through a given curve $z = f(y)$ in the plane of $z\,y$; to find the equation to the surface traced out by the straight line.

The three directrices are

$$\left.\begin{matrix} X = a \\ Y = b \end{matrix}\right\}\ 1 \qquad \left.\begin{matrix} X = a_1 \\ Z = b_1 \end{matrix}\right\}\ 2 \qquad \left.\begin{matrix} Z = f(Y) \\ X = o \end{matrix}\right\}\ (3)$$

And let the equations to the generatrix be

$$x = \alpha\, z + m$$
$$y = \beta\, z + n$$

and consequently $y = \dfrac{\beta}{\alpha}\, x + p$, if $p = n - \dfrac{\beta}{\alpha}\, m$;

Then since this line meets the three given lines, we have the following equations

$$b = \frac{\beta}{\alpha}\, a + p,\quad a_1 = \alpha\, b_1 + m,\quad -\frac{m}{\alpha} = f\left(-\frac{m}{\alpha}\,\beta + n\right)$$

We must now eliminate α, b, m, n from these equations, and that to the generatrix.

By subtraction we have

$$y - b = \frac{\beta}{\alpha}\,(x - a)\,;\ x - a_1 = \alpha\,(z - b_1)\quad \therefore\ \alpha = \frac{x - a_1}{z - b_1},$$

$$\therefore\ \frac{m}{\alpha} = \frac{x}{\alpha} - z = \frac{a_1\, z - b_1\, x}{x - a_1},\ \text{and}$$

$$\frac{n\,\alpha - m\,\beta}{\alpha} = \frac{\alpha\,(y - \beta\, z) - \beta\,(x - \alpha\, z)}{\alpha} = \frac{\alpha\, y - \beta\, x}{\alpha} = y - \frac{\beta\, x}{\alpha} = \frac{b\, x - a\, y}{x - a}.$$

Hence the final equation is

$$\frac{b_1\, x - a_1\, z}{x - a_1} = f\left(\frac{b\, x - a\, y}{x - a}\right)$$

525. The following problem is easily solved in the same manner. To find the equation to a surface formed by a straight line moving parallel to the plane of $x\,z$, and having its extremities in two given curves $z = f(y)$ on $z\,y$, and $x = \phi(y)$ on $x\,y$.

The equation is $$\frac{z}{f(y)} + \frac{x}{\phi(y)} = 1.$$

526. In questions of this kind some care is requisite in selecting the position of the axes and co-ordinate planes, so that the equations, both those given and those to be found, may present themselves in the simplest form. For example, —to find the surface formed by the motion of a straight line constantly passing through three other given straight lines;

Take three lines parallel to the given lines for the axes of co-ordinates; then the equations to the three directrices are

$$\left.\begin{array}{l} X = a_1 \\ Y = b_1 \end{array}\right\} \qquad \left.\begin{array}{l} X = a_2 \\ Z = c_2 \end{array}\right\} \qquad \left.\begin{array}{l} Y = b_3 \\ Z = c_3 \end{array}\right\}$$

and the equations to the generating line in any position are

$$x = \alpha\,z + a,\; y = \beta\,z + b,$$

and consequently $y = \dfrac{\beta}{\alpha}\,x + c$, where $c = b - \dfrac{\beta}{\alpha}a$;

Then since this line meets each of three given lines, we have the following equations:

$$b_1 = \frac{\beta}{\alpha}\,a_1 + c\,;\; a_2 = \alpha\,c_2 + a\,;\; b_3 = \beta\,c_3 + b.$$

We must now eliminate a, b, α, β from these three equations and that to the generatrix; by subtraction we have

$$y - b_1 = \frac{\beta}{\alpha}(x - a_1)\,;\; x - a_2 = \alpha\,(z - c_2)\,;\; y - b_3 = \beta\,(z - c_3)$$

hence, eliminating α and β, we have the required equation

$$(x - a_1)\,(y - b_3)\,(z - c_2) = (x - a_2)\,(y - b)\,(z - c_3)$$

which is of the second order, since the term $x\,y\,z$ disappears. See Hymers's Anal. Geom. p. 23, Cambridge, 1830.

CHAPTER IX.

ON CURVES OF DOUBLE CURVATURE.

527. Definition. A curve of double curvature is one whose generating point is perpetually changing not only the direction of its motion, as in plane curves, but also the plane in which it moves.

