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CALCULUS OF OPERATIONS.

CARMICHAEL.

CALCULUS OF OPERATIONS.

A TREATISE

ON

THE CALCULUS OF OPERATIONS:

DESIGNED TO FACILITATE THE PROCESSES

OF THE

DIFFERENTIAL AND INTEGRAL CALCULUS

AND THE

CALCULUS OF FINITE DIFFERENCES.

BY THE

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THE AUTHOR.

PREFACE.

The want of a text-book on the Calculus of Operations has long been felt by mathematicians. The extensive practical bearings of the Differential and Integral Calculus, and the theoretic interest which is associated with the Calculus of Finite Differences, render it desirable that the processes required in these branches of analysis should be reduced and simplified as far as possible. To the student, the Calculus of Operations proposes to facilitate and abbreviate his labours, while, to the advanced mathematician, it offers a method which will enable him not only to arrive at known results with ease, and express them with elegance, but also to extend his investigations with certainty and rapidity.

To illustrate, however inadequately, the power of this Calculus, is the object of the following Treatise. In its preparation all prolixity of detail has been studiously avoided, as alike unnecessary and wearisome. It has been sought also to exclude, as far as possible, metaphysical subtleties, which might perhaps lend an air of learned mystery, but which serve only to embarrass the reader and weaken his confidence in the results at which he may have arrived.

With a view to the partial indication of the nature of the subjects discussed in the Fellowship Lectures of this University, and more particularly in so far as the development of this branch of analysis is concerned, I have been requested to append notes to the various articles derived from this source. For greater facility of reference, and as a contribution towards the history of this department of science, I have been induced to adopt the same course with regard to assistance derived from other sources. Wherever the subject of any article has been originated by another, and the investigation or method of treatment is, so far as I am at present aware, my own, the reference is appended immediately after the statement of the subject of the article. Wherever the subject and method of treatment are both due to another, the reference is given at the end of the article. It is not, of course, necessary, nor would it, indeed, be possible, to extend this system of reference to the case of those articles which will be at once recognised as common property.

Where any of the results contained in the following pages have been already published by myself, either in the "Cambridge and Dublin Mathematical Journal," or in the "Philosophical Magazine," I have in such cases simply stated the name of the periodical in which such results may have appeared. With the exceptions stated, the remainder of the book is, I believe, new.

My first and largest acknowledgments are due to the Rev. John Hewitt Jellett, Professor of Natural Philosophy in this University, whose Treatise on the "Calculus of Variations" first led me to independent and original investigation. My next acknowledgments are due to the Rev. Charles Graves, Professor of Mathematics in this University, whose investigations in this branch of analysis have largely contributed to illustrate its elegance and power, as the following pages will abundantly testify, and to whom I am indebted for acquaintance with many valuable sources of information. The amount of assistance which I have derived from the valuable collection of Examples illustrative of the processes of the Differential and Integral Calculus by the late Mr. Gregory, Fellow of Trinity College, Cambridge, is very considerable, and much of the importance now attributed to the Calculus of Operations is due to the vindication and illustration of its claims by that distinguished mathematician. My acknowledgments are also due to Sir John Herschel, whose Supplement to the translation of "Lacroix's Differential and Integral Calculus," so remarkable for the subtlety of its reasonings and the breadth of its conceptions, I have studied with much advantage.

I would offer my best thanks to Mr. Arthur Curtis, to whom I am indebted for many valuable suggestions and much kind assistance in the revision of the sheets of this work during its progress through the press.

In dedicating my book to the Board of Trinity College, I have endeavoured to show my appreciation and respect for the enlightened liberality with which they invariably support every genuine effort for the advancement of learning.

TRINITY COLLEGE, DUBLIN,
March, 1855.

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CALCULUS OF OPERATIONS.

CHAPTER I.

INTRODUCTION.

1. THE Calculus of Operations, in the greatest extension of the phrase, may be regarded as that science which treats of the combinations of symbols of operation, conformably to certain given laws, and of the relations by which these symbols are connected with the subjects on which they operate.

As the principal object of the present work is to reduce and simplify the labours of the student, and as the great practical utility of the Calculus of Operations, at present, arises from its bearing upon the Differential and Integral Calculus, and the Calculus of Finite Differences, the symbols of operation employed in illustration, and the laws by which they are governed, are those belonging to the branches of analysis named, and are consequently familiar to the reader.

The differential equations proposed as examples are classified according to the methods by which it appears that their solutions are most easily derived. After each class of ordinary differential equations, where there is but a single independent variable x, will be found, either in the same Chapter or in that immediately consecutive, the class of corresponding partial differential equations in two independent variables x and

y; and it will be evident that the method employed is wholly irrespective of the number of such independent variables. It seems probable that, by this arrangement, the generality and symmetry of the Calculus of Operations will be best exhibited, and the facility with which it can be employed most completely illustrated.

2. Two circumstances appear to have contributed to the retardation of the progress of the Calculus of Operations, as well in its theory as in its practical application. The first circumstance is, that it has been generally held by mathematicians, whether directly or by implication, that inverse symbols of operation (whether distributive or not), and consequently all inverse functions of such symbols, are in their nature indeterminate. I believe this to be a fundamental misconception, and that any indetermination which may exist is due to a different source, namely, the indetermination of the subject of the direct operation, or, in practice, the dependent variable.

Thus, Φ being any distributive symbol of operation, if we have a linear equation with constant coefficients,

$$A\Phi^{\alpha}.\, u + B\Phi^{\beta}.\, u + \ldots + T\,.\, u = \Omega,$$

or,

$$F(\Phi)\,.\, u = \Omega,$$

we know that, in general, u *is* indeterminate.

Moreover, in general, the solution of this equation is given by the evaluation of the symbolic form,

$$u = \frac{1}{F(\Phi)}\,.\,\Omega + \frac{1}{F(\Phi)}\,.\,0,$$

and it will be seen that the value of the first term in this solution is, in all cases, strictly determinate. The indeterminate or arbitrary portion of the solution is given by the second term, and it will be seen that *here* the indetermination is due to a source quite independent of the character of the functional operator.

For confirmation of this view the reader is referred to the Chapter of this work upon the application of the Calculus of Operations to the integration of Differential Equations, Total and Partial, and to the Chapter upon the application of the same Calculus to the subject of Finite Differences.

The second circumstance which seems to have limited the practical employment of the Calculus of Operations is that, although mathematicians were familiar with the distributive character of certain symbols of operation in constant occurrence, as well as with the distributive character of any direct function of those symbols, they do not appear to have been aware that not only are the corresponding *inverse* symbols distributive, but that any *inverse function* of those symbols is equally distributive.

The former of these two points, namely, that if any symbol be distributive, its inverse is also distributive, is due to Mr. Murphy.* The second point, namely, that not only the simple inverse symbol is distributive, but that also any inverse function of the symbol is distributive, is intimately connected with the cause of the retardation of the Calculus of Operations first specified, and its establishment and practical application forms a distinctive feature of the present treatise.

3. With regard to the subject of integration, many of the methods in ordinary use appear to labour under three capital defects. The first defect in the methods, to which allusion is made, is their extremely artificial character, which occasions much embarrassment to the student at first, and considerable difficulty in his effort to retain them. The second great defect in these methods is, that they seem wholly unsusceptible of useful generalization. The third defect is less common, and

* "Philosophical Transactions," 1837.

consists in this, that some of the processes employed are circuitous, terms being introduced which subsequent operations cause to disappear. The methods here put forward appear to be free from these defects, and, in so far as they are calculated to reduce and simplify the labours of the student, may prove, it is hoped, a practical good.

4. In the Chapter on the application of the Calculus of Operations to the solution of Partial Differential Equations, most of the illustrations are borrowed from "Gregory's Examples of the Processes of the Differential and Integral Calculus," Walton's edition, 1846. The general merits of this manual are very considerable, and I am myself largely indebted to it. There are, however, some deficiencies in it, which I have endeavoured to supply.

By a comparison of certain solutions as given there, in the Chapter on the integration of Partial Differential Equations, with the solutions of the same equations as exhibited in the following pages, it will be seen that many of the former are, in point of symmetry, incomplete. During the last few years our conceptions of the Calculus of Operations and its uses have been corrected, widened, and deepened. The errors particularized appear to have been the necessary result of the methods employed, and the methods employed to have been the only ones recognised in the then existing condition of this branch of analysis. It is *à priori* evident that when the equations to be integrated have a symmetrical character, not only should the solutions be symmetrical, but symmetrical methods should be employed for their deduction. Now in the cases to which reference is made, the methods employed are unsymmetrical, and we might consequently have anticipated that the results would be, as they are, incomplete.

A remark to the same effect as that just made, although in connexion with a different subject, is the following:—

"In algebraical analysis it is frequently useful to observe whether the algebraical expressions under consideration are homogeneous or not; that is, whether the dimensions of every term be the same or not; for, if this homogeneity be found at first, no legitimate operation can destroy it; or, if it be not found at first, it cannot be introduced; and thus an easy test is afforded, to a certain extent, of the accuracy of each succeeding step in the analysis.

"For example, if the equation

$$ax^2 + b^2x + c^3 = 0$$

be proposed for solution, in which every term is of three dimensions, that is, which is homogeneous, every step of the solution will present an homogeneous equation, if it be correct.

"As a simple case, it may be well to observe that, if the proposed equation be homogeneous, the final result must be so. A proper attention to this observation will frequently detect an error in the process of solution."*

5. It has been remarked to me by Professor Boole, that the great difficulty in the study of the Calculus of Operations, as connected with the Integral Calculus, consists in the interpretation of the symbolic results at which we may have arrived. The farther the relation between these two subjects is prosecuted, whether in the solution of Differential Equations, the extension of Definite Integrals, or the reduction of equations in Finite Differences, the more imperative becomes the demand for such interpretation. In all these cases, so long as the solutions are symbolic and not completely evaluated, they are unsatisfactory to the advanced mathematician, and perhaps calculated to lead the younger student to undervalue the utility of prosecuting these branches of analysis in conjunction. A large portion of the present work is dedicated to this sub-

* "Wood's Algebra," Eleventh Edition, by Lund; Appendix II., p. xlix.

ject, and a special Chapter is devoted to the interpretation of certain symbols of operation, which appear to possess peculiar significance. In the Chapter upon the application of the Calculus of Operations to the Theory of Curves and Surfaces will be found also some interesting and elegant interpretations of symbols of Operation in connexion with Geometry, for which I am indebted to the Rev. Professor Graves.

6. In the chapter upon the integration of Partial Differential Equations it will be observed that I have not employed the usual notation for the partial differential coefficients of a function of two independent variables, namely, p, q, r, s, t, &c. I have been induced to abandon this notation mainly from the conviction that, by its employment, the symmetry of many partial differential equations has been disguised, and the likelihood of the discovery of operational methods of solution proportionably diminished. An additional and weighty reason for the rejection of the old notation is the consideration, that it is calculated practically to prevent us from extending our view to the case of functions of three or more independent variables, and even in the case of two independent variables, to preclude the entertainment of partial differential equations of an order higher than the second.

7. Some interesting results will be found in the Chapter upon the application of the Calculus of Operations to the integration of systems of Simultaneous Differential Equations, Total and Partial. The illustrations of this employment of the Calculus of Operations hitherto put forward seem, from their tentative character, and the meagre results which have attended them, perhaps calculated to weaken the confidence of the student in the generality and efficiency of this Calculus.

In this same Chapter will be found also some illustrations of the application of the Calculus of Operations to the extension of Definite Integrals.

8. The advanced student, who is desirous of extending his knowledge of the Calculus of Operations beyond the limits which have, of necessity, been observed in an elementary treatise like the present, is referred to the elaborate Papers published in the "Philosophical Transactions," by Professor Boole, Dr. Hargreave, and Rev Mr. Bronwin; in the "Proceedings of the Royal Irish Academy," by the Rev. Professor Graves; in the "Cambridge and Dublin Mathematical Journal," by Professor Boole, Rev. Mr. Bronwin, Mr. Sylvester, Professor Donkin, and Mr. Spottiswoode; in the "Philosophical Magazine," by Dr. Hargreave and Mr. Sylvester; besides the Papers of various other mathematicians, whose labours in this field of research have contributed to its development.

CHAPTER II.

ELEMENTARY PRINCIPLES.

1. Two symbols Φ and Ψ are said to be *commutative* when, u being the subject on which they operate,

$$\Phi\Psi \, . \, u = \Psi\Phi \, . \, u.$$

A symbol Φ is said to be *distributive* when, u and v being two distinct subjects,

$$\Phi \, (u + v) = \Phi \, . \, u + \Phi \, . \, v.$$

A symbol Φ is said to be *iterative*, or to follow the law of indices, when

$$\Phi^m . \, \Phi^n . \, u = \Phi^{m+n} . \, u = \Phi^n . \, \Phi^m . \, u.$$

It is to be observed that this third formula is not to be regarded as a law of symbolic combination in the same sense as the first, nor as a law of symbolic operation in the same sense as the second. In fact, $\Phi^m . \, u$ is rather to be regarded as a mere abbreviated notation for the result of the operation Φ performed m times successively upon u,

$$\Phi \, . \, \Phi \, . \, \Phi \, . \, . \, . \, . \, u,$$

than as equivalent to the result of any *operation raised to the power m*, performed on u.

The laws stated being the principal ones which occur in the practical employment of the Calculus of Operations, we shall for the present confine our attention to them. They may be called, respectively,

I. THE LAW OF COMMUTATION;
II. THE LAW OF DISTRIBUTION;
III. THE LAW OF INDICES.

2. We may at once observe, that whatever theorem is true for any one symbol which satisfies these laws is true for every symbol which satisfies them.

Now the symbols of *numbers* satisfy them; indeed all algebraical equations may be considered as having the same subject, unity, and the constants, as denoting sums of operations performed on unity. Again, the symbols of *differentiation* satisfy those laws; for if u be a function of the two independent variables x and y, it is known that

$$D_x D_y \,.\, u = D_y D_x \,.\, u;$$

that, if u be a function of x only,

$$D_x (u + v) = D_x u + D_x v;$$

and that

$$D_x^m \,.\, D_x^n \,.\, u = D_x^{m+n} . u.$$

Hence we deduce the important consequence, that every theorem in Algebra, which depends on those laws, has an analogue in the Differential Calculus.

In illustration, if we have a linear equation, with constant coefficients, of the form

$$\Phi^n . u + A_1 \Phi^{n-1} . u + A_2 \Phi^{n-2} . u + \ldots + A_n u = X,$$

where Φ operates solely on u, and is therefore commutative with A_1, A_2, ... A_n, then the symbolical solution is

$$u = \left\{ \begin{array}{c} (\Phi^n + A_1 \Phi^{n-1} + A_2 \Phi^{n-2} + \ldots + A_n)^{-1} . X \\ + \\ (\Phi^n + A_1 \Phi^{n-1} + A_2 \Phi^{n-2} + \ldots + A_n)^{-1} . 0, \end{array} \right.$$

or

$$u = \left\{ \begin{array}{c} N_1 (\Phi - a_1)^{-1} . X + N_2 (\Phi - a_2)^{-1} . X + \ldots + N_n (\Phi - a_n)^{-1} . X \\ + \\ N_1 (\Phi - a_1)^{-1} . 0 + N_2 (\Phi - a_2)^{-1} . 0 + \ldots + N_n (\Phi - a_n)^{-1} . 0. \end{array} \right.$$

N_1, N_2, &c., a_1, a_2, &c., having the same values as in the resolution of the rational fraction,

$$\frac{1}{\xi^n + A_1 \xi^{n-1} + A_2 \xi^{n-2} + \ldots + A_n},$$

supposed resolvable into a similar series of terms. The evaluation of the symbolical solution, of course, depends on the particular form of Φ.

It will be seen that the first group of terms in the solution is in all cases strictly determinate, and that the arbitrary portion of the solution is given by the second group. It will also be seen that there are many cases in which the evaluation of the first member can be obtained without the resolution into factors of the symbolical operator.

3. If a symbol Φ be distributive, any power of the symbol, positive or negative, will be also distributive.

In the case of positive powers, we have

$$\Phi . (u + v) = \Phi . u + \Phi . v,$$

the sign $(\cdot)$ being employed to distinguish Φ as an operational symbol from its usual acceptation as functional of the quantities contained under it. As the student becomes familiar with the use of operational symbols, this sign will be occasionally omitted, as unnecessary.

Operating with Φ a second time, we get

$$\Phi^2 . (u + v) = \Phi . (\Phi . u + \Phi . v) = \Phi^2 . u + \Phi^2 . v;$$

and, by successive operation,

$$\Phi^n . (u + v) = \Phi^n . u + \Phi^n . v.$$

In the case of negative indices, assuming as a definition that

$$\Phi^{-1} . \Phi = \Phi . \Phi^{-1},$$

and that the result of the operation of either side of this equivalence upon the same subject is the same as if the subject had not been operated upon at all; if we operate upon either side of the fundamental equation

$$\Phi . (u + v) = \Phi . u + \Phi . v$$

with Φ^{-1}, we see that

$$u + v = \Phi^{-1} . (\Phi . u + \Phi . v).$$

But the *left-hand* member may be written in the form

$$\Phi^{-1} . \Phi . u + \Phi^{-1} . \Phi . v,$$

and consequently the equation itself, in the form

$$\Phi^{-1} . \Phi . u + \Phi^{-1} . \Phi . v = \Phi^{-1} . (\Phi . u + \Phi . v),$$

or

$$\Phi^{-1} . U + \Phi^{-1} . V = \Phi^{-1} . (U + V),$$

and from hence, by sucessive operation, it appears that the theorem is true for negative powers in general.

4. It may here be observed, that if we operate on both sides of the equation

$$\Phi . (u + v) = \Phi . u + \Phi . v$$

with Φ^{-1}, it might be supposed that, inverse symbols being regarded as in their nature indeterminate, the result should be written

$$u + v = \Phi^{-1} \{\Phi . u + \Phi . v\} + \Phi^{-1} . 0,$$

and not

$$u + v = \Phi^{-1}\{\Phi . u + \Phi . v\},$$

simply.

But it is plain that if the left-hand member of any equation be determinate, as the left-hand member in the above equation is supposed to be, the right-hand member of the same equation ought to be equally determinate, and consequently no such term as $\Phi^{-1}. 0$ should be introduced. There is an obvious difference between this case and that of the solution of differential equations and equations in finite differences, in which the dependent variable, which corresponds with the left-hand member above, is indefinite and indeterminate.

5. Any algebraic function of a distributive symbol Φ is itself also distributive.

For since

$$\Phi \,.\, (u+v) = \Phi \,.\, u + \Phi \,.\, v,$$
$$\Phi^2 .\, (u+v) = \Phi^2 .\, u + \Phi^2 .\, v,$$
$$\Phi^3 .\, (u+v) = \Phi^3 .\, u + \Phi^3 .\, v,$$
$$\&c.$$
$$\Phi^n .\, (u+v) = \Phi^n .\, u + \Phi^n .\, v,$$

if we multiply the first equation by A_1, the second by A_2, the third by A_3, &c., and add, we get

$$(A_0 + A_1\Phi + A_2\Phi^2 + \ldots + A_n\Phi^n)(u+v) = \begin{cases} (A_0 + A_1\Phi + A_2\Phi^2 + \ldots)u \\ \qquad\qquad + \\ (A_0 + A_1\Phi + A_2\Phi^2 + \ldots)v \end{cases}$$

or generally, if F be any algebraic function,

$$\mathrm{F}(\Phi) \,.\, (u+v) = \mathrm{F}(\Phi) \,.\, u + \mathrm{F}(\Phi) \,.\, v.$$

Murphy, *Phil. Trans.*, 1837.

We have here only established the principle for positive indices, but it is obvious that the same demonstration will hold for the case in which the indices are negative, and consequently for any indirect function, or that

$$\frac{1}{\mathrm{F}(\Phi)} \,.\, (u+v) = \frac{1}{\mathrm{F}(\Phi)} \,.\, u + \frac{1}{\mathrm{F}(\Phi)} \,.\, v.$$

6. Any two functions of the same distributive symbol are commutative, and if any quantity be operated on successively by two functions of the same symbol, the result is the same as if the quantity had been operated on originally by the product of those functions.

This principle may be established by *à priori* considerations, but the student will most readily and immediately satisfy himself of its truth by actual trial. For example, he will see that the result of the operation

$$(D_x + a) \,.\, (D_x + b) \,.\, F(x)$$

is the same as the result of the operation

$$(D_x + b) \,.\, (D_x + a) \,.\, F(x),$$

and each is the same as the result of the operation

$$(D_x^2 + \overline{a+b}D_x + ab) \,.\, F(x).$$

7. To legitimate the equivalence

$$e^{\Phi+\Psi} \,.\, u = e^{\Phi} \,.\, e^{\Psi} \,.\, u$$

it is necessary and sufficient that Φ and Ψ should be commutative.

For, expand both sides, and in order that

$$\left\{1+\left(\frac{\Phi+\Psi}{1}\right)+\left(\frac{\Phi+\Psi}{1.2}\right)^2+..\right\}u=\left(1+\frac{\Phi}{1}+\frac{\Phi^2}{1.2}+..\right)\left(1+\frac{\Psi}{1}+\frac{\Psi^2}{1.2}+..\right)u$$

we should have

$$(\Phi + \Psi)^2 = \Phi^2 + 2\Phi\Psi + \Psi^2.$$

But the value of the left-hand member is, in general,

$$\Phi^2 + \Phi\Psi + \Psi\Phi + \Psi^2 \,;$$

and consequently, the equation just stated cannot hold unless

$$\Phi\Psi = \Psi\Phi \,;$$

that is, unless the symbols Φ and Ψ be commutative. Moreover, when these symbols are commutative, it is immediately obvious that the general equivalence does hold.

As an application, the symbols

$$xD_y - yD_x, \qquad aD_x + bD_y,$$

are not commutative, consequently we cannot assert the equivalence

$$e^{xD_y-yD_x+aD_x+bD_y} . F(x, y) = e^{xD_y-yD_x} . e^{aD_x+bD_y} . F(x, y),$$

nor consequently the equivalence

$$e^{xD_y-yD_x} . e^{aD_x+bD_y} . F(x, y) = e^{aD_x+bD_y} . e^{xD_y-yD_x} . F(x, y).$$

An important application of this example will be found in a subsequent Chapter.

CHAPTER III.

APPLICATION TO THE INTEGRATION OF LINEAR TOTAL DIFFERENTIAL EQUATIONS.

SECTION I.—PRELIMINARY THEOREMS.

1. If we operate with the symbol xD_x upon x^m, we find that

$$xD_x \,.\, x^m = m \,.\, x^m.$$

Operating with the same symbol upon both sides of this equation,

$$(xD_x)^2 \,.\, x^m = m^2 \,.\, x^m,$$

and, by successive operation,

$$(xD_x)^p \,.\, x^m = m^p \,.\, x^m.$$

Hence the theorem that, if F be any algebraic function,

$$F(xD_x) \,.\, x^m = F(m) \,.\, x^m.$$

BOOLE, *Phil. Trans.*, 1844.

For the purpose of future application, and of more ready identification with a theorem to be given in a subsequent Chapter, I prefer stating this theorem in the slightly different form

$$F(xD_x) \,.\, A_m x^m = F(m) \,.\, A_m x^m,$$

where A_m is any constant.

The theorem has been only demonstrated for direct powers and any direct function, but it is obvious that the same proof will apply to inverse powers and any inverse function. It is also to be remembered that inverse functions are as well distributive as direct functions.

2. Now, if U be any algebraic function of x, it can, in general, be put under the form

$$U = A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots + A_nx^n,$$

where A_0, A_1, A_2, &c., are constants.

Hence, since $F(xD_x)$ is distributive, we obtain the more general theorem

$$F(xD_x)\,U = F(0)\,A_0 + F(1)\,A_1x + F(2)\,A_2x^2 + \ldots + F(n)\,A_nx^n,$$

with the corresponding theorem for inverse functions

$$\frac{1}{F(xD_x)}\,U = \frac{1}{F(0)}\,A_0 + \frac{1}{F(1)}\,A_1x + \frac{1}{F(2)}\,A_2x^2 + \ldots + \frac{1}{F(n)}\,A_nx^n.$$

Examples.

(1.) Let the results of the operations of a^{xD_x} and $\frac{1}{a^{xD_x}}$ respectively, upon U, be investigated.

They are, respectively,

$$A_0 + aA_1x + a^2A_2x^2 + \ldots + a^nA_nx^n,$$

GRAVES, *Fellowship Lectures*, 1851.

and

$$A_0 + \frac{1}{a}\,Ax + \frac{1}{a^2}\,A_2x^2 + \ldots + \frac{1}{a^n}\,A_nx^n.$$

(2.) Let the results of the operations of $F(xD_x)$ and $\frac{1}{F(xD_x)}$, respectively, upon $e^{A_nx^n}$, be investigated.

The subject being expanded, the results required are, respectively,

$$F(0)\,1 + F(n)\,\frac{A_nx^n}{1} + F(2n)\,\frac{(A_nx^n)^2}{1\,.\,2} + F(3n)\,\frac{(A_nx^n)^3}{1\,.\,2\,.\,3} + \&c.$$

and

$$\frac{1}{F(0)}\,1 + \frac{1}{F(n)}\,\frac{A_nx^n}{1} + \frac{1}{F(2n)}\,\frac{(A_nx^n)^2}{1\,.\,2} + \frac{1}{F(3n)}\,\frac{(A_nx^n)^3}{1\,.\,2\,.\,3} + \&c.$$

3. It is easily demonstrable that, whatever be the subject of operation, u,

$$xD_x(xD_x - 1) \,.\, u = x^2 D_x^2 \,.\, u,$$

$$xD_x(xD_x - 1)(xD_x - 2) \,.\, u = x^3 D_x^3 \,.\, u,$$

&c.

and, generally, that

$$xD_x(xD_x - 1)(xD_x - 2) \,.\,.\, (xD_x - n + 1) \,.\, u = x^n D_x^n \,.\, u.$$

BOOLE, *Phil. Trans.*, 1844.

Hence it appears that

$$xD_x(xD_x - 1) \,.\,.\, (xD_x - n + 1) \,.\, x^m = m(m-1) \,.\,.\, (m-n+1) \,.\, x^m,$$

and consequently that

$$xD_x(xD_x - 1) \,.\,.\, (xD_x - n + 1) \,.\, x^n = 1 \,.\, 2 \,.\, 3 \,.\,.\,.\, n \,.\, x^n,$$

results which might have been deduced at once from the theorem

$$F(xD_x) \,.\, x^m = F(m) \,.\, x^m.$$

4. If we operate with the symbol xD_x upon $x^m v$, we find that

$$xD_x \,.\, x^m v = x^m \,.\, (xD_x + m) v.$$

Operating with the same symbol upon both sides of this equation,

$$(xD_x)^2 \,.\, x^m v = x^m \,.\, (xD_x + m)^2 v,$$

and by successive operation,

$$(xD_x)^p \,.\, x^m v = x^m \,.\, (xD_x + m)^p v.$$

Hence the theorem that, if F be any algebraic function,

$$F(xD_x) \,.\, x^m v = x^m F(xD_x + m) v.$$

BOOLE, *Phil. Trans.*, 1844.

This theorem again holds as well for inverse functions as for direct.

5. By the substitution $x = e^{\theta}$, we obtain a very useful form of the theorem

$$F(xD_x).A_m x^m = F(m).A_m x^m,$$

namely,

$$F(D_\theta).A_m e^{m\theta} = F(m).A_m e^{m\theta}.$$

This latter form is perhaps more simple, and thus naturally would appear to claim precedence of that given first. The first form, however, is more susceptible of generalization, and lends itself equally to practical application.

And here it may be observed, that the student will derive considerable benefit from the habit of employing transformations similar to that above exhibited. The same theorem is thus regarded from so many distinct points of view, and the chances of its susceptibility of generalization, or useful practical application, proportionably multiplied.

By the same substitution the theorem

$$F(xD_x).x^m v = x^m.F(xD_x + m)v$$

assumes the form

$$F(D_\theta).e^{m\theta}v = e^{m\theta}.F(D_\theta + m)v;$$

the following modifications of which are occasionally useful, namely,

$$e^{-m\theta}F(D_\theta)e^{m\theta}v = F(D_\theta + m)v$$

and

$$e^{m\theta}F(D_\theta)e^{-m\theta}v = F(D_\theta - m)v.$$

6. It is known that, if u, v be any functions of θ,

$$D_\theta.uv = u.D_\theta v + D_\theta u.v;$$

or, omitting the suffix for the present,

$$D.uv = u.Dv + Du.v,$$

in connexion with which the student will observe that wherever the symbol (.) immediately follows *a symbol* of operation, it is to be understood that such symbol operates on the entire term to its right; whereas, if the same sign immediately follows a *subject* of operation, it is to be understood that any operation indicated is there terminated.

Operating with D a second time, we get

$$D^2 . uv = u . D^2v + 2Du . Dv + D^2u . v;$$

and operating a third time with the same symbol,

$$D^3 . uv = u . D^3v + 3Du . D^2v + 3D^2u . Dv + D^3u . v.$$

It is evident that, in the same manner, if n be any integer,

$$D^n . uv = u . D^nv + nDu . D^{n-1}v + \frac{n . n-1}{1 . 2} D^2u . D^{n-2}v + \ldots + D^nu . v.$$

Arranging these results, now, in a tabular form, we have

$$\begin{aligned} uv &= uv, \\ D \; . uv &= u . Dv + Du . v, \\ D^2 . uv &= u . D^2v + 2Du . Dv + D^2u . v, \\ D^3 . uv &= u . D^3v + 3Du . D^2v + 3D^2u . Dv + D^3u . v, \\ &\ldots \\ &\ldots \\ &\ldots \\ D^n . uv &= u . D^nv + nDu . D^{n-1}v + \frac{n . n-1}{1 . 2} D^2u . D^{n-2}v + \ldots + D^nu . v. \end{aligned}$$

Multiply the first equation by A_0, the second by A_1, the third by A_2, and so on, and add. Then

$$(A_0 + A_1D + A_2D^2 + A_3D^3 + \ldots + A_nD^n) . uv$$

$$=$$

$$u . (A_0 + A_1D + A_2D^2 + A_3D^3 + \ldots + A_nD^n) . v$$

$$+$$

$$Du . (A_1 + 2A_2D + 3A_3D^2 + 4A_4D^3 + \ldots + nA_nD^{n-1}) . v$$

$$+$$

$$\frac{D^2u}{1 . 2} . (2A_2 + 3 . 2 . A_3D + 4 . 3 . A_4D^2 + \ldots + n . n-1 . A_nD^{n-2}) . v$$

$$+ \text{\&c.}$$

But it will be observed that

$$A_1 + 2A_2D + 3A_3D^2 + 4A_4D^3 + \ldots + nA_nD^{n-1}$$

bears the same relation, in point of character, to

$$A_0 + A_1D + A_2D^2 + A_3D^3 + A_4D^4 + \ldots + A_nD^n,$$

as

$$A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \ldots + nA_nx^{n-1}$$

bears to

$$A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \ldots + A_nx^n.$$

Now, if

$$A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots + A_nx^n = F(x),$$

we write

$$A_1 + 2A_2x + 3A_3x^2 + \ldots + nA_nx^{n-1} = F'(x);$$

and so if

$$A_0 + A_1D + A_3D^2 + \ldots + A_nD^n = F(D),$$

we may write

$$A_1 + 2A_2D + 3A_3D^2 + \ldots + nA_nD^{n-1} = F'(D);$$

where all that is meant is, that $F'(D)$ *bears the same relation in point of character* to $F(D)$, as $F'(x)$ bears to $F(x)$.

It is obvious that

$$2A_2 + 3 \,.\, 2 \,.\, A_3D + 4 \,.\, 3 \,.\, A_4D^2 + \ldots + n \,.\, n - 1 \,.\, A_nD^{n-2}$$

bears a similar relation to

$$A_1 + 2A_2D + 3A_3D^2 + \ldots + nA_nD^{n-1},$$

and may be written

$$F''(D);$$

and so on for the remaining terms.

Hence we derive the important theorem that, if F be any algebraic function,

$$F(D) \,.\, uv = u \,.\, F(D)\, v + \frac{Du}{1} \,.\, F'(D)\, v + \frac{D^2 u}{1\,.\,2} \,.\, F''(D)\, v + \&c.$$

HARGREAVE, *Phil. Trans.*, 1848.

By the substitution $e^\theta = x$, before employed, we obtain a form of this theorem, namely,

$$F(xD_x).uv = u.F(xD_x)\, v + \frac{xD_x u}{1}.F'(xD_x)v + \frac{(xD_x)^2 u}{1\,.\,2}.F''(xD_x)v + \&c.$$

which is equally susceptible of application with that previously given, and at the same time suggests an elegant generalization, which will be exhibited in the following Chapter.

7. It is known that

$$u \,.\, Dv = D \,.\, uv - Du \,.\, v.$$

If for v in this equation we write Dv, we get

$$u \,.\, D^2 v = D \,.\, uDv - Du \,.\, Dv,$$

or, by the previous formula,

$$u \,.\, D^2 v = D(D \,.\, uv - Du \,.\, v) - D \,.\, Du \,.\, v + D^2 u \,.\, v,$$

or

$$u \,.\, D^2 v = D^2 \,.\, uv - 2D \,.\, Du \,.\, v + D^2 u \,.\, v.$$

Substituting again in this equation Dv for v, and employing both the previous formulæ, we get

$$u \,.\, D^3 v = D^3 .\, uv - 3D^2 .\, Du \,.\, v + 3D \,.\, D^2 u \,.\, v - D^3 u \,.\, v,$$

and generally

$$u \,.\, D^n v = D^n .\, uv - nD^{n-1} .\, Du \,.\, v + \frac{n \,.\, n-1}{1\,.\,2} D^{n-2} \,.\, D^2 u \,.\, v - \&c.$$

Hence, in a manner precisely similar to that employed in the last article, we get the corresponding theorem

$$u \,.\, F(D) v = F(D) \,.\, uv - F'(D) \,.\, Du \,.\, v + F''(D) \,.\, \frac{D^2 u}{1\,.\,2} .\, v - \&c.$$

HARGREAVE, *Phil. Trans.*, 1848.

By the substitution $e^\theta = x$, as before, we obtain the corresponding form of this theorem,

$$u.F(xD_x)v = F(xD_x).uv - F'(xD_x).xD_xu.v + F''(xD_x).\frac{(xD_x)^2u}{1.2}.v$$
$$- \&c.$$

8. It is already known that, if Φ be any algebraic function,

$$\Phi(D_r).e^{rx} = \Phi(x).e^{rx};$$

and, similarly, that, if Ψ be any other algebraic function,

$$\Psi(D_x).e^{rx} = \Psi(r).e^{rx}.$$

But it is known also that

$$\Psi(D_x).\Phi(D_r) = \Phi(D_r).\Psi(D_x).$$

Hence we infer that

$$\Psi(D_x).\Phi(x).e^{rx} = \Phi(D_r).\Psi(r).e^{rx}. \qquad (\alpha)$$

This theorem being written in the slightly different form,

$$\Psi(D_x).e^{rx}\,\Phi(x) = \Phi(D_r).e^{rx}\,\Psi(r),$$

if we multiply both sides by e^{-rx}, we obtain the singular result

$$\Psi(D_x + r).\Phi(x) = \Phi(D_r + x).\Psi(r). \qquad (\beta)$$

BRONWIN, *Camb. and Dub. Math. Journal*, 1848.

SECTION II.—APPLICATION OF PRECEDING THEOREMS.

9. All differential equations represented by

$$Ax^\alpha D_x^\alpha.y + Bx^\beta D_x^\beta.y + \&c. = X,$$

where A, B, &c. are constants, and X is any algebraic function of the independent variable x only, may obviously be transformed, by the third article of the previous section, into the shape

$$\left.\begin{array}{c} AxD_x(xD_x-1)\ldots(xD_x-\alpha+1) \\ + \\ BxD_x(xD_x-1)\ldots(xD_x-\beta+1) \\ +\ \&c. \end{array}\right\} . y = X.$$

Consequently, the solutions of all such equations are given by the evaluation of the symbolic form

$$y = \frac{1}{F(xD_x)} . X + \frac{1}{F(xD_x)} . 0,$$

where

$$F(xD_x) = \left\{\begin{array}{c} AxD_x(xD_x-1)\ldots(xD_x-\alpha+1) \\ + \\ BxD_x(xD_x-1)\ldots(xD_x-\beta+1) \\ +\ \&c. \end{array}\right.$$

Now, with regard to the symbolic form of the solution, if we suppose that

$$X = L + Mx + Nx^2 + \ldots + Tx^n,$$

the value of *the first term* in the solution is perfectly definite, and, by the second article of the preceding section, is, instantly,

$$\frac{L}{F(0)} + \frac{Mx}{F(1)} + \frac{Nx^2}{F(2)} + \ldots + \frac{Tx^n}{F(n)};$$

while the evaluation of *the second term* gives the arbitrary portion of the solution.

