A TREATISE

ON THE

DIFFERENTIAL AND INTEGRAL CALCULUS.

BY THE

REV. T. G. HALL, M.A.

PROFESSOR OF MATHEMATICS AT KING'S COLLEGE, LONDON:
AND LATE
FELLOW AND TUTOR OF MAGDALENE COLLEGE, CAMBRIDGE.

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PREFACE.

A KNOWLEDGE of the principles of this branch of the Pure Mathematics is absolutely necessary, before any one can successfully undertake the perusal of works on Natural Philosophy, in which the effects of the observed laws that govern the material world are reduced to calculation.

For Students deficient in this knowledge, yet anxious to obtain as much information as may enable them to master the chief analytical difficulties incident to the study of Elementary Treatises on the Mixed Mathematics, this book has been written: with the hope, too, that by its means a subject of high interest may be rendered accessible to an increased number of readers.

The ample Table of Contents which accompanies this work will sufficiently exhibit its plan—and a very hasty glance will at once shew that its chief object is to treat of Functions of one Variable; at the same time the Theory of Functions of two Variables, and its application to questions of Maxima and Minima, is fully explained. But the Chapters on the Integral Calculus contain rules for the Integration of Explicit Functions only.

A few words may be here added in order to explain the principles adopted in laying down the definitions.

By a method, similar to that of M. Poisson I have shewn that $u_1 = f(x+h)$ can always be put under the form $u + Ah + Uh^2$, whence we obtain the equation

$$u_1 - u = Ah + Uh^2.$$

The term Ah is defined to be the differential of u, and A, or the coefficient of h, is called the differential coefficient.

And from these definitions, the Rules for Differentiation have been in general derived.

But as the algebraical labour of finding A may sometimes be greatly diminished, if, after dividing both sides of the equation $u_1 - u = Ah + Uh^2$ by h, we make $h=0$, this method is in a few instances made use of.

The symbol $\frac{du}{dx}$ for the differential coefficient of $u=f(x)$, invented by Leibnitz, and used almost without exception by the continental writers, is here retained—I mention the fact, since the notation $d_x u$ for the same term has lately been revived by some Cambridge Mathematicians.—I do not pretend to decide the question which of the two $\frac{du}{dx}$ or $d_x u$ estimated by its power of best representing the differential coefficient ought to be preferred, but I see that the latter is, to say the least, an imperfect notation, and is liable to the important objection that the suffix x, in the calculus of finite differences has a meaning entirely different from that indicated by the x in d_x. But the most important objection is that already alluded to, that when the proposed notation has been learned in our own elementary works, the eye must become familiarized with that of Leibnitz, before the works of Lacroix and Laplace can be read with advantage.

Lastly, if it be considered necessary to offer an inducement to any one to enter upon the study of a science—which is the result of one of Newton's most brilliant discoveries, let him know "that it is a high privilege, not a duty, to study this language of pure unmixed truth. The laws by which God has thought good to govern the universe are surely subjects of lofty contemplation, and the study of that symbolical language by which alone these laws can be fully decyphered, is well deserving of his noblest efforts*."

* Professor Sedgwick on the Studies of the University.

PREFACE TO THE SECOND EDITION.

To this edition, many examples have been added, and some alterations made in the early chapters, intended to facilitate the labour of the reader. The Integral Calculus is terminated by a chapter upon the Solution of Differential Equations of Two Variables. The theory of these equations is of necessity briefly explained. Upon the whole, there will be found nearly sixty additional pages, which it is hoped, will increase the usefulness of the work.

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INTEGRAL CALCULUS.

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THE

DIFFERENTIAL CALCULUS.

CHAPTER I.

1. One quantity u is said to be a function of another x when the value of the magnitude of u depends upon the variation of x. Thus the area of a triangle is a function of the base, when the altitude remains unaltered, since the area will increase or decrease with the increase or decrease of the base.

And if $u = ax^2 \pm bx$, where a and b are constant quantities, and x a variable one, u is said to be a function of x, since if x changes, the value of u will be altered: this relation between u and x is usually expressed by writing $u = f(x)$ or $\phi(x)$, the symbols f and ϕ expressing the word function.

The quantities expressed by the letters a and b are omitted in the equation $u = f(x)$. Since, although they determine the particular kind of function, they remain unchanged, while x passes through every degree of magnitude.

The quantity x is called the independent variable, and u the dependent variable.

2. Functions are also named explicit and implicit: an explicit function of x, is when u is known in terms of x, as in the equation $u = ax^2 + bx$. An implicit function is when u and x are involved together, as in the equation $u^2x - aux + bx^2 = 0$. An implicit function is written $f(u, x)$ or $\phi(u, x) = 0$.

3. Functions are also divided into algebraical and transcendental.

Algebraical functions are those where u may be expressed in terms of x, by means of an equation consisting of a finite number of terms.

Thus $u = ax^m + bx^{m-1} + \&c. + qx^2 + rx + s$ where (m) is finite, is an *algebraical* function of x.

A *Transcendental* function is one where u is equal to an infinite series, the sum of which cannot be expressed by a limited number of terms.

Thus $u = \log(1 + x)$, which

$$= \frac{1}{M}\left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c. \text{ to infinity}\right\},$$

and $u = \sin x = x - \dfrac{x^3}{2 \cdot 3} + \dfrac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} - \&c.$ to infinity,

are transcendental functions of x.

4. The equation $u = f(x)$ expresses the relation between the function u and the *single* variable x, and the values of u solely depend upon the change that may take place in x: but if we have an equation between three unknown quantities, such as

$$u = ax^2y - bxy^2,$$

where x and y are independent of each other, i. e. not connected together by any other equation; then the value of u depends upon the change, both of x and y, and u is said to be a function of two variables; this is expressed by writing $u = f(x, y)$.

As an instance, we may again take the area of a triangle the magnitude of which depends upon the rectangle of the base and the altitude, which lines are totally independent of each other.

It is obvious that there may be functions of three, four, or of n variables.

* $u = 1 + x + x^2 + x^3 + \&c.$ to infinity is an algebraical function of x, since the sum of the series is expressed by $\frac{1}{1-x}$.

5. Let us however return to functions of one variable, and let $u = f(x)$ express the general relation between the function and its independent variable x.

Let x increase and become $x + h$, then the value of u will most probably be altered. Let the new value be represented by u_1,

$$\begin{aligned} \text{then} \quad & u_1 = f(x+h), \\ \text{and} \quad & u \;= f(x), \text{ by hypothesis.} \\ \hline \therefore \quad & u_1 - u = f(x+h) - f(x). \end{aligned}$$

Now $u_1 - u$, or the difference between the functions of $x + h$ and x, must depend upon h, and we shall first shew that it may be expressed by a series of the form

$$Ah + Bh^2 + Ch^3 + \text{\&c.}$$

or that $\qquad u_1 = u + Ah + Bh^2 + Ch^3 + \text{\&c.}$

where the powers of h ascend: the primary object of the Differential Calculus is to find the value of the coefficients A, B, C, &c.

6. We will first shew that u_1 may be expressed by a series of the above form by a few particular examples.

(1) Let $u = x^3$;

$$\begin{aligned} \therefore \ u_1 = (x+h)^3 &= x^3 + 3x^2h + 3xh^2 + h^3 \\ &= u + 3x^2h + 3xh^2 + h^3, \end{aligned}$$

which is of the required form.

(2) Next, let $u = x^n$;

$$\therefore \ u_1 = (x+h)^n = x^n + nx^{n-1}h + n\frac{(n-1)}{2}x^{n-2}h^2 + \text{\&c.}$$

by the Binomial Theorem.

Or, putting u for x^n,

$$u_1 = u + nx^{n-1}h + n\frac{n-1}{2}x^{n-2}h^2 + \text{\&c.}$$

a series with ascending powers of h.

(3) Let $u = Ax^m + Bx^n + Cx^p x + \&c.$;

$$\begin{aligned}
\therefore\ u_1 &= A(x+h)^m + B(x+h)^n + C(x+h)^p + \&c. \\
&= A\left(x^m + mx^{m-1}h + m\frac{(m-1)}{2}x^{m-2}h^2 + \&c.\right) \\
&+ B\left(x^n + nx^{n-1}h + n\frac{(n-1)}{2}x^{n-2}h^2 + \&c.\right) \\
&+ C\left(x^p + px^{p-1}h + p\frac{p-1}{2}x^{p-2}h^2 + \&c.\right) \\
&+ \&c. \\
&= Ax^m + Bx^n + Cx^p + \&c. \\
&+ (mAx^{m-1} + nBx^{n-1} + \&c.)\,h \\
&+ \left(m\frac{(m-1)}{2}.\,Ax^{m-2} + n\frac{(n-1)}{2}Bx^{n-2} + \&c.\right)h^2 \\
&+ \&c. \qquad\qquad + \&c. \\
&= u + ph + qh^2 + \&c.
\end{aligned}$$

by writing u for its value, $Ax^m + Bx^n + \&c.$, and putting p, q, &c. for the coefficients of h, h^2, &c.

(4) It may also be shewn that a^{x+h}, $\log(x+h)$, $\sin(x+h)$, can be expanded into series of the form

$$u + Ah + Bh^2 + Ch^3, \ \&c.$$

but we proceed to demonstrate the following general Proposition.

7. Prop. If $u = f(x)$, and u_1 be the value of u when x becomes $x + h$, then

$$u_1 = u + Ah + Uh^2,$$

where u is the original function, and Uh^2 represents all the terms that follow Ah.

(1) u_1 or $f(x+h)$ can contain only such powers of h, as have positive indices. For if

$$u_1 = M + Ah^a + Bh^{-\beta} + \&c. = M + Ah^a + \frac{B}{h^\beta} + \&c.$$

when $h = 0$, u_1 instead of becoming $= u$, would be infinite.

(2) The first term of the expansion must $= u$.

For let u_1 or $f(x+h) = M + Ah^a + \&c.$ then let $h = 0$;

$$\therefore f(x) = u = M, \text{ i.e. } M = u;$$

$$\text{or } u_1 = u + Ah^a + \&c.$$

Let therefore u_1 or $f(x+h) = u + Ah^\alpha + Bh^\beta + \&c.$

where α is the least of all the indices of h, and β the next in magnitude, and A, B, &c. are functions of x.

Now whether x becomes $x+h$, or h becomes $2h$, u_1 will become $f(x+2h)$, and the expansions upon either suppositions will be identical.

(1) Let h become $2h$ or $h+h$, and let u_2 be the value of u_1;

$$\begin{aligned} \therefore u_2 &= f(x+2h) \\ &= u + A(2h)^\alpha + B(2h)^\beta + \&c. \\ &= u + 2^\alpha Ah^\alpha \ + 2^\beta Bh^\beta \ + \&c. \ldots\ldots\ldots (1). \end{aligned}$$

(2) Let x become $x+h$, then u_1 becomes as before $f(x+2h)$ or u_2, and let (u), (A), (B), &c. represent the values of u, A, B, &c.

$$\therefore u_2 = (u) + (A)h^\alpha + (B)h^\beta + \&c.$$

But (u) is the same as u_1, for it is $f(x+h)$,

$$\therefore (u) = u + Ah^\alpha + Bh^\beta + \&c.$$

Also (A), (B), &c. being what A, B, &c. become by putting $x+h$ for x,

$$\therefore (A) = A + A_1 h^{\alpha_1} + A_2 h^{\beta_1} + \&c.$$

$$(B) = B + B_1 h^{\alpha_2} + B_2 h^{\beta_2} + \&c.$$

then multiplying (A) by h^{α} and (B) by h^{β}, and substituting we have

$$\begin{aligned} u_2 = u &+ Ah^{\alpha} + Bh^{\beta} + \&c. \\ &+ Ah^{\alpha} + A_1 h^{\alpha+\alpha_1} + \&c. \\ &+ Bh^{\beta} + \&c. \\ = u &+ 2Ah^{\alpha} + A_1 h^{\alpha+\alpha_1} + 2Bh^{\beta} + \&c. \ldots (2). \end{aligned}$$

Equating the coefficients of the same powers of h in series (1) and (2),

$$2A = 2^{\alpha} A, \text{ i.e. } 2 = 2^{\alpha}; \quad \therefore \alpha = 1,$$

$$\text{and } u_1 = f(x+h) = u + Ah + Bh^{\beta} + \&c.$$

whence it appears that the *second term of the expansion of $f(x+h)$ contains the first power of h only.*

From this it follows that $\alpha_1 = 1$ for $A_1 h^{\alpha_1}$ is the second term of the expression for A when x becomes $x + h$; and therefore

$$\begin{aligned} u_2 &= u + 2Ah + A_1 h^2 + 2Bh^{\beta} + \&c. \text{ from } (2) \\ &= u + 2Ah + 2^{\beta} . Bh^{\beta} + \&c. \ldots\ldots\ldots\ldots (1) \end{aligned}$$

Now, since in series (2) a term is found involving h^2, some corresponding term must be found in series (1); and as β is less than any index that follows it, β must $= 2$. And therefore,

$$\begin{aligned} u_1 = f(x+h) &= u + Ah + Bh^2 + Ch^{\gamma} + \&c. \\ &= u + Ah + (B + Ch^{\gamma-2} + \&c.)h^2 \\ &= u + Ah + Uh^2. \end{aligned}$$

8. The second term of the expansion, or Ah, is called the differential of u: differential being the diminutive of difference.

For Ah is the first term of the difference between u_1 and u, and consequently a part only of the difference: but the difference and differential differ the less, the less h is, and in cases of approximation, the latter is sometimes taken for the former.

Instead of writing differential at full length, the letter d is used, thus du is put for, differential of u, and thus $du = Ah$: but as in this case h is called the differential of x, instead of h dx is written, for symmetry of notation, and thus $du = Adx$.

A is called the *first differential coefficient*, and is expressed by the symbol $\frac{du}{dx}$, when $u = f(x)$.

Hence we define a *differential* to be *the second term of the* expansion of $f(x+h)$, and the differential coefficient to be the *coefficient of the first power of* h.

The process by which A, or $\frac{du}{dx}$ is found is called *differentiation.*

Hence also by our definitions we see that the differential of $u = A$ multiplied by the differential of x; or calling the first differential δu, and the second δx, we have

$$\delta u = A\delta x, \quad \therefore \quad \frac{\delta u}{\delta x} = A = \frac{du}{dx},$$

or the ratio of the differentials of u and x is equal to the ratio of the differential coefficient to unity.

We have used the letter δ to avoid confounding the differentials with the differential coefficients.

9. Again, since $u_1 = u + Ah + Uh^2$,

$$\frac{u_1 - u}{h} = A + Uh;$$

that is, the ratio of the increment $u_1 - u$ of the function to the increment of x, $= A + Uh$, and as h decreases, tends to A as its limit, and when h vanishes actually $= A$.

That is, A or $\frac{du}{dx}$ is the limit of the ratio of the increment of the function to that of the variable upon which it depends.

10. Hence we have a method of finding the differential coefficient, that is, the coefficient of the first power of h. Expand $f(x+h)$, subtract $f(x)$, divide both sides by h, then make $h=0$, and the term or terms remaining of the expansion will be equal to the coefficient required. This method is frequently very convenient.

11. We have seen that if u be any function of x, and x becomes $x+h$,

$$f(x+h) = u + \frac{du}{dx}h + Uh^2.$$

Similarly, if p, q, &c. be functions of x, then they will respectively become when x is made $x+h$,

$$p + \frac{dp}{dx}h + Ph^2,$$

$$\text{and } q + \frac{dq}{dx}h + Qh^2,$$

where Ph^2 and Qh^2 represent all the terms after the first power of h.

12. Hence it appears that in order to find the differential or differential coefficient, we have merely to put $x+h$ for x, and expand $f(x+h)$ according to the powers of h, and the term corresponding to Ah will give us at once both of the objects of our enquiry. But such a direct process would be always tedious, and often almost impracticable.

We proceed to investigate rules which will not only greatly diminish the labour of *differentiation*, but render it a simple algebraical operation.

We will first however apply the general process to the function

$$u = \frac{a+x}{b+x};$$

$$\therefore\ u_1 = \frac{a+x+h}{b+x+h} = \frac{\dfrac{a+x}{b+x} + \dfrac{h}{b+x}}{1 + \dfrac{h}{b+x}}$$

$$= \left(\frac{a+x}{b+x} + \frac{h}{b+x}\right) \cdot \frac{1}{1 + \dfrac{h}{b+x}}$$

$$= \left(\frac{a+x}{b+x} + \frac{h}{b+x}\right) \cdot \left\{1 - \frac{h}{b+x} + \frac{h^2}{(b+x)^2} - \&c.\right\}$$

$$= \frac{a+x}{b+x} + h\left\{\frac{1}{b+x} - \frac{a+x}{(b+x)^2}\right\} + ph^2 + \&c.;$$

$$\therefore\ \frac{du}{dx} = \frac{1}{b+x} - \frac{a+x}{(b+x)^2} = \frac{b-a}{(b+x)^2}.$$

Again. Since $u = \dfrac{a+x}{b+x}$, we shall have by the same process

$$u_1 = u + h \cdot \left(\frac{b-a}{(b+x^2)}\right) + ph^2 + qh^3 + \&c.$$

and $$\frac{u_1 - u}{h} = \frac{b-a}{(b+x)^2} + ph + qh^2 + \&c.;$$

and by making $h = 0$, as in Art. 10,

$$\frac{du}{dx} = \frac{b-a}{(b+x)^2}.$$

RULES FOR FINDING THE DIFFERENTIAL COEFFICIENT.

13. Let $u = ax$ where a is a constant quantity. For x put $x + h$;

$$\therefore\ u \text{ becomes } u + \frac{du}{dx} h + Uh^2;$$

$$\therefore\ u + \frac{du}{dx} h + Uh^2 = a(x+h) = ax + ah = u + ah.$$

Equating the coefficients of h,

$$\frac{du}{dx} = a, \quad \text{or} \quad \frac{d(ax)}{dx} = a.$$

14. Let $u = ax \pm b$, where a and b are constant.

The same substitutions being made,

$$u + \frac{du}{dx}h + Uh^2 = a(x+h) \pm b = ax \pm b + ah$$

$$= u + ah;$$

$$\therefore \frac{du}{dx} = a,$$

that is, $$\frac{d(ax \pm b)}{dx} = a.$$

But by the preceding Article,

$$\frac{d(ax)}{dx} = a;$$

$$\therefore \frac{d(ax \pm b)}{dx} = \frac{d(ax)}{dx};$$

that is, constant quantities connected with a variable one by the signs $\pm$ disappear in differentiation.

15. Let $u = ax^m$. Then making x become $x + h$,

$$u + \frac{du}{dx}h + Uh^2 = a(x+h)^m = a\,.\,(x^m + mx^{m-1}h + \&c.)$$

$$= ax^m + max^{m-1}\,.\,h + \&c.;$$

$$\therefore \frac{du}{dx} = max^{m-1},$$

or to find the differential coefficient of ax^m, we must multiply by the index and then diminish the index by unity.

Ex. $u = 5x^7$; $\therefore \dfrac{du}{dx} = 35x^6$.

16. Let $u = ap$ where p is a function of (x);

therefore if x becomes $x + h$,

$$u \text{ becomes } u + \frac{du}{dx} h + Uh^2,$$

$$p \;\ldots\ldots\ldots\; p + \frac{dp}{dx} h + Ph^2;$$

$$\therefore\; u + \frac{du}{dx} h + Uh^2 = ap + a\frac{dp}{dx} h + aP \,.\, h^2;$$

$$\therefore\; \frac{du}{dx} = a \,.\, \frac{dp}{dx},$$

$$\text{that is, } \frac{d(ap)}{dx} = a\frac{dp}{dx}.$$

17. If $u = ap + b$, a and b being constant quantities,

$$\text{then } \frac{du}{dx} = a \,.\, \frac{dp}{dx},$$

$$\therefore\; \frac{d(ap+b)}{dx} = \frac{a\,dp}{dx} = \frac{d(ap)}{dx}.$$

18. Let $u = p + q + r + \&c.$ where p, q, r are each functions of x;

$$\therefore\; u + \frac{du}{dx} h + \&c. = p + \frac{dp}{dx} h + q + \frac{dq}{dx} h + r + \frac{dr}{dx} h + \&c.;$$

$$\therefore\; \frac{du}{dx} = \frac{dp}{dx} + \frac{dq}{dx} + \frac{dr}{dx} + \&c.;$$

$$\therefore\; \frac{d\,.\,(p + q + r + \&c.)}{dx} = \frac{dp}{dx} + \frac{dq}{dx} + \frac{dr}{dx} + \&c.$$

Hence the differential coefficient of the sum of any functions equals the sum of the differential coefficient of each function taken separately.

19. Let $u = pq$,

$$\therefore\ u + \frac{du}{dx}h + Uh^2 = (p + \frac{dp}{dx}h + Ph^2) \times (q + \frac{dq}{dx}h + Qh^2)$$

$$= pq + \left(p\frac{dq}{dx} + q\cdot\frac{dp}{dx}\right)h + Bh^2 + \&c.;$$

$$\text{where } B = Pq + Qp + \frac{dp}{dx}\cdot\frac{dq}{dx};$$

$$\therefore\ \frac{du}{dx} = p\frac{dq}{dx} + q\cdot\frac{dp}{dx},$$

or the differential coefficient of the product of two quantities equals the sum of the products of each quantity into the differential coefficient of the other.

20. Let $u = \frac{p}{q}$: Here $\frac{du}{dx}$ may be found by substituting the values which u, p, and q have when x becomes $x + h$; but it may be deduced in an easy manner from the preceding Article.

$$\text{Since } u = \frac{p}{q},$$

$$\therefore\ qu = p; \quad \text{and } q\frac{du}{dx} + u\frac{dq}{dx} = \frac{dp}{dx};$$

$$\therefore\ \frac{du}{dx} = \frac{1}{q}\cdot\frac{dp}{dx} - \frac{u}{q}\cdot\frac{dq}{dx}$$

$$= \frac{1}{q}\cdot\frac{dp}{dx} - \frac{p}{q^2}\cdot\frac{dq}{dx}$$

$$= \frac{q\cdot\dfrac{dp}{dx} - p\cdot\dfrac{dq}{dx}}{q^2}.$$

A simple expression, the form of which is more easily to be remembered than its enunciation.

21. Let $u = pqr$, writing qr for q in Art. 19;

$$\therefore \frac{du}{dx} = p \cdot \frac{d(qr)}{dx} + qr \frac{dp}{dx}.$$

$$\text{But } \frac{d \cdot (qr)}{dx} = q \cdot \frac{dr}{dx} + r \cdot \frac{dq}{dx};$$

$$\therefore \frac{du}{dx} = pq \frac{dr}{dx} + pr \cdot \frac{dq}{dx} + qr \cdot \frac{dp}{dx}.$$

Similarly may the differential coefficient be found for the product of n functions, and it will be equal to the sum if the n products of the differential coefficient of each of the quantities multiplied by the remaining $n - 1$ factors. Thus

$$\frac{d \cdot \{p \cdot q \cdot r \cdot s \ldots (n)\}}{dx} = q \cdot r \cdot s \ldots (n-1) \frac{dp}{dx} + prs \ldots (n-1) \cdot \frac{dq}{dx}$$

$$+ pqs \ldots (n-1) \frac{dr}{dx} + \&c.$$

22. Let $u = p^n$, p being a function of x;

$$\therefore u + \frac{du}{dx} h + Uh^2 = (p + \frac{dp}{dx} h + Ph^2)^n.$$

$$\text{Let } P_1 = \frac{dp}{dx} + Ph;$$

$$\therefore (p + \frac{dp}{dx} h + Ph^2)^n = (p + P_1 h)^n$$

$$= p^n + np^{n-1} P_1 h + Bh^2 + \&c.$$

$$= p^n + np^{n-1} \left\{ \frac{dp}{dx} + Ph \right\} h + Bh^2 + \&c.$$

$$= p^n + np^{n-1} \frac{dp}{dx} h + np^{n-1} \cdot Ph^2 + Bh^2 + \&c.$$

therefore, equating the coefficients of h,

$$\frac{du}{dx} = n \cdot p^{n-1} \cdot \frac{dp}{dx};$$

or, to find the differential coefficient of p^n, multiply by the index, diminish the index by unity, and then multiply by the differential coefficient of p.

Ex. If $u = (a^2 + x^2)^n$ then $p = a^2 + x^2$ and $\frac{dp}{dx} = 2x$;

$$\therefore \frac{du}{dx} = n (a^2 + x^2)^{n-1} \cdot 2x.$$

23. The rule for finding the differential coefficient of p^n is perfectly general, but where $n = \frac{1}{2}$, it has a value which it is useful to remember. Thus

$$\frac{d(\sqrt{p})}{dx} = \tfrac{1}{2} p^{\frac{1}{2}-1} \frac{dp}{dx} = \frac{\frac{dp}{dx}}{2\sqrt{p}}.$$

whence this rule. To find the differential coefficient of the square root of any quantity, divide the differential coefficient of the quantity under the square root by twice the square root of the quantity itself.

Ex. Let $u = \sqrt{a + bx + cx^2}$;

$$\therefore p = a + bx + cx^2; \quad \therefore \frac{dp}{dx} = b + 2cx;$$

$$\text{and } \frac{du}{dx} = \frac{b + 2cx}{2\sqrt{a + bx + cx^2}}.$$

24. $u = \frac{1}{p^n}$. Here $\frac{du}{dx}$ may be deduced from the general form $u = p^n$; but as its form ought to be remembered, we shall deduce it separately.

If $u = \dfrac{1}{p^n}$; $\quad \therefore\ u \cdot p^n = 1.$

$$p^n \cdot \frac{du}{dx} + nu \cdot p^{n-1} \cdot \frac{dp}{dx} = 0.$$

But $up^{n-1} = \dfrac{1}{p}$;

$$\therefore\ p^n \cdot \frac{du}{dx} = -\frac{n}{p} \cdot \frac{dp}{dx}.$$

and $\dfrac{du}{dx} = -\dfrac{n}{p^{n+1}}\dfrac{dp}{dx}.$

Cor. If $p = x$; $\quad \therefore\ \dfrac{d \cdot \left(\dfrac{1}{x^n}\right)}{dx} = -\dfrac{n}{x^{n+1}}.$

Examples of the differentiation of algebraical functions.

(1) $\quad u = x^{\frac{7}{3}}$; $\quad \therefore\ \dfrac{du}{dx} = 3 \cdot \tfrac{7}{3} \cdot x^{\frac{7}{3}-1} = 7x^{\frac{4}{3}}.$

(2) $\quad u = x^3 + x^2 + x + 1,$

$$\frac{du}{dx} = 3x^2 + 2x + 1.$$

(3) $\quad u = (x + a) \cdot (x + b),$

$$\frac{du}{dx} = (x + a)\frac{d(x + b)}{dx} + (x + b) \cdot \frac{d(x + a)}{dx}. \quad \text{Art. (19)}$$

$$= x + a + x + b = 2x + (a + b).$$

(4) $\quad u = x(1 + x^2)(1 + x^3),$

$$\frac{du}{dx} = (1 + x^2)(1 + x^3)\frac{dx}{dx} + x(1 + x^3) \cdot \frac{d(1 + x^2)}{dx} + x(1 + x^2) \cdot \frac{d(1 + x^3)}{dx}$$

$$= (1 + x^2)(1 + x^3) + x(1 + x^3) \cdot 2x + x(1 + x^2)\,3x^2$$

$$= 1 + 3x^2 + 4x^3 + 6x^5.$$

(5) $u = \dfrac{1}{x^n} = x^{-n}$,

$$\frac{du}{dx} = -\,n x^{-n-1} = -\,\frac{n}{x^{n+1}};$$

$$\therefore \text{ if } u = \frac{1}{x}, \quad \frac{du}{dx} = -\,\frac{1}{x^2}.$$

(6) $$u = \frac{x + a}{x + b};$$

$$\therefore \frac{du}{dx} = \frac{(x+b)\,.\,\dfrac{d(x+a)}{dx} - (x+a)\,.\,\dfrac{d(x+b)}{dx}}{(x+b)^2}$$

$$= \frac{x + b - (x + a)}{(x+b)^2} = \frac{b - a}{(x+b)^2}.$$

(7) $$u = \frac{x^m}{(x+1)^m};$$

$$\therefore \frac{du}{dx} = \frac{(x+1)^m\,.\,m x^{m-1} - x^m\,.\,m\,.\,\overline{x+1}|^{m-1}}{(x+1)^{2m}}$$

$$= \frac{(x+1)\,.\,m x^{m-1} - m x^m}{\overline{x+1}|^{m+1}} = \frac{m x^{m-1}}{\overline{x+1}|^{m+1}}.$$

(8) $$u = \sqrt{1 + x^2} = \overline{1 + x^2}|^{\frac{1}{2}},$$

$$\frac{du}{dx} = \tfrac{1}{2}\,.\,(1 + x^2)^{-\frac{1}{2}}\,.\,2x = \frac{x}{\sqrt{1 + x^2}}.$$

(9) $$u = \sqrt{x + \sqrt{1 + x^2}},$$

$$\frac{du}{dx} = \tfrac{1}{2}\,(x + \sqrt{1 + x^2})^{-\frac{1}{2}}\,.\,\frac{d\,.\,(x + \sqrt{1 + x^2})}{dx},$$

and $$\frac{d(x + \sqrt{1 + x^2})}{dx} = 1 + \frac{x}{\sqrt{1 + x^2}} = \frac{x + \sqrt{1 + x^2}}{\sqrt{1 + x^2}}.$$

$$\frac{du}{dx} = \frac{1}{2} \cdot \frac{\sqrt{x + \sqrt{1 + x^2}}}{\sqrt{1 + x^2}}.$$

(10) $u = (2ax + x^2)^m$,

$$\frac{du}{dx} = m(2ax + x^2)^{m-1} \cdot \frac{d(2ax + x^2)}{dx}$$

$$= 2m \cdot (2ax + x^2)^{m-1} \cdot (a + x).$$

(11) $u = \dfrac{\sqrt{a^2 + x^2}}{\sqrt{a^2 - x^2}}$,

since $\dfrac{d\sqrt{a^2 + x^2}}{dx} = \dfrac{x}{\sqrt{a^2 + x^2}}$, and $\dfrac{d\sqrt{a^2 - x^2}}{dx} = \dfrac{-x}{\sqrt{a^2 - x^2}}$;

$$\therefore \frac{du}{dx} = \frac{\sqrt{a^2 - x^2} \cdot \dfrac{x}{\sqrt{a^2 + x^2}} + \sqrt{a^2 + x^2} \cdot \dfrac{x}{\sqrt{a^2 - x^2}}}{(a^2 - x^2)}$$

$$= \frac{(a^2 - x^2)x + (a^2 + x^2)x}{(a^2 + x^2)^{\frac{1}{2}}(a^2 - x^2)^{\frac{3}{2}}}$$

$$= \frac{2a^2x}{(a^2 + x^2)^{\frac{1}{2}}(a^2 - x^2)^{\frac{3}{2}}}.$$

(12) $u = \dfrac{\sqrt{1 + x} + \sqrt{1 - x}}{\sqrt{1 + x} - \sqrt{1 - x}} = \dfrac{(\sqrt{1 + x} + \sqrt{1 - x})^2}{2x}$

$$= \frac{1 + \sqrt{1 - x^2}}{x};$$

$$\therefore \frac{du}{dx} = \frac{\dfrac{-x^2}{\sqrt{1 - x^2}} - (1 + \sqrt{1 - x^2})}{x^2} = -\frac{x^2 + \sqrt{1 - x^2} + 1 - x^2}{x^2\sqrt{1 - x^2}}$$

$$= -\frac{1 + \sqrt{1 - x^2}}{x^2\sqrt{1 - x^2}}.$$

(13) $u = x(1+x^2) \cdot \sqrt{1-x^2} = (x+x^3)\sqrt{1-x^2}$,

$$\frac{du}{dx} = (1+3x^2)\sqrt{1-x^2} - \frac{x^2+x^4}{\sqrt{1-x^2}}$$

$$= \frac{1+3x^2-x^2-3x^4-x^2-x^4}{\sqrt{1-x^2}}$$

$$= \frac{1+x^2-4x^4}{\sqrt{1-x^2}}.$$

(14) $u = (a+x)(b+x)(c+x)$:

$$\frac{du}{dx} = 3x^2 + 2(a+b+c)x + ab + ac + bc.$$

(15) $u = \dfrac{x}{\sqrt{1+x^2}}$: $\quad \dfrac{du}{dx} = \dfrac{1}{(1+x^2)^{\frac{3}{2}}}$.

(16) $u = \dfrac{x^2-2}{3}\sqrt{1+x^2}$: $\quad \dfrac{du}{dx} = \dfrac{x^3}{\sqrt{1+x^2}}$.

(17) $u = \dfrac{x^4-x^2+1}{x^4+x^2+1}$: $\quad \dfrac{du}{dx} = \dfrac{4x(x^4-1)}{(x^4+x^2+1)^2}$.

(18) $u = \dfrac{x^3}{\sqrt{1+x^6}}$: $\quad \dfrac{du}{dx} = \dfrac{3x^2}{(1+x^6)^{\frac{3}{2}}}$.

(19) $u = \dfrac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}$: $\quad \dfrac{du}{dx} = \dfrac{2(\sqrt{1+x^2}-1)^2}{x^3\sqrt{1+x^2}}$.

(20) $u = \dfrac{(x+1)^{\frac{3}{2}}}{\sqrt{x-1}}$: $\quad \dfrac{du}{dx} = \dfrac{(x-2)\sqrt{x+1}}{(x-1)^{\frac{3}{2}}}$.

CHAPTER II.

DIFFERENTIATION OF CIRCULAR, EXPONENTIAL, AND LOGARITHMIC FUNCTIONS.

25. To find the differential coefficient of u, when

$$u = \sin x, \quad \cos x, \quad \tan x, \quad \sec x, \quad \&c.$$

The following Proposition must first be proved.

If h be an arc, $\dfrac{\sin h}{h}$ and $\dfrac{\tan h}{h}$ are each = unity, when $h = 0$*.

* This Proposition may be thus proved,

Since $\sin h$ vanishes when $h = 0$, it must depend upon the positive powers of h.

Let therefore

$$\sin h = a h^m + b h^n + \&c. = a h^m + N h^n,$$

where $N h^n$ includes all the terms after $a h^m$. Therefore, writing $2h$ for h,

$$\sin 2h = a\,.\,(2h)^m + N(2h)^n = 2^m a h^m + 2^n N h^n;$$

$$\therefore \quad \frac{\sin 2h}{\sin h} = 2\cos h = \frac{2^m a h^m + 2^n N h^n}{a h^m + N h^m} = \frac{2^m + 2^n \dfrac{N}{a}\,.\,h^{n-m}}{1 + \dfrac{N}{a}\,h^{n-m}};$$

let $h = 0$; $\therefore \cos h = 1$;

$$\therefore \quad 2 = 2^m; \quad \therefore \quad m = 1;$$

and $\sin h = a h + N h^n$,

and $\cos h = \sqrt{1 - \sin^2 h} = \sqrt{1 - (ah + Nh^n)^2} = 1 - \dfrac{a^2 h^2}{2} + \&c.$

and $\tan h = \dfrac{\sin h}{\cos h} = \dfrac{a h + N h^n}{1 - \dfrac{a^2 h^2}{2} + \&c.} = a h + \dfrac{a^3 h^3}{2} + \&c.$

$$\text{Now} \quad h > \sin h < \tan h,$$

$$\text{or} \quad h > a h + N h^n < a h + \frac{a^3 h^3}{2} + \&c.$$

$$\text{or} \quad 1 > a + N h^{n-1} < a + \frac{a^3 h^2}{2} + \&c.$$

It is proved, see *Trigonometry*, Art. 53,

$$\text{that } h > \sin h < \tan h,$$

or h lies between $\sin h$ and $\tan h$.

If therefore

$$\frac{\sin h}{\tan h} = 1\,; \qquad \therefore \text{ also } \frac{\sin h}{h} = 1, \quad \text{and } \frac{\tan h}{h} = 1.$$

$$\text{Now } \frac{\sin h}{\tan h} = \frac{\cos h}{1} = \frac{1}{1} = 1, \quad \text{when } h = 0,$$

$\therefore \dfrac{\sin h}{h}$ and $\dfrac{\tan h}{h}$ are also respectively equal to unity.

26. Let $u = \sin x$.

For x, put $x + h$; therefore u becomes

$$u + \frac{du}{dx} h + Uh^2.$$

$$\text{and } \quad u + \frac{du}{dx} h + Uh^2 = \sin(x + h),$$

$$\text{and } \quad u \qquad\qquad\qquad = \sin x\,;$$

$$\therefore \frac{du}{dx} h + Uh^2 = \sin(x + h) - \sin x$$

$$= 2\cos\left(x + \frac{h}{2}\right) . \sin\frac{h}{2}.$$

$$\text{But } \frac{a + Nh^{n-1}}{a + \frac{a^3 h^2}{2} + \&c.} = 1, \text{ when } h = 0\,;$$

$$\therefore \frac{a + Nh^{n-1}}{1} \text{ also} = 1, \text{ when } h = 0\,;$$

$$\therefore \frac{a}{1} = 1, \quad \text{i. e. } a = 1\,;$$

$$\therefore \sin h = h + Nh^n, \text{ and } \frac{\sin h}{h} = 1 + Nh^{n-1} = 1, \text{ when } h = 0,$$

$$\text{and } \tan h = h + \frac{h^3}{2} + \&c. \text{ and } \frac{\tan h}{h} = 1 + \frac{h^2}{2} + \&c. = 1, \text{ when } h = 0.$$

Since $\sin A - \sin B = 2 \sin \dfrac{A-B}{2} \cdot \cos \dfrac{A+B}{2}$.

$$\therefore \quad \frac{du}{dx} + Uh = \cos\left(x + \frac{h}{2}\right) \cdot \frac{\sin \frac{h}{2}}{\frac{h}{2}}.$$

and making $h = 0$,

$$\therefore \quad \frac{\sin \frac{h}{2}}{\frac{h}{2}} = 1,$$

$$\frac{du}{dx} = \cos x,$$

$$\text{or} \quad \frac{d \cdot \sin x}{dx} = \cos x.$$

27. $u = \cos x$; putting $x + h$ for x,

$$u + \frac{du}{dx} h + Uh^2 = \cos(x + h);$$

$$\therefore \quad \frac{du}{dx} h + Uh^2 = \cos(x+h) - \cos x$$

$$= -2 \sin\left(x + \frac{h}{2}\right) \sin \frac{h}{2};$$

$$\therefore \quad \frac{du}{dx} + Uh = -\sin\left(x + \frac{h}{2}\right) \frac{\sin \frac{h}{2}}{\frac{h}{2}},$$

and making $h = 0$,

$$\frac{du}{dx} = \frac{d \cdot \cos x}{dx} = -\sin x.$$

28. $u = \tan x$;

$$\therefore \quad u + \frac{du}{dx} h + U h^2 = \tan (x + h);$$

$$\therefore \quad \frac{du}{dx} h + U h^2 = \tan (x + h) - \tan x$$

$$= \frac{\tan h \, (1 + \tan^2 x)}{1 - \tan x \, . \tan h};$$

$$\therefore \quad \frac{du}{dx} + U h = \frac{\tan h}{h} \, . \, \frac{(1 + \tan^2 x)}{1 - \tan x \, . \tan h};$$

make $h = 0$, $\tan h = 0$, and $\dfrac{\tan h}{h} = 1$.

$$\frac{du}{dx} = \frac{d \, . \tan x}{dx} = 1 + \tan^2 x = \sec^2 x = \frac{1}{\cos^2 x}.$$

29. $u = \sec x = \dfrac{1}{\cos x}$;

$\therefore$ by differentiation,

$$\frac{du}{dx} = \frac{\;\frac{- d \, . \cos x}{dx}\;}{(\cos x)^2} = \frac{\sin x}{(\cos x)^2} = \frac{\sin x}{\cos x} \, . \, \frac{1}{\cos x},$$

$$\text{or} \quad \frac{d \, . \sec x}{dx} = \tan x \, . \sec x.$$

30. $u = v \, . \sin x = 1 - \cos x$;

$$\therefore \quad \frac{du}{dx} = - \frac{d \, . \cos x}{dx} = \sin x.$$

31. $u = \operatorname{cotan} x = \dfrac{\cos x}{\sin x}$,

$$\frac{du}{dx} = \frac{\sin x \, . \frac{d \, . \cos x}{dx} - \cos x \, . \frac{d \sin x}{dx}}{(\sin x)^2} = - \frac{(\sin x)^2 + \overline{\cos x}|^2}{\overline{\sin x}|^2}$$

$$= \frac{1}{(\sin x)^2} = - (\operatorname{cosect} x)^2 = - (1 + \overline{\cot x}|^2).$$

32. $u = \text{cosect}\, x = \dfrac{1}{\sin x}$;

$$\therefore \quad \frac{du}{dx} = \frac{-\dfrac{d \sin x}{dx}}{\overline{\sin x}|^2} = \frac{-\cos x}{\overline{\sin x}|^2} = -\cot x \,\text{cosect}\, x.$$

33. Hence collecting the results.

$$\text{If} \quad u = \sin x, \qquad \frac{du}{dx} = \cos x,$$

$$u = \cos x, \qquad \frac{du}{dx} = -\sin x,$$

$$u = \tan x, \qquad \frac{du}{dx} = 1 + \tan^2 x = \frac{1}{\cos^2 x},$$

$$u = \sec x, \qquad \frac{du}{dx} = \sec x \,.\, \tan x,$$

$$u = v \sin x; \quad \therefore \quad \frac{du}{dx} = \sin x,$$

$$u = \cot x, \qquad \frac{du}{dx} = -(1 + \cot^2 x) = -\frac{1}{\sin^2 x},$$

$$u = \text{cosec}\, x, \qquad \frac{du}{dx} = -\text{cosec}\, x \,.\, \cot x.$$

34. To find the differential coefficients of the arc in terms of the sine, cosine, tangent, &c.

Before we do this it will be necessary to shew that if u be a function of x, or if $u = f(x)$, and consequently x a function of u (since it is a matter of convention which of the two is the independent variable), or as it is written $f^{-1}(u)$, where f^{-1} is called the inverse function,

$$\frac{du}{dx} = \frac{1}{\dfrac{dx}{du}},$$

or that if A be the coefficient of h in the expansion of $u_1 = f(x+h)$, and if B be the coefficient of m in the expansion of $x + h = f^{-1}(u+m)$, where $u + m = u_1$,

$$A = \frac{1}{B}.$$

Let δu, δx, be the differentials of u and x.

Then since $\frac{a}{b} = \frac{1}{\frac{b}{a}}$,

$$\therefore \frac{\delta u}{\delta x} = \frac{1}{\frac{\delta x}{\delta u}}.$$

But since the ratio of the differentials is equal to the ratio of the differential coefficient to unity,

$$\therefore \frac{\delta u}{\delta x} = \frac{du}{dx} \text{ and } \frac{\delta x}{\delta u} = \frac{dx}{du}.$$

$$\text{And } \therefore \frac{du}{dx} = \frac{1}{\frac{dx}{du}}.$$

But of this Proposition we add another proof which may appear to some more satisfactory.

Let $u = f(x)$, and when x becomes $x + h$ let $u + m$ be the value of u;

$$\therefore u + m = u + \frac{du}{dx} h + Uh^2;$$

$$\therefore m = \frac{du}{dx} h + Uh^2 \ldots\ldots\ldots(1).$$

Let now x be required in terms of u, or $x = f^{-1}(u)$;

$$\therefore\ x + h = x + \frac{dx}{du}m + Xm^2;$$

$$\text{or } h = \frac{dx}{du}m + Xm^2;$$

therefore, by substituting for h in equation (1),

$$m = \frac{du}{dx} \times \frac{dx}{du}m + Xm^2\frac{du}{dx} + U.\left(\frac{dx}{du}m + Xm^2\right)^2.$$

Equating the coefficients of (m),

$$1 = \frac{du}{dx} \times \frac{dx}{du};$$

$$\text{that is, } \frac{du}{dx} = \frac{1}{\frac{dx}{du}}.$$

Ex.* When $u = \sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, &c. find $\frac{du}{dx}$.

(1) $u = \sin^{-1}x$; $\therefore\ x = \sin u$;

$$\therefore\ \frac{dx}{du} = \cos u = \sqrt{1 - \sin^2 u} = \sqrt{1 - x^2};$$

$$\therefore\ \frac{du}{dx} = \frac{1}{\frac{dx}{du}} = \frac{1}{\sqrt{1 - x^2}}.$$

(2) $u = \cos^{-1}x$, or $x = \cos u$; $\therefore\ \frac{dx}{du} = -\sin u$;

$$\therefore\ \frac{du}{dx} = -\frac{1}{\sin u} = -\frac{1}{\sqrt{1 - x^2}}.$$

* By $u = \sin^{-1}x$ is meant, u is an arc whose sine is x. Similarly, $u = \tan^{-1}x$ is an arc u of which the tangent is x; these are called inverse functions. Thus, if $u = \log x$, then $u = \log^{-1}x$ expresses that u is a number of which the logarithm is x.

(3) $u = \tan^{-1}x$; $\therefore\ x = \tan u$; $\therefore\ \dfrac{dx}{du} = (1 + \tan^2 u)$;

$$\therefore\ \frac{du}{dx} = \frac{1}{1 + \tan^2 u} = \frac{1}{1 + x^2}.$$

(4) $u = \sec^{-1}x$; $\therefore\ x = \sec u$; $\therefore\ \dfrac{dx}{du} = \sec u \tan u$;

$$\therefore\ \frac{du}{dx} = \frac{1}{\sec u \tan u} = \frac{1}{x\sqrt{x^2 - 1}}.$$

(5) $u = \cot^{-1}x$; $\therefore\ x = \cot u$; $\therefore\ \dfrac{dx}{du} = -(1 + \cot^2 u)$;

$$\therefore\ \frac{du}{dx} = -\frac{1}{1 + \cot^2 u} = -\frac{1}{1 + x^2}.$$

(6) $u = \operatorname{cosec}^{-1}x$; $\therefore\ x = \operatorname{cosec} u$; $\therefore\ \dfrac{dx}{du} = -\operatorname{cosec} u . \cot u$;

$$\therefore\ \frac{du}{dx} = \frac{-1}{\operatorname{cosec} u \cot u} = -\frac{1}{x\sqrt{x^2 - 1}}.$$

(7) $u = v . \sin^{-1} x$; $\therefore\ x = v \sin . u$;

$$\therefore\ \frac{dx}{du} = \sin u = \sqrt{1 - \cos^2 u} = \sqrt{(1 - \cos u)(1 + \cos u)}.$$

But $1 - \cos u = x$; $1 + \cos u = 2 - x$,

$$\therefore\ \frac{dx}{du} = \sqrt{2x - x^2} \quad \text{and} \quad \frac{du}{dx} = \frac{1}{\sqrt{2x - x^2}}.$$

Hence recapitulating;

$$\therefore\ \frac{d . \sin^{-1}x}{dx} = \frac{1}{\sqrt{1 - x^2}},$$

$$\frac{d . \cos^{-1}x}{dx} = \frac{-1}{\sqrt{1 - x^2}},$$

$$\frac{d \,.\, \tan^{-1} x}{dx} = \frac{1}{1 + x^2},$$

$$\frac{d \,.\, \sec^{-1} x}{dx} = \frac{1}{x\sqrt{x^2 - 1}},$$

$$\frac{d \,.\, \cot^{-1} x}{dx} = \frac{-1}{1 + x^2},$$

$$\frac{d \,.\, \operatorname{cosec}^{-1} x}{dx} = \frac{-1}{x\sqrt{x^2 - 1}}.$$

$$\frac{d \,.\, v\sin^{-1} x}{dx} = \frac{1}{\sqrt{2x - x^2}}.$$

35. Again, if $u = \sin^{-1}\frac{x}{a}$;

$$\therefore \; \frac{x}{a} = \sin u;$$

$$\therefore \; \frac{dx}{du} = a\cos u = a\sqrt{1 - \frac{x^2}{a^2}} = \sqrt{a^2 - x^2};$$

$$\therefore \; \frac{du}{dx} = \frac{1}{\frac{dx}{du}} = \frac{1}{\sqrt{a^2 - x^2}}.$$

(1) Similarly, if $u = \cos^{-1}\frac{x}{a}$, $\frac{du}{dx} = -\frac{1}{\sqrt{a^2 - x^2}}$.

(2) If $u = \tan^{-1}\frac{x}{a}$;

$$\therefore \; \frac{x}{a} = \tan u,$$

and $\frac{dx}{du} = a\,(1 + \tan^2 u) = a\left(1 + \frac{x^2}{a^2}\right) = \frac{a^2 + x^2}{a}$;

$$\therefore \; \frac{du}{dx} = \frac{a}{a^2 + x^2}.$$

$$\text{Or}\quad \frac{d.\left(\sin^{-1}\dfrac{x}{a}\right)}{dx} = \frac{1}{\sqrt{a^2 - x^2}},$$

$$\frac{d.\cos^{-1}\left(\dfrac{x}{a}\right)}{dx} = \frac{-1}{\sqrt{a^2 - x^2}},$$

$$\frac{d.\left(\tan^{-1}\dfrac{x}{a}\right)}{dx} = \frac{a}{a^2 + x^2},$$

$$\text{and similarly,}\quad \frac{d.\left(\sec^{-1}\dfrac{x}{a}\right)}{dx} = \frac{a}{x\sqrt{x^2 - a^2}}.$$

These expressions should be carefully remembered.

EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

36. Let $u = a^x$, an equation which in general expresses the relation between a number u and its logarithm x, find $\dfrac{du}{dx}$.

LEMMA. $a^x = 1 + Ax + Bx^2 + \&c.$

where $A = (a - 1) - \frac{1}{2}(a - 1)^2 + \frac{1}{3}(a - 1)^3 - \&c.$

For $a^x = (1 + a - 1)^x = (1 + b)^x$, if $b = a - 1$.

$$\text{Now}\quad (1 + b)^x = 1 + xb + \frac{x(x-1)}{2}b^2 + \frac{x(x-1)(x-2)}{2.3}b^3 + \&c.$$

$$= 1 + xb + \left(\frac{x^2}{2} - \frac{x}{2}\right)b^2 + \left(\frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{3}\right)b^3 + \&c.$$

$$= 1 + \left(b - \frac{b^2}{2} + \frac{b^3}{3} - \&c.\right)x + \left(\frac{b^2}{2} - \frac{b^3}{2} + \&c.\right)x^2 + \&c.$$

$$= 1 + Ax + Bx^2 + \&c.$$

where $A = b - \frac{b^2}{2} + \frac{b^3}{3} - \&c.$

$$= (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.$$

and B is also a function of $(a-1)$.

This being proved,

$$u + \frac{du}{dx} h + Uh^2 = a^{x+h} = a^x \cdot a^h$$

$$= a^x \cdot \{1 + Ah + Bh^2 + \&c.\} \text{ by the Lemma;}$$

$\therefore$ equating the coefficients of h,

$$\frac{du}{dx} = Aa^x.$$

Cor. Let e be that value of a which makes $A = 1$,

$$\text{or } (e-1) - \tfrac{1}{2}(e-1)^2 + \tfrac{1}{3}(e-1)^3 - \&c. = 1;$$

$$\therefore \frac{de^x}{dx} = e^x,$$

e is found to be $= 2.71828$, &c. and is the base of what is called the hyperbolic system of logarithms.

37. Next let $u = \log x$; $\therefore x = a^u$; $\therefore \frac{dx}{du} = Aa^u = A \cdot x$;

$$\therefore \frac{du}{dx} = \frac{1}{\frac{dx}{du}} = \frac{1}{A} \cdot \frac{1}{x}.$$

If the base be (e), $A = 1$ and $\frac{du}{dx} = \frac{1}{x}$, or $d \cdot \log x = \frac{dx}{x}$.

38. We next proceed to find the value of $\frac{du}{dx}$ when $u = f(z)$, where z is a function of x, so that $z = \phi(x)$.

In fact, the labour of differentiation is often much lessened by the substitution of z for some function of x, when we want to find the differential coefficient of a complicated function.

As in Art. 34, let δu, δz, δx, be the corresponding differentials of u, z, x.

Then since $\dfrac{A}{B} = \dfrac{A}{C} \times \dfrac{C}{B}$.

$$\therefore \frac{\delta u}{\delta x} = \frac{\delta u}{\delta z} \times \frac{\delta z}{\delta x}.$$

But $\dfrac{\delta u}{\delta x} = \dfrac{du}{dx}$, $\quad \dfrac{\delta u}{\delta z} = \dfrac{du}{dz}$, $\quad$ and $\dfrac{\delta z}{\delta x} = \dfrac{dz}{dx}$.

$$\therefore \frac{du}{dx} = \frac{du}{dz} \times \frac{dz}{dx}.$$

Or if m and h be the increments of z and x, the coefficient of h in $f(x+h) =$ the product of the coefficients of m and h in the expansions of $f(z+m)$ and $\phi(x+h)$.

But we give another proof of this important proposition.

Let u become $f(z)$ where $z = \phi(x)$,

and let $z + m$ be the value of z when x becomes $x + h$.

Then since u is a function of x as well as of z, the value of u will become

$$u + \frac{du}{dx} h + Uh^2,$$

$$\text{and } f(z) \text{ becomes } u + \frac{du}{dz} m + Zm^2;$$

$$\therefore \frac{du}{dx} h + Uh^2 = \frac{du}{dz} m + Zm^2 \ldots\ldots\ldots (1).$$

$$\text{But } z + m = \phi(x + h) = z + \frac{dz}{dx} h + Z_1 h^2;$$

$$\therefore m = \frac{dz}{dx} h + Z_1 h^2,$$

where $Z_1 h^2$ represents the terms after $\dfrac{dz}{dx} . h$, substituting this value of m, in equation (1),

$$\frac{du}{dx}h + Uh^2 = \frac{du}{dz}\cdot\frac{dz}{dx}h + \frac{du}{dz}\cdot Z_1h^2 + Z\left(\frac{dz}{dx}h + Z_1h^2\right)^2,$$

equating the coefficients of h,

$$\frac{du}{dx} = \frac{du}{dz}\cdot\frac{dz}{dx}.$$

Ex. Let $u = 2az + bz^2$, where $z = \sqrt{1 + x^2}$,

$$\frac{du}{dz} = 2a + 2bz, \quad \text{and} \quad \frac{dz}{dx} = \frac{x}{\sqrt{1 + x^2}};$$

$$\therefore \frac{du}{dx} = \frac{2(a + bz)\,.\,x}{\sqrt{1 + x^2}} = \frac{2(a + b\sqrt{1 + x^2})x}{\sqrt{1 + x^2}}.$$

39. Next let $u = \sin z$, where z is a function of x.

Then $\dfrac{du}{dx} = \dfrac{du}{dz}\cdot\dfrac{dz}{dx}$.

But $\dfrac{du}{dz} = \cos z$; $\therefore \dfrac{du}{dx} = \cos z\,.\,\dfrac{dz}{dx}$.

40. Let $u = \cos z$; find $\dfrac{du}{dx}$.

Here $\dfrac{du}{dz} = -\sin z$;

$$\therefore \frac{du}{dx} = \frac{du}{dz}\cdot\frac{dz}{dx} = -\sin z\,.\,\frac{dz}{dx}.$$

41. Let $u = \tan z$.

$$\therefore \frac{du}{dz} = (1 + \tan^2 z); \quad \therefore \frac{du}{dx} = (1 + \tan^2 z)\frac{dz}{dx}.$$

Similarly,

If $u = \sec . z$, $\qquad \dfrac{du}{dx} = \sec z . \tan z . \dfrac{dz}{dx}$,

$u = v . \sin z$, $\qquad \dfrac{du}{dx} = \sin z . \dfrac{dz}{dx}$,

$u = \cot . z$, $\qquad \dfrac{du}{dx} = -(1 + \cot^2 z) . \dfrac{dz}{dx}$,

$u = \operatorname{cosec} . z$, $\qquad \dfrac{du}{dx} = -\operatorname{cosec} z . \cot z . \dfrac{dz}{dx}$.

42. Again if $u = a^z$, find $\dfrac{du}{dx}$.

Since $\dfrac{du}{dz} = A a^z$,

$$\therefore \frac{du}{dx} = \frac{du}{dz} . \frac{dz}{dx} = A . a^z . \frac{dz}{dx}.$$

Cor. If $A = 1$, $\therefore a = e$, $\therefore \dfrac{d . e^z}{dx} = e^z . \dfrac{dz}{dx}$.

43. And if $u = \log(z)$, find $\dfrac{du}{dx}$.

Then $\dfrac{du}{dz} = \dfrac{1}{A} . \dfrac{1}{z}$;

$$\therefore \frac{du}{dx} = \frac{du}{dz} . \frac{dz}{dx} = \frac{1}{A} . \frac{1}{z} . \frac{dz}{dx}.$$

If $A = 1$, $\dfrac{d . \log(z)}{dx} = \dfrac{\;\dfrac{dz}{dx}\;}{z}$.

Or the differential coefficient of the logarithm of any quantity, is equal to the differential coefficient of the quantity, divided by the quantity itself.

EXAMPLES OF DIFFERENTIATION.

(1) $$u = \sin 3x \, . \cos 2x,$$

$$\frac{du}{dx} = 3\cos 3x \cos 2x - 2\sin 3x \, . \sin 2x$$

$$= \cos 3x \cos 2x + 2(\cos 3x \cos 2x - \sin 3x \sin 2x)$$

$$= \cos 3x \cos 2x + 2\cos 5x.$$

(2) $$u = \sin(\cos x) = \sin z, \text{ if } z = \cos x;$$

$$\therefore \frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx} = \cos z \, . (-\sin x) = -\sin x \cos(\cos x).$$

(3) $$u = \sin^{-1} \frac{x}{\sqrt{1+x^2}} = \sin^{-1} z, \text{ if } z = \frac{x}{\sqrt{1+x^2}},$$

$$\frac{du}{dx} = \frac{\frac{dz}{dx}}{\sqrt{1-z^2}},$$

$$\frac{dz}{dx} = \frac{\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2} = \frac{1}{(1+x^2)^{\frac{3}{2}}},$$

$$\sqrt{1-z^2} = \sqrt{1 - \frac{x^2}{1+x^2}} = \frac{1}{\sqrt{1+x^2}};$$

$$\therefore \frac{du}{dx} = \frac{1}{1+x^2}.$$

(4) $$u = \text{h. l.}\,(x + \sqrt{1+x^2}) = \text{h. l.}\, z,$$

$$\therefore \frac{du}{dx} = \frac{\frac{dz}{dx}}{z}, \quad \text{and } z = x + \sqrt{1+x^2};$$

$$\therefore \frac{dz}{dx} = 1 + \frac{x}{\sqrt{1+x^2}} = \frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}} = \frac{z}{\sqrt{1+x^2}};$$

$$\therefore \frac{du}{dx} = \frac{1}{\sqrt{1+x^2}}.$$

(5) $u = \text{h. l.}\,(x+1+\sqrt{2x+x^2}) = \text{h. l.}\,\{x+1+\sqrt{(x+1)^2-1}\}$,

whence, from the last example,

$$\frac{du}{dx} = \frac{1}{\sqrt{(x+1)^2-1}} = \frac{1}{\sqrt{2x+x^2}}.$$

(6) $u = \text{h. l.}\dfrac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}+x} = \text{h. l.}\dfrac{1}{(\sqrt{x^2+1}+x)^2}$

$= -2\,\text{h. l.}(\sqrt{x^2+1}+x) = -\dfrac{2}{\sqrt{x^2+1}}$, by Example (4).

(7) $u = \text{h. l.}\dfrac{x}{\sqrt{x^2+1}+x} = \text{h. l.}\,x - \text{h. l.}\,(\sqrt{x^2+1}+x)$;

$$\therefore \frac{du}{dx} = \frac{1}{x} - \frac{1}{\sqrt{x^2+1}}.$$

(8) $u = \text{h. l.}\dfrac{x}{\sqrt{x^2+1}+1}$, $\qquad \dfrac{du}{dx} = \dfrac{1}{x\sqrt{x^2+1}}$.

(9) $u = \overline{\log x}|^n$; $\quad\therefore \dfrac{du}{dx} = n\,.\,(\log x)^{n-1}\,.\,\dfrac{1}{x}$.

(10) $u = \log(\log x) = \log z$;

$$\therefore \frac{du}{dx} = \frac{du}{dz}\cdot\frac{dz}{dx} = \frac{1}{z}\cdot\frac{1}{x} = \frac{1}{x\log x}.$$

(11) $u = x^{f(x)} = x^z$, suppose.

$\text{h. l.}\,u = z\,\text{h. l.}\,x$; $\quad\therefore \dfrac{du}{dx}\cdot\dfrac{1}{u} = \dfrac{dz}{dx}\log x + z\,.\,\dfrac{1}{x}$;

$$\therefore \frac{du}{dx} = x^z\left\{\frac{z}{v} + \frac{dz}{dx}\,.\,\log x\right\}$$

If $z = x$, $\quad \dfrac{du}{dx} = x^x\,\{1+\log x\} = x^x\log(ex)$.

(12) $u = z^v$, z and v being functions of x.

$\text{h. l.}\,u = v\,\text{h. l.}\,z$; $\quad\therefore \dfrac{du}{dx}\cdot\dfrac{1}{u} = \dfrac{dv}{dx}\,.\,\text{h. l.}\,z + v\,.\,\dfrac{dz}{dx}\cdot\dfrac{1}{z}$;

$$\therefore \quad \frac{du}{dx} = z^v \left\{ \frac{dv}{dx} \text{ h. l. } z + \frac{v}{z} \cdot \frac{dz}{dx} \right\}.$$

Let $z = \sin x$, and $v = \cos x$;

$$\therefore \quad \frac{du}{dx} = \overline{\sin x}|^{\cos x} \left\{ -\sin x \text{ h. l. } \sin x + \frac{\cos^2 x}{\sin x} \right\}.$$

(13) $u = e^{e^x} = e^z$, if $z = e^x$;

$$\therefore \quad \frac{du}{dx} = e^z \cdot \frac{dz}{dx} = e^z \cdot e^x = e^{e^x} e^x.$$

(14) $u = z^{v^y}$, where z, v, and y are functions of x.

Let $v^y = v_1$; $\therefore \; u = z^{v_1}$,

$$\text{and} \quad \frac{du}{dx} = z^{v_1} \left\{ \text{h. l. } z \cdot \frac{dv_1}{dx} + \frac{v_1}{z} \cdot \frac{dz}{dx} \right\}.$$

$$\text{But } \because \; v_1 = v^y; \quad \therefore \; \frac{dv_1}{dx} = v^y \left\{ \text{h. l. } v \times \frac{dy}{dx} + \frac{y}{v} \cdot \frac{dv}{dx} \right\};$$

$$\therefore \quad \frac{du}{dx} = z^{v^y} \left\{ v^y \cdot \text{h. l. } z \left(\text{h. l. } v \cdot \frac{dy}{dx} + \frac{y}{v} \cdot \frac{dv}{dx} \right) + \frac{v^y}{z} \cdot \frac{dz}{dx} \right\}$$

$$= z^{v^y} \cdot v^y \left\{ \text{h. l. } z \times \text{h. l. } v \cdot \frac{dy}{dx} + \frac{y}{v} \text{h. l. } z \cdot \frac{dv}{dx} + \frac{1}{z} \cdot \frac{dz}{dx} \right\}.$$

(15) $u = \text{h. l. } \tan x$;

$$\therefore \quad \frac{du}{dx} = \frac{1 + \tan^2 x}{\tan x} = \frac{\sec^2 x}{\tan x} = \frac{1}{\sin x \cos x} = \frac{2}{\sin 2x}.$$

(16) $u = \text{h. l. } \sqrt{\dfrac{1 + \sin x}{1 - \sin x}}$

$$= \text{h. l. } \frac{\cos \frac{x}{2} + \sin \frac{x}{2}}{\cos \frac{x}{2} - \sin \frac{x}{2}} = \text{h. l. } \frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}}$$

$$= \text{h. l. } \tan\left(45 + \frac{x}{2}\right) = \text{h. l. } \tan z, \text{ if } z = 45 + \frac{x}{2}:$$

$$\therefore \frac{du}{dx} = \frac{2 \cdot \frac{dz}{dx}}{\sin 2z} = \frac{1}{\sin(90 + x)} = \frac{1}{\sin(90 - x)} = \frac{1}{\cos x}.$$

(17) $u = \text{h. l.} \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \text{h. l.} \tan \frac{x}{2}$;

$$\therefore \frac{du}{dx} = \frac{2 \frac{1}{2}}{\sin 2 \cdot \frac{x}{2}} = \frac{1}{\sin x}$$

(18) $u = \text{h. l.} (\cos x + \sqrt{-1} \sin x)$;

$$\therefore \frac{du}{dx} = \frac{-\sin x + \sqrt{-1} \cos x}{\cos x + \sqrt{-1} \sin x} = \sqrt{-1} \frac{\cos x + \sqrt{-1} \sin x}{\cos x + \sqrt{-1} \sin x}$$

$$= \sqrt{-1}.$$

(19) $u = \frac{1}{\sqrt{a^2 - b^2}} \cdot \cos^{-1} \left(\frac{b + a \cos x}{a + b \cos x}\right)$.

Let $z = \frac{b + a \cos x}{a + b \cos x}$;

$$\therefore \frac{du}{dx} = \frac{1}{\sqrt{a^2 - b^2}} \cdot \frac{-\frac{dz}{dx}}{\sqrt{1 - z^2}}.$$

But $-\frac{dz}{dx} = \frac{a \sin x \,.\, (a + b \cos x) - b \sin x \,(b + a \cos x)}{(a + b \cos x)^2}$

$$= \frac{(a^2 - b^2) \,.\, \sin x}{(a + b \cos x)^2},$$

$$1 - z^2 = 1 - \left(\frac{b + a \cos x}{a + b \cos x}\right)^2.$$

$$= \frac{(a + b \cos x)^2 - (b + a \cos x)^2}{(a + b \cos x)^2}$$

$$= \frac{(a^2 - b^2) - (a^2 - b^2) \cos^2 x}{(a + b \cos x)^2} = \frac{(a^2 - b^2) \sin^2 x}{(a + b \cos x)^2};$$

$$\therefore \sqrt{1-z^2} = \frac{\sqrt{a^2-b^2}\,.\sin x}{a+b\cos x};$$

$$\therefore \frac{-\dfrac{dz}{dx}}{\sqrt{1-z^2}} = \frac{\sqrt{a^2-b^2}}{a+b\,.\cos x};$$

$$\text{and } \therefore \frac{du}{dx} = \frac{1}{\sqrt{a^2-b^2}}\,.\,\frac{-\dfrac{dz}{dx}}{\sqrt{1-z^2}}$$

$$= \frac{1}{a+b\,.\cos x}$$

(20) $u = \sqrt{1-x^2} + \sin^{-1} x: \quad \dfrac{du}{dx} = \sqrt{\dfrac{1-x}{1+x}}.$

(21) $u = \dfrac{1}{\sqrt{2}} \sin^{-1}.\dfrac{x\sqrt{2}}{1+x^2}: \quad \dfrac{du}{dx} = \dfrac{1-x^2}{1+x^2}\,.\,\dfrac{1}{\sqrt{1+x^4}}.$

(22) $u = x^{\frac{1}{x}}: \quad \dfrac{du}{dx} = \dfrac{x^{\frac{1}{x}}}{x^2}\log\left(\dfrac{e}{x}\right)$

(23) $u = \log.\dfrac{(x+2)^2}{\sqrt{x+1}\,.\,(x+3)^{\frac{3}{2}}}; \quad \dfrac{du}{dx} = \dfrac{x}{x^3+6x^2+11x+6}.$

(24) $u = \dfrac{e^{ax}(a\sin x - \cos x)}{a^2+1}: \quad \dfrac{du}{dx} = e^{ax}\,.\sin x.$

CHAPTER III.

SUCCESSIVE DIFFERENTIATION. MACLAURIN'S THEOREM.

44. If $u = f(x)$, $\frac{du}{dx}$ the differential coefficient may also be a function of x, suppose it to be equal to p, or that $\frac{du}{dx} = p$; then p is also capable of being differentiated, and $\frac{dp}{dx}$ will be the differential coefficient; this again may be a function of x, or $\frac{dp}{dx}$ may $= q$ a function of x; then q may be differentiated, and so on.

This process is called successive differentiation, and $\frac{du}{dx}$, $\frac{dp}{dx}$, $\frac{dq}{dx}$, &c. are called the first, second, third, &c. differential coefficients.

A convenient notation is readily found.

Since $p = \frac{du}{dx}$;

$$\therefore\ q = \frac{dp}{dx} = \frac{d \cdot \left(\frac{du}{dx}\right)}{dx} \text{ which is written } \frac{d^2u}{dx^2};$$

indicating that the function u and $\frac{du}{dx}$ have both been differentiated. The third differential coefficient or $\frac{dq}{dx}$

$$= \frac{d \cdot \left(\frac{d^2u}{dx^2}\right)}{dx} = \frac{d^3u}{dx^3},$$

and the n^{th} differential coefficient is written $\frac{d^nu}{dx^n}$.

Ex. 1. Let $u = x^4 + x^3 + x^2 + x + 1$,

$$\frac{du}{dx} = 4x^3 + 3x^2 + 2x + 1,$$

$$\frac{d^2u}{dx^2} = 3 \cdot 4x^2 + 2 \cdot 3x + 2,$$

$$\frac{d^3u}{dx^3} = 2 \cdot 3 \cdot 4x + 2 \cdot 3,$$

$$\frac{d^4u}{dx^4} = 2 \cdot 3 \cdot 4,$$

$$\frac{d^5u}{dx^5} = 0.$$

Ex. 2. Let $u = \dfrac{1}{x} = x^{-1}$,

$$\frac{du}{dx} = -x^{-2} = -\frac{1}{x^2},$$

$$\frac{d^2u}{dx^2} = 2x^{-3} = \frac{2}{x^3},$$

$$\frac{d^3u}{dx^3} = -2 \cdot 3x^{-4} = -\frac{2 \cdot 3}{x^4},$$

$$\frac{d^4u}{dx^4} = 2 \cdot 3 \cdot 4x^{-5} = \frac{2 \cdot 3 \cdot 4}{x^5},$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{d^nu}{dx^n} = (-1)^n 2 \cdot 3 \cdot 4 \cdot 5 \ldots n \cdot x^{-(n+1)}.$$

45. If u can be expanded into a series of the form

$$u = A + Bx + Cx^2 + Dx^3 + Ex^4, \text{ \&c.}$$

where A, B, C, &c. are all constant, to find these coefficients.

This is Maclaurin's Theorem.

Since $u = A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{\&c.}$

Then by successive differentiation we have

$$\frac{du}{dx} = B + 2Cx + 3Dx^2 + 4Ex^3 + \&c.$$

$$\frac{d^2u}{dx^2} = 2C + 2 . 3 Dx + 3 . 4 Ex^2 + \&c.$$

$$\frac{d^3u}{dx^3} = 2 . 3 . D + 2 . 3 . 4 Ex + \&c.$$

$$\frac{d^4u}{dx^4} = 2 . 3 . 4 E + \&c.$$

$$\&c. = \&c.$$

Make $x = 0$ in these several equations, and let U_0, U_1, U_2, U_3, &c. represent the values of u, $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, &c. on this supposition;

$$\therefore\ U_0 = A,\quad U_1 = B,\quad U_2 = 2C;\quad \therefore\ C = \frac{U_2}{1 . 2},$$

$$U_3 = 2 . 3 . D;\quad \therefore\ D = U_3 \frac{1}{2 . 3},$$

$$E = U_4 . \frac{1}{2 . 3 . 4},\quad \&c. = \&c.;$$

$$\therefore\ u = U_0 + U_1 x + U_2 . \frac{x^2}{1 . 2} + U_3 . \frac{x^3}{2 . 3} + U_4 . \frac{x^4}{2 . 3 . 4} + \&c.$$

Cor. The general term is obviously $= U_n . \frac{x^n}{1 . 2 . 3 \ldots n}$.

(1) $u = (x + a)^4$; $\therefore\ U_0 = a^4$, i. e. when $x = 0$,

$$\frac{du}{dx} = 4 . \overline{x + a}|^3;\qquad \therefore\ U_1 = 4a^3,$$

$$\frac{d^2u}{dx^2} = 4 . 3 \overline{x + a}|^2;\qquad \therefore\ U_2 = 3 . 4a^2,$$

$$\frac{d^3u}{dx^3} = 2 . 3 . 4 (x + a);\qquad \therefore\ U_3 = 2 . 3 . 4a,$$

$$\frac{d^4u}{dx^4} = 2 \cdot 3 \cdot 4; \qquad \therefore\ U_4 = 2 \cdot 3 \cdot 4,$$

$$\frac{d^5u}{dx^5} = 0; \quad \therefore\ U_5 = 0, \text{ and } U_6,\ U_7, \text{ \&c. all} = 0;$$

$$\therefore\ u = (x+a)^4 = a^4 + 4a^3x + \frac{3 \cdot 4}{1 \cdot 2}a^2x^2 + \frac{2 \cdot 3 \cdot 4}{1 \cdot 2 \cdot 3}ax^3 + \frac{2 \cdot 3 \cdot 4}{2 \cdot 3 \cdot 4}x^4$$

$$= a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4.$$

(2) Expand $(a + bx + cx^2)^n$,

$$u = (a + bx + cx^2)^n; \quad \therefore\ U_0 = a^n,$$

$$\frac{du}{dx} = n \cdot (a + bx + cx^2)^{n-1}(b + 2cx); \quad \therefore\ U_1 = nba^{n-1},$$

$$\frac{d^2u}{dx^2} = n(n-1)(a+bx+cx^2)^{n-2}(b+2cx)^2 + 2cn \cdot (a+bx+cx^2)^{n-1};$$

$$\therefore\ U_2 = n \cdot (n-1)a^{n-2}b^2 + n \cdot 2c \cdot a^{n-1},$$

$$\frac{d^3u}{dx^3} = n \cdot (n-1) \cdot (n-2)(a + bx + cx^2)^{n-3}(b + 2cx)^3$$

$$+ 2n \cdot (n-1)\overline{a + bx + cx^2}|^{n-2} 2c \cdot \overline{b + 2cx}|$$

$$+ 2c \cdot n(n-1) \cdot (a + bx + cx^2)^{n-2}(b + 2cx);$$

$$\therefore\ U_3 = n(n-1)(n-2) \cdot a^{n-3}b^3 + 2 \cdot 3n \cdot (n-1)a^{n-2}bc,$$

&c. = &c.

$$\therefore\ (a+bx+cx^2)^n = a^n + na^{n-1}bx + \left\{n \cdot \frac{(n-1)}{2} \cdot a^{n-2}b^2 + na^{n-1}c\right\}x^2$$

$$+ \left(\frac{n \cdot (n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + n \cdot (n-1) \cdot a^{n-2}bc\right)x^3$$

$+$ &c.

(3) Expand $\sin x$ and $\cos x$ in terms of the arc x.

If $u = \sin x$, If $u = \cos x$,

then $\dfrac{du}{dx} = \cos x$, $\dfrac{du}{dx} = -\sin x$,

$$\frac{d^2u}{dx^2} = -\sin x, \qquad \frac{d^2u}{dx^2} = -\cos x,$$

$$\frac{d^3u}{dx^3} = -\cos x, \qquad \frac{d^3u}{dx^3} = +\sin x,$$

$$\frac{d^4u}{dx^4} = +\sin x, \qquad \frac{d^4u}{dx^4} = \cos x,$$

$$\&c. = \&c. \qquad \&c. = \&c.$$

after the 4th differentiation the values of the differential coefficients recur.

Now make $x = 0$, then in the series for $\sin x$,

$U_0 = 0$, $U_1 = 1$, $U_2 = 0$, $U_3 = -1$, $U_4 = 0$, $U_5 = +1$, &c.

and for the $\cos x$,

$U_0 = 1$, $U_1 = 0$, $U_2 = -1$, $U_3 = 0$, $U_4 = 1$, &c.;

$$\therefore \sin x = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \&c.$$

$$\text{and } \cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{2.3.4} - \&c.$$

(4) Similarly, if $u = \tan x$, we may find $\tan x$ in terms of x.

But more readily in the following manner*,

$$\text{let } u = \tan x = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + \&c.$$

$$\therefore \frac{du}{dx} = (1 + \tan^2 x) = a_1 + 3a_3 x^2 + 5a_5 x^4 + 7a_7 x^6 + \&c.$$

$$\text{But } 1 + \tan^2 x = 1 + (a_1 x + a_3 x^3 + a_5 x^5 + \&c.)^2$$
$$= 1 + a_1^2 x^2 + 2a_1 a_3 x^4 + (a_3^2 + 2a_1 a_5) x^6 + \&c.;$$

therefore equating coefficients of the like powers of x

* That $\tan x$ can only involve odd powers of (x) may be thus shewn:

Let $\tan x = a_1 x + b_2 x^2 + a_3 x^3 + b_4 x^4 + a_5 x^5 + \&c.$;

$\therefore \tan(-x) = -a_1 x + b_2 x^2 - a_3 x^3 + b_4 x^4 - a_5 x^5 + \&c.$;

$\therefore \tan x - \tan(-x) = 2a_1 x + 2a_3 x^3 + 2a_5 x^5 + \&c.$

But $\tan(-x) = -\tan(x)$; $\therefore \tan x - \tan(-x) = 2\tan x$;

$\therefore \tan x = a_1 x + a_3 x^3 + a_5 x^5 + \&c.$

$$a_1 = 1, \quad 3a_3 = a_1^2; \quad \therefore \; a_3 = \frac{1}{3},$$

$$5a_5 = 2a_1 a_3 = \frac{2}{3}; \quad \therefore \; a_5 = \frac{2}{3 \,.\, 5},$$

$$7a_7 = a_3^2 + 2a_1 a_5 = \frac{1}{9} + \frac{4}{3 \,.\, 5} = \frac{17}{5 \,.\, 9}; \quad \therefore \; a_7 = \frac{17}{5 \,.\, 7 \,.\, 9};$$

$$\therefore \; u = x + \frac{x^3}{1 \,.\, 3} + \frac{2x^5}{3 \,.\, 5} + \frac{17x^7}{5 \,.\, 7 \,.\, 9} + \text{\&c.}$$

(5) $u = \sin^{-1} x$, whence if $x = 0$, $U_0 = \sin^{-1} 0 = 0$,

and $\dfrac{du}{dx} = \dfrac{1}{\sqrt{1 - x^2}} = \overline{1 - x^2}\big|^{-\frac{1}{2}} = 1 + \dfrac{1}{2}x^2 + \dfrac{1 \,.\, 3}{2 \,.\, 4}x^4 + \dfrac{1 \,.\, 3 \,.\, 5}{2 \,.\, 4 \,.\, 6}x^6 + \text{\&c.}$

but from Maclaurin's Theorem,

$$\frac{du}{dx} = U_1 + 2U_2 \,.\, \frac{x}{1 \,.\, 2} + 3U_3 \,.\, \frac{x^2}{2 \,.\, 3} + 4U_4 \,.\, \frac{x^3}{2 \,.\, 3 \,.\, 4} + 5U_5 \,.\, \frac{x^4}{2 \,.\, 3 \,.\, 4 \,.\, 5} + \text{\&c.}$$

Equating coefficients of the same powers of x;

$$\therefore \; U_1 = 1, \quad U_2 = 0, \quad \frac{3U_3}{2 \,.\, 3} = \frac{1}{2}; \quad \therefore \; U_3 = 1, \quad U_4 = 0,$$

$$\frac{5U_5}{2 \,.\, 3 \,.\, 4 \,.\, 5} = \frac{1 \,.\, 3}{2 \,.\, 4}, \quad U_5 = 1 \,.\, 3^2, \quad U_6 = 0,$$

$$\frac{7U_7}{2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, 7} = \frac{1 \,.\, 3 \,.\, 5}{2 \,.\, 4 \,.\, 6}; \quad \therefore \; U_7 = 1^2 \,.\, 3^2 \,.\, 5^2;$$

$$\therefore \; \sin^{-1} x = x + \frac{x^3}{1 \,.\, 2 \,.\, 3} + \frac{1 \,.\, 3^2 x^5}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5} + \frac{1^2 \,.\, 3^2 \,.\, 5^2 \,.\, x^7}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, 7} + \text{\&c.}$$

$$= x + \frac{1}{1 \,.\, 2} \,.\, \frac{x^3}{3} + \frac{1 \,.\, 3}{2 \,.\, 4} \,.\, \frac{x^5}{5} + \frac{1 \,.\, 3 \,.\, 5}{2 \,.\, 4 \,.\, 6} \,.\, \frac{x^7}{7} + \text{\&c.}$$

the general term of which is obviously

$$\frac{1 \,.\, 3 \,.\, 5 \,\ldots\ldots\, (2n - 3)}{2 \,.\, 4 \,.\, 6 \,\ldots\ldots\, 2n - 2} \,.\, \frac{x^{2n-1}}{2n - 1}.$$

By this series, the length of a circular arc may be found; thus, let $\sin^{-1}x = 30$; $\therefore\ x = \frac{1}{2}$;

$$\therefore\ 30^0 = \frac{1}{2} + \frac{1}{1\,.\,2}\cdot\frac{1}{3\times 8} + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{32\times 5} + \&c.$$

(6) The same series may be thus obtained without the use of Maclaurin's Theorem.

Let $\quad u = \sin^{-1}x = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + \&c.$;

for that $\sin^{-1}x$ cannot contain even powers of x, may be proved by the method used in the note.

Differentiating

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} = a_1 + 3a_3x^2 + 5a_5x^4 + 7a_7x^6 + \&c.$$

$$\text{But } \frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1\,.\,3}{2\,.\,4}\,.\,x^4 + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}x^6 + \&c.$$

equating coefficients,

$$a_1 = 1, \quad 3a_3 = \frac{1}{2}; \quad \therefore\ a_3 = \frac{1}{2}\cdot\frac{1}{3},$$

$$5a_5 = \frac{1\,.\,3}{2\,.\,4}; \quad \therefore\ a_5 = \frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{5},$$

$$7a_7 = \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}; \quad \therefore\ a_7 = \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{1}{7};$$

$$\therefore\ \sin^{-1}x = \frac{x}{1} + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{x^5}{5} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{x^7}{7} + \&c.$$

(7) $\quad u = \tan^{-1}x, \quad x = 0, \quad U_0 = \tan^{-1}0 = 0$;

$$\therefore\ \frac{du}{dx} = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \&c.$$

$$= U_1 + 2U_2\frac{x}{2} + \frac{3U_3x^2}{1\,.\,2\,.\,3} + \frac{4U_4x^3}{2\,.\,3\,.\,4} + \frac{5U_5x^4}{2\,.\,3\,.\,4\,.\,5} + \&c.$$

$$\therefore\ U_1 = 1, \quad U_2 = 0, \quad \frac{U_3}{2} = -1; \quad \therefore\ U_3 = -2, \quad U_4 = 0,$$

$$\frac{U_5}{2.3.4} = 1; \quad \therefore\ U_5 = 2.3.4, \text{ \&c. \&c.}$$

$$\therefore\ u = x - \frac{2x^3}{2.3} + \frac{2.3.4x^5}{2.3.4.5} - \text{\&c.}$$

$$\text{or} \quad \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \text{\&c.}$$

a series for the arc in terms of the tangent.

46. Hence may be found approximate expressions for the length of the arc of a circle.

$$\text{Let } x = \frac{\pi}{4}; \quad \therefore\ \tan\frac{\pi}{4} = 1,$$

$$\text{and } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{\&c.}$$

$$\text{Again, since } \frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3},$$

$$\text{and } \tan^{-1}\frac{1}{2} = \frac{1}{2} - \frac{1}{3}.\frac{1}{2^3} + \frac{1}{5.2^5} - \text{\&c.}$$

$$\tan^{-1}\frac{1}{3} = \frac{1}{3} - \frac{1}{3}.\frac{1}{3^3} + \frac{1}{5.3^5} - \text{\&c.};$$

$$\therefore\ \frac{\pi}{4} = \left(\frac{1}{2} + \frac{1}{3}\right) - \frac{1}{3}.\left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \frac{1}{5}.\left(\frac{1}{2^5} + \frac{1}{3^5}\right) - \text{\&c}$$

Machin having found that

$$\frac{\pi}{4} = 4\tan^{-1}.\frac{1}{5} - \tan^{-1}.\frac{1}{239},$$

invented a series which is rapidly convergent.

The formula may be thus proved:

$$\text{let } A = 4\tan^{-1}\frac{1}{5} = 4a.$$

$$\text{But }\tan A = \tan 4a = \frac{4\tan a - 4\tan^3 a}{1 - 6\tan^2 a + \tan^4 a} = \frac{\frac{4}{5} - \frac{4}{125}}{1 - \frac{6}{25} + \frac{1}{625}}$$

$$= \frac{4(125-5)}{625 - 150 + 1} = \frac{4 \times 120}{476} = \frac{120}{119}; \quad \therefore > 1,$$

$$\text{and }\tan(A - 45^0) = \frac{\tan A - 1}{\tan A + 1} = \frac{\frac{120}{119} - 1}{\frac{120}{119} + 1} = \frac{1}{239};$$

$$\therefore\ A - 45^0 = \tan^{-1}\frac{1}{239};$$

$$\therefore\ 45^0 = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

$$= 4\left\{\frac{1}{5} - \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} - \&c.\right\}$$

$$-\left\{\frac{1}{239} - \frac{1}{3}\cdot\frac{1}{(239)^3} + \frac{1}{5}\cdot\frac{1}{(239)^5} - \&c.\right\}.$$

47. The logarithm of x cannot be found by Maclaurin's Theorem, since if $x = 0$, U_0, U_1, U_2, &c. become infinite: but $u = \log(1 + x)$ may be easily found.

Suppose the logarithms to be hyperbolic, i. e. let $A = 1$.

$$u = \text{h. l. }(1 + x); \quad \therefore\ U_0 = \text{h. l. }(1) = 0,$$

$$\frac{du}{dx} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \&c.$$

by division.

But from the theorem,

$$\frac{du}{dx} = U_1 + U_2 x + \frac{U_3 x^2}{2} + \frac{U_4 x^3}{2 \cdot 3} + \frac{U_5 x^4}{2 \cdot 3 \cdot 4} + \&c.$$

therefore equating coefficients,

$$U_1 = 1, \quad U_2 = -1, \quad U_3 = 2, \quad U_4 = -2 \cdot 3, \quad U_5 = 2 \cdot 3 \cdot 4;$$

$$\therefore \; u = \log(1+x) = x - \frac{x^2}{2} + \frac{2x^3}{2 \cdot 3} - \frac{2 \cdot 3 x^4}{2 \cdot 3 \cdot 4} + \frac{2 \cdot 3 \cdot 4 x^5}{2 \cdot 3 \cdot 4 \cdot 5} + \&c.$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \&c.$$

Cor. Had a been the base of the system, then

$$\frac{du}{dx} = \frac{1}{A} \cdot \frac{1}{x+1}, \text{ where } A = (a-1) - \tfrac{1}{2}\overline{a-1}|^2 + \tfrac{1}{3}\overline{a-1}|^3 - \&c.$$

$$\text{and } \log(1+x) = \frac{1}{A}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \&c.\right)$$

$$= \frac{1}{A} \text{hyp. log. } (1+x).$$

Hence, if we know the hyp. log. we may find the log to a base a by multiplying the hyp. log. by $\frac{1}{A}$. The factor $\frac{1}{A}$ is called the *modulus*.

48. The series for $\log(1+x)$ does not converge, and is useless in actual computation; but from it a number of series may be derived which are rapidly convergent.

Let $-x$ be written for x in the series for $\log(1+x)$;

$$\therefore \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \&c.$$

subtract this from $\log(1+x)$, then since $\log a - \log b = \log \frac{a}{b}$,

$$\log\left(\frac{1+x}{1-x}\right) = 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \&c.\right\},$$

$$\text{for } \frac{1+x}{1-x} \text{ put } \frac{M}{N}; \quad \therefore \; x = \frac{M-N}{M+N},$$

$$\text{and } \log\frac{M}{N} = 2\left\{\frac{M-N}{M+N} + \tfrac{1}{3}\left(\frac{M-N}{M+N}\right)^3 + \&c.\right\}.$$

Suppose $M = N + z$; $\therefore M + N = 2N + z$;

$$\therefore \log(N+z) = \log N + 2\left\{\frac{z}{2N+z} + \frac{1}{3}\frac{z^3}{(2N+z)^3} + \frac{1}{5}\frac{z^5}{(2N+z)^5} + \&c.\right\},$$

and finally, if $z = 1$,

$$\log(N+1) = \log N + 2\left\{\frac{1}{2N+1} + \frac{1}{3}\frac{1}{(2N+1)^3} + \&c.\right\},$$

a formula from which logarithms may be calculated.

Thus, since $\log 1 = 0$,

$$\log 2 = \quad 2\left\{\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \&c.\right\},$$

$$\log 3 = \log 2 + 2\left\{\frac{1}{5} + \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} + \&c.\right\},$$

$$\log 4 = \log 3 + 2\left\{\frac{1}{7} + \frac{1}{3}\cdot\frac{1}{7^3} + \frac{1}{5}\cdot\frac{1}{7^5} + \&c.\right\},$$

&c. &c.

49. Expand a^x in ascending powers of x.

$$u = a^x \qquad x = 0; \quad \therefore U_0 = 1$$

$$\frac{du}{dx} = A \cdot a^x \ldots\ldots\ldots\ldots \therefore U_1 = A$$

$$\frac{d^2u}{dx^2} = A^2 a^x \ldots\ldots\ldots\ldots \therefore U_2 = A^2$$

$$\frac{d^3u}{dx^3} = A^3 a^x \ldots\ldots\ldots\ldots \therefore U_3 = A^3$$

$$\vdots$$

$$\frac{d^nu}{dx^n} = A^n \cdot a^x \ldots\ldots\ldots\ldots \therefore U_n = A^n;$$

$$\therefore a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{2.3} + \frac{A^4x^4}{2.3.4} + \&c.$$

where $A = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.$

COR. 1. Let $x = \frac{1}{A}$, or $Ax = 1$;

$$\therefore\ a^{\frac{1}{A}} = 1 + 1 + \tfrac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \&c. = 2.7182818, \&c. \quad \therefore\ = e.$$

$$\therefore\ \frac{1}{A}\,\text{h. l.}\, a = \text{h. l.}\, e; \quad \therefore\ A = \frac{\log a}{\log e};$$

$$\therefore\ (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c. = \frac{\log a}{\log e}.$$

Hence in the system of hyperbolic logarithms of which the base is e,

$$A = (e-1) - \tfrac{1}{2}(e-1)^2 + \tfrac{1}{3}\overline{e-1}|^3 - \&c. = \frac{\log e}{\log e} = 1,$$

$$\text{and } e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \&c.$$

COR. 2. To compute A.

$$\text{Since } e = a^{\frac{1}{A}}; \quad \therefore\ e^{+A} = a, \text{ and } e^{-A} = \frac{1}{a};$$

$$\therefore\ -A = \log\left(\frac{1}{a}\right) \text{ in the Napierian system.}$$

$$\text{Now } \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$$

for $1 + x$ put n;

$$\therefore\ \log n = (n-1) - \tfrac{1}{2}(n-1)^2 + \tfrac{1}{3}\overline{n-1}|^3 - \&c.;$$

$$\therefore\ -A = \log\left(\frac{1}{a}\right) = \left(\frac{1}{a} - 1\right) - \tfrac{1}{2}\left(\frac{1}{a} - 1\right)^2 + \tfrac{1}{3}\left(\frac{1}{a} - 1\right)^3 - \&c.;$$

$$\text{or } A = \left(\frac{a-1}{a}\right) + \tfrac{1}{2}\left(\frac{a-1}{a}\right)^2 + \tfrac{1}{3}.\left(\frac{a-1}{a}\right)^3 + \&c.$$

Let $a = 10$.

$$A = (.9) + \tfrac{1}{2}(.9)^2 + \tfrac{1}{3}(.9)^3 + \&c. = 2.302585\ldots\ldots$$

and $\dfrac{1}{A} = .43429448$.

This is the number by which we must multiply the Napierian logarithms to obtain those calculated to a base 10, or Brigg's logarithms.

