ANALYTICAL STATICS.

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A TREATISE ON

ANALYTICAL STATICS

WITH NUMEROUS EXAMPLES.

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PREFACE TO THE SECOND EDITION.

In this work will be found all the Propositions which usually appear in treatises on Theoretical Statics. To the different chapters Examples are appended, which have been principally selected from the University and College Examination Papers; these will furnish ample exercise in the application of the principles of the subject.

Some of the Examples in the earlier chapters assume results which are obtained at a later part of the book; the student who has no previous acquaintance with the subject may therefore, on his first perusal of the book, omit the more difficult Examples of the first six chapters.

In the first three chapters and in the ninth chapter I have made considerable use of Mr Pratt's Treatise on Mechanical Philosophy, which was placed at my disposal by the Publishers.

In the second edition the work has been thoroughly revised and has received large additions; these additions have been made with the view of rendering the subject more readily intelligible by explaining and illustrating those parts which were found by the experience of teachers to be difficult for beginners.

I. TODHUNTER.

St John's College,
Aug. 28, 1858.

CONTENTS.

STATICS.

CHAPTER I.

INTRODUCTION.

1. A BODY is a portion of matter limited in every direction, and is consequently of a determinate form and volume. A material particle is a body indefinitely small in every direction; we shall speak of it for shortness as a particle.

2. A body is in motion when the body or its parts occupy successively different positions in space. But we cannot judge of the state of rest or motion of a body without comparing it with other bodies, and for this reason all motions which come under our observation are necessarily *relative* motions.

3. Force is that which produces or tends to produce motion in a body.

4. When several forces act simultaneously on a body, it may happen that they neutralise each other; when a body remains at rest though acted on by forces, it is said to be in equilibrium; or, in other words, the forces are said to maintain equilibrium.

5. Mechanics is the science which treats of the laws of rest and motion of bodies. Statics treats of the laws of the equilibrium of bodies, and Dynamics of the laws of motion of bodies.

6. There are three things to consider in a force acting upon a particle: the *position* of the particle: the *direction*

of the force, that is, the direction in which it tends to make the particle start; and the *intensity* of the force. As the dimensions of a particle are indefinitely small its position may be determined in the same manner as that of a point in geometry, and the direction of the force may be determined in the same manner as that of a line in geometry. We proceed then to consider the magnitude or intensity of a force.

7. Forces can be measured by taking some force as the unit, and expressing by numbers the ratios which other forces bear to this unit. Two forces are *equal* when being applied in opposite directions to a particle they maintain equilibrium. If we take two *equal* forces and apply them to a particle in the *same* direction we obtain a force *double* of either; if we unite *three* equal forces we obtain a *triple* force; and so on.

When we say then that a force applied to a particle is a certain multiple of another force, we mean that the first may be supposed to be composed of a certain number of forces equal to the second and all acting in the same direction. In this way forces become measurable quantities, which can be expressed by numbers, like all other quantities, by referring them to a unit of their own kind. They may also be represented by straight lines proportional in length to these numbers, drawn from the point at which they act and in the directions in which they act.

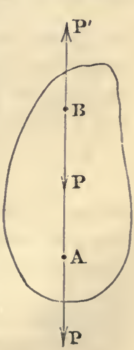
8. Experience teaches us that if a body be let free from the hand, it will fall downwards in a certain direction; however frequently the experiment be made, the result is the same, the body strikes the same spot on the ground in each trial, provided the place from which it is dropped remain the same. The cause of this undeviating effect is assumed to be an affinity which all bodies have for the earth, and is termed the force of attraction. If the body be prevented from falling by the interposition of a table or of the hand, the body exerts a *pressure* on the table or hand. *Weight* is the name given to the pressure which the attraction of the earth causes a body to exert on another with which it is in contact.

9. A solid body is conceived to be an aggregation of material particles which are held together by their mutual affinities. This appears to be a safe hypothesis, since experiments shew that any body is divisible into successively smaller and smaller portions without limit, if sufficient force be exerted to overcome the mutual action of the parts of the body.

10. A rigid body is one in which the particles retain invariable positions with respect to each other. No body in nature is perfectly rigid; every body yields more or less to the forces which act on it. If, then, in any case this compressibility is of a sensible magnitude, we shall suppose that the body has assumed its figure of equilibrium, and then consider the points of application of the forces as a system of invariable form. By body, hereafter, we mean *rigid* body.

11. When a force acts upon a body its effect will be unchanged at whatever point of its direction we suppose it applied, provided this point be either one of the points of the body or be invariably connected with it. This principle is known by the name of the *transmissibility of a force to any point in its line of action;* it is assumed as an axiom or as an experimental fact. We may shew the amount of assumption involved in the axiom, by the following process.

Suppose a body to be kept in equilibrium by a system of forces, one of which is the force P applied at the point A. Take any point B which lies on the direction of this force, and suppose B so connected with A that the distance AB is unchangeable. Then, if at B we introduce two forces, P and P', equal in magnitude and acting in opposite directions along the line AB, it seems evident that no change is made in the effect of the force P at A. Let us now *assume that P at A and P' at B will neutralise each other, and may therefore be removed without disturbing the equilibrium of the body;* then there remains the force P at B producing the same effect as when it acted at A.



12. We shall have occasion hereafter to assume what may be called the *converse* of the principle of the transmissibility of force, namely, that if a force can be transferred from its point of application to a second point without altering its effect, then the second point must be in the direction of the force. See Art. 17.

13. When we find it useful to change the point of application of a force, we shall for shortness not always state that the new point is *invariably connected* with the old point, but this may be always understood.

CHAPTER II.

THE COMPOSITION AND EQUILIBRIUM OF FORCES ACTING UPON A PARTICLE.

14. When a particle is acted on by forces which do not maintain equilibrium it will begin to move in some determinate direction. It is clear then that a *single* force may be found of such a magnitude, that if it acted in the direction *opposite* to that in which the motion would take place this force would prevent the motion, and consequently would be in equilibrium with the other forces which act upon the particle. If then we were to remove the original forces and replace them by a single force, equal in magnitude to that described above, but acting in the opposite direction, the particle would still remain at rest. This force, which is equivalent in its effect to the combined effect of the original forces, is called their *resultant*, and the original forces are called the *components* of the resultant.

It will be necessary then to begin by deducing rules for the *composition of forces;* that is, for finding their resultant force. After we have determined these, it will be easy to deduce the analytical relations which forces must satisfy when in equilibrium.

15. *To find the resultant of a given number of forces acting upon a particle in the same straight line; and to find the condition which they must satisfy that they may be in equilibrium.*

When two or more forces act on a particle in the same direction it is evident that the resultant force is equal to their sum and acts in the same direction.

When two forces act in different directions, but in the same straight line, on a particle, it is equally clear that their resultant is equal to their difference and acts in the direction of the greater component.

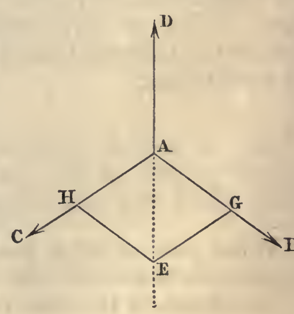
When several forces act in different directions, but in the same straight line, on a particle, the resultant of the forces acting in one direction is equal to the sum of these forces, and acts in the same direction; and so of the forces acting in the opposite direction. The resultant, therefore, of all the forces is equal to the difference of these sums, and acts in the direction of the greater.

If the forces acting in one direction are reckoned positive, and those in the opposite direction negative, then their resultant is equal to their *algebraical* sum; its sign determines the direction in which it acts.

In order that the forces may be in equilibrium, their resultant, and therefore their algebraical sum, must equal zero.

16. There is another case in which we can easily determine the magnitude and direction of the resultant.

Let AB, AC, AD be the directions of three equal forces acting on the particle A; suppose these forces all in the same plane and the three angles BAC, CAD, DAB each equal to 120°; the particle will remain at rest, for there is no reason why it should move in one direction rather than another. Each of the forces is therefore equal and opposite to the resultant of the other two. But if we take on the directions of two of them, AB, AC, two equal lines AG, AH to represent the forces, and complete the parallelogram $GAHE$, the diagonal AE will lie in the same straight line with AD. Also the triangle AGE will be equilateral, and therefore $AE = AG$. Hence, the diagonal AE of the parallelogram constructed on AG, AH represents



the resultant of the two forces which AG and AH respectively represent.

This proposition is a particular case of one to which we now proceed.

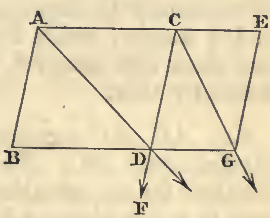
17. *To find the resultant of two forces acting upon a particle not in the same straight line.*

I. To find the *direction* of the resultant.

When the forces are equal it is clear that the direction of the resultant will *bisect* the angle between the directions of the forces; or, if we represent the forces in magnitude and direction by two lines drawn from the point where they act, and describe a parallelogram on these lines, that diagonal of the parallelogram which passes through the point will be the direction of the resultant.

Let us assume that this is true for forces p and m inclined at any angle, and also for forces p and n inclined at the same angle; we can prove that it must then be true for two forces p and $m+n$ also inclined at the same angle.

Let A be the point on which the forces p and m act; AB, AC their directions and proportional to them in magnitude: complete the parallelogram BC, and draw the diagonal AD; then, by hypothesis, the resultant of p and m acts along AD.



Again, take CE in the same ratio to AC that n bears to m. By Art. 11 we may suppose the force n which acts in the direction AE to be applied at A or C; and therefore the forces p, m, and n, in the lines AB, AC, and CE, are the same as p and $m+n$ in the lines AB and AE.

Now replace p and m by their resultant and transfer its point of application from A to D; then resolve this force at D into two parallel to AB and AC respectively: these resolved parts must evidently be p and m, the former acting in the direction DF, and the latter in the direction DG. Then transfer p to C and m to G.

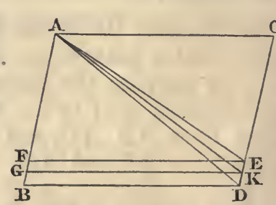
But, by the hypothesis, p and n acting at C have a resultant in the direction CG; therefore p and n may be replaced by their resultant and its point of application transferred to G. And m has also been transferred to G. Hence by this process we have removed the forces which acted at A to the point G without altering their effect. We may infer then (see Art. 12) that G is a point in the line of action of the resultant of p and $m+n$ at A; that is, the resultant of p and $m+n$ acts in the direction of the diagonal AG, provided the hypothesis is correct. But the hypothesis is correct for equal forces, as p, p, and therefore it is true for forces p, $2p$; consequently for p, $3p$, and so on; hence it is true for p, $r.p$.

Hence it is true for p, $r.p$, and p, $r.p$, and consequently for $2p$, $r.p$, and so on; and it is finally true for $s.p$ and $r.p$, where r and s are positive integers.

We have still to shew that the Proposition is true for *incommensurable* forces.

This may be inferred from the fact that when two magnitudes are incommensurable, so that the ratio of one to the other cannot be expressed *exactly* by a fraction, we can still find a fraction which differs from the true ratio by a fraction less than any assigned fraction. Or it may be established indirectly thus.

Let AB, AC represent two such forces. Complete the parallelogram BC. Then if their resultant do not act along AD suppose it to act along AE; draw EF parallel to BD. Divide AC into a number of equal portions, each less than DE; mark off from CD portions equal to these, and let K be the last division; this evidently falls between D and E; draw GK parallel to AC. Then two forces represented by AC, AG have a resultant in the direction AK, because they are commensurable; therefore the forces AC and AB are equivalent to AK together with a force equal to GB applied at A along AB. And we may

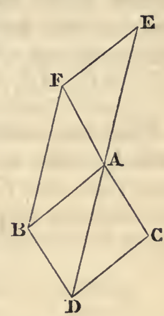


assume as obvious that the resultant of these forces must lie *between* AK and AB; but by supposition the resultant is AE which is *not* between AK and AB. This is absurd.

In the same manner we may shew that every direction besides AD leads to an absurdity, and therefore the resultant must act along AD, whether the forces be commensurable or incommensurable.

II. To find the *magnitude* of the resultant.

Let AB, AC be the directions of the given forces, AD that of their resultant; take AE opposite to AD, and of such a length as to represent the *magnitude* of the resultant. Then the forces represented by AB, AC, AE, balance each other. Complete the parallelogram BE; then the diagonal AF is the direction of the resultant of AE and AB.



Hence AC is in the same straight line with AF; hence FD is a parallelogram; and therefore $AE = FB = AD$. Hence the resultant is represented in *magnitude* as well as in direction by the diagonal of the parallelogram. This proposition is called the *parallelogram of forces.*

18. Hence if P and Q represent two component forces acting at an angle α on a particle, the resultant R is given by the equation

$$R^2 = P^2 + Q^2 + 2PQ \cos \alpha.$$

19. When three forces acting on a particle are in equilibrium they are respectively in the same proportion as the sines of the angles included by the directions of the other two.

For if we refer to the third figure of Art. 17 we have

$$P : Q : R :: AB : AC \text{ (or } BD) : AD$$
$$:: \sin ADB : \sin BAD : \sin ABD$$
$$:: \sin CAE : \sin BAE : \sin BAC.$$

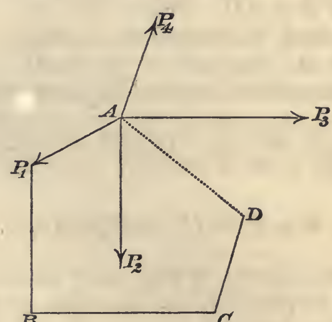
20. Any force acting on a particle may be replaced by two others, if the sides of a triangle drawn parallel to the

directions of the forces have the same relative proportion that the forces have. For by the parallelogram of forces the resultant of the latter two forces is equal to the given force.

This is called the *resolution* of a force.

21. Since the resultant of two forces acting on a particle is represented in magnitude and direction by the diagonal of the parallelogram constructed upon the lines which represent these forces in magnitude and direction, it follows that, in order to obtain the resultant of the forces $P_1, P_2, P_3, \ldots$ which act on a particle A, and are represented by the lines $AP_1, AP_2, AP_3, \ldots$ we may proceed as follows.

Find the resultant of P_1 and P_2, compound this resultant with P_3, this new resultant with P_4, and so on. It follows from this, that if we construct a polygon AP_1BCD, of which the sides are respectively equal and parallel to the lines AP_1,



AP_2, &c., and join A with the last vertex D, the line AD will represent in magnitude and direction the resultant of all the forces.

We may conclude that the necessary and sufficient condition for the equilibrium of a number of forces acting on a particle is, that the point D should coincide with A; that is, that the figure $AP_1B \ldots D$ should be a *complete* polygon. The forces in the figure are not necessarily all in one plane.

The direction and magnitude of the resultant may also be determined analytically, as in the following Articles.

22. *Any number of forces act on a particle in one plane; required to find the magnitude and direction of their resultant.*

Let P_1, P_2, P_3,... be the forces, and α_1, α_2, α_3,... the angles their directions make with a fixed line drawn through the proposed point. Take this fixed line for the axis of x, and one perpendicular to it for that of y. Then, by Art. 20, P_1 may be resolved into $P_1 \cos \alpha_1$ and $P_1 \sin \alpha_1$ acting along the axes of x and y respectively. The other forces may be similarly resolved. By algebraical addition of the forces which act in the same line, we have

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \ldots \text{ along the axis of } x,$$

$$P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 + \ldots \text{ along the axis of } y.$$

We shall express the former by $\Sigma P \cos \alpha$ and the latter by $\Sigma P \sin \alpha$, where the symbol Σ denotes that we take the sum of all the quantities of which the quantity before which it is placed is the type.

If we put $P_1 \cos \alpha_1 = X_1$ and $P_1 \sin \alpha_1 = Y_1$, and use a similar notation for the other components, we have two forces replacing the whole system, namely ΣX along the axis of x and ΣY along that of y. If R denote the resultant of these forces and a the angle at which it is inclined to the axis of x, we have, by Art 17,

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2,$$

$$\tan a = \frac{\Sigma Y}{\Sigma X}.$$

Also $$\cos a = \frac{\Sigma X}{R}; \quad \sin a = \frac{\Sigma Y}{R}.$$

23. *To find the conditions of equilibrium when any number of forces act upon a particle in one plane.*

When the forces are in equilibrium we must have $R=0$; therefore

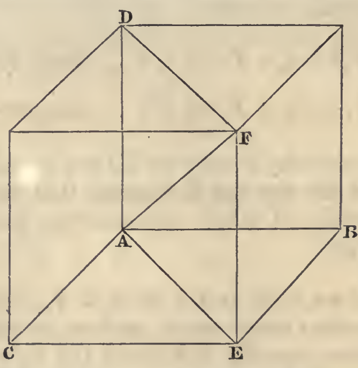
$$(\Sigma X)^2 + (\Sigma Y)^2 = 0;$$

therefore $$\Sigma X = 0; \quad \Sigma Y = 0;$$

and these are the conditions among the forces that they may be in equilibrium.

24. *Three forces act upon a particle in directions making right angles with each other; required to find the magnitude and direction of their resultant.*

Let AB, AC, AD represent the three forces X, Y, Z in magnitude and direction. Complete the parallelogram BC,



and draw the diagonal AE; then AE represents the resultant of X and Y in magnitude and direction, by Art. 17. Now the resultant of this force and Z, that is, of the forces represented by AE, AD, is represented in magnitude and direction by AF, the diagonal of the parallelogram DE. Hence the resultant of X, Y, Z is represented in magnitude and direction by AF. Let R be the magnitude of the resultant, and a, b, c the angles the direction of R makes with those of X, Y, Z. Then, since

$$AF^2 = AE^2 + AD^2 = AB^2 + AC^2 + AD^2,$$

therefore $$R^2 = X^2 + Y^2 + Z^2.$$

Also $\cos a = \frac{AB}{AF} = \frac{X}{R}$, $\cos b = \frac{AC}{AF} = \frac{Y}{R}$, $\cos c = \frac{AD}{AF} = \frac{Z}{R}$.

Thus the magnitude and direction of the resultant are determined.

25. It follows from the last article that any force R the direction of which makes the angles a, b, c with three rectangular axes fixed in space, may be replaced by the three forces $R\cos a$, $R\cos b$, $R\cos c$, acting simultaneously on the particle on which R acts, and having their directions parallel to the axes of coordinates respectively.

26. *Any number of forces act upon a particle in any directions; required to find the magnitude and direction of their resultant.*

Let P_1, P_2, P_3, ... be the forces; let α_1, β_1, γ_1 be the angles which the direction of P_1 makes with three rectangular axes drawn through the proposed point; let α_2, β_2, γ_2 be the angles which the direction of P_2 makes with the same axes; and so on.

Then, by Art. 25, the components of P_1 in the directions of the axes are

$$P_1\cos\alpha_1,\quad P_1\cos\beta_1,\quad P_1\cos\gamma_1,\quad \text{(or } X_1,\ Y_1,\ Z_1, \text{ suppose).}$$

Resolve each of the other forces in the same way, and reduce the system to three forces, by adding those which act in the same line, Art. 15; we thus have

$$P_1\cos\alpha_1 + P_2\cos\alpha_2 + \ldots \quad \text{or } \Sigma P\cos\alpha, \quad \text{or } \Sigma X,$$

$$P_1\cos\beta_1 + P_2\cos\beta_2 + \ldots \quad \text{or } \Sigma P\cos\beta, \quad \text{or } \Sigma Y,$$

$$P_1\cos\gamma_1 + P_2\cos\gamma_2 + \ldots \quad \text{or } \Sigma P\cos\gamma, \quad \text{or } \Sigma Z,$$

acting in the directions of the axes of x, y, and z respectively.

If we call the resultant R, and the angles which its direction makes with the axes a, b, c, we have, by Art. 24,

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2,$$

and $\cos a = \frac{\Sigma X}{R}, \quad \cos b = \frac{\Sigma Y}{R}, \quad \cos c = \frac{\Sigma Z}{R}.$

27. *To find the conditions of equilibrium when any number of forces act upon a material particle.*

When the forces are in equilibrium, we must have $R = 0$; therefore

$$(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2 = 0,$$

therefore $\Sigma X = 0; \ \Sigma Y = 0; \ \Sigma Z = 0;$

and these are the conditions among the forces that they may be in equilibrium.

28. The expression for the magnitude of the resultant in Art. 26 may be rendered independent of the position of the axes. For, from Art. 26,

$$R^2 = (P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \ldots)^2 + (P_1 \cos \beta_1 + P_2 \cos \beta_2 + \ldots)^2$$
$$+ (P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + \ldots)^2.$$

When the expressions on the right-hand side are developed, we shall find that the coefficient of P_1^2 is

$$\cos^2 \alpha_1 + \cos^2 \beta_1 + \cos^2 \gamma_1,$$

and that the coefficient of $P_1 P_2$ is

$$2 (\cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2).$$

Now we know from analytical geometry of three dimensions that

$$\cos^2 \alpha_1 + \cos^2 \beta_1 + \cos^2 \gamma_1 = 1;$$

and that

$$\cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2$$

is equal to the cosine of the angle between the directions of the forces P_1 and P_2, which we may denote by $\cos (P_1, P_2)$.

Similar values will be found for the coefficients of the other terms; and the result may be expressed thus

$$R^2 = \Sigma P^2 + 2\Sigma PP' \cos(P, P')$$

where by P, P' we mean any two of the forces.

29. The equation $R \cos a = \Sigma P \cos \alpha$, in Art. 26, shews that the *resolved part of the resultant in any direction is equal to the sum of the resolved parts of the components in the same direction;* for since the axes were taken arbitrarily, that of x might have been made to coincide with any assigned direction. Or we may establish the proposition thus. Suppose a line drawn through the point of application of the forces, and inclined to the axes at angles α', β', γ'. Take the three equations of Art. 26,

$$R \cos a = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \ldots\ldots$$
$$R \cos b = P_1 \cos \beta_1 + P_2 \cos \beta_2 + \ldots\ldots$$
$$R \cos c = P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + \ldots\ldots$$

Multiply the first by $\cos \alpha'$, the second by $\cos \beta'$, and the third by $\cos \gamma'$, and add. Then, if $\theta_1, \theta_2, \ldots$ denote the angles which $P_1, P_2, \ldots$ make with the arbitrarily drawn line, and θ the angle which the resultant R makes with it, we have, by the formula quoted in Art. 28 for the cosine of the angle between two lines

$$R \cos \theta = P_1 \cos \theta_1 + P_2 \cos \theta_2 + \ldots\ldots$$

30. From Art. 20 it is obvious that a given force may be resolved into two others in an infinite number of ways. When we speak of the resolved part of a force in a given direction, as in the preceding article, *we shall always suppose, unless the contrary is expressed*, that the given force is resolved into two forces, one in the *given direction* and the other in a direction *perpendicular to the given direction.* The former component we shall call the resolved force in the given direction.

When forces act on a particle it will be in equilibrium, provided the sums of the forces resolved along any three directions not lying in one plane are zero. For if the forces do not balance, they must have a single resultant; and as a line cannot be at right angles to three lines which meet in a point and are not in the same plane, the resolved part of the resultant, and therefore the sum of the resolved parts of the given forces, along these three lines, could not vanish, which is contrary to the hypothesis.

31. In Art. 26 we resolved each force of a system into three others along three *rectangular* axes. In the same way we may, if we please, resolve each force along three lines forming a *system of oblique axes.* For whether the figure in Art. 24 represent an *oblique* or *rectangular* parallelopiped, the force AF may be resolved into AD and AE, and the latter again resolved into AB and AC. Hence the resultant of a system of forces may be represented by the diagonal of an oblique parallelopiped, and for equilibrium it will be necessary that this diagonal should vanish, and therefore that the edges of the parallelopiped should vanish.

The following three articles are particular cases of the equilibrium of a particle.

32. *To determine the condition of equilibrium of a particle acted on by any forces and constrained to remain on a given smooth curve.*

By a *smooth* curve we understand a curve that can only exert force on the particle in a direction *normal* to the curve at the point of contact.

Let X, Y, Z denote the forces acting on the particle in directions parallel to three rectangular axes, exclusive of the action of the curve. Let x, y, z denote the co-ordinates of the particle, and s the length of the arc measured from some fixed point up to the point (x, y, z). Then by analytical geometry of three dimensions the cosines of the angles which the tangent to the curve at the point (x, y, z) makes with the axes are $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, respectively. The forces acting on the particle being resolved along the tangent to the curve, their sum is

$$X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}.$$

Unless this vanishes, there will be nothing to prevent the particle from moving; for equilibrium then we must have

$$X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}=0.$$

Conversely if this relation holds the particle will remain at rest, for there is no force to make it move *along the curve*, which is the only motion of which it is capable.

We have supposed the particle to be placed inside a tube which has the form of the curve. If, however, the particle be merely placed in *contact with a curve*, it will be *further* necessary for equilibrium that the resultant of the forces should press the particle *against* the curve and not move it *from* the curve.

33. *To determine the conditions of equilibrium of a particle acted on by any forces and constrained to remain on a given smooth surface.*

A *smooth* surface is one which can exert no force on the particle except in a direction normal to the surface.

Let X, Y, Z denote the forces acting on the particle in directions parallel to three rectangular axes, exclusive of the action of the surface. The resultant of X, Y, Z must act in a direction normal to the surface at the point where the particle is situated; for if it did not, we might decompose it into two forces, one in the normal and one perpendicular to the normal, of which the latter would set the particle in motion. The cosines of the angles which the resultant of X, Y, Z makes with the axes are proportional to X, Y, Z respectively; and if $F(x, y, z)=0$ be the equation to the surface, the cosines of the angles which the normal to the surface at the point (x, y, z) makes with the axes, are by analytical geometry of three dimensions proportional to $\frac{dF}{dx}$, $\frac{dF}{dy}$ and $\frac{dF}{dz}$ respectively. Hence for equilibrium we must have

$$\frac{X}{\frac{dF}{dx}}=\frac{Y}{\frac{dF}{dy}}=\frac{Z}{\frac{dF}{dz}}.$$

If these relations are satisfied, the resultant force is directed along the normal; hence, if we suppose the particle incapable of leaving the surface, the above conditions will be *sufficient* to ensure its equilibrium; but if it be merely placed *on* a surface, it will be further necessary that X, Y, Z should act so that their resultant may press the particle *against the surface.* For example, if the particle be placed on the outside of a sphere, the resultant of X, Y, and Z must act *towards* the centre of the sphere.

34. Suppose it required to determine the action which the curve or the surface exerts on the particle in the preceding cases. Denote it by R, and let α, β, γ be the angles its direction makes with the axes. Since R and the forces X, Y, Z maintain the particle in equilibrium, we have by Art. 27,

$$R\cos\alpha + X = 0,\ \ R\cos\beta + Y = 0,\ \ R\cos\gamma + Z = 0 \ldots\ldots(1).$$

Also when the particle rests on a curve surface whose equation is $F(x, y, z) = 0$, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are known in terms of the co-ordinates of the particle, since they are proportional to $\frac{dF}{dx}$, $\frac{dF}{dy}$, $\frac{dF}{dz}$ respectively. Hence the equations (1) and that to the surface will determine x, y, z, and R, if X, Y, Z be given.

If the particle rest on a curve line, then, since the direction of R is perpendicular to that of the tangent to the curve, we have the following equation from analytical geometry of three dimensions

$$\cos\alpha\frac{dx}{ds} + \cos\beta\frac{dy}{ds} + \cos\gamma\frac{dz}{ds} = 0 \ldots\ldots\ldots\ldots\ldots(2).$$

Since $\frac{dx}{ds}$, $\frac{dy}{ds}$ and $\frac{dz}{ds}$ can be expressed, theoretically at least, in terms of x, y, and z, the equation (2) gives a relation between $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, and x, y, and z. Thus (1) and (2) together with the two equations to the curve and the equation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

are sufficient to determine the seven quantities R, x, y, z, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$.

We may observe that from (1)

$$R^2 = X^2 + Y^2 + Z^2.$$

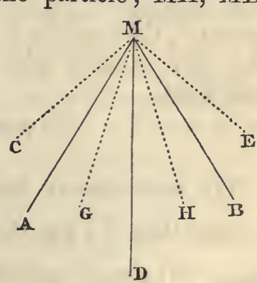
35. Duchayla's proof of the parallelogram of forces which we have given in Art. 17, rests on the principle of the *transmissibility of force;* see Art. 11. We shall give another proof which does not involve this principle; this proof is Poisson's with a slight modification. We assume that if two equal forces act on a particle, the direction of the resultant bisects the angle between the directions of the components. Also, if P denote the magnitude of each of two equal forces, $2x$ the angle between their directions, and R the magnitude of the resultant; then R must be some function of P and x; suppose

$$R = f(P, x).$$

In this equation, if we change our unit of force, the numerical values of P and R will change; but as the above equation must be true, whatever unit of force we adopt, it follows that the function $f(P, x)$ must be of the form $P\phi(x)$. Hence we have

$$R = P\phi(x).$$

Let M represent the position of the particle; MA, MB the directions of the equal forces acting on it; MD the direction of the resultant. Draw the four lines MC, MG, MH, ME, making the angles CMA, GMA, HMB, EMB all equal, and let z denote the magnitude of each angle. Suppose the force P acting along MA to be resolved into two equal forces acting along MC and MG respectively; denote each of these components by Q; then



$$P = Q\phi(z).$$

Resolve P acting along MB in like manner into two forces each equal to Q, acting along ME and MH respectively. Thus the two forces P are replaced by the four forces Q; and consequently the resultant of these four forces must coincide in magnitude and direction with the resultant R of the two forces P.

Let Q' denote the resultant of the two forces Q, acting along MG and MH; since $GMD = HMD = x - z$, we have

$$Q' = Q\phi(x - z),$$

and MD is the direction of Q'.

Similarly, the resultant Q'' of the other forces Q will act along MD; and since $CMD = EMD = x + z$, we have

$$Q'' = Q\phi(x + z).$$

Since Q' and Q'' both act along the line MD, their resultant, which is also the resultant of the four forces Q, must be equal to their sum; hence

$$R = Q' + Q''.$$

But we have $\quad R = P\phi(x) = Q\phi(z)\,\phi(x).$

Hence $$\phi(x)\,\phi(z) = \phi(x + z) + \phi(x - z) \qquad \text{............(1).}$$

This equation admits of more than one solution; for example, if $\phi(x) = 2\cos cx$, or if $\phi(x) = e^{cx} + e^{-cx}$, where c is any constant, the equation is satisfied; we shall however shew that the only solution admissible in the present question is the following

$$\phi(x) = 2\cos x \qquad \text{.........................(2).}$$

We may observe that we need not consider any value of x greater than $\frac{\pi}{2}$, for the directions of two forces acting at a point will always include an angle less than π; we may then assume it as obvious that $\phi(x)$ must be a positive quantity.

We shall first shew that if $\phi(x) = 2\cos x$ when x has any value α, then $\phi(x)$ must $= 2\cos x$ when x has the value $\frac{\alpha}{2}$. In

(1) put x and z each equal to $\frac{\alpha}{2}$, so that $\phi(x+z)$ becomes equal to $2\cos\alpha$; thus

$$\phi\left(\frac{\alpha}{2}\right)\phi\left(\frac{\alpha}{2}\right)=\phi(0)+2\cos\alpha \ldots\ldots\ldots\ldots (3).$$

But the resultant of two equal forces acting in the same straight line is equal to twice either of the component forces; thus $\phi(0)=2$; therefore by (3)

$$\phi\left(\frac{\alpha}{2}\right)\phi\left(\frac{\alpha}{2}\right)=2(1+\cos\alpha)=4\cos^2\frac{\alpha}{2}.$$

Hence $\phi\left(\frac{\alpha}{2}\right)=\pm 2\cos\frac{\alpha}{2}$; but by supposition $\frac{\alpha}{2}$ is less than $\frac{\pi}{2}$, and $\phi\left(\frac{\alpha}{2}\right)$ must be a *positive* quantity; thus

$$\phi\left(\frac{\alpha}{2}\right)=2\cos\frac{\alpha}{2}.$$

Similarly if $\phi(x)=2\cos x$ when $x=\frac{\alpha}{2}$, then $\phi(x)=2\cos x$ when $x=\frac{\alpha}{4}$; and so on. Thus we conclude that if $\phi(x)=2\cos x$ when $x=\alpha$, then $\phi(x)=2\cos x$ when $x=\frac{\alpha}{2^n}$, where n is any positive integer.

We shall next shew that if $\phi(x)=2\cos x$ when $x=\beta$, and when $x=\gamma$, and when $x=\beta-\gamma$, then $\phi(x)=2\cos x$ when $x=\beta+\gamma$. From (1)

$$\begin{aligned}\phi(\beta+\gamma)&=\phi(\beta)\,\phi(\gamma)-\phi(\beta-\gamma)\\&=4\cos\beta\cos\gamma-2\cos(\beta-\gamma)\\&=2\cos(\beta+\gamma).\end{aligned}$$

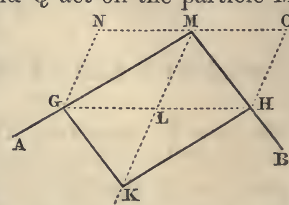
Thus if (2) holds when $x=\beta$, it will hold when $x=2\beta$; this we obtain by supposing $\gamma=\beta$. Then if (2) holds when $x=\beta$ and when $x=2\beta$, it also holds when $x=3\beta$; and so on; that is, if (2) holds when $x=\beta$ it will hold when $x=m\beta$. Thus we conclude that if (2) holds when $x=\alpha$ it will hold when $x=\frac{m\alpha}{2^n}$, where m and n are any integers.

But since the numbers m and n may be as great as we please, we can take them such that the expression $\frac{m\alpha}{2^n}$ may differ as little as we please from any assigned value of x. We may therefore consider (2) as completely demonstrated if it holds for any value of x different from zero. But by Art. 16, it does hold when $x=\frac{1}{3}\pi$, for then $\phi(x)=1=2\cos\frac{1}{3}\pi$; hence it holds always. Hence

$$R=2P\cos x.$$

If then the forces P be represented by lines drawn from their point of application, the resultant R will be represented by that diagonal of the parallelogram described on these lines which passes through the point of application.

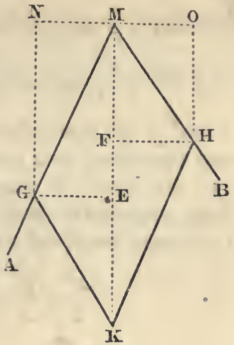
Next, let two *unequal* forces P and Q act on the particle M along the lines MA and MB; represent their intensities by the lines MG and MH taken on their directions, and complete the parallelogram $MGKH$.



First suppose AMB a right angle. Draw the two diagonals MK and GH, which meet in L; through G and H draw GN and HO parallel to ML, meeting in N and O the parallel to GH drawn through M. Then

$$GL=LH=LM.$$

Hence NL and OL are equilateral parallelograms, and therefore, by what has been already proved, the force MG may be regarded as the resultant of MN and ML, and the force MH as the resultant of MO and ML. Hence we may substitute for MG and MH the forces MN, MO, and the two forces ML; MN and MO, since they are equal and opposite, destroy each other, and we have remaining the two forces ML, which together give a force represented in magnitude and direction by MK.



Secondly, suppose the angle AMB not a right angle. Through G and H draw GE and HF perpendicular to the diagonal MK, and GN and HO parallel to this line. Through M draw NMO perpendicular to MK. Then we have $GE = HF$. As we have already shewn, the force MG may be replaced by MN and ME, and the force MH by MO and MF. Since MN and MO are equal and opposite, they will destroy each other, and MF and ME remain; since $MF = KE$, we have MK as the resultant in magnitude and direction of MG and MH.

Hence the *parallelogram of forces* is completely proved.

36. A proof of the parallelogram of forces has been given by Laplace (*Mécanique Céleste*, Liv. I. Chap. 1). In this proof the component forces are at first supposed to be at right angles; the magnitude of their resultant is then determined and afterwards its direction. The first part of the proof is so simple, that it may be conveniently introduced here; it is substantially as follows. Let x and y denote two forces which are inclined at a right angle, and let z denote their resultant; we propose to find the value of z. It is obvious that if the components instead of being x and y were $2x$ and $2y$ respectively, the resultant would be $2z$ and would have the same direction as before; so if the components were $3x$ and $3y$ respectively, the resultant would be $3z$ and would have the same direction as before; and so on. We may therefore assume conversely, that if the inclination of the resultant to each component remains unchanged, the ratio of each component to the resultant will also remain unchanged. Now consider the force x as the resultant of two forces x' and x'', of which x' is in the direction of z, and x'' is perpendicular to that direction. Then by the principle just assumed, we have

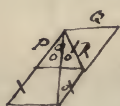
$$\frac{x'}{x} = \frac{x}{z}, \text{ and } \frac{x''}{x} = \frac{y}{z};$$

so that

$$x' = \frac{x^2}{z}, \text{ and } x'' = \frac{xy}{z}.$$

Similarly y may be resolved into $\frac{y^2}{z}$ along the direction of z and $\frac{yx}{z}$ perpendicular to that direction. Thus the forces x and y are equivalent to four forces, two in the direction of z and the other two perpendicular to that direction; the latter two are equal in magnitude and opposite in direction, so that they counteract each other; hence the resultant of the former two must be equal to z. Thus

$$\frac{x^2}{z}+\frac{y^2}{z}=z; \quad \text{therefore } z^2=x^2+y^2.$$



EXAMPLES.

1. Two forces P and Q have a resultant R which makes an angle α with P; if P be increased by R while Q remains unchanged, shew that the new resultant makes an angle $\frac{\alpha}{2}$ with P.

2. Two forces in the ratio of 2 to $\sqrt{(3)}-1$, are inclined to each other at an angle of 60^0; what must be the direction and magnitude of a third force which produces equilibrium?

Result. The required force must be to the first of the given forces as $\sqrt{6}$ to 2; and its direction produced makes an angle of 15^0 with that force.

3. The resultant of two forces P and Q is equal to $Q\sqrt{(3)}$, and makes an angle of 30^0 with P; find P in terms of Q.

Result. Either $P=Q$ or $P=2Q$; in the former case the angle between P and Q is 60^0, in the latter 120^0.

4. If D, E, F be the middle points of the sides of the triangle ABC and O any other point, shew that the system of forces represented by OD, OE, OF is equivalent to that represented by OA, OB, OC.

5. The resultant of two forces is 10lbs., one of them is equal to 8lbs., and the direction of the other is inclined to the resultant at an angle of 36°. Find the angle between the two forces.

Result. $\text{Sin}^{-1}\frac{5}{16}(10-2\sqrt{5})^{\frac{1}{2}}$.

6. The resultant of two forces P, Q, acting at an angle θ, is equal to $(2m+1)\sqrt{(P^2+Q^2)}$; when they act at an angle $\frac{1}{2}\pi-\theta$, it is equal to $(2m-1)\sqrt{(P^2+Q^2)}$; shew that $\tan\theta=\dfrac{m-1}{m+1}$.

7. Two forces F and F' acting in the diagonals of a parallelogram keep it at rest in such a position that one of its edges is horizontal, shew that

$$F\sec\alpha=F'\sec\alpha'=W\operatorname{cosec}(\alpha+\alpha'),$$

where W is the weight of the parallelogram, α and α' the angles between its diagonals and the horizontal side.

8. If a particle be placed on a sphere, and be acted on by three forces represented in magnitude and direction by three chords mutually at right angles drawn through the particle, it will remain at rest.

9. Three forces P, Q, R acting upon a point and keeping it at rest are represented by lines drawn from that point. If P be given in magnitude and direction, and Q in magnitude only, find the locus of the extremity of the line which represents the third force R.

Result. A sphere.

10. A circle whose plane is vertical has a centre of constant repulsive force equal to gravity at one extremity of the horizontal diameter; find the position of equilibrium of a particle within the circle.

Result. The line joining the particle with the centre of the circle makes an angle of 60° with the horizon.

11. A particle is placed on a smooth square table whose side is a at distances c_1, c_2, c_3, c_4 from the corners, and to it are attached strings passing over smooth pulleys at the corners and supporting weights P_1, P_2, P_3, P_4; shew that if there is equilibrium,

$$\left(\frac{P_1}{c_1}+\frac{P_2}{c_2}+\frac{P_3}{c_3}+\frac{P_4}{c_4}\right)^2\frac{c_1^{\,2}}{a^2}=\left(\frac{P_2}{c_2}+\frac{P_3}{c_3}\right)^2+\left(\frac{P_3}{c_3}+\frac{P_4}{c_4}\right)^2.$$

Shew also that

$$\left(\frac{P_1}{c_1}+\frac{P_2}{c_2}+\frac{P_3}{c_3}+\frac{P_4}{c_4}\right)\frac{c_1^{\,2}-c_3^{\,2}}{a^2}=2\left(\frac{P_3}{c_3}-\frac{P_1}{c_1}\right).$$

12. Two small rings slide on the arc of a smooth vertical circle; a string passes through both rings, and has three equal weights attached to it, one at each end and one between the rings; find the position of the rings when they are in equilibrium. The rings are supposed without weight.

Result. Each of the rings must be 30° distant from the highest point of the circle.

13. The extremities of a string without weight are fastened to two equal heavy rings which slide on smooth fixed rods in the same vertical plane and equally inclined to the vertical; and to the middle point of the string a weight is fastened equal to twice the weight of each ring; find the position of equilibrium and the tension of the string.

If the point to which the weight is fastened be not the middle point of the string, shew that in the position of equilibrium the tensions of its two portions will be equal.

14. A light cord with one end attached to a fixed point passes over a pulley in the same horizontal line with the fixed point and supports a weight hanging freely at its other end. A heavy ring being fastened to the cord in different places between the fixed point and the pulley, it is required to find the locus of its positions of equilibrium. If the weight of the ring be small compared with the other weight, the locus will be approximately a parabola.

15. If two forces acting along chords of a circle are inversely proportional to the length of the chords, their resultant will pass through one or other of the points of intersection of lines drawn through the extremities of the chords.

16. A particle rests on an ellipse acted on by forces λx^n, μy^n, parallel to the axes of x and y respectively; find its position of equilibrium. Explain the case in which $n=1$.

17. A particle is placed on the outer surface of a smooth fixed sphere and is acted on by a fixed centre of force lying vertically above the centre of the sphere, at a distance c from it and attracting directly as the distance. Shew that the particle will rest on any part of the sphere if the weight of the particle equals the attraction on it by the fixed centre of force when at a distance c from it.

18. A particle is placed on the surface of an ellipsoid in the centre of which is resident an attractive force, determine the direction in which the particle will begin to move.

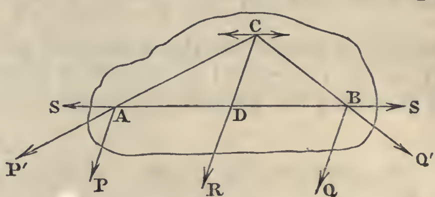
19. Find the point on the surface $\frac{x^3}{a^3}+\frac{y^3}{b^3}+\frac{z^3}{c^3}=1$, where a particle attracted to the origin by a force varying as the distance will rest in equilibrium.

CHAPTER III.

RESULTANT OF TWO PARALLEL FORCES. COUPLES.

37. *To find the magnitude and direction of the resultant of two parallel forces acting in the same plane on a rigid body.*

Let P and Q be the forces; A and B their points of ap-



plication: let P and Q act in the *same* direction, making angles α with AB. The effect of the forces will not be altered if we apply two forces equal in magnitude and acting in opposite directions along the line AB. Let S denote each of these forces, and suppose one to act at A and the other at B.

Then P and S acting at A are equivalent to some force P' acting in some direction AP' inclined to AP (Art. 17); and Q and S acting at B are equivalent to some force Q' acting in some direction BQ' inclined to BQ.

Produce $P'A$, $Q'B$ to cut each other in C, and draw CD parallel to AP, meeting AB in D; suppose C rigidly connected with AB.

Transfer P' and Q' to C (Art. 11), and resolve them along CD and a line parallel to AB; the latter parts will each be equal to S but act in opposite directions, and the sum of the former is $P+Q$. Hence R the resultant of P and $Q=P+Q$ and acts parallel to P and Q in the line CD. We shall now determine the point where this line cuts AB.

The sides of the triangle ACD are parallel to the directions of the forces P, S, P'; therefore by Art. 19

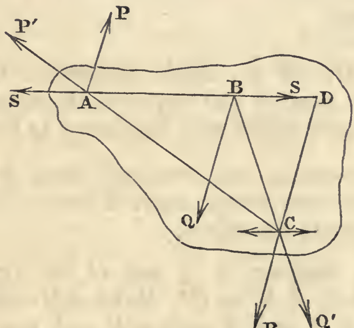
$$\frac{P}{S}=\frac{CD}{DA}, \text{ and similarly } \frac{S}{Q}=\frac{DB}{CD};$$

therefore $\frac{P}{Q}=\frac{DB}{DA}=\frac{a-x}{x}$, if $AB=a$ and $AD=x$;

therefore $$\frac{x}{a}=\frac{Q}{P+Q};$$

this determines the point D through which the direction of the resultant passes. It will be observed that AB is divided in D into segments which are inversely as the forces at A and B respectively.

If the force P act in a direction *opposite* to that of Q,



a similar process will lead us to

$$R=Q-P, \text{ and } \frac{x}{a}=\frac{Q}{Q-P};$$

which may be derived from the formulæ of the preceding article by changing P into $-P$.

It will be observed that AB produced is divided in D into segments which are inversely as the forces at A and B respectively.

38. The point D possesses this remarkable property: that however P and Q are turned about their points of application A and B their directions remaining parallel, D determined as above remains fixed. This point is in consequence called the *centre* of the parallel forces P and Q.

39. If $P=Q$ in the second case of Art. 37, then $R=0$ and $x=\infty$, a result perfectly nugatory. It shews us that the method fails by which we have attempted to compound two equal and opposite parallel forces. In fact the addition of the two forces S still gives, in this case, two equal forces parallel and opposite in their directions.

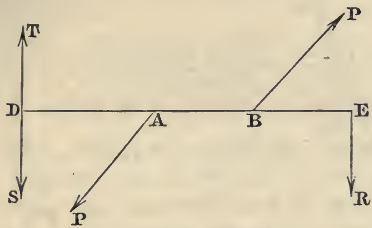
Such a system of forces is called a *Couple.*

We shall investigate the laws of the composition and resolution of couples, since to these we shall reduce the composition and resolution of forces of every description acting upon a rigid body.

40. From Art. 39 we might conjecture that two equal forces acting in parallel and opposite directions do *not admit of a single resultant*, which may be shewn as follows.

Suppose, if possible, that the single force R will maintain equilibrium with two forces, each denoted by P, acting in parallel and opposite directions.

Draw a line meeting in A and B the directions of the forces P, and that of R in E. Make $AD=BE$, and apply at D two forces T and S each $=R$ and parallel to R but in opposite directions; this will not disturb the equilibrium. Hence the five forces R, P, P, S, T are in equilibrium. But since P, P and R form a system in equilibrium so by symmetry do P, P and T. Hence if we remove the last three we shall not disturb the equilibrium, and we accordingly have R and S left maintaining equilibrium. But this is obviously impossible, since they act in the same direction. Hence the two parallel forces P cannot be balanced by a single force, and therefore do not admit of a single resultant.



41. A *couple* consists of two equal forces acting in parallel and opposite directions.

The *arm* of a couple is the perpendicular distance between the directions of its forces.

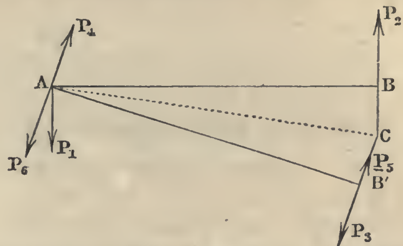
The *moment* of a couple is the product of either of its forces into the perpendicular distance between them.

The *axis* of a couple is a straight line perpendicular to the plane of the couple and proportional in length to the moment.

Two couples in the same plane may differ with respect to direction. For suppose the middle point of the arm of a couple to be fixed, and the arm to move in the direction in which the two forces of the couple tend to urge it; there are two different directions in which the arm may rotate. Suppose a perpendicular drawn to the plane of the couple through the middle point of its arm, so that when an observer is placed along this line with his feet against the plane, the rotation which the forces give to the arm appears to take place from *left* to *right;* the perpendicular so drawn we shall take for the *axis* of the couple.

42. *The effect of a couple is not altered if its arm be turned through any angle about one extremity in the plane of the couple.*

Let the plane of the paper be the plane of the couple, AB the arm, and AB' its new position; the forces P_1, P_2 are equal



and act on the arm AB. At B' and A let the equal and opposite forces P_3P_5, P_4P_6, each equal to P_1 or P_2 be applied, acting perpendicularly to AB'; this will not affect the action of P_1 and P_2.

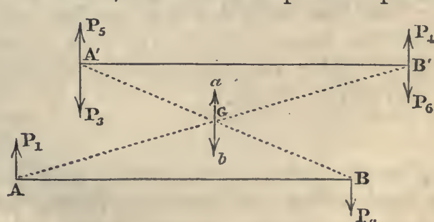
Let BP_2, $B'P_3$ meet in C; join AC; AC manifestly bisects the angle BAB'.

Now P_2 and P_3 are equivalent to some force in the direction CA, and P_1 and P_4 are equivalent to the *same* force in the direction AC. Therefore P_1, P_2, P_3, P_4 are in equilibrium with each other; therefore the remaining forces P_5, P_6 acting at B', A respectively produce the same effect as P_2, P_1 acting at B, A respectively. Hence the proposition is true.

We may now turn the arm of the couple through any angle about B'; and by proceeding in this way may transfer the couple to any position in its own plane.

43. *The effect of a couple is not altered if we transfer the couple to any plane parallel to its own, the arm remaining parallel to itself.*

Let AB be the arm, $A'B'$ its new position parallel to AB.



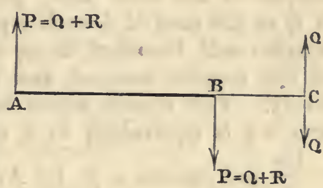
Join AB', $A'B$ bisecting each other in G. At A', B' apply two equal and opposite forces each $= P_1$ or P_2 and parallel to them; and let these forces be P_3, P_4, P_5, P_6; this will not alter the effect of the couple.

But P_1 and P_4 are equivalent to $2P_1$ acting at G in direction Ga parallel to the direction of P_1, and P_2 and P_3 are equivalent to $2P_1$ acting at G in the opposite direction Gb.

Hence P_1, P_2, P_3, P_4 are in equilibrium with each other; therefore the remaining forces P_5 and P_6 acting at A' and B' respectively produce the same effect as P_1 and P_2 acting at A and B respectively. Hence the proposition is true.

44. *The effect of a couple will not be altered if we replace it by another of which the moment is the same; the plane remaining the same and the arms being in the same line and having a common extremity.*

Let AB be the arm; let P, P be the forces, and suppose $P = Q + R$; let $AB = a$, and let the new arm $AC = b$; at C apply two opposite forces each $= Q$ and parallel to P; this will not alter the effect of the couple.



Now R at A and Q at C will balance $Q + R$ at B,

$$\text{if } AB : BC :: Q : R, \quad \text{(Art. 37)},$$

$$\text{or if } AB : AC :: Q : Q + R,$$

$$\text{that is, if } Q \cdot b = P \cdot a;$$

we have then remaining the couple Q, Q acting on the arm AC. Hence the couple P, P acting on AB may be replaced by the couple Q, Q acting on AC, if $Q \cdot b = P \cdot a$, that is, if their moments are the same.

45. From the last three articles it appears that, without altering the effect of a couple, we may change it into another

of equal moment, and transfer it to any position, either in its own plane or in a plane parallel to its own. The couple must remain unchanged so far as concerns the direction of the rotation which its forces would tend to give the arm, supposing its middle point fixed as in Art. 41. In other words, the line which we have called the axis, measured as indicated in that article, must always remain on the *same side* of the plane of the couple.

46. We may infer from Art. 44 that couples may be measured by their moments. Let there be two couples, one in which each force $= P$, and one in which each force $= Q$, the arms of the couples being *equal;* these couples will be in the ratio of P to Q. For suppose, for example, that P is to Q as 3 to 5; then each of the forces P may be divided into 3 equal forces and Q into 5 such equal forces. Then the couple of which each force is P may be considered as the sum of 3 equal couples of the same kind, and the couple of which each force is Q as the sum of 5 such equal couples. The effects of the couples will therefore be as 3 to 5. Next, suppose the arms of the couples *unequal*, and denote them by p and q respectively. The couple which has each of its forces $= Q$ and its arm $= q$ is equivalent to a couple having each of its forces $= \frac{Qq}{p}$ and its arm $= p$, by Art. 44. The couples are therefore by the first case in the ratio of P to $\frac{Qq}{p}$, that is of Pp to Qq.

47. With respect to the effect of a couple, we may observe that it is shewn in works on rigid dynamics that if a couple act on a free rigid body it will set the body in rotation about an axis passing through a certain point in the body called its centre of gravity, *but not necessarily perpendicular to the plane of the couple.*

48. *To find the resultant of any number of couples acting upon a body, the planes of the couples being parallel to each other.*

First, suppose all the couples transferred to the same plane (Art. 43); next, let them be all transferred so as to have their arms in the same straight line, and one extremity common (Art. 42); and lastly, let them be replaced by others having the same arm (Art. 44).

Thus if $P, Q, R, \ldots\ldots\ldots$ be the forces, and

$a, b, c, \ldots\ldots\ldots$ be their arms,

we shall have them replaced by the following forces (supposing α the common arm),

$$P.\frac{a}{\alpha},\quad Q.\frac{b}{\alpha},\quad R.\frac{c}{\alpha}, \ldots\ldots\ldots \text{ acting on the arm } \alpha.$$

Hence their resultant will be a couple of which each force equals

$$P.\frac{a}{\alpha}+Q.\frac{b}{\alpha}+R.\frac{c}{\alpha}+\ldots\ldots\ldots,$$

and arm $=\alpha$,

or of which the moment equals

$$P.a+Q.b+R.c+\ldots\ldots\ldots$$

Hence the moment of the resultant couple is equal to the sum of the moments of the original couples.

If one of the couples, as Q, Q, act in a direction *opposite* to the couple P, P, then the force at each extremity of the arm of the resultant couple will be

$$P.\frac{a}{\alpha}-Q.\frac{b}{\alpha}+R.\frac{c}{\alpha}+\ldots\ldots\ldots,$$

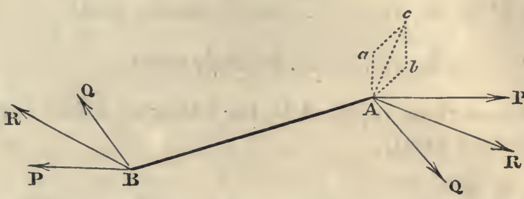
and the moment of the resultant couple will be

$$P.a-Q.b+R.c+\ldots\ldots\ldots,$$

or the algebraical sum of the moments of the original couples; the moments of those couples which tend in the direction opposite to the couple P, P being reckoned negative.

49. *To find the resultant of two couples not acting in the same plane.*

Let the planes of the couples intersect in the line AB,



which is perpendicular to the plane of the paper, and let the couples be referred to the common arm AB, and let their forces thus altered be P and Q.

In the plane of the paper draw Aa, Ab perpendicular to the planes of the couples P, P and Q, Q; and equal in length to their *axes*.

Let R be the resultant of the forces P and Q at A, acting in the direction AR; and of P and Q at B, in the direction BR.

Since AP, AQ are parallel to BP, BQ respectively, therefore AR is parallel to BR.

Hence the two couples are equivalent to the single couple R, R acting on the arm AB.

Draw Ac perpendicular to the plane of R, R, and in the same proportion to Aa, Ab that the moment of the couple R, R is to those of P, P and Q, Q respectively. Then Ac is the *axis* of R, R. Now the three lines Aa, Ac, Ab make the same angles with each other that AP, AR, AQ make with each other; also they are in the same proportion in which $AB.P$, $AB.R$, $AB.Q$ are; that is in which P, R, Q are.

But R is the resultant of P and Q; therefore Ac is the diagonal of the parallelogram on Aa, Ab (see Art. 17).

Hence if two straight lines, having a common extremity, represent the axes of two couples, that diagonal of the parallelogram described on these lines which passes through their common extremity is equal in magnitude and direction to the axis of the resultant couple.

50. *To find the magnitude and position of the couple which is the resultant of three couples which act in planes at right angles to each other.*

Let AB, AC, AD be the axes of the given couples (see fig. to Art. 24). Complete the parallelogram CB, and draw AE the diagonal. Then AE is the axis of the couple which is the resultant of the two couples of which the axes are AB, AC. Complete the parallelogram DE, and draw AF the diagonal. Then AF is the axis of the couple which is the resultant of the couples of which the axes are AE, AD, or of those of which the axes are AB, AC, AD.

Now $\quad AF^2 = AE^2 + AD^2 = AB^2 + AC^2 + AD^2.$

Let G be the moment of the resultant couple; L, M, N those of the given couples;

therefore $$G^2 = L^2 + M^2 + N^2;$$

and if λ, μ, ν be the angles the axis of the resultant makes with those of the components,

$$\cos\lambda = \frac{AB}{AF} = \frac{L}{G}; \quad \cos\mu = \frac{M}{G}; \quad \cos\nu = \frac{N}{G}.$$

51. Hence conversely any couple may be replaced by three couples acting in planes at right angles to each other; their moments being $G\cos\lambda$, $G\cos\mu$, $G\cos\nu$; where G is the moment of the given couple, and λ, μ, ν the angles its axis makes with the axes of the three couples.

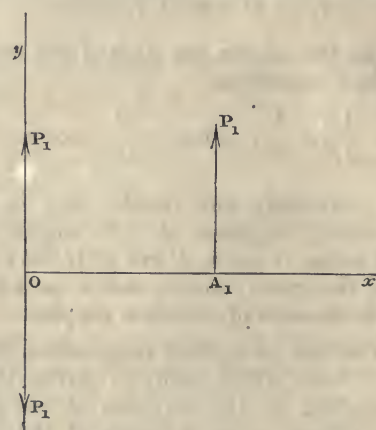
Thus *couples* follow, as to their composition and resolution, laws similar to those which apply to *forces*, the *axis* of the couple corresponding to the *direction* of the force and the *moment* of the couple to the *intensity* of the force. Hence for example, by Art. 29, the resolved part of a resultant couple in any direction is equal to the sum of the resolved parts of the component couples in the same direction.

CHAPTER IV.

RESULTANT OF FORCES IN ONE PLANE. CONDITIONS OF EQUILIBRIUM. MOMENTS.

52. *To find the resultant of any number of parallel forces acting on a rigid body in one plane.*

Let P_1, P_2, P_3...... denote the forces. Take any point in



the plane of the forces as origin and draw rectangular axes Ox, Oy, the latter parallel to the forces. Let A_1 be the point where Ox meets the direction of P_1, and let $OA_1 = x_1$.

Apply at O two forces each equal and parallel to P_1, in opposite directions. Thus the force P_1 is replaced by P_1 at O along Oy, and a couple of which the moment is $P_1 . OA_1$, that is $P_1 . x_1$. Transform the other forces in a similar manner, using a similar notation, and the whole system will be reduced to a force

$$P_1 + P_2 + P_3 + \dots\dots\dots \text{ or } \Sigma P \text{ along } Oy,$$

and a couple

$$P_1x_1 + P_2x_2 + P_3x_3 + \dots\dots\dots \text{ or } \Sigma Px$$

in the plane of the forces and tending to turn the body from the axis of x to the axis of y.

53. *To find the conditions of equilibrium of a system of parallel forces acting on a rigid body in one plane.*

A system of parallel forces can be reduced to a single force and a couple. If neither of these vanish equilibrium is impossible, because a single force cannot neutralize a couple (Art. 40). If the single force alone vanish equilibrium is impossible, because there remains an unbalanced couple. If the couple alone vanish equilibrium is impossible, because there remains an unbalanced force. Hence, for equilibrium it is necessary that both the force and the couple should vanish; that is

$$\Sigma P = 0 \text{ and } \Sigma Px = 0.$$

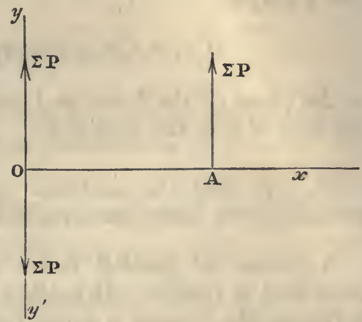
54. The product of a force into the perpendicular drawn upon it from any point, is called the moment of the force with respect to that point. Hence the conditions of equilibrium which have just been obtained for a system of parallel forces acting in one plane may be thus enunciated.

(1) The sum of the forces must vanish.

(2) The sum of the moments of the forces with respect to any point in the plane of the forces must vanish.

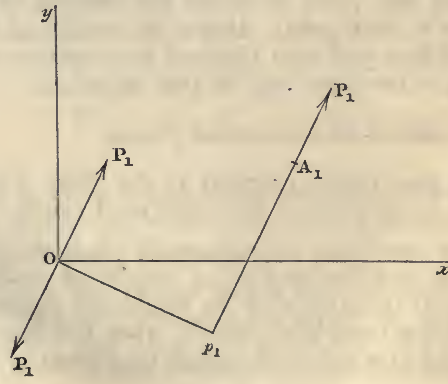
The word *sum* must be understood *algebraically;* forces which act in one direction being considered *positive,* those in the opposite direction must be considered *negative.* Also a moment being considered *positive* when the force tends to urge the point at which it is applied from *right* to *left* when the eye is placed at the origin and looks along the perpendicular on the force, a force tending to urge the point of application from *left* to *right* will have a negative moment.

55. When the sum of the forces vanishes in Art. 52, the forces reduce to a couple. When ΣP is not zero, the forces can be reduced to a single resultant. For if $\Sigma Px = 0$, then ΣP acting at O is *the single* resultant. If ΣPx be not $= 0$, let the couple be transformed to one in which each of the forces is equal to ΣP, and consequently, by Art. 44, the arm is $\frac{\Sigma Px}{\Sigma P}$. Let ΣP acting at A and ΣP acting along Oy' form this couple. The latter force is destroyed by the force ΣP along Oy. Hence the single resultant is ΣP acting at A, that is, at a point the distance of which from O is $\frac{\Sigma Px}{\Sigma P}$.



56. *To find the resultant of any number of forces which act upon a rigid body in one plane.*

Let the system be referred to any rectangular axes Ox, Oy in the plane of the forces.



Let P_1, P_2, P_3, ... denote the forces; α_1, α_2, α_3, ... the angles which their directions make with the axis of x; let x_1, y_1 be the co-ordinates of the point of application of P_1; let x_2, y_2 be those of the point of application of P_2, and so on.

Let A_1 be the point of application of P_1. At O suppose two forces applied in opposite directions each equal and parallel to P_1. Draw Op_1 perpendicular to P_1A_1.

Hence P_1 acting at A_1 is equivalent to P_1 acting at O and a couple of which Op_1 is the arm and each force is P_1, which tends to turn the body from the axis of x to that of y. Now

$$Op_1 = x_1 \sin \alpha_1 - y_1 \cos \alpha_1.$$

Hence the moment of the couple is

$$P_1 (x_1 \sin \alpha_1 - y_1 \cos \alpha_1).$$

The other forces may be similarly replaced. Hence the system is equivalent to the forces

$$P_1, P_2, P_3, \ldots\ldots \text{ acting at } O,$$

in directions parallel to those of the original forces; and the couples of which the moments are

$$P_1 (x_1 \sin \alpha_1 - y_1 \cos \alpha_1),$$
$$P_2 (x_2 \sin \alpha_2 - y_2 \cos \alpha_2),$$
$$P_3 (x_3 \sin \alpha_3 - y_3 \cos \alpha_3),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

acting in the plane of the forces. It will be found that any one of the above expressions for the moments of the couples is positive or negative, according as that couple tends to turn the body from the axis of x towards that of y, or in the contrary direction.

Let R be the resultant of the forces acting at O, let a be the angle which R makes with the axis of x, and G the moment of the resultant couple; then (by Art. 22)

$$R \cos a = \Sigma P \cos \alpha; \quad R \sin a = \Sigma P \sin \alpha;$$

and (by Art. 48)

$$G = \Sigma P(x \sin \alpha - y \cos \alpha).$$

If $P_1 \cos \alpha_1 = X_1$ and $P_1 \sin \alpha_1 = Y_1$, and a similar notation be used for the other forces, the above equations may be written

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2; \ \tan \alpha = \frac{\Sigma Y}{\Sigma X};$$

and $$G = \Sigma (Yx - Xy).$$

57. *To find the conditions for the equilibrium of a system of forces acting on a rigid body in one plane.*

Any system of forces acting in one plane may be reduced to a single force R, and a couple whose moment is G. If neither R nor G vanish equilibrium is impossible, since a single force cannot balance a couple. If R alone vanish equilibrium is impossible, because there remains an unbalanced couple G; if G alone vanish equilibrium is impossible, because there remains an unbalanced force. Hence, for equilibrium we must have $R = 0$ and $G = 0$. Also $R = 0$ requires that $\Sigma X = 0$ and $\Sigma Y = 0$.

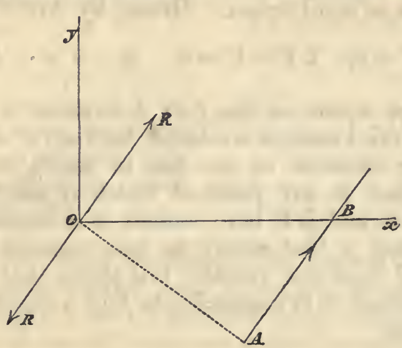
Since G is equal to the sum of the moments of the forces with respect to O, if a system of forces acting in one plane upon a body is in equilibrium, *the sum of the resolved parts of the forces parallel to any two rectangular axes in the plane must vanish, and the sum of the moments of the forces with respect to any origin in the plane must vanish.*

58. *If three forces acting in one plane maintain a rigid body in equilibrium their directions either all meet in a point or are all parallel.*

For suppose two of the directions to meet in a point, and take this point for the origin; then the moment of each of these two forces vanishes, and the equation $G = 0$ requires that the moment of the third force should vanish, that is, the third force must also pass through the origin. Hence, if any two of the forces meet, the third must pass through their point of intersection, which proves the proposition. This proposition may also be established without referring to Art. 57. For if two of the forces meet in a point, they may be supposed both to act at that point and may be replaced by their resultant acting at the same point; this resultant and the third force must keep the body on which they act in equilibrium, and must therefore be equal and opposite; that is, the third force must pass through the point of intersection of the first two.

59. If $R=0$ in Art. 56, the forces reduce to a couple; if R be not $=0$, the forces can be reduced to a single resultant. For if $G=0$, the resultant force is R acting at the origin.

If the couple G be not $=0$, let it be transformed into



one having each of its forces $=R$ and its arm consequently $=\frac{G}{R}$ (Art. 44). Let this couple be turned in its own plane, until one of its forces acts at the origin exactly opposite to the force R, which by hypothesis acts at the origin. Hence these forces destroy each other and we have left R acting at the extremity of the arm OA, in a direction inclined to the axis of x at an angle a, found by the equation $\tan a=\frac{\Sigma Y}{\Sigma X}$ (Art. 56). If this direction meet the axis of x in B, we have

$$OB = OA \operatorname{cosec} a = \frac{G}{R} \cdot \frac{R}{\Sigma Y} = \frac{G}{\Sigma Y},$$

and the equation to the line of action of the single resultant is

$$y' = \frac{\Sigma Y}{\Sigma X}\left(x' - \frac{G}{\Sigma Y}\right);$$

or, $$x'\Sigma Y - y'\Sigma X = \Sigma\,(Yx - Xy),$$

x', y' being the variable co-ordinates.

60. The result of the last article may also be obtained thus. Suppose that the given forces have a single resultant acting at the point (x', y'), and equivalent to the components X' and Y' parallel to the co-ordinate axes. It follows that the given forces will, with $-X'$, $-Y'$ acting at the point (x', y'), form a system in equilibrium. Hence, by Art. 57,

$$\Sigma X - X' = 0, \quad \Sigma Y - Y' = 0, \quad G - Y'x' + X'y' = 0.$$

Of these three equations the first determines X', the second Y', and the third assigns a relation between x' and y', which is in fact the equation to the line in which the single resultant acts and at any point of which it may be supposed to act. If ΣX and ΣY both vanish, it is impossible to find values of x' and y' that satisfy the last equation of the three, so long as G does not vanish; this shews that if the forces reduce to a couple, it is impossible to find a single force equivalent to them.

61. In Art. 56, we have for the moment of the force P_1 about the origin the expression

$$P_1\,(x_1 \sin\alpha_1 - y_1 \cos\alpha_1),$$

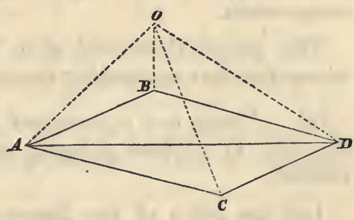
and this we may express by

$$Y_1 x_1 - X_1 y_1.$$

Since X_1 and Y_1 are the rectangular components of P_1, we see by comparing the two expressions that the moment of a force about any origin is equal to the algebraical sum of the moments of its rectangular components about the same origin. (See Art. 54.) There are many such theorems connected with moments, and the demonstration of some of them is facilitated by observing that according to the definition of a moment, it may be geometrically represented by twice the area of the triangle having for its base the line which represents the force and for its vertex the point about which moments are taken. For example, we may prove the theorem which we have already deduced.

62. *The algebraical sum of the moments of two component forces with respect to any point in the plane containing the two forces is equal to the moment of the resultant of the two forces.*

Let AB, AC represent two component forces; complete the parallelogram and draw the diagonal AD representing the resultant force.



(1) Let O, the point about which the moments are to be taken, fall *without* the angle BAC and that which is vertically opposite to it. Join OA, OB, OC, OD.

The triangle OAC having for its base AC and for its height the perpendicular from O on AC is equivalent to a triangle having AC for its base and for its height the perpendicular from B on AC, together with a triangle having BD for its base and for its height the perpendicular from O on BD. This is obvious since BD is equal and parallel to AC, and the perpendicular from O on AC is equal to the perpendicular from O on BD together with the perpendicular from B on AC. Hence we have

$$\Delta\, AOC = \Delta\, BOD + \Delta\, ACD.$$

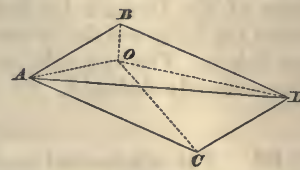
Hence, adding the triangle AOB, we have

$$\Delta AOC + \Delta AOB = \Delta BOD + \Delta ABD + \Delta AOB = \Delta AOD;$$

that is, the moment of AC + the moment of AB = the moment of AD.