When the roots of the equation

$$xD_x(xD_x-1)\ldots(xD_x-\alpha+1) + \frac{B}{A}xD_x(xD_x-1)\ldots(xD_x-\beta+1)$$
$$+\ \&c. = 0$$

are all real and unequal, the arbitrary portion of the solution is given by

$$\frac{1}{(xD_x - a)\,(xD_x - b)\,\ldots + (xD_x - i)} \cdot 0,$$

a, b, &c., i, being the values of the roots.

Now if A_m be any constant, we know that

$$(xD_x - m) \cdot A_m x^m = 0.$$

Consequently the general value of

$$\frac{N}{xD_x - a} \cdot 0 = \frac{1}{xD_x - a} \cdot 0 = C_a x^a$$

where C_a is any *arbitrary* constant.

Hence, the general expression above being decomposed into a system of rational fractions, the arbitrary portion of the solution is, in the case of *real and unequal* roots, given by

$$C_a x^a + C_b x^b + C_c x^c + \text{\&c.} + C_i x^i,$$

where C_a, C_b, C_c, &c., C_i, are arbitrary constants.

It can be readily seen that

$$(xD_x - m)^2 \cdot A_m x^m \cdot \log x = 0,$$
$$(xD_x - m)^3 \cdot A_m x^m (\log x)^2 = 0,$$
$$\text{\&c.}$$

and generally that

$$(xD_x - m)^p \cdot A_m x^m (\log x)^{p-1} = 0.$$

Consequently, if the equation above contain p *equal roots, whose common value is* a, the arbitrary portion of the solution is given by

$$C_a x^a \cdot (\log x)^{p-1} + C_a' x^a \cdot (\log x)^{p-2} + \ldots + C_i x^i,$$

where C_a, C_a', &c., C_i, are distinct arbitrary constants.

Finally, when this equation contains *pairs of imaginary roots*, the form of the arbitrary portion of the solution is

$$C_{a+b\sqrt{-1}}\, x^{a+b\sqrt{-1}} + C_{a-b\sqrt{-1}}\, x^{a-b\sqrt{1}} + \ldots + C_i\, x^i.$$

Camb. and Dub. Math. Journal, 1851.

10. Adopting the transformation before employed, $x = e^\theta$, we see that, without the first reduction, the same method of solution as that just exhibited will apply to the class of equations represented by

$$AD_\theta^a.y + BD_\theta^b.y + \ldots + T.y = f(e^\theta, \sin\theta, \cos\theta),$$

and thus, this class can be integrated by a process simple and uniform, equally susceptible of employment in equations of the higher orders as in those of the lower, and, as will be presently seen, directly indicative, in either case, of a corresponding class of partial differential equations with the appropriate form of solution.

In fact, the right-hand member being reduced to the form, now abbreviated,

$$\Sigma A_m\, e^{m\theta},$$

where m may be positive or negative, fractional or integer, real or imaginary, the equation becomes

$$F(D_\theta) \cdot y = \Sigma A_m \,.\, e^{m\theta},$$

and consequently the solution required is

$$y = \Sigma A_m \cdot \frac{e^{m\theta}}{F(m)} + \frac{1}{F(D_\theta)} \cdot 0,$$

and when m is imaginary we may restore the circular function.

As regards the various forms which the arbitrary portion of the solution may assume, according as the roots of

$$F(D_\theta) = 0$$

are all real and unequal, some equal, or some imaginary, they are given, respectively, by

$$C_a e^{a\theta} + C_b e^{b\theta} + \ldots + C_i e^{i\theta}, \qquad \text{(I.)}$$

$$C_a e^{a\theta} . \theta^{p-1} + C'_a e^{a\theta} . \theta^{p-2} + \ldots + C_i e^{i\theta}, \qquad \text{(II.)}$$

$$C_{a+b\sqrt{-1}}\, e^{(a+b\sqrt{-1})\theta} + C_{a-b\sqrt{-1}}\, e^{(a-b\sqrt{-1})\theta} + \ldots + C_i e^{i\theta}, \qquad \text{(III.)}$$

The germ of this method is to be found in the Chapter of Gregory's *Examples* which is devoted to the integration of linear differential equations with constant coefficients. That it was never matured seems to have been due to the circumstance that the distributive character of inverse functions was not then recognised, and consequently the method was only applied to the case in which the right-hand member consists of but a single term, $A_m e^{m\theta}$.

11. By a single very obvious reduction the solution of the class of differential equations represented by

$$A(m + \lambda x)^{\alpha} D_x^{\alpha} . y + B(m + \lambda x)^{\beta} D_x^{\beta} . y + \&\text{c.} = X$$

may now be obtained.

In fact, assume

$$m + \lambda x = \lambda x',$$

and the differential equation becomes

$$A\lambda^{\alpha} . x'^{\alpha} D_{x'}^{\alpha} . y + B\lambda^{\beta} . x'^{\beta} D_{x'}^{\beta} . y + \&\text{c.} = X,$$

or

$$F(x' D_{x'}) . y = X;$$

the solution of which is had, at once and without any further transformation, in terms of x', and therefore the solution of the given equation by the substitution for x' of

$$\frac{m + \lambda x}{\lambda}.$$

Examples.—First Type.

(1.) $$x^2 D_x^2 y = ax^m + bx^n.$$

This is equivalent to

$$xD_x(xD_x - 1)y = ax^m + bx^n.$$

Consequently the symbolic solution is

$$y = \frac{1}{xD_x(xD_x - 1)}(ax^m + bx^n) + \frac{1}{xD_x(xD_x - 1)} 0,$$

and the evaluated solution is

$$y = \frac{ax^m}{m(m-1)} + \frac{bx^n}{n(n-1)} + C_0 + C_1 x.$$

(2.) $$x^3 D_x^3 y = ax^m + bx^n.$$

This again is equivalent to

$$xD_x(xD_x - 1)(xD_x - 2)y = ax^m + bx^n.$$

Consequently the symbolic solution is, omitting the suffix,

$$y = \frac{1}{xD(xD-1)(xD-2)}(ax^m + bx^n) + \frac{1}{xD(xD-1)(xD-2)} 0,$$

and the evaluated solution

$$y = \frac{ax^m}{m(m-1)(m-2)} + \frac{bx^n}{n(n-1)(n-2)} + C_0 + C_1 x + C_2 x^2.$$

(3.) $$x^2 D_x^2 y - nxD_x y + ny = ax^m.$$

This is equivalent to

$$(xD_x - 1)(xD_x - n)y = ax^m;$$

whence

$$y = \frac{ax^m}{(m-1)(m-n)} + C_1 x + C_2 x^n.$$

(4.) $$x^2 D_x^2 y + 3xD_x y + y = \frac{1}{(1-x)^2}.$$

Expanding the right-hand member, this becomes

$$(xD_x + 1)^2 . y = 1 + 2x + 3x^2 + \&c.$$

Therefore

$$y = \left(1 + \frac{x}{2} + \frac{x^2}{3} + \&c.\right) + \frac{C_1}{x}.\log x + \frac{C_2}{x},$$

or

$$y = \log\left(\frac{1}{1-x}\right)^{\frac{1}{x}} + \frac{C_1}{x}\log x + \frac{C_2}{x}.$$

Examples.—Second Type.

(1.) $$D_\theta y + ay = e^{m\theta},$$

$$y = \frac{e^{m\theta}}{m+a} + Ce^{-a\theta}.$$

(2.) $$D_\theta y - ay = e^{m\theta}\cos r\theta.$$

Reduced to the shape prescribed, this becomes

$$(D_\theta - a).y = \tfrac{1}{2}\{e^{(m+r\sqrt{-1})\theta} + e^{(m-r\sqrt{-1})\theta}\},$$

and the solution is

$$y = \tfrac{1}{2}\left\{\frac{e^{(m+r\sqrt{-1})\theta}}{m+r\sqrt{(-1)}-a} + \frac{e^{(m-r\sqrt{-1})\theta}}{m-r\sqrt{(-1)}-a}\right\} + Ce^{a\theta};$$

or, restoring the circular function,

$$y = e^{m\theta}.\frac{(m-a)\cos r\theta + r\sin r\theta}{(m-a)^2 + r^2} + Ce^{a\theta}.$$

(3.) $$D_\theta^2 y + a^2 y = \cos m\theta.$$

Reduced to the shape prescribed, this becomes

$$(D_\theta + a\sqrt{-1}).(D_\theta - a\sqrt{-1}).y = \tfrac{1}{2}(e^{m\theta\sqrt{-1}} + e^{-m\theta\sqrt{-1}}),$$

and the solution is

$$y = \tfrac{1}{2}\left(\frac{e^{m\theta\sqrt{-1}}}{a^2-m^2} + \frac{e^{-m\theta\sqrt{-1}}}{a^2-m^2}\right) + C_1 e^{-a\theta\sqrt{-1}} + C_2 e^{a\theta\sqrt{-1}};$$

or, restoring the circular function, and substituting for C_1, C_2 suitable equivalents,

$$y = \frac{\cos m\theta}{a^2 - m^2} + C_1' \cos a\theta + C_2' \sin a\theta.$$

(4) $$D_\theta^4 y + 5D_\theta^2 y + 6y = \sin m\theta.$$

This is equivalent to

$$(D_\theta^2 + 2)(D_\theta^2 + 3) \,.\, y = \sin m\theta,$$

which gives

$$y = \frac{\sin m\theta}{m^4 - 5m^2 + 6} + C_1' \cos(2^{\frac{1}{2}}x + \alpha) + C_2' \cos(3^{\frac{1}{2}}x + \beta).$$

(5.) $$D_\theta^2 y - 2aD_\theta y + a^2 y = \sin m\theta.$$

Reduced to a symbolic shape, this becomes

$$(D_\theta - a)^2 \,.\, y = \frac{1}{2\sqrt{-1}} (e^{m\theta\sqrt{1}} - e^{-m\theta\sqrt{-1}}),$$

and consequently the solution is, at once,

$$y = \frac{1}{2\sqrt{-1}} \left(\frac{e^{m\theta\sqrt{-1}}}{\{m\sqrt{(-1)} - a\}^2} - \frac{e^{-m\theta\sqrt{-1}}}{\{m\sqrt{(-1)} + a)^2\}} \right) + e^{a\theta}(C_1\theta + C_2)$$

or, the circular function being restored,

$$y = \frac{(a^2 - m^2)\sin m\theta + 2am\cos m\theta}{(a^2 + m^2)^2} + e^{a\theta}(C_1\theta + C_2).$$

CHAPTER IV.

APPLICATION TO THE INTEGRATION OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS.

SECTION I.—PRELIMINARY THEOREMS.

IT is the principal object of the present Chapter to show that, by a generalization of the principles contained in the previous Chapter, and a suitable development of the consequences of the higher principles, we can obtain similar general methods of solution of corresponding classes of partial differential equations. The solutions of such partial differential equations will be found to be unaffected by the number of independent variables which the equations may contain; but more especial reference is made to those in most common occurrence, containing but two independent variables, x and y.

In the course of the investigation, extensions of many familiar and elementary theorems are furnished, which seem to possess practical utility.

1. It is known that if

$$u_m = f(x, y, z, \&c.)$$

be a homogeneous function of the m^{th} degree in the independent variables x, y, z, &c.,

$$xD_x u_m + yD_y u_m + zD_z u_m + \&c. = m \,.\, u_m.$$

EULER, *Calc. Diff.*, p. 188.

For conciseness, putting the operating symbol,

$$xD_x + yD_y + zD_z + \&c. = \nabla,$$

we have then

$$\nabla \cdot u_m = m \,.\, u_m.$$

Operating with ∇ upon both sides of this equation,

$$\nabla^2 . u_m = m^2 . u_m ;$$

and, by successive operation,

$$\nabla^p . u_m = m^p . u_m .$$

Hence the theorem that, if F be any algebraic function,

$$F(\nabla) . u_m = F(m) . u_m .$$

It is obvious that this theorem holds as well for inverse functions as for direct, and that if we suppose the number of independent variables reduced to one, we obtain the fundamental theorem of the Third Chapter, namely,

$$F(xD_x) . A_m x^m = F(m) . A_m x^m .$$

By the substitutions

$$x = e^{\phi}, \qquad y = e^{\psi}, \text{ \&c.}$$

and

$$D_{\phi} + D_{\psi} + \text{\&c.} = \square,$$

we obtain the form of this theorem,

$$F(\square) . \Theta_m (e^{\phi}, e^{\psi}, \text{\&c.}) = F(m) . \Theta_m (e^{\phi}, e^{\psi}, \text{\&c.}),$$

corresponding to

$$F(D_{\theta}) . A_m e^{m\theta} = F(m) . A_m e^{m\theta} .$$

Camb. and Dub. Math. Journal, 1851.

2. Now if U be any algebraic function of x, y, z, &c., it can be broken up into sets of homogeneous terms, and put under the form

$$U = u_0 + u_1 + u_2 + \ldots + u_n .$$

Hence, since $F(\nabla)$ is distributive, we obtain the theorem

$$F(\nabla) . U = F(0) . u_0 + F(1) . u_1 + F(2) . u_2 + \ldots + F(n) . u_n ,$$

with the corresponding theorem for inverse functions,

$$\frac{1}{F(\nabla)} . U = \frac{1}{F(0)} . u_0 + \frac{1}{F(1)} . u_1 + \frac{1}{F(2)} . u_2 + \ldots + \frac{1}{F(n)} . u_n .$$

Camb. and Dub. Math. Journal, 1851.

Examples.

(1.) Let the results of the operations of a^{∇} and $\frac{1}{a^{\nabla}}$, respectively, upon U, be investigated.

They are, respectively,

$$u_0 + a.u_1 + a^2.u_2 + \ldots + a^n.u_n,$$

and

$$u_0 + \frac{1}{a}.u_1 + \frac{1}{a^2}.u_2 + \ldots + \frac{1}{a^n}.u_n.$$

(2.) Let the results of the operations of $F(\nabla)$ and $\frac{1}{F(\nabla)}$, respectively, upon ϵ^{u_n}, be investigated.

The subject being expanded, the results required are, respectively,

$$F(0).1 + F(n).\frac{u_n}{1} + F(2n).\frac{u_n^2}{1.2} + F(3n).\frac{u_n^3}{1.2.3} + \&c.$$

and

$$\frac{1}{F(0)}.1 + \frac{1}{F(n)}.\frac{u_n}{1} + \frac{1}{F(2n)}.\frac{u_n^2}{1.2} + \frac{1}{F(3n)}.\frac{u_n^3}{1.2.3} + \&c.$$

3. Since y, z, &c. are constant relative to x, and therefore D_x, D_y, &c., commutative, writing

$$\nabla_2 = x^2 D_x^2 + y^2 D_y^2 + \ldots + 2xy\,D_x D_y + \ldots$$

$$\nabla_3 = x^3 D_x^3 + y^3 D_y^3 + \ldots + 3x^2y\,D_x^2 D_y + 3xy^2 D_x D_y^2 + \ldots$$

&c.

we have

$$\nabla(\nabla - 1) = \nabla_2,$$

$$\nabla(\nabla - 1)(\nabla - 2) = \nabla_3,$$

&c.

and generally

$$\nabla(\nabla - 1)(\nabla - 2)\ldots(\nabla - n + 1) = \nabla_n.$$

It appears at once from the fundamental theorem of this Chapter that

$$\nabla(\nabla-1)(\nabla-2)\ldots(\nabla-n+1).u_n = 1.2.3\ldots n.u_n.$$

4. It may be well here to investigate a general proof of a theorem first given by Euler (*Calc. Diff.*, p. 188), namely,

$$\frac{n(n-1)\ldots(n-m+1)}{\overline{m}}.u_n = \Sigma\,\frac{x^\alpha D_x^\alpha . y^\beta D_y^\beta . z^\gamma D_z^\gamma \ldots}{\overline{\alpha}.\overline{\beta}.\overline{\gamma}\ldots}.u_n.$$

where

$$\alpha+\beta+\gamma+\ldots = m.$$

Now as

$$(1+a)^\nabla U = (1+a)^{xD_x}.(1+a)^{yD_y}.(1+a)^{zD_z}\ldots U;$$

expanding and equating the coefficients of a^m on both sides, and then condensing by the formula above,

$$\frac{\nabla(\nabla-1)\ldots(\nabla-m+1)}{\overline{m}}.U = \Sigma\,\frac{x^\alpha D_x^\alpha . y^\beta D_y^\beta . z^\gamma D_z^\gamma \ldots}{\overline{\alpha}.\overline{\beta}.\overline{\gamma}}.U,$$

and when $U = u_n$, we get Euler's theorem.

Philosophical Magazine, 1852.

5. By a process precisely similar to that employed in the fourth article of the Third Chapter, we obtain the theorem

$$F(\nabla).u_m V = u_m.F(\nabla+m)V,$$

with its transformed shape,

$$F(\square).\Theta_m(e^\phi, e^\psi, \&c.)V = \Theta_m(e^\phi, e^\psi, \&c.).F(\square+m)V$$

analogous, respectively to

$$F(xD_x).x^m v = x^m.F(xD_x+m)v,$$

with its transformed shape

$$F(D_\theta).e^{m\theta}v = e^{m\theta}.F(D_\theta+m)v.$$

Philosophical Magazine, 1852.

6. It is easily seen that if U, V be any functions of ϕ, ψ, &c.

$$\square . UV = U . \square V + \square U . V.$$

Operating with $\square$ a second time, we get

$$\square^2 . UV = U . \square^2 V + 2 \square U . \square V + \square^2 U . V;$$

and operating a third time with the same symbol,

$$\square^3 . UV = U . \square^3 V + 3 \square V . \square^2 V + 3 \square^2 U . \square V + \square^3 U . V.$$

It is evident that, in the same manner, if n be any integer,

$$\square^n . UV = U . \square^n V + n \square U . \square^{n-1} V + \frac{n . n-1}{1 . 2} \square^2 U . \square^{n-2} V + \&c. + \square^n U . V.$$

Arranging these results as in the sixth article of the previous Chapter, and employing a process identical with that there exhibited, we obtain the theorem

$$F(\square) . UV = U . F(\square) V + \frac{\square U}{1} . F'(\square) V + \frac{\square^2 U}{1 . 2} . F''(\square) V + \&c.,$$

with its transformation

$$F(\nabla) . UV = U . F(\nabla) V + \frac{\nabla U}{1} . F'(\nabla) V + \frac{\nabla^2 U}{1 . 2} . F''(\nabla) V + \&c.$$

where U and V are now functions of the variables x, y, z, &c. and ∇ is the symbol before employed.

7. Again, by a process identical with that employed in the seventh article of the preceding Chapter, we obtain the theorem, that if U, V be any functions of ϕ, ψ, &c.,

$$U . F(\square) V = F(\square) . UV - F'(\square) . \square U . V + F''(\square) . \frac{\square^2 U}{1 . 2} . V - \&c.$$

with its transformation

$$U . F(\nabla) V = F(\nabla) . UV - F'(\nabla) . \nabla U . V + F''(\nabla) . \frac{\nabla^2 U}{1 . 2} . V - \&c.$$

Philosophical Magazine, 1852.

8. It has been observed that, as

$$D_\theta^2 . \cos m\theta = - m^2 . \cos m\theta,$$

and

$$D_\theta^2 . \sin m\theta = - m^2 . \sin m\theta;$$

so

$$f(D_\theta^2) . \cos m\theta = f(-m^2) . \cos m\theta,$$

and

$$f(D_\theta^2) . \sin m\theta = f(-m^2) . \sin m\theta.$$

GREGORY, *Examples.*

More generally, it is plain that

$$F(\nabla^2) . \{u_{m\sqrt{-1}} + u_{-m\sqrt{-1}}\} = F(-m^2) . \{u_{m\sqrt{-1}} + u_{-m\sqrt{-1}}\}.$$

SECTION II.—APPLICATION OF PRECEDING THEOREMS.

9. All partial differential equations represented by

$$A\nabla_\alpha z + B\nabla_\beta z + \dots = \Omega,$$

or, in the expanded form, by

$$\left.\begin{array}{c} A\left(x^\alpha D_x^\alpha z + \alpha x^{\alpha-1} y D_x^{\alpha-1} D_y z + \dfrac{\alpha(\alpha-1)}{1.2} x^{\alpha-2} y^2 D_x^{\alpha-2} D_y^2 z + ..\right) \\ + \\ B\left(x^\beta D_x^\beta z + \beta x^{\beta-1} y D_x^{\beta-1} D_y z + \dfrac{\beta(\beta-1)}{1.2} x^{\beta-2} y^2 D_x^{\beta-2} D_y^2 z + ..\right) \\ + \&c. \end{array}\right\} = \Omega,$$

where A, B, &c. are constants, and Ω is any algebraic function of the independent variables x and y, may obviously be transformed, by the third article of the previous section, into the shape

$$\left.\begin{array}{c} A\nabla(\nabla-1)\dots(\nabla-\alpha+1) \\ + \\ B\nabla(\nabla-1)\dots(\nabla-\beta+1) \\ + \&c. \end{array}\right\} . z = \Omega.$$

Consequently, the solutions of all such equations are given by the evaluation of the symbolic form

$$z = \frac{1}{F(\nabla)}.\Omega + \frac{1}{F(\nabla)}.0,$$

where

$$F(\nabla) = A\nabla(\nabla-1)\ldots(\nabla-\alpha+1) + B\nabla(\nabla-1)\ldots(\nabla-\beta+1).$$

Now the value of the first term in the solution is perfectly definite, and can be had at once by the second article of this Chapter. It will appear, moreover, that the number and character of the arbitrary functions in a solution, which are due solely to the second term, are unaffected by the number of independent variables which the equation may contain, and are dependent solely on its order.

In fact, when the roots of the equation

$$\nabla(\nabla-1)\ldots(\nabla-\alpha+1) + \frac{B}{A}\nabla(\nabla-1)\ldots(\nabla-\beta+1) + \&c. = 0.$$

are all real and unequal, the arbitrary portion of the solution is of the form

$$u_a + u_b + u_c + \ldots + u_i,$$

$a, b, c, \ldots i$ being the values of the roots, and u_a, u_b, u_c, &c. being homogeneous functions in x, y, &c. of the given degrees a, b, c, &c., respectively, but whose forms are arbitrary.*

---

* In connexion with the appearance of arbitrary functions in the solutions of partial differential equations, it is observed in Gregory's *Examples* (*Int. Cal.*, Chapter VI.), that, "as in the solution of ordinary differential equations we continually meet with expressions of the form Ce^{ax} ($= e^{ax}C$), so in partial differential equations we shall find expressions of the form

$$e^{aD_y \cdot x}\,\phi(y),$$

in which the arbitrary function takes the place of the arbitrary constant."

I cannot regard this view as at all satisfactory, but rather as an inversion of the real state of the case. In fact, as it appears to me, the terms including arbitrary constants are only particular cases of arbitrary functions, in which the variables are reduced to one.

When it contains p equal roots, whose common value is a, the form of the arbitrary portion of the solution is

$$u_a.(\log x + \log y)^{p-1} + v_a.(\log x + \log y)^{p-2} + \&c. + u_b + u_c + \&c.,$$

where u_a, v_a, &c. are different arbitrary homogeneous functions of the same degree.

Finally, when this equation contains pairs of imaginary roots, the form of the arbitrary portion of the solution is

$$u_{a+b\sqrt{-1}} + u_{a-b\sqrt{-1}} + u_c + \ldots + u_i.$$

Camb. and Dub. Math. Journal, 1851.

10. Thus it appears that the solution of an ordinary linear differential equation of the class represented by

$$Ax^\alpha D_x^\alpha y + Bx^\beta D_x^\beta y + \&c. = Mx^m + Nx^n + \&c.$$

being given, we can at once write down the solution of a partial differential equation of the class represented by

$$\left.\begin{array}{c} A(x^\alpha D_x^\alpha.z + \alpha x^{\alpha-1} y D_x^{\alpha-1} D_y.z + \&c.) \\ + \\ B(x^\beta D_x^\beta.z + \beta x^{\beta-1} y D_x^{\beta-1} D_y.z + \&c.) \\ + \&c. \end{array}\right\} = \Theta_m + \Theta_n + \&c.,$$

by substituting for Mx^m, Nx^n, &c. the corresponding known homogeneous functions Θ_m, Θ_n, &c., and by introducing for each term in the solution of the ordinary linear differential equation in which *an arbitrary constant* enters, such as $C_m x^m$, *a homogeneous function of the same degree, but of arbitrary form in x and y.*

11. It is obvious that the same method of solution as that just exhibited will apply to the class of partial differential equations represented by

$$A\,\square^a z + B\,\square^b z + .. + Tz = f(e^\phi, e^\psi, \sin\phi, \cos\phi, \sin\psi, \cos\psi),$$

where, as before,

$$\square = D_\phi + D_\psi.$$

In fact, this equation being thrown into the symbolic shape,

$$F(\square)z = \Sigma A_{m,n}\, e^{m\phi+n\psi},$$

where m, n may be positive or negative, fractional or integer, real or imaginary, the solution is

$$z = \Sigma A_{m,n} \frac{e^{m\phi+n\psi}}{F(m+n)} + \frac{1}{F(\square)} \cdot 0,$$

and where m or n is imaginary, we may restore the circular functions.

As regards the various forms which the arbitrary portion of the solution may assume, according as the roots of

$$F(\square) = 0$$

are all real and unequal, some equal, or some imaginary, they are given, respectively, by

$$u_a(e^{\phi}, e^{\psi}) + u_b(e^{\phi}, e^{\psi}) + \&c. + u_i(e^{\phi}, e^{\psi}), \qquad (\text{I.})$$

$$u_a(e^{\phi}, e^{\psi}).(\phi+\psi)^{p-1} + v_a(e^{\phi}, e^{\psi}).(\phi+\psi)^{p-2} + \&c. + u_i(e^{\phi}, e^{\psi}), \qquad (\text{II.})$$

$$u_{a+b\sqrt{-1}}(e^{\phi}, e^{\psi}) + u_{a-b\sqrt{-1}}(e^{\phi}, e^{\psi}) + \&c. + u_i(e^{\phi}, e^{\psi}). \qquad (\text{III.})$$

Camb. and Dub. Math. Journal, 1853.

12. By a similar reduction to that employed in the eleventh article of the Third Chapter, we can obtain at once the solution of the class of partial differential equations represented by

$$\left.\begin{array}{c} A\{(m+\lambda x)^{\alpha} D_x^{\alpha}.z + \alpha\,(m+\lambda x)^{\alpha-1}(n+\lambda y) D_x^{\alpha-1}D_y.z \\ + \dfrac{\alpha(\alpha-1)}{1.2}(m+\lambda x)^{\alpha-2}(n+\lambda y)^2 D_x^{\alpha-2}D_y^2.z + \&c.\} \\ + \\ B\{(m+\lambda x)^{\beta} D_x^{\beta}.z + \beta\,(m+\lambda x)^{\beta-1}(n+\lambda y) D_x^{\beta-1}D_y.z \\ + \dfrac{\beta(\beta-1)}{1.2}(m+\lambda x)^{\beta-2}(n+\lambda y)^2 D_x^{\beta-2}D_y^2.z + \&c.\} \\ + \&c. \end{array}\right\} = \Omega.$$

In fact, assume

$$m + \lambda x = \lambda x', \;\; n + \lambda y = \lambda y',$$

and the equation becomes of the form

$$F(\nabla') \cdot z = \Omega\,;$$

and, without any further transformation, we get the solution in terms of x' and y', for which the proper values being substituted from the assumptions stated, the solution required is had.

Philosophical Magazine, 1852.

Examples.—First Type.

(1) $$x^2 D_x^2 . z + 2xy D_x D_y . z + y^2 D_y^2 . z = \Theta_m + \Theta_n,$$

where Θ_m, Θ_n, are given homogeneous functions in x and y of the m^{th} and n^{th} degrees respectively.

The symbolic solution is

$$z = \frac{1}{\nabla(\nabla - 1)}(\Theta_m + \Theta_n) + \frac{1}{\nabla(\nabla - 1)}\, 0,$$

and the evaluated solution

$$z = \frac{\Theta_m}{m(m-1)} + \frac{\Theta_n}{n(n-1)} + u_0 + u_1.$$

(2) $$\left.\begin{array}{c} x^3 D_x^3 . u + y^3 D_y^3 . u + z^3 D_z^3 . u \\ + \\ 3\,(x^2 y D_x^2 D_y u + x^2 z D_x^2 D_z u + xy^2 D_x D_y^2 u + \&c.) \end{array}\right\} = \Phi_m + \Phi_n,$$

where, as before, Φ_m, Φ_n are given homogeneous functions in x, y, z.

The evaluated solution is

$$u = \frac{\Phi_m}{m(m-1)(m-2)} + \frac{\Phi_n}{n(n-1)(n-2)} + u_0 + u_1 + u_2.$$

(3) $$x^2 D_x^2 z + 2xy D_x D_y z + y^2 D_y^2 z - n\,(x D_x z + y D_y z) + nz = \Theta_m.$$

This is equivalent to

$$(\nabla - 1)(\nabla - n) \,.\, z = \Theta_m.$$

Consequently,

$$z = \frac{\Theta_m}{(m-1)(m-n)} + u_1 + u_n.^*$$

(4) $$x^2 D_x^2 . z + 2xy D_x D_y . z + y^2 D_y^2 . z + 3(x D_x . z + y D_y . z) + z = \frac{1}{(1-\Theta_1)^2},$$

where Θ_1 is a given homogeneous function of the first degree in x and y.

Then

$$(\nabla + 1)^2 . z = 1 + 2\Theta_1 + 3\Theta_1^2 + \&c.$$

and consequently

$$z = (1 + \frac{\Theta_1}{2} + \frac{\Theta_1^2}{3} + \&c.) + u_{-1} . (\log x + \log y) + v_{-1},$$

or

$$z = \log \left(\frac{1}{1-\Theta_1}\right)^{\frac{1}{\Theta_1}} + u_{-1} . (\log x + \log y) + v_{-1},$$

where u_{-1}, v_{-1} are different arbitrary homogeneous functions in x, y of the degree -1.

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* The solution, then, of

$$x^2 D_x^2 z + 2xy D_x D_y z + y^2 D_y^2 z - n(x D_x z + y D_y z) + nz = 0,$$

is, simply,

$$z = u_n + u_1.$$

If $n = -\dfrac{3}{m-1}$, it is shown by the Rev. Professor Jellett, in his masterly Treatise on the Calculus of Variations (*Dublin*, 1850, p. 253), that this value of z renders the integral

$$\iint (px + qy - z)^m \, dx \, dy\,;$$

or, as I would prefer writing it, the integral

$$\iint (x D_x z + y D_y z - z)^m \, dx \, dy:$$

a maximum or a minimum, within certain assigned limits. The investigation of the relation between the form of the differential solution as above given, and the form of its integral, together with the accidental discovery of the fundamental theorem of the Third Chapter, furnished the germs of the present Treatise. See Appendix A, *On the Calculus of Variations.*

(5) $x^2D_x^2.z + 2xyD_xD_y.z + y^2D_y^2.z - m(m-1)z = \Theta_n$.

Then

$$z = \frac{\Theta_n}{n(n-1) - m(m-1)} + u_\alpha + u_\beta,$$

where α and β are the roots of the equation

$$w^2 - w - m(m-1) = 0.$$

(6.) $$xD_x.w + yD_y.w + zD_z.w - aw = \frac{xy}{z}.$$

GREGORY, *Examples.*

Thrown into the symbolic shape, this equation becomes

$$(\nabla - a)w = \frac{xy}{z},$$

and therefore the solution is

$$w = \frac{1}{1-a}.\frac{xy}{z} + u_a,$$

where u_a is an arbitrary homogeneous function in x, y, z of the degree a.

More generally, the solution of

$$xD_x.w + yD_y.w + zD_z.w - aw = \frac{\Theta_m}{\Theta_n}$$

is

$$w = \frac{1}{(m-n)-a}.\frac{\Theta_m}{\Theta_n} + u_a.$$

(7) $x^nD_x^n.z + nx^{n-1}yD_x^{n-1}D_y.z + \frac{n.n-1}{1.2}x^{n-2}y^2D_x^{n-2}D_y^2.z + \&c. = 0.$

GREGORY, *Examples.*

The symbolic shape of this equation is, by the third article of this Chapter,

$$\nabla(\nabla - 1)(\nabla - 2)\ldots(\nabla - n + 1).z = 0.$$

Consequently its solution is, at once,

$$z = u_0 + u_1 + u_2 + \ldots + u_{n-1}.$$

More generally, the solution of

$$x^n D_x^n z + n x^{n-1} y D_x^{n-1} D_y z + \frac{n(n-1)}{1 \cdot 2} x^{n-2} y^2 D_x^{n-2} D_y^2 z + \ldots = \Theta_a + \Theta_b$$

is

$$z = \frac{\Theta_a}{a(a-1)\ldots(a-n+1)} + \frac{\Theta_b}{b(b-1)\ldots(b-n+1)} + u_0 + u_1 + \ldots + u_{n-1}.$$

The simplicity of the method exhibited in this last example, when compared with the artificial and laborious processes which have been employed for its solution, seems to illustrate, in a remarkable degree, the power of the Calculus of Operations as an instrument of integration, and the facility with which it admits of manipulation.

(8.) $$xD_x z + yD_y z = 2xy\sqrt{(a^2 - z^2)}.$$

GREGORY, *Examples.*

This assumes the symbolic shape

$$\nabla \cdot \sin^{-1} \frac{z}{a} = 2xy,$$

and its solution is therefore

$$\frac{z}{a} = \sin(xy + u_0).$$

More generally, the solution of

$$xD_x w + yD_y w + zD_z w = m\Theta_m \cdot \sqrt{(a^2 - w^2)}$$

is

$$\frac{w}{a} = \sin(\Theta_m + u_0).$$

(9.) $$axD_x w + byD_y w + czD_z w - nw = 0.$$

This equation is reducible to

$$x^{\frac{1}{a}} D_{x^{\frac{1}{a}}} \cdot w + y^{\frac{1}{b}} D_{y^{\frac{1}{b}}} \cdot w + z^{\frac{1}{c}} D_{z^{\frac{1}{c}}} \cdot w - nw = 0,$$

whence

$$w = u_n(x^{\frac{1}{a}}, y^{\frac{1}{b}}, z^{\frac{1}{c}}).$$

And, generally, since

$$F(xD_x, yD_y, zD_z, \&c.) \,.\, Ax^m y^n z^p \ldots = F(m, n, p, \&c.) \,.\, Ax^m y^n z^p \ldots$$

it appears that the solution of all partial differential equations of the type

$$(axD_x + byD_y + \&c. - \alpha)\,(a'xD_x + b'yD_y + \&c. - \beta) \ldots u = V,$$

where

$$V = \Sigma Ax^m y^n \ldots$$

is given by

$$u = \begin{cases} \Sigma \dfrac{Ax^m y^n \ldots}{(am + bn + \&c. - \alpha)\,(a'm + b'n + \&c. - \beta) \ldots} \\ \quad + u_\alpha(x^{\frac{1}{a}}, y^{\frac{1}{b}}, \&c.) + u_\beta(x^{\frac{1}{a}}, y^{\frac{1}{b}}, \&c.) \end{cases}$$

CURTIS, *Camb. and Dub. Math. Journal*, 1854.