50. In the expansion for e^x

$$\text{or } e^x = 1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{2\,.\,3} + \frac{x^4}{2\,.\,3\,.\,4} + \frac{x^5}{2\,.\,3\,.\,4\,.\,5} + \&c.;$$

put successively for x, $x\sqrt{-1}$, and $-x\sqrt{-1}$;

$$\therefore\ e^{x\sqrt{-1}} = 1 + x^{\sqrt{-1}} - \frac{x^2}{2} - \frac{x^3\sqrt{-1}}{2:3} + \frac{x^4}{2\,.\,3\,.\,4} + \frac{x^5\sqrt{-1}}{2\,.\,3\,.\,4\,.\,5} - \&c.$$

$$e^{-x\sqrt{-1}} = 1 - x\sqrt{-1} - \frac{x^2}{2} + \frac{x^3\sqrt{-1}}{2\,..\,3} + \frac{x^4}{2\,.\,3\,.\,4} - \frac{x^5\sqrt{-1}}{2\,.\,3\,.\,4\,.\,5} - \&c.$$

by addition and then by subtraction.

$$e^{x\sqrt{-1}} + e^{-x\sqrt{-1}} = 2\left\{1 - \frac{x^2}{2} + \frac{x^4}{2\,.\,3\,.\,4} - \&c.\right\} = 2\cos x.$$

$$e^{x\sqrt{-1}} - e^{-x\sqrt{-1}} = 2\sqrt{-1}\left\{x - \frac{x^3}{2\,.\,3} + \frac{x^5}{2\,.\,3\,.\,4\,.\,5} - \&c.\right\}$$

$$= 2\sqrt{-1}\sin x.$$

Again adding and dividing by 2,

$$e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\,.\,\sin x.$$

Also by subtraction and dividing by 2,

$$e^{-x\sqrt{-1}} = \cos x - \sqrt{-1}\sin x.$$

COR. 1. Hence $\cos x = \dfrac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}$,

and $\sin x = \dfrac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}$;

$$\therefore \tan x = \frac{1}{\sqrt{-1}} \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}} = \frac{1}{\sqrt{-1}} \left(\frac{e^{2x\sqrt{-1}} - 1}{e^{2x\sqrt{-1}} + 1} \right).$$

COR. 2. These equations have been proved independently of the value of x, we may therefore put mx for x;

$$\therefore \cos mx + \sqrt{-1} \sin mx = e^{mx\sqrt{-1}} = \overline{e^{x\sqrt{-1}}}\big|^{m}$$

$$= (\cos x + \sqrt{-1} \sin x)^m,$$

the formula of De Moivre.

51. We have seen, that,

$$\log (1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \&c.;$$

$$\therefore \log \left(1 + \frac{1}{u}\right) = u^{-1} - \frac{u^{-2}}{2} + \frac{u^{-3}}{3} - \frac{u^{-4}}{4} + \&c.;$$

$$\therefore \log \left\{ \left(\frac{1 + u}{1 + \frac{1}{u}} \right) \right\} = \log u = (u - u^{-1}) - \tfrac{1}{2} (u^2 - u^{-2})$$

$$+ \tfrac{1}{3} (u^3 - u^{-3}) - \&c.$$

For u write $e^{x\sqrt{-1}}$; $\therefore \log u = x\sqrt{-1}$;

$$\therefore x\sqrt{-1} = (e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}) - \tfrac{1}{2} (e^{2x\sqrt{-1}} - e^{-2x\sqrt{-1}}) + \&c.$$

$$= 2\sqrt{-1} \left\{ \sin x - \tfrac{1}{2} \sin 2x + \tfrac{1}{3} \sin 3x - \&c. \right\};$$

$$\therefore \frac{x}{2} = \sin x - \tfrac{1}{2} \sin 2x + \tfrac{1}{3} \sin 3x - \&c.;$$

therefore, differentiating,

$$\tfrac{1}{2} = \cos x - \cos 2x + \cos 3x - \cos 4x + \&c.$$

52. By division, $\dfrac{e^{x\sqrt{-1}}}{e^{-x\sqrt{-1}}} = e^{2x\sqrt{-1}} = \dfrac{\cos x + \sqrt{-1}\sin x}{\cos x - \sqrt{-1}\sin x}$

$$= \frac{1 + \sqrt{-1}\tan x}{1 - \sqrt{-1}\tan x}.$$

$$2x\sqrt{-1} = \text{h. l. } (1 + \sqrt{-1}\tan x) - \text{h. l. } (1 - \sqrt{-1}\tan x).$$

But $\text{h. l. } \{(1+u)\} - \text{h. l. } (1-u) = 2\left\{u + \dfrac{u^3}{3} + \dfrac{u^5}{5} + \&\text{c.}\right\}$;

$$\therefore\ 2x\sqrt{-1} = 2\left\{\sqrt{-1}\tan x + \tfrac{1}{3}\overline{\sqrt{-1}\tan x}\,\big|^3 + \tfrac{1}{5}\overline{\sqrt{-1}\tan x}\,\big|^5 + \&\text{c.}\right\}$$

$$= 2\sqrt{-1}\left\{\tan x - \tfrac{1}{3}\overline{\tan x}\,\big|^3 + \tfrac{1}{5}\overline{\tan x}\,\big|^5 - \&\text{c.}\right\};$$

$$\therefore\ x = \tan x - \tfrac{1}{3}\overline{\tan}\,\big|^3 + \tfrac{1}{5}\overline{\tan x}\,\big|^5 - \&\text{c.},$$

which we have obtained before.

53. From the expressions for $\sin x$ and $\cos x$ some series may be deduced; which, although not strictly examples of the application of Maclaurin's Theorem, may find a place here.

$$\text{Since } \sin x = x - \frac{x^3}{2\,.\,3} + \frac{x^5}{2\,.\,3\,.\,4\,.\,5} - \&\text{c.},$$

and that $\sin x$ vanishes whenever $x = 0$, $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$, &c.; $\therefore$ x, $(\pi^2 - x^2)$, $(2^2\pi^2 - x^2)$, $(3^2\pi^2 - x^2)$, &c. are factors of the equation $\sin x = 0$; and therefore

$$\sin x = Ax\,.\,(\pi^2 - x^2)(2^2\pi^2 - x^2)(3^2\pi^2 - x^2)(4^2\pi^2 - x^2), \&\text{c.}$$

$$= k\,.\,x\,.\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right), \&\text{c.},$$

$$\text{where } k = A\,.\,\pi^2 \times 2^2\pi^2 \times 3^2\pi^2, \&\text{c.};$$

$$\therefore \quad \frac{\sin x}{x} = k\left\{\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\right\} \&c.$$

Let $x = 0$; $\therefore \dfrac{\sin x}{x} = 1$, and the right hand side of the equation is reduced to k; $\therefore k = 1$;

$$\therefore \sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right) \&c.$$

$$= x\left\{1 - \frac{x^2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \&c.\right) + Bx^4 - \&c.\right\}.$$

$$\text{But } \sin x = x\left\{1 - \frac{x^2}{2.3} + \frac{x^4}{2.3.4.5} - \&c.\right\};$$

therefore, equating coefficients of like powers,

$$\frac{1}{\pi^2}\cdot\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \&c.\right\} = \frac{1}{6}.$$

$$\therefore \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \&c. = \frac{\pi^2}{6}.$$

Also, since $\cos x = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{2.3.4} - \&c.$ vanishes, when

$$x = \pm\frac{\pi}{2}, \quad \pm\frac{3\pi}{2}, \quad \pm\frac{5\pi}{2}, \quad \&c.$$

$$\operatorname{Cos} x = A\left(\frac{\pi^2}{2^2} - x^2\right).\left(\frac{3^2\pi^2}{2^2} - x^2\right).\left(\frac{5^2\pi^2}{2^2} - x^2\right).\&c.$$

$$= k\left\{1 - \frac{2^2x^2}{\pi^2}\right\}\left(1 - \frac{2^2x^2}{3^2\pi^2}\right)\left(1 - \frac{2^2x^2}{5^2\pi^2}\right) \&c.$$

whence making $x = 0$, $k = 1$;

$$\therefore \cos x = \left(1 - \frac{2^2 x^2}{\pi^2}\right)\left(1 - \frac{2^2 x^2}{3^2\pi^2}\right)\left(1 - \frac{2^2 x^2}{5^2\pi^2}\right) \&c$$

$$= 1 - \frac{2^2 x^2}{\pi^2}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \&c.\right\} + Bx^4 - \&c.$$

But $\cos x = 1 - \dfrac{x^2}{1 \cdot 2} + \dfrac{x^4}{2 \cdot 3 \cdot 4} - \&c.$;

$$\therefore \frac{2^2}{\pi^2}\left\{\frac{1^2}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \&c.\right\} = \frac{1}{2};$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \&c. = \frac{\pi^2}{8}.$$

54. Since

$$\sin x = x \cdot \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right) \&c.;$$

$\therefore$ h. l. $\sin x =$ h. l. $x +$ h. l. $\left(1 - \dfrac{x^2}{\pi^2}\right) +$ h. l. $\left(1 - \dfrac{x^2}{2^2\pi^2}\right) + \&c.$;
in differentiating,

$$\therefore \frac{\cos x}{\sin x} = \frac{1}{x} - \frac{\frac{2x}{\pi^2}}{1 - \frac{x^2}{\pi^2}} - \frac{\frac{2x}{2^2\pi^2}}{1 - \frac{x^2}{2^2\pi^2}} - \frac{\frac{2x}{3^2\pi^2}}{1 - \frac{x^2}{3^2\pi^2}}$$

$$= \frac{1}{x} - \frac{\frac{2x}{\pi^2}}{1 - \frac{x^2}{\pi^2}} - \frac{\frac{2x}{\pi^2}}{2^2 - \frac{x^2}{\pi^2}} - \frac{\frac{2x}{\pi^2}}{3^2 - \frac{x^2}{\pi^2}} - \&c.$$

Let $\dfrac{x^2}{\pi^2} = \theta^2$; $\quad \therefore x = \pi\theta$, and $\dfrac{2x}{\pi^2} = \dfrac{2\theta}{\pi}$;

$$\therefore \frac{1}{\tan \pi\theta} = \frac{1}{\pi\theta} = \frac{2\theta}{\pi}\left\{\frac{1}{1^2 - \theta^2} + \frac{1}{2^2 - \theta^2} + \frac{1}{3^2 - \theta^2} + \&c.\right\};$$

$$\therefore\ \frac{1}{1^2-\theta^2}+\frac{1}{2^2-\theta^2}+\frac{1}{3^2-\theta^2}+\&c.=\frac{1}{2\theta^2}-\frac{\pi}{2\theta\tan\pi\theta}.$$

Again, let $\frac{x^2}{\pi^2}=-\theta^2$; $\therefore\ x=\pi\theta\sqrt{-1}$, and $\frac{2x}{\pi^2}=\frac{2\theta\sqrt{-1}}{\pi}$,

and $\frac{\cos x}{\sin x}=\frac{1+e^{-2x\sqrt{-1}}}{1-e^{-2x\sqrt{-1}}}\sqrt{-1}=\frac{1+e^{+2\pi\theta}}{1-e^{+2\pi\theta}}\sqrt{-1}$;

$$\therefore\ \frac{2\theta\sqrt{-1}}{\pi}\left\{\frac{1}{1^2+\theta^2}+\frac{1}{2^2+\theta^2}+\frac{1}{3^2+\theta^2}+\&c.\right\}$$

$$=\frac{1}{\pi\theta\sqrt{-1}}-\frac{1+e^{2\pi\theta}}{1-e^{2\pi\theta}}\sqrt{-1};$$

$$\therefore\ \frac{1}{1^2+\theta^2}+\frac{1}{2^2+\theta^2}+\&c.=\frac{\pi}{2}\cdot\frac{e^{2\pi\theta}+1}{e^{2\pi\theta}-1}-\frac{1}{2\theta^2}.$$

Again, $\cos x=\left(1-\frac{2^2x^2}{\pi^2}\right)\left(1-\frac{2^2x^2}{3^2\pi^2}\right)\left(1-\frac{2^2x^2}{5^2\pi^2}\right)$ &c.

$$\text{h. l. }\cos x=\text{h. l. }\left(1-\frac{2^2x^2}{\pi^2}\right)+\text{h. l. }\left(1-\frac{2^2x^2}{3^2\pi^2}\right)$$

$$+\text{h. l. }\left(1-\frac{2^2x^2}{5^2\pi^2}\right),\ \&c.;$$

$$\therefore\ \frac{\sin x}{\cos x}=\frac{2x\cdot\frac{2^2}{\pi^2}}{1-\frac{2^2x^2}{\pi^2}}+\frac{2x\cdot\frac{2^2}{3^2\pi^2}}{1-\frac{2^2x^2}{3^2\pi^2}}+\&c.$$

$$=2x\left\{\frac{\frac{2^2}{\pi^2}}{1-\frac{2^2\pi^2}{\pi^2}}+\frac{\frac{2^2}{\pi^2}}{3^2-\frac{2^2x^2}{\pi^2}}+\&c.\right\}.$$

Let $\frac{2^2x^2}{\pi^2}=\theta^2$; $\quad\therefore\ x=\frac{\pi\theta}{2}$; $\quad\therefore\ 2x\times\frac{2^2}{\pi^2}=\frac{4\theta}{\pi}$;

$$\therefore\ \tan\frac{\pi\theta}{2}=\frac{4\theta}{\pi}\cdot\left\{\frac{1}{1^2-\theta^2}+\frac{1}{3^2-\theta^2}+\frac{1}{5^2-\theta^2}+\&c.\right\};$$

$$\therefore \frac{1}{1^2 - \theta^2} + \frac{1}{3^2 - \theta^2} + \frac{1}{5^2 - \theta} + \&c. = \frac{\pi}{4\theta} \tan \frac{\pi\theta}{2}.$$

$$\text{Next, let } \frac{2^2 x^2}{\pi^2} = -\theta^2;$$

$$\therefore x = \frac{\pi\theta}{2}\sqrt{-1}, \quad \text{and } 2x . \frac{2^2}{\pi^2} = \frac{4\theta\sqrt{-1}}{\pi},$$

$$\frac{\sin x}{\cos x} = \frac{1}{\sqrt{-1}} \frac{1 - e^{-2x\sqrt{-1}}}{1 + e^{-2x\sqrt{-1}}} = \frac{1}{\sqrt{-1}} \frac{1 - e^{\pi\theta}}{1 + e^{\pi\theta}} = \sqrt{-1} . \frac{e^{\pi\theta} - 1}{e^{\pi\theta} + 1};$$

$$\therefore \frac{1}{1^2 + \theta^2} + \frac{1}{3^2 + \theta^2} + \frac{1}{3^2 + \theta^2} + \&c. = \frac{\pi}{4\theta} . \frac{e^{\pi\theta} - 1}{e^{\pi\theta} + 1}.$$

Other similar series may be readily deduced.

55. From the expression

$$e^x = 1 + x + \frac{x^2}{1 . 2} + \frac{x^3}{2 . 3} + \&c.$$

Lagrange in the Calcul. des Fonctions has derived an expression for the general term of the polynomial $(a+b+c+d+\&c.)^m$.

Thus for x put $(a + b + c + d + \&c.) x$;

$$\therefore e^{(a+b+c+d+\&c.)x} = 1 + (a + b + c + \&c.) x$$

$$+ \frac{(a + b + c + d + \&c.)^2 x^2}{1 . 2} + \&c.$$

$$+ \frac{(a + b + c + d + \&c.)^m . x^m}{1 . 2 . 3 \ldots\ldots\ldots m} + \&c.$$

But $e^{(a+b+c+\&c.)x} = e^{ax} \times e^{bx} \times e^{cx} + \&c.$

$$= \left(1 + ax + \frac{a^2 x^2}{1 . 2} + \frac{a^3 x^3}{2 . 3} + \&c.\right)$$

$$\times \left(1 + bx + \frac{b^2 x^2}{1 . 2} + \frac{b^3 x^3}{2 . 3} + \&c.\right)$$

$$\times \left(1 + cx + \frac{c^2 x^2}{1 . 2} + \frac{c^3 x^3}{2 . 3} + \&c.\right)$$

&c.

Now the m^{th} term of this expansion will be the product of

$$\frac{a^p \cdot x^p}{1 \cdot 2 \ldots p} \times \frac{b^q \cdot x^q}{1 \cdot 2 \ldots q} \times \frac{c^r \cdot x^r}{1 \cdot 2 \ldots r} \times \text{\&c.},$$

$$\text{where } p + q + r + \text{\&c.} \ldots\ldots = m,$$

whence $(a + b + c + d + \text{\&c.})^m$ will consist of terms included under the general expression

$$\frac{1 \cdot 2 \cdot 3 \ldots m \times a^p \cdot b^q \cdot c^r \ldots}{1 \cdot 2 \ldots p \times 1 \cdot 2 \ldots q \times 1 \cdot 2 \cdot 3 \ldots r \times \text{\&c.}},$$

subject to the condition that $p + q + r + \text{\&c.} = m$.

EXAMPLES.

(1) If $u = x^n \cdot e^x$, $\dfrac{d^4 u}{dx^4} = e^x \{x^n + 4n x^{n-1} + 6n(n-1)x^{n-2}$

$+ 4n \cdot (n-1) \cdot (n-2) x^{n-3} + n(n-1)(n-2)(n-3) \cdot x^{n-4}\}$.

(2) If $u = e^x \sin x$, $\dfrac{d^2 u}{dx^2} = 2e^x \cos x$,

$$\frac{d^4 u}{dx^4} = -2u, \quad \frac{d^8 u}{dx^8} = 4u, \quad \frac{d^{12} u}{dx^{12}} = -8u.$$

(3) Shew that

$$(\cos . x)^3 = 1 - \frac{3x^2}{4} + \frac{7x^4}{8} - \text{\&c.}$$

$$(\tan . x)^4 = x^4 + \frac{4}{3}x^6 + \frac{6}{5}x^8 + \text{\&c.}$$

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{3x^4}{2 \cdot 3 \cdot 4} - \frac{8x^5}{2 \cdot 3 \cdot 4 \cdot 5} - \text{\&c.}$$

(4) Expand $\sin(a + bx + cx^2)$, and $\log(a + bx + cx^2)$ according to the powers of x.

(5) If $\cos(m) - \cos(m + y) = x$, shew that

$$y = \frac{x}{\sin m} - \frac{1}{2}\cot m \, . \left(\frac{x}{\sin m}\right)^2 + \&c.$$

(6) If $\sin u = m \sin x$, prove that

$$u = mx + \frac{m(m^2 - 1)x^3}{2 \, . \, 3} + \frac{m(9m^4 - 10m^2 + 1)x^5}{2 \, . \, 3 \, . \, 4 \, . \, 5} + \&c.$$

CHAPTER IV.

TAYLOR'S THEOREM.

56. If $u = f(x)$, and u_1 be the value of u when x becomes $x + h$,

$$u_1 = u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{2.3} + \&c. + \frac{d^nu}{dx^n}\cdot\frac{h^n}{2.3....n} + \&c.$$

The proof of this theorem may be made to depend upon the following proposition.

$$\text{If } u_1 = f(x+h),\quad \frac{du_1}{dx} = \frac{du_1}{dh},$$

or the coefficient of h is the same in the expansion $f(x+2h)$, whether we suppose in $f(x+h)$, x to become $x+h$, or h to become $h+h$, i.e. $2h$.

Let $x + h = x_1$;

$$\therefore\ u_1 = f(x_1),$$

$$\text{and } \frac{du_1}{dx_1} = \text{some function of } x_1 = \phi(x_1).$$

$$\text{Now } \frac{du_1}{dx_1} = \frac{du_1}{dx}\cdot\frac{dx}{dx_1}, \quad \text{or} = \frac{du_1}{dh}\cdot\frac{dh}{dx_1}.$$

$$\text{But } \because\ x_1 = x + h,\quad \frac{dx_1}{dx} = 1, \text{ if } h \text{ be constant,}$$

$$\text{and } \frac{dx_1}{dh} = 1, \text{ if } x \text{ be constant;}$$

$$\therefore\ \frac{du_1}{dx} = \frac{du_1}{dh}.$$

Hence also it is obvious, that $\dfrac{d\cdot\left(\dfrac{du_1}{dx}\right)}{dx} = \dfrac{d\cdot\left(\dfrac{du_1}{dh}\right)}{dh}$;

$$\text{i. e. } \frac{d^2u_1}{dx^2} = \frac{d^2u_1}{dh^2},$$

$$\text{and that } \frac{d^nu_1}{dx^n} = \frac{d^nu_1}{dh^n}$$

$$\text{Let } \therefore u_1 = f(x+h) = u + \frac{du}{dx}\cdot h + Ph^\alpha + Qh^\beta + Rh^\gamma + \&c.$$

restoring the terms after $\frac{du}{dx}h$, and arranging the indices of h in order and magnitude, beginning with the least;

$$\therefore \frac{du_1}{dx} = \frac{du}{dx} + \frac{d^2u}{dx^2}h + \frac{dP}{dx}h^\alpha + \frac{dQ}{dx}h^\beta + \frac{dR}{dx}h^\gamma + \&c.,$$

$$\text{and } \frac{du_1}{dh} = \frac{du}{dx} + \alpha Ph^{\alpha-1} + \beta Qh^{\beta-1} + \gamma Rh^{\gamma-1} + \&c.;$$

$\therefore$ taking away the common term $\frac{du}{dx}$, and dividing by h, we have

$$\frac{d^2u}{dx^2} + \frac{dP}{dx}h^{\alpha-1} + \frac{dQ}{dx}h^{\beta-1} + \&c.\ldots\ldots$$

$$= \alpha Ph^{\alpha-2} + \beta Qh^{\beta-2} + \gamma Rh^{\gamma-2} + \&c.$$

Now since in the upper series there is a term $\frac{d^2u}{dx^2}$ independent of h, there must also be a term in the lower series equal to $\frac{d^2u}{dx^2}$, that does not involve h; let this term be the first, as we have supposed the indices α, β, γ, &c. to be taken in order of magnitude and increasing;

$$\therefore \frac{d^2u}{dx^2} = a\,P h^{a-2}, \quad \text{and } a - 2 = 0; \quad \therefore a = 2,$$

$$\text{and } 2P = \frac{d^2u}{dx^2}; \quad \therefore P = \frac{d^2u}{dx^2} \cdot \frac{1}{1.2}.$$

Similarly,

$$\frac{dP}{dx} h^{a-1}, \text{ or } \frac{dP}{dx} h, \text{ must } = \beta Q h^{\beta-2};$$

$$\text{i.e. } \frac{dP}{dx} = \beta Q h^{\beta-3}; \quad \therefore \beta - 3 = 0, \text{ and } \beta = 3;$$

$$\therefore 3Q = \frac{dP}{dx} = \frac{d^3u}{dx^3} \cdot \frac{1}{1.2}, \quad \text{and } Q = \frac{d^3u}{dx^3} \cdot \frac{1}{2.3},$$

$$\text{and } \frac{dQ}{dx} h^{\beta-1} = \frac{dQ}{dx} h^2 = \gamma R h^{\gamma-2};$$

$$\therefore \frac{dQ}{dx} = \gamma R h^{\gamma-4}; \quad \therefore \gamma - 4 = 0, \text{ or } \gamma = 4;$$

$$\therefore R = \tfrac{1}{4} \frac{dQ}{dx} = \frac{d^4u}{dx^4} \cdot \frac{1}{2.3.4},$$

and similarly may the other coefficients be found;

$$\therefore u_1 = u + \frac{du}{dx} h + \frac{d^2u}{dx^2} \frac{h^2}{1.2} + \frac{d^3u}{dx^3} \frac{h^3}{2.3} + \&c.$$

$$+ \frac{d^nu}{dx^n} \frac{h^n}{1.2.3....n} + \&c.$$

a theorem which will give the expansion of $f(x+h)$ in all cases, if x remain indeterminate.

Cor. We may now deduce the theorem of Maclaurin which we have proved by an independent process in the preceding chapter.

For by making $x=0$, u_1 becomes $f(h)$ and u, $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, $\frac{d^3u}{dx^3}$, &c. become U_0, U_1, U_2, U_3, &c.

$$\therefore f(h) = U_0 + U_1 h + U_2 \frac{h^2}{1.2} + U_3 \frac{h^3}{2.3} + \&c.$$

or putting x for h, in which case u may be put for $f(x)$

$$= U_0 + U_1 x + U_2 \frac{x^2}{1.2} + U_3 \frac{x^3}{2.3} + \&c.$$

the theorem required.

EXAMPLES.

57. To expand $\sin(x+h)$, $\cos(x+h)$, $\log(x+h)$ and $(x+h)^n$, by Taylor's Theorem,

$$u_1 = u + \frac{du}{dx} h + \frac{d^2u}{dx^2} \frac{h^2}{1.2} + \frac{d^3u}{dx^3} \frac{h^3}{2.3} + \&c.$$

(1) $u = \sin x$;

$$\therefore \frac{du}{dx} = \cos x, \quad \frac{d^2u}{dx^2} = -\sin x, \quad \frac{d^3u}{dx^3} = -\cos x, \quad \frac{d^4u}{dx^4} = \sin x,$$

after which the values recur;

$$\therefore u_1 = \sin(x+h) = \sin x + \cos x \,.\, h - \sin x \frac{h^2}{1.2} - \cos x \frac{h^3}{2.3}$$

$$+ \sin x \frac{h^4}{2.3.4} + \cos x \frac{h^5}{2.3.4.5} - \&c.$$

(2) $u = \cos x$,

$$\frac{du}{dx} = -\sin x, \quad \frac{d^2u}{dx^2} = -\cos x, \quad \frac{d^3u}{dx^3} = \sin x, \quad \frac{d^4u}{dx^4} = \cos x,$$

after which the values recur;

$$\therefore\ u_1 = \cos(x+h) = \cos x - \sin x \cdot \frac{h}{1} - \cos x \frac{h^2}{1 \cdot 2}$$

$$+ \sin x \cdot \frac{h^3}{2 \cdot 3} + \cos x \frac{h^4}{2 \cdot 3 \cdot 4} - \&c.$$

Cor. If in the two expansions we make $x = 0$, we have

$$\sin h = h - \frac{h^3}{2 \cdot 3} + \frac{h^5}{2 \cdot 3 \cdot 4 \cdot 5} - \&c.$$

$$\cos h = 1 - \frac{h^2}{1 \cdot 2} + \frac{h^4}{2 \cdot 3 \cdot 4} - \&c.$$

(3) $u = \log(x)$;

$$\therefore\ \frac{du}{dx} = \frac{1}{x} = x^{-1},\quad \frac{d^2u}{dx^2} = -x^{-2},\quad \frac{d^3u}{dx^3} = 2x^{-3},\quad \frac{d^4u}{dx^4} = -2 \cdot 3x^{-4};$$

$$\therefore\ u_1 = \log(x+h) = \log x + \frac{h}{x} - \tfrac{1}{2} \cdot \frac{h^2}{x^2} + \tfrac{1}{3} \cdot \frac{h^3}{x^3} - \tfrac{1}{4} \cdot \frac{h^4}{x^4} + \&c.$$

let $x = 1$; $\therefore\ \log 1 = 0$;

$$\therefore\ \log(1+h) = h - \tfrac{1}{2}h^2 + \tfrac{1}{3}h^3 - \tfrac{1}{4}h^4 + \tfrac{1}{5}h^5 - \&c.$$

(4) $u = x^n$;

$$\therefore\ \frac{du}{dx} = nx^{n-1},\quad \frac{d^2u}{dx^2} = n(n-1)x^{n-2},\quad \frac{d^3u}{dx^3} = n \cdot (n-1)(n-2) \cdot x^{n-3};$$

$$\therefore\ u_1 = (x+h)^n = x^n + n \cdot x^{n-1}h + \frac{n \cdot (n-1)}{1 \cdot 2} \cdot x^{n-2}h^2$$

$$+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \cdot x^{n-3}h^3 + \&c.$$

(5) The following Proposition which is used in some demonstrations of the parallelogram of forces, is a good application of the Theorem. Given that

$$f(x) \cdot f(h) = f(x+h) + f(x-h),$$

find the form of $f(x)$.

Let u be put for $f(x)$,

$$\therefore u \,.\, f(h) = 2\left\{u + \frac{d^2 u}{dx^2}\frac{h^2}{1\,.\,2} + \frac{d^4 u}{dx^4}\frac{h^4}{2\,.\,3\,.\,4} + \&c.\right\};$$

$$\therefore f(h) = 2\left\{1 + \frac{1}{u}\frac{d^2 u}{dx^2}\frac{h^2}{1\,.\,2} + \frac{1}{u}\,.\,\frac{d^4 u}{dx^4}\frac{h^4}{2\,.\,3\,.\,4} + \&c.\right\}.$$

Now since h is entirely independent of x, the coefficients

$$\frac{1}{u}\,.\,\frac{d^2 u}{dx^2},\quad \frac{1}{u}\frac{d^4 u}{dx^4},\quad \&c.,$$

which cannot contain h, must be constant.

Let therefore $\dfrac{1}{u}\dfrac{d^2 u}{dx^2} = -a^2$; $\therefore \dfrac{d^2 u}{dx^2} = -a^2 u$;

$$\therefore \frac{d^4 u}{dx^4} = -a^2\frac{d^2 u}{dx^2} = a^4 u, \quad \text{also } \frac{d^6 u}{dx^6} = -a^6 u,$$

$$\text{Hence } f(h) = 2\left\{1 - \frac{a^2 h^2}{1\,.\,2} + \frac{a^4 h^4}{2\,.\,3\,.\,4} - \&c.\right\} = 2\cos ah,$$

and $\therefore f(x) = 2\cos ax$; and $f(x \pm h) = 2\cos a(x \pm h)$,

and the Proposition is verified by the well known trigonometrical formula

$$2\cos A \,.\, 2\cos B = 2\cos(A + B) + 2\cos(A - B).$$

58. Taylor's Theorem may be used to approximate to the roots of equations.

Let $X = 0$ be an equation, of which x is one of the roots, and a an approximate value of x, so that $x = a + h$, h being a very small quantity, hence since $X = 0$ is a function of x;

$$\therefore X = 0 = f(a + h) = f(a) + \frac{d\,.\,f(a)}{da}h + \frac{d^2\,.\,f(a)}{da^2}\frac{h^2}{1\,.\,2} + \&c.$$

but since h is assumed very small, we may neglect the terms after the second, and so obtain an approximate value of h;

$$\therefore\ 0 = f(a) + \frac{d\,.f(a)}{da}h; \quad \therefore\ h = \frac{-f(a)}{\frac{d\,.f(a)}{da}} = \frac{-f(a)}{p},$$

$$\text{and } x = a - \frac{f(a)}{p}.$$

If this value of x be not sufficiently near the true one, let it be put $= a_1$, and let the above process be again made use of, and we shall at length arrive at results more and more near the true one.

Ex. 1. $x^3 - 3x + 1 = 0$. By trial 1.5 is found to be near one of the roots.

$$f(a) = a^3 - 3a + 1 = (1.5)^3 - 3 \times (1.5) + 1 = -.125,$$

$$\frac{d\,.f(a)}{da} = 3a^2 - 3 = 6.75 - 3 = 3.75;$$

$$\therefore\ h = +\frac{.125}{3.75} = +.033;$$

$$\therefore\ x = 1.5 + .033 = 1.533.$$

Ex. 2. $x^x = 100$. Since $3^3 = 27$ and $4^4 = 256$, it is clear that x lies between 3 and 4; let $a = (3.5)$.

$$\text{Now } x\,\text{h. l. } x - \text{h. l. } 100 = 0 = u;$$

$$\therefore\ 1 + \log x = \frac{du}{dx};$$

$$\therefore\ f(a) = 3.5\,\text{h. l. }(3.5) - \text{h. l. } 100.$$

$$\frac{d\,.f(a)}{da} = 1 + \log(3.5).$$

But in the Napierian system,

$$\log 100 = 4.60517.$$

$$\log 3.5 = 1.25276;$$

$$\therefore f(a) = 3.5 \times 1.25276 - 4.60517 = -.22051,$$

$$\frac{d \, . f(a)}{da} = 1 + \log 3.5 = 2.25276;$$

$$\therefore h = \frac{.22051}{2.25276} = .09832;$$

$$\therefore x = a + h = 3.59832;$$

a more exact value may be obtained by writing 3.59832 for a, and proceeding as above.

The Napierian logarithms may be obtained from a table of the common logarithms by dividing each logarithm by the number .43429.

$$\text{Thus N.} \log 100 = \frac{2}{.43429} = 4.60517.$$

52. Transform the equation

$$x^n - px^{n-1} + qx^{n-2} - \&c. = 0, \quad \text{or} \quad X = 0,$$

into one whose roots shall be diminished by a constant quantity z.

Let $x = z + y$;

$$\therefore X = f(z + y), \quad \text{and let } Z = f(z);$$

$$\therefore X = Z + \frac{dZ}{dz} y + \frac{d^2 Z}{dz^2} \cdot \frac{y^2}{1 \, . \, 2} + \frac{d^3 Z}{dz^3} \cdot \frac{y^3}{2 \, . \, 3} + \&c. = 0,$$

by Taylor's Theorem.

Or if Z_1, Z_2, Z_3, &c. Z_n, be put for the differential coefficients, the transformed equation becomes

$$Z + Z_1 y + \frac{Z_2 y^2}{1 \cdot 2} + \frac{Z_3 y^3}{2 \cdot 3} + \&c. + \frac{Z_{n-1} y^{n-1}}{1 \cdot 2 \ldots (n-1)} + \frac{Z_n y^n}{2 \cdot 3 \ldots n} = 0,$$

where Z is the value of X, when z is put for x;

$$\therefore\ Z \quad = z^n - pz^{n-1} + qz^{n-2} - \&c.$$

$$\text{and } Z_1 \quad = nz^{n-1} - (n-1)pz^{n-2} + (n-2)qz^{n-3} - \&c.$$

$$\vdots$$

$$Z_{n-1} = n(n-1)(n-2)\ldots 3 \cdot 2z - (n-1) \cdot (n-2) \ldots 2 \cdot p,$$

$$\text{and } Z_n \quad = n(n-1)(n-2)\ldots 3 \cdot 2;$$

therefore by substitution, the transformed equation will become, after writing the terms in an inverse order,

$$y^n + (nz - p)y^{n-1} + \&c. + Z = 0.$$

COR. This transformed equation may be used to take away any particular term of an equation, by putting any of the coefficients Z_1, Z_2, &c. $= 0$, and substituting in the others the value of z derived from it.

Ex. Take away the second term from the equation

$$3x^3 + 15x^2 + 25x - 3 = 0.$$

The transformed equation is, when $x = z + y$,

$$Z + Z_1 y + \frac{Z_2 y^2}{1 \cdot 2} + \frac{Z_3 y^3}{2 \cdot 3} = 0, \quad (\text{for } Z_1 = 0),$$

$$Z \ = \ 3z^3 + 15z^2 + 25z - 3,$$

$$Z_1 = \ 9z^2 + 30z + 25,$$

$$Z_2 = 18z + 30,$$

$$Z_3 = 18, \quad \text{and } Z_1 = 0.$$

But $Z_2 = 0$; $\therefore\ z = -\dfrac{30}{18} = \dfrac{-5}{3}$,

$$Z_1 = 25 - 50 + 25 = 0,$$

$$Z = -\frac{125}{9} + \frac{125}{3} - \frac{225}{3} - 3 = \frac{-152}{9}.$$

$$\therefore\ -\frac{152}{9} + \frac{18y^3}{6} = 0,$$

$$\therefore\ y^3 - \frac{152}{27} = 0.$$

60. Let $u = (a + bx + cx^2)^r$. Find $\dfrac{d^n u}{dx^n}$.

Since the coefficient of h^n in Taylor's Theorem is $\dfrac{d^n u}{dx^n}$ divided by $1.2\ldots n$, if we expand

$$\{a + b(x + h) + c(x + h)^2\}^r,$$

and collect the terms which are multiplied by h^n, these when multiplied by $1.2.3\ldots n$ will give $\dfrac{d^n u}{dx^n}$;

$$\because\ \{a + b(x+h) + c(x+h)^2\}^r = \{a + bx + cx^2 + (b + 2cx)h + ch^2\}^r.$$

Let $a + bx + cx^2 = p$, and $b + 2cx = q$;

$$\therefore\ u_1 = (p + qh + ch^2)^r = p^r\left\{1 + \frac{q}{p}h + \frac{4pc}{4p^2}h^2\right\}^r.$$

But $4pc = 4ac + 4bcx + 4c^2x^2 = (b + 2cx)^2 + 4ac - b^2$

$$= q^2 + e^2, \quad \text{if } e^2 = 4ac - b^2;$$

$$\therefore\ u_1 = p^r\left\{1 + \frac{q}{p}h + \frac{q^2}{4p^2}h^2 + \frac{e^2h^2}{4p^2}\right\}^r = p^r\left\{\left(1 + \frac{q}{2p}h\right)^2 + \frac{e^2h^2}{4p^2}\right\}^r$$

$$= p^r\{(1 + q_1h)^2 + e_1^2h^2\}^r$$

$$= p^r\{\overline{1 + q_1h}|^{2r} + r.(1 + q_1h)^{2r-2}.e_1^2h^2$$

$$+ r.\frac{(r-1)}{2}(1 + q_1h)^{2r-4}e_1^4h^4 + \&c.\}.$$

Then writing down the coefficient of h^n in $(1+q_1h)^{2r}$,

.... h^{n-2}... $(1+q_1h)^{2r-2}$,

.... h^{n-1}... $(1+q_1h)^{2r-1}$,

&c......&c.

we shall have by addition the coefficient of h^n, which multiplied by $1.2.3...n$, will give $\dfrac{d^nu}{dx^n}$.

Now by the Binomial Theorem, the coefficient of

$$h^n \text{ in } (1+q_1h)^{2r} = \frac{2r.(2r-1).(2r-2)...(2r-n+1)}{1\;.\;2\;.\;3\;......\;n}q_1^n...(1),$$

$$\text{of } h^{n-2} \text{ in } \overline{1+q_1h_1}|^{2r-2} = \frac{(2r-2).(2r-3)...(2r-n+1)}{1\;.\;2\;......\;n-2}q_1^{n-2}....(2),$$

$$\text{of } h^{n-4} \text{ in } (1+q_1h)^{2r-4} = \frac{(2r-4).(2r-5)...(2r-n+1)}{1\;.\;2\;......\;n-4}q_1^{n-4}....(3);$$

therefore, substituting $\dfrac{q}{2p}$ for q_1, $\dfrac{e^2}{4p^2}$ for e_1^2, and multiplying (2) by $r\times e_1^2$, and (3) by $\dfrac{r.(r-1)}{1.2}e_1^4$, &c. and the sum of (1), (2), (3), &c. by $1.2.3...n.p^r$,

$$\frac{d^nu}{dx^n} = 1.2.3...np^r\left\{\frac{2r.(2r-1).(2r-2)...(2r-n+1)}{1\;.\;2\;.\;3\;......\;n}\frac{q^n}{2^np^n}\right.$$

$$\left.+\frac{r.(2r-2)...(2r-n+1)}{1\;.\;2\;......\;n-2}\cdot\frac{q^n}{2^np^n}\cdot\frac{e^2}{q^2}+\text{\&c.}\right\}$$

$$=2r.(2r-1).(2r-2)...(2r-n+1)\left(\frac{q}{2}\right)^n.p^{r-n}\left\{1+r.\frac{n.(n-1)}{2r.(2r-1)}\cdot\frac{e^2}{q^2}\right.$$

$$+\frac{r.(r-1)}{1\;2}\cdot\frac{n.(n-1)\;...\;(n-3)}{2r.(2r-1)...2r-3}\cdot\frac{e^4}{q^4}$$

$$\left.+\frac{r.(r-1)(r-2)}{1\;.\;2\;\;.\;3}\cdot\frac{n.(n-1)...(n-5)}{2r\;......\;(2r-5)}\cdot\frac{e^6}{q^6}+\text{\&c.}\right\}.$$

Ex. 1. Let $u = \dfrac{1}{\sqrt{1 - x^2}}$, the example given by Euler.

Here $r = -\frac{1}{2}$, $a = 1$, $b = 0$, $c = -1$, $p = 1 - x^2$, $q = -2x$, $e^2 = -4$;

$$\therefore \frac{d^n (1 - x^2)^{-\frac{1}{2}}}{dx^n} = \frac{1\,.\,2\ldots n\,.\,x^n}{(1 - x^2)^{n+\frac{1}{2}}} \,.\, \Big\{1 + \tfrac{1}{2}\,.\,\frac{n\,.\,(n-1)}{1\,.\,2}\,.\,\frac{1}{x^2}$$

$$+ \frac{1\,.\,3}{2\,.\,4}\,.\,\frac{n.(n-1)\ldots(n-3)}{1\ \,.\ 2\ \ldots\ 4}\,.\,\frac{1}{x^4} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\,.\,\frac{n\,.\,(n-1)\ldots n-5}{1\ .\ 2\ \ldots\ 6}\,.\,\frac{1}{x^6} + \&c.,\Big\}$$

the law of which is obvious.

Ex. 2. Let $u = \sqrt{1 - x^2}$.

Here $r = \frac{1}{2}$; $\therefore 2r - 1 = 0$, and some of the coefficients will be $\frac{0}{0}$. The example however can be easily put under a proper form.

For, since $u = \sqrt{1 - x^2}$,

$$\frac{du}{dx} = \frac{-x}{\sqrt{1 - x^2}},$$

$$\frac{d^2u}{dx^2} = -\frac{1}{(1 - x^2)^{\frac{3}{2}}},$$

$$\text{or}\quad \frac{d^2\sqrt{1 + x^2}}{dx^2} = -\frac{1}{(1 - x^2)^{\frac{3}{2}}};$$

$$\therefore \frac{d^n\sqrt{1 - x^2}}{dx^n} = -\frac{d^{n-2}}{dx^{n-2}}\,.\,(1 - x^2)^{-\frac{3}{2}};$$

and writing $n - 2$ for n, and $-\dfrac{3}{2}$ for r, we shall have the required term.

Ex. 3. Let $u = \sqrt{\cos z}$,

$$\cos z = 1 - 2\sin^2\frac{z}{2} = 1 - x^2,$$

by substituting x^2 for $2\sin^2\frac{z}{2}$;

$$\therefore\ u = \sqrt{\cos z} = \sqrt{1 - x^2},$$

which is reduced to the preceding case.

EXAMPLES.

(1) $\operatorname{Tan}(x + h) = \tan x + h \cdot \sec^2 x + h^2 \cdot \tan x \cdot \sec^2 x + \&c.$

(2) $$\operatorname{Sin}^{-1}(x + h) = \sin^{-1}x + \frac{h}{\sqrt{1 - x^2}} + \frac{h^2 x}{2(1 - x^2)^{\frac{3}{2}}}$$

$$+ \frac{(1 + 2x^2)h^3}{2 \cdot 3(1 - x^2)^{\frac{5}{2}}} + \&c.$$

(3) Prove that if $u = f(x)$

$$f\left(\frac{x}{2}\right) = u - \frac{du}{dx}\frac{x}{2} + \frac{d^2u}{dx^2} \cdot \frac{x^2}{2 \cdot 2^2} - \frac{d^3u}{dx^3} \cdot \frac{x^3}{2 \cdot 3 \cdot 2^3} + \&c.$$

$$f\left(\frac{x}{1 + x}\right) = u - \frac{du}{dx} \cdot \frac{x^2}{1 + x} + \frac{d^2u}{dx^2} \cdot \frac{x^4}{2(1 + x)^2} - \frac{d^3u}{dx^3} \cdot \frac{x^6}{2 \cdot 3(1 + x)^3} + \&c.$$

(4) Find the 6th differential coefficient of $\sqrt{\cos x}$.

(5) Approximate to a root of the equations

(1) $x^3 - 12x - 28 = 0$. Ans. $x = 4.302$.

(2) $x^4 + x - 3 = 0$. Ans. $x = 1.165$.

CHAPTER V.

FAILURE OF TAYLOR'S THEOREM; LIMITS OF THE SAME THEOREM.

61. By the Theorem of Taylor we are enabled to expand the $f(x+h)$ into a series of the form

$$f(x) + ph + qh^2 + rh^3 + \&c.$$

where the powers of h are integral and ascend.

Indeed we may prove *à priori*, that so long as x retains its general value, the expansion of $f(x+h)$ cannot contain any fractional powers of h.

For, suppose that

$$f(x+h) = U + P\sqrt{h} + \&c.$$

where U represents the sum of the terms involving integral powers of h.

Then since $x+h$ enters $f(x+h)$ in the same manner as x enters $f(x)$, it is plain that both functions (undeveloped) have the same number of values, and that the developement of $f(x+h)$ ought to contain no more than $f(x)$ or $f(x+h)$ does.

Now if particular values be given to x, which will neither make P infinite nor evanescent; then to each value of P there will correspond two values of $P\sqrt{h}$, since $\sqrt{h}$ has two values $+a$ or $-a$; and consequently the expanded function will contain twice as many values as the unexpanded one; and therefore twice as many as $f(x)$, which is manifestly contradictory.

Similar reasoning will apply when the index of h is $\frac{m}{n}$

62. If then we give such a value a to x in $f(x+h)$ as will make the unexpanded function $f(a+h)$ to contain fractional powers of h, we cannot expect that Taylor's Theorem will give the required developement. Now the hypothesis that $x=a$ introduces a fractional index of h into $f(x+h)$, supposes that in the original function there must have been some such terms as $(u-a)^{\frac{m}{n}}$, which becomes $(x-a+h)^{\frac{m}{n}}$ in u_1, or $h^{\frac{m}{n}}$ when $x=a$. In such a case it is clear that some of the differential coefficients will become infinite, when $x=a$.

As an illustration, let us suppose that

$$y = b + (x-a)^{\frac{m}{n}};$$

$$\therefore \frac{du}{dx} = \frac{m}{n}(x-a)^{\frac{m}{n}-1},$$

$$\text{and } \frac{d^2u}{dx^2} = \frac{m}{n}\left(\frac{m}{n}-1\right)(x-a)^{\frac{m}{n}-2},$$

$$\text{and } u_1 = b + (x-a)^{\frac{m}{n}} + \frac{m}{n}.(x-a)^{\frac{m}{n}-1}h$$

$$+\frac{m}{n}.\left(\frac{m}{n}-1\right).(x-a)^{\frac{m}{n}-2}\frac{h^2}{1.2} + \&c.$$

$$+\frac{m}{n}.\left(\frac{m}{n}-1\right)\ldots\left(\frac{m}{n}-p+1\right)(x-a)^{\frac{m}{n}-p}\frac{h^p}{1.2\ldots p} - \&c.$$

where if $\frac{m}{n} < p$, that term and all that follow will become infinite when $x=a$.

This circumstance of the differential coefficients becoming infinite when $x=a$ is called the *Failure* of Taylor's Theorem, an improper term, since it rather may be taken as an index that the function cannot be expanded according to the integral powers of h.

63. Again, as the general expansion of $f(x+h)$ can never contain negative powers of h, for if $f(x+h)$ could

$$= A + Bh^{-a} + \&c.$$

if $h = 0$, $f(x+h)$ instead of becoming $f(x)$, would be infinite, we may be led to expect that if $x = a$ introduces into the unexpanded function $f(x+h)$ a term involving h^{-n}, the expansion by Taylor's Theorem will indicate some absurdity. Now it is clear that to have such a term dependent on h^{-n}, we must originally have had such a term as $\dfrac{M}{(x-a)^n}$; for putting $x+h$ for x,

$$\frac{M}{\overline{x-a}|^n} \text{ becomes } \frac{M}{(x+h-a)^n} = \frac{M}{h^n},$$

when $x = a$. M not being supposed to vanish when $x = a$.

Here all the differential coefficients of $\dfrac{1}{(x-a)^n}$ are infinite when $x = a$.

64. The theorem therefore fails whenever $x = a$ makes some radical disappear from $u = f(x)$, and therefore introduces into $u_1 = f(x+h)$, some term involving a fractional power of h; or when $x = a$ renders the original function infinite.

As a simple example of the first case, let $u = b + \sqrt{x-a}$;

$$\therefore u_1 = b + \sqrt{x+h-a}$$

$$= b + \sqrt{x-a} + \tfrac{1}{2} \cdot \frac{1}{\sqrt{x-a}} h - \tfrac{1}{4} \cdot \frac{1}{(x-a)^{\frac{3}{2}}} \frac{h^2}{1 \cdot 2} + \&c.$$

make $x = a$;

$$\therefore u = b, \quad u_1 = b + \sqrt{h},$$

and the expanded function contains infinite terms.

As an example of the second case, let $u = \dfrac{1}{x-a}$;

$$\therefore u_1 = \frac{1}{x-a+h} = \frac{1}{x-a} - \frac{h}{(x-a)^2} + \frac{h^2}{(x-a)^3} + \&c.$$

where $u = \infty$, $u_1 = \dfrac{1}{h}$, and the terms of the expanded functions are infinite when x is put $= a$.

65. Should however $f(a+h)$ contain, when expanded, integral powers of h as far as the n^{th}, and afterwards fractional powers, the first (n) coefficients may be found by means of Taylor's Theorem. The following proposition will establish this fact:

Let $u_1 = f(x+h)$ when expanded according to the powers of h, contain when $x = a$, fractional powers of h after the $(n-1)^{\text{th}}$ power. Then the differential coefficients as far as the $(n-1)^{\text{th}}$ can be assigned, but all the succeeding ones will be infinite.

Let $f(a+h) = A + Bh + Ch^2 + \&c. + Nh^{n-1} + Ph^{\alpha} + \&c.$

where α is a fraction between $n-1$ and n.

Now since the coefficients A, B, C, N, do not contain h we may obtain their values in the same manner as we determined the coefficients of Maclaurin's Theorem, by finding the successive differential coefficients of $f(a+h)$ with respect to h, and then making $h = 0$.

Thus,

$$\frac{df(a+h)}{dh} = B + 2Ch + \&c. + (n-1)Nh^{n-2} + \alpha Ph^{\alpha-1} + \&c.$$

$$\frac{d^2f(a+h)}{dh^2} = 2C + \&c. + (n-1)(n-2)Nh^{n-3} + \alpha(\alpha-1)Ph^{\alpha-2} + \&c.$$

$$\vdots$$

$$\frac{d^{n-1}f(a+h)}{dh^{n-1}} = \ldots(n-1)(n-2)\ldots2.1N + \alpha.(\alpha-1)\ldots(\alpha-n+2)Ph^{\alpha-n+1},$$

$$\frac{d^nf(a+h)}{dh^n} = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1).Ph^{\alpha-n} + \&c.$$

Now if h be made $= 0$, since $\alpha > n-1$, but $< n$, the terms involving P will vanish from the first $(n-1)$ equations, and the $(n-1)$ differential coefficients will be found.

But since $a - n$ is negative,

$$\frac{d^n f(a+h)}{dh^n} = \frac{a(a-1)(a-2)\ldots(a-n+1).P}{h^{n-a}} \text{ is infinite,}$$

when $h = 0$.

66. Again, should the substitution of $x = a$ introduce negative powers of h, all the differential coefficients will be infinite. This case, to which we have previously alluded, is when $u = f(x)$ contains a term $\frac{1}{(x-a)^m}$, for then if x becomes

$$x + h, \quad \frac{1}{(x-a)^m} = \frac{1}{(x+h-a)^m} = \frac{1}{h^m} \text{ when } x = a.$$

Let then $f(a+h) = Ah^{-m} + \&c.$

$$\therefore \frac{df(a+h)}{dh} = \frac{-mA}{h^{m+1}} + \&c.$$

$$\vdots \qquad \vdots$$

$$\text{and } \frac{d^n f(a+h)}{dh^n} = \frac{-m(m+1)(m+2)\ldots(m+n-1).A}{h^{m+n}},$$

where it is manifest that if $h = 0$, $\frac{d^n f(a+h)}{dh^n}$ will become infinite.

From this reasoning it is obvious that if the n^{th} differential coefficient become infinite when $x = a$, the true expansion contains a fractional power of h lying between $n - 1$ and (n) and that if $x = a$ makes $f(x) = \infty$ the true expansion contains negative powers of h.

Thus, let $u = f(x) = b + (x-a)^{\frac{3}{2}}$, find $f(a+h)$,

$$\frac{du}{dx} = \tfrac{3}{2}(x-a)^{\frac{1}{2}},$$

$$\frac{d^2u}{dx^2} = \tfrac{3}{2}\cdot\tfrac{1}{2}\frac{1}{(x-a)^{\frac{1}{2}}} \text{ which } = \infty \text{ if } x = a;$$

$\therefore$ the fractional index of h is $< 2 > 1$.

But $u_1 = b + (x + h - a)^{\frac{3}{2}} = b + h^{\frac{3}{2}}$ when $x = a$,

and $\frac{3}{2}$ lies between 2 and 1.

Again, if $u = bx^m + c(x - a)^{\frac{p}{q}}$,

$$\frac{du}{dx} = m \, . \, bx^{m-1} + \frac{cp}{q} \, . \, (x - a)^{\frac{p}{q} - 1},$$

$$\vdots$$

$$\frac{d^n u}{dx^n} = m(m-1) \dots (m-n+1) b x^{m-n}$$

$$+ c \, . \, \frac{p}{q} \, . \left(\frac{p}{q} - 1\right) \dots \left(\frac{p}{q} - n + 1\right) . \, x^{\frac{p}{q} - n},$$

and let $\frac{p}{q} < n$ but $> n - 1$. Then $\frac{d^n u}{dx^n}$ is the first differential coefficient which becomes infinite, and there ought in the true expansion to be a term involving $h^{\frac{p}{q}}$ which there is, for by putting $x + h$ for x, and afterwards writing a for x, we have $f(a + h) = b \, . \, (a + h)^m + ch^{\frac{p}{q}}$.

If $m < n$, the values of the differential coefficients will disappear when $x = a$, until we come to the n^{th}, which is infinite when $x = a$.

67. In functions of this description we must have recourse to the common algebraical methods, first writing $x + h$ for x, and then putting a for x.

Thus, suppose $u = 2ax + a\sqrt{x^2 - a^2}$;

$$\therefore f(a + h) = 2a(a + h) + a\sqrt{(a + h)^2 - a^2}$$

$$= 2a(a + h) + a\sqrt{2ah + h^2}$$

$$= 2a(a + h) + a\sqrt{2ah} \, . \left(1 + \frac{h}{2a}\right)^{\frac{1}{2}},$$

and then expand $\left(1 + \frac{h}{2a}\right)^{\frac{1}{2}}$ by the Binomial Theorem.

68. Limits of Taylor's Theorem.

If $f(x+h)$ be expanded by Taylor's Theorem, and we stop at the n^{th} term, the sum of the first n terms may differ widely from the true value of $f(x+h)$; it is therefore necessary to calculate the amount or the limit of the error which arises from neglecting the remaining terms before we make use of the preceding terms as an approximation to the value of $f(x+h)$.

The object of the following pages is to ascertain these limits; but the following proposition must precede the investigation.

69. Prop. If $u=f(x)$ vanish when $x=0$, then u and $\dfrac{du}{dx}$ will have the same sign while x increases from 0 to a, if a be positive, but contrary signs if a be negative; $\dfrac{du}{dx}$ being supposed neither to change its sign, nor to become infinite while x increases from 0 to a.

Let a be divided into n equal parts, each $=h$ or $a=nh$.

Then since $f(x+h)=f(x)+\dfrac{du}{dx}h+Ph^2$ (1);

therefore making $x=0$; and therefore u or $f(x)=0$; and if U_1 and P_1 be the value of $\dfrac{du}{dx}$ and P,

$$f(h)=U_1h+P_1h^2.$$

Now if $\left.\begin{matrix} U_2, & U_3, & U_4\ldots U_n \\ P_2, & P_3, & P_4\ldots P_n \end{matrix}\right\}$ be the values of $\dfrac{du}{dx}$ and $P\ldots$

When h, $2h$, $3h\ldots(n-1)h$ are put for x, we have from (1)

$$f(h+h)-f(h)=U_2h+P_2h^2,$$

$$f(2h+h)-f(h+h)=U_3h+P_3h^2,$$

$$\vdots \qquad \vdots$$

$$f\{(n-1)h+h\}-f\{(n-2)h+h\}=U_nh+P_nh^2,$$

whence, by addition,

$f(nh)$ or $f(a) = (U_1 + U_2 + U_3 + \text{\&c.} + U_n)h + (P_1 + P_2 + \text{\&c.} + P_n)h^2$;

and by diminishing h, the first term $(U_1 + U_2 + U_3 + \text{\&c.} + U_n)h$ may be rendered greater than the second, and therefore the algebraical sign of $f(a)$ will depend alone on the first term.

Also $f(h)$ will have the same sign as U_1, which is $\dfrac{du}{dx}$ when $x = 0$.

Or, since $\dfrac{du}{dx}$ does not change its sign,

$$f(h) \text{ will have the same sign as } \frac{du}{dx}.$$

Also $f(2h) - f(h)$ will have the same sign as U_2, which is the value of $\dfrac{du}{dx}$ when $x = h = \dfrac{a}{n}$; and therefore the same sign as $\dfrac{du}{dx}$.

And therefore $f(a)$ which has the same sign as the sum of the products $(U_1 + U_2 + U_3 + \text{\&c.} + U_n)\dfrac{a}{n}$ will have the same sign as $\dfrac{du}{dx}$, if a be positive, but the contrary sign if a be negative.

70. This proposition being premised, let it be applied to find the limit of the error incurred in neglecting any terms of Taylor's Theorem.

Let us now assume that the true value of $f(x + h)$ or u_1 lies between the values

$$u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1 \cdot 2} + \text{\&c.} + \frac{d^nu}{dx^n} \cdot \frac{h^n}{1 \cdot 2 \ldots n} + \frac{mh^{n+1}}{1 \cdot 2 \cdot 3 \ldots (n+1)},$$

and $$u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1 \cdot 2} + \text{\&c.} + \frac{d^nu}{dx^n} \cdot \frac{h^n}{1 \cdot 2 \ldots n} + \frac{Mh^{n+1}}{1 \cdot 2 \ldots (n+1)},$$

where m and M are the least and greatest values of which the remaining part of Taylor's Theorem

$$\frac{d^{n+1}u}{dx^{n+1}} + \frac{d^{n+2}u}{dx^{n+2}} \cdot \frac{h}{n+2} + \frac{d^{n+3}u}{dx^{n+3}} \frac{h^2}{(n+2)(n+3)} + \&c.$$

is capable of;

$$\therefore\ u_1 - \left(u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \&c. + \frac{d^nu}{dx^n}\frac{h^n}{1.2\ldots n}\right) > \frac{mh^{n+1}}{1.2\ldots(n+1)}$$

$$< \frac{Mh^{n+1}}{1.2.3\ldots(n+1)}.$$

$$\text{or } u_1 - \left(u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \&c.\right.$$

$$\left. + \frac{d^nu}{dx^n}\frac{h^n}{1\ldots n}\right) - \frac{mh^{n+1}}{1.2\ldots(n+1)} > 0,$$

$$\text{and } \frac{Mh^{n+1}}{1.2.3\ldots(n+1)} - u_1 + u + \frac{du}{dx}h + \&c. + \frac{d^nu}{dx^n}\frac{h^n}{1\ldots n} > 0.$$

Now both these quantities vanish when $h = 0$, since then $u_1 = u$.

Therefore by the Lemma, their first differential coefficient will also have the same sign. Now differentiating with respect to h,

$$\frac{du_1}{dh} - \left\{\frac{du}{dx} + \frac{d^2u}{dx^2}.h + \&c. + \frac{d^nu}{dx^n}.\frac{h^{n-1}}{1\ldots(n-1)}\right\} - \frac{mh^n}{1.2\ldots n} > 0,$$

$$\text{and } \frac{Mh^n}{1\ldots n} - \frac{du_1}{dh} + \frac{du}{dx} + \frac{d^2u}{dx^2}\frac{h}{1} + \&c. + \frac{d^nu}{dx^n}\frac{h^{n-1}}{1\ldots(n-1)} > 0.$$

Again, considering these expressions as functions of h which vanish when $h = 0$, for then $\frac{du_1}{dh} = \frac{du}{dx}$; their first differential coefficients will have the same sign as the functions have, or be both greater than zero; whence again differentiating, we have

$$\frac{d^2u_1}{dh^2} - \left\{\frac{d^2u}{dx^2} + \&c. + \frac{d^{n-2}u}{dx^{n-1}} \frac{h^{n-2}}{1.2...(n-2)}\right\} - \frac{mh^{n-1}}{1.2...n-1} > 0,$$

$$\text{and } \frac{Mh^{n-1}}{1.2...n-1} - \frac{d^2u_1}{dh^2} + \frac{d^2u}{dx^2} + \&c. + \frac{d^{n-2}u}{dx^{n-2}} \cdot \frac{h^{n-2}}{1.2...n-2} > 0,$$

which are both functions of h, which vanish when $h = 0$, since then, $\frac{d^2u_1}{dh^2} = \frac{d^2u}{dx^2}$.

Now if this process be continued $(n+1)$ times, we shall at length obtain

$$\frac{d^{n+1}u_1}{dh^{n+1}} - m > 0,$$

$$\text{and } M - \frac{d^{n+1}u_1}{dh^{n+1}} > 0;$$

$$\text{or since } \frac{d^{n+1}u_1}{dh^{n+1}} = \frac{d^{n+1}u_1}{dx^{n+1}},$$

$$M - \frac{d^{n+1}u_1}{dx^{n+1}} > 0, \quad \text{and } \frac{d^{n+1}u_1}{dx^{n+1}} - m > 0,$$

a condition which is satisfied by taking M equal the greatest value of the $(n+1)^{\text{th}}$ differential coefficient, and m equal the least value.

$$\text{or } M = \frac{d^{n+1}f(x+h)}{dx^{n+1}}, \quad \text{and } m = \frac{d^{n+1}f(x)}{dx^{n+1}},$$

and therefore the true value of u_1 lies between

$$u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \&c. + \frac{d^{n+1}f(x)}{dx^{n+1}} \cdot \frac{h^{n+1}}{1.2.3(n+1)},$$

$$\text{and } u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \&c. + \frac{d^{n+1}f(x+h)}{dx^{n+1}} \cdot \frac{h^{n+1}}{1.2(n+1)},$$

and the error made by omitting the terms after the n^{th} is less than

$$\frac{(M-m)h^{n+1}}{1.2.3...(n+1)},$$

Ex. 1. Let $u = x^p$;

$$\therefore \frac{d^{n+1}u}{dx^{n+1}} = p.(p-1).(p-2)....(p-n)\,x^{p-n-1} = m,$$

and $M = p.(p-1).(p-2)....(p-n).(x+h)^{p-n-1}$;

therefore error committed by omitting the terms of $(x+h)^p$, after the n^{th}

$$\text{is} < \frac{p.(p-1)(p-2)......(p-n)}{1.2.3......(n-1)}\left\{(x+h)^{p-n-1} - x^{p-n-1}\right\}.$$

Ex. 2. Let $u = a^x$;

$$\therefore \frac{d^n u}{dx^n} = A^n a^x, \quad \text{and} \quad \frac{d^n u_1}{dx^n} = A^n a^{x+h};$$

therefore the true expansion of a^{x+h} lies between the series,

$$a^x.\left\{1 + Ah + \frac{A^2h^2}{1.2} + \&c. + \frac{A^n h^n}{1.2...n} + \frac{A^{n+1}h^{n+1}}{1.2...n+1}\right\},$$

$$\text{and } a^x.\left\{1 + Ah + \frac{A^2h^2}{1.2} + \&c. + \frac{A^n h^n}{1.2...n} + \frac{A^{n+1}h^{n+1}a^h}{1.2.3...n+1}\right\},$$

and the error committed by omitting the terms after the n^{th},

$$\text{is} < \frac{A^{n+1}h^{n+1}}{1.2...(n+1)}(a^h - 1),$$

$$< \frac{U_n A.h}{n+1}(a^h - 1), \text{ if } U_n \text{ be the } n^{\text{th}} \text{ term.}$$

Again, if U_n be the first term that converges,

$$\frac{U_n}{U_{n+1}} > 1, \text{ i.e. } \frac{n+1}{Ah} > 1.$$

Let $(n+1) = 2Ah$,

$$\text{therefore error} < \frac{U_n}{2}.(a^h - 1), \quad < \frac{U_n}{2}.(a^{\frac{n+1}{2A}} - 1).$$

Ex. 3. Let $u = \log x$;

$$\therefore \frac{d^{n+1}u}{dx^{n+1}} = (-1)^n . 1 . 2 . 3 \ldots n . x^{-(n+1)} = m,$$

and the error, by omitting the terms after the n^{th}, is

$$< (-1)^n . \frac{h^{n+1}}{n+1} . \{(x+h)^{-(n+1)} - x^{-(n+1)}\}.$$

71. From this reasoning, it is plain that there is some one term which is exactly equal to the sum of the terms after the n^{th}. Let N be this term, therefore $f(x+h)$ becomes

$$u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \&\text{c.} + \left(\frac{d^nu}{dx^n} + \frac{Nh}{n+1}\right)\frac{h^n}{1.2\ldots n},$$

and to find the value of h, which shall make $\frac{d^nu}{dx^n}\frac{h^n}{1.2\ldots n}$ greater than the remaining terms of the series, we must merely have

$$\frac{d^nu}{dx^n} > \frac{Nh}{n+1}, \quad \text{or } h < \frac{d^nu}{dx^n} . \frac{n+1}{N},$$

it is not necessary that N should be known, we may substitute for it a greater quantity, as M.

72. We may here add some remarks upon a method of notation, by which the Theorems of Taylor and Maclaurin may be put under very simple forms.

We have hitherto considered the letter d prefixed to u, as in du, d^2u, d^3u, &c. to be a symbol of *operation* and not of *quantity*, thus d, d^2, d^3 &c. indicate that u has been differentiated, once, twice, &c. But we may separate the d and its powers from u; and if we treat it as an algebraical quantity, no error can arise, so long as we bear in mind its original signification.

Thus suppose in Taylor's Theorem where we have

$$u_1 = u + \frac{du}{dx} \cdot h + \frac{d^2u}{dx^2}\frac{h^2}{1\,.\,2} + \frac{d^3u}{dx^3}\frac{h^3}{2\,.\,3},$$

we look upon d as a factor of u we shall have

$$u_1 = u\left\{1 + \frac{d}{dx} \cdot h + \frac{d^2}{dx^2}\frac{h^2}{1\,.\,2} + \frac{d^3}{dx^3}\frac{h^3}{2\,.\,3} + \&c.\right\},$$

let $\dfrac{d}{dx} = t$, and $\therefore \dfrac{d^2}{dx^2} = t^2$, &c.,

$$\therefore u_1 = u\left\{1 + th + \frac{t^2h^2}{1\,.\,2} + \frac{t^3h^3}{2\,.\,3} + \&c.\right\}.$$

$$= ue^t \;;$$

for e^{th} when expanded will produce a series of the required form, and so long as we take care that the powers of d be referred to operation and not to quantity, no error can be produced, and thence Taylor's Theorem may be concisely written

$$u_1 = u\,e^{\frac{d}{dx}\cdot h}.$$

Again since Maclaurin's Theorem is

$$u = U_0 + U_1x + \frac{U_2x^2}{1\,.\,2} + \frac{U_3x^3}{2\,.\,3} + \&c.$$

if we may be allowed to treat the coefficients U_0, U_1, U_2, U_3, &c. as powers U^0, U^1, U^2, U^3, &c. we have

$$u = 1 + Ux + \frac{U^2x^2}{1\,.\,2} + \frac{U^3x^3}{2\,.\,3} + \&c.$$

$$= e^{Ux}.$$

Nor can error arise, if we keep in mind the original meaning of the coefficients U_0, U_1, U_2, &c. and if when we expand e^{Ux} we change the indices of U into suffixes, putting U_0 instead of unity. But the utility of this method of notation will be chiefly apparent when the reader enters upon the study of the Calculus of Finite Differences.

CHAPTER VI

VANISHING FRACTIONS.

73. SOMETIMES the substitution of a particular value for the unknown quantity, will make both the numerator and denominator of a fraction vanish, such a fraction is called a vanishing fraction.

Thus $\dfrac{x^2-1}{x-1}$ becomes $=\dfrac{0}{0}$ when $x=1$, but since by division $\dfrac{x^2-1}{x-1}=(x+1)$, the true value of the fraction when $x=1$ is $1+1=2$.

In this example the numerator and denominator vanish when $x=1$, because both contain the factor $(x-1)$, which is $=0$ on the supposition of $x=1$.

74. We proceed to shew that the values of these fractions may be finite, nothing or infinite.

Let $u=\dfrac{P}{Q}$ be the fraction, and let $x=a$ be the value of x, which makes $P=0$ and $Q=0$.

Then P and Q must be both divisible by $(x-a)$ or the powers of $(x-a)$,

$$\text{let } P=p\,.\,\overline{x-a}|^m, \quad \text{and } Q=q\,.\,\overline{x-a}|^n;$$

$$\therefore\ u=\frac{p}{q}\cdot\frac{\overline{x-a}|^m}{\overline{x-a}|^n}.$$

(1) Let $m = n$; $\therefore\ u = \frac{p}{q}$, and $= \frac{p'}{q'}$ when $x = a$,

which is finite, since neither p nor q contain $(x - a)$.

(2) Let $m > n$; $\therefore\ u = \frac{p}{q} \cdot \overline{x - a}|^{m-n} = 0$, if $x = a$.

(3) Let $m < n$; $\therefore\ u = \frac{p}{q} \cdot \frac{1}{\overline{x - a}|^{n-m}} = \frac{1}{0} = \infty$, if $x = a$.

75. From the preceding example it appears that the true value of the fraction is found by getting rid of the factor $(x - a)^m$, which is common both to the numerator and denominator.

When m and n are whole numbers, the value may be found by successive differentiations.

For since P and Q are each functions of x; when x becomes $x + h$, the fraction u will become

$$\frac{f(x+h)}{\phi(x+h)} = \frac{P + dP \cdot h + d^2P\frac{h^2}{1 \cdot 2} + d^3P\frac{h^3}{2 \cdot 3} + \&c.}{Q + dQ \cdot h + d^2Q\frac{h^2}{1 \cdot 2} + d^3Q\frac{h^3}{2 \cdot 3} + \&c.},$$

writing dP, dQ, &c. for $\frac{dP}{dx}$, $\frac{dQ}{dx}$, &c.

Let $x = a$; $\therefore\ P = 0$, and $Q = 0$,

and the fraction, by dividing each term by h, becomes

$$\frac{f(a+h)}{\phi(a+h)} = \frac{dP + d^2P\frac{h}{1 \cdot 2} + d^3P \cdot \frac{h^2}{2 \cdot 3} + \&c.}{dQ + d^2Q\frac{h}{1 \cdot 2} + d^3Q \cdot \frac{h^2}{2 \cdot 3} + \&c.} = \frac{dP}{dQ}, \text{ when } h = 0,$$

which is the value of $u = \frac{P}{Q}$; when $x = a$.

Should, however, $x = a$ make all the differential coefficients of P to the m^{th} order, and those of Q to the n^{th} order disappear, we have

$$u_1 = \frac{f(a+h)}{\phi(a+h)} = \frac{\dfrac{d^m P}{dx^m} \cdot \dfrac{h^m}{1 \,.\, 2 \,.\, 3 \ldots m} + \&c.}{\dfrac{d^n Q}{dx^n} \cdot \dfrac{h^n}{1 \,.\, 2 \,.\, 3 \ldots n} + \&c.}.$$

If $m = n$, dividing numerator and denominator by $\dfrac{h^m}{1 \,.\, 2 \,.\, 3 \ldots m}$, and then making $h = 0$,

$$u = \frac{\dfrac{d^m P}{dx^m}}{\dfrac{d^m Q}{dx^m}}.$$

If $m > n$, u is $= 0$.

If $m < n$, u is $= \dfrac{1}{0}$. $= \infty$

76. If m be a fraction, this method is inapplicable. Since $x = a$ will make some one of the differential coefficients infinite.

Thus, if $u = \dfrac{(x^2 - a^2)^{\frac{1}{2}}}{\sqrt{x - a}} = \sqrt{x + a} = \sqrt{2a}$, $\quad \dfrac{dP}{dx} = \dfrac{x}{\sqrt{x^2 - a^2}}$,

and $\dfrac{dQ}{dx} = \dfrac{1}{2\sqrt{x - a}}$,

both of which become infinite when $x = a$.

In such a case we must have recourse to a method, which is perfectly general, and not difficult in its application.

Let $\dfrac{P}{Q}$ be the fraction, where P and Q both vanish when $x = a$. For x put $a + h$, and let the numerator and denominator be expanded according to the powers of h, the indices increasing, so that the fraction becomes

$$\frac{Ah^\alpha + Bh^\beta + Ch^\gamma + \&c.}{A_1 h^{\alpha_1} + B_1 h^{\beta_1} + C_1 h^{\gamma_1} + \&c.},$$

which is of the proper form, since when $h = 0$ the fraction becomes $\frac{0}{0}$.

There will obviously be three cases, $\alpha = \alpha_1$, $\alpha > \alpha_1$, and $\alpha < \alpha_1$.

(1) If $\alpha = \alpha_1$ divide each term by h^α, and we have

$$\therefore \frac{A + Bh^{\beta-\alpha} + Ch^{\gamma-\alpha} + \&c.}{A_1 + B_1 h^{\beta_1-\alpha} + C_1 h^{\gamma_1-\alpha} + \&c.} = \frac{A}{A_1}, \quad \text{when } h = 0,$$

which is infinite.

(2) $\alpha > \alpha_1$, then the fraction

$$= \frac{Ah^{\alpha-\alpha_1} + Bh^{\beta-\alpha_1} + \&c.}{A_1 + B_1 h^{\beta_1-\alpha_1} + \&c.} = 0, \quad \text{when } h = 0.$$

(3) $\alpha < \alpha_1$, we have then

$$\frac{A + Bh^{\beta-\alpha} + \&c.}{A_1 h^{\alpha_1-\alpha} + B_1 h^{\beta_1-\alpha} + \&c.} = \frac{A}{0} = \infty, \quad \text{when } h = 0:$$

Cor. 1. If $u = \frac{P}{Q}$ becomes $\frac{\infty}{\infty}$, when $x = a$ it may be reduced to the form $\frac{0}{0}$.

$$\text{For } \frac{P}{Q} = \frac{\frac{1}{Q}}{\frac{1}{P}} = \frac{\frac{1}{\infty}}{\frac{1}{\infty}} = \frac{0}{0}, \quad \text{when } x = a.$$

COR. 2. If $u = \frac{1}{P} - \frac{1}{Q} = \frac{1}{0} - \frac{1}{0}$, or $\infty - \infty$, when $x = a$, u may be reduced to the form $\frac{0}{0}$.

$$\text{For } \frac{1}{P} - \frac{1}{Q} = \frac{Q - P}{PQ} = \frac{0}{0}, \text{ when } x = a.$$

COR. 3. If $P \times Q = 0 \times \infty$, when $x = a$, it may be put under the form $\frac{0}{0}$.

$$\text{For } Q = \frac{1}{Q_1}, \text{ if } Q_1 = 0, \text{ when } x = a;$$

$$\therefore P \times Q = P \times \frac{1}{Q_1} = \frac{P}{Q_1} = \frac{0}{0}, \text{ if } x = a.$$

EX. 1. Find the value of $u = \dfrac{x^3 - 1}{x^3 + 2x^2 - x - 2}$, when $x = 1$.

$$P = x^3 - 1; \quad \therefore \frac{dP}{dx} = 3x^2 = 3, \text{ when } x = 1.$$

$$Q = x^3 + 2x^2 - x - 2; \quad \therefore \frac{dQ}{dx} = 3x^2 + 4x - 1 = 6, \text{ if } x = 1;$$

$$\therefore u = \frac{0}{0} = \frac{3}{6} = \frac{1}{2}.$$

EX. 2. Find the value of $\dfrac{a^x - b^x}{x}$, when $x = 0$,

$$u = \frac{1 - 1}{0} = \frac{0}{0}, \text{ when } x = 0.$$

$$P = a^x - b^x, \text{ and } Q = x,$$

$$\frac{dP}{dx} = a^x \log a - b^x \log b = \log a - \log b = \log\frac{a}{b}, \text{ when } x = 0,$$

$$\text{and } \frac{dQ}{dx} = 1; \quad \therefore u = \log\left(\frac{a}{b}\right).$$

Ex. 3. $u = \dfrac{x^x - x}{1 - x + \log x} = \dfrac{0}{0}$, if $x = 1$,

$$P = x^x - x, \text{ and } Q = 1 - x + \log x,$$

$$\frac{dP}{dx} = x^x (1 + \log x) - 1 = 0, \text{ if } x = 1,$$

$$\frac{dQ}{dx} = -1 + \frac{1}{x} = 0, \text{ if } x = 1,$$

$$\frac{d^2P}{dx^2} = x^x (1 + \log x)^2 + \frac{x^x}{x} = 2, \text{ if } x = 1,$$

$$\frac{d^2Q}{dx^2} = -\frac{1}{x^2} = -1, \text{ if } x = 1;$$

$$\therefore u = \frac{2}{-1} = -2.$$

Ex. 4. $u = \dfrac{1 - \sin x + \cos x}{\sin x + \cos x - 1} = 1$, if $x = \dfrac{\pi}{2}$.

Ex. 5. $u = \dfrac{a - x - a \text{ h. l. } a + a \text{ h. l. } x}{a - \sqrt{2ax - x^2}} = -1$, if $x = a$.

Ex. 6. $u = \dfrac{e^x - 1 - \log(1 + x)}{x^2} = \dfrac{0}{0}$, when $x = 0$,

$$e^x - 1 = x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{2 \cdot 3} + \&c.$$

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.;$$

$$\therefore \frac{e^x - 1 - \log(1 + x)}{x^2} = \frac{x^2 - \dfrac{x^3}{6} + \&c.}{x^2}$$

$$= \frac{1 - \dfrac{x}{6} + \&c.}{1} = 1, \text{ if } x = 0.$$

Ex. 7. $$u = \frac{\sqrt{a^2 - x^2} + a - x}{\sqrt{a - x} + \sqrt{a^3 - x^3}} = \frac{0}{0}, \text{ when } x = a.$$

Here $\frac{dP}{dx}$ and $\frac{dQ}{dx}$ are both infinite when $x = a$.

We must have recourse to the second method, and since if x be $> a$, $\sqrt{a^2 - x^2}$ is impossible, let $x = a - h$, and making the substitutions

$$u = \frac{\sqrt{2ah - h^2} + h}{\sqrt{h} + \sqrt{h\,(a^2 + ax + x^2)}} = \frac{\sqrt{2a - h} + \sqrt{h}}{1 + \sqrt{a^2 + ax + x^2}}.$$

Let $x = a$, or $h = 0$. Then $u = \dfrac{\sqrt{2a}}{1 + \sqrt{3a^2}}$.

We might have divided the numerator and denominator at once by $\sqrt{a - x}$, and then

$$u = \frac{\sqrt{a + x} + \sqrt{a - x}}{1 + \sqrt{a^2 + ax + x^2}} = \frac{\sqrt{2a}}{1 + \sqrt{3a^2}}, \text{ when } x = a.$$

Ex 8. If $u = \dfrac{1}{1 - x} - \dfrac{2}{1 - x^2} = \infty - \infty$, when $x = 1$,

$$\frac{1}{1 - x} - \frac{2}{1 - x^2} = \frac{1 + x - 2}{1 - x^2} = -\frac{(1 - x)}{1 - x^2} = -\frac{1}{1 + x} = -\tfrac{1}{2}, \text{ if } x = 1.$$

Ex. 9. If $u = (1 - x)\tan\dfrac{\pi x}{2} = \dfrac{1 - x}{\cot\dfrac{\pi x}{2}} = \dfrac{0}{0}$,

when $x = 1$, $u = \dfrac{2}{\pi}$.

Ex. 10. If $u = \dfrac{\pi}{4x} - \dfrac{\pi}{2x\,(e^{\pi x} + 1)}$, find its value when $x = 0$,

$$u = \frac{\pi}{4x} \cdot \frac{e^{\pi x} - 1}{e^{\pi x} + 1} = \frac{0}{0}, \text{ if } x = 0.$$

Now by expanding $e^{\pi x}$ by the formula $e^z = 1 + z + \frac{z^2}{1\,.\,2} + \&c.$

$$u = \frac{\pi}{4x} \cdot \left(\frac{\pi x + \frac{\pi^2 x^2}{1\,.\,2} + \&c.}{2 + \pi x + \frac{\pi^2 x^2}{1\,.\,2} + \&c.} \right) = \frac{\pi}{4} \cdot \left(\frac{\pi + \frac{\pi^2 x}{1\,.\,2} + \&c.}{2 + \pi x + \frac{\pi^2 x^2}{1\,.\,2} + \&c.} \right).$$

Let $x = 0$:

$$\therefore\ u = \frac{0}{0} = \frac{\pi^2}{8}.$$

Ex. 11. $u = \frac{\log x}{x}$, when $x = \infty$.

Let $\log x = y$; $\therefore\ x = e^y$,

$$u = \frac{y}{e^y} = \frac{y}{1 + y + \frac{y^2}{1\,.\,2} + \frac{y^3}{2\,.\,3} + \&c.}$$

$$= \frac{1}{\frac{1}{y} + 1 + \frac{y}{2} + \frac{y^2}{2\,.\,3} + \&c.}$$

$$= \frac{1}{0 + 1 + \infty} = \frac{1}{\infty} = 0, \text{ when } y = \infty.$$

Similarly, if $u = \frac{\log x}{x^n}$, we have, if $y = \log x$,

$$u = \frac{y}{e^{ny}} = 0, \text{ when } y = \infty,$$

by expanding e^{ny}, and dividing the numerator and denominator by y.

Ex. 12. $u = \frac{1}{2x^2} - \frac{\pi}{2x \tan \pi x}$, when $x = 0$; $u = \frac{\pi^2}{6}$.

Ex. 13. $$u = \frac{e^x - e^{\sin x}}{x - \sin x}, \quad x = 0; \quad u = 1.$$

Ex. 14. $$u = \frac{\tan x - \sin x}{(\sin x)^3}, \quad x = 0; \quad u = \frac{1}{2}.$$

Ex. 15. $$u = \frac{1 - (n+1) \cdot x^n + n \cdot x^{n+1}}{(1 - x)^2}, \quad x = 1; \quad u = \frac{n(n+1)}{2}.$$

Ex. 16. $$u = \frac{a - x - a \log\left(\frac{a}{x}\right)}{a - \sqrt{a^2 - (a - x)^2}}, \quad x = a, \quad u = -1.$$

CHAPTER VII.

MAXIMA AND MINIMA.

77. If $u = f(x)$ express the relation between the function u, and the variable x, then if $x = a$ make $f(a)$ greater than both $f(a+h)$ and $f(a-h)$; $u = f(a)$ is said to be a maximum: but if $f(a)$ be less than both $f(a+h)$ and $f(a-h)$, it is called a minimum.

Hence the value of a function is said to be a *maximum* or *minimum*, according as the particular value is greater or less than the values which immediately precede and follow it.

From this definition it appears, that if a quantity either continually increases or constantly decreases, it does not possess the property of a maximum or minimum. Also, as the words maximum or minimum are used in a relative and not in an absolute sense, functions may possess many maxima or minima.

78. In the circle the sine which $= 0$, when the arc $= 0$, increases as the arc increases, till the arc $= 90^0$, when the sine $=$ radius, from this value it decreases, till at the end of the second quadrant it becomes $= 0$.

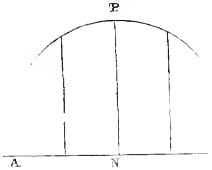
At 90^0, therefore, it is a maximum; for any two sines drawn on opposite sides of the sin 90^0, and equidistant from it, will be both less than the radius.

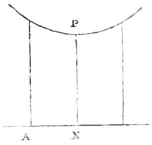
In the parabola, the line drawn from the focus to the vertex, is less than either of two focal distances which can be drawn to the curve on opposite sides of it; it is therefore a minimum.

By reference to figures 1 and 2, we perceive that

NP in fig. 1. is a maximum,

NP.........2. is a minimum.





79. One of the most important applications of the Differential Calculus, is that which affords rules for the discovery of these values.

But the following proposition must first be established.

If $y = A_1 h + A_2 h^2 + A_3 h^3 + \&c. + A_n h^n + A_{n+1} h^{n+1} + \&c.$, where the ratio of any coefficient to the one immediately preceding is finite, i. e. $\frac{A_{n+1}}{A_n}$ is finite, h may be so assumed that any one term shall be greater than the sum of all the terms that follow it.

Let r be greater than the greatest ratio between the coefficients;

$$\therefore \frac{A_2}{A_1} < r, \quad \text{or} \quad A_2 < A_1 r,$$

$$\frac{A_3}{A_2} < r; \quad \therefore A_3 < A_1 r^2,$$

$$\frac{A_4}{A_3} < r; \quad \therefore A_4 < A_1 r^3,$$

&c.

$$\begin{aligned}\therefore A_1 h + A_2 h^2 + A_3 h^3 + \&c. &< A_1 h + A_1 r h^2 + A_1 r^2 h^3 + \&c. \\ &< A_1 h \left\{1 + rh + r^2 h^2 + \&c.\right\} \\ &< A_1 h \frac{1}{1 - rh}.\end{aligned}$$

$$\text{Let } rh = \tfrac{1}{2}, \quad \text{or } h = \frac{1}{2r}; \quad \therefore \frac{1}{1 - rh} = 2;$$

$$\therefore A_1 h + A_2 h^2 + A_3 h^3 + \&c. < 2 A_1 h;$$

$$\therefore A_2 h^2 + A_3 h^3 + \&c. < A_1 h;$$

and in the same manner may $A_2 h^2$ be shewn to be greater than

$$A_3 h^3 + A_4 h^4 + \&c.$$

We have here supposed the series to proceed to infinity: if it extend to n terms, it is evident, *à fortiori* that any one term is greater than the sum of all that follow it.

80. Prop. If $u = f(x)$ be a maximum or minimum when $x = a$. Then on the same supposition, $\dfrac{du}{dx} = 0$.

Let $u_1 = f(x + h)$, and $u_2 = f(x - h)$.

Now at a maximum or minimum, $u = f(x)$ must be greater or less than both $f(x + h)$, and $f(x - h)$, or greater or less than both u_1 and u_2.

Hence, $u_1 - u$ and $u_2 - u$ must both have the same algebraical sign.

$$\text{But } u_1 - u = \frac{du}{dx} h + \frac{d^2u}{dx^2} \frac{h^2}{1 . 2} + \frac{d^3u}{dx^3} \frac{h^3}{2 . 3} + \&c.$$

$$\text{and } \therefore u_2 - u = -\frac{du}{dx} h + \frac{d^2u}{dx^2} \frac{h^2}{1 . 2} - \frac{d^3u}{dx^3} \frac{h^3}{2 . 3} + \&c.$$

by writing $-h$ for h in the value of $u_1 - u$.

Hence, since the first term of the expansion can be made greater than the sum of all the terms that follow it, (if the supposition of $x = a$, does not make any of the differential coefficients infinite,) it is clear that so long as the term $\dfrac{du}{dx} h$ exists, so long will $u_1 - u$ and $u_2 - u$ have a different algebraical sign: i. e. u_1 and u_2 cannot be both greater or both less than u. Therefore, if there be a maximum or minimum, $\frac{du}{dx} = 0$, therefore

$$u_1 - u = \frac{d^2u}{dx^2}\frac{h^2}{1 \,.\, 2} + \frac{d^3u}{dx^3}\frac{h^3}{2 \,.\, 3} + \&c.$$

$$\text{and} \quad u_2 - u = \frac{d^2u}{dx^2}\frac{h^2}{1 \,.\, 2} - \frac{d^3u}{dx^3}\frac{h^3}{2 \,.\, 3} + \&c.$$

Now if $x = a$ does not make $\frac{d^2u}{dx^2} = 0$, the sign of $u_1 - u$ and $u_2 - u$, since h^2 is positive, will depend upon that of $\frac{d^2u}{dx^2}$.

If therefore $\frac{d^2u}{dx^2}$ be positive, $u_1 - u$ and $u_2 - u$ are positive

If $\frac{d^2u}{dx^2}$ be negative, $u_1 - u$ and $u_2 - u$ are negative.

If therefore $\frac{d^2u}{dx^2}$ be positive, u_1 and u_2 are both greater than u, or u is a minimum, and if $\frac{d^2u}{dx^2}$ be negative, then u_1 and u_2 are both less than u; or u is a maximum. Hence this rule: to find whether $u = f(x)$ contains any maxima or minima, put $\frac{du}{dx} = 0$, substitute the values of x thus found in $\frac{d^2u}{dx^2}$, if the results be positive, we have minima; if negative, maxima.

81. Should however $\frac{d^2u}{dx^2} = 0$ when $x = a$,

$$u_1 - u = +\frac{d^3u}{dx^3}\frac{h^3}{2 \,.\, 3} + \&c.$$

$$u_2 - u = -\frac{d^3u}{dx^3}\frac{h^3}{2 \,.\, 3} + \&c.$$

and $u_1 - u$ and $u_2 - u$ have again different signs; and therefore there will be no maxima or minima if $\frac{d^3u}{dx^3}$ exist. Hence it is obvious that we can have a maximum or minimum only when the first differential coefficient that does not vanish is of an even order.

Cor. 1. If u = maximum or minimum, any constant multiple of u is a maximum or minimum.

$$\text{For if } \frac{du}{dx} = 0, \quad a\frac{du}{dx} \text{ is also} = 0;$$

and therefore if u = maximum, au is also a maximum.

Cor. 2. If $f(x)$ be a maximum or minimum, $\overline{f(x)}|^n$, where n is integral, is also a maximum or minimum.

$$\text{For let } u = f(x),$$

$$\text{and } \quad U = \overline{f(x)}|^n;$$

$$\therefore \frac{du}{dx} = f'(x) = 0,$$

if u be a maximum or minimum,

$$\text{and } \frac{dU}{dx} = n\,.\,\overline{f(x)}|^{n-1} f'(x) = 0; \quad \because f'(x) = 0;$$

and therefore U is a maximum or minimum.

Cor. 3. If $u = f(x)$ be a maximum or minimum, $\log u$ is sometimes a maximum or minimum.

$$\text{Let } U = \log u; \quad \therefore \frac{dU}{dx} = \frac{1}{u} \cdot \frac{du}{dx}.$$

$$\text{But } \frac{du}{dx} = 0; \quad \therefore \frac{dU}{dx} = 0,$$

or U is a maximum or minimum, unless $x = a$ makes $u = 0$.

COR. 4. If u = maximum, $\frac{1}{u}$ is a minimum, and conversely.

For let $v = \frac{1}{u}$; $\therefore \frac{dv}{dx} = -\frac{1}{u^2}\frac{du}{dx}$,

$$\frac{d^2v}{dx^2} = +\frac{2}{u^3}\cdot\frac{du^2}{dx^2} - \frac{1}{u^2}\cdot\frac{d^2u}{dx^2} = -\frac{1}{u^2}\cdot\frac{d^2u}{dx^2}, \text{ when } u = \text{maximum}.$$

Therefore, if $\frac{d^2u}{dx^2}$ be negative, $\frac{d^2v}{dx^2}$ is positive, or if u be a maximum, $\frac{1}{u}$ is a minimum.

EXAMPLES.

(1) Let $u = x^3 - 6x^2 + 11x - 6$; find the values of x which make u a maximum or minimum.

$$\frac{du}{dx} = 3x^2 - 12x + 11 = 0;$$

$$\therefore x^2 - 4x + 4 = \tfrac{1}{3}; \quad \therefore x = 2 \pm \frac{1}{\sqrt{3}} = 2 \pm \frac{\sqrt{3}}{3},$$

$$\frac{d^2u}{dx^2} = 6x - 12.$$

Let $x = 2 + \frac{\sqrt{3}}{3}$; $\therefore \frac{d^2u}{dx^2} = 2\sqrt{3}$ indicates a minimum.

$x = 2 - \frac{\sqrt{3}}{3}$; $\therefore \frac{d^2u}{dx^2} = -2\sqrt{3}$.........a maximum.

(2) Let $y = x\tan\theta - \frac{x^2}{4h\cos^2\theta}$; find x that y may be a maximum or minimum.

$$\frac{dy}{dx} = \tan\theta - \frac{x}{2h\cos^2\theta},$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2h\cos^2\theta}.$$

From $\frac{dy}{dx} = 0$, $x = 2h \tan\theta \cos^2\theta = 2h \sin\theta \cos\theta$;

also $\frac{d^2 y}{dx^2}$ is negative; $\therefore$ y is a maximum,

$$\text{and } y = 2h \tan\theta \,.\, \sin\theta \cos\theta - \frac{4h^2 \sin^2\theta \cos^2\theta}{4h \cos^2\theta}$$

$$= 2h \sin^2\theta - h \sin^2\theta = h \sin^2\theta.$$

This equation is that of the path of the projectile, and the maximum value of y is the greatest altitude above the horizontal plane.