(2) Let O fall *within* the angle BAC or its vertically opposite angle.

$$\Delta AOC = \Delta ABD - \Delta BOD$$
$$= \Delta AOB + \Delta AOD.$$



Therefore

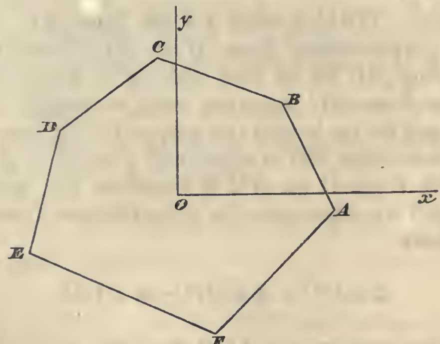
$$\Delta AOD = \Delta AOC - \Delta AOB;$$

that is, the moment of $AD =$ the moment of $AC -$ the moment of AB. As the moments of AC and AB about O are now of *opposite* characters, the moment of the resultant is still equal to the algebraical sum of the moments of the components.

The proposition may also be readily shewn in the case where the two component forces are *parallel;* see Art. 37.

63. *Forces are represented in magnitude and position by the sides of a plane polygon taken in order; required the resultant.*

Let the sides of the figure $ABCDEF$ represent the forces in magnitude and position; the first force being supposed to



act in the line AB from A towards B, the second in the line BC from B towards C, and so on.

As in Art. 56, the forces may be replaced by a resultant force at an arbitrary origin O and a couple. The former is composed of all the forces AB, BC,...... moved each parallel to itself up to O; the resultant consequently vanishes by Art. 21.

The moment of the resultant couple is the sum of the moments of the component couples, and is therefore represented by twice the triangle AOB + twice the triangle BOC + &c....; that is, by *twice the area of the polygon.* Hence the forces reduce to a resultant couple measured by twice the area of the polygon.

We may observe that the algebraical sum of the moments of the two forces which form a couple is the same about whatever point it be taken; it is in fact equal to the *moment of the couple.*

64. If the sum of the moments of the forces $P_1, P_2, P_3, \ldots$ be required about a point whose co-ordinates are h, k instead of about the origin, we must in the expression for G, in Art. 56, put $x_1 - h$, $x_2 - h, \ldots$ for $x_1, x_2, \ldots$ respectively, and $y_1 - k$, $y_2 - k, \ldots$ for $y_1, y_2, \ldots$ respectively. Hence, denoting the result by G_1, we have

$$G_1 = \Sigma\{Y(x-h) - X(y-k)\},$$

$$= k\Sigma X - h\Sigma Y + \Sigma(Yx - Xy),$$

$$= k\Sigma X - h\Sigma Y + G.$$

Hence the value of G_1 depends in general upon the situation of the point about which we take moments. If, however,

$$k\Sigma X - h\Sigma Y = \text{a constant},$$

that is, if the point (h, k) move along any line parallel to the direction of the resultant force R, then G_1 remains unchanged.

If three different points exist with respect to which the sum of the moments vanishes, we have three equations

$$k_1\Sigma X - h_1\Sigma Y + G = 0,$$

$$k_2\Sigma X - h_2\Sigma Y + G = 0,$$

$$k_3\Sigma X - h_3\Sigma Y + G = 0.$$

Hence we deduce

$$(k_1 - k_2)\,\Sigma X = (h_1 - h_2)\,\Sigma Y,$$

$$(k_2 - k_3)\,\Sigma X = (h_2 - h_3)\,\Sigma Y.$$

Unless the point (h_1, k_1), the point (h_2, k_2), and the point (h_3, k_3) lie in a straight line, it is impossible that

$$\frac{k_1 - k_2}{h_1 - h_2} = \frac{k_2 - k_3}{h_2 - h_3};$$

we must therefore have

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad G = 0.$$

Hence if the sum of the moments of a system of forces in one plane vanish with respect to three points in the plane not in a straight line, that system is in equilibrium.

When a system of forces in one plane can be reduced to a single resultant, we have found in Art. 59 that the equation to the direction of the resultant is

$$x'\Sigma Y - y'\Sigma X = \Sigma\,(Yx - Xy).$$

This may be written

$$\Sigma\,\{Y(x' - x) - X(y' - y)\} = 0.$$

The equation to the direction of the resultant thus in fact determines the locus of the points for which the *algebraical sum* of the moments of the forces is zero.

65. Hitherto we have supposed our axes rectangular. If they are oblique and inclined at an angle ω, we may shew, as in Art. 56, that a system of forces in one plane may be reduced to ΣX along the axis of x, ΣY along the axis of y, and a couple the moment of which is $\sin \omega \Sigma\,(Yx - Xy)$. The latter part will be easily obtained, since the moment of the force P_1 is equivalent to the algebraical sum of the moments of its components X_1 and Y_1; and the perpendicular on the former from the origin is $y_1 \sin \omega$, and on the latter $x_1 \sin \omega$.

The conditions for equilibrium are, as before,

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma (Yx - Xy) = 0.$$

The following Examples may be solved by means of the principles given in the preceding articles. When different rigid bodies occur in a question, the equations of Art. 57 must hold with respect to *each*, in order that there may be equilibrium. In cases where only *three* forces act on a body, it is often convenient to use the proposition of Art. 58. Since by Art. 57 the moments of the forces with respect to *any* origin must vanish, we may, if we please, take different origins and form the corresponding equation for each. See Art. 64.

In some of the examples we anticipate the results of the subsequent chapters so far as to assume that the weight of any body acts through a definite and known point, which is the centre of gravity of the body. When two bodies are in contact it is assumed that whatever force one exerts on the other the latter exerts an equal and opposite force on the former; if the bodies are smooth this force acts in the direction of the common normal to the surfaces at the point of contact. We restrict ourselves to the supposition of smooth bodies until Chapter X.

In attempting to solve the problems the student will find it advisable when the system involves more than one body to confine his attention to one at a time of those bodies which are capable of motion, and to be careful to take into consideration *all* the forces which act on that body. When bodies are in contact some letter should be used to denote the mutual force between them, and the magnitude of this force must be found from the equations of equilibrium of the body or bodies which are capable of motion. And when two of the bodies are connected by a string a letter should be used to denote the tension of the string, and the magnitude of the tension must be found from the conditions of equilibrium of the body or bodies which are capable of motion. Beginners often fall into error by *assuming* incorrect values for the tensions of strings and the mutual forces between bodies in contact, instead of determining the correct values from the equations of equilibrium.

EXAMPLES.

1. *ABCD* is a quadrilateral and is acted on by forces which are represented in magnitude and direction by *AB*, *AD*, *CB*, *CD*; shew that the resultant coincides in direction with the line which joins the middle points of the diagonals *AC*, *BD*, and is represented in magnitude by four times this line.

2. Forces whose intensities are proportional to the sides of an isosceles triangle act along the sides of the triangle, those acting along the equal sides tending from the vertex; find the magnitude and position of their resultant.

Result. The required resultant is represented by a line which passes through the middle point of the base of the triangle, is parallel to one of the sides, and double that side in length.

3. The upper end of a uniform heavy rod rests against a smooth vertical wall; one end of a string is fastened to the lower end of the rod and the other end of the string is fastened to the wall; the position of the rod being given, find the point of the wall to which the string must be fastened.

4. A uniform heavy rod is placed across a smooth horizontal rail, and rests with one end against a smooth vertical wall, the distance of which from the rail is $\frac{1}{16}$th of the length of the rod; find the position of equilibrium.

Result. The rod makes an angle of 60° with the horizon.

5. Forces act on a triangle at the middle points of its sides; they are perpendicular to the sides and proportional to them in magnitude; if they all act inwards or all act outwards they will keep the triangle in equilibrium.

6. Forces act on any polygon at the middle points of its sides; they are perpendicular to the sides and proportional to them in magnitude; if they all act inwards or all act outwards they will keep the polygon in equilibrium.

7. An elliptic lamina is acted on at the extremities of pairs of conjugate diameters by forces in its own plane tending outwards, and normal to its edge; there will be equilibrium if the force at the end of every diameter be proportional to the conjugate.

8. A heavy sphere hangs from a peg by a string whose length is equal to the radius, and it rests against another peg vertically below the former, the distance between the two being equal to the diameter. Find the tension of the string and the pressure on the lower peg.

Results. The tension is equal to the weight of the sphere and the pressure to half the weight of the sphere.

9. Two equal rods without weight are connected at their middle points by a pin which allows free motion in a vertical plane; they stand upon a horizontal plane, and their upper extremities are connected by a thread which carries a weight. Shew that the weight will rest half way between the pin and the horizontal line joining the upper ends of the rods.

10. Two equal circular discs with smooth edges, placed on their flat sides in the corner between two smooth vertical planes inclined at a given angle, touch each other in the line bisecting the angle. Find the radius of the least disc which may be pressed between them without causing them to separate.

11. A flat semicircular board with its plane vertical and curved edge upwards rests on a smooth horizontal plane, and is pressed at two given points of its circumference by two beams which slide in smooth vertical tubes; find the ratio of the weights of the beams that the board may be in equilibrium.

12. Two smooth cylinders of equal radii just fit in between two parallel vertical walls, and rest on a smooth horizontal plane without pressing against the walls; if a third be placed on the top of them, find the resulting pressure against either wall.

13. A smooth circular ring rests on two pegs not in the same horizontal plane; find the pressure on each peg.

14. Two spheres are supported by strings attached to a given point, and rest against one another; find the tensions of the strings.

15. Two equal smooth spheres, connected by a string, are laid upon the surface of a cylinder, the string being so short as not to touch the cylinder; determine the position of rest and the tension of the string.

16. A heavy regular polygon is attached to a smooth vertical wall by a string which is fastened to the middle point of one of its sides; the plane of the polygon is vertical and perpendicular to the wall, and one of the extremities of the side to which the string is attached rests against the wall; shew that whatever be the length of the string when the polygon is in equilibrium, the tension of the string and the pressure on the wall are constant.

17. A straight rod without weight is placed between two pegs and forces P and Q act at its extremities in parallel directions inclined to the rod; required the conditions under which the rod will be at rest and the pressures on the pegs.

18. Forces P, Q, R, S act along the sides of a rectangle; find the direction of the resultant force.

√ 19. Two weights P, P are attached to the ends of two strings which pass over the same smooth peg and have their other extremities attached to the ends of a beam AB, the weight of which is W; shew that the inclination of the beam to the horizon $= \tan^{-1}\left(\dfrac{a-b}{a+b}\tan\alpha\right)$; a, b being the distances of the centre of gravity of the beam from its ends, and $\sin\alpha = \dfrac{W}{2P}$.

20. A square is placed with its plane vertical between two small pegs which are in the same horizontal line; shew that it will be in equilibrium when the inclination of one of its edges to the horizon $= \frac{1}{2}\sin^{-1}\dfrac{a^2-c^2}{c^2}$, $2a$ being the length of a side of the square, and c the distance between

the pegs. Shew that the equilibrium will not be affected by the application of any force which bisects the line joining the pegs and passes through the lowest point of the square.

21. One end of a string is fixed to the extremity of a smooth uniform rod, and the other to a ring without weight which passes over the rod, and the string is hung over a smooth peg. Determine the least length of the string for which equilibrium is possible, and shew that the inclination of the rod to the vertical cannot be less than 45°.

22. A string 9 feet long has one end attached to the extremity of a smooth uniform heavy rod two feet in length, and at the other end carries a ring without weight which slides upon the rod. The rod is suspended by means of the string from a smooth peg; prove that if θ be the angle which the rod makes with the horizon, then

$$\tan\theta = 3^{-\frac{1}{3}} - 3^{-\frac{2}{3}}.$$

$9\tan^3\theta + 9\tan\theta = 2$

23. A square rests with its plane perpendicular to a smooth wall, one corner being attached to a point in the wall by a string whose length is equal to a side of the square; shew that the distances of three of its angular points from the wall are as 1, 3, and 4.

24. One end of a beam, whose weight is W, is placed on a smooth horizontal plane; the other end, to which a string is fastened, rests against another smooth plane inclined at an angle α to the horizon; the string passing over a pulley at the top of the inclined plane hangs vertically, supporting a weight P. Shew that the beam will rest in all positions if a certain relation hold between P, W, and α. — Ans $W\sin\alpha = 2P$

25. If a weight be suspended from one extremity of a rod moveable about the other extremity A, which remains fixed, and a string of given length be attached to any point B in the rod, and also to a fixed point C above A, and in the same vertical line with it, then the tension of the string varies inversely as the distance AB.

26. One end of a uniform beam is placed on the ground against a fixed obstacle, and to the other is attached a string

which runs in a horizontal direction to a fixed point in the same vertical line as the obstacle, and passing freely over it, is kept in tension by a weight W suspended at its extremity, the beam being thus held at rest at an inclination of 45° to the horizon. Shew that if the string were attached to the centre instead of to the end of the beam, and passed over the same fixed point, a weight $=\sqrt{2}\,W$ would keep the beam in the same position.

27. Two equal beams AB, AC connected by a hinge at A are placed in a vertical plane with their extremities B, C resting on a horizontal plane; they are kept from falling by strings connecting B and C with the middle points of the opposite sides; shew that the ratio of the tension of each string to the weight of each beam

$$=\tfrac{1}{8}\sqrt{(8\cot^2\theta+\operatorname{cosec}^2\theta)},$$

θ being the inclination of each beam to the horizon.

28. One end of a string is attached to a beam at the point B, and the other end is fastened to the highest point A of a fixed sphere of radius r. If the points of contact of the beam and string trisect the quadrant AC, shew that the distance between B and the centre of gravity of the beam must be $2r(2-\sqrt{3})$.

29. A heavy rod can turn freely about a fixed hinge at one extremity, and it carries a heavy ring which is attached to a fixed point in the same horizontal plane with the hinge by means of a string of length equal to the distance between the point and the hinge. Find the position in which the beam will rest.

30. Two equal heavy beams of sufficient length, and connected by a hinge, are supported by two smooth pegs in the same horizontal line; a sphere is placed between them, determine the position of equilibrium.

31. Forces P, Q, R act along the sides BC, CA, AB of a triangle and their resultant passes through the centres of the inscribed and circumscribed circles; shew that

$P : Q : R :: \cos B-\cos C : \cos C-\cos A : \cos A-\cos B.$

32. Find the position of equilibrium of a uniform beam resting in a vertical plane with one end pressing against a vertical wall, and the other end supported by the convex arc of a vertical parabola whose vertex is at the foot of the wall and axis horizontal.

33. A uniform beam PQ of given weight and length rests in contact with a fixed vertical circle whose vertical diameter is AB, in such a manner that strings AP, BQ attached to the rod and circle are tangents to the circle at the points A and B. Find the tensions of the strings, and shew that the conditions of the problem require that the inclination of the beam to the vertical must be less than $\sin^{-1}\dfrac{\sqrt{(5)}-1}{2}$.

34. Shew that no uniform rod can rest partly within and partly without a fixed smooth hemispherical bowl at an inclination to the horizon greater than $\sin^{-1}\dfrac{1}{\sqrt{3}}$.

35. The sides of a rigid plane polygon are acted on by forces perpendicular to the sides and proportional to them in magnitude, all the forces acting in the plane of the polygon, and being inwards; also the sides taken in the same order are severally divided by the points of application in a constant ratio of p to q; prove that the system of forces is equivalent to a couple whose moment is

$$\frac{\mu(p-q)}{2(p+q)}\Sigma a^2,$$

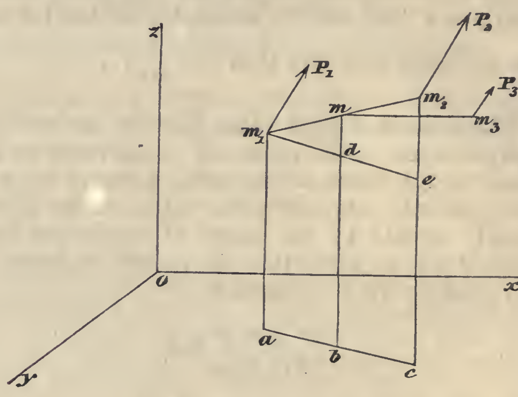
where μa represents the force applied to any side a of the polygon.

CHAPTER V.

FORCES IN DIFFERENT PLANES.

66. *To find the magnitude and direction of the resultant of any number of parallel forces acting upon a rigid body, and to determine the centre of parallel forces.*

Let the points of application of the forces be referred to a system of rectangular co-ordinate axes. Let m_1, m_2,... be the



points of application; let x_1, y_1, z_1, be the co-ordinates of the first point, x_2, y_2, z_2 those of the second, and so on; let P_1, P_2,... be the forces acting at these points, those being reckoned positive which act in the direction of P_1, and those negative which act in the opposite direction.

Join m_1m_2; and take the point m on m_1m_2 such that

$$m_1m = \frac{P_2}{P_1 + P_2} \cdot m_1m_2;$$

then the resultant of P_1 and P_2 is P_1+P_2, and it acts through m parallel to P_1. (Art. 37).

Draw m_1a, mb, m_2c perpendicular to the plane of (x, y), meeting that plane in a, b, c; draw m_1de parallel to abc meeting mb in d and m_2c in e. Then, by similar triangles,

$$\frac{m_1m}{m_1m_2}=\frac{md}{m_2e}=\frac{mb-z_1}{z_2-z_1},$$

therefore $$mb-z_1=\frac{P_2}{P_1+P_2}(z_2-z_1);$$

therefore $$mb=\frac{P_1z_1+P_2z_2}{P_1+P_2}.$$

This gives the ordinate parallel to the axis of z of the point of application of the resultant of P_1 and P_2.

Then supposing P_1 and P_2 to be replaced by P_1+P_2 acting at m, the resultant of P_1+P_2 and P_3 is $P_1+P_2+P_3$, and the ordinate of its point of application

$$=\frac{(P_1+P_2)\,mb+P_3z_3}{P_1+P_2+P_3}=\frac{P_1z_1+P_2z_2+P_3z_3}{P_1+P_2+P_3};$$

and this process may be extended to any number of parallel forces. Let R denote the resultant force and $\bar{z}$ the ordinate of its point of application; then

$$R=\Sigma P,\quad \bar{z}=\frac{\Sigma Pz}{\Sigma P}.$$

Similarly, if $\bar{x}$, $\bar{y}$ be the other co-ordinates of the point of application of the resultant,

$$\bar{x}=\frac{\Sigma Px}{\Sigma P};\quad \bar{y}=\frac{\Sigma Py}{\Sigma P}.$$

The values of $\bar{x}$, $\bar{y}$, $\bar{z}$ are independent of the angles which the directions of the forces make with the axes. Hence if these directions be turned about the points of application of the forces, their parallelism being preserved, the point of application of the resultant will not move. For this reason this point is called the *centre of the parallel forces.*

67. The moment of a force with respect to a plane is the product of the force into the perpendicular distance of its point of application from the plane.

In consequence of this definition, the equations for determining the position of the centre of parallel forces shew that *the sum of the moments of any number of parallel forces with respect to any plane is equal to the moment of their resultant.*

68. If the parallel forces all act in the *same* direction the expression ΣP cannot vanish; hence the values of the co-ordinates of the *centre of parallel forces* found in Art. 66 cannot become infinite or indeterminate, and we are certain that the centre exists. But if some of the forces are positive and some negative, ΣP *may* vanish, and the results of Art. 66 become nugatory. In this case, since the sum of the *positive* forces is equal to the sum of the *negative* forces, the resultant of the former will be equal to the resultant of the latter. Hence the resultant of the whole system of forces is a *couple*, unless the resultant of the positive forces should happen to lie in the same straight line as the resultant of the negative forces.

We shall give another method of reducing a system of parallel forces.

69. *To find the resultant of a system of parallel forces acting upon a rigid body.*

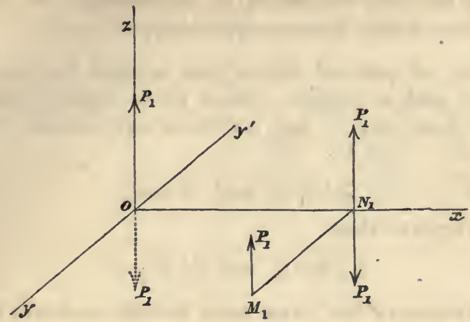
Let P_1, P_2,... denote the forces. Take the axis of z parallel to the forces. Let the plane of (x, y) meet the direction of P_1 in M_1, and suppose x_1, y_1 the co-ordinates of this point.

Draw M_1N_1 perpendicular to the axis of x meeting it in N_1. At the origin O, and also at N_1, apply two forces each equal and parallel to P_1 and in opposite directions. Hence the force P_1 at M_1 is equivalent to the following system,

(1) P_1 at O;

(2) a couple formed of P_1 at M_1 and P_1 at N_1;

(3) a couple formed of P_1 at N_1 and P_1 at O.



The moment of the first couple is P_1y_1, and this couple, without altering its effect, may be transferred to the plane of (y, z), which is parallel to its original plane. The moment of the second couple is P_1x_1, and it is in the plane (x, z).

If we effect a similar transformation of all the forces, we have, as the resultant of the system the following system,

(1) a force ΣP acting at O;

(2) a couple ΣPy in the plane of (y, z);

(3) a couple ΣPx in the plane of (x, z).

The first couple tends to turn the body from the axis of y to that of z, and the second from the axis of x to that of z. We may therefore take Ox as the *axis* of the first couple according to the definition in Art. 41. For the second couple, however, we must either take Oy' as the axis, or consider it as a couple turning from z to x, of which the moment is $-\Sigma Px$ and the axis Oy. Adopting the latter method, we may replace the two couples by a single couple of which the moment is G, where

$$G^2 = (\Sigma Px)^2 + (\Sigma Py)^2,$$

and the axis is inclined to the axis of x at an angle a given by the equations

$$\cos a = \frac{\Sigma Py}{G}; \quad \sin a = \frac{-\Sigma Px}{G}.$$

70. *To find the conditions of equilibrium of a system of parallel forces acting upon a rigid body.*

A system of parallel forces can always be reduced to a single force and a couple. Since these cannot balance, and neither of them singly can maintain equilibrium, they must both vanish. That is,

$$\Sigma P = 0, \text{ and } G = 0;$$

the latter requires that

$$\Sigma Px = 0 \text{ and } \Sigma Py = 0.$$

Hence the sum of the forces must vanish, and also the sum of the moments must vanish with respect to any two planes at right angles to each other and parallel to the forces.

71. When $\Sigma P = 0$, the forces reduce to a couple of which the moment is G. When ΣP is not $=0$, the forces can always be reduced to a single force; this has already appeared in Art. 66, and may also be shewn thus. The forces will reduce to a resultant R acting at the point (x', y'), parallel to the original forces, provided a force $-R$ acting at such point will with the given forces maintain equilibrium. The necessary and sufficient conditions for this are, by Art. 70,

$$\Sigma P - R = 0, \quad \Sigma Px - Rx' = 0, \quad \Sigma Py - Ry' = 0.$$

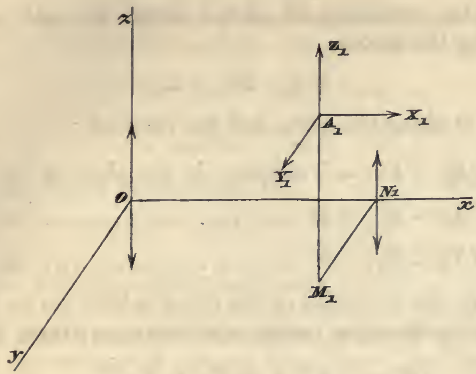
Hence $$R = \Sigma P, \quad x' = \frac{\Sigma Px}{\Sigma P}, \quad y' = \frac{\Sigma Py}{\Sigma P}.$$

These results agree with those of Art. 66.

72. *To find the resultants of any number of forces acting upon a rigid body in any directions.*

Let the forces be referred to three rectangular axes Ox, Oy, Oz; and suppose P_1, P_2, P_3,... the forces; let x_1, y_1, z_1 be the co-ordinates of the point of application of P_1; let x_2, y_2, z_2 be the co-ordinates of the point of application of P_2; and so on.

Let A_1 be the point of application of P_1; resolve P_1 into components X_1; Y_1, Z_1, parallel to the co-ordinate axes. Let



the direction of Z_1 meet the plane of (x, y) in M_1, and draw M_1N_1 perpendicular to Ox. Apply at N_1 and also at O two forces each equal and parallel to Z_1, and in opposite directions. Hence Z_1 at A_1 or M_1 is equivalent to Z_1 at O, and two couples, the former having its moment $= Z_1 . N_1M_1$, and which may be supposed to act in the plane of (y, z), and the latter having its moment $= Z_1 . ON_1$ and acting in the plane of (z, x).

We shall consider those couples as positive which tend to turn the body round the axis of x from y to z, also those which tend to turn the body round the axis of y from z to x, and those which tend to turn the body round the axis of z from x to y.

Hence Z_1 is replaced by Z_1 at O, a couple Z_1y_1 in the plane of (y, z), and a couple $-Z_1x_1$ in the plane of (z, x). Similarly X_1 may be replaced by X_1 at O, a couple X_1z_1 in the plane of (z, x), and a couple $-X_1y_1$ in the plane of (x, y). And Y_1 may be replaced by Y_1 at O, a couple Y_1x_1 in the plane of (x, y), and a couple $-Y_1z_1$ in the plane of (y, z). Therefore the force P_1 may be replaced by X_1, Y_1, Z_1 acting at O, and the couples of which the moments are, by Art. 48,

$$Z_1y_1 - Y_1z_1 \text{ in the plane of } (y, z),$$

$$X_1z_1 - Z_1x_1 \text{ .... } (z, x),$$

$$Y_1x_1 - X_1y_1 \text{ .... } (x, y).$$

By a similar resolution of all the forces we shall have them replaced by the forces

$$\Sigma X, \quad \Sigma Y, \quad \Sigma Z,$$

acting at O along the axes, and the couples

$$\Sigma(Zy - Yz) = L \text{ suppose, in the plane of } (y, z),$$

$$\Sigma(Xz - Zx) = M \text{........, ...} \quad (z, x),$$

$$\Sigma(Yx - Xy) = N \text{........, ...} \quad (x, y).$$

Let R be the resultant of the forces which act at O; a, b, c the angles its direction makes with the axes; then, by Art. 24,

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2,$$

$$\cos a = \frac{\Sigma X}{R}, \quad \cos b = \frac{\Sigma Y}{R}, \quad \cos c = \frac{\Sigma Z}{R}.$$

Let G be the moment of the couple which is the resultant of the three couples L, M, N; λ, μ, ν the angles its axis makes with the co-ordinate axes; then, by Art. 50,

$$G^2 = L^2 + M^2 + N^2,$$

$$\cos \lambda = \frac{L}{G}, \quad \cos \mu = \frac{M}{G}, \quad \cos \nu = \frac{N}{G}.$$

The convention adopted in the present article for distinguishing the signs of couples agrees with that in Art. 41 when the axes of x, y, and z are drawn as in the present figure, but the conventions will not necessarily coincide if the figure be modified; for example, if the axes of y and z be retained as in the figure, but the positive part of the axis of x directed to the left instead of the right, they will not coincide. The convention of the *present* article is that which we shall hereafter always retain.

73. *To find the conditions of equilibrium of any number of forces acting upon a rigid body in any directions.*

A system of forces acting upon a rigid body can always be reduced to a single force and a couple. Since these cannot balance each other and cannot separately maintain equilibrium they must both vanish. Hence $R=0$, and $G=0$;

therefore $$(\Sigma X)^2+(\Sigma Y)^2+(\Sigma Z)^2=0,$$

and $$L^2+M^2+N^2=0.$$

These lead to the six conditions

$$\Sigma X=0,\quad \Sigma Y=0,\quad \Sigma Z=0,$$

$$\Sigma(Zy-Yz)=0,\quad \Sigma(Xz-Zx)=0,\quad \Sigma(Yx-Xy)=0.$$

74. A verbal enunciation may be given of the last three equations by means of a new definition. For the sake of convenience, we repeat two definitions already given in Arts. 54 and 67.

Moment of a force with respect to a *point*. The moment of a force with respect to a point is the product of the force into the perpendicular from the point on the direction of the force.

Moment of a force with respect to a *plane*. The moment of a force with respect to a plane is the product of the force into the distance of its point of application from the plane.

Moment of a force with respect to a *line*. Resolve the force into two components respectively parallel and perpendicular to the line; the product of the component perpendicular to the line into the shortest distance between the line and the direction of this component is called the *moment* of the force with respect to the line.

Hence the moment of a force with respect to a line is equal to the moment of the component of the force perpendicular to the line with respect to the *point* in which a plane drawn through this component perpendicular to the line meets the line. Hence, by Art. 62, the moment of the force may be found by taking the sum of the moments of any two forces into which the perpendicular component may be resolved.

If the force is *parallel* to the given line, its moment about the line is zero. If the force is *perpendicular* to the given line, its moment about the line is the product of the force into the shortest distance between it and the given line.

75. Suppose we require the moment of the force P_1 about the axis of z; we resolve P_1 into the forces Z_1 parallel to the axis of z and Q_1 perpendicular to the axis of z, where Q_1 is itself the resultant of X_1 and Y_1. The moment of Q_1 with respect to the axis of z is equal to the algebraical sum of the moments of its components X_1 and Y_1; that is, to $Y_1x_1 - X_1y_1$. Hence N in Art. 72 denotes the *sum of the moments of the forces round the axis of z*, and similar meanings arise for L and M.

Hence, for the equilibrium of the forces acting on a rigid body, *the sums of the resolved parts of the forces parallel to any three lines at right angles to each other must vanish, and the sums of the moments of the forces with respect to these lines must also vanish.*

76. In order to interpret the meaning of G we observe that if we keep to the same origin, the moment of this couple and the direction of its axis must be independent of the directions of the co-ordinate axes. For R, being the resultant of all the given forces, supposing them applied at a point, is of course independent of the directions of the axes. If by a new choice of axes we obtain G' as the resultant couple, then R and G must be equivalent to R and G', and therefore R, G, $-R$, $-G'$ must form a system in equilibrium. But this is impossible unless $G = G'$ and the axes of G and G' are coincident or parallel.

Since the direction of the co-ordinate axes is arbitrary, suppose the axis of x to coincide with the axis of G; then $M = 0$, $N = 0$, and L and G are identical.

Hence *G is equal to the sum of the moments of the given forces with respect to the line which is the axis of G.*

77. Suppose a force P acting at the point (x, y, z), and let X, Y, Z be its components parallel to the axes. Then, by Art. 72, P at the point (x, y, z) is equivalent to P at the origin, together with the couples $Zy - Yz$, $Xz - Zx$, $Yx - Xy$ round the axes of x, y, z respectively. Let H be the resultant couple, r the distance of the point (x, y, z) from the origin, and α the angle between r and P; then

$$H^2 = (Zy - Yz)^2 + (Xz - Zx)^2 + (Yx - Xy)^2$$
$$= (x^2 + y^2 + z^2)(X^2 + Y^2 + Z^2) - (xX + yY + zZ)^2$$
$$= r^2P^2 - r^2P^2\left(\frac{x}{r}\frac{X}{P} + \frac{y}{r}\frac{Y}{P} + \frac{z}{r}\frac{Z}{P}\right)^2$$
$$= r^2P^2(1 - \cos^2\alpha),$$
$$\therefore\ H = rP\sin\alpha.$$

Thus, as we might have anticipated, H is the moment of the couple formed by P at the point (x, y, z), and a force at the origin equal to P and acting in a parallel and opposite direction. Hence G is the couple formed by compounding the couples similar to H arising from all the forces of the system.

78. As an example of Art. 73 we may take the case in which all the forces are *parallel.* Let α, β, γ be the angles which the direction of the forces P_1, P_2,...... makes with the axes. Then the equations of equilibrium reduce to

$$\Sigma P = 0,$$
$$\Sigma P(y\cos\gamma - z\cos\beta) = 0,$$
$$\Sigma P(z\cos\alpha - x\cos\gamma) = 0,$$
$$\Sigma P(x\cos\beta - y\cos\alpha) = 0.$$

Hence we can deduce the conditions that a system of *parallel* forces may maintain a body in equilibrium, *however they may be turned about their points of application.* For the preceding equations must then hold whatever α, β, γ may be. Thus we must have

$$\Sigma P = 0, \quad \Sigma Px = 0, \quad \Sigma Py = 0, \quad \Sigma Pz = 0.$$

79. In Art. 72 we have reduced the forces acting on a body to a force R and a couple G. If G vanish there remains a single force; and if R vanish, a single couple. If neither R nor G vanish the forces *may* reduce to a single force; we proceed to shew when this is possible.

To find the condition among the forces that they may have a single resultant.

Any system of forces can be reduced to a single force R and a couple G; if then the forces can be reduced to a single resultant S, it follows that G, R, and $-S$ are in equilibrium. If R and $-S$ do not form a couple, they can be reduced to a couple G' and a force R'; therefore R' must balance the couple compounded of G and G'. This is impossible by Art. 40. Hence R and $-S$ must form a couple, and this couple must have its plane coincident with that of G, or parallel to that of G, in order that it may balance G. Therefore that the forces may have a single resultant, the direction of R must be parallel to the plane of G, or coincident with it; that is, must be at *right angles to the axis* of G. Hence, using the notation of Art. 72,

$$\cos a \cos \lambda + \cos b \cos \mu + \cos c \cos \nu = 0,$$

therefore $$L\Sigma X + M\Sigma Y + N\Sigma Z = 0.$$

80. Conversely, if $L\Sigma X + M\Sigma Y + N\Sigma Z = 0$, and $\Sigma X, \Sigma Y, \Sigma Z$ *do not all vanish*, the forces can be reduced to a single force. For the plane of the couple G may be made to contain the force R, and the couple may be supposed to have each of its forces $= R$ and its arm consequently $= \frac{G}{R}$; the couple may then be turned round in its own plane until the force at one end of its arm balances the resultant force R, and there remains R at the other end of the arm.

81. *When the forces are reducible to a single resultant, to find the equations to the line in which it acts.*

Let L, M, N denote the moments of the forces round the co-ordinate axes; L', M', N' the moments of the forces round axes parallel to the co-ordinate axes drawn through the point (x', y', z'). Then L' is found by writing $y_1 - y'$ for y_1, $y_2 - y'$ for $y_2, \ldots\ldots z_1 - z'$ for z_1, $z_2 - z'$ for $z_2, \ldots\ldots$ in the expression $\Sigma (Zy - Yz)$. Therefore

$$L' = \Sigma \{Z(y-y') - Y(z-z')\} = L - y'\Sigma Z + z'\Sigma Y.$$

Similarly

$$M' = \Sigma \{X(z-z') - Z(x-x')\} = M - z'\Sigma X + x'\Sigma Z,$$

$$N' = \Sigma \{Y(x-x') - X(y-y')\} = N - x'\Sigma Y + y'\Sigma X.$$

If x', y', z' can be so taken as to make L', M', and N' vanish, the forces reduce to a single resultant passing through the point (x', y', z'). The three equations

$$L - y'\Sigma Z + z'\Sigma Y = 0 \quad\text{.....................(1)},$$

$$M - z'\Sigma X + x'\Sigma Z = 0 \quad\text{.....................(2)},$$

$$N - x'\Sigma Y + y'\Sigma X = 0 \quad\text{.....................(3)},$$

are equivalent to two independent equations; for if we eliminate z' from (1) and (2), we have

$$L\Sigma X + M\Sigma Y + \Sigma Z(x'\Sigma Y - y'\Sigma X) = 0.$$

But $\qquad L\Sigma X + M\Sigma Y + N\Sigma Z = 0$, by Art 79,

therefore $\qquad N - x'\Sigma Y + y'\Sigma X = 0.$

Thus (3) is a necessary consequence of (1) and (2). Hence (1) and (2) will determine a line at every point of which the resultant couple vanishes; that is, the line in which the single resultant force acts.

82. By the following method we may determine at once the condition for the existence of a single resultant and the equations to its direction.

Suppose that the forces can be reduced to a single force acting at the point (x', y', z'). Let the single force be resolved into components X', Y', Z' parallel to the co-ordinate axes; then if we add to the given system $-X'$, $-Y'$, and $-Z'$, acting at the point (x', y', z') parallel to the axes respectively, there will be equilibrium. Hence, by Art. 73,

$$\Sigma X - X' = 0, \quad \Sigma Y - Y' = 0, \quad \Sigma Z - Z' = 0 \quad\text{..........(1)},$$

$$L - Z'y' + Y'z' = 0,\ M - X'z' + Z'x' = 0,\ N - Y'x' + X'y' = 0 \quad\text{...(2)}.$$

Equations (1) determine X', Y', Z'. It might at first appear that equations (2) would determine x', y', z'; but if we proceed to solve them, we find that they *cannot be simultaneously true unless*

$$L\Sigma X + M\Sigma Y + N\Sigma Z = 0;$$

and if this condition be satisfied, and ΣX, ΣY, ΣZ do not all vanish, then any one of the equations may be derived from the other two, so that there are only *two independent* equations.

Hence that the forces may have a single resultant the above condition must be satisfied, and then any two of equations (2) will determine the locus of points at which this single resultant may be supposed to act. From the form of equations (2) it is obvious that this locus is a straight line, and that its direction cosines are proportional to X', Y', Z', as might have been anticipated.

In order that the force which replaces the system may pass through the origin, we must have

$$L=0, \quad M=0, \quad N=0.$$

83. Although a system of forces cannot always be reduced to a single resultant, it can always be reduced to *two* forces. For we have shewn that the system may be replaced by a force R at the origin, and a couple G lying in a plane through the origin; one of the forces of G may be supposed to act at the origin, and may be compounded with R so that this resultant and the other force of G are equivalent to the whole system. Since the origin is arbitrary, we see that when a system of forces is not reducible to a single force it can be reduced to two forces, one of which can be made to pass through any assigned point.

84. *When three forces maintain a body in equilibrium, they must lie in the same plane.*

Draw any line intersecting the directions of two of the forces and not parallel to the third force, and take this line for the axis of x. Then the first two forces have no moment round the axis of x; therefore the equation $L=0$ requires that the third force should have no moment round the axis of x; that is, the direction of the third force must pass through the axis of x. Since then *any* line, which meets the directions of two of the forces, and is not parallel to the direction of the third, meets that direction, the three forces must lie in one plane.

Combining this proposition with that in Art. 58, we see that if three forces keep a body in equilibrium, they must all lie in the same plane and must meet in a point or be parallel.

85. If the axes of co-ordinates be oblique, suppose l, m, n to denote the sines of the angles between the axes of y and

z, z and x, x and y, respectively; then we may shew, as in Art. 72, that any system of forces can be reduced to ΣX, ΣY, ΣZ, acting at the origin along the axes of x, y, z respectively, and three couples in the three co-ordinate planes, having their moments equal to lL, mM, nN respectively, where, as before, $L = \Sigma (Zy - Yz)$, &c. Also for equilibrium, we must have, as before,

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0;$$

$$L = 0, \quad M = 0, \quad N = 0.$$

That the forces may admit of a single resultant we must have, as before,

$$L\Sigma X + M\Sigma Y + N\Sigma Z = 0,$$

and ΣX, ΣY, ΣZ not all vanishing.

EXAMPLES.

1. Four parallel forces act at the angles of a plane quadrilateral and are inversely proportional to the segments of its diagonals nearest to them; shew that the point of application of their resultant lies at the intersection of the diagonals.

2. Parallel forces act at the angles A, B, C of a triangle and are respectively proportional to a, b, c; shew that their resultant acts at the centre of the inscribed circle.

3. A cone whose vertical angle is 30°, and whose weight is W is placed with its vertex on a smooth horizontal plane; shew that it may be kept with its slant side in a vertical position by a couple whose arm is equal to the length of the slant side of the cone, and each force $\frac{3W}{16}$.

4. Six equal forces act along the edges of a cube which do not meet a given diagonal, taken in order; find their resultant.

Result. A couple, the moment of which is $2Pa\sqrt{3}$, where P denotes each force and a the edge of the cube.

5. A cube is acted on by four forces; one force is in a diagonal, and the others in edges no two of which are in the

same plane and which do not meet the diagonal; find the condition that the forces may have a single resultant.

Result. $(XY+YZ+ZX)\sqrt{3}+P(X+Y+Z)=0$; where X, Y, Z denote the forces along the edges, and P the force along the diagonal.

6. If a triangle is suspended from a fixed point by strings attached to the angles, the tension of each string is proportional to its length.

7. A uniform heavy triangle is supported in a horizontal position by three parallel strings attached to the three sides respectively; shew that there is an infinite number of ways in which the strings may be relatively disposed so that their tensions may be equal, but that the situation of one being given, that of each of the other two is determinate.

8. A sphere of given weight rests upon three planes whose equations are $lx+my+nz=0$, $l_1x+m_1y+n_1z=0$, $l_2x+m_2y+n_2z=0$, the axis of z being vertical; find the pressure upon each plane.

9. A heavy triangle ABC is suspended from a point by three strings, mutually at right angles, attached to the angular points of the triangle; if θ be the inclination of the triangle to the horizon in its position of equilibrium, then

$$\cos\theta=\frac{3}{\sqrt{(1+\sec A\sec B\sec C)}}.$$

10. An equilateral triangle without weight has three unequal particles placed at its angular points; the system is suspended from a fixed point by three equal strings at right angles to each other fastened to the corners of the triangle; find the inclination of the plane of the triangle to the horizon.

Result. The cosine of the angle is

$$\frac{W_1+W_2+W_3}{\sqrt{\{3(W_1^2+W_2^2+W_3^2)\}}}$$

where W_1, W_2, W_3 represent the weights of the particles.

11. Four smooth equal spheres are placed in a hemispherical bowl. The centres of three of them are in the same

horizontal plane, and that of the other is above it. If the radius of each sphere be one-third that of the bowl, shew that the mutual pressures of the spheres are all equal; and find the pressure of each of the lower spheres on the bowl.

Results. Let W be the weight of each of the spheres; then each of the mutual pressures between the spheres is $\frac{W}{\sqrt{6}}$; the pressure of each of the lower spheres on the bowl is $\frac{4W}{\sqrt{6}}$.

12. Three equal spheres hang in contact from a fixed point by three equal strings; find the heaviest sphere of given radius that may be placed upon them without causing them to separate.

Result. Let W be the weight of each of the equal spheres, θ the angle which each string makes with the vertical, ϕ the angle which the line joining the centre of one of the three equal spheres with the centre of the upper sphere makes with the vertical; then the weight of the upper sphere must not exceed $\frac{3W\tan\theta}{\tan\phi - \tan\theta}$.

13. Four forces act on a tetrahedron perpendicular to the faces and proportional to their areas, the points of application of the forces being the centres of the circles circumscribing the faces; the forces act all inwards or all outwards; shew that the tetrahedron will be in equilibrium.

14. Extend the proposition in the preceding example to the case of any polyhedron bounded by triangular faces.

CHAPTER VI.

EQUILIBRIUM OF A CONSTRAINED BODY.

86. *To find the conditions of equilibrium of forces acting upon a rigid body when one point is fixed.*

Let the fixed point be taken as the origin of co-ordinates. The action of the forces on the body will produce a pressure on the fixed point; let X', Y', Z' be the resolved parts of this pressure parallel to the axes. Then the fixed point will exert forces $-X'$, $-Y'$, $-Z'$, against the body; and if we take these forces in connexion with the given forces, we may suppose the body to be free, and the equations of equilibrium are

$$\Sigma X - X' = 0, \quad \Sigma Y - Y' = 0, \quad \Sigma Z - Z' = 0,$$

$$L = 0, \qquad M = 0, \qquad N = 0.$$

The first three equations give the resolved parts of the pressure on the fixed point; and the last three are the only conditions to be satisfied by the given forces.

From the equations $X' = \Sigma X$, $Y' = \Sigma Y$, $Z' = \Sigma Z$, it follows that the pressure on the fixed point is equal to the resultant of all the given forces of the system moved parallel to themselves up to the fixed point.

If all the forces are parallel, we may take the axis of z passing through the fixed point parallel to the forces. Then all the forces included in ΣX vanish, and so do all the forces included in ΣY; thus N vanishes, M reduces to $-\Sigma Zx$, and L reduces to ΣZy. Hence X' and Y' vanish and the equations of equilibrium reduce to

$$\Sigma Z - Z' = 0, \quad \Sigma Zy = 0, \quad \Sigma Zx = 0;$$

the first determines the pressure on the fixed point, and the other two are conditions which must be satisfied by the given forces.

If all the forces act in one plane passing through the fixed point, and we take this plane for that of (x, y), all the forces included in ΣZ vanish; also the ordinate parallel to the axis of z of the point of application of each force is zero. Thus L and M vanish; also Z' vanishes, and the equations of equilibrium reduce to

$$\Sigma X - X' = 0, \quad \Sigma Y - Y' = 0, \quad \Sigma (Yx - Xy) = 0;$$

the first two determine the pressure on the fixed point, and the third is the only condition which the forces must satisfy.

87. *To find the conditions of equilibrium of a body which has two points in it fixed.*

Let the axis of z pass through the two fixed points; and let the distances of the points from the origin be z' and z''. Also let X', Y', Z' be the resolved parts of the pressures on one point, and X'', Y'', Z'' those on the other point.

Then, as in Art. 86, the equations of equilibrium will be

$$\Sigma X - X' - X'' = 0, \ \Sigma Y - Y' - Y'' = 0, \ \Sigma Z - Z' - Z'' = 0,$$

$$L + Y'z' + Y''z'' = 0, \quad M - X'z' - X''z'' = 0,$$

$$N = 0.$$

The first, second, fourth, and fifth of these equations will determine X', X'', Y', Y''; the third equation gives $Z' + Z''$, shewing that the pressures on the fixed points in the direction of the line joining them are indeterminate, being connected by one equation only. The last is the only condition of equilibrium, namely $N = 0$.

88. The indeterminateness which occurs as to the values of Z' and Z'' might have been expected; for if two forces, $-Z'$ and $-Z''$, act upon a rigid body *in the same straight line*, their effect will be the same at whatever point in their line of action we suppose them applied, and consequently

they may be supposed both to act at the same point, or one of them to be increased provided the other be equally diminished. If it be objected that in any experimental case there really would be some *definite* pressure at each fixed point, we must reply, that no body on which we can experiment fulfils the condition of *perfect rigidity*, on which our conclusions depend. See *Poisson*, Art. 270; *Poinsot*, Arts. 128—132.

The case which we have been considering is that of a body which is capable of turning round a *fixed axis;* for an axis will be fixed if two of its points are fixed.

89. If the body, instead of having two fixed points, can turn round an axis and *also slide along it*, then in addition to the condition $N=0$, we must have $\Sigma Z=0$, supposing the axis of z directed along the line on which the body can turn and slide. For the axis will not be able, as in the last case, to furnish any forces $-Z'$ and $-Z''$ to counteract ΣZ, and therefore ΣZ must $=0$.

90. *To find the conditions of equilibrium of a rigid body resting on a smooth plane.*

Let this plane be the plane of (x, y); and let x', y' be the co-ordinates of one of the points of contact, R' the pressure which the body exerts against the plane at that point. Then the force $-R'$, and similar forces for the other points of contact, taken in connexion with the given forces, ought to satisfy the equations of equilibrium; hence

$$\Sigma X=0, \quad \Sigma Y=0, \quad \Sigma Z-R'-R''-\ldots=0,$$

$$L-R'y'-R''y''-\ldots=0, \quad M+R'x'+R''x''+\ldots=0, \quad N=0.$$

If only one point be in contact with the plane, then the third equation gives the pressure, and we have five equations of condition,

$$\Sigma X=0, \quad \Sigma Y=0, \quad L-y'\Sigma Z=0, \quad M+x'\Sigma Z=0, \quad N=0.$$

If two points be in contact, then the equations

$$R'y'+R''y''=L, \quad R'x'+R''x''=-M,$$

give $$R' = \frac{Lx'' + My''}{y'x'' - x'y''}, \quad R'' = -\frac{Lx' + My'}{y'x'' - x'y''};$$

and the equations of condition are

$$\Sigma X = 0, \ \Sigma Y = 0, \ \Sigma Z - \frac{L(x'' - x') + M(y'' - y')}{y'x'' - x'y''} = 0, \text{ and } N = 0.$$

If three points are in contact, then the pressures are determined from the equations

$$R' + R'' + R''' = \Sigma Z,$$

$$R'y' + R''y'' + R'''y''' = L,$$

$$R'x' + R''x'' + R'''x''' = -M,$$

and the conditions of equilibrium are

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad N = 0.$$

If more than three points are in contact, then the pressures are indeterminate, since they are connected by only three equations; but the conditions of equilibrium are still

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad N = 0.$$

91. The equations at the commencement of the preceding article shew that if a body rests in equilibrium against a plane, the forces which press it against the plane must reduce to a *single force* acting perpendicular to the plane, for the condition

$$L\Sigma X + M\Sigma Y + N\Sigma Z = 0$$

is satisfied, since ΣX, ΣY, and N vanish. Hence the forces reduce to a single force; and since ΣX and ΣY vanish, this force must be perpendicular to the fixed plane.

Also, this single force must counterbalance the forces $-R'$, $-R''$ ..., which are all parallel and all act in the same direction. Hence, from considering the construction given in Art. 66 for determining the centre of a system of parallel forces, it follows that the point where this resultant cuts the plane must be within a polygon, formed by so joining the points of contact as to include them all and to have no *re-entering angle.*

MISCELLANEOUS EXAMPLES.

1. The lid $ABCD$ of a cubical box, moveable about hinges at A and B, is held at a given angle to the horizon by a horizontal string connecting C with a point vertically over A: find the pressure on each hinge.

2. Two equal forces act on a cube whose centre is fixed, along diagonals which do not meet of two adjacent faces: find the couple which will keep the cube at rest.

Result. Let P denote each force, a the edge of the cube; the moment of the required couple is either $\frac{Pa\sqrt{3}}{2}$ or $\frac{Pa}{2}$ according to the directions of the two given forces.

3. Three equal heavy rods in the position of the three edges of an inverted triangular pyramid are in equilibrium under the following circumstances. Their upper extremities are connected by strings of equal lengths, and their lower extremities are attached to a hinge about which the rods may move freely in all directions. Find the tension of the strings.

4. A given number of uniform heavy rods, all of the same weight, have their extremities jointed together at a common hinge, about which they can turn freely; and being introduced through a circular hole in a horizontal plane with their hinge end downwards, are spread out symmetrically along the circumference of the hole like the ribs of a conical basket. If a heavy sphere be now placed in the interior of the system of rods, so as to be supported by them, determine the position of rest.

5. A cylinder rests with its base on a smooth inclined plane; a string attached to its highest point, passing over a pully at the top of the inclined plane, hangs vertically and supports a weight; the portion of the string between the cylinder and the pully is horizontal. Determine the conditions of equilibrium.

Results. Let W be the weight of the cylinder, W' the weight attached to the string, α the inclination of the plane to the horizon; then $W' = W \tan \alpha$, and $\tan \alpha$ must not exceed the ratio of the diameter of the base of the cylinder to the height of the cylinder.

6. A cone of given weight W is placed with its base on an inclined plane, and supported by a weight W' which hangs by a string fastened to the vertex of the cone and passing over a pully in the inclined plane at the same height as the vertex. Determine the conditions of equilibrium.

Results. Let α be the inclination of the plane to the horizon, θ the semi-vertical angle of the cone; then

$W' = W \tan \alpha$, and $\tan \theta$ must not be less than $\frac{3}{8} \sin 2\alpha$.

7. A cylinder with its base resting against a smooth vertical plane is held up by a string fastened to it at a point of its curved surface whose distance from the vertical plane is h. Shew that h must be greater than $b - 2a \tan \theta$ and less than b, where $2b$ is the altitude of the cylinder, a the radius of the base, and θ the angle which the string makes with the vertical.

8. A smooth hemispherical shell whose base is closed includes two equal spheres whose radii are one third of that of the shell. The shell is fixed with its base vertical; find the mutual pressures at all the points of contact.

Results. Let R_1 be the pressure between the upper sphere and the shell, R_2 that between the two spheres, R_3 that between the lower sphere and the base of the shell, R_4 that between the lower sphere and the curved part of the shell; then

$$R_1 = \frac{W}{\sqrt{3}}, \quad R_2 = \frac{2W}{\sqrt{3}}, \quad R_3 = \frac{3W}{\sqrt{3}}, \quad R_4 = \frac{4W}{\sqrt{3}}.$$

9. A rectangular table is supported in a horizontal position by four legs at its four angles: a given weight W being placed upon a given point of it, shew that the pressure on each leg is indeterminate, and find the greatest and least value it can have for a given position of the weight.

CHAPTER VII.

GENERAL THEOREMS ON A SYSTEM OF FORCES.

92. In Art. 72 it is proved that the forces acting on a rigid body may be reduced to a force R and a couple G, and that $G^2 = L^2 + M^2 + N^2$, where L, M, N are the moments of the forces round three rectangular axes arbitrarily chosen. It is obvious that neither L, M, nor N can be greater than G; hence, for a given origin, *the resultant moment G is greater than the moment of the forces about any other axis.* For this reason G is called the *principal moment* of the forces.

From the equations in Art. 72, which determine the direction of the axis of G, it follows that $G \cos \phi$ is the moment of the forces about an axis which passes through the given origin, and makes an angle ϕ with the axis of principal moment.

93. The value of R in Art. 72 is independent of the position of the origin of co-ordinates; R is in fact the resultant of the given forces, supposing each of them moved parallel to itself until they are all brought to act at the same point. The value of G, however, depends on the origin we assume. If we take a point whose co-ordinates are x', y', z', and denote by L', M', N' the moments of the forces round lines through this point parallel to the co-ordinate axes, and by G' the *principal moment* of the forces with respect to this point, we have, by Art. 81,

$$\left.\begin{aligned} L' &= L - y'\Sigma Z + z'\Sigma Y, \\ M' &= M - z'\Sigma X + x'\Sigma Z, \\ N' &= N - x'\Sigma Y + y'\Sigma X, \\ G'^2 &= L'^2 + M'^2 + N'^2. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

We proceed to apply these equations to find the least value of G'.

To find the locus of the origins which give the least principal moments, the magnitude of those moments, and the position of their axes.

Multiply the first of equations (1) by ΣX, the second by ΣY, and the third by ΣZ, and add; thus

$$L'\Sigma X + M'\Sigma Y + N'\Sigma Z = L\Sigma X + M\Sigma Y + N\Sigma Z \ldots (2).$$

Also

$$R^2 G'^2 = \{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2\}\{L'^2 + M'^2 + N'^2\}$$
$$= (N'\Sigma Y - M'\Sigma Z)^2 + (L'\Sigma Z - N'\Sigma X)^2$$
$$+ (M'\Sigma X - L'\Sigma Y)^2 + (L'\Sigma X + M'\Sigma Y + N'\Sigma Z)^2 \ldots (3).$$

Of these four terms the last is *constant* for all values of x', y', z' by (2); hence we obtain the *least* value of G' by making the three preceding terms vanish, which gives

$$\frac{L'}{\Sigma X} = \frac{M'}{\Sigma Y} = \frac{N'}{\Sigma Z} \ldots\ldots\ldots\ldots\ldots\ldots (4);$$

that is,

$$\frac{L - y'\Sigma Z + z'\Sigma Y}{\Sigma X} = \frac{M - z'\Sigma X + x'\Sigma Z}{\Sigma Y} = \frac{N - x'\Sigma Y + y'\Sigma X}{\Sigma Z} \ldots (5).$$

Hence the required locus is a *straight line.*

From (4) it appears that L', M', N' are proportional to ΣX, ΣY, ΣZ respectively, which shews that the axis of the principal moment at any point on the line (5) is parallel to the direction of the resultant R. By (3) the value of the least principal moment is

$$\frac{L\Sigma X + M\Sigma Y + N\Sigma Z}{R}.$$

Each of the fractions in (5) is, by a known theorem, equal to

$$\frac{L\Sigma X + M\Sigma Y + N\Sigma Z}{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2};$$

that is, to $$\frac{L\Sigma X+M\Sigma Y+N\Sigma Z}{R^2}.$$

The equations (5) may by suitable transformations be reduced to the ordinary symmetrical equations to a straight line. We have

$$\frac{L-y'\Sigma Z+z'\Sigma Y}{\Sigma X}=\frac{L\Sigma X+M\Sigma Y+N\Sigma Z}{R^2};$$

therefore

$$L\{(\Sigma Y)^2+(\Sigma Z)^2\}+(z'\Sigma Y-y'\Sigma Z)\,R^2=(M\Sigma Y+N\Sigma Z)\,\Sigma X;$$

therefore

$$(z'R^2-M\Sigma X+L\Sigma Y)\,\Sigma Y=(y'R^2-L\Sigma Z+N\Sigma X)\Sigma Z;$$

therefore

$$\frac{1}{\Sigma Y}\left(y'-\frac{L\Sigma Z-N\Sigma X}{R^2}\right)=\frac{1}{\Sigma Z}\left(z'-\frac{M\Sigma X-L\Sigma Y}{R^2}\right).$$

Hence we conclude that the equations (5) may be written

$$\frac{1}{\Sigma X}\left(x'-\frac{N\Sigma Y-M\Sigma Z}{R^2}\right)=\frac{1}{\Sigma Y}\left(y'-\frac{L\Sigma Z-N\Sigma X}{R^2}\right)$$
$$=\frac{1}{\Sigma Z}\left(z'-\frac{M\Sigma X-L\Sigma Y}{R^2}\right):$$

from which we see that the straight line determined by (5) is parallel to the direction of R. Hence this straight line has the following properties: *at every point of it the value of the principal moment is the same, and is less than it is for any point not in the line; also for every point in the line the position of the axis of principal moment is the same, being the line itself.* This line is called the *central axis.*

94. The equation (2) of Art. 93 may be written

$$L'\frac{\Sigma X}{R}+M'\frac{\Sigma Y}{R}+N'\frac{\Sigma Z}{R}=L\frac{\Sigma X}{R}+M\frac{\Sigma Y}{R}+N\frac{\Sigma Z}{R}.$$

This shews that if we resolve L', M', N' along a line parallel to the direction of R, and add the resolved parts, we obtain the same result *whatever origin be chosen.* Thus the *resolved part of any principal moment in the direction of R is constant.* By the resolved part of the principal moment in the direction of R we mean that part of the moment which has *its axis* in the direction of R.

95. From equations (1) of Art. 93 it appears that $L' = L$, $M' = M$, and $N' = N$, provided

$$\frac{x'}{\Sigma X} = \frac{y'}{\Sigma Y} = \frac{z'}{\Sigma Z};$$

that is, if the point (x', y', z') be on a line through the origin parallel to the direction of R. Since the origin is arbitrary, we may therefore assert that the *principal moment remains unchanged, when the point to which it relates moves along any straight line parallel to the direction of R.*

96. The equation to the plane through the origin perpendicular to the direction of R is

$$x'\Sigma X + y'\Sigma Y + z'\Sigma Z = 0 \quad\text{..................(1)}.$$

If we combine this equation with equations (5) of Art. 93, we obtain the co-ordinates of the point of intersection of this plane with the *central axis.*

We thus find for these co-ordinates

$$\frac{N\Sigma Y - M\Sigma Z}{R^2}, \quad \frac{L\Sigma Z - N\Sigma X}{R^2}, \quad \frac{M\Sigma X - L\Sigma Y}{R^2},$$

which we will denote by h, k, l respectively.

If x', y', z' satisfy (1), then $N'\Sigma Y - M'\Sigma Z$

or $$(N - x'\Sigma Y + y'\Sigma X)\,\Sigma Y - (M - z'\Sigma X + x'\Sigma Z)\Sigma Z$$

$$= N\Sigma Y - M\Sigma Z - x'R^2 = R^2\,(h - x').$$

Similarly $$L'\Sigma Z - N'\Sigma X = R^2\,(k - y'),$$

$$M'\Sigma X - L'\Sigma Y = R^2\,(l - z').$$

Therefore from equation (3) of Art. 93

$$G'^2 = R^2\{(h-x')^2 + (k-y')^2 + (l-z')^2\} + \left(\frac{L\Sigma X + M\Sigma Y + N\Sigma Z}{R}\right)^2 \quad \text{.........(2).}$$

Hence G' remains constant for all points in the plane (1) for which $(h-x')^2 + (k-y')^2 + (l-z')^2$ is constant; that is, for all points in (1) which are at a constant distance from the *central axis*. From this and Art. 93 it follows, that if a right cylinder be described round the *central axis*, *the principal moment has the same value for any point on the surface of this cylinder.*

97. Of the two expressions which compose G' in equation (2) of Art. 96, the latter, by Art. 94, is the resolved part of G' *parallel* to the direction of R; hence the former part is the resolved part of G' *perpendicular* to the direction of R. Call the former part Q, and ϕ the angle which the direction of the axis of G' makes with that of R; then $\sin\phi = \frac{Q}{G'}$, and this is constant so long as G' is, that is, *for every point on the surface of the cylinder in the preceding article.*

98. The propositions already given in this chapter admit of other modes of proof, which we proceed to indicate.

To shew that any system of forces can always be reduced to a force and a couple, the axis of the latter being parallel to the direction of the former.

The forces can be always reduced to a force R and a couple G, and the angle ϕ between the former and the axis of the latter is given by the equation

$$\cos\phi = \frac{L\Sigma X + M\Sigma Y + N\Sigma Z}{G \cdot R}.$$

Resolve the couple G into two others; one having its axis *parallel* to the direction of R and its moment equal to $G\cos\phi$, the other having its axis *perpendicular* to the direction of R and its moment equal to $G\sin\phi$. The forces of the latter couple are therefore in a plane parallel to R; and by properly placing this couple in its own plane, and making each of its forces equal to R, one of its forces may be made to balance the force R. We shall then have remaining the couple $G\cos\phi$ and a force R, the direction of which is parallel to the axis of the couple, and which is moved to a distance $\frac{G\sin\phi}{R}$ from its original position. The system is thus reduced to a force R and a couple $\frac{L\Sigma X+M\Sigma Y+N\Sigma Z}{R}$, the axis of the latter being parallel to R, and therefore its plane perpendicular to R.

Since the resultant couple must be independent of the direction of the axes of co-ordinates we conclude that

$$\frac{L\Sigma X+M\Sigma Y+N\Sigma Z}{R}$$

must be constant whatever be the direction of the axes; and as R is constant it follows that $L\Sigma X+M\Sigma Y+N\Sigma Z$ must be constant whatever be the direction of the axes. The expression also remains the same whatever *origin* be chosen, as appears from equation (2) of Art. 93.

99. PROP. *When a system of forces is reduced to a force and a couple in a plane perpendicular to the force, the position and magnitude of the force are always the same.*

The *magnitude* of the force is always the same, for it is the resultant of the given forces supposing each of them moved parallel to itself until they are all brought to act at the same point. We shall now shew that there is a definite straight line along which the resultant force must act.

Let x', y', z' be the co-ordinates of an origin such that the axis of the resultant couple coincides with the direction of the resultant force. Then with the notation of Art. 93 we have

$$\frac{L'}{\Sigma X}=\frac{M'}{\Sigma Y}=\frac{N'}{\Sigma Z},$$

for the direction cosines of the axis of the couple are proportional to L', M', and N', and those of the direction of the force are proportional to ΣX, ΣY, ΣZ. Hence the locus of the origins is the straight line determined by equations (5) of Art. 93.

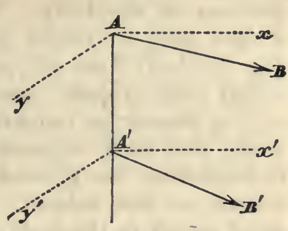
100. It appears from the last article that there is only *one* position of the resultant force in which it is perpendicular to the plane of the resultant couple. If we wish to transfer the resultant force to any other point, we can do it by introducing two forces, R and $-R$, at that point; the latter with the original force R will form a couple; and if this couple be compounded with the original couple we have a new couple, the moment of which is $\sqrt{(K^2 + R^2p^2)}$, K being the original moment and p the distance to which R has been moved. This moment is greater than K; and hence the line in which R acts when perpendicular to the plane of the resultant couple is the *axis of least principal moment.* It is therefore the *central axis.*

K is shewn in Art. 98 to be $= \dfrac{L\Sigma X + M\Sigma Y + N\Sigma Z}{R}$.

101. The principal moment will be the same for every point of the central axis, since when we have reduced the forces to a single force and a couple in a plane perpendicular to the force, the force may be supposed to act at any point in its line of application, and the plane of the couple may be moved parallel to itself into any new position. See also Art. 95. Hence if we draw any plane perpendicular to the central axis, and describe a circle in the plane with radius p, and having its centre at the intersection of the central axis, then, by the last article, the principal moment for any point in this circle will be $\sqrt{(K^2 + R^2p^2)}$, and the angle ϕ at which the direction of its axis is inclined to the direction of R is given by the equation $\tan\phi = \dfrac{Rp}{K}$.

102. *When a system of forces acting upon a rigid body is reduced to two forces, and these are represented by two straight lines which do not meet and are not parallel, the volume of the tetrahedron of which the two straight lines are opposite edges is constant.*

Let the lines AB and $A'B'$ represent the two forces, AA' being a line perpendicular to both. Suppose two parallel lines Ax, $A'x'$ drawn, each perpendicular to AA', and Ay, $A'y'$, respectively perpendicular to Ax, $A'x'$, and also perpendicular to AA'. Let $BAx = \phi$, $B'A'x' = \phi'$, and let T and T' denote the intensities of the forces in AB and $A'B'$ respectively. Then



T may be resolved into $T\cos\phi$ and $T\sin\phi$ acting at A along Ax and Ay respectively, and T' into $T'\cos\phi'$, $T'\sin\phi'$ acting at A' along $A'x'$ and $A'y'$ respectively. Let α be the inclination of AB and $A'B'$, so that $\phi' = \phi + \alpha$. Now determine ϕ by the equation

$$T\cos\phi + T'\cos\phi' = 0 \quad\text{.................. (1)},$$

that is $$T\cos\phi + T'\cos(\phi+\alpha) = 0.$$

Then by (1) the forces $T\cos\phi$ and $T'\cos\phi'$ *will form a couple* in the plane $xAA'x'$; and $T\sin\phi$ and $T'\sin\phi'$ will have a single resultant perpendicular to the plane of this couple, for they cannot form a couple since then the whole system of forces would reduce to a single couple which is contrary to supposition. Let P denote the intensity of this single force so that

$$P = T\sin\phi + T'\sin\phi' \quad\text{.................. (2)}.$$

The moment of the couple is $AA' \times T\cos\phi$. Hence, by the latter part of Art. 98, $AA' \times P \times T\cos\phi$ is constant whatever be the position and magnitude of the forces T and T', so long as they are equivalent to a given system of forces. Now the volume of the tetrahedron of which AB and $A'B'$ are opposite edges is $\frac{1}{6} AB.A'B'.AA'.\sin\alpha$.

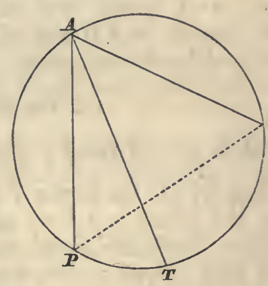
(See Hymers's *Geometry of Three Dimensions*, p. 213.)

But from (1) and (2) we have $T'\sin\alpha = P\cos\phi$. Hence the volume of the tetrahedron becomes $\frac{1}{6} AA'.T.P\cos\phi$, which we have just seen to be constant.

103. When a system of *parallel* forces acting on a rigid body has a single resultant, that resultant always passes

through a fixed point in the body whatever may be the position of the body. When *any* system of forces acts on a rigid body we might investigate the consequences of turning the body from one position into another while the forces retain their original directions, or of turning the forces in such a manner as to leave their *relative* directions unchanged while the body remains fixed. We shall here give some examples of the general theorems that have been demonstrated on this subject. The forces are supposed to act at fixed points in the body.

104. Let PA and QA be the directions of two forces lying in one plane, acting at the points P and Q respectively; TA the direction of their resultant. Suppose the forces in PA, QA to be turned round the points P and Q respectively through the same angle α towards the same direction; since PA and QA will include the same angle as before, their point of intersection will move on a circle passing through P and Q. And



as the magnitudes of the forces are supposed unchanged, the magnitude of the resultant and the angles which it makes with the components remain unchanged. Hence if T be the intersection of the resultant and the circle originally, it will always be so, since the arcs PT and QT are proportional to the angles PAT and QAT; the resultant will therefore have turned through the angle α round the point T.

The same result holds if instead of supposing the body to be fixed and the forces to revolve, we suppose each force to remain parallel to itself and the body to be turned through any angle round a perpendicular to the plane of the forces.

The point T through which the resultant always passes may be called the *centre* of the forces which act at P and Q. It is evident, in like manner, that if a third force pass through a fixed point S and meet the line TA, we may find the *centre* of the forces at T and S, that is, the *centre* of the forces at P, Q, and S; and generally we may infer that *every system of forces in one plane which is reducible to a single resultant has a centre;* or, in other words, if there be a system of forces acting in a plane and having a single resultant, and we know the magnitude of each force, the angles the directions of the forces make with each other, and one point in the direction of each, then we can determine the magnitude of the resultant, the angle its direction makes with those of the component forces, and one point in its direction.

105. If a system of forces maintain a body in equilibrium, and equilibrium also subsist after the body has been turned through *any given* angle which is not a multiple of two right angles, about any axis, then equilibrium will still subsist when the body is turned about the *same* axis through *any angle whatever*, the forces being supposed to act with the same intensity and in parallel directions throughout.

Take the axis of z to coincide with the line about which the body is turned. Since there is equilibrium in its first position, we have

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0 \ldots\ldots\ldots\ldots\ldots(1),$$

$$\Sigma(Zy - Yz) = 0, \ \Sigma(Xz - Zx) = 0, \ \Sigma(Yx - Xy) = 0 \ldots(2).$$

If equilibrium subsist when the body is turned through an angle θ, the equations (1) and (2) must hold when we put $x\cos\theta - y\sin\theta$ for x, and $x\sin\theta + y\cos\theta$ for y. Hence (2) become

$$\sin\theta\,\Sigma(Zx) + \cos\theta\,\Sigma(Zy) - \Sigma(Yz) = 0 \ldots\ldots (3),$$

$$\Sigma(Xz) - \cos\theta\,\Sigma(Zx) + \sin\theta\,\Sigma(Zy) = 0 \ldots\ldots (4),$$

$$\cos\theta\,\Sigma(Yx - Xy) - \sin\theta\,\Sigma(Xx + Yy) = 0 \ldots\ldots (5).$$

By means of (2), equations (3) and (4) become

$$\sin\theta\,\Sigma(Xz) - (1 - \cos\theta)\,\Sigma(Yz) = 0,$$

$$(1 - \cos\theta)\,\Sigma(Xz) + \sin\theta\,\Sigma(Yz) = 0.$$

As these equations hold for some value of $\sin\theta$ different from zero, we must have

$$\Sigma(Xz) = 0, \text{ and } \Sigma(Yz) = 0 \ldots\ldots\ldots\ldots\ldots(6).$$

Then, by (2), we infer

$$\Sigma(Zx) = 0, \text{ and } \Sigma(Zy) = 0 \text{ .... } (7).$$

And from (2) and (5),

$$\Sigma(Yx - Xy) = 0, \text{ and } \Sigma(Xx + Yy) = 0 \text{ .... } (8).$$

And when (6), (7), and (8) are true, (3), (4), and (5) are true for *all* values of θ.