More generally, the solution of all partial differential equations of the type

$$F(xD_x, yD_y, zD_z, \&c.) \,.\, u = \Sigma Ax^m y^n z^p \ldots$$

is given by

$$u = \Sigma \frac{Ax^m y^n z^p \ldots}{F(m, n, p, \&c.)} + \frac{1}{F(xD_x, yD_y, zD_z, \&c.)}\, 0.$$

(10.) $x^2 D_x^2 z - 2xy D_x D_y z + y^2 D_y^2 z + xD_x z + yD_y z - nz = 0.$

This equation is reducible to

$$(xD_x - yD_y)^2 z - nz = 0,$$

whence

$$z = u_{\sqrt{n}}\left(x, \frac{1}{y}\right) + u_{-\sqrt{n}}\left(x, \frac{1}{y}\right).$$

(11.) $$x^2D_x^2z - y^2D_y^2z + xD_xz - yD_yz = 0.$$

GREGORY, *Examples.*

This equation is easily reducible to

$$(xD_x + yD_y) \,.\, (xD_x - yD_y) \,.\, z = 0,$$

whence

$$z = u_0(x, y) + v_0\left(x, \frac{1}{y}\right).$$

(12.) $$D_x^2z - a^2D_y^2z + 2abD_xz + 2a^2bD_yz = 0.$$

GREGORY, *Examples.*

This equation may be obviously thrown into the form

$$(D_x - D_{\frac{y}{a}} + 2ab)(D_x + D_{\frac{y}{a}}) \,.\, z = 0,$$

and the solution is

$$z = u_{-2ab}(e^x, e^{-\frac{y}{a}}) + u_0(e^x, e^{\frac{y}{a}})$$

(13.) $$D_x^2z - 2aD_xD_yz + a^2D_y^2z = 0.$$

GREGORY, *Examples.*

This equation is equivalent to

$$(D_x - aD_y)^2 \,.\, z = 0,$$

and consequently the solution is

$$z = \left(x - \frac{y}{a}\right) \,.\, u_0(e^x, e^{-\frac{y}{a}}) + v_0(e^x, e^{-\frac{y}{a}});$$

or, as it may be written,

$$z = \left(x - \frac{y}{a}\right) \,.\, \phi\left(x + \frac{y}{a}\right) + \psi\left(x + \frac{y}{a}\right).$$

(14.) $$D_t^2z = \frac{d^2z}{dt^2} = a^2D_x^2z,$$

the equation which represents the motion of vibrating chords, and of the pulses produced by a disturbance in a fine cylindrical column of air.

This equation is equivalent to

$$(D_t - aD_x)\,(D_t + aD_x) \,.\, z = 0,$$

and its solution is, at once,

$$z = u_0\left(e^t,\ e^{-\frac{x}{a}}\right) + v_0\left(e^t,\ e^{+\frac{x}{a}}\right),$$

the ordinary form of which is

$$z = \Phi(x + at) + \Psi(x - at).$$

(15.) $D_t^2 z = \dfrac{d^2 z}{dt^2} = a^2(D_x^2 z + 2D_x D_y z + D_y^2 z).$

This equation is equivalent to

$$\{D_t - a(D_x + D_y)\}.\{D_t + a(D_x + D_y)\}.z = 0,$$

the solution of which is

$$z = u_0\left(e^t,\ e^{-\frac{x}{a}},\ e^{-\frac{y}{a}}\right) + v_0\left(e^t,\ e^{+\frac{x}{a}},\ e^{+\frac{y}{a}}\right),$$

the more ordinary form of which would be

$$z = \Phi(x + at,\ y + at) + \Psi(x - at,\ y - at).$$

In the investigation of the physical interpretation of the differential equation, it must be observed that, although, in plane geometry,

$$D_x^2 + D_y^2$$

and, in geometry of three dimensions,

$$D_x^2 + D_y^2 + D_z^2$$

are unaffected by transformation of coordinates; or are, in fact, reproduced; this does not hold in the case of

$$(D_x + D_y)^2, \quad \text{and} \quad (D_x + D_y + D_z)^2.$$

Examples.—Second Type.

(1.) $D_\phi z + D_\psi z + az = e^{m\phi + n\psi},$

$$z = \frac{e^{m\phi + n\psi}}{m + n + a} + u_{-a}(e^\phi,\ e^\psi).$$

(2.) $$D_\phi z + D_\psi z - az = e^{m\phi+n\psi}\cos(r\phi+s\psi).$$

Reduced to the shape prescribed, this becomes

$$(\square - a) \, . \, z = \tfrac{1}{2}\left\{e^{(m+r\sqrt{-1})\phi+(n+s\sqrt{-1})\psi} + e^{(m-r\sqrt{-1})\phi+(n-s\sqrt{-1})\psi}\right\},$$

and the solution is

$$z = \tfrac{1}{2}\left\{\frac{e^{(m+r\sqrt{-1})\phi+(n+s\sqrt{-1})\psi}}{m+n-a+(r+s)\sqrt{-1}} + \frac{e^{(m-r\sqrt{-1})\phi+(n-s\sqrt{-1})\psi}}{m+n-a-(r+s)\sqrt{-1}}\right\} + u_a(e^\phi, e^\psi);$$

or, restoring the circular function,

$$z = e^{m\phi+n\psi}.\,\frac{(m+n-a)\cos(r\phi+s\psi)+(r+s)\sin(r\phi+s\psi)}{(m+n-a)^2+(r+s)^2} + u_a(e^\phi, e^\psi).$$

(3.) $$D_\phi^2 z + 2D_\phi D_\psi z + D_\psi^2 z + a^2 z = \cos(m\phi+n\psi).$$

Reduced to the shape prescribed, this becomes

$$(\square + a\sqrt{-1}) \, . \, (\square - a\sqrt{-1}) \, . \, z = \tfrac{1}{2}\left\{e^{(m\phi+n\psi)\sqrt{-1}} + e^{-(m\phi+n\psi)\sqrt{-1}}\right\},$$

and the solution is

$$z = \tfrac{1}{2}\left\{\frac{e^{(m\phi+n\psi)\sqrt{-1}}}{a^2-(m+n)^2} + \frac{e^{-(m\phi+n\psi)\sqrt{-1}}}{a^2-(m+n)^2}\right\} + u_{-a\sqrt{-1}}(e^\phi, e^\psi) + u_{a\sqrt{-1}}(e^\phi, e^\psi);$$

or, restoring the circular function,

$$z = \frac{\cos(m\phi+n\psi)}{a^2-(m+n)^2} + u_{-a\sqrt{-1}}(e^\phi, e^\psi) + u_{a\sqrt{-1}}(e^\phi, e^\psi).$$

It may be observed that pairs of conjugate arbitrary functions, such as those just exhibited, are imaginary only in appearance, being equivalent to

$$\cos\tfrac{1}{2}a(\phi+\psi) \, . \, \Phi(\phi-\psi) + \sin\tfrac{1}{2}a(\phi+\psi) \, . \, \Psi(\phi-\psi).$$

(4.) $$D_\phi^2 z + 2D_\phi D_\psi z + D_\psi^2 z - 2a(D_\phi z + D_\psi z) + a^2 z = \sin(m\phi+n\psi).$$

Reduced to the symbolic shape, this becomes

$$(\square - a)^2 . \, z = \frac{1}{2\sqrt{-1}}\left\{e^{(m\phi+n\psi)\sqrt{-1}} - e^{-(m\phi+n\psi)\sqrt{-1}}\right\};$$

and consequently the solution is, at once,

$$z = \left\{ \begin{array}{l} \dfrac{1}{2\sqrt{-1}} \left\{ \dfrac{e^{(m\phi+n\psi)\sqrt{-1}}}{\{(m+n)\sqrt{-1}-a\}^2} - \dfrac{e^{-(m\phi+n\psi)\sqrt{-1}}}{\{(m+n)\sqrt{-1}+a\}^2} \right\} \\ \qquad + u_a(e^\phi, e^\psi) \, . \, (\phi+\psi) + v_a(e^\phi, e^\psi); \end{array} \right.$$

or, restoring the circular functions,

$$z = \left\{ \begin{array}{l} \dfrac{\{a^2 - (m+n)^2\} \sin (m\phi + n\psi) + 2a\,(m+n) \cos (m\phi + n\psi)}{\{a^2 + (m+n)^2\}^2} \\ \qquad + u_a(e^\phi, e^\psi) \, . \, (\phi+\psi) + v_a(e^\phi, e^\psi). \end{array} \right.$$

13. It is indispensable that we should discuss an exceptional case, which will sometimes occur in the employment of the present, as of any other, method of integration.

This arises from the circumstance that the inverse process may generate an infinite coefficient, and can be illustrated by the partial differential equation

$$xD_xz + yD_yz - az = \Theta_m.$$

The solution of this equation, as given by our method, is

$$z = \frac{\Theta_m}{m-a} + u_a;$$

in which, when $a = m$, the first term becomes infinite.

To clear away this difficulty, assume in the general solution

$$u_a = v_a - \frac{\Theta_a}{m-a},$$

which gives

$$z = \frac{\Theta_m - \Theta_a}{m-a} + v_a.$$

This becomes indeterminate when $a = m$; therefore, differentiating with respect to a both numerator and denominator, and remembering that

$$\Theta_a = x^a f\left(\frac{y}{x}\right) = y^a F\left(\frac{x}{y}\right),$$

and, therefore,

$$\Theta_a = \frac{x^a f\left(\frac{y}{x}\right) + y^a F\left(\frac{x}{y}\right)}{2},$$

we find for the solution, in the exceptional case,

$$z = \Theta_m \frac{\log x + \log y}{2} + v_m.$$

By an obvious extension it appears that the solution of

$$xD_x w + yD_y w + zD_z w - mw = \Theta_m$$

is

$$w = \Theta_m \frac{\log x + \log y + \log z}{3} + v_m;$$

which of course can be generalized for n independent variables.

Examples.

(1.) $$xD_x z + yD_y z = c.$$

The solution is

$$z = c\frac{\log x + \log y}{2} + u_0.$$

(2.) $$aD_\phi z + bD_\psi z = c.$$

The solution is

$$z = \frac{c}{2}\left(\frac{\phi}{a} + \frac{\psi}{b}\right) + u_0\,(e^{\frac{\phi}{a}}, e^{\frac{\psi}{b}}).$$

(3.) $$x^2 D_x^2 z - y^2 D_y^2 z = xy.$$

GREGORY, *Examples.*

This equation is reducible to

$$(xD_x + yD_y - 1)\,(xD_x - yD_y)\,.\,z = xy,$$

whence

$$z = \frac{1}{xD_x - yD_y}\,.\,xy + u_1(x, y) + u_0\left(x, \frac{1}{y}\right),$$

or

$$z = xy\,\frac{lx + l\dfrac{1}{y}}{2} + u_1(x,y) + u_0\left(x, \frac{1}{y}\right) = xy\log\left(\frac{x}{y}\right)^{\frac{1}{2}} + u_1(x,y) + u_0\left(x, \frac{1}{y}\right).$$

(4.) $$D_\theta y - my = Me^{m\theta}.$$

The solution is

$$y = Me^{m\theta}.\,\theta + Ce^{m\theta}.$$

(5.) $$D_\theta^2 y + m^2 y = \cos m\theta.$$

The general solution of this equation is, in its primary form,

$$y = \tfrac{1}{2}\left(\frac{e^{m\theta\sqrt{-1}}}{2m\sqrt{-1}}.\theta - \frac{e^{-m\theta\sqrt{-1}}}{2m\sqrt{-1}}.\theta\right) + C_1 e^{m\theta\sqrt{-1}} + C_1' e^{-m\theta\sqrt{-1}};$$

and, restoring the circular function in the first term, the solution is

$$y = \frac{\theta}{2m}.\sin m\theta + C_2 \cos m\theta + C_2' \sin m\theta.$$

Equations of the type exhibited in this example frequently occur in the application of analysis to Physics: as, for instance, in the Lunar Theory, and in that of the perturbed motion of pendulums. In these cases, the independent variable is the time, and it is known that the value of y is then not simply periodic, but increases indefinitely with the time.

(6.) $$D_\phi^2.\,z + 2D_\phi D_\psi.\,z + D_\psi^2.\,z + (m+n)^2 z = \cos(m\phi + n\psi).$$

The solution is

$$z = \frac{\phi + \psi}{4(m+n)}.\sin(m\phi + n\psi) + u_{(m+n)\sqrt{-1}}(e^\phi, e^\psi) + u_{-(m+n)\sqrt{-1}}(e^\phi, e^\psi).$$

CHAPTER V.

INTEGRATION OF VARIOUS ADDITIONAL CLASSES OF DIFFERENTIAL EQUATIONS, TOTAL AND PARTIAL.

1. THERE is an extensive class of differential equations, for the solution of which various methods have been proposed, but all more or less embarrassing to the student. Amongst these methods the most usual is that of Integration by Series, in which an expression is assumed for the dependent variable in terms of the independent variable with indeterminate coefficients and indices, and these are subsequently determined by substitution, in the given equation, of the expression so assumed. This method is, from its indirect and tentative character, unsatisfactory to the student, in actual practice unpleasantly tedious, and, as a process, unsusceptible of generalization.

Let the following examples be proposed for solution :—

$$D^2y + ax^ny = 0, \qquad \text{(I.)}$$

GREGORY, *Examples*, p. 340.

$$xD^2y + Dy + y = 0, \qquad \text{(II.)}$$

GREGORY, *Examples*, p. 343.

$$D^2y + \frac{2}{x}Dy + \left(n^2 - \frac{2}{x^2}\right)y = 0. \qquad \text{(III.)}$$

GREGORY, *Examples*, p. 313.

If we multiply the first equation by x^2, the second by x, and the third by x^2, they become, respectively,

$$xD(xD-1).y + ax^{n+2}y = 0, \qquad \text{(I$'$.)}$$

$$(xD)^2.y + xy = 0, \qquad \text{(II$'$.)}$$

$$(xD-1)\,(xD+2).y + n^2x^2y = 0 \qquad \text{(III$'$.)}$$

2. Now the common type of these equations is

$$F(xD)y + Mx^m y = 0;$$

or, more generally,

$$F(xD)y + Mx^m y = X.$$

Let us suppose that

$$X = \Sigma A_a x^a,$$

and proceed to solve the more general type.

Operating on both sides with the inverse of $F(xD)$, we get

$$y + \frac{1}{F(xD)} Mx^m . y = \frac{1}{F(xD)} X + \frac{1}{F(xD)} 0,$$

or

$$y + \frac{1}{F(xD)} Mx^m . y = \Sigma \frac{A_a x^a}{F(a)} + \Sigma C_a x^a,$$

where the last term is the ordinary complementary function, upon the supposition that all the roots of

$$F(u) = 0$$

are real and unequal, and in which, if any modification should arise from the existence of equal or imaginary roots, the generality of the method is not affected.

Now, dissecting the operator in the left-hand member from its subject, and operating with the expansion of its inverse upon the right-hand member, we get

$$y = \left\{ \begin{array}{l} \left\{1 - \frac{1}{F(xD)} Mx^m + \frac{1}{F(xD)} Mx^m \frac{1}{F(xD)} Mx^m - \&c. \right\} \Sigma \frac{A_a x^a}{F(a)} \\ \qquad\qquad\qquad\qquad + \\ \left\{1 - \frac{1}{F(xD)} Mx^m + \frac{1}{F(xD)} Mx^m \frac{1}{F(xD)} Mx^m - \&c. \right\} \Sigma C_a x^a, \end{array} \right.$$

or the required solution is at once

$$y = \begin{cases} \Sigma \dfrac{A_a x^a}{F(a)} \{1 - \dfrac{Mx^m}{F(a+m)} + \dfrac{(Mx^m)^2}{F(a+2m)\, F(a+m)} - \&c.\} \\ \qquad + \\ \Sigma C_\alpha x^\alpha \{1 - \dfrac{Mx^m}{F(\alpha+m)} + \dfrac{(Mx^m)^2}{F(\alpha+2m)\, F(\alpha+m)} - \&c.\}, \end{cases}$$

the coefficients within brackets in the first and second great terms differing merely in the substitution of α for a. When $M = 0$, it is evident that we fall back upon a class of equations already discussed in the Third Chapter.

That the method admits of easy generalization can be readily now shown. For let the partial differential equation to be solved be represented by the type

$$F(\nabla) z + \Theta_m z = \Omega,$$

where Θ_m is an homogeneous function in x and y of the m^{th} degree, Ω a mixed function of x, y, and ∇ the symbol

$$xD_x + yD_y.$$

Operate with

$$\frac{1}{F(\nabla)},$$

having broken up Ω into sets of homogeneous functions; there results

$$\{1 + \frac{1}{F(\nabla)} \Theta_m\} z = \Sigma \frac{\Theta_a}{F(a)} + \Sigma u_\alpha,$$

where u_α is a homogeneous function of the given degree α, but arbitrary in form; and operating on both sides with the inverse of

$$\{1 + \frac{1}{F(\nabla)} \Theta_m\},$$

we get at once, as before,

$$z = \begin{cases} \Sigma \dfrac{\Theta_a}{F(a)} \{1 - \dfrac{\Theta_m}{F(a+m)} + \dfrac{\Theta_m^2}{F(a+2m)\, F(a+m)} - \&c.\} \\ \qquad + \\ \Sigma u_\alpha \{1 - \dfrac{\Theta_m}{F(\alpha+m)} + \dfrac{\Theta_m^2}{F(\alpha+2m)\, F(\alpha+m)} - \&c.\} \end{cases}$$

Camb. and Dub. Math. Journal, 1854.

3. Let us now apply this method to the first example proposed, in its modified form, namely,

$$xD(xD-1)y + ax^{n+2}y = 0.$$

Operating on both sides of this equation with

$$\frac{1}{xD(xD-1)},$$

we get

$$y + \frac{1}{xD(xD-1)}ax^{n+2}y = C_1x + C_2;$$

whence, putting for conciseness $n+2=m$,

$$y = \left\{1 - \frac{1}{xD(xD-1)}ax^m + \frac{1}{xD(xD-1)}ax^m\frac{1}{xD(xD-1)}ax^m - \&c.\right\}(C_1x + C_2),$$

or

$$y = \left\{\begin{array}{c} C_1x\left\{1 - \dfrac{ax^m}{(m+1)m} + \dfrac{(ax^m)^2}{(2m+1)\,2m(m+1)m} - \&c.\right\} \\ + \\ C_0\left\{1 - \dfrac{ax^m}{m(m-1)} + \dfrac{(ax^m)^2}{2m(2m-1)\,m(m-1)} - \&c.\right\} \end{array}\right.$$

and the solution of the given equation is had by replacing for m its value $n+2$.

When $n=-2$, these series fail, but the solution is seen, by the Third Chapter, to be, in this case,

$$y = Ax^{\alpha} + Bx^{\beta},$$

where α and β are the roots of the quadratic

$$p(p-1) + a = 0.$$

In any other conceivable cases of failure of the above series, the solution can be had with equal facility.

4. If it be proposed to integrate the partial differential equation

$$x^2D_x^2z + 2xyD_xD_yz + y^2D_y^2z + \Theta_m.z = 0,$$

which we know to be reducible to the shape

$$\nabla(\nabla - 1)z + \Theta_m z = 0,$$

we proceed to operate on both sides of this form with

$$\frac{1}{\nabla(\nabla - 1)},$$

which gives

$$z + \frac{1}{\nabla(\nabla - 1)}\Theta_m z = u_1 + u_0,$$

from which we derive, as above,

$$z = \begin{cases} u_1\left\{1 - \dfrac{\Theta_m}{(m+1)m} + \dfrac{\Theta_m^2}{(2m+1)\,2m(m+1)\,m} - \&c.\right\} \\ \qquad + \\ u_0\left\{1 - \dfrac{\Theta_m}{m(m-1)} + \dfrac{\Theta_m^2}{2m(2m-1)\,m(m-1)} - \&c.\right\} \end{cases}$$

5. Let us now apply the method to the second example proposed, in its modified form, namely,

$$(xD)^2 . y + xy = 0,$$

premising that its susceptibility of some such method of integration was suggested in the year 1847 by the Rev. Professor Graves.

Operating on both sides of this equation with $\frac{1}{(xD)^2}$, we get

$$y + \frac{1}{(xD)^2}xy = C_1 \log x + C_2;$$

whence

$$y = \left\{1 - \frac{1}{(xD)^2}x + \frac{1}{(xD)^2}x\frac{1}{(xD)^2}x - \&c.\right\}(C_1 \log x + C_2),$$

or

$$y = \begin{cases} C_1\left\{1 - x\dfrac{1}{(1+xD)^2} + x^2\dfrac{1}{(2+xD)^2\,(1+xD)^2} - \&c.\right\}\log x, \\ \qquad + \\ C_2\left\{1 - \dfrac{x}{1^2} + \dfrac{x^2}{1^2.2^2} - \dfrac{x^3}{1^2.2^2.3^2} + \dfrac{x^4}{1^2.2^2.3^2.4^2} - \&c.\right\}. \end{cases}$$

But

$$\frac{1}{(1+xD)^2}\log x = \frac{1}{1^2}\left(1-\frac{2}{1}xD\right)\log x = \frac{1}{1^2}(\log x - 2),$$

$$\frac{1}{(2+xD)^2(1+xD)^2}\log x = \frac{1}{1^2.2^2}\left\{1-2\left(\frac{1}{1}+\frac{1}{2}\right)xD\right\}\log x$$

$$= \frac{1}{1^2.2^2}\left\{\log x - 2\left(\frac{1}{1}+\frac{1}{2}\right)\right\}, \text{ \&c.}$$

Hence, finally,

$$y = \begin{cases} (C_1 \log x + C_2)\left\{1 - \dfrac{x}{1^2} + \dfrac{x^2}{1^2.2^2} - \dfrac{x^3}{1^2.2^2.3^2} + \text{\&c.}\right\} \\ + \\ 2C_1\left\{\dfrac{1}{1^2}x - \dfrac{1}{1^2.2^2}\left(\dfrac{1}{1}+\dfrac{1}{2}\right)x^2 + \dfrac{1}{1^2.2^2.3^2}\left(\dfrac{1}{1}+\dfrac{1}{2}+\dfrac{1}{3}\right)x^3 - \text{\&c.}\right\} \end{cases}$$

6. If the equation to be integrated had been

$$xD^2y + Dy + Mx^{m-1}y = 0,$$

its solution is obviously had by the same general method, and is

$$y = \begin{cases} C_1\left\{1 - Mx^m\dfrac{1}{(m+xD)^2} + (Mx^m)^2\dfrac{1}{(2m+xD)^2(m+xD)^2} - \text{\&c.}\right\}\log x \\ + \\ C_2\left\{1 - \dfrac{Mx^m}{m^2} + \dfrac{(Mx^m)^2}{m^2.(2m)^2} - \dfrac{(Mx^m)^3}{m^2.(2m)^2.(3m)^2} + \text{\&c.}\right\}; \end{cases}$$

or

$$y = \begin{cases} (C_1\log x + C_2)\left\{1 - \dfrac{Mx^m}{m^2} + \dfrac{(Mx^m)^2}{m^2.(2m)^2} - \dfrac{(Mx^m)^3}{m^2.(2m)^2.(3m)^2} + \text{\&c.}\right\} \\ + \\ 2C_1\left\{\dfrac{Mx^m}{m^3} - \dfrac{m(2+1)\,(Mx^m)^2}{m^3(2m)^3} + \dfrac{m^2(6+3+2)(Mx^m)^3}{m^3(2m)^3(3m)^3} - \text{\&c.}\right\} \end{cases}$$

7. Similarly, if it be proposed to integrate the partial differential equation

$$\nabla_2 . z + \nabla . z + \Theta_m . z = 0,$$

where

$$\nabla_2 = x^2 D_x^2 + 2xyD_xD_y + y^2D_y^2,$$

a corresponding reduction gives

$$z + \frac{1}{\nabla^2}\Theta_m . z = u_0 \frac{\log x + \log y}{2} + v_0,$$

and the symbolic solution is

$$z = (1 - \frac{1}{\nabla^2}\Theta_m + \frac{1}{\nabla^2}\Theta_m \frac{1}{\nabla^2}\Theta_m - \&c.)\ (u_0 \frac{\log x + \log y}{2} + v_0).$$

Hence

$$z = \begin{cases} u_0\{1 - \Theta_m \dfrac{1}{(m+\nabla)^2} + \Theta_m^2 \dfrac{1}{(m+\nabla)^2(2m+\nabla)^2} - \&c.\}\dfrac{\log x + \log y}{2}, \\ \qquad + \\ v_0\{1 - \dfrac{\Theta_m}{m^2} + \dfrac{\Theta_m^2}{m^2.(2m)^2} - \dfrac{\Theta_m^3}{m^2.(2m)^2(3m)^2} + \&c.\}; \end{cases}$$

and finally,

$$z = \begin{cases} (u_0 \dfrac{\log x + \log y}{2} + v_0)\{1 - \dfrac{\Theta_m}{m^2} + \dfrac{\Theta_m^2}{m^2.(2m)^2} - \dfrac{\Theta_m^3}{m^2.(2m)^2.(3m)^2} + \&c.\} \\ \qquad + \\ 2u_0\left\{\dfrac{1}{m^3}\Theta_m - \dfrac{m(2+1)}{m^3.(2m)^3}\Theta_m^2 + \dfrac{m^2(6+3+2)}{m^3.(2m)^3.(3m)^3}\Theta_m^3 - \&c.\right\}. \end{cases}$$

8. Proceeding now to apply the same method to the modified form of the third example,

$$(xD - 1)\ (xD + 2)\,y + n^2x^2.y = 0,$$

we get

$$y + \frac{1}{(xD-1)\ (xD+2)}\, n^2x^2.y = C_1x + \frac{C_2}{x^2}.$$

Consequently the solution is given by

$$y = \{1 - \frac{1}{(xD-1)\ (xD+2)}\, n^2x^2$$

$$+ \frac{1}{(xD-1)\ (xD+2)}\, n^2x^2 \frac{1}{(xD-1)\ (xD+2)}\, n^2x^2 - \&c.\}\left(C_1x + \frac{C_2}{x^2}\right),$$

or

$$y = \begin{cases} C_1 x \{1 - \dfrac{(nx)^2}{2.5} + \dfrac{(nx)^4}{2.4.5.7} - \&c.\} \\ \qquad\qquad + \\ \dfrac{C_2}{x^2}\{1 + \dfrac{(nx)^2}{1.2} - \dfrac{(nx)^4}{1.2.4} + \&c.\} \end{cases}$$

For the condensation of such series as this, the following general method has been kindly suggested by Mr. Curtis :—

Convert each of the great terms in the right-hand member into the shape

$$\Sigma f(m)\, u_m;$$

and since this is known to be equivalent to

$$f(\nabla)\,\Sigma u_m,$$

the question is reduced to the condensation of

$$\Sigma u_m,$$

which is, in general, practicable by known methods.

Thus, the right-hand member of the above serial form is reducible to

$$\frac{C_1'}{x^2}\{2\frac{(nx)^3}{\overline{3}} - 4\frac{(nx)^5}{\overline{5}} + 6\frac{(nx)^7}{\overline{7}} - \&c.\} + \frac{C_2'}{x^2}\{1 + 1\frac{(nx)^2}{\overline{2}} - 3\frac{(nx)^4}{\overline{4}} + \&c.\}$$

or

$$-\frac{C_1'}{x^2}(xD - 1)\{nx - \frac{(nx)^3}{\overline{3}} + \frac{(nx)^5}{\overline{5}} - \frac{(nx)^7}{\overline{7}} + \&c.\}$$

$$-\frac{C_2'}{x^2}(xD - 1)\{1 - \frac{(nx)^2}{\overline{2}} + \frac{(nx)^4}{\overline{4}} - \frac{(nx)^6}{\overline{6}} + \&c.\}$$

or

$$-\frac{1}{x^2}(xD - 1)\{C_1' \sin nx + C_2' \cos nx\} = -\frac{1}{x^2}(xD - 1)A\cos(nx + B).$$

The solution of the equation consequently is

$$y = \frac{A}{x^2}\cos(nx + B) + \frac{nA}{x}\sin(nx + B).$$

9. The student will find no difficulty in applying this method to the integration of the equations

$$D^2y - \frac{c^2}{x^4}y = 0,$$

GREGORY, *Examples*, p. 344.

$$D^2y + \frac{c^2}{x^4}y = 0,$$

GREGORY, *Examples*, p. 345.

$$D^2y + c^2y = \frac{6y}{x^2},$$

GREGORY, *Examples*, p. 347.

The primary forms of the solutions of these equations are, respectively,

$$y = \begin{cases} C_0\{1 + \frac{1}{2.3}\left(\frac{c}{x}\right)^2 + \frac{1}{2.3.4.5}\left(\frac{c}{x}\right)^4 + \&c.\} \\ \qquad + \\ C_1x\{1 + \frac{1}{1.2}\left(\frac{c}{x}\right)^2 + \frac{1}{1.2.3.4}\left(\frac{c}{x}\right)^4 + \&c.\} \end{cases}$$

$$y = \begin{cases} C_0\{1 - \frac{1}{2.3}\left(\frac{c}{x}\right)^2 + \frac{1}{2.3.4.5}\left(\frac{c}{x}\right)^4 - \&c.\} \\ \qquad + \\ C_1x\{1 - \frac{1}{1.2}\left(\frac{c}{x}\right)^2 + \frac{1}{1.2.3.4}\left(\frac{c}{x}\right)^4 - \&c.\} \end{cases}$$

$$y = \begin{cases} C_1x^3\{1 - \frac{(cx)^2}{2.7} + \frac{(cx)^4}{2.4.7.9} - \&c.\} \\ \qquad + \\ C_2x^{-2}\{1 + \frac{(cx)^2}{2.3} + \frac{(cx)^4}{1.2.3.4} - \&c.\}. \end{cases}$$

These forms of solution are evidently susceptible of reduction, and, confining our attention for a moment to the first, it is evidently equivalent to

$$y = \frac{C_0 x}{2c}\left(e^{\frac{c}{x}} - e^{-\frac{c}{x}}\right) + \frac{C_1 x}{2}\left(e^{\frac{c}{x}} + e^{-\frac{c}{x}}\right),$$

or

$$y = x\left(Ae^{\frac{c}{x}} + Be^{-\frac{c}{x}}\right).$$

Similarly the solution of the second equation is reducible to the form

$$y = x\left(A' \sin\frac{c}{x} + B' \cos\frac{c}{x}\right).$$

Employing the method proposed by Mr. Curtis, the solution of the third equation is obviously reducible to the form

$$\frac{C_1'}{(cx)^2}\left\{\frac{(cx)^5}{1.3.5} - \frac{(cx)^7}{1.2.3.5.7} + \frac{(cx)^9}{1.2.3.4.5.7.9} - \&c.\right\}$$
$$+$$
$$\frac{C_2'}{(cx)^2}\left\{3 + \frac{(cx)^2}{1.2} + \frac{(cx)^4}{1.2.4} - \frac{(cx)^6}{1.2.4.6} + \&c.\right\}.$$

Now the general term in each of the above series is

$$\pm\frac{(cx)^m}{1.2 \ldots . (m-4)(m-2)m} = \pm(m-3)(m-1)\frac{(cx)^m}{\overline{m}},$$

and the solution is

$$y = \frac{1}{(cx)^2}(xD-3)(xD-1)(C_1' \sin cx + C_2' \cos cx),$$

or

$$y = A\left\{\left(1 - \frac{3}{(cx)^2}\right)\sin(cx+B) + \frac{3}{cx}\cos(cx+B)\right\}.$$

10. It must be allowed that although, in the method of integration here put forward, no mathematical artifice is employed, and although the result appears to be obtained in the most direct manner, yet the ultimate reduction of the solution to its most compact form often demands considerable analytical skill.

The following important remark has been made by Gregory (*Examples*, p. 314) in connexion with the integration of equations with variable coefficients, and more particularly the second example of this Chapter :—"After all, however, when these equations are of the second or higher orders, the number of cases in which they are integrable is very limited, and there seems to be no great prospect of the number being much increased. A little consideration will point out the reason of this. When we speak of an equation being integrable, we mean that the dependent variable can be expressed in terms of the independent variable by means of a finite series of functions of that quantity, the forms of such functions being limited to those known as algebraical and transcendental. Now it has been seen that the simplest forms of differential equations involve the highest transcendents which we recognise as known functions, such as e^{ax} or $\cos nx$; and it is to be expected, that when the equations become more complicated, their integrals must involve higher transcendents to which we have not affixed particular names, and which we do not look on as known forms. This, indeed, is found to be the case, as, for example, in the equation

$$xD^2y + Dy + y = 0,$$

which in its integral involves the transcendent

$$\psi(x) = 1 - \frac{x}{1^2} + \frac{x^2}{1^2.2^2} - \frac{x^3}{1^2.2^2.3^2} + \&c.$$

"It would appear, then, that before we are able to make any further progress in the solution of differential equations, we must create new transcendents in the same way as the ordinary transcendents e^x, $\cos x$, $\log x$, &c. have been created; we must study their properties, and endeavour to express the integrals of differential equations by means of them. The first part of this task has for some time past occupied the attention of mathematicians, and great progress has been made in it, though much still remains to be done. The second part has also been the object of study, though not to the same extent as the other; and several mathematicians have applied themselves with success to the expression of the integrals of differential equations by means of definite integrals which are the representatives of new transcendents. Thus, for instance, in the case cited above, the transcendent

$$1 - \frac{x}{1^2} + \frac{x^2}{1^2.2^2} - \frac{x^3}{1^2.2^2.3^3} + \&c. = \frac{1}{\pi}\int_0^\pi d\theta \cos(2\sin\theta x^{\frac{1}{2}}).$$

Examples of such integrals will be found in Crelle's Journal, vol. x. p. 92; vol. XII. p. 144; vol. XVII. p. 363."

Now it appears to me that, until evaluated, the integral

$$\frac{C_2}{\pi}\int_0^\pi d\theta \cos(2\sin\theta x^{\frac{1}{2}})$$

must be considered to be quite as symbolic as the equivalent

$$\{1 + \frac{1}{(xD)^2} x\}^{-1}. C_2.$$

Indeed, we may regard all symbolic condensations, *as well as definite integrals*, in the light of representatives of new transcendents.

For instance, if

$$U = A_0 + A_1 x + A_2 x^2 + \&c.,$$

where A_0, A_1, A_2, &c. are constants, it has been seen in the Third Chapter that

$$F(xD).U = F(0)\, A_0 + F(1)\, A_1 x + F(2)\, A_2 x^2 + \&c.$$

$$\frac{1}{F(xD)}.U = \frac{1}{F(0)} A_0 + \frac{1}{F(1)} A_1 x + \frac{1}{F(2)} A_2 x^2 + \&c.,$$

and it seems that the left-hand members may fairly be regarded as such representatives.

11. It has been shown that the solution of such an equation as

$$xD_x.z + yD_y.z - az = \Theta_m(x, y) + \Theta_n(x, y) + \&c.,$$

where Θ_m, Θ_n, &c. are homogeneous functions in x and y of the degrees m, n, &c., is at once

$$z = \frac{\Theta_m}{m-a} + \frac{\Theta_n}{n-a} + \text{\&c.} + u_a(x, y),$$

where u_a is an homogeneous function in x, y of given degree, but arbitrary in form.

By the substitutions

$$x = e^{\phi}, \quad y = e^{\psi},$$

it, consequently, appeared that the solution of such an equation as

$$D_\varphi . z + D_\psi . z - az = \Theta_m(e^{\phi}, e^{\psi}) + \Theta_n(e^{\phi}, e^{\psi}) + \text{\&c.}$$

is

$$z = \frac{\Theta_m}{m-a} + \frac{\Theta_n}{n-a} + \text{\&c.} + u_a(e^{\phi}, e^{\psi}),$$

the variables being (except in the case of equal roots in the operator on the left-hand member of the original equation, which case is provided for in the previous Chapter) grouped in the fixed portion of the solution as in the differential equation to be solved.

So far is the simplest deduction. If, however, we suppose the right-hand member of the latter form of the differential equation to be no longer a function of e^{ϕ}, e^{ψ}, but of ϕ, ψ, simply, our method is practically inapplicable, since ϕ and ψ are not exponible in a finite number of terms of e^{ϕ}, e^{ψ}.

12. From this difficulty we are released by the generalization of a method, which may be exhibited upon the ordinary differential equation

$$D_\theta y - ay = \theta^4.$$

The solution of this equation is given by

$$y = (D_\theta - a)^{-1} . \theta^4 + Ce^{a\theta}.$$

Now we may write this in the form

$$y = -(a - D_\theta)^{-1}.\theta^4 + Ce^{a\theta},$$

the arbitrary portion of the solution being in all cases independent of the character of the right-hand member of the equation to be solved.

Expanding the first term, and stopping at the fourth power of D_θ, further expansion being obviously needless, we get

$$-\frac{1}{a}\left(1 + \frac{1}{a}D_\theta + \frac{1}{a^2}D_\theta^2 + \frac{1}{a^3}D_\theta^3 + \frac{1}{a^4}D_\theta^4\right).\theta^4,$$

and actually performing the operations indicated, we obtain for the solution

$$y = -\left(\frac{\theta^4}{a} + \frac{4.\theta^3}{a^2} + \frac{4.3.\theta^2}{a^3} + \frac{4.3.2.\theta}{a^4} + \frac{4.3.2.1}{a^5}\right) + Ce^{a\theta},$$

and the *integration* of the equation proposed is due to a process of *differentiation*.