(3) $u = \overline{\sin x}\rceil^m . \{\sin(a - x)\}^n$; find x that u may be a maximum or minimum.

$$\frac{du}{dx} = m \overline{\sin x}\rceil^{m-1} \overline{\sin a - x}\rceil^n . \cos x$$

$$- n \overline{\sin x}\rceil^m . \overline{\sin a - x}\rceil^{n-1} \cos(a - x) = 0;$$

$$\therefore m \sin(a - x) . \cos x - n \sin x \cos(a - x) = 0;$$

$$\therefore \frac{\sin(a - x) . \cos x}{\cos(a - x) . \sin x} = \frac{n}{m};$$

$$\therefore \frac{\sin(a - x)\cos x + \cos(a - x)\sin x}{\sin(a - x)\cos x - \sin x \cos \overline{a - x}} = \frac{n + m}{n - m},$$

$$\text{or } \frac{\sin a}{\sin(a - 2x)} = \frac{n + m}{n - m},$$

$$\text{or } \sin(a - 2x) = \frac{n - m}{n + m} . \sin a,$$

whence $a - 2x$ may be found from the tables; and therefore x.

(4) $u = \dfrac{\log x}{x}$; find x that u may be a maximum.

$$\frac{du}{dx} = \frac{1 - \log x}{x^2} = 0;$$

$$\therefore \log x = 1 = \log e;$$

$$\therefore x = e,$$

$$\text{and } u = \frac{1}{e}.$$

(5) Find that fraction which exceeds its second power by the greatest possible number.

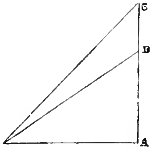
Let x be the fraction;

$$\therefore u = x - x^2 \text{ is a maximum};$$

$$\therefore \frac{du}{dx} = 1 - 2x = 0; \quad \therefore x = \tfrac{1}{2},$$

$$\frac{d^2u}{dx^2} = -2, \text{ or } x = \tfrac{1}{2}, \text{ is a maximum.}$$

(6) Find the distance of P from A, that $\angle CPB$ may be a maximum.



$$AB = a, \qquad AP = x,$$

$$AC = b, \qquad \angle CPB = \theta;$$

$$\therefore \theta = \angle CPA - \angle BPA$$

$$= \tan^{-1}\frac{b}{x} - \tan^{-1}\frac{a}{x},$$

$$\tan\theta = \frac{\dfrac{b}{x} - \dfrac{a}{x}}{1 + \dfrac{ab}{x^2}} = \frac{(b-a)\,.\,x}{x^2 + ab};$$

and because θ is a maximum, $\tan\theta$ is also a maximum, and $\frac{1}{\tan\theta}$ is a minimum;

$$\therefore\ u = \frac{x^2 + ab}{x} = x + \frac{ab}{x},$$

$$\frac{du}{dx} = 1 - \frac{ab}{x^2} = 0; \quad \therefore\ x = \sqrt{ab};$$

and therefore AP is a tangent to a circle circumscribing the triangle PBC.

(7) Of all triangles upon the same base, and having the same perimeter, the isosceles has the greatest area:

$2P$ the perimeter and a the given base,

x and y the remaining sides;

$$\therefore\ \text{area} = \sqrt{P\,.\,(P-a)\,.\,(P-x)\,.\,(P-y)};$$

and since P and $P-a$ are invariable, and if $\sqrt{u}$ be a maximum, u is also a maximum.

$$\text{Let } u = (P-x)\,.\,(P-y).$$

$$\text{But } P - y = P - (2P - a - x) = a + x - P;$$

$$\therefore\ u = (P-x)\,.\,(a+x-P),$$

$$\frac{du}{dx} = -(a + x - P) + P - x - 0;$$

$$\therefore\ -a - 2x + 2P = 0;$$

$$\therefore\ x = P - \frac{a}{2},$$

$$y = 2P - a - x = 2P - a - P + \frac{a}{2}$$

$$= P - \frac{a}{2};$$

and therefore $x = y$, or the triangle is isosceles.

Since $\dfrac{d^2u}{dx^2} = -2$, the triangle is a maximum,

$$\text{and area} = \frac{a}{2}\sqrt{P.(P-a)}.$$

(8) Divide a number a into two such parts, that the product of the m^{th} power of the one into the n^{th} power of the other may be a maximum.

x one part; therefore $a - x$ is the other;

$$u = x^m . \overline{a-x}|^n,$$

$$\frac{du}{dx} = m x^{m-1} . \overline{a-x}|^n - x^m . n \overline{a-x}|^{n-1}$$

$$= x^{m-1} . \overline{a-x}|^{n-1} \{ma - (m+n) . x\} = 0,$$

whence $x = 0$, $x = a$, and $x = \dfrac{ma}{m+n}$;

$$\frac{d^2u}{dx^2} = \{(m-1) . x^{m-2} . \overline{a-x}|^{n-1} - (n-1) . x^{m-1} . \overline{a-x}|^{n-2}\}$$

$$\{ma - (m+n) . x\} - (m+n) . x^{m-1} . \overline{a-x}|^{n-1},$$

which vanishes when $x = 0$ and $x = a$, but if $x = \dfrac{ma}{m+n}$,

$$\frac{d^2u}{dx^2} = -(m+n) . \left(\frac{ma}{m+n}\right)^{m-1} . \left(\frac{na}{m+n}\right)^{n-1};$$

$$\therefore\ x = \frac{ma}{m+n} \text{ gives } u = \text{maximum}.$$

$x = 0$ and $x = a$ will give no results unless m and n are even.

And then $\dfrac{d^m u}{dx^m} = m \, . \, (m-1)(m-2) \ldots 2 \, . \, 1 \, . \, \overline{a - x}|^n + \phi(x)$,

$$\frac{d^n u}{dx^n} = n \, . \, (n-1)(n-2) \ldots 2 \, . \, 1 \, . \, x^m + \phi(a-x),$$

and $\dfrac{d^m u}{dx^m} = m \, . \, (m-1)(m-2) \ldots 2 \, . \, 1 \, . \, a^n$, when $x = 0$,

and $\dfrac{d^n u}{dx^n} = n \, . \, (n-1)(n-2) \ldots 2 \, . \, 1 \, . \, a^m$, when $x = a$;

both of which correspond to minima.

(9) $u^3 - 3aux + x^3 = 0$; find x that u may be a maximum.

Instead of solving the equation with respect to u, differentiate the implicit function; and we have

$$\frac{du}{dx} \, . \, (u^2 - ax) - au + x^2 = 0.$$

But $\dfrac{du}{dx} = 0$; $\therefore \; x^2 - au = 0$, or $u = \dfrac{x^2}{a}$;

whence, by substitution in the original equation,

$$\frac{x^6}{a^3} - 3x^3 + x^3 = 0 \ldots (1); \quad \therefore \; x^3 = 2a^3; \quad \therefore \; x = a \, . \sqrt[3]{2}.$$

Differentiating a second time,

$$\frac{d^2 u}{dx^2}(u^2 - ax) + \frac{du}{dx} \, . \, \left(2u \frac{du}{dx} - a\right) - a \frac{du}{dx} + 2x = 0.$$

But $\dfrac{du}{dx} = 0$, and $u^2 - ax = \dfrac{x^4}{a^2} - ax = \dfrac{x}{a^2} \, . \, (x^3 - a^3) = + ax$;

$$\therefore \; \frac{d^2 u}{dx^2} = \frac{-2x}{ax} = -\frac{2}{a}.$$

From equation (1) we also have $x = 0$; and therefore $u = 0$.

$$\text{Now } \frac{d^2u}{dx^2} = \frac{-2x}{u^2 - ax} = \frac{0}{0}, \text{ if } x = 0.$$

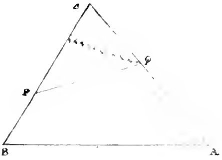
Treating the fraction as a vanishing one,

$$\frac{d^2u}{dx^2} = \frac{-2}{2u\frac{du}{dx} - a} = \frac{2}{a}, \quad \text{when } x = 0;$$

$$\therefore\ x = 0 \text{ gives } u = 0, \text{ a minimum.}$$

$$\text{Also } x = a\sqrt[3]{2} \text{ gives } u = a\sqrt[3]{4}, \text{ a maximum.}$$

(10) Bisect a triangle by the shortest line.



ABC the triangle, and PQ the shortest line.

$\left.\begin{array}{l} CP = x \\ CQ = y \\ PQ = u \end{array}\right\}$ a, b, c the three sides of the triangle, C the $\angle BCA$.

Then $\because \Delta ABC = 2\, \Delta CPQ$;

$$\therefore\ \frac{ab \sin C}{2} = 2 \cdot \frac{xy \sin C}{2} = xy \sin C;$$

$$\therefore\ ab = 2xy,$$ 4 $\frac{ab}{2x}$

$$u^2 = x^2 + y^2 - 2xy \cos C = x^2 + \frac{a^2b^2}{4x^2} - ab \cos C = \text{minimum};$$

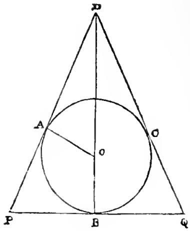
$$\therefore\ 2u\frac{du}{dx} = 0 = 2x - \frac{a^2b^2}{2x^3};$$

$$\therefore\ x^4 = \frac{a^2b^2}{4}, \text{ or } x = \sqrt{\frac{ab}{2}}, \text{ and } y = \frac{ab}{2x} = \sqrt{\frac{ab}{2}};$$

$$\therefore\ u^2 = \frac{ab}{2} + \frac{ab}{2} - ab \cos C = ab\,.\,(1 - \cos C) = ab\,.\left(\frac{2ab + c^2 - (a^2 + b^2)}{2ab}\right)$$

$$= \frac{c^2 - (a-b)^2}{2};$$

$$\therefore\ u = \sqrt{\frac{(c - a + b)(c + a - b)}{2}}.$$



(11) Describe about a given circle ABC, the least isosceles triangle.

DPQ the triangle, DP touching the circle at A.

$DO = x$; $\therefore DA = \sqrt{x^2 - a^2}$;

$OA = a$.

Now $DB : PB :: DA : OA$;

$$\therefore PB = \frac{DB}{DA} OA = \frac{(x+a) \times a}{\sqrt{x^2 - a^2}};$$

$$\therefore \Delta DPQ = PB \times DB$$

$$= \frac{a(x+a)}{\sqrt{x^2 - a^2}} \times (x+a) = a \cdot \frac{\overline{x+a}|^{\frac{3}{2}}}{\sqrt{x-a}} = \text{minimum}.$$

Whence, if $u = \dfrac{(x+a)^3}{x-a}$, $x = 2a$, and $u = a^2 . 3\sqrt{3}$.

(12) Find the greatest area that can be included by four given straight lines.

Let a, b, c, d = the four lines,

θ the $\angle$ included by a, b,

ϕ ... $\angle$ .... c, d,

D the diagonal subtending the two angles, and dividing the quadrilateral into two Δs;

$$\therefore u = \text{area} = \frac{ab \cdot \sin\theta}{2} + \frac{cd \sin\phi}{2} = \text{maximum};$$

$$\therefore \frac{du}{d\theta} = \tfrac{1}{2} \cdot \left(ab \cdot \cos\theta + cd \cos\phi \cdot \frac{d\phi}{d\theta}\right) = 0.$$

But $c^2 + d^2 - 2cd \cos\phi = D^2 = a^2 + b^2 - 2ab \cdot \cos\theta$;

$$\therefore cd \cdot \frac{d\phi}{d\theta} = ab \cdot \frac{\sin\theta}{\sin\phi};$$

$\therefore$ substituting this value, and dividing by $\frac{ab}{2}$.

$$\cos\theta + \cos\phi \cdot \frac{\sin\theta}{\sin\phi} = 0;$$

$$\therefore \sin\phi \cdot \cos\theta + \sin\theta\cos\phi = \sin(\phi+\theta) = 0 = \sin\pi;$$

$$\therefore \phi + \theta = \pi,$$

or the quadrilateral is one which may be inscribed in a circle,

$$\text{and } u = \frac{ab+cd}{2}\sin\theta = \sqrt{(P-a)(P-b)(P-c)(P-d)},$$

$$\text{where } P = \frac{a+b+c+d}{2}.$$

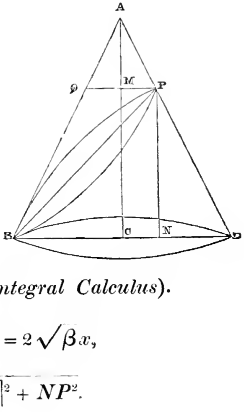
(13) Cut the greatest ellipse from a given cone. ABD the cone. PB the elliptic section,

$$AC = a, \quad CN = x,$$

$$BC = \beta, \quad NP = y,$$

PB the axis-major $= 2a$,

and axis-minor $= 2b$.



Now area of ellipse $= \pi ab$, (*Integral Calculus*).

$$\text{And } 2b = \sqrt{PQ \times BD} = \sqrt{2x \times 2\beta} = 2\sqrt{\beta x},$$

$$2a = \sqrt{BN^2 + NP^2} = \sqrt{\overline{\beta + x}|^2 + NP^2}.$$

$$\text{But } NP = CA \times \frac{DN}{CD} = a \cdot \frac{\beta - x}{\beta};$$

$$\therefore 2a = \sqrt{(\beta+x)^2 + \frac{a^2}{\beta^2}(\beta - x)^2};$$

$$\therefore \text{area} = u = \pi\frac{\sqrt{\beta x}}{2} \cdot \sqrt{\overline{\beta + x}|^2 + \frac{a^2}{\beta^2}(\beta - x)^2} = \text{maximum},$$

$$\therefore \frac{du}{dx} = \frac{\pi\sqrt{\beta}}{2} \cdot \left\{ \sqrt{\overline{\beta + x}|^2 + \frac{a^2}{\beta^2}\overline{\beta - x}|^2} \cdot \frac{1}{2\sqrt{x}} \right.$$

$$\left. + \sqrt{x} \cdot \frac{(\beta + x) - \frac{a^2}{\beta^2}(\beta - x)}{\sqrt{(\beta + x)^2 + \frac{a^2}{\beta^2} \cdot (\beta - x)^2}} \right\} = 0;$$

$$\therefore (\beta + x)^2 + \frac{a^2}{\beta^2}\overline{\beta - x}|^2 + 2x \cdot \left\{(\beta + x) - \frac{a^2}{\beta^2}(\beta - x)\right\} = 0,$$

$$\text{whence } \frac{3(a^2 + \beta^2)x^2}{\beta^2} - \frac{4(a^2 - \beta^2)x}{\beta^2} = -(a^2 + \beta^2);$$

$$\therefore x^2 - \frac{4(a^2 - \beta^2)}{3(a^2 + \beta^2)}\beta x = -\frac{\beta^2}{3};$$

$$\therefore x = \frac{2}{3} \cdot \beta \cdot \frac{(a^2 - \beta^2)}{a^2 + \beta^2} \pm \sqrt{\frac{4}{9}\beta^2 \cdot \left(\frac{a^2 - \beta^2}{a^2 + \beta^2}\right)^2 - \frac{\beta^2}{3}}$$

$$= \frac{2\beta(a^2 - \beta^2) \pm \beta\sqrt{a^4 - 14\beta^2 a^2 + \beta^4}}{3(a^2 + \beta^2)},$$

and the problem is possible so long as $a^4 - 14\beta^2 a^2 + \beta^4$ is positive. The limit of possibility is when the radical disappears.

$$\text{Then } a^4 - 14\beta^2 a^2 + 49\beta^4 = 48\beta^4;$$

$$\therefore a^2 = 7\beta^2 \pm \sqrt{48\beta^4} = \beta^2\{7 \pm 4\sqrt{3}\};$$

$$\therefore a = \beta(2 \pm \sqrt{3}),$$

$$\text{and } x = \frac{2\beta}{3} \cdot \frac{6 \pm 4\sqrt{3}}{8 \pm 4\sqrt{3}} = \frac{\beta}{3} \cdot \frac{3 + 2\sqrt{3}}{2 + \sqrt{3}} = \frac{\beta}{\sqrt{3}}.$$

(14) The content of a cone being given, find its form when its surface is a maximum.

x the altitude, and y the radius of the base,

$$\frac{\pi a^3}{3} \text{ the given content } = \frac{\pi y^2 x}{3},$$

$$u = \text{surface} = \text{convex surface} + \text{base}.$$

And convex surface = sector of circle of which the radius is the slant side and the arc the circumference of the base of cone;

$$\therefore\ u = \pi y \sqrt{x^2 + y^2} + \pi y^2.$$

$$\text{But } y^2 = \frac{a^3}{x}; \quad \therefore\ y^2 + x^2 = \frac{a^3 + x^3}{x};$$

$$\therefore\ u = \pi a^{\frac{3}{2}} \left\{ \frac{\sqrt{a^3 + x^3} + a^{\frac{3}{2}}}{x} \right\};$$

$$\text{whence because } \frac{du}{dx} = 0; \quad x^3 - 2a^3 = 2a^{\frac{3}{2}} \sqrt{x^3 + a^3};$$

$$\therefore\ (x^3 + a^3) - 2a^{\frac{3}{2}} \sqrt{x^3 + a^3} + a^3 = 4a^3;$$

$$\therefore\ \sqrt{x^3 + a^3} = 3a^{\frac{3}{2}};$$

$$\therefore\ x^3 = 8a^3, \quad \text{and } x = 2a,$$

$$y^2 = \frac{a^3}{x} = \frac{a^2}{2}, \quad \text{or } y = \frac{a}{\sqrt{2}},$$

$$\text{and } u = \frac{\pi a}{\sqrt{2}} \cdot \sqrt{4a^2 + \frac{a^2}{2}} + \frac{\pi a^2}{2} = 2\pi a^2.$$

$$\text{Since } y^2 = \frac{a^2}{2}, \quad \text{and } x^2 = 4a^2;$$

$$\therefore\ \frac{y^2}{x^2} = \frac{1}{8},$$

$$\text{and } y = \frac{x}{2\sqrt{2}},$$

the equation to the generating line.

82. The following examples are added by way of exercise, the principal steps to the solution of the questions being indicated.

(1) Divide a line into two such parts, that their product multiplied by the difference of their squares shall be a maximum.

Let $2a$ be the line, $a+x$ and $a-x$ the parts;

$$\therefore \; u = (a^2 - x^2) \cdot 4ax = \text{maximum}, \quad \text{whence } x = \frac{a}{\sqrt{3}}.$$

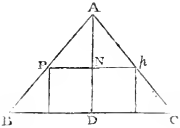
(2) Let $u = (mx+n) \cdot (ny+m)$ be a maximum, and let $a^{mx} \cdot b^{ny} = c$; find x.

$$\text{Here } \frac{du}{dx} = m(ny+m) + n(mx+n)\frac{dy}{dx} = 0;$$

and because $mx \log a + ny \log b = \log c$, $\dfrac{dy}{dx}$ may be found,

$$\text{and } x = \frac{\log\left(\dfrac{cb^m}{a^n}\right)}{\log(a^{2m})}.$$

(3) Inscribe the greatest rectangle in a given triangle.



$AD = a$, $BC = b$, $AN = x$; $\therefore \; Pp = \dfrac{bx}{a}$;

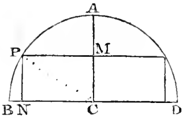
$$\therefore \; u = Pp \cdot ND = \frac{bx}{a}(a-x), \text{ whence } x = \frac{a}{2}, \text{ and } u = \frac{1}{2} ABC.$$

(4) Inscribe the greatest isosceles triangle in a given circle.

Let a = radius, the triangle is equilateral, side $= a\sqrt{3}$,

$$\text{area} = \frac{a^2 3\sqrt{3}}{4}.$$

(5) Inscribe the greatest rectangle in a semicircle.



$$CN = x, \quad CA = a, \quad NP = \sqrt{a^2 - x^2}.$$

$$\therefore\ u = 2PM\,.\,CM = 2x\sqrt{a^2 - x^2}\,; \quad \therefore\ x = \frac{a}{\sqrt{2}}, \quad \text{and } u = a^2\sqrt{2}.$$?

COR. The same construction applies to any curve.

Let $AC = b$, $AM = x$; $\therefore\ PM = f(x)$, and $u = 2(b - x)\,.f(x)$.

Ex. 1. BAD a parabola; then

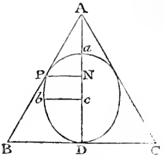
$$y = 2\sqrt{mx}, \quad \text{and } u = 4(b - x)\sqrt{mx}.$$

Ex. 2. BAD a segment of a circle;

$$AM = x, \quad \text{radius} = a\,;$$

$$\therefore\ PM = \sqrt{2ax - x^2}, \quad \text{and } u = 2(b - x)\,.\sqrt{2ax - x^2}.$$

(6) Inscribe the greatest ellipse in a given isosceles triangle.



If $Da = 2x$, $cb = y$; $\therefore\ u = \pi\,.\,yx$.

Let $AD = a$, $DB = b$.

$$\text{Now } cN = \frac{ca^2}{cA} = \frac{x^2}{a - x}; \quad \therefore\ aN = \frac{ax - 2x^2}{a - x}, \quad DN = \frac{ax}{a - x}.$$

$$\text{But } \frac{BD^2}{AD^2}\,.\,AN^2 = PN^2 = \frac{y^2}{x^2}(Na\,.\,ND)\,;$$

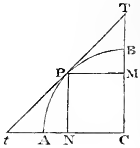
$$\therefore\ b^2\left(\frac{a - 2x}{a - x}\right)^2 = y^2\,.\,\frac{(a - 2x)a}{(a - x)^2},$$

$$\text{whence } y^2 = \frac{b^2}{a}(a - 2x)\,;$$

$$\therefore\ \pi yx = \frac{\pi b}{\sqrt{a}}\,.\,x\sqrt{a - 2x}\,; \quad \therefore\ x = \frac{a}{3}.$$

(7) Inscribe the greatest parabola in a given isosceles triangle.

(8) Cut the greatest parabola from a given cone.



(9) Required the least triangle TCt, which can be described about a given quadrant.

$$u = \tfrac{1}{2} CT \times Ct = ma,$$

$$CA = a, \quad CM = x, \quad CN = y;$$

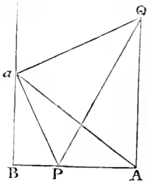
$$\therefore\ CT = \frac{a^2}{x}, \quad Ct = \frac{a^2}{y};$$

and if $u =$ maximum, $x = y$ and $\angle ACP = 45^0$.

(10) The same when APB is a parabolic arc and C the focus.

$$AN = x, \quad AC = a, \quad u = \frac{(x+a)^2\sqrt{a}}{2\sqrt{x}}, \quad \text{whence } x = \frac{a}{3}.$$

(11) Within a given parabola inscribe the greatest parabola, the vertex of the latter being at the bisection of the base of the former.



(12) The corner of a leaf is turned back, so as just to reach the other edge of the page: find when the length of the crease is a minimum.

$$AP = x, \quad AB = a; \quad \therefore\ Aa = \sqrt{2ax},$$

$$\text{also } Aa \, . \, PQ = 2AQ \, . \, AP.$$

Since $AQaP$ may be inscribed in a circle;

$$\therefore\ u^2 = PQ^2 = \frac{2x^3}{2x-a}; \quad \therefore\ x = \frac{3a}{4}.$$

83. If $x = a$, make $\dfrac{du}{dx} = \infty$, the preceding rules are inapplicable, since they are founded upon the supposition that $f(a+h)$ is expanded according to the ascending integral powers of h by means of Taylor's Theorem; but when the differential coefficients become infinite, the developement cannot be effected by it.

Let then $f(a+h)$ be expanded by the ordinary methods, and assume

$$f(a+h) = f(a) + Ph^\alpha + Qh^\beta + Rh^\gamma + \&c.$$

where α is the least of all the indices of h;

$$\therefore\ f(a+h) - f(a) = Ph^\alpha + Qh^\beta + Rh^\gamma + \&c. \ldots (1),$$

$$\text{and } f(a-h) - f(a) = P(-h)^\alpha + Q(-h)^\beta + \&c.$$

by writing $-h$ for h in series (1).

Now if h be made very small, the algebraical sign of the developements will depend on that of their first term. If therefore we have a maximum or minimum, since

$f(a+h) - f(a)$, and $f(a-h) - f(a)$ must have the same signs, Ph^α and $P(-h)^\alpha$; and $\therefore$ h^α and $(-h)^\alpha$ must have the same sign, or α must either be an even number or a fraction with an even number for its numerator.

(1) If α be an even number, it shews that at a maximum or minimum the first existent term of the developement must involve an even power of h, a conclusion we have already come to in the preceding pages.

(2) If α be a fraction, it must be of the form $\dfrac{2n}{2m+1}$.

Ex. Let $u = b + c(x-a)^{\frac{2}{3}}$.

Here $\dfrac{du}{dx} = \dfrac{2c}{3}\dfrac{1}{\overline{x-a}|^{\frac{1}{3}}}$, which is infinite, if $x = a$.

But $x = a$, gives $u = b$,

$x = a + h$ gives $u = b + c h^{\frac{2}{3}}$,

$x = a - h$ ...... $u = b + c h^{\frac{2}{3}}$,

and $f(a + h)$ and $f(a - h)$ are both $> f(a)$, if c be positive,

$< f(a)$, if c be negative.

If $\therefore$ c be positive $x = a$, makes $u = b$ a minimum,

c be negative $x = a$, ... $u = b$ a maximum.

For other examples, see Collection of Examples on the Differential and Integral Calculus.

CHAPTER VIII.

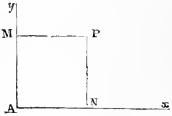
EQUATIONS TO CURVES.

84. We proceed to treat briefly of the equations to a straight line, to the circle, the conic sections, and some other curves, which will be frequently referred to in the succeeding pages.

For complete investigations of the properties of the conic sections and curves in general, we must refer to works expressly written on these subjects.

The object of this Chapter is to furnish the student with such a knowledge of the nature of certain curves, as may make the applications of the Differential Calculus to them obvious and interesting.

We must first premise some elementary remarks before we explain the nature of these equations.



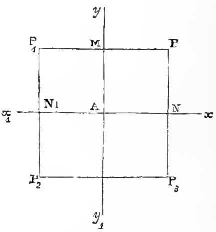
85. From a point A, assumed at pleasure, draw two lines Ay, Ax, perpendicular to each other; then the position of a point P, situated within the angle yAx, will be known if the perpendicular distances PN and PM from P upon Ax, and Ay be known.

Ay and Ax are called axes, and PM and PN the *ordinates* of the point P: but since $AN = PM$, the line PM is seldom drawn, but the position of P determined by taking AN equal to it, and then from N drawing NP perpendicular to Ax.

NP is then called the ordinate, and AN the abscissa; and AN and NP the co-ordinates of P.

Ax is termed the axis of abscissas, and Ay the axis of ordinates.

86. Now a curve is a series of points, and if their respective distances from two such axes as Ay and Ax be known, the curve itself may be drawn. Also every possible equation $y = f(x)$ may be represented by a series of points; for if it be assumed that the values of x may be taken along the line Ax, and those of y be drawn perpendicular to the axis, we shall have, when $x = AN$, $y = f(AN)$, which may be represented by some line as NP. Hence Ax is called the axis of x, and Ay the axis of y and the point A the origin of the co-ordinates; since the values of x and y begin at A, and are measured from it.



87. If $AN = a$, and $NP = b$,

then $x = a$, and $y = b$,

are the equations to a point P.

If however we take $AN_1 = AN$, and $MP_1 = MP$, and complete the rectangle PP_2,

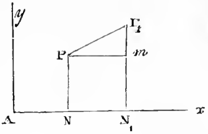
Then unless some assumption be made with respect to the algebraical sign of a and b, we shall be unable to tell in which of the angles yAx, yAx_1, x_1Ay_1, or y_1Ax the point is, since P, P_1, P_2, P_3, in the annexed figure, are each at the same distances from the lines yAy_1 and xAx_1.

If however the values of x, when measured from A along Ax, or to the right hand of A, be reckoned positive, and those in the direction of Ax_1, or to the left of A, be negative; and if the ordinates which are above xAx_1 be called positive, and negative when measured below the same line, no difficulty can arise, and then $\left.\begin{array}{l} x = +a \\ y = +b \end{array}\right\}$ determines the point P,

$\left.\begin{array}{l} x = -a \\ y = +b \end{array}\right\}$ .... P_1,

$\left.\begin{array}{l} x = -a \\ y = -b \end{array}\right\}$ .... P_2,

$\left.\begin{array}{l} x = +a \\ y = -b \end{array}\right\}$ .... P_3,



COR. 1. The distance of a point P from the origin, or

$$AP = \sqrt{AN^2 + PN^2} = \sqrt{x^2 + y^2}.$$

COR. 2. The distance between two points P and P_1 is thus found.

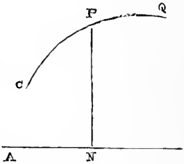
A the origin. $AN = x$, $AN_1 = x_1$,

$NP = y$, $NP_1 = y_1$;

$$\therefore\ PP_1 = \text{distance} = \sqrt{Pm^2 + P_1m^2}$$

$$= \sqrt{(AN_1 - AN)^2 + (P_1N_1 - PN)^2}$$

$$= \sqrt{(x_1 - x)^2 + (y_1 - y)^2}.$$



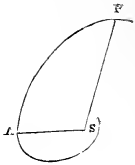
88. If in the curve CPQ the relation between PN and AN be known, and PN be a function of AN, the equation to the curve is said to be known; and from this equation, the curve itself may be drawn. Sometimes the equation $y = f(x)$ is to be found from some given property of the curve; as in the circle, of which the property is that all lines drawn from the centre to the circumference are equal. To questions of this description our attention in this Chapter will be solely directed.

Since AN and NP are drawn at right angles to each other, the equation $y = f(x)$ is called an equation to rectangular co-ordinates.

89. Many curves however cannot be expressed by an equation between rectangular co-ordinates.

Such are the Spirals, which may be conceived to be described by the extremity of a straight line of variable length, which revolves round a fixed point, called the pole of the spiral.

The revolving line (called the radius vector) may be considered as a function of the angle described.

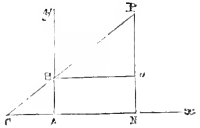


Thus if S be the pole, SP the radius vector, SA the original position of SP, and $\angle ASP = \theta$,

$r = f(\theta)$ is the equation to the spiral.

r and θ are called polar co-ordinates.

We now proceed to investigate the equation to the straight line.



90. The equation to the straight line.

Ax, Ay the two axes of x and y.

$$\left.\begin{array}{r} AN = x \\ NP = y \\ \angle PCA = \theta \end{array}\right\}, \qquad \begin{array}{l} Bn \perp \text{ to } PN, \\ AB = b. \end{array}$$

$$\text{Then } \frac{Pn}{Bn} = \frac{BA}{CA} \tan\theta,$$

$$\text{or } \frac{y-b}{x} = \tan\theta = a, \text{ by writing } a \text{ for } \tan\theta;$$

$$\therefore\ y = ax + b.$$

Cor. 1. If the line be drawn through a given point, let α and β be the co-ordinates of the point.

Then, when $x = \alpha$, $y = \beta$;

$$\therefore\ \beta = a\alpha + b.$$

$$\text{and } y = ax + b;$$

$$\therefore\ y - \beta = a(x - \alpha).$$

Cor. 2. If the line be drawn through the origin,

$$AB = 0; \quad \text{and } \therefore\ b = 0;$$

and $y = ax$ is the equation to a line drawn through A.

91. If two lines intersect, find the point of intersection.

Let $y = ax + b$, and $y = a_1 x + b_1$ be the equations of the two lines.

Then, at the point of intersection, the values of the co-ordinates are the same for both lines;

$$\therefore\ ax + b = a_1 x + b_1,$$

$$\text{and } x = \frac{b_1 - b}{a - a_1},$$

$$\text{and } y = \frac{ab_1 - ab}{a - a_1} + b = \frac{ab_1 - a_1 b}{a - a_1}.$$

92. Find the equation to the line which passes through two given points.

Let $y = ax + b$ be the equation to the line where a and b are to be determined.

α and β, α_1 and β_1 the co-ordinates of the two points;

$$\therefore\ \beta = a\alpha + b \ldots\ldots\ldots\ldots(1),$$

$$\text{and } \beta_1 = a\alpha_1 + b \ldots\ldots\ldots\ldots(2);$$

$$\therefore\ \beta - \beta_1 = a\,.\,(\alpha - \alpha_1);$$

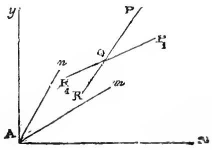
$$\therefore\ a = \frac{\beta - \beta_1}{\alpha - \alpha_1}.$$

But $\because\ y = ax + b$,

and $\beta = a\alpha + b$;

$$\therefore\ y - \beta = a\,.\,(x - \alpha) = \frac{\beta - \beta_1}{\alpha - \alpha_1}\,.\,(x - \alpha).$$

93. To find the angle which two straight lines make with each other at the point of intersection.



$y = ax + b$, and $y = a_1 x + b_1$,

the equations to the two lines.

PQR and P_1QR_1 the lines.

From A draw An parallel to PR,

and Am parallel to $P'R'$;

$$\therefore \angle nAm = PQP';$$

$$\therefore PQP' = nAx - mAx = \tan^{-1}a - \tan^{-1}a_1,$$

$$\text{and } \tan PQP' = \frac{a - a_1}{1 + aa_1}.$$

Cor. 1. If the lines be parallel, $PQP' = 0$, and $a - a_1 = 0$; $\therefore a_1 = a$,

and $\left.\begin{array}{l} y = ax + b \\ y = ax + b_1 \end{array}\right\}$, are the equations to two parallel lines.

Cor. 2. If the lines be perpendicular,

$$\tan PQP' = \frac{1}{0} = \frac{a - a_1}{1 + aa_1};$$

$$\therefore 1 + aa_1 = 0, \quad \text{and } a_1 = -\frac{1}{a},$$

therefore, if $y = ax + b$ be the equation to a line,

$$y = -\frac{1}{a}x + b_1$$

is the equation to a line perpendicular to it.

94. Find the equation to a line drawn through a given point perpendicular to a given line.

$y = ax + b$, the equation to the given line, α and β the co-ordinates of the given point;

$\therefore y = -\frac{1}{x}x + b_1$ is the equation to the perpendicular,

also $\beta = -\frac{1}{a}\alpha + b_1$, since it passes through (α, β);

$\therefore (y - \beta) = -\frac{1}{a}(x - \alpha)$ is the equation required.

95. Find the perpendicular distance of a given point from a given line.

$y = ax + b$ the equation to the given line, and (β, α) the given point;

$$\therefore \ (y - \beta) = -\frac{1}{a}(x - \alpha)$$

is the equation to a perpendicular from a given point upon the given line.

Then if δ be the distance required, and y_1 and x_1 the co-ordinates of the point of intersection of the given line with the perpendicular,

$$\delta = \sqrt{(x_1 - \alpha)^2 + (y_1 - \beta)^2} = (x_1 - \alpha) \, . \, \frac{\sqrt{a^2 + 1}}{a} \, .$$

$$\text{But } ax_1 + b = \beta - \frac{x_1 - \alpha}{a};$$

$$\therefore \ x_1 (a^2 + 1) = a\beta + \alpha - ab;$$

$$\therefore \ x_1 = \frac{a\beta + \alpha - ab}{a^2 + 1},$$

$$x_1 - \alpha = \frac{a\beta - ab - a^2\alpha}{a^2 + 1} = \frac{a}{a^2 + 1}(\beta - b - a\alpha);$$

$$\therefore \ \delta = \frac{\beta - b - a\alpha}{\sqrt{a^2 + 1}} = \frac{\beta - \beta^1}{\sqrt{a^2 + 1}},$$

where β_1 is the value of y when $x = \alpha$.

96. Find the equation to a straight line which passes through a given point, and makes with a given line a given angle.

Let $y = ax + b$ be the equation to the given line,

$y = a_1 x + b_1$ the required equation,

β and α the co-ordinates of the given point;

$\therefore \ (y - \beta) = a_1 (x - \alpha)$ is the equation to the line.

Let $\tan^{-1}m$ be the given angle;

$$\therefore \tan^{-1}m = \tan^{-1}a - \tan^{-1}a_1;$$

$$\therefore \tan^{-1}a_1 = \tan^{-1}a - \tan^{-1}m;$$

$$\therefore a_1 = \frac{a-m}{1+ma};$$

$$\therefore (y-\beta) = \frac{a-m}{1+ma} \cdot (x-a)$$

is the equation required.

97. Find the equation to a straight line, which cuts the axis of y at a distance B from the origin, and the axis of x at a distance A from the origin, in terms of B and A.

$y = ax + b$ the equation to the line,

when $x = 0$, $y = B = \therefore b$,

and $y = 0$, $x = A$; $\therefore aA + B = 0$; $\therefore a = -\frac{B}{A}$;

$$\therefore y = -\frac{B}{A}x + B;$$

$$\therefore \frac{y}{B} + \frac{x}{A} = 1 \text{ is the equation.}$$

THE CIRCLE.

98. The circle is a curve of which the property is, that every point in its circumference is equi-distant from the centre.

Let a and β be the co-ordinates of the centre,

x and y of a point in the circumference,

a the radius.

Then distance between two points a, β, and x, y

$$= \sqrt{(x-a)^2 + (y-\beta)^2} = a, \text{ the radius};$$

$$\therefore y^2 + x^2 - 2\beta y - 2ax + a^2 + \beta^2 - a^2 = 0,$$

is the equation to the circle.

Cor. 1. If the origin be in the circumference, and the axis of x pass through the centre,

$$\beta = 0, \text{ and } \alpha = a;$$

$$\therefore\ y^2 + x^2 - 2ax = 0,$$

$$\text{or } y^2 = 2ax - x^2.$$

Cor. 2. If the origin be in the centre,

$$\alpha = 0, \text{ and } \beta = 0;$$

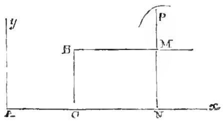
$$\therefore\ y^2 + x^2 - a^2 = 0,$$

$$\text{and } y^2 = a^2 - x^2.$$

TRANSFORMATION OF CO-ORDINATES.

99. In some problems it is necessary to change the position of the axes, the place of the origin, or the inclination of the axes; these cases will be separately treated.

(1) Let the origin be changed, but the axes remain parallel.



A the origin at first.

B the new origin.

P a point in the curve.

$$\left.\begin{array}{l} AC = \alpha \\ BC = \beta \end{array}\right\} \text{ the co-ordinates of } B.$$

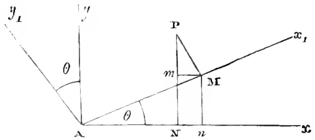
$$AN = x, \quad BM = x_1,$$

$$NP = y, \quad MP = y_1.$$

$$\text{Then } x = x_1 + \alpha, \quad \text{and } y = y_1 + \beta.$$

Substitute these values for x and y, and the equation is transformed, and the co-ordinates are measured from B.

(2) Let the axes be changed but still rectangular.



Ax, Ay, the old axes,

Ax_1, Ay_1, the new ones.

$AN = x$, $AM = x_1$, $\angle xAx_1 = \theta$,

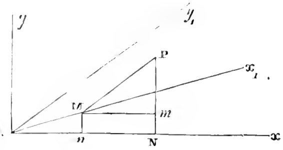
$NP = y$, $MP = y_1$.

Draw Mm perpendicular to PN, and Mn perpendicular to Ax;

$$\therefore\ x = An - Nn = x_1 \cos\theta - y_1 \sin\theta,$$

$$y = Nm + Pm = x_1 \sin\theta + y_1 \cos\theta. \quad \text{For } \angle mPM = \theta.$$

(3) New axes not rectangular, but the origin the same.



Ay_1, Ax_1, the new axes,

$\angle y_1 A x_1 = A$,

$\angle x_1 A x = \theta$,

PM parallel to Ay_1,

$AM = x_1$, $AN = x$,

$MP = y_1$, $NP = y$,

$$x = An + nN = x_1 \cos\theta + y_1 \cos(A + \theta),$$

$$y = Nm + Pm = x_1 \sin\theta + y_1 \sin(A + \theta).$$

COR. 1. If we wish to transform from oblique to rectangular. Since

$$x = x_1 \cos\theta + y_1 \cos(A+\theta),$$

$$y = x_1 \sin\theta + y_1 \sin(A+\theta);$$

$$\therefore\ x\sin\theta = x_1\cos\theta\sin\theta + y_1\cos(A+\theta)\sin\theta,$$

$$y\cos\theta = x_1\cos\theta\sin\theta + y_1\sin(A+\theta)\cos\theta;$$

$$\therefore\ y\cos\theta - x\sin\theta = y_1\{\sin(A+\theta)\,.\cos\theta - \cos(A+\theta)\sin\theta\}$$

$$= y_1\sin A;$$

$$\therefore\ y_1 = \frac{y\cos\theta - x\sin\theta}{\sin A}.$$

Again,

$$x\sin(A+\theta) = x_1\sin(A+\theta)\cos\theta + y_1\sin(A+\theta)\cos(A+\theta),$$

$$y\cos(A+\theta) = x_1\cos(A+\theta)\sin\theta + y_1\sin(A+\theta)\cos(A+\theta);$$

$$\therefore\ x\sin(A+\theta) - y\cos(A+\theta) = x_1\sin A;$$

$$\therefore\ x_1 = \frac{x\sin(A+\theta) - y\cos(A+\theta)}{\sin A}.$$

Let $(\theta_1 = \angle y_1 A x)$; $\therefore\ A = (\theta_1 - \theta)$, and $A + \theta = \theta_1$;

$$\therefore\ y_1 = \frac{y\cos\theta - x\sin\theta}{\sin(\theta_1 - \theta)},$$

$$\text{and } x_1 = \frac{x\sin\theta_1 - y\cos\theta_1}{\sin(\theta - \theta_1)}.$$

COR. 2. If $A = 90^\circ$, $\cos(A+\theta) = -\sin\theta$

$$\text{and } \sin(A+\theta) = \cos\theta,$$

$$\text{and } x = x_1\cos\theta - y_1\sin\theta,$$

$$y = x_1\sin\theta + y_1\cos\theta,$$

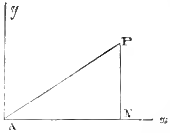
as in the preceding case.

(4) If the origin and inclination of the axes be changed. Let a and β be the co-ordinates of the origin, and then we must put

$$x = a + x_1 \cos\theta + y_1 \cos(A + \theta),$$

$$y = \beta + x_1 \sin\theta + y_1 \sin(A + \theta).$$

100. To transform rectangular co-ordinates into polar, the origin being the pole.



$$AP = r,$$

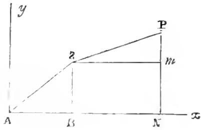
$$\angle PAN = \theta;$$

$$\therefore\ x = r\cos\theta,$$

$$y = r\sin\theta,$$

which put for x and y and the equation will be transformed.

But if the point S be the pole, draw SB perpendicular to Ax, and Sm perpendicular to PN,



$$AB = a, \qquad SP = r,$$

$$BS = \beta, \qquad \angle PSm = \theta;$$

$$\therefore\ x = AB + BN = a + r\cos\theta,$$

$$y = BS + Pm = \beta + r\sin\theta.$$

Ex. 1. Find the polar equation to the circle round a point S, co-ordinates a and β,

$$x^2 + y^2 = a^2;$$

$$\therefore\ (a + r\cos\theta)^2 + (\beta + r\sin\theta)^2 = a^2;$$

$$\therefore\ r^2 + 2r\,.\,(a\cos\theta + \beta\sin\theta) + a^2 + \beta^2 - a^2 = 0.$$

Ex. 2. Transform $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ into polar co-ordinates, the origin being the pole,

$$x^2 + y^2 = r^2, \quad \text{and } x = r\cos\theta, \quad y = r\sin\theta;$$

$$\therefore\ r^4 = a^2 r^2\,.\,(\cos^2\theta - \sin^2\theta);$$

$$\therefore\ r^2 = a^2\cos 2\theta.$$

Ex. 3. Transform the equation $x^2 - y^2 = a^2$ into another, the co-ordinates of which are rectangular, but the axis of y is inclined at an $\angle 45^0$ to the axis of x,

$$x = x_1 \cos\theta - y_1 \sin\theta,$$

$$y = x_1 \sin\theta + x_1 \cos\theta,$$

$$\theta = 2\pi - 45; \quad \therefore \cos\theta = \cos 45 = \frac{1}{\sqrt{2}},$$

$$\text{and } \sin\theta = -\sin 45 = \frac{-1}{\sqrt{2}};$$

$$\therefore x = \frac{x_1 + y_1}{\sqrt{2}},$$

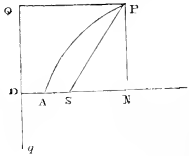
$$y = \frac{x_1 - y_1}{\sqrt{2}};$$

$$\therefore x^2 - y^2 = \frac{(x_1 + y_1)^2 - (x_1 - y_1)^2}{2} = \frac{4x_1y_1}{2} = 2x_1y_1 = a^2;$$

$$\therefore x_1y_1 = \frac{a^2}{2}.$$

THE PARABOLA.

101. If from a fixed line QDq perpendicular lines, as QP, are drawn intersecting lines equal in length, but drawn from a fixed point S, the locus of P is the parabola.



Draw SD perpendicular to Qq. Bisect SD in A, then the curve passes through A.

Let $SA = AD = a$,

$AN = x$,

$NP = y$.

Now QP or $DN = SP$;

$$\therefore DA + AN = \sqrt{NP^2 + SN^2};$$

$$\therefore a + x = \sqrt{y^2 + (x-a)^2};$$

$$\therefore (a+x)^2 \text{ or } (x-a)^2 + 4ax = y^2 + (x-a)^2;$$

$$\therefore y^2 = 4ax.$$

Cor. Let $SP = r$, and $\angle ASP = \theta$.

$$\text{Then } r = DN = 2a + SN = 2a + r\cos PSN$$

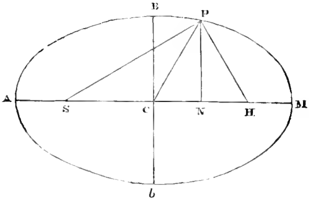
$$= 2a - r\cos\theta;$$

$$\therefore r = \frac{2a}{1+\cos\theta} = \frac{a}{\cos^2\frac{\theta}{2}}.$$

The polar equation.

THE ELLIPSE.

102. If from two fixed points S and H two lines SP



and PH be drawn and intersect, and $SP+PH$=a constant line, The locus of P is an ellipse.

Let $SP + PH = 2a$.

Bisect SH in C, and take $CA = CM = a$, the curve passes through A and M.

Through C draw BCb perpendicular to SH.

With centre S and radius $= a$ cut this line in the points B and b the curve will pass through B and b, since HB and Hb each $= a$.

$$\text{Let } CS : CA :: e : 1;$$

$$\therefore\ CS = ae, \text{ which is called the eccentricity.}$$

$$\text{Make } CN = x, \quad \text{and } CB = b, \quad SP = D,$$

$$NP = y \ldots\ldots\ldots\ldots\ldots\ldots HP = D_1;$$

$$\therefore\ D^2 = SN^2 + NP^2 = (ae + x)^2 + y^2,$$

$$D_1^2 = HN^2 + NP^2 = (ae - x)^2 + y^2;$$

$$\therefore\ D^2 + D_1^2 = 2(a^2e^2 + x^2 + y^2),$$

$$\text{and } D^2 - D_1^2 = 4aex.$$

$$\text{But } D + D_1 = 2a;$$

$$\therefore\ D - D_1 = 2ex;$$

$$\therefore\ D = a + ex, \text{ and } D_1 = a - ex;$$

$$\therefore\ D^2 + D_1^2 = 2a^2 + 2e^2x^2 = 2(a^2e^2 + x^2 + y^2);$$

$$\therefore\ y^2 = a^2.(1 - e^2) - x^2.(1 - e^2)$$
$$= (1 - e^2)(a^2 - x^2).$$

$$\text{But } 1 - e^2 = 1 - \frac{CS^2}{a^2} = \frac{a^2 - CS^2}{a^2} = \frac{SB^2 - CS^2}{a^2} = \frac{b^2}{a^2};$$

$$\therefore\ y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

$$\text{and } \frac{y^2}{b^2} + \frac{x^2}{a^2} = 1.$$

Cor. 1. If A be the origin.

$$\text{Make } AN = x_1;$$

$$\therefore\ x_1 = a + x, \quad \text{or } x = x_1 - a;$$

$$\therefore \ x^2 = x_1^2 - 2ax_1 + a^2;$$

$$\therefore \ a^2 - x^2 = 2ax_1 - x_1^2;$$

$$\therefore \ y^2 = \frac{b^2}{a^2}(2ax_1 - x_1^2).$$

Cor. 2. If S be the pole, and $ASP = \theta$, and $SP = r$;

$$\therefore \ (2a - r)^2 = HP^2 = HN^2 + NP^2 = (2ae - SN)^2 + r^2 \sin^2\theta,$$

and $SN = r \cos PSH = - r\cos\theta$;

$$\therefore \ 4a^2 - 4ar + r^2 = (2ae + r\cos\theta)^2 + r^2\sin^2\theta$$

$$= 4a^2e^2 + 4aer\cos\theta + r^2;$$

$$\therefore \ r.(1 + e\cos\theta) = a(1 - e^2),$$

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}.$$

Cor. 3. If C be the pole, $CP = r$, and $PCM = \theta$.

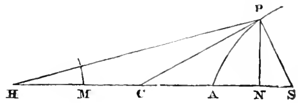
Then $x = r\cos\theta$, and $y = r\sin\theta$;

$$\therefore \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2.\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) = 1;$$

$$\therefore \ r = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} = \frac{ab}{\sqrt{a^2(1 - e^2)\cos^2\theta + a^2\sin^2\theta}}$$

$$= \frac{b}{\sqrt{1 - e^2\cos^2\theta}}.$$

103. If the difference between SP and PH be constant, the locus of P is the hyperbola.



Let the difference be $2a$.

Bisect SH in C.

Take $CA = a = CM$, and the curve passes through A,

$$\left.\begin{array}{l} CN = x \\ NP = y \end{array}\right\}. \quad \text{Let } CS = e.CA = ea, \text{ where } e > 1.$$

Then $HP^2 = HN^2 + NP^2 = (ea + x)^2 + y^2 = D_1^2$,

$$SP^2 = SN^2 + NP^2 = (ea - x)^2 + y^2 = D^2;$$

whence $D_1^2 + D^2 = 2 . (a^2 e^2 + x^2 + y^2)$,

and $D_1^2 - D^2 = 4aex$.

Also $D_1 - D = 2a$;

$$\therefore\ D_1 + D = 2ex;$$

$$\therefore\ D_1 = a + ex, \text{ and } D = ex - a;$$

$$\therefore\ 2a^2 + 2e^2 x^2 = 2(a^2 e^2 + x^2 + y^2),$$

$$\text{and } y^2 = (e^2 - 1) . x^2 - (e^2 - 1)\, a^2$$
$$= (e^2 - 1) . (x^2 - a^2)$$
$$= \frac{b^2}{a^2} . (x^2 - a^2).$$

Making $b^2 = a^2 (e^2 - 1)$;

$$\therefore\ \frac{y^2}{b^2} - \frac{x^2}{a^2} = -1.$$

Cor. 1. If A be the origin, and $AN = x_1$,

$$x = x_1 + a;$$

$$\therefore\ x + a = x_1 + 2a,$$

$$\text{and } x - a = x_1;$$

$$\therefore\ x^2 - a^2 = x_1^2 + 2ax_1,$$

$$\text{and } y^2 = \frac{b^2}{a^2} (2ax_1 + x_1^2).$$

Cor. 2. To find the polar equation, S being the pole,

$$SP = r,$$

$$\angle ASP = \theta.$$

Then $(2a + r)^2 = HP^2 = PN^2 + HN^2$

$$= PN^2 + (2\,CS - SN)^2$$

$$= r^2 \sin^2\theta + (2ae - r\cos\theta)^2;$$

$$\therefore\ 4a^2 + 4ar + r^2 = r^2 + 4a^2e^2 - 4aer\cos\theta;$$

$$\therefore\ r.(1 + e\cos\theta) = a(e^2 - 1),$$

$$r = \frac{a(e^2 - 1)}{1 + e\cos\theta}.$$

Cor. 3. If C be the pole,

$$CP = r,$$

$$\angle\, PCA = \theta;$$

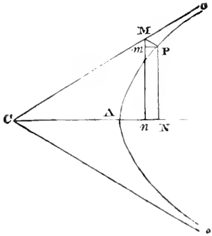
$$\therefore\ x = r\cos\theta, \text{ and } y = r\sin\theta,$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = r^2\left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) = 1,$$

$$r^2 = \frac{a^2b^2}{b^2\cos^2\theta - a^2\sin^2\theta},$$

$$r = \frac{ab}{\sqrt{b^2\cos^2\theta - a^2\sin^2\theta}} = \frac{b}{\sqrt{e^2.\cos^2\theta - 1}}.$$

104. The asymptotes being the axes, and the centre the origin, find the equation to the hyperbola.



The asymptotes are lines, as CO and Co, drawn through the centre, making an angle $= \tan^{-1}\dfrac{b}{a}$ with the axis of the hyperbola.

$CN = x$, $CM = x_1$, and $OCA = oCA = \theta$,

$NP = y$, $MP = y_1$.

Draw Mn perpendicular to CAN, and Pm perpendicular to Mn.

Since MP is parallel to Co, and Pm is parallel to CAN,

$$\therefore \angle MPm = \theta.$$

Now $x = Cn + nN = x_1 \cos\theta + y_1 \cos\theta = (x_1 + y_1)\cos\theta$,

$$y = Mn - Mm = x_1 \sin\theta - y_1 \sin\theta = (x_1 - y_1)\sin\theta;$$

$$\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{(x_1 + y_1)^2}{a^2}\cos^2\theta - \frac{(x_1 - y_1)^2}{b^2}\sin^2\theta = 1.$$

But $\tan\theta = \dfrac{b}{a}$; $\therefore 1 + \tan^2\theta = \dfrac{1}{\cos^2\theta} = \dfrac{b^2 + a^2}{a^2}$;

$$\therefore \frac{\cos^2\theta}{a^2} = \frac{1}{b^2 + a^2},$$

$$\text{and } \frac{\sin^2\theta}{b^2} = \frac{\cos^2\theta}{a^2} = \frac{1}{b^2 + a^2};$$

$$\therefore \frac{(x_1 + y_1)^2 - (x_1 - y_1)^2}{b^2 + a^2} = 1;$$

i. e. $4x_1y_1 = a^2 + b^2$,

$$x_1y_1 = \frac{a^2 + b^2}{4}.$$

Cor. If the hyperbola be rectangular, $b = a$,

and $x_1y_1 = \dfrac{a^2}{2}$.

105. The curves whose equations we have just investigated are termed Conic Sections, since they may be supposed to arise from the intersection of a cone by a plane.

The Conic Sections, exclusive of the straight line, are also called curves of the second degree, since the sum of the indices of the unknown quantities does not exceed two.

The general equation of the second degree is of the form

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0.$$

Now if the centre be the origin, the equation to the curve is the same when $(-x)$ and $(-y)$ are put for x and y: consequently the origin of the co-ordinates of the general equation is not in the centre; since Dy and Ex will both change their signs, when $(-y)$ and $(-x)$ are put for y and x.

To get rid of these terms, transform the equation to the centre by putting $x+\alpha$ and $y+\beta$ for x and y, and making the coefficients of x and y respectively $=0$, we shall have two equations for determining α and β;

$$\text{and } \alpha = \frac{2AE - BD}{B^2 - 4AC},$$

$$\beta = \frac{2CD - BE}{B^2 - 4AC}.$$

The equation is now reduced to

$$Ay^2 + Bxy + Cx^2 + F_1 = 0.$$

Next, to get rid of the term Bxy; let the axes be changed to others, making an angle θ with the axis of x, by putting

$$x = x\cos\theta - y\sin\theta,$$

$$\text{and } y = x\sin\theta + y\cos\theta.$$

Therefore the coefficient of xy becomes

$$2A\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) - 2C\sin\theta\cos\theta = 0,$$

$$\text{or } (A-C)\sin 2\theta = -B\cos 2\theta;$$

$$\therefore \tan 2\theta = \frac{-B}{A-C},$$

an equation which is always possible, since the tangent passes through all degrees of magnitude from zero to infinity. The reduced equation finally becomes

$$My^2 + Nx^2 + F_1 = 0,$$

which may be made to coincide with the equations to the circle, the ellipse, or the hyperbola, by giving proper values to M, N, and F_1.

Cor. 1. If $B^2 = 4AC$, α and β are infinite, and the curve has not a centre.

The equation without the term Bxy becomes

$$My^2 + Nx^2 + Px + Ry + F = 0.$$

$$\text{Now } 4M . N = 4AC - B^2 = 0;$$

therefore either M or $N = 0$.

Let $N = 0$; then the equation becomes

$$My^2 + Px + Ry + F = 0.$$

Again, to get rid of the terms Ry and F, make

$$x = x + a, \quad \text{and } y = y + b,$$

and we have

$$My^2 + (2Mb + R)y + Px + Pa + Rb + Mb^2 + F = 0;$$

to determine a and b,

$$\text{let } 2Mb + R = 0, \quad \text{or } b = \frac{-R}{2M},$$

$$\text{and } Pa + Rb + Mb^2 + F = 0; \quad \therefore a = -\frac{Rb + Mb^2 + F}{P};$$

and the equation becomes

$$My^2 + Px = 0,$$

the equation to the parabola.

If $M = 0$, then we shall have

$$Nx^2 + Ry = 0.$$

Cor. 2. If $My^2 + Nx^2 = F \ldots (1)$ be an ellipse, find the axes.

The equation to the ellipse is

$$a^2y^2 + b^2x^2 = a^2b^2 \ldots\ldots (2);$$

let h be such a quantity as multiplied into the equation (1), will make the terms identical with those of equation (2);

$$\therefore\ hM = a^2,\quad hN = b^2,\quad \text{and } hF = a^2 b^2;$$

$$\therefore\ h^2 MN = a^2 b^2 = hF;\quad \therefore\ h = \frac{F}{MN};$$

$$a^2 = \frac{F}{N},\ \text{and } a = \sqrt{\frac{F}{N}},\quad b^2 = \frac{F}{M};\quad \therefore\ b = \sqrt{\frac{F}{M}}.$$

106. We have assumed that

$$4MN = 4AC - B^2;$$

to prove this we must find M and N in terms of A, C, and B. By putting

$$x = x\cos\theta - y\sin\theta \text{ in the general equation,}$$

$$\text{and } y = x\sin\theta + y\cos\theta;$$

$$M = A\cos^2\theta - B\sin\theta\cos\theta + C\sin^2\theta;$$

$$N = A\sin^2\theta + B\sin\theta\cos\theta + C.\cos^2\theta;$$

$$\therefore\ M + N = A + C,$$

$$M - N = (A - C).\cos 2\theta - B.\sin 2\theta.$$

But since $\tan 2\theta = \dfrac{-B}{A - C}$,

$$\cos 2\theta = \frac{A - C}{\sqrt{(A - C)^2 + B^2}},\ \text{and } \sin 2\theta = \frac{-B}{\sqrt{(A - C)^2 + B^2}};$$

$$\therefore\ M - N = \frac{(A - C)^2 + B^2}{\sqrt{(A - C)^2 + B^2}} = \sqrt{(A - C)^2 + B^2};$$

$$\therefore\ 2M = A + C + \sqrt{(A - C)^2 + B^2},$$

$$2N = A + C - \sqrt{(A - C)^2 + B^2};$$

$$\therefore\ 4MN = (A + C)^2 - (A - C)^2 - B^2$$

$$= 4AC - B^2.$$

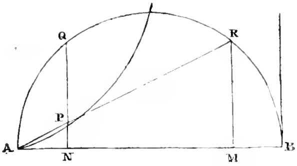
Whence, if $4AC > B^2$, M and N have the same sign,

if $4AC < B^2$, ... different signs,

$= B^2$, either M or N must $= 0$.

CISSOID.

107. AQB is a semicircle.



Take AN and BM equal.

Draw the ordinates NQ, MR.

Join AR cutting NQ in P. The locus of P is the cissoid.

$$\left.\begin{aligned} AN &= x \\ NP &= y \\ AB &= 2a \end{aligned}\right\}.$$

Now $$\frac{AN^2}{NP^2} = \frac{AM^2}{MR^2} = \frac{AM^2}{AM \cdot MB} = \frac{AM}{MB},$$

or $$\frac{x^2}{y^2} = \frac{2a - x}{x};$$

$$\therefore\ y^2 = \frac{x^3}{2a - x}.$$

Cor. The Polar Equation.

$$AP = r, \quad \angle PAN = \theta,$$

$$x = r\cos\theta, \quad y = r\sin\theta,$$

$$\frac{y^2}{x^2} = \frac{\sin^2\theta}{\cos^2\theta} = \frac{x}{2a - x} = \frac{r\cos\theta}{2a - r\cos\theta};$$

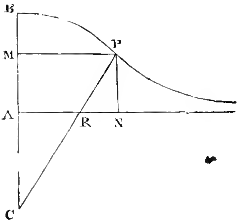
$$2a\sin^2\theta = r\cos\theta\,(\sin^2\theta + \cos^2\theta);$$

$$\therefore\ r = 2a\frac{\sin\theta}{\cos\theta}.\sin\theta$$

$$= 2a\tan\theta . \sin\theta.$$

THE CONCHOID OF NICOMEDES.

108. The line CP revolves round a fixed point C, cutting the line ARN: RP is always of the same length; then the point P will trace out the conchoid.



Let $RP = AB = a$, $AN = x$,

$CA = b$, $PN = y$.

$$\frac{MP^2}{CM^2} = \frac{AR^2}{CA^2} = \frac{RN^2}{NP^2} = \frac{RP^2 - NP^2}{NP^2},$$

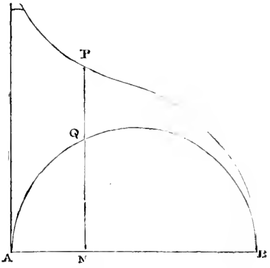
$$\frac{x^2}{(b+y)^2} = \frac{a^2 - y^2}{y^2};$$ $x^2y^2 = (a^2 - y^2)\cdot(b+y)^2$. P. 157

$$\therefore\ y^4 + 2by^3 + (b^2 + x^2 - a^2)\,y^2 - 2a^2by - a^2b^2 = 0.$$

Cor. Let $CP = r$, $\angle PCM = \theta$,

$$r = CP = PR + CR = a + \frac{b}{\cos\theta}.$$

109. AQB is a semi-circle, and NP is taken a fourth proportional to AN, AB, and NQ.



The locus of P is the "witch."

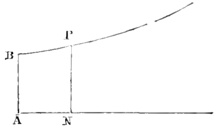
$AN = x$, $AB = 2a$,

$NP = y$; $\therefore NQ = \sqrt{2ax - x^2}$,

and $x : 2a :: \sqrt{2ax - x^2} : y$;

$$\therefore\ y = \frac{2a\sqrt{2ax - x^2}}{x} = 2a\sqrt{\frac{2a - x}{x}}.$$

110. The Logarithmic Curve.



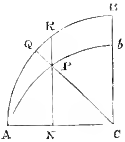
In this curve, the abscissa is the logarithm of the ordinate, or if a be the base of the system, the equation to the curve is $y = a^x$,

$$\therefore\ AB = a^0 = 1,$$

or the ordinate through the origin is always unity.

It is obvious that as the abscissa increases arithmetically, the ordinate increases geometrically.

111. The Quadratrix of Dinostratus.



While the ordinate RN of the quadrant AQB moves uniformly from A to BC, the radius revolves from CA to CB, cutting RN in P: the locus of P is the curve required.

$$AN = x, \qquad CB = 1,$$

$$NP = y, \quad \angle\ QCA = \theta.$$

Then $\theta : \dfrac{\pi}{2} :: x : 1;\ \therefore\ \theta = \dfrac{\pi x}{2}$,

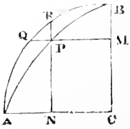
$$\frac{PN}{CN} = \tan\theta;$$

$$\text{or}\quad \frac{y}{1 - x} = \tan\frac{\pi x}{2};$$

$$\therefore\ y = (1 - x) . \tan\frac{\pi x}{2}.$$

Cor. When $x = 1$, $y = Cb = \dfrac{2}{\pi}$.

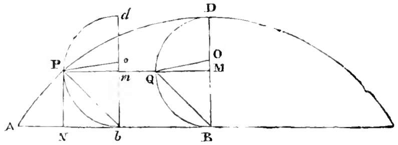
112. If RN move as before, and a line as QPM parallel to AC move uniformly from AC, the intersection P of RN and QM will trace the Quadratrix of Tschirnhausen.



Here $AQ = \frac{\pi x}{2}$, and $NP = \sin AQ$;

$$\therefore\ y = \sin \frac{\pi x}{2} \text{ is the equation.}$$

113. *The Cycloid* is the curve described by a point in the circumference of a circle, which is made to roll along a horizontal line.



Let BQD be the circle, O the centre; and when its diameter is perpendicular to the horizontal line at A, let the point P, which generates the curve, also be at A.

Then Ab must $= Pb$, since each point of Pb has been in contact with each successive point of Ab.

Let $AN = x$, $\quad BD = 2a$,

$NP = y$, $\quad \angle QOB = \theta$:

$$\therefore\ x = Ab - Nb = a\theta - a\sin\theta = a(\theta - \sin\theta);$$

$$\therefore\ y = bm \qquad = a \text{ ver. } \sin\theta = a(1 - \cos\theta);$$

θ cannot be eliminated between these equations.

Cor. 1. To find the differential equations.

$$\frac{dy}{d\theta} = a\sin\theta, \qquad \frac{dx}{d\theta} = a(1 - \cos\theta);$$

$$\therefore\ \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin\theta}{1 - \cos\theta} = \frac{a\sin\theta}{y},$$

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{(1 - \cos\theta) \times (1 + \cos\theta)} = \sqrt{\frac{y}{a} \times \frac{2a - y}{a}};$$

$$\therefore\ a \sin\theta = \sqrt{2ay - y^2};$$

$$\therefore\ \frac{dy}{dx} = \frac{\sqrt{2ay - y^2}}{y}.$$

Cor. 2. To find the equation from D.

$DM = x_1$; $\therefore\ x_1 = 2a - y$,

$MP = y_1$, $\quad y_1 = AB - x$;

$$\therefore\ \frac{dx_1}{dy_1} = \frac{dy}{dx} = \sqrt{\frac{2ay - y^2}{y^2}} = \sqrt{\frac{2a - y}{y}}$$

$$= \sqrt{\frac{x_1}{2a - x_1}}.$$

Cor. 3. The equation from D may be also found from the properties of the curve.

Join Pb and QB, then these being equal and parallel,

$$PQ = Bb = AB - Ab = AB - Pb = DQ.$$

For AB is equal to the semi-circumference DQB.

Let $DOQ = \phi$.

Then $y = PM = MQ + PQ = a \sin\phi + a\phi = a(\phi + \sin\phi)$,

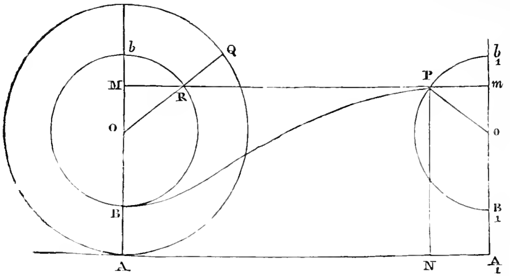
$x = DM = a$ ver. $\sin\phi \quad = a(1 - \cos\phi)$,

and eliminating ϕ by differentiation we have the equation previously obtained.

THE TROCHOID.

114. The trochoid is the curve traced out by a point B in the circumference of the circle BRb, which is carried through space by the rolling of the outer circle AQ upon the horizontal line.

P a point in the trochoid. Through P draw a horizontal



line $MRPm$. Take O and o the centres of the circles.

Draw ORQ and oP.

Then $Pm = RM$, and $\angle AOQ = \angle A_1 oP$.

$$\text{Let } \begin{matrix} OA = a, & AN = x \\ OB = b, & NP = y \end{matrix}\Bigg\}, \quad \angle AOR = \theta.$$

Then it is obvious that arc $AQ = AA_1$;

$$\therefore \; x = AA_1 - NA_1 = a\theta - b \sin\theta,$$

$$y = NP = oA_1 + om = a - b\cos\theta.$$

$$\text{Let } \frac{b}{a} = e;$$

$$\therefore \; x = a(\theta - e\sin\theta),$$

$$y = a(1 - e\cos\theta).$$

If $e = 1$, that is $b = a$, the trochoid becomes the common cycloid and their equations coincide.

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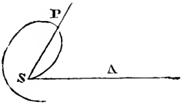
115. SPIRALS.

(1) The spiral of Archimedes. In this spiral the radius vector varies directly as the angle described,

$$\text{or } r \propto \theta; \quad \therefore \; r = a\theta \text{ is the equation.}$$

Its equation may be found from the following mechanical construction:



Let the line SA revolve uniformly round S, while a point P moves uniformly from S along SA, then P will trace the spiral of Archimedes.

$$\text{Let } \angle\, ASP = \theta,$$

$$SP = r;$$

and let a = value of r when $\theta = 2\pi$;

$$\therefore\ r : a :: \theta : 2\pi;$$

$$\therefore\ r = \frac{a}{2\pi}\theta = m\theta \text{ by putting } m = \frac{a}{2\pi}.$$

(2) The logarithmic spiral. Here the angle described is the logarithm of the radius vector, its equation is $r = a^\theta$.

This curve is also called the equiangular spiral, since the angle at which it cuts the radius is constant.

(3) The hyperbolic spiral. In this spiral as the angle increases the radius vector decreases, and its equation is

$$r = \frac{a}{\theta}, \quad \text{or } \theta r = a.$$

(4) The *lituus* so called from its form.

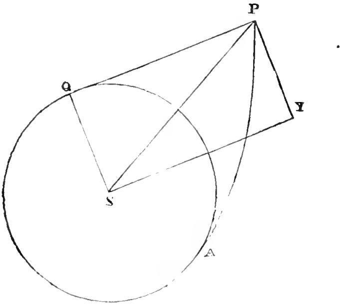
$$\text{Here } r \propto \frac{1}{\sqrt{\theta}}, \quad \text{or } \theta = \frac{a^2}{r^2}.$$

(5) The spiral of Archimedes, the hyperbolic, and the lituus, are included under the general equation

$$r = a\theta^n,$$

as we shall see by putting $n = 1, -1$, or $\frac{1}{2}$.

(6) The *involute* of the circle is described by the extremity of a string which is unwound from the circumference of a circle*.



A the point from which the string began to be unwrapped, QP the string once coincident with the arc AQ, and therefore $= AQ$; PY a tangent to the curve AP or to the involute, SY perpendicular to the tangent, join SP.

$$\left.\begin{array}{l} SP = r \\ SY = p \end{array}\right\}, \quad SQ = a;$$

$$\therefore\ SQ = PY = \sqrt{SP^2 - SY^2};$$

$$\therefore\ a^2 = r^2 - p^2;$$

$$\therefore\ p^2 = r^2 - a^2, \text{ is the equation.}$$

Cor. If $\phi = \sec^{-1}\dfrac{r}{a} = PSQ$, and $\theta = \angle ASP$,

$$\theta + \phi = \frac{\sqrt{r^2 - a^2}}{a}; \quad \therefore\ \theta = \frac{\sqrt{r^2 - a^2}}{a} - \sec^{-1}\frac{r}{a}.$$

* The figure is drawn inaccurately, AP should be perpendicular to the circle at A.

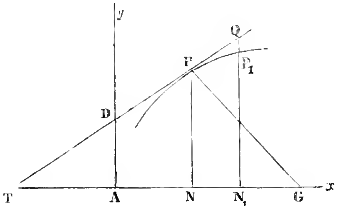
CHAPTER IX.

TANGENTS TO CURVES.

116. Def. A tangent is a line which has a point in common with a curve, and which, of all the straight lines that can be drawn through the point, approaches nearest to the curve.

PP_1 the curve.

QPT the tangent of which the equation is required.



A the origin of co-ordinates.

Ay and Ax the axes of y and x respectively.

$\left.\begin{array}{l} AN = x \\ NP = y \end{array}\right\}$, and $y = f(x)$ the equation to the curve,

and $y_1 = Ax_1 + B$ the equation to the line;

$\therefore\ y = Ax + B$, because it passes through P;

$\therefore\ (y_1 - y) = A\,.\,(x_1 - x)$ ...... is the equation to the line by eliminating B.

$$\text{Let } NN_1 = h; \ \therefore \ N_1P_1 = f(x+h) = y + \frac{dy}{dx}h + Ph^2,$$

$$\text{and } N_1Q = A(x+h) + B = y + Ah;$$

$$\therefore \ QP_1 = \left(A - \frac{dy}{dx}\right)h - Ph^2.$$

And QP_1 the distance between the curve and the line, will be the least when the term involving h vanishes;

$$\text{that is, when } A = \frac{dy}{dx}.$$

For if Δ be the distance between the curve and any other line where $A - \dfrac{dy}{dx}$ does not $= 0$,

$$\Delta = \pm mh - Ph^2, \text{ where } m = \pm\left(A_1 - \frac{dy}{dx}\right);$$

when $QP_1 = (A + \frac{dy}{dx} = 0)h - Ph^2$

$$\therefore \frac{\Delta}{QP_1} = \frac{\pm mh - Ph^2}{-Ph^2} = \frac{\pm m - Ph}{-Ph} = \frac{m}{0} = \infty, \text{ when } h = 0,$$

or Δ is infinitely greater than QP_1;

$$\text{therefore when } PT \text{ is a tangent } A = \frac{dy}{dx};$$

$$\text{and } \therefore \ (y_1 - y) = \frac{dy}{dx}(x_1 - x) \text{ is the equation to the tangent.}$$

Cor. 1. From P draw PG perpendicular to the tangent and meeting the axis of x in G, it is called the normal, and since if $y = ax + b$ be the equation to a line, $y = -\dfrac{1}{a}x + b_1$ is the equation to a line perpendicular to it;

$$\therefore \ y_1 = -\frac{dx}{dy}x_1 + b_1 \text{ is the equation to the normal,}$$

$$\text{and } \therefore \ (y_1 - y) = -\frac{dx}{dy}(x_1 - x) \text{ since it passes through } P.$$

COR. 2. To construct the equation to the tangent,

$$\text{let } x_1 = 0\,; \quad \therefore\ y_1 = AD = y - x\frac{dy}{dx},$$

$$y_1 = 0\,; \quad \therefore\ -x_1 = AT = y\frac{dx}{dy} - x.$$

Find therefore from the given equation $y = f(x)$ the value of $\frac{dy}{dx}$ in terms of x or y, one or both; substitute this value, and we shall find AD and AT. Join TD; this line produced is the tangent.

COR. 3. Hence since $(y_1 - y) = \frac{dy}{dx}(x_1 - x)\ldots\ldots$ is the equation to the tangent, and

$$y_1 - y = -\frac{dx}{dy}(x_1 - x)\ldots\ldots$$

is the equation to the normal;

$$\therefore\ \frac{dy}{dx} = \tan PTN, \text{ and } -\frac{dx}{dy} = \tan PGx,$$

$$\text{or } \frac{dx}{dy} = \tan PGN.$$

COR. 4. NT and NG are respectively called the sub-tangent and sub-normal, and are useful in drawing the tangent and normal,

$$\text{and } NT = AN + AT = x + y\frac{dx}{dy} - x = y\frac{dx}{dy},$$

and from similar triangles NTP, PGN,

$$NG = \frac{NP^2}{NT} = \frac{y^2}{y\cdot\frac{dx}{dy}} = y\frac{dy}{dx}.$$

Hence to draw a tangent or normal, find the values of NT or NG. Join P, T, or P, G, and we have the tangent or normal required.

Cor. 5. The length PT of the tangent

$$= \sqrt{PN^2 + NT^2} = \sqrt{y^2 + y^2 \frac{dx^2}{dy^2}} = y \sqrt{1 + \frac{dx^2}{dy^2}};$$

the length PG of the normal

$$= \sqrt{PN^2 + NG^2} = \sqrt{y^2 + y^2 \frac{dy^2}{dx^2}} = y \sqrt{1 + \frac{dy^2}{dx^2}}.$$

Cor. 6. The tangent of the angle which the tangent makes with the axis of x is $\frac{dy}{dx}$; whence the angle at which the curve cuts the axis may be found.

For the angle which the tangent makes with the axis at the point of section will be the same that the curve makes.

Find therefore the co-ordinates of the point of section, and substitute them in the expression for $\frac{dy}{dx}$, and the resulting value will be the tangent of the angle required.

Ex. 1. Let $y = \frac{x}{1+x}$ be the equation to the curve.

Here if $x = 0$, $y = 0$;

and therefore the origin is the point of section,

$$\text{and } \frac{dy}{dx} = \frac{1}{(1+x)^2} = \frac{1}{1}, \text{ when } x = 0;$$

$$\therefore \tan\theta = 1 = \tan 45^0; \quad \therefore \theta = 45^0.$$

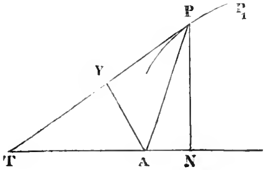
Ex. 2. Let the curve be the cycloid.

$$\text{Here } \frac{dy}{dx} = \frac{\sqrt{2a-y}}{\sqrt{y}} = \sqrt{\frac{2a}{y} - 1},$$

which is infinite if $y = 0$; or at the origin the curve cuts the axis of x at an angle of 90^0.

117. Find the length of the perpendicular from the origin upon the tangent, and the angle which the line from the origin to the point of contact makes with the tangent.

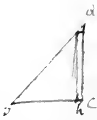
Draw AY perpendicular to PT.



Then from similar triangles AYT, NPT,

$$AY = \frac{AT \times NP}{PT} = \left(y\frac{dx}{dy} - x\right) \cdot \frac{y}{y\sqrt{1 + \frac{dx^2}{dy^2}}} = \frac{y - x\frac{dy}{dx}}{\sqrt{1 + \frac{dy^2}{dx^2}}}$$

$$= \frac{y - px}{\sqrt{1+p^2}}, \quad \text{if } \frac{dy}{dx} = p.$$



$$\text{And } \angle APT = \angle TPN - \angle APN = \tan^{-1} \cdot \frac{dx}{dy} - \tan^{-1} \cdot \frac{x}{y};$$

$$\therefore \tan APT = \frac{\frac{dx}{dy} - \frac{x}{y}}{1 + \frac{x}{y} \cdot \frac{dx}{dy}} = \frac{y - px}{x + py}.$$

$\frac{PC}{CQ} = \frac{dx}{dy}$ $\therefore \tan R =$
$\frac{CQ}{CP} = \frac{dy}{dx}$ $\therefore \tan P =$

Ex. Let the curve be the circle, and the origin in the centre,

$$y^2 = a^2 - x^2,$$

$$\frac{dy}{dx} = -\frac{x}{y}, \quad 1 + \frac{dy^2}{dx^2} = \frac{y^2 + x^2}{y^2} = \frac{a^2}{y^2},$$

$$y - px = y + \frac{x^2}{y} = \frac{a^2}{y},$$

$$py + x = -x + x = 0;$$

$$\therefore AY = \frac{\frac{a^2}{y}}{\frac{a}{y}} = a,$$

$$\text{and } \tan APT = \frac{\frac{a^2}{y}}{0} = \infty; \quad \therefore \angle APT = 90^0.$$

118. To draw a tangent through a given point.

Let α and β be the co-ordinates of the given point;

x and y be the co-ordinates of the curve.

and $(y_1 - y) = \frac{dy}{dx}(x_1 - x)$ is the general equation to the tangent.

But because it passes through a point where $y_1 = \beta$ and $x_1 = \alpha$,

$$\therefore \beta - y = \frac{dy}{dx}(\alpha - x);$$

from which, and the given equation to the curve, the point to which the tangent is to be drawn may be found.

In the circle the equation to the tangent will be

$$(\beta - y) = -\frac{x}{y} \cdot (\alpha - x),$$

$$\text{or } y\beta - y^2 = -x\alpha + x^2 = -x\alpha + r^2 - y^2;$$

$$\therefore y\beta = r^2 - \alpha x,$$

$$\text{or } \beta\sqrt{r^2 - x^2} = r^2 - \alpha x,$$

$$\beta^2 r^2 - \beta^2 x^2 = r^4 - 2r^2 \alpha x + \alpha^2 x^2,$$

$$(a^2 + \beta^2)\,x^2 - 2\,r^2 a\,x = r^2 \,.\, (\beta^2 - r^2)\,;$$

$$\text{whence } x = \frac{r\,.\{ar \pm \beta\sqrt{a^2 + \beta^2 - r^2}\}}{a^2 + \beta^2}$$

$$= \frac{r\{ar \pm \beta\sqrt{\delta^2 - r^2}\}}{\delta^2}, \quad \text{if } \delta^2 = a^2 + \beta^2.$$

The double sign shews that two tangents can be drawn from the same point.

Cor. If $\delta = r$, or let the point be in the circumference;

$$\therefore\ x = \frac{a r^2}{r^2} = a, \quad \text{and } y\beta = y^2; \quad \therefore\ y = \beta,$$

or the tangent touches the circle at the given point.

119. Draw a tangent parallel to a given line.

Let A = tangent of the angle which the given line makes with the axis of x;

$$\therefore\ \frac{dy}{dx} = A, \text{ since tangent and line are parallel;}$$

and $y_1 - y = A\,.\,(x_1 - x)$ is the equation required.

If it pass through a given point, the co-ordinates of the point may be put for x_1 and y_1, and then from the given equation to the curve, and from that of the tangent, the point to which the tangent is to be drawn may be found.

120. Asymptotes are tangents to the curve at a point infinitely distant from the origin.

These may be drawn, if the values of AD or AT, or of both remain finite, when either x or y, or x and y, are infinite.

Asymptotes may be thus constructed:

(1) If AD and AT be finite, join T, D, and the line produced is the asymptote.

(2) If AD be infinite, and AT finite, the asymptote is perpendicular to the axis of x, passing through T.

(3) If AD be infinite, and $AT = 0$, the asymptote coincides with the axis of y.

(4) If AD be finite, and AT infinite, the asymptote is parallel to the axis of x; and if $AD = 0$ is coincident with it.

EXAMPLE. Draw an asymptote to the hyperbola.

Here $y = \frac{b}{a}\sqrt{2ax + x^2}$, and $\frac{dy}{dx} = \frac{b}{a}\frac{a+x}{\sqrt{2ax+x^2}}$,

$$AD = y - x\frac{dy}{dx} = \frac{b}{a}\cdot\left\{\sqrt{2ax+x^2} - \frac{a+x}{\sqrt{2ax+x^2}}\right\} = \frac{bx}{\sqrt{2ax+x^2}}$$

$$= \frac{b}{\sqrt{1+\frac{2a}{x}}} = b \quad \text{if } x = \infty;$$

$$AT = y\frac{dx}{dy} - x = \frac{2ax+x^2}{a+x} - x = \frac{ax}{a+x} = \frac{a}{1+\frac{a}{x}} = a,$$

when $x = \infty$;

$\therefore$ $AT = \frac{1}{2}$ major-axis, or T and C coincide.

Join CD, it produced, is the asymptote.

121. This method is frequently difficult of application, and the following is more generally useful.

If possible, let the equation to the curve be put under the form

$$y = Ax + B + \frac{C}{x} + \frac{D}{x^2} + \frac{E}{x^3} + \&c.$$

then it is obvious, that as x increases, the terms after B decrease; and when x becomes infinitely great, they vanish, and the equation to the infinite branch of the curve is

$$y = Ax + B.$$

But this is the equation to a straight line cutting the axis of y at a point $y = B$, and $x = 0$, and making an angle $= \tan^{-1} A$, with the axis of x. Hence it appears that the infinite branch of the curve is coincident with the line determined by the equation $y = Ax + B$;

$\therefore$ if $y = Ax + B + \dfrac{C}{x} + \dfrac{D}{x^2} + \&c.$ be the equation to a curve,

$y = Ax + B$ is the equation to the asymptote.

Cor. If the form of the expanded $f(x)$ be

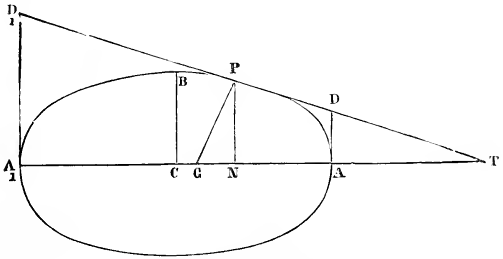
$$y = Ax^2 + Bx + C + \frac{D}{x} + \frac{E}{x^2} + \&c.$$

the asymptote is a parabolic curve, of which the equation is

$$y = Ax^2 + Bx + C.$$

EXAMPLES.

(1) Find the equation to the tangent in the ellipse,



The centre being the origin.

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1;$$

$$\therefore \frac{dy}{dx} = -\frac{b^2}{a^2} \cdot \frac{x}{y};$$

$$\therefore\ y_1 - y = -\frac{b^2}{a^2}\frac{x}{y}(x_1 - x),$$

$$\therefore\ yy_1 - y^2 = \frac{-b^2}{a^2}xx_1 + \frac{b^2}{a^2}x^2 = -\frac{b^2}{a^2}xx_1 + b^2 - y^2\ ;$$

$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$ $\therefore \frac{b^2x^2}{a^2} = b$

$$\therefore\ yy_1 = \frac{b^2}{a^2}.(a^2 - xx_1),$$

$$\text{or } y_1 = \frac{b}{a}.\frac{a^2 - xx_1}{\sqrt{a^2 - x^2}}.$$

Let $y_1 = 0$; $\therefore\ x_1 = CT = \frac{a^2}{x} = \frac{CA^2}{CN}$;

$$\therefore\ CT \times CN = CA^2, \quad \text{(See } \textit{Conic Sections.})$$

and $NT = CT - CN = \frac{a^2 - x^2}{x}$ = sub-tangent,

or $NT \times CN = (a + x)(a - x) = A_1N \times AN$.

Cor. 1. Make $x_1 = a$; $\therefore\ y_1 = AD = \frac{b.(a - x)}{\sqrt{a^2 - x^2}}$,

$x_1 = -a$; $\therefore\ y_1 = A_1D_1 = \frac{b.(a + x)}{\sqrt{a^2 - x^2}}$;

$$\therefore\ AD.A_1D_1 = b^2 = CB^2,$$

$$NG = \frac{NP^2}{NT} = \frac{b^2}{a^2}(a^2 - x^2) \times \frac{x}{a^2 - x^2} = \frac{b^2}{a^2}CN.$$

Cor. 2. The equation to the tangent may be written

$$a^2yy_1 + b^2xx_1 = a^2b^2.$$

In the hyperbola, the equation is

$$a^2yy_1 - b^2xx_1 = -a^2b^2.$$

(2) Find the sub-tangent and sub-normal, &c. in the cissoid.

Here $y^2 = \dfrac{x^3}{2a - x}$;

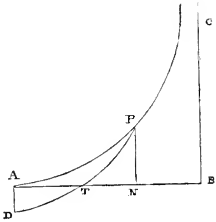
$$\therefore\ y\frac{dy}{dx} = \tfrac{1}{2} \cdot \frac{3x^2 . (2a - x) + x^3}{(2a - x)^2} = \frac{x^2}{2} \cdot \frac{(6a - 2x)}{(2a - x)^2};$$

$$\therefore\ \text{sub-normal} = y\frac{dy}{dx} = \frac{x^2(3a - x)}{(2a - x)^2};$$

dividing $y\dfrac{dy}{dx}$ by y^2,

$$\frac{1}{y}\frac{dy}{dx} = \frac{2a - x}{x^3} \times \frac{x^2(3a - x)}{(2a - x)^2} = \frac{(3a - x)}{x . (2a - x)};$$

$$\therefore\ \text{sub-tangent} = \frac{x(2a - x)}{3a - x}.$$



Also $\because\ \dfrac{dy}{dx} = \dfrac{x^2 . (3a - x)}{y . (2a - x)^2}$.

The equation to the tangent is

$$(y_1 - y) = \frac{x^2}{y} \cdot \frac{3a - x}{(2a - x)^2}(x_1 - x),$$

$$\text{or } yy_1 - y^2 = \frac{x^2(3a - x)}{(2a - x)^2} . (x_1 - x);$$

$$\therefore\ yy_1 = \frac{x^2}{2a - x} \cdot \left\{\frac{(3a - x)(x_1 - x)}{2a - x} + x\right\}$$

$$= \frac{x^2}{(2a - x)^2} . \{(3a - x) . x_1 - ax\};$$

$$\therefore\ y_1 = \frac{\sqrt{x}}{(2a - x)^{\frac{3}{2}}} . \{(3a - x)x_1 - ax\}.$$

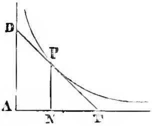
Making y_1 and x_1 successively $= 0$,

$$AT = \frac{a . x}{3a - x}, \text{ and } AD = -y_1 = a . \left(\frac{x}{2a - x}\right)^{\frac{3}{2}}.$$

Note—PTD in the figure should be a straight line.

If $x = 2a$, $\dfrac{dy}{dx}$ and y_1 are infinite; there is therefore an asymptote through B perpendicular to the axis of x.

(3) Rectangular hyperbola referred to the asymptotes.



Here $yx = \dfrac{a^2}{2}$; $\therefore\ y = \dfrac{a^2}{2x}$,

$$\frac{dy}{dx} = -\frac{a^2}{2} \cdot \frac{1}{x^2} = -\frac{y}{x};$$

$$\therefore\ y_1 - y = -\frac{y}{x}(x_1 - x),$$

$$xy_1 - xy = -yx_1 - yx,$$

$$xy_1 + yx_1 = 2yx = a^2;$$

$$x_1 = 0,\quad y_1 = AD = \frac{a^2}{x},$$

$$y_1 = 0,\quad x_1 = AT = \frac{a^2}{y}.$$

The $\Delta DAT = \dfrac{AT \cdot AD}{2} = \dfrac{a^4}{2xy} = a^2$, which is constant.

(4) Let $\sqrt{y} = \sqrt{a} - \sqrt{x}$; find the equation to the tangent

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}};$$

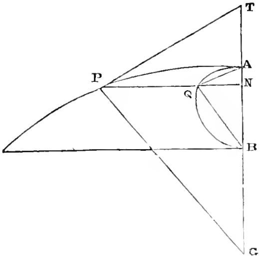
$$\therefore\ y_1 - y = -\frac{\sqrt{y}}{\sqrt{x}}(x_1 - x).$$

Let $x_1 = 0$; $\therefore\ AD = y + \dfrac{x\sqrt{y}}{\sqrt{x}} = y + \sqrt{xy}$,

$y_1 = 0$; $\therefore\ AT = x + \dfrac{y\sqrt{x}}{\sqrt{y}} = x + \sqrt{xy}$;

$$\therefore\ AD + AT = x + 2\sqrt{xy} + y = (\sqrt{x} + \sqrt{y})^2 = a.$$

(5) Draw a tangent to the cycloid



$AN = x,$

$NP = y,$

$AB = 2a.$

Then $\dfrac{dy}{dx} = \dfrac{\sqrt{2ax - x^2}}{x};$

$\therefore\ y \cdot \dfrac{dx}{dy} = \dfrac{y \cdot x}{\sqrt{2ax - x^2}};$

or $NT = \dfrac{NP \cdot AN}{NQ};$

i. e. $NT : NP :: AN : NQ;$

and $\angle N$ is common to Δs ANQ, TNP; $\therefore$ they are similar,

and $\angle PTN = \angle QAN;$

$\therefore$ the tangent TP is parallel to the chord AQ.

Also since $\angle AQB$ is always $= 90$, PG is parallel to BQ.

(6) Draw a tangent to the conchoid

$$xy = (a + y)\sqrt{b^2 - y^2};$$

$$\therefore\ x = \left(\frac{a}{y} + 1\right)\sqrt{b^2 - y^2},$$

$$\frac{dx}{dy} = -\frac{a}{y^2}\sqrt{b^2 - y^2} - \frac{(a + y)}{\sqrt{b^2 - y^2}}$$

$$= -\frac{ab^2 + y^3}{y^2\sqrt{b^2 - y^2}};$$

$$\therefore\ NT = -y\frac{dx}{dy} = \frac{ab^2 + y^3}{y\sqrt{b^2 - y^2}}.$$

(7) Draw an asymptote to the hyperbola

$$y = \pm \frac{b}{a}\sqrt{2ax + x^2} = \pm \frac{b}{a}x\left(1 + \frac{2a}{x}\right)^{\frac{1}{2}}$$

$$= \pm \frac{b}{a}x\left\{1 + \tfrac{1}{2}\frac{2a}{x} + \frac{\frac{1}{2}\cdot(\frac{1}{2}-1)}{1\,.\,2}\cdot\frac{4a^2}{x^2} + \frac{B}{x^3} + \&c.\right\}$$

$$= \pm \frac{b}{a}\cdot\left\{x + a - \frac{1}{2}\cdot\frac{a^2}{x} + \frac{B}{x^2} + \&c.\right\},$$

and therefore $y = \pm\frac{b}{a}(x + a)$ is the equation to two asymptotes;

and since if $x = 0$, $y = \pm b$;

and if $y = 0$, $x = -a$,

both will pass through the centre, and they will be equally inclined to the axis of x.