106. It appears from the preceding article that when forces act *in one plane* on a rigid body and maintain equilibrium, the necessary and sufficient additional condition in order that equilibrium may subsist after the body has been turned round an axis perpendicular to the plane while the forces remain parallel to their original directions, is

$$\Sigma(Xx + Yy) = 0.$$

107. We have remarked in Art. 9 that the property of the divisibility of matter leads us to the supposition that every body consists of an assemblage of material particles or molecules which are held together by their mutual attraction. Now we are totally unacquainted with the nature of these molecular forces; if, however, we assume the two hypotheses that the action of any two molecules on each other is the same, and also that its direction is the line joining them, then we shall be able to deduce the conditions of equilibrium of a rigid body from those of a single particle.

To find the conditions of equilibrium of a rigid body from those of a single molecule.

Let the body be referred to three rectangular axes; and let x_1, y_1, z_1 be the co-ordinates of one of its constituent particles; X_1, Y_1, Z_1 the resolved parts, parallel to the axes, of the forces which act upon this particle exclusive of the molecular forces; $P_1, P_2, P_3, \ldots\ldots$ the molecular forces acting on this particle; $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$;...... the angles their respective directions make with the three axes of co-ordinates.

Then, since this particle is held in equilibrium by the above forces, we have, by Art. 27,

$$X_1 + P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \ldots\ldots = 0 \ldots\ldots (1),$$

$$Y_1 + P_1 \cos \beta_1 + P_2 \cos \beta_2 + \ldots\ldots = 0 \ldots\ldots (2),$$

$$Z_1 + P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + \ldots\ldots = 0 \ldots\ldots (3).$$

We shall have a similar system of equations for each particle in the body; if there be n particles there will be $3n$ equations. These $3n$ equations will be connected one with another, since any molecular force which enters into one system of equations must enter into a second system; this is in consequence of the *mutual* action of the particles.

There are two conditions which will enable us to deduce from these $3n$ equations *six* equations of condition, independent of the molecular forces. These will be the equations which the other forces must satisfy, in order that equilibrium may be maintained.

The first condition is this, that the molecular actions are *mutual;* and that, consequently, if $P_1 \cos \alpha_1$ represent the resolved part parallel to the axis of x of any one of the molecular forces involved in the $3n$ equations, we shall likewise meet with the term $-P_1 \cos \alpha_1$ in another of those equations which have reference to the axis of x. Consequently, if we add all those equations together which have reference to the same axis, we have the three following equations of condition *independent of the molecular forces*,

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0.$$

The second consideration is this: that the straight lines joining the different particles are the directions in which the molecular forces act.

Thus, let P_1 be the molecular action between the particles whose co-ordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2),

$$P_1 \cos \alpha_1, \quad P_1 \cos \beta_1, \quad P_1 \cos \gamma_1,$$

$$-P_1 \cos \alpha_1, \quad -P_1 \cos \beta_1, \quad -P_1 \cos \gamma_1,$$

the corresponding resolved parts of P_1 for the two particles. Then

$$\cos\alpha_1 = \frac{x_2 - x_1}{r}, \quad \cos\beta_1 = \frac{y_2 - y_1}{r}, \quad \cos\gamma_1 = \frac{z_2 - z_1}{r},$$

where $$r = \sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}}.$$

These enable us to obtain three more equations free from molecular forces; for if we multiply (1) and (2) by y_1 and x_1 respectively, and then subtract, we have

$$Y_1 x_1 - X_1 y_1 + \ldots + P_1\{x_1 \cos\beta_1 - y_1 \cos\alpha_1\} + \ldots = 0 \ldots (4).$$

By the same process we obtain from the system of equations which refer to the particle (x_2, y_2, z_2),

$$Y_2 x_2 - X_2 y_2 + \ldots - P_1\{x_2 \cos\beta_1 - y_2 \cos\alpha_1\} + \ldots = 0 \ldots (5).$$

But the values of $\cos\alpha_1$ and $\cos\beta_1$ given above lead to the condition

$$(x_2 - x_1)\cos\beta_1 - (y_2 - y_1)\cos\alpha_1 = 0.$$

Wherefore the equation

$$Y_1 x_1 - X_1 y_1 + \ldots\ldots + Y_2 x_2 - X_2 y_2 + \ldots\ldots = 0$$

will not involve P_1, the molecular action between the particles whose co-ordinates are x_1, y_1, z_1 and x_2, y_2, z_2 respectively.

It follows readily from what we have shewn, that if we form all the equations similar to (4) and (5), and add them together, we shall have a final equation

$$\Sigma(Yx - Xy) = 0,$$

independent of the molecular forces.

In like manner we should obtain

$$\Sigma(Zy - Yz) = 0, \quad \Sigma(Xz - Zx) = 0.$$

Moreover we can shew that these six equations are the *only* equations free from the molecular forces, supposing the body to be rigid, and consequently the molecules to retain their mutual distances invariable. For if a body consist of three molecules, there must evidently be three independent molecular forces to keep them invariable; if to these a fourth be added, we must introduce three new forces to hold it to the others; if we add a fifth, we must introduce three forces to hold this invariably to any three of those which are already rigidly connected; and so on; from which we see that there must be at least $3 + 3(n - 3)$ or $3n - 6$ forces. Hence the $3n$ equations resembling (1), (2), and (3) contain at least $3n - 6$ independent quantities to be eliminated; and therefore there cannot be more than *six* equations of condition connecting the external forces and the co-ordinates of their points of application.

CHAPTER VIII.

CENTRE OF GRAVITY.

108. WEIGHT is measured like other quantities by means of an arbitrary unit. If a certain upward force be necessary to prevent a body from falling, then another body which requires an equal force to sustain it is said to have a weight equal to that of the first. When two weights have been recognised to be equal, a body which requires to sustain it a force equal to the sum of the two equal forces which would sustain the two equal weights, is said to have a weight *double* that of either of the two equal weights; and so on.

It appears from experiment that the weight of a given body is invariable so long as the body remains at the same place on the earth's surface, but changes when the body is taken to a different place. We shall suppose therefore when we speak of the weight of a body that the body remains at one place.

When a body is such that the weight of any portion of it is proportional to the volume of that portion it is said to be of *uniform density;* the density of such a body is measured by the ratio which the weight of any volume of it bears to the weight of an equal volume of some arbitrarily chosen body of uniform density.

The product of the density of a body into its volume is called its *mass.*

When a body is not of uniform density its density at any point is measured thus: find the ratio of the weight of a volume of the body taken so as to include that point to the weight of an equal volume of the standard body; the *limit* of this ratio, when the volume is indefinitely diminished, is the density of the body at the assumed point.

109. It was shewn in Art. 66 that there is a point in every body such that, if the particles of the body be acted on by parallel forces and this point be fixed, the body will rest in whatever position it be placed.

Now the weight of a body may be considered as the resultant of the weights of the different elementary portions of the body, acting in parallel and vertical lines. In this case the point above described as the centre of parallel forces is called the *centre of gravity of the body.* We may define the centre of gravity of any system of heavy particles as a point such that if it be supported and the particles rigidly connected with it, the system will rest in any position.

In the present chapter we shall determine the position of the centre of gravity in bodies of various forms. We shall first give a few elementary examples.

(1) *Given the centres of gravity of two parts which compose a body, to find the centre of gravity of the whole body.*

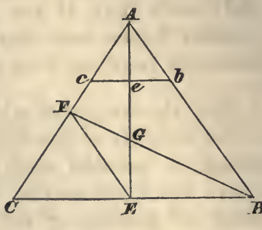
Let G_1 denote the centre of gravity of one part, and G_2 the centre of gravity of the other part; let m_1 denote the mass of the first part and m_2 the mass of the second part. Join $G_1 G_2$ and divide it in G so that $\frac{GG_1}{GG_2} = \frac{m_2}{m_1}$, then G is the centre of gravity of the whole body (Art. 37).

(2) *Given the centre of gravity of a body and also the centre of gravity of a part of the body, to find the centre of gravity of the remainder.*

Let G denote the centre of gravity of the body, and G_1 the centre of gravity of a part of the body; let m denote the mass of the body, and m_1 the mass of the part. Join $G_1 G$ and produce it through G to G_2, so that $\frac{GG_2}{GG_1} = \frac{m_1}{m - m_1}$, then G_2 is the centre of gravity of the remainder.

(3) *To find the centre of gravity of a triangular figure of uniform thickness and density.*

Let ABC be one surface of the triangular figure; bisect BC in E; join AE; draw ceb parallel to CEB cutting AE in e. Then, by similar triangles,



$$ce : CE :: Ae : AE,$$

and $$be : BE :: Ae : AE,$$

$$\therefore ce : CE :: be : BE;$$

but $$CE = BE, \ \therefore ce = be.$$

Hence AE bisects every line parallel to BC. Therefore each of the strips similar to ceb, into which we may suppose the triangle to be divided, will balance on AE, and therefore the centre of gravity must be in the line AE.

Bisect AC in F and join BF; let this cut AE in G. Then, as before, the centre of gravity must be in BF; but it must be in AE; and therefore G is the centre of gravity.

Join EF. Then, because $CE = BE$ and $CF = AF$, therefore EF is parallel to AB and $AB = 2FE$; and by similar triangles,

$$EG : EF :: AG : AB, \ \therefore EG = \tfrac{1}{2}AG.$$

Hence to find the centre of gravity of a triangle, bisect any side, join the point of bisection with the opposite angle, and the centre of gravity lies a third of the way up this line.

The centre of gravity of any plane polygon may be found by dividing it into triangles, determining the centre of gravity of each triangle, and then by Art. 66 deducing the centre of gravity of the whole figure.

We may observe that the centre of gravity of a triangle coincides with the centre of gravity of three *equal* particles placed at the angular points of the triangle. For to find the centre of gravity of three equal particles placed at A, B, C respectively, we join CB and bisect it in E; then E is the centre of gravity of the particles at C and B; suppose these particles collected at E; then join AE and divide AE in G so that EG may be to AG as the mass of the one particle at A to that of the two at E, that is, as 1 to 2; then G is the centre of gravity of the three equal particles. From the construction

G is obviously also the centre of gravity of the triangle ABC.

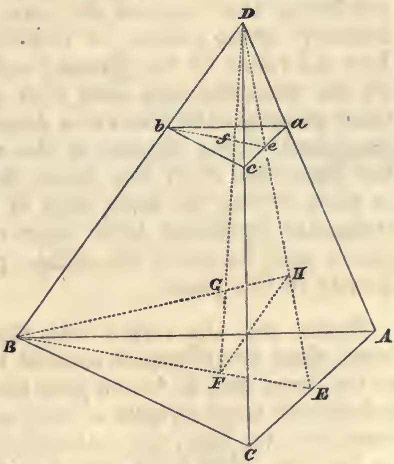
Let the co-ordinates of A referred to any axes be x_1, y_1, z_1; those of B, x_2, y_2, z_2; and those of C, x_3, y_3, z_3; then, by Art. 66, the co-ordinates $\bar{x}, \bar{y}, \bar{z}$ of the centre of gravity of three equal particles placed at A, B, C respectively, are

$$\bar{x} = \tfrac{1}{3}(x_1 + x_2 + x_3); \quad \bar{y} = \tfrac{1}{3}(y_1 + y_2 + y_3); \quad \bar{z} = \tfrac{1}{3}(z_1 + z_2 + z_3).$$

By what we have just proved, these are also the co-ordinates of the centre of gravity of the *triangle* ABC.

It may be remarked that in Art. 66 the co-ordinates may be *rectangular or oblique.*

(4) *To find the centre of gravity of a pyramid on a triangular base.*



Let ABC be the base, D the vertex; bisect AC in E; join BE, DE; take $EF = \frac{1}{3}EB$, then F is the centre of gravity of ABC. Join FD; draw ab, bc, ca parallel to AB, BC, CA respectively, and let DF meet the plane abc in f; join bf and produce it to meet DE in e. Then, by similar triangles, $ae = ec$; also

$$\frac{bf}{BF} = \frac{Df}{DF} = \frac{ef}{EF};$$

$$\text{but } EF = \tfrac{1}{2}BF, \quad \therefore ef = \tfrac{1}{2}bf;$$

therefore f is the centre of gravity of the triangle abc; and if we suppose the pyramid to be made up of an indefinitely great number of indefinitely thin triangular slices parallel to the base, each of these slices has its centre of gravity in DF. Hence the centre of gravity of the pyramid is in DF.

Again, take $EH = \frac{1}{3}ED$; join HB cutting DF in G. Then, as before, the centre of gravity of the pyramid must be in

BH; but it is in DF; hence G, the point of intersection of these lines, is the centre of gravity.

Join FH; then FH is parallel to DB. Also because $EF = \frac{1}{3}EB$, therefore $FH = \frac{1}{3}DB$, and

$$\frac{FG}{FH} = \frac{DG}{DB}; \text{ but } FH = \tfrac{1}{3}DB, \quad \therefore FG = \tfrac{1}{3}DG = \tfrac{1}{4}DF.$$

Hence the centre of gravity is one-fourth of the way up the line joining the centre of gravity of the base with the vertex.

(5) *To find the centre of gravity of any pyramid having a plane base.*

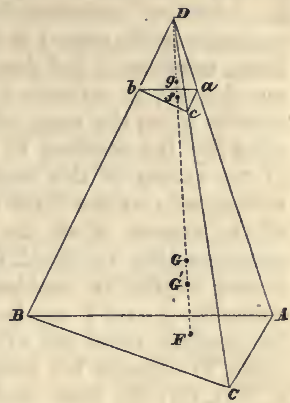
Divide the base into triangles; if any part of the base is curvilinear then suppose the curve to be divided into an indefinitely great number of indefinitely short straight lines. Join the vertex of the pyramid with the centres of gravity of all the triangles, and also with all their angles. Draw a plane parallel to the base at a distance from the base equal to one-fourth of the distance of the vertex from the base; then this plane cuts every line drawn from the vertex to the base in parts having the same ratio of 3 to 1; and therefore the triangular pyramids have their centres of gravity in this plane, and therefore the whole pyramid has its centre of gravity in this plane.

Again, join the vertex with the centre of gravity of the base; then every section parallel to the base will be similar to the base, and if we suppose the pyramid divided into an indefinitely large number of indefinitely thin slices by planes parallel to the base, the centre of gravity of each slice will lie on the line joining the vertex with the centre of gravity of the base. Hence the whole pyramid has its centre of gravity in this line.

Therefore the centre of gravity is one-fourth of the way up the line joining the centre of gravity of the base with the vertex.

(6) *To find the centre of gravity of the frustum of a pyramid formed by parallel planes.*

Let $ABCabc$ be the frustum; G, g the centres of gravity of the pyramids $DABC$, $Dabc$; it is clear that the centre of gravity of the frustum must be in gG produced; suppose it at G'.



Let $Ff = c$,

$AB = a$,

$ab = b$.

Since the whole pyramid $DABC$ is made up of the frustum and the small pyramid, therefore,

$$\frac{GG'}{Gg} = \frac{\text{weight of small pyramid}}{\text{weight of frustum}}$$

$$= \frac{\text{vol. of small pyr.}}{\text{vol. of large pyr.} - \text{vol. of small pyr.}} = \frac{b^3}{a^3 - b^3},$$

since similar solids are as the cubes of their homologous edges;

and $$Gg = DG - Dg = \tfrac{3}{4}(DF - Df) = \tfrac{3}{4}c;$$

$$\therefore\ GG' = \frac{3c}{4} \cdot \frac{b^3}{a^3 - b^3}.$$

Also $GF = \frac{1}{4}DF = \frac{1}{4}(DF - Df)\dfrac{a}{a-b}$ by similar figures,

$$= \frac{c}{4} \cdot \frac{a}{a-b};$$

$$\therefore\ FG' = FG - G'G = \frac{c}{4}\left\{\frac{a}{a-b} - \frac{3b^3}{a^3 - b^3}\right\}.$$

$$= \frac{c}{4}\,\frac{a^2 + 2ab + 3b^2}{a^2 + ab + b^2}.$$

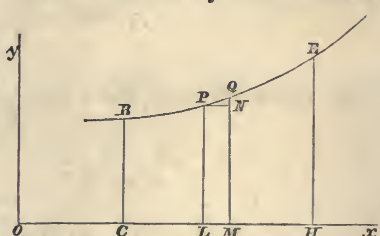
This is true of a frustum of a pyramid on any base, a and b being homologous sides of the two ends.

We proceed now to the analytical calculations.

110. In all the cases in which the Integral Calculus is employed to ascertain the centre of gravity of a body the principle is the same; the body is divided into an indefinitely large number of indefinitely small elements; the volume of an element is estimated, and this being multiplied by the density gives the mass of the element. The mass is multiplied by the abscissa of the element, and we find the sum of the values of this product for all the elements; the result corresponds to the ΣPx of Art. 66. Also we find the sum of the masses of all the elements and thus obtain a result corresponding to the ΣP of the same article. Divide the former result by the latter and we have the value of $\bar{x}$; similarly $\bar{y}$ and $\bar{z}$ can be found. In the following examples the student must not allow the details of the Integral Calculus to obscure his recognition of the fundamental formula of Art. 66; he must consider in every case what corresponds to the P, x, y, z of that article, that is, he must carefully ascertain into what elements the body is decomposed.

Plane Area.

111. Let $CBEH$ be an area bounded by the ordinates BC and EH, the curve BE, and the portion CH of the axis of x; it is required to find the centre of gravity of the area. Or instead of the area we may ask for the centre of gravity of a solid bounded by two planes parallel to



the plane of the paper and equidistant from it, and by a line which moves round the boundary $CBEH$ remaining always perpendicular to the plane of the paper. Divide CH into n portions, and suppose ordinates drawn at the points of division. Let LP and MQ represent two consecutive ordinates, and draw PN parallel to LM.

$$\text{Let } OL = x, \quad LP = y, \quad LM = \Delta x, \quad OC = c, \quad OH = h.$$

The area of the rectangle PM is $y\Delta x$; suppose u to denote the area of PQN, and let x' be the abscissa of the centre of gravity of the area $PQML$. Then if k denote the thickness of the solid and ρ its density, $k\rho\,(y\Delta x + u)$ is the mass of the element $PQML$. Hence, if $\bar{x}$ be the abscissa of the centre of gravity of the whole figure $CBEH$, by Art. 66,

$$\bar{x} = \frac{\Sigma k\rho x'\,(y\Delta x + u)}{\Sigma k\rho\,(y\Delta x + u)} = \frac{\Sigma x'\,(y\Delta x + u)}{\Sigma\,(y\Delta x + u)},$$

supposing the thickness and density uniform. The summation is to include all the figures like $PQML$, which are comprised in $CBEH$.

Now suppose n to increase without limit, and each of the portions LM to diminish without limit; then the term Σu in the denominator of $\bar{x}$ vanishes; for it expresses the sum of all the figures like PQN, and is therefore less than a rectangle having for its breadth Δx and for its height the greatest ordinate comprised between CB and HE. Also the term $\Sigma x'u$ in the numerator of $\bar{x}$ vanishes, for it is less than the product $h\Sigma u$, and as we have just shewn, this ultimately vanishes. Hence the expression for $\bar{x}$ becomes, when the number of divisions is indefinitely increased and each term indefinitely diminished,

$$\frac{\Sigma x' y\Delta x}{\Sigma y\Delta x}.$$

Moreover, x' must lie between x and $x + \Delta x$: suppose it equal to $x + v$, where v is less than Δx; then the numerator of $\bar{x}$ may be written

$$\Sigma xy\Delta x + \Sigma vy\Delta x;$$

and as the latter term cannot be so great as $\Delta x\Sigma y\Delta x$, it ultimately vanishes. Hence we have

$$\bar{x} = \frac{\Sigma xy\Delta x}{\Sigma y\Delta x};$$

that is, the above formula will give the correct value of $\bar{x}$ *when we increase the number of divisions indefinitely and diminish each term indefinitely, and extend the summation over* *the space CBEH.* This will be expressed according to the ordinary notation of the Integral Calculus thus,

$$\bar{x} = \frac{\int_c^h xydx}{\int_c^h ydx} \text{ .... } (1).$$

In the same manner we may shew that

$$\bar{y} = \frac{\int_c^h y'ydx}{\int_c^h ydx},$$

where y' is the limiting value of the ordinate of the centre of gravity of the element $PQML$ when its breadth is indefinitely diminished; y' is therefore $= \frac{1}{2}y$; hence

$$\bar{y} = \frac{\frac{1}{2}\int_c^h y^2dx}{\int_c^h ydx} \text{ .... } (2).$$

We have now only to substitute in (1) and (2) for y its value in terms of x, and then to effect the integration by the ordinary methods.

112. It will not be necessary for the student in solving an example to repeat the whole of the preceding process. When he understands how the necessary exactness may be given, if required, he may proceed shortly thus. The figure $PQML = y\Delta x$ ultimately, and the co-ordinates of its centre of gravity are x and $\frac{1}{2}y$ ultimately. Hence

$$\bar{x} = \frac{\int xydx}{\int ydx} \text{ and } \bar{y} = \frac{\int \frac{1}{2}yydx}{\int ydx},$$

the integrations being taken between proper limits.

Unless the contrary be specified, we shall hereafter suppose the bodies we consider to be of *uniform density*, and shall therefore not introduce any factor to represent the density, because, as in the preceding article, the factor will disappear.

113. Ex. 1. Let the curve be a parabola whose equation is

$$y = 2\sqrt{(ax)}.$$

$$\text{Here}\ \bar{x} = \frac{\int_c^h yx\,dx}{\int_c^h y\,dx} = \frac{\int_c^h 2\sqrt{(ax)}\,x\,dx}{\int_c^h 2\sqrt{(ax)}\,dx} = \frac{\int_c^h x^{\frac{3}{2}}\,dx}{\int_c^h x^{\frac{1}{2}}\,dx}$$

$$= \frac{\frac{2}{5}(h^{\frac{5}{2}} - c^{\frac{5}{2}})}{\frac{2}{3}(h^{\frac{3}{2}} - c^{\frac{3}{2}})}.$$

If $c = 0$, $\bar{x} = \frac{3}{5}h$, which determines the abscissa of the centre of gravity of a portion of a parabolic area beginning at the vertex. Also

$$\bar{y} = \frac{\frac{1}{2}\int_c^h y^2\,dx}{\int_c^h y\,dx} = \frac{2a\int_c^h x\,dx}{2\sqrt{a}\int_c^h x^{\frac{1}{2}}\,dx} = \frac{\sqrt{a}\int_c^h x\,dx}{\int_c^h x^{\frac{1}{2}}\,dx} = \frac{\frac{1}{2}\sqrt{a}\,(h^2 - c^2)}{\frac{2}{3}(h^{\frac{3}{2}} - c^{\frac{3}{2}})}.$$

When $c = 0$, $\bar{y} = \frac{3}{4}\sqrt{(ah)}$.

Ex. 2. Let the curve be an ellipse whose equation is

$$y = \frac{b}{a}\sqrt{(a^2 - x^2)}.$$

$$\text{Here}\ \bar{x} = \frac{\int_c^h yx\,dx}{\int_c^h y\,dx} = \frac{\int_c^h \frac{bx}{a}\sqrt{(a^2 - x^2)}\,dx}{\int_c^h \frac{b}{a}\sqrt{(a^2 - x^2)}\,dx} = \frac{\int_c^h x\sqrt{(a^2 - x^2)}\,dx}{\int_c^h \sqrt{(a^2 - x^2)}\,dx}.$$

Now $$\int x\sqrt{(a^2 - x^2)}\,dx = -\tfrac{1}{3}(a^2 - x^2)^{\frac{3}{2}};$$

therefore $$\int_c^h x\sqrt{(a^2 - x^2)}\,dx = \tfrac{1}{3}(a^2 - c^2)^{\frac{3}{2}} - \tfrac{1}{3}(a^2 - h^2)^{\frac{3}{2}}.$$

And $$\int\sqrt{(a^2 - x^2)}\,dx = \frac{x\sqrt{(a^2 - x^2)}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a};$$

therefore

$$\int_c^h \sqrt{(a^2 - x^2)}\,dx = \frac{h\sqrt{(a^2 - h^2)} - c\sqrt{(a^2 - c^2)}}{2} + \frac{a^2}{2}\left(\sin^{-1}\frac{h}{a} - \sin^{-1}\frac{c}{a}\right).$$

Hence $\bar{x}$ is known.

Also $$\bar{y} = \frac{\int_c^h \frac{y^2}{2}\,dx}{\int_c^h y\,dx} = \frac{\frac{b}{2a}\int_c^h (a^2 - x^2)\,dx}{\int_c^h \sqrt{(a^2 - x^2)}\,dx}$$

$$= \frac{\frac{b}{2a}\left\{a^2(h-c) - \frac{h^3 - c^3}{3}\right\}}{\frac{h\sqrt{(a^2-h^2)} - c\sqrt{(a^2-c^2)}}{2} + \frac{a^2}{2}\left(\sin^{-1}\frac{h}{a} - \sin^{-1}\frac{c}{a}\right)}.$$

If we require the centre of gravity of the quadrant of the ellipse, we must put $c = 0$ and $h = a$. Hence

$$\bar{x} = \frac{4a}{3\pi}, \quad \bar{y} = \frac{4b}{3\pi}.$$

Ex. 3. Let the curve be a cycloid whose equation is

$$y = \sqrt{(2ax - x^2)} + a \operatorname{vers}^{-1}\frac{x}{a};$$

and suppose we require the centre of gravity of half the area of the curve; then

$$\bar{x} = \frac{\int_0^{2a} yx\,dx}{\int_0^{2a} y\,dx}, \quad \bar{y} = \frac{\frac{1}{2}\int_0^{2a} y^2\,dx}{\int_0^{2a} y\,dx}.$$

Now $$\int yx\,dx = \frac{yx^2}{2} - \int \frac{x^2}{2}\frac{dy}{dx}\,dx$$

$$= \frac{yx^2}{2} - \int \frac{x^2}{2}\sqrt{\left(\frac{2a - x}{x}\right)}\,dx.$$

Also, when $x = 0$, $y = 0$, and when $x = 2a$, $y = \pi a$;

therefore $\int_0^{2a} yx\,dx = \frac{1}{2}\{\pi a\,(2a)^2\} - \frac{1}{2}\int_0^{2a} x\sqrt{(2ax - x^2)}\,dx$:

and as $\int_0^{2a} x\sqrt{(2ax - x^2)}\,dx$ will be found $= \frac{1}{2}\pi a^3$, we have

$$\int_0^{2a} yx\,dx = 2\pi a^3 - \tfrac{1}{4}\pi a^3 = \tfrac{7}{4}\pi a^3.$$

Again, $$\int y\,dx = yx - \int x\frac{dy}{dx}\,dx$$

$$= yx - \int \sqrt{(2ax - x^2)}\,dx;$$

therefore $$\int_0^{2a} y\,dx = 2\pi a^2 - \int_0^{2a} \sqrt{(2ax - x^2)}\,dx$$

$$= 2\pi a^2 - \tfrac{1}{2}\pi a^2 = \tfrac{3}{2}\pi a^2;$$

therefore $$\bar{x} = \frac{\frac{7}{4}\pi a^3}{\frac{3}{2}\pi a^2} = \tfrac{7}{6}a.$$

Also

$$\int y^2 dx = y^2 x - 2\int yx \frac{dy}{dx}\,dx$$

$$= y^2 x - 2\int y\sqrt{(2ax - x^2)}\,dx$$

$$= y^2 x - 2\int (2ax - x^2)\,dx - 2a\int \sqrt{(2ax - x^2)} \operatorname{vers}^{-1}\frac{x}{a}\,dx$$

$$= y^2 x - 2ax^2 + \frac{2x^3}{3} - 2a\int \sqrt{(2ax - x^2)} \operatorname{vers}^{-1}\frac{x}{a}\,dx;$$

$$\therefore \int_0^{2a} y^2 dx = 2\pi^2 a^3 - \frac{8a^3}{3} - 2a\int_0^{2a} \sqrt{(2ax - x^2)} \operatorname{vers}^{-1}\frac{x}{a}\,dx.$$

By assuming $\operatorname{vers}^{-1}\frac{x}{a} = \theta$, we may shew that

$$\int_0^{2a} \sqrt{(2ax - x^2)} \operatorname{vers}^{-1}\frac{x}{a}\,dx = \frac{\pi^2 a^2}{4}.$$

Hence $$\int_0^{2a} y^2 dx = \tfrac{3}{2}\pi^2 a^3 - \tfrac{8}{3}a^3;$$

therefore $$\bar{y} = \frac{\frac{1}{2}a\left(\frac{3}{2}\pi^2 - \frac{8}{3}\right)}{\frac{3}{2}\pi} = \frac{a}{3\pi}\left(\tfrac{3}{2}\pi^2 - \tfrac{8}{3}\right).$$

114. If a curve have a branch below the axis of x symmetrical with one above the axis, and we require the centre of gravity of the area bounded by the two branches and ordinates drawn at the distances c and h from the origin, we have

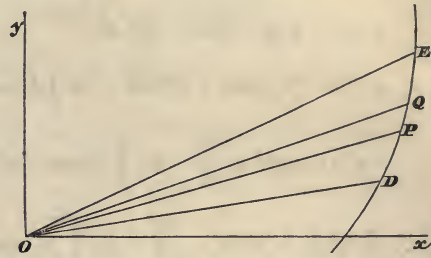
$$\bar{x} = \frac{2\int_c^h yx\,dx}{2\int_c^h y\,dx} = \frac{\int_c^h yx\,dx}{\int_c^h y\,dx},$$

and $$\bar{y} = 0.$$

115. We have hitherto supposed the axes *rectangular;* if they are oblique and inclined at an angle ω, then the figure $PQML$ (see fig. to Art. 111) will $= \sin \omega y \Delta x$ ultimately. Hence the formulæ (1) and (2) of Art. 111 remain true, for $\sin \omega$ occurs as a factor in the numerator and denominator, and may therefore be cancelled.

116. It is sometimes convenient to use polar formulæ.

Let DE be the arc of a curve; and suppose we require the centre of gravity of the area comprised between the arc DE and the radii OD, OE drawn from the pole O.



Divide the angle DOE into a number of angles, of which POQ represents one; let $OP = r$, $POx = \theta$, $POQ = \Delta\theta$. The area $POQ = \frac{1}{2}r^2\Delta\theta$ ultimately (*Diff. Calc.*, Art. 313). Also the centre of gravity of the figure POQ will be ultimately, like that of a triangle, on a line drawn from O bisecting the chord PQ, and at a distance of two-thirds of this line from O. Hence the abscissa and ordinate of the centre of gravity of POQ will be ultimately

$$\tfrac{2}{3}r\cos\theta, \text{ and } \tfrac{2}{3}r\sin\theta \text{ respectively.}$$

Hence
$$\bar{x} = \frac{\int \frac{2}{3}r\cos\theta \frac{1}{2}r^2 d\theta}{\int \frac{1}{2}r^2 d\theta} = \frac{\frac{2}{3}\int r^3 \cos\theta d\theta}{\int r^2 d\theta},$$

$$\bar{y} = \frac{\int \frac{2}{3}r\sin\theta \frac{1}{2}r^2 d\theta}{\int \frac{1}{2}r^2 d\theta} = \frac{\frac{2}{3}\int r^3 \sin\theta d\theta}{\int r^2 d\theta}.$$

In these formulæ we must put for r its value in terms of θ given by the equation to the curve; we must then integrate from $\theta = \alpha$ to $\theta = \beta$, supposing α and β the angles which OD and OE respectively make with the fixed line Ox.

117. Ex. Let O be the focus of a parabola, and the fixed line Ox pass through the vertex; then

$$r = \frac{a}{\cos^2 \frac{1}{2}\theta},$$

where $4a$ is the *latus rectum* of the parabola.

Hence
$$\bar{x} = \frac{\frac{2}{3}a \int_\alpha^\beta \frac{\cos\theta}{\cos^6\frac{1}{2}\theta}\, d\theta}{\int_\alpha^\beta \frac{d\theta}{\cos^4\frac{1}{2}\theta}}.$$

Now
$$\int \frac{\cos\theta}{\cos^6\frac{1}{2}\theta}\, d\theta = \int \frac{\cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta}{\cos^4\frac{1}{2}\theta} \sec^2\tfrac{1}{2}\theta d\theta$$

$$= \int (1 - \tan^2\tfrac{1}{2}\theta)(1 + \tan^2\tfrac{1}{2}\theta) \sec^2\tfrac{1}{2}\theta d\theta$$

$$= \int (1 - \tan^4\tfrac{1}{2}\theta) \sec^2\tfrac{1}{2}\theta d\theta = 2(\tan\tfrac{1}{2}\theta - \tfrac{1}{5}\tan^5\tfrac{1}{2}\theta);$$

$$\therefore \int_\alpha^\beta \frac{\cos\theta}{\cos^6\frac{1}{2}\theta}\, d\theta = 2(\tan\tfrac{1}{2}\beta - \tan\tfrac{1}{2}\alpha) - \tfrac{2}{5}(\tan^5\tfrac{1}{2}\beta - \tan^5\tfrac{1}{2}\alpha).$$

Also
$$\int \frac{d\theta}{\cos^4\frac{1}{2}\theta} = \int (1 + \tan^2\tfrac{1}{2}\theta) \sec^2\tfrac{1}{2}\theta d\theta = 2\tan\tfrac{1}{2}\theta + \tfrac{2}{3}\tan^3\tfrac{1}{2}\theta;$$

$$\therefore \int_\alpha^\beta \frac{d\theta}{\cos^4\frac{1}{2}\theta} = 2(\tan\tfrac{1}{2}\beta - \tan\tfrac{1}{2}\alpha) + \tfrac{2}{3}(\tan^3\tfrac{1}{2}\beta - \tan^3\tfrac{1}{2}\alpha);$$

$$\therefore \bar{x} = \tfrac{2}{3}a \cdot \frac{\tan\frac{1}{2}\beta - \tan\frac{1}{2}\alpha - \frac{1}{5}(\tan^5\frac{1}{2}\beta - \tan^5\frac{1}{2}\alpha)}{\tan\frac{1}{2}\beta - \tan\frac{1}{2}\alpha + \frac{1}{3}(\tan^3\frac{1}{2}\beta - \tan^3\frac{1}{2}\alpha)}.$$

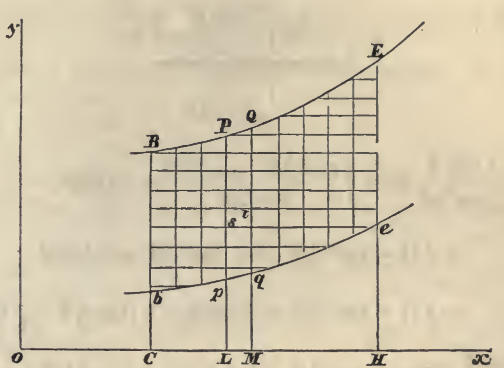
Again,
$$\int \frac{\sin\theta}{\cos^6\frac{1}{2}\theta}\, d\theta = 2\int \frac{\sin\frac{1}{2}\theta}{\cos^5\frac{1}{2}\theta}\, d\theta = \frac{1}{\cos^4\frac{1}{2}\theta};$$

$$\therefore \int_\alpha^\beta \frac{\sin\theta}{\cos^6\frac{1}{2}\theta}\, d\theta = \sec^4\tfrac{1}{2}\beta - \sec^4\tfrac{1}{2}\alpha;$$

$$\therefore \bar{y} = \tfrac{1}{3}a \cdot \frac{\sec^4\frac{1}{2}\beta - \sec^4\frac{1}{2}\alpha}{\tan\frac{1}{2}\beta - \tan\frac{1}{2}\alpha + \frac{1}{3}(\tan^3\frac{1}{2}\beta - \tan^3\frac{1}{2}\alpha)}.$$

Plane Area. Double Integration.

118. There is another method of dividing a plane area into elements, to which we now proceed.



Let a series of lines be drawn parallel to the axis of y, and another series of lines parallel to the axis of x. Let st represent one of the rectangles formed by these lines; and suppose x and y to be the co-ordinates of s, and $x + \Delta x$ and $y + \Delta y$ the co-ordinates of t. Then the area of the rectangle st is $\Delta x \Delta y$, and the co-ordinates of its centre of gravity are ultimately x and y. Hence, to find the abscissa of the centre of gravity of any plane area, we can take the sum of the values of $x\Delta x \Delta y$ for the numerator, and the sum of the values of $\Delta x \Delta y$ for the denominator, Δx and Δy being indefinitely diminished. This is expressed thus,

$$\bar{x} = \frac{\iint x\,dx\,dy}{\iint dx\,dy}.$$

Similarly,

$$\bar{y} = \frac{\iint y\,dx\,dy}{\iint dx\,dy}.$$

119. Suppose, for example, that the area is bounded by the two ordinates BbC, EeH, and the two curves $BPQE$, $bpqe$. Let $y=\phi(x)$ be the equation to the upper curve, and $y=\psi(x)$ the equation to the lower curve; let $OC=c$, $OH=h$. The sum of the product $x\Delta x\,\Delta y$ for all the rectangles similar to st, which are contained in the strip $PQqp$, is equal to $x\Delta x$ multiplied by the sum of the values of Δy, for $x\Delta x$ has the same value for each of these rectangles. Since the sum of the values of Δy is Pp or $\phi(x)-\psi(x)$, we have $x\Delta x\,.\,\{\phi(x)-\psi(x)\}$ as the result obtained by considering all the rectangles in the strip $PQqp$. We have then to sum up the values of $x\Delta x\{\phi(x)-\psi(x)\}$ for all the strips similar to $PQqp$ comprised between Bb and Ee; that is, we must determine the value of $\int_c^h x\{\phi(x)-\psi(x)\}\,dx$. Considerations of a similar kind apply to the denominator of $\bar{x}$, and we obtain

$$\bar{x}=\frac{\int_c^h x\{\phi(x)-\psi(x)\}\,dx}{\int_c^h \{\phi(x)-\psi(x)\}\,dx}.$$

In the numerator of $\bar{y}$ we observe that $y\Delta y\,\Delta x$ represents that portion of it which arises from the element st; hence we shall find the result obtained from all the elements in the strip $PQqp$, if we determine the sum of all the values of $y\Delta y$, and multiply the result by Δx. Now the sum of the values of $y\Delta y$ is $\int_{\psi(x)}^{\phi(x)} y\,dy$, or $\frac{1}{2}[\{\phi(x)\}^2-\{\psi(x)\}^2]$. If we multiply by Δx, and find the sum of the values of the product for all the strips between Bb and Ee, we obtain the numerator of $\bar{y}$. Hence

$$\bar{y}=\frac{\frac{1}{2}\int_c^h [\{\phi(x)\}^2-\{\psi(x)\}^2]\,dx}{\int_c^h \{\phi(x)-\psi(x)\}\,dx}.$$

The value of $\bar{y}$ may be written thus

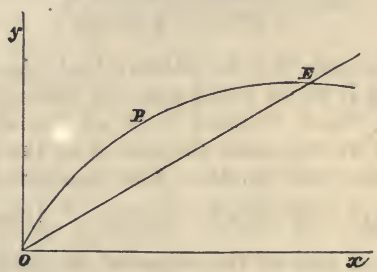
$$\bar{y}=\frac{\int_c^h \frac{1}{2}\{\phi(x)+\psi(x)\}\{\phi(x)-\psi(x)\}\,dx}{\int_c^h \{\phi(x)-\psi(x)\}\,dx}.$$

The meaning of the factors in the numerator is now apparent; for $\{\phi(x)-\psi(x)\}\Delta x$ ultimately represents the area of the strip $PQqp$, and $\frac{1}{2}\{\phi(x)+\psi(x)\}$, which is the ordinate of the middle point of Pp, will ultimately be the ordinate of the centre of gravity of $PQqp$. Hence the above equation agrees with that given in Art. 66,

$$\bar{y} = \frac{\Sigma Py}{\Sigma P}.$$

The process and the figure in the preceding two articles would have been unnecessary if our only object had been to establish the formulæ for $\bar{x}$ and $\bar{y}$, since these formulæ can be obtained more simply as we have just shewn. But we shall require hereafter other formulæ involving double integration, and have therefore directed the reader's attention to these in order to accustom him to the subject.

120. Ex. Let OPE be a parabola having for its equation $y^2 = 4ax$, and OE a straight line having for its equation $y = kx$; find the centre of gravity of the area OPE between the curve and the straight line.



Here $\phi(x) = 2\sqrt{(ax)}$, $\psi(x) = kx$, $c = 0$; h is to be found from the equation $2\sqrt{(ah)} = kh$;

therefore $$h = \frac{4a}{k^2}.$$

Thus $$\bar{x} = \frac{\int_0^h x\{2\sqrt{(ax)} - kx\}\,dx}{\int_0^h \{2\sqrt{(ax)} - kx\}\,dx}$$

$$= \frac{\frac{4}{5}\sqrt{a}h^{\frac{5}{2}} - \frac{1}{3}kh^3}{\frac{4}{3}\sqrt{a}h^{\frac{3}{2}} - \frac{1}{2}kh^2} = h\,\frac{\frac{4}{5}\sqrt{a} - \frac{1}{3}k\sqrt{h}}{\frac{4}{3}\sqrt{a} - \frac{1}{2}k\sqrt{h}} = h\,\frac{\frac{4}{5} - \frac{2}{3}}{\frac{4}{3} - 1}$$

$$= \frac{2h}{5} = \frac{8a}{5k^2}.$$

Similarly, $\bar{y} = \dfrac{\frac{1}{2}\int_0^h (4ax - k^2x^2)\,dx}{\int_0^h \{2\sqrt{(ax)} - kx\}\,dx}$

$$= \frac{\frac{1}{2}(2ah^2 - \frac{1}{3}k^2h^3)}{\frac{4}{3}\sqrt{a}h^{\frac{3}{2}} - \frac{1}{2}kh^2} = \frac{\frac{1}{2}\sqrt{h}\,(2a - \frac{1}{3}k^2h)}{\frac{4}{3}\sqrt{a} - \frac{1}{2}k\sqrt{h}}$$

$$= \frac{\frac{1}{2}\sqrt{h}\,(2a - \frac{4}{3}a)}{\frac{4}{3}\sqrt{a} - \sqrt{a}} = \sqrt{(ah)} = \frac{2a}{k}.$$

121. Sometimes it will be more convenient to integrate the formulæ in Art. 118, *first* with respect to x and then with respect to y. For example, if the given area is comprised between the lines $y = c'$, and $y = h'$, and the curves $x = \psi(y)$, and $x = \phi(y)$, we obtain

$$\bar{x} = \frac{\frac{1}{2}\int_{c'}^{h'} [\{\phi(y)\}^2 - \{\psi(y)\}^2]\,dy}{\int_{c'}^{h'} \{\phi(y) - \psi(y)\}\,dy},$$

$$\bar{y} = \frac{\int_{c'}^{h'} y\,\{\phi(y) - \psi(y)\}\,dy}{\int_{c'}^{h'} \{\phi(y) - \psi(y)\}\,dy}.$$

If we apply these to the example given in Art. 120, we have $\psi(y) = \dfrac{y^2}{4a}$, $\phi(y) = \dfrac{y}{k}$, $c' = 0$, and h' is to be found from the equation $\dfrac{h'}{k} = \dfrac{h'^2}{4a}$; therefore $h' = \dfrac{4a}{k}$.

Hence $$\bar{x} = \frac{\frac{1}{2}\int_0^{h'} \left(\frac{y^2}{k^2} - \frac{y^4}{16a^2}\right) dy}{\int_0^{h'} \left(\frac{y}{k} - \frac{y^2}{4a}\right) dy},$$

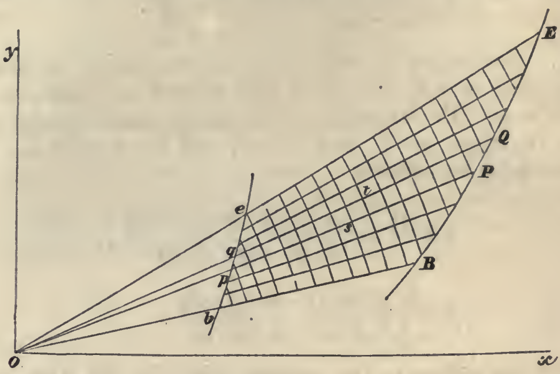
$$\bar{y} = \frac{\int_0^{h'} y\left(\frac{y}{k} - \frac{y^2}{4a}\right) dy}{\int_0^{h'} \left(\frac{y}{k} - \frac{y^2}{4a}\right) dy}.$$

The results will of course be the same as before.

For fuller explanations and illustrations of double integrations the student is referred to treatises on the Integral Calculus. (See especially *Integral Calculus*, Art. 141 and Art. 152.)

122. We will now give polar formulæ involving double integration.

Let a series of lines be drawn from a pole O, also a series of circles be described from O as a centre. Let st be one of the elements formed by this mode of dividing a plane area; let r and θ be the polar co-ordinates of s, $r+\Delta r$ and $\theta+\Delta\theta$ the co-ordinates of t; then the area of the element st will be ulti-



mately $r\Delta\theta\,\Delta r$, and the abscissa and ordinate of its centre of gravity will be $r\cos\theta$ and $r\sin\theta$ respectively. Hence we obtain

$$\bar{x}=\frac{\iint r\cos\theta\, r\,d\theta\,dr}{\iint r\,d\theta\,dr}=\frac{\iint r^2\cos\theta\,d\theta\,dr}{\iint r\,d\theta\,dr}.$$

Similarly $\bar{y}=\dfrac{\iint r^2\sin\theta\,d\theta\,dr}{\iint r\,d\theta\,dr}$.

Suppose the area bounded by the curves $BPQE$, $bpqe$, and the radii ObB, OeE. Let $r=\phi(\theta)$ be the equation to the first curve, $r=\psi(\theta)$ that to the second; and let α and β be the angles which OB and OE make respectively with Ox.

The sum of the values of $r^2\cos\theta\,\Delta r\,\Delta\theta$ for all the elements comprised in the strip $PQqp$, will be found by multiplying the sum of the values of $r^2\Delta r$ by $\cos\theta\Delta\theta$; the former sum is ultimately

$$\int_{\psi(\theta)}^{\phi(\theta)} r^2 dr \quad\text{or}\quad \tfrac{1}{3}\left[\{\phi(\theta)\}^3-\{\psi(\theta)\}^3\right].$$

Hence the numerator of the value of $\bar{x}$ is

$$\tfrac{1}{3}\int_\alpha^\beta \cos\theta\,[\{\phi(\theta)\}^3 - \{\psi(\theta)\}^3]\,d\theta,$$

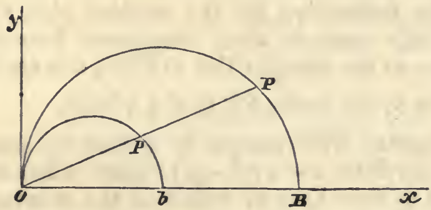
and the denominator, in like manner, is

$$\tfrac{1}{2}\int_\alpha^\beta [\{\phi(\theta)\}^2 - \{\psi(\theta)\}^2]\,d\theta;$$

therefore $$\bar{x} = \frac{\tfrac{2}{3}\int_\alpha^\beta \cos\theta\,[\{\phi(\theta)\}^3 - \{\psi(\theta)\}^3]\,d\theta}{\int_\alpha^\beta [\{\phi(\theta)\}^2 - \{\psi(\theta)\}^2]\,d\theta}.$$

Similarly, $$\bar{y} = \frac{\tfrac{2}{3}\int_\alpha^\beta \sin\theta\,[\{\phi(\theta)\}^3 - \{\psi(\theta)\}^3]\,d\theta}{\int_\alpha^\beta [\{\phi(\theta)\}^2 - \{\psi(\theta)\}^2]\,d\theta}.$$

123. Ex. 1. Find the centre of gravity of the area comprised between two semicircles *Opb* and *OPB*.



Let $Ob = c$, $OB = h$; $\phi(\theta) = h\cos\theta$, $\psi(\theta) = c\cos\theta$; $\alpha = 0$, $\beta = \frac{1}{2}\pi$; thus

$$\bar{x} = \frac{\tfrac{2}{3}(h^3 - c^3)\int_0^{\frac{1}{2}\pi}\cos^4\theta d\theta}{(h^2 - c^2)\int_0^{\frac{1}{2}\pi}\cos^2\theta d\theta}$$

$$= \frac{2}{3}\cdot\frac{3}{4}\frac{(h^3 - c^3)}{h^2 - c^2}$$

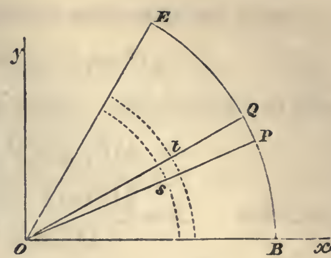
$$= \frac{1}{2}\frac{h^2 + hc + c^2}{h + c}.$$

(See *Integ. Calc.* Art. 35).

Also $$\bar{y} = \frac{\tfrac{2}{3}(h^3 - c^3)\int_0^{\frac{1}{2}\pi}\sin\theta\cos^3\theta d\theta}{(h^2 - c^2)\int_0^{\frac{1}{2}\pi}\cos^2\theta d\theta}$$

$$= \frac{\tfrac{2}{3}(h^3 - c^3)\tfrac{1}{4}}{(h^2 - c^2)\tfrac{1}{4}\pi} = \frac{2(h^2 + hc + c^2)}{3(h + c)\pi}.$$

Ex. 2. The sector of a circle.



Let BOE be the sector, subtending an angle β, $OB = a$.

In this example we may with equal facility integrate first with respect to θ and then with respect to r, or first with respect to r and then with respect to θ.

$$\bar{x} = \frac{\int_0^a \int_0^\beta r^2 \cos\theta \, dr \, d\theta}{\int_0^a \int_0^\beta r \, dr \, d\theta} = \frac{\sin\beta \int_0^a r^2 dr}{\beta \int_0^a r \, dr} = \frac{2a \sin\beta}{3\beta},$$

$$\bar{y} = \frac{\int_0^a \int_0^\beta r^2 \sin\theta \, dr \, d\theta}{\int_0^a \int_0^\beta r \, dr \, d\theta} = \frac{(1 - \cos\beta) \int_0^a r^2 dr}{\beta \int_0^a r \, dr} = \frac{2a(1 - \cos\beta)}{3\beta}.$$

It will be instructive for the student also to notice the solution of this example when rectangular formulæ are used. The equation to the straight line OE is $y = x \tan\beta$.

The equation to the circle EB is $x^2 + y^2 = a^2$.

If we integrate with respect to x first we must integrate from $x = y \cot\beta$ to $x = \sqrt{(a^2 - y^2)}$; since when we integrate with respect to x we have to collect all the elements in a strip which is parallel to the axis of x, and is bounded by OE at one end and by EB at the other. These strips extend from the axis of x up to E, and the ordinate of E is $a \sin\beta$. Hence we integrate with respect to y from $y = 0$ to $y = a \sin\beta$. Therefore

$$\bar{x} = \frac{\int_0^{h'} \int_{\psi(y)}^{\phi(y)} x \, dy \, dx}{\int_0^{h'} \int_{\psi(y)}^{\phi(y)} dy \, dx}, \qquad \bar{y} = \frac{\int_0^{h'} \int_{\psi(y)}^{\phi(y)} y \, dy \, dx}{\int_0^{h'} \int_{\psi(y)}^{\phi(y)} dy \, dx},$$

where $\psi(y) = y \cot\beta$, $\phi(y) = \sqrt{(a^2 - y^2)}$, $h' = a \sin\beta$.

The integrations may be easily effected.

If we wish to integrate with respect to y first, we shall have to divide the figure into two parts by a straight line drawn from E perpendicular to OB. For the part to the left of the dividing line the limits of y are 0 and $x \tan \beta$, and those of x are 0 and $a \cos \beta$. For the part to the right of the dividing line the limits of y are 0 and $\sqrt{(a^2 - x^2)}$, and those of x are $a \cos \beta$ and a. Hence

$$\bar{x} = \frac{\int_0^{a\cos\beta}\int_0^{x\tan\beta} x\,dx\,dy + \int_{a\cos\beta}^{a}\int_0^{\sqrt{(a^2-x^2)}} x\,dx\,dy}{\int_0^{a\cos\beta}\int_0^{x\tan\beta} dx\,dy + \int_{a\cos\beta}^{a}\int_0^{\sqrt{(a^2-x^2)}} dx\,dy}.$$

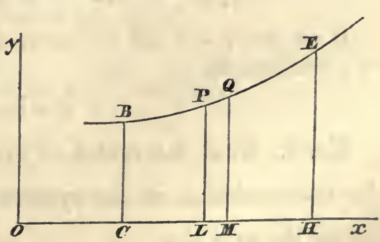
Similarly $\bar{y}$ may be expressed.

We have treated this example as an illustration of integration rather than for the purpose of obtaining the result in the simplest form. We might proceed thus; the centre of gravity must lie on the line which bisects the angle EOB. Hence taking this line for the initial line and using polar co-ordinates, we have $\bar{y} = 0$, and

$$\bar{x} = \frac{\int_0^a\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} r^2 \cos\theta\,dr\,d\theta}{\int_0^a\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} r\,dr\,d\theta} = \frac{4a \sin\frac{1}{2}\beta}{3\beta}.$$

Solid of Revolution.

124. Let a solid be generated by the revolution of the curve $BPQE$ round the axis of x, and suppose we require the centre of gravity of a portion of it intercepted between planes perpendicular to the axis of revolution.



Let the co-ordinates of a point P in the curve be x and y, and $x + \Delta x$ the abscissa of an adjacent point Q. As the curve revolves round the axis of x, the area $PQML$ will generate a volume which is ultimately equal to $\pi y^2 \Delta x$.

Also the abscissa of its centre of gravity will be x ultimately. Hence

$$\bar{x} = \frac{\int \pi y^2 x dx}{\int \pi y^2 dx} = \frac{\int y^2 x dx}{\int y^2 dx}.$$

The centre of gravity of the solid is obviously in the line Ox, so that we only require the value of $\bar{x}$ in order to determine its position.

125. Ex. 1. Let it be required to find the centre of gravity of a portion of a paraboloid. Suppose $y^2 = 4ax$ the equation to the generating parabola, and that the solid is bounded by planes distant c and h respectively from the vertex; then

$$\bar{x} = \frac{\int_c^h 4ax^2 dx}{\int_c^h 4ax dx} = \frac{2}{3} \cdot \frac{h^3 - c^3}{h^2 - c^2}.$$

If we put $c = 0$ we find for the centre of gravity of a segment of a paraboloid commencing at the vertex

$$\bar{x} = \frac{2h}{3}.$$

Ex. 2. Required the centre of gravity of a portion of a sphere intercepted between two parallel planes.

Let $y^2 = a^2 - x^2$ be the equation to the generating circle;

$$\bar{x} = \frac{\int_c^h (a^2 - x^2)\, x dx}{\int_c^h (a^2 - x^2)\, dx} = \frac{\frac{1}{2} a^2 (h^2 - c^2) - \frac{1}{4}(h^4 - c^4)}{a^2 (h - c) - \frac{1}{3}(h^3 - c^3)}.$$

If we put $c = 0$ and $h = a$, we find for the centre of gravity of a hemisphere

$$\bar{x} = \tfrac{3}{8} a.$$

Ex. 3. Find the centre of gravity of the solid generated by the revolution of the cycloid $y = \sqrt{(2ax - x^2)} + a \operatorname{vers}^{-1} \dfrac{x}{a}$ round the axis of x.

Here
$$\bar{x} = \frac{\int_0^{2a} y^2 x dx}{\int_0^{2a} y^2 dx}.$$

Now $y^2 = 2ax - x^2 + 2a\sqrt{(2ax - x^2)} \text{ vers}^{-1}\frac{x}{a} + a^2\left(\text{vers}^{-1}\frac{x}{a}\right)^2$.

Thus the numerator of $\bar{x}$ consists of three integrals of which we will give the values; these values may be obtained without difficulty by transforming the integrals where $\text{vers}^{-1}\frac{x}{a}$ occurs by the assumption $\text{vers}^{-1}\frac{x}{a} = \theta$, so that $x = a(1 - \cos\theta)$, and then integrating by parts. We shall find

$$\int_0^{2a} (2ax - x^2)\, x dx = \frac{4a^4}{3},$$

$$2a\int_0^{2a} x\sqrt{(2ax - x^2)} \text{ vers}^{-1}\frac{x}{a}\, dx = 2a\left(\frac{4a^3}{9} + \frac{\pi^2 a^3}{4}\right),$$

$$a^2\int_0^{2a} x\left(\text{vers}^{-1}\frac{x}{a}\right)^2 dx = \left(\frac{5\pi^2}{4} - 4\right)a^4.$$

Hence the numerator of $\bar{x}$ is $\left(\frac{7\pi^2}{4} - \frac{16}{9}\right)a^4$.

Also the denominator of $\bar{x}$ consists of three integrals which have the following values,

$$\int_0^{2a} (2ax - x^2)\, dx = \frac{4a^3}{3},$$

$$2a\int_0^{2a} \sqrt{(2ax - x^2)} \text{ vers}^{-1}\frac{x}{a}\, dx = 2a\,\frac{\pi^2 a^2}{4},$$

$$a^2\int_0^{2a} \left(\text{vers}^{-1}\frac{x}{a}\right)^2 dx = (\pi^2 - 4)\, a^3.$$

Hence the denominator of $\bar{x}$ is $\left(\frac{3\pi^2}{2} - \frac{8}{3}\right)a^3$.

Therefore
$$\bar{x} = \frac{\left(\frac{7\pi^2}{4} - \frac{16}{9}\right)a^4}{\left(\frac{3\pi^2}{2} - \frac{8}{3}\right)a^3}$$

$$= \frac{(63\pi^2 - 64)\, a}{6\,(9\pi^2 - 16)}.$$

126. If a solid of revolution be formed by revolving a curve round the axis of y, we find for the position of the centre of gravity

$$\bar{y} = \frac{\int \pi x^2 y dy}{\int \pi x^2 dy} = \frac{\int x^2 y dy}{\int x^2 dy}.$$

For example, suppose the cycloid

$$y = \sqrt{(2ax - x^2)} + a \operatorname{vers}^{-1} \frac{x}{a},$$

to revolve round the axis of y, and that we require the centre of gravity of the volume generated by that half of the curve for which y is positive. Here

$$\bar{y} = \frac{\int_0^{\pi a} x^2 y dy}{\int_0^{\pi a} x^2 dy}.$$

Now $\int x^2 y dy = \int x^2 y \frac{dy}{dx} dx$; thus in the present case,

$$\int_0^{\pi a} x^2 y dy = \int_0^{2a} x^2 y \frac{dy}{dx} dx.$$

Similarly $$\int_0^{\pi a} x^2 dy = \int_0^{2a} x^2 \frac{dy}{dx} dx.$$

Thus $$\bar{y} = \frac{\int_0^{2a} x^2 y \left(\frac{2a-x}{x}\right)^{\frac{1}{2}} dx}{\int_0^{2a} x^2 \left(\frac{2a-x}{x}\right)^{\frac{1}{2}} dx}$$

$$= \frac{\int_0^{2a} yx \sqrt{(2ax - x^2)} dx}{\int_0^{2a} x \sqrt{(2ax - x^2)} dx}.$$

The numerator of y consists of two integrals which have the following values,

$$\int_0^{2a} x (2ax - x^2) dx = \frac{4a^4}{3},$$

$$a \int_0^{2a} x \sqrt{(2ax - x^2)} \operatorname{vers}^{-1} \frac{x}{a} dx = a \left(\frac{4a^3}{9} + \frac{\pi^2 a^3}{4}\right).$$

The value of the denominator of $\bar{y}$ is $\frac{\pi}{2} a^3$.

Therefore $$\bar{y} = \frac{\frac{4a^4}{3} + \frac{4a^4}{9} + \frac{\pi^2 a^4}{4}}{\frac{\pi}{2} a^3}$$

$$= \left(\frac{16}{9} + \frac{\pi^2}{4}\right) \frac{2a}{\pi}.$$

127. We may also find it convenient in some cases to use formulæ involving double integration.

Suppose the figure in Art. 118 to revolve round the axis of x; let x, y be the co-ordinates of s; and $x + \Delta x$, $y + \Delta y$ those of t. The area st generates by revolution an elementary ring, the volume of which is

$$\pi (y + \Delta y)^2 \Delta x - \pi y^2 \Delta x;$$

this may be put ultimately equal to $2\pi y \Delta y \Delta x$. The centre of gravity of this ring is on the axis of x, and its abscissa is ultimately x. Hence by proceeding as before we shall have ultimately

$$\bar{x} = \frac{\int_c^h \int_{\psi(x)}^{\phi(x)} yx \, dx \, dy}{\int_c^h \int_{\psi(x)}^{\phi(x)} y \, dx \, dy},$$

where $y = \psi(x)$ is the equation to the lower bounding curve and $y = \phi(x)$ to the upper, and c and h are the abscissæ of the planes which bound the solid of revolution perpendicularly to its axis.

Similarly, if the solid is formed by revolving the area included between two curves round the axis of y, we shall have

$$\bar{y} = \frac{\int_{c'}^{h'} \int_{\psi(y)}^{\phi(y)} xy \, dy \, dx}{\int_{c'}^{h'} \int_{\psi(y)}^{\phi(y)} x \, dy \, dx}.$$

Or we may use polar formulæ. Suppose the figure in Art. 122 to revolve round the axis of x; let r, θ be the polar co-ordinates of s; and $r + \Delta r$, $\theta + \Delta\theta$ those of t. The volume of the ring generated by the revolution of the area *st* is ultimately $2\pi r \sin\theta\, r\,\Delta r\Delta\theta$; and the abscissa of the centre of gravity of the ring is ultimately $r\cos\theta$. Hence

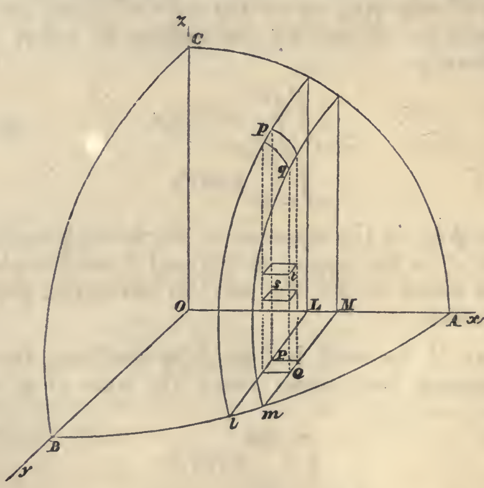
$$\bar{x} = \frac{\iint r^3 \sin\theta \cos\theta\, d\theta\, dr}{\iint r^2 \sin\theta\, d\theta\, dr}.$$

Similarly, if the figure revolve round the axis of y

$$\bar{y} = \frac{\iint r^3 \cos\theta \sin\theta\, d\theta\, dr}{\iint r^2 \cos\theta\, d\theta\, dr}.$$

Any Solid.

128. To find the centre of gravity of a solid we divide it into elements as follows: draw a series of planes perpen-



dicular to the axis of x, then two consecutive planes will include between them a slice such as $LplmqM$ in the figure; draw a second series of planes perpendicular to the axis of y, then each slice is divided into strips such as $PpqQ$ in the figure; lastly, draw planes perpendicular to the axis of z, then each strip is divided into parallelopipeds such as *st* in the figure. Let x, y, z be the co-ordinates of s and $x+\Delta x$, $y+\Delta y$, $z+\Delta z$ those of t; then $\Delta x \Delta y \Delta z$ is the volume of *st*, and as the co-ordinates of its centre of gravity are ultimately x, y, and z, we have

$$\bar{x} = \frac{\iiint x\,dx\,dy\,dz}{\iiint dx\,dy\,dz},$$

$$\bar{y} = \frac{\iiint y\,dx\,dy\,dz}{\iiint dx\,dy\,dz},$$

$$\bar{z} = \frac{\iiint z\,dx\,dy\,dz}{\iiint dx\,dy\,dz}.$$

129. In applying the above formulæ to examples, great care is necessary in assigning proper limits to the integrations; this we shall illustrate by Examples.

Ex. 1. Find the centre of gravity of the eighth part of an ellipsoid cut off by three principal planes.

Let the equation to the surface be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Then the equation to the curve in which the surface meets the plane of (x, y) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Integrate first with respect to z, and take for the limits $z = 0$ and $z = c\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}$; we thus include all the elements like *st* which form the strip $PpqQ$. Next integrate with respect to y, and take for the limits $y = 0$ and $y = b\sqrt{\left(1 - \frac{x^2}{a^2}\right)}$; we thus include all the strips like $PpqQ$ which form the slice $LplmqM$. Lastly integrate with respect to x, and take for the limits $x=0$ and $x=a$; we thus include all the slices $LplmqM$ which form the solid we are considering. Hence

$$\bar{x}=\frac{\int_0^a\int_0^{y_1}\int_0^{z_1}x\,dx\,dy\,dz}{\int_0^a\int_0^{y_1}\int_0^{z_1}dx\,dy\,dz},$$

where we put z_1 for $c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}$,

and y_1 for $b\sqrt{\left(1-\frac{x^2}{a^2}\right)}$.

Now $$\int_0^{z_1}dz=z_1=c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)};$$

therefore $$\bar{x}=\frac{\int_0^a\int_0^{y_1}x\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}dx\,dy}{\int_0^a\int_0^{y_1}\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}dx\,dy}.$$

And $\int_0^{y_1}\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}dy$, or $\frac{1}{b}\int_0^{y_1}\sqrt{(y_1^2-y^2)}\,dy=\frac{\pi y_1^2}{4b}$;

therefore $$\bar{x}=\frac{\int_0^a xy_1^2dx}{\int_0^a y_1^2dx}=\frac{\int_0^a\left(1-\frac{x^2}{a^2}\right)x\,dx}{\int_0^a\left(1-\frac{x^2}{a^2}\right)dx}=\frac{3a}{8}.$$

Similarly $\bar{y}=\frac{3b}{8}$, $\bar{z}=\frac{3c}{8}$.

We may in this example effect the integrations with equal simplicity in any order we please; if we integrate first for x, then for y, and lastly for z, we shall have

$$\bar{x}=\frac{\int_0^c\int_0^{y_1}\int_0^{x_1}x\,dz\,dy\,dx}{\int_0^c\int_0^{y_1}\int_0^{x_1}dz\,dy\,dx};$$

where x_1 stands for $a\sqrt{\left(1-\frac{z^2}{c^2}-\frac{y^2}{b^2}\right)}$,

and y_1 stands for $b\sqrt{\left(1-\frac{z^2}{c^2}\right)}$.

This will be easily seen by drawing a figure so as to make the planes bounding the *slice* parallel to that of (x, y), and the edges of the *strip* parallel to the axis of x.

Ex. 2. Let it be required to find the centre of gravity of the solid bounded by the planes $z=\beta x$, $z=\gamma x$, and the cylinder $y^2=2ax-x^2$. We shall have

$$\bar{x}=\frac{\int_0^{2a}\int_{-y_1}^{y_1}\int_{\beta x}^{\gamma x} x\,dx\,dy\,dz}{\int_0^{2a}\int_{-y_1}^{y_1}\int_{\beta x}^{\gamma x} dx\,dy\,dz},$$

where y_1 is put for $\sqrt{(2ax-x^2)}$.

Now $$\int_{\beta x}^{\gamma x} dz=(\gamma-\beta)\,x,$$

therefore $$\bar{x}=\frac{\int_0^{2a}\int_{-y_1}^{y_1} x^2\,dx\,dy}{\int_0^{2a}\int_{-y_1}^{y_1} x\,dx\,dy}.$$

Also $$\int_{-y_1}^{y_1} dy=2\sqrt{(2ax-x^2)};$$

therefore $$\bar{x}=\frac{\int_0^{2a} x^2\sqrt{(2ax-x^2)}\,dx}{\int_0^{2a} x\sqrt{(2ax-x^2)}\,dx}=\frac{5a}{4}.$$

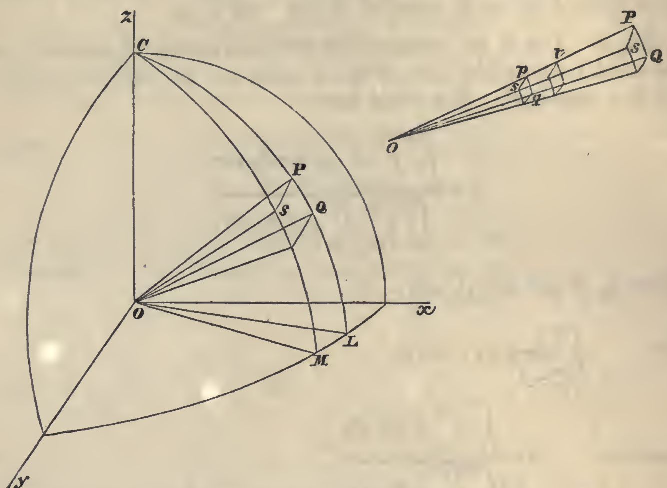
See *Integral Calculus*, Ex. 5 to Chap. III.

Similarly we may find

$$\bar{y}=0,\quad \bar{z}=\frac{5a(\beta+\gamma)}{8}.$$

130. It is often convenient to divide a solid into polar elements.

Let a series of planes be drawn through the axis of z; the solid is thus divided into wedge-shaped slices such as $COML$. Let a series of right cones be described round the axis of z having their vertices at O; thus each slice is divided into pyramidal solids like $OPQS$. Lastly, let a series of concentric



spheres be described round O as centre; thus each pyramid is divided into elements similar to $pqst$.

Let $\quad xOL = \phi, \qquad COP = \theta, \qquad Op = r,$

$\qquad LOM = \Delta\phi, \qquad POQ = \Delta\theta, \qquad pt = \Delta r.$

Then pq is the arc of a circle of which the radius is r and the angle $\Delta\theta$; therefore $pq = r\Delta\theta$.

Also ps is the arc of a circle of which the radius is $r \sin\theta$ and the angle $\Delta\phi$; therefore $ps = r \sin\theta\Delta\phi$.

Hence, since the element $pqst$ is ultimately a parallelopiped, its volume is $r^2 \sin\theta\Delta\theta\Delta\phi\Delta r$.

Also the co-ordinates of its centre of gravity are ultimately $r\cos\phi\sin\theta$, $r\sin\phi\sin\theta$, and $r\cos\theta$. Hence supposing its density to be ρ, we have

$$\bar{x} = \frac{\iiint \rho r^3 \sin^2\theta \cos\phi \, d\phi \, d\theta \, dr}{\iiint \rho r^2 \sin\theta \, d\phi \, d\theta \, dr},$$

$$\bar{y} = \frac{\iiint \rho r^3 \sin^2\theta \sin\phi \, d\phi \, d\theta \, dr}{\iiint \rho r^2 \sin\theta \, d\phi \, d\theta \, dr},$$

$$\bar{z} = \frac{\iiint \rho r^3 \sin\theta \cos\theta \, d\phi \, d\theta \, dr}{\iiint \rho r^2 \sin\theta \, d\phi \, d\theta \, dr}.$$

131. Ex. 1. Apply the preceding formulæ to find the centre of gravity of a hemisphere whose density varies as the n^{th} power of the distance from the centre.