If a circle be described on a flat sheet of paper, it is a plane curve; let the sheet of paper be rolled into a cylindrical form, then the circle has two curvatures, that which it originally had, and that which it has acquired by the flexion of the paper, hence in this situation it is called a curve of double curvature.

528. Curves of double curvature arise from the intersection of two surfaces; for example, place one foot of a pair of compasses on a cylindrical surface, let the other in revolving constantly touch the surface, it will describe a curve of double curvature, which, though not a circle, has yet all

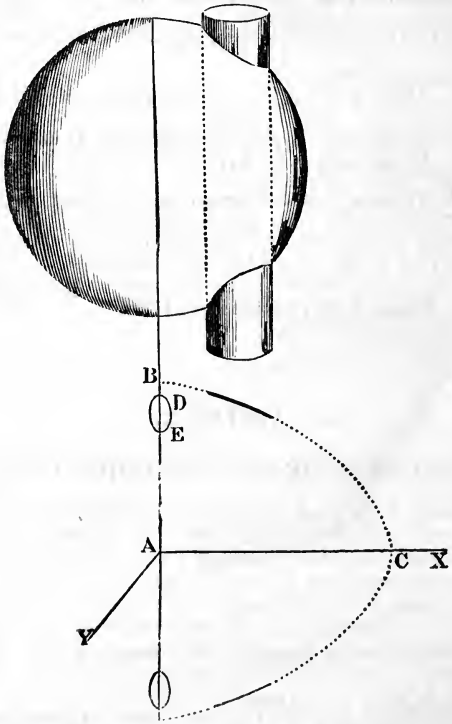
its points at equal distances from the fixed foot of the compasses. The curve is then part of a spherical surface, whose radius is equal to the distance between the feet of the compasses, and consequently is the intersection of this sphere with the cylinder.

529. The equations to the two surfaces taken together are the equations to their intersection, and consequently are the equations to the curve of double curvature.

By the separate elimination of the variables in the two equations, we obtain the respective projections of the curve upon the co-ordinate planes. Two of these are sufficient to define the curve of double curvature; for we may pass two cylinders through two projections of the curve, at right angles to each other, and to the co-ordinate planes, the intersection of these cylinders is the required curve. This is analogous to the consideration of a straight line, being the intersection of two planes.

We proceed to examine curves of double curvature arising from the intersections of surfaces.

530. Let the curve arise from the intersection of a sphere and right cylinder; the origin of co-ordinates being at the centre of the sphere, the axis of the cylinder in the plane $x\,z$ and parallel to the axis of z.



Let the distance between the centres of the sphere and cylinder $= c$, then the equation to the sphere is $x^2 + y^2 + z^2 = a^2$, and the equation to the cylinder is $(x - c)^2 + y^2 = b^2$, (507.)

eliminating y, $z^2 = a^2 + c^2 - b^2 - 2\,c\,x$ (1),

eliminating x, $z^2 = a^2 - b^2 - c^2 \mp 2\,c\,\sqrt{b^2 - y^2}$ (2).

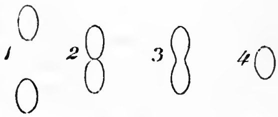
From (1) the projection of the curve on $x\,z$ is a portion of a parabola B C whose vertex is C, where $\mathrm{A\,C} = \dfrac{a^2 + c^2 - b^2}{2\,c}$. and $\mathrm{A\,B} = \sqrt{a^2 + c^2 - b^2}$.

From (2) the projection on $y\,z$ consists of two ovals, whose positions are determined by the two extreme values of z,

$$\mathrm{A\,D} = \pm \sqrt{a^2 - (b - c)^2}$$

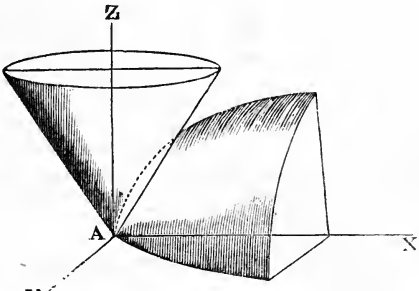
$$\mathrm{A\,E} = \pm \sqrt{a^2 - (b + c)^2}.$$

As c increases, that is, as the cylinder moves further from A, A E decreases, and the ovals approach nearer to each other, as in fig. (1); when $c = a - b$, that is, when the sphere but just encloses the cylinder A E $= 0$, and the ovals meet, fig. (2). As c increases, we obtain fig. (3), which gradually approaches fig. (4); and lastly, when $c = a$ vanishes entirely.