(8) Draw the asymptote to the curve

$$y^3 = x^3 + ax^2 = x^3\left(1 + \frac{a}{x}\right);$$

$$\therefore\ y = x\left(1 + \frac{a}{x}\right)^{\frac{1}{3}} = x\left\{1 + \tfrac{1}{3}\cdot\frac{a}{x} + \frac{A}{x^2} + \frac{B}{x^2} + \&c.\right\}$$

$$= x + \frac{a}{3} + \frac{A}{x} + \frac{B}{x^2} + \&c.;$$

$\therefore\ y = x + \frac{a}{3}$ is the equation to the asymptote which cuts the axis of x at an $\angle = 45^0$, and at a point $x = -\frac{a}{3}$.

(9) Let $y\,.\,(ax + b^2) = x^3$, draw an asymptote,

$$y = \frac{x^3}{ax + b^2} = \frac{1}{a}\,\frac{x^3}{x + c} \text{ by putting } \frac{b^2}{a} = c$$

$$= \frac{1}{a} \cdot \frac{x^2}{1 + \frac{c}{x}} = \frac{1}{a} x^2 \left\{1 - \frac{c}{x} + \frac{c^2}{x^2} - \frac{c^3}{x^2} + \&c.\right\}$$

$$= \frac{x^2}{a} - \frac{cx}{a} + \frac{c^2}{a} - \frac{c^2}{ax} + \&c.;$$

$\therefore\ ay = x^2 - cx + c^2$ is the equation to the asymptotic curve.

This being put under the form

$$ay - \tfrac{3}{4}c^2 = x^2 - cx + \frac{c^2}{4} = \left(x - \frac{c}{2}\right)^2;$$

$$\text{or } \left(x - \frac{c}{2}\right)^2 = a\left(y - \tfrac{3}{4}\frac{c^2}{a}\right).$$

shews that the curve is a parabola, the axis of which is perpendicular to the axis of x, and the position of the vertex determined by making

$$x_1 = \frac{c}{2} \text{ and } y_1 = \tfrac{3}{4}\frac{c^2}{a}; \text{ the latus rectum} = a.$$

(10) Let the equation be $ay^4 - bx^4 + c^3xy = 0$.

$$\text{Let } y = zx;$$

$$\therefore\ ax^4z^4 - bx^4 + c^3x^2z = 0;$$

$$\therefore\ ax^2z^4 - bx^2 + c^3z = 0;$$

$$\therefore\ x^2 = \frac{c^3z}{b - az^4},$$

and x will be infinite when $b - az^4 = 0$,

$$\text{or } z = \sqrt[4]{\frac{b}{a}},$$

$$\text{and then } y = xz = x\sqrt[4]{\frac{b}{a}},$$

is the equation to the asymptote.

(11) Find when the curve, which is the locus of the general equation of the second order, has an asymptote.

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0$$

is the general equation,

$$\text{or } y^2 + 2(ax + b)y + cx^2 + ex + f = 0,$$

dividing by A, and making the proper substitutions;

$$\therefore\ y^2 + 2(ax+b)y + (ax+b)^2 = (a^2 - c)x^2 + (2ab - e)x + b^2 - f,$$

$$\text{and } y = -(ax+b) \pm \sqrt{(a^2-c)x^2 + (2ab-e)x + (b^2-f)};$$

$$\therefore\ = -x\left\{\left(a + \frac{b}{x}\right) \pm \sqrt{(a^2-c) + \frac{2ab-e}{x} + \frac{b^2-f}{x^2}}\right\}$$

$$= -x\left\{a + \frac{b}{x} \pm \sqrt{a^2-c}\left\{1 + \frac{1}{2x}\frac{2ab-e}{(a^2-c)} + \frac{A}{x^2} + \frac{B}{x^3}\right\}\right.$$

$$= -\left\{ax + b \pm \sqrt{a^2-c}\left(x + \tfrac{1}{2}\frac{2ab-e}{a^2-c} + \frac{A}{x} + \&c.\right)\right\};$$

and therefore the equation to the asymptote, which is of the form $y = mx + n$, is

$$y = -(a \pm \sqrt{a^2-c})x - \left(b \pm \frac{2ab-e}{2\sqrt{a^2-c}}\right),$$

which is possible when $a^2 > c$, or $\dfrac{B^2}{4A^2} > \dfrac{C}{A}$,

$$\text{or } B^2 - 4AC > 0,$$

which is the case in the hyperbola.

Cor. If $a^2 = c$, the equation is of the form

$$y = -(ax + b) \pm \sqrt{mx}\,.\sqrt{1 + \frac{n}{x} \ldots}$$

which cannot be reduced to the form $y = Ax + B$.

122. Find the locus of the intersections of perpendiculars drawn from the origin upon the tangent, with the tangent.

Let $y = f(x)$ be the equation to the curve;

$\therefore\ (y_1 - y) = \dfrac{dy}{dx}(x_1 - x)$ is the equation to the tangent,

and $y_1 = -\dfrac{dx}{dy}x_1$ is the equation to the perpendicular from the origin upon the tangent.

Between these three equations eliminate y, x, and $\dfrac{dy}{dx}$, and the resulting equation will contain y_1 x_1 and constant quantities, which will be the equation to the curve required.

Ex. Let the curve be the hyperbola, and the origin the centre;

$$\therefore\ \frac{y^2}{b^2} = \frac{x^2}{a^2} - 1; \quad \therefore\ \frac{dy}{dx} = \frac{b^2}{a^2}\cdot\frac{x}{y};$$

and $\therefore\ y_1 = -\dfrac{a^2}{b^2}\cdot\dfrac{y}{x}x_1$ from equation to perpendicular;

$$\therefore\ \frac{y_1^2}{x_1^2} = \frac{a^4}{b^4}\cdot\frac{y^2}{x^2} = \frac{a^4}{b^4}\left(\frac{b^2}{a^2} - \frac{b^2}{x^2}\right) = \frac{a^2}{b^2} - \frac{a^4}{b^2x^2};$$

$$\therefore\ x^2 = \frac{a^4x_1^2}{a^2x_1^2 - b^2y_1^2};$$

$$\therefore\ x = \frac{a^2x_1}{\sqrt{a^2x_1^2 - b^2y_1^2}},$$

$$\text{and } y = -\frac{b^2}{a^2}\cdot\frac{y_1}{x_1}x = \frac{-b^2y_1}{\sqrt{a^2x_1^2 - b^2y_1^2}}.$$

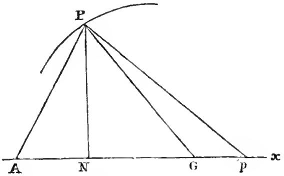
But $\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1$ is equation to tangent;

$$\therefore\ \frac{x_1^2 + y_1^2}{\sqrt{a^2x_1^2 - b^2y_1^2}} = 1.$$

$$\therefore\ (x_1^2 + y_1^2)^2 = a^2x_1^2 - b^2y_1^2.$$

COR. If $a = b$, or the hyperbola be the equilateral, $(x_1^2 + y^2)^2 = a^2 (x_1^2 - y_1^2)$. The equation to the lemniscata.

PROB. P any point in a curve, PG a normal; let Pp make with PG the angle $pPG = \angle APG$.



Find the equation to Pp.

$$\left.\begin{array}{l} AN = x \\ NP = y \end{array}\right\} x_1 \text{ and } y_1 \text{ the co-ordinates of } Pp;$$

$$\left.\frac{dy}{dx} = p\right\} \therefore (y_1 - y) = A \, . \, (x_1 - x) \text{ is its equation,}$$

where $A = \tan Ppx$.

But $Ppx = APp + PAG = 2APG + PAG$,

and $2APG = 2(\pi - PAG - AGP)$;

$\therefore 2APG + PAG = 2\pi - PAG - 2AGP$;

or $Ppx = 2\pi - \tan^{-1}\dfrac{y}{x} - 2\tan^{-1}\dfrac{1}{p} = (2\pi - M)$;

$\therefore A = \tan(2\pi - M) = -\tan M$

$$= -\frac{\dfrac{y}{x} + \dfrac{\frac{2}{p}}{1 - \frac{1}{p^2}}}{1 - \dfrac{y}{x} \, . \, \dfrac{\frac{2}{p}}{1 - \frac{1}{p^2}}} = -\frac{\dfrac{y}{x} + \dfrac{2p}{p^2 - 1}}{1 - \dfrac{y}{x}\dfrac{2p}{p^2 - 1}}$$

$$= \frac{2px - y(1-p^2)}{2py + x(1-p^2)};$$

$$\text{and } \therefore\ y_1 - y = \frac{2px - y(1-p^2)}{2py + x(1-p^2)}(x_1 - x).$$

COR. If $y_1 = 0$ $x_1 = Ap = \dfrac{2.(py+x)(px-y)}{2px - y(1-p^2)}$.

This is an optical problem, giving the focus A and the equation to ray AP, to find the equation to the reflected ray; p is the intersection of the reflected ray with the axis.

EXAMPLES.

(1) Let $y^n = a^{n-1}x$; $NT = nx$; $NG = \dfrac{y^2}{nx}$.

If $n = 2$, the curve is the parabola; $NT = 2x$, $NG = \dfrac{a}{2}$.

(2) Let the curve be the witch:

$$NT = -\frac{2ax - x^2}{a}; \quad NG = -\frac{4a^3}{x^2}.$$

(3) The focus of a parabola is in the centre of a given circle, its vertex bisects the radius, find the point and angle of intersection of circle and parabola.

(4) Shew that the normal to the curve defined by $y^2 = 4ax$, is a tangent to the curve defined by $y^2 = \dfrac{4}{27a}(x-2a)^3$; and that when the curves intersect $x = 8a$.

(5) If $y^2 = 4a(x+a)$ be the equation to a parabola, the origin in the focus, shew that the points of intersection of the tangents, and perpendiculars from the focus, are determined by the equations

$$x_1 = -a, \text{ and } y_1 = \frac{y}{2}.$$

CHAPTER X.

THE DIFFERENTIALS OF THE AREAS AND LENGTHS OF CURVES: OF THE SURFACES AND VOLUMES OF SOLIDS OF REVOLUTION: SPIRALS.

123. ONE of the applications of the Integral Calculus is to find the areas of curves included between given ordinates, the lengths of their arcs, and the surfaces and contents of solids.

The solids of which we shall treat are called solids of revolution, since they may be supposed to be generated by the revolution of a plane figure round a line, thus termed an axis. Hence it follows that every section perpendicular to the axis will be a circle, the radius of which is the revolving ordinate, and every section made by a plane passing through the axis will reproduce the original area.

Considering the areas and lengths of curves, and the contents and surfaces of solids, to be functions of one of the quantities x or y, we can, by the Differential Calculus, find equations between the differential coefficients of these functions, and expressions containing x or y, by which we shall hereafter obtain the values of the functions themselves.

We shall find it useful first to establish the truth of the following Proposition.

124. If $A + Bx$, $A_1 + B_1x$, and $A + bx$, be three algebraical expressions taken in order of magnitude, viz.

$$A_1 + B_1x < A + Bx, \text{ but } > A + bx,$$

then shall $A_1 = A$.

For if A do not equal A_1,

$$\text{since } A + Bx > A_1 + B_1 x,$$

$$\text{and } A_1 + B_1 x > A + bx;$$

$$\therefore\ A - A_1 + (B_1 - B)x \text{ is } > 0,$$

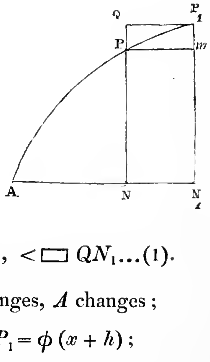
$$\text{and } A_1 - A + (B_1 - b)x \quad \text{is} > 0,$$

whatever x be; but if we make $x = 0$, we have

$$A - A_1 > 0,$$

$$\text{and } A_1 - A > 0;$$

or both $(A - A_1)$ and $-(A - A_1)$ are at the same time >0: an absurdity, unless $A_1 = A$.



125. Let AP be a curve, and $y = f(x)$, the equation to it, where $AN = x$, $NP = y$; and let A = area ANP.

$$\text{Then } \frac{dA}{dx} = y.$$

Let $NN_1 = h_1$. Complete the parallelograms QN_1 and PN_1.

Then the area P_1PNN_1 is $> \square\, PN_1$, $< \square\, QN_1 \ldots (1)$.

Now A depends upon x, for as x changes, A changes;

$$\therefore\ A = ANP = \phi(x); \text{ and } \therefore\ AN_1P_1 = \phi(x + h);$$

$$\therefore\ PP^1N_1N = \phi(x + h) - \phi(x) = \frac{dA}{dx}h + \frac{d^2A}{dx^2}\frac{h^2}{1.2} + \&c.$$

by Taylor's Theorem;

and $\square\, PN_1 = yh$,

$$\square\, QN_1 = h \times P_1N_1 = h.f(x+h) = h\,\{y + ph + Ph^2\},\ p = \frac{dy}{dx};$$

therefore, dividing by h, we have by (1),

$$\frac{dA}{dx} + \frac{d^2A}{dx^2}\frac{h}{1.2} + \&c. > y < y + ph + Ph^2;$$

i. e. $y + ph + Ph^2$, $\dfrac{dA}{dx} + \dfrac{d^2A}{dx^2}\dfrac{h}{1\,.\,2} + \text{\&c.}$ and y

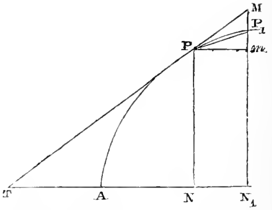
are in order of magnitude; whence, by the Lemma,

$$\frac{dA}{dx} = y.$$

126. If s = length of the curve AP;

$$\frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}}.$$

Draw the tangent PM, and chord PP'.



Then

$$\text{arc } PP' > \text{chord } PP' < PM + MP_1.$$

But arc $PP' = AP_1 - AP = \phi(x+h) - \phi(x) = \dfrac{ds}{dx}h + \dfrac{d^2s}{dx^2}\dfrac{h^2}{1\,.\,2} + \text{\&c.}$

$$\text{chord } PP' = \sqrt{Pm^2 + (P'm)^2}^{\,*} = \sqrt{h^2 + (ph + Ph^2)^2}$$

$$= h\sqrt{(1 + p^2) + 2Pph + P^2h^2},$$

$$PM = \sqrt{Pm^2 + Mm^2}^{\,*} = \sqrt{h^2 + p^2h^2} = h\sqrt{1 + p^2},$$

$$MP_1 = MN_1 - N_1P_1 = (y + ph) - (y + ph + Ph^2) = -Ph^2;$$

whence, dividing by h

$$\frac{ds}{dx} + \frac{d^2s}{dx^2}\cdot\frac{h}{1\,.\,2} + \text{\&c.} \overset{+}{>} \sqrt{1 + p^2 + 2Pph + P^2h^2} < \sqrt{1 + p^2} - Ph$$

$$> \sqrt{1 + p^2} + \frac{Pp}{\sqrt{1 + p^2}}h + \text{\&c.} < \sqrt{1 + p^2} - Ph;$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + p^2} = \sqrt{1 + \frac{dy^2}{dx^2}}.$$

* $P_1m = P_1N_1 \quad PN = y + ph + Ph^2 - y = ph + Ph^2$,

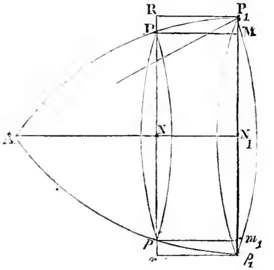
$Mm = Pm\,.\,\tan MPm = h\,.\,\tan PTN = h\,.\,\dfrac{dy}{dx}.$

$\sqrt{1+p^2+2Pph+P^2h^2} = \{\overline{(1+p^2)+2Pph} + P^2h^2\}^{\frac{1}{2}} = ((1+p^2)+2Pph)^{\frac{1}{2}} + \text{\&c}.$

$(\sqrt{1+p^2} + \dfrac{2Pph}{2\,.\,\sqrt{1+p^2}} + \text{\&c}) + \text{\&c} = \sqrt{1+\dfrac{dy^2}{dx^2}} + \dfrac{Pph}{\sqrt{\ldots}} + \text{\&c}$

127. If V be the volume of a solid of revolution APp,

$$\frac{dV}{dx} = \pi y^2.$$



Let $\left.\begin{array}{l} AN = x \\ NP = y \\ NN_1 = h \end{array}\right\}$; $\therefore AP_1p_1 = f(x+h) = V + \frac{dV}{dx}h + \frac{d^2V^2}{dx^2}\frac{h^2}{1.2} + \&c.$

Then the solid Ppp_1P_1 is $>$ cylinder PMm_1p,

$$< \text{cylinder } RP_1p_1r;$$

$$\text{i. e. } \frac{dV}{dx}h + \frac{d^2V}{dx^2}\frac{h^2}{1.2} + \&c. > \pi y^2 h,$$

$$< \pi(y + ph + Ph^2)^2 h,.$$

$$\text{or } \frac{dV}{dx} + \frac{d^2V}{dx^2}\frac{h}{1.2} + \&c. > \pi y^2 < \pi(y + ph + Ph^2)^2,$$

$$\text{or } > \pi y^2 < \pi y^2 + 2\pi pyh + \&c.$$

$$\text{whence } \frac{dV}{dx} = \pi y^2.$$

PROP. The surface of a truncated cone, of which the radii of the greater and smaller ends are a, b, and the slant side s, $= \pi s(a + b)$.

Let l = length of cone, radius of the base $= a$,

$l_1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = b$;

therefore, surface* of frustum

$$= \pi l a - \pi l_1 b = \pi \{sa + l_1 (a - b)\},$$

but l or $l_1 + s : l_1 :: a : b$;

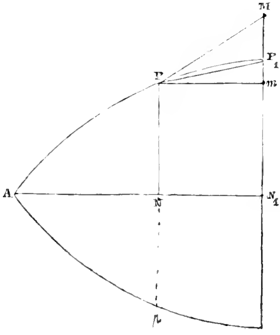
$$\therefore s : l_1 :: a - b : b;$$

$$\therefore sb = l_1 (a - b);$$

$\therefore$ surface of frustum $= \pi s . (a + b)$.

128. If $S =$ surface of the solid of revolution APp,

$$\frac{dS}{dx} = 2\pi y \sqrt{1 + \frac{dy^2}{dx^2}}.$$



$$\left.\begin{aligned} AN &= x \\ NP &= y \\ AP &= s \\ NN_1 &= h \end{aligned}\right\}.$$ Draw the tangent PM, and chord PP'.

* The surface of a cone when unwrapped coincides with the sector of a circle, the centre of which is the vertex of the cone, and radius the slant side, and arc or base, the circumference of the base of the cone.

But area of sector $= \frac{\text{base} \times \text{rad.}}{2} = \frac{1}{2}$ circumference of the base of cone $\times$ slant side;

or if (s) be the slant side, and (a) the radius of the cone's base,

convex surface of cone $= \frac{1}{2} . 2\pi as = \pi as$.

Then, surface generated by arc PP' will be

$>$ than that by the chord PP',

$<$ by PM and MP'.

Now chords PP_1 and PM generate truncated cones, of which the surfaces respectively are

$$\pi\{PN + P_1N_1\} PP', \quad \text{and } \pi\{PN + MN_1\} PM;$$

and MP' will generate a circular zone $= \pi(MN_1^2 - N_1P_1^2)$; and the surface generated by arc PP_1

$$= \frac{dS}{dx}h + \frac{d^2S}{dx^2}\frac{h^2}{1.2} + \&c.$$

But $\{PN + P_1N_1\} PP'$

$$= (2y + ph + Ph^2) . \sqrt{h^2 + (ph + Ph^2)^2}$$

$$= (2y + ph + Ph^2)h\sqrt{1 + p^2 + Mh},$$

$Mh =$ terms involving h;

and $(PN + MN_1) PM$

$$= (2y + ph) . \sqrt{h^2 + p^2h^2} = (2y + ph)h\sqrt{1 + p^2},$$

also $MN_1^2 - N_1P_1^2$

$$= (y + ph)^2 - (y + ph + Ph^2)^2 = - Ph^2(2y + 2py + Ph^2)$$

$$= - Nh^2, \text{ by substitution;}$$

$$\therefore \frac{dS}{dx} + \frac{d^2S}{dx^2}\frac{h}{1.2} + \&c. > \pi(2y + ph + Ph^2)\sqrt{1 + p^2 + Mh}$$

$$< \pi(2y + ph)\sqrt{1 + p^2 - Nh}$$

$$> 2\pi y\sqrt{1 + p^2 + Mh} + \text{terms involving } h,$$

$$< 2\pi y\sqrt{1 + p^2} + ph\sqrt{1 + p^2 - Nh}$$

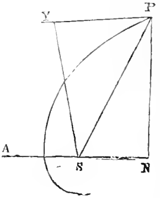
$$> 2\pi y\sqrt{1 + p^2} + \frac{yMh}{\sqrt{1 + p^2}} + \&c.$$

$$< 2\pi y\sqrt{1 + p^2} + ph\sqrt{1 + p^2} - Nh;$$

$$\therefore \frac{dS}{dx} = 2\pi y \sqrt{1+p^2} = 2\pi y \sqrt{1+\frac{dy^2}{dx^2}}.$$

129. The expressions that we have just obtained, and those of the preceding Chapter, are only applicable to the cases where the equation to the curve is known in terms of the rectangular co-ordinates; we shall now find corresponding expressions for the perpendicular upon the tangent, the area and length of a curve, &c. when referred to polar co-ordinates; that is, when $r = f(\theta)$, or $p = f(r)$, p being the perpendicular on the tangent, r the radius vector, and θ the angle traced out by r.

First, to find the expression for the perpendicular on the tangent in polar curves.



$SN = x$, $SP = r$,

$NP = y$, $SY = p$, and $\angle ASP = \theta$.

$$\text{Now, Art. 117, } SY = p = \frac{y - x\dfrac{dy}{dx}}{\sqrt{1+\dfrac{dy^2}{dx^2}}}.$$

$$\text{But } x = SP \cos PSN = -r\cos\theta = f(\theta),$$

$$y = SP \sin PSN = +r\sin\theta = \phi(\theta),$$

$$\text{and } \frac{dy}{dx} = \left(\frac{dy}{d\theta}\right)\left(\frac{d\theta}{dx}\right) = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}};$$

$$\therefore p = \frac{y\,.\dfrac{dx}{d\theta} - x\dfrac{dy}{d\theta}}{\sqrt{\dfrac{dx^2}{d\theta^2}+\dfrac{dy^2}{d\theta^2}}}.$$

$$\text{But } \frac{dx}{d\theta} = +r\sin\theta - \cos\theta\,.\frac{dr}{d\theta},$$

$$\text{and } \frac{dy}{d\theta} = r\cos\theta + \sin\theta\,.\frac{dr}{d\theta};$$

$$\therefore\ y\frac{dx}{d\theta} - x\frac{dy}{d\theta} = r^2\sin^2\theta + r^2\cos^2\theta = r^2,$$

$$\frac{dx^2}{d\theta^2} + \frac{dy^2}{d\theta^2} = r^2(\sin^2\theta + \cos^2\theta) + \frac{dr^2}{d\theta^2}.(\sin^2\theta + \cos^2\theta)$$

$$= r^2 + \frac{dr^2}{d\theta^2};$$

$$\therefore\ p = \frac{r^2}{\sqrt{r^2 + \dfrac{dr^2}{d\theta^2}}};$$

whence p may be found in terms of r and θ; but the formula may be put under a form more convenient for practice. Thus,

$$\because\ \frac{1}{p^2} = \frac{r^2 + \dfrac{dr^2}{d\theta^2}}{r^4} = \frac{1}{r^2} + \frac{1}{r^4}.\frac{dr^2}{d\theta^2}.$$

Let $r = \dfrac{1}{u}$; $\therefore\ u = \dfrac{1}{r}$; $\therefore\ \dfrac{du}{d\theta} = -\dfrac{dr}{r^2.d\theta}$;

$$\therefore\ \frac{1}{p^2} = u^2 + \frac{du^2}{d\theta^2}.$$

EXAMPLE. Find the value of p in the Conic Sections.

$$r = \frac{m}{1 + e\cos\theta},\ \text{where } m = \tfrac{1}{2}\text{ latus rectum};$$

$$\therefore\ u = \frac{1}{m} + \frac{e}{m}.\cos\theta,$$

$$\frac{du}{d\theta} = -\frac{e}{m}.\sin\theta;$$

$$\therefore\ u^2 + \frac{du^2}{d\theta^2} = \frac{1}{m^2}.\{1 + 2e\cos\theta + e^2\}$$

$$= \frac{1}{m^2}.(2mu - 1 + e^2);\quad \because\ e\cos\theta = mu - 1,$$

$$= \frac{1}{m^2} \cdot \left\{ \frac{2m - r(1 - e^2)}{r} \right\};$$

$$\therefore \; p^2 = \frac{m^2 . r}{2m - r(1 - e^2)}.$$

(1) In parabola, $e = 1$; $\therefore \; p^2 = \frac{mr}{2}$, and $m = 2SA$;

$$\therefore \; SY^2 = SP . SA.$$

(2) In ellipse, $e < 1$; $m = \frac{b^2}{a}$; $1 - e^2 = \frac{b^2}{a^2}$;

$$\therefore \; p^2 = \frac{\frac{b^4}{a^2} . r}{2\frac{b^2}{a} - \frac{b^2}{a^2} . r} = \frac{b^2 r}{2a - r}.$$

(3) In hyperbola, $e^2 > 1$; $e^2 - 1 = \frac{b^2}{a^2}$;

$$\therefore \; p^2 = \frac{m^2 r}{2m + r(e^2 - 1)} = \frac{b^2 r}{2a + r};$$

and therefore in ellipse and hyperbola, $SY^2 = \frac{BC^2 . SP}{HP}$.

Cor. 1. Since if $s =$ arc of a curve,

$$\frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}}.$$

If s, x, y be functions of θ,

$$\frac{ds}{dx} = \frac{ds}{d\theta} \cdot \frac{d\theta}{dx}, \quad \text{and} \quad \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx};$$

$$\therefore \; \frac{ds}{d\theta} = \sqrt{\frac{dx^2}{d\theta^2} + \frac{dy^2}{d\theta^2}} = \sqrt{r^2 + \frac{dr^2}{d\theta^2}},$$

which is the differential coefficient of $s = f(\theta)$.

Cor. 2. Assuming the expression $p = \dfrac{r^2}{\sqrt{r^2 + \dfrac{dr^2}{d\theta^2}}}$,

$$\therefore \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\,\frac{dr^2}{d\theta^2};$$

$$\therefore \frac{dr^2}{d\theta^2} = r^4\left(\frac{1}{p^2} - \frac{1}{r^2}\right) = r^2.\frac{r^2 - p^2}{p^2};$$

$$\therefore \frac{d\theta}{dr} = \frac{p}{r\sqrt{r^2 - p^2}};$$

whence given $\theta = f(r)$, we may find $p = \phi(r)$.

Cor. 3. Since $\dfrac{ds}{d\theta} = \dfrac{ds}{dr} \times \dfrac{dr}{d\theta}$,

and $\dfrac{ds}{d\theta} = \sqrt{r^2 + \dfrac{dr^2}{d\theta^2}} = \dfrac{dr}{d\theta}\sqrt{1 + r^2.\dfrac{d\theta^2}{dr^2}}$;

$$\therefore \frac{ds}{dr} = \sqrt{1 + r^2\frac{d\theta^2}{dr^2}} = \sqrt{1 + \frac{p^2}{r^2 - p^2}}$$

$$= \frac{r}{\sqrt{r^2 - p^2}}.$$

Cor. 4. If A = area ASP, $\dfrac{dA}{d\theta} = \dfrac{r^2}{2}$.

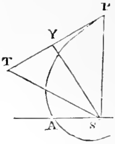
For $ASP = ANP - SNP = ANP - \dfrac{yx}{2}$;

$$\therefore \frac{dA}{d\theta} = y.\frac{dx}{d\theta} - \tfrac{1}{2}.\left(y\frac{dx}{d\theta} + x\frac{dy}{d\theta}\right) = \tfrac{1}{2}\left(y\frac{dx}{d\theta} - x\frac{dy}{d\theta}\right)$$

$= \frac{1}{2}r^2$. (Art. 129.)

130. To draw a tangent to a spiral.

P the point to which the tangent is to be drawn.



S the pole. Join SP. Suppose PT to be the tangent. Draw SY perpendicular to PT, and ST perpendicular to PS.

ST is called the sub-tangent.

$$\text{And } ST = SP \cdot \frac{SY}{PY} = \frac{pr}{\sqrt{r^2 - p^2}} = \text{also } r^2 \cdot \frac{d\theta}{dr}.$$

Find therefore from the equation to the spiral $\dfrac{pr}{\sqrt{r^2 - p^2}}$, or $r^2 \cdot \dfrac{d\theta}{dr}$, according as the equation is $\theta = f(r)$, or $p = f(r)$.

Draw ST perpendicular to SP and equal to either of these values.

Join TP, it is the tangent.

Cor. Since $ST = \pm r^2 \dfrac{d\theta}{dr} = \mp \dfrac{d\theta}{du}$,

$\because r = \dfrac{1}{u} \therefore \dfrac{r d\theta}{dr} = \dfrac{d\theta}{du} \cdot \dfrac{du}{dr}$ & $\dfrac{du}{dr} = \dfrac{\mp 1}{r^2}$ $\therefore r^2 \dfrac{d\theta}{dr} = \mp \dfrac{d\theta}{du}$.

$$\therefore \frac{1}{ST^2} = \left(\frac{du}{d\theta}\right)^2;$$

$$\therefore \frac{1}{SY^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{1}{SP^2} + \frac{1}{ST^2}.$$

131. Asymptotes to Spirals.

If ST remain finite when SP is infinite, a tangent may be drawn which will touch the curve at a point infinitely distant from S, and is therefore an asymptote. And since those lines are said to be parallel which coincide only at an infinite distance; the asymptote must be drawn parallel to the infinite line SP.

Hence to construct, we must find θ and $r^2 \dfrac{d\theta}{dr}$ when r is infinite. Draw SP at the angle found by making $r = \infty$, ST perpendicular to SP, and from T draw TP parallel to the infinite radius vector, TP produced is the asymptote.

EXAMPLES.

(1) Find the equation between p and r, when $\theta = \dfrac{a^n}{r^n}$,

$$\theta = \frac{a^n}{r^n} = a^n u^n ;$$

$$\therefore\ u^n = \frac{\theta}{a^n};$$

$$\therefore\ \frac{du}{d\theta} = \frac{1}{na^n . u^{n-1}} = \frac{r^{n-1}}{b^n} \text{ by substitution};$$

$$\therefore\ u^2 + \frac{du^2}{d\theta^2} = \frac{1}{r^2} + \frac{r^{2n-2}}{b^{2n}} = \frac{1}{r^2}\left\{\frac{b^{2n} + r^{2n}}{b^{2n}}\right\};$$

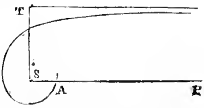
$$\therefore\ p = \frac{b^n . r}{\sqrt{b^{2n} + r^{2n}}}.$$

(2) Draw a tangent and asymptote to the spiral; where

$$\theta = \frac{a}{r} = au;$$

$$\therefore\ \frac{1}{ST} = \frac{du}{d\theta} = \frac{1}{a}; \quad \therefore\ ST = a;$$

or the locus of T is a circle radius $= a$.



Since ST is constant, and $\theta = 0$ when $r = \infty$.

Produce SA indefinitely. Draw ST perpendicular to it and $= a$. Then a line from T parallel to SP will be the asymptote required.

(3) If $r^2 = a^2 \cos 2\theta$ which is the polar equation to the Lemniscata; find equation between p and r.

$$\text{Here } u^2 = \frac{1}{a^2 \cos 2\theta};$$

$$\therefore \cos 2\theta = \frac{1}{u^2 a^2};$$

$$\therefore \frac{d\theta}{du} = \frac{1}{u^3 a^2 \sin 2\theta},$$

$$\text{and } \sin 2\theta = \sqrt{1 - \frac{1}{a^4 u^4}} = \frac{\sqrt{a^4 u^4 - 1}}{a^2 u^2};$$

$$\therefore \frac{d\theta}{du} = \frac{1}{u\sqrt{a^4 u^4 - 1}};$$

$$\therefore \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 = u^2 + a^4 u^6 - u^2$$

$$= a^4 u^6 = \frac{a^4}{r^6};$$

$$\therefore p = \frac{r^3}{a^2}.$$

(4) Let $r = a^\theta$, the equation to the logarithmic spiral;

$$\therefore \text{h. l. } r = \theta \text{ h. l. } a = A\theta;$$

$$\therefore \frac{dr}{d\theta} = Ar;$$

$$\therefore \frac{d\theta}{dr} \text{ or } \frac{p}{r\sqrt{r^2 - p^2}} = \frac{1}{Ar};$$

$$\therefore \frac{p}{\sqrt{r^2 - p^2}} = \frac{1}{A} \text{ or } \frac{SY}{PY} = \tan SPY = \frac{1}{A};$$

that is, $\angle SPY$ is constant, and on this account the curve is called the equiangular spiral.

Cor. 1. Since $\dfrac{\sqrt{r^2 - p^2}}{p} = A$; $\therefore \dfrac{r^2}{p^2} = 1 + A^2$;

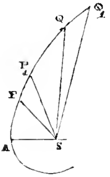
$$\therefore \frac{p}{r} = \sin SPY = \frac{1}{\sqrt{1 + A^2}};$$

$\therefore p = \dfrac{r}{\sqrt{1 + A^2}} = mr$, by substitution.

Cor. 2. The radii including equal angles are proportional.

Let SP and SP_1 including an $\angle\, a$,

and SQ and SQ_1 include the same angle.



$$\text{Let } \angle\, ASP = \theta,$$

$$\text{and } ASQ = \phi;$$

$$\therefore SP = a^\theta, \qquad SQ = a^\phi,$$

$$SP_1 = a^{\theta + a}, \quad SQ_1 = a^{\phi + a};$$

$$\therefore \frac{SP_1}{SP} = a^a, \quad \text{and } \frac{SQ_1}{SQ} = a^a;$$

$$\therefore \frac{SP}{SP_1} = \frac{SQ}{SQ_1},$$

$$\text{or } SP : SP_1 :: SQ : SQ_1.$$

Cor. 3. Given the ratio of SP and SP_1, which include an angle a, find a.

$$\text{Let } SP : SP_1 :: 1 : 1 + c.$$

$$\text{But } SP = a^\theta, \text{ and } SP_1 = a^{\theta + a};$$

$$\therefore \frac{SP_1}{SP} = 1 + c = a^a;$$

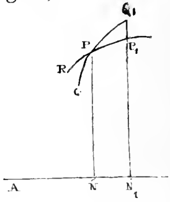
$\therefore$ h. l. $(1 + c) = a$ h. l. $a = aA = \dfrac{a}{\tan\beta}$, if $\beta =$ constant $\angle\, SPY$,

$$\text{or } a = \tan\beta \,.\, \text{h. l. } (1 + c).$$

CHAPTER XI.

CURVATURE AND OSCULATING CURVES.

132. When two curves, as QPQ_1, RPP_1, cut each other in the manner represented in the figure, the values of y and x are the same for both curves at the point of intersection; i. e. if $y=f(x)$ be the equation to the curve RPP_1, and $y=\phi(x)$ the equation to QPP_1, and $AN=a$, and $NP=b$, the values a, and b put for x and y will make the equations $b=f(a)$ and $b=\phi(a)$ true equations, and $\therefore f(a)=\phi(a)$.



133. But if for x, $a+h$, be written, (or as we shall put it, $x+h$,) the values of the ordinates of the two curves no longer become equal, and their difference, which is represented in the figure by P_1Q_1, is equal to the difference between $f(x+h)$ and $\phi(x+h)$, and will therefore be some function of h, and its value will depend upon the relations existing between the differential coefficients of $f(x)$ and $\phi(x)$.

For, let $y_1=N_1P_1$, $y_2=N_1Q_1$, $z=f(x)$, and $v=\phi(x)$;

$$\therefore y_1 = y + \frac{dz}{dx}h + \frac{d^2z}{dx^2}\frac{h^2}{1.2} + \frac{d^3z}{dx^3}\frac{h^2}{2.3} + \&c.,$$

$$\text{and } y_2 = y + \frac{dv}{dx}h + \frac{d^2v}{dx^2}\frac{h^2}{1.2} + \frac{d^2v}{dx^3}\frac{h^3}{2.3} + \&c.;$$

$$\therefore P_1Q_1 = \left(\frac{dv}{dx} - \frac{dz}{dx}\right)h + \left(\frac{d^2v}{dx^2} - \frac{d^2z}{dx^2}\right)\frac{h^2}{1.2} + \&c.;$$

or putting $A_1\,A_2\,A_3$, &c. A_n for the coefficients of h, h^2, h^3, &c.

The distance Δ between the curves, or the difference between the ordinates, is represented by a series with ascending powers of h, so that

$$\Delta = A_1 h + A_2 h^2 + A_3 h^3 + A_4 h^4 + \&c. + A_n h^n + \&c.$$

Cor. 1. First, let $A_1 = 0$; $\therefore \dfrac{dv}{dx} = \dfrac{dz}{dx}$, or the first differential coefficients are equal.

But $\dfrac{dv}{dx}$ and $\dfrac{dz}{dx}$ represent the trigonometrical tangents of the angles which the tangents of the two curves at the point P make with the axis of x.

Hence at such a point the ordinates are equal, and the tangents are coincident.

This is called a contact of the *first* order.

Cor. 2. Let not only $A_1 = 0$, but $A_2 = 0$, therefore we have

$$f(x) = \phi(x), \quad \frac{df(x)}{dx} = \frac{d\phi(x)}{dx}, \text{ and } \frac{d^2 f(x)}{dx^2} = \frac{d^2\phi(x)}{dx^2},$$

This is called a contact of the *second* order.

And in general the curves are said to have a contact of the n^{th} order when the first power of h, in the expression for Δ is h^{n+1}; i. e. when all the differential coefficients as far as the $(n+1)^{\text{th}}$ are respectively equal in both series.

134. To find the degree of contact which a proposed curve of given species has with a given curve of known dimensions.

Let $y = f(x)$ be the equation to the given curve, and $y_1 = \phi(x_1)$ the equation to the proposed curve, which is supposed to contain n arbitrary constants.

Then, to determine these n constants, we must have the n equations

$$y = y_1, \quad \frac{dy}{dx} = \frac{dy_1}{dx_1}, \quad \frac{d^2y}{dx^2} = \frac{d^2y_1}{dx^2_1}, \quad \text{and} \quad \frac{d^{n-1}y}{dx^{n-1}} = \frac{d^{n-1}y_1}{dx_1^{n-1}};$$

or the contact must be of the $(n-1)^{\text{th}}$ order.

Thus, let it be required to find the degree of contact which a straight line may have with a given curve; we observe that the equation to the line is $y_1 = ax_1 + b$, and contains two arbitrary constants a, b, or the contact may be of the first order.

Next to determine the line which has a contact of the first order with a curve.

In this example $\frac{dy}{dx} = \frac{dy_1}{dx_1} = a$; and $\therefore\ y = y_1$, and $x = x_1$,

$$\therefore\ y = ax + b, \text{ or } b = (y - ax) = y - \frac{dy}{dx}x;$$

therefore substituting for a, and b,

$$y_1 = \frac{dy}{dx}x_1 + y - x\frac{dy}{dx},$$

or $y_1 - y = \frac{dy}{dx}(x_1 - x)$, which is the equation to the tangent, or the tangent has a contact of the first order, with the curve which it touches.

135. In the circle of which the equation is

$$R^2 = (x_1 - a)^2 + (y_1 - \beta)^2$$

there are three arbitrary constants, the radius R and the co-ordinates of the centre a and β. The circle therefore may have a contact of the second order, and the constants may be determined by means of the equations

$$y = y_1, \quad \frac{dy}{dx} = \frac{dy_1}{dx_1}, \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d^2y_1}{dx_1^2}.$$

The circle so found is called the *circle of curvature*, and its radius the radius of curvature of any point in a given curve.

For since the curvature in the same circle is uniform, while it varies inversely as the radius in different circles, and that curves are *geometrically* said to have the same curvature, when at a common point, they have the same tangent, and ultimately the same deflection from the tangent, which conditions are both fulfilled by the circle that has a contact of the second order; this circle is assumed to be the proper measure of curvature, and curves are said to have the same or different curvature, according as the radii of these circles are the same or different, and the curvature in general

$$\propto \frac{1}{\text{radius of curvature}}.$$

The circle of curvature is also called the osculating circle.

136. To find the radius of curvature, and co-ordinates of the centre of the osculating circle to any proposed curve.

Let $y = f(x)$ be the equation to a given curve,

$$R^2 = (x_1 - a)^2 + (y_1 - \beta)^2 \text{ the equation to the circle;}$$

$$\therefore\ 0 = (x_1 - a) + (y_1 - \beta) \cdot \frac{dy_1}{dx_1} \dots\dots\dots\dots (1),$$

$$\therefore x_1 - a = -(y_1 - \beta) \frac{dy_1}{dx_1}$$

$$\text{and } 0 = 1 + \frac{dy_1^2}{dx_1^2} + (y_1 - \beta) \cdot \frac{d^2 y_1}{dx_1^2} \dots\dots\dots\dots (2).$$

$$\therefore y_1 - \beta = -\frac{1 + \frac{dy_1^2}{dx_1^2}}{\frac{d^2 y_1}{dx_1^2}}$$

$$\text{But } y = y_1,\quad x = x_1,\ \frac{dy}{dx} = \frac{dy_1}{dx_1},\quad \text{and } \frac{d^2 y}{dx^2} = \frac{d^2 y_1}{dx_1^2};$$

$\therefore$ changing x_1 into x, and y_1 into y;

$$\therefore\ R^2 = (x - a)^2 + (y - \beta)^2$$

$$= (y - \beta)^2 \cdot \left\{1 + \frac{dy^2}{dx^2}\right\} \text{ from} \dots\dots\dots (1).$$

$$= \frac{\left(1 + \frac{dy^2}{dx^2}\right)^2}{\left(\frac{d^2y}{dx^2}\right)^2} \cdot \left(1 + \frac{dy^2}{dx^2}\right) \text{ from......(2).}$$

$$= \frac{\left(1 + \frac{dy^2}{dx^2}\right)^3}{\left(\frac{d^2y}{dx^2}\right)^2};$$

$$\therefore\ R = \pm \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

This expression has two signs; but if we call the radius positive, where the curve is concave to the axis, or when $\frac{d^2y}{dx^2}$ is negative; and if, on the contrary, when the curve is convex, or when $\frac{d^2y}{dx^2}$ is positive, the radius be reckoned negative, we shall always have

$$R = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}}.$$

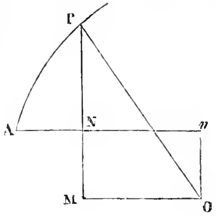
The co-ordinates α and β may be found from the equations

$$y - \beta = -\frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}};$$

and $x - \alpha = -(y - \beta) \cdot \frac{dy}{dx} = + \frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}} \cdot \frac{dy}{dx};$

and the circle is thus completely determined.

In the annexed figure, let AP be the given curve, PO the radius of curvature, and O therefore the centre of the osculating circle.



$$An = a,$$

$$nO = -\beta;$$

$$\therefore\ PM = y - \beta = -\frac{\left(1 + \frac{dy^2}{dx^2}\right)}{\frac{d^2y}{dx^2}},$$

$$OM = a - x = -\frac{\left(1 + \frac{dy^2}{dx^2}\right)}{\frac{d^2y}{dx^2}} \cdot \frac{dy}{dx}$$

PM and OM are respectively called the semi-chords perpendicular and parallel to the axis of x.

For if we describe the circle, of which the radius is OP and centre O, PM is half the chord of an arc, since OM is perpendicular to it, and OM is equal to half the chord drawn from P parallel to AN.

137. The point O changes its position with the change in the place of P, and traces out a curve, which is termed the *evolute* of the original curve. Hence we may define the evolute to be the locus of the centre of the circle of curvature, and its co-ordinates are a and β;

$$\text{and from } y - \beta = -\frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}},$$

$$x - a = -(y - \beta) \cdot \frac{dy}{dx} = \frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}} \frac{dy}{dx},$$

and $y = f(x)$; y, x, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ may be eliminated, and there will be an equation between a, β and constant quantities which is that of the evolute.

138. Since $x - a + (y - \beta) \cdot \frac{dy}{dx} = 0$;

$$\therefore\ y - \beta = -\frac{dx}{dy} \cdot (x - a);$$

but this is the equation to the normal of the original curve, drawn from a point of which the co-ordinates are x, y, and passing through a point whose co-ordinates are a and β. Hence the normal passes through the centre of the circle of curvature, and therefore the radius is coincident with the normal.

139. The radius of curvature is a tangent to the evolute.

Resuming the equation $(x - a) + (y - \beta) \cdot \frac{dy}{dx} = 0$.

Differentiate it, considering y, β and a as functions of x;

$$\therefore\ 1 - \frac{da}{dx} + (y - \beta) \cdot \frac{d^2y}{dx^2} + \frac{dy^2}{dx^2} - \frac{d\beta}{dx} \cdot \frac{dy}{dx} = 0.$$

$$\text{But } 1 + \frac{dy^2}{dx^2} + (y - \beta) \cdot \frac{d^2y}{dx^2} = 0;$$

$$\therefore\ -\frac{da}{dx} - \frac{d\beta}{dx} \cdot \frac{dy}{dx} = 0;$$

$$\therefore\ \frac{dy}{dx} = -\frac{\frac{da}{dx}}{\frac{d\beta}{dx}} = -\frac{da}{d\beta};$$

$$\therefore\ (x - a) - (y - \beta) \cdot \frac{da}{d\beta} = 0;$$

$$\text{or } (\beta - y) = \frac{d\beta}{da} \cdot (a - x),$$

which is the equation to the tangent drawn to a point, of which the co-ordinates are β and α, and passing through a point co-ordinates y, x.

$$\text{But } (\beta - y) = \frac{d\beta}{d\alpha}(\alpha - x) \text{ is identical with}$$

$$(y - \beta) = -\frac{dx}{dy}(x - \alpha),$$

or the equation to the normal of the original curve.

Hence the normal to the curve, i. e. the radius of curvature is the tangent to the evolute.

140. To find the length of the evolute.

$$\text{Since } R^2 = (x - \alpha)^2 + (y - \beta)^2;$$

differentiating, considering y, α, β, R, as functions of x,

$$R\frac{dR}{dx} = (x - \alpha)\left(1 - \frac{d\alpha}{dx}\right) + (y - \beta)\left(\frac{dy}{dx} - \frac{d\beta}{dx}\right)$$

$$= (x - \alpha) + (y - \beta)\frac{dy}{dx} - \left\{(x - \alpha)\cdot\frac{d\alpha}{dx} + (y - \beta)\frac{d\beta}{dx}\right\}.$$

$$\text{But } (x - \alpha) + (y - \beta)\cdot\frac{dy}{dx} = 0;$$

$$\therefore\ -R\frac{dR}{dx} = (x - \alpha)\cdot\frac{d\alpha}{dx} + (y - \beta)\cdot\frac{d\beta}{dx},$$

$$\text{or } -R\,dR = (x - \alpha)\,d\alpha + (y - \beta)\,d\beta.$$

$$\text{But } x - \alpha = -(y - \beta)\frac{dy}{dx} = +(y - \beta)\frac{d\alpha}{d\beta};$$

$$\therefore\ -R\,dR = (x - \alpha)\,d\alpha + (y - \beta)\,d\beta = \frac{(y - \beta)}{d\beta}\cdot(d\alpha^2 + d\beta^2)\ldots(1);$$

$$\text{and } R = \sqrt{(x - \alpha^2) + (y - \beta)^2} = (y - \beta)\sqrt{\frac{d\alpha^2}{d\beta^2} + 1},$$

$$= \frac{(y - \beta)}{d\beta}\sqrt{d\alpha^2 + d\beta^2}\ldots(2);$$

$\therefore$ dividing (1) by (2),

$$-dR = \sqrt{d\alpha^2 + d\beta^2} = ds\,;$$

$$\therefore\ dR \pm ds = 0\,;$$

$$\therefore\ R \pm s = \text{a constant.}$$

Hence if the curve be algebraical, R may be found in finite terms, and the length of the evolute known; that is, the evolutes of algebraic curves are rectifiable.

Cor. Let s_1 and s_2 be the lengths of the arcs of the evolute from its commencement to the points where the radii are γ_1 and γ_2;

$$\therefore\ \gamma_1 \pm s_1 = c,$$

$$\text{and}\ \ \gamma_2 \pm s_2 = c,$$

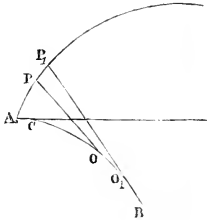
$$\gamma_1 - \gamma_2 \pm (s_1 - s_2) = 0,$$

$$\text{let}\ \ s_1 - s_2 = a\,;$$

$$\therefore\ \pm a = (\gamma_2 - \gamma_1),$$

or the difference in length between two radii of curvature equals the length of the arc of the evolute intercepted by them.

From this property of the curve it has derived the name of evolute.



For if we take a string of constant length, one end of which is fastened at B, and the remainder is made to coincide with the curve COO_1B, then if the string be unwrapped or evolved from COO_1B, it will describe the curve APP_1.

COB is called the evolute, and

APP_1 .... involute.

From this construction it is obvious,

(1) That the arc OO_1 is equal to $P_1O_1 - PO$.

(2) That O is the centre of a circle of which the radius is OP, and is consequently the radius of curvature to the point P.

(3) That PO is a tangent to the evolute.

(4) That PO is a normal to the curve.

141. Another geometrical method of finding the radius of curvature and the co-ordinates of the centre of the osculating circle is to assume that centre to be the limit of the intersections of two consecutive normals.

The truth of this assumption may be thus shewn:

$(y - \beta) = -\dfrac{dx}{dy} \cdot (x - \alpha)$ is the equation to the normal,

$$\text{or } (x - \alpha) + (y - \beta) \cdot \frac{dy}{dx} = 0.$$

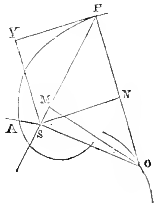
Now at the point of intersection, α and β remain the same for the two normals, while x, y and $\dfrac{dy}{dx}$ vary, since at a consecutive point, x and y become $x + dx$ and $y + dy$; therefore differentiating, considering α and β as constant,

$$1 + \frac{dy^2}{dx^2} + (y - \beta) \cdot \frac{d^2 y}{dx^2} = 0.$$

The same equation we have before obtained to find the co-ordinate β of the centre, and α is then known from

$$x - \alpha = -(y - \beta)\frac{dy}{dx}.$$

142. Hence may we find the radius of curvature in spirals.



AP the spiral, S the pole.

PO a normal, and O the point of ultimate intersection of two consecutive normals.

O is the centre of the circle of curvature.

$$\left.\begin{array}{l} SP = r,\ \ PO = R \\ SY = p,\ \ SO = r_1 \end{array}\right\},\ \ SN \perp \text{ on } PO = p_1.$$

$$\text{Now } SO^2 = SP^2 + PO^2 - 2\,PO\,.\,PN,$$

$$\text{or } r_1^{\,2} = r^2 + R^2 - 2R\,.\,p \text{ for } PN = SY.$$

Then since SO and OP remain constant, while SP and SY vary, and since $p = f(r)$;

$$\therefore\ 0 = r - R\frac{dp}{dr};$$

$$\therefore\ R = r\,.\frac{1}{\frac{dp}{dr}} = r\,.\frac{dr}{dp}.$$

If OM be drawn perpendicular to PS, or PS produced,

$$PM = \tfrac{1}{2} \text{ the chord of curvature through } S,$$

$$\text{and } PM = PO \times \frac{SY}{SP} = r\,.\frac{dr}{dp}\,.\frac{p}{r} = p\,.\frac{dr}{dp}.$$

143. Evolutes to spirals.

The point O will trace out the evolute, and PO is always a tangent to it, and SN is perpendicular to PO, we must therefore find the relation between SO and SN.

$$\text{Now } r_1^{\,2} = r^2 + R^2 - 2Rp \ldots\ldots (1),$$

$$\text{and } p_1 = PY = \sqrt{r^2 - p^2} \ldots\ldots (2).$$

$$\text{and } p = f(r) \ldots\ldots\ldots\ldots (3),$$

$$\text{and } R = r \cdot \frac{dr}{dp} \ldots\ldots\ldots\ldots (4),$$

between these equations p, r and R may be eliminated, and the resulting equation will involve r_1, p_1, and constant quantities, which will be the equation required.

Ex. Let the spiral be the equiangular.

$$\text{Here } p = r \sin \beta = mr;$$

$$\therefore R = \frac{rdr}{dp} = \frac{r}{m},$$

$$\text{and } p_1 = \sqrt{r^2 - p^2} = r\sqrt{1 - m^2};$$

$$\therefore R = \frac{p_1}{m\sqrt{1 - m^2}},$$

$$\text{and } p = mr = m^2 R = \frac{mp_1}{\sqrt{1 - m^2}}.$$

$$\text{But } r_1^2 = r^2 + R^2 - 2Rp;$$

$$\therefore r_1^2 = \frac{p_1^2}{1 - m^2} + \frac{p_1^2}{m^2(1 - m^2)} - \frac{2p_1^2}{1 - m^2}$$

$$= \frac{p_1^2}{1 - m^2}\left\{\frac{1}{m^2} - 1\right\} = \frac{p_1^2}{m^2};$$

$$\therefore p_1 = mr_1,$$

or the evolute is a spiral similar and equal to the original, and described round the same pole S.

144. When two curves intersect, we have seen that the distance between them, measured along the ordinate is, (when x becomes $x + h$) expressed by the equation

$$\Delta = A_1 h + A_2 h^2 + A_3 h^3 + A_4 h^4 + A_5 h^5 + \&c.$$

If therefore we put $(-h)$ for h, we shall have an expression for the distance between them at a point where the abscissa is $x - h$: let Δ_1 be this distance;

$$\therefore \Delta_1 = -A_1 h + A_2 h^2 - A_3 h^3 + A_4 h^4 - A_5 h^5 + \&c.$$

Now assuming that h may be taken so small that any one term shall exceed the sum of all that follow it. We observe *First*, that if $A_1 = 0$, Δ and Δ_1 have the same sign, or that in a contact of the first order, the curves touch, but do not intersect.

Thus the tangent does not cut the curve, unless $A_2 = 0$, or at a point of contrary flexure.

Secondly. Let both $A_1 = 0$ and $A_2 = 0$, or the contact be of the second order. Then

$$\Delta = A_3 h^3 + A_4 h^4 + \&c.$$

$$\Delta_1 = -A_3 h^3 + A_4 h^4 - \&c.,$$

which have different signs, and therefore if the osculating curve be below the given curve at a point where the abscissa is $x + h$, it will be above it at a point where x becomes $x - h$. Hence the circle of curvature both cuts and touches the curve.

There is an exception to this rule, which is when the radius of curvature is a maximum or minimum; for then $A_3 = 0$, and the expressions for Δ and Δ_1 have the same sign.

If the contact be of the third order,

$$\Delta = A_4 h^4 + A_5 h^5 + \&c.$$

$$\Delta_1 = A_4 h^4 - A_5 h^5 + \&c.;$$

that is, Δ and Δ_1 have the same sign, and therefore the osculating curve does not cut the given curve.

From this reasoning it is obvious that, when the contact is of an even order, the osculating curve both touches and cuts the given curve, but when the contact is of an odd order, it merely touches it.

145. When the radius of curvature is a maximum or minimum, the contact is of the third order, or $A_3 = 0$.

Let $p = \frac{dy}{dx}$, $q = \frac{d^2y}{dx^2}$, and $r = \frac{d^3y}{dx^3}$.

Then $R = \frac{(1+p^2)^{\frac{3}{2}}}{-q}$.

But if R be the maximum or minimum, $\frac{dR}{dx} = 0$;

$$\therefore\ 3\sqrt{1+p^2}\,.\,p - \frac{(1+p^2)^{\frac{3}{2}}}{q^2}\,r = 0,$$

$$\text{or}\ \ r = \frac{d^3y}{dx^3} = \frac{3pq^2}{1+p^2}.$$

But from the circle,

$$1 + p^2 + (y - \beta)\,q = 0,$$

and if there be a contact of the third order, we may differentiate this equation again, and put the co-ordinates of the curve for those of the circle;

$$\therefore\ 2pq + pq + (y - \beta)\,r = 0;$$

$$\therefore\ 3pq = -(y - \beta)\,.\,r,$$

$$\text{and}\ \ 1 + p^2 = -(y - \beta)\,.\,q;$$

$$\therefore\ \frac{3pq}{1+p^2} = \frac{r}{q};$$

$$\therefore\ r = \frac{3pq^2}{1+p^2}.$$

The same result as before, and therefore when A_3, which is the difference between the third differential coefficient of $y = f(x)$, and of $R^2 = (x - \alpha)^2 + (y - \beta)^2$, equals 0, or when the contact is of the third order, the radius of curvature is either a maximum or minimum.

EXAMPLES.

(1) Find the radius of curvature and evolute of the common parabola.

$$y^2 = 4ax,$$

$$\frac{dy}{dx} = +\frac{2a}{y},$$

$$\frac{d^2y}{dx^2} = -\frac{2a}{y^2} \times \frac{dy}{dx} = -\frac{4a^2}{y^3},$$

$$1 + \frac{dy^2}{dx^2} = 1 + \frac{4a^2}{y^2} = \frac{4a^2+y^2}{y^2} = \frac{4a\,(a+x)}{y^2}.$$

$$\text{But } R = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}} = \frac{\{4a\,.(a+x)\}^{\frac{3}{2}}}{4a^2} = \frac{2(a+x)^{\frac{3}{2}}}{\sqrt{a}}$$

$$= \frac{2}{\sqrt{a}}\,.\,SP^{\frac{3}{2}}.$$

$$\text{But } (y-\beta) = \frac{1+\frac{dy^2}{dx^2}}{-\frac{d^2y}{dx^2}} = \frac{4a^2+y^2}{y^2} \times \frac{y^3}{4a^2} = y\,.\,\frac{(a^2+ax)}{a^2}$$

$$= y + \frac{yx}{a};$$

$$\therefore\ -\beta = \frac{yx}{a} = \frac{y^3}{4a^2};$$

$$\therefore\ -y = (4a^2\beta)^{\frac{1}{3}}$$

$$x - \alpha = -\frac{dy}{dx}\,.\,(y-\beta) = -\frac{2a}{y}\,\frac{y\,(x+a)}{a} = -2\,(x+a);$$

$$\therefore\ 3x = \alpha - 2a, \quad \text{or } x = \left(\frac{\alpha - 2a}{3}\right).$$

But $y^2 = 4ax$;

$$\therefore\ (4a^2\beta)^{\frac{2}{3}} = \frac{4a}{3}(\alpha - 2a;)$$

$$\therefore\ 16a^4\beta^2 = \frac{64a^3}{27}(\alpha - 2a)^3;$$

$$\therefore\ \beta^2 = \frac{4}{27a}\cdot(\alpha - 2a)^3 = \frac{4}{27a}x_1^3,\ \text{by putting } \alpha - 2a = x_1.$$

The equation to the semi-cubical parabola.

(2) In the Conic Sections, the radius of curvature $\propto$ (normal)^3.

$$\text{Length of normal} = N = y\sqrt{1 + \frac{dy^2}{dx^2}};$$

$$\therefore\ \sqrt{1 + \frac{dy^2}{dx^2}} = \frac{N}{y};$$

$$\therefore\ R = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}} = \frac{N^3}{-y^3\frac{d^2y}{dx^2}}.$$

Now if the vertex be the origin, and the axis the axis of x,

$$y^2 = 2mx + nx^2;$$

$$\therefore\ y\frac{dy}{dx} = m + nx;$$

$$\therefore\ y\frac{d^2y}{dx^2} + \frac{dy^2}{dx^2} = n;$$

$$\therefore\ y^3\frac{d^2y}{dx^2} = ny^2 - y^2\frac{dy^2}{dx^2}$$

$$= n\,.\,(2mx + nx^2) - (m + nx)^2$$

$$= -m^2;$$

$$\therefore\ R = \frac{N^3}{m^2}.$$

(3) Find the radius of curvature to any point of an ellipse.

$$y = \frac{b}{a}\sqrt{a^2 - x^2};$$

$$\therefore \frac{dy}{dx} = -\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}}$$

$$-\frac{d^2y}{dx^2} = \frac{b}{a}\left(\frac{\sqrt{a^2 - x^2} + \dfrac{x^2}{\sqrt{a^2 - x^2}}}{a^2 - x^2}\right) = \frac{b}{a}\cdot\frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}} = \frac{ba}{(a^2 - x^2)^{\frac{3}{2}}},$$

$$1 + \frac{dy^2}{dx^2} = 1 + \frac{b^2x^2}{a^2(a^2 - x^2)} = \frac{a^4 - (a^2 - b^2)x^2}{a^2(a^2 - x^2)} = \frac{a^2 - e^2x^2}{a^2 - x^2};$$

$$\therefore R = \frac{(a^2 - e^2x^2)^{\frac{3}{2}}}{ba}.$$

Cor. Let R_1 be the radius at the vertex, and R_2 the radius at the extremity of the minor axis;

$$\therefore R_1 = \frac{(a^2 - a^2e^2)^{\frac{3}{2}}}{ba} = \frac{b^2}{a}, \text{ and } R_2 = \frac{a^3}{ba} = \frac{a^2}{b};$$

therefore, the length of the evolute of the elliptic quadrant

$$= R_2 - R_1 = \frac{a^2}{b} - \frac{b^2}{a} = \frac{a^3 - b^3}{ab};$$

and therefore the length of the whole evolute

$$= \frac{4}{ab}\cdot(a^3 - b^3).$$

If R be a maximum or minimum, $\dfrac{dR}{dx} = 0$;

$$\therefore -3\sqrt{a^2 - e^2x^2}\,.\,e^2x = 0;$$

$\therefore x = 0$, and $x^2 = \dfrac{a^2}{e^2}$; but $x = \dfrac{a}{e}$ or $> a$ is impossible,

$$\text{and} \quad \frac{d^2 R}{dx^2} = -3e^2\sqrt{a^2 - e^2x^2} + 3e^2x\frac{d}{dx}\sqrt{a^2 - e^2x^2}$$

$$= -3e^2a, \quad \text{if } x = 0;$$

therefore R is a maximum, when $x = 0$, or $y = \pm b$.

Hence, at the extremities of the minor axis the circle of curvature touches the ellipse.

(4) To find the equation to the evolute.

$$y - \beta = \frac{1 + \frac{dy^2}{dx^2}}{-\frac{d^2y}{dx^2}} = \frac{(a^2 - e^2x^2)\sqrt{a^2 - x^2}}{ba} = \frac{y(a^2 - e^2x^2)}{b^2};$$

$$\therefore \beta = -y \cdot \left\{\frac{a^2 - e^2x^2}{b^2} - 1\right\} = -\frac{ye^2}{b^2}(a^2 - x^2) = -\frac{y^3e^2 \cdot a^2}{b^4};$$

$$\therefore \left(\frac{y}{b}\right)^3 = -\frac{b\beta}{(ae)^2};$$

$$\therefore \left(\frac{y}{b}\right)^2 = \frac{(b\beta)^{\frac{2}{3}}}{(ae)^{\frac{4}{3}}},$$

$$x - \alpha = -(y - \beta) \cdot \frac{dy}{dx} = \frac{(a^2 - e^2x^2)\sqrt{a^2 - x^2}}{ba} \cdot \frac{b}{a} \cdot \frac{x}{\sqrt{a^2 - x^2}}$$

$$= \frac{(a^2 - e^2x^2)}{a^2}x = x - \frac{e^2x^3}{a^2};$$

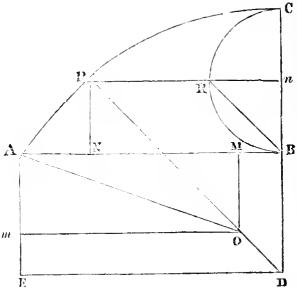
$$\therefore \left(\frac{x}{a}\right)^3 = \frac{\overset{+}{-}\alpha}{ae^2}; \quad \therefore \frac{x^2}{a^2} = \frac{(\alpha)^{\frac{2}{3}}}{a^{\frac{2}{3}}e^{\frac{4}{3}}}.$$ + ?

$$\text{But} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

$$\therefore \frac{\alpha^{\frac{2}{3}}}{a^{\frac{2}{3}}e^{\frac{4}{3}}} + \frac{(b\beta)^{\frac{2}{3}}}{(ae)^{\frac{4}{3}}} = 1,$$

$$\text{or} \quad (a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (ae)^{\frac{4}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

(5) Radius of curvature of cycloid.



$$\left.\begin{aligned} AN &= x \\ NP &= y \\ AB &= 2a \end{aligned}\right\}; \quad \therefore \frac{dy}{dx} = \frac{\sqrt{2a-y}}{\sqrt{y}};$$

$$\therefore 1 + \frac{dy^2}{dx^2} = \frac{2a}{y};$$

$$\therefore \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} = \frac{-a}{y^2}\frac{dy}{dx};$$

$$\therefore -\frac{d^2y}{dx^2} = \frac{a}{y^2};$$

$$\therefore R = \left(\frac{2a}{y}\right)^{\frac{3}{2}} \cdot \frac{y^2}{a} = \frac{2a}{y}\sqrt{\frac{2a}{y}} \cdot \frac{y^2}{a} = 2\sqrt{2ay} = 2RB.$$

Evolute.

$$y - \beta = \frac{2a}{y} \times \frac{y^2}{a} = 2y; \quad \therefore \beta = -y,$$

$$x - \alpha = -(y - \beta)\frac{dy}{dx} = -2y\frac{\sqrt{2ay - y^2}}{y} = -2\sqrt{2ay - y^2};$$

$$\therefore \alpha = x + 2\sqrt{2ay - y^2}.$$

Take therefore,

$$AM = x + 2\sqrt{2ay - y^2} = AN + 2Rn,$$

and $MO = NP$, and O is a point in the evolute.

Its identity with the cycloid may be thus shewn:

$$\frac{d\alpha}{dx} = 1 + \frac{2(a-y)}{\sqrt{2ay - y^2}} \cdot \frac{dy}{dx} = 1 + \frac{2(a-y)}{y} = \frac{2a - y}{y}.$$

$$\text{But } \frac{d\alpha}{dx} = \frac{d\alpha}{d\beta} \cdot \frac{d\beta}{dx} = -\frac{d\alpha}{d\beta} \cdot \frac{dy}{dx} = -\frac{d\alpha}{d\beta}\sqrt{\frac{2a-y}{y}};$$

$$\therefore -\frac{d\alpha}{d\beta} = \sqrt{\frac{2a-y}{y}} = \sqrt{\frac{2a - (-\beta)}{(-\beta)}}.$$

* AOD in the figure should be a curve.

Take $Am = x_1 = -\beta$, and $mO = a = y_1$;

$$\therefore \ -\frac{da}{d\beta} = \frac{dy_1}{dx_1} = \sqrt{\frac{2a - x_1}{x_1}} = \frac{\sqrt{2ax_1 - x_1^2}}{x_1}.$$

The equation to a cycloid, of which the vertex is A, and the diameter of the generating circle $= 2a$.

(6) Find the chords of curvature drawn through the centre and focus of an ellipse.

Since by Conic Sections

$$CD^2 + CP^2 = AC^2 + BC^2,$$

$$\text{and } CD \times PF = AC \cdot BC.$$

If $CP = r$, and $PF = p$,

$$p^2 = \frac{a^2 b^2}{a^2 + b^2 - r^2}$$

is the equation between p and r, measuring from the centre;

$$\therefore \ 2 \text{ h. l. } p = \text{h. l. } a^2 b^2 - \text{h. l. } (a^2 + b^2 - r^2);$$

$$\therefore \ \frac{dp}{pdr} = \frac{r}{a^2 + b^2 - r^2};$$

$$\therefore \text{ chord through centre} = \frac{2pdr}{dp} = \frac{2(a^2 + b^2 - r^2)}{p} = \frac{2CD^2}{CP},$$

$$\text{diameter} = 2r\frac{dr}{dp} = \frac{2(a^2 + b^2 - r^2)}{p} = \frac{2CD^2}{PF}.$$

Chord through the focus.

$$\text{Here } p^2 = \frac{b^2 r}{2a - r};$$

$$\therefore \ 2 \text{ h. l. } p = \text{h. l. } b^2 + \text{h. l. } r - \text{h. l. } (2a - r);$$

$$\therefore \ \frac{2dp}{pdr} = \frac{1}{r} + \frac{1}{2a - r} = \frac{2a}{r \cdot (2a - r)};$$

$$\therefore \ p\frac{dr}{dp} = \frac{r(2a - r)}{a} = \frac{SP \cdot HP}{AC} = \frac{CD^2}{AC};$$

$$\therefore \text{ chord} = \frac{2CD^2}{AC}.$$

(7) Find the form of the parabola

$$y = a + bx + cx^2,$$

which has a contact of the second order, with a given curve at a given point.

Make the given point the origin: then the equation becomes

$$y = bx + cx^2;$$

$$\therefore \frac{dy}{dx} = p = b + 2cx, \text{ and } \frac{d^2y}{dx^2} = q = 2c.$$

But at the origin $x = 0$; $\therefore b = p$, and $c = \frac{q}{2}$;

$$\therefore y = px + \frac{qx^2}{2} = \frac{q}{2} \cdot \left(x^2 + \frac{2p}{q} x + \frac{p^2}{q^2}\right) \frac{p^2}{2q};$$

$$\therefore \left(y + \frac{p^2}{2q}\right) = \frac{q}{2} \cdot \left(x + \frac{p}{q}\right)^2.$$

The equation to a parabola, of which the axis is perpendicular to the axis of x, and the co-ordinates of the vertex

$$-\frac{p^2}{2q}, \text{ and } -\frac{p}{q};$$

the latus rectum $= \frac{2}{q}$.

Cor. The general equation to the second degree, or

$$y^2 + (ax + b)\, y + cx^2 + ex + f = 0,$$

containing five constants, may have a contact of the fourth order, with a curve. And should there be a point at which $a^2 - 4c = 0$, the osculating curve is a parabola.

Immediately before and after this point, a^2 must be greater or less than $4c$; and therefore the osculating parabola is intermediate between an osculating ellipse and hyperbola.

EXAMPLES.

(1) If $y^2 + x^2 = ax - ay$; which is an equation to the circle, $R = \frac{a}{\sqrt{2}}$.

(2) In the cubical parabola where $a^2 y = x^3$;

$$R = -\frac{(a^4 + 9x^4)^{\frac{3}{2}}}{6a^2 x};$$

and in the semicubical parabola where $ay^2 = x^3$;

$$R = -\frac{(4a + 9x)^{\frac{3}{2}}}{6a}.$$

(3) The equation to the hyperbola being $y^2 = \frac{b^2}{a^2}(x^2 - a^2)$;

$R = \frac{(e^2 x^2 - a^2)^{\frac{3}{2}}}{ab}$; and the equation to the evolute is

$$(a\alpha)^{\frac{2}{3}} - (b\beta)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}.$$

(4) In the parabola the chord of curvature through the focus $= 4SP$.

(5) If $yx = a^2$, $R = -\frac{(x^4 + a^4)^{\frac{3}{2}}}{2a^2 x^3}$, and equation to evolute is

$$(\alpha + \beta)^{\frac{2}{3}} - (\alpha - \beta)^{\frac{2}{3}} = (4a)^{\frac{2}{3}}.$$

(6) The equation to the catenary is $2y = a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$; shew that the radius of curvature is equal, but opposite, to the normal. $R = -\frac{y^2}{a}$.

(7) If $r = a(1 + \cos\theta)$; find equation between p and r; and shew that radius of curvature $= \frac{2\sqrt{ar}}{3}$; and chord $= \frac{2r}{3}$.

(8) In the spiral of Archimedes, if $r = \dfrac{a}{\sqrt{3}}$, shew that the radius = the chord of curvature.

(9) Find the evolute of the spiral of which the equation is $p^2 = e^2 \cdot \dfrac{r^2 - a^2}{e^2 - a^2}$. The evolute is a similar curve,

$$p_1^{\,2} = e^2 \cdot \frac{r_1^{\,2} - a_1^{\,2}}{e^2 - a^2}; \quad \text{and } a_1 = \frac{a^2}{e}.$$

(10) Find the chords of curvature drawn through the centre and focus of an hyperbola.

(11) If $y\sqrt{1 + \dfrac{dy^2}{dx^2}} = a$, be the equation to a curve (the Tractrix); the equation to the evolute is $\dfrac{d\beta}{d\alpha} = \dfrac{\sqrt{\beta^2 - a^2}}{a}$ (the Catenary).

(12) The length of an arc of the evolute of the parabola is expressed by

$$\frac{2(SP^{\frac{3}{2}} - SA^{\frac{3}{2}})}{\sqrt{SA}}.$$

CHAPTER XII.

SINGULAR POINTS IN CURVES.

146. If in the equation to a curve expressed by $y = f(x)$, where y is the ordinate, and x the abscissa; some value of x as a makes any of the differential coefficients 0, $\frac{1}{0}$, or $\frac{0}{0}$, the point so determined is called a singular point.

(1) Let the values of the first differential coefficient be considered;

Since $\frac{dy}{dx}$ represents the tangent of the angle which the tangent makes with the axis of x, if $\frac{dy}{dx} = 0$, the tangent is parallel to the axis of x, and this circumstance generally indicates a maximum or minimum value of the ordinate.

If $\frac{dy}{dx} = \frac{1}{0}$, the tangent is perpendicular to the axis of x.

If $y = 0$ when $\frac{dy}{dx} = 0$, then the axis of x is a tangent to the curve at the origin.

If $x = 0$ when $\frac{dy}{dx} = \frac{1}{0}$, then the tangent passes through the origin, and is coincident with the axis of y.

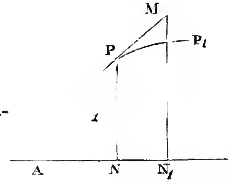
When $\frac{dy}{dx} = \frac{0}{0}$. Many branches may pass through the point, as we shall see in the succeeding pages.

If $\frac{d^2y}{dx^2}$ have a real value when $\frac{dy}{dx} = 0$, the ordinate is a maximum or minimum, as in the annexed figures.

Before we proceed to investigate the values of $\frac{d^2y}{dx^2}$ at these points, we must establish the following proposition:

147. Prop. If the ordinate y be reckoned positive, a curve is convex or concave to the axis, according as $\frac{d^2y}{dx^2}$ is positive or negative.

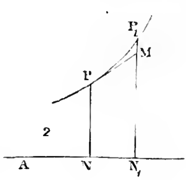
In the annexed figures, let



$$\left.\begin{array}{l} AN = x \\ NP = y \\ NN_1 = h \end{array}\right\},$$ and $y = f(x)$ be the equation to the curve.

Draw the tangent PM, its equation is

$$y_1 - y = \frac{dy}{dx}(x_1 - x).$$



Now at the point P_1, the equation to the curve becomes $N_1P_1 = f(x + h)$, or

$$N_1P_1 = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{2.3} + \&c.;$$

and for the tangent, putting $x + h$ for x_1, and N_1M for y_1,

$$N_1M = y + \frac{dy}{dx}.h;$$

therefore the deflection from the tangent, or MP_1

in figure (1) $= N_1 M - N_1 P_1 = -\dfrac{d^2y}{dx^2}\dfrac{h^2}{1 \cdot 2} - \dfrac{d^3y}{dx^3}\dfrac{h^3}{2 \cdot 3} -$ &c.

in figure (2) $= N_1 P_1 - N_1 M = +\dfrac{d^2y}{dx^2}\dfrac{h^2}{1 \cdot 2} + \dfrac{d^3y}{dx^3}\dfrac{h^3}{2 \cdot 3} +$ &c.;

and since h^2 is positive, and that h may be taken so small, that the first term of the expansion may be made greater than the sum of all the terms that follow it, the algebraical sign of MP_1 will depend upon that of $\dfrac{d^2y}{dx^2}$.

But we have seen that when the curve is concave to the axis, $MP_1 = -\dfrac{d^2y}{dx^2}\dfrac{h^2}{2} -$ &c.; and when convex to the axis, it $= +\dfrac{d^2y}{dx^2}\dfrac{h^2}{1 \cdot 2} +$ &c. Hence when y is positive, a curve is convex or concave to the axis, according as $\dfrac{d^2y}{dx^2}$ is positive or negative, or generally according as y and $\dfrac{d^2y}{dx^2}$ have the same or different signs.

148. Sometimes the curve after being convex to the axis suddenly changes its curvature, and becomes concave, the point at which the change takes place is called a point of inflexion, or of *contrary flexure.*

If the tangent at this point be produced, one branch of the curve will be above it and the other below it, consequently on one side of the point in question $\dfrac{d^2y}{dx^2}$ will be positive, and on the other side negative. Hence at the point itself $\dfrac{d^2y}{dx^2}$ must $= 0$, or ∞, for no quantity can change its sign without passing through zero or infinity.

There is not however a point of inflexion corresponding to every value of x, that makes $\dfrac{d^2y}{dx^2} = 0$, for not only must this equation be satisfied, but $\frac{d^2y}{dx^2}$ must change its sign after having passed through the point under consideration.

Also if the same value of x that makes $\frac{d^2y}{dx^2}=0$, also makes $\frac{d^3y}{dx^3}=0$, there will not be a point of contrary flexure.

For since $\frac{d^2y}{dx^2}$ is a function of x, write $x+h$ and $x-h$ for x, and then $\frac{d^2y}{dx^2}$ becomes, on these two suppositions, either

$$f(x+h)=\frac{d^2y}{dx^2}+\frac{d^3y}{dx^3}h+\frac{d^4y}{dx^4}\frac{h^2}{2}+\&c.;$$

$$\text{or } f(x-h)=\frac{d^2y}{dx^2}-\frac{d^3y}{dx^3}h+\frac{d^4y}{dx^4}\frac{h^2}{2}-\&c.$$

But at a point of inflexion $\frac{d^2y}{dx^2}=0$;

$\therefore$ the deflections from the tangent at points $x+h$ and $x-h$ are respectively proportional to

$$+\frac{d^3y}{dx^3}h+\frac{d^4y}{dx^4}\frac{h^2}{2}+\&c.,$$

$$\text{and } -\frac{d^3y}{dx^3}h+\frac{d^4y}{dx^4}\frac{h^2}{2}-\&c.,$$

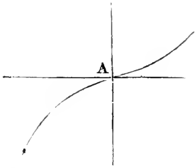
which have contrary signs if $\frac{d^3y}{dx^3}$ do not $=0$; but if $\frac{d^3y}{dx^3}=0$, and $\frac{d^4y}{dx^4}$ do not vanish, the deflections before and after the point will have the same algebraical sign, and the branches are both concave, or both convex, to the axis.

And hence in general there may be a point of contrary flexure, when the first differential coefficient which does not vanish is of an odd order.

Hence, to find whether a curve has a point of inflexion, put $\frac{d^2y}{dx^2} = 0$, or $\frac{1}{0}$, and if a be one of the values of x so determined, substitute $a + h$, and $a - h$ for x in the expression for $\frac{d^2y}{dx^2}$. Then if $\frac{d^2y}{dx^2}$ be affected with different signs, $x = a$ gives a point of contrary flexure.

Ex. 1. The cubical parabola $a^2y = x^3$,

$$y = \frac{x^3}{a^2}; \text{ and if } x = 0,\ y = 0,$$



$$\frac{dy}{dx} = \frac{3x^2}{a^2}$$

$$\frac{d^2y}{dx^2} = \frac{6x}{a^2}.$$

If x be positive or negative, y and $\frac{d^2y}{dx^2}$ are positive or negative; the curve is therefore always convex to the axis.

$$\text{If } x = 0,\ \frac{d^2y}{dx^2} = 0.$$

$$\text{If } x = h,\ \frac{d^2y}{d^2x} = \frac{6h}{a^2} \text{ is positive.}$$

$$\text{If } x = -h,\ \frac{d^2y}{dx^2} = -\frac{6h}{a^2} \text{ is negative.}$$

The origin is therefore a point of contrary flexure; also, since $x = 0$ makes $\frac{dy}{dx} = 0$ and $y = 0$, the axis of x is a tangent to the curve.

Ex. 2. The Witch. $y = \frac{2a}{x}\sqrt{2ax - x^2}$,

$$\frac{dy}{dx} = 2a\left(\frac{\frac{ax - x^2}{\sqrt{2ax - x^2}} - \sqrt{2ax - x^2}}{x^2}\right) = \frac{-2a^2}{x\sqrt{2ax - x^2}},$$

$$\frac{d^2y}{dx^2} = 2a^2 \frac{\sqrt{2ax - x^2} + \dfrac{ax - x^2}{\sqrt{2ax - x^2}}}{x^2(2ax - x^2)} = 2a^2 \frac{(3a - 2x)}{x \,.\, (2ax - x^2)^{\frac{3}{2}}},$$

which $= 0$ if $x = \frac{3a}{2}$, and changes its algebraical sign, when $\frac{3a}{2} + h$ and $\frac{3a}{2} - h$ are successively put for x.

There are therefore two points of contrary flexure when

$$x = \frac{3a}{2}, \quad \text{and } y = \pm \frac{2a}{\sqrt{3}}.$$

Ex. 3. In the trochoid, find the point of contrary flexure.

$$y = a(1 - e\cos\theta),$$

$$x = a(\theta - e\sin\theta);$$

$$\therefore \frac{dy}{d\theta} = ae\sin\theta,$$

$$\text{and } \frac{dx}{d\theta} = a(1 - e\cos\theta);$$

$$\therefore \frac{dy}{dx} = \frac{e\sin\theta}{1 - e\cos\theta},$$

$$\frac{d^2y}{dx^2} = \frac{e\cos\theta(1 - e\cos\theta) - e\sin^2\theta}{(1 - e\cos\theta)^2} \cdot \frac{d\theta}{dx}$$

$$= \frac{e\cos\theta - e^2}{(1 - e\cos\theta)^2} \times \frac{1}{a(1 - e\cos\theta)}$$

$$= \frac{e(\cos\theta - e)}{a(1 - e\cos\theta)^3},$$

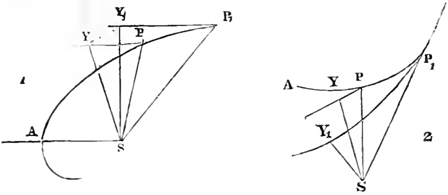
$$\frac{d^2y}{dx^2} = 0 \quad \text{if } \cos\theta = e,$$

and $\cos(\theta + h)$ is $< e$, and $\cos(\theta - h) > e$;

$\therefore \cos\theta = e$ gives the point of contrary flexure,

$$\text{and} \quad y = a(1 - e^2) = a\left(1 - \frac{b^2}{a^2}\right) = \frac{a^2 - b^2}{a}.$$

149. Points of contrary flexure in spirals.



Let there be two spirals, one concave and the other convex to the pole.

Take two points P and P_1 in each near to each other, and draw SY and SY_1 perpendiculars on the tangents at P and P_1,

and let $SY = p$, $SP = r$, and $SP_1 = r + h$,

and $p = f(r)$ the equation to the spiral;

therefore if Δ be the difference between SY_1 and SY, we have in figure (1), where the curve is concave to the pole,

$$\Delta = f(r+h) - f(r) = \frac{dp}{dr}.h + \frac{d^2p}{dr^2}\frac{h^2}{1.2} + \&c.;$$

but in figure (2), where the spiral is convex to S,

$$\Delta = f(r) - f(r+h) = -\frac{dp}{dr}.h - \frac{d^2p}{dr^2}\frac{h^2}{1.2} - \&c.;$$

and as h may be taken so small that $\dfrac{dp}{dr}h$ may be greater than all the terms that follow, we see that the spiral is concave or convex to S, according as $\dfrac{dp}{dr}$ is positive or negative.

Hence at a point of contrary flexure $\frac{dp}{dr} = 0$, and changes its sign immediately before and after the point under consideration.

EXAMPLES.

Let $r = a\theta^n$, find the point of contrary flexure.

$$\text{h.l. } r = \text{h.l. } a + n \text{ h.l. } \theta;$$

$$\therefore \frac{1}{r} = n \cdot \frac{d\theta}{dr} \cdot \frac{1}{\theta};$$

$$\therefore \frac{d\theta}{dr} = \frac{\theta}{nr} = \frac{\sqrt[n]{\frac{r}{a}}}{nr} = \frac{r^{\frac{1}{n}}}{na^{\frac{1}{n}}r}.$$

174. $$\text{But } \frac{d\theta}{dr} = \frac{p}{r\sqrt{r^2 - p^2}};$$

$$\therefore \frac{p}{\sqrt{r^2 - p^2}} = \frac{r^{\frac{1}{n}}}{na^{\frac{1}{n}}}$$

$$\therefore \frac{r^2}{p^2} = 1 + \frac{n^2 a^{\frac{2}{n}}}{r^{\frac{2}{n}}} = \frac{r^{\frac{2}{n}} + n^2 a^{\frac{2}{n}}}{r^{\frac{2}{n}}}$$

$$\therefore p = \frac{r^{\frac{n+1}{n}}}{\sqrt{r^{\frac{2}{n}} + n^2 a^{\frac{2}{n}}}};$$

$$\therefore \frac{dp}{dr} = \frac{n+1}{n} \cdot r^{\frac{1}{n}} \sqrt{r^{\frac{2}{n}} + n^2 a^{\frac{2}{n}}} - r^{\frac{n+1}{n}} \times \frac{r^{-\frac{n-2}{n}}}{n\sqrt{r^{\frac{2}{n}} + n^2 a^{\frac{2}{n}}}} = 0,$$

omitting the denominator;

$$\therefore\ (n+1)r^{\frac{1}{n}}(r^{\frac{2}{n}}+n^2a^{\frac{2}{n}})-r^{\frac{3}{n}}=0\,;$$

$$\therefore\ r^{\frac{1}{n}}\{n\,r^{\frac{2}{n}}+n^2(n+1)a^{\frac{2}{n}}\}=0,$$

whence $r=0$, and $r=a\{-n\,.\,(n+1)\}^{\frac{n}{2}}$.

If $\frac{n}{2}$ be a fraction with an even denominator, it is obvious that $n\,.\,(n+1)$ must be a negative number;

$$\therefore\ \text{let } n^2+n=-p\,;\quad \therefore\ n+\tfrac{1}{2}=\sqrt{\tfrac{1}{4}-p}\,;$$

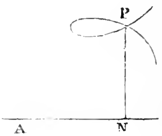
$$\therefore\ n=-\tfrac{1}{2}\pm\sqrt{\tfrac{1}{4}-p}\,;\quad \therefore\ p \text{ must never exceed } \tfrac{1}{4}.$$

If $p=\frac{1}{4}$, $n=-\frac{1}{2}$, and $r=\dfrac{a}{\sqrt{\theta}}$, or $\theta=\dfrac{a^2}{r^2}$ the equation to the lituus.

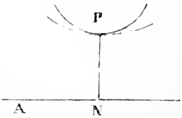
MULTIPLE POINTS.

150. Whenever two or more branches of a curve pass through a point, it is called a multiple point; and a double, triple, or quadruple point, according as two, three, or four branches pass through it.

If the branches intersect, as in figure (1), which represents a double point, there will be at P two tangents, inclined at different angles to the axis, and thus $\frac{dy}{dx}$ will have two values corresponding to one of x or y.



Should however the branches pass through P, as in fig. 2. and touch each other, and the contact be only of the first order, there will be but one value of $\frac{dy}{dx}$; but as there are two deflections from the tangent, there will be two values of $\frac{d^2y}{dx^2}$.



151. PROBLEM. If $u = f(x, y) = 0$ be the equation to a curve, cleared of radicals, and there be a point where two or more branches intersect, $\frac{dy}{dx} = \frac{0}{0}$ at the point of intersection.

Let the equation be differentiated, the result will be of the form

$$M.\frac{dy}{dx} + N = 0, \quad \text{where } M = \frac{du}{dy} \quad \text{and } N = \frac{du}{dx}.$$

Then since two branches intersect, $\frac{dy}{dx}$ will have two values, but M and N will be the same for both. Let α and β be the two values of $\frac{dy}{dx}$;

$$\therefore\ M\alpha + N = 0,$$

$$\text{and } M\beta + N = 0;$$

$$\therefore\ M(\alpha - \beta) = 0, \text{ and } \alpha - \beta \text{ does not} = 0;$$

$$\therefore\ M = 0; \quad \text{and } \therefore\ N = 0, \quad \text{and } \frac{dy}{dx} = -\frac{N}{M} = \frac{0}{0}.$$

COR. 1. Hence the value of p or $\frac{dy}{dx}$ may be found, by the same method as that by which the values of vanishing fractions are determined.

Thus, since $p = -\frac{N}{M} = \frac{0}{0}$ when $x = a$ and $y = b$; differentiating numerator and denominator,

$$p = -\frac{M_1 p + N_2}{M_1 + M_2 p}, \quad \text{where } M_1 = \frac{dN}{dy} = \therefore \frac{dM}{dx},$$

$$\text{or } M_2 p^2 + 2M_1 p + N_2 = 0;$$

a quadratic equation, from which two values of p may be found, and which, if possible values, indicate that the curve has a double point.

COR. 2. If, however, $M_1 = 0$, $N_1 = 0$, and $M_2 = 0$, when $x = a$ and $y = b$, differentiate numerator and denominator a second time, and putting $q = \frac{d^2 y}{dx^2}$, we shall have p of the form

$$p = -\frac{M_1 q + N_2 p^2 + N_3 p + N_4}{M_2 q + M_3 p^2 + M_4 p + M_5}$$

$$= -\frac{N_2 p^2 + N_3 p + N_4}{M_3 p^2 + M_4 p + M_5}, \text{ when } x = a \text{ and } y = b;$$

whence we have a cubic equation of the form

$$p^3 + ap^2 + bp + c = 0 \text{ to determine } p;$$

and if there be three possible roots, there is a triple point.

This process must be continued, if the numerator and denominator again vanish.

EX. 1. Find the species of point at the origin of the curve,

$$ay^2 - x^3 - bx^2 = 0.$$

Differentiating and putting p for $\frac{dy}{dx}$,

$$2ayp - 3x^2 - 2bx = 0;$$

$$\therefore p = \frac{3x^2 + 2bx}{2ay} = \frac{0}{0}, \text{ if } x = 0 \text{ and } y = 0,$$

there may be a multiple point.

Differentiating the numerator and denominator,

$$p = \frac{6x + 2b}{2ap} = \frac{2b}{2ap}, \text{ when } x = 0;$$

$$\therefore p^2 = \frac{b}{a}, \text{ and } p = \pm\sqrt{\frac{b}{a}};$$

which shews that the origin is a double point, and that the tangents cut the axis at angles,

$$\tan^{-1}\sqrt{\frac{b}{a}}, \text{ and } \tan^{-1}\left(-\sqrt{\frac{b}{a}}\right).$$

This example will be useful in shewing another method by which multiple points may be found. Thus, if there be a surd quantity which disappears from the equation $y = f(x)$ by making $x = a$, but which is found in the equation

$$\frac{dy}{dx} = \phi(x),$$

then $\frac{dy}{dx}$ will have two values, while y has but one, and a double point is indicated. For resuming the example, and solving it with respect to y,

$$y = \pm \frac{x}{\sqrt{a}} \sqrt{x + b},$$

$$\frac{dy}{dx} = \pm \frac{\sqrt{x+b}}{\sqrt{a}} + \frac{x}{2\sqrt{a}\sqrt{x+b}}.$$

Make $x = 0$. Then $y = 0$, and $\frac{dy}{dx} = \pm\sqrt{\frac{b}{a}}$, as before.

Ex. 2. Find the point at the origin of the Lemniscata.

$$(x^2 + y^2)^2 = a^2(x^2 - y^2).$$

Here $2(x + py) \cdot (x^2 + y^2) = a^2(x - py)$;

$$\therefore\ p = \frac{a^2 x - 2x(x^2 + y^2)}{a^2 y + 2y(x^2 + y^2)} = \frac{0}{0}, \text{ if } x \text{ and } y = 0,$$

$$= \frac{a^2 - 2(x^2 + y^2) - 2x(2x + 2yp)}{a^2 p + 2p(x^2 + y^2) + 2y(x + 2yp)}$$

$$= \frac{a^2}{a^2 p}, \quad \text{if } x = 0 \text{ and } y = 0;$$

$$\therefore\ p^2 = 1, \quad \text{and } p = \pm 1,$$

and the values of $\frac{dy}{dx}$ are $+1$ and -1, or $\frac{dy}{dx} = \tan 45$, and $\tan 135$.