Similarly the solution of the partial differential equation

$$D_\phi.z + D_\psi.z - az = \Theta_4(\phi, \psi)$$

is given by

$$z = -\frac{1}{a}\left(1 + \frac{\square}{a} + \frac{\square^2}{a^2} + \frac{\square^3}{a^3} + \frac{\square^4}{a^4}\right).\Theta_4 + u_a(e^\phi, e^\psi),$$

and the solution is had by actual performance of the operations indicated upon the particular form of Θ_4. It is, of course, evident that in stopping the expansion at the fourth power of the symbol $\square$ we suppose the homogeneous function Θ_4 to contain no inverse powers of ϕ or ψ, for in such case the expansion should be continued *ad infinitum*.

13. As a second example of this method, let it be proposed to integrate the equation

$$D_\theta^2 y - 4D_\theta y + 4y = \theta^2.$$

Being thrown into the form

$$(D_\theta - 2)^2.y = \theta^2,$$

the solution is given by

$$y = (2 - D_\theta)^{-2}.\theta^2 + Ce^{2\theta}.\theta + C'e^{2\theta};$$

or, expanding the first term to the second order of D_θ,

$$y = \frac{1}{2^2}\{1 + \frac{2}{2}D_\theta + \frac{3}{2^2}D_\theta^2\}.\theta^2 + Ce^{2\theta}.\theta + C'e^{2\theta};$$

or, actually performing the operations indicated,

$$y = \frac{1}{2^2}\left\{\theta^2 + \frac{2.2.\theta}{2} + \frac{3.2.1}{2^2}\right\} + Ce^{2\theta}.\theta + C'e^{2\theta}.$$

In a manner precisely similar we can obtain the solution of the partial differential equation

$$D_\phi^2 z + 2D_\phi D_\psi z + D_\psi^2 z - 4(D_\phi z + D_\psi z) + 4z = \Theta_2(\phi, \psi).$$

It is, in fact, given by

$$z = \frac{1}{2^2}\{1 + \frac{2}{2}\,\square + \frac{3}{2^2}\,\square^2\}.\Theta_2(\phi,\psi) + u_2(e^\phi, e^\psi).(\phi + \psi) + v_2(e^\phi, e^\psi),$$

in which the operations indicated are to be actually performed upon the particular form of Θ_2, supposed, as before, to contain no inverse powers of the independent variables.

From the nature of the cases exhibited it is obvious that the value of this method of integration is ultimately attributable to the circumstance, that there is needed but a finite number of terms in the expansion of the symbolic operator. There are, indeed, other conceivable cases, in which this method could be employed with advantage; for instance, if the results of the operations indicated above were periodic, or if the different resultant terms, after the operations, conformed to some discoverable law.

14. It is obvious that we may generalize this method of integration still more completely. In fact, by a process in

every respect identical with that just exhibited, we may integrate the partial differential equation

$$F_1(\phi, \psi) . D_\phi z + F_2(\phi, \psi) . D_\psi z - az = \Omega(\phi, \psi),$$

and the solution is given by

$$z = \begin{cases} - \{a - F_1(\phi, \psi) . D_\phi - F_2(\phi, \psi) . D_\psi\}^{-1} . \Omega \\ + \{F_1(\phi, \psi) . D_\phi + F_2(\phi, \psi) . D_\psi - a\}^{-1} . 0, \end{cases}$$

in which Ω may be broken up into sets of homogeneous terms, the degree and character of which will regulate the extent to which the expansion of the operating symbol is to be carried.

15. Upon reference to the solution of the equation

$$D_\phi z + D_\psi z - az = \Theta_4(\phi, \psi)$$

it will be seen that the method of integration there proposed fails when

$$a = 0,$$

and we are obliged, in such cases, to have recourse to other means.

Thus, let it be proposed to integrate the partial differential equation

$$D_x z + D_y z = x^m y^n.$$

Now, since D_y is constant relative to D_x, we have, by the fifth article of the Third Chapter,

$$\frac{1}{D_x + D_y} = e^{-xD_y} . \frac{1}{D_x} . e^{xD_y},$$

therefore

$$z = e^{-xD_y} . \frac{1}{D_x} . x^m (x + y)^n + u_0 (e^x, e^y)$$

or, one particular solution of the proposed equation is,

$$z = \frac{x^{m+n+1}}{m+n+1} + n\frac{x^{m+n}}{m+n}(y-x) + \frac{n . n-1}{1.2}\frac{x^{m+n-1}}{m+n-1}(y-x)^2 + \&c. + u_0(e^x, e^y).$$

Combining this with the corresponding particular solution in y, the general solution is,

$$z = \begin{cases} \dfrac{x^{m+n+1}+y^{m+n+1}}{2(m+n+1)} + \dfrac{nx^{m+n}-my^{m+n}}{2(m+n)}(y-x) + \\ \dfrac{\dfrac{n(n-1)}{1.2}x^{m+n-1} + \dfrac{m(m-1)}{1.2}y^{m+n-1}}{2(m+n-1)}(y-x)^2 + \&c. + \phi(x-y). \end{cases}$$

16. As a second example of this method of integration, let it be proposed to investigate the solution of the partial differential equation

$$aD_x w + bD_y w + cD_z w = xyz.$$

This equation being transformed into the shape

$$D_{\frac{x}{a}} w + D_{\frac{y}{b}} w + D_{\frac{z}{c}} w = abc \, . \frac{x}{a}\frac{y}{b}\frac{z}{c},$$

it is obvious that, after abc in the right-hand member, the solution ought to be symmetrical in $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$.

By a method similar to that in the last article, the integral is found to be

$$w = \frac{abc}{3}\left[\frac{1}{12}\left(\frac{x^4}{a^4}+\frac{y^4}{b^4}+\frac{z^4}{c^4}\right) - \frac{1}{6}\left\{\frac{x^3}{a^3}\left(\frac{y}{b}+\frac{z}{c}\right)+\frac{y^3}{b^3}\left(\frac{z}{c}+\frac{x}{a}\right)+\frac{z^3}{c^3}\left(\frac{x}{a}+\frac{y}{b}\right)\right\}\right.$$
$$\left. + \frac{x}{a}\frac{y}{b}\frac{z}{c}\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)\right] + u_0\left(e^{\frac{x}{a}}, e^{\frac{y}{b}}, e^{\frac{z}{c}}\right).$$

It may be observed that the solution of the equation resembling this in Gregory's *Examples* is unsymmetrical. Indeed, such a result might have been anticipated from the unsymmetrical method there employed. It cannot be too frequently observed that, where the equations to be solved are symmetrical, the solutions should be symmetrical; and, not only so, but symmetrical methods should be employed for their deduction.

This latter consideration renders it highly desirable that we were in possession of some general and direct operational method for obtaining the solutions of such equations as have been just now discussed.

17. Let it be proposed to investigate the form of the solution of the general equation,

$$D_x^n . z + A_1 D_x^{n-1} D_y . z + A_2 D_x^{n-2} D_y^2 . z + \ldots + A_n D_y^n . z = V,$$

in which the left-hand member is an homogeneous function of the n^{th} order in the symbols D_x, D_y, the coefficients constants, and V a given function of x and y.

If $a_1, a_2, a_3, \ldots a_n$ be the roots of the equation

$$u^n + A_1 u^{n-1} + A_2 u^{n-2} + \ldots + A_n = 0,$$

the above equation can evidently be thrown into the form

$$(D_x - a_1 D_y)(D_x - a_2 D_y) \ldots\ldots (D_x - a_n D_y) . z = V,$$

the solution of which is

$$z = \frac{1}{(D_x - a_1 D_y) \ldots (D_x - a_n D_y)} . V + u_0 (e^x, e^{-\frac{y}{a_1}}) + v_0 (e^x, e^{-\frac{y}{a_2}}) + \&c.$$

The arbitrary portion might obviously be written in the form

$$\Phi\left(x + \frac{y}{a_1}\right) + \Psi\left(x + \frac{y}{a_2}\right) + \&c.$$

If p of the quantities a_1, a_2, a_3, &c. be equal, the arbitrary portion of the solution is, in general,

$$\left(x - \frac{y}{a_1}\right)^{p-1} . u_0\left(e^x, e^{-\frac{y}{a_1}}\right) + \left(x - \frac{y}{a_1}\right)^{p-2} . v_0\left(e^x, e^{-\frac{y}{a_1}}\right) + \ldots + w_0\left(e^x, e^{-\frac{y}{a_n}}\right).$$

And if the equation in u should contain pairs of imaginary roots, the arbitrary portion of the solution is of the form

$$u_0(e^x, e^{-\frac{y}{(a_1 + a_2\sqrt{-1})}}) + v_0(e^x, e^{-\frac{y}{(a_1 - a_2\sqrt{-1})}}) + w_0(e^x, e^{-\frac{y}{a_3}}) + \ldots$$

CHAPTER VI.

Section I.—Integration of Systems of Simultaneous Differential Equations.

1. Let it be proposed to integrate the system of simultaneous differential equations of the first order, containing n dependent variables,

$$\left.\begin{array}{l} D_t x = a_1 x + b_1 y + c_1 z + \ldots \\ D_t y = a_2 x + b_2 y + c_2 z + \ldots \\ D_t z = a_3 x + b_3 y + c_3 z + \ldots \\ \qquad\qquad \&c. \end{array}\right\}$$

Multiply the first equation by λ, the second by μ, the third by ν, &c.; then, adding all together,

$$\lambda D_t x + \mu D_t y + \nu D_t z + \ldots = \left\{\begin{array}{c} (a_1\lambda + a_2\mu + a_3\nu + \ldots)\, x \\ + \\ (b_1\lambda + b_2\mu + b_3\nu + \ldots)\, y \\ + \\ (c_1\lambda + c_2\mu + c_3\nu + \ldots)\, z \\ + \&c. \end{array}\right.$$

Now, as we have introduced n arbitrary constants, we are at liberty to subject them to n conditions, which we may suppose to be

$$a_1\lambda + a_2\mu + a_3\nu + \ldots = k\lambda,$$

$$b_1\lambda + b_2\mu + b_3\nu + \ldots = k\mu,$$

$$c_1\lambda + c_2\mu + c_3\nu + \ldots = k\nu,$$

$$\&c.$$

k being a new constant.

The preceding equation is thus reduced to the form

$$D_t(\lambda x + \mu y + \nu z + \ldots) = k(\lambda x + \mu y + \nu z + \ldots),$$

the solution of which is

$$\lambda x + \mu y + \nu z + \ldots = Ce^{kt},$$

where C is an arbitrary constant.

Now, with regard to the quantity k, it is to be observed that if $(n-1)$ of the quantities λ, μ, ν, &c. be eliminated between the assumed equations of connexion, the n^{th} quantity will disappear of itself, and we obtain an equation of the n^{th} degree in k and the known quantities a_1, b_1, c_1, &c.: consequently, in the above solution, k may be supposed to have any one of n known values.

Hence, writing down the series of solutions corresponding to the several roots k_1, k_2, k_3, &c., it is obvious that the general solution of the given system of simultaneous differential equations is exponible in the form,

$$\left.\begin{array}{l} x = C_1 e^{k_1 t} + C_2 e^{k_2 t} + C_3 e^{k_3 t} + \ldots + C_n e^{k_n t} \\ y = D_1 e^{k_1 t} + D_2 e^{k_2 t} + D_3 e^{k_3 t} + \ldots + D_n e^{k_n t} \\ z = E_1 e^{k_1 t} + E_2 e^{k_2 t} + E_3 e^{k_3 t} + \ldots + E_n e^{k_n t} \\ \qquad\qquad \&c. \end{array}\right\}$$

where, of the constants C_1, D_1, E_1, &c., n only are arbitrary.

When some of the roots k_1, k_2, k_3, &c., are equal, or when there are pairs of imaginary roots, modifications sufficiently obvious must be introduced in the general form of solution. Thus, in the case of a single pair of imaginary roots, the general form of solution becomes

$$\left.\begin{array}{l} x = e^{k_1 t}.\, C_1 \cos(k_2 t + C_2) + C_3 e^{k_3 t} + \ldots + C_n e^{k_n t} \\ y = e^{k_1 t}.\, D_1 \cos(k_2 t + D_2) + D_3 e^{k_3 t} + \ldots + D_n e^{k_n t} \\ z = e^{k_1 t}.\, E_1 \cos(k_2 t + E_2) + E_3 e^{k_3 t} + \ldots + E_n e^{k_n t} \\ \qquad\qquad \&c. \end{array}\right\}$$

Examples.

(1.) Let it be proposed to integrate the system

$$\left.\begin{aligned} D_t x &= by \\ D_t y &= ax \end{aligned}\right\}$$

in which, for simplicity, the suffixes to the constants are omitted.

The solution is

$$\left.\begin{aligned} x &= C_1 e^{k_1 t} + C_2 e^{k_2 t} \\ y &= D_1 e^{k_1 t} + D_2 e^{k_2 t} \end{aligned}\right\}$$

where k_1, k_2 are the roots of the quadratic equation obtained by the elimination of λ, μ between the equations

$$\left.\begin{aligned} a\mu &= k\lambda \\ b\lambda &= k\mu \end{aligned}\right\}, \quad \text{or} \quad \left\{\begin{aligned} k_1 &= +\sqrt{(ab)}, \\ k_2 &= -\sqrt{(ab)}. \end{aligned}\right.$$

Thus the solution is

$$x = C_1 e^{+\sqrt{(ab)}t} + C_2 e^{-\sqrt{(ab)}t},$$

$$y = C_1\left(\frac{a}{b}\right)^{\frac{1}{2}} e^{+\sqrt{(ab)}t} - C_2\left(\frac{a}{b}\right)^{\frac{1}{2}} e^{-\sqrt{(ab)}t}.$$

(2.) Let it be proposed to integrate the system

$$\left.\begin{aligned} D_t x &= a_1 x + b_1 y \\ D_t y &= a_2 x + b_2 y \end{aligned}\right\}.$$

The solution is

$$\left.\begin{aligned} x &= C_1 e^{k_1 t} + C_2 e^{k_2 t} \\ y &= C_1\left(\frac{k_1 - a_1}{b_1}\right) e^{k_1 t} + C_2\left(\frac{k_2 - a_1}{b_1}\right) e^{k_2 t} \end{aligned}\right\},$$

where k_1, k_2 are the roots of the quadratic equation

$$(k - a_1)\ (k - b_2) = b_1 a_2.$$

(3.) Let it be proposed to integrate the system of three simultaneous equations,

$$\left.\begin{aligned} D_t x &= \frac{dx}{dt} = b_1 y + c_1 z \\ D_t y &= \frac{dy}{dt} = a_2 x + c_2 z \\ D_t z &= \frac{dz}{dt} = a_3 x + b_3 y \end{aligned}\right\}$$

The solution is

$$\left.\begin{aligned} x &= C_1 e^{k_1 t} + C_2 e^{k_2 t} + C_3 e^{k_3 t} \\ y &= D_1 e^{k_1 t} + D_2 e^{k_2 t} + D_3 e^{k_3 t} \\ z &= E_1 e^{k_1 t} + E_2 e^{k_2 t} + E_3 e^{k_3 t} \end{aligned}\right\}$$

where k_1, k_2, k_3, are the roots of the cubic equation obtained by the elimination of λ, μ, ν between the equations

$$\left.\begin{aligned} a_2 \mu + a_3 \nu &= k\lambda \\ b_1 \lambda + b_3 \nu &= k\mu \\ c_1 \lambda + c_2 \mu &= k\nu \end{aligned}\right\}$$

or

$$k^3 - (b_1 a_2 + c_1 a_3 + c_2 b_3)\, k - (c_2 b_1 a_3 + c_1 a_2 b_3) = 0.$$

It will be observed that the left-hand members, in the given system of simultaneous differential equations, represent the rectangular components of the velocity of a material point, the coordinates of whose initial position determine the three arbitrary quantities in the above solution.

2. It is plain that we may employ a similar method for the integration of the system of simultaneous differential equations of the m^{th} order,

$$\left.\begin{aligned} D_t^m x &= a_1 x + b_1 y + c_1 z + \ldots \\ D_t^m y &= a_2 x + b_2 y + c_2 z + \ldots \\ D_t^m z &= a_3 x + b_3 y + c_3 z + \ldots \\ &\&c. \end{aligned}\right\}$$

The reduct equation is, in this case,

$$D_t^m (\lambda x + \mu y + \nu z + \ldots) = k^m (\lambda x + \mu y + \nu z + \ldots),$$

the equations of condition being

$$\left.\begin{array}{c} a_1\lambda + a_2\mu + a_3\nu + \ldots = k^m\lambda \\ b_1\lambda + b_2\mu + b_3\nu + \ldots = k^m\mu \\ c_1\lambda + c_2\mu + c_3\nu + \ldots = k^m\nu \\ \&c. \end{array}\right\}$$

and the solution of the reduct equation is, if α, α', α'', &c., be the m several roots of unity,

$$\lambda x + \mu y + \nu z + \ldots = Ce^{\alpha kt} + C'e^{\alpha' kt} + C''e^{\alpha'' kt} + \ldots$$

Example.

Let it be proposed to integrate the system

$$\left.\begin{array}{l} D_t^2 x = \dfrac{d^2x}{dt^2} = b_1 y + c_1 z \\ D_t^2 y = \dfrac{d^2y}{dt^2} = a_2 x + c_2 z \\ D_t^2 z = \dfrac{d^2z}{dt^2} = a_3 x + b_3 y \end{array}\right\}$$

in which the left-hand members represent the rectangular components of the accelerating force operating at any instant upon a material point.

The equations of condition in this case are

$$\left.\begin{array}{l} a_2\mu + a_3\nu = k^2\lambda \\ b_1\lambda + b_3\nu = k^2\mu \\ c_1\lambda + c_2\mu = k^2\nu \end{array}\right\};$$

the reduct equation,

$$D_t^2(\lambda x + \mu y + \nu z) = k^2(\lambda x + \mu y + \nu z);$$

the solution of this equation

$$\lambda x + \mu y + \nu z = Ce^{kt} + C'e^{-kt};$$

while the equation to determine k is

$$k^6 - (b_1a_2 + c_1a_3 + c_2b_3)\,k^2 - (c_2b_1a_3 + c_1a_2b_3) = 0.$$

As this equation is of the sixth degree, it might be supposed by the student that the complete solution of the problem should consist of six equations, each involving two arbitrary constants. It will be observed, however, that since the roots of the equation in k are of the form

$$\pm k_1, \pm k_2, \pm k_3,$$

and λ, μ, ν, depend only on k^2, these six equations, each of which is of the form

$$\lambda x + \mu y + \nu z = Ce^{kt} + C'e^{-kt},$$

are reducible to three, and there are not virtually more than six arbitrary constants. These constants are, in general, determined by given initial co-ordinates and given initial constituent velocities. The student will find no difficulty in extending this observation to the general case as above stated.

3. If the system of simultaneous differential equations, proposed for integration, were given in the form

$$\left.\begin{array}{c} a_1 D_t^m x + b_1 D_t^m y + c_1 D_t^m z + \ldots = x \\ a_2 D_t^m x + b_2 D_t^m y + c_2 D_t^m z + \ldots = y \\ a_3 D_t^m x + b_3 D_t^m y + c_3 D_t^m z + \ldots = z \\ \&c. \end{array}\right\}$$

a method somewhat similar may be employed.

The first equation being multiplied by λ, the second by μ, the third by ν, &c., and, all being added together, subject to the conditions

$$\left.\begin{array}{c} \lambda a_1 + \mu a_2 + \nu a_3 + \ldots = \dfrac{\lambda}{k^m} \\ \lambda b_1 + \mu b_2 + \nu b_3 + \ldots = \dfrac{\mu}{k^m} \\ \lambda c_1 + \mu c_2 + \nu c_3 + \ldots = \dfrac{\nu}{k^m} \\ \&c. \end{array}\right\}$$

the reduct equation is, as before,

$$D_t^m (\lambda x + \mu y + \nu z + \ldots) = k^m (\lambda x + \mu y + \nu z + \ldots),$$

and its solution

$$\lambda x + \mu y + \nu z + \ldots = Ce^{akt} + C'e^{a'kt} + C''e^{a''kt} + \ldots$$

It is, perhaps, unnecessary to observe, that the values of the constants, and of the several roots of the equation in k, are wholly different from those occurring in the previous article, in which a notation similar to the above was employed.

4. Let it be proposed to integrate the system of simultaneous differential equations,

$$\left.\begin{array}{l} \Phi(D)\, x + \Psi(D)\, y = F_1(t) \\ \Phi(D)\, y - \Psi(D)\, x = F_2(t) \end{array}\right\}$$

in which, for the sake of simplicity, we omit the suffix t to the symbol of differentiation.

Operating upon the first equation with $\Phi(D)$, and making substitution from the second equation, we get

$$\Phi(D)^2.\, x + \Psi(D).\{\Psi(D)\, x + F_2(t)\} = \Phi(D).F_1(t).$$

The operations susceptible of execution being performed, this equation is obviously reducible to the form

$$\{\Phi(D)^2. + \Psi(D)^2.\}\, x = F_3(t),$$

in which now there is but a single dependent variable.

This last equation, in general, admits of solution, and the value of x being had, that of y is obtained by substitution in either of the given equations.

Example.

$$(a_0 + a_2 D^2 + a_4 D^4)\, x + (a_1 D + a_3 D^3)\, y = m \sin nt,$$
$$(a_0 + a_2 D^2 + a_4 D^4)\, y - (a_1 D + a_3 D^3)\, x = m \cos nt.$$

Then

$$x = +\,\Sigma(A\cos\lambda^{\frac{1}{2}}t + B\sin\lambda^{\frac{1}{2}}t) + \frac{m\sin nt}{a_0 - a_1 n - a_2 n^2 + a_3 n^3 + a_4 n^4},$$

$$y = -\,\Sigma(A\sin\lambda^{\frac{1}{2}}t - B\cos\lambda^{\frac{1}{2}}t) + \frac{m\cos nt}{a_0 - a_1 n - a_2 n^2 + a_3 n^3 + a_4 n^4},$$

where all values are to be assigned to λ, which satisfy the biquadratic equation

$$(a_0 - a_2\lambda + a_4\lambda^2)^2 - \lambda(a_1 - a_3\lambda)^2 = 0.$$

GREGORY, *Examples*, p. 390.

5. We may, in some cases, employ the Calculus of Operations to great advantage in the investigation of the solutions of systems of simultaneous *partial* differential equations, and the results will be found to exhibit themselves in a remarkably symmetrical form.

Thus, if we had the system

$$\left.\begin{aligned} D_x^2 . z &= r = f_1(x, y) \\ D_x D_y . z &= s = f_2(x, y) \\ D_y^2 . z &= t = f_3(x, y) \end{aligned}\right\}$$

multiply the first equation by x^2, the second by $2xy$, and the third by y^2, and adding, we get

$$x^2 D_x^2 . z + 2xy D_x D_y . z + y^2 D_y^2 . z = x^2 f_1 + 2xy f_2 + y^2 f_3.$$

Break up the right-hand member, as before, into sets of homogeneous functions, and the whole assumes the symbolic shape

$$\nabla(\nabla - 1)\, z = \Theta_m + \Theta_n + \Theta_p + \&c.;$$

and the required solution is

$$z = \frac{\Theta_m}{m(m-1)} + \frac{\Theta_n}{n(n-1)} + \&c. + u_0 + u_1,$$

where u_0 and u_1 are arbitrary homogeneous functions in x and y, of the degrees 0 and 1, respectively.

The *primâ facie* method of solving such a system would be, to integrate the first equation twice with respect to x, supposing y constant, thereby introducing two arbitrary functions of y; to integrate the second equation successively with respect to x and y, thereby introducing two more arbitrary functions, the one of y, and the other of x; to integrate the third equation twice with respect to y, thereby introducing a further pair of arbitrary functions of x; and finally, by a comparison of the solutions thus obtained, to determine the characters of the resultant arbitrary functions as far as possible.

It is obvious that our method of solution will apply to the system

$$\left.\begin{aligned} D_\phi^2 . z &= f_1(e^\phi, e^\psi, \sin\phi, \sin\psi, \cos\phi, \cos\psi) \\ D_\phi D_\psi . z &= f_2(e^\phi, e^\psi, \sin\phi, \sin\psi, \cos\phi, \cos\psi) \\ D_\psi^2 . z &= f_3(e^\phi, e^\psi, \sin\phi, \sin\psi, \cos\phi, \cos\psi) \end{aligned}\right\}$$

the functions f_1, f_2, f_3 being severally reduced to the form

$$\Sigma A_{m,n} e^{m\phi+n\psi}.$$

Many other similar applications will readily suggest themselves.

It is important to observe, that the conditions of the above questions render it necessary to introduce a limitation upon the forms of the arbitrary functions. It is, in fact, evident that these functions must include no inverse powers of the independent variables, otherwise, although the result of the aggregation of the partial differential equations might be correctly solved, yet such a solution would not satisfy the equations separately.

Camb. and Dub. Math. Journal, 1853.

6. If it be required to eliminate the arbitrary functions from the equation

$$xF_1(z) + yF_2(z) = 1,$$

it is known that the result is obtained by differentiating the equation twice with respect to x and y, respectively, the first differentiation giving

$$\left.\begin{array}{l} F_1(z)+\{xF_1'(z)+yF_2'(z)\}.D_x z=0 \\ F_2(z)+\{xF_1'(x)+yF_2'(z)\}.D_y z=0 \end{array}\right\}$$

whence, by division,

$$\frac{F_1'(z)}{F_2'(z)}=f(z)=\frac{p}{q}=\frac{D_x z}{D_y z},$$

and the second differentiation giving

$$q^2 r-2pqs+p^2 t=0,$$

or

$$(D_y z)^2.D_x^2 z-2D_x z.D_y z.D_x D_y z+(D_x z)^2.D_y^2 z=0,$$

the partial differential equation of the gauche surface generated by a right line, which, gliding upon two fixed directrices, remains constantly parallel to the plane of the axes x and y.

Similarly, it appears that the solution of the system of $\frac{p(p-1)}{1.2}$ simultaneous partial differential equations, containing p independent variables,

$$\left.\begin{array}{c} (D_y w)^2.D_x^2 w-2D_x w.D_y w.D_x D_y w+(D_x w)^2.D_y^2 w=0 \\ (D_x w)^2.D_z^2 w-2D_z w.D_x w.D_z D_x w+(D_z w)^2.D_x^2 w=0 \\ (D_z w)^2.D_y^2 w-2D_y w.D_z w.D_y D_z w+(D_y w)^2.D_z^2 w=0 \\ \&c. \end{array}\right\}$$

is

$$\left.\begin{array}{c} xf_1(w,z,\&c.)+yf_2(w,z,\&c.)+\ldots=1 \\ zf_{1'}(w,y,\&c.)+xf_{2'}(w,y,\&c.)+\ldots=1 \\ yf_{1''}(w,x,\&c.)+zf_{2''}(w,x,\&c.)+\ldots=1 \\ \&c. \end{array}\right\}$$

by a comparison of which we get for the ultimate solution

$$xF_1(w)+yF_2(w)+zF_3(w)+\&c.=1,$$

exhibiting only p arbitrary functions.

7. Let it be proposed to integrate the system of equations which determine the small motions of homogeneous elastic gases, namely,

$$\left.\begin{aligned} D_t^2 u = \frac{d^2u}{dt^2} = a^2 D_x (D_x u + D_y v + D_z w) \\ D_t^2 v = \frac{d^2v}{dt^2} = a^2 D_y (D_x u + D_y v + D_z w) \\ D_t^2 w = \frac{d^2w}{dt^2} = a^2 D_z (D_x u + D_y v + D_z w) \end{aligned}\right\}$$

GREGORY, *Examples*, p. 392.

Let

$$D_x u + D_y v + D_z w = V,$$

and, differentiating the first equation with respect to x, the second with respect to y, the third with respect to z, and adding, we get

$$D_t^2 V = a^2 (D_x^2 V + D_y^2 V + D_z^2 V),$$

the integral of which is

$$V = e^{at(D_x^2 + D_y^2 + D_z^2)^{\frac{1}{2}}} . \Phi(x, y, z) + e^{-at(D_x^2 + D_y^2 + D_z^2)^{\frac{1}{2}}} . \Psi(x, y, z),$$

which, by an elaborate process of transformation, has been given by Poisson (Mémoires de l'Institut, 1818), in the shape

$$4\pi V =$$

$$\int_0^\pi \int_0^{2\pi} t\phi(x + at\cos\theta, y + at\sin\theta\cos\phi, z + at\sin\theta\sin\phi)\sin\theta\, d\theta\, d\phi +$$

$$D_t \int_0^\pi \int_0^{2\pi} t\psi(x + at\cos\theta, y + at\sin\theta\cos\phi, z + at\sin\theta\sin\phi)\sin\theta\, d\theta\, d\phi.$$

The value of V being thus found, u, v, w are to be obtained by its substitution in the given system of equations.

8. If the system to be integrated were that representing the small motions of homogeneous elastic solids and homogeneous incompressible liquids, namely,

$$\left.\begin{aligned} D_t^2 u = PD_x (D_x u + D_y v + D_z w) + Q(D_x^2 u + D_y^2 u + D_z^2 u) \\ D_t^2 v = PD_y (D_x u + D_y v + D_z w) + Q(D_x^2 v + D_y^2 v + D_z^2 v) \\ D_t^2 w = PD_z (D_x u + D_y v + D_z w) + Q(D_x^2 w + D_y^2 w + D_z^2 w) \end{aligned}\right\}$$

where P and Q are constants, we may proceed in a manner somewhat similar. The first equation being differentiated with respect to x, the second with respect to y, the third with respect to z, and all being then added together, we get

$$D_t^2 V = P(D_x^2 V + D_y^2 V + D_z^2 V) + Q(D_x^2 V + D_y^2 V + D_z^2 V)$$

or

$$D_t^2 V = (P + Q)(D_x^2 V + D_y^2 V + D_z^2 V),$$

the solution of which has been just given.

9. Let the system to be integrated be

$$\left.\begin{aligned} AD_t p + (C-B)\,qr &= A\frac{dp}{dt} + (C-B)\,qr = 0 \\ BD_t q + (A-C)\,rp &= B\frac{dq}{dt} + (A-C)\,rp = 0 \\ CD_t r + (B-A)\,pq &= C\frac{dr}{dt} + (B-A)\,pq = 0 \end{aligned}\right\}$$

the well-known equations which serve to determine the relation between the angular velocity of instantaneous rotation and the time, in the case of a rigid body rotating about its centre of gravity, and not subject to the action of any forces, A, B, C being the three principal moments of inertia.

These equations being multiplied, respectively, by p, q, r, all then added together, and the result integrated, we get

$$Ap^2 + Bq^2 + Cr^2 = V,$$

a constant, which is known to represent the *vis viva* of the body.

The original equations being again multiplied, respectively, by Ap, Bq, Cr, all then added together, and the result integrated, we get

$$A^2p^2 + B^2q^2 + C^2r^2 = M^2,$$

M being another constant, which is known to represent the principal moment of the quantities of motion.

Now if ω be the angular velocity of instantaneous rotation, and we investigate the values of p^2, q^2, r^2, from the equations,

$$\left.\begin{aligned} Ap^2 + Bq^2 + Cr^2 &= V \\ A^2p^2 + B^2q^2 + C^2r^2 &= M^2 \\ p^2 + q^2 + r^2 &= \omega^2 \end{aligned}\right\}$$

we obtain

$$\left.\begin{aligned} p^2 &= \frac{M^2 - (B+C)V + BC\omega^2}{(C-A)(B-A)} \\ q^2 &= \frac{M^2 - (C+A)V + CA\omega^2}{(A-B)(C-B)} \\ r^2 &= \frac{M^2 - (A+B)V + AB\omega^2}{(B-C)(A-C)} \end{aligned}\right\}$$

If we substitute these values in any one of the original equations, we obtain the following relation:

$$dt = \frac{ABC\,.\,\omega d\omega}{\{(B+C)V - BC\omega^2 - M^2\}^{\frac{1}{2}}.\{(C+A)V - CA\omega^2 - M^2\}^{\frac{1}{2}}.\{(A+B)V - AB\omega^2 - M^2\}^{\frac{1}{2}}}.$$

LAGRANGE, *Mec. Anal.*, Seconde Partie, p. 247.

10. Let the system to be integrated be

$$\left.\begin{aligned} D_t^2 u - D_u R &= \frac{d^2u}{dt^2} - \frac{dR}{du} = 0 \\ D_t^2 v - D_v R &= \frac{d^2v}{dt^2} - \frac{dR}{dv} = 0 \\ D_t^2 w - D_w R &= \frac{d^2w}{dt^2} - \frac{dR}{dw} = 0 \\ &\&c., \end{aligned}\right\}$$

the number of variables u, v, w, &c., being n, and R being a function of r, where

$$r = (u^2 + v^2 + w^2 + \ldots)^{\frac{1}{2}}.$$

The system may be obviously written in the shape,

$$\left.\begin{array}{l} D_t^2 u - D_r R \,.\, \dfrac{u}{r} = 0 \\ D_t^2 v - D_r R \,.\, \dfrac{v}{r} = 0 \\ D_t^2 w - D_r R \,.\, \dfrac{w}{r} = 0 \\ \qquad \&c. \end{array}\right\}$$

Eliminating $D_r R$ between these equations in pairs, and integrating the corresponding results, we obtain $\dfrac{n \,.\, n-1}{2}$ first integrals, namely,

$$\left.\begin{array}{l} vD_t u - uD_t v = C_1 \\ uD_t w - wD_t u = C_2 \\ wD_t v - vD_t w = C_3 \\ \qquad \&c. \end{array}\right\}$$

Multiplying the given equation by $2D_t u$, $2D_t v$, $2D_t w$, &c., adding them all together, and integrating the result, we get

$$(D_t u)^2 + (D_t v)^2 + (D_t w)^2 + \&c. = 2(R + A),$$

A being an arbitrary constant.

But the equations composing the previous system of integrals being severally squared, and all then added together, we obtain

$$(u^2 + v^2 + \&c.)\,\{(D_t u)^2 + (D_t v)^2 + \&c.\} - (uD_t u + v\,D_t v + \&c.)^2 = B^2.$$

Hence there results

$$(D_t r)^2 = 2(R + A) - \frac{B^2}{r^2},$$

and consequently

$$dt = \frac{rdr}{\{2r^2\,(R + A) - B^2\}^{\frac{1}{2}}}.$$

By means of this equation eliminating $D_r R$ from the first equation of the modified form of the given system, we have

$$rD_t^2 u - uD_t^2 r + \frac{B^2}{r^2} \cdot \frac{u}{r} = 0,$$

or

$$\left(\frac{r^2}{B} D_t\right)^2 \cdot \frac{u}{r} + \frac{u}{r} = 0.$$

Hence if ϕ be a quantity determined by the equation

$$d\phi = \frac{Bdt}{r^2} = \frac{Bdr}{r\{2r^2(R+A) - B^2\}^{\frac{1}{2}}}$$

we obtain for the required system of second integrals

$$\left.\begin{array}{l} u = r(g_1 \cos\phi + h_1 \sin\phi) \\ v = r(g_2 \cos\phi + h_2 \sin\phi) \\ w = r(g_3 \cos\phi + h_3 \sin\phi) \\ \quad \&c. \\ \phi + \beta = \displaystyle\int \frac{Bdr}{r\{2r^2(R+A) - B^2\}^{\frac{1}{2}}} \\ t + \alpha = \displaystyle\int \frac{rdr}{\{2r^2(R+A) - B^2\}^{\frac{1}{2}}} \end{array}\right\}$$

By means of these we obtain ϕ in terms of r, and r in terms of $t + \alpha$, and therefore ϕ in terms of $t + \alpha$. Thus we have u, v, w, &c., expressed in terms of t, α, β, A, B, g_1, h_1, &c.

It would appear that the ultimate number of arbitrary constants is $2n + 4$; but since β only tends to alter g_1, h_1, &c., it may be neglected. And since upon squaring the first group of the resulting system, and adding, we get

$$1 = \cos^2\phi \ \Sigma(g^2) + 2\sin\phi\cos\phi \ \Sigma(gh) + \sin^2\phi \ \Sigma(h^2),$$

which can only subsist for all values of ϕ by the fulfilment of the conditions

$$\Sigma(g^2) = 1, \quad \Sigma(gh) = 0, \quad \Sigma(h^2) = 1,$$

it is plain that there are not virtually more than $2n$ independent arbitrary constants.