Different values, as c, $\frac{a}{2}$, &c., may be given to b, and we may then trace the projections: they offer no difficulty, but we recommend their investigation, as the complete examination of one example greatly facilitates the comprehension of all others.

531. Ex. 2. A right cone and a paraboloid of revolution have their vertices coincident, the axis of the cone being perpendicular to the axis of the paraboloid.



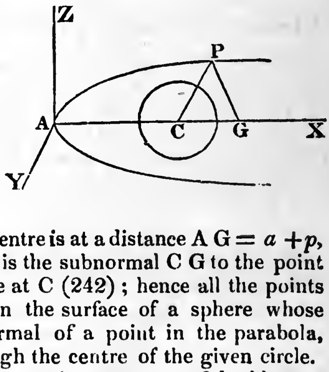
The equation to the cone is $x^2 + y^2 = e^2\,z^2$, (468) and that to the paraboloid, $y^2 + z^2 = p\,x$ (469); hence the projection on $x\,z$ is $x^2 + p\,x = (1 + e^2)\,z^2$, which is an hyperbola, whose axes are p and $\dfrac{p}{\sqrt{e^2 + 1}}$ (157).

Again, $x^2 + y^2 = e^2 (p\,x - y^2) \therefore (1 + e^2)\,y^2 = e^2\,p\,x - x^2$; hence the projection on $x\,y$ is an ellipse, whose vertex is A and axes $e^2\,p$ and $\frac{e^2\,p}{\sqrt{1 + e^2}}$ (103).

The equation to the projection on $y\,z$ is $(y^2 + z^2)^2 + p^2\,y^2 = e^2\,p^2\,z^2$; this is the equation to a Lemniscata, and becomes the Lemniscata of Bernouilli, when $e = 1$, that is, when the cone is right-angled (314).

532. To find the curve of intersection of two surfaces, we have eliminated the variables separately, and thus obtained the equations to the projections on the co-ordinate planes; conversely, by combining these last equations either by addition or multiplication, &c., so as to have an equation between the three variables, we may obtain the surface on which the curve of double curvature may be described. This surface does not at all define the curve of double curvature; since an infinite number of curves may be traced on this individual surface, to all of which the general equation to the surface belongs.

The results of the above combination are often interesting. For example: Let the curve be the intersection of a parabolic cylinder on $x\,y$, with a circular cylinder on $x\,z$, the origin being the vertex of the parabola, and the centre of the circle being in the axis of the parabola, which is also the axis of x.



Let $y^2 = 2\,p\,x$ be the equation to the parabola A P on $x\,y$,
$(x - a)^2 + z^2 = r^2$.... circle on $x\,z$,
Combining these equations by addition,
$(x - a)^2 - 2\,p\,x + y^2 + z^2 = r^2$,
or $(x - a - p)^2 + y^2 + z^2 = r^2 + p^2 + 2\,a\,p$.

Which is the equation to a sphere whose centre is at a distance A G $= a + p$, measured from A along A X. Now, p is the subnormal C G to the point P of the parabola, P C being the ordinate at C (242); hence all the points of the curve of double curvature are on the surface of a sphere whose centre is at the extremity of the subnormal of a point in the parabola, the ordinate of which point passes through the centre of the given circle.

533. The intersections of surfaces are not always curves of double curvature, but often they are plane curves. We proceed, then, to show how plane curves may be detected, and their equations determined.

Whenever we obtain a straight line for a projection, the curve cannot be one of double curvature.

Ex. Let the curve be the intersection of two parabolic cylinders, whose equations are

$$x^2 = a\,z$$
$$b\,y = x^2.$$

Eliminating x, we have $b\,y = a\,z$, hence the projection on $y\,z$ is a straight line; and as no projection of a curve of double curvature can be a straight line, it follows that the curve of intersection is a plane curve.