Take the axis of z perpendicular to the plane base of the hemisphere. Let a be the radius of the hemisphere, and $\rho = \mu r^n$, where μ is a constant. First integrate with respect to r from 0 to a; we thus include all the elements like $pqst$ comprised in the pyramid $OPQS$. Next integrate with respect to θ from 0 to $\frac{1}{2}\pi$, we thus include all the pyramids in the slice $COML$. Finally, integrate from $\phi = 0$ to $\phi = 2\pi$; we thus include all the slices. Thus

$$\bar{z} = \frac{\int_0^{2\pi}\int_0^{\frac{1}{2}\pi}\int_0^a r^{n+3} \sin\theta \cos\theta \, d\phi \, d\theta \, dr}{\int_0^{2\pi}\int_0^{\frac{1}{2}\pi}\int_0^a r^{n+2} \sin\theta \, d\phi \, d\theta \, dr},$$

$$= \frac{n+3}{n+4} a \frac{\int_0^{2\pi}\int_0^{\frac{1}{2}\pi} \sin\theta \cos\theta \, d\phi \, d\theta}{\int_0^{2\pi}\int_0^{\frac{1}{2}\pi} \sin\theta \, d\phi \, d\theta} = \frac{n+3}{n+4} \cdot \frac{a}{2};$$

$\bar{x}$ and $\bar{y}$ each $= 0$.

Ex. 2. A right cone has its vertex on the surface of a sphere and its axis coincident with a diameter of the sphere, find the centre of gravity of the solid included between the cone and sphere. Take the axis of z coincident with that of the cone; suppose a the radius of the sphere, β the semi-vertical angle of the cone. The polar equation to the sphere is $r = 2a\cos\theta$, and to the cone $\theta = \beta$. Hence we have

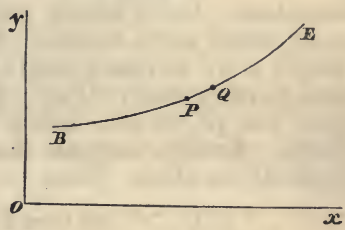
$$\bar{z} = \frac{\int_0^{2\pi}\int_0^{\beta}\int_0^{2a\cos\theta} r^3 \cos\theta \sin\theta \, d\phi \, d\theta \, dr}{\int_0^{2\pi}\int_0^{\beta}\int_0^{2a\cos\theta} r^2 \sin\theta \, d\phi \, d\theta \, dr};$$

$\bar{x}$ and $\bar{y}$ each $= 0$.

Curve.

132. Suppose a circle of variable radius to move so that its centre describes a given curve and its plane is always perpendicular to the tangent line of the curve, we may require the centre of gravity of the solid generated. The simplest case is that in which the radius is constant and the solid of uniform density; the result depends *solely* on the nature of the curve described by the centre of the circle, and for shortness the process is called finding *the centre of gravity of a curve.*

Let $BPQE$ be a *plane* curve; BP the length measured from some fixed point B, $BP=s$, $PQ=\Delta s$; x, y the co-ordinates of P. Let k denote the area of a transverse section; then the volume of the element PQ is $k\Delta s$, and the co-ordinates of its centre of gravity are ultimately x and y. Hence



$$\bar{x}=\frac{\int kxds}{\int kds}=\frac{\int xds}{\int ds} \ldots (1) \text{ if } k \text{ be constant,}$$

$$\bar{y}=\frac{\int kyds}{\int kds}=\frac{\int yds}{\int ds} \ldots (2).$$

Since $\frac{ds}{dx}=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$, we may also write

$$\bar{x}=\frac{\int x\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}dx}{\int\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}dx},\quad \bar{y}=\frac{\int y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}dx}{\int\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}dx} \ldots (3).$$

From the equation to the curve y and $\frac{dy}{dx}$ are known in terms of x; their values must be substituted in the preceding expressions and the integrations then effected.

If we use polar co-ordinates we have $x = r\cos\theta$, $y = r\sin\theta$, and $\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}$.

Hence

$$\bar{x} = \frac{\int r\cos\theta \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta}{\int \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta}, \quad \bar{y} = \frac{\int r\sin\theta \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta}{\int \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta} \quad \ldots\ldots (4);$$

for r and $\frac{dr}{d\theta}$ we must substitute their values in terms of θ given by the equation to the curve.

133. Ex. 1. A straight rod of uniform thickness and density.

Taking the origin on the line we have $y = \beta x$, where β is constant; hence, by equations (3) of Art. 132, supposing the origin to be at one end of the rod and h the abscissa of the other end,

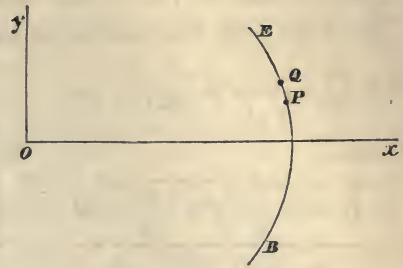
$$\bar{x} = \frac{\int_0^h x\,dx}{\int_0^h dx} = \frac{h}{2}, \quad \bar{y} = \frac{\beta\int_0^h x\,dx}{\int_0^h dx} = \frac{\beta h}{2}.$$

That is, the centre of gravity is the middle point of the rod.

Ex. 2. Suppose the transverse section of the rod to vary as the n^{th} power of the distance from one end. Take the origin at this end, and suppose the axis of x to coincide with the axis of the rod; then $\bar{y} = 0$, and in equation (1) of Art. 132 we put μx^n for k, where μ is constant. Hence, if h be the length of the rod,

$$\bar{x} = \frac{\int_0^h x^{n+1} ds}{\int_0^h x^n ds} = \frac{\int_0^h x^{n+1} dx}{\int_0^h x^n dx} = \frac{n+1}{n+2} h.$$

Ex. 3. An arc of a circle.



Take the origin at the centre of the circle, and the axis of x bisecting the arc. Then $\bar{y}=0$; and supposing 2α to be the angle subtended at O by the given arc, and a the radius of the circle, we have, by Art. 132, equation (4),

$$\bar{x}=\frac{a^2\int_{-\alpha}^{\alpha}\cos\theta\,d\theta}{a\int_{-\alpha}^{\alpha}d\theta}=\frac{a\sin\alpha}{\alpha}.$$

Ex. 4. The arc of a semicycloid.

Take the origin at the vertex, and the axis of y a tangent there; then $\left(\frac{dy}{dx}\right)^2=\frac{2a-x}{x}$: hence

$$\bar{x}=\frac{\int_0^{2a}x\sqrt{\left(\frac{2a}{x}\right)}dx}{\int_0^{2a}\sqrt{\left(\frac{2a}{x}\right)}dx}=\frac{\int_0^{2a}x^{\frac{1}{2}}dx}{\int_0^{2a}x^{-\frac{1}{2}}dx}=\frac{\frac{2}{3}(2a)^{\frac{3}{2}}}{2(2a)^{\frac{1}{2}}}=\frac{2a}{3},$$

$$\bar{y}=\frac{\int_0^{2a}y\sqrt{\left(\frac{2a}{x}\right)}dx}{\int_0^{2a}\sqrt{\left(\frac{2a}{x}\right)}dx}=\frac{\int_0^{2a}\frac{y}{\sqrt{x}}dx}{\int_0^{2a}\frac{dx}{\sqrt{x}}}.$$

Now $\int\frac{y}{\sqrt{x}}dx=2\int y\frac{d\sqrt{x}}{dx}dx=2y\sqrt{x}-2\int\sqrt{x}\frac{dy}{dx}dx$

$$=2y\sqrt{x}-2\int\sqrt{(2a-x)}\,dx=2y\sqrt{x}+\tfrac{4}{3}(2a-x)^{\frac{3}{2}};$$

therefore $$\int_0^{2a}\frac{y}{\sqrt{x}}dx=2\pi a(2a)^{\frac{1}{2}}-\tfrac{4}{3}(2a)^{\frac{3}{2}};$$

therefore $$\bar{y}=\frac{2\pi a(2a)^{\frac{1}{2}}-\frac{4}{3}(2a)^{\frac{3}{2}}}{2(2a)^{\frac{1}{2}}}=(\pi-\tfrac{4}{3})a.$$

Ex. 5. The curve $y = \frac{1}{2}c\,(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$.

If s' denote the length of an arc of the curve measured from the point whose co-ordinates are 0, c, to the point (x', y'), we have for the co-ordinates of its centre of gravity

$$\bar{x} = \frac{\int_0^{x'} x\frac{ds}{dx}\,dx}{s'}, \quad \bar{y} = \frac{\int_0^{x'} y\frac{ds}{dx}\,dx}{s'}.$$

Now $$\frac{dy}{dx} = \frac{1}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}}),$$

therefore $$1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}(e^{\frac{x}{c}} + e^{-\frac{x}{c}})^2,$$

and $$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \frac{1}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}});$$

thus $$s = \frac{c}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}}), \text{ and } s' = \frac{c}{2}(e^{\frac{x'}{c}} - e^{-\frac{x'}{c}}).$$

Also $$\int x\frac{ds}{dx}\,dx = \frac{1}{2}\int x\,(e^{\frac{x}{c}} + e^{-\frac{x}{c}})\,dx$$

$$= \frac{cx}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}}) - \frac{c}{2}\int(e^{\frac{x}{c}} - e^{-\frac{x}{c}})\,dx$$

$$= \frac{cx}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}}) - \frac{c^2}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}});$$

therefore $$\int_0^{x'} x\frac{ds}{dx}\,dx = \frac{cx'}{2}(e^{\frac{x'}{c}} - e^{-\frac{x'}{c}}) - \frac{c^2}{2}(e^{\frac{x'}{c}} + e^{-\frac{x'}{c}}) + c^2$$

$$= x's' - cy' + c^2,$$

and $$\bar{x} = x' - \frac{c\,(y' - c)}{s'}.$$

Also $$\int y\frac{ds}{dx}\,dx = \frac{c}{4}\int(e^{\frac{x}{c}} + e^{-\frac{x}{c}})\,(e^{\frac{x}{c}} + e^{-\frac{x}{c}})\,dx$$

$$=\frac{c}{4}\int\left(e^{\frac{2x}{c}}+2+e^{-\frac{2x}{c}}\right)dx=\frac{c^2}{8}\left(e^{\frac{2x}{c}}-e^{-\frac{2x}{c}}\right)+\frac{cx}{2};$$

therefore
$$\int_0^{x'} y\frac{ds}{dx}\,dx=\frac{c^2}{8}\left(e^{\frac{2x'}{c}}-e^{-\frac{2x'}{c}}\right)+\frac{cx'}{2}$$

$$=\frac{y's'}{2}+\frac{cx'}{2},$$

and
$$\bar{y}=\frac{y'}{2}+\frac{cx'}{2s'}.$$

134. If the curve be of double curvature, the formulæ (1) and (2) of Art. 132 still hold; in order to effect the integrations we may use the formula

$$\frac{ds}{dz}=\sqrt{\left\{1+\left(\frac{dx}{dz}\right)^2+\left(\frac{dy}{dz}\right)^2\right\}};$$

and from the two equations to the curve we must find $\frac{dx}{dz}$ and $\frac{dy}{dz}$ in terms of z. (See *Integral Calculus*, Art. 120.) For example, in the helix

$$x=a\cos nz,\quad y=a\sin nz;$$

therefore
$$\frac{ds}{dz}=\sqrt{(1+n^2a^2)},$$

$$\bar{x}=\frac{\int\sqrt{(1+n^2a^2)}\,x\,dz}{\int\sqrt{(1+n^2a^2)}\,dz}=\frac{\int a\cos nz\,dz}{\int dz}.$$

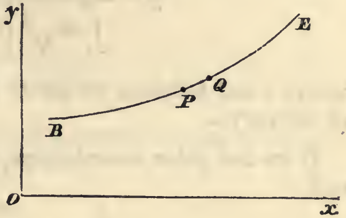
If we take for the limits $z=0$ and $z=h$, we have

$$\bar{x}=\frac{a\sin nh}{nh}.$$

Similarly
$$\bar{y}=\frac{a(1-\cos nh)}{nh},\quad \bar{z}=\tfrac{1}{2}h.$$

Surface of Revolution.

135. Let $BPQE$ be a curve which by revolving round the axis of x generates a surface. Suppose a shell of which this surface is the exterior boundary, and of which the interior boundary is another surface of revolution round the axis of x indefinitely near to the former. Required the centre of gravity of a portion of this shell cut off by planes perpendicular to the axis of x.



Let P, Q, be adjacent points in the exterior generating curve; suppose B a fixed point in the curve, let $BP = s$, and $PQ = \Delta s$; let x, y be the co-ordinates of P; k the thickness of the shell at P. The volume of the element contained between two planes perpendicular to the axis of x through P and Q respectively is ultimately $2\pi yk\Delta s$, and the abscissa of the centre of gravity of this element is ultimately x; hence

$$\bar{x} = \frac{\int 2\pi ykx\,ds}{\int 2\pi yk\,ds} = \frac{\int yx\,ds}{\int y\,ds},$$

if k be constant.

Since $\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$, we have

$$x = \frac{\int_c^h yx\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\,dx}{\int_c^h y\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\,dx},$$

where c and h are the distances from the origin of the bounding planes.

Since the centre of gravity required is on the axis of x, we need only the value of $\bar{x}$ in order to determine its position.

Similarly, if the curve $BPQE$ generates a surface by revolving round the axis of y, we have

$$\bar{y} = \frac{\int_c^h xy\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx}{\int_c^h x\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx},$$

where c and h denote as before the abscissæ of the extremities of the curve.

If we use polar co-ordinates, we have $x = r\cos\theta$, $y = r\sin\theta$, and

$$\frac{ds}{d\theta} = \sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}};$$

thus if the curve revolves round the axis of x, we have

$$\bar{x} = \frac{\int r^2\sin\theta\cos\theta\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta}{\int r\sin\theta\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta},$$

and if the curve revolves round the axis of y, we have

$$\bar{y} = \frac{\int r^2\cos\theta\sin\theta\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta}{\int r\cos\theta\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta}.$$

The limits of the integrations are the values of θ which correspond to the extremities of the curve.

Ex. 1. A cylindrical surface.

Take the axis of the cylinder as the axis of x; then y = the radius of the cylinder, and is constant; hence

$$\bar{x} = \frac{\int_c^h x\,dx}{\int_c^h dx} = \frac{\frac{1}{2}(h^2-c^2)}{h-c} = \frac{h+c}{2}.$$

Ex. 2. A spherical surface.

Here $$y = \sqrt{(a^2 - x^2)},$$

$$\frac{dy}{dx} = -\frac{x}{\sqrt{(a^2 - x^2)}},$$

$$\frac{ds}{dx} = \frac{a}{\sqrt{(a^2 - x^2)}} = \frac{a}{y};$$

therefore $$\bar{x} = \frac{\int_c^h ax\,dx}{\int_c^h a\,dx} = \frac{c+h}{2}.$$

Hence in both these examples the centre of gravity is equidistant from the two bounding planes.

Ex. 3. The surface of a cone.

Here $y = x \tan \alpha$, where α is the semivertical angle,

$$\frac{ds}{dx} = \sec \alpha,$$

$$\bar{x} = \frac{\int_c^h x \tan \alpha\, x \sec \alpha\, dx}{\int_c^h x \tan \alpha \sec \alpha\, dx} = \frac{2(h^3 - c^3)}{3(h^2 - c^2)} = \frac{2(h^2 + hc + c^2)}{3(h + c)}.$$

Ex. 4. Suppose the cycloid

$$y = \sqrt{(2ax - x^2)} + a \text{ vers}^{-1} \frac{x}{a}$$

to revolve round the axis of x.

Here $$\frac{dy}{dx} = \sqrt{\left(\frac{2a - x}{x}\right)}, \quad \frac{ds}{dx} = \sqrt{\left(\frac{2a}{x}\right)};$$

thus $$\bar{x} = \frac{\int_0^{2a} yx\sqrt{\left(\frac{2a}{x}\right)} dx}{\int_0^{2a} y\sqrt{\left(\frac{2a}{x}\right)} dx} = \frac{\int_0^{2a} yx^{\frac{1}{2}}\,dx}{\int_0^{2a} yx^{-\frac{1}{2}}\,dx}.$$

Now $$\int yx\; dx = \frac{2yx^{\frac{3}{2}}}{3} - \frac{2}{3}\int x^{\frac{3}{2}} \frac{dy}{dx} dx$$

$$= \frac{2yx^{\frac{3}{2}}}{3} - \frac{2}{3}\int x\sqrt{(2a - x)}\,dx;$$

therefore $$\int_0^{2a} yx^{\frac{1}{2}}\,dx = \frac{2\pi a\,(2a)^{\frac{3}{2}}}{3} - \frac{2}{3}\int_0^{2a} x\sqrt{(2a-x)}\,dx;$$

and $$\int x\sqrt{(2a-x)}\,dx = -\frac{2x\,(2a-x)^{\frac{3}{2}}}{3} + \frac{2}{3}\int (2a-x)^{\frac{3}{2}}\,dx$$
$$= -\frac{2x\,(2a-x)^{\frac{3}{2}}}{3} - \frac{4}{15}(2a-x)^{\frac{5}{2}},$$

therefore $$\int_0^{2a} x\sqrt{(2a-x)}\,dx = \frac{4}{15}(2a)^{\frac{5}{2}};$$

thus $$\int_0^{2a} yx^{\frac{1}{2}}\,dx = \frac{2\pi a\,(2a)^{\frac{3}{2}}}{3} - \frac{8}{45}(2a)^{\frac{5}{2}}.$$

Also $$\int_0^{2a} yx^{-\frac{1}{2}}\,dx = 2\pi a\,(2a)^{\frac{1}{2}} - \frac{4}{3}(2a)^{\frac{3}{2}},$$ (see page 126),

therefore $$\bar{x} = \frac{\dfrac{2\pi a\,(2a)^{\frac{3}{2}}}{3} - \dfrac{8}{45}(2a)^{\frac{5}{2}}}{2\pi a\,(2a)^{\frac{1}{2}} - \dfrac{4}{3}(2a)^{\frac{3}{2}}}$$

$$= a\,\frac{\dfrac{4\pi}{3} - \dfrac{32}{45}}{2\pi - \dfrac{8}{3}} = \frac{\dfrac{2a}{3}\left(\pi - \dfrac{8}{15}\right)}{\pi - \dfrac{4}{3}}.$$

Ex. 5. Suppose the cycloid

$$y = \sqrt{(2ax - x^2)} + a\,\text{vers}^{-1}\frac{x}{a}$$

to revolve round the axis of y, and that we require the centre of gravity of the surface generated by that half of the curve for which y is positive.

Here $$\bar{y} = \frac{\displaystyle\int_0^{2a} yx\sqrt{\left(\frac{2a}{x}\right)}\,dx}{\displaystyle\int_0^{2a} x\sqrt{\left(\frac{2a}{x}\right)}\,dx} = \frac{\displaystyle\int_0^{2a} yx^{\frac{1}{2}}dx}{\displaystyle\int_0^{2a} x^{\frac{1}{2}}dx}.$$

The value of the numerator was found in the preceding example; and

$$\int_0^{2a} x^{\frac{1}{2}}dx = \frac{2}{3}(2a)^{\frac{3}{2}},$$

therefore $$\bar{y} = \frac{\frac{2\pi a}{3}(2a)^{\frac{3}{2}} - \frac{8}{45}(2a)^{\frac{5}{2}}}{\frac{2}{3}(2a)^{\frac{3}{2}}}$$

$$= a\left(\pi - \frac{8}{15}\right).$$

Ex. 6. Find the centre of gravity of the surface formed by revolving the curve $r = a(1 + \cos\theta)$ round the initial line. Here

$$\frac{dr}{d\theta} = -a\sin\theta, \quad r^2 + \left(\frac{dr}{d\theta}\right)^2 = 2a^2(1 + \cos\theta),$$

therefore $$\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} = 2a\cos\frac{\theta}{2}.$$

Thus $$\bar{x} = \frac{\int_0^\pi r^2 \sin\theta\cos\theta\, 2a\cos\frac{\theta}{2}\, d\theta}{\int_0^\pi r\sin\theta\, 2a\cos\frac{\theta}{2}\, d\theta}$$

$$= \frac{2a\int_0^\pi \cos^6\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2} - 1\right)\sin\frac{\theta}{2}\, d\theta}{\int_0^\pi \cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\, d\theta}.$$

Now $$\int \cos^6\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2} - 1\right)\sin\frac{\theta}{2}\, d\theta = -\frac{4}{9}\cos^9\frac{\theta}{2} + \frac{2}{7}\cos^7\frac{\theta}{2};$$

therefore $$\int_0^\pi \cos^6\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2} - 1\right)\sin\frac{\theta}{2}\, d\theta = \frac{4}{9} - \frac{2}{7}.$$

Similarly $$\int_0^\pi \cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\, d\theta = \frac{2}{5},$$

therefore $$\bar{x} = \frac{2a\left(\frac{4}{9} - \frac{2}{7}\right)}{\frac{2}{5}} = \frac{50a}{63}.$$

Any Surface.

136. Let there be a shell having any given surface for one of its boundaries, and suppose its thickness indefinitely

small. Let x, y, z be the co-ordinates of any point of the given surface, k the thickness at that point, ΔS the area of an element of the surface there, then $k\Delta S$ is ultimately the volume of this element, and x, y, z the co-ordinates of its centre of gravity; hence

$$\bar{x} = \frac{\int kx\, dS}{\int k\, dS},$$

and similar expressions hold for $\bar{y}$ and $\bar{z}$.

It may be shewn (see *Integral Calculus*, Art. 170) that if we take ΔS such that its projection on the plane of (x, y) is the rectangle $\Delta x\, \Delta y$,

$$\frac{\Delta S}{\Delta x\, \Delta y} = \sqrt{\left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}} \text{ ultimately.}$$

Hence $$\bar{x} = \frac{\iint kx \sqrt{\left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}}\, dx\, dy}{\iint k \sqrt{\left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}}\, dx\, dy}.$$

Ex. The surface of the eighth part of a sphere.

Here $$x^2 + y^2 + z^2 = a^2,$$

$$\sqrt{\left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}} = \frac{a}{\sqrt{(a^2 - x^2 - y^2)}}.$$

Hence $$\bar{x} = \frac{\iint \frac{x\, dx\, dy}{\sqrt{(a^2 - x^2 - y^2)}}}{\iint \frac{dx\, dy}{\sqrt{(a^2 - x^2 - y^2)}}}.$$

First integrate with respect to y from $y = 0$ to $y = \sqrt{(a^2 - x^2)}$; we thus include all the elements that form the strip of surface of which $LlmM$ is the projection on the plane of (x, y); see fig. to Art. 128.

Now $$\int_0^{\sqrt{(a^2 - x^2)}} \frac{dy}{\sqrt{(a^2 - x^2 - y^2)}} = \tfrac{1}{2}\pi;$$

therefore $$\bar{x} = \frac{\int \frac{1}{2}\pi x\, dx}{\int \frac{1}{2}\pi\, dx} = \frac{\int x\, dx}{\int dx}.$$

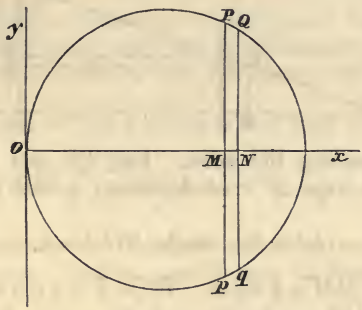
The limits of the integration for x are 0 and a;

therefore $$\bar{x} = \tfrac{1}{2}a.$$

Similarly $$\bar{y} = \tfrac{1}{2}a, \quad \bar{z} = \tfrac{1}{2}a.$$

137. In the preceding articles we have given the usual formulæ for finding the centres of gravity of bodies, but particular cases may occur which may be most conveniently treated by special methods. We add some examples.

(1) A circle revolves round a tangent line through an



angle of 180°; find the centre of gravity of the solid generated. Let Oy be the tangent line about which the circle revolves, and let the plane of the paper bisect the solid; the centre of gravity will therefore lie in the axis of x. Let $OM=x$, $MP=y=\sqrt{(2ax-x^2)}$, $MN=\Delta x$. The figure $PQqp$ will by its revolution generate a semi-cylindrical shell, whose volume is ultimately $2y\pi x\Delta x$; the centre of gravity of this shell will be in the axis of x at a distance $\frac{2x}{\pi}$ from O (see Art. 133, Ex. 3);

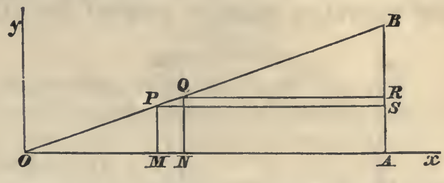
therefore
$$\bar{x}=\frac{\int_0^{2a}\frac{2x}{\pi}2y\pi x\,dx}{\int_0^{2a}2y\pi x\,dx}=\frac{\frac{2}{\pi}\int_0^{2a}yx^2\,dx}{\int_0^{2a}yx\,dx}$$

$$=\frac{2}{\pi}\frac{\int_0^{2a}x^2\sqrt{(2ax-x^2)}\,dx}{\int_0^{2a}x\sqrt{(2ax-x^2)}\,dx}.$$

It will be found that $\bar{x}=\frac{5a}{2\pi}$.

(2) The density of a right cone varies as the n^{th} power of the distance from the axis; find the centre of gravity of the cone.

Let OAB be the right-angled triangle which by revolving



round Ox generates the cone. Let PS and QR be drawn parallel to the axis of x at distances y and $y + \Delta y$ respectively. Let

$$OA = h, \quad \text{angle } BOA = \alpha.$$

Then $$OM = y \cot \alpha, \quad PS = h - y \cot \alpha.$$

The volume of the cylindrical shell generated by the revolution of $PQRS$ round Ox is ultimately

$$2\pi y \, \Delta y \, (h - y \cot \alpha).$$

Its density is μy^n, where μ is constant; therefore, its mass is

$$2\pi \mu y^{n+1} \Delta y \, (h - y \cot \alpha).$$

The distance of its centre of gravity from O is ultimately (see Art. 135, Ex. 1)

$$\tfrac{1}{2}(OM + OA), \text{ that is } \tfrac{1}{2}(h + y \cot \alpha);$$

therefore $$\bar{x} = \frac{\int_0^{h \tan \alpha} 2\pi \mu y^{n+1} (h - y \cot \alpha) \tfrac{1}{2}(h + y \cot \alpha) \, dy}{\int_0^{h \tan \alpha} 2\pi \mu y^{n+1} (h - y \cot \alpha) \, dy}$$

$$= \frac{\frac{1}{2}\int_0^{h \tan \alpha} y^{n+1} (h^2 - y^2 \cot^2 \alpha) \, dy}{\int_0^{h \tan \alpha} y^{n+1} (h - y \cot \alpha) \, dy};$$

and the integrations can be easily performed.

(3) A shell has for its outer and inner boundaries two similar and similarly situated ellipsoids; required the centre of gravity of the eighth part of it included between three principal planes. Let a, b, c be the semiaxes of the exterior ellipsoid, ra, rb, rc those of the inner ellipsoid, r being a quantity less than unity.

If a, b, c be the semiaxes of an ellipsoid; the volume of the eighth part is $\frac{1}{6}\pi abc$, and the co-ordinates of its centre of gravity are $\frac{3}{8}a$, $\frac{3}{8}b$, and $\frac{3}{8}c$ (see Art. 129). Hence

$$\tfrac{3}{8}a \,.\, \tfrac{1}{6}\pi abc = \tfrac{3}{8}ra \,.\, \tfrac{1}{6}\pi r^3 abc + \bar{x}\left(\tfrac{1}{6}\pi abc - \tfrac{1}{6}\pi r^3 abc\right);$$

therefore $$\bar{x} = \frac{\frac{3}{8}a(1-r^4)}{1-r^3} = \tfrac{3}{8}a \,.\, \frac{1+r+r^2+r^3}{1+r+r^2}.$$

If we suppose the shell indefinitely thin, we must put $r=1$, and then $\bar{x} = \frac{1}{2}a$. Similar results may be found for $\bar{y}$ and $\bar{z}$.

(4) An ellipsoid is composed of an infinite number of indefinitely thin shells; each shell has for its outer and inner boundaries two similar and similarly situated ellipsoids; the density of each shell is constant, but the density varies from shell to shell according to a given law; determine the centre of gravity of the eighth part of the ellipsoid included between three principal planes.

Let x, y, z represent the three semi-axes of an ellipsoid; then the volume of the ellipsoid is $\frac{4\pi}{3}xyz$. Suppose that $y = mx$ and $z = nx$, where m and n are constants, then the volume becomes $\frac{4\pi mn}{3}x^3$, and if there be a *similar* ellipsoid having $x + \Delta x$ for the semi-axis corresponding to the semi-axis x of the first ellipsoid, the volume of the second ellipsoid will be $\frac{4\pi mn}{3}(x+\Delta x)^3$. Hence the volume of a shell bounded by two similar and similarly situated ellipsoids may be denoted by $\frac{4\pi mn}{3}\{(x+\Delta x)^3 - x^3\}$, and therefore by $4\pi mnx^2\Delta x$ when the thickness is indefinitely diminished. Let $\phi(x)$ denote the density of the shell, then its mass is $4\pi mn\phi(x)\,x^2\Delta x$.

Thus the mass of the eighth part of the shell is $\frac{\pi mn}{2}\phi(x)\,x^2\Delta x$. And the abscissa of the centre of gravity of the shell measured along the semi-axis x is $\frac{x}{2}$, by the preceding example. Thus for the abscissa $\bar{x}$ of the centre of gravity we have

$$\bar{x} = \frac{\int_0^a \frac{\pi mn}{2}\phi(x)\,\frac{x}{2}\,x^2\,dx}{\int_0^a \frac{\pi mn}{2}\phi(x)\,x^2\,dx}$$

$$= \frac{\frac{1}{2}\int_0^a x^3\phi(x)\,dx}{\int_0^a x^2\phi(x)\,dx},$$

where a is the semi-axis of the external surface corresponding to the semi-axis x. When $\phi(x)$ is given the integrations may be completed; and when $\bar{x}$ is known, the other co-ordinates of the centre of gravity may be inferred from symmetry.

(5) A chord of an ellipse cuts off a segment of constant area; determine the locus of the centre of gravity of the segment.

If a chord cuts off a segment of constant area from a *circle*, it is evident from the symmetry of the figure that the locus of the centre of gravity of the segment is a *concentric circle.* Now if the circle be projected orthogonally upon a plane inclined to the plane of the circle the circle projects into an ellipse; and the segments of the circle of constant area project into segments of the ellipse of constant area; also the *concentric circle* projects into a second ellipse similar to the first ellipse and similarly situated.

Thus the required locus is an ellipse similar to the given ellipse and similarly situated.

This problem might have been solved without making use of projections, in the manner shewn in the next example.

(6) A plane cuts off from an ellipsoid a segment of constant volume; determine the locus of the centre of gravity of the segment.

Let the cutting plane have any position; and refer the ellipsoid to conjugate semi-diameters as axes; let the plane of (y, z) be parallel to the position of the cutting plane, and suppose the equation to the ellipsoid to be

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1.$$

Now suppose the segment cut off by the plane to be divided into an indefinitely large number of indefinitely thin slices by planes parallel to the plane of (y, z). By the properties of the ellipsoid these slices will be bounded by ellipses which have their centres on the axis of x; and thus we see that the centre of gravity of the segment cut off will be on the axis of x. Consider one of the slices bounded by planes which have for their abscissæ x and $x + \Delta x$ respectively; then it will be found that the volume of the slice is ultimately

$$\pi b'c' \left(1 - \frac{x^2}{a'^2}\right) \sin \omega \sin \alpha \Delta x,$$

where ω is the angle between the axes of y and z, and α is the angle which the axis of x makes with the plane of (y, z). Suppose V to denote the constant volume, and $\lambda a'$ the abscissa of the plane cutting off the segment; then

$$V = \pi b'c' \sin \omega \sin \alpha \int_{\lambda a'}^{a'} \left(1 - \frac{x^2}{a'^2}\right) dx$$

$$= \pi a'b'c' \sin \omega \sin \alpha \left\{1 - \lambda - \frac{1}{3}(1 - \lambda^3)\right\}.$$

Now by the properties of the ellipsoid

$$\pi a'b'c' \sin \omega \sin \alpha = \pi abc,$$

where a, b, c are the semi-axes of the ellipsoid; thus

$$V = \pi abc \left\{1 - \lambda - \frac{1}{3}(1 - \lambda^3)\right\} \quad \text{.................. (1)}$$

And, if $\bar{x}$ be the abscissa of the centre of gravity of the segment cut off,

$$\bar{x} = \frac{\pi b'c' \sin\omega \sin\alpha \int_{\lambda a'}^{a'} x\left(1 - \frac{x^2}{a'^2}\right) dx}{V}$$

$$= \frac{\pi a'^2 b'c' \sin\omega \sin\alpha}{V} \left\{\frac{1}{2}(1-\lambda^2) - \frac{1}{4}(1-\lambda^4)\right\}$$

$$= \frac{\pi abc}{V} \left\{\frac{1}{2}(1-\lambda^2) - \frac{1}{4}(1-\lambda^4)\right\} a' \ldots\ldots\ldots\ldots\ldots (2).$$

Now (1) gives a constant value for λ, and then (2) shews that $\bar{x}$ *bears a constant ratio to* a'.

Thus the locus of the centre of gravity of segments of an ellipsoid of constant volume is an ellipsoid similar to the original ellipsoid and similarly situated.

(7) Find the centre of gravity of a portion of an ellipsoid comprised between two cones whose common vertex is at the centre of the ellipsoid and whose bases are parallel.

The volume between the two cones may be divided into an indefinitely large number of shells which have the centre of the ellipsoid as their common vertex, and their bases in planes parallel to the bases of the two cones. We shall first shew that if the planes which contain the bases of the shells are equidistant *the shells are all equal.* Take conjugate semi-diameters as axes, and let the plane of (y, z) be parallel to the bases of the two cones. The volume of the cone which has the centre of the ellipsoid as vertex, and for its base the plane curve formed by the intersection of the ellipsoid with the plane which has x for its abscissa, is

$$\tfrac{1}{3}\pi b'c' \sin\omega \sin\alpha \left(1 - \frac{x^2}{a'^2}\right) x,$$

where the notation is the same as in the preceding example. The volume of the cone which has the centre of the ellipsoid as vertex, and for its base the plane curve formed by the intersection of the ellipsoid with the plane which has $x + \Delta x$ for its abscissa, is

$$\tfrac{1}{3}\pi b'c' \sin\omega \sin\alpha \left\{1 - \frac{(x+\Delta x)^2}{a'^2}\right\} (x + \Delta x).$$

The volume of the slice between the planes whose abscissæ are x and $x+\Delta x$ respectively is ultimately

$$\pi b'c' \sin\omega \sin\alpha \left(1-\frac{x^2}{a'^2}\right)\Delta x.$$

Hence we obtain for the volume of one of the shells ultimately the product of $\pi b'c' \sin\alpha \sin\omega$ by

$$\left[\tfrac{1}{3}\left(1-\frac{x^2}{a'^2}\right)x+\left(1-\frac{x^2}{a'^2}\right)\Delta x-\tfrac{1}{3}\left\{1-\frac{(x+\Delta x)^2}{a'^2}\right\}(x+\Delta x)\right];$$

this product is ultimately

$$\frac{2\pi b'c' \sin\omega \sin\alpha \Delta x}{3}, \text{ or } \frac{2\pi bc \sin\alpha \Delta x}{3}.$$

The centre of gravity of each shell is on the axis of x at a distance from the vertex of the cone, which is equal to three fourths of the abscissa of the plane in which the base of the cone is situated (see Ex. (5) of Art. 109). Let $\bar{x}$ denote the abscissa of the centre of gravity of the proposed solid; then if h and k be the abscissæ of the plane bases of the two cones,

$$\bar{x}=\frac{\frac{3}{4}\int_h^k \frac{2\pi bc\sin\alpha}{3}x\,dx}{\int_h^k \frac{2\pi bc\sin\alpha}{3}dx}=\frac{3\,(k^2-h^2)}{8\,(k-h)}=\frac{3}{8}(k+h).$$

We shall conclude this chapter with a few general propositions involving properties of the centre of gravity.

138. *If the mass of each of a system of particles be multiplied into the square of its distance from a given point, the sum of the products is least when the given point is the centre of gravity of the system.*

Let the centre of gravity of the system be made the origin; let α, β, γ, be the co-ordinates of the given point; x_1, y_1, z_1, the co-ordinates of the first particle; x_2, y_2, z_2, those of the second; and so on; m_1, m_2, ... the masses of the particles; ρ_1, ρ_2, ... the distances of the particles from their centre of gravity; r_1, r_2, ... the distances of the particles from the fixed point; then

$$r_1^2 = \alpha^2 + \beta^2 + \gamma^2 - 2(\alpha x_1 + \beta y_1 + \gamma z_1) + \rho_1^2,$$
$$r_2^2 = \alpha^2 + \beta^2 + \gamma^2 - 2(\alpha x_2 + \beta y_2 + \gamma z_2) + \rho_2^2,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Multiply these equations by m_1, m_2, m_3, ... respectively, and add; then

$$\Sigma mr^2 = (\alpha^2 + \beta^2 + \gamma^2)\,\Sigma m - 2(\alpha\Sigma mx + \beta\Sigma my + \gamma\Sigma mz) + \Sigma m\rho^2.$$

But, since the origin is the centre of gravity of the system,

$$\Sigma mx = 0, \quad \Sigma my = 0, \quad \Sigma mz = 0,$$

therefore $\quad \Sigma mr^2 = (\alpha^2 + \beta^2 + \gamma^2)\,\Sigma m + \Sigma m\rho^2.$

Now $\Sigma m\rho^2$ is independent of the position of the given point; hence the least value of Σmr^2 is that which it has when $\alpha^2 + \beta^2 + \gamma^2$ vanishes, that is, when the given point is at the centre of gravity of the system.

139. Let α_1, β_1, γ_1, be the angles which ρ_1 makes with the axes; α_2, β_2, γ_2, the angles ρ_2 makes with the axes; and so on; then we have, supposing the origin the centre of gravity of the system,

$$\Sigma m\rho \cos\alpha = 0, \quad \Sigma m\rho \cos\beta = 0, \quad \Sigma m\rho \cos\gamma = 0.$$

Square each of these equations and add the results; then if m, m' represent any two masses, and (ρ, ρ') the angle between the lines which join them with the centre of gravity,

$$\Sigma m^2\rho^2 + 2\Sigma mm'\rho\rho' \cos(\rho, \rho') = 0.$$

But $\quad 2\rho\rho' \cos(\rho, \rho') = \rho^2 + \rho'^2 - u^2,$

where u denotes the distance of m and m'. Hence

$$\Sigma m^2\rho^2 + \Sigma mm'(\rho^2 + \rho'^2 - u^2) = 0.$$

If we select the coefficient of ρ_1^2, we find it to be

$$m_1^2 + m_1(m_2 + m_3 + \dots), \text{ or } m_1\Sigma m,$$

and the other coefficients are similar. Hence the above equation may be written

$$\Sigma m\Sigma m\rho^2 = \Sigma mm'u^2.$$

140. *If a particle be acted on by a number of forces each passing through a fixed point and proportional to the distance from that point, the resultant force will pass through a fixed point and be proportional to the distance from that point.*

Take any position of the particle as the origin; let x_1, y_1, z_1, be the co-ordinates of a fixed point; r_1 the distance of this point from the origin; $\mu_1 r_1$ the force which acts on the particle from this fixed point. Similarly let x_2, y_2, z_2, be the co-ordinates of a second fixed point; r_2 its distance from the origin, and $\mu_2 r_2$ the corresponding force on the particle, and so on. Let X, Y, Z denote the whole force acting on the particle along the axes of x, y, z; then, by Art. 26,

$$X = \mu_1 r_1 \times \frac{x_1}{r_1} + \mu_2 r_2 \times \frac{x_2}{r_2} + \mu_3 r_3 \times \frac{x_3}{r_3} + \ldots\ldots$$

$$= \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 + \ldots\ldots$$

Similarly

$$Y = \mu_1 y_1 + \mu_2 y_2 + \mu_3 y_3 + \ldots\ldots$$

$$Z = \mu_1 z_1 + \mu_2 z_2 + \mu_3 z_3 + \ldots\ldots$$

Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of the centre of gravity of a system of particles, whose masses are proportional to μ_1, μ_2, μ_3, ... placed at the respective fixed points; then

$$\bar{x} = \frac{\Sigma \mu x}{\Sigma \mu}, \quad \bar{y} = \frac{\Sigma \mu y}{\Sigma \mu}, \quad \bar{z} = \frac{\Sigma \mu z}{\Sigma \mu};$$

therefore $\quad X = \bar{x}\Sigma\mu, \quad Y = \bar{y}\Sigma\mu, \quad Z = \bar{z}\,\Sigma\mu.$

These equations shew that the resultant force is equal to $\bar{r}\Sigma\mu$, where $\bar{r}$ is the distance of the centre of gravity from the origin, and that its direction passes through the centre of gravity. Hence when the particle is situated at the centre of gravity the resultant force vanishes and the particle is in equilibrium.

141. *A body is placed on a horizontal plane, to find when it will be supported.*

The only force acting on it besides the resistance of the plane is its own weight, and this acts in a vertical direction through the centre of gravity of the body. Hence, by Art. 91, the body will not be in equilibrium unless the vertical through the centre of gravity of the body falls within a polygon formed by so joining the points of contact of the body and the plane as to include them all and have no re-entering angle.

142. *When a body is suspended from a point round which it can move freely, it will not rest unless its centre of gravity be in the vertical line passing through the point of suspension.*

For the body is acted on by two forces, its own weight which acts vertically through its centre of gravity and the force arising from the fixed point; for equilibrium these forces must act in the same straight line and in opposite directions; thus the centre of gravity must be in the vertical line passing through the point of suspension.

Hence if a body be suspended successively from two points the vertical lines drawn through the points of suspension will both pass through the centre of gravity; therefore the point in which they intersect is the centre of gravity.

If a body be capable of revolving round an axis it will not rest unless the centre of gravity be in the vertical plane passing through the axis. For the body is acted on by its own weight and the forces arising from the fixed points; by Art. 87, the moment of the weight round the fixed axis must vanish, this requires the centre of gravity to be in the vertical plane through the fixed axis.

The student will readily perceive as an experimental fact that there is an important difference between the position of equilibrium in which the centre of gravity is vertically *above* the fixed point or fixed axis, and that in which it is vertically *below* it. In the former case, if the body be slightly disturbed from its equilibrium position and then left to itself, it will begin to recede from its original position. In the latter case, if the body be slightly disturbed from its equilibrium position and then left to itself, it will begin to return to its original position. The former position of equilibrium is called *unstable*, and the latter *stable*. We shall return to this point in Chap. XIV.

143. The volume (V) of a portion of a cylinder intercepted between two planes, one of which is perpendicular to the axis of the cylinder, is given by the equation

$$V = \iint z\,dx\,dy,$$

where the plane of (x, y) is supposed perpendicular to the axis, and z is the ordinate of a point in the other plane. The limits of the integrations depend on the curve in which the plane of (x, y) cuts the surface. This follows from the Integral Calculus.

Let ϕ denote the angle between the two planes; the area of an element of the other section of which $\Delta x \Delta y$ is the projection on the plane of (x, y) is $\Delta x\,\Delta y \sec\phi$. Let A denote the area of the section of the cylinder by the plane of (x, y), and consequently $A \sec\phi$ the area of the other section; let $\bar{z}$ denote the ordinate of the centre of gravity of the plane area formed by the intersection of the cylinder by the second plane; then

$$A \sec\phi . \bar{z} = \iint z \sec\phi\,dx\,dy,$$

or

$$A\bar{z} = \iint z\,dx\,dy,$$

therefore

$$V = A\bar{z}.$$

The volume is therefore equal to the area of the base multiplied by the perpendicular upon it from the centre of gravity of the other section.

The centres of gravity of the two plane sections are on the same line parallel to the generating lines. For the coordinates of the centre of gravity of the section by the plane of (x, y) are

$$\frac{\iint x\,dx\,dy}{A} \text{ and } \frac{\iint y\,dx\,dy}{A},$$

and those of the upper section are

$$\frac{\iint x \sec\phi\,dx\,dy}{A\sec\phi} \text{ and } \frac{\iint y \sec\phi\,dx\,dy}{A\sec\phi},$$

which agree with the former values.

Thus the centres of gravity of all plane sections of a cylinder are situated on a line parallel to the generating lines of the cylinder.

If a portion of a cylinder be cut off by two planes, neither of which is perpendicular to the axis, we may suppose it to be the difference of two portions which have for their common base a section perpendicular to the axis. The difference of the lines drawn from the centres of gravity of the oblique sections perpendicular to the orthogonal section will be the line joining those centres of gravity. Hence the volume of a portion of a cylinder contained between any two planes is equal to the product of the area of an orthogonal section by the line joining the centres of gravity of the oblique sections.

Guldinus's Properties.

144. *If any plane figure revolve about an axis lying in its plane, the content of the solid generated by this figure in revolving through any angle is equal to a prism, of which the base is the revolving figure and height the length of the path described by the centre of gravity of the area of the plane figure.*

The axis of revolution in this and the following proposition is supposed not to cut the generating curve.

Let the axis of revolution be the axis of x, and the plane of the revolving figure in its initial position the plane of (x, y); let β be the angle through which the figure revolves.

The elementary area $\Delta x \Delta y$ of the plane figure in revolving through an angle $\Delta\theta$ generates the elementary solid whose volume is $y \Delta\theta\, \Delta x \Delta y$; therefore the whole solid

$$= \iiint_0^\beta y\, dx\, dy\, d\theta = \beta \iint y\, dx\, dy.$$

The limits of x and y depend upon the nature of the curve. But if $\bar{y}$ be the ordinate to the centre of gravity of the plane figure, then, by Art. 118,

$$\bar{y} = \frac{\iint y\, dx\, dy}{\iint dx\, dy},$$

the limits being the same as before.

Therefore the whole solid $=\beta\iint y\,dx\,dy=\bar{y}\beta\iint dx\,dy=$ arc described by centre of gravity multiplied by the area of the figure.

If any figure revolve about an axis lying in its own plane, the surface of the solid generated is equal in area to the rectangle, of which the sides are the length of the perimeter of the generating figure and the length of the path of the centre of gravity of the perimeter.

The surface generated by the arc Δs of the figure revolving through an angle $\Delta\theta$ is $y\Delta\theta\,\Delta s$; therefore the whole surface

$$=\iint_0^\beta y\,ds\,d\theta=\beta\int y\,ds.$$

The limits depend on the nature of the curve. But if $\bar{y}$ be the ordinate to the centre of gravity of the perimeter,

$$\bar{y}=\frac{\int y\,ds}{\int ds},$$

the limits being the same as before.

Therefore the whole surface $=\bar{y}\beta\int ds=$ arc described by centre of gravity, multiplied by the length of the perimeter.

Ex. 1. *To find the solid content and the surface of the ring formed by the revolution of a circle round a line in its own plane which it does not meet.*

Let the distance of the centre of the circle from the axis of revolution be a; let b be the radius of the circle; then the length of the path of the centre of gravity of the area of the figure is $2\pi a$, and the area of the figure is πb^2;

therefore content of the solid $=2\pi^2ab^2$.

Also the length of the path of the centre of gravity of the perimeter is $2\pi a$, and the length of the perimeter is $2\pi b$;

therefore surface of the solid $=4\pi^2ab$.

Ex. 2. *To find the centre of gravity of the area and also of the arc of a semicircle.*

A semicircle by revolving about its diameter generates a sphere; the content of the sphere is $\frac{4}{3}\pi a^3$, and the surface

$4\pi a^2$, the radius being a; the area of the semicircle is $\frac{1}{2}\pi a^2$, and the perimeter πa; therefore, distance of centre of gravity of *area* from the diameter

$$=\frac{\text{content of sphere}}{2\pi\,.\,\text{area of semicircle}}=\frac{4a}{3\pi};$$

distance of centre of gravity of arc from diameter

$$=\frac{\text{surface of sphere}}{2\pi\,.\,\text{arc of semicircle}}=\frac{2a}{\pi}.$$

Ex. 3. *To find the surface and the solid content of the solid formed by the revolution of a cycloid round the tangent at its vertex.*

In Art. 133 we have found $\frac{2a}{3}$ for the distance of the centre of gravity of the arc of a cycloid from its vertex; and the whole length of the arc is $8a$. Therefore the surface of the solid generated is

$$2\pi\times\frac{2a}{3}\times 8a;\ \text{that is } \frac{32}{3}\pi a^2.$$

And in Art. 113 we have found that the distance of the centre of gravity of the area included between the cycloid and its base from the vertex is $\frac{7}{6}a$; and the area so included is $3\pi^2a^2$. Hence the area of the portion which in the present case revolves round the tangent is $4\pi a^2-3\pi a^2$, that is πa^2. And the centre of gravity of this area may be shewn to be at a distance $\frac{a}{2}$ from the vertex. (See Ex. (2) of Art. 109). Therefore the solid content of the figure generated is $2\pi\frac{a}{2}\pi a^2$, that is π^2a^3.

EXAMPLES.

1. Find the centre of gravity of five equal heavy particles placed at five of the angular points of a regular hexagon.

2. Five pieces of a uniform chain are hung at equidistant points along a rigid rod without weight, and their lower ends are in a straight line passing through one end of the rod; find the centre of gravity of the system.

3. A plane quadrilateral $ABCD$ is bisected by the diagonal AC, and the other diagonal divides AC into two parts in the ratio of p to q; shew that the centre of gravity of the quadrilateral lies in AC and divides it into two parts in the ratio of $2p+q$ to $p+2q$.

4. From the fact that any system of heavy particles has one centre of gravity and only one, deduce the property that the lines joining the middle points of the opposite sides of any quadrilateral figure bisect each other.

5. Given the co-ordinates of the angular points of a pyramid, determine the co-ordinates of its centre of gravity.

6. ABC is a triangle; D, E, F are the middle points of its sides; shew that the centre of gravity of the *sides* of ABC coincides with the centre of the circle inscribed in DEF.

7. A piece of wire is formed into a triangle; find the distance of the centre of gravity from each of the sides, and shew that if x, y, z be the three distances, and r the radius of the inscribed circle, then

$$4xyz - r^2(x+y+z) - r^3 = 0.$$

8. If the centre of gravity of a four-sided figure coincide with one of its angular points, shew that the distances of this point and the opposite angular point from the line joining the other two angular points are as 1 to 2.

9. Shew that the common centre of gravity of a right-angled isosceles triangle, and the squares described on the two equal sides, is at a distance $= \frac{\sqrt{2}}{15}a$ from the point in which those sides meet, a being the length of one of them.

10. Prove the following construction for the centre of gravity of any quadrilateral. Let E be the intersection of

the diagonals, and F the middle point of the line which joins their middle points; draw the line EF and produce it to G, making $FG = \frac{1}{3}EF$; then G shall be the centre of gravity required.

11. A triangle ABC is successively suspended from the angles A and B, and the two positions of any side are at right angles to each other; shew that

$$5c^2 = a^2 + b^2.$$

12. A right-angled triangular lamina ABC is suspended from a point D in its hypothenuse AB; prove that in the position of equilibrium AB will be horizontal if

$$AD : DB :: AB^2 + AC^2 : AB^2 + BC^2.$$

13. A given isosceles triangle is inscribed in a circle; find the centre of gravity of the remaining area of the circle.

14. If three uniform rods be rigidly united so as to form half of a regular hexagon, prove that if suspended from one of the angles, one of the rods will be horizontal.

15. If ABC be an isosceles triangle having a right angle at C, and D, E be the middle points of AC, AB respectively, prove that a perpendicular from E upon BD will pass through the centre of gravity of the triangle BDC.

16. $ABCD$ is any plane quadrilateral figure, and a, b, c, d are respectively the centres of gravity of the triangles BCD, CDA, DAB, ABC; shew that the quadrilateral $abcd$ is similar to $ABCD$.

17. A, B, C, D, E, F are six equal particles at the angles of *any* plane hexagon, and a, b, c, d, e, f are the centres of gravity respectively of ABC, BCD, CDE, DEF, EFA, and FAB. Shew that the opposite sides and angles of the hexagon $abcdef$ are equal, and that the lines joining opposite angles pass through one point, which is the centre of gravity of the particles A, B, C, D, E, F.

18. A straight line ED cuts off $\frac{1}{n}$th part of the right-angled triangle ABC of which A is the right angle. $AB=a$, $AC=b$. Shew that the centre of gravity of $CEDB$ describes the curve whose equation is

$$\frac{ab}{n}=\{3(n-1)y-nb\}\{3(n-1)x-na\}.$$

19. The distance of the centre of gravity of any number of sides AB, BC, CD ...... KL of a regular polygon from the centre of the inscribed circle

$$=\frac{AL\times\text{radius}}{AB+BC+CD+\ldots\ldots+KL}.$$

20. A frustum is cut from a right cone by a plane bisecting the axis and parallel to the base; shew that it will rest with its slant side on a horizontal table if the height of the cone bear to the diameter of its base a greater ratio than $\sqrt{7}$ to $\sqrt{17}$.

21. If particles of unequal weights be placed at the angular points of a triangular pyramid, and G_1 be their common centre of gravity; G_2, G_3, ... their common centres of gravity for every possible arrangement of the particles; shew that the centre of gravity of equal particles placed at G_1, G_2, ... is the centre of gravity of the pyramid.

22. If a cone have its base united concentrically to the base of a hemisphere of equal radius, find the height of the cone that the solid may rest on a horizontal table on any point of its spherical surface. *Result.* $r\sqrt{3}$.

23. If any polygon circumscribe a circle, the centre of gravity of the *area* of the polygon, the centre of gravity of the *perimeter* of the polygon, and the centre of the circle, are in the same straight line; also the distance of the first point from the third is two-thirds of the distance of the second point from the third.

24. If any polyhedron circumscribe a sphere, the centre of gravity of the *volume* of the polyhedron, the centre of

gravity of the *surface* of the polyhedron, and the centre of the sphere, are in the same straight line; also the distance of the first point from the third is three-fourths of the distance of the second point from the third.

25. From a right cone the diameter of whose base is equal to its altitude is cut a right cylinder the diameter of whose base is equal to its altitude, their axes being in the same line and the base of the cylinder lying in the base of the cone; from the remaining cone a similar cylinder is cut, and so on, indefinitely; shew that the distance of the centre of gravity of the remaining portion from the base of the cone is $\frac{17}{80}$ of the altitude of the cone.

26. A square is cut from an equilateral triangle, a side of the square coinciding with a side of the triangle; from the equilateral triangle which remains another square is cut, and so on, *ad infinitum:* find the centre of gravity of the sum of the squares.

27. Find the centre of gravity of the area contained between the curves $y^2 = ax$ and $y^2 = 2ax - x^2$, which is above the axis of x.

Results. $\bar{x} = a \cdot \frac{15\pi - 44}{15\pi - 40}$; $\bar{y} = \frac{a}{3\pi - 8}$.

28. Find the centre of gravity of the area enclosed by the curve $r = a(1 + \cos\theta)$. *Result.* $\bar{x} = \frac{5}{6}a$.

29. Find the centre of gravity of the area included by a loop of the curve $r = a\cos 2\theta$.

Result. $\bar{x} = \frac{128a\sqrt{2}}{105\pi}$.

30. Find the centre of gravity of the area included by a loop of the curve $r = a\cos 3\theta$.

Result. $\bar{x} = \frac{81\sqrt{3}a}{80\pi}$.

31. The locus of the centre of gravity of all equal segments cut off from a parabola is an equal parabola.

32. Find the centre of gravity of a segment of a circle.

33. Find the centre of gravity of the area included by the curves $y^2 = ax$ and $x^2 = by$.

Result. $\bar{x} = \frac{9}{20} a^{\frac{1}{3}} b^{\frac{2}{3}}$, $\bar{y} = \frac{9}{20} a^{\frac{2}{3}} b^{\frac{1}{3}}$.

34. Find the centre of gravity of a portion of an equilateral hyperbola bounded by the curve, the transverse axis, and a radius vector drawn from the centre.

Results. $$\bar{x} = \frac{2y'}{3 \log (x' + y') - 3 \log a};$$

$$\bar{y} = \frac{2(x' - a)}{3 \log (x' + y') - 3 \log a};$$

where x', y' are the co-ordinates of the point of intersection of the curve and the bounding radius vector.

35. Two equal circles (radius a) are drawn, each passing through the centre of the other, and a third circle touches both, having one of their points of intersection for its centre; the distance of the centre of gravity of the smaller area included between the outer and inner circles from the common radius of the first two is

$$\frac{12 - 2\pi \sqrt{3}}{2\pi - 3\sqrt{3}} a.$$

36. The density of a triangle varies as the n^{th} power of the distance from the base; determine n when the centre of gravity of the triangle divides the line joining the vertex with the middle point of the base in the ratio of 3 to 1.

Result. $n = -\frac{1}{3}$.

37. Find the centre of gravity of the volume formed by the revolution round the axis of x of the area of the curve $y^4 - axy^2 + x^4 = 0$.

Result. $\bar{x} = \frac{3a\pi}{32}$.

38. Find the centre of gravity of the volume generated by the revolution of the area in Ex. 27 round the axis of y.

Result. $\bar{y} = \frac{5a}{2(15\pi - 44)}$.

39. Find the centre of gravity of a hemisphere when the density varies as the square of the distance from the centre.

Result. $\bar{x} = \frac{5a}{12}$.

40. Find the centre of gravity of the solid generated by a semiparabola bounded by the latus rectum revolving round the latus rectum.

Result. Distance from focus $= \frac{5}{32}$ of latus rectum.

41. The solid included between the surfaces of a continuous hyperboloid and its conical asymptote is cut by two planes perpendicular to their common axis; find the position of the centre of gravity of that portion which lies between the planes.

Result. Midway between the planes.

42. A solid sector of a sphere hangs from a point in its circular rim with its axis horizontal, find its vertical angle.

Result. The cosine of the semi-vertical angle is $\frac{3}{5}$.

43. Find the centre of gravity of the solid generated by the revolution of a semicircle about a line perpendicular to the diameter, and which does not meet the semicircle.

Result. Distance from the plane generated by the diameter

$$= \frac{4r}{3\pi}.$$

44. A is a point in the generating line of a right cylinder on a circular base, and B, C are two others in the generating line diametrically opposite. The cylinder is bisected by a plane ABC, and one of the semicylinders is cut by two planes at right angles to ABC, passing through AB and AC. Shew that if the solid ABC be placed with its convex side on a horizontal plane, the plane ABC will be inclined to the horizon at an angle $\tan^{-1} (\frac{3}{16}\pi)$, when there is equilibrium.

45. A solid cone is cut by two planes perpendicular to the same principal section, one through its axis, and the other parallel to a slant side; find the limiting value of the vertical angle of the cone, that the piece cut out may rest on its curved surface on a horizontal plane.

Result. The cosine of the vertical angle must not be greater than $\frac{5}{9}$.

46. A quadrant of a circle revolves round one of its extreme radii through an angle of 30°; find the centre of gravity of the solid traced out, the density being supposed to vary as the distance from the centre.

Results. $\bar{x} = \frac{3a}{5}$; $\bar{y} = \frac{3a}{5}(2 - \sqrt{3})$; $\bar{z} = \frac{2a}{5}$. The axis of x is supposed to coincide with the initial position of the revolving radius.

47. A solid is formed by the revolution of the area of the curve

$$y^{2n-2} = ax^{2-n}$$

round the axis of x; shew that the distance of the centre of gravity of any segment of this solid from the vertex bears to the height of the segment the ratio of 1 to n. The segment is supposed cut off by a plane perpendicular to the axis.

48. Find the centre of gravity of the *surface* of the solid $z^2 + y^2 = 2ax$, cut off by the plane $x = c$.

Result. $\bar{x} = \dfrac{(3c - a)(a + 2c)^{\frac{3}{2}} + a^{\frac{5}{2}}}{5\{(a + 2c)^{\frac{3}{2}} - a^{\frac{3}{2}}\}}$.

49. Apply Guldinus's theorem to find the volume of the frustum of a right cone in terms of its altitude and the radii of its ends.

Result. $\frac{h\pi}{3}(R^2 + Rr + r^2)$.

50. Find the surface and the volume of the solid formed by the revolution of a cycloid round its base.

Result. $\frac{64\pi a^2}{3}$; $5\pi^2 a^3$.

51. A segment of a circle revolves round its chord, which subtends an angle of 90° at the centre; find the surface and volume of the solid generated.

Results. $\dfrac{\pi a^2 (4 - \pi)}{\sqrt{2}}$; $\dfrac{a^3 (10 - 3\pi)\pi}{6\sqrt{2}}$.

52. An ellipse whose excentricity is $\frac{4}{3\pi}$ revolves about any tangent line. Prove that the volume generated by one portion into which the ellipse is divided by its minor axis varies inversely as the volume generated by the other portion.

53. A plane area moves in such a manner as to be always normal to the curve along which its centre of gravity moves; prove that the volume generated is equal to the given area multiplied by the length of the path of the centre of gravity.

Hence find the volume of a cycloidal tube whose normal section is of constant area.

54. Extend Guldinus's theorem for finding the volume of a ring to the case in which the ring is formed by the revolution of a plane area about a straight line *parallel* to its plane.

A ring is formed by the revolution of the lemniscate (whose equation is $r^2 = a^2 \cos 2\theta$) about a straight line parallel to its plane situated in a plane drawn through its double point and perpendicular to its axis; shew that the volume of this ring is $\frac{\pi^2 a^3}{4\sqrt{2}}$.

CHAPTER IX.

MACHINES.

145. A Machine is an instrument, or a system of solid bodies, for the purpose of transmitting force from one part to another of the system.

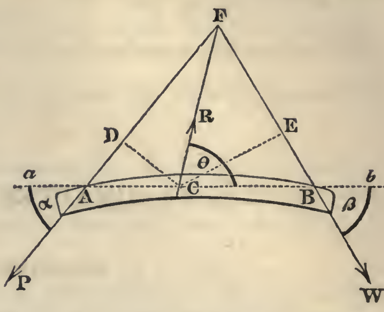
It would be endless to describe all the machines that have been invented; we shall consequently confine ourselves to those of simple construction. The most simple machines are denominated the Mechanical Powers. These we shall explain, and also a few combinations of them.

146. A *Lever* is an inflexible rod moveable only about a fixed axis, which is called the *fulcrum*. The portions of the lever into which the fulcrum divides it are called the *arms* of the lever: when the arms are in the same straight line, it is called a *straight lever*, and in other cases a *bent lever*.

Two forces act upon the lever about the fulcrum, called the *power* and the *weight:* the power is the force applied by the hand (or other means) to sustain or overcome the other force, or the weight. There are three species of levers: the first has the fulcrum between the power and weight; in the second the weight acts between the fulcrum and the power; and in the third the power acts between the fulcrum and the weight.

147. *To find the conditions of equilibrium of two forces acting in the same plane on a lever.*

Let the plane of the paper be the plane in which the forces act, and also be perpendicular to the axis, of which C is the projection, and about which the lever can move; A, B the points of application of the forces P, W; α, β the angles which the directions of the forces make with any line aCb drawn through C on the paper. Let R be the pressure upon the fulcrum, and θ the angle which it makes with the line aCb; then if we apply a force R in the direction CR, we may suppose the fulcrum removed, and the body to be held in equilibrium by the forces P, W, R.



We shall resolve these forces in directions parallel and perpendicular to aCb; and also take their moments about C; then by Art. 57 we have the following equations:

$$P\cos\alpha - W\cos\beta - R\cos\theta = 0 \quad\ldots\ldots\ldots\ldots(1),$$

$$P\sin\alpha + W\sin\beta - R\sin\theta = 0 \quad\ldots\ldots\ldots\ldots(2),$$

and

$$P.CD - W.CE = 0 \quad\ldots\ldots\ldots\ldots(3),$$

CD and CE being drawn perpendicular to the directions of P and W.

These three equations determine the ratio of P to W when there is equilibrium; and the magnitude and direction of the pressure on the fulcrum.

For equation (3) gives

$$\frac{P}{W} = \frac{CE}{CD} = \frac{\text{perpendicular on direction of } W}{\text{perpendicular on direction of } P} \quad\ldots\ldots(4).$$

Also by transposing the last terms of (1) and (2), we have

$$R\cos\theta = P\cos\alpha - W\cos\beta,$$

$$R\sin\theta = P\sin\alpha + W\sin\beta.$$

Add their squares; therefore

$$R^2 = P^2 + W^2 - 2PW\cos(\alpha + \beta),$$

which gives the magnitude of R.

From (1) and (2) by transposition and division

$$\tan\theta = \frac{P\sin\alpha + W\sin\beta}{P\cos\alpha - W\cos\beta},$$

which gives the direction of the pressure.

If we suppose B to be the fulcrum and take the moments about B instead of C, we have instead of equation (4) the following:

$$\frac{P}{R} = \frac{\text{perpendicular on direction of } R}{\text{perpendicular on direction of } P} \quad\ldots\ldots\ldots\ldots(5).$$

This is not a new equation of condition; but is a consequence of the three already given, (1), (2), (3). To shew this, imagine AD and BE produced to meet CR: they will meet this line in the same point, since the distances by these two constructions are $CD\operatorname{cosec}(\theta - \alpha)$ and $CE\operatorname{cosec}(\theta + \beta)$; and these are made equal, by equations (1), (2), (3), if we eliminate P, W. Suppose, then, F to be the point in which these lines meet. By multiplying (1), (2), respectively by $\sin\beta$ and $\cos\beta$, and adding, we have

$$\frac{P}{R} = \frac{\sin(\theta + \beta)}{\sin(\alpha + \beta)} = \frac{FB\sin(\theta + \beta)}{FB\sin(\alpha + \beta)}$$

$$= \frac{\text{perpendicular on direction of } R}{\text{perpendicular on direction of } P}:$$

therefore this equation is a consequence of the equations (1), (2), (3), as might have been anticipated.

It follows, then, that *the condition of equilibrium in a lever of any species is that the two forces must be inversely as the perpendiculars drawn upon their directions from the fulcrum and the forces must act so as to tend to turn the lever in opposite directions round the fulcrum.*

148. This property of the lever renders it a useful instrument in multiplying the efficacy of a force. For any two

forces, however unequal in magnitude, may be made to balance each other simply by fixing the fulcrum so that the ratio of its distances from the directions of the forces shall be equal to the inverse ratio of the forces. If the fulcrum be moved from this position, then that force will preponderate from which the fulcrum is moved and the equilibrium will be destroyed. We are thus led to understand how mechanical advantage is gained by using a crow-bar to move heavy bodies, as large blocks of stone: a poker to raise the coals in a grate: scissors, shears, nippers, and pincers; these last consisting of two levers of the first kind. The brake of a pump is a lever of the first kind. In the Stanhope printing-press we have a remarkable illustration of the mechanical advantage that can be gained by levers. The frame-work in which the paper to be printed is fixed, is acted upon by the shorter arm of a lever, the other arm being connected with a second lever, the longer arm of which is worked by the pressman. These levers are so adjusted that at the instant the paper comes in contact with the types, the perpendiculars from the fulcra upon the directions of the forces acting at the shorter arms are exceedingly short, and consequently the levers multiply the force exerted by the pressman to an enormous extent.

As examples of levers of the second kind, we may mention a wheelbarrow, an oar, a chipping-knife, a pair of nutcrackers.

It must be observed, however, that as the lever moves about the fulcrum the space through which the weight is moved is, in the first and second species of lever, smaller than the space passed through by the power: and therefore what is gained in power is lost in despatch. For example in the case of the crow-bar: to raise a block of stone through a given space by applying the hand at the further extremity of the lever, we must move the hand through a greater space than that which the weight describes.

But in the third species of lever the reverse is the case. The power is nearer the fulcrum than the weight, and is consequently greater; but the motion of the weight is greater than that of the power. In this kind of lever despatch is gained at the expense of power. An excellent example is the treddle of a turning lathe. But the most striking example of levers of the third kind is found in the animal frame, in the construction of which it seems to be a prevailing principle to sacrifice power to readiness and quickness of action. The limbs of animals are generally levers of this description. The condyle of the bone rests in its socket as the fulcrum; a strong muscle attached to the bone near the condyle is the power, and the weight of the limb together with any resistance opposed to its motion is the weight. A slight contraction of the muscle gives a considerable motion to the limb.

149. The lever is applied to determine the weight of substances. Under this character it is called a Balance. The Common Balance has its two arms equal, with a scale suspended from each extremity; the fulcrum being *above* the centre of gravity of the beam and scales. The substance to be weighed is placed in one scale, and weights placed in the other till the beam remains in equilibrium in a perfectly horizontal position; in which case the weight of the substance is indicated by the weights by which it is balanced. If the weights differ ever so slightly the horizontality of the beam will be disturbed, and after oscillating for some time (in consequence of the fulcrum being placed *above* the centre of gravity of the beam and scales) it will, on attaining a state of rest, form an angle with the horizon, the extent of which is a measure of the sensibility of the balance.

When we take the weight in the other scale as a measure of the weight of the substance we are weighing, we assume that the arms of the lever are of equal length and that the beam would be itself in equilibrium if the scales were empty. We can ascertain if these conditions are satisfied by observing whether equilibrium still subsists when the substance is transferred to the scale which the weight originally occupied and the weight to that which the substance originally occupied.

150. In the construction of a balance the following requisites should be attended to.

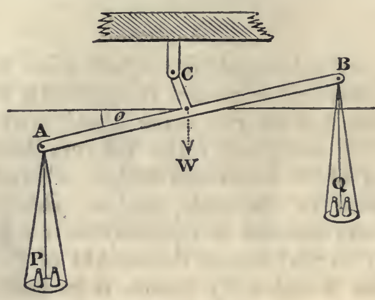
(1) When loaded with equal weights the beam should be perfectly horizontal.

(2) When the weights differ, even by a slight quantity, the *sensibility* should be such as to detect this difference.

(3) When the balance is disturbed it should readily return to its state of rest, or it should have *stability*. We shall now consider how these may be fulfilled.

To find how the requisites of a good balance may be satisfied.

Let P and Q be the weights in the scales; $AB = 2a$:



C the fulcrum; h its distance from the line joining A, B: W the weight of the beam and scales: k the distance of the centre of gravity of these (*i. e.* of the point of application of W) from C measured downwards; θ the angle the beam makes with the horizon when there is equilibrium.

Let us take the moments of P, Q, W about C: their sum equals zero since there is equilibrium (Art. 57). Then

the distance of P's direction from $C = a \cos\theta - h \sin\theta$

.... Q's .... $= a \cos\theta + h \sin\theta$

.... W's...................... $= k \sin\theta$,

we have, therefore,

$$P(a\cos\theta - h\sin\theta) - Q(a\cos\theta + h\sin\theta) - Wk\sin\theta = 0;$$

therefore $$\tan\theta = \frac{(P - Q)a}{(P + Q)h + Wk}.$$

This determines the position of equilibrium. The first requisite—the horizontality when P and Q are equal—is satisfied by making the arms equal.