Ex. 3. Find the same, when $x^4 - ayx^2 + by^3 = 0$.

Here $4x^3 - ax^2p - 2ayx + 3by^2p = 0$;

$$\therefore\ p = \frac{2ayx - 4x^3}{3by^2 - ax^2} = \frac{0}{0}, \text{ when } x \text{ and } y = 0,$$

$$= \frac{2apx + 2ay - 12x^2}{6bpy - 2ax} = \frac{0}{0}, \text{ if } x = 0,$$

$$= \frac{2ap + 2axq + 2ap - 24x}{6bp^2 + 6byq - 2a}, \quad q = \frac{d^2y}{dx^2},$$

$$= \frac{4ap}{6bp^2 - 2a}, \text{ if } x = 0;$$

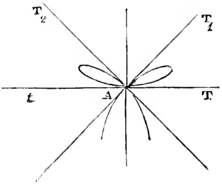
$$\therefore\ 6bp^3 - 2ap = 4ap, \quad \text{or } p\,.\{bp^2 - a\} = 0;$$

$$\therefore\ p = 0, \quad \text{and } p = \pm\sqrt{\frac{a}{b}};$$

there is a triple point at the origin, and the axis of x is one of the tangents.

The triple point at A is represented in the annexed figure; TAt is the axis of x, AT_1 and AT_2 are the tangents of the angles

$$\tan^{-1}\sqrt{\frac{a}{b}}, \text{ and } \tan^{-1}\left(-\sqrt{\frac{a}{b}}\right).$$



Ex. 4. Find the species of point at the origin of the curve

$$y^3 - 3axy + x^3 = 0.$$

Here if $x = 0$, $y = 0$; therefore, differentiating,

$$y^2\frac{dy}{dx} - ax\frac{dy}{dx} - ay - x^2 = 0;$$

$$\therefore\ \frac{dy}{dx} = p = \frac{ay - x^2}{y^2 - ax} = \frac{0}{0}, \text{ if } x = 0 \text{ and } y = 0;$$

$$\therefore\ p = \frac{ap - 2x}{2yp - a} = \frac{ap}{2yp - a},\ \text{if } x = 0\,;$$

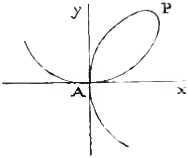
$$\therefore\ 2yp^2 - ap = ap,$$

$$\text{or } p\,(yp - a) = 0\,;$$

$$\therefore\ p = 0,\quad \text{and } p = \frac{a}{y} = \frac{a}{0} = \infty\,.$$

The origin is therefore a double point, and the two axes are the tangents.

The curve is represented in the annexed figure.



152. If the branches touch, then $\frac{dy}{dx}$ will have but one value, and yet at the same time be of the form $\frac{0}{0}$.

For, supposing the contact to be of the n^{th} order between two branches of the curve; then the values of the differential coefficients, as far as the $(n+1)^{\text{th}}$ coefficient, when $x = a$, and $y = b$, will be the same for both branches; but after the n^{th} will be different.

Let $M\frac{dy}{dx} + N = 0$ be the equation after the first differentiation, the original equation being previously freed from the said quantities.

Then, repeating the differentiation (n) times, we have

$$M\,.\,\frac{d^{n+1}y}{dx^{n+1}} + N_1 = 0,$$

M being the same as before, and N_1 being the sum of the differential coefficients below the $(n+1)^{\text{th}}$, together with functions of x or y.

But $\dfrac{d^{n+1}y}{dx^{n+1}}$ has two values, as α and β, while M and N_1 remain unchanged;

$$\therefore\ M.(\alpha - \beta) = 0; \quad \therefore\ M = 0.$$

But $M\dfrac{dy}{dx} + N = 0$; $\therefore$ if $x = a$, and $y = b$, $N = 0$;

and $\therefore\ \dfrac{dy}{dx} = \dfrac{0}{0}$.

The analytical character of double points of this description is, that when $\dfrac{dy}{dx} = \dfrac{0}{0}$ has but one value, $\dfrac{d^2y}{dx^2}$, which also $= \dfrac{0}{0}$, has two.

CONJUGATE OR ISOLATED POINTS.

153. Conjugate or isolated points are those which have real existence, and are determined by the equation to the curve: but from which no branches extend.

Hence if $x = a$ and $y = b$ give such a point, then $x = a + h$, and $x = a - h$, will make y impossible, as well as $\dfrac{dy}{dx}$, and many of the differential coefficients.

At such a point $\dfrac{dy}{dx} = \dfrac{0}{0}$, if the equation be cleared of radicals.

In differentiating the equation $u = f(xy) = 0$ it will be of the form $M\dfrac{dy}{dx} + N = 0$.

Now at a conjugate point $\frac{dy}{dx}$ is impossible, let it $= (a + \beta\sqrt{-1})$;

$$\therefore \; Ma + N + M\beta\sqrt{-1} = 0;$$

whence $M = 0$, and $Ma + N = 0$; $\therefore \; N = 0$,

$$\text{and } \frac{dy}{dx} = \frac{0}{0};$$

whence the value of $\frac{dy}{dx}$ may be determined by the method used for finding the multiple points.

Ex. $ay^2 - x^3 + bx^2 = 0$,

$$2ayp - 3x^2 + 2bx = 0;$$

$$\therefore \; p = \frac{3x^2 - 2bx}{2ay} = \frac{0}{0}, \text{ if } x = 0 \text{ and } \therefore \; y = 0$$

$$= \frac{6x - 2b}{2ap} = -\frac{b}{ap}, \quad \text{if } x = 0;$$

$$\therefore \; p^2 = -\frac{b}{a}; \quad \text{and } \therefore \; p = \sqrt{\frac{-b}{a}}.$$

Now $x = 0$ gives $y = 0$, while $p = \sqrt{\dfrac{-b}{a}}$.

Also since $y = x\sqrt{\dfrac{x-b}{a}}$, if $x = 0 + h$, or $0 - h$, the values of y are impossible, and the origin is therefore a conjugate point. The same result may be obtained by differentiating the equation

$$y = x\sqrt{\frac{x-b}{a}}.$$

$$\text{For } \frac{dy}{dx} = \sqrt{\frac{x-b}{a}} + \frac{x}{2\sqrt{a}\sqrt{x-b}},$$

$$\text{and if } x = 0,\ \frac{dy}{dx} = \sqrt{\frac{-b}{a}}.$$

And in general, if there be a surd which vanishes from the equation $y = f(x)$ if $x = a$, but which becomes impossible in $\frac{dy}{dx} = \phi(x)$, there will be a conjugate point. If at the same time the values of y are impossible, both before and after the point.

CUSPS.

154. When two branches of a curve touch each other at a point through which the branches do not extend, the point is called a Cusp.

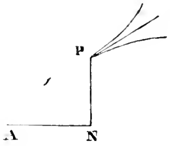
The branches have at this point but one tangent, and the cusp is of the *first species* when the branches lie on opposite sides of the common tangent, and of the *second species* when upon the same side.

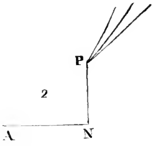
Hence if $x = a$ and $y = b$ give the point in question, $\frac{dy}{dx}$ will have but one value at the point: and when either $a + h$, or $a - h$ is put for x, $\frac{d^2y}{dx^2}$ will have two values.

If the values of $\frac{d^2y}{dx^2}$ be both positive or both negative, the cusp is of the second species, and if one value be positive and the other negative, the cusp is of the first species, for the deflection of the tangent from the curve is measured by $\frac{d^2y}{dx^2}$.

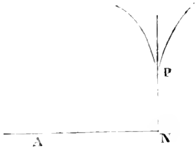
Since by the definition the branches suddenly stop at the cusp, either $a + h$, or $a - h$, when put for x will make the ordinate and the differential coefficients impossible.

Figures (1) and (2) exhibit cusps of the first and second species.





Sometimes the cusp is of the form represented in the above



figure, in which $(a+h)$ and $(a-h)$ when put for x will give real values for the ordinate.

These are discoverable by observing, that if $x = a$ and $y = b$ give the point P, that $y = b - k$ makes x impossible. Or we may at once transform the equation to the axis of y making $x = f^{-1}(y)$; and find the values of x, $\frac{dx}{dy}$, and $\frac{d^2x}{dy^2}$ at and near the point where $y = b$.

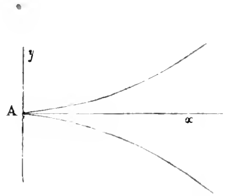
Ex. 1. The semi-cubical parabola.

$$a y^2 = x^3,$$

$$y = \frac{x^{\frac{3}{2}}}{\sqrt{a}},$$

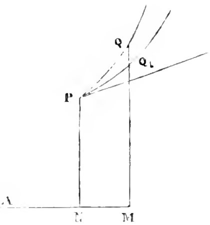
$$\frac{dy}{dx} = \frac{3}{2}\sqrt{\frac{x}{a}}.$$

$$\frac{d^2y}{dx^2} = \frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{ax}}.$$



If $x=0$, y and $\frac{dy}{dx}=0$, if $x=-h$, they are both impossible. But if $x=h$, y and $\frac{d^2y}{dx^2}$ have two values, one positive and the other negative; the axis of x is therefore a tangent: there are two branches to the curve, one above and the other below the axis of x, and both convex to it, but the curve does not extend through the origin to the negative axis of the abscissas. The origin is a cusp of the first species.

Ex. 2. Find the point, when $x=a$ in the curve of which the equation is



$$y = b + cx^2 + (x-a)^{\frac{5}{2}};$$

$$\therefore \frac{dy}{dx} = 2cx + \frac{5}{2}(x-a)^{\frac{3}{2}},$$

$$\text{and } \frac{d^2y}{dx^2} = 2c + \frac{5 \cdot 3}{2 \cdot 2}(x-a)^{\frac{1}{2}}.$$

Let $x=a$;

$$\therefore y = b + ca^2,$$

$$\text{and } \frac{dy}{dx} = 2ca; \quad \text{and } \frac{d^2y}{dx^2} = 2c,$$

$$x = a + h;$$

$$\therefore y = b + c(a+h)^2 + h^{\frac{5}{2}}, \quad \text{and } \frac{d^2y}{dx^2} = 2c + \frac{15}{4}h^{\frac{1}{2}};$$

whence in consequence of the index $\frac{1}{2}$, y and $\frac{d^2y}{dx^2}$ have each two values, but those of $\frac{d^2y}{dx^2}$ are both positive, and since $a-h$ put for x makes y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ impossible, the point is a cusp, and of the second species.

Take $x = a$, $y = b + ca^2$, and draw a tangent through P inclined to the axis of abscissas at an $\angle = \tan^{-1} 2ca$, and then draw two branches above the line through two points Q and Q_1, where

$$MQ = b + c(a+h)^2 + h^{\frac{5}{2}}, \quad \text{and } MQ_1 = b + c(a+h)^2 - h^{\frac{5}{2}},$$

and the curves will be represented.

For additional illustrations of these points, the student is referred to *Peacock's Collection of Examples of the Applications of the Differential and Integral Calculus.*

155. We shall conclude this Chapter by a few remarks upon the subject of tracing curves by means of their equations.

(1) If it be possible, let the equation be solved with respect to one of the unknown quantities as y, and let it be put under the form $y = f(x)$.

Then give to x all the possible *positive* values the equation admits of, and so determine the branches above and below the axis of *positive* abscissas.

Next put $(-x)$ for x in the equation $y = f(x)$, and in the equation, thus transformed, again substitute for x all its possible *positive* values, and the branches above and below the axis of *negative* abscissas will be determined.

(2) Ascertain whether the curve has asymptotes, and if it has, draw them.

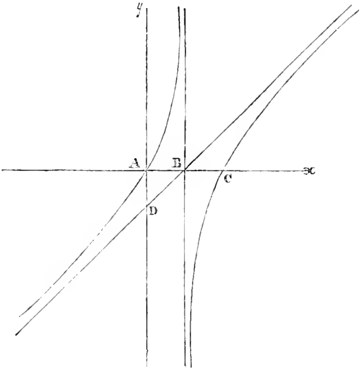
(3) Find whether the branches be concave or convex to the axis, and determine the nature and situation of the singular points.

The preceding remarks refer to curves having rectangular co-ordinates, but if the equation be between r and θ, we must give to θ all values from $\theta = 0$ to $\theta = 2\pi$, and draw the corresponding values of r.

Ex. 1. Let $y = x \cdot \dfrac{x - 2a}{x - a}$, trace the curve.

Take A the origin.

Draw Ax and Ay the two axes.



Let $x = 0$; $\therefore\ y = 0$,

$x < a$; $\therefore\ y$ is positive,

$x = a$, $y = \infty$,

$x > a < 2a$, y is negative,

$x = 2a$, $y = 0$,

$x > 2a$, y is positive,

$x = \infty$, y is ∞.

Again, let $-x$, be put for x;

$$\therefore\ y = -x\frac{x + 2a}{x + a} \text{ is always negative.}$$

To draw the asymptote,

$$y = x \cdot \left(\frac{1 - \frac{2a}{x}}{1 - \frac{a}{x}}\right) = x\left(1 - \frac{2a}{x}\right) \cdot \left\{1 + \frac{a}{x} + \&c.\right\};$$

$$\therefore\ y = x\left\{1 - \frac{a}{x} - \frac{2a^2}{x^2} - \&c.\right\}$$

$$= x - a - \frac{2a^2}{x} - \&c.\,;$$

$\therefore\ y = x - a$ is the equation to the asymptote.

Take therefore $AB = a$, and $AD = a$, and the line BD produced is the asymptote. Also take $AC = 2a$.

Then since $y = 0$, both when $x = 0$ and $x = 2a$, the curve cuts the axis at A and C.

Between A and B the curve is above the axis, and at B the ordinate is infinite; from B to C the curve is below the axis, and from C to infinity is above Ax. Again, since if x be negative y is negative; the branch on the left hand of A is entirely below the axis.

To find the value of $\frac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{x - 2a}{x - a} + x \cdot \frac{(x - a) - (x - 2a)}{(x - a)^2}$$

$$= \frac{x^2 - 3ax + 2a^2}{(x - a)^2} + \frac{ax}{(x - a)^2}$$

$$= \frac{x^2 - 2ax + 2a^2}{(x - a)^2}.$$

$$\text{Let } x = a\,; \quad \therefore\ \frac{dy}{dx} = \infty\,;$$

therefore the infinite ordinate through B is an asymptote;

$$x = 0\,; \quad \therefore\ \frac{dy}{dx} = 2,$$

or angle at which the curve cuts the axis at A is $= \tan^{-1}(2)$,

$x = 2a$, and $\frac{dy}{dx}$ again $= 2$, or angle at which the curve cuts the axis at C is $= \angle$ at A.

Since $(x^2 - 2ax + 2a^2) = (x-a)^2 + a^2$, the roots are impossible; therefore $\frac{dy}{dx}$ does not $= 0$, or there is no maximum or minimum ordinate.

$$\text{Again, } \frac{d^2y}{dx^2} = \frac{2\,.\,(x-a)^2 - 2\left\{(x-a)^2 + a^2\right\}}{(x-a)^3} = \frac{-2a^2}{(x-a)^3};$$

$$\therefore\ \frac{d^2y}{dx^2} \text{ is positive if } x < a, \text{ and is negative if } x > a.$$

But $x < a$, y is positive, and $x > a < 2a$, y is negative;

and $x > 2a$, y is positive;

therefore from A to B, and from B to C the curve is convex, and from C concave to the axis.

If x be negative, $\frac{d^2y}{dx^2} = \frac{2a^2}{(x+a)^3}$ is positive, but y is negative; therefore the branch from A to the left hand is concave to the axis.

Ex. 2. Let $y^2 = \frac{x^3+1}{x-1} = \frac{x+1}{x-1}(x^2 - x + 1)$;

$$\therefore\ y = \pm\sqrt{\frac{x+1}{x-1}}\,.\sqrt{x^2 - x + 1}.$$

A the origin; Ax, Ay the axes.

If $x = 0$, $y = \frac{1}{\sqrt{-1}}$ is impossible,

$x < 1$, y is impossible,

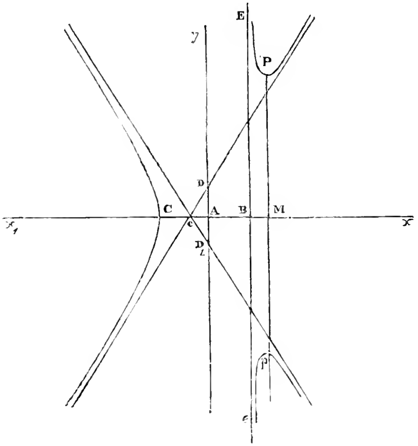
$x = 1$, y is $\pm\infty$,

$x > 1$, y is possible $\pm$,

$x = \infty$, y is $\infty \pm$;

therefore there are two infinite branches extending above and below the axis of positive abscissas.

For x put $-x$; $\therefore\ y = \pm\sqrt{\dfrac{x-1}{x+1}}(x^2+x+1)$, which is impossible, if x be <1, and $=0$, if $x=1$.



If $x>1$, and increase to infinity, y is possible $\pm$ and increases to infinity; therefore there are two infinite branches which meet the axis Ax_1 in a point C, if $AC=1$.

To find the asymptote:

$$y = \pm\sqrt{\frac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}}}\,.\,x\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}$$

$$= \pm\frac{1+\dfrac{1}{2x}+\&c.}{1-\dfrac{1}{2x}+\&c.}\,x\left(1-\frac{1}{2x}+\frac{1}{2x^2}+\&c.\right)$$

$$= \pm \left\{1 + \frac{1}{x} + \&c.\right\} x \left(1 - \frac{1}{2x} + \&c.\right)$$

$$= \pm x \left\{1 + \frac{1}{2x} + \frac{A}{x^2} + \&c.\right\};$$

$\therefore\ y = \pm (x + \frac{1}{2})$ gives the two asymptotes.

Take $AD = AD_1 = \frac{1}{2}$, and $AC = \frac{1}{2}$. Join CD and CD_1, these lines are the asymptotes, and if through B an infinite ordinate be drawn, two branches of the curve will lie within the angular space formed by the intersections of this line with CD and CD_1 produced.

For these branches of the curve will always lie above the asymptotes. Since the ordinate of the asymptote is always less than the ordinate of the curve.

This may be thus shewn. Let y_1 be the ordinate of the asymptote;

$$\therefore\ y^2 = \frac{x^3 + 1}{x - 1}, \quad \text{and } y_1{}^2 = x^2 + x + \tfrac{1}{4};$$

$$\therefore\ y^2 - y_1{}^2 = \frac{x^3 + 1 - (x - 1) . (x^2 + x + \frac{1}{4})}{x - 1},$$

$$\text{or } (y + y_1)(y - y_1) = \frac{3x + 1}{4 . (x - 1)}.$$

But $y + y_1$ is essentially positive; therefore $y - y_1$ is positive; or $y > y_1$.

By a similar mode of reasoning it may be shewn that the branches which extend from C, above and below the axis Cx_1, lie between the lines D_1C, and DC (asymptotes) produced.

To find the values of $\dfrac{dy}{dx}$.

$$2\,\text{h. l. } y = \text{h. l. } (x^3 + 1) - \text{h. l. } (x - 1);$$

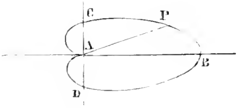
$$\therefore \frac{dy}{dx} = \frac{y}{2} \cdot \left(\frac{3x^2}{x^3+1} - \frac{1}{x-1}\right) = \frac{2x^3 - 3x^2 - 1}{2(x-1)^{\frac{3}{2}} \cdot \sqrt{x^3+1}},$$

which is ∞, if $x = 1$, or $x = -1$.

Hence the infinite ordinate through B touches the curve at an infinite distance from Ax, or is an asymptote: and the curve at C where $y = 0$ cuts the axis at right angles. Also since the numerator $2x^3 - 3x^2 - 1$ is $= -2$ when $x = 1$, and is $= 3$ when $x = 2$, there is some value of x between 1 and 2 which will make $\frac{dy}{dx} = 0$, or y a minimum. Take AM this value, and MP and Mp will be minimum ordinates.

Ex. 3. As a last example, let the equation be between polar co-ordinates.

Trace the curve defined by the equation $r = a(1 + \cos\theta)$,



$$\theta = 0; \quad \therefore r = a(1+1) = 2a,$$

$$\theta = \frac{\pi}{2}; \quad \therefore r = a.$$

$$\text{Let } \theta = \frac{\pi}{2} + \alpha; \quad \therefore \cos\theta = -\sin\alpha,$$

and $r = a(1 - \sin\alpha)$, or $r < a$,

$$\alpha = \frac{\pi}{2}, \text{ or } \theta = \pi; \quad \therefore r = 0.$$

Again, let $\theta = (\pi + \alpha)$; $\therefore \cos\theta = -\cos\alpha$,

and $r = a(1 - \cos\alpha)$, which increases as α increases,

and $r = a$ when $\alpha = 90$.

Again, let $\theta = \frac{3\pi}{2} + \alpha$; $\therefore \cos\left(\frac{3\pi}{2} + \alpha\right) = + \sin\alpha$;

$\therefore r = a(1 + \sin\alpha)$, which increases as α increases, and when $\alpha = \frac{\pi}{2}$, or $\theta = 2\pi$, $r = 2a$.

It is obvious that the curves in the first and fourth quadrants are the same, and those in the second and third resemble each other in every respect.

Take $AB = 2a$, $AC = AD = a$, and the points at which it cuts the axes are determined.—This curve is called the *Cardioide*.

EXAMPLES.

(1) If $y = ax + bx^2 + cx^3$; there is a point of inflexion if $x = -\frac{b}{3c}$.

(2) $y^2 = \frac{a^4}{a^2 + x^2}$; trace the curve: there are inflexions if $x = \pm \frac{a}{\sqrt{2}}$.

(3) $y = (x - 2)\sqrt{\frac{x - 9}{x}}$; trace the curve; there is a conjugate point if $x = 2$; and y is a minimum if $x = -\frac{3}{2}$.

(4) $y^2 = ax^2 - x^3$, a cusp of the first species at the origin, an inflexion if $x = a$; a maximum ordinate if $x = \frac{2a}{3}$.

(5) If $y = \frac{x}{1 + x^2}$, there are two points of flexure; the curve cuts the axis at 45^0, and the axis of x is an asymptote to the two infinite branches.

(6) $(x^2 + y^2)^2 = 4axy$; a double point at the origin, $p = 0$ and $= \infty$.

(7) $(x^2 + y^2)^3 = 4a^2x^2y^2$; a quadruple point at the origin.

(8) $y^4 + 2axy^2 - ax^3 = 0$; a triple point at the origin, $p = \pm \frac{1}{\sqrt{2}}$, and $= \infty$.

CHAPTER XIII.

CHANGE OF THE INDEPENDENT VARIABLE.

156. From the equation $y=f(x)$, we have in the preceding pages derived the values of the differential coefficients

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \&c.$$

considering x to be the independent variable: but as it is left entirely to our own choice which quantity of the two we may assume to be a function of the other, let us see what substitutions we ought to make for

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \&c.$$

when y is changed into the independent variable, and x becomes the function.

Again, it is frequently convenient to make a substitution for x in the equation $y=f(x)$, such that

$$x=\phi(\theta), \text{ and } \therefore\ y=f(\theta);$$

we must therefore find the values of

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \&c.$$

on this new supposition.

We have indeed already seen that

$$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}},$$

and that $\dfrac{dy}{d\theta} = \dfrac{dy}{dx} \cdot \dfrac{dx}{d\theta}$;

and in the following Propositions these equations will be again proved, and other important results obtained.

157. Prop. If $y = f(x)$, and therefore $x = f^{-1}(y)$, find the value of the differential coefficients $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, &c. in terms of

$$\frac{dx}{dy}, \quad \frac{d^2x}{dy^2}, \quad \frac{d^3x}{dy^3}, \text{ \&c.}$$

Let $y + k$ be the value of y when x becomes $x + h$;

$$\therefore\ k = \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{2.3} + \text{\&c.} \ldots\ldots (1).$$

And since $x + h = f^{-1}(y + k)$,

$$\therefore\ h = \frac{dx}{dy}k + \frac{d^2x}{dy^2}\frac{k^2}{1.2} + \frac{d^3x}{dy^3}\cdot\frac{k^3}{2.3} + \text{\&c.}$$

substituting for h in equation (1),

$$k = \frac{dy}{dx}\left\{\frac{dx}{dy}k + \frac{d^2x}{dy^2}\frac{k^2}{1.2} + \frac{d^3x}{dy^3}\frac{k^3}{2.3} + \text{\&c.}\right\}$$

$$+ \frac{d^2y}{dx^2}\left\{\frac{dx^2}{dy^2}k^2 + \frac{dx}{dy}\cdot\frac{d^2x}{dy^2}k^3 + \text{\&c.}\right\}\cdot\frac{1}{1.2}$$

$$+ \frac{d^3y}{dx^3}\left\{\frac{dx^3}{dy^3}k^3 + \text{\&c.}\right\}\frac{1}{1.2.3}$$

$$+ \text{\&c.}$$

$$= \frac{dy}{dx}\cdot\frac{dx}{dy}k + \left(\frac{dy}{dx}\cdot\frac{d^2x}{dy^2} + \frac{d^2y}{dx^2}\cdot\frac{dx^2}{dy^2}\right)\frac{k^2}{1.2}$$

$$+ \left\{\frac{d^3x}{dy^3}\frac{dy}{dx} + 3\frac{dx}{dy}\cdot\frac{d^2y}{dx^2}\cdot\frac{d^2x}{dy^2} + \frac{d^3y}{dx^3}\cdot\frac{dx^3}{dy^3}\right\}\frac{k^3}{2.3}$$

$$+ \text{\&c.}$$

therefore, equating coefficients,

$$1 = \frac{dy}{dx} \cdot \frac{dx}{dy}; \quad \therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}, \qquad \therefore p = \frac{1}{p_1}$$

and $\dfrac{dy}{dx} \cdot \dfrac{d^2x}{dy^2} + \dfrac{d^2y}{dx^2} \cdot \dfrac{dx^2}{dy^2} = 0$;

$$\therefore \frac{d^2x}{dy^2} + \frac{d^2y}{dx^2} \cdot \frac{dx^3}{dy^3} = 0;$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-\frac{d^2x}{dy^2}}{\frac{dx^3}{dy^3}},$$

and, putting

p, q, r, &c. for the differential coefficients when $y = f(x)$,

and p_1, q_1, r_1, &c.................................. when $x = f^{-1}(y)$;

$$\therefore q = \frac{-q_1}{p_1^3}.$$

Also, since $\dfrac{d^3x}{dy^3} \cdot \dfrac{dy}{dx} + 3\dfrac{dx}{dy} \cdot \dfrac{d^2y}{dx^2} \cdot \dfrac{d^2x}{dy^2} + \dfrac{d^3y}{dx^3} \cdot \dfrac{dx^3}{dy^3} = 0$;

$$\therefore r_1 p + 3p_1 q q_1 + r p_1^3 = 0,$$

$$\text{or } \frac{r_1}{p_1} - \frac{3q_1^2}{p_1^2} + r p_1^3 = 0;$$

$$\therefore r = \frac{3q_1^2 - r_1 p_1}{p_1^5};$$

and similarly may the other coefficients be found.

EXAMPLE. Take the expression for the radius of curvature

$$R = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}} = \frac{(1 + p^2)^{\frac{3}{2}}}{-q}, \quad \text{and if } x = f^{-1}(y);$$

$$\therefore\ R = \frac{\left(1 + \frac{1}{p_1^2}\right)^{\frac{3}{2}}}{\frac{q_1}{p_1^3}} = \frac{(p_1^2 + 1)^{\frac{3}{2}}}{q_1}.$$

Let $y^2 = 4mx$; $\quad x = \frac{y^2}{4m}$,

$$\frac{dx}{dy} = p_1 = \frac{y}{2m},$$

$$\frac{d^2x}{dy^2} = q_1 = \frac{1}{2m};$$

$$\therefore\ R = \frac{\left(1 + \frac{y^2}{4m^2}\right)^{\frac{3}{2}}}{\frac{1}{2m}} = \frac{(4m^2 + y^2)^{\frac{3}{2}}}{4m^2} = \frac{2\,.\,(m + x)^{\frac{3}{2}}}{\sqrt{m}}.$$

158. Again: if $y = f(\theta)$, and $x = \phi(\theta)$, express $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c. in terms of

$$\frac{dy}{d\theta},\quad \frac{dx}{d\theta},\quad \frac{d^2y}{d\theta^2},\quad \frac{d^2x}{d\theta^2},\ \text{\&c.}$$

Let $y + k$, $\theta + m$, and $x + h$, be corresponding values of y, θ, and x; therefore, by Taylor's theorem,

$$k = \frac{dy}{d\theta} m + \frac{d^2y}{d\theta^2} \cdot \frac{m^2}{1\,.\,2} + \frac{d^3y}{d\theta^3} \cdot \frac{m^3}{2\,.\,3} + \text{\&c.}$$

$$h = \frac{dx}{d\theta} m + \frac{d^2x}{d\theta^2} \cdot \frac{m^2}{1\,.\,2} + \frac{d^3x}{d\theta^3} \cdot \frac{m^3}{2\,.\,3} + \text{\&c.}$$

also, since $y + k$ is a function of $x + h$,

$$k = \frac{dy}{dx} h + \frac{d^2y}{dx^2} \cdot \frac{h^2}{1\,.\,2} + \frac{d^3y}{dx^3} \cdot \frac{h^3}{2\,.\,3} + \text{\&c.}$$

therefore, substituting for k and h,

$$\frac{dy}{d\theta} m + \frac{d^2y}{d\theta^2} \cdot \frac{m^2}{1\,.\,2} + \text{\&c.}$$

$$= \frac{dy}{dx} \cdot \left(\frac{dx}{d\theta} m + \frac{d^2x}{d\theta^2} \cdot \frac{m^2}{1 \cdot 2} + \&c.\right)$$

$$+ \frac{1}{1 \cdot 2} \cdot \frac{d^2y}{dx^2} \left(\frac{dx^2}{d\theta^2} m^2 + \frac{dx}{d\theta} \cdot \frac{d^2x}{d\theta^2} m^3 + \&c.\right)$$

$$+ \&c.$$

$$= \frac{dy}{dx} \cdot \frac{dx}{d\theta} m + \left(\frac{dy}{dx} \cdot \frac{d^2x}{d\theta^2} + \frac{d^2y}{dx^2} \cdot \frac{dx^2}{d\theta^2}\right) \cdot \frac{m^2}{1 \cdot 2} + \&c.;$$

$$\therefore \frac{dy}{d\theta} = \frac{dy}{dx} \cdot \frac{dx}{d\theta}, \quad \text{or} \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}},$$

and $$\frac{d^2y}{d\theta^2} = \frac{dy}{dx} \cdot \frac{d^2x}{d\theta^2} + \frac{d^2y}{dx^2} \cdot \frac{dx^2}{d\theta^2};$$

$$\therefore \frac{d^2y}{dx^2} = \frac{\frac{d^2y}{d\theta^2} - \frac{dy}{dx} \cdot \frac{d^2x}{d\theta^2}}{\frac{dx^2}{d\theta^2}}$$

$$= \frac{\frac{dx}{d\theta} \cdot \frac{d^2y}{d\theta^2} - \frac{dy}{d\theta} \cdot \frac{d^2x}{d\theta^2}}{\left(\frac{dx}{d\theta}\right)^3},$$

and similarly may $\frac{d^3y}{dx^3}$ be found.

159. The expression for the radius of curvature being

$$R = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}},$$

we have, when $x = \phi(\theta)$ and $y = f(\theta)$,

$$\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}} = \frac{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{dx}{d\theta}\right)^3},$$

$$\text{and } -\frac{d^2y}{dx^2} = \frac{\dfrac{dy}{d\theta}\cdot\dfrac{d^2x}{d\theta^2} - \dfrac{dx}{d\theta}\cdot\dfrac{d^2y}{d\theta^2}}{\left(\dfrac{dx}{d\theta}\right)^3};$$

$$\therefore\ R = \frac{\left\{\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{dy}{d\theta}\cdot\dfrac{d^2x}{d\theta^2} - \dfrac{dx}{d\theta}\cdot\dfrac{d^2y}{d\theta^2}}.$$

Let $x = -r\cos\theta$ and $y = r\sin\theta$ be the equations between x and θ, and y and θ;

$$\therefore\ \frac{dx}{d\theta} = +r\sin\theta - \cos\theta\frac{dr}{d\theta}, \text{ and } \frac{dy}{d\theta} = r\cos\theta + \sin\theta\,.\,\frac{dr}{d\theta};$$

$$\therefore\ \frac{d^2x}{d\theta^2} = +r\cos\theta + 2\sin\theta\frac{dr}{d\theta} - \cos\theta\,.\,\frac{d^2r}{d\theta^2};$$

$$\text{and } \frac{d^2y}{d\theta^2} = -r\sin\theta + 2\cos\theta\frac{dr}{d\theta} + \sin\theta\,.\,\frac{d^2r}{d\theta^2};$$

$$\therefore\ \frac{dx^2}{d\theta^2} + \frac{dy^2}{d\theta^2} = r^2 + \frac{dr^2}{d\theta^2};$$

$$\text{and } \frac{dy}{d\theta}\frac{d^2x}{d\theta^2} - \frac{dx}{d\theta}\cdot\frac{d^2y}{d\theta^2}$$

$$= r^2 + 2\,.\,\frac{dr^2}{d\theta^2} - r\,.\,\frac{d^2r}{d\theta^2};$$

$$\therefore\ R = \frac{\left(r^2 + \dfrac{dr^2}{d\theta^2}\right)^{\frac{3}{2}}}{r^2 + 2\,.\,\dfrac{dr^2}{d\theta^2} - r\dfrac{d^2r}{d\theta^2}}.$$

Ex. 1. Let $r = a\sin\theta$, the equation to the circle from a point in the circumference, θ being the angle between the tangent at origin and the radius vector r, and a being the diameter;

$$\therefore \frac{dr}{d\theta} = a\cos\theta = \sqrt{a^2 - r^2},$$

$$\frac{d^2 r}{d\theta^2} = -a\sin\theta = -r;$$

$$\therefore R = \frac{(r^2 + a^2 - r^2)^{\frac{3}{2}}}{r^2 + 2a^2 - 2r^2 + r^2} = \frac{a^3}{2a^2} = \frac{a}{2}.$$

Ex. 2. Let $r^2 = a^2 \cos 2\theta$, the equation to the Lemniscata;

$$\therefore \frac{dr}{d\theta} = -\frac{a^2}{r} \,.\, \sin 2\theta = -\frac{a^2}{r} \,.\, \sqrt{1 - \frac{r^4}{a^4}};$$

$$\therefore r^2 + \frac{dr^2}{d\theta^2} = r^2 + \frac{a^4 - r^4}{r^2} = \frac{a^4}{r^2},$$

$$\frac{d^2 r}{d\theta^2} = \frac{a^2}{r^2} \,.\, \sin 2\theta \,.\, \frac{dr}{d\theta} - \frac{2a^2}{r} \,.\, \cos 2\theta$$

$$= -\frac{a^4}{r^3}\left(1 - \frac{r^4}{a^4}\right) - \frac{2a^2}{r} \,.\, \frac{r^2}{a^2};$$

$$\therefore -r\frac{d^2 r}{d\theta^2} = \frac{a^4 - r^4}{r^2} + 2r^2 = \frac{a^4 + r^4}{r^2};$$

$$\therefore r^2 + 2\frac{dr^2}{d\theta^2} - r\frac{d^2 r}{d\theta^2} = r^2 + 2 \,.\, \frac{a^4 - r^4}{r^2} + \frac{a^4 + r^4}{r^2}$$

$$= \frac{3a^4}{r^2};$$

$$\therefore R = \frac{\left(\dfrac{a^4}{r^2}\right)^{\frac{3}{2}}}{\dfrac{3a^4}{r^2}} = \frac{a^6}{3a^4 r} = \frac{a^2}{3r}.$$

Ex. 3. As another exemplification of the formulas of the change of the independent variable, let the equation

$\dfrac{d^2 y}{dx^2} - \dfrac{x}{1 - x^2} \,.\, \dfrac{dy}{dx} + \dfrac{y}{1 - x^2} = 0$, where x is the independent variable, be transformed into one where $\theta = \cos^{-1} x$ shall be the independent variable.

$$x = \cos\theta\,;\quad \therefore\ \frac{dx}{d\theta} = -\sin\theta,\ \text{and}\ \frac{d^2x}{d\theta^2} = -\cos\theta = -x,$$

$$\text{and}\ \frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = -\frac{1}{\sin\theta}\cdot\frac{dy}{d\theta},$$

$$\text{and}\ \frac{d^2y}{dx^2} = \frac{\dfrac{d^2y}{d\theta^2} - \dfrac{dy}{dx}\cdot\dfrac{d^2x}{d\theta^2}}{\dfrac{dx^2}{d\theta^2}} = \frac{\dfrac{d^2y}{d\theta^2} - \dfrac{1}{\sin\theta}\cdot\cos\theta\,\dfrac{dy}{d\theta}}{\cancel{\sin 2\theta}}\,;\ \sin^2\theta$$

$\therefore$ substituting for x, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, and for $1 - x^2 = \sin^2\theta$,

$$\frac{1}{\sin^2\theta}\cdot\frac{d^2y}{d\theta^2} - \frac{\cos\theta}{\sin^3\theta}\cdot\frac{dy}{d\theta} + \frac{\cos\theta}{\sin^3\theta}\cdot\frac{dy}{d\theta} + \frac{y}{\sin^2\theta} = 0\,;$$

$$\therefore\ \frac{d^2y}{d\theta^2} + y = 0,$$

an equation which is satisfied by making

$$y = A\cos(\theta + B).$$

160. Find the radius of curvature, where the arc is the independent variable.

$$R = \frac{\left(1 + \dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\dfrac{d^2y}{dx^2}},\ \text{and}\ \frac{ds^2}{dx^2} = 1 + \frac{dy^2}{dx^2}.$$

But if x and y be functions of s,

$$1 + \frac{dy^2}{dx^2} = 1 + \frac{\left(\dfrac{dy}{ds}\right)^2}{\left(\dfrac{dx}{ds}\right)^2} = \left(\frac{ds}{dx}\right)^2\left\{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right\} = \frac{ds^2}{dx^2}\,;$$

$$\therefore \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1,$$

$$\text{and } \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}} = \left(\frac{ds}{dx}\right)^3,$$

$$\text{and } -\frac{d^2y}{dx^2} = \frac{\dfrac{dy}{ds}\cdot\dfrac{d^2x}{ds^2} - \dfrac{dx}{ds}\cdot\dfrac{d^2y}{ds^2}}{\left(\dfrac{dx}{ds}\right)^3};$$

$$\therefore R = \frac{1}{\dfrac{dy}{ds}\cdot\dfrac{d^2x}{ds^2} - \dfrac{dx}{ds}\cdot\dfrac{d^2y}{ds^2}},$$

which, by multiplying numerator and denominator by ds^3, may be written $R = \dfrac{ds^3}{dy\,d^2x - dx\,d^2y}$,
where dy, dx, d^2y, and d^2x are the first and second differentials of y and x with respect to s.

$$\text{Again, } \frac{1}{R} = \frac{dy}{ds}\cdot\frac{d^2x}{ds^2} - \frac{dx}{ds}\cdot\frac{d^2y}{ds^2}.$$

$$\text{But } \frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} = 1;$$

$$\therefore \frac{dx}{ds}\cdot\frac{d^2x}{ds^2} + \frac{dy}{ds}\cdot\frac{d^2y}{ds^2} = 0;$$

$$\therefore -\frac{dx}{ds} = \frac{dy}{ds}\cdot\frac{\dfrac{d^2y}{ds^2}}{\dfrac{d^2x}{ds^2}},$$

$$\text{and } -\frac{dx}{ds}\,\frac{d^2y}{ds^2} = \frac{dy}{ds}\,\frac{\left(\dfrac{d^2y}{ds^2}\right)^2}{\dfrac{d^2x}{ds^2}};$$

$$\therefore \frac{dy}{ds} \cdot \frac{d^2x}{ds^2} - \frac{dx}{ds} \cdot \frac{d^2y}{ds^2} = \frac{dy}{ds} \cdot \frac{\left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2x}{ds^2}\right)^2}{\frac{d^2x}{ds^2}}.$$

$$\text{But } \left(\frac{d^2y}{ds^2}\right)^2 = \frac{\frac{dx^2}{ds^2}}{\frac{dy^2}{ds^2}} \cdot \left(\frac{d^2x}{ds^2}\right)^2;$$

$$\therefore \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2x}{ds^2}\right)^2 = \frac{\left(\frac{dx^2}{ds^2} + \frac{dy^2}{dx^2}\right)}{\left(\frac{dy}{ds}\right)^2} \cdot \left(\frac{d^2x}{ds^2}\right)^2 = \frac{\left(\frac{d^2x}{ds^2}\right)^2}{\left(\frac{dy}{ds}\right)^2};$$

$$\therefore \sqrt{\left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2x}{ds^2}\right)^2} = \frac{\frac{d^2x}{ds^2}}{\frac{dy}{ds}};$$

$$\therefore \frac{1}{R} = \sqrt{\left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2x}{ds^2}\right)^2};$$

$$\text{and } R = \frac{1}{\sqrt{\left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2x}{ds^2}\right)^2}};$$

or multiplying numerator and denominator by ds^2, and using d^2y, and d^2x, for the second differentials of y and x,

$$R = \frac{ds^2}{\sqrt{(d^2y)^2 + (d^2x)^2}}.$$

Ex. Find the radius of curvature of the catenary.

Here $x = \sqrt{c^2 + s^2}$, and $y = c$, h. l. $\dfrac{s + \sqrt{s^2 + c^2}}{e}$;

$$\therefore \quad \frac{dx}{ds} = \frac{s}{\sqrt{c^2 + s^2}}, \text{ and } \frac{d^2x}{ds^2} = \frac{c^2}{(c^2+s^2)^{\frac{3}{2}}},$$

$$\frac{dy}{ds} = c\,\frac{1}{\sqrt{s^2 + c^2}}, \text{ and } \frac{d^2y}{ds^2} = \frac{-cs}{(c^2+s^2)^{\frac{3}{2}}};$$

$$\therefore \quad \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 = \frac{c^4 + c^2s^2}{(c^2+s^2)^3} = \frac{c^2}{(c^2 + s^2)^2};$$

$$\therefore \quad R = \frac{c^2 + s^2}{c} = \frac{x^2}{c}.$$

CHAPTER XIV.

FUNCTIONS OF TWO OR MORE VARIABLES.—IMPLICIT FUNCTIONS.

161. As yet we have only treated of functions of a single variable, we next proceed to the case in which $u = f(xy)$, where x and y are independent of each other, and the value of u correspondent to new values $x + h$, and $y + k$, of x and y are required.

Now if u is a function of x and y, or $u = f(xy)$, u may vary on three suppositions; 1st, x may vary, and y remain constant; 2d, y may vary, and x remain constant; and 3d, x and y may both vary together.

Suppose $u = xy^2$, and let x become $x + h$, and y remain constant; therefore if u' be the value of u,

$$u^1 = (x + h)\, y^2 = xy^2 + y^2 h.$$

Next let y become $y + k$, and x be constant, and let u_1 be the value of u;

$$\therefore\ u_1 = x\,(y + k)^2 = xy^2 + 2xyk + xk^2.$$

Again, in the equation $u = xy^2$ write $x + h$, and $y + k$ for y, and let u_2 be the value of y, that is, $u_2 = f(x + h,\ y + k)$;

$$\therefore\ u_2 = (x + h)\,(y + k)^2 = xy^2 + y^2 h + 2xyk + 2ykh + xk^2 + k^2 h,$$

the same result as would have been obtained had we put

$$y + k \text{ for } y \text{ in } u^1, \text{ or } x + h \text{ for } x \text{ in } u_1.$$

162. Next *considering* the question in a general point of *view*.

Let $u = f(x, y)$, then if y remain constant, while x becomes $x + h$, we have by Taylor's Theorem,

$$f(x + h, y) = u + \frac{du}{dx} h + \frac{d^2 u}{dx^2} \frac{h^2}{1 \cdot 2} + \frac{d^3 u}{dx^3} \frac{h^3}{2 \cdot 3} + \&c.;$$

or, if x remain constant while y becomes $y + k$,

$$f(x, y + k) = u + \frac{du}{dy} k + \frac{d^2 u}{dy^2} \frac{k^2}{1 \cdot 2} + \frac{d^3 u}{dy^3} \cdot \frac{k^3}{2 \cdot 3} + \&c.$$

Suppose now that x and y both vary; or x become $x + h$, and y become $y + k$; it is not possible to make both these assumptions at once: but if we use either of the two series, for $f(x + h, y)$ or $f(x, y + k)$, and in the former put $y + k$ for y, or in the latter $x + h$ for x, we shall in either case have $f(x + h, y + k)$, and its true developement.

Assuming the first expansion,

$$f(x + h \cdot y) = u + \frac{du}{dx} \cdot h + \frac{d^2 u}{dx^2} \frac{h^2}{1 \cdot 2} + \frac{d^3 u}{dx^3} \frac{h^3}{2 \cdot 3} + \&c.$$

But $u = f(xy)$, and therefore $\frac{du}{dx}$, $\frac{d^2 u}{dx^2}$, are also functions of x and y, if therefore y become $y + k$; u, $\frac{du}{dx}$, $\frac{d^2 u}{dx^2}$, &c. will become functions of $y + k$, and may be expanded by Taylor's Theorem, x being considered constant.

Let therefore y become $y + k$;

$$\therefore\ u \text{ becomes } u + \frac{du}{dy} \cdot k + \frac{d^2 u}{dy^2} \frac{k^2}{1 \cdot 2} + \frac{d^3 u}{dy^3} \cdot \frac{k^3}{2 \cdot 3} + \&c. \ldots\ldots (a),$$

and to obtain the values of $\frac{du}{dx}$, $\frac{d^2 u}{dx^2}$, &c. we must write $\frac{du}{dx}$, $\frac{d^2 u}{dx^2}$, &c. for u in the series (a);

$$\therefore\ \frac{du}{dx} \text{ becomes } \frac{du}{dx} + \frac{d\,.\left(\frac{du}{dx}\right)}{dy}k + \frac{d^2.\left(\frac{du}{dx}\right)}{dy^2}\cdot\frac{k^2}{1\,.\,2} + \&c.$$

$$\frac{d^2u}{dx^2} \cdots\cdots\cdots \frac{d^2u}{dx^2} + \frac{d\,.\left(\frac{d^2u}{dx^2}\right)}{dy}.k + \&c.$$

$$\frac{d^3u}{dx^3} \cdots\cdots\cdots \frac{d^3u}{dx^3} + \frac{d\,.\left(\frac{d^3u}{dx^3}\right)}{dy}.k + \&c.$$

But it has been agreed to write $\frac{d^2u}{dy\,.\,dx}$ for $\frac{d\,.\left(\frac{du}{dx}\right)}{dy}$, which expresses that the function has been differentiated twice, 1st considering x as variable, and then making y variable: and $\frac{d\,.\left(\frac{d^2u}{dx^2}\right)}{dy}$ is written $\frac{d^3u}{dydx^2}$, and $\frac{d^n.\left(\frac{d^mu}{dx^m}\right)}{dy^n}$ is written $\frac{d^{m+n}.u}{dy^n.dx^m}$, denoting the differential coefficient when the function has been differentiated m times with regard to x, and n times with regard to y.

Making these substitutions, and multiplying the expansion of $\frac{du}{dx}$ by h, that of $\frac{d^2u}{dx^2}$ by $\frac{h^2}{1\,.\,2}$, &c. we shall have

$$f(x+h,\ y+k) = u + \frac{du}{dy}k + \frac{d^2u}{dy^2}\frac{k^2}{1\,.\,2} + \frac{d^3u}{dy^3}\frac{k^3}{2\,.\,3} + \&c.$$

$$+ \frac{du}{dx}h + \frac{d^2u}{dy\,.\,dx}hk + \frac{d^3u}{dy^2.\,dx}\cdot\frac{hk^2}{1\,.\,2} + \&c.$$

$$+ \frac{d^2u}{dx^2}\frac{h^2}{1\,.\,2} + \frac{d^3u}{dy\,.\,dx^2}\cdot\frac{h^2k}{1\,.\,2} + \&c.$$

$$+ \frac{d^3u}{dx^3}\frac{h^3}{2\,.\,3} + \&c.$$

$$+ \&c.\ldots\ldots + \&c.$$

163. But this developement was obtained, by first supposing x, and then y to vary; but manifestly we should have had an equal result, had y first become $y+k$, and then x to have increased to $x+h$. On this supposition we have

$$f(x, y+k) = u + \frac{du}{dy}k + \frac{d^2u}{dy^2}\frac{k^2}{1.2} + \frac{d^3u}{dy^3}\cdot\frac{k^3}{2.3} + \&c.;$$

and then putting $x+h$ for x,

$$u \text{ becomes } u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{2.3} + \&c.,$$

$$\frac{du}{dy} \cdots\cdots\cdots\cdots \frac{du}{dy} + \frac{d^2u}{dxdy}h + \frac{d^3u}{dx^2dy}\cdot\frac{h^2}{1.2} + \&c.,$$

$$\frac{d^2u}{dy^2} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \frac{d^2u}{dy^2} + \frac{d^3u}{dxdy^2}\frac{h}{1} + \&c.,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \&c.;$$

whence by substitution the total developement becomes

$$\begin{aligned} f(x+h, y+k) = u &+ \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{2.3} + \&c. \\ &+ \frac{du}{dy}k + \frac{d^2u}{dxdy}kh + \frac{d^3u}{dx^2dy}\frac{h^2k}{1.2} + \&c. \\ &+ \frac{d^2u}{dy^2}\frac{k^2}{1.2} + \frac{d^3u}{dxdy^2}\frac{k^2h}{1.2} + \&c. \\ &+ \frac{d^3u}{dy^3}\frac{k^3}{2.3} + \&c. \\ &+ \&c. \end{aligned}$$

Cor. 1. Since the series are equal, the coefficients of the same powers of h and k ought to be equal;

$$\therefore \frac{d^2u}{dydx} = \frac{d^2u}{dxdy},$$

$$\frac{d^3u}{dy^2dx} = \frac{d^3u}{dxdy^2},$$

$$\frac{d^3u}{dydx^2} = \frac{d^3u}{dx^2dy},$$

$$\&c. = \&c.$$

$$\text{and } \frac{d^{m+n}.u}{dy^m dx^n} = \frac{d^{m+n}u}{dx^n dy^m}.$$

Whence it appears that the order of differentiation produces no alteration; or that the differential coefficient of u differentiated m times with respect to x, and n times with respect to y, is equal to the differential coefficient when u has been first differentiated n times with regard to y, and then m times with regard to x.

Cor. 2. Again, since $\dfrac{d^2u}{dy.dx} = \dfrac{d^2u}{dxdy}$;

therefore by writing $\dfrac{du}{dx}$ for u we have

$$\frac{d^2.\left(\frac{du}{dx}\right)}{dy.dx} = \frac{d^2.\left(\frac{du}{dx}\right)}{dxdy},$$

$$\text{or } \frac{d^3u}{dydx^2} = \frac{d^3u}{dxdydx},$$

$$\text{and } \frac{d^2.\left(\frac{du}{dy}\right)}{dydx} = \frac{d^2.\left(\frac{du}{dy}\right)}{dxdy};$$

$$\therefore \frac{d^3.u}{dydxdy} = \frac{d^3.u}{dx.dy^2}.$$

164. Since $\dfrac{du}{dx}, \dfrac{d^2u}{dx^2}, \dfrac{d^3u}{dx^3}$, &c. have been obtained by the consideration of x alone being the independent variable, such differential coefficients have been called *partial* differ-

ential coefficients, and for the same reason $\frac{du}{dy}$, $\frac{d^2u}{dy^2}$, &c. are also called partial differential coefficients, and these partial differential coefficients are frequently included within brackets, thus $\left(\frac{du}{dx}\right)$ is the partial differential coefficient with respect to x, and $\left(\frac{du}{dy}\right)$ is the partial differential coefficient with respect to y, and $\left(\frac{du}{dx}\right)dx$, and $\left(\frac{du}{dy}\right)dy$, are the partial differentials of u, with regard to x and y respectively.

165. The term $\frac{du}{dx}h + \frac{du}{dy}k$, which involves only the first powers of h and k is called the total differential of u, and putting dx for h, and dy for k, is thus written;

$$du = \left(\frac{du}{dx}\right). dx + \left(\frac{du}{dy}\right) dy,$$

or the total differential of $u = f(xy)$ is the sum of the partial differentials.

Ex. 1. $u = x^m y^n$. Find du and shew that $\frac{d^2u}{dydx} = \frac{d^2u}{dxdy}$

$$\left(\frac{du}{dx}\right) = mx^{m-1}y^n, \quad \left(\frac{du}{dy}\right) = nx^m y^{n-1};$$

$$\therefore\ du = mx^{m-1}.y^n.dx + nx^m y^{n-1}dy$$

$$= x^{m-1}y^{n-1}(mydx + nxdy),$$

$$\text{and } \frac{d^2u}{dydx} = nmx^{m-1}.y^{n-1} = \frac{d^2u}{dxdy}.$$

Ex. 2. $u = \sin x^2 y$,

$$\frac{du}{dx} = 2xy\cos x^2y, \quad \frac{du}{dy} = x^2\cos x^2 y,$$

$$\frac{d^2u}{dy.dx} = +2x\cos x^2y - 2x^3y\sin x^2y = \frac{d^2u}{dxdy}.$$

166. Having given the first differential of u, we may form by differentiation the successive differentials d^2u, d^3u, &c. For

$$du = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy \text{ .... } (\beta).$$

And differentiating, considering $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ as functions of x and y, and dx and dy constant, we have, by writing successively, $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ for u in (β),

$$d.\left(\frac{du}{dx}\right) = \left(\frac{d^2u}{dx^2}\right).dx + \frac{d^2u}{dy\,dx}.dy,$$

$$d.\left(\frac{du}{dy}\right) = \frac{d^2u}{dx\,dy}.dx + \left(\frac{d^2u}{dy^2}\right)dy.$$

Then substituting these values, since

$$d^2u = d.\left(\frac{du}{dx}\right).dx + d\left(\frac{du}{dy}\right).dy,$$

$$d^2u = \frac{d^2u}{dx^2}.dx^2 + 2.\frac{d^2u}{dx.dy}.dy.dx + \frac{d^2u}{dy^2}.dy^2.$$

Again, to find d^3u, substituting as before

$$d.\left(\frac{d^2u}{dx^2}\right) = \left(\frac{d^3u}{dx^3}\right).dx + \frac{d^3u}{dy\,dx^2}.dy,$$

$$d.\left(\frac{d^2u}{dx\,dy}\right) = \frac{d^3u}{dx^2dy}.dx + \frac{d^3u}{dx\,dy\,dx}.dy,$$

$$d.\left(\frac{d^3u}{dy^2}\right) = \frac{d^3u}{dx\,dy^2}.dx + \left(\frac{d^3u}{dy^3}\right)dy;$$

$$\therefore\ d^3u = \left(\frac{d^3u}{dx^3}\right)dx^3 + 3.\frac{d^3u}{dx^2dy}dx^2dy$$

$$+ 3.\frac{d^3u}{dy^2dx}.dy^2.dx + \left(\frac{d^3u}{dy^3}\right)dy^3.$$

167. The law of continuity is almost obvious; for the numerical coefficients appear to be those of the terms of the expansion of the binomial $(h+k)^n$: but to prove that the coefficients of the successive differentials of u do follow the law of the coefficients of the binomial theorem, let us assume that

$$d^n u = \frac{d^n u}{dx^n}dx^n + n \cdot \frac{d^n u}{dx^{n-1}dy} \cdot dx^{n-1}dy$$

$$+ n \cdot \frac{n-1}{2} \frac{d^n u}{dx^{n-2}dy^2} dx^{n-2}dy^2 + \&c.$$

Then, differentiating the successive terms by means of

$$du = \frac{du}{dx}dx + \frac{du}{dy} \cdot dy,$$

$$d \cdot \left(\frac{d^n u}{dx^n}\right) = \frac{d^{n+1}u}{dx^{n+1}} \cdot dx + \frac{d^{n+1}u}{dx^n dy} \cdot dy \ldots\ldots\ldots\ldots (1),$$

$$d \cdot \left(\frac{d^n u}{dx^{n-1}dy}\right) = \frac{d^{n+1}u}{dx^n dy} \cdot dx + \frac{d^{n+1}u}{dx^{n-1}dy^2} \cdot dy \ldots\ldots\ldots (2),$$

$$d \cdot \left(\frac{d^n u}{dx^{n-2}dy^2}\right) = \frac{d^{n+1}u}{dx^{n-1}dy^2} \cdot dx + \frac{d^{n+1}u}{dx^{n-2}dy^3} \cdot dy \ldots\ldots (3),$$

$$\&c. = \&c.$$

Then, multiply (1) by dx^n, (2) by $n \cdot dx^{n-1}dy$, (3) by $n \cdot \left(\frac{n-1}{2}\right) \cdot dx^{n-2}dy^2$, and adding

$$d^{n+1}u = \frac{d^{n+1}u}{dx^{n+1}} \cdot dx^{n+1} + (n+1) \cdot \left(\frac{d^{n+1}u}{dx^n dy}\right) \cdot dx^n dy$$

$$+ \frac{(n+1) \cdot n}{2} \cdot \frac{d^{n+1}u}{dx^{n-1}dy^2} \cdot dx^{n-1}dy^2 + \&c.$$

or if the formula be true for the index n, it is true for $n+1$, and we have seen that it is true when $n = 3$; it is therefore always true.

Cor. If instead of dx and dy we write h and k, we have

$$du = \frac{du}{dx} h + \frac{du}{dy} k,$$

$$d^2u = \frac{d^2u}{dx^2} h^2 + 2\frac{d^2u}{dy\,dx} hk + \frac{d^2u}{dy^2} k^2,$$

$$d^3u = \frac{d^3u}{dx^3} h^3 + 3\frac{d^3u}{dx^2\,dy} h^2k + 3\frac{d^3u}{dy^2\,dx} hk^2 + \frac{d^3u}{dy^3} k^3,$$

&c.

and therefore $u_2 = f\{(x+h), (y+k)\}$

$$= u + du + \frac{d^2u}{1.2} + \frac{d^3u}{2.3} + \text{\&c.}$$

$$+ \frac{d^nu}{1.2.3\ldots n} + \text{\&c.}$$

or the expansion of $f(x+h, y+k)$ may be found from the successive differentiation of $u = f(x, y)$.

168. Again, if $u = f(x, y, z)$, and if $x+h$, $y+k$, $z+m$, be new values of x, y, z, and u_2 the value of u,

$$u_2 = u + \frac{du}{dx} h + \frac{du}{dy} k + \frac{du}{dz} m$$

$$+ Ah^2 + Bk^2 + Cm^2 + \text{\&c.}$$

For, supposing z to be constant while x and y become $x+h$, and $y+k$ respectively;

$$\therefore f(x+h, y+k, z) = u + \frac{du}{dx} h + \frac{du}{dy} k + \text{\&c.}$$

Now, let z become $z+m$;

$$\therefore u \text{ becomes } u + \frac{du}{dz} m + \text{\&c.}$$

$$\frac{du}{dx} \cdots\cdots\cdots \frac{du}{dx} + \frac{d^2u}{dz\,dx} m + \text{\&c.}$$

$$\frac{du}{dy} \text{ becomes } \frac{du}{dy} + \frac{d^2u}{dz\,dy}m + \&c.$$

$$\&c. \quad = \quad z,$$

$$u_2 = f(x+h,\ y+k,\ z+m) = u + \frac{du}{dz}m + \frac{du}{dx}h + \frac{du}{dy}k + \&c.$$

and in a similar manner may the expansion of a function of four or more variables be effected.

COR. Hence, if for m, h, and k, we put dz, dx, and dy,

$$du = d\,.f(x,\ y,\ z) = \left(\frac{du}{dx}\right)dz + \left(\frac{du}{dy}\right)dy + \left(\frac{du}{dz}\right)dz.$$

This result may however be obtained in the following manner:

169. Let $u = f(x,\ y,\ z)$; find du.

Let $n = \phi\,(y,\ z)$, so that we may put

$$u = f(x,\ n);$$

$$\therefore\ du = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dn}\right)dn.$$

But $n = \phi\,(y,\ z)$,

$$\therefore\ dn = \frac{dn}{dy}\,.\,dy + \frac{dn}{dz}\,.\,dz;$$

$$\therefore\ du = \frac{du}{dx}dx + \frac{du}{dn}\,.\,\frac{dn}{dy}dy + \frac{du}{dn}\,.\,\frac{dn}{dz}dz.$$

$$\text{But } \frac{du}{dn}\,.\,\frac{dn}{dy} = \frac{du}{dy},$$

$$\text{and } \frac{du}{dn}\,.\,\frac{dn}{dz} = \frac{du}{dz};$$

$$\therefore\ du = \frac{du}{dx}\,.\,dx + \frac{du}{dy}\,.\,dy + \frac{du}{dz}\,.\,dz;$$

and the same method may be extended to any number of variables; whence it appears that the differential of a function of any number of variables = the sum of the partial differentials.

IMPLICIT FUNCTIONS.

170. When y is an implicit function of x, it is frequently very difficult to solve the equation with respect to y, and to obtain $y = f(x)$; but by considering $f(x, y) = 0$ to be a function of two variables, we may from the preceding expansions for such functions obtain rules easy of application.

We shall first shew that if $u = 0 = f(x, y)$, $du = 0$.

Let u_1 represent the function when x becomes $x + h$; and therefore y becomes $y + k$;

$$\begin{aligned} \therefore\ u_1 = u &+ \left(\frac{du}{dx}\right) h + \left(\frac{d^2u}{dx^2}\right) . \frac{h^2}{1 . 2} + \&c. \\ &+ \left(\frac{du}{dy}\right) k + \frac{d^2u}{dx\,dy} . kh + \&c. \\ &+ \left(\frac{d^2u}{dy^2}\right) . \frac{k^2}{1 . 2} + \&c. \\ &+ \&c. \end{aligned}$$

But because $u = 0$, whatever the values of x and y may be; therefore also $u_1 = 0$;

$$\therefore\ 0 = \left(\frac{du}{dx}\right) h + \left(\frac{du}{dy}\right) k + \left(\frac{d^2u}{dx^2}\right) \frac{h^2}{1 . 2} + \frac{d^2u}{dx\,dy} hk + \left(\frac{d^2u}{dy^2}\right) \frac{k^2}{1 . 2}.$$

But $y + k$ is also a function of $x + h$,

$$k = \frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1 . 2} + \&c.$$

therefore, substituting for k,

$$0 = \left\{\left(\frac{du}{dx}\right) + \left(\frac{du}{dy}\right) \frac{dy}{dx}\right\} h + Bh^2 + Ch^3 + \&c.$$

$$\therefore \left(\frac{du}{dx}\right) + \left(\frac{du}{dy}\right)\frac{dy}{dx} = 0,$$

a theorem by which $\frac{dy}{dx}$ may be found from the partial differentials $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$.

Ex. $y^3 - 3axy + x^3 = 0$; find $\frac{dy}{dx}$.

Let $u = y^3 - 3axy + x^3$;

$$\therefore \frac{du}{dx} = -3ay + 3x^2,$$

$$\frac{du}{dy} = 3y^2 - 3ax.$$

$$\text{But } \left(\frac{du}{dx}\right) + \left(\frac{du}{dy}\right) \cdot \frac{dy}{dx} = 0;$$

$$\therefore -3ay + 3x^2 + (3y^2 - 3ax)\frac{dy}{dx} = 0;$$

$$\therefore \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}.$$

Cor. 1. Since $\left(\frac{du}{dx}\right) + \left(\frac{du}{dy}\right) \cdot \frac{dy}{dx} = 0$;

$$\therefore \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) \cdot \frac{dy}{dx} dx = 0,$$

or $\left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy = 0.$ Since $y = f(x)$.

Whence $du = 0$.

Cor. 2. Hence, since if $u = 0$, $du = 0$; $\therefore$ if $du = 0$, $d^2u = 0$; and thus if $u = 0$, $d^n u = 0$.

COR. 3. From the equation

$$\left(\frac{du}{dx}\right) + \left(\frac{du}{dy}\right) \cdot \frac{dy}{dx} = 0,$$

may $d^2u = 0$ be found; for writing p for $\frac{dy}{dx}$, we have

$$\left(\frac{du}{dx}\right) + p\left(\frac{du}{dy}\right) = 0;$$

and $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ are functions of x and y; therefore if v be put

$$= \left(\frac{du}{dx}\right) + p\left(\frac{du}{dy}\right),$$

$v = 0$ will be a function of the three variables x, y, and p; and therefore

$$dv = 0 = d^2u = \left(\frac{dv}{dx}\right) dx + \left(\frac{dv}{dy}\right) dy + \left(\frac{dv}{dp}\right) dp = 0,$$

$$\text{or } \frac{dv}{dx} + \frac{dv}{dy} \cdot \frac{dy}{dx} + \frac{dv}{dp} \cdot \frac{dp}{dx} = 0;$$

$$\text{whence } \frac{dp}{dx} = q = \frac{d^2y}{dx^2} \text{ may be found;}$$

and if d^3u be required, we must put $w = f(x, y, p, q)$; and then we have

$$d^3u = \frac{dw}{dx} dx + \frac{dw}{dy} dy + \frac{dw}{dp} dp + \frac{dw}{dq} dq = 0;$$

$$\text{whence } \frac{dq}{dx} \text{ or } \frac{d^3y}{dx^3} \text{ may be found.}$$

171. Next, let $u = 0$ be a function of three variables x, y, z, or let z be an implicit function of (x, y); and let $z + m$ be the value of z when the independent variables, x and y, become respectively $x + h$ and $y + k$;

$\therefore$ since $u_1 = 0 = f(x+h,\ y+k,\ z+m)$,

$$0 = \left(\frac{du}{dx}\right) h + \left(\frac{du}{dy}\right) k + \left(\frac{du}{dz}\right) m + Ah^2 + Bk^2 + Cm^2 + \&c.$$

But $z + m = \phi(x+h,\ y+k)$;

$$\therefore\ m = \left(\frac{dz}{dx}\right) h + \left(\frac{dz}{dy}\right) k + \&c.$$

therefore, substituting for m, we have

$$0 = \left\{\left(\frac{du}{dx}\right) + \left(\frac{du}{dz}\right) \cdot \frac{dz}{dx}\right\} h + \left\{\left(\frac{du}{dy}\right) + \left(\frac{du}{dz}\right) \cdot \frac{dz}{dy}\right\} k + \&c.$$

whence, since h and k are independent, we have

$$\left(\frac{du}{dx}\right) + \left(\frac{du}{dz}\right) \cdot \frac{dz}{dx} = 0 \ldots\ldots\ldots (1),$$

$$\left(\frac{du}{dy}\right) + \left(\frac{du}{dz}\right) \cdot \frac{dz}{dy} = 0 \ldots\ldots\ldots (2);$$

whence $dz = \dfrac{dz}{dx} dx + \dfrac{dz}{dy} dy$ may be found.

172. If we wish to obtain the differential coefficients of the superior orders, we can find them by differentiating the equations

$$\left(\frac{du}{dx}\right) + \left(\frac{du}{dz}\right) \cdot \frac{dz}{dx} = 0 \ldots\ldots\ldots\ldots (1),$$

$$\left(\frac{du}{dy}\right) + \left(\frac{du}{dz}\right) \cdot \frac{dz}{dy} = 0 \ldots\ldots\ldots\ldots (2),$$

and thus obtain $\left(\dfrac{d^2z}{dx^2}\right)$, $\dfrac{d^2z}{dydx}$ and $\left(\dfrac{d^2z}{dy^2}\right)$.

We must consider these equations as functions of x, y, z, and that $\left(\dfrac{du}{dx}\right)$, $\left(\dfrac{du}{dy}\right)$ and $\left(\dfrac{du}{dz}\right)$ are also functions of the same variables.

Let equation (1) be differentiated with respect to x, it must then be considered as a function of x and z, and the differentials of $\frac{du}{dx}$ and $\frac{du}{dz}$, may be derived from (1), by first putting $\frac{du}{dx}$ for u, and then $\frac{du}{dz}$ for u in equation (1);

$$\therefore \left(\frac{d^2u}{dx^2}\right) + \frac{d^2u}{dx \,.\, dz} \cdot \frac{dz}{dx} + \frac{d^2u}{dz \,.\, dx} \cdot \frac{dz}{dx}$$

$$+ \left(\frac{d^2u}{dz^2}\right) \frac{dz^2}{dx^2} + \left(\frac{du}{dz}\right) \cdot \frac{d^2z}{dx^2} = 0,$$

$$\text{or } \left(\frac{d^2u}{dx^2}\right) + 2 \cdot \frac{d^2u}{dz \,.\, dx} \cdot \frac{dz}{dx} + \left(\frac{d^2u}{dz^2}\right) \cdot \frac{dz^2}{dx^2}$$

$$+ \left(\frac{du}{dz}\right) \cdot \left(\frac{d^2z}{dx^2}\right) = 0 \,\ldots\ldots\ldots\, (3).$$

Again, differentiate (2) with respect to y.

We may obtain the differentials of $\left(\frac{du}{dy}\right)$ and $\left(\frac{du}{dz}\right)$ from (2) by first writing $\left(\frac{du}{dy}\right)$ for u, and then $\left(\frac{du}{dz}\right)$ for u in (2), whence we have

$$\left(\frac{d^2u}{dy^2}\right) + 2 \frac{d^2u}{dz dy} \cdot \frac{dz}{dy} + \left(\frac{d^2u}{dz^2}\right) \cdot \frac{dz^2}{dy^2}$$

$$+ \left(\frac{du}{dz}\right) \cdot \left(\frac{d^2z}{dy^2}\right) = 0 \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\, (4)$$

Now either differentiate (1) with respect to y, or (2) with respect to x; and since in the former case $\frac{dz}{dx}$ becomes $\frac{d^2z}{dy dx}$, and that in the latter $\frac{dz}{dy}$ becomes $\frac{d^2z}{dx dy}$, and that $\frac{d^2z}{dy dx} = \frac{d^2z}{dx dy}$, the results will be identical.

Let equation (1) be differentiated with respect to y; and to do this put $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dz}\right)$ for u in equation (2), whence we have

$$\frac{d^2u}{dx \cdot dy} + \frac{d^2u}{dxdz} \cdot \frac{dz}{dy} + \frac{d^2u}{dz \cdot dy} \cdot \frac{dz}{dx}$$

$$+ \left(\frac{d^2u}{dz^2}\right) \cdot \frac{dz}{dx} \cdot \frac{dz}{dy} + \left(\frac{dx}{dz}\right) \frac{d^2z}{dy \cdot dx} = 0 \ldots\ldots\ldots (5).$$

From the equations (3), (4), (5), $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dy^2}$ and $\frac{d^2z}{dydx}$ may be found.

By a similar process may the differentials of the third order be found.

ELIMINATION BY MEANS OF DIFFERENTIATION.

173. We have seen that if a constant quantity be connected with the function by the signs $\pm$, it disappears from the differential coefficients. Should however it be multiplied into the function or any term of the function, it will still appear in the value of the differential coefficient.

Thus if $u = 0$ be a function of x and y, involving a constant a, both $u = 0$ and $du = 0$ will contain a, but between these two equations it may be eliminated, and an equation will arise independent of a, which is called a differential equation.

Thus, let $y = ax^2$;

$$\therefore \frac{dy}{dx} = 2ax, \quad \text{or } a = \frac{1}{2x} \cdot \frac{dy}{dx};$$

$$\therefore y = \frac{x}{2} \cdot \frac{dy}{dx},$$

an equation from which a has disappeared.

By differentiation also irrational and transcendental quantities may be eliminated.

$$\text{Thus, let } y = (a^2 + x^2)^{\frac{m}{n}};$$

$$\therefore \frac{dy}{dx} = 2x \cdot \frac{m}{n} \cdot (a^2 + x^2)^{\frac{m}{n} - 1} = \frac{2mx\,(a^2 + x^2)^{\frac{m}{n}}}{n\,(a^2 + x^2)} = \frac{2mxy}{n\,(a^2 + x^2)}.$$

If there be two constants as a and b involved in the equation $y = f(x)$, then to eliminate them, the equations $u = 0$, $du = 0$ and $d^2u = 0$ must be combined.

Ex. 1. Let $u = y - ax^2 - bx = 0$,

$$\text{or } y = ax^2 + bx;$$

$$\therefore \frac{dy}{dx} = 2ax + b,$$

$$\frac{d^2y}{dx^2} = 2a;$$

$$\therefore b = \frac{dy}{dx} - x\frac{d^2y}{dx^2};$$

$$\therefore y - \frac{x^2}{2}\frac{d^2y}{dx^2} - x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = 0;$$

$$\therefore \frac{d^2y}{dx^2} - \frac{2}{x} \cdot \frac{dy}{dx} + \frac{2y}{x^2} = 0.$$

Ex. 2. Let $y = a \cdot \cos mx + b \cdot \sin mx$ eliminate a and b.

$$\frac{dy}{dx} = -ma \sin mx + mb \cos mx,$$

$$\frac{d^2y}{dx^2} = -m^2a \cos mx - m^2b \sin mx$$

$$= -m^2 \{a \cos mx + b \sin mx\}$$

$$= -m^2 y;$$

$$\therefore \quad \frac{d^2 y}{dx^2} + m^2 y = 0.$$

Ex. 3. Let $y = ae^{2x} \sin(3x + b)$ eliminate a and b.

$$\frac{dy}{dx} = 2ae^{2x} \sin(3x + b) + 3ae^{2x} \cos(3x + b)$$

$$= 2y + 3y \cot(3x + b),$$

$$\frac{d^2 y}{dx^2} = 4ae^{2x} \sin(3x + b) + 6ae^{2x} \cos(3x + b)$$

$$+ 6ae^{2x} \cos(3x + b) - 9ae^{2x} \sin(3x + b)$$

$$= -5y + 12y \cot(3x + b)$$

$$= -5y + 4\frac{dy}{dx} - 8y;$$

$$\therefore \quad \frac{d^2 y}{dx^2} - 4\frac{dy}{dx} + 13y = 0.$$

174. Again, if $u = f(xyz) = 0$, or $z = f(xy)$.

We may by means of the partial differential coefficients $\frac{dz}{dy}$ and $\frac{dz}{dx}$ eliminate two constants from $z = f(xy)$, and by proceeding to the second differential, we have three other equations for $\frac{d^2 z}{dx^2}$ $\frac{d^2 z}{dydx}$ and $\frac{d^2 z}{dy^2}$, and therefore five constants may be eliminated: and not only constants, but indeterminate functions.

Ex. 1. Let $z = f(ax + by)$; eliminate the arbitrary function.

$$\text{Let } (ax + by) = v;$$

$$\therefore \; z = f(v),$$

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and $\frac{dz}{dx} = \frac{dz}{dv} \cdot \frac{dv}{dx}$.

But $\frac{dz}{dv} = f'(v)$, and $\frac{dv}{dx} = a$;

$$\therefore \frac{dz}{dx} = af'(v),$$

and $\frac{dz}{dy} = \frac{dz}{dv} \cdot \frac{dv}{dy} = f'(v) \cdot b$,

or $b \cdot \frac{dz}{dx} = abf'(v)$,

and $a \cdot \frac{dz}{dy} = ab \cdot f'(v)$;

$$\therefore b \frac{dz}{dx} - a \frac{dz}{dy} = 0.$$

As an example. Let $z = \sin(ax + by)$;

$$\therefore \frac{dz}{dx} = a \cos(ax + by),$$

and $\frac{dz}{dy} = b \cos(ax + by)$;

$$\therefore b \frac{dz}{dx} - a \frac{dz}{dy} = 0,$$

and similarly if $z = (ax + by)^n$, or $z = \log(ax + by)$, the differential equation will be verified.

Ex. 2. Let $z = (x + y)^m \phi(x^2 - y^2)$; eliminate the function.

$$\frac{dz}{dx} = m \cdot (x+y)^{m-1} \phi(x^2 - y^2) + 2(x+y)^m \phi'(x^2 - y^2) \cdot x \ldots (1),$$

$$\frac{dz}{dy} = m \cdot (x+y)^{m-1} \phi(x^2 - y^2) - 2(x+y)^m \phi'(x^2 - y^2) \cdot y \ldots (2).$$

Multiply (1) by y, and (2) by x, and add;

$$\therefore\ y\frac{dz}{dx}+x\frac{dz}{dy}=m(x+y)^m\phi(x^2-y^2)=mz,$$

$$\text{or } py+qx=mz.$$

Ex. 3. Let $zx=f\left(\frac{x}{y}\right)$ eliminate the function.

$$z=\frac{1}{x}f\left(\frac{x}{y}\right);$$

$$\therefore\ \frac{dz}{dx}=-\frac{1}{x^2}f\left(\frac{x}{y}\right)+\frac{1}{xy}.f'.\left(\frac{x}{y}\right),$$

$$\frac{dz}{dy}=\ldots\ldots\ldots\ldots\ldots-\frac{1}{y^2}f'.\left(\frac{x}{y}\right),$$

$$\text{or } p+\frac{z}{x}=\frac{1}{xy}.f'\left(\frac{x}{y}\right),$$

$$q=-\frac{1}{y^2}.f'\left(\frac{x}{y}\right);$$

$$\therefore\ \frac{px+z}{qx}=-\frac{y}{x};$$

$$\therefore\ px+qy+z=0.$$

Ex. 4. Let $z=f(y+ax)+\phi(y-ax)$ eliminate the arbitrary functions.

$$\frac{dz}{dx}=a.f'(y+ax)-a\phi'(y-ax),$$

$$\frac{dz}{dy}=f'(y+ax)+\phi'(y-ax).$$

Differentiating a second time,

$$\frac{d^2z}{dx^2} = a^2 f''(y + ax) + a^2 \phi''(y - ax),$$

$$\frac{d^2z}{dy^2} = f''(y + ax) + \phi''(y - ax)$$

$$= \frac{1}{a^2} \cdot \frac{d^2z}{dx^2};$$

$$\therefore \frac{d^2z}{dx^2} - a^2 \cdot \frac{d^2z}{dy^2} = 0.$$

This equation occurs in some investigations in Natural Philosophy.

CHAPTER XV.

MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES. LAGRANGE'S THEOREM.

175. If $u = f(x, y)$ be an equation between the function u, and the two independent variables, x and y, there may be some particular value of x, and also a value of y, which will make the function greater or less than the values which immediately precede or follow it. It is then a maximum or minimum. We proceed to find the relation between the differential coefficients, when this circumstance takes place.

176. Let u_1 be the value of u, when $x+h$ and $y+k$, are written for x and y respectively; and u_2 the value of u when $x-h$ and $y-k$ are substituted for the same quantities. Also put A for $\dfrac{d^2u}{dx^2}$, B for $\dfrac{d^2u}{dy\,dx}$, and C for $\dfrac{d^2u}{dy^2}$. Then

$$u_1 = u + \frac{du}{dx}h + \frac{du}{dy}k + \tfrac{1}{2}\{Ah^2 + 2Bhk + Ck^2\} + \&c.$$

$$\text{and } u_2 = u - \left(\frac{du}{dx}h + \frac{du}{dy}k\right) + \tfrac{1}{2}\{Ah^2 + 2Bhk + Ck^2\} - \&c.$$

Now since the values of h and k may be assumed so small that, (as long as the differential coefficients $\dfrac{du}{dx}$ and $\dfrac{du}{dy}$ remain finite) the algebraical sign of $u_1 - u$ and $u_2 - u$ will depend upon that of the term

$$\left(\frac{du}{dx}h + \frac{du}{dy}k\right),$$

it is manifest, that if this term exist, $u_1 - u$ and $u_2 - u$ cannot be both positive or both negative, or there cannot be a minimum or maximum of u.

Therefore at a maximum or minimum $\frac{du}{dx}h + \frac{du}{dy}k$ must $= 0$. A condition which can only be fulfilled by making $\frac{du}{dx} = 0$, and $\frac{du}{dy} = 0$.

Therefore at a maximum or minimum,

$$u_1 - u = \tfrac{1}{2}(Ah^2 + 2Bhk + Ck^2) + \&c.$$

$$= \frac{h^2}{1 \cdot 2}\{A + 2Bn + Cn^2\} + \&c.$$

by putting $k = nh$.

Therefore the sign of $u_1 - u$, and also of $u_2 - u$, will depend upon that of the coefficient of $\frac{h^2}{2}$, that is, upon

$$A + 2Bn + Cn^2.$$

Hence, this term must not change its sign whatever be the value of n; which it will not do, if it can be put under the form of the sum of two squares, as $(x + a)^2 + \beta^2$.

Now $A + 2Bn + Cn^2 = \frac{1}{C}\{CA + 2BCn + C^2n^2\}$

$$= \frac{1}{C}\{CA - B^2 + (B + Cn)^2\}$$

$$= \frac{1}{C}\left\{CA - B^2 + C^2\left(\frac{B}{C} + n\right)^2\right\},$$

which will be of the requisite form, if CA be not less than B^2: or to have a maximum or minimum of a function of two variables, we must first have $\frac{du}{dx} = 0$ and $\frac{du}{dy} = 0$; and secondly, $\frac{d^2u}{dx^2} \times \frac{d^2u}{dy^2}$ not less than $\left(\frac{d^2u}{dydx}\right)^2$.

It is obvious that $\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dy^2}$ must have the same sign; and if they be both negative, u is a maximum, if positive, u is a minimum.

If the second differential coefficient of u become $= 0$, when the first does, there will not be a maximum or minimum, unless the third differential coefficient vanishes, and the fourth neither vanishes nor changes its sign whatever be the value of n.

Ex. 1. Let $u = x^3 + y^3 - 3axy$,

$$\frac{du}{dx} = 3x^2 - 3ay = 0; \quad \therefore \ y = \frac{x^2}{a},$$

$$\frac{du}{dy} = 3y^2 - 3ax = 0;$$

$$\therefore \ \frac{x^4}{a^2} - ax = 0;$$

therefore $x = 0$, and $x^3 - a^3 = 0$; whence $x = a$, the other two roots are impossible,

$$\text{and } y = \frac{x^2}{a} = \frac{a^2}{a} = a, \quad \text{or } = 0.$$

$$\text{Also } \frac{d^2u}{dx^2} = 6x, \quad \frac{d^2u}{dy^2} = 6y, \quad \text{and } \frac{d^2u}{dy\,dx} = -3a.$$

$$\text{If } x = 0, \quad A = 0, \quad C = 0, \quad \text{and } B = -3a.$$

$$\text{If } x = a, \quad A = 6a, \ C = 6a, \qquad B = -3a,$$

$$AC = 36a^2, \quad \text{and } B^2 = 9a^2,$$

and $x = a$ gives a minimum,

$$\text{and } u = -a^3,$$

$x = 0$ gives neither a maximum nor minimum.

Ex. 2. $u = x^3y^2(a - x - y)$, find the values of x and y that u may be a maximum or minimum.

$$\frac{du}{dx} = 3x^2y^2(a - x - y) - x^3y^2 = 0,$$

$$\frac{du}{dy} = 2x^3y(a - x - y) - x^3y^2 = 0;$$

$$\therefore\ 3(a - x - y) = x,$$

$$2(a - x - y) = y;$$

$$\therefore\ 2x = 3y;$$

$$\therefore\ 3a - 3x - 2x = x, \quad \text{or } x = \frac{a}{2},$$

$$2a - 3y - 2y = y, \quad \text{or } y = \frac{a}{3};$$

$$\therefore\ a - x - y = a - \frac{a}{2} - \frac{a}{3} = \frac{a}{6},$$

$$\frac{d^2u}{dx^2} = A = 6xy^2(a - x - y) - 6x^2y^2 = 6\left\{\frac{a}{2}\cdot\frac{a^2}{9}\cdot\frac{a}{6} - \frac{a^2}{4}\cdot\frac{a^2}{9}\right\} = -\frac{a^4}{9},$$

$$\frac{d^2u}{dy^2} = C = 2x^3(a - x - y) - 4x^3y = 2\left\{\frac{a^3}{8}\cdot\frac{a}{6} - 2\cdot\frac{a^3}{8}\cdot\frac{a}{3}\right\} = -\frac{a^4}{8},$$

$$\frac{d^2u}{dx\,dy} = B = 6x^2y(a - x - y) - 3x^2y^2 - 2x^3y = \frac{a^4}{12} - \frac{a^4}{12} - \frac{a^4}{12} = -\frac{a^4}{12};$$

$$\therefore\ AC = \frac{a^8}{72}, \quad \text{and } B^2 = \frac{a^8}{144};$$

therefore AC is $> B^2$, and A is negative,

$$\text{and } u = \frac{a^3}{8} \times \frac{a^2}{9} \times \frac{a}{6} = \frac{a^6}{432} \text{ is a maximum.}$$

Ex. 3. $u = (x + 1)\,.\,(y + 1)\,.\,(z + 1)$, where $a^x b^y c^z = A$:

$$\therefore\ x \text{ h. l. } a + y \text{ h. l. } b + z \text{ h. l. } c = \text{h. l. } A.$$

$$\text{Now } \frac{du}{dx} = (y + 1)\,.\left\{z + 1 + (x + 1)\,.\,\frac{dz}{dx}\right\} = 0,$$

$$\frac{du}{dy} = (x + 1)\,.\left\{z + 1 + (y + 1)\,.\,\frac{dz}{dy}\right\} = 0,$$

$$\text{and } \frac{dz}{dx} = -\frac{\text{h. l. } a}{\text{h. l. } c}, \quad \text{and } \frac{dz}{dy} = -\frac{\text{h. l. } b}{\text{h. l. } c};$$

$$\therefore\ z + 1 - (x + 1)\,.\frac{\text{h. l. } a}{\text{h. l. } c} = 0,$$

$$\text{or}\ \ (z + 1)\,\text{h. l. } c = (x + 1)\,\text{h. l. } a, \quad \text{and} \ \therefore\ c^{z+1} = a^{x+1}.$$

$$\text{Also}\ \ z + 1 - (y + 1)\,.\frac{\text{h. l. } b}{\text{h. l. } c} = 0\,;$$

$$\text{and}\ \therefore\ c^{z+1} = b^{y+1} = a^{x+1}.$$

$$\text{Also}\ \because\ z + 1 = (x + 1)\,.\frac{\text{h. l. } a}{\text{h. l. } c} = (y + 1)\,\frac{\text{h. l. } b}{\text{h. l. } c}\,;$$

$$\therefore\ z = \frac{(x + 1)\,\text{h. l. } a - \log c}{\log c},$$

$$y = \frac{(x + 1)\,\text{h. l. } a - \text{h. l. } b}{\log b}\,;$$

$$\therefore\ x \log a + (x + 1) \log a - \log b + (x + 1) \log a - \log c = \log A\,;$$

$$\therefore\ 3x \log a + 2 \log a - \log bc = \log A\,;$$

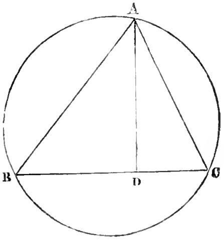
$$\therefore\ x = \frac{\log Abc - 2 \log a}{3 \log (a)}\,;$$

$$\therefore\ x + 1 = \frac{\log Abc + \log a}{3 \log a} = \frac{\log (Abca)}{3 \log a},$$

$$y + 1 = \frac{\log (Abca)}{3 \log b}, \quad \text{and}\ \ z + 1 = \frac{\log (Abca)}{3 \log c}\,;$$

$$\therefore\ u = \frac{(\log Abca)^3}{\log (abc)^3}.$$

Ex. 4. In a circle of given radius, inscribe the greatest triangle.



R the radius.

a, b, c the sides.

$\theta = \angle B, \quad \phi = \angle C.$

$$u = \frac{BC \times AD}{2} = \text{maximum},$$

and $2R \times AD = AB \, . \, AC$, (*Euclid*, Book VI. Prop. C.)

or $2Rc \sin\theta = cb$,

$$\text{or} \quad b = 2R\sin\theta;$$

$$\therefore \; c = b \, . \frac{\sin\phi}{\sin\theta} = 2R\sin\phi,$$

$$a = 2R\sin A = 2R\sin(\phi+\theta).$$

$$\text{But } u = \frac{BC \, . \, AD}{2} = \frac{a \, . \, bc}{4R} = 2R^2\sin\theta\sin\phi\sin(\phi+\theta) = \text{max}.$$

$$\therefore \; \frac{du}{d\theta} = 2R^2\left\{\cos\theta \, . \sin(\phi+\theta) + \sin\theta \, . \cos(\phi+\theta)\right\}\sin\phi = 0,$$

$$\text{and } \frac{du}{d\phi} = 2R^2\left\{\cos\phi \, . \sin(\phi+\theta) + \sin\phi \, . \cos(\phi+\theta)\right\}\sin\theta = 0;$$

$$\therefore \; \sin(\phi + 2\theta) = 0 = \sin\pi,$$

$$\text{and } \sin(\theta + 2\phi) = 0 = \sin\pi;$$

$$\therefore \; \phi + 2\theta = \pi,$$

$$\text{and } \theta + 2\phi = \pi;$$

$$\therefore \; \theta - \phi = 0, \quad \text{or } \theta = \phi;$$

$$\therefore \; 3\theta = \pi, \quad \text{and } \theta = 60^0,$$

$$\text{and } A = \pi - 2\theta = 60^0,$$

and the triangle is equiangular;

$$\text{and } u = 2R^2 . \sin^3 60 = R^2 . \frac{3\sqrt{3}}{4}.$$

Ex. 5. Inscribe the greatest parallelopipedon within a given ellipsoid.

Let $2x$, $2y$, $2z$ be the edges,

$2a$, $2b$, $2c$ the principal diameters of the ellipsoid;

$$\therefore \; u = 8xyz \text{ is a maximum.}$$

$$\text{But } \frac{z^2}{c^2} + \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

$$\therefore \; z^2 = c^2 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right);$$

$$\therefore \; \frac{du}{dx} = 8yz + 8yx\frac{dz}{dx} = 0,$$

$$\frac{du}{dy} = 8xz + 8yx\frac{dz}{dy} = 0.$$

$$\text{But } \frac{dz}{dx} = -\frac{x}{z} \cdot \frac{c^2}{a^2}, \text{ and } \frac{dz}{dy} = -\frac{y}{z} \cdot \frac{c^2}{b^2};$$

$$\therefore \; z - \frac{x^2}{z} \cdot \frac{c^2}{a^2} = 0, \quad \text{and } z - \frac{y^2}{z}\frac{c^2}{b^2} = 0;$$

$$\therefore \; \frac{z^2}{c^2} = \frac{x^2}{a^2}, \quad \text{and } \frac{z^2}{c^2} = \frac{y^2}{b^2} = \; \therefore \frac{x^2}{a^2};$$

$$\therefore \; \frac{3x^2}{a^2} = 1, \quad \text{and } x = \frac{a}{\sqrt{3}};$$

$$\therefore \; \frac{y^2}{b^2} = \tfrac{1}{3}; \quad \therefore \; y = \frac{b}{\sqrt{3}},$$

$$\text{and } \frac{z^2}{c^2} = \tfrac{1}{3}; \quad \therefore \; z = \frac{c}{\sqrt{3}},$$

$$\text{and } u = \frac{8}{3}\frac{abc}{\sqrt{3}}.$$

LAGRANGE'S THEOREM.

177. Let $u = f(y)$, where $y = z + x\phi(y)$, and z is independent of x; required u or $f(y)$ in terms of x.

By Maclaurin,

$$u = U_0 + U_1 x + U_2\frac{x^2}{1 \cdot 2} + U_3\frac{x^3}{2 \cdot 3} + \&c. + \frac{U_n x^n}{1 \cdot 2 \cdot 3 \ldots n} + \&c.$$

where U_0, U_1, U_2, &c. are the values of $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, &c.; when $x = 0$.

First, if $x = 0$, $y = z$; $\therefore$ $U_0 = f(z)$.

Now $\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$, and $\frac{du}{dz} = \frac{du}{dy} \cdot \frac{dy}{dz}$.