534. Again, If we can so combine the equations to the projections as to produce the general equation to a plane, the curve, which is necessarily traced on that plane, is itself a plane curve. For example: let the curve

arise from the intersection of two parabolic cylinders, whose equations are

$$x^2 = a\,z$$
$$b\,y = x^2 + c\,x.$$

In the second equation, substituting $a\,z$ for x^2, we obtain

$$b\,y = a\,z + c\,x;$$

which equation belonging to a plane, the curve is a plane curve.

535. There is another and more general method of detecting plane curves. From the two equations to the surfaces eliminate one of the variables, as z, for example, we obtain an equation $F(x, y) = 0$.

Now, if the curve be plane, it may arise from the intersection of either of the surfaces with a plane whose equation is $z = m\,x + n\,y + p$; eliminate z between this equation to the plane and that to one of the surfaces, the result is $f(x, y) = 0$, which must be identical with $F(x, y) = 0$; therefore, comparing $F(x, y) = 0$, with $f(x, y) = 0$, we may obtain various equations to determine m, n, and p; which values of m, n, and p must satisfy all the equations in which these quantities appear; if not, the curve is one of double curvature.

For example; take the intersection of a sphere and cylinder, art. 530.

The equation to the Sphere is $x^2 + y^2 + z^2 = a^2$ (1)
.... Cylinder $(x - c)^2 + y^2 = b^2$ (2)
.... Plane $z = m\,x + n\,y + p$ (3)

Eliminating z between (1) and (3), we have $f(x, y) = 0$

$$(m^2+1)\,x^2 + (n^2+1)\,y^2 + 2\,m\,n\,x\,y + 2\,m\,p\,x + 2\,n\,p\,y + p^2 - a^2 = 0 \quad (4)$$

Comparing (2) and (4), we have $m = 0$, $n = 0$ from the co-efficients of x^2 and y^2; but the condition of $m = 0$ destroys the coefficient of x in (4); and thereby shows that (4) cannot be made identical with (2). The curve is therefore a curve of double curvature.

But let the equation to the cylinder be $x^2 + y^2 = b^2$, then $m = 0$ and $n = 0$ render (4) and (2) identical; therefore the curve is a plane curve, situated in a plane, whose equation is $z = \sqrt{a^2 - b^2}$; this is clear, also, from geometrical considerations.

536. To find the curve represented by the equations

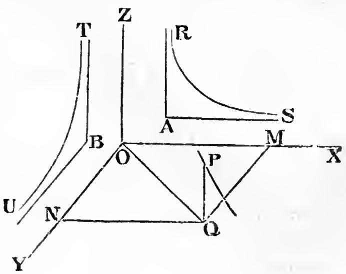
$$\frac{a}{x} + \frac{c}{z} = 1,\ \frac{b}{y} + \frac{c}{z} = 1.$$

These equations, taken separately, belong to two right hyperbolic cylinders; one with the base in $x\,z$, and the other in $y\,z$. (209, Ex. 3.)

R S is the hyperbola on $x\,z$, its centre being at A; T U is the hyperbola on $y\,z$, its centre being at B.

Also, $\dfrac{a}{x} = \dfrac{b}{y}$, or $y = \dfrac{b}{a}\,x$.

Hence the projection of the intersection of the above cylinders on $x\,y$ is a straight line O Q, and therefore the curve is a plane curve, situated in the plane Z O Q, perpendicular to $x\,y$.



537. As we cannot have a very clear notion of the curve itself, merely from the idea of the two hyperbolic cylinders, we shall find the equation to the curve in the plane Z O Q; that is, in its own plane.

Let P be any point in the curve; O M $= x$, M Q $= y$, P Q $= z$. Then, in order to find the relation between O Q ($= u$) and Q P ($= z$), we shall express O M and O N in terms of O Q, and substitute in the given equations.

The equation to O Q is $y = \frac{b}{a} x = x \tan.\theta$ (if $\frac{b}{a} = \tan.\theta$),

$$\therefore \text{O M} = \text{O Q} \cos.\theta, \text{ and O N} = \text{O Q} \sin.\theta.$$

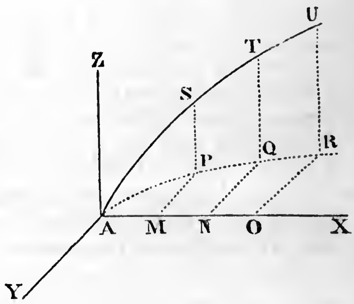
Hence the equation $\frac{a}{x} + \frac{c}{z} = 1$ becomes $\frac{a}{u \cos.\theta} + \frac{c}{z} = 1$,

and the equation $\frac{b}{y} + \frac{c}{z} = 1$ becomes $\frac{b}{u \sin.\theta} + \frac{c}{z} = 1$.