For the second we observe that for a given difference of P and Q the sensibility is greater the greater $\tan\theta$ is; and for a given value of $\tan\theta$ the sensibility is greater the smaller the difference of P and Q is: hence $\frac{\tan\theta}{P-Q}$ is a correct measure of the sensibility: and therefore the second requisite is fulfilled by making $(P+Q)\frac{h}{a}+W\frac{k}{a}$ as small as possible.

The stability is greater the greater the moment of the forces which tend to restore the equilibrium when it is destroyed. Suppose $P=Q$, then P and Q may be placed at the mid-point between A and B: and the moment of the forces tending to restore equilibrium equals

$$\{(P+Q)\,h+Wk\}\sin\theta.$$

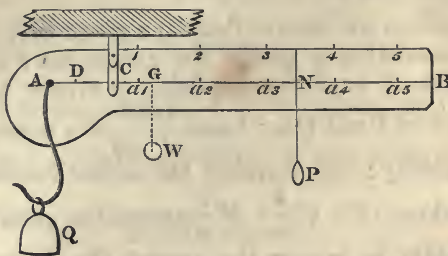
Hence to satisfy the third requisite, this must be made as large as possible. This is, in part, at variance with the second requisite. They may, however, both be satisfied by making $(P+Q)\,h+Wk$ large, and a large also: that is, by increasing the distances of the fulcrum from the beam and from the centre of gravity of the beam and scales, and by lengthening the arms.

It must be remarked that the sensibility of a balance is of more importance than the stability, since the eye can judge pretty accurately whether the index of the beam makes equal oscillations on each side of the vertical line; that is, whether the position of rest would be horizontal: if this be not the case, then the weights must be altered till the oscillations are nearly equal.

151. Another kind of balance is that in which the arms are unequal, and the same weight is used to weigh different substances by varying its point of support, and observing its distance from the fulcrum by means of a graduated scale. The common steelyard is of this description.

152. *To shew how to graduate the common steelyard.*

Let AB be the beam of the steelyard. A the fixed point



from which the substance to be weighed is suspended, Q being its weight; C the fulcrum; W the weight of the beam together with the hook or scale-pan suspended from A; G the centre of gravity of these.

Suppose that P suspended at N balances Q suspended from A; then, taking the moments of P, Q, W about C, we have

$$Q.AC - W.CG - P.CN = 0;$$

$$\therefore\ Q = \frac{CN + \frac{W}{P} . CG}{AC} P.$$

Take the point D, so that $CD = \frac{W}{P} CG$; therefore

$$Q = \frac{CN + CD}{AC} P = \frac{DN}{AC} P.$$

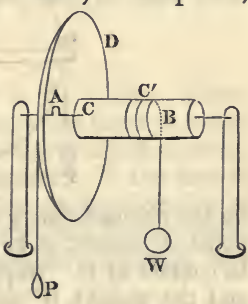
Now let the arm DB be graduated by taking Da_1, Da_2, Da_3,...... equal respectively to AC, $2AC$, $3AC$......; let the figures 1, 2, 3, 4,...... be placed over the points of graduation, and let subdivisions be made between these. Then by observing the graduation at N we know the ratio of Q to P; and this latter being a given weight we know the weight of Q. In this way any substance may be weighed.

153. The second of the Mechanical Powers is the Wheel and Axle. This machine consists of two cylinders fixed

together with their axes in the same line: the larger is called the wheel, and the smaller the axle. The cord by which the weight is suspended is fastened to the axle, and then coiled round it, while the power which supports the weight acts by a cord coiled round the circumference of the wheel, by spokes acted on by the hand, as in the *capstan*, or by the hand acting on a handle, as in the *windlass*.

154. *To find the ratio of the power and weight in the Wheel and Axle when in equilibrium.*

Let AD be the wheel and $CC'B$ the axle; P the power, represented by a weight suspended from the circumference of the wheel at A; W the weight hanging from the axle at B.



Then since the axis of the machine is fixed, the condition of equilibrium is that the sum of the moments of the forces about this axis vanishes, (Art. 87); therefore

$$P \times \text{rad. of wheel} = W \times \text{rad. of axle};$$

$$\text{therefore } \frac{W}{P} = \frac{\text{rad. of wheel}}{\text{rad. of axle}}.$$

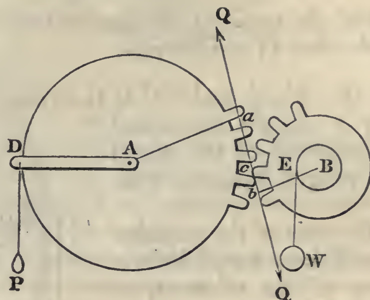
It will be seen that this machine is only a modification of the lever. In short it is an assemblage of levers all having the same axis: and as soon as one has been in action the next comes into play; and in this way an endless leverage is obtained. In this respect, then, the wheel and axle surpasses the common lever in mechanical advantage. It is much used in docks and in shipping.

155. The third Mechanical Power is the Toothed Wheel. It is extensively applied in all machinery; in cranes, steam-engines, and particularly in clock and watch work. If two circular hoops of metal or wood having their outer circumferences indented, or cut into equal teeth all the way round, be so placed that their edges touch, one tooth of one circum-

ference lying between two of the other (as represented in the figure); then if one of them be turned round by any means, the other will be turned round also. This is the simple construction of a pair of toothed wheels.

156. *To find the relation of the power and weight in Toothed Wheels.*

Let A and B be the fixed centres of the toothed wheels



on the circumferences of which the teeth are arranged; C the point of contact of two teeth; QCQ a normal to the surfaces in contact at C. Suppose an axle is fixed on the wheel B, and the weight W suspended from it at E by a cord; also suppose the power P acts by an arm AD; draw Aa, Bb perpendicular to QCQ. Let the mutual pressure at C be Q. Then, since the wheel A is in equilibrium about the fixed axis A, the sum of the moments about A equals zero; therefore

$$P.AD - Q.Aa = 0.$$

Also since the wheel B is in equilibrium about B, the sum of the moments about B equals zero; therefore

$$Q . Bb - W. BE = 0.$$

Then by eliminating Q from these two equations,

$$\frac{P}{W} = \frac{P}{Q} \cdot \frac{Q}{W} = \frac{Aa}{AD} \cdot \frac{BE}{Bb};$$

$$\text{or } \frac{\text{moment of } P}{\text{moment of } W} = \frac{Aa}{Bb}:$$

when the teeth are small this ratio very nearly

$$=\frac{\text{rad. of wheel } A}{\text{rad. of wheel } B}.$$

157. Wheels are in some cases turned by means of straps passing over their circumferences. In such cases the minute protuberances of the surfaces prevent the sliding of the straps, and a mutual action takes place such as to render the calculation exactly analogous to that in the Proposition.

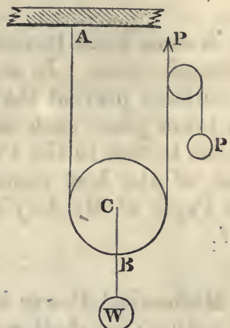
For the calculation of the best forms for the teeth, the reader is referred to a Paper of Mr Airy's, in the *Camb. Phil. Trans.* Vol. II. p. 277.

158. The fourth Mechanical Power is the Pully. There are several species of pullies: we shall mention them in order. The simple pully is a small wheel moveable about its axis: a string passes over part of its circumference. If the axis is fixed the effect of the pully is only to change the direction of the string passing over it: if however the axis be moveable, then, as will be presently seen, a mechanical advantage may be gained.

It is sometimes assumed as axiomatic that if a perfectly flexible string passes over a smooth surface the tension of the string will be the same throughout; we shall see, however, in the Chapter on Flexible Strings that this result admits of demonstration. In the present chapter we shall only require a part of the general proposition. We shall suppose the pullies to be circular, and assume that the tensions of the two portions of any string which are separated by a portion in contact with a pully are equal. And this may be shewn to be necessarily true if we merely admit that the string is a tangent to the circle at the point where it ceases to be in contact with the pully. For since the pully is smooth the directions of all the forces which it exerts on the string must pass through the *centre of the pully;* hence if we take the moments with respect to this point of the forces which act on the string we see that the string cannot be in equilibrium unless the tensions of the two portions are equal.

159. *To find the ratio of the power and weight in the single moveable Pully.*

I. Suppose the parts of the string divided by the pully are parallel.

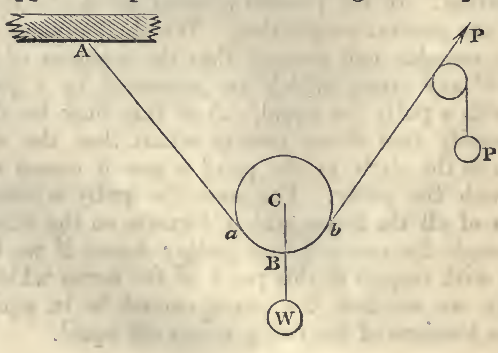


Let the string ABP have one extremity fixed at A, and after passing under the pully at B suppose it held by the hand exerting a force P. The weight W is suspended by a string from the centre C of the pully.

Now the tension of the string ABP is the same throughout. Hence the pully is acted on by three parallel forces, P, P, and W; hence

$$2P - W = 0; \text{ therefore } \frac{W}{P} = 2.$$

II. Suppose the portions of the string are not parallel.



Let α and α' be the angles which Aa and Pb make with the vertical.

Now the pully is held in equilibrium by W in CW, P in aA, P in bP. Hence, resolving the forces horizontally and vertically,

$$P \sin\alpha - P\sin\alpha' = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$P\cos\alpha + P\cos\alpha' - W = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

the equation of moments round C is an identical equation.

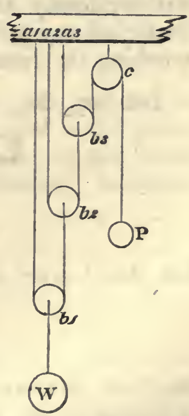
By (1), $\sin\alpha = \sin\alpha'$ and $\alpha = \alpha'$;

therefore, by (2), $\dfrac{W}{P} = 2\cos\alpha$,

which is the relation required.

160. *To find the ratio of the power and weight in a system of pullies, in which each pully hangs from a fixed point by a separate string, one end being fastened in the pully above it and the other end on a fixed beam, and all the strings being parallel.*

Let n be the number of moveable pullies.



I. Let us neglect the weight of the pullies themselves. Then

tension of $b_1W = W$;

tension of $a_1b_1b_2 = \frac{1}{2} W$;

tension of $a_2b_2b_3 = \dfrac{1}{2^2} W$;

tension of $a_3b_3c = \dfrac{1}{2^3} W$;

and so on; and the tension of the string passing under the n^{th} pully $= \dfrac{1}{2^n} W$, and this $= P$; therefore

$$\frac{W}{P} = 2^n.$$

II. Let us suppose the weights of the pullies to be considered; and let $\omega_1, \omega_2, \omega_3, \dots \omega_n$ be these weights.

Then if $p_1, p_2, p_3, \dots p_n$ be the weights which they would sustain at P, and P_1 the weight W would sustain at P, we have

$$p_1 = \frac{\omega_1}{2^n}, \quad p_2 = \frac{\omega_2}{2^{n-1}}, \dots\dots p_n = \frac{\omega_n}{2}, \quad P_1 = \frac{W}{2^n};$$

therefore $$P = p_1 + p_2 + \dots\dots + p_n + P_1,$$

or $$P = \frac{1}{2^n}\{W + \omega_1 + 2\omega_2 + 2^2\omega_3 + \dots\dots + 2^{n-1}\omega_n\}.$$

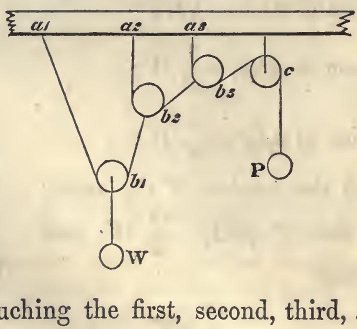
If $\omega_1 = \omega_2 = \omega_3 = \dots\dots = \omega_n$,

$$P = \frac{1}{2^n}\{W + (2^n - 1)\,\omega_1\}.$$

161. *To find the ratio of the power and weight when the system is the same as in the last Proposition, but the strings are not parallel.*

We shall neglect the weights of the blocks. The pullies will evidently so adjust themselves that the string at the centre of any pully will bisect the angle between the strings touching its circumference.

Let $2\alpha_1, 2\alpha_2, 2\alpha_3, \dots 2\alpha_n$ be the angles included between



the strings touching the first, second, third, ... n^{th} pullies respectively.

Then, by Art. 159,

$$\text{tension of } a_1b_1b_2 = \frac{W}{2\cos\alpha_1};$$

$$\text{tension of } a_2b_2b_3 = \frac{W}{2^2\cos\alpha_1\cos\alpha_2};$$

$$\text{tension of } a_3b_3c = \frac{W}{2^3\cos\alpha_1\cos\alpha_2\cos\alpha_3};$$

$$\text{tension of the last string} = \frac{W}{2^n\cos\alpha_1\cos\alpha_2\cos\alpha_3\ldots\cos\alpha_n},$$

and this $= P$; therefore

$$\frac{W}{P} = 2^n\cos\alpha_1\cos\alpha_2\cos\alpha_3\ldots\cos\alpha_n.$$

162. *To find the relation of the power and weight in a system of pullies where the same string passes round all the pullies.*

This system consists of two blocks, each containing a number of the pullies with their axes coincident. The weight is suspended from the lower block, which is moveable, and the power acts at the loose extremity of the string, which passes round the respective pullies of the upper and lower block alternately.

Since the same string passes round all the pullies, its tension will be everywhere the same, and equal to the power P. Let n be the number of portions of string at the lower block; then $n.P$ will be the sum of their tensions; therefore

$$W = n.P.$$

If we take into account the weight of the lower block, and call it B, then

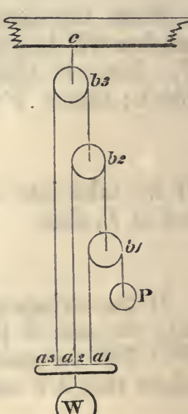
$$W + B = n.P.$$

If the strings at the lower block are not vertical, we must take the sum of the parts resolved vertically, and equate it to W. But in general this deviation from the vertical is so slight that it is neglected.

163. As the weight is rising or falling, it will be observed that in general the pullies move with different angular motions. The degree of angular motion of each pully depends upon the magnitude of its radius. Mr James White took advantage of this, to choose the radii of the pullies in such a manner as to give those in the same block the same angular motion, and so to prevent the wear and resistance caused by the friction of the pullies against each other. This being the case, the pullies in each block might be fastened together, or, instead of this, cut out of one mass.

It will be seen without much difficulty, that if the weight W be raised through a space a, each of the portions of string between the two blocks will be shortened by the length a; and therefore, that the portions of string which move over the pullies in the two blocks, taken alternately, will have their lengths equal to a, $2a$, $3a$, $4a$... Suppose the end of the string fastened to the lower block; then if the radii of the pullies of the upper block be proportional to the odd numbers 1, 3, 5, ... these pullies will move with the same angular velocity, and might be made all in one piece, as mentioned above. And if the radii of the lower pullies be proportional to the even integers 2, 4, 6, ... these also will move with a common angular velocity, and might therefore be cut out of one piece.

164. *To find the ratio of the power to the weight when all the strings are attached to the weight.*



If we neglect the weights of the pullies, the tension of the string $b_1a_1 = P$; the tension of $a_2b_2 = 2P$; and so on: if there be n pullies, then the sum of the tensions of the strings attached to the weight

$$= P + 2P + 2^2P + \dots + 2^{n-1}P = (2^n - 1)P;$$

$$\text{therefore } \frac{W}{P} = 2^n - 1.$$

If we suppose the weights of the pullies are $\omega_1, \omega_2, \omega_3, \dots$ reckoning from the lowest, and $\omega', \omega'', \omega''', \dots$ the portions of W which they respectively support (since they evidently assist P), and W' the portion of W supported by P; then

$$W' = (2^n - 1)\,P,$$
$$\omega' = (2^{n-1} - 1)\,\omega_1,$$
$$\omega'' = (2^{n-2} - 1)\,\omega_2,$$
$$\dots\dots\dots\dots\dots\dots$$
$$\omega^{(n-1)} = (2 - 1)\,\omega_{n-1};$$

therefore $W = W' + \omega' + \dots\dots = (2^n - 1)\,P + (2^{n-1} - 1)\,\omega_1,$
$$+ (2^{n-2} - 1)\,\omega_2 + \dots\dots + (2 - 1)\,\omega_{n-1}.$$

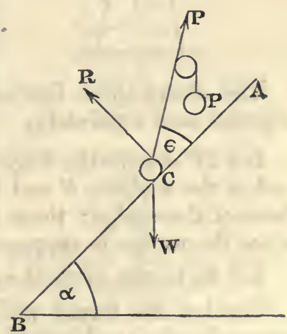
If $\omega_1 = \omega_2 = \omega_3 = \dots\dots$

$$W = (2^n - 1)\,P + \{2^{n-1} + 2^{n-2} + \dots\dots + 2 - (n-1)\}\,\omega_1,$$
$$= (2^n - 1)\,P + (2^n - n - 1)\,\omega_1.$$

165. The fifth Mechanical Power is the Inclined Plane.

By an inclined plane we mean a plane inclined to the horizon. A weight W may be supported on an inclined plane by a power P less than W.

166. *To find the ratio of the power and weight in the inclined plane.*



Let AB be the inclined plane; α the angle it makes with the horizon. Let the power P act on the weight in the direction CP, making an angle ϵ with the plane. Now the weight at C is held at rest by P in CP, W in the vertical CW, and a pressure R in CR, at right angles to the plane.

Hence, by Art. 27, if we resolve these forces perpendicular and parallel to the plane, we have

$$R + P\sin\epsilon - W\cos\alpha = 0 \dots\dots\dots\dots\dots (1),$$
$$P\cos\epsilon - W\sin\alpha = 0 \dots\dots\dots\dots\dots (2).$$

The second equation gives the required relation $\dfrac{P}{W} = \dfrac{\sin\alpha}{\cos\epsilon}$; and the first equation gives the magnitude of the pressure R.

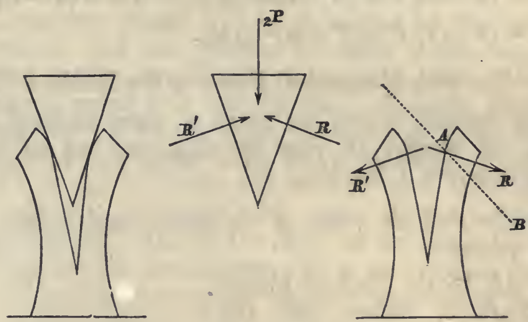
If P act horizontally, $\epsilon = -\alpha$, and $P = W\tan\alpha$.

If P act parallel to the plane, $\epsilon = 0$, $P = W \sin \alpha$.

If P act vertically, $\epsilon = \frac{1}{2}\pi - \alpha$, $P = W$.

167. The sixth Mechanical Power is the Wedge. This is a triangular prism, and is used to separate obstacles by introducing its edge between them and then thrusting the wedge forward. This is effected by the blow of a hammer or other such means, which produces a violent pressure for a short time, sufficient to overcome the greatest forces.

168. *An isosceles wedge is kept in equilibrium by pressures on its three faces; to find the relation between them.*



The above three figures represent the wedge and obstacles together and separately.

Let $2P$ denote the force acting perpendicularly to the thick end of the wedge; R and R' the forces which act on the other faces of the wedge: these forces are perpendicular to the faces since the wedge is supposed smooth.

Let 2α be the vertical angle of the wedge.

Resolve the forces which act on the wedge in directions perpendicular and parallel to the thick end; then for the equilibrium of the wedge we have

$$2P = (R + R') \sin \alpha,$$

$$R \cos \alpha = R' \cos \alpha;$$

therefore

$$R = R',$$

$$P = R \sin \alpha.$$

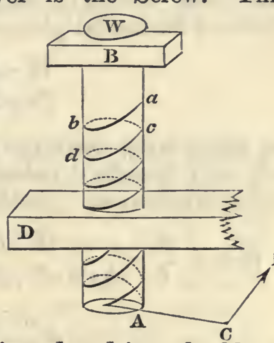
We do not write down the equations of equilibrium of the obstacle, because we do not know the forces exerted on it at different points of its base by the ground on which it rests.

It is usual to resolve the force R which acts on the wedge and obstacle into two components; one along the line in which A, the point of the obstacle in contact with the wedge, would move if the wedge were pushed further into the obstacle, and the other perpendicular to this direction. Let AB be the first direction, making an angle i with the direction of R; then the resolved part of R in this direction is $R\cos i$, which we will call S;

therefore $$\frac{P}{S}=\frac{\sin\alpha}{\cos i}.$$

As however nothing is known about the value of the angle i, the result is of no practical value.

169. The last Mechanical Power is the Screw. This machine in its simple construction consists of a cylinder AB with a uniform projecting thread *abcd*... traced round its surface, and making a constant angle with lines parallel to the axis of the cylinder. This cylinder fits into a block D pierced with an equal cylindrical aperture, on the inner surface of which is cut a groove the exact counterpart of the projecting thread *abcd*......



It is easily seen from this description, that when the cylinder is introduced into the block, the only manner in which it can move is backwards or forwards by revolving about its axis, the thread sliding in the groove. Suppose W to be the weight acting on the cylinder (including the weight of the cylinder itself), and P to be the power acting at the end of an arm AC at right angles to the axis of the cylinder; the block D is supposed to be firmly fixed, and the axis of the cylinder to be vertical.

170. *To find the ratio of the power and weight in the Screw when they are in equilibrium.*

Let the distance of C from the axis of the cylinder $= a$; and the radius of the cylinder $= b$.

Now the forces which hold the cylinder in equilibrium are W, P, and the reactions of the pressures of the various portions of the thread on the corresponding portions of the lower surface of the groove in which the thread rests; these reactions are indeterminate in their number but they all act in directions perpendicular to the surface of the groove, and therefore their directions make a constant angle with the axis of the cylinder. Let $\frac{\pi}{2} - \alpha$ be the angle which the thread of the screw makes with the axis of the cylinder, then α is the angle which the direction of each reaction makes with the axis of the cylinder. If, then, R be one of these reactions, $R \cos \alpha$, $R \sin \alpha$ are the resolved parts vertically and horizontally; the horizontal portions of the reactions act each at right angles to a radius of the cylinder. Hence, resolving the forces vertically, and also taking the moments of the forces in horizontal planes, we have

$$W - \Sigma . R \cos \alpha = 0 \text{ .... (1)},$$

$$Pa - \Sigma . Rb \sin \alpha = 0 \text{ .... (2)}:$$

we might write down the other four equations of equilibrium, but they introduce unknown quantities with which we are unconcerned in our question.

Hence $\frac{W}{P} = \frac{a \cos \alpha \Sigma . R}{b \sin \alpha \Sigma . R}$, because b and α are constant,

$$= \frac{a \cos \alpha}{b \sin \alpha} = \frac{2\pi a}{2\pi b \tan \alpha}$$

$$= \frac{\text{circumference of circle of which the rad. is } a}{\text{vertical dist. between two successive winds of the thread}}.$$

The Screw is used to gain mechanical power in many ways. In excavating the Thames Tunnel, the heavy iron frame-work which supported the workmen was gradually advanced by means of large screws.

MISCELLANEOUS EXAMPLES.

1. If one arm of a common balance be longer than the other, shew that the real weight of any body is the geometrical mean between its apparent weights as weighed first in one scale and then in the other.

2. The arms of a false balance are unequal, and one of the scales is loaded; a body whose true weight is P lbs. appears to weigh W lbs. when placed in one scale, and W' lbs. when placed in the other scale; find the ratio of the arms and the weight with which the scale is loaded.

$$\textit{Results.}\quad \frac{W'-P}{P-W};\qquad \frac{WW'-P^2}{P-W}.$$

3. A triangular lamina ABC, whose weight is W, is suspended by a string fastened at C; find the weight which must be attached at B that the vertical through C may bisect the angle ACB.

$$\textit{Result.}\quad P=\frac{W}{3}\,\frac{b-a}{a}.$$

4. Two equal weights are suspended by a string passing freely over three tacks, which form an isosceles triangle whose base is horizontal; find the vertical angle when the pressure on each tack is equal to one of the weights. *Result.* 120°.

5. A uniform heavy rod, at a given point of which a given weight is attached, is sustained at one end; determine its length when the force which applied at the other end will keep it horizontal is least.

6. $ABGC$, DEF are two horizontal levers without weight; B, F their fulcrums: the end D of one lever rests upon the end C of the other, HK is a rod without weight suspended by two equal parallel strings from the points E, G. Prove that a weight P at A will balance a weight W placed anywhere on the rod HK, provided

$$\frac{EF}{DF}=\frac{BG}{BC}\text{ and }\frac{P}{W}=\frac{BG}{AB}.$$

7. If the axis about which a wheel and axle turns coincide with that of the axle, but not with the axis of the wheel, find the greatest and least ratios of the power and weight necessary for equilibrium, neglecting the weight of the machine.

8. In the system of pullies where each string is attached to the weight, let one of the strings be nailed to the block through which it passes, then shew that the power may be increased up to a certain limit without producing motion. If there be three pullies, and the action of the middle one be checked in the manner described, find the tension of each string for given values of P and W.

9. A weight w is supported on an inclined plane by two forces, each equal to $\frac{w}{n}$, one of which acts parallel to the base. Shew that equilibrium may be possible when the inclination of the plane is not greater than $2\tan^{-1}\left(\frac{1}{n}\right)$, n being a positive integer.

10. A weight is suspended from the two ends of a straight lever without weight whose length is 5 feet, by strings whose lengths are 3 and 4 feet. Find the position of the fulcrum that the lever may rest in a horizontal position.

Result. At a distance $3\frac{1}{5}$ feet from that end of the lever to which the longer string is fastened.

11. A uniform steelyard AB, having a constant weight P, and a scale-pan of weight kP, suspended at B and A respectively, is used as a balance by moving the rod backwards and forwards upon the fulcrum C, on which the whole rests. Shew that the beam must be graduated by the formula

$$AC = \frac{1 + \frac{1}{2}k'}{n + k + k' + 1} \cdot AB,$$

the weight of the rod being $k'P$, and n being each of the natural numbers 1, 2, 3, ... taken in succession.

12. AB is a rod without weight capable of turning freely about its extremity A, which is fixed; CD is another rod equal to $2AB$, and attached at its middle point to the extremity B of the former, so as to turn freely about this point; a given force acts at C in the direction CA, find the force which must be applied at D in order to produce equilibrium.

13. A lever without weight in the form of the arc of a circle, having two weights P and Q suspended from its extremities, rests with its convexity downwards upon a horizontal plane; determine the position of equilibrium.

Result. Let α be the angle which the arc subtends at the centre of the circle, θ the inclination to the vertical of the radius at the extremity of which P is suspended; then

$$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha}.$$

14. The sides of a rhombus $ABCD$ are hinged together at the angles; at A and C are two pulling forces (P, P) acting in the diagonal AC; and at B and D there are two other pulling forces (Q, Q) acting in BD; shew that

$$\cos BAD = \frac{P^2 - Q^2}{P^2 + Q^2}.$$

15. AB, BC are two equal and uniform beams connected by a hinge at B; there is a fixed hinge at A; a string fastened at C passes over a pully at D and is attached to a weight P; AD is horizontal and equal to twice the length of either beam; shew that if P be such as to keep BC horizontal $P = W.\sqrt{\tfrac{3}{2}}$, and $\tan\theta = 2\tan\phi = 2\sqrt{2}$, where θ and ϕ are the angles which AB, CD make with the horizon, and $2W$ the weight of each beam.

16. A string $ABCDEP$ is attached to the centre A of a pully whose radius is r; it then passes over a fixed point B and under the pully which it touches in the points C and D; it afterwards passes over a fixed point E and has a weight P attached to its extremity; BE is horizontal and $= \tfrac{5}{3}r$, and DE is vertical; shew that if the system be in equilibrium the weight of the pully is $\tfrac{5}{2}P$, and find the distance AB.

Result. $AB = \dfrac{8r}{3\sqrt{7}}$.

17. Three uniform rods rigidly connected in the form of a triangle rest on a smooth sphere of radius r; shew that the tangent of the inclination of the plane of the triangle to the horizon is $\frac{a}{\sqrt{(r^2-\rho^2)}}$, where a is the distance of the centres of the circles inscribed in the triangle itself and in the triangle formed by joining the middle points of the rods, and ρ is the radius of the circle inscribed in the triangle.

18. If a steelyard be constructed with a given rod whose weight is inconsiderable compared with that of the sliding weight, the sensibility varies inversely as the sum of the sliding weight and the greatest weight which can be weighed.

19. A heavy equilateral triangle hung up on a smooth peg by a string, the ends of which are attached to two of its angular points, rests with one of its sides vertical; shew that the length of the string is double the height of the triangle.

20. Three equal heavy spheres lying in contact on a horizontal plane are held together by a string. A cube, whose weight is W, is placed with one of its diagonals vertical so that its lower sides touch the spheres; shew that the tension of the string is not less than $\frac{W\sqrt{2}}{3\sqrt{3}}$.

21. A roof of given span is to be constructed of two beams, which are to be connected at the vertex by a single pin, and the weight of the roof would increase in proportion to the length of the beams; what will be the angle of inclination to the horizon, when the whole pressure on the wall is the least possible?

Shew that the direction of the line of pressure will then make the same angle with the vertical line which the beam makes with the horizontal line.

22. An endless string supports a system of equal heavy pullies, the highest of which is fixed, the string passing round every pully and crossing itself between each. If α, β, γ, &c. be the inclinations to the vertical of the successive portions of string, prove that $\cos\alpha$, $\cos\beta$, $\cos\gamma$, &c. are in arithmetical progression.

23. Three equal heavy cylinders, each of which touches the other two, are bound together by a string and laid upon a horizontal table so that their axes are horizontal; the tension of the string being given, find the pressures between the cylinders.

Results. The horizontal pressure $= T - \frac{W}{2\sqrt{3}}$, the other $= T + \frac{W}{\sqrt{3}}$; where T is the tension of the string and W the weight of each cylinder.

24. A string of equal spherical beads is placed upon a smooth cone having its axis vertical, the beads being just in contact with each other, so that there is no mutual pressure between them. Find the tension of the string; and deduce the limiting value when the number of beads is indefinitely great.

25. A smooth cylinder is supported on an inclined plane with its axis horizontal, by means of a string which, passing over the upper surface of the cylinder, has one end attached to a fixed point and the other to a weight W which hangs freely; if α be the inclination of the plane to the horizon, and θ the inclination to the vertical of that part of the string which is fastened to the fixed point, the weight of the cylinder is

$$2W\frac{\sin\frac{1}{2}\theta\cos(\alpha+\frac{1}{2}\theta)}{\sin\alpha}.$$

26. An inextensible string binds tightly together two smooth cylinders whose radii are given; find the ratio of the pressure between the cylinders to the tension by which it is produced.

Result. $\frac{4\sqrt{(r_1 r_2)}}{r_1+r_2}$; where r_1 and r_2 are the given radii.

27. A ball of given weight and radius is hung by a string of given length from a fixed point, to which is also attached another given weight by a string so long that the weight hangs below the ball; find the angle which the string to which the ball is attached makes with the vertical.

Result. Let Q be the weight of the ball, P the weight which hangs below the ball, a the radius of the ball, l the length of the string; then the inclination of the string to the vertical is $\sin^{-1}\left(\frac{P}{P+Q}\cdot\frac{a}{a+l}\right)$.

28. A ring whose weight is P is moveable along a smooth rod inclined to the horizon at an angle α; another ring of weight P' is moveable along a rod in the same vertical plane as the former and inclined at an angle α' to the horizon; a string which connects these rings passes through a third ring of weight $2W$; shew that the system cannot be in equilibrium unless

$$P \tan \alpha - P' \tan \alpha' + W(\tan \alpha - \tan \alpha') = 0.$$

29. A right cone whose axis is a and vertical angle

$$2 \sin^{-1} \sqrt{\left(\frac{3}{7}\right)}$$

is placed with its base in contact with a smooth vertical wall, and its curved surface on a smooth horizontal rod parallel to the wall; shew that it will remain at rest if the distance of the rod from the wall be not greater than a nor less than $\frac{a}{7}$.

30. A paraboloid is placed with its vertex downwards and axis vertical between two planes each inclined to the horizon at an angle of 45°; find the greatest ratio which the height of the paraboloid may have to its latus rectum, so that, if it be divided by a plane through its axis and the line of intersection of the inclined planes, the two parts may remain in equilibrium.

Result. Let h be the height and $4a$ the latus rectum; then the greatest ratio is determined by

$$h + \frac{32}{15\pi}\sqrt{(ah)} = 3a.$$

31. Three bars of given length are maintained in a horizontal position, and tied together at their extremities so as to form a horizontal triangle; and a smooth sphere of given weight and size rests upon them. Find the pressure of the

sphere on each bar, and the magnitude and direction of the tension on each of the connecting bands.

32. One end of a string is fastened to a point in a smooth vertical wall, the other to a point in the circumference of the base of a cylinder; the cylinder is in equilibrium, having a point of its upper end in contact with the wall; find the distance of this point below the point in the wall to which the string is fastened.

Result. Suppose x the required distance, l the length of the string, h the height of the cylinder, b the diameter of its base; then

$$3x^2 = l^2 - h^2 - b^2.$$

33. The ends of a string are fastened to two fixed points, and from knots at given points in the string given weights are hung; shew that the horizontal component of the tension is the same for all the portions into which the string is divided by the knots. Shew also that if the weights are all equal the tangents of the angles which the successive portions of the string make with the horizon are in Arithmetical Progression. (Such a system is called a *Funicular Polygon.*)

34. Two uniform beams loosely jointed at one extremity are placed upon the smooth arc of a parabola, whose axis is vertical and vertex upwards. If l be the semi-latus rectum of the parabola, and a, b, the lengths of the beams, shew that they will rest in equilibrium at right angles to each other, if

$$l(a+b)(a^4+b^4)^{\frac{3}{2}} = a^4b^4;$$

and find the position of equilibrium.

35. A quadrilateral is formed by four rigid rods jointed at the ends; shew that two of its sides must be parallel in order that it may preserve its form when the middle points of either pair of opposite sides are joined together by a string in a state of tension.

36. Four rods, jointed at their extremities, form a quadrilateral, which may be inscribed in a circle; if they be kept in equilibrium by two strings joining the opposite angular points, shew that the tension of each string is inversely proportional to its length.

37. Four equal and uniform heavy rods being jointed by hinges so as to form a square, two opposite angles are connected by a string; this frame-work stands on a fixed point, the string being horizontal; find the tension of the string.

Result. Twice the weight of a rod.

38. Four equal and uniform heavy rods are connected by hinges; the system is suspended by a string attached to one hinge, and the lowest hinge is in contact with a horizontal plane; find the tension of the string and the pressure on the plane.

Result. Each is twice the weight of a rod.

39. A regular hexagon, composed of six equal heavy rods moveable about their angular points, is suspended from one angle which is connected by threads with each of the opposite angles. Shew that the tensions of the threads are as $\sqrt{3} : 2$. Find also the strain along each rod.

40. A regular hexagon is composed of six equal heavy rods moveable about their angular points; one rod is fixed in a horizontal position, and the ends of this rod are connected by vertical strings with the ends of the lowest rod; find the tension of each string.

Result. $\frac{3}{2}W$; where W is the weight of a rod.

41. Suppose that in the preceding example each end of the fixed rod is connected with the more remote end of the lowest rod, so that the strings instead of being parallel are inclined at an angle of 60°; find the tension of each string.

Result. $W\sqrt{3}$.

42. A regular hexagon is composed of six equal heavy rods moveable about their angular points, and two opposite angles are connected by a horizontal string; one rod is placed on a horizontal plane, and a weight is placed at the middle point of the highest rod; find the tension of the string.

Result. Let W be the weight of each rod, and W' the weight placed on the highest rod; then the tension is

$$\frac{3W + W'}{\sqrt{3}}.$$

CHAPTER X.

FRICTION.

171. In the investigations of the preceding chapter, we have supposed that the surfaces of the bodies in contact are perfectly smooth. By a *smooth* surface is meant a surface which opposes no resistance whatever to the motion of a body upon it. A surface which does oppose a resistance to the motion of a body upon it is said to be *rough*. In practice it is found that all bodies are more or less rough.

The friction of a body on a surface is measured by the least force which will put the body in motion along the surface.

172. Coulomb made a series of experiments upon the friction of bodies against each other and deduced the following laws. *Mémoires......des Savans Etrangers*, Tom. x.

(1) *The friction varies as the pressure when the materials of the surfaces in contact remain the same.* When the pressures are very great indeed, it is found that the friction is somewhat less than this law would give.

(2) *The friction is independent of the extent of the surfaces in contact so long as the pressure remains the same.* When the surfaces in contact are extremely small, as for instance a cylinder resting on a surface, this law gives the friction much too great.

These two laws are true when the body is on the point of moving and also when it is actually in motion; but in the case of motion the magnitude of the friction is much less than when the body is in a state bordering on motion.

(3) *The friction is independent of the velocity when the body is in motion.*

It follows from these laws that if P be the normal pressure between two surfaces, then the friction is μP, where μ is a constant quantity for the same materials and is called the *coefficient of friction.*

In the state bordering on motion and when the surfaces in contact are of finite extent, we have the following results from experiment;

$\mu = \frac{1}{2}$, surfaces wood, the grain being in the same direction;
$= \frac{1}{4}$, ... opposite.........;
$= \frac{1}{4}$, metallic surfaces;
$= \frac{1}{5}$, one surface wood and the other metal.

Oil and grease considerably diminish friction; fresh tallow reduces it to half its value.

In the state bordering on motion and when the surfaces in contact are single lines, then $\mu = \frac{1}{12}$ for wood. When the surface in contact is a physical point the statical friction is inconsiderable.

For full particulars on this subject we refer the reader to Coulomb's papers, and to the Memoirs published in the *Mémoires de l'Institut.* by M. Morin.

173. *Angle of Friction.* Suppose a body acted on only by its weight to be placed upon a horizontal plane and the plane to be turned round a horizontal line until the body begins to slide. Let W be the weight of the body and α the angle the plane makes with the horizon. The pressure of the body on the plane will be equal to the resolved part of its weight perpendicular to the plane, that is to $W \cos \alpha$. The friction is equal to the resolved part of the weight parallel to the plane, that is to $W \sin \alpha$. If μ be the coefficient of friction, we have

$$W \sin \alpha = \mu W \cos \alpha;$$

therefore $$\tan \alpha = \mu.$$

This experiment will enable us to determine the value of the coefficient of friction for different substances. The inclination of the plane when the body is just about to slide is called the *angle of friction.*

174. In Art. 32 we have found the condition of equilibrium of a particle constrained to rest on a *smooth* curve; we proceed

to the case of a particle on a *rough* curve. Suppose the curve a plane curve; let X, Y be the forces which act on the particle parallel to the axes of x and y exclusive of the action of the curve. The sum of the resolved parts of X and Y along the tangent to the curve is

$$X\frac{dx}{ds} + Y\frac{dy}{ds}.$$

The sum of the resolved parts along the normal is

$$\pm\left(X\frac{dy}{ds} - Y\frac{dx}{ds}\right).$$

If μ be the coefficient of friction the greatest friction capable of being called into action is

$$\pm\mu\left(X\frac{dy}{ds} - Y\frac{dx}{ds}\right).$$

Hence, the condition of equilibrium will be that the *numerical* value of $X\frac{dx}{ds} + Y\frac{dy}{ds}$ must be less than the *numerical* value of $\mu\left(X\frac{dy}{ds} - Y\frac{dx}{ds}\right)$, without regard to sign in either case. This may be conveniently expressed thus,

$$\left(X\frac{dx}{ds} + Y\frac{dy}{ds}\right)^2 \text{ must be less than } \mu^2\left(X\frac{dy}{ds} - Y\frac{dx}{ds}\right)^2.$$

We may exhibit this condition in a different form, as will be seen in the following article.

175. Next let the curve be of double curvature. Let P denote the resultant force acting on the particle exclusive of the action of the curve; X, Y, Z the components of P parallel to the axes; l, m, n the direction cosines of the tangent to the curve at the point where the particle is placed; θ the angle between this tangent and the direction of P. The resolved part of P along the tangent is $P\cos\theta$, and that perpendicular to the tangent is $P\sin\theta$. Hence, if μ be the coefficient of friction, we must have for equilibrium

$$P\cos\theta < \mu P\sin\theta;$$

therefore $$\cos^2\theta < \mu^2 (1 - \cos^2\theta);$$

therefore $$\cos^2\theta < \frac{\mu^2}{1+\mu^2};$$

therefore $$\left(\frac{Xl + Ym + Zn}{P}\right)^2 < \frac{\mu^2}{1+\mu^2}.$$

It is easy to shew that this result includes that of the former article by putting $n = 0$, $m = \frac{dy}{ds}$, $l = \frac{dx}{ds}$.

176. *A particle is constrained to remain on a rough surface; determine the condition of equilibrium.*

Let P be the resultant force on the particle exclusive of the action of the surface; ϕ the angle between the direction of P and the normal to the surface at the point where the particle is placed; $u = 0$ the equation to the surface; x, y, z the coordinates of the particle. The resolved part of P along the normal is $P \cos \phi$, and that perpendicular to the normal is $P \sin \phi$. Hence, for equilibrium we must have

$$P \sin \phi < \mu P \cos \phi;$$

therefore $$\sin^2\phi < \mu^2 \cos^2 \phi;$$

therefore $$\cos^2 \phi > \frac{1}{1+\mu^2};$$

therefore $$\frac{\left(X\frac{du}{dx} + Y\frac{du}{dy} + Z\frac{du}{dz}\right)^2}{P^2\left\{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2\right\}} > \frac{1}{1+\mu^2}.$$

177. In the following articles of this chapter we shall investigate certain equations which hold when the equilibrium of different machines is on the point of being disturbed. The equations in such cases will involve the forces acting on the machine and μ the coefficient of friction. When we have found one of these *limiting equations*, we can draw the following inferences:

(1) If in order to satisfy the equation for a given set of forces it is necessary to ascribe to μ a value greater than its

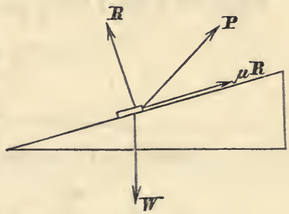
extreme value for the substances in question, which is known by experiment, equilibrium is impossible.

(2) If the limiting equation can be satisfied by ascribing to μ values less than its extreme value, equilibrium *may* be possible. We say *may* be possible, because the limiting equation may not be the *only* equation of equilibrium, and of course for equilibrium it is necessary that *all* the appropriate equations be satisfied.

Equilibrium of Machines with Friction.

178. Inclined Plane.

Let α be the inclination of the plane to the horizon. Suppose a force P_1 acting at an inclination θ to the plane and the body on the point of moving *down* the plane. Let R be the normal action of the plane, μR the friction which acts *up* the plane, W the weight of the body. Resolve the forces along and perpendicular to the plane; then, for equilibrium we have



$$P_1 \cos\theta + \mu R - W \sin\alpha = 0 \quad\ldots\ldots\ldots\ldots(1),$$

$$R + P_1 \sin\theta - W\cos\alpha = 0 \quad\ldots\ldots\ldots\ldots(2).$$

Substitute in (1) the value of R from (2); thus

$$P_1 = \frac{W\sin\alpha - \mu W\cos\alpha}{\cos\theta - \mu\sin\theta}.$$

Next, suppose P_2 a force acting at an inclination θ to the plane, such that the body is on the point of moving *up* the plane. Friction now acts *down* the plane, and we shall find

$$P_2 = \frac{W\sin\alpha + \mu W\cos\alpha}{\cos\theta + \mu\sin\theta}.$$

This result may be deduced from the former by changing the sign of μ.

There will be equilibrium if the body be acted on by a force P, the magnitude of which lies between P_1 and P_2.

Suppose ϵ to be the *angle of friction*, so that

$$\tan \epsilon = \mu,$$

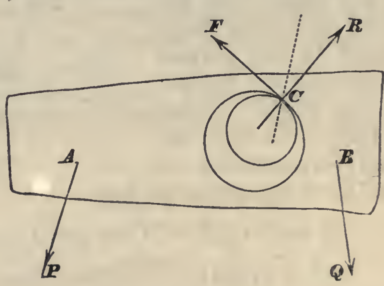
then $$P_1 = \frac{W \sin \alpha - \tan \epsilon \, W \cos \alpha}{\cos \theta - \tan \epsilon \sin \theta}$$

$$= \frac{W \sin (\alpha - \epsilon)}{\cos (\theta + \epsilon)};$$

Similarly, $$P_2 = \frac{W \sin (\alpha + \epsilon)}{\cos (\theta - \epsilon)}.$$

179. Lever with Friction.

Suppose a solid body pierced with a cylindrical hole through



which passes a solid fixed cylindrical axis. Let the outer circle in the figure represent a section of the cylindrical hole made by a plane perpendicular to its axis, and the inner circle the corresponding section of the solid axis. In the plane of this section, we suppose two forces P and Q to act on the solid body at the points A and B. Also at the point of contact C there is a normal force R and a tangential force F. These four forces keep the body in equilibrium.

Since R and F have a resultant passing through C, it follows that the resultant of P and Q must also pass through C (Art. 58). Let γ be the angle between the directions of P and Q, and S the resultant of P and Q; then

$$S^2 = P^2 + Q^2 + 2PQ \cos \gamma.$$

Let the direction of S be represented by the dotted line making an angle θ with R. Then since F, R, and S are in equilibrium,

$$R = S\cos\theta \dots\dots\dots\dots\dots\dots (1),$$

$$F = S\sin\theta \dots\dots\dots\dots\dots\dots (2).$$

For the limiting position of equilibrium $F = \mu R$; therefore

$$\tan\theta = \mu \dots\dots\dots\dots\dots\dots (3)$$

We may now find the relation between P and Q, by taking moments round the centre of the exterior circle; let r be the radius of this circle; a and b the distances of A and B from its centre; α and β the angles the directions of P and Q make with these distances; then

$$Pa\sin\alpha + Fr = Qb\sin\beta;$$

or by (2) and (3),

$$Pa\sin\alpha + \frac{r\mu}{(1+\mu^2)^{\frac{1}{2}}}(P^2 + Q^2 + 2PQ\cos\gamma)^{\frac{1}{2}} = Qb\sin\beta \dots (4).$$

If we suppose the friction to act in the opposite direction to that in the figure, we shall obtain

$$Pa\sin\alpha - \frac{r\mu}{(1+\mu^2)^{\frac{1}{2}}}(P^2 + Q^2 + 2PQ\cos\gamma)^{\frac{1}{2}} = Qb\sin\beta \dots (5).$$

Equilibrium will not subsist unless P, Q, a, b, α, β, γ are so adjusted that (4) or (5) can be satisfied without giving to μ a value greater than its limit known by experiment.

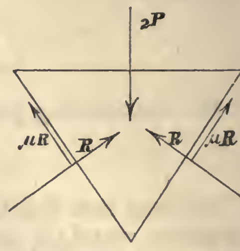
The following form may be given to the limiting equation. Let s be the length of the perpendicular from the centre of the outer circle on the dotted line. Since F, R, and S are in equilibrium, we have by taking moments

$$Fr = Ss;$$

therefore $$\frac{r\mu}{(1+\mu^2)^{\frac{1}{2}}} = s.$$

180. Wedge with Friction. (See Art. 168.)

Suppose the wedge to be on the point of moving in the direction in which $2P$ urges it, and assume for simplicity that each face is similarly acted on by the obstacle. The forces which maintain the wedge in equilibrium are $2P$ perpendicular to the thick end, R perpendicular to each face, and μR along each face towards the thick end. Hence, resolving the forces parallel to the direction of $2P$,



$$P = R \sin \alpha + \mu R \cos \alpha \quad \text{.............. (1).}$$

Forces equal and opposite to R and μR act on the obstacle at each point of contact. If R' be the resultant of R and μR, we have

$$R' = R \sqrt{(1 + \mu^2)} \quad \text{.................. (2).}$$

Let S be the resolved part of R' along a direction making an angle i with that of R and i' with that of R' (see Art. 168); then

$$S = R' \cos i'$$
$$= R \cos i + \mu R \sin i \quad \text{.................. (3).}$$

(1), (2), and (3) will give the ratio of P to R' and of P to S.

181. Screw with friction. (See Arts. 169, 170.)

If the surfaces of the screw are rough it is kept in equilibrium by W, P, a system of forces perpendicular to the surface of the groove, and a system of forces arising from friction. Let $R_{,}$ denote one of the forces perpendicular to the surface of the groove, $\mu R_{,}$ the corresponding friction; then $R_{,}$ makes an angle α with the axis of the cylinder on which the screw is raised, and $\mu R_{,}$ an angle $\frac{1}{2}\pi - \alpha$ with the axis of the cylinder. Suppose W about to prevail over P; then resolving the forces parallel to the axis of the cylinder, and taking moments round it, we have

$$W - \Sigma R (\cos \alpha + \mu \sin \alpha) = 0,$$

$$Pa - \Sigma R (\sin \alpha - \mu \cos \alpha)\, b = 0.$$

Therefore
$$\frac{P}{W}=\frac{b(\sin\alpha-\mu\cos\alpha)}{a(\cos\alpha+\mu\sin\alpha)}$$
$$=\frac{b}{a}\frac{\tan\alpha-\mu}{1+\mu\tan\alpha}$$
$$=\frac{b}{a}\tan(\alpha-\epsilon),$$
if
$$\mu=\tan\epsilon.$$

If we suppose P about to prevail over W, we shall find similarly
$$\frac{P}{W}=\frac{b}{a}\tan(\alpha+\epsilon).$$

EXAMPLES.

1. A rectangular prism, whose breadth is $2.8\dot{3}$ feet and thickness less than 2 inches, is laid with its axis horizontal, and with its smaller face on an inclined plane where the coefficient of friction is $\frac{1}{17}$. Shew that if the inclination of the plane is gradually increased, the prism will roll before it will slide.

2. If the roughness of a plane which is inclined to the horizon at a known angle be such that a body will just rest supported on it, find the least force requisite to draw the body up.

Results. Let α be the inclination of the plane, W the weight of the body; then the least force is $W\sin 2\alpha$, and it acts at an inclination α to the plane.

3. Two rough bodies rest on an inclined plane, and are connected by a string which is parallel to the plane; if the coefficient of friction be not the same for both, find the greatest inclination of the plane which is consistent with equilibrium.

Result. $\tan\theta=\dfrac{\mu_1 W_1+\mu_2 W_2}{W_1+W_2}$.

4. A rectangular table stands on a rough inclined plane, and has two sides horizontal; if the coefficient of friction of the lowest feet be μ and that of the others be μ', find the inclination of the plane when the table is on the point of sliding.

5. Two unequal weights on a rough inclined plane are connected by a string which passes through a fixed pully in the plane; find the greatest inclination of the plane consistent with the equilibrium of the weights.

$$\textit{Result.} \quad \tan\alpha = \frac{\mu(W_1 + W_2)}{W_1 - W_2}.$$

6. A heavy uniform rod whose length is $2a$ is supported by resting on a rough peg, a string of length l being attached to one end of the rod and fastened to a given point in the same horizontal plane with the peg. If when the rod is on the point of sliding the string is perpendicular to it the coefficient of friction is $\frac{l}{a}$.

7. Two weights P, Q of similar material rest on a rough double inclined plane, and are connected by a fine string passing over the common vertex; Q is on the point of motion down the plane, shew that the weight which may be added to P without producing motion is

$$\frac{P\sin 2\phi \sin(\alpha+\beta)}{\sin(\beta-\phi)\sin(\alpha-\phi)},$$

α, β being the angles of inclination of the planes and $\tan\phi$ the coefficient of friction.

8. A weight P is attached to a point in the circumference of a rough circular ring whose weight is W: shew that the ring will hang on a horizontal rod in a plane perpendicular to it with any point of the ring in contact with the rod, if the coefficient of friction be not less than

$$\frac{1}{\sqrt{(n^2+2n)}}, \quad \text{where } n = \frac{W}{P}.$$

9. Two equal heavy rings are moveable on a horizontal rough rod; a string of given length which passes through

them has both ends attached to a given weight; find the greatest possible distance between the rings.

10. Three equal hemispheres, having their bases downwards, are placed in contact with each other upon a horizontal table; if a smooth sphere of the same substance and equal radius be placed upon them, shew that there will be equilibrium or not, according as the coefficient of friction between the hemispheres and the table is greater or less than $\frac{1}{6}\sqrt{2}$.

11. A uniform rod rests wholly within a rough hemispherical bowl in a vertical plane through its centre, prove that the limiting position of equilibrium will be given by the equation

$$\tan\theta = \frac{\sin 2\epsilon}{2\cos(\beta+\epsilon)\cos(\beta-\epsilon)},$$

θ being the inclination of the rod to the horizon, 2β the angle it subtends at the centre, and $\tan\epsilon$ the coefficient of friction.

12. A thin rod rests in a horizontal position between two rough planes equally inclined to the horizon, and whose inclination to each other is 2α; if μ be the coefficient of friction, then the greatest possible inclination of the line of intersection of the planes to the horizon is $\tan^{-1}\frac{\mu}{\sin\alpha}$.

13. A surface is formed by the revolution of an equilateral hyperbola about one of its asymptotes which is vertical; shew that a particle will rest upon it, supposing it rough, anywhere beyond the intersection of the surface with a certain circular cylinder.

14. A heavy particle under the action of gravity will rest on a rough paraboloid $\frac{x^2}{\alpha}+\frac{y^2}{\beta}=2z$, if it be placed on the surface at any point above the curve of intersection of the surface with the cylinder $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=\mu^2$; the axis of the paraboloid being vertical, its vertex upwards, and μ the coefficient of friction.

15. A rough elliptic pully (weight W) can turn freely about one extremity of its major axis, and two weights, P, Q, are suspended by a string which passes over the pully; when in equilibrium its plane is vertical, and its axis inclined at 60° to the horizon, prove that the excentricity of the ellipse is equal to

$$\frac{\sqrt{\{(3Q+W-P)(Q-W-3P)\}}}{(Q-P)\sqrt{3}}.$$

16. A heavy hemisphere rests with its convex surface on a rough inclined plane. Find the greatest possible inclination of the plane.

17. One end A of a heavy rod ABC rests against a rough vertical plane; and a point B of the rod is connected with a point in the plane by a string, the length of which is equal to AB; determine the position of equilibrium of the rod, and shew how the direction in which the friction acts depends upon the position of B.

18. Three equal balls, placed in contact on a horizontal plane, support a fourth of the same size. Determine the least values of the coefficients of friction of the balls with each other and with the plane, that the equilibrium may be possible.

Results. The coefficient of the friction between the balls $=\sqrt{3}-\sqrt{2}$; the coefficient of the friction between the balls and plane $=\frac{1}{4}(\sqrt{3}-\sqrt{2})$.

19. Determine the curve on the rough surface of an ellipsoid, at every point of which a particle acted on by three equal forces whose directions are parallel to the axes of the ellipsoid, will rest in a limiting position of equilibrium.

20. $BCDE$ is a square board; a string is fixed to a point A in a rough wall and to the corner B of the board. Shew that the board will rest with its plane perpendicular to the wall, and its side CD resting against it, if AC be not greater than μBC.

21. A rectangular parallelopiped of given dimensions is placed with one face in contact with a rough inclined plane;

determine the limits of its position in order that equilibrium may exist.

22. A board, moveable about a horizontal line in its own plane, is supported by resting on a rough sphere which lies on a horizontal table; find the greatest inclination at which the board can rest.

Result. Let θ be the inclination of the board to the horizon; then $\tan\frac{\theta}{2}=\mu$, where μ is the coefficient of the friction between the board and the sphere.

23. A string PCB passes over a smooth pully C, and has a given weight P attached to one extremity, while the other extremity B is attached to one end of a heavy uniform beam AB at B. The other end A of the beam rests upon a rough horizontal plane; determine the position of the beam when in equilibrium.

24. A hemisphere is supported by friction with its curved surface in contact with a horizontal and vertical plane; find the limiting position of equilibrium.

Result. If θ be the inclination of the plane base to the horizon $\sin\theta=\dfrac{8\mu(1+\mu)}{3(1+\mu^2)}$.

25. When a person tries to pull out a two-handled drawer by pulling one of its handles in a direction perpendicular to its front, find the condition under which the drawer will stick fast.

26. Determine the condition under which a given weight may be supported upon a rough vertical screw without the action of any force; for example, if the coefficient of friction be $\frac{1}{2}$, find the least number of turns which may be given to a thread upon a cylinder 2 feet long and 6 inches in circumference. *Result.* Eight.

27. Two uniform beams of the same length and material placed in a vertical plane, are in a state of rest bordering on motion under the following circumstances: their upper ends are connected by a smooth hinge, about which they can move freely; their other ends rest on a rough horizontal

plane, and the beams are perpendicular to each other: find the coefficient of friction between the beams and the horizontal plane. *Result.* $\mu = \frac{1}{2}$.

28. A straight uniform beam is placed upon two rough planes whose inclinations to the horizon are α and α', and the coefficients of friction $\tan\lambda$ and $\tan\lambda'$; shew that if θ be the limiting value of the angle of inclination of the beam to the horizon at which it will rest, W its weight, and R, R' the pressures upon the planes

$$2\tan\theta = \cot(\alpha' + \lambda') - \cot(\alpha - \lambda),$$

$$\frac{R}{\cos\lambda \sin(\alpha' + \lambda')} = \frac{R'}{\cos\lambda' \sin(\alpha - \lambda)} = \frac{W}{\sin(\alpha - \lambda + \alpha' + \lambda')}.$$

29. A heavy right cylinder is placed with its base on a rough horizontal plane, and is capable of motion round its axis; find the least couple in a horizontal plane which will move it.

30. Two weights of different material are laid on an inclined plane connected by a string extended to its full length, inclined at an angle θ to the line of intersection of the inclined plane with the horizon; if the lower weight be on the point of motion find the magnitude and direction of the force of friction on the upper weight.

31. A carriage stands upon four equal wheels; given the coefficient of friction between the axles and the wheels find the greatest slope on which it can remain at rest neglecting the weights of the wheels.

CHAPTER XI.

FLEXIBLE STRINGS. INEXTENSIBLE.

182. A STRING is said to be *perfectly flexible* when any force, however small, which is applied otherwise than along the direction of the string will change its form. For shortness, we use the word *flexible* as equivalent to *perfectly flexible*. Sometimes the word *chain* is used as synonymous with *string*.

If a flexible string be kept in equilibrium by two forces, one at each end, we assume as self-evident that those forces must be equal and act in opposite directions, so that the string assumes the form of a straight line in the direction of the forces. In this case the *tension* of the string is measured by the force applied at one end.

Let ABC represent a string kept in equilibrium by a force T at one end A and an equal force T at the other end C acting in opposite directions along the line AC. Since any portion AB of the string is in equilibrium it follows that a force T must act on AB at B from B towards C in order to balance the force acting at A; and similarly, T must act on BC from B towards A in order that BC may be in equilibrium. This result is expressed by saying that *the tension of the string is the same throughout.*

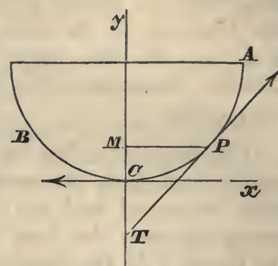
Unless the contrary be expressed, a string is supposed *inextensible* and the boundary of a transverse section of it is supposed to be a curve, every chord of which is indefinitely small.

183. When a flexible string is acted on by other forces besides one at each end it may in equilibrium assume a *curvilinear* form. If at any point of the curve we suppose

a section made by a plane perpendicular to the tangent, the mutual action of the portions on opposite sides of this plane must be in the direction of the tangent, or else equilibrium would not hold, since the string is perfectly flexible.

184. *A heavy string of uniform density and thickness is suspended from two given points; required to find the equation to the curve in which the string hangs when it is in equilibrium.*

Let A, B, be the fixed points to which the ends are attached; the string will rest in a *vertical plane* passing through A and B, because there is no reason why it should deviate to one side rather than the other of this vertical plane. Let ACB be the form it assumes, C being the lowest point; take this as the origin of co-ordinates; let P be any point in the curve; CM, which is vertical, $=y$; MP, which is horizontal, $=x$; $CP=s$.



The equilibrium of any portion CP will not be disturbed if we suppose it to become rigid. Let c and t be the lengths of portions of the string of which the weights equal the tensions at C and P. Then CP is a rigid body acted on by three forces which are proportional to c, s, and t, and act respectively, horizontally, vertically, and along the tangent at P. Draw PT the tangent at P meeting the axis of y in T; then the forces holding CP in equilibrium have their directions parallel to the sides of the triangle PMT, and therefore bear the same proportion one to another that these sides do (see Art. 19); therefore

$$\frac{PM}{MT}=\frac{\text{tension at lowest point}}{\text{weight of the portion } CP}, \text{ or } \frac{dx}{dy}=\frac{c}{s},$$

therefore $$\frac{dy}{dx}=\frac{s}{c} \text{ and } \frac{dy}{ds}=\frac{s}{\sqrt{(c^2+s^2)}};$$

therefore $$y+c=\sqrt{(c^2+s^2)}\text{.....................}(1);$$

the constant added being such that $y=0$ when $s=0$; therefore

$$s^2 = y^2 + 2yc \quad \text{.......................(2)}.$$

Also $$\frac{dx}{dy} = \frac{c}{s} = \frac{c}{\sqrt{(y^2+2yc)}},$$

therefore $$x = c \log \frac{y + c + \sqrt{(y^2+2yc)}}{c} \quad \text{...............(3)},$$

the constant being chosen so that x and y vanish together. The last equation gives

$$ce^{\frac{x}{c}} = y + c + \sqrt{(y^2+2yc)}.$$

Transpose and square; thus

$$c^2 e^{\frac{2x}{c}} - 2(y+c)\, ce^{\frac{x}{c}} + c^2 = 0;$$

therefore $$y + c = \tfrac{1}{2}c\,(e^{\frac{x}{c}} + e^{-\frac{x}{c}}) \quad \text{..................(4)}.$$

Also $$s = \sqrt{\{(y+c)^2 - c^2\}} \text{ by (2)}$$

$$= \tfrac{1}{2}c\,(e^{\frac{x}{c}} - e^{-\frac{x}{c}}) \quad \text{.........................(5)}.$$

Any one of these five equations may be taken as the equation to the curve. If in equation (4) we write y' for $y+c$, which amounts to moving the origin to a point vertically below the lowest point of the curve at a distance c from it, we have

$$y' = \tfrac{1}{2}c\,(e^{\frac{x}{c}} + e^{-\frac{x}{c}}).$$

When the string is uniform in density and thickness, as in the present instance, the curve is called the *common catenary.*

185. *To find the tension of the string at any point.*

Let the tension at P be equal to the weight of a length t of the string; then, as shewn in the last article,

$$\frac{\text{tension at } P}{\text{weight of } CP} = \frac{PT}{TM}, \quad \text{therefore } \frac{t}{s} = \frac{ds}{dy}.$$

But $s^2 = y^2 + 2yc$ by equation (2) of Art. 184, therefore

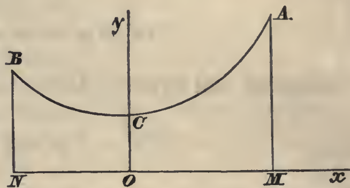
$$t = y + c = y'.$$

This shews that the tension at any point is the weight of a portion of string whose length is the ordinate at that point, the origin being at a distance c below the lowest point.

Hence, if a uniform string hang freely over any two points, the extremities of the string will lie in the same horizontal line when the string is in equilibrium.

186. *To determine the constant* c, *the points of suspension and the length of the string being given.*

Let A and B be the fixed extremities, C the lowest point of the curve.



$$OC = c, \quad OM = a,$$
$$ON = a', \quad MA = b,$$
$$NB = b', \quad CA = l, \quad CB = l'.$$

Also let

$$\left.\begin{array}{l} a + a' = h \\ b - b' = k \\ l + l' = \lambda \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

then h, k, λ are known quantities, since the length of the string and the positions of its ends are given. From Art. 184

$$\left.\begin{array}{l} b = \frac{1}{2}c\,(e^{\frac{a}{c}} + e^{-\frac{a}{c}}) \\ b' = \frac{1}{2}c\,(e^{\frac{a'}{c}} + e^{-\frac{a'}{c}}) \\ l = \frac{1}{2}c\,(e^{\frac{a}{c}} - e^{-\frac{a}{c}}) \\ l' = \frac{1}{2}c\,(e^{\frac{a'}{c}} - e^{-\frac{a'}{c}}) \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Equations (1) and (2) are theoretically sufficient to enable us to eliminate a, a', b, b', l, and l' and to determine c. We may deduce from them

$$\lambda = \frac{1}{2}c\,(e^{\frac{a}{c}} - e^{-\frac{a}{c}} + e^{\frac{a'}{c}} - e^{-\frac{a'}{c}}),$$

$$k = \frac{1}{2}c\,(e^{\frac{a}{c}} + e^{-\frac{a}{c}} - e^{\frac{a'}{c}} - e^{-\frac{a'}{c}});$$

therefore $$\lambda + k = c\,(e^{\frac{a}{c}} - e^{-\frac{a'}{c}}),$$

$$\lambda - k = c\,(e^{\frac{a'}{c}} - e^{-\frac{a}{c}});$$

therefore $$\lambda^2 - k^2 = c^2\,(e^{\frac{a+a'}{c}} + e^{-\frac{a+a'}{c}} - 2)$$

$$= c^2\,(e^{\frac{h}{c}} + e^{-\frac{h}{c}} - 2);$$

therefore $$\sqrt{(\lambda^2 - k^2)} = c\,(e^{\frac{h}{2c}} - e^{-\frac{h}{2c}}) \text{..................} (3).$$

This is the equation from which c is to be found, but on account of its transcendental form it can only be solved by approximation. If the exponents of e are small, we may expand by the exponential theorem and thus obtain the approximate value of c. In order that the exponents may be small, c must be large compared with h; since $\frac{dy}{ds} = \frac{s}{\sqrt{(c^2+s^2)}}$ by Art. 184 it follows that when c is large, compared with the length of the string, $\frac{dy}{ds}$ is small, and therefore the curve does not deviate much from a straight line. Hence, when the two points of support are nearly in a horizontal line and the distance between them nearly equal to the given length of the string, we may conclude that $\frac{h}{c}$ will be small. In this case, we have from (3)

$$\sqrt{(\lambda^2 - k^2)} = 2c\left\{\frac{h}{2c} + \frac{1}{\lfloor 3}\left(\frac{h}{2c}\right)^3 + \frac{1}{\lfloor 5}\left(\frac{h}{2c}\right)^5 + \dots\dots\right\};$$

therefore $$\sqrt{(\lambda^2 - k^2)} = h + \frac{h^3}{24c^2} \text{ approximately.}$$

187. *To find the equations of equilibrium when a flexible string is acted on by any forces.*

Let x, y, z be the co-ordinates of a point P of the string; let s denote the length of the curve BP measured from some fixed point B up to P, and δs the length of the arc PQ between P and an adjacent point Q. Let κ be the area of a section of the string at P, and ρ the density at P; let T be the tension of the string at P; then $T\frac{dx}{ds}$, $T\frac{dy}{ds}$, and $T\frac{dz}{ds}$ are the resolved parts of T parallel to the co-ordinate axes; and the resolved parts of the tension at Q parallel to the axes will be, by Taylor's Theorem,

$$T\frac{dx}{ds}+\frac{d}{ds}\left(T\frac{dx}{ds}\right)\delta s+\text{terms in }(\delta s)^2,\ \&\text{c.}$$

$$T\frac{dy}{ds}+\frac{d}{ds}\left(T\frac{dy}{ds}\right)\delta s+\ldots\ldots\ldots\ldots\ldots\ldots$$

$$T\frac{dz}{ds}+\frac{d}{ds}\left(T\frac{dz}{ds}\right)\delta s+\ldots\ldots\ldots\ldots\ldots\ldots$$

Let $X\rho\kappa\delta s$, $Y\rho\kappa\delta s$, $Z\rho\kappa\delta s$ be the external forces which act on the element PQ parallel to the axes. The equilibrium of the element will not be disturbed by supposing it to become rigid; hence, by Art. 27, the sum of the forces parallel to the axis of x must vanish; thus

$$T\frac{dx}{ds}+\frac{d}{ds}\left(T\,\frac{dx}{ds}\right)\delta s+\ldots+X\rho\kappa\delta s-T\frac{dx}{ds}=0,$$

or $$\frac{d}{ds}\left(T\,\frac{dx}{ds}\right)+X\rho\kappa=0 \text{ ultimately.}$$

Similarly $$\frac{d}{ds}\left(T\,\frac{dy}{ds}\right)+Y\rho\kappa=0,$$

and $$\frac{d}{ds}\left(T\,\frac{dz}{ds}\right)+Z\rho\kappa=0.$$

The product $\kappa\rho$ may be conveniently replaced by m, so that if m be constant ml represents the mass of a length l of the string, and therefore m the mass of a unit of length of the string. If m be not constant, then, if we conceive a string having its length equal to the unit of length and its section and density throughout the same as those of the given string at the point (x, y, z), m will be the mass of such supposed string.

The element δs of the string, the equilibrium of which we have considered, becomes more nearly a *particle* the more we diminish δs; hence it is sufficient to consider the three equations of Art. 27 instead of the six equations of Art. 73.

The three equations which we have found are theoretically sufficient for determining T, y, and z as functions of x, remembering that $\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right\}}$; and when we know the values of y and z in terms of x, we know the equations to the curve which the string forms.

188. The equations for the equilibrium of a flexible string may be written thus;

$$\left.\begin{aligned} T\frac{d^2x}{ds^2} + \frac{dT}{ds}\frac{dx}{ds} + mX = 0 \\ T\frac{d^2y}{ds^2} + \frac{dT}{ds}\frac{dy}{ds} + mY = 0 \\ T\frac{d^2z}{ds^2} + \frac{dT}{ds}\frac{dz}{ds} + mZ = 0 \end{aligned}\right\} \text{..............} (1).$$

Multiply these equations by $\frac{dx}{ds}$, $\frac{dy}{ds}$, and $\frac{dz}{ds}$ respectively and add; then, since

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1,$$

and
$$\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2} = 0,$$

we have
$$\frac{dT}{ds} + m\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) = 0 \text{ ....} (2);$$

therefore $T + \int m\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) ds = \text{constant} \ldots (3).$

If the forces are such that $m(Xdx + Ydy + Zdz)$ is the immediate differential of some function of x, y, z, as $f(x, y, z)$, then

$$T + f(x, y, z) = \text{constant}.$$

If the forces are such that their resultant at every point of the curve is perpendicular to the tangent at that point, we have

$$X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} = 0,$$

therefore, by (3), T is constant.