The integrals for determining t and ϕ are not independent, for if we assume a function

$$S = \int \frac{dr}{r} \{2r^2 (R + A) - B^2\}^{\frac{1}{2}},$$

it is readily seen that

$$t + \alpha = \frac{dS}{dA}, \quad \phi + \beta = - \frac{dS}{dB}.$$

BINET, as quoted by GREGORY, *Examples*, p. 396.

11. Let it be proposed to integrate the system of equations

$$\left.\begin{array}{l} AD_x u + BD_y u + CD_z u = E \\ A'D_x u + B'D_y u + C'D_z u = E' \end{array}\right\}$$

A, A', B, B', &c., being constants.

JELLETT, *Law's Mathematical Prize*, 1854.

These equations being integrated separately, we get

$$u = \frac{E}{3}\left(\frac{x}{A} + \frac{y}{B} + \frac{x}{C}\right) + v_0\left(e^{\frac{x}{A}}, e^{\frac{y}{B}}, e^{\frac{z}{C}}\right),$$

$$u = \frac{E'}{3}\left(\frac{x}{A'} + \frac{y}{B'} + \frac{z}{C'}\right) + w_0\left(e^{\frac{x}{A'}}, e^{\frac{y}{B'}}, e^{\frac{z}{C'}}\right),$$

and the result required is obtained by the identification of these solutions.

Now, it is evident that they may be exhibited under the shape

$$u = \frac{E}{3}\left(\frac{x}{A} + \frac{y}{B} + \frac{z}{C}\right) + \Phi\left\{\frac{y}{B} - \frac{z}{C}, \frac{z}{C} - \frac{x}{A}, \frac{x}{A} - \frac{y}{B}\right\},$$

$$u = \frac{E'}{3}\left(\frac{x}{A'} + \frac{y}{B'} + \frac{z}{C'}\right) + \Psi\left\{\frac{y}{B'} - \frac{z}{C'}, \frac{z}{C'} - \frac{x}{A'}, \frac{x}{A'} - \frac{y}{B'}\right\}.$$

By a comparison of these solutions it is obvious that the forms of Φ and Ψ must be linear; in other words, that the equations just stated should be exponible in the shape

$$u=\frac{E}{3}\left(\frac{x}{A}+\frac{y}{B}+\frac{z}{C}\right)+l\left(\frac{y}{B}-\frac{z}{C}\right)+m\left(\frac{z}{C}-\frac{x}{A}\right)+n\left(\frac{x}{A}-\frac{y}{B}\right)+\alpha,$$

$$u=\frac{E'}{3}\left(\frac{x}{A'}+\frac{y}{B'}+\frac{z}{C'}\right)+l'\left(\frac{y}{B'}-\frac{z}{C'}\right)+m'\left(\frac{z}{C'}-\frac{x}{A'}\right)+n'\left(\frac{x}{A'}-\frac{y}{B}\right)+\beta,$$

where l, l', m, m', &c., are unknown constants. Either of these solutions is identified with the other by the suppositions

$$\frac{E}{A}+3\,\frac{n-m}{A}=\frac{E'}{A'}+3\,\frac{n'-m'}{A'}$$

$$\frac{E}{B}+3\,\frac{l-n}{B}=\frac{E'}{B'}+3\,\frac{l'-n'}{B'}$$

$$\frac{E}{C}+3\,\frac{m-l}{C}=\frac{E'}{C'}+3\,\frac{m'-l'}{C'}$$

$$\alpha=\beta.$$

12. Let the system to be integrated be

$$\left.\begin{array}{c} D_x^2A+2pD_xD_zA+p^2D_z^2A=0 \\ D_xD_yA+pD_yD_zA+qD_xD_zA+pqD_z^2A=0 \\ D_y^2A+2qD_yD_zA+q^2D_z^2A=0 \end{array}\right\}$$

JELLETT, *Calculus of Variations*, p. 345.

where A is some function of x, y, z, p, and q, the ordinary notation for the partial differential coefficients of z with respect to x and y being retained in this particular case for the sake of simplicity.

These equations may be thrown into the form

$$(D_x+pD_z)^2\,.\,A=0$$

$$(D_x+pD_z)\,(D_y+qD_z)\,.\,A=0$$

$$(D_y+qD_z)^2\quad A=0,$$

under the condition that the quantities p and q are regarded as independent variables, and consequently not differentiated, unless with respect to themselves.

The integration of the equations severally gives

$$A = \left(x + \frac{z}{p}\right) \phi_1 (z - px, p, q, y) + \psi_1 (z - px, p, q, y),$$

$$A = \phi_2 (z - px, p, q, y) + \psi_2 (z - qy, p, q, x),$$

$$A = \left(y + \frac{z}{q}\right) \phi_3 (z - qy, p, q, x) + \psi_3 (z - qy, p, q, x),$$

and by identification of these results we ultimately obtain

$$A = \left\{ \begin{array}{l} x\Phi(z - px - qy, p, q) + yX(z - px - qy, p, q) \\ \qquad + \Psi(z - px - qy, p, q). \end{array} \right.$$

13. If the system to be integrated were

$$D_x^2 A + 2pD_x D_z A + pD_z^2 A = -\frac{1 + p^2}{(1 + p^2 + q^2)^{\frac{3}{2}}}$$

$$D_x D_y A + pD_y D_z A + qD_x D_z A + pqD_z^2 A = -\frac{pq}{(1 + p^2 + q^2)^{\frac{3}{2}}}$$

$$D_y^2 A + 2qD_y D_z A + q^2 D_z^2 A = -\frac{1 + q^2}{(1 + p^2 + q^2)^{\frac{3}{2}}},$$

JELLETT, *Calculus of Variations*, p. 374.

we assume

$$(D_x + pD_z) A = (D_x A) = u, \quad (D_y + qD_z) A = (D_y A) = v,$$

and the first and last of the given equation are thus reduced, respectively, to the forms

$$(D_x + pD_z) u = (D_x u) = -\frac{1 + p^2}{(1 + p^2 + q^2)^{\frac{3}{2}}}.$$

$$(D_y + qD_z) v = (D_y v) = -\frac{1 + q^2}{(1 + p^2 + q^2)^{\frac{3}{2}}}.$$

The integration of these gives

$$u = -\frac{(1 + p^2) x}{(1 + p^2 + q^2)^{\frac{3}{2}}} + \phi (z - px, p, q, y),$$

$$v = -\frac{(1 + q^2) y}{(1 + p^2 + q^2)^{\frac{3}{2}}} + \psi (z - qy, p, q, x).$$

Multiplying these respectively by dx, dy, integrating, and taking into account the second given equation, we get ultimately

$$A = -\tfrac{1}{2}\frac{\{x^2 + y^2 + (px + qy)^2\}}{(1 + p^2 + q^2)^{\frac{3}{2}}} + x\Phi + yX + \Psi,$$

where Φ, X, and Ψ are the functions of the last example.

14. Let the system to be integrated be that which determines the azimuthal motion of the plane of a freely-suspended pendulum, namely,

$$\left.\begin{aligned} D_t^2 x &= \frac{d^2x}{dt^2} = -\frac{gx'}{r+l} + N\frac{(x'-x)}{l} \\ D_t^2 y &= \frac{d^2y}{dt^2} = -\frac{gy'}{r+l} + N\frac{(y'-y)}{l} \\ D_t^2 z &= \frac{d^2z}{dt^2} = -\frac{gz'}{r+l} + N\frac{(z'-z)}{l} \end{aligned}\right\}$$

where the origin is the centre of the earth, the positive axis of z the axis of rotation of the earth directed upwards, the positive axis of y directed towards the spectator, and the positive axis of x to the right hand; x, y, z denoting the co-ordinates of the centre of oscillation of the pendulum; x', y', z' the co-ordinates of the point of suspension; g the attraction of the earth; l the length of the pendulum; r the radius of the earth; and N the tension of the string.

If these equations be transformed to the point of suspension as origin, the positive axis of z being vertically downwards, the positive axes of x and y being in the horizon, and directed towards the east and north respectively, we shall obtain the following:

$$\left.\begin{aligned} D_t^2 x + \frac{Nx}{l} &= \qquad + 2k\sin\lambda D_t y + 2k\cos\lambda D_t z + k^2 x \\ D_t^2 y + \frac{Ny}{l} &= -k^2 r\cos\lambda\sin\lambda - 2k\sin\lambda D_t x + k^2\sin\lambda(y\sin\lambda + z\cos\lambda) \\ D_t^2 z + \frac{Nz}{l} &= -k^2 r\cos^2\lambda + g - 2k\cos\lambda D_t x + k^2\cos\lambda(y\sin\lambda + z\cos\lambda) \end{aligned}\right\}$$

λ being the latitude of the place of observation; or, supposing the axes of co-ordinates transformed to the vertical and horizon of the actual spheroid,

$$\left.\begin{aligned} D_t^2 x + \frac{Nx}{l} &= \quad + 2k \sin\lambda D_t y + 2k \cos\lambda D_t z + k^2 x \\ D_t^2 y + \frac{Ny}{l} &= \quad - 2k \sin\lambda D_t x + k^2 \sin\lambda (y \sin\lambda + z \cos\lambda) \\ D_t^2 z + \frac{Nz}{l} &= g - 2k \cos\lambda D_t x + k^2 \cos\lambda (y \sin\lambda + z \cos\lambda) \end{aligned}\right\}$$

If the terms depending on k^2 be neglected in these equations, we obtain

$$\left.\begin{aligned} D_t^2 x + \frac{Nx}{l} &= \quad + 2k \sin\lambda D_t y + 2k \cos\lambda D_t z \\ D_t^2 y + \frac{Ny}{l} &= \quad - 2k \sin\lambda D_t x \\ D_t^2 z + \frac{Nz}{l} &= g - 2k \cos\lambda D_t x \end{aligned}\right\}$$

Eliminating N between the first two equations, we find

$$(yD_t^2 x - xD_t^2 y) = 2k \sin\lambda\ (yD_t y + xD_t x) + 2k \cos\lambda\ yD_t z.$$

Integrating this equation, we obtain

$$yD_t x - xD_t y = k \sin\lambda (x^2 + y^2) + 2k \cos\lambda \int y dz.$$

Transforming this equation to polar co-ordinates by the formulæ

$$\begin{aligned} x &= l \sin\phi \sin\theta, \\ y &= l \cos\phi \sin\theta, \\ z &= l \cos\theta, \end{aligned}$$

in which ϕ denotes the azimuth measured from the north, and θ the deviation of the pendulum from the vertical, we find

$$D_t\phi = \frac{d\phi}{dt} = k \sin\lambda - \frac{2k \cos\lambda}{\sin\theta} \int \cos\phi \sin^2\theta d\theta. \qquad (x)$$

This equation proves that the azimuthal velocity consists of two parts: one uniform, and equal $k \sin\lambda$, directed from the north to the east; the other periodic, and passing through all its changes in the time of an oscillation of the pendulum, and depending on the amplitude of the vibration. As the azimuth ϕ may be considered constant during the time of an oscillation, the second term in the last equation may be integrated. Hence we obtain

$$D_t\phi = \frac{d\phi}{dt} = k\sin\lambda - \tfrac{2}{3}k\cos\lambda\cos\phi\,.\,\theta,$$

θ being a small arc, the powers of which above the second may be neglected, and vanishing twice during each oscillation.

From the equation just found it is easy to see that the plane of oscillation undergoes a periodic variation in azimuth; in consequence of which the projection of the centre of oscillation of the pendulum on the horizon will describe a curve resembling a figure of eight, in which, if the pendulum be in the meridian, the motion in the northern loop is retrograde; and in the southern loop progressive.

The variation in azimuth produced by the second term of equation (a) will be insensible, unless θ become nearly equal to π, in which case the change in azimuth will become indefinitely great; for, integrating this equation, we find the initial motion being in the meridian,

$$D_t\phi = \frac{d\phi}{dt} = k\sin\lambda - k\cos\lambda\,\frac{\theta - \sin\theta\cos\theta}{\sin^2\theta}.$$

If, in this equation, θ be equal to π, the second term will be infinite and negative, denoting that the plane of vibration swings round suddenly to the west. This result is evident without analysis: for, if the pendulum be started in the meridian, so as to pass the lowest point with a velocity due to twice its length, it will reach the top of the circle without velocity, and fall suddenly to the west, in the prime vertical.

If the pendulum were to perform a complete revolution with a high velocity, the time of revolution in azimuth of the plane of its motion would tend to the limit $23^h\,56^m$; but when the motion is oscillatory, the theoretical time of revolution in azimuth will be $23^h\,56^m \times \operatorname{cosec} \lambda$, as has been proved for small arcs of vibration by M. Binet ("Comptes rendus de l'Acad. des Sciences," Feb. 17, 1851).

GALBRAITH AND HAUGHTON, *Proc. of Royal Irish Academy*, 1851.

SECTION II.—EVALUATION AND EXTENSION OF DEFINITE INTEGRALS.

15. The Calculus of Operations is obviously susceptible of application to the subject of single and multiple Definite Integrals, and an interesting field is thus opened for investigation to the student. It would be impossible here, and, so far as our present purpose is considered, unnecessary, to follow up such an investigation in its details; and a few general theorems, with examples illustrative of their application, must suffice.

16. It has been proved in the eighth article of the Third Chapter, that, if Φ and Ψ be any algebraic functions,

$$\Psi(D_x)\,.\,\Phi(x)\,.\,e^{rx} = \Phi(D_r)\,.\,\Psi(r)\,.\,e^{rx}.$$

If r be changed into $-r$, we get

$$\Phi(-D_r)\,.\,\Psi(-r)\,.\,e^{-rx} = \Psi(D_x)\,.\,\Phi(x)\,.\,e^{-rx}.$$

Now it is easily proved that

$$\int_0^\infty dx \sin mx\, e^{-rx} = \frac{m}{m^2+r^2}.$$

Hence operating on each side of this equation with

$$\Phi(-D_r)\,.\,\Psi(-r)$$

we obtain the theorem

$$\int_0^\infty dx \sin mx\,.\,\Psi(D_x)\,.\,\Phi(x)\,e^{-rx} = \Phi(-D_r)\,\frac{m\Psi(-r)}{m^2+r^2},$$

and, by a similar process, the theorem

$$\int_0^\infty dx \cos mx\,.\,\Psi(D_x)\,.\,\Phi(x)\,e^{-rx} = \Phi(-D_r)\,\frac{r\Psi(-r)}{m^2+r^2}.$$

BRONWIN, *Camb. and Dub. Math. Journal*, 1848.

17. If

$$\int dx \int dy \int dz \, .. \, \Omega\,(x, y, z, \&c.) \, . \, a^{\phi(xyz\,\&c.)} \, . \, b^{\chi(xyz\,\&c.)} \, . \, c^{\psi(xyz\,\&c.)} \, .. = K,$$

the quantities a, b, c, &c., being unconnected with the limits, then will

$$\int dx \int dy \int dz \, .. \, \Omega(x,y,z,\&c.) F(\phi+\chi+\psi+\&c.) . a^{\phi} . b^{\chi} . c^{\psi} .. = F(\nabla) . K,$$

where,

$$\nabla = aD_a + bD_b + cD_c + \&c.$$

This is obvious, since, from the supposition made relative to a, b, c, &c., we can operate with the symbol ∇ under the integral signs. It will be observed that the result bears a resemblance to Liouville's well-known extension of Dirichlet's integral.

Conversely, if it be required to investigate the value of the multiple definite integral

$$\int dx \int dy \int dz \ldots \Omega\,(x, y, z, \&c) \, . \, F(\phi + \chi + \psi + \&c.) \, . \, a^{\phi} . \, b^{\chi} . \, c^{\psi} \ldots$$

in which the quantities a, b, c, &c., are unconnected with the limits, the inquiry is reducible to the investigation of the value of the simpler multiple definite integral

$$\int dx \int dy \int dz \ldots \Omega\,(x, y, z) \, . \, a^{\phi} . \, b^{\chi} . \, c^{\psi} \ldots$$

and subsequent operation upon the result with the symbolic form $F(\nabla)$, where ∇ is the symbol above defined.

We seem to have here made a step towards the solution of that which has been long a difficulty in the treatment of multiple definite integrals, namely, the generalization of those in which the variables enter *as complicated functions in the indices* of known quantities. The most valuable extensions yet obtained are those in which the element of the primary multiple definite integral exhibits the variables under finite forms only.

Examples.

(1.) It can be easily proved that

$$\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \,.\, a^{-x^2} \,.\, b^{-y^2} \,.\, c^{-z^2} = \frac{1}{8}\pi^{\frac{3}{2}} \frac{1}{\{\log a \,.\, \log b \,.\, \log c\}^{\frac{1}{2}}};$$

hence

$$\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \,.\, \mathrm{F}\,(x^2 + y^2 + z^2)\, a^{-x^2} \,.\, b^{-y^2} \,.\, c^{-z^2}$$
$$= \frac{1}{8}\pi^{\frac{3}{2}} \,.\, \mathrm{F}(-\nabla) \frac{1}{\{\log a \,.\, \log b \,.\, \log c\}^{\frac{1}{2}}}.$$

Conversely, the investigation of the value of

$$\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \,.\, \mathrm{F}\,(x^2 + y^2 + z^2)\, a^{-x^2} \,.\, b^{-y^2} \,.\, c^{-z^2}$$

is reduced to the investigation of the value of

$$\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \,.\, a^{x^2} \,.\, b^{-y^2} \,.\, c^{-z^2},$$

which is known to be

$$\frac{1}{8}\pi^{\frac{3}{2}} \frac{1}{\{\log a \,.\, \log b \,.\, \log c\}^{\frac{1}{2}}},$$

and operation upon this quantity with $\mathrm{F}\,(-\nabla)$.

(2.) Again, we readily see that

$$\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \,.\, a^{-px} \,.\, b^{-qy} \,.\, c^{-rz} \,.\, x^{l-1} \,.\, y^{m-1} \,.\, z^{n-1}$$
$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{p^l q^m r^n} \,.\, \frac{1}{(\log a)^l \,.\, (\log b)^m \,.\, (\log c)^n};$$

and hence

$$\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \,.\, \Phi\,(px + qy + rz)\, a^{-px} \,.\, b^{-qy} \,.\, c^{-rz}\, x^{l-1} \,.\, y^{m-1} \,.\, z^{n-1}$$
$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{p^l\, q^m\, r^n}\, \Phi\,(-\nabla) \frac{1}{(\log a)^l . (\log b)^m . (\log c)^n}.$$

Conversely, the investigation of the value of the integral

$$\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \,.\, \Phi\,(px + qy + rz)\, a^{-px} \,.\, b^{-qy} \,.\, c^{-rz}\, x^{l-1} y^{m-1} z^{n-1}$$

is reduced to the investigation of the value of

$$\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \,.\, a^{-px}\, b^{-qy}\, c^{-rz}\, x^{l-1}\, y^{m-1}\, z^{n-1},$$

which is known to be

$$\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{p^l\, q^m\, r^n} \cdot \frac{1}{(\log a)^l \,.\, (\log b)^m \,.\, (\log c)^n},$$

and operation upon the result with $F\,(-\nabla)$.

Philosophical Magazine, 1852.

18. More generally, since

$$F(xD_x, yD_y, zD_z, \&\text{c.}) \,.\, x^m y^n z^p \ldots = F(m, n, p, \&\text{c.}) \,.\, x^m y^n z^p \ldots,$$

we have

$$F(aD_a, bD_b, cD_c, \&\text{c.}) \,.\, a^{\phi(x,y,z,\&\text{c.})} \,.\, b^{\chi(x,y,z,\&\text{c.})} \,.\, c^{\psi(x,y,z,\&\text{c.})} \ldots$$
$$=$$
$$F(\phi, \chi, \psi, \&\text{c.}) \,.\, a^{\phi(x,y,z,\&\text{c.})} \,.\, b^{\chi(x,y,z,\&\text{c.})} \,.\, c^{\psi(x,y,z,\&\text{c.})}.$$

Hence, if

$$\int dx \int dy \int dz \ldots \Omega(x, y, z, \&\text{c.})\, a^{\phi(x,y,z,\&\text{c.})} . b^{\chi(x,y,z,\&\text{c.})} . c^{\psi(x,y,z,\&\text{c.})} \ldots = K,$$

the quantities a, b, c, &c., being unconnected with the limits, then will

$$\int dx \int dy \ldots \Omega(x, y, \&\text{c.})\, F(\phi, \chi, \&\text{c.})\, a^\phi b^\chi \ldots = F(aD_a, bD_b, \&\text{c.}) . K.$$

Conversely, if it be required to investigate the value of the multiple definite integral

$$\int dx \int dy \int dz \ldots \Omega\,(x, y, z, \&\text{c.}) \,.\, F(\phi, \chi, \psi, \&\text{c.})\, a^\phi b^\chi c^\psi \ldots$$

in which the quantities a, b, c, &c., are unconnected with the limits, the inquiry is reducible to the investigation of the value of the simpler multiple definite integral

$$\int dx \int dy \int dz \ldots \Omega(x, y, z, \&\text{c.}) \,.\, a^\phi . b^\chi . c^\psi, \ldots$$

and subsequent operation upon the result with the symbolic form

$$F(aD_a, bD_b, cD_c, \&\text{c.}).$$

CHAPTER VII.

INTERPRETATION OF SYMBOLS OF OPERATION.

1. It appears from a theorem in the second article of the Third Chapter, that if

$$U = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots + A_n x^n,$$

where A_0, A_1, A_2, &c., are constants, then

$$a^{xD_x}.U = A_0 + aA_1 x + a^2 A_2 x^2 + a^3 A_3 x^3 + \ldots + a^n A_n x^n.$$

The interpretation of this result is readily seen to be, that the operation of a^{xD_x} upon the mixed rational function U converts the variable x throughout it into ax.

Graves, *Fellowship Lectures*, 1851.

2. We saw, moreover, in the corresponding article of the Fourth Chapter, that if U contain any number of independent variables, and be expressed as a sum of a number of homogeneous functions, thus

$$U = u_0 + u_1 + u_2 + \ldots + u_n,$$

that

$$a^{xD_x + yD_y + zD_z + \ldots}.U,$$

or

$$a^{\nabla}.U = u_0 + au_1 + a^2 u_2 + \ldots + a^n u_n.$$

The interpretation of this result is, again, seen to be, that the operation of a^{∇} upon U converts the variables x, y, z, &c., throughout it, into ax, ay, az, &c., respectively.

3. Let it be proposed to investigate the value of the symbolic quantity

$$e^{\phi(x)D_x+\psi(y)D_y+\chi(z)D_z+\&c.}.U,$$

where

$$U=f(x,\ y,\ z,\ \&c.).$$

Now, if we put

$$\frac{dx}{\phi(x)}=d\xi,\quad \frac{dy}{\psi(y)}=d\eta,\quad \frac{dz}{\chi(z)}=d\zeta,\ \&c.,$$

the given symbolic quantity becomes

$$e^{D_\xi+D_\eta+D_\zeta+\&c.}.U;$$

and as U, from being a function of x, y, z, &c., can be transformed into a function of ξ, η, ζ, &c., by the aid of the assumptions just made, the question is reduced to a shape which admits of obvious solution.

Thus, as

$$U=f(x,\ y,\ z,\ \&c.)$$

and

$$\left.\begin{aligned} \xi+c &= \int\frac{dx}{\phi(x)}=\Phi(x)\\ \eta+d &= \int\frac{dy}{\psi(y)}=\Psi(y)\\ \zeta+e &= \int\frac{dz}{\chi(z)}=X(z)\\ &\&c. \end{aligned}\right\}$$

we get

$$U=f\{\Phi^{-1}(\xi+c),\ \Psi^{-1}(\eta+d),\ X^{-1}(\zeta+e),\ \&c.\},$$

and, therefore,

$$e^{D_\xi+D_\eta+\&c.}.U=f\{\Phi^{-1}(\xi+c+1),\ \Psi^{-1}(\eta+d+1),\ \&c.\}$$

whence, finally,

$$e^{\phi(x)D_x+\psi(y)D_y+\&c.}.f(x,y,\&c.)=f\{\Phi^{-1}(\Phi x+1),\Psi^{-1}(\Psi x+1),\&c.\},$$

or, the result of the operation of the symbol

$$e^{\phi(x)D_x+\psi(y)D_y+\&c.}$$

upon any function of x, y, &c., is to change x into $\Phi^{-1}(\Phi x + 1)$, y into $\Psi^{-1}(\Psi y + 1)$, &c.

In the practical application of this theorem, the difficulties with which we have to contend are, the deduction of the integrals

$$\int \frac{dx}{\phi(x)}, \quad \int \frac{dy}{\psi(y)}, \text{ \&c.,}$$

and the inversion of the functions Φ, Ψ, &c.

This article is mainly a generalization of results obtained, for a single variable, by the Rev. Professor Graves, and communicated by him to the Royal Irish Academy, April, 1852.

Examples.

(1.) The simplest and most obvious illustration of the valuable theorem contained in the previous article is afforded by the suppositions

$$\phi(x) = x, \quad \psi(y) = y, \quad \chi(z) = z, \text{ \&c.}$$

In this case, in fact, the operative symbol

$$\phi(x) D_x + \psi(y) D_y + \chi(z) D_z + \text{\&c.}$$

becomes

$$xD_x + yD_y + zD_z + \text{\&c.} = \nabla,$$

and therefore

$$e^{\nabla} . f(x, y, \text{\&c.}) = f\{\log^{-1}(1 + \log x), \log^{-1}(1 + \log y), \text{\&c.}\},$$

or

$$e^{\nabla} . f(x, y, z, \text{\&c.}) = f(ex, ey, ez, \text{\&c.}).$$

If we break up f into sets of homogeneous terms, it is evident that this result is identical with that given in the second article of the Fourth Chapter, already cited, namely

$$e^{\nabla} U = u_0 + eu_1 + e^2u_2 + \text{\&c.} + e^n u_n.$$

(2.) More generally, let

$$\phi(x) = x^m, \quad \psi(y) = y^n, \text{ \&c.,}$$

and the result of the evaluation of

$$e^{x^m D_x + y^n D_y + \&c.} f(x, y, \&c.)$$

is

$$f\left\{\frac{x}{\{1-(m-1)\,x^{m-1}\}^{\frac{1}{m-1}}},\ \frac{y}{\{1-(n-1)\,y^{n-1}\}^{\frac{1}{n-1}}},\ \&c.\right\}.$$

It is not difficult to verify this formula for the particular cases

$$m = n = \&c. = 0,$$

$$m = n = \&c. = 1.$$

In that in which

$$m = n = \&c. = 2,$$

we get the result

$$e^{x^2 D_x + y^2 D_y + \&c.} f(x, y, \&c) = f\left(\frac{x}{1-x}, \frac{y}{1-y}, \&c.\right);$$

and when

$$m = n = \&c. = 3,$$

we get

$$e^{x^3 D_x + y^3 D_y + \&c.}. f(x, y, \&c.) = f\left\{\frac{x}{(1-2x^2)^{\frac{1}{2}}}, \frac{y}{(1-2y^2)^{\frac{1}{2}}}, \&c.\right\}.$$

(3.) Let

$$\phi(x) = (a^2 - x^2)^{\frac{1}{2}}, \quad \psi(y) = (b^2 - y^2)^{\frac{1}{2}}, \ \&c.$$

and the result of the evaluation of

$$e^{(a^2-x^2)^{\frac{1}{2}} D_x + (b^2-y^2)^{\frac{1}{2}} D_y + \&c.} f(x, y, \&c.)$$

is

$$f\left\{a \sin\left(1 + \sin^{-1}\frac{x}{a}\right), \ b \sin\left(1 + \sin^{-1}\frac{y}{b}\right), \ \&c.\right\}.$$

(4.) Let

$$\phi(x) = a^2 + x^2, \quad \psi(y) = b^2 + y^2, \ \&c.,$$

and the result of the evaluation of

$$e^{(a^2+x^2)D_x + (b^2+y^2)D_y + \&c.}. f(x, y, \&c.)$$

is

$$f\left\{a \tan a \left(1 + \frac{1}{a} \tan^{-1} \frac{x}{a}\right), \quad b \tan b \left(1 + \frac{1}{b} \tan^{-1} \frac{y}{b}\right) \&\text{c.}\right\}.$$

(5.) Let

$$\phi(x) = x \log x, \quad \psi(y) = y \log y, \quad \&\text{c.},$$

and the result of the evaluation of

$$e^{x \log x D_x + y \log y D_y + \&\text{c.}} . f(x, y, \&\text{c.})$$

is

$$f(x^e, y^e, \&\text{c.}).$$

Thus the result of the operation of the symbol

$$e^{x \log x D_x + y \log y D_y + \&\text{c.}}$$

upon any function of x, y, &c., is the change of x into x^e, y into y^e, &c.

This theorem may be deduced directly with great facility, from the case of a single variable, by putting $x = e^\theta$, since then

$$e^{x \log x . D_x} . x^m = e^{\theta D_\theta}\left(1 + m\theta + \frac{m^2\theta^2}{1 . 2} + \&\text{c.}\right),$$

whence by the second article of the Third Chapter

$$e^{x \log x D_x} . x^m = 1 + e . m\theta + e^2 . \frac{m^2\theta^2}{1 . 2} + \&\text{c.} = e^{em\theta} = (x^e)^m.$$

(6.) Selecting now a particular form for the function operated on, we shall suppose that it is linear in x, y, &c. Then

$$e^{\phi(x) D_x + \psi(y) D_y + \&\text{c.}}(ax + by + \&\text{c.}) = a\,\Phi^{-1}(\Phi x + 1) + b\Psi^{-1}(\Psi y + 1) + \&\text{c.},$$

and we may introduce the values of $\phi(x)$, $\psi(y)$, &c., employed in the previous examples.

4. There are certain cases in which the evaluation of the quantity

$$e^{\phi(x) D_x + \psi(y) D_y + \&\text{c.}}\, U$$

may be considerably facilitated. Thus, if U consist of the product of certain minor functions u, v, w, &c., we may avail ourselves of the theorem given in the following article, namely, that

$$e^{\Psi}.\, uvw \ldots = e^{\Psi} u \,.\, e^{\Psi} v \,.\, e^{\Psi} w \ldots$$

if Ψ be such a symbol that

$$\Psi \,.\, uv = u\Psi v + v\Psi u\,;$$

since it is obvious that

$$\phi\,(x)\; D_x + \psi\,(y)\; D_y + \text{\&c.}$$

satisfies the required condition.

5. Ψ being a distributive symbol, such that

$$\Psi.\, uv = u\Psi v + v\Psi u,$$

it can be readily proved that

$$e^{\Psi}.\, uv = e^{\Psi} u \,.\, e^{\Psi} v.$$

For

$$e^{\Psi}.\, uv = \left(1 + \frac{\Psi}{1} + \frac{\Psi^2}{1.2} + \ldots\right).\, uv\,;$$

or, by the given condition,

$$e^{\Psi}.\, uv = uv + \frac{u\Psi v + v\Psi u}{1} + \frac{u\Psi^2 v + 2\Psi u.\Psi v + v\Psi^2 u}{1.2} + \text{\&c.},$$

a result coincident with the expansion of

$$e^{\Psi} u \,.\, e^{\Psi} v.$$

Hence it follows that

$$e^{\Psi}.\, uvw \ldots . = e^{\Psi} u \,.\, e^{\Psi} v \,.\, e^{\Psi} w \ldots .$$

and, consequently, that

$$e^{\Psi}.\, u^n = (e^{\Psi}.\, u)^n.$$

GRAVES, *Fellowship Lectures*, 1851.

Hence we derive the theorem that, if F denote any algebraic function,

$$e^{\Psi}.F(u) = F(e^{\Psi}u),$$

or the deduction of the result of the operation of e^{Ψ} upon any function of u is reduced to the deduction of the result of the operation of the same symbol upon u, simply.

Thus the distributive symbol

$$xD_x + yD_y + zD_z + \&c. = \nabla$$

satisfies the above law, and, therefore,

$$e^{\Theta\nabla}.F(U) = F(e^{\Theta\nabla}.U),$$

where Θ and U are any functions whatsoever of x, y, z, &c.

Example.

Investigate the algebraic value of the symbolic quantity

$$e^{\Theta_m\nabla}.F(\Theta_n),$$

where Θ_m, Θ_n are known homogeneous functions of the degrees m, n, respectively, in x, y, z, &c.

Now, by the fifth article of the Fourth Chapter,

$$e^{\Theta_m\nabla}.\Theta_n = \{1 + n\Theta_m + \frac{n(n+m)}{1.2}\Theta_m^2 + \&c.\}\Theta_n,$$

or

$$e^{\Theta_m\nabla}.\Theta_n = \frac{\Theta_n}{(1-m\Theta_m)^{\frac{n}{m}}},$$

and, therefore,

$$e^{\Theta_m\nabla}.F(\Theta_n) = F\left\{\frac{\Theta_n}{(1-m\Theta_m)^{\frac{n}{m}}}\right\}.$$

Thus

$$e^{(ax+by+cz)\nabla}.F(x^2+y^2+z^2) = F\left\{\frac{x^2+y^2+z^2}{\{1-(ax+by+cz)\}^2}\right\}.$$

Camb. and Dub. Math. Journal, 1853.

6. If it be proposed to investigate the value of the symbolic quantity

$$e^{\phi(x,y,z,\&c.)D_x+\chi(x,y,z,\&c.)D_y+\psi(x,y,z,\&c.)D_z+\&c.}.U,$$

where the coefficients of the symbols of differentiation are mixed functions of all the variables, and

$$U=F(x,\ y,\ z,\ \&c.),$$

a method completely different must be adopted, and the question seems to be, in general, insoluble.

If, however, we suppose that the result may be represented by

$$F\{\Phi(x,\ y,\ z,\ \&c.),\ X(x,\ y,\ z,\ \&c.),\ \Psi(x,\ y,\ z,\ \&c.),\ \&c.\},$$

or that the effect of the operating symbol is, while the form of F remains the same, to convert x into $\Phi(x, y, z, \&c.)$, y into $X(x, y, z, \&c.)$, and so on: the forms of these functions still unknown may be thus investigated.

Let

$$\phi(x,y,z,\&c.)D_x+\chi(x,y,z,\&c.)D_y+\psi(x,y,z,\&c.)D_z+\&c.=\omega.$$

Then

$$\omega\,.\,x=\phi(x,y,z,\&c.)$$

and

$$\omega\,.\,e^{\omega}.\,x=e^{\omega}.\,\omega\,.\,x=e^{\omega}.\,\phi(x,\ y,\ z,\ \&c.)$$

But, by hypothesis,

$$e^{\omega}.\,x=\Phi(x,\ y,\ z,\ \&c.),$$

and

$$e^{\omega}.\,\phi(x,y,z,\&c.)=\phi\{\Phi(x,y,z,\&c.),X(x,y,z,\&c.),\Psi(x,y,z,\&c.),\&c.\}$$

Hence the first equation of the following system, and, by processes precisely similar, the remainder:

$$(\phi D_x+\chi D_y+\psi D_z+\&c.)\,.\,\Phi=\phi(\Phi,\ X,\ \Psi,\ \&c.)$$
$$(\phi D_x+\chi D_y+\psi D_z+\&c.)\,.\,X=\chi(\Phi,\ X,\ \Psi,\ \&c.)$$
$$(\phi D_x+\chi D_y+\psi D_z+\&c.)\,.\,\Psi=\psi(\Phi,\ X,\ \Psi,\ \&c.)$$

Thus, to find the forms of Φ, X, Ψ, &c., we have a number of simultaneous partial differential equations equal to the number of functions whose forms are to be found.

If the converse of this question had been proposed for investigation, that is, if the forms of Φ, X, Ψ, &c. being given, we had been required to investigate the forms of ϕ, χ, ψ, &c., we may regard the previous system as one of simultaneous *functional* equations, from which the forms of ϕ, χ, ψ, &c. are to be determined.

GRAVES, *Fellowship Lectures*, 1852.

7. Let it be proposed to investigate the effect of the operation of the symbol $\sin\left(\frac{\pi}{2}xD\right)$ upon any rational and integer function of x, suppose

$$A_0 + A_1x + A_2x^2 + A_3x^3 + \&c. = X.$$

Then

$$\sin\left(\frac{\pi}{2}xD\right)X = \sin\left(\frac{\pi}{2}\,.\,0\right)A_0 + \sin\left(\frac{\pi}{2}\,.\,1\right)A_1x + \sin\left(\frac{\pi}{2}\,.\,2\right)A_2x^2 + \&c.,$$

or

$$\sin\left(\frac{\pi}{2}xD\right)X = A_1x - A_3x^3 + A_5x^5 - A_7x^7 + \&c.$$

Thus the effect of the operation $\sin\left(\frac{\pi}{2}xD\right)$ upon any rational and integer function of x is, to cancel the terms whose indices are *even* numbers, and to change the signs alternately of those terms whose indices are *odd*.