Since $b = a \tan.\theta$, or $b \cos.\theta = a \sin.\theta$, these two equations are the same, and either of them belongs to the required curve; hence the curve is an hyperbola, whose equation referred to its centre is

$$u z = \frac{a c}{\cos.\theta} = \frac{b c}{\sin.\theta}. \quad (209)$$

538. To describe a curve of double curvature by points



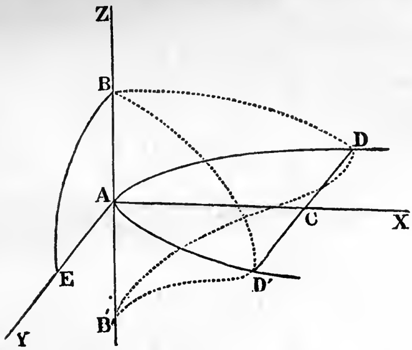
Let $f(x, y) = 0$, and $\phi(x\ z) = 0$, be two of its projections.

Upon $x\,y$ trace the curve A P Q R, whose equation is $f(x, y) = 0$.

For any value of x, as A M, we obtain a corresponding value M P of y; from $\phi(x, z) = 0$, we can also obtain a corresponding value of z. From P draw P S perpendicular to $x\ y$, and equal to this value of z; then S is a point in the curve. By repeating this process we may obtain any number of points S T U, &c., in the curve.

It is evident, that if any value given to x or y renders z imaginary, no part of the curve can be constructed corresponding to such values of x or y. Also, that if z be negative, P S must be drawn below the plane $x\ y$.

539. Ex. 1. Let the curve arise from the intersection of a parabolic cylinder on $x\ y$, and a circular cylinder on $y\ z$, the axes perpendicular to each other; and the vertex of the parabola together with the centre of the circle at the origin of co-ordinates.



Let $y^2 = a\,x$ be the equation to the parabola D A D′,

$y^2 + z^2 = a^2$ . . . . circle E B,

$\therefore\ z^2 + a\,x = a^2$ is a parabola on $x\,z$.

Let A B $= a$, A C $= a$, and let the ordinate C D $= a$.

To trace the curve, we have the three equations on the co-ordinate planes,

$$z = \pm\sqrt{a\,(a - x)}$$

$$z = \pm\sqrt{a^2 - y^2}$$

$$y = \pm\sqrt{a\,x}.$$

If $x = 0$, $y = 0$, and $z = a$, $\therefore$ the curve passes through B ; as x increases, y increases, and z diminishes ;

When $x = a$, $y = a$, and $z = 0$, therefore the curve decreases in altitude from B down to meet the parabola in D. This gives the dotted branch BD.

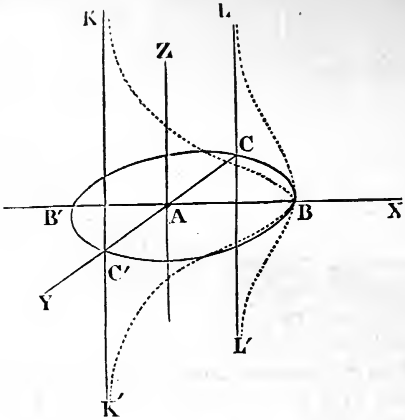
If x is greater than a, z is imaginary; therefore the curve does not extend beyond D.

But since $z = \pm\sqrt{a\,(a-x)}$ there is another ordinate corresponding to every value of x between o and a; hence there is another branch, equal and opposite to B D, but below the plane $x\,y$. This is represented by D B′.

Again, since when y is negative, the values of z do not change, there is another arc, B D′ B′, represented by the double dotted line, which is exactly similar to B D B′.

Therefore, the curve is composed of four parts, B D, D B′, B D′, and D′ B′, equal to one another, and described upon the surface of the parabolic cylinder, whose base is D A D′. These branches form altogether a figure something like that of an ellipse, of which the plane is bent to coincide with the cylinder.