In the equations (1) transpose the terms mX, mY, mZ to the right-hand side, then square and add; thus

$$T^2\left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right\} + \left(\frac{dT}{ds}\right)^2 = m^2(X^2 + Y^2 + Z^2).$$

Hence if ρ be the radius of absolute curvature of the curve formed by the string, and $Fm\delta s$ the resultant external force on the element δs, so that $F^2 = X^2 + Y^2 + Z^2$,

$$\left(\frac{T}{\rho}\right)^2 + \left(\frac{dT}{ds}\right)^2 = m^2F^2 \text{..................... (4)}.$$

If T be constant $\frac{dT}{ds} = 0$; hence in this case mF varies as $\frac{1}{\rho}$.

From the equations of equilibrium in Art. 187, we deduce by integration,

$$T\frac{dx}{ds} = -\int mXds,$$

$$T\frac{dy}{ds} = -\int mYds,$$

$$T\frac{dz}{ds} = -\int mZds.$$

Square and add; then

$$T^2 = \{\int mXds\}^2 + \{\int mYds\}^2 + \{\int mZds\}^2 \text{......... (5)}.$$

The constants that enter when we integrate the differential equations of equilibrium must be determined from the special circumstances of each particular problem. Thus the co-ordinates of fixed points to which the ends of the string are attached may be given, and the length of the string. Or, besides the forces represented by $mX\delta s$, $mY\delta s$, and $mZ\delta s$ acting on each element, given forces F_1 and F_2 may act at the extremities of the string; in this case if T_1 and T_2 denote the values of T at the two extremities of the string, we must have T_1 equal in magnitude to F_1 and opposite to it in direction, and similarly for T_2 and F_2.

189. From equations (1) of Art. 188, eliminate T and $\frac{dT}{ds}$; then we have

$$X\left(\frac{d^2y}{ds^2}\frac{dz}{ds}-\frac{d^2z}{ds^2}\frac{dy}{ds}\right)+Y\left(\frac{d^2z}{ds^2}\frac{dx}{ds}-\frac{d^2x}{ds^2}\frac{dz}{ds}\right)$$
$$+Z\left(\frac{d^2x}{ds^2}\frac{dy}{ds}-\frac{d^2y}{ds^2}\frac{dx}{ds}\right)=0,$$

this shews that the resultant external force which acts on an element δs of the string lies in the osculating plane at the point (x, y, z).

190. The general equations of equilibrium become, when all the forces are in one plane, namely, that of (x, y),

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)+mX=0,\quad \frac{d}{ds}\left(T\frac{dy}{ds}\right)+mY=0 \;\ldots\ldots\; (1).$$

Suppose $X=0$, so that the external force is parallel to the axis of y; the first equation gives

$$T\frac{dx}{ds}=\text{a constant, } C \text{ say},$$

therefore
$$T=\frac{C}{\frac{dx}{ds}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

Hence the second equation becomes

$$C\frac{d}{ds}\left(\frac{dy}{dx}\right)+mY=0,$$

or
$$C\frac{d^2y}{dx^2}\frac{dx}{ds}+mY=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (3).$$

For example; required the form of the curve when its weight is the only force acting on it and the area of the section at any point is proportional to the tension at that point. Here Y is constant and may be denoted by $-g$, the axis of y being vertically upwards. Let μ be the value of m at the lowest point of the curve, and μag the tension at that point; then

$$\frac{T}{\mu ag}=\frac{m}{\mu}, \text{ therefore } m=\frac{T}{ag}=\frac{C}{ag}\frac{ds}{dx}.$$

Hence (3) gives $$\frac{d^2y}{dx^2}\left(\frac{dx}{ds}\right)^2-\frac{1}{a}=0,$$

therefore $$\frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}=\frac{1}{a},$$

therefore $$\tan^{-1}\frac{dy}{dx}=\frac{x}{a}+\text{constant}.$$

The constant vanishes if we suppose the origin at the lowest point of the curve; therefore

$$\frac{dy}{dx}=\tan\frac{x}{a};$$

therefore $$\frac{y}{a}=-\log\cos\frac{x}{a}\text{ ....(4).}$$

Since in this case the area of the section at any point is proportional to the tension at that point, the curve determined by (4) is called the *Catenary of equal strength.*

191. The equations (1) of the preceding article may be written

$$T\frac{d^2x}{ds^2}+\frac{dT}{ds}\frac{dx}{ds}+mX=0\text{ ....(1).}$$

$$T\frac{d^2y}{ds^2}+\frac{dT}{ds}\frac{dy}{ds}+mY=0\text{ ....(2).}$$

Multiply (1) by $\frac{dy}{ds}$ and (2) by $\frac{dx}{ds}$, and subtract; thus

$$T\left(\frac{d^2x}{ds^2}\frac{dy}{ds}-\frac{d^2y}{ds^2}\frac{dx}{ds}\right)+m\left(X\frac{dy}{ds}-Y\frac{dx}{ds}\right)=0,$$

from which, since $\frac{dx}{ds}\frac{d^2x}{ds^2}+\frac{dy}{ds}\frac{d^2y}{ds^2}=0$, we find

$$T=\frac{m\frac{dx}{ds}}{\frac{d^2y}{ds^2}}\left(X\frac{dy}{ds}-Y\frac{dx}{ds}\right)\text{..............}(3).$$

Again, multiply (1) by $\frac{dx}{ds}$ and (2) by $\frac{dy}{ds}$ and add; then

$$\frac{dT}{ds}+m\left(X\frac{dx}{ds}+Y\frac{dy}{ds}\right)=0\text{................}(4).$$

From (3) and (4) by eliminating T, we deduce

$$m\left(X\frac{dx}{ds}+Y\frac{dy}{ds}\right)+\frac{d}{ds}\left\{\frac{m\frac{dx}{ds}}{\frac{d^2y}{ds^2}}\left(X\frac{dy}{ds}-Y\frac{dx}{ds}\right)\right\}=0,$$

which is the general equation to the curve when given forces act in one plane.

192. In Art. 188 we have found the equations

$$\frac{dT}{ds}+m\left(X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}\right)=0\text{ ....}(1),$$

$$\left(\frac{T}{\rho}\right)^2+\left(\frac{dT}{ds}\right)^2=m^2F^2\text{.......................}(2).$$

Let ϕ be the angle which the resultant force $mF\delta s$ makes with the tangent at the point (x, y, z); then

$$F\cos\phi=X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds};$$

therefore, by (1),

$$\frac{dT}{ds} = -mF\cos\phi \quad\text{.......................}\quad (3),$$

and therefore, by (2),

$$\left(\frac{T}{\rho}\right)^2 = m^2F^2\sin^2\phi \quad\text{.....................}\quad (4).$$

If the force be such that its direction always passes through a fixed point, the whole string will lie in a plane passing through its ends and through the fixed point, for there is no reason why it should lie on one side rather than the other of this plane. Let r be the distance of the point (x, y, z) of the curve from the fixed point, p the perpendicular from the fixed point on the tangent at (x, y, z); then (3) and (4) may be written

$$\frac{dT}{ds} = -mF\frac{dr}{ds} \quad\text{........................}\quad (5),$$

$$\frac{T}{\rho} = mF\frac{p}{r} \quad\text{...........................}\quad (6).$$

Hence $$\frac{1}{T}\frac{dT}{ds} = -\frac{r}{\rho p}\frac{dr}{ds} = -\frac{1}{p}\frac{dp}{ds};$$

therefore $$\log T = \text{constant} - \log p,$$

or $$Tp = C.$$

Also, from (5), $$T = -\int mFdr.$$

Therefore $$\frac{C}{p} = -\int mFdr.$$

Put $\phi(r)$ for $-\int mFdr$; then

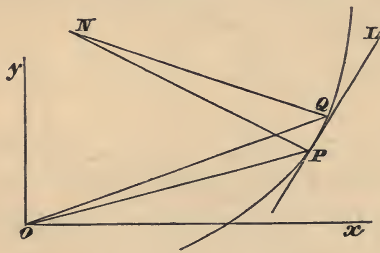
$$\phi(r) = \frac{C}{p} = C\left\{\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}},$$

and from this differential equation the relation between r and θ must be found.

The equation $Tp = C$ may also be obtained simply thus; suppose a finite portion of the string to become rigid; this portion is acted on by the tensions at its two ends and by *other forces which all pass through a fixed point;* take moments round this fixed point; hence the product of the tension into the perpendicular from the fixed point on the tangent must have the same value at the two ends of the finite portion of the string. Thus Tp = constant.

193. The results of the last article give us the form of a string when acted on by any *central force;* these results may also be obtained directly in the following manner.

Let O be the centre of force, P a point in the curve, Q an



adjacent point; r, θ the polar co-ordinates of P; let s be the length of the curve measured from some fixed point up to P, and $PQ = \delta s$. Draw PL the tangent at P; and PN, QN normals at P and Q respectively, then PN is ultimately the radius of curvature at P. Let T denote the tension at P, $T + \delta T$ the tension at Q, $Fm\delta s$ the force acting on the element PQ, which will ultimately be in the direction OP produced.

Let $PNQ = \psi$, and ϕ be the angle between PL and OP produced. Resolve the forces acting on the element along PL and PN; then

$$(T + \delta T) \cos \psi + Fm\delta s \cos \phi - T = 0,$$

$$(T + \delta T) \sin \psi - Fm\delta s \sin \phi = 0.$$

Now $$\sin \psi = \frac{\delta s}{\rho} \text{ ultimately, and } \cos \psi = 1.$$

Hence the equations become

$$\delta T + Fm\delta s \cos\phi = 0,$$

or
$$\frac{dT}{ds} + Fm\cos\phi = 0,$$

and
$$\frac{T}{\rho} - Fm\sin\phi = 0,$$

and the solution may be continued as in the last article.

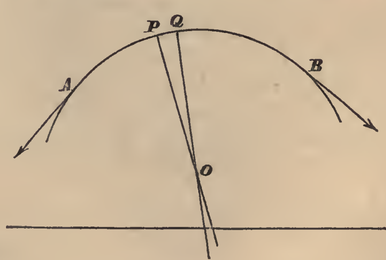
We have supposed the force repulsive, that is, tending *from* O; if it act *towards* O the figure will be *convex* towards O and we shall have the results

$$\frac{dT}{ds} - mF\cos\phi = 0, \quad \frac{T}{\rho} - mF\sin\phi = 0.$$

194. *A string is stretched over a smooth plane curve; to find the tension at any point and the pressure on the curve.*

First suppose the weight of the string neglected.

Let $APQB$ be the string, A and B being the points where



it leaves the curve. Let P, Q be adjacent points in the string; let the normals to the curve at P and Q meet in O; let θ be the angle which PO makes with some fixed line, and $\theta + \delta\theta$ the angle which QO makes with the same line. The element PQ is acted on by a tension at P along the tangent at P, a tension at Q along the tangent at Q, and the resistance of the smooth curve which will be ultimately along PO.

Let s be the length of the curve measured from some fixed point up to P, and $PQ = \delta s$; let $R\delta s$ denote the resistance of the curve on PQ, T the tension at P, $T + \delta T$ the tension at Q. Suppose the element PQ to become rigid, and resolve the forces acting on it along the tangent and normal at P; then

$$T - (T + \delta T) \cos \delta\theta = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$R\delta s - (T + \delta T) \sin \delta\theta = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Now $$\cos \delta\theta = 1 - \frac{(\delta\theta)^2}{1 . 2} + \frac{(\delta\theta)^4}{\underline{|4}} - \ldots$$

hence (1) gives by division by $\delta\theta$

$$\frac{\delta T}{\delta\theta} - (T + \delta T)\left\{\frac{\delta\theta}{2} - \frac{(\delta\theta)^3}{\underline{|4}} + \ldots\right\} = 0;$$

therefore ultimately

$$\frac{dT}{d\theta} = 0,$$

therefore $$T = \text{constant} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Also $\delta s = \rho\delta\theta$ ultimately, ρ being the radius of curvature at P, therefore, from (2), we have

$$R = \frac{T}{\rho} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Since T is constant, the string will not be in equilibrium unless the forces pulling at its two ends are equal; this is usually assumed as self-evident in the theory of the pully.

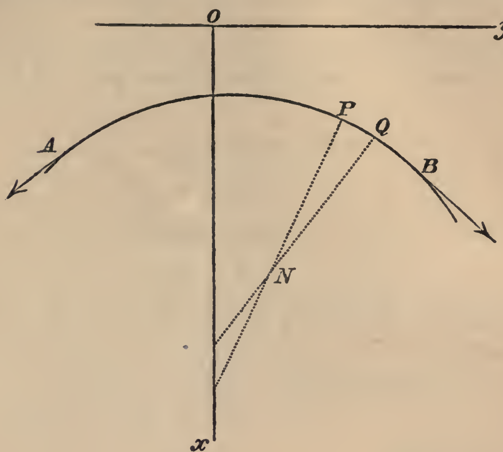
The *whole* pressure on the curve will be $\int R ds$; therefore by (4), the whole pressure

$$= \int \frac{T}{\rho} ds = \int T d\theta.$$

Since T is constant, $\int T d\theta = T\theta + \text{constant}$;

therefore the whole pressure $= T(\theta_2 - \theta_1)$, supposing θ_1 the value of θ at A, and θ_2 at B.

Next suppose the weight of the string taken into account.



Take the axis of y horizontal and that of x vertically downwards. The element PQ is acted on by a tension at P along the tangent at P, a tension at Q along the tangent at Q, the weight of the element vertically downwards, and the resistance of the smooth curve which will be ultimately along the normal at P. Let θ be the acute angle which the normal PN makes with the axis of x, $\theta + \delta\theta$ the angle which the normal QN makes with the axis of x. Let s be the length of the curve measured from some fixed point up to P, and $PQ = \delta s$; let T be the tension at P, and $T + \delta T$ the tension at Q; let $mg\delta s$ be the weight of the element, and $R\delta s$ the resistance of the smooth curve on the element. Suppose the element PQ to become rigid, and resolve the forces acting on it along the tangent and normal at P; then

$$T - (T + \delta T)\cos\delta\theta - mg\delta s \sin\theta = 0 \text{............}(5).$$

$$R\delta s - (T + \delta T)\sin\delta\theta - mg\delta s\cos\theta = 0 \text{............}(6).$$

From (5) we obtain ultimately

$$\frac{dT}{ds} = -mg\sin\theta \text{....................}(7),$$

and from (6)

$$R = \frac{T}{\rho} + mg\cos\theta \text{ .... (8)},$$

where ρ is the radius of cnrvature of the curve at P.

Since the curve is supposed to be a known curve, s and ρ may be supposed known functions of θ; thus (7) and (8) will enable us to find T and R in terms of θ. Or we may express T and R in terms of the rectangular co-ordinates of the point P; for if we denote these co-ordinates by x and y, we have

$$\sin\theta = \frac{dx}{ds}, \quad \cos\theta = \frac{dy}{ds};$$

thus (7) may be written

$$\frac{dT}{ds} = -mg\frac{dx}{ds};$$

therefore, if m be constant,

$$T = -mgx + C,$$

where C is some constant; the value of this constant will be known if the tension of the string be known at some given point, for example at A or at B.

Also from (8)

$$R = \frac{C - mgx}{\rho} + mg\frac{dy}{ds};$$

and ρ and $\frac{dy}{ds}$ will be known in terms of x and y since the curve is known.

195. In the above investigations we stated that the resistance of the curve on the element PQ acts *ultimately* along the normal at P; and in forming the equations of equilibrium of the element of the string we supposed the resistance to act strictly along the normal at P. It is easy to shew that no error is thus introduced. For the resistance at P is along the normal at P, and at Q it is along the normal at Q, hence

the resistance on the element PQ may be taken to be a force which acts in some direction intermediate between the directions of these two normals; suppose ψ the angle which its direction makes with that of the normal at P. We should then write $R\delta s \cos\psi$ instead of $R\delta s$ in the equations (2) and (6), where ψ is an angle less than $\delta\theta$; hence in the limit $\cos\psi = 1$ and equations (4) and (8) remain unchanged. Also the term $R\delta s \sin\psi$ must be introduced into equations (1) and (5); thus equation (1) becomes

$$T - (T + \delta T)\cos\delta\theta - R\delta s \sin\psi = 0;$$

$$\text{therefore } \frac{\delta T}{\delta\theta} - (T + \delta T)\left\{\frac{\delta\theta}{2} - \frac{(\delta\theta)^3}{\lfloor 4} + \ldots\right\} + R\frac{\delta s}{\delta\theta}\sin\psi = 0;$$

and ultimately $\frac{\delta s}{\delta\theta} = \rho$ and $\sin\psi = 0$; hence as before

$$\frac{dT}{d\theta} = 0.$$

Similarly we may shew that equation (7) remains true after the introduction of the term $R\delta s \sin\psi$ into equation (5).

196. *A string is stretched over a rough plane curve; to find the tension at any point and the pressure on the curve in the limiting position of equilibrium.*

First suppose the weight of the string neglected. See the first figure of Article 194.

The element PQ is acted on by a tension at P along the tangent at P, a tension at Q along the tangent at Q, the resistance of the curve which will be ultimately along the normal at P, and the friction which will be ultimately along the tangent at P and in the direction opposite to that in which the element is about to move. Let T denote the tension at P, $T + \delta T$ that at Q, $R\delta s$ the resistance, $\mu R\delta s$ the friction; suppose the string about to move from A towards B. Suppose the element PQ to become rigid, and resolve the forces acting on it along the tangent and normal at P; then

$$T + \mu R\delta s - (T + \delta T)\cos\delta\theta = 0 \ldots\ldots\ldots\ldots(1),$$

$$R\delta s - (T + \delta T)\sin\delta\theta = 0 \ldots\ldots\ldots\ldots(2).$$

From (1) we have ultimately

$$\frac{dT}{ds} = \mu R \dots\dots\dots\dots\dots\dots\dots\dots (3),$$

and from (2),

$$\frac{T}{\rho} = R \dots\dots\dots\dots\dots\dots\dots\dots (4);$$

therefore $\qquad \frac{\rho}{T}\frac{dT}{ds} = \mu, \quad \text{or} \quad \frac{1}{T}\frac{dT}{d\theta} = \mu,$

therefore $\qquad \log T = \mu\theta + \text{constant},$

therefore $\qquad T = Ce^{\mu\theta}.$

Let T_1 be the force which acts on the string at the end A, and therefore the value of T at this point; and let T_2 be the force at B; let θ_1 and θ_2 be the corresponding values of θ;

then $\qquad T_1 = Ce^{\mu\theta_1}, \quad T_2 = Ce^{\mu\theta_2};$

therefore $\qquad \frac{T_2}{T_1} = e^{\mu(\theta_2 - \theta_1)},$

and $\qquad T = T_1 e^{\mu(\theta-\theta_1)} = T_2 e^{\mu(\theta-\theta_2)}.$

Also $\qquad \int R\,ds = \int \frac{T}{\rho}\,ds = \int T\,d\theta = T_1 \int e^{\mu(\theta-\theta_1)}\,d\theta,$

$$= \frac{T_1}{\mu} e^{\mu(\theta-\theta_1)} + \text{constant};$$

therefore the whole pressure on the curve

$$= \frac{T_1 e^{-\mu\theta_1}}{\mu} \left(e^{\mu\theta_2} - e^{\mu\theta_1}\right) = \frac{T_2 - T_1}{\mu}.$$

Next suppose the weight of the string taken into account. Proceeding as in the second case of Art. 194, and supposing the string about to move from A to B, we have

$$T - (T + \delta T)\cos\delta\theta - mg\delta s \sin\theta + \mu R\delta s = 0 \dots\dots\dots (5).$$

$$R\delta s - (T + \delta T)\sin\delta\theta - mg\delta s \cos\theta = 0 \dots\dots\dots\dots (6)$$

From (5) we obtain ultimately

$$\frac{dT}{ds} = \mu R - mg \sin \theta,$$

and from (6) $$R = \frac{T}{\rho} + mg \cos \theta;$$

therefore $$\frac{dT}{ds} = \frac{\mu T}{\rho} + mg (\mu \cos \theta - \sin \theta),$$

therefore $$\rho \frac{dT}{ds} - \mu T = mg (\mu \cos \theta - \sin \theta) \rho,$$

that is $$\frac{dT}{d\theta} - \mu T = mg (\mu \cos \theta - \sin \theta) \rho.$$

Thus we have a *differential equation* for finding T, and we may proceed in the ordinary way to obtain the solution. Multiply both sides of the last equation by $e^{-\mu\theta}$; thus

$$\frac{d}{d\theta} (Te^{-\mu\theta}) = mge^{-\mu\theta} (\mu \cos \theta - \sin \theta) \rho$$

therefore $$Te^{-\mu\theta} = \int mge^{-\mu\theta} (\mu \cos \theta - \sin \theta) \rho \, d\theta.$$

Hence when ρ is known in terms of θ we shall only have to integrate a known function of θ in order to obtain the value of T in terms of θ.

197. *To form the equations of equilibrium of a string stretched over a smooth surface and acted on by any forces.*

Let s be the length of the string measured from some fixed point B to the point P; x, y, z the co-ordinates of P; δs the length of the element of the string between P and an adjacent point Q; $m\delta s$ the mass of the element; $R\delta s$ the resistance of the surface on this element, the direction of which will be ultimately the normal to the surface at P; let α, β, γ be the angles which the normal at P makes with the axes; $Xm\delta s$, $Ym\delta s$, $Zm\delta s$ the forces parallel to the axes acting on the element, exclusive of the resistance $R\delta s$. Hence, in the equations of Art. 187, for Xm we must put $Xm + R\cos\alpha$, and make similar substitutions for Ym and Zm; therefore

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + Xm + R\cos\alpha = 0 \quad\ldots\ldots\ldots\ldots(1),$$

$$\frac{d}{ds}\left(T\frac{dy}{ds}\right) + Ym + R\cos\beta = 0 \quad\ldots\ldots\ldots\ldots(2),$$

$$\frac{d}{ds}\left(T\frac{dz}{ds}\right) + Zm + R\cos\gamma = 0 \quad\ldots\ldots\ldots\ldots(3).$$

Multiply (1) by $\frac{dx}{ds}$, (2) by $\frac{dy}{ds}$, and (3) by $\frac{dz}{ds}$, and add; then, since

$$\frac{dx}{ds}\cos\alpha + \frac{dy}{ds}\cos\beta + \frac{dz}{ds}\cos\gamma = 0,$$

because a tangent to the surface at any point is perpendicular to the normal at that point, we have, as in Art. 188,

$$\frac{dT}{ds} + m\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) = 0 \quad\ldots\ldots\ldots (4).$$

Again, multiply (1) by $\cos\alpha$, (2) by $\cos\beta$, and (3) by $\cos\gamma$, and add; then

$$T\left\{\frac{d^2x}{ds^2}\cos\alpha + \frac{d^2y}{ds^2}\cos\beta + \frac{d^2z}{ds^2}\cos\gamma\right\}$$

$$+ m\{X\cos\alpha + Y\cos\beta + Z\cos\gamma\} + R = 0 \ldots (5).$$

Let $Fm\delta s$ be the resultant of $Xm\delta s$, $Ym\delta s$, $Zm\delta s$, and ψ the angle its direction makes with the normal to the surface at the point (x, y, z); then

$$X\cos\alpha + Y\cos\beta + Z\cos\gamma = F\cos\psi.$$

Let ρ be the radius of absolute curvature of the curve formed by the string at the point (x, y, z); α', β', γ' the angles its direction makes with the axes; ϕ the angle its direction makes with the normal to the surface; then

$$\frac{d^2x}{ds^2}=\frac{\cos\alpha'}{\rho},\quad \frac{d^2y}{ds^2}=\frac{\cos\beta'}{\rho},\quad \frac{d^2z}{ds^2}=\frac{\cos\gamma'}{\rho}.$$

Hence (5) becomes

$$\frac{T}{\rho}\cos\phi+Fm\cos\psi+R=0\ldots\ldots\ldots\ldots\ldots(6).$$

Let $u=0$, be the equation to the surface; then

$$\frac{\cos\alpha}{\frac{du}{dx}}=\frac{\cos\beta}{\frac{du}{dy}}=\frac{\cos\gamma}{\frac{du}{dz}}=N \text{ suppose.}$$

Hence (1) may be written

$$T\frac{d^2x}{ds^2}+Xm+\frac{dT}{ds}\frac{dx}{ds}+RN\frac{du}{dx}=0,$$

and (2) and (3) may be similarly expressed.

Eliminate $\frac{dT}{ds}$ and RN, and we obtain

$$\left(T\frac{d^2x}{ds^2}+Xm\right)\left(\frac{dy}{ds}\frac{du}{dz}-\frac{dz}{ds}\frac{du}{dy}\right)$$
$$+\left(T\frac{d^2y}{ds^2}+Ym\right)\left(\frac{dz}{ds}\frac{du}{dx}-\frac{dx}{ds}\frac{du}{dz}\right)$$
$$+\left(T\frac{d^2z}{ds^2}+Zm\right)\left(\frac{dx}{ds}\frac{du}{dy}-\frac{dy}{ds}\frac{du}{dx}\right)=0.$$

If we put for T its value from (4), the resulting equation, together with $u=0$, will determine the curve formed by the string.

It appears from Art. 189 that the resultant of $Fm\delta s$ and $R\delta s$ must lie in the osculating plane of the curve at the point

(x, y, z). If the direction of $Fm\delta s$ be always normal to the surface $u=0$, then, since that of $R\delta s$ is also normal to the surface, it follows that the *normal to the surface lies in the osculating plane to the curve.* This we know to be a property of the lines of maximum or minimum length that can be drawn on a surface between two given points. Hence, when a string is stretched over a smooth surface and acted on only by forces which are in the direction of normals to the surface at their points of application, it forms the line of maximum or minimum length that can be drawn on the surface between the extreme points of its contact with the surface.

When $Fm\delta s$ is always normal to the surface, it follows from (4) that T is constant.

EXAMPLES.

1. In the common catenary shew that the weight of the string between the lowest point and any other point is the geometrical mean between the sum and difference of the tensions at the two points.

2. If α and β are the inclinations to the horizon of the tangents at the extremities of a portion of a common catenary and l the length of the portion, shew that the height of one extremity above the other is

$$l\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}};$$

the portion is supposed to be all on the same side of the lowest point.

3. A uniform heavy chain 110 feet long is suspended from two points in the same horizontal plane 108 feet asunder; shew that the tension at the lowest point is 1.477 times the weight of the chain nearly.

4. A uniform chain of length $2l$ is suspended from two fixed points in the same horizontal plane; $2a$ is the distance between the fixed points and c the length of chain whose weight is equal to the tension at the lowest point; shew that

when l is such that the tension at the points of suspension is the least possible that tension is equal to the weight of a length $\frac{al}{c}$ of the chain, and l and c are determined by

$$l = \tfrac{1}{2}c\left(e^{\frac{a}{c}} - e^{-\frac{a}{c}}\right), \quad (l^2 + c^2)\, c^2 = a^2 l^2.$$

5. If a uniform chain be fixed at two points, and any number of links A, B, C, ... be at liberty to move along smooth horizontal lines in the same vertical plane, prove that the loops AB, BC, CD, &c. will form themselves into curves which will all be arcs of the same catenary.

6. Three links of a chain A, B, and C are moveable freely along three rigid horizontal lines in the same vertical plane. If when A and C are pulled as far apart as possible, their horizontal distances from B are equal, shew that this will always be the case when they are held in any other position.

7. A chain hangs in equilibrium over two smooth points which are in a horizontal line and at a given distance apart; find the least length of the chain that equilibrium may be possible.

Result. The least length is ae, where $2a$ is the given distance.

8. Prove that the exertion necessary to hold a kite diminishes as the kite rises higher, the force of the wind being independent of the height, and the pressure of the wind on the string being neglected.

9. A uniform heavy string rests on an arc of a smooth curve whose plane is vertical, shew that the tension at any point is proportional to its vertical height above the lowest point of the string. If the string rests on a parabola whose axis is vertical, determine the vertical distance of its ends below the highest point so that the pressure at this point may be equal to twice the weight of a unit of length of the string.

Result. The vertical distance is equal to half the latus rectum of the parabola.

10. One end of a uniform heavy chain hangs freely over the edge of a smooth table, and the other end passing over a fixed pully reaches to the same distance below the table as the pully is above it. Supposing half the chain to be on the table in the position of equilibrium, compare its whole length with the height of the pully.

Result. The length is to the height as $6 + 2\sqrt{3}$ to 1.

11. A uniform heavy chain is fastened at its extremities to two rings of equal weights which slide on smooth rods intersecting in a vertical plane and inclined at the same angle α to the vertical; find the condition that the tension at the lowest point may be equal to half the weight of the chain; and in that case shew that the vertical distance of the rings from the point of intersection of the rods is

$$\frac{l}{2} \cot \alpha \log (1 + \sqrt{2}),$$

where l is the length of the chain.

12. The density at any point of a catenary of variable density varies as the radius of curvature; determine the equation to the catenary.

Result. The curve in Art. 190.

13. A heavy cord with one end fixed to a point in the surface of a smooth horizontal cylinder is passed below the cylinder and carried round over the top, the other end being allowed to hang freely. Shew that unless the portion which hangs vertically be longer than the diameter of the cylinder, the cord will slip off, so as to hang down from the fixed point without passing below the cylinder.

14. If a uniform string hang in the form of a parabola by the action of normal forces only, the force at any point P varies as $(SP)^{-\frac{3}{2}}$, S being the focus.

15. If a string without weight touch a given cylinder in $\frac{1}{4}$th part of its circumference and in a plane perpendicular to its axis, what tension at one extremity will support a weight of 100lbs. suspended at the other, friction being supposed to be $\frac{1}{10}$th part of the pressure? To what will this tension be reduced if the string is wound round $1\frac{1}{4}$th circumferences?

16. If $\mu = \frac{1}{4}$, and a string without weight passes twice round a post, prove, by taking approximate values of e and π, that any force will support another more than twenty times as great.

17. If two scales, one containing a weight P and the other a weight Q, be suspended by a string without weight over a rough sphere, and if Q be on the point of descending, then the weight $\frac{Q^2 - P^2}{P}$ put into the opposite scale will make that scale be on the point of descending.

18. Two equal weights P, P' are connected by a string without weight which passes over a rough fixed horizontal cylinder; compare the forces required to raise P according as P is pushed up or P' pulled down.

19. A, B, C are three rough pegs in a vertical plane: P, Q, R are the greatest weights which can be severally supported by a weight W, when connected with it by strings without weight passing over A, B, C, over A, B, and over B, C respectively; shew that the coefficient of friction at B is $\frac{1}{\pi} \log \frac{Q \,.\, R}{P \,.\, W}$.

20. A light thread, whose length is $7a$, has its extremities fastened to those of a uniform heavy rod whose length is $5a$, and when the thread is passed over a thin round peg, it is found that the rod will hang at rest, provided the point of support be anywhere within a space a in the middle of the thread; determine the coefficient of friction between the thread and the peg when the rod hangs in a position bordering upon motion, and find its inclination to the horizon and the tensions of the different parts of the string.

Results. The coefficient of friction is determined by the equation $e^{\frac{1}{2}\mu\pi} = \frac{4}{3}$. The inclination of the rod to the horizon is $\cos^{-1} \frac{24}{25}$.

21. From a fixed point a heavy uniform chain hangs down so that part of the chain rests on a rough horizontal

plane; find the least length of chain that may be in contact with the plane.

22. A heavy chain of weight W rests entirely in contact with the arc of a rough vertical circle in a state bordering on motion. If $\tan\alpha$ be the coefficient of friction, shew that the resultant normal pressure on the circle is equal to $W\cos\alpha$, and that its direction makes an angle α with the vertical.

23. A heavy chain of length l rests partly on a rough horizontal table, and the remainder passing over the smooth edge of the table, (which is rounded off into the form of a semicylinder of radius a) hangs freely down; shew that if z be the least length on the table consistent with equilibrium,

$$z(\mu+1) = l - \tfrac{1}{2}\pi a + a.$$

24. A heavy uniform chain is hung round the circumference of a rough vertical circle of given radius. How much lower must one end of the chain hang than the other when it is on the point of motion?

Result. Let a be the length of the longer piece which hangs down, b the length of the shorter piece, r the radius of the circle, $\tan\beta$ the coefficient of friction; then

$$e^{\pi\tan\beta} = \frac{a - r\sin 2\beta}{b + r\sin 2\beta}.$$

25. A uniform beam of weight W is moveable about a hinge at one extremity, and has the other attached to a string without weight which, passing over a very small rough peg placed vertically above the hinge, and at a distance from it equal to the length of the beam, supports a weight P; shew that if θ be the inclination of the beam to the vertical when it is just upon the point of falling, then

$$W\sin\tfrac{1}{2}\theta = Pe^{\frac{\mu(\pi+\theta)}{2}}.$$

Find also the strain on the hinge.

26. One end of a heavy chain is attached to a fixed point A, and the other end to a weight which is placed on a rough horizontal plane passing through A, and the chain hangs through a slit in the horizontal plane. Shew that if l be

the length of the chain, a the greatest distance of the weight from A at which equilibrium is possible, μ the coefficient of friction, and n twice the ratio of the given weight to the weight of the chain,

$$\mu(1+n)\,e^{\frac{a}{l\mu(1+n)}} = 1 + \sqrt{\{1 + \mu^2(1+n)^2\}}.$$

27. A uniform string acted on by a central force assumes the form of an arc of a circle; determine the law of the force, the centre of force being on the circumference of the circle.

Result. The force varies inversely as the cube of the distance.

28. A smooth sphere rests upon a string without weight fastened at its extremities to two fixed points; shew that if the arc of contact of the string and sphere be not less than $2\tan^{-1}\frac{48}{55}$, the sphere may be divided into two equal portions by means of a vertical plane without disturbing the equilibrium.

29. Shew that if a chain exactly surrounds a smooth vertical circle, so as to be in contact at the lowest point without pressing, the whole pressure on the circle is double the weight of the chain, and the tension at the highest point is three times that at the lowest.

30. Two strings without weight of the same length have each of their ends fixed at each of two points in the same horizontal plane. A smooth sphere of radius r and weight W is supported upon them at the same distance from each of the given points. If the plane in which each string lies make an angle α with the horizon, prove that the tension of each is $\frac{Wa}{8r}\operatorname{cosec}\alpha$; a being the distance between the points.

31. A uniform heavy chain hangs over two smooth pegs at a distance $2a$ apart in the same horizontal plane. When there is equilibrium, $2s$ is the length of the chain between the pegs, which hangs in the form of a catenary, c is the length of a portion of the chain whose weight is equal to the tension at the lowest point, and h the length of the end that hangs

down vertically. If δs and δh be the small increments of s and h corresponding to a small uniform expansion of the chain, shew that $\delta s : \delta h = s.c - h.a : h.c - s.a$.

32. A uniform heavy chain is placed on a rough inclined plane; what length of chain must hang over the top of the plane, in order that the chain may be on the point of slipping up the plane?

33. A uniform rod of length b has its ends attached to the ends of a flexible string without weight of length a; this string is passed over a very small cylindrical peg, and when the rod hangs in its limiting position of equilibrium, the parts of the string on opposite sides of the peg are inclined to each other at an angle α. Shew that the coefficient of friction between the string and peg is

$$\frac{1}{\pi - \alpha} \log \frac{a + \sqrt{\{b^2 - (a^2 - b^2) \tan^2 \frac{1}{2}\alpha\}}}{a - \sqrt{\{b^2 - (a^2 - b^2) \tan^2 \frac{1}{2}\alpha\}}}.$$

34. AB, AC are two equal and uniform rods moveable about a fixed hinge at A, CB a uniform chain, equal in length to AB or AC and $\left(\frac{1}{n}\right)^{\text{th}}$ of its weight, connects the ends B and C; shew that in the position of equilibrium, the angle θ which either rod makes with the horizon is given approximately by the equation

$$\cos\theta = \frac{1}{2} - \frac{1}{4(n+1)^2},$$

n being large compared with unity.

35. A heavy uniform beam has its extremities attached to a string which passes round the arc of a rough vertical circle; if in the limiting position of equilibrium the beam be inclined at an angle of 60° to the vertical, and the portion of string in contact with the circle cover an arc of 270°, shew that the coefficient of friction is $\frac{1}{3\pi} \log 3$.

36. A uniform string just circumscribes a given smooth circle, and is attracted by a force varying as the distance to

a point within the circle. Find the tension at any point, supposing it to vanish at the point nearest to the centre of force, and shew that the force at the greatest distance

$$=\frac{\text{whole pressure on circle}}{\text{mass of the string}}.$$

37. A heavy string whose length is $\frac{\pi}{2}a$ rests on the circumference of a rough vertical circle of radius a; if the string be in a position of limiting equilibrium, and if β be the angular distance of its highest extremity from the vertex of the circle, shew that

$$\tan\beta=\frac{(1-\mu^2)\,e^{\frac{\mu\pi}{2}}-2\mu}{1-\mu^2+2\mu e^{\frac{\mu\pi}{2}}},$$

and explain this result when $(1-\mu^2)\,e^{\frac{\mu\pi}{2}}-2\mu$ is negative.

Also if μ be such that $\beta=0$, shew that the *whole pressure* on the curve is to the weight of the string as 2 to $\pi\mu$.

CHAPTER XII.

FLEXIBLE STRINGS. EXTENSIBLE.

198. In the preceding chapter we considered the equilibrium of flexible *inextensible* strings; we now proceed to some propositions relative to flexible *extensible* strings. Such strings are also called *elastic* strings.

When a uniform extensible string is stretched by a force, it is found by experiment that the extension varies as the product of the original length and the stretching force. Thus if T represent the force, l' the original length, l the stretched length,

$$l - l' = \frac{l'T}{\lambda},$$

where λ is some constant depending on the nature of the string.

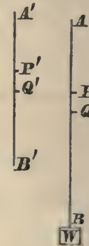
The fact expressed by this equation is called Hooke's law, from the name of its discoverer.

The quantity λ is sometimes called the *modulus of elasticity*.

In the equation $l - l' = \dfrac{l'T}{\lambda}$ if we put $T = \lambda$ we obtain $l = 2l'$; thus the modulus of elasticity for any uniform elastic string is equal to the tension required to stretch that string to double its natural length.

199. *An elastic string has a weight attached to one end, it is fastened at the other and hangs vertically; determine the extension of the string, taking its own weight into account.*

Let $A'B'$ represent the natural length of the string; AB the stretched length. Let $A'P' = x'$, $P'Q' = \delta x'$. Suppose $A'P'$ stretched into AP, and $P'Q'$ into PQ; let $AP = x$, $PQ = \delta x$. Let $A'B' = a'$, w = the weight of the string, and W be the attached weight.



Let T be the tension at P, and $T + \delta T$ the tension at Q. Then the element PQ is acted on by the forces T and $T + \delta T$ at its ends, and by its own weight; its weight is the same as that of $P'Q'$, that is $\frac{\delta x'}{a'} w$;

therefore $$T + \delta T - T + \frac{\delta x'}{a'} w = 0,$$

or $$\frac{dT}{dx'} = -\frac{w}{a'} \text{ ultimately} \quad\text{..................(1)};$$

therefore $$T = -\frac{wx'}{a'} + \text{constant}.$$

The value of the constant must be found by observing that when $x' = a'$, $T = W$; therefore

$$W = -w + \text{constant};$$

therefore $$T = W + w\left(1 - \frac{x'}{a'}\right) \quad\text{.................. (2).}$$

Also the element PQ may be considered ultimately uniform and stretched by a tension T; hence, by the experimental law,

$$\delta x = \delta x' \left(1 + \frac{T}{\lambda}\right) \quad\text{......................(3)};$$

therefore $$\frac{dx}{dx'} = 1 + \frac{T}{\lambda},$$

$$= 1 + \frac{W}{\lambda} + \frac{w}{\lambda}\left(1 - \frac{x'}{a'}\right).$$

Integrate; thus

$$x = x'\left(1 + \frac{W + w}{\lambda}\right) - \frac{wx'^2}{2\lambda a'}.$$

No constant is required because $x = 0$ when $x' = 0$.

Let a denote the stretched length of the string; then putting $x'=a'$, we have

$$a = a'\left(1+\frac{W+w}{\lambda}\right) - \frac{wa'}{2\lambda} = a'\left(1+\frac{W+\frac{1}{2}w}{\lambda}\right).$$

Thus the extension is the same as would be produced if an elastic string of length a', the weight of which might be neglected, were stretched by a weight $W+\frac{1}{2}w$ at its end.

200. In the solution of the preceding problem we might have arrived at equation (2) by observing that the tension at any point must be equal to the weight of the string below that point together with W; but the method we adopted is more useful as a guide to the solution of similar problems. It is perhaps not superfluous to notice an error into which students often fall; since the element δx is acted on by a tension T at one end, and $T+\delta T$ or ultimately T at the other end, $2T$ is considered the stretching force, and instead of (3)

$$\delta x = \delta x'\left(1+\frac{2T}{\lambda}\right)$$

is used. This would be of no consequence if *uniformly* adopted, for it would only amount to using $\frac{1}{2}\lambda$ instead of λ in (3); but mistakes arise from not adhering to one system or the other. It should be observed that if a string without weight be acted on by a force T at each end, it is in the same state of tension as if it were *fastened* at one end and acted on by a force T at the other.

201. The equations of Art. 187, and Art. 197 may be applied to an elastic string in equilibrium. They may also be modified as follows, if we wish to introduce the unstretched length of the string instead of the stretched length.

Let s' and $\delta s'$ represent the natural lengths which become s and δs by stretching; let $m'\delta s'$ be the mass of an element before stretching, and $m\delta s$ the mass of the same element after stretching; then

$$m\delta s = m'\delta s',$$

$$\delta s = \delta s'\left(1+\frac{T}{\lambda}\right);$$

therefore $$m\left(1+\frac{T}{\lambda}\right)=m'.$$

Hence the first equation of equilibrium of Art. 187 may be written

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)+\frac{m'X}{1+\frac{T}{\lambda}}=0;$$

and the other two equations may be written similarly.

Equation (2) of Art. 188, or equation (4) of Art. 197 becomes

$$\left(1+\frac{T}{\lambda}\right)\frac{dT}{ds}+m'\left(X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}\right)=0; \text{ therefore}$$

$$\lambda\left(1+\frac{T}{\lambda}\right)^2+2m'\int\left(X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}\right)ds=\text{constant},$$

provided m' be constant; that is, provided the string in its unstretched state be uniform.

Since $\left(1+\frac{T}{\lambda}\right)^2=\left(\frac{ds}{ds'}\right)^2$, the last equation may be used to connect s and s', and thus find the extension of the string.

202. We may apply the preceding article to the case in which the weight of the string is the only force acting on it, the string being supposed originally uniform, and fixed at two points.

In this case $X=0$, $Y=-g$, $Z=0$, as in Art. 190; therefore

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)=0 \quad\text{.........................}(1).$$

$$\left(1+\frac{T}{\lambda}\right)\frac{d}{ds}\left(T\frac{dy}{ds}\right)-m'g=0 \quad\text{..................}(2).$$

From (1) $T\frac{dx}{ds}=$ a constant $=m'cg$ suppose;

therefore $$T=m'cg\sec\psi \quad\text{.........................}(3),$$

where ψ is the angle which the tangent to the curve at the point (x, y) makes with the axis of x.

Hence (2) gives

$$\left(1+\frac{m'cg}{\lambda}\sec\psi\right)\frac{d\tan\psi}{ds}=\frac{1}{c}\ \ldots\ldots\ldots\ldots\ldots(4),$$

therefore $$\left(1+\frac{m'cg}{\lambda}\sec\psi\right)\frac{d\tan\psi}{dx}\frac{dx}{ds}=\frac{1}{c},$$

thus $$\cos\psi\left(1+\frac{m'cg}{\lambda}\sec\psi\right)\frac{d\tan\psi}{dx}=\frac{1}{c},$$

that is $$\frac{1}{\cos\psi}\frac{d\psi}{dx}+\frac{m'cg}{\lambda}\frac{d\tan\psi}{dx}=\frac{1}{c};$$

therefore, by integration,

$$\int\frac{1}{\cos\psi}\frac{d\psi}{dx}dx+\frac{m'cg}{\lambda}\tan\psi=\frac{x}{c};$$

and $$\int\frac{1}{\cos\psi}\frac{d\psi}{dx}dx=\int\frac{d\psi}{\cos\psi}=\log\frac{1+\sin\psi}{\cos\psi};$$ thus

$$\log\frac{1+\sin\psi}{\cos\psi}+\frac{m'cg}{\lambda}\tan\psi=\frac{x}{c}\ \ldots\ldots\ldots\ldots\ldots(5).$$

No constant is required in the integration if we suppose the axis of y to pass through the lowest point of the curve, for there $\psi=0$.

From (4) we may deduce

$$\sin\psi\left(1+\frac{m'cg}{\lambda}\sec\psi\right)\frac{d\tan\psi}{dy}=\frac{1}{c}\ \ldots\ldots\ldots\ldots(6);$$

therefore, by integration,

$$\sec\psi+\frac{m'cg}{2\lambda}\tan^2\psi=\frac{y}{c}\ \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

No constant is required in the integration if we suppose the origin of co-ordinates to be at the distance c below the lowest point of the curve.

We cannot obtain an equation between x and y in a *finite form;* but in a particular case we can obtain such an equation which is approximately true. Let $\lambda = m'gl$; then (5) may be written

$$\frac{1+\sin\psi}{\cos\psi} = e^{\frac{x}{c} - \frac{c}{l}\tan\psi},$$

therefore
$$\frac{\cos\psi}{1+\sin\psi} = e^{-\frac{x}{c} + \frac{c}{l}\tan\psi};$$

therefore by addition and reduction

$$\frac{2}{\cos\psi} = e^{\frac{x}{c} - \frac{c}{l}\tan\psi} + e^{-\frac{x}{c} + \frac{c}{l}\tan\psi},$$

therefore
$$\tan^2\psi = \tfrac{1}{4}\left(e^{\frac{x}{c} - \frac{c}{l}\tan\psi} - e^{-\frac{x}{c} + \frac{c}{l}\tan\psi}\right)^2.$$

Now suppose $\frac{c}{l}$ is a very small quantity, put u for $\frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)$ and v for $\frac{1}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$; then the last equation gives

$$\tan\psi = u - \frac{cv}{l}\tan\psi + \frac{c^2u}{1.2\,l^2}\tan^2\psi - \frac{c^3v}{\underline{|3}\,l^3}\tan^3\psi + \dots;$$

from this we can find $\tan\psi$ approximately, and then $\sec\psi$ will be known approximately, and by substituting in (7) we shall obtain approximately y in terms of x.

Equation (2) may also be written

$$\frac{d}{ds'}\left(T\frac{dy}{ds}\right) = m'g,$$

therefore
$$\frac{d}{ds'}\left(m'cg\frac{dy}{dx}\right) = m'g;$$

therefore, by integration,

$$\frac{dy}{dx} = \frac{s'}{c};$$

here s' denotes the natural length of that portion of the string which is between the lowest point and the point (x, y).

Hence for $\tan\psi$ in (5) and (7) we may put $\frac{s'}{c}$, and make corresponding substitutions for $\sin\psi$ and $\cos\psi$. Thus (7) becomes

$$\sqrt{(c^2+s'^2)}+\frac{m'gs'^2}{2\lambda}=y \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

As an example of these formulæ suppose that a heavy uniform elastic string hangs in equilibrium over two smooth pegs in a horizontal plane, and let it be required to find the depth of the ends of the string below the vertex of the curved portion.

From (3) the tension at any point of the curve is

$$m'g\sqrt{(c^2+s'^2)}.$$

Let l' be the natural length of the portion which hangs over one of the pegs; then the weight of this portion is $m'gl'$. Let s' denote the unstretched length of the portion between the vertex and one peg; then by equating the two expressions for the tension, we have

$$m'gl'=m'g\sqrt{(c^2+s'^2)},$$

therefore $$l'=\sqrt{(c^2+s'^2)} \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

Now by (8) $$c^2+s'^2+\frac{2\lambda}{m'g}\sqrt{(c^2+s'^2)}=\frac{2\lambda y}{m'g}+c^2;$$

therefore

$$\sqrt{(c^2+s'^2)}+\frac{\lambda}{m'g}=\frac{1}{m'g}\sqrt{(c^2m'^2g^2+\lambda^2+2\lambda m'gy)} \ldots\ldots (10).$$

Suppose l to be the length to which the string hanging vertically, of which the natural length is l', is stretched; then by Art. 199

$$l=l'\left(1+\frac{m'gl'}{2\lambda}\right);$$

therefore $$l'+\frac{\lambda}{m'g}=\frac{1}{m'g}\sqrt{(2l\lambda m'g+\lambda^2)};$$

hence from this result combined with (9) and (10)

$$2l\lambda m'g + \lambda^2 = c^2 m'^2 g^2 + \lambda^2 + 2\lambda m'gy;$$

therefore $$l = y + \frac{m'gc^2}{2\lambda}.$$

Thus the end of the string descends to the depth $\frac{m'gc^2}{2\lambda}$ below the axis of x, and therefore to the depth $c\left(1 + \frac{m'cg}{2\lambda}\right)$ below the vertex of the curve.

EXAMPLES.

1. Two equal heavy beams, AB, CD, are connected diagonally by similar and equal elastic strings AD, BC; determine the position of equilibrium when AB is held horizontal; and shew that if the natural length of each string equals AB, and the elasticity be such that the weight of AB would stretch the string to three times its natural length, then

$$\frac{1}{AB} = \frac{1}{BC} + \frac{1}{AC}.$$

2. An elastic string will just reach round two pegs in a horizontal plane; a ring whose weight would double the length of the string hanging from a point is slung on it: shew that if θ be the inclination of two portions of the string to the horizon,

$$\sin 2\theta = 2(\sqrt{2} - 1).$$

3. An elastic string has its ends attached to those of a uniform beam of the same length as the unstretched string, the weight of the beam being such as would stretch the string to twice its natural length; shew that when the system is hung up by means of the string on a smooth peg, the inclination θ of the string to the vertical will be given by the equation

$$\tan\theta + 2\sin\theta - 2 = 0.$$

4. Three equal circular discs are placed in contact in a vertical plane with their centres in the same horizontal line, and an endless elastic cord wound alternately above and below them, so as to touch every point of their circumferences without being stretched beyond its natural length. When the support of the middle disc is removed, the centres of the three form a right-angled triangle. Shew that the modulus of elasticity of the cord is $\frac{W}{2} \cdot \frac{3\pi}{4-\pi}$, W being the weight of the disc.

5. A fine elastic string is tied round two equal cylinders whose surfaces are in contact and axes parallel, the string not being stretched beyond its natural length; one of the cylinders is turned through two right angles, so that the axes are again parallel: find the tension of the string, supposing a weight of 1 lb. would stretch it to twice its natural length.

Result. $\frac{\pi-2}{\pi+2}$ of a lb.

6. Two equal and similar elastic strings AC, BC, fixed at two points A, B in the same horizontal line, support a given weight at C. The extensibility and original lengths of the strings being given, find an equation for determining the angle at which each string is inclined to the horizon, and deduce an approximate value of the angle when the extensibility is very small.

7. Six equal rods are fastened together by hinges at each end, and one of the rods being supported in a horizontal position the opposite one is fastened to it by an elastic string joining their middle points. Supposing the modulus of elasticity is equal to the weight of each rod, find the original length of the string in order that the hexagon may be equiangular in its position of equilibrium.

Result. $\frac{a\sqrt{3}}{4}$, where a is the length of a rod.

8. An unstretched elastic string without weight has n equal weights attached to it at equal distances, and is then suspended from one end. Prove that the increase of length is

half what it would be if the same string were stretched by a weight equal to $n+1$ of the former hanging at one end.

9. Three equal cylindrical rods are placed symmetrically round a fourth of the same radius, and the bundle is then surrounded by two equal elastic bands at equal distances from the two ends; if each band when unstretched would just pass round one rod, and a weight of 1 lb. would just stretch it to twice its natural length, shew that it would require a force of 9 lbs. to extract the middle rod, the coefficient of friction being equal to $\frac{1}{6}\pi$.

10. Two elastic strings are just long enough to fit on a sphere without stretching; they are placed in two planes at right angles to each other, and the sphere is suspended at their point of intersection. If 2θ be the angle subtended at the centre by the arc which is unwrapped, shew that

$$\theta^4 = \frac{3\pi}{4}\frac{W}{\lambda},$$

θ being supposed small.

11. In the common catenary, if the string be slightly extensible, shew that its whole extension will be proportional to the product of its length and the height of its centre of gravity above the directrix.

12. A uniform rough cylinder is supported with its axis horizontal by an elastic string without weight; the string lies in the plane which is perpendicular to the axis of the cylinder, and passes through its centre of gravity; the ends of the string are attached to points which are in the same horizontal plane above the cylinder and at a distance equal to the diameter of the cylinder. Find how much the string is stretched.

Result. Let $2W$ be the weight of the cylinder, a the radius of the cylinder, b' the natural length of each vertical portion of the string; then the extension is

$$\frac{2b'W}{\lambda} + \frac{2a}{\mu}\log\frac{\lambda + We^{\frac{\mu\pi}{2}}}{\lambda + W}.$$

13. A heavy string very slightly elastic is suspended

from two points in the same horizontal plane; shew that if c, l be the lengths of unstretched string whose weights are respectively equal to the tension at the lowest point and the modulus of elasticity, the equation to the catenary will be very approximately

$$y = \frac{c}{2}\{e^{\frac{x}{c}} + e^{-\frac{x}{c}} - \frac{c}{4l}(e^{\frac{x}{c}} - e^{-\frac{x}{c}})^2\}.$$

14. A weight P just supports another weight Q by means of a fine elastic string passing over a rough cylinder whose axis is horizontal. If λ be the modulus of elasticity, μ the coefficient of friction, and a the radius of the cylinder, shew that the extension of that part of the string which is in contact with the cylinder is

$$\frac{a}{\mu}\log\frac{Q+\lambda}{P+\lambda}.$$

15. A sphere placed on a horizontal plane is divided by a vertical plane into two equal parts, which are just held together by an elastic string, which passes round the greatest horizontal section; find the original length of the string.

Result. $\dfrac{32\lambda\pi a}{16\lambda + 3W}$.

16. Four equal heavy rods are fastened to one another by hinges so as to form a square $ABCD$; A and C are connected by an elastic string whose natural length is equal to the diagonal AC, and the system is suspended from the point A; find the position of equilibrium.

Result. Let W be the weight of a rod, θ the inclination of each rod to the vertical; then

$$\cos\theta = \frac{1}{\sqrt{2}}\left(1 + \frac{2W}{\lambda}\right).$$

17. An elastic band, whose unstretched length is $2a$, is placed round four rough pegs A, B, C, D, which constitute the angular points of a square whose side is a; if it be taken hold of at a point P, between A and B, and pulled in the

direction AB, shew that it will begin to slip round A and B at the same time if

$$AP = \frac{a}{1 + e^{\frac{\mu\pi}{2}}}.$$

18. An elastic string without weight of variable thickness is extended by a given force; find the whole extension.

19. An elastic string whose density varies as the distance from one end, is suspended by that end and stretched by its own weight. If W be the weight of the string, l' its unstretched length, l its stretched length, shew that

$$l = l'\left(1 + \frac{2W}{3\lambda}\right).$$

20. A circular elastic string is placed on a smooth sphere so that the whole string is in one horizontal plane; the string subtends when unstretched an angle 2α at the centre, and an angle 2θ when in a position of equilibrium; shew that

$$\sin\theta = \sin\alpha\left(1 + \frac{a}{c}\sin\alpha\tan\theta\right),$$

where a = radius of sphere, and c depends on the nature of the string.

21. A circular elastic string is placed over a smooth right cone, and repelled by a force in the vertex varying inversely as the distance. Shew that if l' be the unstretched length, l the stretched length of the string, F the repulsive force on a unit of string at a unit of distance, then

$$l' = l\left(1 + \frac{F}{\lambda}\right).$$

22. A heavy elastic string surrounds a smooth horizontal cylinder, so that the surface of the cylinder is subject to no pressure at the lowest point; find the pressure at any point of the cylinder, and the tension of the string; its modulus of elasticity being equal to the weight of a portion of string the natural length of which is $\frac{9}{8}$ of the diameter of the cylinder.

23. A uniform heavy elastic string, whose natural length is a, is in equilibrium upon a rough inclined plane; find the tension at any point, and shew that the direction of the friction changes at a point of the string, the natural distance of which from the upper end is

$$\frac{a}{2}\left(1 + \frac{\tan \alpha}{\mu}\right),$$

where α is the inclination of the plane to the horizon.

24. A heavy elastic cord is passed through a number of fixed smooth rings. Shew that in the position of equilibrium its extremities will lie in the same horizontal plane. The same will also be the case if the cord rest upon any smooth surface.

25. An elastic string is laid on a cycloidal arc, the plane of which is vertical and vertex upwards, and when stretched by its own weight is in contact with the whole of the cycloid; the modulus of elasticity is the weight of a portion of the string whose natural length is twice the diameter of the generating circle; find the natural length of the string.

Result. It is equal to the circumference of the generating circle.

CHAPTER XIII.

ATTRACTIONS.

203. It appears from considerations which are detailed in works on Physical Astronomy, that two particles of matter placed at any sensible distance apart attract each other with a force directly proportional to the product of their masses, and inversely proportional to the square of their distance.

Suppose then a particle to be attracted by all the particles of a body; if we resolve the attraction of each particle of the body into components parallel to fixed rectangular axes, and take the sum of the components which act in a given direction, we obtain the resolved attraction of the whole body on the particle in that direction, and can thus ascertain the resultant attraction of the body in magnitude and direction. We shall give some particular examples, and then proceed to general formulæ.

204. *To find the attraction of a uniform straight line on an external point.*

By a straight line we understand a cylinder such that the section perpendicular to its axis is a curve, every chord of which is indefinitely small.

Let AB be the line, P the attracted particle; take A for the origin, and AB for the direction of the axis of x. Draw PL perpendicular to Ax; let $AB = l$, $AL = a$, $PL = b$. Let M and N be adjacent points in the line, $AM = x$, $MN = \delta x$. If ρ be the density of the line, and κ the area of a section perpendicular

to its length, the mass of the element is $\rho\kappa\delta x$. Let m be the mass of P; then the attraction of the element MN on P is (Art. 203)

$$\frac{\mu m\,\rho\kappa\,\delta x}{(PM)^2},$$

where μ is some constant quantity. Hence, the resolved part of the attraction of the element parallel to the axis of x, is

$$\frac{\mu m\,\rho\kappa\,\delta x}{PM^2}\cdot\frac{ML}{PM} \text{ or } \frac{\mu m\,\rho\kappa\,(a-x)\,\delta x}{\{b^2+(a-x)^2\}^{\frac{3}{2}}}.$$

Also the resolved part of the attraction of the element parallel to the axis of y, is

$$\frac{\mu m\,\rho\kappa\,\delta x}{PM^2}\cdot\frac{PL}{PM} \text{ or } \frac{\mu m\,\rho\kappa b\,\delta x}{\{b^2+(a-x)^2\}^{\frac{3}{2}}}.$$

Let X and Y be the resolved parts of the attraction of the line, parallel to the axes of x and y respectively; then

$$X=\mu m\,\rho\kappa\int_0^l\frac{(a-x)\,dx}{\{b^2+(a-x)^2\}^{\frac{3}{2}}},$$

$$Y=\mu m\,\rho\kappa\int_0^l\frac{b\,dx}{\{b^2+(a-x)^2\}^{\frac{3}{2}}}.$$

Now $$\int\frac{(a-x)\,dx}{\{b^2+(a-x)^2\}^{\frac{3}{2}}}=\frac{1}{\{b^2+(a-x)^2\}^{\frac{1}{2}}};$$

therefore $$\int_0^l\frac{(a-x)\,dx}{\{b^2+(a-x)^2\}^{\frac{3}{2}}}=\frac{1}{\{b^2+(a-l)^2\}^{\frac{1}{2}}}-\frac{1}{\{b^2+a^2\}^{\frac{1}{2}}}\ \ldots\ldots(1);$$

$$\int\frac{dx}{\{b^2+(a-x)^2\}^{\frac{3}{2}}}=-\frac{a-x}{b^2\,\{b^2+(a-x)^2\}^{\frac{1}{2}}};$$

therefore

$$\int_0^l\frac{b\,dx}{\{b^2+(a-x)^2\}^{\frac{3}{2}}}=\frac{1}{b}\left[\frac{a}{(b^2+a^2)^{\frac{1}{2}}}-\frac{a-l}{\{b^2+(a-l)^2\}^{\frac{1}{2}}}\right]\ \ldots\ldots(2).$$

Hence, putting f for $\mu\rho\kappa$, we have

$$X = fm\left\{\frac{1}{PB} - \frac{1}{PA}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$Y = \frac{fm}{PL}\left\{\frac{AL}{PA} - \frac{BL}{PB}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Let $APL = \alpha$, $BPL = \beta$, $APB = \gamma$; then

$$X = \frac{fm}{PL}(\cos\beta - \cos\alpha),$$

$$Y = \frac{fm}{PL}(\sin\alpha - \sin\beta);$$

therefore $\sqrt{(X^2 + Y^2)} = \frac{fm}{PL}\sqrt{\{(\cos\beta - \cos\alpha)^2 + (\sin\alpha - \sin\beta)^2\}}$

$$= \frac{fm}{PL}\sqrt{(2 - 2\cos\gamma)} = \frac{2fm}{PL}\sin\tfrac{1}{2}\gamma \ldots(5).$$

This gives the magnitude of the resultant attraction. Also

$$\frac{X}{Y} = \frac{\cos\beta - \cos\alpha}{\sin\alpha - \sin\beta} = \tan\frac{\alpha+\beta}{2} \quad\ldots\ldots\ldots\ldots\ldots(6).$$

This shews that the direction of the resultant attraction bisects the angle APB.

If L fall *between* A and B, it will be seen from (1) and (2) that the expression for X in (3) remains unchanged, but that for Y in (4) is changed to

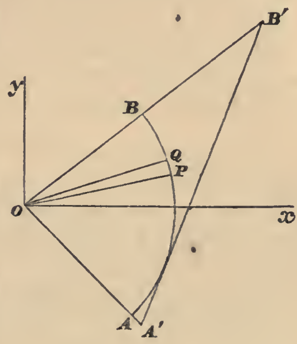
$$\frac{fm}{PL}\left\{\frac{AL}{PA} + \frac{BL}{PB}\right\}.$$

This will not affect the result in (5), and the direction of the resultant will still bisect the angle APB.

From the investigation it appears that X is the resolved attraction parallel to the axis of x directed *towards* the axis of y, and Y the resolved attraction parallel to the axis of y and *towards* the axis of x.

205. In the above investigation we have taken m to denote the mass of the attracted particle; in future we shall always suppose the mass of the attracted particle to be denoted by unity. In order to form a precise idea of the quantity μ, we may suppose two particles each having its mass equal to the unit of mass, then μ will be the whole force which one of these exerts on the other when the distance between them is the unit of length. As, however, by properly choosing the unit of mass we may make $\mu = 1$, we shall not in future introduce μ.

206. *To find the attraction of a circular arc on a particle situated at the centre of the circle.*



Let AB be any circular arc; through O the centre of the circle draw a line bisecting the angle AOB, and take this line for the axis of x. Let $POx = \theta$, $QOP = \delta\theta$, $AOB = 2\alpha$, $OB = r$. The attraction of the element PQ resolved parallel to the axes of x and y respectively is, if ρ and κ have the same meaning as in Art. 204,

$$\frac{\kappa\rho r\delta\theta}{r^2}\cos\theta \text{ and } \frac{\kappa\rho r\delta\theta}{r^2}\sin\theta;$$

therefore $$X = \frac{\kappa\rho}{r}\int_{-\alpha}^{+\alpha}\cos\theta d\theta = \frac{2\kappa\rho}{r}\sin\alpha,$$

$$Y = \frac{\kappa\rho}{r}\int_{-\alpha}^{+\alpha}\sin\theta d\theta = 0.$$

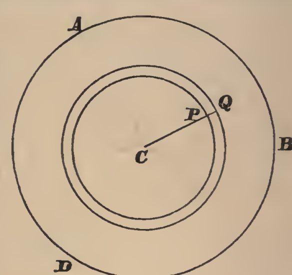
By comparing these results with those in Art. 204, it appears that the attraction of a circular arc on a particle at the centre is the same in magnitude and direction as that of any straight line $A'B'$ which *touches* the arc AB and is terminated by the lines OA and OB produced, the arc and line being supposed to have the same density, and the areas of their transverse sections equal.

If OP and OQ be produced to meet the line $A'B'$ in points P' and Q' respectively, it may be shewn that the attraction of the element $P'Q'$ on a particle at O is equal to that of PQ, and in this manner we might prove what we have just shewn, that the attractions of AB and $A'B'$ on a particle at O are equal and coincident. This proposition is given in Earnshaw's *Dynamics*, p. 326.

It easily follows, that if a particle be attracted by the three sides of a triangle, it will be in equilibrium if it be placed at the centre of the circle inscribed in the triangle.

207. *To find the attraction of a uniform circular lamina on a particle situated in a straight line drawn through the centre of the lamina perpendicular to its plane.*

Suppose C the centre of the circle DAB, the plane of the paper coinciding with one face of the lamina, and the attracted particle being in a straight line drawn through C perpendicular to the lamina and at a distance a from C. Describe from the centre C two adjacent concentric circles, one with radius $CP = r$, and the other with radius $CQ = r + \delta r$. Let κ denote the thickness of the lamina, which is supposed to be an indefinitely small quantity, then the mass of the circular ring contained between the adjacent circles is $2\pi\rho\kappa r\delta r$. Every particle in this circular ring is at a distance $\sqrt{(a^2+r^2)}$ from the attracted particle; also the resultant attraction of the ring is in the line through C perpendicular to the lamina, and is equal to



$$\frac{2\pi\rho\kappa r\delta r}{a^2+r^2} \cdot \frac{a}{\sqrt{(a^2+r^2)}},$$

the factor $\frac{a}{\sqrt{(a^2+r^2)}}$ being the multiplier necessary in order to resolve the attraction of any element of the ring along the normal to the lamina through C.

Hence, the resultant attraction of the whole lamina is

$$2\pi\rho\kappa a\int_0^b \frac{rdr}{(a^2+r^2)^{\frac{3}{2}}},$$

where b is the radius of the boundary of the lamina.

Now $$\int\frac{rdr}{(a^2+r^2)^{\frac{3}{2}}} = -\frac{1}{\sqrt{(a^2+r^2)}};$$

therefore $$\int_0^b\frac{rdr}{(a^2+r^2)^{\frac{3}{2}}} = \frac{1}{a} - \frac{1}{\sqrt{(a^2+b^2)}};$$

therefore the resultant attraction

$$= 2\pi\rho\kappa a\left\{\frac{1}{a} - \frac{1}{\sqrt{(a^2+b^2)}}\right\}.$$

If we suppose b to become infinite, we obtain for the attraction of an infinite lamina on an external particle, the expression $2\pi\rho\kappa$, which is independent of the distance of the attracted particle from the lamina.

From the last result we can deduce the resultant attraction of a uniform plate of finite thickness, but of infinite extent, on an external particle. For, suppose the plate divided into an indefinitely large number of laminæ, each of the thickness κ; then the attraction of each lamina acts in a line through the attracted particle perpendicular to the surfaces of the plate, and is equal to $2\pi\rho\kappa$. Hence, the resultant attraction will be found by adding the attractions of the laminæ, and will be $2\pi\rho h$, if h be the thickness of the plate.

If a particle be placed *on* the exterior surface of an infinite plate, the result just found will express the attraction of the plate on the particle. If it be placed in the interior of the plate at a distance h from one of the bounding planes and h' from the other, the resultant attraction will be $2\pi\rho\,(h'-h)$ *towards* the latter plane.

208. By means of the preceding article we can find the resultant attraction of a uniform cylinder on a particle

situated on its axis. Suppose the cylinder divided into an indefinitely large number of laminæ by planes perpendicular to its axis; let x be the distance of a lamina from the attracted particle, δx the thickness of the lamina, b the radius of the cylinder; then the attraction of the lamina is

$$2\pi\rho\left\{1-\frac{x}{\sqrt{(x^2+b^2)}}\right\}\delta x,$$

Suppose the attracted particle *outside* the cylinder at a distance c from it; let h be the height of the cylinder; then the resultant attraction of the cylinder

$$=2\pi\rho\int_c^{c+h}\left\{1-\frac{x}{\sqrt{(x^2+b^2)}}\right\}dx$$

$$=2\pi\rho\left[h-\sqrt{\{(c+h)^2+b^2\}}+\sqrt{(c^2+b^2)}\right].$$

If we suppose $c=0$ so that the particle is on the surface of the cylinder the resultant attraction is

$$2\pi\rho\{h-\sqrt{(h^2+b^2)}+b\}.$$

209. To find the attraction of a uniform *cone* on a particle at its vertex, we begin with the expression

$$2\pi\rho\left\{1-\frac{x}{\sqrt{(x^2+b^2)}}\right\}\delta x,$$

for the attraction of a lamina of the cone. Also, if α be the semivertical angle of the cone, we have

$$\frac{x}{\sqrt{(x^2+b^2)}}=\cos\alpha;$$

hence, the resultant attraction

$$=2\pi\rho(1-\cos\alpha)\int_0^h dx=2\pi\rho(1-\cos\alpha)\,h;$$

where h is the height of the cone.

It is easily seen that the same expression holds for the attraction of the frustum of a cone on a particle situated at the vertex of the complete cone, h representing in this case the height of the frustum.

If the cone be an *oblique* cone the base of which is any plane figure it is still true that the attraction of a frustum on a particle at the vertex varies as the thickness of the frustum. Consider two indefinitely thin parallel laminæ at different distances from the vertex of such a cone, then the attractions of these laminæ on the particle at the vertex will be the same. For take any indefinitely small element of area on the surface of one of the laminæ, and let a conical surface be formed by lines which pass through the perimeter of this area and through the attracted particle; this conical surface will intercept elements in the two laminæ which are bounded by similar plane figures. Now, supposing the laminæ of the same thickness, the masses of the elements will vary as the squares of their distances from the attracted particle, and thus they will exert *equal attractions* on this particle. The same result holds for every corresponding pair of elements in the two laminæ, and thus the two laminæ exert on the particle at the vertex attractions which are equal in amount and coincident in direction. From this it follows that the attraction of a frustum varies as its thickness.

210. We have hitherto considered the attracting body to be of uniform density, but considerable variety may be introduced into the questions by various suppositions as to the law of density. Suppose, for instance, that in the case of the circular lamina in Art. 207 the density at any point of the lamina is $\phi(r)$, where r is the distance of that point from the centre; $\phi(r)$ must then be put instead of ρ in Art. 207 and must be placed under the integral sign. Therefore the attraction of the lamina will be

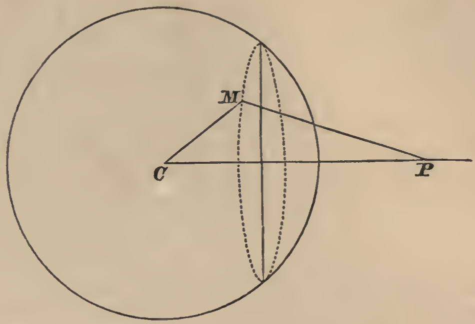
$$2\pi a\kappa\int_0^b \frac{\phi(r)\,r\,dr}{(a^2+r^2)^{\frac{3}{2}}}.$$

If $\phi(r)=\dfrac{\sigma}{r}$, where σ is a constant, the result is

$$2\pi a\kappa\sigma\int_0^b \frac{dr}{(a^2+r^2)^{\frac{3}{2}}},\quad\text{or}\quad \frac{2\pi\kappa\sigma b}{a\,(a^2+b^2)^{\frac{1}{2}}}.$$

211. *To find the resultant attraction of an assemblage of particles constituting a homogeneous spherical shell of very small thickness upon a particle outside the shell.*

Let C be the centre of the shell, M any particle of it, P the



attracted particle. Let $CM = r$, $PM = y$, $CP = c$, $\theta =$ the angle PCM, $\phi =$ the angle which the plane PCM makes with the plane of the paper, $\delta r =$ the thickness of the shell, and let ρ denote the density of the shell.