Similarly,

$$\cos\left(\frac{\pi}{2}xD\right)X = A_0 - A_2x^2 + A_4x^4 - A_6x^6 + \&c.;$$

and thus the effect of the operation $\cos\left(\frac{\pi}{2}xD\right)$ upon any rational and integer function of x is, to cancel those terms whose indices are *odd* numbers, and to change the signs alternately of those terms whose indices are *even*.

GRAVES, *Law's Mathematical Prize*, 1853.

8. In a manner precisely similar it may be proved that, if U be any rational and integer function of x, y, z, &c., broken up into sets of homogeneous functions, thus

$$U = u_0 + u_1 + u_2 + u_3 + \&c.,$$

the effect of the operation $\sin\left(\frac{\pi}{2}\nabla\right)$ upon U is, to cancel the homogeneous functions whose order is *even*, and to change the signs alternately of the functions whose order is *odd*, or that

$$\sin\left(\frac{\pi}{2}\nabla\right).U = u_1 - u_3 + u_5 - u_7 + \&c.$$

Again it may be proved that

$$\cos\left(\frac{\pi}{2}\nabla\right).U = u_0 - u_2 + u_4 - u_6 + \&c.,$$

and the interpretation is obvious.

By combining the last two results we find that

$$\left\{\sin\left(\frac{\pi}{2}\nabla\right) + \cos\left(\frac{\pi}{2}\nabla\right)\right\}.U = u_0 + u_1 - u_2 - u_3 + u_4 + u_5 - u_6 - u_7 + \&c.,$$

the interpretation of which is, that the effect of the operation

$$\sin\left(\frac{\pi}{2}\nabla\right) + \cos\left(\frac{\pi}{2}\Delta\right)$$

upon U, is simply to change the sign of each alternate *pair* of homogeneous functions, the series being supposed to ascend regularly from the order zero.

9. From what has been just said it appears that the solution of the linear total differential equation

$$\sin\left(\frac{\pi}{2}xD\right).y = 0$$

is

$$y = C_0 + C_2x^2 + C_4x^4 + \&c.,$$

and that the solution of the equation

$$\cos\left(\frac{\pi}{2}xD\right).y = 0$$

is

$$y = C_1x + C_3x^3 + C_5x^5 + \&c.,$$

where C_0, C_1, C_2, C_3, &c. are arbitrary constants.

10. In the same manner it appears that the solution of the partial differential equation

$$\sin\left(\frac{\pi}{2}\nabla\right).z = 0$$

is

$$z = u_0 + u_2 + u_4 + \&c.,$$

and that the solution of

$$\cos\left(\frac{\pi}{2}\nabla\right).z = 0$$

is

$$z = u_1 + u_3 + u_5 + \&c.,$$

where u_0, u_1, u_2, u_3, &c. are arbitrary homogeneous functions of the orders 0, 1, 2, 3, &c.

CHAPTER VIII.

APPLICATION TO ANALYTIC GEOMETRY.

1. If a plane curve, whose equation is

$$F(x, y) = 0,$$

be subjected to a simple *translation* in its plane, its equation in general assumes the form

$$F(x + a,\ y + b) = 0.$$

Now the symbolic equation

$$e^{aD_x + bD_y} . F(x, y) = 0$$

is exactly equivalent to this, or the operation of the symbol

$$e^{aD_x + bD_y}$$

upon the equation of a plane curve, is equivalent to the simple translation of the curve in its plane to a position determined by the values of the quantities a and b. We may also, of course, regard it as equivalent to a translation, in the opposite direction, of the axes of co-ordinates, to an origin whose co-ordinates are a and b.

Graves, *Fellowship Lectures*, 1851.

2. With regard to this theorem it may be observed, that a slight consideration of its form leads to the development of an unexpected and interesting result.

Thus, if the symbolic expression

$$e^{aD_x + bD_y} . F(x, y) = 0$$

be expanded, there is found for the equation, in full, of the translated curve, which I would propose to call the *transferée*,

$$\left\{1 + \frac{(aD_x + bD_y)}{1} + \frac{(aD_x + bD_y)^2}{1.2} + \&c.\right\}.F(x, y) = 0;$$

and, consequently, the points of intersection of the *transferée* with the original curve, lie on the curve

$$\left\{\frac{(aD_x + bD_y)}{1} + \frac{(aD_x + bD_y)^2}{1.2} + \&c.\right\}.F(x, y) = 0.$$

If the equation of the original curve be given as a direct algebraic function of x and y, as it generally is, the equation just found is evidently terminable, and of one degree lower than that of the given curve.

Examples.

(1.) Let it be proposed to investigate the character of the points of intersection of a curve of the *second* degree with any *transferée.*

Then, the equation of the curve being given in the shape

$$u_2 + u_1 + u_0 = 0,$$

the points of intersection with the *transferée,* determined by the co-ordinates a, b, lie on the curve

$$\frac{(aD_x + bD_y)}{1}.(u_2 + u_1) + \frac{(aD_x + bD_y)^2}{1.2}.u_2 = 0,$$

which is, of course, *a right line.*

(2.) Let it be proposed to investigate the character of the points of intersection of a curve of the *third* degree with any *transferée.*

The equation of the curve being given in the shape

$$u_3 + u_2 + u_1 + u_0 = 0,$$

the points of intersection with the *transferée,* determined by the co-ordinates a, b, lie on the curve

$$\frac{(aD_x + bD_y)}{1}(u_3 + u_2 + u_1) + \frac{(aD_x + bD_y)^2}{1.2}(u_3 + u_2) + \frac{(aD_x + bD_y)^3}{1.2.3}u_3 = 0,$$

which is plainly of the *second* degree.

3. In a manner precisely similar it appears that the operation of the symbol

$$e^{aD_x + bD_y + cD_z}$$

upon the equation of a surface

$$F(x, y, z) = 0$$

is equivalent to a simple translation of the surface to a position determined by the values of the quantities a, b, c.

4. The student will observe also that the curves of intersection of any surface with a *transferée* determined by the quantities a, b, c, lie upon the surface

$$\left\{\frac{(aD_x + bD_y + cD_z)}{1} + \frac{(aD_x + bD_y + cD_z)^2}{1.2} + \&c.\right\}.F(x, y, z) = 0,$$

which is, in general, of one degree lower than the given surface.

Thus, it appears that the curve of intersection of a surface of the second degree with any *transfereé* lies upon the plane

$$\frac{(aD_x + bD_y + cD_z)}{1}(u_2 + u_1) + \frac{(aD_x + bD_y + cD_z)^2}{1.2}u_2 = 0;$$

and that the curves of intersection of a surface of the third degree with any *transfereé* lie upon the surface of the second degree,

$$\frac{(aD_x + bD_y + cD_z)}{1}(u_3 + u_2 + u_1) + \frac{(aD_x + bD_y + cD_z)^2}{1.2}(u_3 + u_2) + \frac{(aD_x + bD_y + cD_z)^3}{1.2.3}u_3 = 0.$$

5. If the equation of a plane curve referred to polar co-ordinates be

$$f(r, \theta) = 0,$$

the equation of the same curve, after *rotation* in its plane through any angle ω round an axis passing through the origin and perpendicular to the plane, is, since any radius vector r is unaltered by the rotation,

$$f(r, \theta + \omega) = 0,$$

or

$$e^{\omega D_\theta} . f(r, \theta) = 0.$$

But, expressing the equation of the curve in rectangular co-ordinates,

$$f(r, \theta) = F(x, y),$$

and

$$D_\theta = D_\theta x . D_x + D_\theta y . D_y = xD_y - yD_x.$$

Consequently the equation of the curve after the rotation is

$$e^{\omega(xD_y - yD_x)} . F(x, y) = 0,$$

or the operation of the symbol

$$e^{\omega(xD_y - yD_x)}$$

upon the equation of a plane curve

$$F(x, y) = 0$$

is equivalent to the rotation of the curve in its plane through an angle ω round an axis passing through the origin and perpendicular to the plane.

GRAVES, *Fellowship Lectures*, 1851.

Moreover, we know from other sources that the equation of this new curve is

$$F(x \cos \omega - y \sin \omega,\ x \sin \omega + y \cos \omega) = 0.$$

Hence we derive the second inference, that the result of the evaluation of the symbolic quantity

$$e^{\omega(xD_y - yD_x)} . F(x, y)$$

is

$$F(x\cos\omega - y\sin\omega,\ x\sin\omega + y\cos\omega).$$

6. It has been remarked already in the second Chapter that, upon *à priori* grounds, we cannot write

$$e^{\omega(xD_y - yD_x) + aD_x + bD_y} . F(x, y)$$

as equivalent to

$$e^{\omega(xD_y - yD_x)} . e^{aD_x + bD_y} . F(x, y);$$

or, again, as equivalent to

$$e^{aD_x + bD_y} . e^{\omega(xD_y - yD_x)} . F(x, y),$$

since, in fact, the symbols

$$\omega(xD_y - yD_x), \quad aD_x + bD_y,$$

are not commutative.

In consistency with this result, it may be observed that, *a posteriori*, the equations

$$e^{\omega(xD_y - yD_x)} . e^{aD_x + bD_y} . F(x, y) = 0,$$

and

$$e^{aD_x + bD_y} . e^{\omega(xD_y - yD_x)} . F(x, y) = 0,$$

plainly represent distinctly posited curves, the axis of rotation in the one case being nearer to the curve than in the other; and the angle through which the curve is rotated being the same in both cases.

This appears again from analytical considerations, the first equation being equivalent to

$$F(x\cos\omega - y\sin\omega + a,\ x\sin\omega + y\cos\omega + b) = 0,$$

and the second to

$$F\{(x + a)\cos\omega - (y + b)\sin\omega,\ (x + a)\sin\omega + (y + b)\cos\omega\} = 0.$$

7. If the angle ω be very small, we may neglect terms of the second order in the expansion of

$$e^{\omega(xD_y - yD_x)} . F(x, y) = 0 ;$$

or, we may consider the virtual expansion of this expression to be

$$F(x, y) + \omega\,(xD_y - yD_x) . F(x, y) = 0.$$

Hence, if

$$F(x, y) = 0$$

be the equation of a plane curve, and this curve receive a very small rotation in its plane round an axis passing through the origin and perpendicular to the plane, the points of intersection of the curve so rotated with the original curve, lie on the curve

$$(xD_y - yD_x) . F(x, y) = 0,$$

which, of course, also passes through the origin.

GRAVES, *Fellowship Lectures*, 1851.

The student will observe that, in this case, the points of intersection of the two curves lie on a third curve of the same degree.

Example.

Let the first curve be a conic section, or let

$$F(x, y) = Ax^2 + A'y^2 + 2Bxy + 2\,(Cx + C'y) + D = 0.$$

The equation

$$(xD_y - yD_x) . F(x, y) = 0,$$

in this case, is

$$x\,(A'y + Bx + C') - y\,(Ax + By + C) = 0,$$

or

$$B\,(x^2 - y^2) + (A' - A)\,xy + C'x - Cy = 0 ;$$

and, consequently, if a conic section receive a very small rotation in its plane round an axis passing through the origin, and perpendicular to the plane, the points of intersection of the rotated conic, with the original, lie on *an equilateral hyperbola* passing through the origin.

If the conic section be a circle, the points of intersection of the rotated circle, with the original, lie on a right line passing through the origin, as from *à priori* considerations they should.

GRAVES, *Fellowship Lectures*, 1851.

8. It is possible that F may be of such a form that the equation

$$(xD_y - yD_x) . F(x, y) = 0$$

is satisfied identically, and in that case the original curve coincides with the rotated. This form is had by the solution of the equation

$$(xD_y - yD_x) . u = 0$$

which is

$$u = \phi_0(e^{-x^2}, e^{y^2}) = \Psi(x^2 + y^2);$$

and as the solution of

$$\Psi(x^2 + y^2) = 0$$

is

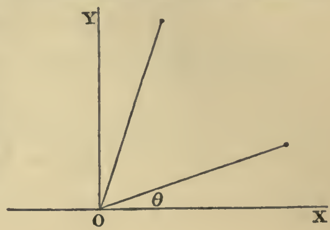
$$x^2 + y^2 = C,$$

it follows that the circle is the only curve which coincides with itself rotated through a small angle, and that only when the axis of rotation passes through its centre perpendicularly to its plane. It is plain, too, that this coincidence, once established, holds even when the rotation is supposed finite.

9. Let it be proposed to investigate a symbol, which, ope-

rating on a function of x and y, will interchange x and y, or transform $F(x, y)$, into $F(y, x)$.

This result will evidently be obtained if, the equation being transformed from rectangular into polar co-ordinates, a symbol be found, which, operating on the function of r and θ, changes θ into $\frac{1}{2}\pi - \theta$, r being left unaltered.



Now, we know that

$$e^{-\frac{1}{2}\pi D_\theta}.f(r, \theta) = f(r, \theta - \tfrac{1}{2}\pi),$$

and that the sign of $\theta - \frac{1}{2}\pi$ is altered throughout (see the second article of the Third Chapter) by operating with

$$(-1)^{(\theta - \frac{1}{2}\pi)D_{(\theta - \frac{1}{2}\pi)}} = (-1)^{(\theta - \frac{1}{2}\pi)D_\theta};$$

or that

$$(-1)^{(\theta - \frac{1}{2}\pi)D_\theta}.f(r, \theta - \tfrac{1}{2}\pi) = f(r, \tfrac{1}{2}\pi - \theta).$$

Hence it appears, by substituting for (–1) its exponential form, and transforming back to rectangular co-ordinates, that

$$e^{\pm\pi\sqrt{-1}(\tan^{-1}\frac{y}{x} - \frac{1}{2}\pi)(xD_y - yD_x)}.F(x, y) = F(y, x).$$

GRAVES, *Fellowship Lectures*, 1853.

10. If the equation of a surface referred to polar co-ordinates be

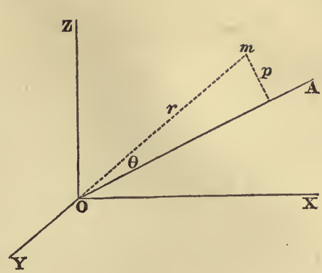
$$f(r, \theta, \phi) = 0,$$

the equation of the same surface, after rotation through any angle ψ round the axis from which θ is reckoned, is

$$f(r, \theta, \phi + \psi) = 0,$$

or

$$e^{\psi D_\phi}.f(r, \theta, \phi) = 0.$$



But if m be any point of the original surface, AO the axis of revolution, and ds an element perpendicular to the plane of $\overline{om, OA}$ at m, since

$$ds = pd\phi,$$

p being the perpendicular from m upon the axis OA, we have,

$$D_\phi = pD_s = p\,(D_s x\,.\,D_x + D_s y\,.\,D_y + D_s z\,.\,D_z).$$

Hence if l, m, n be the angles made by the axis of rotation OA with the axes of co-ordinates,

$$D_\phi = (y\cos n - z\cos m)D_x + (z\cos l - x\cos n)D_y + (x\cos m - y\cos l)D_z$$

or

$$D_\phi = \cos l(zD_y - yD_z) + \cos m(xD_z - zD_x) + \cos n(yD_x - xD_y).$$

Consequently, the equation of the surface after the rotation is

$$e^{\psi[\cos l(zD_y - yD_z) + \cos m(xD_z - zD_x) + \cos n(yD_x - xD_y)]}\,.\,F(x,\,y,\,z) = 0,$$

or the operation of the symbol

$$e^{\psi[\cos l(zD_y - yD_z) + \cos m(xD_z - zD_x) + \cos n(yD_x - xD_y)]}$$

upon the equation of the surface

$$F(x,\,y,\,z) = 0,$$

is equivalent to the rotation of the surface through an angle ψ round an axis passing through the origin, the angles made by which with the axes of co-ordinates are l, m, n.

GRAVES, *Fellowship Lectures*, 1851.

11. Again, from the illegitimacy of writing

$$e^{\psi[\cos l(zD_y - yD_z) + \cos m(xD_z - zD_x) + \cos n(yD_x - xD_y)]}\,.\,F(x,\,y,\,z)$$

as equivalent to

$$e^{\psi\cos l(zD_y-yD_z)} \cdot e^{\psi\cos m(xD_z-zD_x)} \cdot e^{\psi\cos n(yD_x-xD_y)} \cdot F(x, y, z),$$

we conclude that, in general, the principle of composition of rotations does not hold. When the angle of rotation ψ is *infinitely small*, second powers may be neglected in the expansion of the symbol of operation, the equivalence before illegitimate in this case becomes just, and we know independently that, in this case, the principle of composition of rotations does hold.

12. If the angle ψ be very small, we may, as before, neglect terms of the second order in the expansion of

$$e^{\psi[\cos l(zD_y-yD_z)+\cos m(xD_z-zD_x)+\cos n(yD_x-xD_y)]} \cdot F(x, y, z) = 0.$$

Hence if

$$F(x, y, z) = 0$$

be the equation of a surface, and this surface receive a very small rotation round an axis, the angles made by which with the axes of co-ordinates are l, m, n, the equation

$$\{\cos l\,(zD_y - yD_z) + \cos m\,(xD_z - zD_x) + \cos n\,(yD_x - xD_y)\} \cdot F(x, y, z) = 0$$

represents a third surface passing through the intersection of the first two, and, of course, through the origin.

Example.

If the given surface be of the second order, and its equation of the form

$$Ax^2 + A'y^2 + A''z^2 + 2(Byz + B'zx + B''xy) + 2(Cx + C'y + C''z) + D = 0,$$

it can be easily deduced that the curves of intersection of this surface, with itself rotated through a very small angle, round an axis, the angles made by which with the axes of co-ordinates are l, m, n, lie upon a third surface of the second order whose equation is

$$(B'\cos m - B''\cos n)\, x^2 + (B''\cos n - B\cos l)\, y^2 + (B\cos l - B'\cos m)\, z^2 + \&c. = 0,$$

and which passes through the origin. The reader will observe that the sum of the coefficients of x^2, y^2, and z^2, is equal to zero, in which circumstance consists the analogy with the corresponding theorem in plane geometry, before stated.

13. It is obvious, both geometrically and analytically, that if any surface coincide with itself rotated through a very small angle round any axis, the coincidence will hold when the angle through which the rotation takes place becomes finite; in other words, the surface must be one of revolution.

Hence it appears that the general differential equation of a surface of revolution described round an axis passing through the origin, and making with the axes of co-ordinates the angles l, m, n, is

$$\{\cos l\,(zD_y - yD_z) + \cos m\,(xD_z - zD_x) + \cos n\,(yD_x - xD_y)\}\,.u = 0,$$

or

$$\{(y\cos n - z\cos m)D_x + (z\cos l - x\cos n)D_y + (x\cos m - y\cos l)D_z\}\,.u = 0,$$

a form somewhat more symmetrical than those given by Monge and Leroy, and whose geometric interpretation is that a perpendicular to the plane of the radius vector and the axes of rotation is at right angles with the normal to the surface.

The student will observe that the equation is satisfied by

$$\left.\begin{array}{c}(y\cos n - z\cos m)^2 + (z\cos l - x\cos n)^2 + (x\cos m - y\cos l)^2 \\ - \Phi(x\cos l + y\cos m + z\cos n)\end{array}\right\} = 0$$

the geometrical interpretation of which is obvious.

14. The theorem of the last article, together with the corresponding one in plane geometry, may be derived with, perhaps, greater facility from the consideration of the forms of their respective equations in polar co-ordinates.

Thus, it appears that the only plane curve which coincides with itself rotated through any very small angle round an axis passing through the origin, and perpendicular to the plane of the curve, is given generally by

$$D_\theta . f(r, \theta) = 0;$$

or, the polar equation of the curve should be independent of θ, or be of the simple form

$$f(r) = 0,$$

which gives

$$r = \text{const.},$$

the general equation of a circle.

Similarly it appears that the only surface which coincides with itself rotated through a very small angle round the axis of polar co-ordinates, is generally given by

$$D_\phi . f(r, \theta, \phi) = 0;$$

or, the polar equation* of the surface should be independent of ϕ, or be of the simple form

$$f(r, \theta) = 0.$$

Thus, the general equation of a surface of revolution in rectangular co-ordinates would be

$$f\left\{(x^2 + y^2 + z^2)^{\frac{1}{2}},\ \cos^{-1} \frac{x \cos l + y \cos m + z \cos n}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}\right\} = 0,$$

which gives

$$x \cos l + y \cos m + z \cos n = \Phi(x^2 + y^2 + z^2).$$

15. It can be easily proved directly that the symbols

$$xD_x + yD_y, \qquad xD_y - yD_x,$$

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* For illustrations of the employment of polar co-ordinates in the investigation of the properties of surfaces, see Appendix B, *On the Quadrature of Surfaces, and the Rectification of Curves.*

and, consequently, any functions of those symbols, are commutative; a result which can also be easily derived geometrically from the considerations that

$$xD_x + yD_y = rD_r, \qquad xD_y - yD_x = D_\theta.$$

More generally it appears geometrically that the symbols

$$xD_x + yD_y + zD_z$$

and

$$\cos l(zD_y - yD_z) + \cos m(xD_z - zD_x) + \cos n(yD_x - xD_y),$$

and, consequently, any functions of those symbols, are commutative.

Hence it follows that the surface

$$a^{xD_x+yD_y+zD_z} \,.\, e^{\psi[\cos l(zD_y-yD_z)+\&c.]} \,.\, F(x, y, z) = 0,$$

is identical with the surface

$$e^{\psi[\cos l(zD_y-yD_z)+\&c.]} \,.\, a^{xD_x+yD_y+zD_z} \,.\, F(x, y, z) = 0.$$

This result, interpreted, shows that if any surface

$$F(x, y, z) = 0$$

be rotated through any angle round an axis passing through the origin, and a surface be taken similar to this, and similarly placed, the origin being the common centre of similitude, and a the ratio of their linear magnitudes, the resulting surface is the same as that which would be obtained if the steps of this process were inverted.

GRAVES, *Fellowship Lectures*, 1854.

16. If the equation to any plane curve be

$$U = F(x, y) = 0,$$

it is known that the general differential equation of the tangent line at any point (x, y) is

$$\alpha D_x U + \beta D_y U = xD_x U + yD_y U,$$

α, β being the *current* co-ordinates of the line.

But, if U be broken up into sets of homogeneous functions, thus,

$$U = u_n + u_{n-1} + \ldots + u_2 + u_1 + u_0,$$

we know that

$$(xD_x + yD_y).U = nu_n + (n-1)u_{n-1} + \ldots + 2u_2 + u_1,$$

and, by the nature of the given curve,

$$u_n + u_{n-1} + \ldots + u_2 + u_1 + u_0 = 0;$$

therefore

$$(xD_x + yD_y).U = -(u_{n-1} + 2u_{n-2} + \ldots + nu_0).$$

Hence the above form of the equation of the tangent line to the plane curve $U = 0$ is, in general, susceptible of reduction to the shape

$$\alpha D_x U + \beta D_y U + u_{n-1} + 2u_{n-2} + \ldots + nu_0 = 0,$$

in which α, β are the current co-ordinates of the line.*

17. It is known that, if P be the perpendicular from the origin upon the tangent at the point (x, y) of the plane curve

$$U = 0,$$

then is

$$P = \frac{xD_x U + yD_y U}{\{(D_x U)^2 + (D_y U)^2\}^{\frac{1}{2}}}.$$

From the previous article it appears that this expression is, in general, reducible to the shape

$$P = -\frac{u_{n-1} + 2u_{n-2} + \ldots + nu_0}{\{(D_x U)^2 + (D_y U)^2\}^{\frac{1}{2}}}.$$

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* It may be well to state, that the principle of the method here employed, with its results, as exhibited in the remainder of this Chapter, first suggested themselves in the month of December, 1851, and were communicated to the Dublin University Philosophical Society in the month of April, 1852.

From this expression we may derive the following theorem—

Given a plane curve of the n^{th} degree, the points thereon for which perpendiculars from the origin upon the corresponding tangents have the same constant value k, are determined by the intersection with the given curve, of one whose degree is $2(n-1)$, *and whose equation is*

$$k^2\{(D_xU)^2 + (D_yU)^2\} = (u_{n-1} + 2u_{n-2} + \ldots + nu_0)^2.$$

18. If a, β be taken as the *fixed* co-ordinates of a point not on the plane curve, the equation

$$aD_xU + \beta D_yU + u_{n-1} + 2u_{n-2} + \ldots + nu_0 = 0$$

expresses the polar curve of the fixed point (a, β) with respect to the given curve

$$U = 0.$$

Hence it appears that the polar curve to the origin with respect to the curve

$$u_n + u_{n-1} + \ldots + u_2 + u_1 + u_0 = 0,$$

is represented, in general, by the equation

$$u_{n-1} + 2u_{n-2} + 3u_{n-3} + \ldots + nu_0 = 0.$$

Examples.

(1.) Let

$$U = u_2 + u_1 + u_0 = 0,$$

or let the curve selected be of the *second* order.

Then, the general expression for the tangent at any point is

$$aD_xU + \beta D_yU + u_1 + 2u_0 = 0,$$

a, β being the current co-ordinates of the line.

The general expression for the perpendicular from the origin upon the tangent at any point is

$$P = -\frac{u_1 + 2u_0}{\{(D_xU)^2 + (D_yU)^2\}^{\frac{1}{2}}}.$$

The points on the given curve of the second order, for which perpendiculars from the origin upon the corresponding tangents have the same constant value k, are determined by the intersection with the given curve of another of the *second* order, whose equation is

$$k^2\{(D_xU)^2 + (D_yU^2\} = (u_1 + 2u_0)^2.$$

The general expression for the polar curve of a fixed point (α, β), with respect to the given curve, is

$$\alpha D_xU + \beta D_yU + u_1 + 2u_0 = 0,$$

which represents, of course, a right line, as it ought.

Finally, the general expression for the polar of the origin with respect to the given curve is

$$u_1 + 2u_0 = 0.$$

(2.) Let

$$U = u_3 + u_2 + u_1 + u_0 = 0,$$

or the curve selected be of the *third* order.

Then, the general expression for the tangent at any point is

$$\alpha D_xU + \beta D_yU + u_2 + 2u_1 + 3u_0 = 0,$$

α, β being the current co-ordinates of the line.

The general expression for the perpendicular from the origin upon the tangent at any point is

$$P = -\frac{u_2 + 2u_1 + 3u_0}{\{(D_xU)^2 + (D_yU)^2\}^{\frac{1}{2}}}.$$

The points on the given curve of the third order, for which the perpendiculars from the origin upon the corresponding tangents have the same constant value k, are determined by the intersection with the given curve of another of the *fourth* order, whose equation is

$$k^2\{(D_xU)^2 + (D_yU)^2\} = (u_2 + 2u_1 + 3u_0)^2.$$

The polar curve of any point (a, β), with respect to the given curve of the third order, is a curve of the *second* order, whose equation is

$$aD_xU + \beta D_yU + u_2 + 2u_1 + 3u_0 = 0.$$

Finally, the general expression for the polar of the origin, with respect to the given curve of the third order, is the curve of the second order,

$$u_2 + 2u_1 + 3u_0 = 0.$$

19. There is no difficulty in extending the results of the preceding articles to the case of geometry of three dimensions.

Thus the general differential equation of the tangent plane to any surface

$$U = F(x, y, z) = 0,$$

where

$$U = u_n + u_{n-1} + \ldots + u_2 + u_1 + u_0,$$

is reducible to the shape

$$aD_xU + \beta D_yU + \gamma D_zU + u_{n-1} + 2u_{n-2} + \ldots + nu_0 = 0,$$

where a, β, γ are the current co-ordinates of the plane.

Again, the general expression for the perpendicular from the origin upon the tangent plane at any point of the surface

$$U = 0$$

is

$$P = -\frac{u_{n-1} + 2u_{n-2} + \ldots + nu_0}{\{(D_xU)^2 + (D_yU)^2 + (D_zU)^2\}^{\frac{1}{2}}}.$$

Hence, again, the theorem—

Given a surface of the n^{th} degree, the points thereon for which perpendiculars from the origin upon the corresponding tangent planes have the same constant value k, are determined by the intersection with the given surface of another surface, whose degree is $2(n-1)$, and whose equation is

$$k^2\{(D_xU)^2 + (D_yU)^2 + (D_zU)^2\} = (u_{n-1} + 2u_{n-2} + \ldots + nu_0)^2.$$

Again, if a, β, γ be taken as the fixed co-ordinates of a point not on the surface, the equation

$$aD_xU + \beta D_yU + \gamma D_zU + u_{n-1} + 2u_{n-2} + \ldots + nu_0 = 0$$

expresses the polar surface of the fixed point with respect to the given surface.

Hence, again, it appears that the polar surface to the origin with respect to the given surface is represented in general by the equation

$$u_{n-1} + 2u_{n-2} + 3u_{n-3} + \ldots + nu_0 = 0.$$

The student may for himself apply these results to the cases of surfaces of the second and third orders, and also to the case of the surface whose equation is represented by

$$u_n = c.$$

20. If the point (a, β, γ) be supposed capable of motion on the surface of the m^{th} degree,

$$V = v_m + v_{m-1} + \ldots + v_1 + v_0 = 0,$$

let it assume various consecutive positions on this surface. The corresponding successive polars, taken with respect to $U = 0$, will by their intersections generate a third surface, whose relation to $V = 0$ is commonly expressed by the distinctive appellation of Reciprocal Polar, for the case in which $U = 0$ is of the second order.

To find the equation of this third surface, differentiating $V = 0$ and the general equation of the polar surface with respect to a, β, γ, we get

$$D_aV.da + D_\beta V.d\beta + D_\gamma V.d\gamma = 0,$$

$$D_xU.da + D_yU.dy + d_zU.dz = 0.$$

Multiplying the latter equation by the indeterminate quantity λ, adding, and putting the coefficients of da, $d\beta$, $d\gamma$, respectively, equal to zero, there results the system

$$\left.\begin{array}{l} D_\alpha V + \lambda D_x U = 0 \\ D_\beta V + \lambda D_y U = 0 \\ D_\gamma V + \lambda D_z U = 0 \end{array}\right\};$$

and between this system, the equation of the polar surface, and $V = 0$, we have to eliminate α, β, γ, and λ.

To accomplish this, we multiply the three equations of the last system by α, β, γ, respectively, and remembering that

$$\alpha D_\alpha V + \beta D_\beta V + \gamma D_\gamma V = -\{v_{m-1} + 2v_{m-2} + \ldots + mv_0\},$$
$$\alpha D_x U + \beta D_y U + \gamma D_z U = -\{u_{n-1} + 2u_{n-2} + \ldots + nu_0\},$$

we find that

$$\lambda = -\frac{v_{m-1} + 2v_{m-2} + \ldots + mv_0}{u_{n-1} + 2u_{n-2} + \ldots + nu_0} = -\frac{(V)}{(U)}.$$

Thus it remains for us to eliminate α, β, γ between the four equations,

$$V = 0,$$

and

$$\left.\begin{array}{l} \dfrac{1}{(V)} \cdot D_\alpha V = \dfrac{1}{(U)} \cdot D_x U \\ \dfrac{1}{(V)} \cdot D_\beta V = \dfrac{1}{(U)} \cdot D_y U \\ \dfrac{1}{(V)} \cdot D_\gamma V = \dfrac{1}{(U)} \cdot D_z U \end{array}\right\}$$

where the left-hand members contain only α, β, and γ; and the right-hand only x, y, and z.

Such an elimination, in the present state of analysis, is, I believe, impossible, and the general question therefore insoluble. Thus the only general representation of the envelope of the successive polar surfaces is the *system* of four equations last mentioned.

21. Upon communicating the above result to Mr. Spottiswoode, it was observed by him that the three last equations

may be written in a new form, possibly leading to interesting consequences, and I am indebted to the Rev. Richard Townsend, Fellow of Trinity College, Dublin, for a valuable modification of his suggestions.

If we remember that the point (α, β, γ) lies on the surface $V = 0$, it is obvious that, P being the perpendicular from the origin on the tangent plane at this point, and l, m, n the angles made by it with the co-ordinate axes, we may write those three last equations in the form

$$\left.\begin{aligned} -\frac{\cos l}{P} &= \frac{1}{(U)}.D_x U \\ -\frac{\cos m}{P} &= \frac{1}{(U)}.D_y U \\ -\frac{\cos n}{P} &= \frac{1}{(U)}.D_z U \end{aligned}\right\}.$$

It is evident that the right-hand members of the system do not admit of a modification similar to that which we have employed on the left hand, since the point (x, y, z) is *not* necessarily on the surface $U = 0$.

22. In one case, the general question of the envelope of the successive polars not only admits of solution, but the resultant equation of the envelope appears to possess both elegance and utility. It is that in which $V = 0$ assumes the symmetrical form

$$\frac{\alpha^m}{a^m} + \frac{\beta^m}{b^m} + \frac{\gamma^m}{c^m} = 1,$$

while $U = 0$ still retains all its generality.*

The three last equations of the third article, in this case, become

* The ordinary reciprocal of this last equation was given many years ago by the Rev. George Salmon, Fellow of Trinity College, and Donegal Lecturer on Mathematics in the University of Dublin.

$$\frac{\alpha^{m-1}}{a^m} + \frac{1}{(U)} \cdot D_x U = 0,$$

$$\frac{\beta^{m-1}}{b^m} + \frac{1}{(U)} \cdot D_y U = 0,$$

$$\frac{\gamma^{m-1}}{c^m} + \frac{1}{(U)} \cdot D_z U = 0,$$

and eliminating α, β, γ between these equations and $V = 0$, we get the equation of the envelope required, in the symmetrical form

$$(aD_x U)^{\frac{m}{m-1}} + (bD_y U)^{\frac{m}{m-1}} + (cD_z U)^{\frac{m}{m-1}} = [-(U)]^{\frac{m}{m-1}},$$

where

$$(U) = u_{n-1} + 2u_{n-2} + 3u_{n-3} + \ldots + nu_0.$$

The discussion of some particular cases will be found to lead to interesting results.

(1) When $m = 2$, or when the pole is confined to a central surface of the second degree, then will the degree of the envelope of the successive polars with respect to a surface of the n^{th} degree be, in general,

$$2(n-1).$$

(2) When, moreover, the surface, with respect to which the polar is taken, is also of the second degree and central, the envelope, or now *Reciprocal Polar*, to

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = 1$$

will be a third surface of the second degree, and its equation takes the symmetrical form

$$a^2(D_x U)^2 + b^2(D_y U)^2 + c^2(D_z U)^2 = (u_1 + u_0)^2.$$

(3) When the pole is confined to a central *curve* of the *second* degree, and the polar taken with respect to any curve

of the *third*, the envelope of the successive polars is a curve of the *fourth* degree, which is symmetrically represented by the equation

$$a^2(D_x U)^2 + b^2(D_y U)^2 = (u_2 + 2u_1 + 3u_0)^2,$$

a result which seems susceptible of elegant application to the theory of curves of the third degree.

23. For additional information with regard to the application of the Calculus of Operations to Geometry, the reader is referred to the valued treatise on the *Higher Plane Curves*, by the Rev. George Salmon; and to the elaborate papers on the Calculus of Forms, published in the Cambridge and Dublin Mathematical Journal, by Professor Boole and Mr. Sylvester.

CHAPTER IX.

MISCELLANEOUS APPLICATIONS IN THE DIFFERENTIAL AND INTEGRAL CALCULUS.

1. If $a_1, a_2, a_3, \ldots a_n$ be any given functions of x, then will

$$a_1 D.\, a_2 D.\, a_3 D \ldots D.a_n + (-)^n a_n D \ldots a_3 D \,.\, a_2 D \,.\, a_1,$$

the suffix to the symbol of differentiation being omitted for simplicity, be a perfect differential.

This theorem is readily established by induction. Thus, in the case of two functions, we have

$$a_1 D.\, a_2 = D.\, a_1 a_2 - a_2 D.\, a_1,$$

therefore

$$a_1 D.a_2 + (-)^2 a_2 D.a_1 = D.\, a_1 a_2,$$

a perfect differential.