But $\frac{dy}{dx} = x \cdot \phi'(y) \cdot \frac{dy}{dx} + \phi(y)$, where $\phi' y = \frac{d\phi(y)}{dy}$;

$$\therefore \frac{dy}{dx} = \frac{\phi(y)}{1 - x\phi'(y)},$$

$$\frac{dy}{dz} = 1 + x \cdot \phi'(y) \cdot \frac{dy}{dz}; \quad \therefore \frac{dy}{dz} = \frac{1}{1 - x\phi'(y)};$$

$$\therefore \frac{dy}{dx} = \phi(y) \cdot \frac{dy}{dz};$$

$$\therefore \frac{du}{dx} = \frac{du}{dy} \cdot \phi(y) \cdot \frac{dy}{dz} = \phi(y) \cdot \frac{du}{dy} \cdot \frac{dy}{dz} = \phi(y) \cdot \frac{du}{dz}.$$

Make $x = 0$; $\therefore$ $U_1 = \phi(z) \cdot \frac{d \cdot f(z)}{dz}$.

Next, let $\phi(y) \frac{du}{dz} = \frac{du_1}{dz}$;

$$\therefore \frac{du}{dx} = \frac{du_1}{dz};$$

$$\therefore \frac{d^2u}{dx^2} = \frac{d^2u_1}{dx\,dz} = \frac{d^2u_1}{dz\,dx} = \frac{d \cdot \left(\frac{du_1}{dx}\right)}{dz} = \frac{d \cdot \left\{\phi(y) \cdot \left(\frac{du_1}{dz}\right)\right\}}{dz}$$

$$= \frac{d \cdot \left\{\{\phi(y)\}^2 \cdot \frac{du}{dz}\right\}}{dz}.$$

$$U_2 = \frac{d\left\{\{\phi(z)\}^2 \cdot \frac{d \cdot f(z)}{dz}\right\}}{dz}.$$

And so may U_3 be found, but to find U_n

$$\text{assume } \frac{d^{n-1}u}{dx^{n-1}} = \frac{d^{n-2}\left\{\{\phi(y)\}^{n-1}.\dfrac{du}{dz}\right\}}{dz^{n-2}};$$

$$\text{let } \{\phi(y)\}^{n-1}.\frac{du}{dz} = \frac{du_{n-1}}{dz};$$

$$\therefore \frac{d^{n-1}u}{dx^{n-1}} = \frac{d^{n-1}u_{n-1}}{dz^{n-1}};$$

$$\therefore \frac{d^n u}{dx^n} = \frac{d^n u_{n-1}}{dx \,.\, dz^{n-1}} = \frac{d^n u_{n-1}}{dz^{n-1}dx} = \frac{d^{n-1}.\left(\dfrac{du_{n-1}}{dx}\right)}{dz^{n-1}}$$

$$= \frac{d^{n-1}.\left\{\{\phi(y)\}.\dfrac{du_{n-1}}{dz}\right\}}{dz^{n-1}}$$

$$= \frac{d^{n-1}.\left\{\{\phi(y)\}^n.\dfrac{du}{dz}\right\}}{dz^{n-1}};$$

$$\therefore U_n = \frac{d^{n-1}.\left\{\{\phi(z)\}^n.\dfrac{d.f(z)}{dz}\right\}}{dz^{n-1}}.$$

Hence if the assumption be true for $n-1$, it is true for n; and it is true for $n=1$ and $n=2$; therefore it is universally true, and writing Z for $\dfrac{d.f(z)}{dz}$, we have

$$u = f(z) + \{\phi(z).Z\}.\frac{x}{1} + \frac{d.\{[\phi(z)]^2.Z\}}{dz}.\frac{x^2}{1.2}$$

$$+ \frac{d^2.\{[\phi(z)]^3.Z\}}{dz^2}.\frac{x^3}{2.3} + \&c. + \frac{d^{n-1}.\{[\phi(z)]^n.Z\}}{dz^{n-1}}.\frac{x^n}{1.2.3.n}$$

$$+ \&c. \ldots\ldots\ldots (1),$$

which is the theorem required.

Cor. If $f(y) = y$ or y be required, then

$$f(z) = z, \text{ and } Z = \frac{df(z)}{dz} = 1;$$

$$\therefore\ y = z + \phi(z) \cdot \frac{x}{1} + \frac{d \cdot \{\phi(z)\}^2}{dz^2} \cdot \frac{x^2}{1 \cdot 2}$$

$$+ \frac{d^2\{\phi(z)\}^3}{dz^2} \cdot \frac{x^2}{2 \cdot 3} + \&c. \ldots\ldots\ldots (2).$$

Ex. 1. $y^3 - ay + b = 0$; find y or the root of the cubic equation.

Here $y = \frac{b}{a} + \frac{1}{a} \cdot y^3$, comparing this with $y = z + x\phi(y)$, and taking series (2),

$$z = \frac{b}{a}, \quad x = \frac{1}{a}, \quad \phi(y) = y^3;$$

$$\therefore\ \phi(z) = z^3,\ \{\phi(z)\}^2 = z^6,\ \{\phi(z)\}^3 = z^9,\ \{\phi(z)\}^4 = z^{12},\ \&c.;$$

$$\therefore\ \frac{d \cdot \{\phi(z)\}^2}{dz} = 6z^5, \quad \frac{d^2\{\phi(z)\}^3}{dz^2} = 8 \cdot 9z^7,$$

$$\frac{d^3\{\phi(z)\}^4}{dz^3} = 10 \cdot 11 \cdot 12z^9;$$

$$\therefore\ y = z + z^3 \cdot \frac{x}{1} + 6 \cdot z^5 \cdot \frac{x^2}{1 \cdot 2} + 8 \cdot 9z^7 \cdot \frac{x^3}{2 \cdot 3}$$

$$+ \frac{10 \cdot 11 \cdot 12}{2 \cdot 3 \cdot 4} \cdot z^9 \cdot x^4 + \&c.$$

$$= \frac{b}{a}\left\{1 + \frac{b^2}{a^3} + 3 \cdot \frac{b^4}{a^6} + 12\,\frac{b^6}{a^9} + 55 \cdot \frac{b^8}{a^{12}} + \&c.\right\}.$$

Ex. 2. In the same example find y^n.

Here $Z = nz^{n-1}$, $\phi(z) = z^3$ and using series (1);

$$\therefore\ \phi(z) \cdot Z = nz^{n+2},$$

$$\{\phi(z)\}^2 Z = n \cdot z^{n+5}; \quad \therefore \quad \frac{d\{\phi(z)\}^2 . Z}{dz} = n \cdot (n+5) \cdot z^{n+4},$$

$$\{\phi(z)\}^3 Z = n \cdot z^{n+8}; \quad \therefore \quad \frac{d^2\{\phi(z)\}^3 . Z}{dz^2} = n \cdot (n+8) \cdot (n+7) \cdot z^{n+6};$$

$$\therefore \quad y^n = z^n + n z^{n+2} . \frac{x}{1} + \frac{n(n+5)}{1 \cdot 2} . z^{n+4} . x^2$$

$$+ \frac{n \cdot (n+8) \cdot (n+7)}{1 \cdot 2 \cdot 3} z^{n+6} . x^3 + \&c.$$

$$= \frac{b^n}{a^n} \left\{ 1 + n \cdot \frac{b^2}{a^2} \cdot \frac{1}{a} + \frac{n(n+5)}{1 \cdot 2} \cdot \frac{b^4}{a^4} \cdot \frac{1}{a^2} \right.$$

$$\left. + \frac{n \cdot (n+8)(n+7)}{1 \cdot 2 \cdot 3} \cdot \frac{b^6}{a^6} \cdot \frac{1}{a^3} + \&c. \right\}.$$

Ex. 3. Find $\log y$, when $1 - y + a^y = 0$,

$$y = 1 + a^y, \text{ and } u = \log y;$$

$$\therefore \quad z = 1, \quad x = 1, \quad \phi(y) = a^y, \quad f(z) = \log z; \quad \therefore \quad Z = \frac{1}{z};$$

$$\therefore \quad f(z) = \log(1) = 0,$$

$$\phi(z) \cdot Z = a^z \cdot \frac{1}{z} = a,$$

$$\{\phi(z)\}^2 Z = a^{2z} \frac{1}{z};$$

$$\therefore \quad \frac{d\{\phi(z)\}^2 . Z}{dz} = 2A \cdot a^{2z} . \frac{1}{z} - \frac{a^{2z}}{z^2} = 2Aa^2 - a^2, \quad z = 1,$$

and $\{\phi(z)\}^3 . Z = a^{3z} . \frac{1}{z}$.

$$\frac{d . \{\phi(z)\}^3 Z}{dz} = 3Aa^{3z} . \frac{1}{z} - \frac{a^{3z}}{z^2},$$

$$\frac{d^2 . \{\phi(z)\}^3 . Z}{dz^2} = 9A^2 a^{3z} . \frac{1}{z} - \frac{6A . a^{3z}}{z^2} + \frac{2a^{3z}}{z^3}$$

$$= 9A^2 a^3 - 6Aa^3 + 2a^3, \quad z = 1,$$

$$= a^3 (9A^2 - 6A + 2);$$

$$\therefore \log y = a + (2A - 1)\frac{a^2}{1 . 2} + (9A^2 - 6A + 2) . \frac{a^3}{1 . 2 . 3} + \&c.$$

Ex. 4. Let $y = m + e \sin y$, find y.

Comparing this with $y = z + x\phi(y)$,

$$z = m, \quad x = e, \quad \phi(y) = \sin y, \text{ and } f(z) = z.$$

$$\text{Here } Z = \frac{df(z)}{dz} = 1;$$

$$\therefore y = z + \phi(z)\frac{x}{1} + \frac{d\{\phi(z)\}^2}{dz} . \frac{x^2}{1 . 2} + \frac{d^2\{\phi(z)\}^3}{dz^2} \frac{x^3}{2 . 3} + \&c.$$

$$\phi(z) = \sin z = \sin m \text{ if } x = 0;$$

$$\therefore \{\phi(z)\}^2 = \sin^2 z;$$

$$\therefore \frac{\{d\phi(z)\}^2}{dz} = 2 \sin z . \cos z = \sin 2z = \sin 2m \text{ if } x = 0,$$

$$\{\phi(z)\}^3 = \sin^3 z;$$

$$\therefore \frac{\{d\phi(z)\}^3}{dz} = 3 \sin^2 z \cos z,$$

$$\frac{d^2\{\phi(z)\}^3}{dz^3}$$

$$= 6 \sin z \cos^2 z - 3 \sin^3 z$$

$$= 6 \sin z - 9 \sin^3 z$$

$$= \tfrac{3}{4}(3 \sin 3z - \sin z),$$

(putting for $\sin^3 z$, $\tfrac{3}{4}\sin z - \tfrac{1}{4}\sin 3z$;)

$$\therefore z = m + \sin m . \frac{e}{1} + \sin 2m . \frac{e^2}{1 . 2} + \tfrac{3}{4}(3 \sin 3m - \sin m)\frac{e^3}{2 . 3}$$

$$+ \&c.$$

178. To determine the curve which touches any number of curves of a given form, and described after a given law.

For the better explanation of this application of the Differential Calculus to curves, let us take a particular case, and suppose it were required to find the equation to the curve, that shall touch any number of circles, having a constant radius r, but the centres of which are placed in a curve whose equation is known.

Then if y and x be the co-ordinates of the touching curve, a and β the co-ordinates of the centre of one of the circles

$$(y-\beta)^2+(x-a)^2=r^2.$$

But β and a are the co-ordinates of the curve in which the centres of the circles are found; therefore β is a known function of a, or $\beta=\phi(a)$;

$$\therefore \{y-\phi(a)\}^2+(x-a)^2=r^2 \ldots\ldots\ldots\ldots (1.)$$

Now if we suppose a to receive an infinitely small increment, the equation (1) will belong to an equal circle, the centre of which is infinitely near to that denoted by equation (1), and the two circles will intersect at a point of which the co-ordinates are ultimately x and y; and similarly proceeding with a third and other circles we may conceive the touching curve to be formed by the continual intersections of these circles.

And to determine its equation, which must be independent of a, a must be eliminated between the equations $\{y-\phi(a)\}^2+(x-a)^2=r^2$, and the equation which indicates that we have passed from the consideration of one circle to the other, that is, the differential of the equation (1), taken with respect to a.

Hence we may conclude, that if $V=f(xya)=0$ represent the equation of one of the given curves, the touching curve may be found by eliminating a between the equations

$$V=0, \text{ and } \frac{dV}{da}=0.$$

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That $V=0$, and $\dfrac{dV}{da}=0$, are simultaneous equations, may be thus shewn. Resuming the equation to the circle,

Let $a+\delta a$, and $\beta+\delta\beta$ be the values of a and β in the consecutive circle;

$$\therefore \{x-(a+\delta a)\}^2+\{y-(\beta+\delta\beta)\}^2=r^2;$$

therefore by subtraction,

$$(x-a)^2-\{x-(a+\delta a)\}^2+(y-\beta)^2-\{y-(\beta+\delta\beta)\}^2=0,$$

$$\text{or } \delta a\{2.(x-a)-\delta a\}+\delta\beta\{2.(y-\beta)-\delta\beta\}=0,$$

$$\text{or } 2(x-a)+2.(y-\beta)\frac{\delta\beta}{\delta a}-\{\delta\beta\frac{\delta\beta}{\delta a}+\delta a\}=0.$$

Now make $\delta a=0$, and $\delta\beta=0$, in which case the point of intersection of the two circles becomes a point in the touching curve, and $\dfrac{\delta\beta}{\delta a}$ becomes the differential coefficient of β with respect to a;

$\therefore\ 2(x-a)+2(y-\beta)\dfrac{\delta\beta}{\delta a}=0$, which is the differential coefficient of $(x-a)^2+(y-\beta)^2=r^2$ with respect to a, between which two equations a may be eliminated.

Prob. I. Find the equation to the curve which shall touch all the straight lines defined by the equation

$$y=ax+r\sqrt{a^2+1},$$

where r is a perpendicular of constant length from the origin upon the lines.

Differentiating with respect to a, x and y being constant,

$$x+\frac{ra}{\sqrt{a^2+1}}=0;\quad \therefore\ \frac{r}{x}=-\frac{\sqrt{a^2+1}}{a};$$

$$\therefore \frac{1}{a^2} = \frac{r^2}{x^2} - 1 = \frac{r^2 - x^2}{x^2};$$

$$\therefore a = \frac{x}{\sqrt{r^2 - x^2}},$$

$$\text{and } \sqrt{a^2 + 1} = -a\frac{r}{x} = -\frac{r}{\sqrt{r^2 - x^2}};$$

$$\therefore y = ax + r\sqrt{a^2 + 1}$$

$$= \frac{x^2}{\sqrt{r^2 - x^2}} - \frac{r^2}{\sqrt{r^2 - x^2}} = -\frac{r^2 - x^2}{\sqrt{r^2 - x^2}} = -\sqrt{r^2 - x^2};$$

$\therefore y^2 + x^2 = r^2$, the equation to a circle.

Prob. II. A straight line of given length slides down between two rectangular axes, find the curve to which it is always a tangent.

Let c be the length of the line,

a and b the co-ordinates of its extremities, or the parts of the axes cut off in any given pósition of the line;

$$\therefore \frac{x}{a} + \frac{y}{b} = 1, \text{ and } a^2 + b^2 = c^2;$$

$$\therefore \frac{x}{a^2} + \frac{y}{b^2}\frac{db}{da} = 0,$$

$$\text{and } a + b\frac{db}{da} = 0, \quad \therefore \frac{db}{da} = -\frac{a}{b};$$

$$\therefore \frac{x^2}{a^2} - \frac{ya}{b^3} = 0, \text{ and } b = a\sqrt[3]{\frac{y}{x}};$$

$$\therefore a^2 + b^2 = a^2\left\{\frac{y^{\frac{2}{3}} + x^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right\} = c^2;$$

$$\therefore \; a = c\,\frac{x^{\frac{1}{3}}}{\sqrt{y^{\frac{2}{3}} + x^{\frac{2}{3}}}},$$

$$\text{and } b = c\,\frac{y^{\frac{1}{3}}}{\sqrt{x^{\frac{2}{3}} + x^{\frac{2}{3}}}};$$

$$\therefore \; \frac{x}{a} + \frac{y}{b} = \frac{\sqrt{y^{\frac{2}{3}} + x^{\frac{2}{3}}}}{c}\left\{x^{\frac{2}{3}} + y^{\frac{2}{3}}\right\} = 1;$$

$$\therefore \; (x^{\frac{2}{3}} + y^{\frac{2}{3}})^{\frac{3}{2}} = c,$$

$$\text{and } x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

If the equation of condition be $a^n + b^n = c^n$, then the equation to the touching curve will be

$$x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} = c^{\frac{n}{n+1}}.$$

Prob. III. Find the curve which touches all the ellipses described round the same centre and with coincident axes, the rectangle of the axes being a constant area.

$$\text{Here } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots\ldots\ldots (1).$$

$$ab = m^2 = \text{the constant area} \ldots\ldots\ldots (2).$$

Differentiating, a variable and $b = f(a)$,

$$\frac{x^2}{a^3} + \frac{y^2}{b^3}\cdot\frac{db}{da} = 0 \text{ from } (1),$$

$$b + a\frac{db}{da} = 0 \text{ from } (2),$$

$$\therefore \; \frac{db}{da} = -\frac{b}{a};$$

$$\therefore \frac{x^2}{a^3} - \frac{y^2}{b^3}\frac{b}{a} = 0, \text{ or } \frac{y^2}{b^2} = \frac{x^2}{a^2};$$

$$\therefore b = a\frac{y}{x}, \text{ and } ab = m^2 = a^2\frac{y}{x};$$

$$\therefore a^2 = m^2\frac{x}{y},$$

$$b^2 = m^2\frac{y}{x};$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{m^2}\{xy + xy\} = 1;$$

$$\therefore xy = \frac{m^2}{2},$$

the equation to the rectangular hyperbola.

Prob. IV. Find the equation to the curve whose tangent cuts off from the axes, two lines the sum of which $= c$,

$$\sqrt{x} + \sqrt{y} = \sqrt{c}.$$

Prob. V. Find the curve which touches all the curves included under the equation

$$y = x\tan\theta - \frac{x^2}{4h\cos^2\theta}:$$

θ being supposed variable:

$$x^2 = 4h(h - y).$$

Prob. VI. Find the curve when $AD^m = a^{m-1}. AT$.

Prob. VII. Find the curve, so that the rectangle contained by two lines, drawn perpendicular to the axis of x, one from the origin, the other from a given point in it to meet the tangent, may $= b^2$.

Prob. VIII. Find the curve whose tangent cuts off from the axes a constant area.

THE

INTEGRAL CALCULUS.

CHAPTER I.

1. THE Integral Calculus is the inverse of the Differential, its object being to discover the original function from a given relation between the differential coefficients and functions of x and u.

In this treatise, we shall solely confine ourselves to the case in which the first differential coefficient $\frac{du}{dx}$ is an *explicit* function of x, as $\phi'(x)$, and $u = \phi(x)$ is required.

2. The process by which u is found from $\frac{du}{dx}$ is called *integration*, and when to be performed is expressed by prefixing the symbol $\int_x$.

$$\text{Thus if } \frac{du}{dx} = \phi(x),$$

$$u = \int_x . \phi(x) + C.$$

The letter C, representing a constant quantity, is added, since constant quantities connected with the original function by the signs $\pm$ disappear in differentiation: and therefore, when we return to the original value u, an arbitrary quantity as C must be added, the value of which will be determined by the nature of the Problem.

3. The simplest case is when $\frac{du}{dx} = ax^m$, a monomial.

Let $u = Ax^n + C$;

$$\therefore \frac{du}{dx} = nAx^{n-1} = ax^m;$$

$$\therefore a = nA, \quad \text{and} \quad m = n - 1; \quad \therefore n = (m+1),$$

$$\text{and } A = \frac{a}{n} = \frac{a}{m+1};$$

$$\int_x ax^n = \frac{a}{m+1} . x^{m+1} + C;$$

or to integrate a monomial, *add unity to the index, and divide by the index so increased*, and add a constant.

COR. 1. Thus also if $\dfrac{du}{dx} = ax^{-m} = \dfrac{a}{x^m}$,

$$u = -\frac{a}{m-1} . \frac{1}{x^{m-1}} + C,$$

which is derived from the preceding by writing $-m$ for m.

COR. 2. The general formula fails when $m = -1$, for then

$$u = \frac{a . x^{1-1}}{1-1} + C = \frac{a}{0} + C.$$

But if $m = -1$, $\quad \dfrac{du}{dx} = \dfrac{a}{x} = a . \dfrac{1}{x}$.

$$\text{Now } \frac{1}{x} = \frac{d . \log x}{dx}; \quad \therefore \frac{a}{x} = a . \frac{d . (\log x)}{dx};$$

$$\text{and } \therefore a . \int_x \frac{1}{x} = a . \log x + C;$$

however, the true value of $\dfrac{du}{dx}$ may be derived from the general expression, if C be first determined.

For, suppose $u = 0$ when $x = b$;

$$\therefore 0 = \frac{ab^{m+1}}{m+1} + C, \quad \text{or} \quad C = -\frac{ab^{m+1}}{m+1};$$

$$\therefore\ u = a\,.\,\frac{x^{m+1} - b^{m+1}}{m+1},\ \text{a fraction of the form } \frac{0}{0}$$

when $m = -1$, and of which the real value is

$$a\,\text{h. l.}\,\frac{x}{b} = a\,\text{h. l.}\,x - a\,\text{h. l.}\,b = a\,\text{h. l.}\,x + C;$$

the same value as that which has been just obtained by a different process.

4. Since if $u = \log\{f(x)\} = \log(z)$, where $z = f(x)$,

$$\frac{du}{dx} = \frac{\dfrac{dz}{dx}}{z}.$$

$$\therefore\ \int_x \frac{\dfrac{dz}{dx}}{z} = \log(z) + C.$$

Hence, if we have a fractional expression, such that the numerator is the differential coefficient of the denominator, the integral is the logarithm of the denominator.

Ex. 1. Let $\dfrac{du}{dx} = \dfrac{x}{1+x^2} = \dfrac{1}{2}\,.\,\dfrac{2x}{1+x^2}$;

$$\therefore\ u = \tfrac{1}{2}\,.\,\text{h. l.}\,(1+x^2) = \text{h. l.}\,\sqrt{1+x^2}.$$

Ex. 2. Let $\dfrac{du}{dx} = \dfrac{2x-1}{x^2-x+1}$;

$$\therefore\ u = \text{h. l.}\,(x^2 - x + 1).$$

5. Again, since

$$\frac{dp}{dx} + \frac{dq}{dx} + \frac{dr}{dx} + \&c. = \frac{d}{dx}(p+q+r+\&c.);$$

$$\therefore\ \int_x\left\{\frac{dp}{dx} + \frac{dq}{dx} + \frac{dr}{dx} + \&c.\right\} = \int_x \frac{d}{dx}(p+q+r+\&c.)$$

$$= p + q + r + \&c.$$

or the integral of the sum of any number of differential coefficients = sum of the integrals of each differential coefficient taken separately.

Ex. Let $\frac{du}{dx} = Ax^m + Bx^n + Cx^p + \&c.$;

$$\therefore\ u = A\int_x x^m + B\int_x x^n + C\int_x x^p + \&c.$$

$$= \frac{A}{m+1}x^{m+1} + \frac{B}{n+1}x^{n+1} + \frac{C}{p+1}x^{p+1} + \&c.$$

6. If $\frac{du}{dx} = z^m . \frac{dz}{dx}$, where z is a function of x, find u.

Since if $u = z^{m+1} + C$,

$$\frac{du}{dx} = (m+1)\, z^m . \frac{dz}{dx};$$

$$\therefore\ \int_x z^m . \frac{dz}{dx} = \frac{z^{m+1}}{m+1} + C;$$

or to integrate a function of this description, increase the index by unity, divide by the index so increased, and by the differential coefficient of the quantity under the index.

EXAMPLES OF SIMPLE INTEGRATION.

(1) Let $\frac{du}{dx} = ax^3$; $\quad\therefore\ u = \frac{ax^4}{4}$.

(2) Let $\frac{du}{dx} = \frac{a}{x^2} = ax^{-2}$; $\quad\therefore\ u = \frac{ax^{-1}}{-1} = -\frac{a}{x}$.

(3) Let $\frac{du}{dx} = ax^{\frac{m}{n}}$; $\quad\therefore\ u = \frac{n}{m+n} . ax^{\frac{m+n}{n}}$.

(4) Let $\frac{du}{dx} = (ax^n + b)^m x^{n-1}$.

$$\text{Let } z = ax^n + b;$$

$$\therefore \frac{dz}{dx} = nax^{n-1};$$

$$\therefore (ax^n + b)^m \cdot x^{n-1} = \frac{1}{na} \cdot z^m \cdot \frac{dz}{dx};$$

$$\therefore u = \frac{1}{na} \cdot \int_x \cdot z^m \cdot \frac{dz}{dx} = \frac{z^{m+1}}{na \cdot (m+1)}$$

$$= \frac{(ax^n + b)^{m+1}}{na \cdot (m+1)}.$$

(5) $\dfrac{du}{dx} = (ax + b)^m; \quad \therefore u = \dfrac{(ax+b)^{m+1}}{a \cdot (m+1)}.$

(6) $\dfrac{du}{dx} = (ax^n + b)^m \cdot x^r$, m being a whole number.

Expand $(ax^n + b)^m$ by the binomial, and after having multiplied each term by x^r, integrate them separately.

(7) $\dfrac{du}{dx} = \dfrac{x^m}{(a + bx)^n}$, m and n being whole numbers.

$$\text{Let } a + bx = z; \quad \therefore x = \frac{z - a}{b};$$

$$\therefore \frac{x^m}{(a + bx)^n} = \frac{(z-a)^m}{b^m \cdot z^n};$$

$$\text{and } \frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx} = \frac{du}{dz} \cdot b,$$

$$\text{or } \frac{du}{dz} = \frac{1}{b} \cdot \frac{du}{dx} = \frac{1}{b^{m+1}} \cdot \frac{(z-a)^m}{z^n};$$

$$\therefore u = \frac{1}{b^{m+1}} \cdot \int_z \frac{(z-a)^m}{z^n}.$$

Expand $(z - a)^m$ by the binomial, and integrate each term separately, first dividing by z^n.

(8) $\dfrac{du}{dx} = \dfrac{1}{x^m(a+bx)^n}$, m and n being integers.

For x put $\dfrac{1}{z}$; $\therefore \dfrac{dx}{dz} = -\dfrac{1}{z^2}$,

and $\dfrac{du}{dx} = -\dfrac{du}{dz} \cdot z^2 = \dfrac{z^{m+n}}{(az+b)^n}$;

$$\therefore u = -\int_z \frac{z^{m+n-2}}{(az+b)^n},$$

which resolves itself into the preceding case.

(9) $\dfrac{du}{dx} = \dfrac{1}{a+bx^2} = \dfrac{1}{a} \cdot \dfrac{1}{1+\dfrac{b}{a}x^2}$.

Let $z^2 = \dfrac{b}{a} \cdot x^2$; $\therefore z = \sqrt{\dfrac{b}{a}} \cdot x$, and $\dfrac{dz}{dx} = \sqrt{\dfrac{b}{a}}$;

$$\therefore \frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx} = \frac{du}{dz} \cdot \sqrt{\frac{b}{a}} = \frac{1}{a} \cdot \frac{1}{1+z^2},$$

$$\text{or } u = \frac{1}{\sqrt{ab}} \int_z \cdot \frac{1}{1+z^2}.$$

But if $u = \tan^{-1} z$, $\dfrac{du}{dz} = \dfrac{1}{1+z^2}$;

$$\therefore u = \frac{1}{\sqrt{ab}} \cdot \tan^{-1} z = \frac{1}{\sqrt{ab}} \cdot \tan^{-1} x \sqrt{\frac{b}{a}}.$$

Ex. $\dfrac{du}{dx} = \dfrac{1}{2+3x^2}$; $\therefore u = \dfrac{1}{\sqrt{6}} \tan^{-1} x \sqrt{\dfrac{3}{2}}$.

(10) $\dfrac{du}{dx} = \dfrac{1}{a+x}$; $\therefore u = \text{h.l.}\ (a+x)$.

(11) $\dfrac{du}{dx} = \dfrac{x^2}{1+x^3}$; $\therefore u = \frac{1}{3}\,\text{h.l.}\,(1+x^3) = \text{h.l.}\,\sqrt[3]{1+x^3}$.

(12) $\dfrac{du}{dx} = (a+bx+cx^2)^m \cdot (b+2cx)$; $\therefore\ u = \dfrac{(a+bx+cx^2)^{m+1}}{m+1}$.

Integrate by the preceding methods the differential coefficients

(1) $ax^{\frac{5}{2}}$. (2) $ax^3 + bx^7 + cx^9$. (3) $(ax^3 + b)^2 \cdot x^2$.

(4) $(2ax+x^2)^7 \cdot (a+x)$. (5) $\dfrac{x^2}{(a+bx)}$. (6) $\dfrac{3x^2+2x+1}{x^3+x^2+x+2}$.

(7) $\dfrac{1}{1+5x^2}$. (8) $\dfrac{1}{x^2(a+bx)}$. (9) $\left(\dfrac{a}{x} + \dfrac{b}{x^2} + \dfrac{c}{x^3} + \dfrac{e}{x^4}\right)$.

(10) $(ax + bx^2)^3$. (11) $\left(a + bx + \dfrac{c}{x}\right)^2$.

(12) $\left(ax^2 + \dfrac{b}{x^2}\right)^3$. (13) $(1 + x^2)(1 + x)^2 \cdot x^2$.

(14) $\dfrac{(1+x)^2 \cdot (1-x)}{x^2}$. (15) $\dfrac{x^4}{1+x^2}$. (16) $\dfrac{x^3}{(2+x)^2}$.

7. These simple integrals being found, it will be convenient to classify the remaining functions in the following order.

(1) Rational fractions of the form

$$\frac{Ax^{m-1} + Bx^n + Cx^p + \&c.}{A_1x^m + B_1x^{n_1} + C_1x^{p_1} + \&c.}.$$

(2) Irrational quantities.

(3) Exponential and logarithmic functions which are of the forms

$$a^x f(a^x),\quad \log(x),\quad \log(p),\quad p^m \log(q).$$

(4) Circular functions which are of the form

$$\sin p,\ f(\sin p),\ \&c.$$

The methods for the integration of such functions will be given in the four succeeding Chapters.

CHAPTER II.

RATIONAL FRACTIONS.

8. Every rational fraction may be represented by

$$\frac{Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \&c.}{A_1x^m + B_1x^{m-1} + C_1x^{m-2} + \&c.},$$

for it is manifest that the index of x in the numerator can by division be made less by unity at least, than that of x in the denominator.

To integrate this fraction we must first separate it into fractions of a more simple form.

Now the denominator may be composed 1st of simple factors all different. 2d. Some of the factors may be equal. 3rd. It may contain quadratic factors, the roots of which are impossible. 4th. It may be an assemblage of all these.

9. We shall first consider the case where the factors are all different.

Let therefore $\frac{U}{V}$ be a fraction where V is the product of n factors all different, so that

$$V = (x - a_1)(x - a_2)(x - a_3)\dots(x - a_n).$$

Assume $\frac{U}{V} = \frac{A_1}{(x - a_1)} + \frac{A_2}{(x - a_2)} + \frac{A_3}{(x - a_3)} + \&c. + \frac{A_n}{(x - a_n)}$;

$$\therefore\ U = A_1(x-a_2)\,.\,(x-a_3)\dots(x-a_n) + A_2\,.\,(x-a_1)\,.\,(x-a_3)\,.\,(x-a_4)$$
$$+ \&c. + A_n\,.\,(x-a_1)(x-a_2)\dots(x-a_{n-1})\ \dots\dots$$

Successively make $x = a_1, a_2, a_3$, &c.; and let U_{a_1}, U_{a_2}, U_{a_3}, &c. be the corresponding values of U;

$$\therefore\ U_{a_1} = A_1 (a_1 - a_2)(a_1 - a_3) \dots (a_1 - a_n),$$

$$\text{or } A_1 = \frac{U_{a_1}}{(a_1 - a_2)(a_1 - a_3) \dots\dots}.$$

Similarly, $A_2 = \dfrac{U_{a_2}}{(a_2 - a_1)(a_2 - a_3), \text{ \&c.}}$, and $A_3 = \dfrac{U_{a_3}}{(a_3 - a_1)(a_3 - a_2)\dots}$,

$$\int_x \left(\frac{U}{V}\right) = A_1 \int_x \frac{1}{x - a_1} + A_2 \int_x \frac{1}{x - a_2} + A_3 \int_x \frac{1}{x - a_3} + \text{\&c.}$$

$$= A_1 \text{ h. l. } (x - a_1) + A_2 \text{ h. l. } (x - a_2) + A_3 \,.\, \text{h. l. } (x - a_3) + \text{\&c.}$$

$$= \text{h. l. } (x - a_1)^{A_1} (x - a_2)^{A_2} (x - a_3)^{A_3} \dots (x - a_n)^{A_n}.$$

10. Let some of the roots be equal, viz. m of them $= a$, or let $(x - a)^m$ be a factor of V.

$$\text{Let } V = (x - a)^m Q.$$

$$\text{Assume } \frac{U}{V} = \frac{A}{(x - a)^m} + \frac{B}{(x - a)^{m-1}} + \frac{C}{(x - a)^{m-2}} + \text{\&c.} + \frac{P}{Q};$$

$$\therefore\ U = AQ + \{B \,.\, (x - a) + C \,.\, (x - a)^2 + \text{\&c.}\}\, Q + P (x - a)^m.$$

Let $x = a$, and let U_a, Q_a be the values of U and Q;

$$\therefore\ U_a = AQ_a, \text{ and } A = \frac{U_a}{Q_a};$$

$$\therefore\ U - \frac{U_a}{Q_a} \,.\, Q = (x - a) \{[B + C \,.\, (x - a) + D (x - a)^2 + \text{\&c.}]$$
$$Q + P (x - a)^{m-1}\}.$$

Hence, as the right-hand side of the equation is divisible by $(x - a)$, the left-hand side is also, let the division be effected, and let U^1 be the quotient;

$$\therefore\ U^1 = \{B + C \,.\, (x - a) + D (x - a)^2 + \text{\&c.}\}\, Q + P \,.\, (x - a)^{m-1}.$$

Again, make $x = a$, and we have $B = \frac{U'_a}{Q_a}$, and proceeding in the same manner we at length arrive at P, which is either constant, or a function of x; if the latter, the case is reduced to that of the preceding article.

To illustrate these methods, we will take two examples*.

(1) Integrate $\dfrac{du}{dx} = \dfrac{x^2 - 7x + 1}{x^3 - 6x^2 + 11x - 6}$.

The denominator is $= (x - 1)(x - 2)(x - 3)$.

$$\text{Let } \frac{x^2 - 7x + 1}{x^3 - 6x^2 + 11x - 6} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 2};$$

$$\therefore\ x^2 - 7x + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C.(x-1)(x-2).$$

$$\text{Let } x = 1;\ \therefore\ 1 - 7 + 1 = -5 = A(1-2)(1-3) = 2A;\ \therefore\ A = -\frac{5}{2}.$$

$$x = 2;\ \therefore\ 4 - 14 + 1 = -9 = B(2-1)(2-3) = -B;\ \therefore\ B = 9,$$

$$x = 3;\ \therefore\ 9 - 21 + 1 = -11 = C(3-1)(3-2) = 2C;\ \therefore\ C = -\frac{11}{2},$$

$$\therefore\ \int_x \frac{U}{V} = -\frac{5}{2}.\int_x \frac{1}{x-1} + 9\int_x \frac{1}{x-2} - \frac{11}{2}.\int_x \frac{1}{x-3}$$

$$= -\frac{5}{2}\,\text{h. l. }(x-1) + 9\,\text{h. l. }(x-2) - \frac{11}{2}\,\text{h. l. }(x-3)$$

$$= \text{h. l. } \frac{(x-2)^9}{\sqrt{(x-1)^5(x-3)^{11}}}.$$

(2) Integrate $\dfrac{du}{dx} = \dfrac{2x - 5}{(x + 3)(x + 1)^2}$.

$$\text{Let } \frac{2x - 5}{(x+3)(x+1)^2} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{P}{x+3};$$

$$\therefore\ 2x - 5 = A.(x + 3) + B(x + 1)(x + 3) + P.(x + 1)^2.$$

* In these and the following examples the constant will be omitted.

$$\text{Let } x = -1; \quad \therefore \ -7 = A(3-1) = 2A; \quad \therefore \ A = -\frac{7}{2};$$

$$\therefore \ 2x - 5 + \frac{7}{2}(x+3) = B(x+1)(x+3) + P(x+1)^2;$$

$$\therefore \ \frac{11x+11}{2}, \text{ or } \frac{11}{2}(x+1) = B(x+1)(x+3) + P(x+1)^2;$$

$$\therefore \ \frac{11}{2} = B(x+3) + P(x+1).$$

$$\text{Let } x + 1 = 0; \quad \therefore \ \frac{11}{2} = 2B; \quad \therefore \ B = \frac{11}{4},$$

$$x + 3 = 0; \quad \therefore \ \frac{11}{2} = -2P; \quad \therefore \ P = -\frac{11}{4};$$

$$\therefore \ \int_x \frac{U}{V} = -\frac{7}{2}\int_x \frac{1}{(x+1)^2} + \frac{11}{4}\int_x \frac{1}{x+1} - \frac{11}{4}\int_x \frac{1}{x+3}$$

$$= \frac{7}{2} \cdot \frac{1}{x+1} + \frac{11}{4} \text{ h. l. } x + 1 - \frac{11}{4} \text{ h. l. } (x+3)$$

$$= \frac{7}{2} \cdot \frac{1}{x+1} + \frac{11}{4} \text{ h. l. } \left(\frac{x+1}{x+3}\right).$$

11. Next, let V contain quadratic factors having impossible roots.

(1) Let V contain two impossible roots only, and let $(x-a)^2 + \beta^2$ be the quadratic factor;

$$\therefore \ V = Q \cdot \{(x-a)^2 + \beta^2\}.$$

$$\text{Assume } \therefore \ \frac{U}{V} = \frac{Mx + N}{(x-a)^2 + \beta^2} + \frac{P}{Q};$$

$$\therefore \ U = (Mx + N) Q + P\{(x-a)^2 + \beta^2\}.$$

$$\text{Put } x = a + \beta\sqrt{-1}; \quad \therefore \ (x-a)^2 + \beta^2 = 0.$$

Then U becomes $U_1 + U_2\sqrt{-1}$, and Q becomes $Q_1 + Q_2\sqrt{-1}$.

Substituting and making the sum of the possible quantities $= 0$, and also the coefficient of $\sqrt{-1}$, $= 0$, M and N may be found.

Or if P be first found, subtract $P\{(x-a)^2+\beta^2\}$ from each side of the equation;

$$\therefore\ U - P\{(x-a)^2+\beta^2\} = (Mx+N)\,.\,Q.$$

Divide both sides by Q, and then

$$Mx+N = \frac{U - P\,.\,\{(x-a)^2+\beta^2\}}{Q} \text{ is known;}$$

$$\therefore\ \int_x \frac{U}{V} = \int_x \frac{Mx+N}{(x-a)^2+\beta^2} + \int_x \frac{P}{Q}.$$

To integrate $\dfrac{du}{dx} = \dfrac{Mx+N}{(x-a)^2+\beta^2}$, let $x - a = z$;

$$\therefore\ \frac{du}{dx} = \frac{du}{dz} = \frac{Mz + Ma + N}{z^2+\beta^2}$$

$$= \frac{Mz}{z^2+\beta^2} + \frac{Ma+N}{z^2+\beta^2};$$

$$\therefore\ u = M\int_z \frac{z}{z^2+\beta^2} + (Ma+N)\int_z \frac{1}{z^2+\beta^2}$$

$$= M \text{ h. l. } \sqrt{z^2+\beta^2} + \frac{Ma+N}{\beta}\tan^{-1}\left(\frac{z}{\beta}\right)$$

$$= M \text{ h. l. } \sqrt{(x-a)^2+\beta^2} + \frac{Ma+N}{\beta}\tan^{-1}\left(\frac{x-a}{\beta}\right).$$

COR. If $a = 0$, or $\dfrac{du}{dx} = \dfrac{Mx+N}{x^2+\beta^2}$.

$$u = M \text{ h. l. } \sqrt{x^2+\beta^2} + \frac{N}{\beta}\tan^{-1}\frac{x}{\beta}.$$

Ex. 1. Let $\dfrac{du}{dx} = \dfrac{x-3}{x^3+1} = \dfrac{x-3}{(x+1)(x^2-x+1)}$.

Let $\dfrac{x-3}{x^3+1} = \dfrac{A}{x+1} + \dfrac{Mx+N}{(x^2-x+1)}$;

$$\therefore\ x-3 = A(x^2-x+1) + (x+1)(Mx+N),$$

$$x = -1;\quad \therefore\ -4 = 3A, \text{ or } A = -\frac{4}{3};$$

$$\therefore\ x - 3 + \frac{4}{3}(x^2-x+1) = \frac{4x^2-x-5}{3} = \frac{(4x-5)(x+1)}{3}$$

$$= (x+1)(Mx+N);$$

$$\therefore\ \frac{4x-5}{3} = Mx+N;$$

$$\therefore\ \int_x \frac{x-3}{x^3+1} = -\frac{4}{3}\int_x \frac{1}{x+1} + \frac{1}{3}\int_x \frac{4x-5}{x^2-x+1}.$$

To integrate $\dfrac{du}{dx} = \dfrac{4x-5}{x^2-x+1} = \dfrac{4x-5}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}$.

Let $x - \frac{1}{2} = z$; $\quad\therefore\ 4x-5 = 4z-3$;

$$\therefore\ u = \int_z \frac{4z-3}{z^2+\frac{3}{4}} = 4\int_z \frac{z}{z^2+\frac{3}{4}} - 3\int_z \frac{1}{z^2+\frac{3}{4}}$$

$$= 2\text{ h. l. }\left(z^2+\frac{3}{4}\right) - 2\sqrt{3}\tan^{-1}\frac{2z}{\sqrt{3}};$$

$$\therefore\ \int_x \frac{x-3}{x^3+1} = -\frac{4}{3}\text{h.l. }(x+1) + \frac{2}{3}\text{h.l. }(x^2-x+1) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$$

$$= \text{h. l. }\left(\frac{\sqrt{x^2-x+1}}{x+1}\right)^{\frac{4}{3}} - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}.$$

Ex. 2. Let $\frac{du}{dx} = \frac{1}{(x+1)(x+2)^2(x^2+1)}$, which includes the three cases.

Let $\frac{U}{V} = \frac{A}{x+1} + \frac{B}{(x+2)^2} + \frac{C}{x+2} + \frac{Mx+N}{x^2+1}$,

$$1 = A \,.\, (x+2)^2(x^2+1) + \{B + C\,.\,(x+2)\}(x^2+1)(x+1)$$
$$+ (Mx+N)\,.\,(x+1)(x+2)^2,$$

$$x = -2; \quad \therefore \; 1 = B \,.\, 5 \,.\, (1-2) = -5B, \text{ i. e. } B = -\frac{1}{5},$$

$$x = -1; \quad \therefore \; 1 = A \,.\, 2 = 2A; \quad \therefore \; A = \frac{1}{2}.$$

$$1 - \frac{(x+2)^2(x^2+1)}{2} + \frac{(x^2+1)\,.\,(x+1)}{5} = C\,.\,(x+2)(x^2+1)(x+1)$$
$$+ (Mx+N)\,.\,(x+1)(x+2)^2,$$

$$\text{or } -\frac{(5x^4 + 18x^3 + 23x^2 + 18x + 8)}{10} = (x+2)\,.\,(x+1)$$
$$\{C\,.\,(x^2+1) + (Mx+N)(x+2)\}.$$

Divide both sides by $(x+2)\,.\,(x+1)$, or x^2+3x+2

$$-\frac{5x^2+3x+4}{10} = C(x^2+1) + (Mx+N)(x+2).$$

$$\text{Let } x = -2; \quad \therefore \; -\frac{9}{5} = 5C; \quad \therefore \; C = -\frac{9}{25};$$

$$\therefore \; \frac{9(x^2+1)}{25} - \frac{5x^2+3x+4}{10} = -\frac{(7x^2+15x+2)}{50} = (Mx+N)(x+2),$$

$$\text{or } -\frac{(7x+1)\,.\,(x+2)}{50} = (Mx+N)(x+2);$$

$$\therefore \; Mx + N = -\frac{7x+1}{50};$$

$$\therefore \; \int_x \frac{U}{V} = \frac{1}{2}\,.\int_x \frac{1}{x+1} - \frac{1}{5}\int_x \frac{1}{(x+2)^2} - \frac{9}{25}\int_x \frac{1}{x+2} - \frac{1}{50}\,.\int_x \frac{7x+1}{x^2+1}$$

$$= \frac{1}{2}\text{h. l. }(x+1) + \frac{1}{5}\frac{1}{x+2} - \frac{9}{25}\text{h. l. }(x+2) - \frac{7}{50}$$
$$\text{h. l. }\sqrt{x^2+1} - \frac{1}{50}\tan^{-1} x.$$

12. If there be m quadratic factors, each $= (x-a)^2 + \beta^2$, we must assume

$$\frac{U}{V} = \frac{Mx+N}{\{(x-a)^2+\beta^2\}^m} + \frac{M_1x+N_1}{\{(x-a)^2+\beta^2\}^{m-1}} + \&c. + \frac{P}{Q};$$

$$\therefore\ U = \{Mx + N + (M_1x+N_1)[(x-a)^2+\beta^2] + \&c.\}$$

$$Q + P\{(x-a)^2+\beta^2\}^m;$$

and after determining $(Mx+N)$, by putting $(x-a)^2+\beta^2 = 0$, and subtracting $(Mx+N) \cdot Q$ from U; divide both sides by the factor $(x-a)^2+\beta^2$, and then proceed in a similar manner to find M_1 and N_1.

Ex. Let $\frac{U}{V} = \frac{1}{(x^2+1)^2(x+1)}$, resolve it into its partial fractions.

$$\frac{U}{V} = \frac{Mx+N}{(x^2+1)^2} + \frac{M_1x+N_1}{x^2+1} + \frac{P}{x+1};$$

$$\therefore\ U = 1 = \{(Mx+N) + (M_1x+N_1)(x^2+1)\}(x+1) + P(x^2+1)^2.$$

Let $x = \sqrt{-1}$;

$$\therefore\ 1 = (M\sqrt{-1}+N).(\sqrt{-1}+1) = -M + M\sqrt{-1} + N\sqrt{-1} + N;$$

$$\therefore\ N - M = 1, \text{ and } N + M = 0;\ \therefore\ N = \tfrac{1}{2} = -M;\ \therefore\ M = -\tfrac{1}{2},$$

$$1 + \tfrac{1}{2}.(x+1)(x-1) = \frac{x^2+1}{2} = (M_1x+N_1).(x^2+1).(x+1) + P(x^2+1)^2;$$

$$\therefore\ \tfrac{1}{2} = (M_1x+N_1)(x+1) + P(x^2+1).$$

$$\text{Let } x = \sqrt{-1};\quad \therefore\ +\tfrac{1}{2} = (M_1\sqrt{-1}+N_1)(\sqrt{-1}+1)$$
$$= -M_1 + M_1\sqrt{-1} + N_1\sqrt{-1} + N_1;$$

$$\therefore\ N_1 - M_1 = +\tfrac{1}{2}, \text{ and } N_1 + M_1 = 0;\ \therefore\ N_1 = \tfrac{1}{4}, \text{ and } M_1 = -N_1 = -\tfrac{1}{4},$$

$$x = -1;\quad \therefore\ \tfrac{1}{2} = P \times 2;\quad \therefore\ P = \tfrac{1}{4};$$

$$\therefore\ \frac{U}{V} = -\tfrac{1}{2}.\frac{x-1}{(x^2+1)^2} - \tfrac{1}{4}\frac{x-1}{x^2+1} + \tfrac{1}{4}\frac{1}{x-1}.$$

13. The fraction $-\frac{1}{2}\frac{x-1}{(x^2+1)^2}$ to be integrated must be divided into two others, $-\frac{1}{2}\frac{x}{(x^2+1)^2}$, and $\frac{1}{2}\cdot\frac{1}{(x^2+1)^2}$. The former is easily integrated, for

$$\int_x \frac{x}{(x^2+1)^2} = \int_x (x^2+1)^{-2}\,.\,x = -\frac{(x^2+1)^{-1}}{2} = -\frac{1}{2}\cdot\frac{1}{(x^2+1)};$$

but $\int_x \frac{1}{(x^2+1)^2}$ must be integrated by a method which we now proceed to explain.

$$\text{Since } \frac{d\,.\,pq}{dx} = p\cdot\frac{dq}{dx} + q\cdot\frac{dp}{dx};$$

$$\therefore\ \int_x p\cdot\frac{dq}{dx} = pq - \int_x q\cdot\frac{dp}{dx};$$

or if any differential coefficient can be divided into two parts, one of which is a function of x, as p, and the other the differential coefficient of a known function q; then u, the required function, is equal to the product of p and q minus the integral of q multiplied by the differential coefficient of p.

The utility of this method depends upon the function $q\cdot\frac{dp}{dx}$ being less complicated than the original differential coefficient.

Ex. 1. Let $\frac{du}{dx} = x^3(1+x^2)^2 = x^2\,.\,x(1+x^2)^2$.

$$\text{Here } p = x^2, \quad \frac{dq}{dx} = x(1+x^2)^2;$$

$$\therefore\ \frac{dp}{dx} = 2x, \quad q = \int_x x(1+x^2)^2 = \frac{(1+x^2)^3}{2\,.\,3};$$

$$\therefore\ \int_x x^3\,.\,(1+x^2)^2 = \frac{x^2\,.\,(1+x^2)^3}{2\,.\,3} - \frac{1}{3}\cdot\int_x x(1+x^2)^3$$

$$= \frac{x^2\,.\,(1+x^2)^3}{2\,.\,3} - \frac{1}{3}\cdot\frac{(1+x^2)^4}{2\,.\,4}.$$

Ex. 2. Let $\dfrac{du}{dx} = \dfrac{1}{(x^2+1)^n}$.

Now $\dfrac{1}{(x^2+1)^{n-1}} = \dfrac{x^2+1}{(x^2+1)^n} = \dfrac{x^2}{(x^2+1)^n} + \dfrac{1}{(x^2+1)^n}$;

$$\therefore \int_x \frac{1}{(x^2+1)^n} = -\int_x \frac{x^2}{(x^2+1)^n} + \int_x \frac{1}{(x^2+1)^{n-1}}.$$

But $\dfrac{x^2}{(x^2+1)^n} = x \cdot \dfrac{x}{(x^2+1)^n}$.

Here $p = x$, and $\dfrac{dq}{dx} = \dfrac{x}{(x^2+1)^n} = x\,(x^2+1)^{-n}$;

$$\therefore \frac{dp}{dx} = 1, \quad \text{and } q = -\frac{1}{(2n-2)(x^2+1)^{n-1}};$$

$$\therefore \int_x \frac{x^2}{(x^2+1)^n} = \frac{-x}{(2n-2)(x^2+1)^{n-1}} + \frac{1}{2n-2} \cdot \int_x \frac{1}{(x^2+1)^{n-1}};$$

$$\therefore \int_x \frac{1}{(x^2+1)^n} = \frac{1}{2n-2} \cdot \frac{x}{(x^2+1)^{n-1}} - \frac{1}{2n-2} \cdot \int_x \frac{1}{(x^2+1)^{n-1}}$$

$$+ \int_x \frac{1}{(x^2+1)^{n-1}}$$

$$= \frac{1}{2n-2} \cdot \frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2n-2} \cdot \int_x \frac{1}{(x^2+1)^{n-1}};$$

by this process $\int_x \dfrac{1}{(x^2+1)^n}$ is made to depend upon $\int_x \dfrac{1}{(x^2+1)^{n-1}}$, and by substituting $n-1$, $n-2$, and $n-(n-1)$ for n, it will be reduced to $\int_x \dfrac{1}{x^2+1} = \tan^{-1} x$.

Ex. Let $n = 4$, or let $\int \dfrac{1}{(x^2+1)^4}$ be required.

$$\int_x \frac{1}{(x^2+1)^4} = \frac{1}{6} \cdot \frac{x}{(x^2+1)^3} + \frac{5}{6} \cdot \int_x \frac{1}{(x^2+1)^3},$$

$$\int_x \frac{1}{(x^2+1)^3} = \frac{1}{4} \cdot \frac{x}{(x^2+1)^2} + \frac{3}{4} \cdot \int_x \frac{1}{(x^2+1)^2},$$

$$\int_x \frac{1}{(x^2+1)^2} = \frac{1}{2} \cdot \frac{x}{x^2+1} + \frac{1}{2} \cdot \int_x \frac{1}{x^2+1}$$

$$= \frac{1}{2} \cdot \frac{x}{x^2+1} + \frac{1}{2} \tan^{-1} x;$$

$$\therefore \int_x \frac{1}{(x^2+1)^4} = \frac{1}{6} \cdot \frac{x}{(x^2+1)^3} + \frac{5}{4.6} \cdot \frac{x}{(x^2+1)^2} + \frac{3.5}{2.4.6} \cdot \frac{x}{x^2+1}$$

$$+ \frac{3.5}{2.4.6} \tan^{-1} x.$$

14. To the $\int_x \frac{1}{(x^2+1)^n}$ may also be referred the

$$\int_x \frac{Mx+N}{\{(x-a)^2+\beta^2\}^m}.$$

For, let $x - a = z$;

$$\therefore \int_x \frac{Mx+N}{((x-a)^2+\beta^2)^m} = \frac{Mz+(Ma+N)}{(z^2+\beta^2)^m}$$

$$= M\int_z \frac{z}{(z^2+\beta^2)^m} + (Ma+N)\int_z \frac{1}{(z^2+\beta^2)^m},$$

and $M\int_z \frac{z}{(z^2+\beta^2)^m} = -\frac{M}{2m-2} \cdot \frac{1}{(z^2+\beta^2)^{m-1}},$

and $(Ma+N)\int_z \frac{1}{(z^2+\beta^2)^m} = \frac{Ma+N}{\beta^{2m}} \int_z \frac{1}{\left(\frac{z^2}{\beta^2}+1\right)^m}$

$$= \frac{Ma+N}{\beta^{2m}} \cdot \int_y \frac{\beta}{(y^2+1)^m} = \frac{Ma+N}{\beta^{2m-1}} \int_y \frac{1}{(y^2+1)^m},$$

by putting $\frac{z}{\beta} = y$, or $z = \beta y$, which can be integrated by the preceding method.

15. Let $\dfrac{du}{dx} = \dfrac{x^m}{(x^2+1)^n}$,

$$\frac{x^m}{(x^2+1)^n} = x^{m-1}\cdot\frac{x}{(x^2+1)^n}.$$

Here $p = x^{m-1}$, $\dfrac{dq}{dx} = \dfrac{x}{(x^2+1)^n}$;

$$\therefore\ \frac{dp}{dx} = (m-1)\,.\,x^{m-2},\quad q = -\frac{1}{(2n-2)(x^2+1)^{n-1}};$$

$$\therefore\ \int_x \frac{x^m}{(x^2+1)^n} = -\frac{x^{m-1}}{(2n-2)(x^2+1)^{n-1}} + \frac{m-1}{2n-2}\cdot\int_x \frac{x^{m-2}}{(x^2+1)^{n-1}};$$

$$\therefore\ \int_x \frac{x^{m-2}}{(x^2+1)^{n-1}} = -\frac{x^{m-3}}{(2n-4)(x^2+1)^{n-2}} + \frac{m-3}{2n-4}\cdot\int_x \frac{x^{m-4}}{(x^2+1)^{n-2}};$$

and in this manner if $m < n$, the integral is reducible either to $\int_x \dfrac{x}{(x^2+1)^r}$ or $\int \dfrac{1}{(x^2+1)^r}$; the former of which is immediately integrable, and the latter is integrated by the method of Art. 13.

16. Next, to integrate functions of the form

$$\frac{x^m}{(a+bx+cx^2)^n} \text{ and } \frac{1}{x^m(a+bx+cx^2)^n}.$$

In these cases the trinomial cx^2+bx+a must be reduced to a binomial; and then the integration may be effected by methods already given: we will first however shew how the function may be integrated when $m = 0$ and $n = 1$.

17. Integrate $\dfrac{du}{dx} = \dfrac{1}{a+bx+cx^2}$,

$$\frac{1}{a+bx+cx^2} = \frac{1}{c}\,\frac{1}{\left(\dfrac{a}{c}+\dfrac{b}{c}x+x^2\right)}.$$

$$\text{Let } x + \frac{b}{2c} = z; \quad \therefore \frac{dz}{dx} = 1,$$

$$\text{and } x^2 + \frac{bx}{c} + \frac{a}{c} = z^2 + \frac{a}{c} - \frac{b^2}{4c^2};$$

$$\therefore \frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx} = \frac{du}{dz} = \frac{1}{c\left(z^2 + \frac{a}{c} - \frac{b^2}{4c^2}\right)}.$$

(1) Let $\frac{a}{c} > \frac{b^2}{4c^2}$, or $4ac > b^2$;

$$\therefore u = \frac{1}{c} \cdot \int_z \frac{1}{z^2 + \frac{4ac - b^2}{4c^2}}.$$

But $\because \int_z \frac{1}{z^2 + a^2} = \frac{1}{a} . \tan^{-1} \frac{z}{a}$;

$$\therefore u = \frac{1}{c} \cdot \frac{2c}{\sqrt{4ac - b^2}} . \tan^{-1}\left(\frac{2cz}{\sqrt{4ac - b^2}}\right)$$

$$= \frac{2}{\sqrt{4ac - b^2}} . \tan^{-1} \frac{2cx + b}{\sqrt{4ac - b^2}}.$$

(2) Let $\frac{a}{c} < \frac{b^2}{4c^2}$, make $a^2 = \frac{b^2 - 4ac}{4c^2}$;

$$\therefore u = \frac{1}{c} \cdot \int_z \frac{1}{z^2 - a^2} = \frac{1}{2ca} \cdot \int_z \left(\frac{1}{z - a} - \frac{1}{z + a}\right)$$

$$= \frac{1}{2ca} . \text{h. l.} \left(\frac{z - a}{z + a}\right)$$

$$= \frac{1}{\sqrt{b^2 - 4ac}} \text{h. l.} \frac{2cx + b - \sqrt{b^2 - 4ac}}{2cx + b + \sqrt{b^2 - 4ac}}.$$

Ex. 1. Let $a = b = c = 1$; $\therefore \sqrt{4ac - b^2} = \sqrt{3}$,

$$\int_x \frac{1}{1 + x + x^2} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x + 1}{\sqrt{3}}.$$

Ex. 2. Let $c=b=1$, and $a=-1$; $\therefore \sqrt{b^2-4ac}=\sqrt{5}$;

$$\therefore \int_x \frac{1}{x^2+x-1} = \frac{1}{\sqrt{5}}\text{h. l.}\left(\frac{2x+1-\sqrt{5}}{2x+1+\sqrt{5}}\right).$$

18. To integrate $\dfrac{x^m}{(a+bx+cx^2)^n}$,

$$\frac{x^m}{(a+bx+cx^2)^n} = \frac{1}{c^n}\cdot\frac{x^m}{\left(x^2+\dfrac{b}{c}x+\dfrac{a}{c}\right)^n}.$$

Let $x+\dfrac{b}{2c}=z$, or $x+\alpha=z$, if $\alpha=\dfrac{b}{2c}$;

$$\therefore x^2+\frac{b}{c}x+\frac{a}{c} = z^2+\frac{a}{c}-\frac{b^2}{4c^2} = (z^2\pm\beta^2);$$

$$\therefore \int_x \frac{x^m}{(a+bx+cx^2)^n} = \frac{1}{c^n}\int_z \frac{(z-\alpha)^m}{(z^2\pm\beta^2)^n}.$$

Here are two cases:

(1) Let $\dfrac{a}{c}>\dfrac{b^2}{4c^2}$; then $\displaystyle\int_z \frac{(z-\alpha)}{(z^2+\beta^2)^n}$ may be found by the method used in Art. 15.

(2) Let $\dfrac{a}{c}<\dfrac{b^2}{4c^2}$; $\therefore \displaystyle\int_z \frac{(z-\alpha)^m}{(z^2-\beta^2)^n} = \int_z \frac{(z-\alpha)^m}{(z+\beta)^n(z-\beta)^n}$, must be integrated by the method of partial fractions.

19. Again, to integrate $\dfrac{1}{x^m(a+bx+cx^2)^n}$.

Let $x=\dfrac{1}{z}$; $\therefore \dfrac{dz}{dx}=-\dfrac{1}{x^2}=-z^2$;

$$\therefore \frac{du}{dx}=\frac{du}{dz}\cdot\frac{dz}{dx}=-z^2\frac{du}{dz},$$

$$\text{or } \frac{z^{m+2n}}{(az^2+bz+c)^n} = -z^2\frac{du}{dz};$$

$$\therefore \; u = -\int_z \frac{z^{m+2n-2}}{(az^2 + bz + c)^n},$$

which is a case of the preceding article.

20. Next to integrate the differential coefficients

$$\frac{1}{x^n - 1}, \text{ and } \frac{1}{x^n + 1}.$$

Since when n is an even number,

$$x^n - 1 = (x-1)(x+1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right)\ldots$$

continued to the factor

$$x^2 - 2x\cos\left(\frac{n-2}{n}\right)\pi + 1;$$

and when n is odd,

$$x^n - 1 = (x-1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right)\ldots$$

continued to the factor

$$x^2 - 2x\cos\frac{n-1}{n}\pi + 1;$$

and since the factors of $x^n + 1 = 0$ are contained in

$$x^2 - 2x\cos\frac{2m+1}{n}\pi + 1;$$

we may integrate these differential coefficients by resolving them into partial fractions, having simple and quadratic equations for their denominators: to effect this the following process which obviates the necessity of finding the constants of the successive numerators may be used.

CASE I. Let n be even, resolve $\dfrac{1}{x^n - 1}$ into its quadratic factors.

Since $x^n - 1 = (x-1)(x+1)\left(x^2 - 2x\cos\dfrac{2m\pi}{n} + 1\right)$,

where $x^2 - 2x\cos\dfrac{2m\pi}{n} + 1$

represents all the quadratic factors.

$$\therefore \log(x^n - 1) = \log(x-1) + \log(x+1) + \log\left(x^2 - 2x\cos\frac{2m\pi}{n} + 1\right);$$

$$\therefore \frac{nx^{n-1}}{x^n - 1} = \frac{1}{x-1} + \frac{1}{x+1} + \frac{2x - 2\cos\dfrac{2m\pi}{n}}{x^2 - 2x\cos\dfrac{2m\pi}{n} + 1};$$

$$\therefore \frac{nx^n}{x^n - 1} = \frac{x}{x-1} + \frac{x}{x+1} + \frac{2x^2 - 2x\cos\dfrac{2m\pi}{n}}{x^2 - 2x\cos\dfrac{2m\pi}{n} + 1}.$$

Now subtract n from the left hand side of the equation, and what will amount to the same thing, subtract, on the right side, one from each simple factor, and two from each quadratic factor: then

$$\frac{n}{x^n - 1} = \frac{1}{x-1} + \frac{1}{x+1} - \frac{2 - 2x\cos\dfrac{2m\pi}{n}}{x^2 - 2x\cos\dfrac{2m\pi}{n} + 1};$$

$$\therefore \int_x \frac{1}{x^n - 1} = \frac{1}{n}\log(x^2 - 1) - \frac{2}{n}\int_x \frac{1 - x\cos\dfrac{2m\pi}{n}}{x^2 - 2x\cos\dfrac{2m\pi}{n} + 1}.$$

The last integral is of the form $\displaystyle\int_x \frac{1 - \beta x}{x^2 - 2\beta x + 1}$, and is, if we make $x - \beta = z$; $1 - \beta^2 = \delta^2$, and integrate

$$= \delta \tan^{-1}\frac{z}{\delta} - \beta \log\sqrt{z^2 + \delta^2}, \text{ or since } \delta = \sin\frac{2m\pi}{n};$$

$$\int_x \frac{1}{x^n - 1} = \frac{1}{n}\log(x^2 - 1) - \frac{2}{n}\left\{\sin\frac{2m\pi}{n}\tan^{-1}\left(\frac{x - \cos\frac{2m\pi}{n}}{\sin\frac{2m\pi}{n}}\right)\right.$$

$$\left. - \cos\frac{2m\pi}{n}\log\sqrt{x^2 - 2x\cos\frac{2m\pi}{n} + 1}\right\}.$$

The method when n is odd is precisely the same, but there is but one simple factor.

The same method also applies to $\frac{1}{x^n + 1}$: for when n is odd,

$$x^n + 1 = (x + 1)\left(x^2 - 2x\cos\frac{2m+1}{n}\pi + 1\right);$$

and when n is even,

$$x^n + 1 = \left(x^2 - 2x\cos\frac{2m+1}{n}\pi + 1\right);$$

and proper substitutions being made for m, all the factors will be exhibited.

21. Finally, to integrate the functions $\frac{x^r}{x^n - 1}$ and $\frac{x^r}{x^n + 1}$.

Since all the quadratic factors of $x^n - 1$ are included in the general formula $x^2 - 2x\cos\frac{2m\pi}{n} + 1$.

$$\text{Let } \frac{x^r}{x^n - 1} = \frac{Mx + N}{x^2 - 2x\cos\left(\frac{2m\pi}{n}\right) + 1} + \frac{P}{Q};$$

$$\therefore\ x^r = (Mx + N)\,.\,Q + P\left\{x^2 - 2x\cos\left(\frac{2m\pi}{n}\right) + 1\right\}.$$

$$\text{Let } x = \cos\frac{2m\pi}{n} + \sqrt{-1}\,\sin\frac{2m\pi}{n},$$

and let Q_1 be the value of Q; and on this supposition also let z be put for this particular value of x, for the sake of brevity;

$$\therefore\ z^r = (Mz + N)\,Q_1.$$

But $x^n - 1 = Q\left(x^2 - 2x\cos\frac{2m\pi}{n} + 1\right)$;

$$\therefore\ nx^{n-1} = Q\left(2x - 2\cos\frac{2m\pi}{n}\right)$$

$$+ \left(x^2 - 2x\cos\frac{2m\pi}{n} + 1\right).\frac{dQ}{dx};$$

$$\therefore\ nz^{n-1} = Q_1\left(2z - 2\cos\frac{2m\pi}{n}\right);$$

$$\therefore\ nz^n = n = Q_1\left(2z^2 - 2z\cos\frac{2m\pi}{n}\right) = Q_1(z^2 - 1).$$

Since $z^n = 1$, and $z^2 - 2z\cos\frac{2m\pi}{n} + 1 = 0$;

$$\therefore\ Q_1 = \frac{n}{z^2 - 1}.$$

Hence $z^r = (Mz + N).\frac{n}{z^2 - 1}$,

or $n(Mz + N) = z^r(z^2 - 1) = z^{r+1}.\left(z - \frac{1}{z}\right)\ldots\ldots(1)$.

But since $z = \cos\frac{2m\pi}{n} + \sqrt{-1}\sin\frac{2m\pi}{n}$;

$$\therefore\ \frac{1}{z} = \cos\frac{2m\pi}{n} - \sqrt{-1}\sin\frac{2m\pi}{n},$$

and $z - \frac{1}{z} = 2\sqrt{-1}\sin\frac{2m\pi}{n}$,

and $z^{r+1} = \cos\frac{2n(r+1).\pi}{n} + \sqrt{-1}\sin\frac{2m(r+1).\pi}{n}$;

whence substituting in (1), we have

$$Mn \cdot \cos\frac{2m\pi}{n} + Mn\sqrt{-1}\sin\frac{2m\pi}{n} + Nn$$

$$= \left\{\cos\left(\frac{2m.(r+1)\pi}{n}\right) + \sqrt{-1}.\sin\left(\frac{2m(r+1)\pi}{n}\right)\right\} 2\sqrt{-1}\sin\frac{2m\pi}{n}$$

$$= 2\sqrt{-1}\cos\frac{2m.(r+1)\pi}{n}\sin\frac{2m\pi}{n} - 2\sin\frac{2m(r+1)\pi}{n} . \sin\frac{2m\pi}{n};$$

$$\therefore\ Mn\sin\frac{2m\pi}{n} = 2\cos\frac{2m(r+1)\pi}{n} . \sin\frac{2m\pi}{n},$$

$$\text{or } M = \frac{2}{n} . \cos\left(\frac{2m.(r+1)\pi}{n}\right),$$

$$\text{and } N = -\frac{2}{n} . \sin\frac{2m.(r+1)\pi}{n}\sin\frac{2m\pi}{n}$$

$$-\frac{2}{n} . \cos\frac{2m(r+1)\pi}{n} . \cos\frac{2m\pi}{n}$$

$$= -\frac{2}{n} . \cos\frac{2mr.\pi}{n};$$

$$\therefore\ \frac{Mx + N}{x^2 - 2x\cos\frac{2m\pi}{n} + 1} = \frac{2}{n}\,\frac{x . \cos\frac{2m(r+1)\pi}{n} - \cos\frac{2mr.\pi}{n}}{x^2 - 2x\cos\frac{2m\pi}{n} + 1}.$$

Case I. Let n be odd; $\therefore\ Q = x - 1$, and $P = A$;

$$\text{also } x^r = f(Mx + N) . (x - 1) + A . \frac{x^n - 1}{x - 1},$$

where $f . (Mx + N)$ represents the numerator of the fraction, formed by reducing the fractions having quadratic denominators into one.

$$\text{Let } x = 1;\ \therefore\ \frac{x^n - 1}{x - 1} = n, \text{ and } A = \frac{1}{n}.$$

$$\therefore \int_x \frac{x^r}{x^r - 1} = \frac{1}{n} \cdot \int_x \frac{1}{x-1} + \frac{2}{n} \cdot \int_x \frac{x \cos \frac{2m.(r+1)\pi}{n} - \cos \frac{2mr.\pi}{n}}{x^2 - 2x \cos \frac{2m\pi}{n} + 1};$$

the latter integral is of the form $\int_x \frac{Mx + N}{x^2 - 2ax + 1}$

$$= M . \text{h. l.} \sqrt{x^2 - 2ax + 1} + \frac{Ma + N}{\sqrt{1 - a^2}} . \tan^{-1} \left(\frac{x - a}{\sqrt{1-a^2}}\right);$$

putting for M and N their values;

and since $a = \cos \frac{2m\pi}{n}$; $\therefore \sqrt{1 - a^2} = \sin \frac{2m\pi}{n}$

Also $\frac{Ma + N}{\sqrt{1 - a^2}} = - \frac{2}{n} . \sin \frac{2m . (r + 1) . \pi}{n}$, we have

$$\int_x \frac{x^r}{x^n - 1} = \frac{1}{n} . \int \frac{1}{(x - 1)} + \frac{2}{n} \left\{ \cos . \frac{2m (r + 1) \pi}{n} \right.$$

$$\text{h. l.} \sqrt{x^2 - 2x \cos \frac{2m\pi}{n} + 1} - \sin \frac{2m (r + 1) \pi}{n}$$

$$\left. \tan^{-1} \left(\frac{x - \cos \frac{2m\pi}{n}}{\sin \frac{2m\pi}{n}} \right) \right\},$$

where m must be taken from $m = 1$ to $m = \frac{n - 1}{2}$.

Cor. 1. If $r = 0$, we have $\int \frac{1}{x^n - 1}$

$$= \frac{1}{n} . \text{h. l.} (x - 1) + \frac{2}{n} . \cos \frac{2m\pi}{n} . \text{h. l.} \sqrt{x^2 - 2x \cos \frac{2m\pi}{n} + 1}$$

$$- \frac{2}{n} . \sin \frac{2m\pi}{n} . \tan^{-1} \frac{x - \cos \frac{2m\pi}{n}}{\sin \frac{2m\pi}{n}}.$$

COR. 2. If we add to the integral the constant quantity

$$-\sin\left(\frac{2m(r+1).\pi}{n}\right)\tan^{-1}\left(\frac{-\cos\left(\frac{2m\pi}{n}\right)}{\sin\frac{2m\pi}{n}}\right).$$

Since $\tan^{-1}(A-B)=\dfrac{A-B}{1+AB}$;

$$\therefore \int_x \frac{x^r}{x^n-1}=\frac{1}{n}.\text{h. l. }(x-1)+\frac{2}{n}.\cos\left(\frac{2m.(r+1)\pi}{n}\right)$$

$$\text{h. l. }\sqrt{x^2-2x\cos\frac{2m\pi}{n}+1}-\frac{2}{n}.\sin\frac{2m(r+1)\pi}{n}$$

$$\tan^{-1}\left\{\frac{x\sin\frac{2m\pi}{n}}{1-x\cos\frac{2m\pi}{n}}\right\},$$

$$\text{and }\int_x\frac{1}{x^n-1}=\frac{1}{n}\text{ h. l. }(x-1)+\frac{2}{n}\cos\left(\frac{2m\pi}{n}\right)$$

$$\text{h. l. }\sqrt{x^2-2x\cos\frac{2m\pi}{n}+1}-\frac{2}{n}.\sin\frac{2m\pi}{n}$$

$$\tan^{-1}\left\{\frac{x\sin\frac{2m\pi}{n}}{1-x\cos\frac{2m\pi}{n}}\right\}.$$

$$\text{Ex. }\int\frac{1}{x^5-1}=\frac{1}{5}\text{ h. l. }(x-1)+\frac{2}{5}\cos\frac{2\pi}{5}$$

$$\sqrt{x^2-2x\cos\frac{2\pi}{5}+1}-\frac{2}{5}\sin\frac{2\pi}{5}\tan^{-1}\left\{\frac{x\sin\frac{2\pi}{5}}{1-x\cos\frac{2\pi}{5}}\right\}$$

$$+\frac{2}{5}\cos\frac{4\pi}{5}\text{ h. l. }\sqrt{x^2-2x\cos\frac{4\pi}{5}+1}$$

$$-\frac{2}{5}\sin\frac{4\pi}{5}\tan^{-1}\left\{\frac{x\sin\frac{4\pi}{5}}{1-x\cos\frac{4\pi}{5}}\right\}+C.$$

COR. If n be even, there will be two terms of the form $\frac{A}{x-1}$ and $\frac{B}{x+1}$; and A and B may be found by putting $+1$ and -1 for x in the equation

$$x^r = f(Mx+N)(x-1) + A\frac{x^n-1}{x-1},$$

$$\text{and } x^r = f(Mx+N)(x+1) + B\frac{x^n-1}{x+1};$$

$$\therefore\ A=\frac{1}{n},\quad\text{and } B=\pm\frac{1}{n}.$$

22. Also the function $\frac{x^r}{x^n+1}$, since the quadratic factors of the denominator are included in the general formula, $x^2-2x\cos\left(\frac{2m+1}{n}\right)\pi+1$, may be integrated in a similar manner, and will be found to depend upon the terms

$$\frac{2}{n}\left\{\cos\frac{(n-r-1)\{(2m+1)\}\pi}{n}.\text{h.l.}\sqrt{x^2-2x\cos\frac{2m+1}{n}\pi+1}\right.$$

$$\left.+\sin\left\{\frac{(n-r-1)(2m+1)\pi}{n}\right\}\tan^{-1}\frac{x\sin\left(\frac{2m+1}{n}\pi\right)}{1-x\cos\left(\frac{2m+1}{n}\pi\right)}\right\}+C.$$

If n be even, m must be taken from $m=0$ to $m=\frac{n}{2}$.

If n be odd.......................... $m=1$ to $m=\frac{n-1}{2}$, and there will be a simple factor $(x + 1)$, and a fraction $\frac{A}{x+1}$;

where $A = \frac{(-1)^r}{n}$.

Cor. Hence $\int_x \frac{1}{x^n + 1}$ will depend upon the terms

$$\frac{2}{n} . \cos \left\{ \frac{(n-1).(2m+1).\pi}{n} \right\} . \text{h.l.} \sqrt{x^2 - 2x \cos\left(\frac{2m+1}{n}\right)\pi + 1}$$

$$- \frac{2}{n} . \sin \frac{(n-1).(2m+1)\pi}{n} \tan^{-1} \left\{ \frac{x . \sin \frac{(2m+1)}{n} \pi}{1 - x \cos \left(\frac{2m+1}{n} \pi\right)} \right\} + C.$$

23. To integrate $\frac{x^r}{x^{2n} - 2x^n \cos a + 1}$.

The quadratic factors of the denominator will be all comprised in the term $x^2 - 2x \cos\theta + 1$, where $\theta = \frac{2m\pi + a}{n}$.

Let $\frac{x^r}{x^{2n} - 2x^n . \cos a + 1} = \frac{Mx + N}{x^2 - 2x \cos\theta + 1} + \frac{P}{Q}$;

$\therefore\ x^r = (x^2 - 2x\cos\theta + 1) P + (Mx + N) . Q.$

Let $x = \cos\theta + \sqrt{-1} \sin\theta = z$, and Q_1 be the value of Q;

$\therefore\ z^r = (Mz + N) . Q_1.$

Now $x^{2n} - 2x^n \cos a + 1 = (x^2 - 2x\cos\theta + 1) . Q$;

$$\therefore\ 2n . x^{2n-1} - 2n . x^{n-1} \cos a = (2x - 2\cos\theta)\, Q + (x^2 - 2x\cos\theta + 1) \frac{dQ}{dx}$$

$$2nz^{2n-1} - 2nz^{n-1} \cos a = (2z - 2\cos\theta) . Q_1,$$

or $2nz^{2n} - 2nz^n \cos a = n(z^{2n} - 1) = (2z^2 - 2z\cos\theta) . Q_1$

$$= (z^2 - 1) . Q_1;$$

$$\therefore\ Q_1 = \frac{n(z^{2n} - 1)}{z^2 - 1};$$

$$\therefore\ z^r = (Mz + N) \cdot n \frac{z^{2n} - 1}{z^2 - 1} = n \cdot (Mz + N) \cdot \frac{z^{2n-1} - \frac{1}{z}}{z - \frac{1}{z}};$$

$$\therefore\ z^{r-n+1} = n\,(Mz + N) \cdot \frac{\left(z^n - \frac{1}{z^n}\right)}{z - \frac{1}{z}},$$

or $\cos\{(r - n + 1)\}\,\theta + \sqrt{-1}\sin(r - n + 1)\,\theta$

$$= n\,\{M(\cos\theta + \sqrt{-1}\sin\theta) + N\} \times \frac{\sin n\theta}{\sin\theta}$$

$$= n \cdot \left(M\frac{\cos\theta\sin n\theta}{\sin\theta} + M\sqrt{-1} \cdot \sin n\theta + N \cdot \frac{\sin n\theta}{\sin\theta}\right);$$

$$\therefore\ M = \frac{1}{n} \cdot \frac{\sin(r - n + 1)\,\theta}{\sin n\theta} = \frac{1}{n}\,\frac{\sin\left\{\frac{(r - n + 1) \cdot (2m\pi + a)}{n}\right\}}{\sin a},$$

and $\dfrac{N \cdot \sin n\theta}{\sin\theta}$, or $\dfrac{N\sin a}{\sin\theta} = \dfrac{1}{n}\cos(r - n + 1)\,\theta - \dfrac{M \cdot \cos\theta\sin a}{\sin\theta}$

$$= \left\{\frac{1}{n} \cdot \cos(r - n + 1)\,\theta \cdot \sin\theta - \frac{1}{n} \cdot \sin(r - n + 1)\,\theta \cdot \cos\theta\right\}\frac{1}{\sin\theta}$$

$$= -\frac{1}{n} \cdot \frac{\sin(r - n)\,\theta}{\sin\theta} = \frac{1}{n} \cdot \frac{\sin(n - r)\,\theta}{\sin\theta};$$

$$\therefore\ N = +\frac{1}{n}\,\frac{\sin(n - r)\,\theta}{\sin n\theta} = +\frac{1}{n}\,\frac{\sin\left\{\frac{(n - r)\,(2m\pi + a)}{n}\right\}}{\sin a},$$

and the integral is reduced to that of

$$\frac{1}{n\sin a} \cdot \int_x \frac{x \cdot \sin(r - n + 1)\,\theta + \sin(n - r)\,\theta}{x^2 - 2x\cos\theta + 1}.$$

The integral of which is known.

EXAMPLES.

(1) $$\int_x \frac{2x+3}{x^3+x^2-2x} = \frac{1}{3}\log\frac{(x-1)^5}{\sqrt{x^4+2x^3}}.$$

(2) $$\int_x \frac{x}{(x^2+1)\,.\,(x^2+3)} = \log\sqrt{\frac{x^2+1}{x^2+3}}.$$

(3) $$\int_x \frac{x^2}{(x^2+1)\,.\,(x^2+4)} = \frac{1}{3}\left\{2\tan^{-1}\frac{x}{2} - \tan^{-1}x\right\}.$$

(4) $$\int_x \frac{3x^2+x-2}{(x-1)^3.\,(x^2+1)} = -\frac{1}{2}\,\frac{1}{(x-1)^2} - \frac{5}{2}\cdot\frac{1}{x-1}$$
$$+\frac{3}{2}\log\frac{\sqrt{x^2+1}}{x-1} - \tan^{-1}x$$

(5) $$\int_x \frac{x^4}{(1+x^2)^2} = \left(x^3+\frac{3x}{2}\right)\frac{1}{1+x^2} - \frac{3}{2}\tan^{-1}x.$$

(6) $$\int_x \frac{x^5}{(1+x^2)^3} = \left(x^2+\frac{3}{4}\right)\frac{1}{(1+x^2)^2} + \log\sqrt{1+x^2}.$$

(7) $$\int_x \frac{1}{x^5.\,(1+x^2)} = -\frac{1}{5x^5} + \frac{1}{3x^3} - \frac{1}{x} - \tan^{-1}x.$$

(8) $$\int_x \frac{1}{x^4(1+x^2)^2} = -\left(\frac{1}{3x^3} - \frac{5}{3x} - \frac{5x}{2}\right)\frac{1}{1+x^2} + \frac{5}{2}\tan^{-1}x.$$

(9) $$\int_x \frac{x^2}{a+bx+cx^2} = \frac{x}{c} - \frac{b}{c^2}.\log\sqrt{a+bx+cx^2}$$
$$+\left(\frac{b}{2c^2}-\frac{a}{c}\right)\cdot\int\frac{1}{a+bx+cx^2}.$$

(10) $$\int_x \frac{x}{(1+x+x^2)^2} = -\frac{x+2}{3(1+x+x^2)} + \frac{2}{3\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right).$$

(11) $$\int_x \frac{x^5}{x^3+1} = \frac{x^3}{3} - \log\sqrt[3]{x^3+1}.$$

(12) $$\int_x \frac{x^4}{x^3+1} = \frac{x^2}{2} + \frac{1}{3}\log\frac{\sqrt{x^2-x+1}}{(x+1)} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{x\sqrt{3}}{2-x}.$$

(13) $$\int_x \frac{x^2}{x^4-a^4} = \frac{1}{4}\log\frac{x-a}{x+a} + \frac{1}{2}\tan^{-1}\frac{x}{a}.$$

(14) $$\int_x \frac{x^2}{x^4+1} = \frac{1}{4\sqrt{2}}\log\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}.$$

(15) $$\int_x \frac{1}{x^6-1} = \frac{1}{6}\log\frac{(x-1)\sqrt{x^2-x+1}}{(x+1)\sqrt{x^2+x+1}} - \frac{1}{2\sqrt{3}}\tan^{-1}\frac{x\sqrt{3}}{1-x^2}.$$

CHAPTER III.

IRRATIONAL QUANTITIES.

24. The functions of this class will be treated of in the following order:

(1) Those which are the differential coefficients of known functions.

(2) Those which may be reduced to rational functions by means of obvious substitutions.

(3) Those which must be referred by means of *Formulas of Reduction* to known integrals.

25. Class I. Since if $u = \sin^{-1} x$, $\dfrac{du}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$,

and if $u = \cos^{-1} x$, $\dfrac{du}{dx} = -\dfrac{1}{\sqrt{1 - x^2}}$,

.... $u = V\sin^{-1} x$, $\dfrac{du}{dx} = \dfrac{1}{\sqrt{2x - x^2}}$,

.... $u = \sec^{-1} x$, $\dfrac{du}{dx} = \dfrac{1}{x\sqrt{x^2 - 1}}$;

$$\therefore \int_x \frac{1}{\sqrt{1 - x^2}} = \sin^{-1} x,$$

$$\int_x \frac{-1}{\sqrt{1 - x^2}} = \cos^{-1} x,$$

$$\int_x \frac{1}{\sqrt{2x - x^2}} = V\sin^{-1} x,$$

$$\int_x \frac{1}{x\sqrt{x^2 - 1}} = \sec^{-1} x.$$

COR. Since $\int_x \frac{1}{\sqrt{a^2 - x^2}} = \int_x \frac{\frac{1}{a}}{\sqrt{1 - \frac{x^2}{a^2}}} = \int_z \frac{1}{\sqrt{1 - z^2}}$,

$$\text{where } z = \frac{x}{a};$$

$$\therefore \int_x \frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1}(z) = \sin^{-1}\left(\frac{x}{a}\right),$$

and $\int_x \frac{1}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \cdot \int_x \frac{\frac{1}{a}}{\frac{x}{a}\sqrt{\frac{x^2}{a^2} - 1}} = \frac{1}{a} \cdot \int_z \frac{1}{z\sqrt{z^2 - 1}}$

$$= \frac{1}{a} \sec^{-1} z = \frac{1}{a} \cdot \sec^{-1}\left(\frac{x}{a}\right).$$

26. CLASS II. Next, to integrate the differential coefficients:

$$\frac{1}{\sqrt{x^2 \pm a^2}}, \quad \frac{1}{\sqrt{x^2 \pm 2ax}}, \quad \frac{1}{x\sqrt{x^2 + a^2}}, \quad \frac{1}{x\sqrt{a^2 - x^2}}.$$

(1) If $\frac{du}{dx} = \frac{1}{\sqrt{x^2 + a^2}}$, required $\int_x \frac{1}{\sqrt{x^2 + a^2}}$.

$$\text{Let } \sqrt{x^2 + a^2} = xz;$$

$$\therefore x^2 = \frac{a^2}{z^2 - 1},$$

$$2 \text{ h. l. } x = \text{h. l. } a^2 - \text{h. l. } (z^2 - 1);$$

$$\therefore \frac{1}{xz} \text{ or } \frac{du}{dx} = -\frac{1}{z^2 - 1} \frac{dz}{dx}.$$

But $\frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx}$; $\therefore \frac{du}{dz} \cdot \frac{dz}{dx} = -\frac{1}{z^2 - 1} \cdot \frac{dz}{dx}$;

$$\therefore \frac{du}{dz} = -\frac{1}{z^2 - 1},$$

and $$u = -\int_z \frac{1}{z^2 - 1} = +\tfrac{1}{2}.\int_z \left\{\frac{1}{z+1} + \frac{1}{z-1}\right\} = \tfrac{1}{2}.\text{h. l.}\frac{z+1}{z-1}$$

$$= \tfrac{1}{2}.\text{h. l.}\frac{\sqrt{a^2 + x^2} + x}{\sqrt{a^2 + x^2} - x} = \tfrac{1}{2}.\text{h. l.}\frac{(\sqrt{a^2 + x^2} + x)^2}{a^2}$$

$$= \text{h. l.}\frac{\sqrt{a^2 + x^2} + x}{a}.$$

(2) Since $$\frac{1}{\sqrt{x^2 + 2ax}} = \frac{1}{\sqrt{(x+a)^2 - a^2}};$$

$$\therefore \int_x \frac{1}{\sqrt{x^2 + 2ax}} = \text{h. l.}\,(x + a + \sqrt{x^2 + 2ax}).$$

(3) If $$\frac{du}{dx} = \frac{1}{x\sqrt{x^2 + a^2}}.$$

Let $\sqrt{x^2 + a^2} = z$;

$$\therefore x^2 = z^2 - a^2;$$

$$\therefore 2\,\text{h. l.}\,x = \text{h. l.}\,(z^2 - a^2);$$

$$\therefore \frac{1}{xz} \text{ or } \frac{du}{dx} = \frac{1}{z^2 - a^2}.\frac{dz}{dx},$$

and $$\frac{du}{dx} = \frac{du}{dz}.\frac{dz}{dx};$$

$$\therefore \frac{du}{dz}.\frac{dz}{dx} = \frac{1}{z^2 - a^2}.\frac{dz}{dx};$$

$$\therefore \frac{du}{dz} = \frac{1}{z^2 - a^2};$$

$$\therefore u = \int_z \frac{1}{z^2 - a^2} = \frac{1}{2a}.\int_z \left(\frac{1}{z-a} - \frac{1}{z+a}\right) = \frac{1}{2a}.\text{h. l.}\frac{z-a}{z+a}$$

$$= \frac{1}{2a}.\text{h. l.}\frac{\sqrt{x^2 + a^2} - a}{\sqrt{x^2 + a^2} + a} = \frac{1}{2a}.\text{h. l.}\frac{x^2}{(\sqrt{x^2 + a^2} + a)^2}$$

$$= \frac{1}{a}.\text{h. l.}\left(\frac{x}{\sqrt{x^2 + a^2} + a}\right).$$

(4) Similarly, if $\dfrac{du}{dx} = \dfrac{1}{x\sqrt{a^2 - x^2}}$,

$$u = \frac{1}{a}.\,\text{h. l.}\,\left(\frac{x}{\sqrt{a^2 - x^2} + a}\right).$$

27. The function $\dfrac{1}{\sqrt{a + bx + cx^2}}$ is easily reduced to a known form.

For
$$\frac{1}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{c}}.\frac{1}{\sqrt{\frac{a}{c} + \frac{b}{c}x + x^2}}$$

$$= \frac{1}{\sqrt{c}}.\frac{1}{\sqrt{\left(x + \frac{b}{2c}\right)^2 + \frac{a}{c} - \frac{b^2}{4c^2}}}$$

$$= \frac{1}{\sqrt{c}}.\frac{1}{\sqrt{\left(x + \frac{b}{2c}\right)^2 + \frac{4ac - b^2}{4c^2}}},$$

which is of the form $\dfrac{1}{\sqrt{x^2 \pm a^2}}$;

$$\therefore\ u = \frac{1}{\sqrt{c}}.\,\text{h. l.}\ x + \frac{b}{2c} + \sqrt{\frac{a}{c} + \frac{bx}{c} + x^2}\ \frac{2c}{\sqrt{4ac - b^2}}$$

$$= \frac{1}{\sqrt{c}}.\,\text{h. l.}\,\left(\frac{2cx + b + 2\sqrt{c}\sqrt{a + bx + cx^2}}{\sqrt{4ac - b^2}}\right).$$

Let $a = 1$, $b = 1$, $c = 1$:

$$\therefore\ \int_x \frac{1}{\sqrt{1 + x + x^2}} = \text{h. l.}\,\left(\frac{2x + 1 + 2\sqrt{1 + x + x^2}}{\sqrt{3}}\right).$$

Also,
$$\frac{1}{\sqrt{a + bx - cx^2}} = \frac{1}{\sqrt{c}}.\frac{1}{\sqrt{\frac{a}{c} + \frac{bx}{c} - x^2}}$$

$$= \frac{1}{\sqrt{c}} \cdot \frac{1}{\sqrt{\frac{a}{c} + \frac{b^2}{4c^2} - \left(x - \frac{b}{2c}\right)^2}} = \frac{1}{\sqrt{c}} \cdot \frac{1}{\sqrt{\frac{4ac + b^2}{4c^2} - \left(x - \frac{b}{2c}\right)^2}},$$

which is of the form $\frac{1}{\sqrt{a^2 - z^2}} = \frac{d \cdot \sin^{-1}\left(\frac{z}{a}\right)}{dz}$;

$$\therefore \int_x \frac{1}{\sqrt{a + bx - cx^2}} = \frac{1}{\sqrt{c}} \cdot \sin^{-1} \frac{x - \frac{b}{2c}}{\sqrt{\frac{4ac + b^2}{4c^2}}}$$

$$= \frac{1}{\sqrt{c}} \sin^{-1}\left(\frac{2cx - b}{\sqrt{4ac + b^2}}\right).$$

Ex. Let $a = b = c = 1$;

$$\therefore \int_x \frac{1}{\sqrt{1 + x - x^2}} = \sin^{-1}\left(\frac{2x - 1}{\sqrt{5}}\right).$$

28. Integrate $\frac{du}{dx} = \frac{1}{x\sqrt{a + bx + cx^2}}$.

Let $x = \frac{1}{z}$; $\therefore \frac{dx}{dz} = -\frac{1}{z^2} = -x^2$;

$$\therefore \frac{du}{dz} = \frac{du}{dx} \cdot \frac{dx}{dz} = \frac{1}{\frac{1}{z}\sqrt{a + \frac{b}{z} + \frac{c}{z^2}}} \times -\frac{1}{z^2} = \frac{-1}{\sqrt{az^2 + bz + c}},$$

$$\text{or } u = -\int_z \frac{1}{\sqrt{az^2 + bz + c}},$$

which has been just integrated.