The volume of the elementary solid at M is $r^2 \sin\theta\, \delta r\, \delta\theta\, \delta\phi$ (see Art. 130). The attraction of the whole shell acts along PC; the attraction of the element at M resolved along PC is

$$\frac{\rho r^2 \sin\theta\, \delta r\, \delta\theta\, \delta\phi}{y^2} \frac{c - r\cos\theta}{y}.$$

We shall eliminate θ from this expression by means of the equation

$$y^2 = c^2 + r^2 - 2rc\cos\theta;$$

therefore

$$\sin\theta \frac{d\theta}{dy} = \frac{y}{cr},$$

$$c - r\cos\theta = \frac{y^2 + c^2 - r^2}{2c}.$$

Therefore the attraction of M on P along PC

$$= \frac{\rho r \delta r}{2c^2}\left(1 + \frac{c^2 - r^2}{y^2}\right) \delta y\, \delta\phi.$$

Hence the resultant attraction of the whole shell

$$= \frac{\rho r\,\delta r}{2c^2}\int_{c-r}^{c+r}\int_0^{2\pi}\left(1+\frac{c^2-r^2}{y^2}\right)dy\,d\phi = \frac{\pi\rho r\delta r}{c^2}\int_{c-r}^{c+r}\left(1+\frac{c^2-r^2}{y^2}\right)dy,$$

$$= \frac{\pi\rho r\delta r}{c^2}(2r+2r) = \frac{4\pi\rho r^2\delta r}{c^2} = \frac{\text{mass of the shell}}{c^2}.$$

This result shews that the shell attracts the particle at P in the same manner as if the mass of the shell were condensed at its centre.

212. It follows from the preceding article, that a sphere which is either homogeneous or consists of concentric spherical shells of uniform density, will attract the particle at P in the same manner as if the whole mass were collected at its centre.

213. *To find the attraction of a homogeneous spherical shell of small thickness on a particle placed within it.*

We must proceed as in Art. 211; but the limits of y are in this case $r-c$ and $r+c$; hence the resultant attraction of the shell

$$= \frac{\pi\rho r\delta r}{c^2}\int_{r-c}^{r+c}\left(1-\frac{r^2-c^2}{y^2}\right)dy = \frac{\pi\rho r\delta r}{c^2}(2c-2c) = 0.$$

Therefore a particle within the shell is equally attracted in every direction.

Suppose a particle inside a homogeneous sphere at the distance r from its centre; then by what has just been shewn all that portion of the sphere which is at a greater distance from the centre than the particle produces no effect on the particle. Also by Art. 211, the remainder of the sphere attracts the particle in the same manner as if the mass of the remainder were all collected at the centre of the sphere. Thus if ρ be the density of the sphere the attraction on the particle is

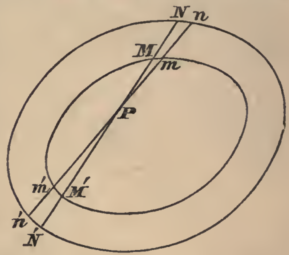
$$\frac{\frac{4}{3}\pi\rho r^3}{r^2}, \text{ that is } \frac{4\pi\rho r}{3}.$$

Thus *inside a homogeneous sphere the attraction varies as the distance from the centre.*

214. The propositions respecting the attraction of a uniform spherical shell on an external or internal particle were given by Newton (*Principia*, Lib. I. Prop. 70, 71). The result with respect to the *internal* particle was afterwards extended by Newton to the case of a shell bounded by similar and similarly situated spheroidal surfaces (*Principia*, Lib. I. Prop. 91, Cor. 3). The proposition is also true when the shell is bounded by similar and similarly situated *ellipsoidal* surfaces, which we proceed to demonstrate in the method given by Newton for spheroidal surfaces.

215. *If a shell of uniform density be bounded by two ellipsoidal surfaces which are concentric, similar, and similarly situated, the resultant attraction on an internal particle vanishes.*

Let the attracted particle P be the vertex of an infinite series of right cones. Let $NMPM'N'$ and $nmPmn'$ be two generating lines of one of these cones, and suppose the curves in the figure to represent the intersection of the surfaces of the shell by a plane containing these generating lines. The curves will be similar and similarly situated ellipses, and by a property of such ellipses,



$$MN = M'N' \text{ and } mn = m'n'.$$

By taking the angle of the cone small enough, each of the two portions of the shell which it intercepts will be ultimately a *frustum of a cone*, and being of equal altitude and having a common vertical angle, they *will exercise equal attractions on P.* (See Art. 209.) Similar considerations hold with respect to each of the infinite series of cones of which P is the vertex, and consequently the resultant attraction of the shell vanishes.

This result being true, whatever be the thickness of the shell, is true when the shell becomes indefinitely thin.

216. In a somewhat similar way we may establish the following proposition which is due to Poisson; *the resultant attraction of an indefinitely thin shell bounded by two ellipsoidal surfaces which are concentric, similar, and similarly*

situated on an external particle is in the direction of the axis of the enveloping cone which has its vertex at the given particle. (*Crelle's Journal*, Vol. XII. p. 141.) Denote the external particle by Q; and suppose P in the preceding figure to be the point where the axis of the enveloping cone intersects the plane of contact of the cone and the ellipsoidal shell. Draw any lines $NMM'N'$ and $nmm'n'$ as in the preceding figure. Let μ denote the mass of the element Mn and μ' the mass of the element $M'n'$.

The attraction of μ is equal to $\frac{\mu}{QM^2}$ and it acts along QM; the attraction of μ' is equal to $\frac{\mu'}{QM'^2}$ and it acts along QM'.

Now
$$\frac{\mu}{PM^2} = \frac{\mu'}{PM'^2};$$

and it is known that QM and QM' make *equal angles* with QP (see *Conic Sections*, Chap. XV., last example); therefore

$$\frac{PM}{QM} = \frac{PM'}{QM'};$$

and therefore
$$\frac{\mu}{QM^2} = \frac{\mu'}{QM'^2}.$$

Thus the elements μ and μ' exert *equal attractions* on Q; and since the directions of these attractions make equal angles with QP, the *resultant* attraction of these two elements acts along QP. A similar result holds for every pair of elements into which the ellipsoidal shell may be decomposed; and thus the proposition follows. It appears from the course of the demonstration that any plane through P divides the shell into two parts which exercise equal attractions on Q.

We shall now give in the next two articles some propositions which will serve as exercises; the approximate results which we shall obtain may be subsequently verified by an exact investigation. (See Art. 226.)

217. *To find the attraction of a homogeneous oblate spheroid of small excentricity on a particle at its pole.*

Let c be the length of the minor axis and a that of the major axis of the generating ellipse. The spheroid may be supposed made up of a concentric sphere, the radius of which is c, and an exterior shell; we shall calculate the attractions of these portions separately.

Let a section be made of the sphere and spheroid by a plane perpendicular to the axis of revolution of the spheroid at a distance x from the attracted particle. This plane cuts the sphere and spheroid in concentric circles; the area of the former being πy^2 and of the latter $\frac{\pi a^2 y^2}{c^2}$, where $y^2 = 2cx - x^2$; the difference of these areas is $\pi \left(\frac{a^2}{c^2} - 1\right) y^2$. If a section be made by a second plane, parallel to the former and at a distance δx from it, the volume of the portion of the *shell* intercepted between the planes will be $\pi \left(\frac{a^2}{c^2} - 1\right) y^2 \delta x$. The distance of every particle of the annulus thus formed from the attracted particle is approximately $\sqrt{(2cx)}$, and, as the resultant attraction of the annulus will act along the axis of the spheroid, it will, approximately,

$$= \pi\rho \left(\frac{a^2}{c^2} - 1\right) \frac{x}{\sqrt{(2cx)}} \frac{y^2 \delta x}{2cx},$$

$$= \pi\rho \left(\frac{a^2}{c^2} - 1\right) \frac{2cx^{\frac{1}{2}} - x^{\frac{3}{2}}}{(2c)^{\frac{3}{2}}} \delta x.$$

Therefore the resultant attraction of the shell

$$= \frac{\pi\rho\,(a^2 - c^2)}{2^{\frac{3}{2}}\,c^{\frac{7}{2}}} \int_0^{2c} (2cx^{\frac{1}{2}} - x^{\frac{3}{2}})\,dx = \frac{8\pi\rho\,(a^2 - c^2)}{15c}.$$

If we suppose $c = a\,(1 - \epsilon)$, ϵ being very small, we have

$$a^2 - c^2 = 2c^2\epsilon \text{ approximately};$$

therefore the resultant attraction of the shell

$$= \frac{16\pi\rho\epsilon c}{15}.$$

Also the attraction of the sphere on the particle, by Art. 212,

$$= \tfrac{4}{3}\pi\rho c;$$

therefore the attraction of the spheroid on the particle

$$=\tfrac{4}{3}\pi\rho\,(1+\tfrac{4}{5}\epsilon)\,c.$$

218. *To find the attraction of a homogeneous oblate spheroid of small excentricity on a particle at its equator.*

Let c be the length of the minor axis, and a that of the major axis of the generating ellipse. The spheroid may be supposed to be the difference between a concentric sphere of radius a and a shell, and the attractions of the sphere and shell may be separately calculated. Let a section be made of the sphere and spheroid by a plane perpendicular to the line joining the attracted particle with the common centre of the sphere and spheroid, and at a distance x from the attracted particle; this plane will cut the sphere in a circle the area of which is πy^2, where $y^2 = 2ax - x^2$, and it will cut the spheroid in an ellipse of which the semiaxes are respectively y and $\frac{cy}{a}$, and the area of which is therefore $\frac{\pi c}{a} y^2$. The difference of the two areas is $\pi\left(1-\frac{c}{a}\right)y^2$. If a section be made by a second plane parallel to the former, and at a distance δx from it, the volume of the portion of the shell intercepted between the planes will be $\pi\left(1-\frac{c}{a}\right)y^2\delta x$. The distance of every particle of the annulus thus formed from the attracted particle is approximately $\sqrt{(2ax)}$; and as the resultant attraction of the annulus will act along the line joining the attracted particle with the centre, it will approximately

$$=\pi\rho\left(1-\frac{c}{a}\right)\frac{x}{\sqrt{(2ax)}}\,\frac{y^2\delta x}{2ax},$$

$$=\pi\rho\left(1-\frac{c}{a}\right)\frac{2ax^{\frac{1}{2}}-x^{\frac{3}{2}}}{(2a)^{\frac{3}{2}}}\,\delta x.$$

Therefore the resultant attraction of the shell

$$=\frac{\pi\rho\,(a-c)}{2^{\frac{3}{2}}a^{\frac{5}{2}}}\int_0^{2a}(2ax^{\frac{1}{2}}-x^{\frac{3}{2}})\,dx=\frac{8\pi\rho\,(a-c)}{15},$$

$$=\frac{8\pi\rho a\epsilon}{15},\ \text{if}\ c=a\,(1-\epsilon).$$

Also the attraction of the sphere, by Art. 212,

$$= \tfrac{4}{3}\pi\rho a;$$

therefore the attraction of the spheroid on the particle

$$= \tfrac{4}{3}\pi\rho a - \frac{8\pi\rho a\epsilon}{15} = \tfrac{4}{3}\pi\rho\,(1 - \tfrac{2}{5}\epsilon)\,a,$$

$$= \tfrac{4}{3}\pi\rho\,(1 + \tfrac{3}{5}\epsilon)\,c.$$

In the same manner it might be shewn that the attractions of a homogeneous *prolate* spheroid of small excentricity on particles at the pole and equator are respectively

$$\tfrac{4}{3}\pi\rho\,(1 - \tfrac{4}{5}\epsilon)\,c \text{ and } \tfrac{4}{3}\pi\rho\,(1 - \tfrac{3}{5}\epsilon)\,c,$$

$2c$ being the axis of revolution of the spheroid, and

$$a = c\,(1 - \epsilon).$$

219. One more example may be given. It is sometimes useful to compare the attraction exerted by the Earth on a particle at the top of a mountain with the attraction exerted by the Earth on the same particle at the ordinary level of the Earth's surface. The investigation is given by Poisson, (*Mécanique*, Tom. I. pp. 492–496). Let r denote the Earth's radius, x the height of the mountain, g the attraction of the Earth on a particle of a unit of mass at the ordinary level of the Earth's surface. If there were no mountain the attraction of the Earth on the particle at a distance x from its surface would be $g\dfrac{r^2}{(r+x)^2}$: we have then to add to this expression the attraction exerted by the mountain itself. Suppose the mountain to be of uniform density ρ, and consider it to be cylindrical in shape, and the particle to be at the centre of its upper surface; then by Art. 208 the resultant attraction is

$$2\pi\rho\,\{x - \sqrt{(x^2 + b^2)} + b\},$$

where b is the radius of the cylinder. If b is so large in comparison with x that the square of $\dfrac{x}{b}$ can be neglected, this expression reduces to $2\pi\rho x$. Thus if g' denote the attraction at the top of the mountain

$$g' = \frac{gr^2}{(r+x)^2} + 2\pi\rho x.$$

Let σ denote the *mean* density of the Earth, so that the mass of the earth is $\frac{4\pi\sigma r^3}{3}$; then

$$g = \frac{4\pi\sigma r^3}{3r^2} = \frac{4\pi\sigma r}{3};$$

thus
$$g' = g\left\{\frac{r^2}{(r+x)^2} + \frac{3\rho x}{2\sigma r}\right\}.$$

Now the mean density of the Earth is known to be about five and a half times that of water, and from what may be conjectured of the density of matter at the Earth's surface, we may suppose $\frac{\rho}{\sigma} = \frac{1}{2}$. And

$$\frac{r^2}{(r+x)^2} = \left(1 + \frac{x}{r}\right)^{-2} = 1 - \frac{2x}{r} \text{ approximately};$$

thus
$$g' = g\left(1 - \frac{2x}{r} + \frac{3x}{4r}\right) = g\left(1 - \frac{5x}{4r}\right).$$

How far the approximations made in this article are allowable might be difficult to estimate; from Article 207, it appears that in taking $2\pi\rho x$ to represent the attraction of the mountain, we do in fact make the mountain to consist of a uniform plate of finite thickness x, but of infinite extent.

We have hitherto confined ourselves to simple examples of the ordinary law of attraction; we now proceed to consider some other laws of attraction, and also some more complex cases of the ordinary law.

220. *If the particles of a body attract with a force varying as the product of the mass into the distance, the resultant attraction of the body is the same as if the whole mass of the body were collected at its centre of gravity.*

Take the centre of gravity of the attracting body as the origin of co-ordinates, and let a, b, c be the co-ordinates of the attracted particle. Divide the attracting body into indefinitely small elements; let x, y, z be the co-ordinates of an element, m its mass, and r its distance from the attracted particle. Then the attraction of this element is mr, and by resolving it parallel to the co-ordinate axes, we obtain

$$mr \,.\, \frac{a-x}{r}, \quad mr \,.\, \frac{b-y}{r}, \quad mr \,.\, \frac{c-z}{r},$$

respectively. Hence, if X, Y, Z denote the resolved parts of the whole attraction, we have

$$X = \Sigma m (a-x), \quad Y = \Sigma m (b-y), \quad Z = \Sigma m (c-z).$$

But, since the origin is the centre of gravity of the attracting body, we have

$$\Sigma mx = 0, \quad \Sigma my = 0, \quad \Sigma mz = 0;$$

therefore $$X = a\Sigma m, \quad Y = b\Sigma m, \quad Z = c\Sigma m.$$

But these expressions are the resolved attractions of a mass Σm placed at the origin, which establishes the proposition.

221. *To find the attraction of a homogeneous spherical shell on a particle without it; the law of attraction being represented by $\phi(y)$, where y is the distance.*

If we proceed as in Art. 211, we find the resultant attraction of the shell on P along PC

$$= \frac{\pi\rho r\delta r}{c^2} \int_{c-r}^{c+r} (y^2 + c^2 - r^2)\, \phi(y)\, dy.$$

Suppose $$\int \phi(y)\, dy = \phi_1(y),$$

and $$\int y\phi_1(y)\, dy = \psi(y).$$

Then, integrating by parts, we have

$$\int (y^2 + c^2 - r^2)\, \phi(y)\, dy = (y^2 + c^2 - r^2)\, \phi_1(y) - 2\int y\phi_1(y) dy$$
$$= (y^2 + c^2 - r^2)\, \phi_1(y) - 2\psi(y);$$

therefore $$\frac{\pi\rho r\delta r}{c^2}\int_{c-r}^{c+r}(y^2+c^2-r^2)\,\phi(y)\,dy$$

$$=2\pi\rho r\delta r\left\{\frac{c+r}{c}\phi_1(c+r)-\frac{c-r}{c}\phi_1(c-r)-\frac{1}{c^2}\psi(c+r)+\frac{1}{c^2}\psi(c-r)\right\}$$

$$=2\pi\rho r\delta r\frac{d}{dc}\left\{\frac{\psi(c+r)-\psi(c-r)}{c}\right\}.$$

This last form is introduced merely as an analytical artifice to simplify the expression.

222. *To find the attraction of the shell on an internal particle.*

The calculation is the same as in the last article, except that the limits of y are $r-c$ and $r+c$. Hence, the attraction of the shell

$$=2\pi\rho r\delta r\left\{\frac{r+c}{c}\phi_1(r+c)+\frac{r-c}{c}\phi_1(r-c)-\frac{1}{c^2}\psi(r+c)+\frac{1}{c^2}\psi(r-c)\right\}$$

$$=2\pi\rho r\delta r\frac{d}{dc}\left\{\frac{\psi(r+c)-\psi(r-c)}{c}\right\}.$$

223. The formulæ of the preceding two articles will give the attraction when the law of attraction is known.

Ex. 1. Let $\phi(r)=\frac{1}{r^2}$; therefore $\phi_1(r)=-\frac{1}{r}+A$,

$$\psi(r)=-r+\tfrac{1}{2}Ar^2+B;$$

A and B being constants.

Therefore the attraction on an external particle

$$=2\pi\rho r\delta r\frac{d}{dc}\left\{\frac{-4r+A(c+r)^2-A(c-r)^2}{2c}\right\}$$

$$=2\pi\rho r\delta r\frac{d}{dc}\left(-\frac{2r}{c}+2Ar\right)=\frac{4\pi\rho r^2\delta r}{c^2},\ \text{(Art. 211).}$$

The attraction on an internal particle

$$= 2\pi\rho r\delta r \frac{d}{dc}\left\{\frac{-4c + A(r+c)^2 - A(r-c)^2}{2c}\right\}$$

$$= 2\pi\rho r\delta r \frac{d}{dc}\{-2 + 2Ar\} = 0, \text{ (Art. 213).}$$

Ex. 2. Let $\phi(r) = r;$

therefore $\phi_1(r) = \frac{1}{2}r^2 + A, \quad \psi(r) = \frac{1}{8}r^4 + \frac{1}{2}Ar^2 + B.$

The attraction on an external particle

$$= 2\pi\rho r\delta r \frac{d}{dc}\left\{\frac{(c+r)^4 - (c-r)^4 + 4A(c+r)^2 - 4A(c-r)^2}{8c}\right\}$$

$$= 2\pi\rho r\delta r \frac{d}{dc}\{c^2 r + r^3 + 2Ar\}$$

$$= 4\pi\rho r^2 c\delta r = \text{mass} \times c.$$

The attraction therefore is the same as if the shell were collected at its centre. This property we discovered for the law of the inverse square. We shall now ascertain whether there are any other laws which give the same property.

224. *To find what laws of attraction allow us to suppose a spherical shell condensed into its centre when attracting an external particle.*

Let $\phi(r)$ be the law of force; then, if c be the distance of the centre of the shell from the attracted particle, r the radius of the shell, and $\psi(r) = \int\{r\int\phi(r)\,dr\}\,dr$, the attraction of the shell

$$= 2\pi\rho r\delta r \frac{d}{dc}\left\{\frac{\psi(c+r) - \psi(c-r)}{c}\right\}.$$

But if the shell be condensed into its centre, the attraction

$$= 4\pi\rho r^2\delta r\phi(c);$$

therefore $$\frac{d}{dc}\left\{\frac{\psi(c+r) - \psi(c-r)}{c}\right\} = 2r\phi(c).$$

Expand $\psi(c+r)$ and $\psi(c-r)$ in powers of r; then using $\psi'(c)$ for $\frac{d\psi(c)}{dc}$, &c., we have

$$2r\phi(c) = 2\frac{d}{dc}\left\{\frac{r}{c}\psi'(c) + \frac{r^3}{c\,\underline{|3}}\psi'''(c) + \ldots\right\}$$

$$= 2r\phi(c) + 2\frac{d}{dc}\left\{\frac{r^3}{c\,\underline{|3}}\psi'''(c) + \ldots\right\};$$

therefore $$\frac{d}{dc}\left\{\frac{r^3}{c\,\underline{|3}}\psi'''(c) + \ldots\right\} = 0,$$

whatever r may be; therefore

$$\frac{d}{dc}\left\{\frac{\psi'''(c)}{c}\right\} = 0, \quad \frac{d}{dc}\left\{\frac{\psi^{(v)}(c)}{c}\right\} = 0, \text{ \&c.}$$

But $$\psi'(c) = c\int\phi(c)\,dc;$$

therefore $$\psi''(c) = \int\phi(c)\,dc + c\phi(c);$$

therefore $$\psi'''(c) = 2\phi(c) + c\phi'(c).$$

Therefore, by the first of the above equations of condition for $\psi(c)$,

$$\frac{2\phi(c)}{c} + \phi'(c) = \text{a constant}.$$

Put $3A$ for this constant; multiply both sides of the equation by c^2 and integrate; thus

$$c^2\phi(c) = Ac^3 + B;$$

therefore $$\phi(c) = Ac + \frac{B}{c^2}.$$

This value satisfies all the other equations of condition for $\psi(c)$; therefore the required laws of attraction are those of the direct distance, the inverse square, and a law compounded of these.

225. *To find for what laws the shell attracts an internal particle equally in every direction.*

When this is the case,

$$\frac{d}{dc}\left\{\frac{\psi(r+c)-\psi(r-c)}{c}\right\}=0;$$

therefore $$\psi'(r)+\frac{c^2}{\lfloor 3}\psi'''(r)+\ldots=A,$$

whatever c is, A being a constant independent of c; therefore

$$\psi'(r)=A, \quad \psi'''(r)=0, \text{ \&c.}$$

From the second condition, we have

$$\psi(r)=B+B'r+B''r^2,$$

where B, B', and B'' are constants.

Hence $$\psi'(r) \text{ or } r\int\phi(r)\,dr=B'+2B''r;$$

therefore $$\int\phi(r)\,dr=\frac{B'}{r}+2B'';$$

therefore $$\phi(r)=-\frac{B'}{r^2}:$$

with this value of $\phi(r)$ all the other equations of condition are satisfied; hence the only law which satisfies the condition is that of the inverse square.

226. *To find the attraction of a homogeneous oblate spheroid upon a particle within its mass, the law of attraction being that of the inverse square of the distance.*

Let a and c be the semiaxes, a being greater than c; and let the equation to the spheroid referred to its centre as origin be

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1 \text{ ....} (1).$$

Let f, g, h be the co-ordinates of the attracted particle; r the distance from the attracted particle of any point of the attracting mass; θ the angle which r makes with a line parallel to the axis of z; ϕ the angle which the plane containing r and a line through the point (f, g, h) parallel to the axis of z makes with the plane of (x, z). The volume of an element of the attracting mass

$$= r^2 \sin\theta\, \delta\theta\, \delta\phi\, \delta r,$$

as in Art. 130. Let ρ be the density of the spheroid; then the attraction of this element on the attracted particle is $\rho \sin\theta\, \delta\theta\, \delta\phi\, \delta r$; and the resolved parts of this parallel to the axes of x, y, z, are

$$\rho \sin^2\theta \cos\phi\, \delta\theta\, \delta\phi\, \delta r, \quad \rho \sin^2\theta \sin\phi\, \delta\theta\, \delta\phi\, \delta r,$$

$$\text{and } \rho \sin\theta \cos\theta\, \delta\theta\, \delta\phi\, \delta r,$$

respectively. Hence the attractions of the whole spheroid will be found by integrating these expressions between proper limits. We proceed to find these limits.

In equation (1) put

$$f + r\sin\theta\cos\phi \text{ for } x,$$
$$g + r\sin\theta\sin\phi \text{ for } y,$$
$$h + r\cos\theta \qquad \text{for } z;$$

then the equation to the spheroid becomes

$$\frac{(f + r\sin\theta\cos\phi)^2 + (g + r\sin\theta\sin\phi)^2}{a^2} + \frac{(h + r\cos\theta)^2}{c^2} = 1,$$

$$\text{or } r^2\left\{\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{c^2}\right\} + 2r\left\{\frac{f\sin\theta\cos\phi + g\sin\theta\sin\phi}{a^2} + \frac{h\cos\theta}{c^2}\right\}$$

$$= 1 - \frac{f^2 + g^2}{a^2} - \frac{h^2}{c^2}.$$

Put $$\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{c^2} = K,$$

$$\frac{f\sin\theta\cos\phi + g\sin\theta\sin\phi}{a^2} + \frac{h\cos\theta}{c^2} = F,$$

$$F^2 + K\left(1 - \frac{f^2 + g^2}{a^2} - \frac{h^2}{c^2}\right) = H;$$

then $$K^2r^2 + 2KFr + F^2 = H \text{..................} (2).$$

Equation (2) will give two values for r, one positive and the other negative; these values we may denote by r_1 and $-r_2$, where

$$r_1 = \frac{-F+\sqrt{H}}{K}, \quad r_2 = \frac{F+\sqrt{H}}{K}.$$

Hence to find the whole attraction of the spheroid parallel to the axis of x, we first integrate the expression

$$\rho \sin^2\theta \cos\phi \, \delta\theta \, \delta\phi \, \delta r$$

with respect to r between the limits $r=0$ and $r=r_1$, and also between the limits $r=0$ and $r=r_2$, and take the difference; we thus obtain

$$\rho \sin^2\theta \cos\phi \, (r_2 - r_1) \, \delta\theta \, \delta\phi ;$$

this must be integrated between 0 and π for ϕ, and 0 and π for θ. If A denote the whole attraction parallel to the axis of x, acting *towards* the origin, we have then

$$A = 2\rho \int_0^\pi \int_0^\pi \frac{F}{K} \sin^2\theta \cos\phi \, d\theta \, d\phi.$$

We may simplify this expression by omitting those terms which vanish by the principles of the Integral Calculus; thus

$$A = 2f\rho c^2 \int_0^\pi \int_0^\pi \frac{\sin^3\theta \cos^2\phi \, d\theta \, d\phi}{c^2 \sin^2\theta + a^2 \cos^2\theta}$$

$$= \pi f\rho c^2 \int_0^\pi \frac{\sin^3\theta \, d\theta}{c^2 \sin^2\theta + a^2 \cos^2\theta}$$

$$= \pi f\rho c^2 \int_0^\pi \frac{(1-\cos^2\theta) \sin\theta \, d\theta}{c^2 + (a^2 - c^2) \cos^2\theta}$$

$$= \frac{\pi f\rho c^2}{a^2 - c^2} \int_0^\pi \left\{ \frac{a^2 \sin\theta}{c^2 + (a^2 - c^2)\cos^2\theta} - \sin\theta \right\} d\theta$$

$$= \frac{2\pi f\rho c^2}{a^2 - c^2} \left\{ \frac{a^2}{c\sqrt{(a^2 - c^2)}} \tan^{-1} \frac{\sqrt{(a^2 - c^2)}}{c} - 1 \right\}.$$

Let $c^2 = a^2(1 - e^2)$; then the result may be written

$$A = 2\pi f\rho \left\{ \frac{\sqrt{(1-e^2)}}{e^3} \sin^{-1} e - \frac{1-e^2}{e^2} \right\}.$$

In the same manner, if B denote the whole attraction parallel to the axis of y,

$$B = 2\pi g\rho \left\{\frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e - \frac{1-e^2}{e^2}\right\}.$$

Let C denote the whole attraction parallel to the axis of z, then

$$C = 2\rho\int_0^\pi\int_0^\pi \frac{F}{K}\sin\theta\cos\theta\,d\theta\,d\phi$$

$$= 2h\rho a^2\int_0^\pi\int_0^\pi \frac{\sin\theta\cos^2\theta\,d\theta\,d\phi}{c^2\sin^2\theta + a^2\cos^2\theta}$$

$$= \frac{2\pi h\rho a^2}{a^2-c^2}\int_0^\pi\left\{\sin\theta - \frac{c^2\sin\theta}{c^2+(a^2-c^2)\cos^2\theta}\right\}d\theta$$

$$= \frac{4\pi h\rho a^2}{a^2-c^2}\left\{1 - \frac{c}{\sqrt{(a^2-c^2)}}\tan^{-1}\frac{\sqrt{(a^2-c^2)}}{c}\right\}$$

$$= 4\pi h\rho\left\{\frac{1}{e^2} - \frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e\right\}.$$

If the spheroid be *prolate* a is less than c. It may be shewn then that

$$A = \frac{2\pi f\rho c^2}{c^2-a^2}\left\{1 - \frac{a^2}{c\sqrt{(c^2-a^2)}}\log\frac{c+\sqrt{(c^2-a^2)}}{a}\right\},$$

$$B = \frac{2\pi g\rho c^2}{c^2-a^2}\left\{1 - \frac{a^2}{c\sqrt{(c^2-a^2)}}\log\frac{c+\sqrt{(c^2-a^2)}}{a}\right\},$$

$$C = \frac{4\pi h\rho a^2}{c^2-a^2}\left\{\frac{c}{\sqrt{(c^2-a^2)}}\log\frac{c+\sqrt{(c^2-a^2)}}{a} - 1\right\}.$$

It may be noticed that in both cases

$$\frac{A}{f} + \frac{B}{g} + \frac{C}{h} = 4\pi\rho.$$

227. From the expressions in the preceding article we see that the attraction is independent of the magnitude of the spheroid and depends solely upon the excentricity.

Hence the attraction of the spheroid similar to the given one and passing through the attracted particle, is the same as that of any other similar and similarly situated concentric spheroid *comprising the attracted particle in its mass*. Hence a spheroidal shell the surfaces of which are similar, similarly situated, and concentric, attracts a particle within it equally in all directions. This has been already established; see Art. 215.

If we put the ellipticity of the spheroid $= \epsilon$, and suppose ϵ very small so that we may neglect its square, we have for the *oblate* spheroid since $c = a(1-\epsilon)$,

$$e^2 = 1 - \frac{c^2}{a^2} = 1 - (1-\epsilon)^2 = 2\epsilon \text{ approximately.}$$

After expansion and reduction we shall obtain approximately

$$A = \tfrac{4}{3}\pi\rho\,(1 - \tfrac{2}{5}\epsilon)\,f,$$
$$B = \tfrac{4}{3}\pi\rho\,(1 - \tfrac{2}{5}\epsilon)\,g,$$
$$C = \tfrac{4}{3}\pi\rho\,(1 + \tfrac{4}{5}\epsilon)\,h;$$

For the *prolate* spheroid since $a = c(1-\epsilon)$,

$$e^2 = 1 - \frac{a^2}{c^2} = 1 - (1-\epsilon)^2 = 2\epsilon.$$

After expansion and reduction we shall obtain approximately

$$A = \tfrac{4}{3}\pi\rho\,(1 + \tfrac{2}{5}\epsilon)\,f,$$
$$B = \tfrac{4}{3}\pi\rho\,(1 + \tfrac{2}{5}\epsilon)\,g,$$
$$C = \tfrac{4}{3}\pi\rho\,(1 - \tfrac{4}{5}\epsilon)\,h.$$

228. If instead of the spheroid we take an ellipsoid whose semiaxes are a, b, c, it may be shewn that

$$C = 4\pi h\rho ab \int_0^{\frac{1}{2}\pi} \frac{\cos^2\theta \sin\theta\, d\theta}{\sqrt{(a^2\cos^2\theta + c^2\sin^2\theta)}\,\sqrt{(b^2\cos^2\theta + c^2\sin^2\theta)}};$$

and the values of A and B may be found by symmetrical changes in the letters a, b, c and f, g, h.

If we change a, b, c into $a(1+n)$, $b(1+n)$, $c(1+n)$ respectively, the expression for C *remains unchanged;* and so also the expressions for A and B remain unchanged. This shews that a shell of any thickness, the internal and external boundaries of which are similar and similarly situated concentric ellipsoids, exerts no attraction on a particle within the inner boundary. This has been already established; see Art. 215.

229. Suppose we require the attraction of a spheroid on an *external* particle.

In the equation (2) of Art. 226, we shall now have $F^2 - H$ a *positive* quantity, and the two roots of that quadratic equation will have the *same* sign. Hence we shall find

$$A = 2\rho \iint \frac{\sqrt{H}}{K} \sin^2\theta \cos\phi \, d\phi \, d\theta.$$

The limits of the integration with respect to θ will involve ϕ, for these limits will be found by putting $H = 0$, and this leads to the following quadratic equation for determining $\tan\theta$,

$$\tan^2\theta \left\{ \left(\frac{f\cos\phi + g\sin\phi}{a^2} \right)^2 + \frac{1}{a^2}\left(1 - \frac{f^2+g^2}{a^2} - \frac{h^2}{c^2}\right) \right\}$$

$$+ \frac{2h\tan\theta}{c^2} \cdot \frac{f\cos\phi + g\sin\phi}{a^2} + \frac{1}{c^2}\left(1 - \frac{f^2+g^2}{a^2}\right) = 0.$$

Then the limits of ϕ are to be determined from the condition that the values of $\tan\theta$ furnished by this quadratic equation must be *equal;* this leads after some reduction to the following equation for determining the limits of ϕ,

$$(f\cos\phi + g\sin\phi)^2 = f^2 + g^2 - a^2.$$

It is however unnecessary to proceed with these complicated integrations, for we can obtain the result indirectly by means of Ivory's theorem, which furnishes a relation between the attractions of *ellipsoids* on *external* and *internal* particles; this theorem will be true for *spheroids* as they are included among *ellipsoids*, and since the attraction of a spheroid on an *internal* particle has been already found, the theorem will enable us to determine the attraction of a spheroid on an *external* particle.

230. *To enunciate and prove Ivory's Theorem.*

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$, be the equations to the surfaces bounding two homogeneous ellipsoids of the same density and having the same centre and foci; then

$$a^2 - b^2 = \alpha^2 - \beta^2, \quad a^2 - c^2 = \alpha^2 - \gamma^2 \,\ldots\ldots\ldots (1).$$

Let f, g, h and f', g', h' be the co-ordinates of two particles so situated on the surfaces of these ellipsoids that

$$\frac{f}{f'} = \frac{a}{\alpha}, \quad \frac{g}{g'} = \frac{b}{\beta}, \quad \frac{h}{h'} = \frac{c}{\gamma} \ldots\ldots\ldots\ldots (2).$$

Also, since (f, g, h) and (f', g', h') are points in the surfaces of the first and second ellipsoids respectively, we have

$$\frac{f^2}{a^2} + \frac{g^2}{b^2} + \frac{h^2}{c^2} = 1, \quad \frac{f'^2}{\alpha^2} + \frac{g'^2}{\beta^2} + \frac{h'^2}{\gamma^2} = 1 \ldots\ldots (3).$$

Then *the attraction of the first ellipsoid parallel to the axis of* z *on the particle situated at the point* (f', g', h') *on the surface of the second is to the attraction of the second ellipsoid on the particle situated at the point* (f, g, h) *on the surface of the first in the same direction as* ab *to* $\alpha\beta$, *the law of attraction being any function of the distance; and similarly with respect to the axes of* y *and* x. This is Ivory's Theorem.

We shall, for convenience, represent the law of attraction by the function $r\phi(r^2)$, r being the distance.

The attraction of the first ellipsoid on the particle (f', g', h') parallel to the axis of z

$$= \rho\iiint (h' - z)\, \phi\,\{(f' - x)^2 + (g' - y)^2 + (h' - z)^2\}\, dx\, dy\, dz\,;$$

the limits of z are

$$-c\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)} \text{ and } c\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)},$$

the limits of y are

$$-b\sqrt{\left(1 - \frac{x^2}{a^2}\right)} \text{ and } b\sqrt{\left(1 - \frac{x^2}{a^2}\right)},$$

the limits of x are $-a$ and a.

Let $\psi(r) = \frac{1}{2}\int \phi(r)\, dr$; then the attraction

$$= \rho \iint \{\psi\,[(f'-x)^2 + (g'-y)^2 + (h'+z)^2] - \psi\,[(f'-x)^2 + (g'-y)^2 + (h'-z)^2]\}\, dx\, dy$$

between the specified limits; it must be remembered that in this expression $z = c\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}$, but we do not substitute this value merely for preserving the functions under as simple a form as possible. Now put $x = ar$, $y = bs$, $z = ct$, then the attraction

$$= \rho ab \iint \{\psi\,[(f'-ar)^2 + (g'-bs)^2 + (h'+ct)^2] - \psi\,[(f'-ar)^2 + (g'-bs)^2 + (h'-ct)^2]\}\, dr\, ds,$$

the limits of s being $-\sqrt{(1-r^2)}$ and $\sqrt{(1-r^2)}$, and those of r being -1 and 1; also $t = \sqrt{(1-r^2-s^2)}$.

Now $\qquad (f'-ar)^2 + (g'-bs)^2 + (h' \mp ct)^2$

$$= f'^2 + g'^2 + h'^2 - 2\,(f'ar + g'bs \pm h'ct) + a^2r^2 + b^2s^2 + c^2t^2;$$

by substituting for h'^2 from (3), and putting $1 - r^2 - s^2$ for t^2, this becomes

$$f'^2\left(1 - \frac{\gamma^2}{\alpha^2}\right) + g'^2\left(1 - \frac{\gamma^2}{\beta^2}\right) + \gamma^2 - 2\,(f'ar + g'bs \pm h'ct) + (a^2 - c^2)\,r^2 + (b^2 - c^2)\,s^2 + c^2;$$

eliminating f', g', h' by (2), and making use of (1), this becomes

$$\frac{f^2}{a^2}(a^2 - c^2) + \frac{g^2}{b^2}(b^2 - c^2) + c^2 - 2\,(f\alpha r + g\beta s \pm h\gamma t) + (\alpha^2 - \gamma^2)\,r^2 + (\beta^2 - \gamma^2)\,s^2 + \gamma^2$$

$= f^2 + g^2 + h^2 - 2\,(f\alpha r + g\beta s \pm h\gamma t) + \alpha^2r^2 + \beta^2s^2 + \gamma^2t^2$, by (3),

$= (f - \alpha r)^2 + (g - \beta s)^2 + (h \mp \gamma t)^2$.

Hence the attraction of the first ellipsoid on (f', g', h') parallel to z

$$= \rho ab \iint \{\psi\,[(f-\alpha r)^2 + (g-\beta s)^2 + (h+\gamma t)^2]$$
$$- \psi\,[(f-\alpha r)^2 + (g-\beta s)^2 + (h-\gamma t)^2]\}\,dr\,ds\,;$$

the limits of s being $-\sqrt{(1-r^2)}$, $\sqrt{(1-r^2)}$, and of r being -1, 1. By symmetry, this expression

$$= \frac{ab}{\alpha\beta} \times \text{attraction of second ellipsoid on } (f, g, h) \text{ parallel to } z.$$

The same may be proved for the attractions parallel to the other axes, and consequently the theorem, as enunciated, is true.

We observe that one of these ellipsoids lies wholly *within* the other; for if not, the points in which they cut each other lie in the line of which the equations are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \text{ and } \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1\,;$$

the points of intersection must therefore satisfy the equation

$$x^2\left(\frac{1}{a^2} - \frac{1}{\alpha^2}\right) + y^2\left(\frac{1}{b^2} - \frac{1}{\beta^2}\right) + z^2\left(\frac{1}{c^2} - \frac{1}{\gamma^2}\right) = 0,$$

and this by (1) becomes

$$\left(\frac{x}{a\alpha}\right)^2 + \left(\frac{y}{b\beta}\right)^2 + \left(\frac{z}{c\gamma}\right)^2 = 0,$$

an equation which can be satisfied solely by $x=0$, $y=0$, $z=0$; but these do not satisfy the equations above, and therefore the surfaces do not intersect in any point.

Hence to find the attraction of an ellipsoid of which the semiaxes are a, b, c on an external particle of which the co-ordinates are f', g', h', we must first calculate the attraction of an ellipsoid of which the semiaxes are α, β, γ parallel to the axes, on an internal particle of which the co-ordinates are f, g, h, these six quantities being determined by the equations

$$\alpha^2 - \beta^2 = a^2 - b^2, \quad \alpha^2 - \gamma^2 = a^2 - c^2,$$

$$\frac{f'^2}{\alpha^2} + \frac{g'^2}{\beta^2} + \frac{h'^2}{\gamma^2} = 1,$$

$$f = \frac{af'}{\alpha}, \quad g = \frac{bg'}{\beta}, \quad h = \frac{ch'}{\gamma},$$

and then the attractions required will be these three calculated attractions multiplied respectively by

$$\frac{bc}{\beta\gamma}, \quad \frac{ac}{\alpha\gamma}, \quad \frac{ab}{\alpha\beta}.$$

231. *To prove that the resultant attraction of the particles of a body of any figure upon a particle of which the distance is very great in comparison with the greatest diameter of the attracting body, is very nearly the same, as if the particles were condensed into their centre of gravity and attracted according to the same law, whatever that law be.*

Let the origin of co-ordinates be taken at the centre of gravity of the attracting body, the axis of x through the attracted particle; let c be its abscissa, and x, y, z the co-ordinates of any particle of the body, ρ the density of that particle.

Then the distance between these two particles, or r,

$$= \sqrt{\{(c-x)^2 + y^2 + z^2\}}.$$

Let $r\phi(r^2)$ be the law of attraction; then the whole attraction parallel to the axis of x

$$= \iiint \rho\,(c-x)\,\phi\,(c^2 - 2cx + x^2 + y^2 + z^2)\,dx\,dy\,dz,$$

the limits being obtained from the equation to the surface of the body. This attraction therefore

$$= \iiint \rho\,(c-x)\,\{\phi\,(c^2) - (2cx - x^2 - y^2 - z^2)\,\phi'(c^2) + \ldots\}\,dx\,dy\,dz$$

$$= c\phi\,(c^2) \iiint \rho \left\{1 - \frac{x}{c}\left(1 + \frac{2c^2\phi'(c^2)}{\phi\,(c^2)}\right) + (y^2 + z^2 + 3x^2)\frac{\phi'(c^2)}{\phi(c^2)} + \ldots\right\} dx\,dy\,dz$$

$$= Mc\phi\,(c^2) + c^3\phi'(c^2) \iiint \rho \frac{y^2 + z^2 + 3x^2}{c^2}\,dx\,dy\,dz + \ldots\ldots\ldots\ldots (A),$$

M being the mass of the body, and $\iiint \rho x\,dx\,dy\,dz = 0$, since x is measured from the centre of gravity of the body.

Now suppose x, y, z to be exceedingly small in comparison with c; then all the terms of (A) after the first are extremely small in comparison with that term, it being observed that $c^3\phi'(c^2)$ is of the same order as $c\phi(c^2)$ in terms of c. Hence the Proposition is true.

232. From Art. 224, it appears that when the law of attraction is that of the inverse square of the distance, a sphere composed of shells, each of which is homogeneous, attracts an external particle with a resultant force, which is the same as if the sphere were condensed at its centre. It may be shewn also that *two such spheres* attract each other in the same manner as if each were condensed at its centre. For consider any element of mass forming part of the first sphere; the attraction of this on the second sphere will be equal and opposite to the resultant attraction of the second sphere upon it, and will therefore be the same as if the second sphere were collected at its centre. Similarly, the attraction of any other element of the first sphere on the second will be the same as if the second were collected at its centre. Proceeding thus, we find that the whole action of the first sphere on the second is the same as if the second were collected at its centre, and therefore the mutual attraction of the spheres is the same as if each were collected at its centre.

If the law of attraction be that of the direct distance, then two bodies of any shape attract each other with a resultant force which is the same as if each were collected at its centre of gravity.

We proceed to general formulæ for the attraction of bodies of any form.

233. Let there be a body of any form; let ρ represent the density of an element, the volume of which is $dx\,dy\,dz$, x, y, z being the co-ordinates of the element. Suppose the attraction between the particles of masses m and m' respectively, at a distance r, to be $mm'F(r)$; then the components X, Y, Z parallel to the axes, and *from* the origin, of the attraction of

the body on a particle whose mass is unity, and co-ordinates a, b, c are found by the equations

$$X = \iiint \rho \frac{x-a}{r} F(r)\, dx\, dy\, dz,$$

$$Y = \iiint \rho \frac{y-b}{r} F(r)\, dx\, dy\, dz,$$

$$Z = \iiint \rho \frac{z-c}{r} F(r)\, dx\, dy\, dz,$$

$$r \text{ being } = \{(x-a)^2 + (y-b)^2 + (z-c)^2\}^{\frac{1}{2}}.$$

The integrations are to be taken so as to include all the elements of the attracting body.

Let $\phi(r)$ be such a function of r that $F(r)$ is its differential coefficient with respect to r, and let

$$U = \iiint \rho\phi(r)\, dx\, dy\, dz,$$

the integrations being extended so as to include all the elements of the attracting body; then will

$$X = -\frac{dU}{da}, \quad Y = -\frac{dU}{db}, \quad Z = -\frac{dU}{dc}.$$

For $\dfrac{d\phi(r)}{da} = \dfrac{d\phi(r)}{dr}\dfrac{dr}{da} = F(r)\dfrac{dr}{da} = -F(r)\dfrac{x-a}{r}$;

therefore
$$X = -\iiint \rho \frac{d\phi(r)}{da}\, dx\, dy\, dz$$

$$= -\frac{d}{da}\iiint \rho\phi(r)\, dx\, dy\, dz$$

$$= -\frac{dU}{da}.$$

Similarly, the equations $Y = -\dfrac{dU}{db}$ and $Z = -\dfrac{dU}{dc}$ may be established.

It may be observed that if in any case, for example that of an infinite solid, the integral U becomes infinite, but the differential coefficients $\frac{dU}{da}$, $\frac{dU}{db}$, $\frac{dU}{dc}$ are finite, the preceding values of X, Y, Z will still be correct.

For suppose we take a finite portion of the solid; the components of its attraction will have for values the differential coefficients of U. Suppose now that we extend without limit the portion of the mass considered, the components of the attraction will always be

$$-\frac{dU}{da}, \quad -\frac{dU}{db}, \quad -\frac{dU}{dc},$$

whether U increase without limit or not. Hence, if these three expressions tend to limits, those limits will be the components of the attraction of the infinite solid. And if they increase indefinitely, we may conclude that the attraction increases without limit as the portion of the body considered increases; this we express by saying that the attraction of the solid is infinite.

234. If the law of attraction be that of the inverse square, we have

$$F(r) = \frac{1}{r^2}, \text{ and } \phi(r) = -\frac{1}{r}.$$

Let $V = -U$, that is, let

$$V = \iiint \frac{\rho dx dy dz}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

then, as in the preceding article, we have for the attractions parallel to the axes of x, y, z respectively, and *from* the origin,

$$X = \frac{dV}{da}, \quad Y = \frac{dV}{db}, \quad Z = \frac{dV}{dc}.$$

The equation which gives V is equivalent to the following operation:—*decompose the attracting mass into indefinitely small elements, and divide the mass of each element by the distance of that element from the attracted particle; the sum of these quotients is V.* Hence, the value of V will be quite independent of the axes, rectangular or polar, which we may find it convenient to employ. Suppose we use the ordinary polar formulæ, and take the position of the attracted particle for the origin; then the element of volume is (Art. 130) $r^2 \sin\theta\, \delta\phi\, \delta\theta\, \delta r$; therefore

$$V = \iiint \rho r \sin\theta\, d\phi\, d\theta\, dr \ldots\ldots\ldots\ldots\ldots (2).$$

Suppose the attracted particle forms part of the attracting mass; then, since r vanishes for those particles of the attracting mass which are in contact with the attracted particle, from equation (1) it would be doubtful if V is *finite* in this case; but from (2) we see that it really is finite.

235. *To express by means of V the attraction resolved along any line.*

Let s be the length of the arc of any curve measured from a fixed point up to P the attracted particle; l, m, n the direction cosines of the tangent to this line at P; R the attraction resolved along this tangent; then

$$R = lX + mY + nZ$$

$$= l\frac{dV}{da} + m\frac{dV}{db} + n\frac{dV}{dc}.$$

Now, if we restrict ourselves to points lying on the line s, V will become a function of s alone; for V is a function of a, b, and c, and each of these may be regarded as a function of s; thus we shall have by the differential calculus,

$$\frac{dV}{ds} = \frac{dV}{da}\frac{da}{ds} + \frac{dV}{db}\frac{db}{ds} + \frac{dV}{dc}\frac{dc}{ds};$$

and since $\frac{da}{ds} = l$, $\frac{db}{ds} = m$, $\frac{dc}{ds} = n$, we get

$$R = \frac{dV}{ds}.$$

236. *To examine the meaning of the function V.*

This function is of so much importance that it will be well to dwell a little on its meaning.

In the first place it may be observed that the equation (1) contains a physical definition of V, which has nothing to do

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with the system of co-ordinates, rectangular, polar, or any other, which may be used to define algebraically the positions of P and of the attracting particles. Thus V is to be contemplated as a function of the position of P in space, if such an expression may be allowed, rather than as a function of the co-ordinates of P; although, in consequence of its depending upon the position of P, V will be a function of the co-ordinates of P, of whatever kind they may be.

Secondly, it may be remarked that although an attracted particle has hitherto been conceived as situated at P, yet V has a definite meaning depending upon the position of the point P, whether any attracted matter exist there or not. Thus V is to be contemplated as having a definite value at each point of space, irrespective of the attracted matter which may exist at some places.

The function V is called the *potential* of the attracting mass.

237. *To calculate the value of V in the case of a spherical shell, the density being a function of the distance from the centre.*

Take for the axis of x the line joining the centre of the sphere with the attracted particle P, which is obviously the direction of the resultant attraction; let a be the distance of P from the centre; u the distance of any point in the attracting shell from the centre; θ and ϕ the other polar co-ordinates of this point; then the mass of the element at this point is $\rho u^2 \sin\theta\, \delta u\, \delta\theta\, \delta\phi$, and

$$V = \int_{u_1}^{u_2}\int_0^{\pi}\int_0^{2\pi} \frac{\rho u^2 \sin\theta\, du\, d\theta\, d\phi}{r},$$

where u_1 and u_2 are the internal and external radii of the shell; hence,

$$V = 2\pi \int_{u_1}^{u_2}\int_0^{\pi} \frac{\rho u^2 \sin\theta\, du\, d\theta}{r}.$$

Now $$r^2 = u^2 - 2au\cos\theta + a^2;$$

therefore $$\sin\theta \frac{d\theta}{dr} = \frac{r}{au},$$

and $$V = \frac{2\pi}{a} \iint \rho u \, du \, dr.$$

We must now distinguish three cases.

I. When P is *beyond* the external surface, the limits of r are $a - u$ and $a + u$; therefore

$$V = \frac{2\pi}{a} \int_{u_1}^{u_2} \int_{a-u}^{a+u} \rho u \, du \, dr$$

$$= \frac{4\pi}{a} \int_{u_1}^{u_2} \rho u^2 du \text{ .... (1).}$$

But if M denote the mass of the spherical shell,

$$M = 4\pi \int_{u_1}^{u_2} \rho u^2 du;$$

therefore $$V = \frac{M}{a}.$$

Hence, $X = \frac{dV}{da} = -\frac{M}{a^2}$, or the attraction is the same as if the mass of the shell were collected at its centre; this was proved in Art. 212.

II. When P is *within* the internal surface, the limits of r are $u - a$ and $u + a$; therefore

$$V = \frac{2\pi}{a} \int_{u_1}^{u_2} \int_{u-a}^{u+a} \rho u \, du \, dr$$

$$= 4\pi \int_{u_1}^{u_2} \rho u \, du \text{ .... (2).}$$

Since this is independent of a, we have

$$\frac{dV}{da} = 0.$$

This is equivalent to the result found in Art. 213.

III. By combining the results contained in equations (1) and (2), we see that if P be *between* the bounding surfaces of the shell,

$$V=\frac{4\pi}{a}\int_{u_1}^{a}\rho u^2du+4\pi\int_{a}^{u_2}\rho u\,du.$$

From this we may deduce a result involved in Arts. 212 and 213, namely, that the *resultant attraction is the same as if all the matter which is nearer to the centre than P were collected at the centre, and the rest of the matter neglected.*

238. *At any point (a, b, c) external to the attracting mass, the function V satisfies the partial differential equation*

$$\frac{d^2V}{da^2}+\frac{d^2V}{db^2}+\frac{d^2V}{dc^2}=0.$$

For since $r=\{(x-a)^2+(y-b)^2+(z-c)^2\}^{\frac{1}{2}}$,

$$\frac{d}{da}\left(\frac{1}{r}\right)=\frac{x-a}{r^3},\quad \frac{d}{db}\left(\frac{1}{r}\right)=\frac{y-b}{r^3},\quad \frac{d}{dc}\left(\frac{1}{r}\right)=\frac{z-c}{r^3},$$

$$\frac{d^2}{da^2}\left(\frac{1}{r}\right)=\frac{3\,(x-a)^2}{r^5}-\frac{1}{r^3},$$

$$\frac{d^2}{db^2}\left(\frac{1}{r}\right)=\frac{3\,(y-b)^2}{r^5}-\frac{1}{r^3},$$

$$\frac{d^2}{dc^2}\left(\frac{1}{r}\right)=\frac{3\,(z-c)^2}{r^5}-\frac{1}{r^3};$$

therefore $$\frac{d^2}{da^2}\left(\frac{1}{r}\right)+\frac{d^2}{db^2}\left(\frac{1}{r}\right)+\frac{d^2}{dc^2}\left(\frac{1}{r}\right)=0.$$

Now $$V=\iiint\frac{\rho\,dx\,dy\,dz}{r};$$

therefore $$\frac{d^2V}{da^2}=\iiint\frac{d^2}{da^2}\left(\frac{1}{r}\right)\rho\,dx\,dy\,dz,$$

and similar expressions hold for $\frac{d^2V}{db^2}$ and $\frac{d^2V}{dc^2}$; therefore

$$\frac{d^2V}{da^2}+\frac{d^2V}{db^2}+\frac{d^2V}{dc^2}=0.$$

This result holds so long as the attracted particle is not in contact with the attracting mass. If, however, the attracted particle is in contact with the attracting mass, r can vanish, and therefore $\frac{1}{r}$ and its differential coefficients become *infinite;* the preceding demonstration does not hold in this case.

239. *At an internal point* (a, b, c) *about which the density is* ρ, *the function* V *satisfies the equation*

$$\frac{d^2V}{da^2}+\frac{d^2V}{db^2}+\frac{d^2V}{dc^2}=-4\pi\rho.$$

To determine the value of $\frac{d^2V}{da^2}+\frac{d^2V}{db^2}+\frac{d^2V}{dc^2}$ in this case, suppose a sphere described in the body so that it shall include the attracted particle, and let $V=V_1+V_2$, where V_2 refers to the sphere and V_1 to the remainder of the attracting body; then

$$\frac{d^2V}{da^2}+\frac{d^2V}{db^2}+\frac{d^2V}{dc^2}=\frac{d^2V_1}{da^2}+\frac{d^2V_1}{db^2}+\frac{d^2V_1}{dc^2}$$

$$+\frac{d^2V_2}{da^2}+\frac{d^2V_2}{db^2}+\frac{d^2V_2}{dc^2}$$

$$=\frac{d^2V_2}{da^2}+\frac{d^2V_2}{db^2}+\frac{d^2V_2}{dc^2},$$

by what has been already proved.

Now the centre of the sphere may be chosen as near the attracted particle as we please, and the radius of the sphere may be taken so small that its density may be considered ultimately uniform, and equal to that at the point (a, b, c).

Let α, β, γ be the co-ordinates of the centre of the sphere; then the attractions of the sphere on the particle parallel to the axes are, by Art. 212,

$$\frac{4}{3}\pi\rho(a-\alpha),\ \frac{4\pi\rho}{3}(b-\beta),\ \frac{4\pi\rho}{3}(c-\gamma);$$

therefore $$\frac{dV_2}{da}=-\frac{4\pi\rho}{3}(a-\alpha),\quad \frac{d^2V_2}{da^2}=-\frac{4\pi\rho}{3},$$

$$\frac{dV_2}{db}=-\frac{4\pi\rho}{3}(b-\beta),\quad \frac{d^2V_2}{db^2}=-\frac{4\pi\rho}{3},$$

$$\frac{dV_2}{dc}=-\frac{4\pi\rho}{3}(c-\gamma),\quad \frac{d^2V_2}{dc^2}=-\frac{4\pi\rho}{3};$$

therefore $$\frac{d^2V_2}{da^2}+\frac{d^2V_2}{db^2}+\frac{d^2V_2}{dc^2}=-4\pi\rho;$$

therefore $$\frac{d^2V}{da^2}+\frac{d^2V}{db^2}+\frac{d^2V}{dc^2}=-4\pi\rho.$$

This result holds when the point (a, b, c) is *in the interior* of the attracting mass; if it be *on the surface* of the attracting mass the result must be modified. At an ordinary point of the surface we may suppose a *hemisphere*, instead of a *sphere* as in the preceding investigation, described so as to include the attracted particle; thus the right hand member of the final equation will be $-2\pi\rho$ instead of $-4\pi\rho$. At a *singular* point of the surface we shall obtain a different result; thus at the corner of a cube for example the result will be

$$-\frac{4\pi\rho}{8},\ \text{that is } -\frac{\pi\rho}{2}.$$

240. *Application to the Sphere.* In Art. 237 we have calculated V by direct integration in the case of a body composed of homogeneous spherical shells. We may also deduce its value by means of the equations in Art. 238 and 239. This we shall now do. If a sphere be composed of homogeneous shells, V will be a function of the distance r of the centre of the sphere from the attracted particle; the resultant

attraction will act along the line which joins these two points, and will be denoted by $\frac{dV}{dr}$.

The equation

$$r^2 = a^2 + b^2 + c^2$$

will give $$\frac{dr}{da} = \frac{a}{r}, \quad \frac{dr}{db} = \frac{b}{r}, \quad \frac{dr}{dc} = \frac{c}{r};$$

hence $$\frac{dV}{da} = \frac{dV}{dr}\frac{dr}{da} = \frac{dV}{dr}\frac{a}{r};$$

therefore $$\frac{d^2V}{da^2} = \frac{a^2}{r^2}\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{a^2}{r^3}\frac{dV}{dr};$$

similarly $$\frac{d^2V}{db^2} = \frac{b^2}{r^2}\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{b^2}{r^3}\frac{dV}{dr},$$

and $$\frac{d^2V}{dc^2} = \frac{c^2}{r^2}\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{c^2}{r^3}\frac{dV}{dr}.$$

By adding these equations we have, if the particle be outside the sphere, by Art. 238,

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0.$$

This may be written

$$\frac{d}{dr}\left\{r^2\frac{dV}{dr}\right\} = 0;$$

therefore $$\frac{dV}{dr} = \frac{C}{r^2},$$

where C is some constant.

Suppose the sphere to be hollow, and that the attracted particle is within the inner surface, the radius of which we shall denote by r_1. Since the attraction ought evidently to vanish when $r = 0$, we must have $C = 0$; therefore $\frac{dV}{dr} = 0$.

Hence the attraction always vanishes, and the particle is in equilibrium whatever be its position within the unoccupied part of the sphere.

Suppose next that the particle forms part of the mass of the sphere; we have, by Art. 239,

$$\frac{d^2V}{dr^2}+\frac{2}{r}\frac{dV}{dr}=-4\pi\rho,$$

ρ being a given function of r.

Multiply by r^2, and integrate from the value r_1 of r; since $\frac{dV}{dr}=0$ for all points in the interior, it is so at the limit r_1;

thus $$r^2\frac{dV}{dr}=-4\pi\int_{r_1}^{r}\rho r^2 dr.$$

But $\int_{r_1}^{r} 4\pi r^2\rho dr$ is the mass comprised within that surface of the sphere which passes through the attracted particle. If we call it M', we have

$$\frac{dV}{dr}=-\frac{M'}{r^2}.$$

The absolute value of the attraction will therefore be $\frac{M'}{r^2}$; it is the same as if the mass M' acted alone and were collected at its centre.

If the attracted particle is on the exterior surface having its radius $=r_2$, we have, if M be the whole mass of the hollow sphere,

$$\frac{dV}{dr}=-\frac{M}{r_2^{\,2}},$$

and the attraction exercised upon this particle will have for its value

$$\frac{M}{r_2^{\,2}}.$$

Lastly, consider a particle outside the sphere; that is, for which r is greater than r_2; we have, as in the first case,

$$\frac{dV}{dr}=\frac{c}{r^2}.$$

But in consequence of the discontinuity arising from the particles of the mass, the constant c is not restricted to have the same value as for the interior points. To determine it we put $r=r_2$; then, from the preceding case, we ought to have

$$\frac{dV}{dr}=-\frac{M}{r_2^2},$$

therefore $$c=-M;$$

and we shall have for external points,

$$\frac{dV}{dr}=-\frac{M}{r^2}.$$

The attraction will therefore have for its value

$$\frac{M}{r^2}.$$

This agrees with Art. 212.

241. *Application to an indefinite cylinder.* Consider next a hollow indefinite cylinder composed of homogeneous shells, the density being a function of the distance from the axis of the cylinder which we take for the axis of z. Its action upon any particle will be directed towards the point where the axis is cut by a perpendicular plane passing through the attracted particle. Take this point of the axis for origin; let r be its distance from the attracted particle; the attraction will depend only on r, and its value will be

$$\frac{dV}{dr}.$$

But for the points which are not part of the mass of the cylinder, we have, by Art. 238, observing that V is independent of c,

$$\frac{d^2V}{da^2}+\frac{d^2V}{db^2}=0,$$

whence $$\frac{d^2V}{dr^2}+\frac{1}{r}\frac{dV}{dr}=0.$$

Multiplying by r, we have

$$\frac{d}{dr}\left(r\frac{dV}{dr}\right)=0;$$

therefore $$\frac{dV}{dr}=\frac{c}{r},$$

c being some constant.

We observe, as in the case of a hollow sphere, that the points exterior to the cylindrical shell and those in the interior being separated by those of the shell, for which the circumstances are different, there is a discontinuity in passing from values of r greater than the radius of the external surface, to those of r less than the radius of the internal surface.

For points of the interior of the shell c is invariable; but it is obviously $=0$ when $r=0$; therefore for all points in the interior

$$\frac{dV}{dr}=0.$$

Hence we conclude, that *an indefinite hollow cylinder composed of homogeneous shells exercises no attraction upon a point situated within the interior of its internal surface.*

Let us now find the value of $\frac{dV}{dr}$ for points belonging to the mass of the cylinder; for these points we have, by Art. 239,

$$\frac{d^2V}{dr^2}+\frac{1}{r}\frac{dV}{dr}=-4\pi\rho,$$

and we find by integration, calling r_1 the radius of the internal surface,

$$r\frac{dV}{dr}=-4\pi\int_{r_1}^{r}\rho r dr.$$

No constant is necessary, because $\frac{dV}{dr}=0$ when $r=r_1$, since it is so for all the points of the interior of the surface of which the radius is r_1. Put $r = r_2$, then

$$\frac{dV}{dr} = -\frac{4\pi}{r_2}\int_{r_1}^{r_2} \rho r dr.$$

For external points we ought to have

$$\frac{dV}{dr} = \frac{C}{r}.$$

Make $r = r_2$, then, by reason of the preceding equation,

$$C = -4\pi \int_{r_1}^{r_2} \rho r dr.$$

The constant being thus determined, we have for all values of r greater than r_2,

$$\frac{dV}{dr} = \frac{C}{r},$$

and the attraction of the cylinder will be

$$\frac{C}{r}.$$

We shall close this chapter with some propositions extracted from an article by Professor Stokes, in the fourth volume of the *Cambridge and Dublin Mathematical Journal*, to which we have been already indebted in Art. 236.

242. *A surface of equilibrium* is one upon which a particle would rest in equilibrium if acted upon by the forces of the system, the surface being supposed fixed.

If V be the potential of an attracting body on a particle, then $V =$ constant, is the equation to a surface of equilibrium with respect to the attraction of the body. For we have shewn in Art. 235, that $\frac{dV}{ds}$ is equal to the attraction resolved along the tangent to a curve drawn through the attracted particle, but if this curve be on the surface $V =$ constant, then $\frac{dV}{ds} = 0$; that is, there is no force acting on P in the direction of any tangent to the surface $V =$ constant. Hence, if P be placed on the surface, it will remain in equilibrium. (Art. 33.)

Lines of force are curves traced so that the tangent at any point is the direction of the resultant force at that point. Hence the *lines of force* are perpendicular to the *surfaces of equilibrium*.

243. *If S be any closed surface to which all the attracting mass is external, dS an element of S, dn an element of the normal drawn outwards at dS, then*

$$\int \frac{dV}{dn} dS = 0,$$

the integral being taken throughout the whole surface S.

Let m' be the mass of any attracting particle which is situated at the point P', P' being by hypothesis external to S. Through P' draw any right line L cutting S, and produce it indefinitely in one direction from P'. The line L will in general cut S in two points; but if the surface S be *re-entrant* (that is, a closed surface which may be cut by a tangent plane), it may cut it in four, six, or any even number of points. Denote the points of section, taken in order, by P_1, P_2, P_3, &c., P_1 being that which lies nearest to P'. With P' for vertex, describe about the line L a conical surface containing an infinitely small solid angle α, and denote by $A_1, A_2, \ldots$ the areas which it cuts out from S about the points $P_1, P_2, \ldots\ldots$ Let θ_1, θ_2, be the angles which the normals drawn outwards at P_1, $P_2, \ldots\ldots$ make with the line L, taken in the direction from P_1 to P'; N_1, $N_2, \ldots\ldots$ the attractions of m' at P_1, $P_2, \ldots\ldots$ resolved along the normals; r_1, $r_2, \ldots\ldots$ the distances of P_1, $P_2, \ldots\ldots$ from P'. It is evident that the angles θ_1, $\theta_2, \ldots\ldots$ will be alternately acute and obtuse. Then we have

$$N_1 = \frac{m'}{r_1^2} \cos\theta_1, \quad N_2 = -\frac{m'}{r_2^2} \cos(\pi - \theta_2), \text{ \&c.}$$

We have also in the limit,

$$A_1 = \alpha r_1^2 \sec\theta_1, \quad A_2 = \alpha r_2^2 \sec(\pi - \theta_2), \text{ \&c.};$$

and therefore

$$N_1 A_1 = \alpha m', \quad N_2 A_2 = -\alpha m', \quad N_3 A_3 = \alpha m', \text{ \&c.};$$

and therefore, since the number of points P_1, P_2,... is even,

$$N_1A_1 + N_2A_2 + N_3A_3 + N_4A_4 \ldots = \alpha m' - \alpha m' + \alpha m' - \alpha m' \ldots = 0.$$

Now the whole solid angle contained within a conical surface described with P' for vertex, so as to circumscribe S, may be divided into an infinite number of elementary solid angles, to each of which the preceding reasoning will apply; and it is evident that the whole surface S will thus be exhausted. We have, therefore,

$$\text{limit of } \Sigma NA = 0;$$

or, by the definition of an integral,

$$\int NdS = 0.$$

The same will be true of each attracting particle m'; and therefore, if N refer to the attraction of the whole attracting mass, we shall still have $\int NdS = 0$. But, by Art. 235, $N = \frac{dV}{dn}$, which proves the proposition.

244. *If V be the potential of any mass M_1, and if M_0 be the portion of M_1 contained within a closed surface S,*

$$\int \frac{dV}{dn} dS = -4\pi M_0,$$

dn and dS having the same meaning as in Art. 243, and the integration being extended to the whole surface S.

Let m' be the mass of an attracting particle situated at the point P' inside S. Through P' draw a right line L, and produce it indefinitely in one direction. This line will in general cut S in one point; but if S be a re-entrant surface, it may be cut by L in three, five, or any odd number of points. About L describe a conical surface containing an infinitely small solid angle α, and having its vertex at P', and let the rest of the notation be as in Art. 243. In this case, the angles θ_1, θ_2,.... will be alternately obtuse and acute, and we shall have

$$N_1 = -\frac{m'}{r_1^2}\cos(\pi - \theta_1) = \frac{m'}{r_1^2}\cos\theta_1,$$

$$A_1 = \alpha r_1^2 \sec(\pi - \theta_1) = -\alpha r_1^2 \sec\theta_1,$$

and therefore $N_1A_1 = -\alpha m'$.

Should there be more than one point of section, the terms N_2A_2, N_3A_3, &c. will destroy each other two and two, as in Art. 243. Now all angular space round P' may be divided into an infinite number of solid angles such as α, and it is evident that the whole surface S will thus be exhausted. We get, therefore,

$$\text{limit of } \Sigma NA = -\Sigma\alpha m' = -m'\Sigma\alpha;$$

or, since $\Sigma\alpha = 4\pi$, $\int NdS = -4\pi m'$.

The same formula will apply to any other internal particle, and it has been shewn in Art. 243, that for an external particle $\int NdS = 0$. Hence, adding together all the results, and taking N now to refer to the attraction of all the particles, both internal and external, we get $\int NdS = -4\pi M_0$. But $N = \frac{dV}{dn}$, which proves the proposition.

245. For the researches of M. Chasles on the attraction of ellipsoids, we refer to Duhamel's *Cours de Mécanique*, or to the original memoirs in the *Journal de l'Ecole Polytechnique*, tom. XV., and the *Mémoires......des Savans Etrangers*, tom. IX. In the original memoirs will be found copious references to preceding writers on the subject.

On the general theory of attractions, the student may consult a memoir by Gauss, translated in Taylor's *Scientific Memoirs*, vol. III., and in Liouville's *Journal de Mathématiques*, tom. VII.; and also a memoir by M. Chasles in the *Connaissance des Temps pour l'année* 1845.

Some further references will be seen in the article by Professor Stokes already cited.

For the application to the theory of electricity, we refer to a series of articles by Professor Thomson in different volumes of the *Cambridge and Dublin Mathematical Journal*. See vol. I. p. 94, and vol. III. p. 140.