540. Ex. 2. Let the circle, whose equation is $x^2 + y^2 = a^2$, be the projection of the curve of double curvature on $x\,y$; and the curve, of which the equation is $a^2\,y^2 = a^2\,z^2 - y^2\,z^2$, be the projection on $y\,z$, to trace the curve.



Let B C B′ C′ be the circle on $x\ y$ whose equation is $x^2 + y^2 = a^2$; then the equation on $y\ z$ being $a^2 y^2 = a^2 z^2 - y^2 z^2$, the equation on $x\ z$ is $x^2 z^2 = a^4 - a^2 x^2$.

$$\therefore z = \pm \frac{a}{x}\sqrt{a^2 - x^2},$$

$$\text{or } z = \pm \frac{a\,y}{\sqrt{a^2 - y^2}},$$

$$\text{and } y = \pm \sqrt{a^2 - x^2}.$$

If $x = 0$, $y = a$, $z =$ infinity, therefore the vertical line C L through C is an asymptote to the curve. As x increases, y decreases, and z decreases, therefore the curve approaches the plane of $x\ y$. If $x = a$, $y = o$, $z = o$, therefore the curve passes through B. If x is greater than a, y and z are each impossible, therefore no part of the curve is beyond B: for any value of y there are two of z, therefore for the values of y in the quadrant A C B, there are two equal and opposite branches, L B, B L′.

Similarly there are two other equal branches, K B, B K′, for the quadrant B A C′; and as the same values of y and z recur for x negative, there are four other branches equal and opposite to those already drawn, which correspond to the semicircle C B′ C′, and which proceed from B′.

These two examples are taken from Clairaut's Treatise on Curves of Double Curvature; a work containing numerous examples and many excellent remarks on this subject.

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ERRATA.

Page 7, line 1, read Let $x = \sqrt{3} = \sqrt{2+1}$. In the last figure let A B, B C, and C D each be equal to the linear unit, then A D $= \sqrt{3}$.

40.....2, read $\therefore\ y' = \pm \dfrac{r}{\sqrt{1+a^2}}$.

40.....3, for $\dfrac{r^2}{y}$ read $\dfrac{r^2}{y'}$.

40....17, for 24 read 25.

48....12, for y^3 read y^2.

110....24, for e read c.

111....17, for tan. θ . tan. $= \dfrac{c}{a}$, read tan. θ . tan. $\theta' = \dfrac{c}{a}$.

112....20, for $\dfrac{y^2}{b^3}$ read $\dfrac{y^2}{b^2}$.

114....18, for conjugate read semi-conjugate.

123....30, for $x' + m\,\mathrm{S} = \mathrm{P}$, read $x' + m = \mathrm{S\,P}$.

153, in the table, column 7, insert c.

190, line 5 from bottom, for $3\,a$, read $3\,a^2$.

209, line 10, read cos. $l\,x$ cos. $l_1\,x$ + cos. $l\,y$ cos. $l_1\,y$.

217, line 13, for (2) read (1).

221, line 27, read cos. $\theta = \dfrac{x\,x_1 + y\,y_1 + z\,z_1 + \&c.}{r\,r_1} =$

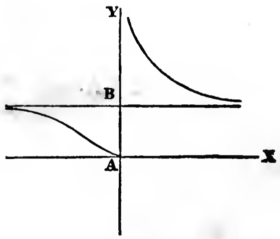
224, line 10, for 397 read 402.

247, line 3, $\therefore\ (x - x_1)^2 + (y - y_1)^2 = r^2$;
and if the axis coincide with A Z, $x^2 + y^2 = r^2,\ z = 0$;

249, line 1. for $r - c$ read z, and for c read $z - c$.

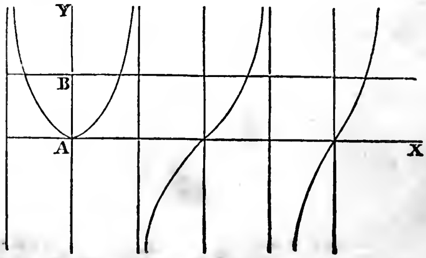
ERRATA IN THE FIGURES.

Art. 352. $y = a^{\frac{1}{x}}$



Art. 353 $y = x^x$. See the figure in the same page just above the Art. 353; the letter B should be at the point where the upper curve meets A Y.

Art. 355 $y = x \tan. x$



Art. 363. The involute of the circle