29. Integrate $\frac{du}{dx} = \frac{1}{x^2\sqrt{a + bx + cx^2}}$.

Let $x = \frac{1}{z}$; $\therefore \frac{du}{dx} = \frac{z^3}{\sqrt{az^2 + bz + c}} = \frac{du}{dz} \cdot \frac{dz}{dx} = -\frac{du}{dz} z^2$;

$$\therefore \; u = -\int_z \frac{z}{\sqrt{az^2+bz+c}} = -\frac{1}{a}\cdot\int_z \frac{az+\frac{b}{2}-\frac{b}{2}}{\sqrt{az^2+bz+c}}$$

$$= -\frac{1}{a}\int_z \left\{ \frac{az+\frac{b}{2}}{\sqrt{az^2+bz+c}} - \frac{b}{2}\cdot\frac{1}{\sqrt{az^2+bz+c}} \right\}$$

$$= -\frac{1}{a}\sqrt{az^2+bz+c} + \frac{b}{2a}\cdot\int_z \frac{1}{\sqrt{az^2+bz+c}};$$

the integral of which depends upon a preceding example.

30. Integrate $\dfrac{du}{dx} = \dfrac{1}{(a+bx)\sqrt{c+ex}}$.

Let $z = \sqrt{c+ex}$;

$$\therefore \; \frac{dz}{dx} = \frac{e}{2z}.$$

Also $x = \dfrac{z^2-c}{e}$;

$$\therefore \; a+bx = \frac{ae-bc+bz^2}{e};$$

$$\therefore \; \frac{du}{dx} = \frac{e}{(ae-bc+bz^2)z} = \frac{du}{dz}\times\frac{e}{2z};$$

$$\therefore \; \frac{du}{dz} = \frac{2}{(ae-bc+bz^2)} = \frac{2}{b\left(z^2+\frac{ae-bc}{b}\right)}.$$

(1) Let $ae > bc$;

$$\therefore \; u = \frac{2}{b}\times\sqrt{\frac{b}{ae-bc}}\,.\tan^{-1}\frac{z\sqrt{b}}{\sqrt{ae-bc}}$$

$$= \frac{2}{\sqrt{b}\sqrt{ae-bc}}\tan^{-1}\left(\frac{\sqrt{b}}{\sqrt{ae-bc}}\sqrt{c+ex}\right).$$

(2) Let $ae < bc$, and let $\dfrac{bc - ae}{b} = \beta^2$;

$$\therefore\ u = \frac{2}{b} \cdot \int_z \frac{1}{z^2 - \beta^2} = \frac{1}{\beta b} \cdot \int_z \left\{\frac{1}{z - \beta} - \frac{1}{z + \beta}\right\}$$

$$= \frac{1}{\beta b} \text{ h. l. } \frac{z - \beta}{z + \beta} = \frac{1}{\beta b} \text{ h. l. } \frac{\sqrt{c + ex} - \beta}{\sqrt{c + ex} + \beta}.$$

Let $a = b = c = 1$, and $e = -1$; $\therefore\ \beta = \sqrt{2}$;

$$\therefore\ \int_x \frac{1}{(1 + x)\sqrt{1 - x}} = \frac{1}{\sqrt{2}} \cdot \text{h. l. } \frac{\sqrt{1 - x} - \sqrt{2}}{\sqrt{1 - x} + \sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \cdot \text{h. l. } \frac{-3 + x + 2\sqrt{2}\sqrt{1 - x}}{1 + x}.$$

31. Integrate $\dfrac{du}{dx} = \dfrac{1}{(a + bx)\sqrt{c + ex^2}}$.

Let $a + bx = z$; $\therefore\ \dfrac{dx}{dz} = \dfrac{1}{b}$,

and $x = \dfrac{z - a}{b}$;

$$\therefore\ c + ex^2 = \frac{cb^2 + e(z - a)^2}{b^2} = \frac{cb^2 + ea^2 - 2aez + ez^2}{b^2}$$

$$= \frac{e}{b^2} \cdot \{\beta^2 - 2az + z^2\} \text{ by substitution;}$$

$$\therefore\ \frac{du}{dx} = \frac{1}{z\dfrac{\sqrt{e}}{b}\sqrt{\beta^2 - 2az + z^2}} = \frac{du}{dz} \cdot b;$$

$$\therefore\ u = \frac{1}{\sqrt{e}} \cdot \int_z \frac{1}{z\sqrt{z^2 - 2az + \beta^2}}, \text{ a known integral.}$$

32. Integrate $\dfrac{du}{dx} = \dfrac{1}{(a + bx^2)\sqrt{c + ex^2}}$.

$$\text{Let } x = \frac{1}{z}; \quad \therefore \frac{dx}{dz} = -\frac{1}{z^2}.$$

$$\text{And } \frac{z^3}{(az^2 + b)\sqrt{cz^2 + e}} = \frac{du}{dz} \cdot \frac{dz}{dx} = -z^2 \cdot \frac{du}{dz};$$

$$\therefore \frac{du}{dz} = -\frac{z}{(az^2 + b\sqrt{cz^2 + e}}.$$

$$\text{Again, make } \sqrt{cz^2 + e} = v,$$

$$\therefore \frac{z}{\sqrt{cz^2 + e}} = \frac{1}{c} \cdot \frac{dv}{dz}.$$

$$\text{and } az^2 + b = a\left(\frac{v^2 - e}{c}\right) + b = \frac{av^2 - ae + bc}{c};$$

$$\therefore -\frac{z}{(az^2 + b)\sqrt{cz^2 + e}} = -\frac{1}{av^2 - ae + bc} \cdot \frac{dv}{dz} = \frac{du}{dv} \cdot \frac{dv}{dz};$$

$$\therefore \frac{du}{dv} = -\frac{1}{av^2 - ae + bc};$$

$$\therefore u = -\int_v \frac{1}{av^2 - ae + bc} = -\frac{1}{a}\int_v \frac{1}{v^2 \pm \beta^2},$$

which will be a circular arc, or a logarithm, according as the positive or negative sign is taken.

33. Integrate $\dfrac{du}{dx} = \dfrac{1}{(a + bx)\sqrt{cx^2 + ex + m}}$,

$$\frac{1}{(a + bx)\sqrt{cx^2 + ex + m}} = \frac{1}{\sqrt{c}} \cdot \frac{1}{(a + bx)\sqrt{x^2 + \frac{e}{c}x + \frac{m}{c}}}$$

$$= \frac{1}{\sqrt{c}} \cdot \frac{1}{(a + bx)\sqrt{\left(x + \frac{e}{2c}\right)^2 + \frac{m}{c} - \frac{e^2}{4c^2}}}.$$

$$\text{Let } x + \frac{e}{2c} = z; \quad \therefore \frac{dx}{dz} = 1.$$

$$\text{And } bx + a = bz + a - \frac{be}{2c} = bz + \beta b.$$

$$\text{Let } \frac{m}{c} - \frac{e^2}{4c^2} = \alpha^2;$$

$$\therefore u = \frac{1}{b\sqrt{c}} \cdot \int_z \frac{1}{(z+\beta)\sqrt{z^2+\alpha^2}}.$$

Again, make $z + \beta = \frac{1}{v}$;

$$\therefore z^2 + \alpha^2 = \frac{1}{v^2} - \frac{2\beta}{v} + \alpha^2 + \beta^2 = \frac{1}{v^2} - \frac{2\beta}{v} + \frac{1}{\delta^2};$$

$$\therefore u = -\frac{1}{b\sqrt{c}} \cdot \int_v \frac{\frac{1}{v^2}}{\frac{1}{v}\sqrt{\frac{1}{v^2} - \frac{2\beta}{v} + \frac{1}{\delta^2}}}$$

$$= -\frac{\delta}{b\sqrt{c}} \cdot \int_v \frac{1}{\sqrt{\delta^2 - 2\beta\delta^2 v + v^2}},$$

a known form.

34. Next, to integrate $X(a+bx)^{\frac{p}{q}}$, where X is a rational function of x.

Make $a + bx = z^q$;

$$\therefore x = \frac{z^q - a}{b}, \text{ and } \frac{dx}{dz} = \frac{q}{b} z^{q-1},$$

$$\text{and } \int_x X \cdot (a+bx)^{\frac{p}{q}} = \int_z Z \cdot z^p \cdot \frac{q}{b} z^{q-1} = \frac{q}{b} \int_z Z \cdot z^{p+q-1},$$

where Z is the value of X, when $\frac{z^q - a}{b}$ is put for x.

35. Again, to integrate $X.(x + \sqrt{1+x^2})^{\frac{p}{q}}$, where x is either a rational function of x, or of x and $\sqrt{1+x^2}$.

$$\text{Make } x + \sqrt{1+x^2} = z^q;$$

$$\therefore\ 1 + x^2 = z^{2q} - 2xz^q + x^2;$$

$$\therefore\ x = \tfrac{1}{2}\left\{z^q - z^{-q}\right\};$$

$$\therefore\ \frac{dx}{dz} = \frac{q}{2}\left\{z^{q-1} + z^{-q-1}\right\} = \frac{q}{2}\,\frac{z^{2q}+1}{z^{q+1}},$$

$$\text{and } \sqrt{1+x^2} = z^q + x = \tfrac{1}{2}(z^q + z^{-q});$$

$$\therefore\ \textstyle\int_x X(x+\sqrt{1+x^2})^{\frac{p}{q}} = \int_z Z\,.\,z^p\,.\,\dfrac{q}{2}\,.\,\dfrac{z^{2q}+1}{z^{q+1}}$$

$$= \frac{q}{2}\textstyle\int_z Z\,.\,\dfrac{z^{p+2q}+z^p}{z^{q+1}},$$

Z being the value of X, when $\frac{1}{2}(z^q - z^{-q})$ is put for x.

36. Lastly, to integrate

$$\frac{1}{(1-x^m)\sqrt[2m]{2x^m - 1}} \quad\text{and}\quad \frac{x^{m-1}}{(1-x^m)\sqrt[2m]{2x^m-1}}.$$

In the former, make $2x^m - 1 = z^{2m}x^{2m}$;

$$\therefore\ x^{2m} - 2x^m + 1 = x^{2m}(1 - z^{2m}),$$

$$\text{or } \left(\frac{1-x^m}{x^m}\right)^2 = 1 - z^{2m}\ldots\ldots\ldots(1),$$

$$\text{or } \frac{1-x^m}{x^{2m+1}}\,.\,\frac{dx}{dz} = z^{2m-1}\ldots\ldots\ldots(2);$$

therefore by dividing (2) by (1),

$$\frac{1}{(1-x^m)\,xz}\,\frac{dx}{dz} = \frac{z^{2m-2}}{1-z^{2m}}.$$

But $\int_x \frac{1}{(1-x^m)\sqrt[2m]{2x^m-1}} = \int_z \frac{1}{(1-x^m)\,xz}\,\frac{dx}{dz} = \int_z \frac{z^{2m-2}}{1-z^{2m}}$,

which is rationalized.

In the latter, let $2x^m - 1 = z^{2m}$;

$$\therefore\ x^{m-1}\frac{dx}{dz} = z^{2m-1},$$

$$\text{and } 1 - x^m = 1 - \tfrac{1}{2}(z^{2m}+1) = \tfrac{1}{2}.(1 - z^{2m});$$

$$\therefore\ \frac{x^{m-1}}{(1-x^m)z}\,\frac{dx}{dz} = \frac{2z^{2m-2}}{1-z^{2m}};$$

$$\therefore\ u = \int_z \frac{2.z^{2m-2}}{1-z^{2m}},\ \text{which is rational.}$$

These formulas were integrated by Lexell.

BINOMIAL DIFFERENTIAL COEFFICIENTS.

37. Next, to integrate $\frac{du}{dx} = x^{m-1}(a+bx^n)^{\frac{p}{q}}$.

This function may be rationalized whenever $\frac{m}{n}$ or $\frac{m}{n}+\frac{p}{q}$ is an integer.

(1) Let $a + bx^n = z^q$;

$$\therefore\ x^n = \frac{z^q - a}{b},$$

$$x^m = \left(\frac{z^q-a}{b}\right)^{\frac{m}{n}};$$

$$\therefore\ x^{m-1} = \frac{q}{nb}z^{q-1}\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}-1}\frac{dz}{dx};$$

$$\therefore \frac{du}{dx} = \frac{q}{nb} z^{p+q-1} \left(\frac{z^q - a}{b}\right)^{\frac{m}{n}-1} \frac{dz}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx};$$

$$\therefore \frac{du}{dz} = \frac{q}{nb} z^{p+q-1} \left(\frac{z^q - a}{b}\right)^{\frac{m}{n}-1},$$

which is rational if $\frac{m}{n}$ be an integer, and easily integrable, by expanding the binomial.

(2) If $\frac{m}{n}$ be a fraction.

$$\text{Let } a + bx^n = x^n z^q;$$

$$\therefore x^n = \frac{a}{z^q - b},$$

$$x^m = \frac{a^{\frac{m}{n}}}{(z^q - b)^{\frac{m}{n}}};$$

$$\therefore x^{m-1} = -\frac{qa^{\frac{m}{n}}}{n} \cdot \frac{z^{q-1}}{(z^q - b)^{\frac{m}{n}+1}} \frac{dz}{dx},$$

$$\text{and } (a + bx^n)^{\frac{p}{q}} = \frac{a^{\frac{p}{q}} z^p}{(z^q - b)^{\frac{p}{q}}};$$

$$\therefore \frac{du}{dx} = -\frac{qa^{\frac{m}{n}+\frac{p}{q}}}{n} \times \frac{z^{p+q-1}}{(z^q - b)^{\frac{m}{n}+\frac{p}{q}+1}} \cdot \frac{dz}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx};$$

$$\therefore \frac{du}{dz} = -\frac{qa^{\frac{m}{n}+\frac{p}{q}}}{n} \cdot \frac{z^{p+q-1}}{(z^q - b)^{\frac{m}{n}+\frac{p}{q}+1}},$$

which is rational whenever $\frac{m}{n} + \frac{p}{q}$ is an integer, and easily integrable if $\frac{m}{n} + \frac{p}{q}$ be a *negative* integer.

COR. We have assumed that m and n are integers, but if they be fractions as $\frac{r}{s}$ and $\frac{r_1}{s_1}$.

$$\text{Make } v^{ss_1} = x; \quad \therefore \; x^{\frac{r}{s}} = v^{rs_1}, \text{ and } x^{\frac{r_1}{s_1}} = v^{r_1 s}.$$

Also n is assumed positive, for if not, let

$$x = \frac{1}{u}; \quad \therefore \; x^{-n} = u^n.$$

Ex. 1. Let $\frac{du}{dx} = x^3\sqrt{1 + x^2}$.

$$\text{Here } m - 1 = 3, \text{ and } n = 2; \quad \therefore \; \frac{m}{n} = \frac{4}{2} = 2.$$

$$\text{Let } 1 + x^2 = z^2; \quad \therefore \; x^2 = z^2 - 1,$$

$$x^4 = (z^2 - 1)^2;$$

$$\therefore \; x^3 = (z^2 - 1)\, z \,.\, \frac{dz}{dx},$$

$$\text{and } \frac{du}{dx} = \frac{du}{dz} \,.\, \frac{dz}{dx};$$

$$\therefore \; (z^2 - 1)\, z^2 \,\frac{dz}{dx} = \frac{du}{dz} \,.\, \frac{dz}{dx};$$

$$\therefore \; \frac{du}{dz} = z^2 (z^2 - 1) = z^4 - z^2;$$

$$\therefore \; u = \frac{z^5}{5} - \frac{z^3}{3} + C$$

$$= \frac{(1 + x^2)^{\frac{5}{2}}}{5} - \frac{(x^2 + 1)^{\frac{3}{2}}}{3} + C.$$

Ex. 2. Let $\frac{du}{dx} = \frac{1}{x^4\sqrt{1 + x^2}}$.

$$\text{Here } \frac{m}{n} = -\frac{3}{2}, \text{ and } \frac{p}{q} = -\tfrac{1}{2}; \quad \therefore \; \frac{m}{n} + \frac{p}{q} = -2.$$

$$\text{And } \frac{1}{x^4\sqrt{1+x^2}} = \frac{1}{x^5\sqrt{x^{-2}+1}} = \frac{x^{-5}}{\sqrt{x^{-2}+1}}.$$

$$\text{Let } x^{-2}+1 = z^2.$$

$$\therefore\ x^{-3} = -z\,.\,\frac{dz}{dx},$$

$$x^{-2} = z^2 - 1\,;$$

$$\therefore\ \frac{x^{-5}}{\sqrt{x^{-2}+1}} = -\frac{z(z^2-1)}{z}\frac{dz}{dx} = \frac{du}{dz}\cdot\frac{dz}{dx};$$

$$\therefore\ u = -\int_z(z^2-1) = -\left(\frac{z^3}{3}-z\right)$$

$$= -z\,.\left(\frac{z^2}{3}-1\right)$$

$$= -\frac{\sqrt{1+x^2}}{x}\left\{\frac{1+x^2}{3x^2}-1\right\}$$

$$= \frac{(2x^2-1)\sqrt{1+x^2}}{3x^3}.$$

Ex. 3. Let $\dfrac{du}{dx} = \dfrac{1}{\sqrt[n]{1-x^n}} = (1-x^n)^{-\frac{1}{n}}$.

$$\text{Here } \frac{m}{n}+\frac{p}{q} = \frac{1}{n}-\frac{1}{n} = 0.$$

Let $\therefore\ 1-x^n = x^n z^n;\quad \therefore\ x^n = \dfrac{1}{1+z^n};$

$$\therefore\ n\,\text{h.l.}\,x = -\,\text{h.l.}\,(1+z^n)\,;$$

$$\therefore\ \frac{1}{x} = -\frac{z^{n-1}}{1+z^n}\frac{dz}{dx};$$

$$\therefore\ \frac{1}{xz} = \frac{1}{\sqrt[n]{1-x^n}} = -\frac{z^{n-2}}{1+z^n}\cdot\frac{dz}{dx};$$

$$\text{or } \frac{du}{dx} = \frac{du}{dz}\frac{dz}{dx} = -\frac{z^n}{1+z^n}\frac{dz}{dx};$$

$$\therefore \frac{du}{dz} = -\frac{z^n}{1+z^n},$$

which may be integrated by partial fractions.

38. These methods of substitution are seldom adopted, the formula of reduction $\int p\frac{dq}{dx} = pq - \int q \cdot \frac{dp}{dx}$ being more generally *useful.*

Instead however of integrating the formula $\int_x x^{m-1}(a+bx^n)^{\frac{p}{q}}$ for every value of n, we shall confine ourselves to the cases in which $n = 2$, and where a, b, have particular values, the integrals thus found will be those which are of frequent occurrence in physical problems.

These are $\int_x \frac{x^m}{\sqrt{1 \pm x^2}}$, $\int_x \frac{x^m}{\sqrt{x^2 - 1}}$, $\int_x \frac{1}{x^m\sqrt{x^2 \pm 1}}$.

Having integrated these functions, we shall next integrate

$$\int_x \frac{x^m}{\sqrt{2ax - x^2}}, \text{ and } \int_x \frac{1}{x^m\sqrt{2ax - x^2}}, \text{ and } \int_x \frac{x^m}{\sqrt{a + bx + cx^2}}.$$

39. $\frac{du}{dx} = \frac{x^m}{\sqrt{1 - x^2}}$, (m) an integer,

$$\int_x \frac{x^m}{\sqrt{1 - x^2}} = \int_x x^{m-1}\frac{x}{\sqrt{1 - x^2}}.$$

Here $p = x^{m-1}$, and $\frac{dq}{dx} = \frac{x}{\sqrt{1 - x^2}}$;

$$\therefore \frac{dp}{dx} = (m-1)x^{m-2}, \text{ and } q = -\sqrt{1 - x^2};$$

$$\therefore \int_x \frac{x^m}{\sqrt{1-x^2}} = -x^{m-1}\sqrt{1-x^2} + (m-1)\int_x x^{m-2}\sqrt{1-x^2}$$

$$= -x^{m-1}\sqrt{1-x^2} + (m-1)\int \frac{x^{m-2}}{\sqrt{1-x^2}} - (m-1)\int \frac{x^m}{\sqrt{1-x^2}},$$

putting $\dfrac{1-x^2}{\sqrt{1-x^2}}$ for $\sqrt{1-x^2}$;

$$\therefore m\int_x \frac{x^m}{\sqrt{1-x^2}} = -x^{m-1}\sqrt{1-x^2} + (m-1)\int_x \frac{x^{m-2}}{\sqrt{1-x^2}};$$

$$\therefore \int \frac{x^m}{\sqrt{1-x^2}} = -\frac{x^{m-1}\sqrt{1-x^2}}{m} + \frac{m-1}{m}\int_x \frac{x^{m-2}}{\sqrt{1-x^2}};$$

and by putting $m-2$, $m-4$, $m-6$, &c. for m, the integral will be reduced either to

$$\int_x \frac{x}{\sqrt{1-x^2}}, \quad \text{or} \quad \int_x \frac{1}{\sqrt{1-x^2}};$$

that is, to $-\sqrt{1-x^2}$, or $\sin^{-1}x$, according as m is odd or even.

Ex. Let $\int_x \dfrac{x^4}{\sqrt{1-x^2}}$ be required. Here $m = 4$,

$$\int_x \frac{x^4}{\sqrt{1-x^2}} = -\frac{x^3\sqrt{1-x^2}}{4} + \frac{3}{4}\int_x \frac{x^2}{\sqrt{1-x^2}},$$

$$\int_x \frac{x^2}{\sqrt{1-x^2}} = -\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\int_x \frac{1}{\sqrt{1-x^2}}$$

$$= -\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x + C;$$

$$\therefore \int_x \frac{x^4}{\sqrt{1-x^2}} = -\sqrt{1-x^2}\left\{\frac{x^3}{4} + \frac{3x}{2\,.\,4}\right\} + \frac{1\,.\,3}{2\,.\,4}\sin^{-1}x + C.$$

40. To find the general value of the integral.

(1) Let m be even $= 2n$,

and let $P_{2n} = \int_x \frac{x^{2n}}{\sqrt{1-x^2}}$, and $Q_{2n-1} = x^{2n-1}\sqrt{1-x^2}$;

$$\therefore\ P_{2n} = -\frac{1}{2n} Q_{2n-1} + \frac{2n-1}{2n} P_{2n-2},$$

$$P_{2n-2} = -\frac{1}{2n-2} Q_{2n-3} + \frac{2n-3}{2n-2} P_{2n-4},$$

$$P_{2n-4} = -\frac{1}{2n-4} Q_{2n-5} + \frac{2n-5}{2n-4} P_{2n-6},$$

$$\vdots$$

$$P_2 = -\tfrac{1}{2} Q_1 + \tfrac{1}{2} P_0, \text{ where } P_0 = \sin^{-1} x;$$

$$\therefore\ P_{2n} = -\left\{\frac{Q_{2n-1}}{2n} + \frac{2n-1}{2n.(2n-2)} Q_{2n-3} + \frac{(2n-1).(2n-3)}{2n.(2n-2)(2n-4)} Q_{2n-5} + \&c.\right\}$$

$$+ \frac{(2n-1)(2n-3)(2n-5)\ldots\ldots 3.1}{2n\ .\ (2n-2)(2n-4)\ldots\ldots 4.2} \sin^{-1} x + C.$$

If the integral be assumed $= 0$, when $x = 0$. Then $C = 0$, for Q_{2n-1}, Q_{2n-3}, &c. each $= 0$.

If $x = 1$, Q_{2n-1}, Q_{2n-3}, &c. each $= 0$, and $\sin^{-1} x = \frac{\pi}{2}$;

$$\therefore \int_x \frac{x^{2n}}{\sqrt{1-x^2}}, \left.\begin{matrix}\text{from } x=0\\ \text{to } x=1\end{matrix}\right\} = \frac{(2n-1).(2n-3).(2n-5)\ldots 3.1}{2n\ .\ (2n-2).(2n-4)\ldots 4.2} \cdot \frac{\pi}{2}.$$

(2) Let m be odd and $= 2n+1$;

$$\therefore\ P_{2n+1} = -\frac{1}{2n+1} Q_{2n} + \frac{2n}{2n+1} P_{2n-1},$$

$$P_{2n-1} = -\frac{1}{2n-1} Q_{2n-2} + \frac{2n-2}{2n-1} P_{2n-3},$$

$$\vdots$$

$$P_3 = -\tfrac{1}{3} Q_2 + \tfrac{2}{3} P_1,$$

$$\text{and } P_1 = \int_x \frac{x}{\sqrt{1-x^2}} = -\sqrt{1-x^2};$$

$$\therefore\ P_{2n+1} = -\left\{\frac{1}{2n+1} Q_{2n} + \frac{2n}{(2n+1)(2n-1)} Q_{2n-2}\right.$$

$$\left. + \frac{2n.(2n-2)}{(2n+1)(2n-1)(2n-3)} Q_{2n-4} + \&c.\right\}$$

$$- \frac{2n\ .\ (2n-2)(2n-4)\ldots 4.2}{(2n+1)(2n-1)(2n-3)\ldots 5.3}\sqrt{1-x^2} + C.$$

If $P_{2n+1} = 0$ when $x = 0$, since then Q_{2n}, Q_{2n-2}, &c. each $= 0$;

$$\therefore\ 0 = -\frac{2n\ .\ (2n-2)(2n-4)\ldots 4.2}{(2n+1)(2n-1)(2n-3)\ldots 5.3} + C.$$

whence by subtraction,

$$P_{2n+1} = \frac{2n\ .\ (2n-2)(2n-4)\ldots 4.2}{(2n+1)(2n-1)(2n-3)\ldots 5.3}$$

$$-\left\{\frac{1}{2n+1} Q_{2n} + \frac{2n}{(2n+1)(2n-1)} Q_{2n-2} + \&c.\right\}$$

Let $x = 1$;

$$\therefore\ \int_x \frac{x^{2n+1}}{\sqrt{1-x^2}},\ \left.\begin{matrix}\text{from } x = 0 \\ \text{to } x = 1\end{matrix}\right\} = \frac{2n\ .\ (2n-2)\ldots 6.4.2}{(2n+1)(2n-1)\ldots 7.5.3}.$$

COR. If n be infinite, then we may make $P_{2n} = P_{2n+1}$,

$$\text{or } \frac{\pi}{2} \cdot \frac{1.3.5.7,\ \&c.}{2.4.6.8,\ \&c.} = \frac{2.4.6.8,\ \&c.}{3.5.7.9,\ \&c.},$$

$$\text{or } \frac{\pi}{2} = \frac{2.2.4.4.6.6.8.8,\ \&c.}{1.3.3.5.5.7.7.9,\ \&c.},$$

which is Wallis's theorem for the length of the circle

41. Integrate $\dfrac{du}{dx} = \dfrac{1}{x^m\sqrt{1+x^2}}$.

$$\int_x \frac{1}{x^m\sqrt{1+x^2}} = \int_x \frac{x}{x^{m+1}\sqrt{1+x^2}}.$$

Here $p = \dfrac{1}{x^{m+1}}$, and $\dfrac{dq}{dx} = \dfrac{x}{\sqrt{1+x^2}}$;

$$\therefore \frac{dp}{dx} = -\frac{m+1}{x^{m+2}}, \text{ and } q = \sqrt{1+x^2};$$

$$\therefore \int_x \frac{1}{x^m\sqrt{1+x^2}} = \frac{\sqrt{1+x^2}}{x^{m+1}} + (m+1)\,.\int_x \frac{\sqrt{1+x^2}}{x^{m+2}}$$

$$= \frac{\sqrt{1+x^2}}{x^{m+1}} + (m+1)\,.\int_x \frac{1}{x^{m+2}\sqrt{1+x^2}} + (m+1)\,.\int_x \frac{1}{x^m\sqrt{1+x^2}};$$

$$\therefore \int_x \frac{1}{x^{m+2}\sqrt{1+x^2}} = -\frac{1}{m+1}\cdot\frac{\sqrt{1+x^2}}{x^{m+1}} - \frac{m}{m+1}\cdot\int_x \frac{1}{x^m\sqrt{1+x^2}}.$$

For $m+2$ put m;

$$\therefore \int_x \frac{1}{x^m\sqrt{1+x^2}} = -\frac{1}{m-1}\cdot\frac{\sqrt{1+x^2}}{x^{m-1}} - \frac{m-2}{m-1}\cdot\int_x \frac{1}{x^{m-2}\sqrt{1+x^2}};$$

a formula of reduction by which the integral may be reduced either to

$$\int_x \frac{1}{x\sqrt{1+x^2}}, \text{ or } \int_x \frac{1}{x^2\sqrt{1+x^2}},$$

according as m is odd or even,

$$\text{and } \int_x \frac{1}{x\sqrt{1+x^2}} = \text{h. l. } \frac{x}{1+\sqrt{1+x^2}},$$

$$\text{and } \int_x \frac{1}{x^2\sqrt{1+x^2}} = \int_x \frac{1}{x^3\sqrt{x^{-2}+1}} = -\sqrt{x^{-2}+1} = -\frac{\sqrt{1+x^2}}{x}.$$

42. Integrate $\dfrac{1}{x^m\sqrt{x^2-1}}$.

$$\int_x \frac{1}{x^m\sqrt{x^2-1}} = \int_x \frac{x}{x^{m+1}\sqrt{x^2-1}} = \int_x \frac{1}{x^{m+1}} \cdot \frac{d\,.\sqrt{x^2-1}}{dx}$$

$$= \frac{\sqrt{x^2-1}}{x^{m+1}} + (m+1)\,.\int_x \frac{\sqrt{x^2-1}}{x^{m+2}}$$

$$= \frac{\sqrt{x^2-1}}{x^{m+1}} + (m+1)\,.\int_x \frac{1}{x^m\sqrt{x^2-1}} - (m+1)\,.\int_x \frac{1}{x^{m+2}\sqrt{x^2-1}}\,;$$

$$\therefore \int_x \frac{1}{x^{m+2}\sqrt{x^2-1}} = \frac{1}{m+1}\cdot\frac{\sqrt{x^2-1}}{x^{m+1}} + \frac{m}{m+1}\cdot\int_x \frac{1}{x^m\sqrt{x^2-1}}\,;$$

therefore, writing m for $(m+2)$,

$$\int_x \frac{1}{x^m\sqrt{x^2-1}} = \frac{1}{m-1}\cdot\frac{\sqrt{x^2-1}}{x^{m-1}} + \frac{m-2}{m-1}\cdot\int_x \frac{1}{x^{m-2}\sqrt{x^2-1}},$$

a formula by which the integral may be reduced, when m is odd, to $\displaystyle\int \frac{1}{x\sqrt{x^2-1}} = \sec^{-1} x$, and when m is even, to

$$\int_x \frac{1}{x^2\sqrt{x^2-1}} = -\frac{\sqrt{x^2+1}}{x}.$$

EXAMPLE. Find $\displaystyle\int_x \frac{1}{x^5\sqrt{x^2-1}}$,

$$\int_x \frac{1}{x^5\sqrt{x^2-1}} = \tfrac{1}{4}\cdot\frac{\sqrt{x^2-1}}{x^4} + \tfrac{3}{4}\cdot\int \frac{1}{x^3\sqrt{x^2-1}},$$

$$\int_x \frac{1}{x^3\sqrt{x^2-1}} = \tfrac{1}{2}\cdot\frac{\sqrt{x^2-1}}{x^2} + \tfrac{1}{2}\,.\sec^{-1} x\,;$$

$$\therefore \int_x \frac{1}{x^5\sqrt{x^2-1}} = \tfrac{1}{4}\cdot\frac{\sqrt{x^2-1}}{x^4} + \frac{3}{2\,.\,4}\cdot\frac{\sqrt{x^2-1}}{x^2} + \frac{1\,.\,3}{2\,.\,4}\sec^{-1} x.$$

43. Integrate $\dfrac{du}{dx} = \dfrac{x^m}{\sqrt{2ax - x^2}}$.

$$\int_x \frac{x^m}{\sqrt{2ax - x^2}} = \int_x \frac{-x^{m-1}.(a-x) + ax^{m-1}}{\sqrt{2ax - x^2}}$$

$$= -\int_x \frac{x^{m-1}(a-x)}{\sqrt{2ax - x^2}} + a.\int_x \frac{x^{m-1}}{\sqrt{2ax - x^2}}.$$

Now $\int_x x^{m-1}\dfrac{a-x}{\sqrt{2ax-x^2}} = x^{m-1}\sqrt{2ax-x^2} - (m-1).\int_x \sqrt{2ax-x^2}.x^{m-2}$

$$= x^{m-1}\sqrt{2ax-x^2} - 2.(m-1)a.\int_x \frac{x^{m-1}}{\sqrt{2ax-x^2}} + (m-1).\int_x \frac{x^m}{\sqrt{2ax-x^2}};$$

therefore, substituting

$$m\int_x \frac{x^m}{\sqrt{2ax-x^2}} = -x^{m-1}\sqrt{2ax-x^2} + (2m-1)a.\int_x \frac{x^{m-1}}{\sqrt{2ax-x^2}};$$

$$\therefore \int_x \frac{x^m}{\sqrt{2ax-x^2}} = -\frac{x^{m-1}\sqrt{2ax-x^2}}{m} + \frac{2m-1}{m}.a.\int \frac{x^{m-1}}{\sqrt{2ax-x^2}},$$

a formula by which the integral may be reduced to

$$\int_x \frac{1}{\sqrt{2ax - x^2}} = V\sin^{-1}\frac{x}{a}.$$

The last term $= a^m.\dfrac{(2m-1)(2m-3)(2m-5)\ldots3.1}{m \quad m-1 \quad m-2 \ldots 2.1} V\sin^{-1}\dfrac{x}{a}$.

Cor. Suppose the value of the integral be 0 when $x = 0$, and its value be required when $x = 2a$.

Then, since all the terms on the right-hand side vanish when $x = 0$; therefore $C = 0$: and when $x = 2a$, all the terms of the form $x^{m-1}\sqrt{2ax - x^2}$ vanish, but $V\sin^{-1}\dfrac{x}{2a} = \pi$;

$$\therefore \int_x \frac{x^m}{\sqrt{2ax - x^2}} \text{ from } x = 0 \text{ to } x = 2a,$$

$$\pi.a^m.\frac{1.3.5\ldots(2m-3).(2m-1)}{1.2.3\ldots(m-1) \qquad m}$$

44. Let $\dfrac{du}{dx} = \dfrac{1}{x^m\sqrt{2ax - x^2}}$.

Make $z = \dfrac{1}{x}$; $\therefore\ \dfrac{dz}{dx} = -\dfrac{1}{x^2} = -z^2$;

$$\therefore\ \frac{du}{dx} = -z^2 \cdot \frac{du}{dz} = \frac{-z^{m+1}}{\sqrt{2az-1}};$$

$$\therefore\ u = -\int_z \frac{z^{m-1}}{\sqrt{2az-1}} = -\frac{1}{\sqrt{2a}} \cdot \int_z \frac{z^{m-1}}{\sqrt{z - \dfrac{1}{2a}}} = -\sqrt{\beta} \cdot \int \frac{z^{m-1}}{\sqrt{z-\beta}},$$

where $\beta = \dfrac{1}{2a}$.

$$\text{Now } \int_z \frac{z^{m-1}}{\sqrt{z-\beta}} = 2z^{m-1}\sqrt{z-\beta} - 2(m-1) \cdot \int_z z^{m-2}\sqrt{z-\beta}$$

$$= 2z^{m-1}\sqrt{z-\beta} - 2(m-1) \cdot \int_z \frac{z^{m-1}}{\sqrt{z-\beta}} + 2(m-1) \cdot \int_z \frac{z^{m-2}\beta}{\sqrt{z-\beta}};$$

$$\therefore\ \int_z \frac{z^{m-1}}{\sqrt{z-\beta}} = \frac{2}{2m-1} z^{m-1}\sqrt{z-\beta} + \frac{2m-2}{2m-1}\beta \cdot \int_z \frac{z^{m-2}}{\sqrt{z-\beta}},$$

a formula by which the integral may be reduced to

$$\int_z \frac{1}{\sqrt{z-\beta}} = 2\sqrt{z-\beta}.$$

Example. Let $m = 2$, or $\displaystyle\int_x \frac{1}{x^2\sqrt{2ax - x^2}}$ be required.

Here $m - 1 = 1$;

$$\therefore\ \int_z \frac{z}{\sqrt{z-\beta}} = \tfrac{2}{3} z\sqrt{z-\beta} + \tfrac{2}{3}\beta 2\sqrt{z-\beta};$$

$$\therefore\ u = C - \tfrac{2}{3}\sqrt{\beta} \cdot (z\sqrt{z-\beta} + 2\beta\sqrt{z-\beta})$$

$$= C - \tfrac{2}{3}\frac{1}{\sqrt{2a}} \cdot \left(\frac{1}{x}\sqrt{\frac{2a-x}{2ax}} + \frac{1}{a}\sqrt{\frac{2a-x}{2ax}}\right)$$

$$= C - \tfrac{2}{3}\left(\frac{\sqrt{2ax - x^2}}{2ax^2} + \frac{\sqrt{2ax - x^2}}{2a^2x}\right)$$

$$= C - \frac{\sqrt{2ax - x^2}}{3a^2x^2}(a + x).$$

45. Integrate $\dfrac{du}{dx} = \dfrac{x^m}{\sqrt{a + bx + cx^2}}$.

$$\int_x \frac{x^m}{\sqrt{a + bx + cx^2}} = \int_x \frac{x^m}{\sqrt{c}\,\sqrt{\dfrac{a}{c} + \dfrac{b}{c}x + x^2}}.$$

$$\text{Let } x + \frac{b}{2c} = z;$$

$$\therefore\ x^2 + \frac{b}{c}x + \frac{a}{c} = z^2 + \frac{a}{c} - \frac{b^2}{4c^2} = z^2 \pm \beta^2;$$

and making $\dfrac{b}{2c} = a$,

$$\int_x \frac{x^m}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{c}}\int_z \frac{(z - a)^m}{\sqrt{z^2 \pm \beta^2}},$$

and by expanding $(z - a)^m$, the integral may be made to depend upon $\displaystyle\int_z \frac{z^m}{\sqrt{z^2 \pm \beta^2}}$.

Example. Let $m = 2$;

$$\therefore\ \int_x \frac{x^2}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{c}}\cdot\int_z \frac{z^2 - 2az + a^2}{\sqrt{z^2 \pm \beta^2}}$$

$$= \frac{1}{\sqrt{c}}\int_x \frac{z^2}{\sqrt{z^2 \pm \beta^2}} - \frac{2a}{\sqrt{c}}\cdot\sqrt{z^2 \pm \beta^2} + \frac{a^2}{\sqrt{c}}\cdot \text{h. l.}\,(z + \sqrt{z^2 \pm \beta^2}),$$

$$\text{and } \int \frac{z^2}{\sqrt{z^2 \pm \beta^2}} = \frac{z\sqrt{z^2 \pm \beta^2}}{2} - \frac{\beta^2}{2}\cdot \text{h. l.}\,(z + \sqrt{z^2 \pm \beta^2});$$

$$\therefore \int_x \frac{x^2}{\sqrt{a+bx+cx^2}} = \frac{z\sqrt{z^2 \pm \beta^2}}{2\sqrt{c}} - \frac{2a}{\sqrt{c}}\sqrt{z^2 \pm \beta^2}$$

$$+ \frac{a^2 - \frac{\beta^2}{2}}{\sqrt{c}} \text{h. l.} (z + \sqrt{z^2 \pm \beta^2})$$

$$= \frac{\sqrt{a+bx+cx^2}}{c} \cdot \left(\frac{x}{2} + \frac{b}{4c} - \frac{b}{c}\right) - \frac{1}{\sqrt{c}} \cdot \left(\frac{3b^2}{8c^2} - \frac{a}{2c}\right) \text{h. l.} (z + \sqrt{z^2 + \beta})$$

$$= \sqrt{a+bx+cx^2} \left\{\frac{2cx - 3b}{4c^2}\right\}$$

$$- \frac{3b^2 - 4ac}{8c^2\sqrt{c}} \cdot \text{h. l.} \left(\frac{2cx + b + 2\sqrt{c} \cdot \sqrt{a+bx+cx^2}}{2c}\right).$$

46. Integrate $\dfrac{du}{dx} = \dfrac{1}{x^m\sqrt{a+bx+cx^2}}$.

Let $x = \dfrac{1}{z}$;

$$\therefore \frac{du}{dx} = -z^2\frac{du}{dz} = \frac{z^{m+1}}{\sqrt{az^2+bz+c}};$$

$$\therefore u = C - \int_z \frac{z^{m-1}}{\sqrt{az^2+bz+c}},$$

which may be integrated by the preceding method.

47. Lastly, integrate $\dfrac{1}{\sqrt{c-x}\sqrt{2ax-x^2}}$, which is met with in Mechanics,

$$\frac{1}{\sqrt{c-x}\sqrt{2ax-x^2}} = \frac{1}{\sqrt{c}} \frac{\left(1 - \frac{x}{c}\right)^{-\frac{1}{2}}}{\sqrt{2ax-x^2}}$$

$$= \frac{1}{\sqrt{c}} \cdot \frac{1}{\sqrt{2ax-x^2}} \cdot \left\{1 + \frac{1}{2} \cdot \frac{x}{c} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^2}{c^2} + \&c.\right\},$$

and thus the integral depends upon $\int_x \frac{x^m}{\sqrt{2ax - x^2}}$, which has been found already.

48. We shall conclude this chapter by proving Bernoulli's series for the integration of u.

Since $\int_x u = ux - \int_x x \frac{du}{dx}$,

$$\text{and } \int_x x \frac{du}{dx} = \frac{x^2}{2} \cdot \frac{du}{dx} - \tfrac{1}{2} \cdot \int_x x^2 \frac{d^2u}{dx^2},$$

$$\int_x x^2 \frac{d^2u}{dx^2} = \frac{x^3}{3} \cdot \frac{d^2u}{dx^2} - \tfrac{1}{3} \cdot \int_x x^3 \frac{d^3u}{dx^3}.$$

$$\int_x x^3 \frac{d^3u}{dx^3} = \frac{x^4}{4} \cdot \frac{d^3u}{dx^3} - \tfrac{1}{4} \cdot \int_x x^4 \frac{d^4u}{dx^4};$$

&c.... = &c.

$$\therefore \int_x u = ux - \frac{x^2}{1.2} \cdot \frac{du}{dx} + \frac{x^3}{2.3} \cdot \frac{d^2u}{dx^2} - \frac{x^4}{2.3.4} \cdot \frac{d^3u}{dx^3} + \&c.$$

Ex. Let $u = ax^3 + bx^2 + cx + e$;

$$\therefore \frac{du}{dx} = 3ax^2 + 2bx + c,$$

$$\frac{d^2u}{dx^2} = 6ax + 2b,$$

$$\frac{d^3u}{dx^3} = 6a, \quad \text{and} \quad \frac{d^4u}{dx^4} = 0;$$

$$\therefore \int_x u = ax^4 + bx^3 + cx^2 + ex - \frac{3ax^4 + 2bx^3 + cx^2}{2} + ax^4 + \frac{bx^3}{3} - \frac{ax^4}{4}$$

$$= \frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + ex.$$

EXAMPLES.

(1) $\int_x x^2 . \sqrt{a+bx} = \left\{ \frac{(a+bx)^2}{7} - \frac{2a(a+bx)}{5} + \frac{a^2}{3} \right\} \frac{2(a+bx)^{\frac{3}{2}}}{b^4}.$

(2) $\int_x \frac{\sqrt{a+bx}}{x^2} = -\frac{\sqrt{a+bx}}{x} + \frac{b}{2\sqrt{a}} \log \left\{ \frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right\}.$

(3) $\int_x \frac{x^3}{\sqrt{a+bx}} = \left\{ \frac{(a+bx)^3}{7} - \frac{3}{5} a(a+bx)^2 + a^2.(a+bx) - a^3 \right\}$

$$\frac{2\sqrt{a+bx}}{b^4}.$$

(4) $\int_x \frac{1}{x^2\sqrt{4+3x}} = -\frac{\sqrt{4+3x}}{4x} - \frac{3}{8} \log \frac{\sqrt{4+3x} - 2}{\sqrt{4+3x} + 2}.$

(5) $\int_x \frac{x^2}{(1+x)^{\frac{3}{2}}} = \left\{ \frac{(1+x)^2}{3} - 2(1+x) - 1 \right\} \frac{2}{\sqrt{1+x}}.$

(6) $\int_x x^5 \sqrt{1+x^2} = \left(x^4 - \frac{4x^2}{5} + \frac{8}{35} \right) \frac{(1+x^2)^{\frac{3}{2}}}{7}.$

(7) $\int_x \frac{\sqrt{1+x^2}}{x^6} = -\left\{ \frac{1}{x^5} - \frac{2}{3x^2} \right\} \frac{(1+x^2)^{\frac{3}{2}}}{7}.$

(8) $\int_x x^3 (1+x^2)^{\frac{3}{2}} = \frac{5x^2 - 2}{35} (1+x^2)^{\frac{5}{2}}.$

(9) $\int_x x^3 (1+x^2)^{\frac{5}{2}} = \frac{7x^2 - 2}{63} (1+x^2)^{\frac{7}{2}}.$

(10) $\int_x \frac{x^5}{\sqrt{1-x^2}} = -\left\{ \frac{x^4}{5} + \frac{4x^2}{15} + \frac{8}{15} \right\} \sqrt{1-x^2}.$

(11) $\int_x \frac{x^6}{\sqrt{1-x^2}} = -\left\{ \frac{x^5}{6} + \frac{5x^3}{24} + \frac{5x}{16} \right\} \sqrt{1-x^2} + \frac{5}{16} \sin^{-1} x.$

(12) $\int_x \frac{1}{x^6\sqrt{1+x^2}} = -\left\{\frac{1}{5x^5} - \frac{4}{15x^3} + \frac{8}{15x}\right\}\sqrt{1+x^2}.$

(13) $\int_x \frac{1}{(a+bx^2)^{\frac{5}{2}}} = \left\{\frac{1}{3a(a+bx^2)} + \frac{2}{3a^2}\right\}\frac{x}{\sqrt{a+bx^2}}.$

(14) $\int_x \frac{1}{(2ax+x^2)^{\frac{3}{2}}} = -\frac{x+a}{a^2\sqrt{2ax+x^2}}.$

(15) $\int_x \frac{1}{(2ax+x^2)^{\frac{5}{2}}} = -\left\{\frac{1}{3(2ax+x^2)} - \frac{2}{3a^2}\right\}\frac{x+a}{a^2\sqrt{2ax+x^2}}.$

(16) $\int_x \frac{1}{(1+x+x^2)^{\frac{3}{2}}} = \frac{2.(2x+1)}{3\sqrt{1+x+x^2}}.$

(17) $\int_x \frac{1}{(1+x+x^2)^{\frac{5}{2}}} = \left\{\frac{1}{1+x+x^2} + \frac{8}{3}\right\}\frac{2(2x+1)}{9\sqrt{1+x+x^2}}.$

(18) $\int_x \frac{1}{(1+x)^2\sqrt{x}} = \frac{\sqrt{x}}{(1+x)} + \tan^{-1}(\sqrt{x}).$

(19) $\int_x \frac{x\sqrt{x}}{1+x} = \left(\frac{x}{3} - 1\right) 2\sqrt{x} + 2\tan^{-1}\sqrt{x}.$

(20) $\int_x \frac{x\sqrt{x}}{1+x^2} = 2\sqrt{x} + \frac{1}{\sqrt{2}}\left\{\log\left(\frac{x+1+\sqrt{2x}}{\sqrt{1+x^2}}\right) - \tan^{-1}\frac{\sqrt{2x}}{1-x}\right\}.$

Rationalize the integrals

(1) $\int_x \frac{x^3 + 2x^{\frac{3}{2}} + x^{\frac{1}{3}}}{x + 3x^{\frac{1}{4}}}$, (2) $\int_x \frac{x^4}{x^5 + \sqrt[4]{(1+x)^3}}$,

in (1) make $x = z^{12}$, and in (2) make $(1+x) = z^4$.

CHAPTER IV.

INTEGRALS OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS.

49. THESE functions are of the form $X (\log x)^n$, $X . \log Y$, $X . a^x$, where X and Y are functions of x.

50. Integrate $\int_x X . (\log x)^n$.

$$\text{Let } \int_x X = P, \quad \int_x P . \frac{1}{x} = Q, \quad \text{and } \int_x Q . \frac{1}{x} = R.$$

$$\text{Then } \int_x X (\log x)^n = P (\log x)^n - n . \int_x P . (\log x)^{n-1} . \frac{1}{x},$$

$$\text{and } \int_x \frac{P}{x} . (\log x)^{n-1} = Q (\log x)^{n-1} - (n-1) . \int_x Q . (\log x)^{n-2} . \frac{1}{x},$$

$$\int_x \frac{Q}{x} . (\log x)^{n-2} = R (\log x)^{n-2} - (n-2) . \int_x R . (\log x)^{n-3} . \frac{1}{x};$$

&c.

$$\therefore \int_x X (\log x)^n = P (\log x)^n - n . Q (\log x)^{n-1}$$
$$+ n . (n-1) . R . (\log x)^{n-2} - \&c.$$

Ex. $\int_x x^m (\log x)^n$,

$$\int_x x^m (\log x)^n = \frac{x^{m+1} (\log x)^n}{m+1} - \frac{n}{m+1} . \int_x x^{m+1} . (\log x)^{n-1} . \frac{1}{x}$$

$$= \frac{x^{m+1} (\log x)^n}{m+1} - \frac{n}{m+1} . \int_x x^m . (\log x)^{n-1},$$

$$\int_x x^m (\log x)^{n-1} = \frac{x^{m+1} (\log x)^{n-1}}{m+1} - \frac{n-1}{m+1} . \int_x x^m . (\log x)^{n-2},$$

and in this manner may the integral be reduced to

$$\int_x x^m = \frac{x^{m+1}}{m+1}, \text{ if } n \text{ be a whole number,}$$

$$\text{and } \therefore \int_x x^m \,.\, (\log x)^n = \frac{x^{m+1}}{m+1} \{(\log x)^n - \frac{n}{m+1} (\log x)^{n-1}$$

$$+ \frac{n\,.\,(n-1)}{\{(m+1)\}^2} \log x)^{n-2} - \&c.\} \pm \frac{n\,.\,(n-1)\,.\,(n-2)\ldots.2\,.\,1}{(m+1)^{m+1}} x^{m+1}.$$

Every term of the integral vanishes both when $x = 0$ and $x = 1$, except the last, which vanishes only when $x = 0$.

$$\therefore \int_x x^m (\log x)^n, \left.\begin{matrix}\text{from } x = 0 \\ \text{to } x = 1\end{matrix}\right\} = \pm \frac{1\,.\,2\,.\,3\ldots.(n-1)\,.\,n}{(m+1)^{m+1}}.$$

51. Integrate $\int_x \frac{X}{(\log x)^n}$, n a whole number,

$$\int_x \frac{X}{(\log x)^n} = \int_x \frac{X\,.\,x\,.\,\frac{d\,.\log x}{dx}}{(\log x)^n}. \quad \text{Since } x\frac{d\,.\log x}{dx} = x\frac{1}{x} = 1$$

$$= \frac{-X\,.\,x}{(n-1)(\log x)^{n-1}} + \frac{1}{n-1}\,.\int \frac{\frac{d\,.\,(Xx)}{dx}}{(\log x)^{n-1}}.$$

Let $\frac{d\,.\,(Xx)}{dx} = P$;

$$\therefore \int_x \frac{X}{(\log x)^n} = \frac{-Xx}{(n-1)(\log x)^{n-1}} + \frac{1}{n-1}\,.\int_x \frac{P}{(\log x)^{n-1}};$$

$$\text{and } \int_x \frac{P}{(\log x)^{n-1}} = \frac{-P\,.\,x}{(n-2)(\log x)^{n-2}} + \frac{1}{n-2}\,.\int_x \frac{Q}{(\log x)^{n-1}},$$

$$\text{where } Q = \frac{d\,.\,(P\,.\,x)}{dx};$$

$$\therefore \int_x \frac{X}{(\log x)^n} = \frac{-Xx}{(n-1)(\log x)^{n-1}} - \frac{Px}{(n-1)(n-2)\,.\,(\log x)^{n-2}}$$

$$- \frac{Q\,.\,x}{(n-1)(n-2)(n-3)(\log x)^{n-3}} - \&c.$$

in this manner the integral may be reduced to $\int_x \frac{X_1}{(\log x)}$, where X_1 is a function of x, which cannot be integrated except by a series.

EXAMPLE. $\int_x \frac{x^m}{(\log x)^2}$,

$$\int_x \frac{x^m}{(\log x)^2} = \int_x \frac{x^{m+1} \cdot \frac{1}{x}}{(\log x)^2}$$

$$= \frac{-x^{m+1}}{\log x} + (m+1) \cdot \int_x \frac{x^m}{\log x}.$$

To integrate $\int_x \frac{x^m}{\log x}$.

Let $\log x = z$; $\therefore x = e^z$, and $x^m = e^{mz}$;

$$\therefore \int_x \frac{x^m}{\log x} = \int_z \frac{e^{mz}}{z} \cdot \frac{dx}{dz} = \int \frac{e^{mz}}{z} e^z = \int_z \frac{e^{(m+1)z}}{z}$$

$$= \int_z \left\{1 + (m+1)z + \frac{(m+1)^2 z^2}{1.2} + \frac{(m+1)^3 z^3}{2.3} + \&c.\right\} \cdot \frac{1}{z}.$$

$$= \log z + (m+1) \cdot z + \frac{(m+1)^2 z^2}{1.2^2} + \frac{(m+1)^3 . z^3}{2.3^2} + \&c.$$

$$= \log . \log x + (m+1) \log x + \frac{(m+1)^2 (\log x)^2}{1.2^2} + \frac{(m+1)^3 (\log x)^3}{2.3^2} + \&c.$$

COR. If $m = 0$, we have

$$\int_x \frac{1}{\log x} = \log . \log x + (\log x) + \frac{(\log x)^2}{1.2} + \&c.$$

52. Integrate $\int_x a^x . X$, X being a function of x.

$$\text{Since } \frac{d . a^x}{dx} = a^x \log a = A a^x;$$

$$\therefore \int_x a^x = \frac{a^x}{A}.$$

Therefore, integrating by parts,

$$\int_x a^x . X = \frac{X a^x}{A} - \frac{1}{A} . \int \frac{dX}{dx} . a^x.$$

$$\text{Let } \frac{dX}{dx} = P, \quad \frac{dP}{dx} = Q, \text{ \&c.}$$

$$\therefore \int_x a^x . X = \frac{X a^x}{A} - \frac{1}{A} . \int_x P a^x,$$

$$\int_x a^x . P = \frac{P a^x}{A} - \frac{1}{A} . \int Q a^x ;$$

$$\therefore \int_x a^x . X = a^x . \left\{ \frac{X}{A} - \frac{P}{A^2} + \frac{Q}{A^3} - \text{\&c.} \right\}.$$

EXAMPLE. Let $\int_x x^m . a^x$ be required.

$$\int_x x^m . a^x = \frac{x^m . a^x}{A} - \frac{m}{A} . \int_x x^{m-1} . a^x,$$

$$\int_x x^{m-1} . a^x = \frac{x^{m-1} . a^x}{A} - \frac{m-1}{A} . \int_x x^{m-2} . a^x,$$

&c.

$$\therefore \int_x x^m . a^x = a^x . \left\{ \frac{x^m}{A} - \frac{m x^{m-1}}{A^2} + \frac{m . (m-1) x^{m-2}}{A^3} - \text{\&c.} \right\}.$$

53. Integrate $\dfrac{du}{dx} = \dfrac{a^x}{x^n}$,

$$\int_x \frac{a^x}{x^n} = \frac{-a^x}{(n-1) x^{n-1}} + \frac{A}{n-1} . \int_x \frac{a^x}{x^{n-1}} ;$$

$$\therefore \int_x \frac{a^x}{x^{n-1}} = \frac{-a^x}{(n-2) x^{n-2}} + \frac{A}{n-2} . \int_x \frac{a^x}{x^{n-2}},$$

&c.

$$\therefore \int_x \frac{a^x}{x^n} = \frac{-a^x}{(n-1).x^{n-1}} - \frac{A a^x}{(n-1).(n-2).x^{n-2}}$$

$$- \frac{A^2 a^x}{(n-1).(n-2).(n-3).x^{n-3}} - \&c.$$

$$+ \frac{A^{n-1}}{(n-1).(n-2)\dots.1} . \int_x \frac{a^x}{x},$$

and $\int \frac{a^x}{x}$ can be found by expanding a^x.

$$\text{For } \frac{a^x}{x} = \frac{1 + Ax + \frac{A^2 x^2}{1.2} + \frac{A^3 x^3}{2.3} + \&c.}{x}$$

$$= \frac{1}{x} + A + \frac{A^2 x}{1.2} + \frac{A^3 x^2}{2.3} + \&c.$$

$$\therefore \int_x \frac{a^x}{x} = \text{h. l. } x + Ax + \frac{A^2 x^2}{1.2^2} + \frac{A^3 x^3}{2.3^2} + \frac{A^4 x^4}{2.3.4^2} + \&c.$$

Ex. 1. Find $\int_x \frac{1}{x} \log(a + bx)$.

$$\log(a + bx) = \log a\left(1 + \frac{b}{a}x\right) = \log a + \log\left(1 + \frac{b}{a}x\right)$$

$$= \log a + \left(\frac{b}{a}x - \frac{b^2 x^2}{2a^2} + \frac{b^3 x^3}{3a^3} - \frac{b^4 x^4}{4a^4} + \&c.\right);$$

$$\therefore \int_x \frac{1}{x} \log(a + bx) = \log x . \log a$$

$$+ \left(\frac{b}{a}x - \frac{b^2 x^2}{2^2 a^2} + \frac{b^3 x^3}{3^2 a^3} - \frac{b^4 x^4}{4^2 a^4} + \&c.\right).$$

Ex. 2. Find $\int_x x^{nx}$.

$$x^{nx} = 1 + nx \log x + \frac{(nx \log x)^2}{1.2} + \frac{(nx \log x)^3}{1.2.3} + \&c.$$

$$\therefore \int_x x^{nx} = x + n \cdot \int_x x \log x + \frac{n^2}{1 \cdot 2} \cdot \int_x x^2 (\log x)^2$$

$$+ \frac{n^3}{2 \cdot 3} \cdot \int_x x^3 (\log x)^3 + \&c.$$

The integration of the terms depends upon $\int x^m (\log x)^m$.

$$\text{Now } \int x^m \cdot (\log x)^m = \frac{x^{m+1}}{m+1} \cdot \left\{ (\log x)^m - \frac{m}{m+1} \cdot (\log x)^{m-1} \right.$$

$$\left. + \frac{m \cdot (m-1)}{(m+1)^2} \cdot (\log x)^{m-2} - \&c. \pm \frac{m \cdot (m-1) \cdot (m-2) \ldots 3 \cdot 2 \cdot 1}{(m+1)^m} \right\};$$

$$\therefore \int_x x \log x = \frac{x^2}{2} \cdot (\log x - \tfrac{1}{2}),$$

$$\int_x x^2 (\log x)^2 = \frac{x^3}{3} \left\{ (\log x)^2 - \tfrac{2}{3} \log x + \frac{2 \cdot 1}{3^2} \right\},$$

$$\int_x (x \log x)^3 = \frac{x^4}{4} \left\{ (\log x)^3 - \tfrac{3}{4} (\log x)^2 + \frac{3 \cdot 2}{4^2} (\log x) - \frac{3 \cdot 2}{4^3} \right\},$$

&c.

and arranging the terms according to the powers of $\log x$,

$$\int_x x^{nx} = x - \frac{n x^2}{2^2} + \frac{n^2 x^3}{3^3} - \frac{n^3 x^4}{4^4} + \frac{n^4 x^5}{5^5} - \&c.$$

$$+ \left(\frac{x^2}{2} - \frac{n x^3}{3^2} + \frac{n^2 x^4}{4^3} - \&c. \right) n \log x$$

$$+ \left(\frac{x^3}{3} - \frac{n x^4}{4^2} + \frac{n^2 x^5}{5^3} - \&c. \right) \frac{n^2 (\log x)^2}{1 \cdot 2}$$

$$+ \left(\frac{x^4}{4} - \frac{n x^5}{5^2} + \frac{n^2 x^6}{6^3} - \&c. \right) \frac{n^3 (\log x)^3}{1 \cdot 2 \cdot 3}$$

$$+ \&c.$$

COR. Since $x^m (\log x)^n$ vanishes both when $x = 0$, and $x = 1$,

$$\therefore\ \int_x x^{nx}, \left.\begin{matrix}\text{from } x=0\\ \text{to } x=1\end{matrix}\right\} = 1 - \frac{n}{2^2} + \frac{n^2}{3^3} - \frac{n^3}{4^4} + \frac{n^4}{5^5} - \&c.,$$

and if $n = 1$,

$$\int_x x^x = \frac{1}{1} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \&c.$$

between the same limits.

This last integral gives the area of a curve, of which the equation is $y = x^x$, included between two ordinates, each $= 1$, one drawn through the origin, and the other at a distance $= 1$ from it.

EXAMPLES.

(1) $$\int_x x^2 (\log x)^2 = \frac{x^3}{3} \cdot \left((\log x)^2 - \frac{2}{3}\log x + \frac{1.2}{9}\right).$$

(2) $$\int_x \frac{x^3}{\sqrt{\log x}} = \frac{x^4}{4\sqrt{\log x}} \cdot \left\{1 + \frac{1}{8\log x} + \frac{1.3}{(8.\log x)^2} + \frac{1.3.5}{(8.\log x)^3} + \&c.\right\}.$$

(3) $$\int_x \frac{x^4}{(\log x)^3} = -\frac{x^5}{2(\log x)^2} - \frac{5x^5}{2.\log x} + \frac{25}{2} \cdot \int_x \frac{x^4}{\log x}.$$

(4) $$\int_x a^x . x^3 = a^x \left\{\frac{x^3}{A} - \frac{3x^2}{A^2} + \frac{6x}{A^3} - \frac{6}{A^4}\right\}.$$

(5) $$\int_x \frac{a^x}{x^4} = -a^x \left\{\frac{1}{3x^3} + \frac{A}{2.3x^2} + \frac{A^2}{2.3.x}\right\} + \frac{A^3}{2.3} \cdot \int_x \frac{a^x}{x}.$$

(6) $$\int_x \frac{a^x}{\sqrt{x}} = \frac{a^x}{A\sqrt{x}}\left\{1 + \frac{1}{2xA} + \frac{3}{(2xA)^2} + \frac{3.5}{(2xA)^3} + \&c.\right\},$$

$$\text{also } = \frac{Aa^x}{\sqrt{x}} \cdot \left\{\frac{2xA}{1} - \frac{(2xA)^2}{1.3} + \frac{(2xA)^3}{3.5} - \frac{(2xA)^4}{3.5.7} + \&c.\right\}.$$

(7) $$\int_x x^{nx} . x^m \left\{\begin{matrix}\text{from } x=0\\ \text{to } x=1\end{matrix}\right\} = \frac{1}{m+1} - \frac{n}{(m+2)^2} + \frac{n^2}{(m+3)^3} - \frac{n^3}{(m+4)^4} + \&c.$$

CHAPTER V.

CIRCULAR FUNCTIONS.

54. THESE are of the form $\sin^n\theta$, $\cos^n\theta$, $(\sin\theta)^m(\cos\theta)^n$, $\frac{1}{(\sin\theta)^n}$, &c., and $X.\sin^{-1}x$, where X is a function of x.

Almost all these functions may be integrated by parts, and may be thus reduced either to known or to more simple integrals.

These are

$$\sin\theta,\ \cos\theta,\ \frac{1}{\cos^2\theta},\ \tan\theta,\ \cot\theta,\ \frac{1}{\cos\theta\sin\theta},\ \frac{1}{\sin\theta},\ \text{and}\ \frac{1}{\cos\theta}.$$

(1) $\int_\theta \sin\theta = -\cos\theta.$

(2) $\int_\theta \cos\theta = \sin\theta.$

(3) $\int_\theta \frac{1}{\cos^2\theta} = \tan\theta.$

(4) $\int_\theta \tan\theta = \int_\theta \frac{\sin\theta}{\cos\theta} = -\text{h. l.}\cos\theta.$

(5) $\int_\theta \cot\theta = \int_\theta \frac{\cos\theta}{\sin\theta} = +\text{h. l.}\sin\theta.$

55. Integrate $\frac{1}{\sin\theta}$, $\frac{1}{\cos\theta}$, and $\frac{1}{\sin\theta\cos\theta}$.

(1) $$\int_\theta \frac{1}{\sin\theta} = \int_\theta \frac{\sin\theta}{\sin^2\theta} = \int_\theta \frac{\sin\theta}{1-\cos^2\theta} = \tfrac{1}{2}\int\left(\frac{\sin\theta}{1-\cos\theta}+\frac{\sin\theta}{1+\cos\theta}\right)$$

$$= \tfrac{1}{2}\,\text{h. l.}\left(\frac{1-\cos\theta}{1+\cos\theta}\right) = \tfrac{1}{2}\,\text{h. l.}\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} = \text{h. l.}\left(\tan\frac{\theta}{2}\right).$$

(2) $$\int_\theta \frac{1}{\cos\theta} = \int_\theta \frac{1}{\sin\left(\frac{\pi}{2}-\theta\right)} = \int_\theta \frac{1}{\sin\left(\frac{\pi}{2}+\theta\right)}$$

$$= \int_\phi \frac{1}{\sin\phi} = \text{h. l.}\tan\frac{\phi}{2} = \text{h. l.}\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

(3) $$\int_\theta \frac{1}{\sin\theta\cos\theta} = \int_\theta \frac{2}{\sin 2\theta} = \int_{2\theta} \frac{1}{\sin 2\theta} = \text{h. l.}\,(\tan\theta).$$

56. Find $\int_x X.\sin^{-1}x$, where X is a function of x.

Make $\int_x X = P$, and then integrating by parts,

$$\int_x X.\sin^{-1}x = P\sin^{-1}x - \int_x \frac{P}{\sqrt{1-x^2}},$$

and $\int_x \frac{P}{\sqrt{1-x^2}}$ has been integrated.

Ex. $\int_x \frac{x}{\sqrt{1-x^2}}\sin^{-1}x$. Here $P = -\sqrt{1-x^2}$;

$$\therefore \int_x \frac{x}{\sqrt{1-x^2}} = -\sqrt{1-x^2}.\sin^{-1}x + \int\sqrt{1-x^2}\frac{1}{\sqrt{1-x^2}}$$

$$= -\sqrt{1-x^2}.\sin^{-1}x + x.$$

Similarly may $\int_x X\cos^{-1}x$ be integrated.

57. To integrate $\frac{du}{dx} = X \tan^{-1} x$.

$$\text{Let } \int_x X = P;$$

$$\therefore \int_x X \tan^{-1} x = P \tan^{-1} x - \int_x \frac{P}{1 + x^2}.$$

Ex. $\int_x \frac{x^2}{1 + x^2} \tan^{-1} x$ be required.

$$P = \int_x \frac{x^2}{1 + x^2} = \int_x \left(1 - \frac{1}{1 + x^2}\right) = x - \tan^{-1} x;$$

$$\therefore \int_x \frac{x^2}{1 + x^2} \tan^{-1} x = (x - \tan^{-1} x) \tan^{-1} x - \int_x (x - \tan^{-1} x) \frac{1}{1 + x^2}$$

$$= (x - \tan^{-1} x) \tan^{-1} x - \text{h. l.} \sqrt{1 + x^2} + \frac{(\tan^{-1} x)^2}{2}$$

$$= \left\{x - \tfrac{1}{2}(\tan^{-1} x)\right\} \tan^{-1} x - \text{h. l.} \sqrt{1 + x^2}.$$

58. Integrate $\frac{du}{d\theta} = \sin^n \theta$.

Integrating by parts, since $\sin^n \theta = \sin^{n-1} \theta \,.\, \sin \theta$;

$$\therefore \int_\theta \sin^n \theta = \int \sin^{n-1} \theta \,.\, \sin \theta$$

$$= - \sin^{n-1} \theta \,.\, \cos \theta + (n - 1) \,.\, \int_\theta \sin^{n-2} \theta \cos^2 \theta;$$

and putting $1 - \sin^2 \theta$ for $\cos^2 \theta$

$$= - \sin^{n-1} \theta \cos \theta + (n - 1) \,.\, \int_\theta \sin^{n-2} \theta - (n - 1) \int_\theta \sin^n \theta;$$

$$\therefore \int_\theta \,.\, \sin^n \theta = - \frac{\sin^{n-1} \theta \,.\, \cos \theta}{n} + \frac{n - 1}{n} \int_\theta \sin^{n-2} \theta,$$

a formula by which $\int_\theta \sin^n \theta$ may be reduced to $- \cos \theta$, or θ, according as n is odd or even.

Ex. $\int_\theta (\sin^3 \theta) = - \frac{\sin^2 \theta \cos \theta}{3} + \frac{2}{3} \,.\, \int_\theta \sin \theta$

$$= - \frac{\sin^2 \theta \,.\, \cos \theta}{3} - \frac{2}{3} \cos \theta.$$

59. Integrate $\dfrac{du}{d\theta} = \cos^n\theta$.

$$\int_\theta \cos^n\theta = \int \cos^{n-1}\theta \,.\, \cos\theta$$

$$= + \cos^{n-1}\sin\theta + (n-1)\,.\int_\theta \cos^{n-2}\theta\sin^2\theta$$

$$= \cos^{n-1}\theta\sin\theta + (n-1)\int_\theta \cos^{n-2}\theta - (n-1)\int_\theta \cos^n\theta$$

$$= \frac{\cos^{n-1}\theta\sin\theta}{n} + \frac{n-1}{n}\int_\theta \cos^{n-2}\theta,$$

a formula by which $\int_\theta \cos^n\theta$ may be reduced to $\sin\theta$ or θ, according as n is odd or even.

Ex. $$\int_\theta \cos^3\theta = \frac{\cos^2\theta\sin\theta}{3} + \frac{2}{3}\int_\theta \cos\theta$$

$$= \frac{\cos^2\theta\sin\theta}{3} + \frac{2}{3}\sin\theta.$$

60. Let $\dfrac{du}{d\theta} = \dfrac{1}{(\sin\theta)^n}$.

Since $\sin^2\theta + \cos^2\theta = 1$;

$$\therefore\; u = \int_\theta \frac{1}{(\sin\theta)^n} = \int_\theta \frac{\sin^2\theta + \cos^2\theta}{(\sin\theta)^n}$$

$$= \int_\theta \frac{1}{(\sin\theta)^{n-2}} + \int_\theta \frac{\cos^2\theta}{(\sin\theta)^n},$$

and $$\int_\theta \frac{\cos^2\theta}{(\sin\theta)^n} = -\frac{\cos\theta}{(n-1)(\sin\theta)^{n-1}} - \frac{1}{n-1}\int_\theta \frac{\sin\theta}{(\sin\theta)^{n-1}};$$

$$\therefore\; u = -\frac{\cos\theta}{(n-1)(\sin\theta)^{n-1}} + \left(1 - \frac{1}{n-1}\right)\int_\theta \frac{1}{(\sin\theta)^{n-2}}$$

$$= -\frac{\cos\theta}{(n-1)(\sin\theta)^{n-1}} + \frac{n-2}{n-1}\,.\int_\theta \frac{1}{(\sin\theta)^{n-2}},$$

a formula by which n may be diminished.

Ex. Let $u = \int_\theta \frac{1}{(\sin\theta)^3}$ be required;

therefore here $n = 3$;

$$\therefore \int_\theta \frac{1}{(\sin\theta)^3} = -\frac{\cos\theta}{2(\sin\theta)^2} + \tfrac{1}{2}\int_\theta \frac{1}{\sin\theta}$$

$$= -\frac{\cos\theta}{2(\sin\theta)^2} + \tfrac{1}{2}\log\left(\tan\frac{\theta}{2}\right).$$

61. If $\frac{du}{d\theta} = \frac{1}{(\cos\theta)^n}$, then, as in last article,

$$u = \int_\theta \frac{1}{(\cos\theta)^{n-2}} + \int_\theta \frac{\sin^2\theta}{(\cos\theta)^n},$$

and $$\int_\theta \frac{\sin^2\theta}{(\cos\theta)^n} = \frac{\sin\theta}{(n-1)\,.\,(\cos\theta)^{n-1}} - \frac{1}{n-1}\int_\theta \frac{\cos\theta}{(\cos\theta)^{n-1}};$$

$$\therefore\ u = \frac{\sin\theta}{(n-1)(\cos\theta)^{n-1}} + \frac{n-2}{n-1}\int_\theta \frac{1}{(\cos\theta)^{n-2}}.$$

62. Let $\frac{du}{d\theta} = (\sin\theta)^m(\cos\theta)^n$ m and n both integers,

$$(\sin\theta)^m(\cos\theta)^n = (\sin\theta)^m \cos\theta\,(\cos\theta)^{n-1};$$

$$\therefore\ \int_\theta (\sin\theta)^m(\cos\theta)^n = \frac{(\sin\theta)^{m+1}(\cos\theta)^{n-1}}{m+1} + \frac{n-1}{m+1}\int_\theta (\sin\theta)^{m+2}(\cos\theta)^{n-2}$$

$$= \frac{(\sin\theta)^{m+1}(\cos\theta)^{n-1}}{m+1} + \frac{n-1}{m+1}\left\{\int_\theta (\sin\theta)^m(\cos\theta)^{n-2} - \int_\theta (\sin\theta)^m(\cos\theta)^n\right\};$$

$$\therefore\ \left(1 + \frac{n-1}{m+1}\right)u = \frac{m+n}{m+1}u = \frac{(\sin\theta)^{m+1}(\cos\theta)^{n-1}}{m+1}$$

$$+ \frac{n-1}{m+1}\,.\int_\theta (\sin\theta)^m(\cos\theta)^{n-2};$$

$$\therefore\ u = \frac{(\sin\theta)^{m+1}(\cos\theta)^{n-1}}{m+n} + \frac{n-1}{m+n}\cdot\int(\sin\theta)^m(\cos\theta)^{n-2},$$

a formula by which the integral may be reduced to

$$\int_\theta(\sin\theta)^m, \quad\text{or}\quad \int_\theta(\sin\theta)^m\cos\theta.$$

Ex. Let $m = 3$, and $n = 2$;

$$\therefore\ \int(\sin\theta)^3\cos^2\theta = \frac{(\sin\theta)^4\cos\theta}{5} + \frac{1}{5}\cdot\int_\theta(\sin\theta)^3$$

$$= \frac{(\sin\theta)^4\cos\theta}{5} - \frac{(\sin\theta)^2\cos\theta}{3.5} - \frac{2\cos\theta}{3.5},$$

substituting the value of $\int_\theta(\sin\theta)^3$ from Art. 58.

63. Let $\dfrac{du}{d\theta} = \dfrac{\sin^m\theta}{\cos^n\theta}$,

$$u = \int_\theta\frac{\sin^{m-1}\theta\sin\theta}{(\cos\theta)^n} - \frac{(\sin\theta)^{m-1}}{(n-1)\cdot(\cos\theta)^{n-1}} - \frac{m-1}{n-1}\int_\theta\frac{(\sin\theta)^{m-2}}{(\cos\theta)^{n-2}},$$

a formula by which the integral is easily reducible to a known form.

Let $m = 3$, and $n = 4$;

$$\therefore\ u = \int_\theta\frac{(\sin\theta)^3}{(\cos\theta)^4} = \frac{(\sin\theta)^2}{3(\cos\theta)^3} - \frac{2}{3}\cdot\int\frac{\sin\theta}{(\cos\theta)^2}$$

$$= \frac{(\sin\theta)^2}{3(\cos\theta)^3} - \frac{2}{3}\cdot\frac{1}{\cos\theta}$$

$$= \frac{1}{(\cos\theta)^3}\left\{\frac{(\sin\theta)^2}{3} - \frac{2}{3}\cos^2\theta\right\}$$

$$= \frac{1}{(\cos\theta)^3}\left\{\sin^2\theta - \frac{2}{3}\right\};$$

otherwise thus,

$$\int_\theta \frac{(\sin\theta)^3}{(\cos\theta)^4} = \int_\theta \frac{\sin\theta\,.\,(1-\cos^2\theta)}{(\cos\theta)^4} = \int_\theta \frac{\sin\theta}{(\cos\theta)^4} - \int_\theta \frac{\sin\theta}{(\cos\theta)^2}$$

$$= \tfrac{1}{3}\,.\,\frac{1}{\cos^3\theta} - \frac{1}{\cos\theta}$$

$$= \frac{1}{\cos^3\theta}\,.\,\left\{\tfrac{1}{3} - \cos^2\theta\right\}$$

$$= \frac{1}{\cos^3\theta}\,.\left\{\sin^2\theta - \frac{2}{3}\right\}$$

64. Find $\int_\theta \theta^n \sin\theta$.

$$\int_\theta \theta^n \quad \sin\theta = -\,\theta^n\cos\theta + n\,.\int \theta^{n-1}\cos\theta,$$

$$\int_\theta \theta^{n-1}\cos\theta = +\,\theta^{n-1}\sin\theta - (n-1)\int_\theta \theta^{n-2}\sin\theta,$$

$$\int_\theta \theta^{n-2}\sin\theta = -\,\theta^{n-2}\cos\theta + (n-2)\int_\theta \theta^{n-3}\cos\theta,$$

&c. = &c. &c.

$$\int_\theta \theta^n\sin\theta = -\,\theta^n\cos\theta + n\theta^{n-1}\sin\theta + n(n-1)\theta^{n-2}\cos\theta$$
$$-\,n(n-1)(n-2)\theta^{n-3}\sin\theta - \&c.$$

COR. Similarly may $\int_\theta \theta^n\cos\theta$ be found and shewn to be

$$= \theta^n\sin\theta + n\theta^{n-1}\cos\theta - n(n-1)\theta^{n-2}\sin\theta$$
$$-\,n(n-1)(n-2)\theta^{n-3}\cos\theta + \&c.$$

65. Integrate $\sin m\theta\,.\cos n\theta$, $\sin m\theta\,\sin n\theta$, and $\cos m\theta\,.\cos n\theta$.

Since $\sin A\,.\cos B = \tfrac{1}{2}\,.\,\{\sin(A+B) + \sin(A-B)\}$;

$$\therefore\ \sin m\theta\,.\cos n\theta = \tfrac{1}{2}\,.\,\{\sin(m+n)\theta + \sin(m-n)\theta\};$$

$$\therefore\ \int_\theta(\sin m\theta\,.\cos n\theta) = -\,\tfrac{1}{2}\,.\left\{\frac{\cos(m+n)\theta}{m+n} + \frac{\cos(m-n)\theta}{m-n}\right\}.$$

Also since $\cos m\theta \,.\, \cos n\theta = \frac{1}{2} \,.\, \{\cos (m+n)\theta + \cos (m-n)\theta\}$,

and $\sin m\theta \,.\, \sin n\theta = \frac{1}{2} \,.\, \{\cos (m-n)\theta - \cos (m+n)\theta\}$;

$$\therefore \int_\theta (\cos m\theta \,.\, \cos n\theta) = \tfrac{1}{2} \,.\, \left\{\frac{\sin (m+n)\theta}{m+n} + \frac{\sin (m-n)\theta}{m-n}\right\},$$

$$\text{and } \int_\theta (\sin m\theta \sin n\theta) = -\tfrac{1}{2} \,.\, \left\{\frac{\sin (m+n)\theta}{m+n} - \frac{\sin (m-n)\theta}{m-n}\right\}.$$

66. Integrate $(\tan\theta)^m$, and $(\tan\theta)^{-m}$.

$$(\tan\theta)^m = (\tan\theta)^{m-2}\{1 + \tan^2\theta - 1\}$$

$$= (\tan\theta)^{m-2} \frac{d \,.\, \tan\theta}{d\theta} - (\tan\theta)^{m-2};$$

$$\therefore \int_\theta (\tan\theta)^m = \frac{(\tan\theta)^{m-1}}{m-1} - \int_\theta (\tan\theta)^{m-2},$$

$$\int_\theta (\tan\theta)^{m-2} = \frac{(\tan\theta)^{m-3}}{m-3} - \int_\theta (\tan\theta)^{m-4},$$

&c. &c.

$$\therefore \int_\theta (\tan\theta)^m = \frac{(\tan\theta)^{m-1}}{m-1} - \frac{(\tan\theta)^{m-3}}{m-3} + \frac{(\tan\theta)^{m-5}}{m-5} - \&c.$$

a formula by which the integral may be reduced either to θ, or

$$\int_\theta \tan\theta = \int_\theta \frac{\sin\theta}{\cos\theta} = -\text{h. l.} \cos\theta.$$

$$\text{Ex.} \quad \int_\theta (\tan\theta)^4 = \frac{\tan^3\theta}{3} - \int_\theta (\tan\theta)^2,$$

$$\int_\theta (\tan\theta)^2 = \tan\theta - \int_\theta \left(\frac{d\theta}{d\theta}\right) = \tan\theta - \theta;$$

$$\therefore \int_\theta (\tan\theta)^4 = \frac{\tan^3\theta}{3} - \tan\theta + \theta.$$

67. $\int_\theta \frac{1}{(\tan\theta)^m} = \int_\theta \frac{1+\tan^2\theta - \tan^2\theta}{(\tan\theta)^m}$

$$= \int_\theta \frac{\frac{d.(\tan\theta)}{d\theta}}{(\tan\theta)^m} - \int_\theta \frac{1}{(\tan\theta)^{m-2}} = -\frac{1}{(m-1)(\tan\theta)^{m-1}} - \int_\theta \frac{1}{(\tan\theta)^{m-2}},$$

and $\int_\theta \frac{1}{(\tan\theta)^{m-2}} = -\frac{1}{(m-3)(\tan\theta)^{m-3}} - \int_\theta \frac{1}{(\tan\theta)^{m-4}}$;

$$\therefore \int_\theta \frac{1}{(\tan\theta)^m} = -\frac{1}{(m-1)(\tan\theta)^{m-1}}$$

$$+ \frac{1}{(m-3)(\tan\theta)^{m-3}} - \frac{1}{m-5(\tan\theta)^{m-5}} + \&c.$$

a formula by which the integral is reduced to θ, or

$$\int_\theta \frac{1}{\tan\theta} = \text{h. l. } (\sin\theta).$$

68. $\int_x e^{ax} \sin kx$.

Integrating by parts, and making $p = \sin kx$, and $\frac{dq}{dx} = e^{ax}$,

in the formula $\int_x p\frac{dq}{dx} = pq - \int_x q\frac{dp}{dx}$, we have

$$\int_x e^{ax}\sin kx = \frac{e^{ax}\sin kx}{a} - \frac{k}{a}.\int_x e^{ax}\cos kx \ldots\ldots (1),$$

and $\int_x e^{ax}\cos kx = \frac{e^{ax}\cos kx}{a} + \frac{k}{a}.\int_x e^{ax}\sin kx$.

Multiplying by $\frac{k}{a}$, and transposing

$$\frac{k^2}{a^2}.\int_x e^{ax}\sin kx = -\frac{ke^{ax}\cos kx}{a^2} + \frac{k}{a}.\int_x e^{ax}\cos kx \ldots\ldots (2).$$

Adding (1) and (2),

$$\left(1+\frac{k^2}{a^2}\right)\int_x e^{ax}\sin kx = \frac{(a\,.\sin kx - k\cos kx)\,e^{ax}}{a^2};$$

$$\therefore\ \int_x e^{ax}\sin kx = \frac{(a\,.\sin kx - k\cos kx)\,e^{ax}}{k^2+a^2}.$$

Similarly, $\displaystyle\int_x e^{ax}\cos kx = \frac{(a\,.\cos kx + k\sin kx)\,e^{ax}}{k^2+a^2}.$

69. To integrate $\dfrac{du}{dx} = \dfrac{1}{a+b\,.\cos x}.$

Since $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}.$

Let $z = \tan\frac{x}{2}$;

$$\therefore\ \cos x = \frac{1-z^2}{1+z^2};$$

$$\therefore\ \sin x = \frac{4z}{(1+z^2)^2}\,\frac{dz}{dx}.$$

But $\sin x = \sqrt{1-\left(\dfrac{1-z^2}{1+z^2}\right)^2} = \dfrac{2z}{1+z^2}$;

$$\therefore\ \frac{dz}{dx} = \frac{(1+z^2)}{2};$$

$$\therefore\ \frac{1}{a+b\,\frac{1-z^2}{1+z^2}} = \frac{du}{dz}\cdot\frac{1+z^2}{2};$$

$$\therefore\ \frac{du}{dz} = \frac{2}{a(1+z^2)+b(1-z^2)}.$$

(1) Let $a > b$;

$$\therefore \frac{du}{dz} = \frac{2}{a+b+(a-b)z^2} = \frac{2}{a+b} \cdot \frac{1}{1 + \dfrac{a-b}{a+b} \cdot z^2};$$

$$\therefore u = \frac{2}{a+b} \cdot \sqrt{\frac{a+b}{a-b}} \cdot \tan^{-1}\left(z\sqrt{\frac{a-b}{a+b}}\right)$$

$$= \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}} \cdot \tan\frac{x}{2}\right\}.$$

(2) Let $a < b$;

$$\therefore \frac{du}{dz} = \frac{2}{(b-a)} \cdot \frac{1}{\dfrac{b+a}{b-a} - z^2}.$$

$$\therefore u = \frac{1}{b-a} \cdot \sqrt{\frac{b-a}{b+a}} \cdot \text{h. l.} \frac{\sqrt{\dfrac{b+a}{b-a}} + z}{\sqrt{\dfrac{b+a}{b-a}} - z}$$

$$= \frac{1}{\sqrt{b^2-a^2}} \cdot \text{h. l.} \frac{\sqrt{b+a} + \sqrt{b-a} \cdot \tan\dfrac{x}{2}}{\sqrt{b+a} - \sqrt{b-a} \cdot \tan\dfrac{x}{2}}.$$

70. Similarly may $\int_x \frac{1}{a + b\sin x}$ be found.

Also $\int_x \frac{\sin x}{a + b\cos x}$, and $\int_x \frac{\cos x}{a + b\cos x}$.

For $\int_x \frac{\sin x}{a+b\cos x} = \frac{1}{b}\int_x \frac{-\dfrac{d \cdot \cos x}{dx}}{a + b\cos x} = -\frac{1}{b} \cdot \text{h. l.}\,(a + b\cos x)$, and $\int_x \frac{\cos x}{a + b\cos x} = \int_x \frac{a + b\cos x - a}{b(a + b\cos x)}$

$$= \int_x \left\{\frac{1}{b} - \frac{a}{b} \cdot \frac{1}{a + b\cos x}\right\} = \frac{x}{b} - \frac{a}{b} \cdot \int_x \frac{1}{a + b\cos x}.$$

And to integrate $\frac{1}{a + b\tan x}$. Let $\tan x = z$;

$$\therefore \frac{dx}{dz} = \frac{1}{1 + z^2};$$

$$\therefore \int_x \frac{1}{a + b\tan x} = \int_z \frac{1}{(1 + z^2)(a + bz)},$$

which must be integrated by partial fractions.

EXAMPLES.

(1) $\int_\theta (\sin\theta)^5 = -\cos\theta \left\{\frac{(\sin\theta)^4}{5} + \frac{4(\sin\theta)^2}{15} - \frac{8}{15}\right\}.$

(2) $\int_\theta (\cos\theta)^6 = \sin\theta \left\{\frac{(\cos\theta)^5}{6} + \frac{5(\cos\theta)^3}{24} + \frac{5\cos\theta}{16}\right\} + \frac{5\theta}{16}.$

(3) $\int_\theta (\sin\theta)^2(\cos\theta)^4 = \frac{(\sin\theta \cdot \cos\theta)^3}{6} + \frac{1}{8}\sin^2\theta\cos\theta$

$$- \frac{1}{16}\sin\theta\cos\theta + \frac{\theta}{16}.$$

(4) $\int_\theta (\sin\theta)^6(\cos\theta)^3 = \left(\frac{(\cos\theta)^2}{9} + \frac{2}{63}\right)(\sin\theta)^7.$

(5) $\int_\theta \frac{1}{(\sin\theta)^5} = -\cos\theta \left\{\frac{1}{4(\sin\theta)^4} + \frac{3}{8(\sin\theta)^2}\right\} + \frac{3}{8}\log\tan\frac{\theta}{2}$

(6) $\int_\theta \frac{1}{(\cos\theta)^6} = \sin\theta \left\{\frac{1}{5(\cos\theta)^5} + \frac{4}{15(\cos\theta)^3} + \frac{8}{15 . \cos\theta}\right\}.$

(7) $$\int_\theta \frac{(\sin\theta)^5}{(\cos\theta)^2} = -\frac{1}{\cos\theta}\left\{\frac{(\sin\theta)^4}{3} + \frac{4(\sin\theta)^2}{3} - \frac{8}{3}\right\}.$$

(8) $$\int_\theta \frac{(\cos\theta)^4}{(\sin\theta)^3} = \left\{(\cos\theta)^3 - \frac{3\cos\theta}{2}\right\}\frac{1}{(\sin\theta)^2} - \frac{3}{2}.\log\tan\frac{\theta}{2}.$$

(9) $$\int_\theta \frac{1}{(\sin\theta)^2.(\cos\theta)^3} = \left\{\frac{1}{2(\cos\theta)^2} - \frac{3}{2}\right\}\frac{1}{\sin\theta} + \frac{3}{2}.\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

(10) $$\int_\theta \frac{1}{(\sin\theta)^4(\cos\theta)^2} = -\frac{1}{3\cos\theta(\sin\theta)^3} - \frac{8}{3}\cot 2\theta.$$

(11) $$\int_\theta \theta^3.\cos\theta = \theta^3\sin\theta + 3\theta^2\cos\theta - 6\theta\sin\theta - 6\cos\theta.$$

(12) $$\int_x \frac{x^2}{\sqrt{1-x^2}}\sin^{-1}x = \frac{(\sin^{-1}x)^2}{4} - \frac{x\sqrt{1-x^2}}{2}\sin^{-1}x + \frac{x^2}{4}.$$

(13) $$\int_x \frac{x}{(1-x^2)^{\frac{3}{2}}}\sin^{-1}x = \frac{\sin^{-1}x}{\sqrt{1-x^2}} + \log\sqrt{\frac{1-x}{1+x}}.$$

(14) $$\int_x \frac{x^2}{1+x^2}\tan^{-1}x = x\tan^{-1}x - \tfrac{1}{2}(\tan^{-1}x)^2 - \log\sqrt{1+x^2}.$$

(15) $$\int_x e^{ax}.(\sin x)^2 = \frac{e^{ax}.\sin x\,(a\sin x - 2\cos x)}{a^2+4} + \frac{2.e^{ax}}{a(a^2+4)}.$$

(16) $$\int_x \frac{1}{(a+b\cos x)^2} = \frac{1}{a^2-b^2}\left\{\frac{-b\sin x}{a+b\cos x} + a\int_x \frac{1}{a+b\cos x}\right\}.$$

CHAPTER VI.

APPLICATION OF THE INTEGRAL CALCULUS TO DETERMINE THE AREAS AND LENGTHS OF PLANE CURVES, AND THE VOLUMES AND SURFACES OF SOLIDS OF REVOLUTION.

71. We have seen in the Differential Calculus, that if $y = f(x)$ be the equation to a curve, and A the area of a portion ANP, that $\dfrac{dA}{dx} = y = f(x)$.

Hence, when the equation to a curve is given, its area may be found by finding the value of $\int_x f(x)$, and this integral may in general be found by means of the rules given in the preceding Chapters.

If the equation to the curve be between polar co-ordinates,

$$\text{then } \frac{dA}{d\theta} = \frac{r^2}{2}; \quad \therefore\ A = \int_\theta \frac{r^2}{2}.$$

It is frequently convenient to put $y = f(z)$, i.e. to substitute z for $\phi(x)$; but then, since

$$\frac{dA}{dz} = \frac{dA}{dx} \cdot \frac{dx}{dz} = y\frac{dx}{dz};$$

$$\therefore\ A = \int_z y \cdot \frac{dx}{dz} = \int_z f(z) \cdot \frac{dx}{dz}.$$

72. Again, if s represents the length of a curve, of which the equation is $y = f(x)$,

$$\text{since } \frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}};$$

$$\therefore \; s = \int_x \sqrt{1 + \frac{dy^2}{dx^2}},$$

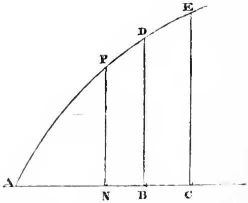
where $\frac{dy}{dx}$ must be found from $y = f(x)$.