EXAMPLES.

In the following Examples the ordinary law of attraction is to be assumed, unless the contrary be stated.

1. A solid is generated by the revolution of a sector of a circle about one of its bounding radii; find the attraction on a particle at the centre. *Result.* $\pi a\rho \sin^2 \beta$.

2. The rim of a hemispherical bowl consists of matter repelling with a force varying directly as the distance; shew that a particle will rest when placed *anywhere* on the concave surface.

3. A tube in the form of a parabola is placed with its axis vertical and vertex downwards; a heavy particle is placed in the tube, and a repulsive force acts along the ordinate upon the particle: find the law of force that it may sustain the particle in any position.

4. A portion of a cylinder of uniform density is bounded by a spherical surface, the radius of which is greater than that of the cylinder, and the centre coincides with the middle point of the base; find the attraction on a particle at this point.

Result. $2\pi\rho a - \frac{\pi\rho a^2}{b}$; where a is the radius of the cylinder and b the radius of the sphere.

5. Find the resultant attraction of a spherical segment on a particle at its vertex.

Result. $$2\pi h\rho \left\{1 - \frac{1}{3}\sqrt{\left(\frac{2h}{a}\right)}\right\},$$

where a is the radius of the sphere and h the height of the segment.

6. Find the resultant attraction of a spherical segment on a particle at the centre of its base.

Result. $$\frac{2\pi h\rho}{3(a-h)^2}\{3a^2 - 3ah + h^2 - (2a-h)^{\frac{3}{2}}h^{\frac{1}{2}}\}.$$

7. Find the locus of a point such that its resultant attraction on a fixed line may always pass through a fixed point in the line. *Result.* A sphere.

8. Find the attraction of a segment of a paraboloid of revolution bounded by a plane perpendicular to its axis on a particle at the focus.

9. Round the circumference of a circle n equal centres of force are ranged symmetrically; each force is repulsive and varies inversely as the m^{th} power of the distance. A particle is placed in the plane of the circle very near its centre; shew that the resultant force on it tends to the centre of the circle and varies approximately as the distance of the particle from the centre, except when $m = 1$.

10. Eight centres of force, resident in the corners of a cube, attract, according to the same law and with the same absolute intensity, a particle placed very near the centre of the cube; shew that their resultant attraction passes through the centre of the cube, unless the law of force be that of the inverse square of the distance.

11. If the law of force in the preceding example be that of the inverse square of the distance find the approximate value of the attraction on a particle placed very near the centre.

Result. Take the centre of the cube as origin and the axes parallel to the edges of the cube; then if x, y, z be the co-ordinates of the particle the attraction parallel to the axis of x is approximately

$$\frac{56x}{9\,(a\sqrt{3})^5}\,(3y^2 + 3z^2 - 2x^2)$$

towards the origin; $2a$ being the length of an edge.

12. The attraction of a uniform rod of indefinite length on an external particle varies as $(\text{distance})^{-1}$ of the point from the rod. Prove this, and supposing the asymptotes of an hyperbola to consist of such material, shew that a particle will be in equilibrium at any point of the hyperbola, and that the pressure

on the curve at any point is proportional to the length of the tangent intercepted by the asymptotes.

13. An elliptic lamina attracts an internal particle (x, y) with a force varying inversely as the distance; shew that if X, Y be the whole attractions parallel to the axes,

$$\frac{X}{x} + \frac{Y}{y} = \text{constant}.$$

14. If A, B, C be the attractions of an ellipsoid in directions parallel to its axes on an internal particle situated at the point (f, g, h), shew that

$$\frac{A}{f} + \frac{B}{g} + \frac{C}{h} = 4\pi\rho.$$

(See Arts. 228 and 239.)

15. The resultant attraction of a particle which attracts according to the inverse cube of the distance upon a plane lamina is the same as upon that part of the spherical shell described about the particle as centre and touching the plane of the lamina, which is cut off by straight lines from the centre to the edge of the lamina.

16. A particle attracted by two centres of force at A and B is placed in a fixed groove. Shew that the particle remains at rest at whatever point it is placed, provided that the form of the groove be such that

$$(AP - c)(BP - c') = cc',$$

where c, c' are constants dependent upon the absolute forces.

17. If a portion of a thin spherical shell, whose projections upon the three co-ordinate planes through the centre are A, B, C, attract a particle at the centre with a force varying as any function of the distance, shew that the particle will begin to move in the direction of a straight line whose equations are

$$\frac{x}{A} = \frac{y}{B} = \frac{z}{C}.$$

18. The particles of a thin hemispherical shell attract with a force $= \mu$ (distance), and those of a right conical shell repel with a force $= \mu$ (distance). The rims of their bases coincide, and their vertices are turned in opposite directions, shew that a particle will rest in the common axis produced at a distance from the vertex of the sphere = length of the axis of the cone, the vertical angle of the cone being $2 \tan^{-1}\frac{4}{3}$.

19. Shew that if the attraction vary inversely as the distance an indefinitely thin plane ring exerts no force on a particle in the plane of the ring within its inner circumference.

20. Shew that if the attraction vary inversely as the distance an indefinitely thin plane ring attracts a particle in the plane of the ring beyond its outer circumference in the same manner as if the mass of the ring were collected at its centre.

21. If a straight line be the attracting body, shew that the *lines of force* are hyperbolas and the *surfaces of equilibrium* spheroids. (*Cambridge and Dublin Mathematical Journal*, Vol. III. p. 94.)

22. From the proposition established in Art. 244, deduce that established in Art. 239. (*Cambridge and Dublin Mathematical Journal*, Vol. V. p. 215.)

CHAPTER XIV.

VIRTUAL VELOCITIES.

246. We proceed to establish a general theorem respecting the equilibrium of a body or system of bodies, called the *Principle of Virtual Velocities.*

When a system of particles is in equilibrium, and we *suppose* each of them placed in a position indefinitely near that which it really occupies, without disturbing the connexion of the parts of the system with each other, the line which joins the first position of a particle with the second is called the *virtual velocity of that particle.*

The term *velocity* is used because we may conceive all the displacements to be made in the *same* indefinitely small time, and then the spaces described are proportional to the velocities. The word *virtual* is used to intimate that the displacements are not really made, but only *supposed.* We retain the established phraseology, but it is evident from these explanations that the words *virtual velocity* might be conveniently replaced by *hypothetical displacement.*

By the words, *without disturbing the connexion of the parts of the system with each other*, we mean, that any rigid body which exists in the system is supposed to remain of invariable form, and that any rods or strings which connect different parts of the system are to remain unbroken. This, at least, will serve for a preliminary statement to assist the reader, and we shall recur to the subject again; see Art. 257. Hence, by reason of this limitation the virtual velocities of the different parts of a system are frequently so connected that when those of a definite number of points are assumed, those of all the rest necessarily follow.

247. The virtual velocity of a particle estimated in a given direction is the projection of the virtual velocity on this direction; it is considered positive when the direction

of the motion of the particle, in passing from its first position to its second, makes an acute angle with that along which we are estimating the velocity. Thus the virtual velocity of a particle estimated along any given line is found both in magnitude and sign, by multiplying the absolute virtual velocity by the cosine of the angle which its direction makes with the given line.

The *virtual moment* of a force is the product of its intensity by the virtual velocity of its point of application *estimated in the direction of the force.*

We can now enunciate the principle of virtual velocities.

If any system of particles is in equilibrium, and we conceive a displacement of all the particles which is consistent with the conditions to which they are subject, *the sum of the virtual moments of all the forces is zero, whatever be the displacement.* And conversely, *if this relation hold for all the virtual displacements, the system is in equilibrium.*

248. The student will derive from the demonstrations which follow a better notion of the meaning of the principle than from the mere enunciation of it; it is, in fact, necessary to obtain a general view of the whole subject before attempting fully to comprehend the preliminary definitions and statements. One remark may be made for the purpose of anticipating a difficulty; each *virtual moment* is by definition an indefinitely small quantity, that is, ultimately vanishes, so that the principle seems to amount only to this, *take each force of the system and multiply it by a quantity which ultimately vanishes, then the sum of these products vanishes.* The principle, however, implies more than this statement, as we shall see.

The convenient term *virtual moment* is given by Duhamel; it may, however, be useful to enunciate the principle of virtual velocities without introducing a new definition, and we therefore give the following.

Suppose a material system held in equilibrium by any forces, and suppose the points of application of the forces moved through very small spaces in a manner consistent with the connexion of the parts of the system with each other. Let perpendiculars be drawn from the new positions of the points upon the directions of the forces acting at the points in their positions of equilibrium. The distance of any perpendicular from the original point of application of the corresponding force, is called the virtual velocity of the point with respect to that force, and is estimated positive or negative, according as the perpendicular falls on the side of the point towards which the force acts or on the opposite side. Then the principle is this, *the algebraical sum of the product of each force of the system and the corresponding virtual velocity vanishes. And conversely, if the sum vanishes for every displacement the system is in equilibrium.*

Before we proceed to a general demonstration, we will consider two simple cases, that of a particle, and that of a rigid rod acted on by forces at its ends.

249. Suppose that forces act on a single particle and maintain it in equilibrium. Let P_1, P_2, ... denote the forces; α_1, α_2, ... the angles which their directions respectively make with *any* fixed line arbitrarily chosen; then, by Art. 29,

$$\Sigma P \cos \alpha = 0.$$

If every term of this equation be multiplied by the arbitrary quantity r, we have $\Sigma Pr \cos \alpha = 0$. But $r \cos \alpha_1$ is the projection of the length r, measured along the fixed line, on the direction of the force P_1; a similar meaning may be assigned to $r \cos \alpha_2$, $r \cos \alpha_3$, ... Also r may be considered as the distance of the first position of the particle from a second position arbitrarily chosen, and therefore, when r is indefinitely diminished, $r \cos \alpha_1$, $r \cos \alpha_2$, ... become the virtual velocities of the particle with respect to P_1, P_2, ... Hence, the principle holds in this case.

Conversely, if $\Sigma Pr \cos \alpha = 0$ for all directions of displacement; then, $\Sigma P \cos \alpha = 0$ for all directions, and the particle is in equilibrium under the action of the given forces.

In this case, we observe that the hypothetical displacement of the particle may be of any magnitude we please, and that the sum of the products of each force into the projection of the displacement upon its direction is not only *ultimately*, but *always* zero.

250. Since when a system of forces acting on a particle is in equilibrium, each force is equal and opposite to the resultant of all the other forces, and, as we have just seen, the sum of the products of each force into its *virtual velocity* is zero, it follows, that the product of any force into its virtual velocity is numerically equal to the sum of such products for any system of forces which it balances, but is of the opposite sign. Hence, the product of any force into its virtual velocity is equal to the sum of such products for any system of forces acting at one point of which it is the resultant.

251. Next, suppose a rigid rod acted on by a force at each end. Let x, y, z be the co-ordinates of one end, and x', y', z' those of the other; l the length of the rod; then

$$(x-x')^2+(y-y')^2+(z-z')^2=l^2 \ldots\ldots\ldots\ldots (1).$$

Suppose the rod displaced; let δx, δy, δz be the changes made in the co-ordinates of one end; $\delta x'$, $\delta y'$, $\delta z'$ those made in the co-ordinates of the other end; then

$$(x+\delta x-x'-\delta x')^2+(y+\delta y-y'-\delta y')^2+(z+\delta z-z'-\delta z')^2=l^2 \ldots. (2).$$

From (1) and (2),

$$2(x-x')(\delta x-\delta x')+2(y-y')(\delta y-\delta y')+2(z-z')(\delta z-\delta z')$$
$$+(\delta x-\delta x')^2+(\delta y-\delta y')^2+(\delta z-\delta z')^2=0 \ldots\ldots\ldots (3).$$

Let α, β, γ be the angles which the original direction of the rod makes with the axes; then

$$x'-x=l\cos\alpha,\quad y'-y=l\cos\beta,\quad z'-z=l\cos\gamma \ldots. (4).$$

If then, in (3), we *neglect the terms* $(\delta x-\delta x')^2$, $(\delta y-\delta y')^2$, $(\delta z-\delta z')^2$ *in comparison with those we retain*, we have

$$(x-x')(\delta x-\delta x')+(y-y')(\delta y-\delta y')+(z-z')(\delta z-\delta z')=0,$$

or, by means of (4),

$$\delta x\cos\alpha+\delta y\cos\beta+\delta z\cos\gamma=\delta x'\cos\alpha+\delta y'\cos\beta+\delta z'\cos\gamma \ldots (5).$$

Suppose P the resultant of the forces acting at one end of the rod, and P' the resultant of those acting at the other end; then, in order that there may be equilibrium, these forces must be equal in magnitude and must act along the rod in opposite directions. This is obvious, or may be easily shewn by Art. 73. Since then $P' = -P$, we have by (5)

$$P(\delta x \cos\alpha + \delta y \cos\beta + \delta z \cos\gamma)$$
$$+ P'(\delta x' \cos\alpha + \delta y' \cos\beta + \delta z' \cos\gamma) = 0 \ldots\ldots (6).$$

Since P acts *along* the rod, the first term is the product of P into the resolved virtual velocity of its point of application, and the second term is a similar product for P'; hence, the principle of virtual velocities holds in this case.

The converse of this theorem is true in this case, but we shall not give a separate demonstration of it; it is of course included in the general demonstration of Art. 252.

If (5) were *absolutely true*, then in the case of a rod, as in that of a single particle, the sum of the products of each force into the projection of the displacement of its point of application on the direction of the force would be zero, whether the *displacement were finite or infinitesimal.* But (5) instead of being absolutely true is obtained from (3) by neglecting *squares and products of the resolved displacements* δx, $\delta x'$, δy,...

252. We proceed to establish the truth of the principle in the case of a rigid body. We shall assume that any possible displacement of a rigid body may be produced, by first making the body rotate about some axis, and then moving all the particles of the body through equal spaces in parallel directions. (For this assumption we may refer to De Morgan's *Differential and Integral Calculus*, page 489; or to the *Philosophical Magazine* for March 1851, page 187.) Suppose, for simplicity, that the axis of z is made to coincide with the axis about which the body is turned; let θ be the angle through which the body is turned, then the co-ordinates of a particle which were originally x and y will become, *if we neglect the square and higher powers of* θ,

$$x - y\theta \text{ and } y + x\theta \text{ respectively;}$$

the co-ordinate z of the particle remains unchanged.

Let the body be now further displaced, so that each particle moves through a space of which a, b, c are the projections on the co-ordinate axes; then, if δx, δy, δz denote the *whole* changes made in the co-ordinates x, y, z of a particle, we have

$$\delta x = a - y\theta, \quad \delta y = b + x\theta, \quad \delta z = c.$$

Since the forces which act on the rigid body are supposed to keep it in equilibrium, we have by Art. 73,

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$

$$\Sigma (Zy - Yz) = 0, \quad \Sigma (Xz - Zx) = 0, \quad \Sigma (Yx - Xy) = 0.$$

Multiply the first of these equations by a, the second by b, the third by c, and the sixth by θ, and add; we then find

$$\Sigma \{X(a - y\theta) + Y(b + x\theta) + Zc\} = 0,$$

or $$\Sigma (X\delta x + Y\delta y + Z\delta z) = 0.$$

Let P_1 denote the force of which X_1, Y_1, Z_1 are the components, and P_2, P_3,...... have similar meanings; and let δp_1, δp_2,...... be the resolved virtual velocities corresponding to these forces; then, by Art. 250, the above equation may be written

$$\Sigma P\delta p = 0.$$

This proves the principle in the case of a rigid body.

Conversely, if the sum of the products of the forces and the resolved virtual velocities vanishes for *every* possible displacement of a rigid body, the forces keep the body in equilibrium.

For suppose, in the first place, the body is so displaced that every point of it moves parallel to the axis of x over a space a; then we have, by hypothesis,

$$\Sigma Xa = 0;$$

therefore $$\Sigma X = 0.$$

Similarly, by suitable displacements, we may prove that

$$\Sigma Y = 0, \text{ and } \Sigma Z = 0.$$

Next, suppose the body turned round the axis of z through a small angle θ; then, by hypothesis,

$$\Sigma (X\delta x + Y\delta y) = 0,$$

and $$\delta x = -y\theta, \quad \delta y = x\theta;$$

therefore $$\theta \Sigma (Xy - Yx) = 0;$$

therefore $$\Sigma (Yx - Xy) = 0.$$

Similarly, by suitable displacements, we may prove

$$\Sigma (Zy - Yz) = 0, \quad \Sigma (Xz - Zx) = 0.$$

Hence, the six equations of equilibrium hold.

253. If there be a system of two or more rigid bodies, then, since the principle of virtual velocities holds for any possible displacement of any one of the bodies, it holds for any possible displacement of the system.

254. In Art. 252 we have simplified the proof of the first part of the principle of virtual velocities, by supposing the axis of z to coincide with that about which the body was made to undergo an angular displacement. The following will be the process, if we suppose the displacement made about a line passing through the origin, and inclined to the axes at angles whose direction cosines are l, m, n.

Let r be the distance of any point (x, y, z) from the origin; ϕ the angle this distance makes with the given line; ρ the perpendicular from (x, y, z) on the given line; then

$$r^2 = x^2 + y^2 + z^2,$$

$$\cos\phi = \frac{lx}{r} + \frac{my}{r} + \frac{nz}{r};$$

therefore $$\rho^2 \text{ or } r^2 \sin^2\phi = x^2 + y^2 + z^2 - (lx + my + nz)^2.$$

Suppose the body turned through a small angle θ round the given line; let $x + \delta x$, $y + \delta y$, $z + \delta z$, be the co-ordinates of that point of the body which was originally at (x, y, z).

Since r and ρ are unchanged by the displacement, we have, by neglecting $(\delta x)^2$, $(\delta y)^2$, $(\delta z)^2$ in comparison with δx, δy, δz,

$$0 = x\delta x + y\delta y + z\delta z,$$

$$0 = l\delta x + m\delta y + n\delta z;$$

therefore $$\frac{\delta x}{yn - zm} = \frac{\delta y}{zl - xn} = \frac{\delta z}{xm - yl} = \lambda \text{ suppose......(1).}$$

And since $\{(\delta x)^2 + (\delta y)^2 + (\delta z)^2\}^{\frac{1}{2}} = 2\rho \sin \frac{1}{2}\theta$,

$$\lambda \{(yn - zm)^2 + (zl - xn)^2 + (xm - yl)^2\}^{\frac{1}{2}} = 2\rho \sin \tfrac{1}{2}\theta,$$

or $$\lambda \{x^2 + y^2 + z^2 - (lx + my + nz)^2\}^{\frac{1}{2}} = 2\rho \sin \tfrac{1}{2}\theta;$$

therefore $$\lambda = \theta \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

neglecting θ^3 and higher powers of θ.

Suppose the body to be further displaced, so that each particle moves over spaces a, b, c parallel to the co-ordinate axes; if δx, δy, δz denote now the *whole* displacement of the particle whose original co-ordinates were x, y, z, we have

$$\delta x = (yn - zm)\,\theta + a,$$
$$\delta y = (zl - xn)\,\theta + b,$$
$$\delta z = (xm - yl)\,\theta + c.$$

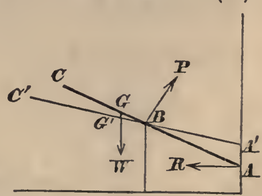
Multiply the six equations in Art. 73 by a, b, c, $-l\theta$, $-m\theta$, $-n\theta$, respectively, and add, then

$$\Sigma\,(X\delta x + Y\delta y + Z\delta z) = 0.$$

255. We shall illustrate the principle of virtual velocities in the solution of the following problem.

A beam in a vertical plane rests on a post B and against a wall at A; required the circumstances of equilibrium.

Let the distance of B from the wall $= b$; let G be the centre of gravity of the beam; $AG = a$; and the inclination of the beam to the wall $= \theta$. The reaction (P) of the post at B is



perpendicular to the surfaces in contact, and therefore to the beam; the reaction (R) of the wall is perpendicular to the wall for the same reason; let W be the weight of the beam. We may consider the beam in equilibrium under the action of P, R, W, and suppose the post and wall removed.

Now the object of the problem might be, solely to determine the position of equilibrium, or also to determine P and not R, or R and not P, or to determine both P and R and also the position of equilibrium. We shall solve the problem by the principle of virtual velocities under these four suppositions, in order to explain the method of proceeding so as to avoid as much trouble as possible according to the nature of the question.

(1) Suppose the position of equilibrium only required. We must then give the beam a small arbitrary geometric motion such that the unknown pressures P and R shall not occur in the equation of virtual velocities; the beam must therefore remain in contact with the wall and the post, as in the figure.

Let $\delta\theta$ be the increase of θ owing to the displacement. Then the height of G above the horizontal line through B, (or z), before displacement

$$= GB \cos\theta = (a - b \operatorname{cosec}\theta)\cos\theta = a\cos\theta - b\cot\theta;$$

the height after displacement is found by changing θ into $\theta + \delta\theta$ in this expression; therefore, the vertical space described by G or δz

$$= a\cos(\theta + \delta\theta) - b\cot(\theta + \delta\theta) - (a\cos\theta - b\cot\theta)$$

$$= \left(\frac{b}{\sin^2\theta} - a\sin\theta\right)\delta\theta;$$

and, by the principle of virtual velocities, $W\delta z = 0$; therefore

$$b - a\sin^3\theta = 0, \quad \sin\theta = \sqrt[3]{\frac{b}{a}},$$

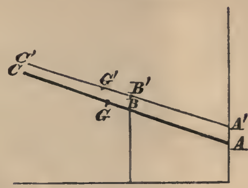
and this determines the *position of equilibrium*.

(2) But suppose we wish to find the pressure P as well as the position of equilibrium.

We must in this case move the beam off the post, in order that the virtual velocity of B with respect to P may not vanish, and consequently P not disappear as in the first case.

Let $AA' = c$, and let, as before, $\delta\theta$ be the change of θ.

We have to find the displacement of B estimated along



the line of action of P. Now conceive the beam brought into its second position by two steps; first let it be moved parallel to itself till the lower end comes to A', and next let it revolve round A' through a small angle $\delta\theta$. By the first step B moves through a space parallel and equal to AA'; by the second step B describes a small arc of a circle the length of which is $AB \,.\, \delta\theta$, that is b cosec $\theta\, \delta\theta$. Thus the displacement of B estimated along the line of action of P is ultimately $c \sin\theta - b \operatorname{cosec}\theta\,\delta\theta$.

Similarly by the first step G moves through a space equal and parallel to AA', and by the second step G describes a small arc of a circle the length of which is $a\delta\theta$. Thus the displacement of G resolved vertically downwards is ultimately $a\delta\theta \sin\theta - c$.

Therefore, by the principle of virtual velocities,

$$W(a \sin\theta\delta\theta - c) + P(c \sin\theta - b \operatorname{cosec}\theta\delta\theta) = 0;$$

therefore, $\delta\theta\,(Wa \sin\theta - Pb \operatorname{cosec}\theta) - c\,(W - P\sin\theta) = 0$;

and, since c and $\delta\theta$ may be any independent small quantities,

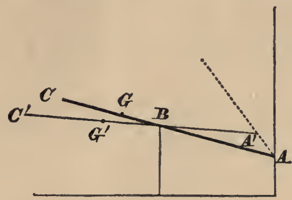
$$Wa \sin\theta - Pb \operatorname{cosec}\theta = 0, \quad W - P\sin\theta = 0;$$

therefore $\qquad \sin\theta = \sqrt[3]{\frac{b}{a}}, \quad \text{and } \frac{P}{W} = \sqrt[3]{\frac{a}{b}}.$

(3) Suppose we wish to know R and the position of equilibrium, and not P.

Then we should displace the beam so as to give to A a virtual velocity with respect to R, but not one to B with respect to P.

The beam must therefore still remain in contact with the peg. Let $AA' = c$, and let α be the angle which AA' makes with the vertical. Now conceive the beam brought into its second position by two steps; first let it be moved parallel



to itself till the lower end comes to A', and next let it revolve round A' through an angle $\delta\theta$ so as to bring the beam again into contact with the peg. The displacement of A estimated along the line of action of R is $c \sin \alpha$. The displacement of G estimated vertically downwards is $a\delta\theta \sin \theta - c \cos \alpha$.

Moreover there is a relation between $\delta\theta$, c, and α, arising from the fact that the whole displacement of the beam is such as to keep the beam still in contact with the peg. From the triangle ABA', we have

$$\frac{\sin \delta\theta}{\sin (\theta - \alpha)} = \frac{AA'}{A'B};$$

hence, $$\delta\theta = \frac{c \sin (\theta - \alpha) \sin \theta}{b} \text{ ultimately.}$$

Therefore by the principle of virtual velocities

$$W \left\{ \frac{ac}{b} \sin^2 \theta \sin (\theta - \alpha) - c \cos \alpha \right\} + Rc \sin \alpha = 0;$$

that is,

$$W \left(\frac{a \sin^3 \theta}{b} - 1 \right) c \cos \alpha + \left(R - \frac{Wa}{b} \sin^2 \theta \cos \theta \right) c \sin \alpha = 0;$$

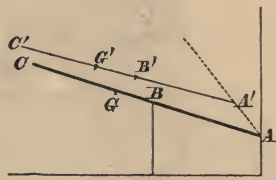
and $c \cos \alpha$ and $c \sin \alpha$ are independent; therefore

$$\frac{a \sin^3 \theta}{b} - 1 = 0, \quad R - \frac{Wa}{b} \sin^2 \theta \cos \theta = 0;$$

therefore $$\sin \theta = \sqrt[3]{\frac{b}{a}}, \quad \text{and } \frac{R}{W} = \frac{\sqrt{(a^{\frac{2}{3}} - b^{\frac{2}{3}})}}{b^{\frac{1}{3}}}.$$

(4) Lastly, suppose we wish to determine P and R and the position of equilibrium.

Then we must give the beam the most general displacement possible in the plane of the forces; let $AA' = c$, and



let α be the angle which AA' makes with the vertical. Now conceive the beam brought into its second position by two steps; first let it be moved parallel to itself till the lower end comes to A', and next let it revolve round A' through an angle $\delta\theta$. The displacement of A estimated along the line of action of R is $c \sin \alpha$. The displacement of G estimated vertically downwards is

$$a\delta\theta \sin \theta - c \cos \alpha.$$

The displacement of B along the line of action of P is

$$c \cos \left(\alpha + \frac{\pi}{2} - \theta\right) - b \operatorname{cosec} \theta\delta\theta,$$

that is, $$c \sin (\theta - \alpha) - b \operatorname{cosec} \theta\delta\theta.$$

Therefore by the principle of virtual velocities

$$W(a\delta\theta \sin \theta - c \cos \alpha) + Rc \sin \alpha$$
$$+ P\{c \sin (\theta - \alpha) - b \operatorname{cosec} \theta\delta\theta\} = 0;$$

that is,

$$(Wa \sin \theta - Pb \operatorname{cosec} \theta)\, \delta\theta + (P \sin \theta - W)\, c \cos \alpha$$
$$+ (R - P \cos \theta)\, c \sin \alpha = 0,$$

and $\delta\theta$, $c \cos \alpha$, and $c \sin \alpha$ are independent; therefore

$$Wa \sin \theta - Pb \operatorname{cosec} \theta = 0, \;\; P \sin \theta - W = 0, \;\; R - P \cos \theta = 0.$$

These three equations are the equations which we should have obtained by the principles of Art. 57; they give by elimination

$$\sin\theta = \sqrt[3]{\frac{b}{a}}, \quad \frac{P}{W} = \left(\frac{a}{b}\right)^{\frac{1}{3}}, \quad \frac{R}{W} = \frac{\sqrt{(a^{\frac{2}{3}} - b^{\frac{2}{3}})}}{b^{\frac{1}{3}}}.$$

We have thus illustrated the method of application of this principle; and we observe, in general, that when the object of the problem does not require certain unknown forces, we must give the body the most arbitrary geometrical motion possible without giving the points of application of these forces any motion in their directions.

256. In applying the principle of virtual velocities to deduce the conditions of equilibrium of any system, it is often convenient to give the body such a displacement as to make the *virtual moments of some of the forces separately vanish.* This has been exemplified in the preceding article, and we will now enumerate some cases in which the virtual moment of a force vanishes.

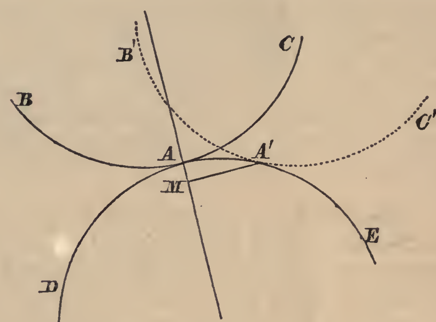
(1) In the hypothetical displacement, if any particles of the system have remained in their original places, the virtual moment of forces acting at such points is obviously zero. If a body, for example, have one point fixed, then the virtual velocity of this point is zero for any hypothetical displacement of the body, which does not break the condition of this point being fixed.

(2) Suppose a body compelled to remain with one point in contact with a smooth fixed plane, so that the plane exerts a force on the body at the point of contact in a direction perpendicular to the plane. Let the body be displaced so as to have the *same* point still in contact with the fixed plane, then the perpendicular drawn from the new position of the point of contact on the old direction of the action of the fixed plane meets that direction at the old position of the point of contact; that is, the virtual velocity of the point of contact relative to the force exerted by the plane is zero.

Similarly, if the body have more than one point in contact with the plane, and be so displaced that the *same points* of the body remain in contact with the fixed plane, the virtual moment of each force which the plane exerts on the body vanishes.

(3) Let two smooth bodies be in contact; then each exerts a force on the other along their common normal. Suppose *one* of them so displaced, that the point in it which was originally in contact with the other body still remains in contact with it; the case is similar to that of a body in contact with a fixed plane; the virtual velocity of the point of contact relative to the normal force is not zero, but is indefinitely small compared with the absolute virtual velocity.

Let BAC be a section of one body made by a plane which



contains the common normal to the surfaces, and DAE the section of the other made by the same plane; A the point of contact. Suppose the body BAC displaced into the position $B'A'C'$, so that the point A is moved to A'. Draw $A'M$ perpendicular to the common normal to the surfaces. Then AM represents the virtual velocity of the point of contact with respect to the normal force, while the line joining A and A' is the absolute virtual velocity. Since MAA' is ultimately a right angle, AM vanishes compared with AA'.

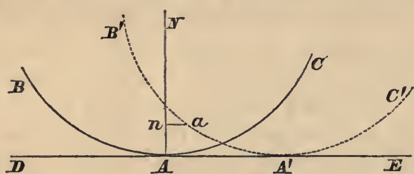
(4) Suppose two bodies in contact at a single point, and let them be *both* displaced so that they still remain with the same point of each body in contact. Let P denote the force in the normal on one body, and therefore $-P$ that on the other; then, if $P\delta p$ denote the virtual moment of the normal

force with respect to the first body, $-P\delta p$ will be the virtual moment with respect to the second body. Hence, by taking the sum of the virtual moments for the two bodies, the mutual action P disappears.

A similar result holds if the bodies be in contact at more points than one.

(5) Suppose a body in contact with a smooth fixed plane at a single point, and let the body be displaced by *rolling* it on the fixed plane.

Let BAC be a section of the body made by a plane through



the point of contact A containing the common normal to the surfaces, and suppose this section a circle. Let DAE be the intersection of this plane with the fixed smooth plane. Suppose $B'A'C'$ the position of the body after displacement, A' being the new point of contact, and let a be the point in the body which was originally in contact with the fixed smooth plane. Draw an perpendicular to the normal AN; then, An represents the resolved virtual velocity of the point of contact with respect to the normal force. Now An is equal to the product of the chord $A'a$ and the sine of the angle between this chord and $A'A$; and as this angle is ultimately indefinitely small, An is indefinitely small compared with the chord $A'a$, and therefore also compared with the arc $A'a$ or AA'. Hence if we neglect powers of AA' higher than the first, the virtual moment of the force along the normal acting at the point of contact is zero.

A similar result holds if BAC, DAE be *any* curves instead of a circle and straight line respectively.

(6) Let us suppose the bodies in contact to be *rough*, and a displacement to be made by *rolling* one upon the other as in the preceding case. The action of each body on the other will not be directed along the normal AN, but may be resolved

into two, one along AN and the other perpendicular to AN. The virtual moment of the former force vanishes, as we have shewn in the preceding case; and since the direction of the line joining A and a ultimately coincides with AN and is therefore perpendicular to the second force, the virtual moment of the second force vanishes in the same manner as in the third case.

(7) Suppose an inextensible string to have one end attached to a fixed point, and the other end to a particle either isolated or forming part of a rigid body; one of the forces of the system is then the tension of this string which acts along its length. Let the particle be so displaced as to keep the string stretched, then it may pass from its first to its second position by moving over an arc of a circle, and in the same manner as in the third case, we see that the virtual velocity of the particle with respect to the tension which the string exerts, is indefinitely small compared with the absolute virtual velocity of the particle. Hence, the tension of the string disappears from the equation of virtual velocities.

(8) Suppose an inextensible string connecting two particles of the system, and let the particles be displaced along the direction of the string, the string being kept stretched. Then, if one particle be displaced through a space δp, and P denote the tension of the string, and therefore the force exerted by the string on this particle, $P\delta p$ is the virtual moment of the force which the string exerts on this particle; also $-P\delta p$ will be the virtual moment of the force which the string exerts on the second particle. Hence, by taking the sum of the virtual moments for the two particles, the tension of the string disappears from the equation of virtual velocities.

(9) If we suppose a further displacement of the system in the preceding case, by keeping one particle fixed and making the other describe an arc of a circle, then, by the seventh case, the tension of the string disappears from the equation of virtual velocities.

By a combination of the displacements considered in the seventh and eighth cases, we can produce any displacement that the two particles can undergo, so long as the string is kept stretched. Hence, the tension of a string connecting two particles disappears from the equation of virtual velocities.

We have supposed the string to pass in a straight line from one particle to the other, but the same result would hold if the string were deflected by passing through one or more smooth fixed rings, supposing it always kept stretched. The demonstration would not hold for an *extensible* string.

257. We can now understand more distinctly the meaning of the words, *without disturbing the connexion of the parts of the system with each other*, which are introduced into the enunciation of the theorem. The theorem is shewn in Art 249 to be true for a particle; if then we consider a rigid body to be a collection of particles held together by molecular forces, the theorem will hold for *every* displacement of the particles of the rigid body, provided we include the molecular forces and estimate their several virtual moments. But from the demonstration in Art. 252 it appears that we need *not* consider the molecular forces, provided we give to the different particles such displacements only as are consistent with the unbroken rigidity of the body. So with respect to such forces as are enunciated in the preceding article, we may, if we take them into consideration, give to the system any displacements we please; but if we do *not* take them into consideration, we must give such displacements only as we can prove will not introduce the virtual moments of these forces. Hence, the words which we are explaining amount to a direction to be careful to include *every* force of the system, except such as we know have their virtual moments zero for the particular displacement we are considering.

258. The following example will shew how the principle of virtual velocities may assist in the solution of problems. Six equal rods are fastened together by hinges at each end, and one of the rods being supported in a horizontal position the opposite one is fastened to it by an elastic string joining their middle points; determine the tension of this string.

Let W denote the weight of each rod, T the tension of the string. Suppose the system displaced slightly so that the lowest rod descends vertically through a space x. Then it will be easily seen that the centre of gravity of each of the two rods which are adjacent to the highest rod descends through a space $\frac{x}{4}$; and the centre of gravity of each of the two rods which are adjacent to the lowest rod descends through a space $\frac{3x}{4}$; the point of application of the tension on the lowest rod descends through a space x. Therefore by the principle of virtual velocities

$$2W\frac{x}{4} + 2W\frac{3x}{4} + Wx - Tx = 0;$$

$$\text{therefore } T = 3W.$$

The mutual actions at the hinges disappear from the equation furnished by the principle of virtual velocities, and thus the required result is readily obtained.

259. The following is the process by which we may deduce the equations of equilibrium of any system from the principle of virtual velocities.

Let P_1, P_2, P_3,... denote the forces which act on a system; $P_1\delta p_1$, $P_2\delta p_2$,... their respective virtual moments for any displacement; then, by the principle,

$$P_1\delta p_1 + P_2\delta p_2 + P_3\delta p_3 + \ldots = 0 \ldots\ldots\ldots\ldots (1).$$

This equation we proceed to develope.

Let α_1, β_1, γ_1 be the angles which the direction of P_1 makes with the co-ordinate axes; x_1, y_1, z_1 the co-ordinates of the point of application of P_1; then

$$\delta p_1 = \cos\alpha_1 \delta x_1 + \cos\beta_1 \delta y_1 + \cos\gamma_1 \delta z_1 \ldots\ldots (2);$$

this is rigorously true, and similar equations hold for δp_2, δp_3, ...

Now, in consequence of the connexion of the system, for example, the rigidity of some parts of it, or the junction of parts by rods or strings, relations will hold between the co-ordinates x_1, y_1, z_1, x_2, y_2, z_2, ... in virtue of which all of them may be expressed in terms of a certain number of them; or all of them may be expressed in terms of certain other independent co-ordinates and angles.

Suppose $\xi_1, \xi_2, \xi_3, \ldots \phi_1, \phi_2, \phi_3, \ldots$ to denote these independent co-ordinates and angles. *Then, if we neglect the squares and products, and higher powers of* $\delta x_1, \delta y_1, \ldots \delta\xi_1, \delta\xi_2, \ldots \delta\phi_1, \delta\phi_2, \ldots$, we shall obtain equations of the form

$$\delta x_1 = A_1\delta\xi_1 + A_2\delta\xi_2 + \ldots + a_1\delta\phi_1 + a_2\delta\phi_2 + \ldots,$$

$$\delta x_2 = B_1\delta\xi_1 + B_2\delta\xi_2 + \ldots + b_1\delta\phi_1 + b_2\delta\phi_2 + \ldots,$$

....

where $A_1, A_2, \ldots B_1, B_2, \ldots a_1, a_2, \ldots b_1, b_2, \ldots$ are functions of the variables, but do not contain the increments $\delta\xi_1, \delta\xi_2, \ldots \delta\phi_1, \delta\phi_2, \ldots$

Let the values of $\delta x_1, \delta y_1 \ldots$ be substituted in the equations of which (2) is the type, and then let the values of $\delta p_1, \delta p_2, \ldots$ be substituted in (1); this equation will take the form

$$Q_1\delta\xi_1 + Q_2\delta\xi_2 + \ldots + q_1\delta\phi_1 + q_2\delta\phi_2 + \ldots = 0 \quad \ldots\ldots (3).$$

The conditions for the equilibrium of the system are

$$Q_1 = 0, \quad Q_2 = 0, \ldots \quad q_1 = 0, \quad q_2 = 0, \quad \ldots\ldots\ldots (4).$$

For since $\delta\xi_1, \delta\xi_2, \ldots \delta\phi_1, \delta\phi_2, \ldots$ are by supposition independent, we might have given the body such a displacement as to leave $\xi_2, \xi_3, \ldots \phi_1, \phi_2, \ldots$ unchanged; and then (3) would reduce to

$$Q_1\delta\xi_1 = 0; \text{ therefore } Q_1 = 0.$$

Similarly, we may shew that the other equations of (4) hold.

260. We will give a simple example in illustration of the method of the preceding article. A string of given length has one end fixed at a point in the line of intersection of two vertical planes at right angles to each other, and at the other end carries a heavy particle which is repelled from these planes by forces of which one is constant and the other varies as the distance from the plane; find the positions of equilibrium.

Take the vertical plane from which the particle is repelled by a constant force as the plane of (x, z), and the other vertical plane as the plane of (y, z); take the point to which the end of the string is fixed as the origin, and let the axis of z be vertically downwards. Let x, y, z denote the co-ordinates of the particle in a position of equilibrium, and l the length of the string. Let W be the weight of the particle, F the constant repulsive force, μx the force which varies as the distance of the particle from the plane of (y, z). Conceive the particle displaced into an adjacent position, the co-ordinates of which are $x + \delta x$, $y + \delta y$, $z + \delta z$. Then by the principle of virtual velocities

$$\mu x \delta x + F \delta y + W \delta z = 0 \dots\dots\dots\dots\dots\dots (1);$$

the tension of the string has no virtual moment by Art. 256.

Also $$x^2 + y^2 + z^2 = l^2 \dots\dots\dots\dots\dots\dots (2);$$

therefore $$x\delta x + y \delta y + z \delta z = 0 \dots\dots\dots\dots\dots\dots (3).$$

By (3) we can express δz in terms of δx and δy; thus (1) becomes

$$\left(\mu x - \frac{Wx}{z}\right) \delta x + \left(F - \frac{Wy}{z}\right) \delta y = 0.$$

Therefore $$\mu x - \frac{Wx}{z} = 0, \text{ and } F - \frac{Wy}{z} = 0.$$

From the first of these equations we obtain either $z = \frac{W}{\mu}$, or else $x = 0$. If we take the former solution we obtain $y = \frac{F}{\mu}$, and then x is known from (2); thus one position of equilibrium is determined. If we take the solution $x = 0$, then y and z must be found from the equations

$$Fz - Wy = 0, \quad y^2 + z^2 = l^2;$$

thus another position of equilibrium is determined.

261. The principle of virtual velocities is useful in Statics in the solution of such problems as that in Art. 255, where forces occur which have their virtual moments zero for certain displacements. The following is an important general proposition to which the principle leads.

A system of rigid bodies under the action of no forces but their weights, mutual pressures, and pressures upon smooth immoveable surfaces, will be in equilibrium, if placed so that the centre of gravity is in the lowest or highest position it can *possibly attain by moving the system consistently with the connexion of its parts with one another.*

Let z_1, z_2,... denote the distances below a fixed horizontal plane of the different particles of the system; w_1, w_2,... the weights of these particles. That the system may be in equilibrium, we must have

$$w_1\delta z_1 + w_2\delta z_2 + w_3\delta z_3 + \ldots = 0 \ldots\ldots\ldots\ldots (1);$$

for by Art. 256 the virtual moments of all the other forces which act on the system vanish. Let $\bar{z}$ denote the depth of the centre of gravity of the system below the fixed horizontal plane; then

$$\bar{z} = \frac{w_1 z_1 + w_2 z_2 + w_3 z_3 + \ldots}{w_1 + w_2 + w_3 + \ldots};$$

therefore $(w_1 + w_2 + w_3 + \ldots)\ \delta\bar{z} = w_1\delta z_1 + w_2\delta z_2 + w_3\delta z_3 + \ldots (2)$.
Now when $\bar{z}$ has a maximum or minimum value,

$$\delta\bar{z} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

(see *Diff. Calc.* Arts. 232, 238).

Hence, when the centre of gravity is at a maximum or minimum distance from the fixed horizontal plane, (1) is satisfied and the system is in equilibrium.

The equation (3) is a *necessary* but not a *sufficient* condition for $\bar{z}$ having a maximum or minimum value; hence, we cannot assert conversely, that when the system is in equilibrium, the centre of gravity must be at a maximum or minimum depth.

262. If the system of rigid bodies be such that the centre of gravity is always in the same horizontal plane, every position is a position of equilibrium. For in this case $\bar{z}$ is a constant, and therefore $\delta\bar{z}$ always $= 0$.

263. Suppose a system in equilibrium, and that an indefinitely small displacement is given to it; if it then tend to return to its original position, that position is said to be one of *stable equilibrium;* if the system tend to move further from its original position, that position is said to be one of *unstable equilibrium.*

To determine in any case whether the equilibrium of a system is *stable* or *unstable*, is a question of dynamics on which we do not enter. The reader may refer to Poisson, Art. 570, or Duhamel, Tom. II. Art. 69; the best investigation of the question, however, will be found in the *Cours Complémentaire d'Analyse et de Mécanique Rationelle, par* J. Vieille, *Paris*, 1851.

The following general theorem is demonstrated. Suppose the forces which act upon a system such that

$$\Sigma(Xdx + Ydy + Zdz)$$

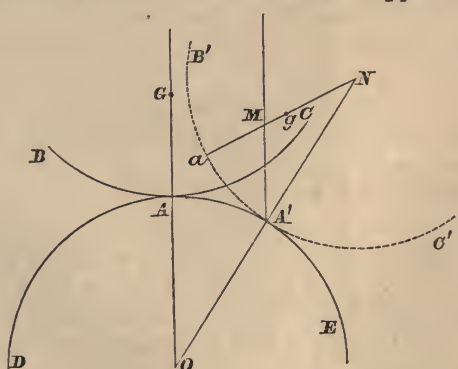
is the immediate differential of some function of the co-ordinates, ϕ; then, for every position of equilibrium, ϕ is, in general, a *maximum* or *minimum;* in the former case the equilibrium is *stable* and in the latter *unstable.*

An important particular case is that of the system in Art. 261, in which the equilibrium is *stable* when the centre of gravity has its *lowest* position, and *unstable* when it has its highest position.

The following is a simple example of distinguishing the nature of equilibrium.

264. *A heavy body rests on a fixed body, to determine the nature of the equilibrium;* the surfaces being supposed rough.

Let BAC be a vertical section of the upper body made



by a plane through its centre of gravity G, and DAE the section of the lower body made by the same plane. We suppose these sections both circular; let r be the radius of the upper section and R that of the lower. Let the upper body be displaced into the position $B'A'C'$, and suppose a that point in the upper body which was originally at A; at A' the new point of contact draw the common normal $OA'N$, meeting in O the radius AO of the lower surface, and in N the radius aN of the upper surface. Draw a vertical line through A' meeting aN in M; let g be the new position of the centre of gravity of the upper body. If we suppose the surfaces rough enough to prevent all sliding, the upper body will turn round A', and the equilibrium will be *unstable* if g falls further from a than M, and *stable* if g be between M and a.

Let $$AOA' = \theta, \quad aNA' = \phi.$$

Since we suppose the upper body displaced by *rolling* on the lower, we have

$$\text{arc } AA' = \text{arc } aA';$$

therefore $$R\theta = r\phi.$$

Also $$\frac{MN}{NA'} = \frac{\sin\theta}{\sin(\theta+\phi)} = \frac{\sin\theta}{\sin\left(1+\frac{R}{r}\right)\theta}$$

$$= \frac{1}{1+\frac{R}{r}} \text{ ultimately};$$

therefore $$MN = \frac{r^2}{r+R},$$

and $$aM = r - \frac{r^2}{r+R} = \frac{Rr}{R+r}.$$

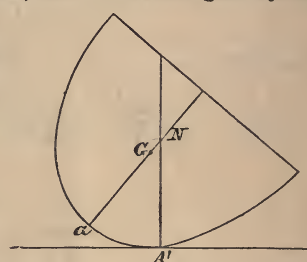
Hence, the equilibrium is stable or unstable according as ag, or AG, is less or greater than $\frac{Rr}{R+r}$.

If the lower surface be *concave* instead of *convex*, it may be shewn in the same way that the equilibrium is stable or unstable according as AG is less or greater than $\frac{Rr}{R-r}$.

The results of this article will hold when the sections BAC and DAE are not circles; r and R will then stand for the radii of curvature of the upper and lower sections at the point A. If the lower surface is plane, R is infinite, and for stable equilibrium AG must be less than r.

265. If $AG = \frac{Rr}{R+r}$ in the first case, or $= \frac{Rr}{R-r}$ in the second case, the equilibrium has been called *neutral*. In this case, a further investigation will have to be made to determine whether the equilibrium is stable or unstable. Suppose, for example, that a portion of a paraboloid rests in neutral equilibrium with its vertex in contact with a horizontal plane, it is required to determine whether the equilibrium is stable or unstable.

Since the equilibrium is neutral, the centre of gravity G must coincide with the centre of curvature of the generating parabola at the vertex; now, if different points be taken in a parabola, the further the assumed point is from the vertex, the further is the point of intersection of the normal and the axis from the vertex. Hence, the normal $A'N$ in the figure meets the axis of the parabola further from a than G is, and the equilibrium is *stable*.



It is easy to shew generally, that if a portion of a solid of revolution rest in neutral equilibrium with its vertex on a horizontal plane, the equilibrium is really *stable* or *unstable*, according as the radius of curvature of the generating curve has a *minimum* or *maximum* value at the vertex.

266. The results of Art. 264, when the sections BAC and DAE are circles, may also be obtained by using the theorem which we have quoted in Art. 263.

Let z denote the height of the centre of gravity g above the horizontal line through O, and let $Ng = c$; then

$$z = (R + r) \cos \theta - c \cos (\theta + \phi)$$

$$= (R + r) \cos \theta - c \cos \left(1 + \frac{R}{r}\right) \theta.$$

Expand the cosines in powers of the angles; thus

$$z = R + r - c + \left\{c \left(1 + \frac{R}{r}\right)^2 - (R + r)\right\} \frac{\theta^2}{2}$$

$$- \left\{c \left(1 + \frac{R}{r}\right)^4 - (R + r)\right\} \frac{\theta^4}{\underline{|4}} + \ldots$$

Suppose the coefficient of θ^2 not to be zero; then when θ is indefinitely small z is greater or less than $R + r - c$, according as the coefficient of θ^2 is positive or negative; in the former case $R + r - c$ is a *minimum* value of z, and in the latter case it is a *maximum* value. Therefore the equilibrium is stable if c be greater than $\frac{r^2}{R + r}$, and unstable if c be less than $\frac{r^2}{R + r}$.

Suppose however that $c = \frac{r^2}{R + r}$, then the coefficient of θ^2 is zero; in this case the equilibrium is said to be *neutral*. We must now examine the coefficient of θ^4 in the value of z; this coefficient is

$$- \frac{1}{\underline{|4}} \left\{c \left(1 + \frac{R}{r}\right)^4 - (R + r)\right\},$$

that is,

$$- \frac{1}{\underline{|4}} \left\{\frac{(R + r)^3}{r^2} - (R + r)\right\},$$

that is,

$$- \frac{R (R + r) (R + 2r)}{r^2 \underline{|4}};$$

since this is a *negative* quantity it follows that $R + r - c$ is a *maximum* value of z and the equilibrium is really *unstable*.

267. The following problem will furnish an instructive example. A frame formed of four uniform rods of the length a connected by smooth hinges, is hung over two smooth pegs

in the same horizontal line at a distance $\frac{a}{\sqrt{2}}$, the two pegs being in contact with different rods; shew that the frame is in equilibrium when each angle is 90^0, and determine whether the equilibrium is stable or unstable.

Denote the pegs by A and B; suppose the beam in contact with A to make an angle θ with the horizon, and the beam in contact with B to make an angle ϕ with the horizon; let u denote the depth of the centre of gravity of the system below AB. Then it may be shewn that

$$u = \frac{a}{2}(\sin\theta + \sin\phi) - \frac{c\sin\theta\sin\phi}{\sin(\theta+\phi)},$$

where $$c = \frac{a}{\sqrt{2}}.$$

Thus u is a function of the two independent variables θ and ϕ, and in order that u may have a maximum or minimum value θ and ϕ must be taken so as to satisfy $\frac{du}{d\theta} = 0$ and $\frac{du}{d\phi} = 0$. It will be found on trial that $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{4}$ are suitable values. But it will be found that with these values for θ and ϕ we get

$$\frac{d^2u}{d\theta^2} = -\frac{c}{2}, \quad \frac{d^2u}{d\theta\,d\phi} = -c, \quad \frac{d^2u}{d\phi^2} = -\frac{c}{2};$$

so that $\left(\frac{d^2u}{d\theta\,d\phi}\right)^2 - \frac{d^2u}{d\theta^2}\frac{d^2u}{d\phi^2}$ is *positive* and u is *neither a maximum nor a minimum* when $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{4}$. All the foregoing is a simple example of the Differential Calculus; we proceed to apply it to the Mechanical Problem in question.

Let δu denote the change in u consequent upon changing the value of θ from $\frac{\pi}{4}$ to $\frac{\pi}{4} + \delta\theta$, and the value of ϕ from

$\frac{\pi}{4}$ to $\frac{\pi}{4}+\delta\phi$; then it follows from the preceding investigations that

$$\delta u = -\frac{c}{4}\{(\delta\theta)^2 + 4\delta\theta\delta\phi + (\delta\phi)^2\} + \&c.,$$

where under the &c. are included terms in $\delta\theta$ and $\delta\phi$ of a higher order than the second. Now although u is neither a maximum nor a minimum when θ and ϕ are each $\frac{\pi}{4}$, yet there is equilibrium then because δu is then zero so far as terms of the first order in $\delta\theta$ and $\delta\phi$. (See Art. 261.) But as u is *neither* a maximum *nor* a minimum the equilibrium cannot be stated to be either stable or unstable *universally;* it is in fact *stable* with respect to *some* displacements and *unstable* with respect to *other* displacements. If, for example, we consider only such displacements as make $\delta\theta = \delta\phi$, then δu is certainly *negative* when $\delta\theta$ and $\delta\phi$ are taken small enough; thus the centre of gravity is *raised* by the displacement and so the equilibrium is *stable*. If, again, we consider only such displacements as make $\delta\theta = -\delta\phi$, then δu is certainly *positive* when $\delta\theta$ and $\delta\phi$ are taken small enough; thus the centre of gravity is *depressed* by the displacement and so the equilibrium is *unstable*.

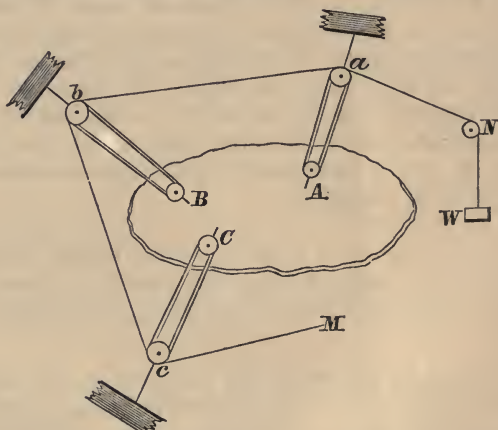
268. *Of all curves of a given length drawn between two fixed points in a horizontal line, the common catenary is that which has its centre of gravity furthest from the line joining the points.*

This proposition belongs to the *Calculus of Variations*, but an imperfect proof of it may be obtained from some of the preceding principles. Since the string which hangs in a common catenary is in equilibrium we conclude that the depth of its centre of gravity from the horizontal line is a maximum or minimum. (See however Art. 261.) And we may infer that the depth is a *maximum* and not a *minimum* from the experimental fact that if the string be slightly displaced it will return to its position of equilibrium so that its equilibrium is *stable*. (See Art. 263.) Hence in any other position of the string than that of equilibrium the centre of gravity will be nearer to the given horizontal line. And as the string which hangs in the common catenary is of uniform density and thickness its centre of gravity coincides with that of the curve. Thus the proposition is established.

269. Lagrange has given a demonstration of the principle of virtual velocities, which does not assume a knowledge of the conditions of equilibrium of any system of forces; this demonstration is difficult and has not been universally received. We shall place it here and refer the reader to Poisson, Art. 337, and to the article 'Virtual Velocities' in the *Penny Cyclopædia*, for further information.

We have first to shew how any system of forces may be replaced by a string in a state of tension passing round a combination of pullies.

Let forces P, Q, R,...... acting at the points A, B, C,......



maintain a system in equilibrium; let pullies be fixed to the system at the points A, B, C,... and let the pullies a, b, c,... be attached to fixed blocks, so that Aa may be the direction of the force P, Bb that of Q, and so on. Let a string have a weight W attached to one end, and be passed round the pully N and then round the pullies a and A a sufficient number of times to render the sum of the tensions equal to P. Let the same string then pass on to the pully b, and be passed round b

and B a sufficient number of times, until the sum of the tensions is equal to Q. The string is then passed on to c, and round cC, and so on; the end of the string is fastened to a fixed point M. Thus the system of forces P, Q, R,... may be replaced by a single string, the tension of which is W. We here assume that the forces P, Q, R,...are commensurable.

We proceed now to the proof, in which we follow Lagrange's words very closely.

It is evident, in order that the system may remain in equilibrium, that the weight W must be incapable of descending when any indefinitely small displacement whatever is given to the points of the system; for since the weight always tends to descend, if there were any displacement of the system which would allow it to descend, it would necessarily descend and produce this displacement in the system.

Let α, β, γ,... denote the indefinitely small spaces, which any displacement would cause the points of the system to describe in the direction of the forces, which respectively act at them, and let p, q, r,... denote the number of parallel strings which are attached to the pullies A, B, C,... It is obvious that the spaces α, β, γ,...are those by which the pullies A, B, C,...will approach a, b, c,...and that the string joining these pullies will thus be diminished by $p\alpha$, $q\beta$, $r\gamma$,... Thus, in consequence of the inextensibility of the string, the weight W would descend through the space $p\alpha + q\beta + r\gamma + ...$ Hence, in order that the system of forces P, Q, R,... may be in equilibrium, we must have

$$p\alpha + q\beta + r\gamma + ... = 0;$$

and therefore, since $P = pW$, $Q = qW$,...

$$P\alpha + Q\beta + R\gamma + ... = 0.$$

This equation is the analytical expression of the principle of virtual velocities.

If the quantity $P\alpha + Q\beta + R\gamma + ...$, instead of being zero, were negative, it might appear that this condition would be sufficient to ensure equilibrium, since it is impossible that the weight could of itself *ascend*. But we must remember, that whatever may be the connexion of the parts of the system, the relations which consequently hold between the indefinitely small quantities α, β, γ, ... can only be expressed by differential equations, and which are therefore *linear* as to these quantities; so that there will be necessarily one or more of them which remain indeterminate and may be taken with a positive or negative sign; thus the values of these quantities will be always such that they can simultaneously change their sign. Hence, it follows that if for a certain displacement of the system, the quantity $P\alpha + Q\beta + R\gamma + \dots$ is negative, it would become positive by changing the signs of α, β, γ, ...; thus the opposite displacement is equally possible, and this would make the weight descend and destroy the equilibrium.

Conversely, if the equation

$$P\alpha + Q\beta + R\gamma + \dots = 0$$

holds for every possible indefinitely small displacement of the system, it will remain in equilibrium. For, the weight remaining unmoved during these displacements, the forces which act upon the system remain in the same condition, and there is no reason why they should produce one, rather than the other, of the two displacements, for which α, β, γ, ... have different signs. This is the case of a balance which remains in equilibrium, because there is no reason why it should incline to one side rather than the other.

The principle of virtual velocities being thus proved for commensurable forces, will also hold when the forces are incommensurable; for we know that any proposition which can be proved for commensurable quantities may be extended by a *reductio ad absurdum* to incommensurable quantities.

EXAMPLES.

1. A cone whose semi-vertical angle is $\tan^{-1}\frac{1}{\sqrt{2}}$ is enclosed in the circumscribing spherical surface; shew that it will rest in any position.

2. A heavy uniform rod of length a moves in a vertical plane about a hinge at one extremity. A string fastened to the other, passes over a pully in a vertical line above the hinge, and is attached to a weight equal to half that of the rod, which rests on a curve. The length of the string and the height of the pully above the hinge are each equal to the length of the rod, and the system is in equilibrium in all positions. Shew that the equation to the curve is

$$r = 4a\sin^2\tfrac{1}{2}\theta,$$

the pully being the origin and the prime radius being vertical.

3. Two rods each of length $2a$ have their ends united at an angle α, and are placed in a vertical plane on a sphere of radius r. Prove that the equilibrium is stable or unstable according as

$$\sin\alpha \text{ is} > \text{or} < \frac{2r}{a}.$$

4. A prolate spheroid rests with its smaller end on a horizontal table. Is the equilibrium stable or unstable?

5. A cylinder rests with the centre of its base in contact with the highest point of a fixed sphere, and four times the altitude of the cylinder is equal to a great circle of the sphere; supposing the surfaces in contact to be rough enough to prevent sliding, shew that the cylinder may be made to rock through an angle of 90^0, but not more, without falling off the sphere.

6. A very small bar of matter is moveable about one extremity which is fixed halfway between two centres of force attracting inversely as the square of the distance; if l be the length of the bar, and $2a$ the distance between the

centres of force, prove that there will be two positions of equilibrium for the bar, or four, according as the ratio of the absolute intensity of the more powerful force to that of the less powerful, is, or is not, greater than $\frac{a+2l}{a-2l}$; and distinguish between the stable and unstable positions.

7. Two particles connected by a string support each other on the arc of a vertical circle; shew that the centre of gravity is in the vertical through the centre of the circle. What is the nature of the equilibrium?

8. A sphere of radius a, loaded so that the centre of gravity may be at a given distance b from the centre of figure, is placed on a rough plane inclined at an angle α to the horizon. Shew that there will be two positions of equilibrium, one stable and the other unstable, in which the distances of the point of contact from the centre of gravity are respectively,

$$a \cos \alpha - \sqrt{(b^2 - a^2 \sin^2 \alpha)},$$

and $$a \cos \alpha + \sqrt{(b^2 - a^2 \sin^2 \alpha)}.$$

Hence, find the greatest inclination of the plane which will allow the sphere to rest. Is the equilibrium stable or unstable in this limiting case?

9. A sphere of radius r rests on a concave sphere of radius R; if the sphere be loaded so that the height of its centre of gravity from the point of contact be $\frac{3}{2}r$, find R so that the equilibrium may be neutral. *Result.* $R = 3r$.

10. A heavy cone rests with the centre of its base on the vertex of a fixed paraboloid of revolution; shew that the equilibrium will be neutral if the height of the cone be equal to twice the latus rectum of the generating parabola. Shew that the equilibrium is really stable.

11. A heavy particle attached to one extremity of an elastic string is placed upon a smooth curve, the string lying upon the

curve and its other extremity being fixed to a point in the curve; find the curve when the particle rests in all positions.

Result. A cycloid.

12. A uniform square board is capable of motion in a vertical plane about a hinge at one of its angular points; a string attached to one of the nearest angular points, and passing over a pully vertically above the hinge at a distance from it equal to the side of the square, supports a weight whose ratio to the weight of the board is 1 to $\sqrt{2}$. Find the positions of equilibrium and determine whether they are respectively stable or unstable.

13. Two small smooth rings of equal weight slide on a fixed elliptical wire of which the major axis is vertical, and are connected by a string passing over a smooth peg at the upper focus; prove that the rings will rest in whatever position they may be placed.

14. A small heavy ring slides on a smooth wire in the form of a curve whose plane is vertical, and is connected by a string passing over a fixed pully in the plane of the curve with another weight which hangs freely; find the form of the curve that the ring may be in equilibrium in any position.

Result. A conic section having its focus at the pully.

15. If an elliptic board be placed, so that its plane is vertical, on two pegs which are in the same horizontal plane, there will be equilibrium if these pegs be at the extremities of a pair of conjugate diameters. What are the limits which the distance between the pegs must not exceed or fall short of, in order that this position of equilibrium may be possible? Shew that the equilibrium is *unstable*.

16. A solid of revolution, whose centre of gravity coincides with the centre of curvature at the vertex, rests on a rough horizontal plane. Shew that the equilibrium is stable or unstable according as the value of $3\left(\frac{d^2y}{dx^2}\right)^3 - \frac{d^4y}{dx^4}$, when x and y vanish, is positive or negative, x and y being co-ordinates of

the generating curve, measured along the tangent and normal at the vertex.

17. If a plane pass through one extremity A of the base of a cylinder and be inclined at an angle of 45^0 to the axis, the piece so cut off will rest in neutral equilibrium, if placed with its circular end on the vertex of a paraboloid whose latus rectum is five-eighths of the diameter of the base, the point of contact being also at this same distance from A.

18. A piece of string is fastened at its extremities to two fixed points; determine from mechanical considerations the form which must be assumed by the string in order that the surface generated by its revolution about the line joining the fixed points may be the greatest possible.

MISCELLANEOUS EXAMPLES.

1. A uniform wire is bent into the form of three sides AB, BC, CD of an equilateral polygon; and its centre of gravity is at the intersection of AC and BD. Shew that the polygon must be a regular hexagon.

2. Three forces act along three lines which may be considered as generating lines in the same system of a hyperboloid of one sheet; shew that if the forces admit of a single resultant, it must act along another generating line of the same system.

3. A gate moves freely about a vertical axis, along which it also slides; while a point in the plane of the gate, and rigidly connected with it, rests on a given rough inclined plane; find the limiting position of equilibrium.

4. Suppose lines to be drawn from one of the centres of the four circles that touch the sides or the sides produced of a given triangle to the other three centres, and let these lines represent three forces in magnitude and direction; then the line joining the first centre with the centre of the circle circumscribing the triangle will represent in magnitude and direction one-fourth of the resultant.

5. A particle rests in equilibrium in a fine groove in the form of a helix, the axis of which is inclined to the horizon at a given angle α. Find the distance of the particle from a vertical plane passing through the axis. Also find the greatest value of α for a given helix in order that there may be a position of equilibrium of the particle.

6. A quadrilateral figure possesses the following property; any point being taken and four triangles formed by joining this point with the angular points of the figure, the centres of gravity of these triangles lie in the circumference of a circle; prove that the diagonals of the quadrilateral are at right angles to each other.

7. A square board is supported in a horizontal position by three vertical strings; if one of them be attached to a corner, where must the others be attached in order that the weight which can be placed on any part of the board without overturning it may be the greatest possible?

8. A triangular plate hangs by three parallel threads attached at the corners, and supports a heavy particle. Prove that if the threads are of equal strength, a heavier particle may be supported at the centre of gravity than at any other point of the disk.

9. If through the centre of gravity of each of the faces of any polyhedron there act a force in the direction perpendicular to the face and in magnitude proportional to its area, the system will be in equilibrium, supposing all the forces to act inwards or all to act outwards.

10. A right cone is cut obliquely and then placed with its section on a horizontal plane; prove that when the angle of the cone is less than $\sin^{-1}\frac{1}{4}$, there will be two sections for which the equilibrium is neutral, and for intermediate sections the cone will fall over.

11. A right cylinder on an elliptic base (the semiaxes of which are a and b) rests with its axis horizontal between two smooth inclined planes inclined at right angles to each other;

determine the positions of equilibrium, (1) when the inclination of one of the planes is greater than $\tan^{-1}\frac{a}{b}$, (2) when the inclination of both planes is less than $\tan^{-1}\frac{a}{b}$.

12. A pack of cards is laid on a table; each projects in the direction of the length of the pack beyond the one below it; if each projects as far as possible, prove that the distances between the extremities of the successive cards will form an harmonic progression.

13. Find the least excentricity of an ellipse in order that it may be capable of resting in equilibrium upon a perfectly rough inclined plane.

Result. $e^2 = \dfrac{2\sin\alpha}{1+\sin\alpha}$.

14. Two mutually repelling particles are placed in a parabolic groove, and connected by a thread which passes through a small ring at the focus; shew that if the particles be at rest, either their abscissæ are equal, or the line AS is a mean proportional between them.

15. Each element of a parabolic arc bounded by the vertex and the latus rectum is acted on by a force in the normal proportional to the distance of the element from the axis of the parabola. Shew that the equation to the line in which the resultant acts, is

$$15y + 10x = 26a.$$

16. Each element of the arc of an elliptic quadrant is acted on by a force in the normal proportional to the ordinate of that point. Shew that the equation to the line in which the resultant acts is

$$6by - 3\pi ax + 4a^2 - 4b^2 = 0.$$

17. A smooth body in the form of a sphere is divided into hemispheres and placed with the plane of division vertical upon a smooth horizontal plane; a string loaded at its extremities with two equal weights hangs upon the sphere, passing over its highest point and cutting the plane of division

at right angles; find the least weight which will preserve the equilibrium.

18. The locus of the centre of gravity of segments of equal area A in an ellipse is a similar concentric ellipse whose minor axis is

$$\frac{4}{3}\frac{ab^2}{A}\sin^3\frac{\phi}{2}, \text{ where } A = \frac{ab}{2}(\phi - \sin\phi).$$

19. The foci of a rough prolate spheroid attract directly as the distance; if a particle without weight be placed on the spheroid, find within what limits it must be placed so as to be in equilibrium. Shew that if the coefficient of friction be greater than $\frac{e}{2\sqrt{(1-e^2)}}$, where e is the excentricity, the particle will rest anywhere on the surface.

20. A circular disc of mass m' and radius c rests in contact with two equal uniform straight rods AB, AC, which are joined at A by a smooth hinge, and which attract each other and the disc with a force varying as the distance; also the disc attracts the rods similarly. Shew that there is equilibrium if

$$m'c(2c\cos\alpha - a\sin\alpha) = ma^2\sin^4\alpha\cos\alpha,$$

where m is the mass of each rod, a the length of each rod, and 2α their inclination to each other.

21. A square picture hangs in a vertical plane by a string, which passing over a smooth nail has its ends fastened to two points symmetrically situated in one side of the frame. Determine the positions of equilibrium, and whether they are stable or unstable.

Results. Let l be the length of the string, c the distance of the two points to which the ends of the string are fastened, h the length of a side of the square; then if lh be greater than $c\sqrt{(c^2+h^2)}$ there is only one position of equilibrium, namely, the ordinary position, and the equilibrium is *stable;* if lh be less than $c\sqrt{(c^2+h^2)}$ there are two oblique positions of *stable* equilibrium, besides the ordinary position of equilibrium, which is *stable* with respect to some displacements and *unstable* with respect to other displacements.

22. A flexible thread is placed in a tube of any form and is acted on by any forces. The diameter of the tube is equal to that of the thread and is infinitesimal. Determine the position of the equilibrium.

23. Two equal particles are connected by two given strings without weight, which are placed like a necklace on a smooth cone with its axis vertical and vertex upwards; find the tensions of the strings.

24. A triangle of area A revolves through an angle ϕ about an axis in its own plane taken parallel to one side; shew that the least amount of surface generated is

$$A \cdot \phi \cdot \frac{(a+b+c)^2 - 2a^2}{2(b+c)a},$$

where a is the greatest side.

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Preparing for Publication.

A

CRITICAL HISTORY

OF THE

PROGRESS OF THE

CALCULUS OF VARIATIONS

DURING THE NINETEENTH CENTURY.

BY

I. TODHUNTER, M.A.,

FELLOW AND ASSISTANT TUTOR OF ST JOHN'S COLLEGE, CAMBRIDGE.

PREFACE.

The present work is constructed on the same plan as my treatise on Plane Trigonometry, to which it is intended as a sequel; it contains all the propositions usually included under the head of Spherical Trigonometry, together with a large collection of examples for exercise. In the course of the work reference is made to preceding writers from whom assistance has been obtained; besides these writers I have consulted the treatises on Trigonometry by Lardner, Lefebure de Fourcy, and Snowball, and the treatise on Geometry published in the Library of Useful Knowledge. The examples have been chiefly selected from the University and College Examination Papers.

In the account of Napier's Rules of Circular Parts an explanation has been given of a method of proof devised by Napier, which seems to have been overlooked by most modern writers on the subject. I have had the advantage of access to an unprinted Memoir on this point by the late R. L. Ellis of Trinity College; Mr Ellis had in fact rediscovered for himself Napier's own method. For the use of this Memoir and for some valuable references on the subject I am indebted to the Dean of Ely.

Considerable labour has been bestowed on the text in order to render it comprehensive and accurate, and the examples have all been carefully verified; and thus I venture to hope that the work will be found useful by Students and Teachers.

I. TODHUNTER.

St John's College,
August 15, 1859.

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