Again

$$a_1 D a_2 D a_3 = D.\, a_1 a_2 D a_3 - D a_1 \,.\, a_2 D a_3,$$

or

$$a_1 D a_2 D a_3 = D.\, a_1 a_2 D a_3 - D.\, a_2 a_3 D a_1 + a_3 D a_2 D a_1,$$

whence

$$a_1 D a_2 D a_3 + (-)^3 a_3 D a_2 D a_1 = D.\, a_1 a_2 D a_3 - D.a_2 a_3 D a_1,$$

the right-hand member of which is evidently a perfect differential. The same conclusion may be established for the higher cases in a manner precisely similar.

From this theorem we infer that the condition necessary in order that, in general,

$$a_1 D a_2 D a_3 \ldots D a_n$$

be a perfect differential, is

$$a_n D a_{n-1} D \ldots D a_2 D a_1 = 0.$$

Graves, *Fellowship Lectures*, 1850.

2. If, n being odd, the series of functions a_1, a_2, a_3, &c., after the middle function, recur, *but in an inverted order*, the theorem admits of an elegant modification, and it is easy to show, by the method just employed, that

$$a_1 D a_2 D \ldots a_n \ldots D a_2 D a_1 + (-)^n a_n (D a_{n-1} D a_{n-2} \ldots D a_2 D a_1)^2$$

is a perfect differential.

GRAVES, *Fellowship Lectures*, 1850.

3. If the expansion of the operating symbol

$$\frac{1}{a_1 a_2 a_3 \ldots . a_n} . D a_1 D a_2 \ldots . D a_n$$

be represented by

$$C_n D^n + C_{n-1} D^{n-1} + \ldots . + C_1 D + C_0,$$

then will

$$e^{\int C_{n-1} dx} = a_1 . a_2^2 . a_3^3 . \ldots . a_n^n.$$

If the coefficient of D^{n-1} in the expansion of the given operating symbol be directly investigated, the truth of this theorem will readily appear. This coefficient is easily found to be the aggregate

$$\frac{D a_1}{a_1} + \frac{2 D a_2}{a_2} + \frac{3 D a_3}{a_3} + \ldots . + \frac{n D a_n}{a_n} = C_{n-1}.$$

Hence

$$\int C_{n-1} dx = \log a_1 + 2 \log a_2 + 3 \log a_3 + \ldots + n \log a_n,$$

and, therefore,

$$e^{\int C_{n-1} dx} = a_1 . a_2^2 . a_3^3 . \ldots . a_n^n.$$

GRAVES, *Fellowship Lectures*, 1850.

4. The expansion of the symbolic operator

$$u_{n+1} D u_n D \ldots \ldots D u_2 D u_1$$

upon any given subject v, being given in the form

$$P_0 + P_1 D + P_2 D^2 + \ldots . + P_n D^n,$$

it may be required to investigate the character of the coefficients P_0, P_1, P_2, &c., and the laws by which they are derived.

The reader will find no difficulty in seeing that

$$P_0 = u_{n+1} D u_n D \ldots\ldots D u_2 D u_1 .,$$

$$P_1 = u_{n+1} D u_n D \ldots D u_2 u_1 . + u_{n+1} D \ldots . D u_3 u_2 D u_1 . \\ + u_{n+1} D \ldots\ldots D u_4 u_3 D u_2 D u_1 . + \&c.$$

$$P_2 = u_{n+1} D u_n D \ldots D u_3 u_2 u_1 . + u_{n+1} D \ldots D u_4 u_3 u_2 D u_1 . + \&c.$$

$$\vdots$$

$$P_n = u_{n+1} u_n \ldots\ldots u_3 u_2 u_1 . ,$$

and he will observe that, while P_0 is simply the result of the operation of the given symbolic form on unity, P_1 is formed from P_0 by taking the sum of all the terms which may be obtained by a single omission of the letter D in P_0; similarly that P_2 is formed from P_1 by taking the sum of all the different terms which may be obtained by a single omission of the letter D in P_1, and so on.

GRAVES, *Fellowship Lectures*, 1850.

5. The expansion of the symbolic operator

$$u_1 D u_2 D \ldots\ldots D u_n D u_{n+1}$$

upon any given subject v, being given in the form

$$Q_0 + D Q_1 + D^2 Q_2 + \ldots + D^n Q_n,$$

let it be required to investigate the character of the coefficients Q_0, Q_1, Q_2, &c., and to trace the relations, if any, between these coefficients and those in the previous article P_0, P_1, P_2, &c.

It is evident, from preceding investigations, that

$$Q_n \ \ = + P_n$$
$$Q_{n-1} = - P_{n-1}$$
$$Q_{n-2} = + P_{n-2}$$
$$\vdots$$
$$Q_0 = (-)^n P_0.$$

GRAVES, *Fellowship Lectures*, 1850.

6. Hence it appears that, if R be any given function of x, the symmetrical symbolic operator

$$u_1 D u_2 D \ldots D u_n D u_{n+1} R u_{n+1} D u_n D \ldots D u_2 D u_1$$

may be written in the form

$$(-)^n \{P_0 - DP_1 + D^2 P_2 - \ldots + (-)^n D^n P_n\} R \{P_0 + P_1 D + P_2 D_2 + \ldots + P_n D^n\}$$

which is easily seen to be reducible to the form

$$A_0 + DA_1 D + D^2 A_2 D^2 + \ldots . + D^n A_n D^n.$$

GRAVES, *Fellowship Lectures*, 1850.

7. The integration of the differential equation of the n^{th} order,

$$D^n y + A_1 D^{n-1} y + A_2 D^{n-2} y + \ldots . + A_{n-1} Dy + A_n y = X, \qquad (1)$$

in which the coefficients A_1, A_2, A_3, &c., are given functions of the independent variable, can always be reduced to the integration of the same equation, without the right-hand member,

$$D^n y + A_1 D^{n-2} u + A_2 D^{n-1} u + \ldots . + A_{n-1} Du + A_n u = 0. \qquad (2)$$

Let

$$y = u_1 \int v_1 \, dx,$$

where u_1 satisfies the second equation, supposed integrated. Then, by substitution in the first equation, we get

$$\left.\begin{array}{c}\{D^n u_1 . \int v_1 dx + nD^{n-1}u_1 . v_1 + \ldots\ldots + nDu_1 . D^{n-2}v_1 \quad + u_1 . D^{n-1}v_1\} \\ + \\ A_1\{D^{n-1}u_1 . \int v_1 dx + (n-1)D^{n-2}u_1 . v_1 + \ldots + (n-1)Du_1 . D^{n-3}v_1 + u_1 . D^{n-2}v_1\} \\ + \\ A_2\{D^{n-2}u_1 . \int v_1 dx + (n-2)D^{n-3}u_1 . v_1 + \ldots + (n-2)Du_1 . D^{n-4}v_1 + u_1 . D^{n-3}v_1\} \\ + \&c.\end{array}\right\} = X$$

or, observing that the aggregate of the terms in the first vertical row disappears, reversing the order of the rows, and dividing by u_1,

$$D^{n-1}v_1 + B_2 D^{n-2}v_1 + B_3 D^{n-3}v_1 + \ldots = \frac{X}{u_1},$$

where

$$u_1 B_2 = nDu_1 + Au_1$$

$$u_1 B_3 = \frac{n . n-1}{1.2} D^2 u_1 + A_1(n-1)Du_1 + A_2 u_2$$

$$\&c.$$

Thus the integration of the equation of the n^{th} order is reduced to the integration of an equation of the same form, but of the order $(n-1)$.

Again, supposing that

$$v_1 = u_2 \int v_2 dx,$$

when u_2 is a particular solution of the equation at which we have just arrived, wanting its second member, we have the question reduced to the integration of the equation of the order $(n-2)$

$$D^{n-2}v_2 + C_3 D^{n-3}v_2 + C_4 D^{n-4}v_2 + \ldots = \frac{X}{u_1 u_2},$$

and finally to the integration of the equation

$$Dw + Pw = \frac{X}{u_1 u_2 \ldots u_{n-1}}.$$

Here, supposing that

$$w = u_n \int v_n dx,$$

we get, by substitution,

$$(Du_n + Pu_n) \cdot \int v_n dx + u_n v_n = \frac{X}{u_1 u_2 \ldots u_{n-1}};$$

and, since u_n is supposed to give

$$Du_n + Pu_n = 0,$$

we have

$$v_n = \frac{X}{u_1 u_2 u_3 \ldots u_{n-1} u_n};$$

and, therefore, the integral required is

$$y = u_1 \int u_2 \int u_3 \int \ldots \int \frac{X}{u_1 u_2 u_3 \ldots u_n} dx.$$

LIBRI, *Crelle's Journal*, vol. x.

8. The general integral of the equation with the right-hand member (1) may be *expressed in terms of the n particular integrals* of the equation without the right-hand member (2).

Let U_1, U_2, U_3, ... U_n be the n particular integrals of the latter equation, then

$$u = C_1 U_1 + C_2 U_2 + C_3 U_3 + \ldots + C_n U_n.$$

Divide both sides of the equation by U_1 and differentiating, we eliminate C_1,

$$D\frac{u}{U_1} = C_2 D\frac{U_2}{U_1} + C_3 D\frac{U_3}{U_1} + \ldots + C_n D\frac{U_n}{U_1}.$$

Dividing both sides again by the coefficient of C_2 and differentiating, we eliminate C_2; and by continuing this process we finally arrive at

$$D\frac{1}{V_n} D\frac{1}{V_{n-1}} D\frac{1}{V_{n-2}} \ldots D\frac{1}{V_2} D\frac{1}{U_1} \cdot u = 0,$$

where

$$V_2 = D\frac{U_2}{U_1},$$

$$V_3 = D\frac{D\dfrac{U_3}{U_1}}{D\dfrac{U_2}{U_1}},$$

&c.

Now the coefficient of $D^n u$ derived from this is evidently

$$\frac{1}{V_n V_{n-1} V_{n-2} \dots U_1};$$

and dividing by this, in order to make the coefficient of $D^n u$ unity, we have

$$V_n V_{n-1} V_{n-2} \dots U_1 D\frac{1}{V_n} D\frac{1}{V_{n-1}} D\frac{1}{V_{n-2}} \dots D\frac{1}{U_1} . u,$$

which must be equivalent to the left-hand member of equation (2), or the result of the operator *upon* y must be equal to X, or

$$y = U_1 \int D\frac{U_2}{U_1} \int D\frac{D\dfrac{U_3}{U_1}}{D\dfrac{U_2}{U_1}} \int \dots \int \frac{X dx}{U_1 D\dfrac{U_2}{U_1}\dots}.*$$

9. By a comparison of the last two articles, it is evident that the solutions of the successive *reduct* equations in the first article are susceptible of expression in terms of the n particular solutions of the equation (2); in fact, that

$$u_2 = D\frac{U_2}{U_1},$$

$$u_3 = D\frac{D\dfrac{U_3}{U_1}}{D\dfrac{U_2}{U_1}}, \text{ \&c.}$$

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* I am indebted for this proof to the "Treatise on Differential Equations and the Calculus of Finite Differences," by the Rev. J. Hymers. Cambridge: 1839.

Again, it is obvious that

$$u_n u_{n-1} \ldots u_2 u_1 D \frac{1}{u_n} D \frac{1}{u_{n-1}} \ldots . D \frac{1}{u_2} D \frac{1}{u_1} . y = X;$$

or that, as in Algebra, any linear differential equation

$$\Phi(D) y = X$$

can be written as a monomial if we can find the particular solutions of

$$\Phi(D) y = 0.$$

The extreme difficulty of applying M. Libri's speculations to practice precludes the introduction of examples; indeed, none have been proposed, so far as I am aware, which are not susceptible of solution by easier means.

10. Let it be proposed to investigate the result of the operation of the symbolic form

$$A_n D^n + A_{n-1} D^{n-1} + \ldots + A_1 D + A_0 = \Phi,$$

when A_n, A_{n-1}, ... A_0, are given *functions of the independent variable*, and when the subject of the operation is the product of two functions u and v.

It has been shown in the Third Chapter that, if $F(D)$ be any pure algebraic function of D, such as

$$A_n D^n + A_{n-1} D^{n-1} + \ldots + A_1 D + A_0,$$

where A_n, A_{n-1}, ... A_0 are *constants*, that

$$F(D) . uv = u . F(D) v + \frac{Du}{1} . F'(D) v + \frac{D^2 u}{1.2} . F''(D) v + \&c.,$$

or, writing this result in an obviously equivalent form, that

$$F(D)\, uv = F(D) v . u + \frac{1}{1} D_D F(D) v . Du + \frac{1}{1.2} D_D^2 F(D) v . D^2 u$$
$$+ \&c.$$

Upon reference to the process by which this theorem was established, it will readily be seen by the reader that when the quantities A_n, A_{n-1}, ... A_0 are no longer supposed constant, but given functions of the independent variable,

$$\Phi \cdot uv = \Phi v \cdot u + \frac{1}{1} D_D \Phi v \cdot Du + \frac{1}{1.2} D_D^2 \Phi v \cdot D^2 u + \&c.$$

Similarly,

$$u\Psi \cdot v = \Psi \cdot uv - \frac{1}{1} D_D \Psi \cdot Du \cdot v + \frac{1}{1.2} D_D^2 \Psi \cdot D^2 u \cdot v - \&c.$$

where

$$\Psi = D^n A_n + D^{n-1} A_{n-1} + \ldots + DA_1 + A_0.$$

GRAVES, *Fellowship Lectures*, 1850.

11. If $\pi, \rho, \rho_1, \rho_2, \ldots \rho_n$ be symbols of operation, such that, u being any subject,

$$\begin{aligned}
(\pi\rho - \rho\pi) \cdot u &= \rho_1 \cdot u \\
(\pi\rho_1 - \rho_1\pi) \cdot u &= \rho_2 \cdot u \\
(\pi\rho_2 - \rho_2\pi) \cdot u &= \rho_3 \cdot u \\
&\cdot \quad \cdot \\
&\cdot \quad \cdot \\
&\cdot \quad \cdot \\
(\pi\rho_{n-1} - \rho_{n-1}\pi) \cdot u &= \rho_n \cdot u;
\end{aligned}$$

then will

$$f(\pi)\rho \cdot u = \rho f(\pi) \cdot u + \frac{\rho_1}{1} \cdot f'(\pi) \cdot u + \frac{\rho_2}{1.2} f''(\pi) \cdot u + \ldots \qquad \text{(I.)}$$

$$\rho f(\pi) \cdot u = f(\pi)\rho \cdot u - f'(\pi) \frac{\rho_1}{1} \cdot u + f''(\pi) \frac{\rho_2}{1.2} \cdot u - \ldots \qquad \text{(II.)}$$

$$f\left(\pi + \frac{1}{\rho}\rho_1\right) \cdot u = f(\pi) \cdot u + \frac{1}{\rho} \frac{\rho_1}{1} f'(\pi)\, u + \frac{1}{\rho} \frac{\rho_2}{1.2} f''(\pi) \cdot u + \ldots \text{(III.)}$$

$$f\left(\pi - \rho_1 \frac{1}{\rho}\right) \cdot u = f(\pi) \cdot u - f'(\pi) \frac{\rho_1}{1} \cdot \frac{1}{\rho} \cdot u + f''(\pi) \frac{\rho_2}{1.2} \cdot \frac{1}{\rho} \cdot u - \ldots \text{(IV.)}$$

where, as before, f represents any rational and integer function of the quantity exhibited under it.

To prove these theorems, we shall write our data in the shape

$$\begin{aligned}
\pi\rho - \rho\pi &= \rho_1 \\
\pi\rho_1 - \rho_1\pi &= \rho_2 \\
\pi\rho_2 - \rho_2\pi &= \rho_3 \\
&\;\;\vdots \\
\pi\rho_{n-1} - \rho_{n-1}\pi &= \rho_n,
\end{aligned}$$

dropping the subject u; and operating on the first equation with π, we get

$$\pi^2\rho = \pi\rho_1 + (\rho_1 + \rho\pi)\pi,$$

or

$$\pi^2\rho = \rho\pi^2 + 2\rho_1\pi + \rho_2.$$

Similarly, we get

$$\pi^3\rho = \rho\pi^3 + 3\rho_1\pi^2 + 3\rho_2\pi + \rho_3,$$

and, generally,

$$\pi^n\rho = \rho\pi^n + n\rho_1\pi^{n-1} + \frac{n \,.\, n-1}{1 \,.\, 2}\rho_2\pi^{n-2} + \ldots,$$

whence

$$f(\pi)\rho = \rho f(\pi) + \frac{\rho_1}{1}f'(\pi) + \frac{\rho_2}{1.2}f''(\pi) + \ldots,$$

which, the subject u being written in its place, is the first theorem.

To deduce the second, we have

$$\rho\pi = \pi\rho - \rho_1;$$

and operating with each side of this equation upon π, we have

$$\rho\pi^2 = \pi^2\rho - 2\pi\rho_1 + \rho_2,$$

and, generally,

$$\rho\pi^n = \pi^n\rho - n\pi^{n-1}\rho_1 + \frac{n \,.\, n-1}{1 \,.\, 2}\pi^{n-2}\rho_2 \mp \ldots,$$

whence

$$\rho f(\pi) = f(\pi)\rho - f'(\pi)\frac{\rho_1}{1} + f''(\pi)\frac{\rho_2}{1.2} - \ldots,$$

which, the subject being introduced, is the theorem required.

In deducing the third theorem, we observe that

$$\pi + \frac{1}{\rho}\rho_1 = \frac{1}{\rho}\pi\rho,$$

and hence we only require to prove that

$$f\left(\frac{1}{\rho}\pi\rho\right) = f(\pi) + \frac{1}{\rho}\frac{\rho_1}{1}f'(\pi) + \frac{1}{\rho}\frac{\rho_2}{1.2}f''(\pi) + \ldots;$$

or that

$$f\left(\frac{1}{\rho}\pi\rho\right) = \frac{1}{\rho}\left\{\rho f(\pi) + \frac{\rho_1}{1}f'(\pi) + \frac{\rho_2}{1.2}f''(\pi) + \ldots\right\}$$

Now this we can easily do, for

$$\left(\frac{1}{\rho}\pi\rho\right)^2 = \left(\frac{1}{\rho}\pi\rho\right).\left(\frac{1}{\rho}\pi\rho\right) = \frac{1}{\rho}\pi^2\rho,$$

$$\left(\frac{1}{\rho}\pi\rho\right)^3 = \ldots\ldots\ldots\ldots = \frac{1}{\rho}\pi^3\rho,$$

and, in general,

$$f\left(\frac{1}{\rho}\pi\rho\right) = \frac{1}{\rho}f(\pi)\rho;$$

and, inserting from the first theorem its value for

$$f(\pi)\rho,$$

we get the theorem required.

The fourth theorem admits of deduction with equal simplicity.

Examples.

(1.) Let

$$\pi = D, \qquad \rho = X.$$

Then

$$DX - XD = X' = \rho_1,$$

$$DX' - X'D = X'' = \rho_2,$$

$$\&c.,$$

and therefore

$$f(D)\,X\,.\,u = Xf(D)\,.\,u + \frac{X'}{1}\,.f'(D)\,.\,u + \frac{X''}{1\,.\,2}\,.f'(D)\,.\,u + \&c.,$$

$$f\left(D + \frac{X'}{X}\right).u = f(D)\,.\,u + \frac{1}{1}\frac{X'}{X}\,.f'(D)\,.\,u + \frac{1}{1.2}\frac{X''}{X}\,.f''(D)\,.\,u + \&c.$$

(2.) Again, let

$$\pi = x, \qquad \rho = \phi(D);$$

and since

$$x\phi(D) - \phi(D)x = -\phi'(D) = \rho_1,$$

$$-x\phi'(D) + \phi'(D)x = +\phi''(D) = \rho_2,$$

$$\&c.,$$

we get

$$f\left(x - \frac{\phi'(D)}{\phi(D)}\right).u = \frac{1}{\phi(D)}\left\{\phi(D)f(x) - \phi'(D)\frac{f'(x)}{1} + \phi''(D)\frac{f''(x)}{1\,.\,2} - \&c.\right\}.u.$$

DONKIN, *Camb. and Dub. Math. Journal*, 1850.

CHAPTER X.

APPLICATION TO THE CALCULUS OF FINITE DIFFERENCES.

1. It may easily be seen, as the result of equivalent expansions, that $f(x+h)$ may be represented by the symbolic quantity

$$e^{hD}.f(x),$$

omitting for the present, for the greater facility in writing, the suffix to the symbol D.

Hence it follows that

$$e^{D}.f(x) = f(x+1),$$

and consequently that

$$e^{D}.f(x) - f(x) = f(x+1) - f(x);$$

or, Δ being the symbol of the Calculus of Finite Differences, that

$$(e^{D} - 1).f(x) = \Delta.f(x).$$

2. By successive operation we find that

$$(e^{D} - 1)^{n}.f(x) = \Delta^{n}.f(x),$$

whence *at once* the expression for the n^{th} difference of a function in terms of the function and its n successive values, namely,

$$\Delta^{n}.u_x = u_{x+n} - \frac{n}{1}.u_{x+n-1} + \frac{n.n-1}{1.2}.u_{x+n-2} - \&c.$$

From the equivalence immediately previous to this we may obviously deduce the general theorem,

$$F(\Delta).u_x = F(e^{D} - 1).u_x,$$

F being any algebraic function.

3. With regard to the equivalence

$$e^D - 1 = \Delta,$$

we may observe that, since two functions of the same symbol are commutative, and since Δ is exhibited in a function of D, Δ and D are commutative.

From this equivalence, or directly, we derive

$$e^D = \Delta + 1,$$

whence

$$e^{nD} . u_x = (1 + \Delta)^n . u_x;$$

and again we obtain the theorem, converse to the previous, by which the n^{th} successive function is represented in terms of its primitive and its n successive differences, namely,

$$u_{x+n} = u_x + \frac{n}{1} . \Delta u_x + \frac{n . n-1}{1 . 2} . \Delta^2 u_x + \&c.$$

It is again obvious that from the equivalence immediately previous to this we may deduce the general theorem

$$F(e^D) . u_x = F(1 + \Delta) . u_x.$$

4. From the equivalence

$$\Delta^n . f(x) = f(x+n) - \frac{n}{1} . f(x+n-1) + \frac{n . n-1}{1 . 2} . f(x+n-2) - \&c.,$$

we derive, by supposing $x = 0$, the singular theorem

$$\Delta^n . f(0) = f(n) - \frac{n}{1} . f(n-1) + \frac{n . n-1}{1 . 2} . f(n-2) - \&c.,$$

and, as a particular case,

$$\Delta^n . 0^m . = n^m - \frac{n}{1} . (n-1)^m + \frac{n . n-1}{1 . 2} . (n-2)^m - \&c.,$$

and as long as m is $< n$, the left-hand member is equal to zero.

Similarly from the equivalence

$$f(x+n) = f(x) + \frac{n}{1} . \Delta f(x) + \frac{n . n-1}{1 . 2} . \Delta^2 f(x) + \&c.$$

we derive the theorem

$$f(n) = f(0) + \frac{n}{1} . \Delta f(0) + \frac{n . n-1}{1 . 2} . \Delta^2 f(0) + \&c.,$$

and as a particular case,

$$n^m = \frac{n . n-1 . \dots . n-m+1}{1 . 2 . \dots . m} . \Delta^m 0^m + \&c.,$$

the previous terms disappearing.

With regard to the expression

$$\Delta^n . 0^m,$$

it may be observed that it is merely a conventional notation implying that, after the operation signified by Δ^n has been performed upon x^m, we put $x = 0$, and take the result. In general

$$\Delta^n . f(0) = \{\Delta^n f(x)\}_{x=0}.$$

5. As Δ is expressed in terms of D by the equivalence

$$\Delta = e^D - 1,$$

so D is expressed in terms of Δ by the equivalence

$$D \doteq \log(1 + \Delta),$$

whence

$$D^n . u_x = \log^n (1 + \Delta) . u_x;$$

or, more generally, F being any algebraic function,

$$F(D) . u_x = F\{\log(1 + \Delta)\} . u_x.$$

It may be observed that all the above theorems hold good for negative powers, since Δ^{-1} is distributive, and D and Δ are commutative.

6. To develope the n^{th} difference of a function in direct powers of the differential coefficients of the function.

We know that

$$\Delta^n = (e^D - 1)^n,$$

and consequently we may evidently write

$$\Delta^n = \{A_0 + A_1 D + A_2 D^2 + A_3 D^3 + \ldots\},$$

where A_0, A_1, &c. are to be determined.

Now it is plain that, as the values of these constants are independent of the particular subject operated on, we may select such subjects as may serve for their determination. But if we suppose each side to operate on x^m, and put $x = 0$ in the result, both sides will vanish as long as m is less than n, and when m is equal or greater than n,

$$A_m = \frac{\Delta^n . 0^m}{\overline{m}}.$$

Consequently

$$\Delta^n . u_x = \left\{ \frac{\Delta^n . 0^n}{\overline{n}} D^n + \frac{\Delta^n . 0^{n+1}}{\overline{n+1}} D^{n+1} + \ldots \right\} . u_x.$$

7. Mr. Curtis has successfully investigated the theorems in the Calculus of Finite Differences corresponding to those in the Differential Calculus discovered by Mr. Hargreave, and given in the Third Chapter of this work, namely,

$$F(D) . uv = u . F(D) v + \frac{Du}{1} . F'(D) v + \frac{D^2 u}{1 . 2} . F''(D) v + \text{\&c.},$$

and

$$u . F(D) v = F(D) . uv - F'(D) . \frac{Du}{1} . v + F''(D) . \frac{D^2 u}{1 . 2} . v - \text{\&c.}$$

The analogues are, respectively,

$$F(\Delta) . u_x v_x = u_x . F(\Delta) v_x + \frac{\Delta u_x}{1} . F'(\Delta) v_{x+1} + \frac{\Delta^2 u_x}{1 . 2} . F''(\Delta) v_{x+2} + \text{\&c.},$$

and

$$u_x . F(\Delta) v_x = F(\Delta) . u_x v_x - F'(\Delta) . \frac{\Delta u_x}{1} . v_{x+1} + F''(\Delta) . \frac{\Delta^2 u_x}{1 . 2} . v_{x+2} - \text{\&c.}$$

8. It is obvious that

$$\Delta . a^x = (a - 1) . a^x.$$

Operate with Δ a second time, and

$$\Delta^2 . a^x = (a-1)^2 . a^x,$$

and, by successive operation,

$$\Delta^p . a^x = (a-1)^p . a^x.$$

Hence, if F be any algebraic function, we derive the theorem

$$F(\Delta) . a^x = F(a-1) . a^x;$$

or more generally, if C be any constant,

$$F(\Delta) . Ca^x = F(a-1) . Ca^x.$$

9. As this theorem holds as well for inverse functions as for direct, it is plain that we may at once employ it for the solution of the class of equations in Finite Differences, with constant coefficients, represented by

$$\Delta^n u_x + P\Delta^{n-1} . u_x + Q\Delta^{n-2} . u_x + \ldots + Tu_x = f(e^x, \sin x, \cos x),$$

where P, Q, &c. are constants; or, the right-hand member being reduced to the form of a sum represented by

$$F(\Delta) . u_x = \Sigma Me^{mx},$$

where m may be positive or negative, fractional or integer, real or imaginary.

The solution, then, is given by the symbolic form

$$u_x = \Sigma M \frac{e^{mx}}{F(e^m - 1)} + \frac{1}{F(\Delta)} . 0;$$

and the complete evaluation of this form depends on the nature of the roots of

$$F(\Delta) = 0.$$

If the roots be all real and unequal, the arbitrary portion of the solution is

$$C_1(\alpha+1)^x + C_2(\beta+1)^x + C_3(\gamma+1)^x + \text{\&c.}$$

If there be p roots whose common value is a, the arbitrary portion of the solution is

$$C_1(\alpha+1)^x . x^{p-1} + C_1'(\alpha+1)^x . x^{p-2} + \ldots + C_2(\beta+1)^x + \&c.$$

Finally, if there be pairs of imaginary roots, the form of the arbitrary portion of the solution is

$$C_1\{\alpha+\beta\sqrt{(-)}+1\}^x + C_2\{\alpha-\beta\sqrt{(-)}+1\}^x + C_3(\gamma+1)^x + \&c.$$

Examples.

(1) $$\Delta^2 . u_x + a^2 . u_x = \cos mx.$$

This is, of course, equivalent to

$$\Delta^2 . u_x + a^2 . u_x = \tfrac{1}{2}\{e^{mx\sqrt{-1}} + e^{-mx\sqrt{-1}}\};$$

and consequently the solution is, at once,

$$u_x = \tfrac{1}{2}\left\{\frac{e^{mx\sqrt{-1}}}{(e^{m\sqrt{-1}}-1)^2+a^2} + \frac{e^{-mx\sqrt{-1}}}{(e^{-m\sqrt{-1}}-1)+a^2}\right\} + C_1(1+a\sqrt{-})^x + C_2(1-a\sqrt{-})^x.$$

(2) $$\Delta^2 . u_x - 2a\Delta . u_x + a^2 u_x = \sin mx.$$

This, again, is equivalent to

$$(\Delta - a)^2 . u_x = \frac{1}{2\sqrt{-1}}\{e^{mx\sqrt{-1}} - e^{-mx\sqrt{-1}}\},$$

and the solution is

$$u_x = \frac{1}{2\sqrt{-1}}\left\{\frac{e^{mx\sqrt{-1}}}{(e^{m\sqrt{-1}}-1-a)^2} - \frac{e^{-mx\sqrt{-1}}}{(e^{-m\sqrt{-1}}-1-a)^2}\right\} + C_1(a+1)^x . x + C_2(a+1)^x.$$

10. It will naturally occur to the reader that some corresponding general method of solution should exist for equations represented by

$$u_{x+n} + Pu_{x+n-1} + Qu_{x+n-2} + \ldots + Tu_x = f(e^x, \sin x, \cos x).$$

Such an equation may obviously be reduced to the symbolic form

$$e^{nD}.\,u_x + Pe^{(n-1)D}.\,u_x + Qe^{(n-2)D}.\,u_x + \ldots + Tu_x = \Sigma Me^{mx},$$

or

$$F(e^D).\,u_x = \Sigma Me^{mx},$$

where m is positive or negative, fractional or integer, real or imaginary.

Now it can be readily proved, as in the previous article, that

$$F(\epsilon^D).C\alpha^x = F(\alpha).C\alpha^x;$$

and, consequently, the symbolic solution of the equation in Finite Differences is

$$u_x = \Sigma M \frac{e^{mx}}{F(e^m)} + \frac{1}{F(e^D)}.0,$$

the evaluation of which, as before, depends upon the nature of the roots of

$$F(e^D) = 0.$$

It will be sufficient to discuss the case in which all the roots are real and unequal, as the remaining cases can then at once be written down by the aid of the last article. In this case, the form of the arbitrary portion of the solution is

$$C_1\alpha^x + C_2\beta^x + C_3\gamma^x + \ldots$$

The relation between this article and the preceding is readily seen from the consideration, that the general equation of the present article may be written down in the form

$$(\Delta+1)^n.\,u_x + P(\Delta+1)^{n-1}.\,u_x + \ldots + Tu_x = f(e^x, \sin x, \cos x),$$

or

$$F(\Delta+1).\,u_x = \Sigma Me^{mx}.$$

Examples.

(1.) $$u_{x+2} + a^2.\,u_x = \cos mx.$$

This is equivalent to

$$e^{2D}.\,u_x + a^2.\,u_x = \tfrac{1}{2}\left\{e^{mx\sqrt{-1}} + e^{-mx\sqrt{-1}}\right\},$$

and the solution is

$$u_x = \tfrac{1}{2}\left\{\frac{e^{mx\sqrt{-1}}}{e^{2m\sqrt{-1}}+a^2} + \frac{e^{-mx\sqrt{-1}}}{e^{-2m\sqrt{-1}}+a^2}\right\} + C_1(a\sqrt{-1})^x + C_2(-a\sqrt{-1})^x.$$

(2.) $$u_{x+2} - 2au_{x+1} + a^2 u_x = \sin mx.$$

This is equivalent to

$$(e^D - a)^2 . u_x = \frac{1}{2\sqrt{-1}}\{e^{mx\sqrt{-1}} - e^{-mx\sqrt{-1}}\},$$

and the solution is

$$u_x = \frac{1}{2\sqrt{-1}}\left\{\frac{e^{mx\sqrt{-1}}}{(e^{m\sqrt{-1}}-a)^2} - \frac{e^{-mx\sqrt{-1}}}{(e^{-m\sqrt{-1}}-a)^2}\right\} + C_1 a^x . x + C_2 a^x.$$

11. If we operate on both sides of the theorem, given in the third article of the Sixth Chapter, with

$$e^{\phi x D_x + \psi y D_y + \&c.}$$

we easily find that

$$e^{2(\phi x D_x + \psi y D_y + \&c.)} f(x, y, \&c.) = f\{\Phi^{-1}(\Phi x + 2), \Psi^{-1}(\Psi y + 2), \&c.\};$$

and hence, in general, that

$$e^{m(\phi x D_x + \psi y D_y + \&c.)} f(x, y, \&c.) = f\{\Phi^{-1}(\Phi x + m), \Psi^{-1}(\Psi y + m), \&c.\}$$

Thus, the form of f being supposed unknown, and those of Φ and Ψ given, the solution of the equation of Finite Differences, with constant coefficients,

$$\left.\begin{array}{c} Af\{\Phi^{-1}(\Phi x + a),\ \Psi^{-1}(\Psi y + a)\} \\ + \\ Bf\{\Phi^{-1}(\Phi x + b),\ \Psi^{-1}(\Psi y + b)\} \\ + \&c. \end{array}\right\} = 0,$$

is reduced to the solution of the symbolic partial differential equation

$$Ae^{a(\phi x D_x + \psi y D_y)} z + Be^{b(\phi x D_x + \psi y D_y)} z + \&c. = 0,$$

which may be written, for brevity,

$$F(e^{\phi x D_x + \psi y D_y}) z = 0;$$

or, *by the previous transformation*, given in the chapter cited,

$$F(e^{D_\xi + D_\eta})\, z = 0.$$

Now, if the roots of

$$F(p) = 0$$

be all real and unequal, the symbolic solution of this equation is

$$z = (e^{D_\xi + D_\eta} - m)^{-1} . 0 + (e^{D_\xi + D_\eta} - n)^{-1} . 0 + \&c.$$

where m, n, &c. are the values of the roots.

But by a previous theorem, given in the first article of the Third Chapter,

$$\chi(D_\xi + D_\eta) . f_m(e^\xi, e^\eta) = \chi(m) . f_m(e^\xi, e^\eta),$$

f_m being a homogeneous function of the m^{th} degree.

Hence the solution of the symbolic equation, and therefore the solution of the equation of finite differences, is, substituting for m, $\log m$,

$$z = u_{\log m}(e^\xi, e^\eta) + u_{\log n}(e^\xi, e^\eta) + \&c.,$$

where *the forms* of $u_{\log m}$, $u_{\log n}$, &c. are arbitrary, but their degrees given by the suffixes.

Finally, introducing the arbitrary constants c, d, &c., as is evidently legitimate, and then substituting their values for $\xi + c$, $\eta + d$, &c., we get the solution in the form

$$z = u_{\log m}(e^{\Phi x}, e^{\Psi y}) + u_{\log n}(e^{\Phi x}, e^{\Psi y}) + \&c.$$

If

$$F(p) = 0$$

contain pairs of imaginary roots, the solution assumes the form

$$z = u_{\log_{(m+n\sqrt{-1})}}(e^{\Phi x}, e^{\Psi y}) + u_{\log_{(m-n\sqrt{-1})}}(e^{\Phi x}, e^{\Psi y}) + \&c. + u_{\log p} + \&c.$$

Finally, if the same equation contain α equal roots, whose common value is m, the form of the solution is

$$z = u_{\log m}(e^{\Phi x}, e^{\Psi y}).(\Phi x + \Psi y)^{\alpha-1} + v_{\log m}(e^{\Phi x}, e^{\Psi y}).(\Phi x + \Psi y)^{\alpha-2} + \&c.$$
$$+ u_{\log n}(e^{\Phi x}, e^{\Psi y}) + \&c.,$$

where $u_{\log m}$, $v_{\log m}$ are different arbitrary homogeneous functions of the same degree.

12. It is now at once obvious that we are prepared to solve such an equation in finite differences as

$$A\phi(x+a,\ y+a,\ \&c.) + B\phi(x+b,\ y+b,\ \&c.) + \&c. = 0,$$

either as an illustration of the previous article, or independently. Adopting the latter course, it is easy to see that we can solve the still higher equation

$$\Sigma A\phi(x+a,\ y+a,\ \&c.) = f(e^x,\ \sin x,\ e^y,\ \sin y,\ \&c.),$$

where we can reduce the right-hand member to the form

$$\Sigma A_{p,q,\,\&c.}\, e^{px+qy+\,\&c.},$$

p, q, &c. being positive or negative, integral or fractional, real or imaginary.

For, throwing the equation into the form

$$F(e^{D_x+D_y})\,\phi(x,\ y,\ \&c.) = \Sigma A_{p,q,\&c.}\, e^{px+qy+\,\&c.},$$

we have the solution in the form

$$\phi = \Sigma A_{p,q,\,\&c.} \frac{e^{px+qy+\,\&c.}}{F(e^{p+q+\,\&c.})} + u_{\log m}(e^x,\ e^y,\ \&c.) + \&c.,$$

the roots of $F(p) = 0$ being supposed, the simplest case, all real and unequal.

It is evident that the solution of the equation in finite differences, in which there is but a single variable, is but a particular case of the form now stated.

Camb. and Dub. Math. Journal, 1853.

13. If the equation to be solved be reducible to the type

$$F(e^D)\,.\,u_x + A_x u_x = B_x,$$

where A_x, B_x are given functions of x, we may proceed by a method analogous to that exhibited in the opening articles of the Fifth Chapter.

Thus, operating on both sides of the representative equation with

$$\frac{1}{F(e^D)}$$

we get

$$u_x + \frac{1}{F(e^D)} A_x u_x = \frac{1}{F(e^D)} B_x + \frac{1}{F(e^D)} 0,$$

or

$$1 + \frac{1}{F(e^D)} A_x u_x = \frac{1}{F(e^D)} B_x + \Sigma C_a a^x,$$

where the last term is the ordinary complementary function upon the supposition that all the roots of

$$F(u) = 0$$

are real and unequal, and in which, if any modification should arise from the existence of equal or imaginary roots, the generality of the method is not affected.

Now, dissecting the operator in the left-hand member from its subject, and operating with the expansion of its inverse upon the right-hand member, we get

$$y = \begin{cases} \{1 - \frac{1}{F(e^D)} A_x + \frac{1}{F(e^D)} A_x \frac{1}{F(e^D)} A_x - \text{\&c.}\} \frac{1}{F(e^D)} B_x \\ + \\ \{1 - \frac{1}{F(e^D)} A_x + \frac{1}{F(e^D)} A_x \frac{1}{F(e^D)} A_x - \text{\&c.}\} \Sigma C_a a^x \end{cases}$$

the further reduction of which depends upon the particular forms of the given functions A_x, B_x.

14. As an illustration of this method, let it be proposed to solve the well-known linear equation of the first order,

$$u_{x+1} - A_x u_x = B_x.$$

Reduced to the symbolic form, this equation becomes

$$e^D u_x - A_x u_x = B_x,$$

whence

$$u_x - \frac{1}{e^D} A_x u_x = B_{x-1} + 0,$$

which gives

$$u_x = \begin{cases} \{1 + \frac{1}{e^D} A_x + \frac{1}{e^D} A_x \frac{1}{e^D} A_x + \&c.\} B_{x-1} \\ + (\textit{solution of } v_{x+1} - A_x v_x = 0), \end{cases}$$

or

$$u_x = \begin{cases} \{B_{x-1} + A_{x-1} B_{x-2} + A_{x-1} A_{x-2} B_{x-3} + \&c.\} \\ \qquad + C A_{x-1} A_{x-2} \ldots A_2 A_1, \end{cases}$$

where C_1 is an arbitrary constant, or, in the conventional notation of this calculus,

$$u_x = PA_{x-1} \{\Sigma \frac{B_x}{PA_x} + C\}.$$

Example.

$$u_{x+1} - a u_x = x^m.$$

Then

$$u_x = \{(x-1)^m + a(x-2)^m + a^2(x-3)^m + \&c.\} + Ca^x.$$

The solution of this equation may also be given by

$$u_x = -\frac{1}{a} \{x^m + \frac{(x+1)^m}{a} + \frac{(x+2)^m}{a^2} + \&c.\} + Ca^x,$$

since we may write the symbolic solution in the form

$$u_x = -\frac{1}{a - e^D} x^m + \frac{1}{e^D - a} 0.$$

Hence we conclude the equivalence of the two series,

$$+ \frac{1}{a} \{a(x-1)^m + a^2(x-2)^m + a^3(x-3)^m + \&c.\}$$

and

$$-\frac{1}{a} \{x^m + \frac{(x+1)^m}{a} + \frac{(x+2)^m}{a^2} + \&c.\}.$$

15. If

$$(x)_m = x(x+1)(x+2)\dots(x+m-1),$$

then will

$$(x+1)_m = (x+1)(x+2)(x+3)\dots(x+m),$$

and consequently

$$\Delta(x)_m = m(x+1)(x+2)\dots(x+m-1).$$

Therefore

$$x\Delta \,.\, (x)_m = m \,.\, (x)_m,$$

and by successive operation,

$$(x\Delta)^p \,.\, (x_m) = (m)^p \,.\, (x)_m.$$

Hence the theorem, analogous to that demonstrated in the Third Chapter,

$$F(xD_x) \,.\, A_m x^m = F(m) \,.\, A_m x^m,$$

that

$$F(x\Delta) \,.\, A_m(x)_m = F(m) \,.\, A_m(x)_m,$$

where F is any algebraic function, and A_m is any constant.

This theorem is virtually given by Professor Boole in the Memoir before quoted ("Philosophical Transactions," 1844), but, for the very elementary demonstration above given I am indebted to the Rev. Professor Graves.

Example.

$$x\Delta(x\Delta-1)(x\Delta-2)\dots(x\Delta-n+1)\,.\,(x)_m = m(m-1)(m-2)\dots(m-n+1)\,.\,(x)_m,$$

and, as a particular case of this,

$$x\Delta(x\Delta-1)(x\Delta-2)\dots(x\Delta-n+1)\,.\,(x)_n = 1\,.\,2\,.\,3\dots n\,.\,(x)_n,$$

the analogues of which may be found in the same Chapter to which reference has just been made.

16. The theorem obtained in the preceding article may be applied with advantage to the investigation of the solution of such equations in Finite Differences as are represented by

$$A(x\Delta)^{\alpha}.u_x + B(x\Delta)^{\beta}.u_x + \ldots + T.u_x = 0,$$

and in fact such solutions are given by the symbolic form

$$u_x = \frac{1}{F(x\Delta)}.0.$$

When the roots of the equation

$$F(x\Delta) = 0$$

are all real and unequal, the evaluated solution required is

$$u_x = C_a(x)_a + C_b(x)_b + \ldots + C_i(x)_i,$$

$a, b, \ldots i$ being the values of the roots, supposed integers, and C_a, C_b, &c. arbitrary constants.

Example.

$$(x\Delta)^2.u_x - 3(x\Delta).u_x + 2.u_x = 0,$$
$$u_x = C_2(x)_2 + C_1(x)_1.$$

17. It is easily demonstrable that

$$x\Delta(x\Delta - 1) = x(x+1)\Delta^2 = (x)_2\Delta^2,$$
$$x\Delta(x\Delta - 1)(x\Delta - 2) = x(x+1)(x+2)\Delta^3 = (x)_3\Delta^3,$$
&c.,

and generally that

$$x\Delta(x\Delta - 1)\ldots(x\Delta - n + 1) = (x)_n\Delta^n.$$

18. By the employment of this law we are enabled to solve, with great ease, all equations in Finite Differences represented by

$$A(x)_{\alpha}\Delta^{\alpha}.u_x + B(x)_{\beta}\Delta^{\beta}.u_x + \text{&c.} + T.u_x = 0.$$

In fact, such equations are at once reducible to the form

$$A'(x\Delta)^{\alpha'}.u_x + B'(x\Delta)^{\beta'}.u_x + \&c. + T'.u_x = 0,$$

and their solutions consequently given by the preceding article of this Chapter.

19. Let it be proposed to solve the equation

$$u_{x+1,y} - D_y \,.\, u_{x,y} = 0.$$

This equation is plainly equivalent to

$$(e^{D_x} - D_y)\,.\,u_{x,y} = 0,$$

whence

$$u_{x,y} = (D_y)^x.\phi(y),$$

where ϕ is an arbitrary function.

Similarly, the integral of

$$u_{x+1,y} - aD_y^n \,.\, u_{x,y} = 0$$

is

$$u_{x,y} = a^x(D_y)^{nx}.\phi(y).$$

More generally, let it be proposed to integrate the equation

$$u_{x+n,y} + aD_y.u_{x+n-1,y} + bD_y^2\,.\,u_{x+n-2,y} + \ldots + kD_y^n\,.\,u_{x,y} = 0.$$

This equation is exponible in the shape

$$(e^{nD_x} + aD_y\,.\,e^{(n-1)D_x} + \ldots + kD_y^n)\,.\,u_{x,y} = 0;$$

consequently the symbolic solution is of the form

$$u_{x,y} = (e^{D_x} - \alpha D_y)^{-1}.(e^{D_x} - \beta D_y)^{-1}\ldots(e^{D_x} - \kappa D_y)^{-1}.0,$$

or

$$u_{x,y} = \alpha^x(D_y)^x.\phi_\alpha(y) + \beta^x(D_y)^x.\phi_\beta(y) + \&c.,$$

where ϕ_α, ϕ_β, &c. are arbitrary functions.

Now it is important to observe, that this process of integration is independent of the nature of the operation D_y, and the same form of integral belongs to

$$u_{x+n,y} + a\Delta_y \cdot u_{x+n-1,y} + b\Delta_y^2 \cdot u_{x+n-2,y} + \ldots = 0,$$

or to any form of the equation in which the operator on the second term is of a distinct kind from e^{D_x}.

HERSCHEL, *Supplement to Translation of Lacroix.*

APPENDIX A.

(Page 39.)

On the Calculus of Variations.

I.

It is shown (Jellett's *Calculus of Variations*, p. 253), that the value of z which, for certain assigned limits, renders the double integral

$$\iint (xD_x z + yD_y z - z)^m \, dxdy,$$

a maximum or a minimum, is given by the partial differential equation,

$$x^2 D_x^2 z + y^2 D_y^2 z + 2xy D_x D_y z + \frac{3}{m-1} (xD_x z + yD_y z - z) = 0,$$

the solution of which, if we put

$$n = -\frac{3}{m-1},$$

is given by

$$z = u_n + u_1,$$

where u_n and u_1 are homogeneous functions of the independent variables of the given degrees n and unity, but whose forms are arbitrary.

By a method precisely similar, it may be proved that the form of the function w which, for certain assigned limits, renders the symmetrical multiple integral, containing p independent variables,

$$\iiint \ldots (xD_x w + yD_y w + zD_z w + \ldots - w)^m \, dxdydz \ldots$$

a maximum or a minimum, is given by the partial differential equation

$$\left.\begin{array}{c} x^2D_x^2 w + y^2D_y^2 w + z^2D_z^2 w + \ldots + 2xyD_xD_y w + \ldots \\ + \\ \dfrac{p+1}{m-1}(xD_x w + yD_y w + zD_z w + \ldots - w) \end{array}\right\} = 0,$$

the solution of which is, if, as before, we put

$$n = -\frac{p+1}{m-1},$$

given by

$$w = u_n + u_1,$$

where the arbitrary functions merely differ from the previous in the number of the independent variables (p) included under them.

II.

It is shown (*Calculus of Variations*, p. 262) that the value of z which, for certain assigned limits, renders the double integral

$$\iint \sqrt{(D_x z)^2 + (D_y z)^2}\, dxdy = \iint Z\, dxdy$$

a maximum or a minimum, is given by the partial differential equation

$$D_x\left(\frac{D_x z}{Z}\right) + D_y\left(\frac{D_y z}{Z}\right) = 0.$$

Similarly it may be proved that the form of the function w which, for certain assigned limits, renders the symmetrical multiple integral

$$\iiint \ldots \sqrt{(D_x w)^2 + (D_y w)^2 + (D_z w)^2 + \ldots}\; dxdydz \ldots$$

or

$$\iiint \ldots W\, dxdydz \ldots$$

a maximum or a minimum, is given by the partial differential equation

$$D_x\left(\frac{D_x w}{W}\right) + D_y\left(\frac{D_y w}{W}\right) + D_z\left(\frac{D_z w}{W}\right) + \ldots = 0.$$

If there be but three independent variables x, y, z, the advanced student will find no difficulty in verifying the following theorem:—

If S be a closed surface enclosing a continuous mass, the density at each point of which is constant, and F the resultant attraction of an external system M′ at any element dm within S, then in order that the value of the triple integral

$$\iiint F dm$$

taken throughout the space included within S, should be a maximum or a minimum, the distribution of M should be such that

$$D_x A + D_y B + D_z C = 0,$$

A, B, C being the direction cosines of the resultant attraction F.

It may be easily seen also that, *if the surfaces of equilibrium belonging to the system M′ were constructed, the portions of those surfaces which lie within S are the surfaces of minimum superficies amongst all those which can be described through the closed curves in which, respectively, they intersect S.*

III.

It is shown (*Calculus of Variations*, p. 240) that in order that the double integral

$$\iint f(x, y, z, D_x z, D_y z)\, dxdy$$

should be reducible to a single integral, it is necessary and sufficient that f should be of the form

$$F_1(x, y, z) D_x w + F_2(x, y, z) D_y w - F_3(x, y, z),$$

the functions F_1, F_2, F_3 being connected by the condition

$$D_x F_1 + D_y F_2 + D_z F_3 = 0.$$

As an example of this theorem it is proved that the double integral

$$\iint \mu (x D_x z + y D_y z - z)\, dxdy$$

is reducible to a single integral, when μ is an homogeneous function in x, y, z, of the order -3.

In the same manner it may be proved that, in order that the symmetrical multiple integral, containing p independent variables

$$\iiint \ldots f(x, y, z, \ldots w, D_x w, D_y w, D_z w, \ldots)\, dxdydz \ldots$$

should be reducible one degree, it is necessary and sufficient that f should be of the form

$$F_1(x, y, z, \ldots w) D_x w + F_2(x, y, z, \ldots w) D_y w + \&c. - F_{p+1}(x, y, z \ldots w)$$

the functions $F_1, F_2, \ldots, F_{p+1}$ being connected by the condition

$$D_x F_1 + D_y F_2 + D_z F_3 + \ldots + D_w F_{p+1} = 0.$$

Again, as an example of this theorem, it may be proved that the multiple integral

$$\iiint \ldots \mu (x D_x w + y D_y w + z D_z w + \ldots - w) \, dxdydz \ldots$$

is instantly reducible one degree, when μ is an homogeneous function in $x, y, z, \ldots w$ of the order $-(p+1)$.

IV.

As illustrations of the employment of the opening theorem of the last article, the reader will accept the following applications to the theory of Attractions, which were communicated to the Dublin University Philosophical Society, December, 1850, and have been since published in the Transactions of that Society.

It has been shown by Gauss, in his celebrated memoir on Attractions, that

(1.) For *all closed surfaces* lying wholly outside, and *including* a given system of masses M any way distributed,

$$\int P d\sigma = const. = 4\pi M,$$

P being the component of the resultant attraction at each point of the surface, in the direction of the normal, and $d\sigma$ the element of the surface.

(2.) For *all closed surfaces* lying wholly outside and *excluding* the same system of masses,

$$\int P d\sigma = const. = 0.$$

Though these theorems, when taken in combination with others given by Gauss, have proved in his hands most fertile, yet in themselves they appear isolated and distinct, unattended by any immediate result, and unconnected by any general law.

They obviously come within the province of the Calculus of Variations, and it becomes an interesting matter of inquiry how they may be reduced to it.

Now, a, β, γ being the normal angles,

$$\int P d\sigma = \int (X\cos a + Y\cos\beta + Z\cos\gamma)\, d\sigma,$$

or

$$\int P d\sigma = -\iint (XD_x z + YD_y z - Z)\, dxdy\,;$$

which evidently fulfils the condition required in order that it may be reducible to a single integral, for

$$D_x X + D_y Y + DZ_z = D_x^2 V + D_y^2 V + D_z^2 V = 0.$$

Hence generally, if M be a system of masses, any way distributed, and C a closed curve lying wholly outside this system, for all surfaces described through C, and which do not intersect M,

$$\int P d\sigma = const.,$$

this constant having either of *two* values, according as the series of surfaces lies at one side or the other of M.

To establish this latter point, let us conceive a series of surfaces described through the closed curve C, and progressively approaching the system of masses, but not so as to intersect it. Then the only legitimate conclusion which we can draw from the results furnished by the Calculus of Variations is, according to the principles of this Calculus, that

$$\int P d\sigma$$

is the same for *all these.* Once the series intersects M, our conclusions cease to be valid, and we cannot apply to the class of surfaces, which arises from the extension of the series *beyond* M, any results at which we may have arrived relative to the series on the side from which the development has commenced, for the sequence of the one series upon the other is attended by an *abrupt change.*

The values of the constants in the theorems given by Gauss are easily deducible.

In the latter case, let us suppose the series of closed surfaces to degenerate into a point, which is evidently legitimate, and it follows that the constant value of the integral in this case is zero. In the former, we may choose for the particular closed surface a sphere with its centre at the centre of gravity of the masses, and its radius so large that the masses, so far as they affect the surface of this sphere, may be supposed to be condensed into their centre of gravity, and the constant value of the integral in this case is

$$4\pi M.$$

The application of the general principle to a particular case will be found to furnish an interesting result.

If the curve C be *plane*, and its plane do not intersect the system M, then for all surfaces described through this curve, and which *neither intersect nor include* M, the constant value of the integral will be

$$\Pi.\Sigma$$

where Σ is the area enclosed within the plane curve, and Π is the sum of the normal components of the attractions exercised at each point of Σ, and applied at its centre of gravity.

It appears, then, that for all plane curves enclosing the same area, the value of the integral for different systems of masses will vary as Π, and that for the same value of Π, perpendicular at a given point to a given plane, the constant value of the integral will vary as the area of each curve described in the plane with its centre of gravity at the given point.

V.

As a second illustration, it may be observed that the condition

$$D_x F_1 + D_y F_2 + D_z F_3 = 0$$

is evidently satisfied, if F_1 be independent of x, F_2 of y, and F_3 of z. Hence we derive the following general theorem, in which it is supposed that a, β, γ denote the normal angles at any point of a surface, a, b, c the radial, and r the radius vector.

For all surfaces passing through the same closed curve the symmetrical double integral

$$\iint r^m \{\cos\alpha . F_1(\cos b, \cos c) + \cos\beta . F_2(\cos c, \cos a) + \cos\gamma . F_3(\cos a, \cos b)\}\, d\sigma$$

taken throughout the extent bounded on each surface by the limiting curve, will be constant, F_1, F_2, F_3 *being homogeneous functions of the* m^{th} *degree, where* m *may be positive or negative, fractional or integral.*

And hence again,

The value of this double integral is the same for all closed surfaces.

In the particular case

$$\iint r^2 \{\cos\alpha.\sin^2 a + \cos\beta.\sin^2 b + \cos\gamma.\sin^2 c\}\, d\sigma,$$

this constant value for all closed surfaces will be found to be zero, its value being investigated for the sphere; and in general, if m be a positive integer, the value of

$$\iint r^m \{\cos\alpha.\sin^m a + \cos\beta.\sin^m b + \cos\gamma.\sin^m c\}\, d\sigma$$

for all closed surfaces, is zero.

VI.

Let S be a closed surface enclosing a continuous stratified mass M, and F the force of attraction exercised at any element dm within S by an external system M'. It is required to investigate the character of the distribution of M, which will render

$$\int F^2 dm,$$

taken throughout the space included within S, a maximum or a minimum. Then

$$\iiint \{(D_x V)^2 + (D_y V)^2 + (D_z V)^2\} \rho\, dxdydz = max.\ or\ min.,$$

and V is given by the equation

$$D_x(\rho D_x V) + D_y(\rho D_y V) + D_z(\rho D_z V) = 0;$$

or, since M' is wholly external to S, by

$$D_x V.D_x\rho + D_y V.D_y\rho + D_z V.D_z\rho = 0;$$

or, if S be a line in the direction of the force,

$$FD_s\rho = 0.$$

But in general F is not zero, therefore

$$D_s\rho = 0\ ;$$

or, *the direction of the force F must be tangential to the surface*

$$\rho = const.$$

It is an obvious consequence that surfaces of constant density described in S,

$$\rho = const.$$

are the *loci of the trajectories* to the surfaces of equilibrium belonging to M',

$$V = const.,$$

which intersect S.

APPENDIX B.

(Page 114.)

On the Quadrature of Surfaces and the Rectification of Curves.

1. It is well known that there are many plane curves whose equations are more easily expressed in polar than in rectangular co-ordinates, and for whose rectification we employ the formula

$$S = \int_{\theta_1}^{\theta_2} \sqrt{\{r^2 + (D_\theta r)^2\}}\, d\theta.$$

Of this class are, the Spiral of Archimedes,

$$r = a\theta;$$

the Lituus,

$$r^2 = \left(\frac{a}{\theta}\right)^2;$$

the Lemniscate,

$$r^2 = a^2 \cos 2\theta;$$

the Logarithmic Spiral,

$$r = ce^{\frac{\theta}{a}};$$

and the Cardioid,

$$r = a(1 - \cos\theta).$$

2. I am not aware that any mathematician has attempted to trace the surfaces analogous to these; but, for the *quadrature* of such surfaces, when discovered, it is absolutely necessary that we should have a general expression *in polar co-ordinates* for the element of any surface. Such an expression is not found in the ordinary works upon the Differential and Integral Calculus. In the elaborate treatise upon this subject by M. L'Abbé Moigno (Paris, 1844, tom. ii. p. 235), the expression is investigated by the usual analytical method, transformation of co-ordinates, from the well-known expression in rectangular co-ordinates,

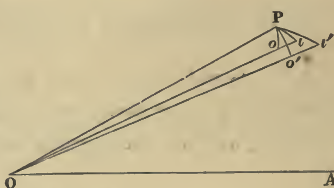
$$d\sigma = \sqrt{(1 + p^2 + q^2)}\, dxdy,$$

and is given in the following shape,

$$d\sigma = \sqrt{\{r^2 \sin^2\theta + \sin^2\theta (D_\theta r)^2 + (D_\phi r)^2\}}\, r\, d\theta\, d\phi.$$

A short geometrical deduction of this expression may not be unacceptable to the student.

Let P be any point on the surface. Through the axis OA and OP describe a plane, and round the axis describe, with the same line, a cone. The surface may then be supposed to be divided into its elements by planes and cones consecutive to these respectively (the planes all passing through the axis and the cones round it), half of one such element being represented by $\iota P \iota'$. Then, remembering that the planes cut the cones orthogonally, we have



$$d\sigma = P\iota . P\iota' . \sin \iota P \iota' = P\iota . P\iota' . \sqrt{(1 - \cos^2 \iota P \iota')},$$

whence

$$d\sigma = P\iota . P\iota' . \sqrt{(1 - \sin^2 \iota P o . \sin^2 \iota' P o')} = \sqrt{(P\iota^2 . P\iota'^2 - o\iota^2 . o'\iota'^2)},$$

o and o' being the points where the sphere described round the origin with radius OP intersects the consecutive radii vectores to the points ι, ι'; or

$$d\sigma = \sqrt{[\{r^2 \sin^2\theta\, d\phi^2 + (D_\phi r)^2 d\phi^2\} . \{r^2 d\theta^2 + (D_\theta r)^2 d\theta^2\} - (D_\theta r)^2 d\theta^2 . (D_\phi r)^2 d\phi^2]},$$

or, finally,

$$d\sigma = \sqrt{\{r^2 \sin^2\theta + \sin^2\theta (D_\theta r)^2 + (D_\phi r)^2\}}\, r\, d\theta\, d\phi.$$

3. From this expression we may readily derive that for the perpendicular from the origin upon the tangent plane, in *polar* co-ordinates. In *rectangular* co-ordinates it is known to be

$$P = \frac{z - px - qy}{\sqrt{(1 + p^2 + q^2)}};$$

but the transformation of this to polar co-ordinates would be troublesome and tedious. We may easily derive the required expression from the volume of the elementary cone, for

$$Pd\sigma = r^3 \sin\theta\, d\theta\, d\phi,$$

and, therefore,

$$P = \frac{r^2 \sin\theta}{\sqrt{\{r^2 \sin^2\theta + \sin^2\theta (D_\theta r)^2 + (D_\phi r)^2\}}}.$$

4. As an example of the application of the formula for the quadrature of surfaces, let us suppose that it is required to investigate the quadrature, between given limits, of the surface

$$r = me^{-\phi} \cos\theta.$$

Then

$$D_\theta r = -me^{-\phi} \sin\theta,\ D_\phi r = -me^{-\phi} \cos\theta;$$

therefore,

$$d\sigma = \sqrt{(m^2 e^{-2\phi} \cos^2\theta \sin^2\theta + m^2 e^{-2\phi} \sin^4\theta + m^2 e^{-2\phi} \cos^2\theta)}\, r\, d\theta\, d\phi,$$

or

$$d\sigma = m^2 e^{-2\phi} \cos\theta\, d\theta\, d\phi;$$

whence

$$\Sigma = m^2 \int e^{-2\phi} (\sin\theta_2 - \sin\theta_1)\, d\phi.$$

Let us suppose the limits to be given by the intersections, with the given surface, of the cones

$$\theta_2 = a\phi,\ \theta_1 = b\phi,$$

and

$$\Sigma = m^2 \int_{\phi_1}^{\phi_2} e^{-2\phi} (\sin a\phi - \sin b\phi)\, d\phi,$$

an integral which is susceptible of easy reduction, since we know that

$$\int e^{-m\phi} \sin a\phi\, d\phi = -e^{-m\phi} \frac{m \sin a\phi + a \cos a\phi}{m^2 + a^2}.$$

5. As a second example, let it be proposed to investigate the quadrature, within given limits, of the surface

$$r = m \cos\phi \sin\theta.$$

Here

$$D_\theta r = m \cos\phi \cos\theta,\ D_\phi r = -m \sin\phi \sin\theta,$$

and

$$d\sigma = m^2 \cos\phi \sin^2\theta \, d\theta \, d\phi;$$

whence

$$\Sigma = m^2 \int_{\theta_1}^{\theta_2} (\sin\phi_2 - \sin\phi_1) \sin^2\theta \, d\theta;$$

and, if the limits be given as before, there is no difficulty in determining the quadrature completely.

6. In the treatise upon the "Calculus of Variations," by the Rev. Professor Jellett, before quoted, it is shown that the surface which, within given limits, renders the double integral,

$$\iint \surd(p^2 + q^2) \, dx dy,$$

or, γ being the angle made by the radius vector with the axis of z,

$$\iint \sin\gamma \, . \, d\sigma,$$

a minimum, is given by the partial differential equation

$$q^2 r - 2pqs + p^2 t = 0,$$

whose integral is known to be

$$xF_1(z) + yF_2(z) = 1$$

representing the gauche surface generated by a right line, which, gliding upon two fixed directrices, remains constantly parallel to the plane of the axes of x and y; as indeed might be anticipated from a consideration of the question in its second form.

In the same manner it might be shown that the surface which, within given limits, renders the double integral

$$\iint \surd \{(D_\theta r)^2 + (D_\phi r)^2\} \, d\theta \, d\phi$$

a minimum, is given by the equation

$$\phi F_1(r) + \theta F_2(r) = 1.$$

If it be proposed to investigate the property of this surface corresponding to the character of the generation of the analogous surface in rectangular co-ordinates, as the latter character is exhibited by the supposition $z = \text{const.}$, so the former property may be investigated by the supposition $r = \text{const.}$ Let then the surface be supposed to intersect a sphere described round the origin, and let the nature of the curve of intersection be examined. If we resolve any element into its rectangular components, one such component is $rd\theta$, and the other $r\sin\theta\, d\phi$. Let i be the inclination of the element to the meridional plane described through its extremity and the fixed axis, and it is evident that

$$\tan i = \frac{r\sin\theta\, d\phi}{r\,d\theta} = -\frac{F_2(c)}{F_1(c)}\sin\theta,$$

c being the radius of the sphere; or, *the tangent of the angle of inclination of the curve to the meridional plane is proportional to the sine of the angle made by the radius vector with the axis.*

7. It may be well here to indicate certain desiderata, the knowledge of which might lead to the discovery of some interesting properties of surfaces.

The *measure of curvature* at any point of a surface is expressed in rectangular co-ordinates by the formula

$$\frac{1}{R_1 R_2} = \frac{rt - s^2}{(1 + p^2 + q^2)^{\frac{3}{2}}}:$$

we have no corresponding expression in polar co-ordinates. Such might be discovered by the investigation of the analogue of the known formula for plane curves,

$$\rho = r\frac{dr}{dp}.$$

Again, the *sum of the curvatures* at any point of a surface is expressed by the formula, in rectangular co-ordinates,

$$\frac{1}{R_1} + \frac{1}{R_2} = -\frac{(1+q^2)\,r - 2pqs + (1+p^2)\,t}{(1+p^2+q^2)^{\frac{3}{2}}}:$$

we have no corresponding expression in polar co-ordinates. Other desiderata will readily suggest themselves.

8. With regard to the rectification of curves, it may be useful to make a few observations upon a subject which has recently attracted much attention among French mathematicians. In the

Notes by M. Liouville to his valuable edition of the *Application de l'Analyse à la Geometrie* of the illustrious Monge, will be found (p. 558) the following remarks :—

"M. Serret a fait usage de certaines variables qu' il avait déjà employées au tome XIII. du *Journal de Mathématiques*, pour resoudre le problème suivant : x, y, z, s, étant quatre fonctions d'une variable indépendente θ assujetties a verifier l'équation

$$dx^2 + dy^2 + dz^2 = ds^2,$$

exprimer sans forme finie et sans aucun signe d'intégration, les valeurs générales de ces fonctions. La solution de ce problème conduit, par exemple, a trouver des courbes à double courbure qui soient à la fois algébriques et rectifiables algébriquement, ou dont l'arc dépende d'une transcendante donnée. Le problème analogue pour les courbes planes dépend de l'equation plus simple

$$dx^2 + dy^2 = ds^2,$$

et se resout, comme on sait, par les formules

$$x = \psi'(\theta)\sin\theta + \psi''(\theta)\cos\theta,$$
$$y = \psi'(\theta)\cos\theta - \psi''(\theta)\sin\theta,$$
$$s = \psi(\theta) + \psi''(\theta),$$

ou la fonction ψ est arbitraire. Les formules de M. Serret pour l'équation

$$dx^2 + dy^2 + dz^2 = ds^2,$$

sont beaucoup plus compliquées, et, partant, beaucoup moins utiles."

It appears to me that the integration of these equations may be effected directly, and with great simplicity, by employing the Calculus of Quaternions.

Thus, in the notation of this Calculus, the first equation

$$dx^2 + dy^2 = ds^2$$

is equivalent to

$$-(idx + jdy)^2 = -(d\rho)^2,$$

or

$$idx + jdy = d\rho\,;$$

whence

$$ix + jy = \rho + \alpha,$$

α being an arbitrary vector; or, between given limits,

$$i(x_2 - x_1) + j(y_2 - y_1) = \rho_2 - \rho_1,$$

an identity, as it ought to be.

Similarly, the second equation

$$dx^2 + dy^2 + dz^2 = ds^2$$

is equivalent to

$$-(idx + jdy + kdz)^2 = -(d\rho)^2,$$

or

$$idx + jdy + kdz = d\rho\,;$$

whence

$$ix + jy + kz = \rho + \alpha,$$

α being an arbitrary vector; or, between given limits,

$$i(x_2 - x_1) + j(y_2 - y_1) + k(z_2 - z_1) = \rho_2 - \rho_1,$$

an identity, as it ought to be.

APPENDIX C.

Additional Applications to Integration.

1. Let it be proposed to integrate the equation

$$xyD_xD_yz + bxD_xz + ayD_yz + abz = V,$$

where V is a function of x and y.

Gregory, *Examples*, p. 366.

This equation being thrown into the form

$$(xD_x + a)(yD_y + b)z = V,$$

its solution is, at once

$$z = \frac{1}{(xD_x + a)(yD_y + b)}V + x^{-a}\Phi(y) + y^{-b}\Psi(x);$$

and if V be supposed to be of the shape

$$\Sigma A_{m,n}x^m y^n,$$

the full evaluated solution is (page 42),

$$z = \Sigma\frac{A_{m,n}x^m y^n}{(m+a)(n+b)} + x^{-a}\Phi(y) + y^{-b}\Psi(x).$$

More generally, the solution of the equation

$$(xD_x + a)(yD_y + b)(zD_z + c)\ldots w = \Sigma A_{m,n,p,\,\&c.}x^m y^n z^p\ldots$$

is

$$w = \Sigma\frac{A_{m,n,p,\,\&c.}x^m y^n z^p\ldots}{(m+a)(n+b)(p+c)\ldots} + x^{-a}\Phi(y, z, \&c.) + y^{-b}\Psi(z, x, \&c.) + \&c.$$

2. Let the equation to be integrated be

$$D_x^2 z + \frac{2}{x}D_x z = a^2 D_y^2 z.$$

Gregory, *Examples*, p. 367.

Multiplying by x^2 and reducing, we get

$$xD_x(xD_x+1)\,z = a^2x^2D_y^2\,z,$$

or

$$D_x^2(xz) = a^2D_y^2(xz),$$

whence

$$z=\frac{1}{x}\left\{\Phi(y+ax)+\Psi(y-ax)\right\}.$$

3. Let the equation to be integrated be

$$D_y^2 z = a^2\left\{D_x^2 z + \frac{2}{x}D_x z - \frac{2}{x^2}z\right\}.$$

GREGORY, *Examples*, p. 367.

Upon reference to page 55 it will be evident that, this equation being thrown into the shape

$$(xD_x-1)(xD_x+2)\,z - \left(\frac{x}{a}D_y\right)^2 z = 0,$$

its solution is

$$z=\frac{1}{x^2}(xD_x-1)\left\{\Phi(x+ay)+\Psi(x-ay)\right\};$$

or, in full,

$$z=\frac{1}{x}\left\{\Phi'(x+ay)+\Psi'(x-ay)\right\}-\frac{1}{x^2}\left\{\Phi(x+ay)+\Psi(x-ay)\right\}.$$

4. Although the system of equations representing the small motions of homogeneous elastic gases (page 77),

$$\left.\begin{aligned} D_t^2 u &= a^2D_x(D_xu+D_yv+D_zw)\\ D_t^2 v &= a^2D_y(D_xu+D_yv+D_zw)\\ D_t^2 w &= a^2D_z(D_xu+D_yv+D_zw)\end{aligned}\right\}$$

cannot be integrated generally, particular integrals have been proposed corresponding to particular cases.

Thus, in the case of *spherical waves* going to and from the centre whose co-ordinates are α, β, γ, these equations are satisfied by

$$u=\frac{x-\alpha}{r^2}\left\{\Phi'(at-r)+\Psi'(at+r)\right\}+\frac{x-\alpha}{r^3}\left\{\Phi(at-r)-\Psi(at+r)\right\}$$

$$v=\frac{y-\beta}{r^2}\left\{\Phi'(at-r)+\Psi'(at+r)\right\}+\frac{y-\beta}{r^3}\left\{\Phi(at-r)-\Psi(at+r)\right\}$$

$$w=\frac{z-\gamma}{r^2}\left\{\Phi'(at-r)+\Psi'(at+r)\right\}+\frac{z-\gamma}{r^3}\left\{\Phi(at-r)-\Psi(at+r)\right\}$$

Again, for *plane waves* moving in the direction of a line which makes with the axes of co-ordinates angles λ, μ, ν, these equations are evidently satisfied by

$$u = \cos\lambda \, . \, \Phi(x \cos\lambda + y \cos\mu + z \cos\nu - at)$$
$$v = \cos\mu \, . \, \Phi(x \cos\lambda + y \cos\mu + z \cos\nu - at)$$
$$w = \cos\nu \, . \, \Phi(x \cos\lambda + y \cos\mu + z \cos\nu - at)$$

AIRY, *Tracts*, p. 267.

THE END.

ERRATA.

Page 3, line 24, *for* forms *read* form.

" 9, " 11, *for* if u be a function *read* if u and v be functions.

" 13, " 8, *for* $\left\{1+\left(\frac{\Phi+\Psi}{1}\right)+\left(\frac{\Phi+\Psi}{1.2}\right)^2+\&c.\right\}.u$

read $\left\{1+\frac{(\Phi+\Psi)}{1}+\frac{(\Phi+\Psi)^2}{1.2}+\&c.\right\}.u.$

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