73. Also, if V and S respectively represent the volume and surface of a solid of revolution, since

$$\frac{dV}{dx} = \pi y^2, \quad \text{and} \quad \frac{dS}{dx} = 2\pi y \sqrt{1 + \frac{dy^2}{dx^2}};$$

$$\therefore \; V = \pi \int_x y^2, \quad \text{and} \quad S = 2\pi \cdot \int_x y \sqrt{1 + \frac{dy^2}{dx^2}}.$$

74. A constant must be added to each of these integrals, the determination of which depends upon the nature of the particular problem.



As an illustration, let the area ABD be required, the nature of the curve ANP being known by the equation $y = f(x)$, where $AN = x$, and $NP = y$.

Let $AB = a$, and $ANP = A$;

$$\therefore \; \frac{dA}{dx} = y = f(x);$$

$$\therefore \; A = ANP = \int_x f(x) = \phi(x) + C \ldots\ldots\ldots (1).$$

Now to find C, we observe that if $x = 0$ the area $= 0$; if therefore at the same time $\phi(x) = 0$; $\therefore \; C = 0$,

$$\text{and } ANP = \phi(x), \quad \text{and } ABD = \phi(a);$$

the same result as would have been obtained had we successively put $x = 0$ and $x = a$ in equation (1), and subtracted the former result from the latter.

This process is called integrating between the limits of $x = 0$ and $x = a$.

To take a second instance, let the area $DBCE$ be required where $AC = b$; putting a for x in equation (1),

$$\text{area } ABD = \phi(a) + C,$$

$$\text{and area } ACE = \phi(b) + C;$$

$$\therefore \text{ area } BDEC = \phi(b) - \phi(a).$$

Hence, if the value of an integral $u = \phi(x)$ be required between two values a and b of x, omit the constant, and having put a and b successively for x in $\phi(x)$, subtract $\phi(a)$ from $\phi(b)$.

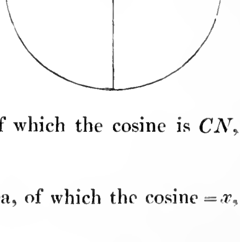
This is called integrating between the limits or values of x, a and b, and the integral so found is called a *definite integral.*

AREAS OF CURVES.

75. To find the areas of curves, or to integrate the function

$$\frac{dA}{dx} = y, \quad \text{or} \quad \frac{dA}{d\theta} = \frac{r^2}{2}$$

Ex. 1. To find the area of the circle.



$$\left.\begin{array}{r} CN = x \\ NP = y \\ CA = a \end{array}\right\}; \quad \therefore\ y = \sqrt{a^2 - x^2};$$

$$\therefore\ A = \textstyle\int_x y = \int_x \sqrt{a^2 - x^2};$$

$$\therefore \text{ area } CBPN = \textstyle\int_x \sqrt{a^2 - x^2}.$$

But $CBPN$ is a circular area, of which the cosine is CN, and radius $= CA$.

Hence $\int_x \sqrt{a^2 - x^2} =$ a circular area, of which the cosine $= x$, and radius $= a$.

Again, let $AN = x$, then $NP = y = \sqrt{2ax - x^2}$;

$$\therefore\ ANP = \textstyle\int_x y = \int_x \sqrt{2ax - x^2}.$$

But ANP is a circular area, of which AN is the versed sine;

$\therefore\ \int_x \sqrt{2ax - x^2}$ = a circular area, of which ver-sin $= x$.

Resuming the expression for $CBPN$, we have

$$CBPN = \textstyle\int_x \sqrt{a^2 - x^2} = \displaystyle\int_x \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} = \int_x \frac{a^2}{\sqrt{a^2 - x^2}} - \int_x \frac{x^2}{\sqrt{a^2 - x^2}}$$

$$= \frac{1}{2}\int \frac{a^2}{\sqrt{a^2 - x^2}} + \frac{x}{2}\sqrt{a^2 - x^2}$$

$$= \frac{a^2}{2} \,.\, \sin^{-1}\frac{x}{a} + \frac{xy}{2}$$

$$= \frac{a^2}{2} \,.\, \sin^{-1}\frac{x}{a} + \Delta\, CPN;$$

$$\therefore\ CBPN - CPN = \text{sector } BCP = \frac{a^2}{2} \,.\, \sin^{-1}\frac{x}{a} = \frac{a \,.\, BP}{2}.$$

Cor. Since $CBPN = \dfrac{a^2}{2} \,.\, \sin^{-1}\dfrac{x}{a} + \dfrac{x\sqrt{a^2 - x^2}}{2}$,

let $x = a$;

$$\therefore \text{ area of the quadrant } ACB = \frac{a^2}{2}\frac{\pi}{2} = \frac{\pi a^2}{4};$$

therefore area of circle $= \pi a^2$.

(2) To find the area of an ellipse.

Here $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$;

$$\therefore\ A = \textstyle\int_x y = \dfrac{b}{a} \,.\, \int_x \sqrt{a^2 - x^2} = \dfrac{b}{a} \,.\, \text{circular area cos} = x + C.$$

But $A = 0$ when $x = 0$; $\therefore\ C = 0$;

$$\therefore\ A = \frac{b}{a} \times \text{circular area } \cos = x, \text{ and rad} = a\,;$$

therefore whole ellipse

$$= \frac{b}{a} \times \text{circle radius } a, = \frac{b}{a}\,\pi a^2 = \pi a b.$$

(3) To find the area of the common parabola.

$$y^2 = 4 m x\,; \quad \therefore\ y = 2\sqrt{m x}.$$

$$\text{area} = \int_x y = 2\int_x \sqrt{m x} = 2\sqrt{m}\,.\,\tfrac{2}{3}\,x^{\frac{3}{2}} + C.$$

And area $= 0$, if $x = 0$; $\therefore\ C = 0$;

$$\therefore\ \text{area} = \frac{4\sqrt{m}}{3}\,x^{\frac{3}{2}} = \tfrac{2}{3}\,2\sqrt{m x}\,.\,x = \tfrac{2}{3}\,y x$$

$$= \tfrac{2}{3} \text{ of circumscribing rectangle.}$$

(4) To find the area of the Witch.

$$y = \frac{2a}{x}\sqrt{2 a x - x^2}\,;$$

$$\therefore\ \text{area} = \int_x y = 2a\int_x \frac{\sqrt{2 a x - x^2}}{x} = 2a\int_x \frac{2a - x}{\sqrt{2 a x - x^2}}$$

$$= 2a\left\{\int_x \frac{a - x}{\sqrt{2 a x - x^2}} + a\int_x \frac{1}{\sqrt{2 a x - x^2}}\right\}$$

$$= 2a\left\{\sqrt{2 a x - x^2} + a \text{ ver-sin}^{-1}\frac{x}{a}\right\} + C.$$

And area $= 0$, if $x = 0$; $\therefore\ C = 0$;

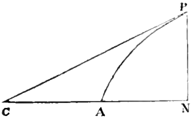
$$\therefore\ \text{area} = 2a\left\{\sqrt{2 a x - x^2} + a \text{ ver-sin}^{-1}\frac{x}{a}\right\}.$$

Let $x = 2a$;

$\therefore$ area $= 2a \times a\,.\,\text{ver-sin}^{-1}(2) = 2\pi a^2 = 2$ area of circle rad $= a$.

(5) Find the area of the hyperbolic sector CAP.

Sector $CAP = \Delta CNP$ – area ANP.



$$\left.\begin{array}{r} \text{Let } CN = x \\ NP = y \\ CA = a \end{array}\right\}; \quad \therefore\ y = \frac{b}{a}\sqrt{x^2 - a^2}.$$

$$ANP = \int_x y = \frac{b}{a}\int_x \sqrt{x^2 - a^2} = \frac{b}{a}\cdot\int_x \frac{x^2 - a^2}{\sqrt{x^2 - a^2}}$$

$$= \frac{b}{a}\cdot\left\{x\sqrt{x^2 - a^2} - \int_x \sqrt{x^2 - a^2} - a^2 .\,\text{h. l.}\,(x + \sqrt{x^2 - a^2})\right\}$$

$$= \frac{b}{a}\cdot\left\{\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}.\,\text{h. l.}\,(x + \sqrt{x^2 - a^2})\right\} + C,$$

$$\text{and } 0 = -\frac{ba}{2}.\,\text{h. l.}\,a + C.$$

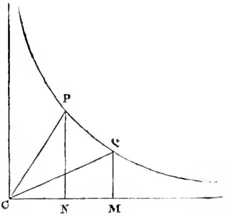
For $ANP = 0$, if $x = a$; therefore, subtracting

$$ANP = \frac{xy}{2} - \frac{ba}{2}.\,\text{h. l.}\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right)$$

$$= \Delta\,CNP - \frac{ba}{2}.\,\text{h. l.}\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right);$$

$$\therefore \text{ sector } CAP = \frac{ba}{2}.\,\text{h. l.}\left(\frac{x}{a} + \frac{y}{b}\right).$$

(6) Find the area of the portion $PNMQ$, PQ being an arc of the rectangular hyperbola.



Here $yx = \frac{a^2}{2}$. Let $CN = \alpha$, and $CM = \beta$,

$$\int_x y = \frac{a^2}{2}\int_x \frac{1}{x} = \frac{a^2}{2}.\,\text{h. l.}\,x + C;$$

$$\therefore\ PQMN = \frac{a^2}{2}\,.\,(\text{h. l. }\beta - \text{h. l. }\alpha) = \frac{a^2}{2}\,.\,\text{h. l. }\left(\frac{\beta}{\alpha}\right),$$

and sector CPQ = area $CPQM - \Delta\, CQM = CNP + PNMQ - CQM$.

$$\text{But } \because \frac{yx}{2} = \frac{a^2}{4}; \quad \therefore\ CNP = CQM;$$

$$\therefore \text{ sector } CPQ = \text{area } PNMQ.$$

(7) Find the area of the cissoid.

$$\text{Here } y^2 = \frac{x^3}{2a - x}; \quad \therefore\ y = \frac{x^{\frac{3}{2}}}{\sqrt{2a - x}},$$

$$\therefore \text{ area} = \textstyle\int_x y = \int_x \frac{x^{\frac{3}{2}}}{\sqrt{2a - x}}$$

$$= -2\sqrt{2a - x}\,.\,x^{\frac{3}{2}} + 3\,.\,\textstyle\int_x x^{\frac{1}{2}}\sqrt{2a - x}$$

$$= -2\sqrt{2a - x}\,.\,x^{\frac{3}{2}} + 3\,.\,\textstyle\int_x \sqrt{2ax - x^2}$$

$$= -2x\sqrt{2ax - x^2} + 3 \text{ circular area ver-sin } x + C,$$

from $x = 0$, to $x = 2a$,

$$\text{area} = 3\,.\,\frac{\pi a^2}{2} = \frac{3\pi}{2}a^2.$$

(8) Find the area of the cycloid;

Measuring from the vertex,

$$\frac{dy}{dx} = \frac{\sqrt{2ax - x^2}}{x},$$

$$\text{area} = \textstyle\int_x y = yx - \int_x x\frac{dy}{dx}$$

$$= yx - \textstyle\int_x \sqrt{2ax - x^2}$$

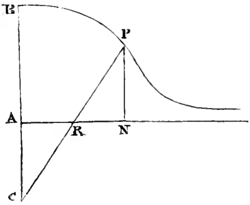
$$= yx - \text{circular area ver-sin} = x + C,$$

from $x = 0$, and $\therefore\ y = 0$, to $x = 2a$, where $y = \pi a$.

Semi-cycloidal area $= 2\pi a^2 - \frac{1}{2}\pi a^2 = \frac{3}{2}\pi a^2$;

$\therefore$ cycloid $= 3\pi a^2 = 3$. area of generating circle.

(9) Area of the conchoid,



$CA = a, \quad AN = x,$

$AB = b, \quad NP = y,$

$\therefore \; xy = (a + y)\sqrt{b^2 - y^2}.$

Now $\dfrac{dA}{dx} = \dfrac{dA}{dy} \cdot \dfrac{dy}{dx}$;

$$\therefore \; \frac{dA}{dy} = \frac{dA}{dx} \cdot \frac{dx}{dy} = y \cdot \frac{dx}{dy},$$

$$\text{and } x = \left(\frac{a}{y} + 1\right)\sqrt{b^2 - y^2},$$

$$\frac{dx}{dy} = -\frac{a}{y^2}\sqrt{b^2 - y^2} - \frac{a + y}{y} \cdot \frac{y}{\sqrt{b^2 - y^2}}$$

$$= -\frac{ab^2 + y^3}{y^2\sqrt{b^2 - y^2}},$$

$$A = \int_y y\frac{dx}{dy} = -\int \frac{ab^2}{y\sqrt{b^2 - y^2}} - \int \frac{y^2}{\sqrt{b^2 - y^2}}$$

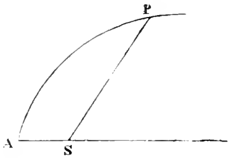
$$= C - ab \,.\, \text{h. l.}\, \frac{y}{b + \sqrt{b^2 - y^2}} + \frac{y}{2}\sqrt{b^2 - y^2} - \frac{b^2}{2}\sin^{-1}\frac{y}{b},$$

and $0 = C - \dfrac{b^2}{2} \cdot \dfrac{\pi}{2}$, since area $= 0$, when $y = b$;

$$\therefore \text{ area} = \frac{b^2}{2} \cdot \left\{\frac{\pi}{2} - \sin^{-1}\frac{y}{b}\right\} - ab \text{ h. l. } \left(\frac{y}{b + \sqrt{b^2 - y^2}}\right) + \frac{y\sqrt{b^2 - y^2}}{2},$$

which is infinite, if $y = 0$.

(10) In the common parabola to find the area ASP.



$SA = a, \quad \angle ASP = \theta,$

$SP = r,$

$$\therefore \; r = \frac{2a}{1+\cos\theta} = \frac{a}{\cos^2\frac{\theta}{2}};$$

$$\therefore \; ASP = \int_\theta \tfrac{1}{2} r^2 = \tfrac{1}{2} \cdot \int_\theta \frac{a^2}{\cos^4\frac{\theta}{2}} = \frac{a^2}{2} \cdot \int_\theta \frac{1}{\cos^2\frac{\theta}{2}} \cdot \sec^2\frac{\theta}{2}$$

$$= \frac{a^2}{2} \cdot \int_\theta \frac{d \cdot \tan\frac{\theta}{2}}{\frac{d\theta}{2}} \cdot \left(1 + \tan^2\frac{\theta}{2}\right)$$

$$= a^2 \left\{\tan\frac{\theta}{2} + \tfrac{1}{3} \cdot \tan^3\frac{\theta}{2}\right\} + C, \text{ and } C = 0,$$

since the area $= 0$, when $\theta = 0$;

$$\therefore \; \text{area } ASP = a^2 \left\{\tan\frac{\theta}{2} + \tfrac{1}{3} \cdot \tan^3\frac{\theta}{2}\right\}.$$

(11) Find the area of a portion of the lemniscata.

Here $r^2 = a^2 \cos 2\theta$;

$$\therefore \; \int_\theta \tfrac{1}{2} r^2 = \frac{a^2}{2} \cdot \int_\theta \cos 2\theta = \frac{a^2}{4} \sin 2\theta + C.$$

There is no area when $\theta = 0$; $\therefore \; C = 0$;

$$\therefore \; \text{area} = \frac{a^2}{4} \sin 2\theta.$$

Let $\theta = 45^0$;

$$\therefore \; \tfrac{1}{4}\text{th of lemniscata} = \frac{a^2}{4};$$

and therefore area of lemniscata $= a^2$.

(12) Find the area of the spiral of which the equation is

$$r = a\theta^n.$$

If A be the area, $\dfrac{dA}{d\theta} = \frac{1}{2}r^2.$

But $\dfrac{dA}{d\theta} = \dfrac{dA}{dr} \cdot \dfrac{dr}{d\theta}$;

$$\therefore \frac{dA}{dr} = \frac{dA}{d\theta} \cdot \frac{d\theta}{dr} = \frac{1}{2} r^2 \frac{d\theta}{dr};$$

$$\therefore A = \tfrac{1}{2} \textstyle\int_r r^2 \dfrac{d\theta}{dr}.$$

Here $\theta = \left(\dfrac{r}{a}\right)^{\frac{1}{n}}$,

$$\frac{d\theta}{dr} = \frac{1}{n a^{\frac{1}{n}}} \cdot r^{\frac{1}{n}-1};$$

$$\therefore \tfrac{1}{2} \textstyle\int_r r^2 \dfrac{d\theta}{dr} = \dfrac{1}{2 n a^{\frac{1}{n}}} \cdot \textstyle\int_r r^{1+\frac{1}{n}},$$

$$\text{or area} = \frac{1}{2 n a^{\frac{1}{n}}} \cdot \frac{n}{2n+1} r^{\frac{2n+1}{n}} + C,$$

and $C = 0$, if area $= 0$, when $r = 0$.

Cor. Let $n = 1$, or the spiral be that of Archimedes;

$$\therefore \text{area} = \frac{r^3}{6a}.$$

But if R be the value of r when $\theta = 2\pi$,

$$a = \frac{R}{2\pi};$$

$$\therefore \text{area} = \frac{2\pi r^3}{6R} = \frac{\pi r^3}{3R}.$$

At the end of the first revolution $r = R$;

therefore area of spiral in first revolution $= \dfrac{\pi R^2}{3}$.

To find the area after two revolutions of the radius vector we must find r when $\theta = 4\pi$.

$$\text{Now } r = \frac{R}{2\pi}\theta = \frac{R}{2\pi}4\pi = 2R.$$

But before $r = 2R$, it will have made two revolutions, and therefore have twice generated the area from $r = 0$ to $r = R$.

Consequently we must subtract the area described in the first revolution from that in the second;

$$\therefore \text{ area} = \frac{\pi \cdot (2R)^3}{3R} - \frac{\pi \cdot R^2}{3} = \frac{7\pi R^2}{3}.$$

And area intercepted between the arcs of the first and second revolution $= \dfrac{7\pi R^2}{3} - \dfrac{\pi R^2}{3} = 2\pi R^2$.

At the n^{th} revolution $r = nR$,

......$(n-1)^{\text{th}}$ ...... $r = (n-1)R$;

$$\therefore \text{ area after } n \text{ revolutions} = \frac{\pi}{3} \cdot \frac{(nR)^3 - \{(n-1)R\}^3}{R}$$

$$= \frac{\pi R^2}{3}\{n^3 - (n-1)^3\}.$$

$$\text{Area after } (n+1) \text{ revolutions} = \frac{\pi R^2}{3}\{(n+1)^3 - n^3\};$$

$\therefore$ space between the arcs after $n+1$ and n revolutions

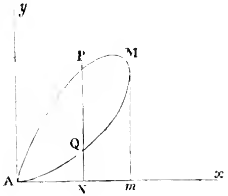
$$= \frac{\pi R^2}{3} \cdot \{(n+1)^3 + (n-1)^3 - 2n^3\} = \frac{\pi R^2}{3} \cdot 6n = 2n\pi R^2$$

$= n$ times the space between the first and second.

(13) Find the area of the curve of which the equation is

$$y^3 - 3axy + x^3 = 0.$$

If the curve be traced there will be found a nodus as $APMQ$, to which the axes Ay and Ax are tangents.



$$\text{Let } y = xz; \quad \therefore\ z = \frac{y}{x} = \tan PAN;$$

$$\therefore\ x^3z^3 - 3ax^2z + x^3 = 0;$$

$$\therefore\ x = \frac{3az}{1+z^3}, \quad \text{and } y = \frac{3az^2}{1+z^3};$$

And since x is $= 0$, for each of the branches APM and AQN, this will happen if $z = \infty$ or $= 0$.

$$\text{Now } \frac{dA}{dz} = \frac{dA}{dx} \cdot \frac{dx}{dz} = y \cdot \frac{dx}{dz},$$

$$\text{and } \frac{dx}{dz} = \frac{3a \cdot \{1 + z^3 - 3z^3\}}{(1+z^3)^2} = 3a \frac{1 - 2z^3}{(1+z^3)^2};$$

$$\therefore\ A = \int_z y \cdot \frac{dx}{dz} = 9a^2 \cdot \int_z \frac{z^2(1-2z^3)}{(1+z^3)^3}$$

$$= 9a^2 \int_z \left\{\frac{z^2[1 - 2(z^3+1) + 2]}{(1+z^3)^3}\right\}$$

$$= 9a^2 \left\{\int_z \frac{3z^2}{(1+z^3)^3} - \int_z \frac{2z^2}{(1+z^3)^2}\right\}$$

$$= 9a^2 \left\{-\frac{1}{2} \cdot \frac{1}{(1+z^3)^2} + \frac{2}{3} \cdot \frac{1}{1+z^3}\right\} + C.$$

Let $z = 0$; $\therefore C = -\frac{3a^2}{2}$, and let $z_1 = \frac{y}{x}$ at M;

$$\therefore \text{ area } AQMm = -\frac{3a^2}{2} + 9a^2\left\{-\frac{1}{2}\cdot\frac{1}{(1+z_1^3)^2} + \frac{2}{3}\frac{1}{1+z_1^3}\right\}.$$

Again, integrating between $z = \infty$ and $z = z_1$ for the branch APM,

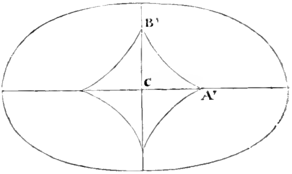
$$\text{area } APMm = 9a^2\left\{-\frac{1}{2}\frac{1}{(1+z_1^3)^2} + \frac{2}{3}\cdot\frac{1}{1+z_1^3}\right\};$$

$$\therefore \text{ the nodus } APMQ = \text{area } APMm - \text{area } AQMm = \frac{3a^2}{2}.$$

(14) Find the area of the evolute of an ellipse

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1,$$

where $CA_1 = \alpha$, and $CB_1 = \beta$.



Let $y = xz$;

$$\therefore x^{\frac{2}{3}} + \left(\frac{a}{\beta}xz\right)^{\frac{2}{3}} = a^{\frac{2}{3}};$$

$$\therefore x^{\frac{2}{3}} = \frac{a^{\frac{2}{3}}}{1+(cz)^{\frac{2}{3}}}, \quad \text{where } c = \frac{a}{\beta};$$

$$\therefore x = \frac{a}{\{1+(cz)^{\frac{2}{3}}\}^{\frac{3}{2}}}, \quad \text{and } y = \frac{az}{\{1+(cz)^{\frac{2}{3}}\}^{\frac{3}{2}}}.$$

For the arc B_1A_1 the limits of x are 0 and a; $\therefore$ of z, they are ∞ and 0.

$$\text{Now } \frac{dA}{dz} = \frac{dA}{dx}\frac{dx}{dz} = y \cdot \frac{dx}{dz},$$

$$\text{and } \frac{dx}{dz} = -\frac{ac^{\frac{2}{3}}z^{-\frac{1}{3}}}{\{1+(cz)^{\frac{2}{3}}\}^{\frac{3}{2}}};$$

$$\therefore A = \int_z y \cdot \frac{dx}{dz} = -a^2 \int_z \frac{(cz)^{\frac{2}{3}}}{\{1+(cz)^{\frac{2}{3}}\}^4}$$

$$= -\frac{3a^2}{c} \cdot \int_v \frac{v^4}{(1+v^2)^4}, \text{ if } cz = v^3,$$

$$\int_v \frac{v^4}{(1+v^2)^4} = -\frac{v^3}{6(1+v^2)^3} + \tfrac{1}{2} \cdot \int_v \frac{v^2}{(1+v^2)^3}.$$

$$\text{and } \int_v \frac{v^2}{(1+v^2)^3} = -\frac{v}{4(1+v^2)^2} + \tfrac{1}{4}\int_v \frac{1}{(1+v^2)^2},$$

$$\text{and } \int_v \frac{1}{(1+v^2)^2} = \tfrac{1}{2}\frac{v}{1+v^2} + \tfrac{1}{2} \cdot \tan^{-1} v;$$

$$\therefore A = -\frac{3a^2}{c}\left\{-\frac{v^3}{6(1+v^2)^3} - \frac{v}{8.(1+v^2)^2} + \frac{1}{2.8}\frac{v}{1+v^2} + \frac{1}{2.8}\tan^{-1} v\right\},$$

from $z = \infty$, that is from $v = \infty$,

to $z = 0$ ...... to ...... $v = 0$.

$$\text{Area } B_1CA_1 = \frac{3a^2}{c} \cdot \frac{1}{16} \cdot \frac{\pi}{2} = \frac{3a\beta \cdot \pi}{32};$$

$$\text{therefore whole area} = 4 \cdot B_1CA_1 = \frac{3}{8}\pi a\beta$$

$$= \frac{3}{8}\pi \frac{a^2-b^2}{a} \times \frac{a^2-b^2}{b}$$

$$= \frac{3}{8}\pi \frac{(a^2-b^2)^2}{ab}.$$

THE LENGTHS OF CURVES.

76. To find the lengths of curves, or to integrate

$$\frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}}, \text{ when } y = f(x).$$

Ex. 1. Find the length of an arc (measured from the vertex) of the common parabola.

$$y^2 = 4mx;$$

$$\therefore \frac{dy}{dx} = \frac{2m}{y}; \quad \therefore \frac{dy^2}{dx^2} = \frac{4m^2}{y^2} = \frac{m}{x}.$$

$$s = \int_x \sqrt{1 + \frac{dy^2}{dx^2}} = \int_x \sqrt{1 + \frac{m}{x}} = \int_x \frac{\sqrt{x+m}}{\sqrt{x}}$$

$$= \int_x \frac{x+m}{\sqrt{x^2+mx}} = \int_x \frac{x + \frac{m}{2}}{\sqrt{x^2+mx}} + \int_x \frac{\frac{m}{2}}{\sqrt{x^2+mx}}$$

$$= \sqrt{x^2+mx} + \frac{m}{2}\text{ h. l. }\left(x + \frac{m}{2} + \sqrt{x^2+mx}\right) + C,$$

$$= \ldots\ldots 0 \ldots + \frac{m}{2}\text{ h. l. }\left(\frac{m}{2}\right)\ldots\ldots\ldots\ldots\ldots\ldots + C.$$

Since $s = 0$, when $x = 0$;

$$\therefore s = \sqrt{x^2+mx} + \frac{m}{2}\text{ h. l. }\left(\frac{2x + m + 2\sqrt{x^2+mx}}{m}\right).$$

Ex. 2. Find when curves included under the general equation $y = ax^{\frac{m}{n}}$ are rectifiable.

$$\frac{dy}{dx} = \frac{m}{n} a x^{\frac{m-n}{n}};$$

$$\therefore s = \int_x \sqrt{1 + \frac{m^2a^2}{n^2} . x^{\frac{2m-2n}{n}}}; \text{ which is integrable.}$$

(1) When $\dfrac{n}{2m-2n}$ is an integer $= r$,

$$\text{or } \frac{m}{n} - 1 = \frac{1}{2r}, \quad \text{or } \frac{m}{n} = \frac{1}{2r} + 1 = \frac{2r+1}{2r}.$$

(2) When $\dfrac{n}{2m-2n} + \frac{1}{2} =$ an integer $= q$,

$$\text{or } \frac{m}{n} - 1 = \frac{1}{2q-1}, \quad \text{or } \frac{m}{n} = \frac{2q}{2q-1}.$$

Let $r = 1, 2, 3$, &c. $q = 1, 2, 3$, &c.;

$$\therefore \frac{m}{n} = \frac{3}{2}, \frac{5}{3}, \frac{7}{6}, \text{ \&c. and } \frac{m}{n} = \frac{2}{1}, \frac{4}{3}, \frac{6}{5}, \text{ \&c.}$$

(3) Let $\dfrac{m}{n} = \dfrac{3}{2}$, or the curve be the semi-cubical parabola;

$$\therefore y = ax^{\frac{3}{2}}, \text{ and } \frac{dy}{dx} = \frac{3a}{2}x^{\frac{1}{2}} = \frac{\sqrt{x}}{\sqrt{c}}, \text{ by putting } \sqrt{c} = \frac{2}{3a};$$

$$\therefore s = \int_x \sqrt{1 + \frac{x}{c}} = \frac{1}{\sqrt{c}} \cdot \int_x \sqrt{x+c} = \frac{1}{\sqrt{c}} \frac{2}{3}(x+c)^{\frac{3}{2}} + C.$$

But if $s = 0$, $x = 0$; $\therefore C = -\dfrac{1}{\sqrt{c}} \dfrac{2}{3} c^{\frac{3}{2}}$;

$$\therefore s = \frac{1}{\sqrt{c}} \cdot \frac{2}{3} \cdot \left\{(x+c)^{\frac{3}{2}} - c^{\frac{3}{2}}\right\}.$$

(4) Find the length of the cycloid.

$$\frac{dy}{dx} = \sqrt{\frac{2a-x}{x}};$$

$$\therefore 1 + \frac{dy^2}{dx^2} = 1 + \frac{2a-x}{x} = \frac{2a}{x};$$

$$\therefore\ s = \int_x \sqrt{\frac{2a}{x}} = \sqrt{2a} \cdot \int_x \frac{1}{\sqrt{x}} = 2\sqrt{2ax} + C,$$

and $C = 0$, since $s = 0$ when $x = 0$;

therefore $s = 2\sqrt{2ax}$ = twice the chord of the arc of the generating circle, corresponding to the arc of the cycloid.

Hence the cycloid is rectifiable.

And if $x = 2a$, $s = 2\sqrt{4a^2} = 4a$,

or the length of the semi-cycloid = twice the diameter of the circle.

(5) Find the length of the arc of an ellipse.

$$y = \frac{b}{a}\sqrt{a^2 - x^2},$$

$$\frac{dy}{dx} = -\frac{b}{a} \cdot \frac{x}{\sqrt{a^2 - x^2}};$$

$$\therefore\ 1 + \frac{dy^2}{dx^2} = 1 + \frac{b^2 x^2}{a^2(a^2 - x^2)} = \frac{a^4 - (a^2 - b^2)x^2}{a^2(a^2 - x^2)} = \frac{a^2 - e^2 x^2}{a^2 - x^2};$$

$$\therefore\ s = \int_x \frac{\sqrt{a^2 - e^2 x^2}}{\sqrt{a^2 - x^2}} = a\int_z \frac{\sqrt{1 - e^2 z^2}}{\sqrt{1 - z^2}}, \quad \text{if } x = za;$$

$\therefore$ expanding $\sqrt{1 - e^2 z^2}$ by the binomial,

$$s = a\int_z \frac{1}{\sqrt{1 - z^2}} \cdot \left\{1 - \tfrac{1}{2} e^2 z^2 - \frac{1 \cdot 1}{2 \cdot 4} e^4 z^4 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \cdot e^6 z^6 - \text{\&c.}\right\},$$

and the integration will depend on $\displaystyle\int_z \frac{z^{2n}}{\sqrt{1 - z^2}}$.

If the quadrant be required, we must integrate between the limits if $x = 0$ and $x = a$, or from $z = 0$ to $z = 1$, but then

$$\int_z \frac{z^{2n}}{\sqrt{1 - z^2}} = \frac{\pi}{2} \cdot \frac{1 \cdot 3 \cdot 5 \ldots\ldots (2n - 1)}{2 \cdot 4 \cdot 6 \ldots\ldots 2n};$$

$$\therefore \int_z \frac{z^2}{\sqrt{1-z^2}} = \frac{\pi}{2}\cdot\frac{1}{2}; \quad \int_z \frac{z^4}{\sqrt{1-z^2}} = \frac{\pi}{2}\cdot\frac{1.3}{2.4};$$

$$\int_z \frac{z^6}{\sqrt{1-z^2}} = \frac{\pi}{2}\cdot\frac{1.3.5}{2.4.6}, \text{ \&c.};$$

$$\therefore \text{ also } \int_z \frac{1}{\sqrt{1-z^2}} = \frac{\pi}{2};$$

therefore elliptic quadrant

$$= \frac{\pi a}{2}\cdot\left\{1 - \frac{1}{2^2}e^2 - \frac{1.3}{2^2.4^2}e^4 - \frac{1.3^2.5}{2^2.4^2.6^2}e^6 - \text{\&c.}\right\},$$

a series which is rapidly convergent when e is a small fraction.

(6) The length of the elliptic quadrant may be found by circular functions. For since x is never $> a$,

$$\text{Let } x = a\cos\theta; \quad \therefore\ y = \frac{b}{a}\sqrt{a^2 - a^2\cos^2\theta} = b\sin\theta.$$

$$\text{Also } \frac{ds}{d\theta} = \sqrt{\frac{dx^2}{d\theta^2} + \frac{dy^2}{d\theta^2}} = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$= \sqrt{a^2 - (a^2 - b^2)\cos^2\theta}$$

$$= a\sqrt{1 - e^2\cos^2\theta}$$

$$= a\left\{1 - \tfrac{1}{2}e^2\cos^2\theta - \frac{1.1}{2.4}e^4\cos^4\theta - \frac{1.1.3}{2.4.6}e^6\cos^6\theta - \text{\&c.}\right\},$$

which must be integrated from $\theta = 0$, to $\theta = \dfrac{\pi}{2}$.

$$\text{Now } \int_\theta \cos^{2n}\theta = +\sin\theta.\cos^{2n-1}\theta + (2n-1).\int_\theta \cos^{2n-2}\theta.\sin^2\theta$$

$$= \frac{\sin\theta\cos^{2n-1}\theta}{2n} + \frac{2n-1}{2n}.\int\cos^{2n-2}\theta,$$

and $\sin\theta\cos^{2n-1}\theta = 0$, when $\theta = 0$, and $\theta = \frac{\pi}{2}$;

$\therefore$ calling $\int_\theta \cos^{2n}\theta = P_{2n}$

$$P_{2n} = \frac{2n-1}{2n} \cdot P_{2n-2},$$

$$P_{2n-2} = \frac{2n-3}{2n-2} \cdot P_{2n-4},$$

$$\vdots$$

$$P_2 = \tfrac{1}{2} \cdot P_0 = \tfrac{1}{2}\theta = \tfrac{1}{2}\frac{\pi}{2} \text{ from } \theta = 0 \text{ to } \theta = \frac{\pi}{2}:$$

$$\therefore P_{2n} = \frac{(2n-1)\cdot(2n-3)\ldots\ldots 3 \cdot 1}{2n\cdot(2n-2)\ldots\ldots 4 \cdot 2} \cdot \frac{\pi}{2};$$

$$\therefore \int_\theta \cos^2\theta = \tfrac{1}{2} \cdot \frac{\pi}{2}; \quad \int_\theta \cos^4\theta = \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2},$$

$$\int_\theta \cos^6\theta = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{\pi}{2} \text{ \&c.};$$

$$\therefore s = \frac{\pi a}{2} \cdot \left\{1 - \frac{1}{2^2} e^2 - \frac{1 \cdot 3}{2^2 \cdot 4^2} e^4 - \frac{1 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6^2} \cdot e^6 - \frac{1 \cdot 3^2 \cdot 5^2 \cdot 7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} e^8 - \text{\&c.}\right\}.$$

(7) Find the length of a hyperbolic arc.

$$y = \frac{b}{a}\sqrt{x^2 - a^2}, \quad \frac{dy}{dx} = \frac{b}{a} \cdot \frac{x}{\sqrt{x^2 - a^2}};$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}} = \sqrt{\frac{(b^2 + a^2)\,x^2 - a^4}{a^2\,(x^2 - a^2)}} = \sqrt{\frac{e^2x^2 - a^2}{x^2 - a^2}};$$

$$\therefore s = \int_x \sqrt{\frac{e^2x^2 - a^2}{x^2 - a^2}} = a \cdot \int_z \sqrt{\frac{e^2z^2 - 1}{z^2 - 1}}, \text{ if } x = az;$$

and as x is to be taken from $x = a$ to $x = \infty$;

therefore z must be taken between $z = 1$, and $z = \infty$;

$$\text{But } a\int_z \sqrt{\frac{e^2z^2-1}{z^2-1}} = ae\int_z z\frac{\sqrt{1-\dfrac{1}{e^2z^2}}}{\sqrt{z^2-1}}$$

$$= a\int_z \frac{ez}{\sqrt{z^2-1}}\cdot\left\{1-\frac{1}{2}\frac{1}{(ez)^2}-\frac{1.1}{2.4}\cdot\frac{1}{(ez)^4}-\frac{1.1.3}{2.4.6}\cdot\frac{1}{(ez)^6}\right.$$

$$\left.-\frac{1.1.3.5}{2.4.6.8}\cdot\frac{1}{(ez)^8}-\&c.\right\},$$

whence, after multiplying every term of the expansion, it appears that every term except the first depends upon the integration of $\int_z \frac{1}{z^m\sqrt{z^2-1}}$, when m is odd.

$$\text{Now } \int \frac{1}{z^m\sqrt{z^2-1}} = \frac{1}{m-1}\cdot\frac{\sqrt{z^2-1}}{z^{m-1}} + \frac{m-2}{m-1}\cdot\int_z \frac{1}{z^{m-2}\sqrt{z^2-1}},$$

and $\frac{\sqrt{z^2-1}}{z^{m-1}}$ vanishes both when $z=1$, and $z=\infty$;

$$\therefore \int_z \frac{1}{z^m\sqrt{z^2-1}} = \frac{m-2}{m-1}\int_z \frac{1}{z^{m-2}\sqrt{z^2-1}}, \quad \left.\begin{array}{r}\text{from } z=1\\ \text{to } z=\infty\end{array}\right\}.$$

$$\text{But } \int \frac{1}{z\sqrt{z^2-1}} = \sec^{-1}z = \frac{\pi}{2} \text{ from } z=1 \text{ to } z=\infty;$$

$$\therefore \int \frac{1}{z^3\sqrt{z^2-1}} = \frac{1}{2}\frac{\pi}{2},$$

$$\text{and } \int \frac{1}{z^5\sqrt{z^2-1}} = \frac{3}{4}\cdot\int \frac{1}{z^3\sqrt{z^2-1}} = \frac{1.3}{2.4}\cdot\frac{\pi}{2};$$

$$\text{and } \int \frac{1}{z^7\sqrt{z^2-1}} = \frac{5}{6}\cdot\int \frac{1}{z^5\sqrt{z^2-1}} = \frac{1.3.5}{2.4.6}\cdot\frac{\pi}{2}.$$

&c. = &c.

$$\therefore\ s = ae\,.\int \frac{z}{\sqrt{z^2-1}} - \frac{\pi a}{2}\,.\left\{\frac{1}{2}\,.\frac{1}{e} + \frac{1\,.\,1}{2^2\,.\,4}\,.\frac{1}{e^3}\right.$$

$$\left. + \frac{1\,.\,1\,.\,3^2}{2^2\,.\,4^2\,.\,6}\,.\frac{1}{e^5} + \frac{1\,.\,1\,.\,3^2\,.\,5^2}{2^2\,.\,4^2\,.\,6^2\,.\,8}\,.\frac{1}{e^7} + \&\text{c.}\right\}.$$

Now the equation to the asymptote is $y = \dfrac{bx}{a}$;

$$\therefore \text{ length of asymptote} = \sqrt{x^2 + \frac{b^2x^2}{a^2}} = x\sqrt{\frac{a^2+b^2}{a^2}} = ex = aez.$$

But $ae\displaystyle\int_z \frac{z}{\sqrt{z^2-1}} = ae\sqrt{z^2-1} = aez$ from $z = 1$ to $z = \infty$.

If therefore A be the length of the asymptote, and H the length of an infinite hyperbolic arc,

$$A - H = \frac{\pi a}{2}\ \left\{\frac{1}{2}\,.\frac{1}{e} + \frac{1\,.\,1}{2^2\,.\,4}\,.\frac{1}{e^3} + \frac{1\,.\,1\,.\,3^2}{2^2\,.\,4^2\,.\,6}\,.\frac{1}{e^5}\right.$$

$$\left. + \frac{1\,.\,1\,.\,3^2\,.\,5^2}{2^2\,.\,4^2\,.\,6^2\,.\,8}\,.\frac{1}{e^7} + \&\text{c.}\right\}.$$

(8) Find the length of an arc of the logarithmic curve.

Here $y = a^x$, and $\dfrac{dy}{dx} = Aa^x = A\,.\,y$,

and $\dfrac{ds}{dy} = \dfrac{ds}{dx}\,.\dfrac{dx}{dy} = \sqrt{1+A^2y^2}\,.\,\dfrac{1}{Ay}$;

$$\therefore\ s = \int_y \frac{\sqrt{1+A^2y^2}}{Ay} + \int_y \frac{Ay}{\sqrt{1+A^2y^2}} + \int_y \frac{1}{Ay\sqrt{1+A^2y^2}}$$

$$= \frac{\sqrt{1+A^2y^2}}{A} + \frac{1}{A}\,\text{h. l.}\ \frac{Ay}{1+\sqrt{1+A^2y^2}} + C,$$

and $s = 0$ if $y = 1$; $\quad\therefore\ C = -\dfrac{\sqrt{1+A^2}}{A} - \dfrac{1}{A}\,\text{h. l.}\,\dfrac{A}{1+\sqrt{1+A^2}}$;

$$\therefore \; s = \frac{1}{A}\left\{\sqrt{1 + A^2 y^2} - \sqrt{1 + A^2} + \text{h. l.}\left[\frac{y\,(1 + \sqrt{1 + A^2})}{1 + \sqrt{1 + A^2 y^2}}\right]\right\}.$$

(9) Find the length of an arc of the Lemniscata.

$$r^2 = a^2 \cos 2\theta, \quad \text{and } s = \int \sqrt{1 + r^2 \frac{d\theta^2}{dr^2}}.$$

$$\text{Now } -r = a^2 \sin 2\theta \, . \frac{d\theta}{dr} = a^2 \sqrt{1 - \frac{r^4}{a^4}} \, . \frac{d\theta}{dr} = \sqrt{a^4 - r^4} \, . \frac{d\theta}{dr};$$

$$\therefore \; \frac{r\,d\theta}{dr} = \frac{-r^2}{\sqrt{a^4 - r^4}};$$

$$\therefore \; 1 + r^2 \frac{d\theta^2}{dr^2} = \frac{a^4}{a^4 - r^4};$$

$$\therefore \; s = \int_r \frac{a^2}{\sqrt{a^4 - r^4}} = a \, . \int_z \frac{1}{\sqrt{1 - z^4}}, \text{ if } r = az$$

$$= a \int_z . \left(\frac{1}{\sqrt{1 - z^2}} \, . \frac{1}{\sqrt{1 + z^2}}\right)$$

$$= a \int_z . \frac{1}{\sqrt{1 - z^2}} \, . \left\{1 - \tfrac{1}{2} z^2 + \frac{1 \, . \, 3}{2 \, . \, 4} z^4 - \frac{1 \, . \, 3 \, . \, 5}{2 \, . \, 4 \, . \, 6} z^6 + \&c.\right\}.$$

Let the integral be required from $\theta = 45^0$ to $\theta = 0$; i. e. from $r = 0$ to $r = a$, or from $z = 0$ to $z = 1$;

$$\therefore \; \int_z \frac{1}{\sqrt{1 - z^2}} = \frac{\pi}{2}, \quad \int_z \frac{z^2}{\sqrt{1 - z^2}} = \tfrac{1}{2} \, . \frac{\pi}{2},$$

$$\int_z \frac{z^4}{\sqrt{1 - z^2}} = \frac{1 \, . \, 3}{2 \, . \, 4} \, . \frac{\pi}{2}, \text{ and } \int_z \frac{z^6}{\sqrt{1 - z^2}} = \frac{1 \, . \, 3 \, . \, 5}{2 \, . \, 4 \, . \, 6} \, \frac{\pi}{2};$$

$$\therefore \; s = \frac{\pi a}{2} \, . \left\{1 - \frac{1}{2^2} + \frac{1^2 \, . \, 3^2}{2^2 \, . \, 4^2} - \frac{1^2 \, . \, 3^2 \, . \, 5^2}{2^2 \, . \, 4^2 \, . \, 6^2} + \frac{1^2 \, . \, 3^2 \, . \, 5^2 \, . \, 7^2}{2^2 \, . \, 4^2 \, . \, 6^2 \, . \, 8^2} - \&c.\right\}.$$

The whole length of the lemniscata $= 4s =$ the circumference of a circle rad $= a$ multiplied into the series between the brackets.

THE VOLUMES AND SURFACES OF SOLIDS OF REVOLUTION.

77. To find the volumes and surfaces of solids, or to integrate the functions

$$\frac{dV}{dx} = \pi y^2, \text{ and } \frac{dS}{dx} = 2\pi y \sqrt{1 + \frac{dy^2}{dx^2}}.$$

Ex. (1) To find the content of a cone with a circular base.

Let a = altitude b = radius of base.

Then if the vertex be the origin and the altitude the axis of x,

$$y = \frac{b}{a} x;$$

$$\therefore\ V = \pi \int_x y^2 = \frac{\pi b^2}{a^2} \int_x x^2 = \frac{\pi b^2}{a^2} \cdot \frac{x^3}{3} + C.$$

And $V = 0$ if $x = 0$; $\therefore\ C = 0$;

$$\therefore\ V = \frac{\pi b^2}{a^2} \cdot \frac{x^3}{3}. \quad \text{Let } x = a;$$

$\therefore$ whole cone $= \dfrac{\pi b^2 a}{3} = \frac{1}{3}$ of a cylinder of the same altitude and on the same base.

(2) Find the volume of the paraboloid.

$y^2 = 4mx$ is the equation to the generating curve;

$\therefore$ volume $= \pi \int_x y^2 = \pi \int_x 4mx = 2\pi m \cdot x^2 + C$, and $C = 0$;

$$\therefore\ \text{volume} = 2\pi m x^2 = \frac{\pi 4mx \cdot x}{2} = \frac{\pi y^2 x}{2}.$$

But $\pi y^2 x$ = volume of a cylinder base $= \pi y^2$ and altitude $= x$;

$\therefore$ paraboloid $= \frac{1}{2}$ circumscribing cylinder.

(3) Find the content or volume of a sphere.

$$y^2 = 2ax - x^2;$$

$$\therefore \text{ content} = \pi \int_x . (2ax - x^2) = \pi\left(ax^2 - \frac{x^3}{3}\right) + C,$$

and content = 0 when $x = 0$; $\therefore C = 0$;

$$\therefore \text{ content of segment} = \pi x^2 \left\{a - \frac{x}{3}\right\}.$$

But $\pi x^2 a$ = content of a cylinder, base = πx^2, and altitude = a, and $\pi x^2 \frac{x}{3}$ = content of a cone of the same base; therefore content of a spherical segment is the difference between the contents of an isosceles cone of the same altitude, and of a cylinder on the same base but altitude equal to the radius.

Let $x = 2a$;

$\therefore$ sphere $= 4\pi a^2 (a - \frac{2}{3} a) = \frac{4}{3} \pi a^3 = \frac{2}{3} \pi a^2 2a = \frac{2}{3}$ of circumscribing cylinder.

(4) Find the content of the prolate spheroid formed by the revolution of an ellipse round its major axis,

$$y^2 = \frac{b^2}{a^2} (a^2 - x^2) \ldots\ldots (1);$$

$$\therefore \text{ solid} = \pi \int_x \frac{b^2}{a^2} (a^2 - x^2) = \pi \frac{b^2}{a^2}\left(a^2 x - \frac{x^3}{3}\right) + C,$$

from $x = -a$, to $x = +a$

$$= \frac{4}{3} \pi b^2 a.$$

If the solid content of the oblate spheroid, which is formed by revolution round the minor axis be required; take the minor axis for the axis of x, and the major axis for that of y.

Then in equation (1) put y for x and x for y, we have

$$x^2 = \frac{b^2}{a^2}(a^2 - y^2); \quad \therefore \ y^2 = \frac{a^2}{b^2}(b^2 - x^2);$$

$$\therefore \ \text{solid} = \pi \cdot \int_x \frac{a^2}{b^2}(b^2 - x^2) = \frac{\pi a^2}{b^2}\left(b^2 x - \frac{x^3}{3}\right),$$

from $x = -b$, to $x = +b$,

$$\therefore \ \text{solid} = \frac{4}{3}\pi a^2 b;$$

therefore prolate spheroid : oblate spheroid

$$:: b^2 a : a^2 b :: b : a.$$

Cor. Hence sphere on major axis : prolate spheroid

$$:: \frac{4}{3}\pi a^3 : \frac{4}{3}\pi b^2 a :: a^2 : b^2,$$

and sphere on minor axis : oblate spheroid :: $b^2 : a^2$.

(5) Content of the solid generated by the conchoid round the axis of x,

$$xy = (a + y)\sqrt{b^2 - y^2},$$

$$\frac{dV}{dy} = \frac{dV}{dx} \cdot \frac{dx}{dy} = \pi y^2 \cdot \frac{dx}{dy},$$

and $$\frac{dx}{dy} = -\frac{a}{y^2}\sqrt{b^2 - y^2} - \frac{(a + y)}{\sqrt{b^2 - y^2}} = -\frac{ab^2 + y^3}{y^2\sqrt{b^2 - y^2}};$$

$$\therefore \ V = -\pi \int_x \left\{\frac{ab^2}{\sqrt{b^2 - y^2}} + \frac{y^3}{\sqrt{b^2 - y^2}}\right\},$$

and $$\int_y \frac{y^3}{\sqrt{b^2 - y^2}} = -y^2\sqrt{b^2 - y^2} + 2\int_y y\sqrt{b^2 - y^2}$$

$$= -y^2\sqrt{b^2 - y^2} - \tfrac{2}{3} \cdot (b^2 - y^2)^{\frac{3}{2}};$$

$$\therefore\ V = C - \pi \cdot \left\{ab^2 \sin^{-1}\frac{y}{b} - y^2\sqrt{b^2 - y^2} - \tfrac{2}{3}(b^2 - y^2)^{\frac{3}{2}}\right\},$$

and $V = 0$, when $y = b$; $\therefore\ 0 = C - \pi\left\{ab^2\frac{\pi}{2}\right\}$; $\therefore\ C = \dfrac{\pi^2 ab^2}{2}$;

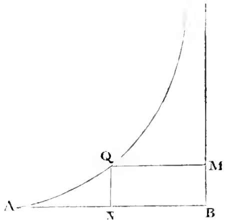
$$\therefore\ V = \frac{\pi^2 ab^2}{2} - \pi\left\{ab^2 \sin^{-1}\frac{y}{b} - \frac{\sqrt{b^2 - y^2}}{3}(y^2 + 2b^2)\right\}.$$

Let $y = 0$;

$$\therefore\ \text{whole volume} = \frac{\pi^2 a b^2}{2} + \frac{2\pi b^3}{3}$$

$$= \pi b^2\left\{\frac{\pi a}{2} + \frac{2b}{3}\right\}.$$

(6) Find the content of the solid generated by the revolution of the cissoid round its asymptote,



$AB = 2a$,

$BM = x$,

$MQ = y$.

Now $NQ^2 = \dfrac{AN^3}{BN}$,

$$\text{or}\ \ x^2 = \frac{(2a - y)^3}{y};$$

$$\therefore\ \text{solid} = \pi \textstyle\int_x y^2 = \pi y^2 x - 2\pi \int_y yx.$$

But $x^2y^2 = y(2a - y)^3$; $\therefore\ xy = \sqrt{y}\,.\,(2a - y)^{\frac{3}{2}}$;

$$\therefore\ \textstyle\int_y xy = \int_y (2a - y)\sqrt{2ay - y^2}$$

$$= \textstyle\int_y (a - y)\sqrt{2ay - y^2} + a\int_y \sqrt{2ay - y^2}$$

$$= \frac{(2ay - y^2)^{\frac{3}{2}}}{3} + a \times \text{circular area, ver-sin} = y;$$

$$\therefore \text{ solid} = \pi \left\{(2ay - y^2)^{\frac{3}{2}} - \tfrac{2}{3}(2ay - y^2)^{\frac{3}{2}}\right.$$
$$\left. - 2a \,.\, \text{circular area, ver-sin} = y + C\right\}$$
$$= \pi \frac{(2ay - y^2)^{\frac{3}{2}}}{3} - 2a\pi \times \text{circular area ver-sin} = y + C.$$

$$\text{Let } y = 2a; \quad \therefore \text{ solid} = 0,$$

$$\text{and therefore } 0 = -2a\pi \frac{\pi a^2}{2} + C; \quad \therefore C = \pi^2 a^3;$$

$$\therefore \text{ solid} = \pi^2 a^3 + \pi(2ay - y^2)^{\frac{3}{2}} - 2a\pi \,.\, \text{circular area ver-sin} = y:$$

therefore let $y = 0$;

therefore whole solid $= \pi^2 a^3$.

(7) Find the solid generated by the revolution of the cycloid round its base.

Make the base the axis of x;

$$\therefore \frac{dy}{dx} = \frac{\sqrt{2ay - y^2}}{y},$$

$$\text{and } \frac{dV}{dx} = \pi y^2 = \frac{dV}{dy} \cdot \frac{dy}{dx} = \frac{dV}{dy} \frac{\sqrt{2ay - y^2}}{y};$$

$$\therefore V = \pi \,.\, \int_y \frac{y^3}{\sqrt{2ay - y^2}}.$$

$$\text{Now } \int_y \frac{y^m}{\sqrt{2ay - y^2}} = -\frac{y^{m-1}\sqrt{2ay - y^2}}{m} + \frac{2m-1}{m} a \int_y \frac{y^{m-1}}{\sqrt{2ay - y^2}};$$

$$\therefore \int_y \frac{y^3}{\sqrt{2ay - y^2}} = -\frac{y^2\sqrt{2ay - y^2}}{3} + \tfrac{5}{3} a \int_y \frac{y^2}{\sqrt{2ay - y^2}},$$

$$\int_y \frac{y^2}{\sqrt{2ay - y^2}} = -\frac{y\sqrt{2ay - y^2}}{y} + \tfrac{3}{2} a \int \frac{y}{\sqrt{2ay - y^2}},$$

$$\int_y \frac{y}{\sqrt{2ay - y^2}} = -\sqrt{2ay - y^2} + a \,.\, \text{ver-sin}^{-1} \frac{y}{a};$$

$$\therefore \int \frac{y^3}{\sqrt{2ay - y^2}} = -\frac{y^2\sqrt{2ay - y^2}}{3} - \frac{5}{2.3}ay\sqrt{2ay - y^2}$$

$$-\frac{3.5}{2.3}a^2\sqrt{2ay - y^2} + \frac{3.5}{2.3}a^3 \text{ ver-sin}^{-1}\frac{y}{a},$$

from $y = 0$ to $y = 2a$;

$$\int \frac{y^3}{\sqrt{2ay - y^2}} = \tfrac{5}{2}a^3.\pi;$$

$$\therefore \quad V = \frac{5\pi^2 a^3}{3}.$$

(8) Find the solid generated by the revolution of the cycloid round its axis.

If V be the volume, $\dfrac{dV}{dx} = \pi y^2$,

$$\text{and } V = \pi \int_x y^2 = \pi \left\{y^2 x - 2\int_x xy \cdot \frac{dy}{dx}\right\},$$

$$\text{and } \frac{dy}{dx} = \frac{\sqrt{2ax - x^2}}{x} \text{ (equation from vertex);}$$

$$\therefore \int_x xy\frac{dy}{dx} = 2\int_x y\sqrt{2ax - x^2}.$$

But if $\theta = \text{ver-sin}^{-1}\dfrac{x}{a}$,

$$y = a(\theta + \sin\theta), \quad \sqrt{2ax - x^2} = a\sin\theta,$$

$$\text{and } x = a(1 - \cos\theta); \quad \therefore \frac{dx}{d\theta} = \sin\theta;$$

$$\therefore \int_x y\sqrt{2ax - x^2} = a^3\int_\theta \sin^2\theta.(\theta + \sin\theta) = a^3\int_\theta(\theta\sin^2\theta + \sin^3\theta).$$

But $\int_\theta \theta.\sin^2\theta = \theta\int(\sin^2\theta) - \int_\theta\int_\theta(\sin^2\theta)$,

$$\text{and } \int_\theta \sin^2\theta = \tfrac{1}{2}\int_\theta(1 - \cos 2\theta) = \frac{\theta}{2} - \frac{\sin 2\theta}{4},$$

$$\text{and } \int_\theta\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) = \frac{\theta^2}{4} + \frac{\cos 2\theta}{8};$$

$$\therefore \int \theta \sin^2\theta = \frac{\theta^2}{2} - \frac{\theta \sin 2\theta}{4} - \frac{\theta^2}{4} - \frac{\cos 2\theta}{8}$$

$$= \frac{\theta^2}{4} - \frac{\theta \sin 2\theta}{4} - \frac{\cos 2\theta}{8} = \frac{\pi^2}{4}, \quad \left.\begin{array}{r}\text{from } \theta = 0\\ \text{to } \theta = \pi\end{array}\right\}$$

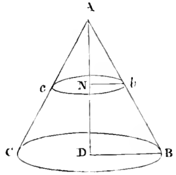
$$\text{and } \int \sin^3\theta = -\frac{\sin^2\theta \cos\theta}{3} - \tfrac{2}{3}\cos\theta$$

$$= \tfrac{4}{3} \text{ from } \theta = 0 \text{ to } \theta = \pi,$$

and $y^2 x = (\pi a)^2 . 2a$ from $x = 0$ to $x = 2a$, or $y = 0$ to $y = \pi a$;

$$\therefore V = \pi\left\{2\pi^2 a^3 - 2a^3\left(\frac{\pi^2}{4} + \frac{4}{8}\right)\right\} = \pi a^3\left\{\frac{3\pi^2}{2} - \frac{8}{3}\right\}.$$

(9) To find the volume of a conical figure, the base of which is bounded by any given curve.



From A draw AD perpendicular to the base, and $= a$.

In AD take $AN = x$, N being a point in a section bc, parallel and similar to the base BC.

Let A = area of the base,

S = area of section bc;

$$\therefore \frac{S}{A} = \frac{bN^2}{BD^2} = \frac{AN^2}{AD^2} = \frac{x^2}{a^2};$$

$$\therefore S = A\frac{x^2}{a^2},$$

$$\text{and } \frac{dV}{dx} = S = A . \frac{x^2}{a^2};$$

$$\therefore V = \frac{A}{a^2}\int_x x^2 = \frac{Ax^3}{3a^2} + C, \quad \text{and } C = 0;$$

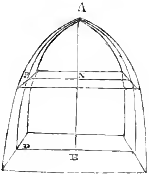
$$\therefore ABC = \frac{Aa^3}{3a^2} = \frac{A . a}{3}$$

$$= \text{base} \times \tfrac{1}{3} \text{ of the altitude.}$$

COR. This proposition is manifestly true for a pyramid of any base.

(10) To find the content of a Groin, a solid of which the sections parallel to the base are squares, and those perpendicular bounded by a given curve.

Let the given curve AD be a quadrant.



$$\left.\begin{array}{l} AN = x \\ NP = y \end{array}\right\}, \quad AB = BD = a;$$

therefore generating area $= (2y)^2 = 4y^2$;

$$\therefore \frac{dV}{dx} = 4y^2 = 4(2ax - x^2);$$

$$\therefore V = 4\left(ax^2 - \frac{x^3}{3}\right), \text{ and from } x = 0 \text{ to } x = a,$$

$$V = \tfrac{8}{3}a^3.$$

To find the surface:

generating surface = perimeter of square $= 8y$;

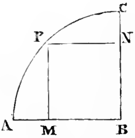
$$\therefore \frac{dS}{dx} = 8y\frac{ds}{dx} = 8\sqrt{a^2 - x^2} \cdot \frac{a}{\sqrt{a^2 - x^2}} = 8a;$$

$$\therefore S = 8ax = 8a^2.$$

And similarly may the content and surface be found, whatever be the curve APD.

Also, if the base be any other figure, of which the area is a function of y as a circle, a parabola, a triangle, &c. and APB be a curve of which the equation is $y = f(x)$, the surface and solid content may be found.

(11) Find the volume of the solid generated by the revolution of a parabolic area round its ordinate.



$$AM = x, \qquad BN = x_1, \qquad AB = a,$$

$$MP = z, \qquad NP = y_1, \qquad BC = b;$$

$$\frac{dV}{dx_1} = \pi y_1{}^2 = \pi (a - x)^2 = \pi \left(a - \frac{y^2}{4m}\right)^2$$

$$= \pi \left(\frac{4ma - y^2}{4m}\right)^2 = \frac{\pi}{(4m)^2} (b^2 - y^2)^2,$$

and $\dfrac{dV}{dx_1} = \dfrac{dV}{dy}$; $\therefore = \dfrac{\pi}{(4m)^2} \{b^4 - 2b^2y^2 + y^4\}$;

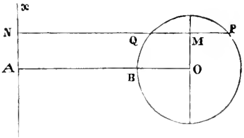
$$\therefore\ V = \frac{\pi}{(4m)^2} \left\{b^4 y - \frac{2b^2y^3}{3} + \frac{y^5}{5}\right\} \text{ from } y = 0 \text{ to } y = b,$$

$$= \frac{\pi b^5}{(4m)^2} \{1 - \tfrac{2}{3} + \tfrac{1}{5}\} = \frac{\pi b^5}{(4m)^2} \cdot \frac{8}{15}.$$

But $b^2 = 4ma$; $\therefore\ \dfrac{1}{(4m)^2} = \dfrac{a^2}{b^4}$;

$$\therefore\ V = \frac{8}{15} \pi a^2 b.$$

(12) Find the volume of the solid generated by the circle BQP which revolves about an axis ANx, in its own plane.



Let $AO = b$, $OB = a$,

$MQ = y$, $OM = x$.

Then surface generated by QP

$$= \pi (NP^2 - NQ^2) = \pi \{(b + y)^2 - (b - y)^2\} = 4\pi b y;$$

$$\therefore\ \frac{dV}{dx} = 4\pi b y;\quad \therefore\ V = 4\pi b \textstyle\int_x y = 4\pi b \dfrac{\pi a^2}{2},$$

or solid $= 2\pi^2 b a^2$.

$$\text{Surface} = 2\pi \cdot \textstyle\int_x (NP + NQ) \cdot \dfrac{ds}{dx} = 4\pi b \cdot \textstyle\int_x \dfrac{ds}{dx} = 4\pi b \cdot \pi a$$

$$= 4\pi^2 b a.$$

(13) The surface of a sphere.

$$y = \sqrt{2ax - x^2}, \text{ and } \frac{dy}{dx} = \frac{a - x}{\sqrt{2ax - x^2}}.$$

$$1 + \frac{dy^2}{dx^2} = 1 + \frac{(a-x)^2}{2ax - x^2} = \frac{a^2}{2ax - x^2} = \frac{a^2}{y^2}.$$

$$\text{Surface} = 2\pi \int_x y \sqrt{1 + \frac{dy^2}{dx^2}} = 2\pi \int_x y \cdot \frac{a}{y} = 2\pi \int_x a = 2\pi a x + C.$$

Surface $= 0$, if $x = 0$; $\therefore C = 0$;

$\therefore$ surface of a segment $= 2\pi a x$;

$\therefore$ surface of sphere $= 2\pi a \cdot 2a = 4\pi a^2$.

(14) Convex surface of a paraboloid.

$$y^2 = 4mx, \quad \frac{dy}{dx} = \frac{2m}{y};$$

$$\therefore 1 + \frac{dy^2}{dx^2} = 1 + \frac{4m^2}{y^2} = 1 + \frac{4m^2}{4mx} = 1 + \frac{m}{x} = \frac{x+m}{x};$$

$$\therefore \text{surface} = \int_x 2\pi y \sqrt{1 + \frac{dy^2}{dx^2}} = 4\pi \sqrt{m} \cdot \int_x \sqrt{x} \sqrt{\frac{x+m}{x}}$$

$$= 4\pi \sqrt{m} \int_x \sqrt{x+m}$$

$$= 4\pi \sqrt{m} \tfrac{2}{3} (x+m)^{\frac{3}{2}} + C,$$

$$0 = 4\pi \sqrt{m} \tfrac{2}{3} m^{\frac{3}{2}} + C;$$

$$\therefore \text{surface} = \frac{8\pi\sqrt{m}}{3} \cdot \left\{ (x+m)^{\frac{3}{2}} - m^{\frac{3}{2}} \right\}.$$

(15) Find the surface generated by the revolution of the cycloid round its base.

$$\text{Here } \frac{dy}{dx} = \frac{\sqrt{2ay - y^2}}{y}; \quad \therefore \sqrt{1 + \frac{dy^2}{dx^2}} = \frac{\sqrt{2ay}}{y};$$

$$\therefore \frac{dS}{dy} = \frac{dS}{dx} \cdot \frac{dx}{dy} = 2\pi y \sqrt{1 + \frac{dy^2}{dx^2}} \cdot \frac{dx}{dy}$$

$$= 2\pi \sqrt{2ay} \cdot \frac{y}{\sqrt{2ay - y^2}};$$

$$\therefore\ S = 2\pi\sqrt{2a}\,.\int_x \frac{y}{\sqrt{2a-y}}$$

$$= 2\pi\sqrt{2a}\,\{-2y\sqrt{2a-y} + 2\int_y \sqrt{2a-y}\}$$

$$= 2\pi\sqrt{2a}\,\{-2y\sqrt{2a-y} - \frac{4}{3}(2a-y)^{\frac{3}{2}}\},$$

from $y = 0$ to $y = 2a$;

$\therefore$ surface generated by semi-cycloid $= 2\pi.\frac{4}{3}(2a)^2 = \frac{32}{3}\pi a^2$.

(16) Find the same when round the axis.

Measuring from the vertex, $\dfrac{dy}{dx} = \sqrt{\dfrac{2a-x}{x}}$.

$$\text{Surface} = 2\pi\int y\frac{ds}{dx} = 2\pi\left\{ys - \int_x s\frac{dy}{dx}\right\},\quad s = 2\sqrt{2ax}$$

$$= 2\pi\left\{2y\sqrt{2ax} - 2\sqrt{2a}\int_x \sqrt{x}\sqrt{\frac{2a-x}{x}}\right\}$$

$$= 4\pi\,\{y\sqrt{2ax} - \sqrt{2a}\int_x\sqrt{2a-x}\}$$

$$= 4\pi\sqrt{2a}\,\{y\sqrt{x} + \frac{2}{3}\{(2a-x)^{\frac{3}{2}}\},$$

from $x = 0$ to $x = 2a$, or $y = 0$ to $y = \pi a$.

$$\text{Surface} = 4\pi\sqrt{2a}\,\{\pi a\sqrt{2a} - \frac{2}{3}.(2a)^{\frac{3}{2}}\}$$

$$= 8\pi a\,\{\pi a - \frac{4}{3}a\}$$

$$= 8\pi a^2.\left\{\pi - \frac{4}{3}\right\}.$$

(17) To find the surface of the prolate spheroid.

$$y = \frac{b}{a}\sqrt{a^2 - x^2},\quad \text{and}\quad 1 + \frac{dy^2}{dx^2} = \frac{a^2 - e^2x^2}{a^2 - x^2},$$

$$\frac{dS}{dx} = 2\pi y\frac{ds}{dx} = 2\pi\frac{b}{a}\sqrt{a^2-x^2}\sqrt{\frac{a^2-e^2x^2}{a^2-x^2}}$$

$$= \frac{2\pi b}{a}\sqrt{a^2 - e^2x^2}$$

$$= 2\pi b.\sqrt{1 - \frac{e^2x^2}{a^2}};$$

$$\therefore \ S = 2\pi b \int_x \sqrt{1 - \frac{e^2 x^2}{a^2}} = \frac{2\pi b a}{e} \cdot \int_x \sqrt{1 - z^2}, \text{ if } z = \frac{ex}{a},$$

$$\int_x \sqrt{1 - z^2} = \int_z \frac{1}{\sqrt{1 - z^2}} - \int_z \frac{z^2}{\sqrt{1 - z^2}}$$

$$= \sin^{-1} z + z\sqrt{1 - z^2} - \int_z \sqrt{1 - z^2}$$

$$= \tfrac{1}{2} \sin^{-1} z + \frac{z}{2}\sqrt{1 - z^2};$$

$$\therefore \ S = \frac{\pi b a}{e} \cdot \left\{\sin^{-1}\left(\frac{ex}{a}\right) + \frac{ex}{a}\sqrt{1 - \frac{e^2 x^2}{a^2}}\right\},$$

from $x = -a$ to $x = +a$.

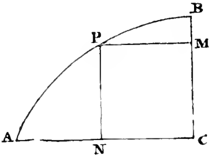
$$\text{Surface} = \frac{2\pi b a}{e} \cdot \left\{\sin^{-1} e + e\sqrt{1 - e^2}\right\}$$

$$= 2\pi a^2 \left\{\sqrt{1 - e^2} \cdot \frac{\sin^{-1} e}{e} + 1 - e^2\right\}.$$

Let $e = 0$, or spheroid become a sphere; $\therefore \dfrac{\sin^{-1} e}{e} = 1$,

and surface $= 2\pi a^2 \{1 + 1\} = 4\pi a^2$.

(18) To find the surface of an oblate spheroid.



$BM = x_1 \quad CN = x$

$MP = y_1 \quad NP = y.$

$$\frac{dS}{dx_1} = 2\pi y_1 \cdot \frac{ds}{dx_1};$$

$$\text{or } \frac{dS}{dy} = 2\pi x \cdot \frac{ds}{dy};$$

$$\therefore \ \frac{dS}{dx} = 2\pi x \frac{ds}{dx} = 2\pi x \sqrt{\frac{a^2 - e^2 x^2}{a^2 - x^2}}.$$

$$\text{Let } \frac{du}{dx} = x\sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}}.$$

$$\text{Make } \sqrt{a^2 - x^2} = z; \quad \therefore \frac{-x}{\sqrt{a^2 - x^2}} = \frac{dz}{dx}$$

$$a^2 - e^2x^2 = a^2 - a^2e^2 + e^2z^2 = b^2 + e^2z^2 = e^2(c^2 + z^2) \text{ if } c = \frac{b}{e};$$

$$\therefore \frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx} = -\frac{dz}{dx} \cdot e\sqrt{c^2 + z^2};$$

$$\therefore u = -e\int_z \sqrt{c^2 + z^2}$$

$$= -\frac{e}{2}\{z\sqrt{c^2 + z^2} + c^2 \text{ h. l. } (z + \sqrt{z^2 + c^2})\}.$$

The integral must be taken from $x = 0$ to $x = a$;

$\therefore z$ must be taken from $z = a$ to $z = 0$.

$$u = \frac{e}{2} \cdot \left\{a\sqrt{c^2 + a^2} + c^2 \text{ h. l. } \frac{a + \sqrt{a^2 + c^2}}{c}\right\};$$

$$\therefore \text{ surface} = 2S = 2\pi e \cdot \left\{a\sqrt{c^2 + a^2} + c^2 \text{ h. l. } \frac{a + \sqrt{a^2 + c^2}}{c}\right\}$$

$$= 2\pi e\left\{\frac{a^2}{e} + \frac{a^2(1 - e^2)}{e^2} \text{ h. l. } \frac{a + \frac{a}{e}}{\frac{a}{e}\sqrt{1 - e^2}}\right\}$$

$$= 2\pi a^2 \cdot \left\{1 + \frac{(1 - e^2)}{2e} \text{ h. l. } \left(\frac{1 + e}{1 - e}\right)\right\}.$$

Let $e = 0$, or let spheroid become a sphere.

$$\text{Then, since } \frac{1}{2e} \text{ h. l. } \left(\frac{1 + e}{1 - e}\right) = 1 \text{ when } e = 0,$$

the surface $= 4\pi a^2$.

CHAPTER VII.

DIFFERENTIAL EQUATIONS.

78. In the integrations which have been performed in the preceding Chapters, the differential coefficient, has been either given as a function of one of the variables, or else in such terms of the two, that by a very evident process, it has been reduced to a function of one only. We now proceed to integrate differentials, when the differential coefficients and the variables x and y are mingled together.

79. Differential equations are divided into classes, dependent upon the *order* and *degree* of the differential coefficient.

Thus an equation involving,

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \quad \&c. \quad \frac{d^ny}{dx^n},$$

is called a differential equation of the n^{th} order and of the first degree, while one containing

$$\frac{dy}{dx}, \quad \left(\frac{dy}{dx}\right)^2, \quad \left(\frac{dy}{dx}\right)^3, \quad \&c. \quad \left(\frac{dy}{dx}\right)^n$$

is said to be of the first order, and of the n^{th} degree: and finally an equation in which are to be found the n^{th} powers of the differential coefficients, and the m^{th} differential coefficient is named an equation of the m^{th} order and the n^{th} degree.

We shall confine ourselves to the more simple classes, beginning with that in which the first power of the first differential coefficient is alone found.

Differential equations of the first order and the first degree.

80. These are included under the formula

$$M + N.\frac{dy}{dx} = 0,$$

where M and N may be any functions of x and y, we shall begin with homogeneous equations.

81. Let $M + N\frac{dy}{dx} = 0$, be a homogeneous equation, in which the sum of the indices of y and x together, is the same in every term.

$$\text{Make } y = xz; \quad \therefore \frac{dy}{dx} = z + x\frac{dz}{dx}.$$

Divide by N and the equation becomes,

$$\frac{M}{N} + \frac{dy}{dx} = 0, \text{ or } \frac{M}{N} + z + x\frac{dz}{dx} = 0.$$

But $\frac{M}{N}$ must be of no dimensions, and will be a function of

$$\frac{y}{x} \text{ or } z; \text{ let } \therefore \frac{M}{N} = f(z);$$

$$\therefore x\frac{dz}{dx} = -\{z + f(z)\};$$

$$\therefore \frac{dx}{xdz} = -\frac{1}{z + f(z)};$$

$$\therefore \text{h. l.}\left(\frac{x}{c}\right) = -\int_z \frac{1}{z + f(z)},$$

the right hand side of the equation may be integrated by the ordinary rules.

We put $x = yz$, or $y = xz$, as may be most convenient, for the solution is more easily effected, when we substitute for that differential coefficient which involves the fewest terms.

We may here remark that the notation $\int X dx$, and $\int_x X$, both of which will be met with, mean the same thing.

Ex. 1. Let $x + y = (x - y)\dfrac{dy}{dx}$.

Here make $y = xz$; $\therefore \dfrac{dy}{dx} = z + x\dfrac{dz}{dx}$;

$$\therefore z + x\frac{dz}{dx} = \frac{x+y}{x-y} = \frac{1+z}{1-z}.$$

$$\therefore x\frac{dz}{dx} = \frac{1+z^2}{1-z};$$

$$\therefore \frac{dx}{x\,dz} = \frac{1-z}{1+z^2} = \frac{1}{1+z^2} - \frac{z}{1+z^2};$$

$$\therefore \log\left(\frac{x}{c}\right) = \tan^{-1} z - \log\sqrt{1+z^2};$$

$$\therefore \log\left(\frac{x}{c}\sqrt{1+z^2}\right), \text{ or } \log\frac{\sqrt{x^2+y^2}}{c} = \tan^{-1}\frac{y}{x}.$$

Ex. 2. Find the curve in which the subtangent is equal to the sum of the abscissa and ordinate.

Here $y\dfrac{dx}{dy} = x + y$; and let $x = yz$;

$$\therefore \frac{dx}{dy} = z + y\frac{dz}{dy} = \frac{x+y}{y} = z + 1;$$

$$\therefore \frac{dy}{y\,dz} = 1; \quad \therefore \log\left(\frac{y}{c}\right) = z = \frac{x}{y}.$$

Ex. 3. Find the curve in which the subnormal $= y - x$,

$$y\frac{dy}{dx} = y - x; \quad \therefore \frac{dy}{dx} = 1 - \frac{x}{y}.$$

Let $y = xz$; $\quad \therefore z + x\dfrac{dz}{dx} = 1 - \dfrac{1}{z} = \dfrac{z-1}{z}$;

$$\therefore \quad \frac{dx}{x\,dz} = \frac{-z}{z^2 - z + 1};$$

$$\therefore \quad \log\left(\frac{\sqrt{y^2 - yx + x^2}}{c}\right) = \frac{1}{\sqrt{3}} \cot^{-1}\left(\frac{2y - x}{x\sqrt{3}}\right).$$

Ex. 4. Find the curve in which the distance from the origin to a point in the curve equals the subtangent.

$$\text{Here } AP = NT, \quad \text{or } \sqrt{y^2 + x^2} = y\frac{dx}{dy}.$$

$$\text{Make } x = yz; \quad \therefore \; z + y\,.\frac{dz}{dy} = \frac{\sqrt{x^2 + y^2}}{y} = \sqrt{1 + z^2};$$

$$\therefore \quad \frac{dy}{y\,dz} = \frac{1}{\sqrt{1 + z^2} - z} = \sqrt{1 + z^2} + z;$$

$$\text{whence } \log\left\{\frac{y^3}{c^2\left(x + \sqrt{x^2 + y^2}\right)}\right\} = \frac{x}{y^2}\left(x + \sqrt{x^2 + y^2}\right).$$

Ex. 5. $(\sqrt{x} - \sqrt{y}) = \sqrt{y}\,.\dfrac{dx}{dy}$. Make $x = yz$.

Ex. 6. $x\dfrac{dy}{dx} - y = \sqrt{x^2 + y^2}$. Make $y = xz$.

Then $x^2 = c^2 + 2cy$.

82. The equation $(a + bx + cy)\,dx + (a_1 + b_1 x + c_1 y)\,dy = 0$ can be rendered homogeneous by making

$$v = a + bx + cy, \quad \text{and } z = a_1 + b_1 x + c_1 y;$$

$$\therefore \; dv = b\,dx + c\,dy, \quad dz = b_1\,dx + c_1\,dy;$$

$$\therefore \; c_1 dv - c\,dz = (bc_1 - b_1 c)\,dx,$$

$$b\,dz - b_1 dv = (bc_1 - b_1 c)\,dy;$$

whence by substitution the equation becomes

$$v(c_1 dv - c dz) + z(b dz - b_1 dv) = 0,$$

$$\text{or } \ (vc_1 - b_1 z) dv + (bz - cv) dz = 0,$$

which is a homogeneous equation.

Cor. This method is inapplicable when $bc_1 = b_1 c$; but since then $c_1 = \dfrac{b_1 c}{b}$, the equation becomes

$$(a + bx + cy) dx + \left(a_1 + b_1 x + b_1 \frac{cx}{b}\right) dy = 0,$$

$$\text{i.e. } (a + bx + cy) dx + \left\{a_1 + \frac{b_1}{b}(bx + cy)\right\} dy = 0,$$

an equation in which the variables may be separated by making $bx + cy = z$; $\therefore\ dx = \dfrac{dz - c dy}{b}$;

$$\therefore\ (a + z)\frac{dz - cdy}{b} + \left(a_1 + \frac{b_1 z}{b}\right) dy = 0;$$

$$\therefore\ (a + z) dz - (ca + cz - a_1 b - b_1 z) dy = 0;$$

$$\therefore\ \frac{dy}{dz} = \frac{(a + z)}{ca - a_1 b + (c - b_1) z} = \frac{(a + z)}{\alpha + \beta z},$$

where $\alpha = ca - a_1 b$ and $\beta = c - b_1$, the integral of which may be readily found.

83. To integrate the *linear* equation, (so called since the first power of y is alone involved).

$$\frac{dy}{dx} + Py = Q,$$

in which P and Q are functions of x.

$$\text{Since } \frac{d}{dx}(ye^{\int_x P}) = \frac{dy}{dx} e^{\int_x P} + e^{\int_x P} . Py$$

$$= e^{\int_x P}\left\{\frac{dy}{dx} + Py\right\}.$$

It is obvious that if we multiply both sides of the equation by $e^{\int_x P}$, the left hand side will be a complete differential, and the right hand a function of x alone; both sides may therefore be integrated.

Multiply therefore by $e^{\int_x P}$.

$$\therefore\ e^{\int_x P}\left\{\frac{dy}{dx} + Py\right\} = e^{\int_x P} . Q;$$

$$\therefore\ \frac{d}{dx}\left(y e^{\int_x P}\right) = e^{\int_x P} . Q;$$

$$\therefore\ y e^{\int_x P} = C + \int e^{\int_x P} . Q;$$

$$\therefore\ y = Ce^{-\int_x P} + e^{-\int_x P}\int e^{\int_x P} . Q.$$

Ex. 1. Let $\dfrac{dy}{dx} + y = ax^3$.

Here $P = 1$, $\int_x P = x$; $\therefore\ e^{\int_x P} = e^x$, $Q = ax^3$;

$$\therefore\ ye^x = C + a\int e^x . x^3 = C + ae^x\{x^3 - 3x^2 + 6x - 6\};$$

$$\therefore\ y = Ce^{-x} + a\{x^3 - 3x^2 + 6x - 6\}.$$

Ex. 2. $(1 + x^2)\dfrac{dy}{dx} - yx = a$;

$$\text{or } \frac{dy}{dx} - y\frac{x}{1 + x^2} = \frac{a}{1 + x^2}.$$

Here $P = -\dfrac{x}{1 + x^2}$; $\int_x P = \log\dfrac{1}{\sqrt{1 + x^2}}$; $e^{\int_x P} = \dfrac{1}{\sqrt{1 + x^2}}$.

$$\therefore\ y \times \frac{1}{\sqrt{1 + x^2}} = a\int_x \frac{1}{\sqrt{1 + x^2}} \times \frac{1}{1 + x^2}$$

$$= a\int_x \frac{1}{(1 + x^2)^{\frac{3}{2}}} = \frac{ax}{\sqrt{1 + x^2}} + c;$$

$$\therefore\ y = ax + c\sqrt{1 + x^2}.$$

84. The equation $y^{m-1}\dfrac{dy}{dx} + Py^m = Qy^n$ may be reduced to the preceding form, in the following manner.

Divide by y^n.

$$\therefore\ y^{m-n-1}\frac{dy}{dx} + Py^{m-n} = Q.$$

Let $y^{m-n} = (m-n)z$, $\quad\therefore\ y^{m-n-1}\dfrac{dy}{dx} = \dfrac{dz}{dx}$.

$$\therefore\ \frac{dz}{dx} + (m-n)Pz = Q;$$

which is of the required form.

Ex. $\quad v\dfrac{dv}{ds} - \dfrac{hv^2}{s} = -\dfrac{m}{s^2}$. (*Whewell's Dynamics*, p. 182.)

$$\text{Let } v^2 = 2z;\quad \therefore\ v\frac{dv}{ds} = \frac{dz}{ds};$$

$$\therefore\ \frac{dz}{ds} - \frac{2hz}{s} = -\frac{m}{s^2}.$$

Here $P = -\dfrac{2h}{s}$; $\therefore\ \int_s P = -2h\log(s) = \log\dfrac{1}{s^{2h}}$; $\therefore\ e^{\int_s P} = \dfrac{1}{s^{2h}}$.

$$\therefore\ zs^{-2h} = -m\int_s s^{-(2h+2)} = c + \frac{ms^{-(2h+1)}}{2h+1};$$

$$\therefore\ z = \frac{v^2}{2} = cs^{2h} + \frac{m}{(2h+1)s}.$$

Integration of exact differentials. The method of finding a factor which will render a function integrable.

85. The equation $Mdx + Ndy = 0$ is not always the result of the differentiation of $f(xy) = c$: for after the differentiation its terms may have been divided by some common factor, or the equation may have arisen from the elimination of an arbitrary constant between the primitive equation and its derivative.

But whenever $Mdx + Ndy = 0$ is the complete differential of a function of two variables, the condition $\frac{d^2u}{dxdy} = \frac{d^2u}{dydx}$ is fulfilled, or since $M = \frac{du}{dx}$ and $N = \frac{du}{dy}$;

$$\therefore \quad \frac{dM}{dy} = \frac{d^2u}{dxdy} = \frac{dN}{dx}.$$

Hence as it is necessary that every equation $Mdx+Ndy=0$ which is a complete differential should fulfil this condition, we have conversely a method by which we may find whether any equation is or is not a complete differential; and since then $\frac{du}{dx} = M$, and $\frac{du}{dy} = N$, we can by integrating these partial differential equations, find the integral.

86. For since $M = \frac{du}{dx}$, M is the partial differential coefficient of u, with regard to x, considering x alone to vary, and its integral will give all the terms in which x is to be found: let the integration be performed. Then

$$u = \int_x M + Y.$$

Here instead of adding a constant C, we have put Y, for as y has been supposed not to vary, the constant will include those terms of the original equation, which are functions of y alone. Next to determine Y: differentiate with regard to y;

$$\therefore \quad \frac{du}{dy} = \frac{d\int_x M}{dy} + \frac{dY}{dy}.$$

$$\text{But } \frac{du}{dy} = N, \quad \therefore \quad \frac{dY}{dy} = N - \frac{d\int_x M}{dy};$$

$$\therefore \quad Y = \int_y \left(N - \frac{d\int_x M}{dy}\right) + C;$$

$$\therefore \quad u = \int_x M + \int_y \left(N - \frac{d\int_x M}{dy}\right) + C.$$

87. Since Y ought to be a function of y only,

$$\int_y \left(N - \frac{d\int_x M}{dy}\right)$$

should be independent of x. To prove this, let $y + \delta y$ be put for y in $\int_x M$, when we have

$$\int_x \left(M + \frac{dM}{dy}\delta y + \&c.\right) = \int_x M + \delta y \int_x \frac{dM}{dy} + \&c.$$

δ is removed from beneath the sign of integration since $\int_x$ refers to the variation of x only.

$$\text{Hence } \frac{d\int_x M}{dy} = \int_x \frac{dM}{dy};$$

$$\therefore\ Y = \int_y \left(N - \int_x \frac{dM}{dy}\right) + C;$$

$$\therefore\ \frac{dY}{dy} = N - \int_x \frac{dM}{dy};$$

Now differentiate with regard to x;

$$\therefore\ \frac{d^2 Y}{dx\,dy} = \frac{dN}{dx} - \frac{dM}{dy} = 0;$$

or $Y = \int_y \left(N - \frac{d\int_x M}{dy}\right)$ contains y only.

We may remark that had the partial differential coefficient N or $\frac{du}{dy}$ been first integrated, the same result would be obtained; and in the application of the theory, that differential must be chosen which appears most likely to facilitate the solution of the equation.

Ex. 1. Let $du = \frac{2dx}{\sqrt{x^2 - y^2}} - \frac{2x\,dy}{y\sqrt{x^2 - y^2}}$.

Here $M = \dfrac{2}{\sqrt{x^2 - y^2}}$; $N = \dfrac{-2x}{y\sqrt{x^2 - y^2}}$

$$\frac{dM}{dy} = \frac{2y}{(x^2 - y^2)^{\frac{3}{2}}}, \quad \frac{dN}{dx} = \frac{-2}{y}\left(\frac{-y^2}{(x^2 - y^2)^{\frac{3}{2}}}\right) = \frac{2y}{(x^2 - y^2)^{\frac{3}{2}}};$$

$$\therefore \ u = \textstyle\int_x M + Y = 2\int_x \sqrt{x^2 - y^2} + Y = 2\log(x + \sqrt{x^2 - y^2}) + Y,$$

$$\frac{du}{dy} = \frac{-2y}{(x + \sqrt{x^2 - y^2})\sqrt{x^2 - y^2}} + \frac{dY}{dy} = \frac{-2x}{y\sqrt{x^2 - y^2}};$$

$$\therefore \ \frac{dY}{dy} = \frac{2}{\sqrt{x^2 - y^2}}\left\{\frac{y}{x + \sqrt{x^2 - y^2}} - \frac{x}{y}\right\}$$

$$= \frac{-2}{\sqrt{x^2 - y^2}}\left\{\frac{x^2 - y^2 + x\sqrt{x^2 - y^2}}{y(x + \sqrt{x^2 - y^2})}\right\} = \frac{-2}{y};$$

$$\therefore \ Y = C - 2\log y;$$

$$\therefore \ u = \log\left(\frac{x + \sqrt{x^2 - y^2}}{y}\right)^2 + C.$$

Ex. 2. Let $du = \dfrac{a(x\,dx + y\,dy)}{\sqrt{x^2 + y^2}} + \dfrac{y\,dx - x\,dy}{x^2 + y^2} + 3by^2\,dy = 0.$

$$M = \frac{ax}{\sqrt{x^2 + y^2}} + \frac{y}{x^2 + y^2}; \quad N = \frac{ay}{\sqrt{x^2 + y^2}} - \frac{x}{x^2 + y^2} + 3by^2.$$

Here $\dfrac{dM}{dy} = \dfrac{-ay}{(x^2 + y^2)^{\frac{3}{2}}} + \dfrac{x^2 - y^2}{(x^2 + y^2)^2} = \dfrac{dN}{dx}$;

$$\therefore \ u = \textstyle\int_x M + Y = a\sqrt{x^2 + y^2} + \tan^{-1}\dfrac{x}{y} + Y,$$

$$\frac{du}{dy} = \frac{ay}{\sqrt{x^2 + y^2}} - \frac{x}{y^2 + x^2} + \frac{dY}{dy}; \quad \therefore \ \frac{dY}{dy} = 3by^2;$$

$$\therefore \ Y = by^3 + C \text{ and } u = a\sqrt{x^2 + y^2} + \tan^{-1}\frac{x}{y} + by^3 + C.$$

Ex. 3. $$\int \frac{x\,dy - y\,dx}{x^2 + y^2} = \tan^{-1}\frac{x}{y} + C;$$

this may be derived from the preceding example by making $a = 0$ and $b = 0$.

88. When the equation $M\,dx + N\,dy = 0$ does not fulfil the condition $\frac{dM}{dy} = \frac{dN}{dx}$; a property which is termed the *criterion of integrability*, it is no longer a complete differential, some factor having disappeared from it. Could however the the factor be restored, every equation of this class might be integrated by the same process: but there is great difficulty in finding this factor; in most cases the differential equation, by which it is to be determined, is more complicated than the original one.

Thus suppose z to be the factor, then $Mz\,dx + Nz\,dy = 0$ is a complete differential, and therefore

$$\frac{d\,(Mz)}{dy} = \frac{d\,(Nz)}{dx};$$

$$\therefore\ z\frac{dM}{dy} + M\frac{dz}{dy} = z\frac{dN}{dx} + N\frac{dz}{dx};$$

whence z is to be found, a problem in most cases impracticable.

It may be determined when z contains only one variable as x, for then $\frac{dz}{dy} = 0$, and then

$$\frac{dz}{z\,dx} = \frac{1}{N}\left(\frac{dM}{dy} - \frac{dN}{dx}\right).$$

The right hand side must be a function of x only, which is the case in the linear equation, for $N = 1$, and M contains only the first power of y; therefore integrating

$$\log\frac{z}{c} = X; \ \therefore\ z = ce^x.$$

89. But to find *a priori* the multiplier which will make the equation $dy + (Py - Q)\,dx = 0$ an exact differential.

Let z be the multiplier: multiply by it;

$$\therefore\ z\,dy + z(Py - Q)\,dx = N\,dy + M\,dx;$$

$$\therefore\ \frac{dN}{dx} = \frac{dz}{dx};\quad \frac{dM}{dy} = (Py - Q)\frac{dz}{dy} + Pz;$$

$$\therefore\ \frac{dz}{dx} = (Py - Q)\frac{dz}{dy} + Pz;$$

$$\therefore\ \frac{dz}{dx}dx = (Py - Q)\,dx\frac{dz}{dy} + Pz\,dx$$

$$= -\frac{dz}{dy}dy + Pz\,dx;\quad \text{since } (Py - Q)\,dx = -dy;$$

$$\therefore\ \frac{dz}{dx}dx + \frac{dz}{dy}dy = dz = Pz\,dx;$$

$$\text{i.e. } \frac{1}{z}\frac{dz}{dx} = P;$$

$$\therefore\ z = e^{\int_x P};$$

which justifies the assumption made, when the linear equation was solved in a preceding article.

90. We shall now add some few problems which illustrate the solution of differential equations.

Find the curve which cuts any number of curves of a given species at a given angle.

Let y and x be the co-ordinates of the curve of given species,

y_1 and x_1 those of the required curve,

m = tangent of given angle.

$$\text{Then } \tan^{-1} m = \tan^{-1}\frac{dy}{dx} - \tan^{-1}\frac{dy_1}{dx_1};$$

$$\therefore\ m = \frac{\dfrac{dy}{dx} - \dfrac{dy_1}{dx_1}}{1 + \dfrac{dy}{dx}\cdot\dfrac{dy_1}{dx_1}};$$

and $\dfrac{dy}{dx}$ may be found from the given curve, and is a function of x and y, or $\phi(xy)$, and since at the point of intersection the co-ordinates of both curves are the same, we may for x_1 and y_1 put x and y; and then the equation to the required curve is

$$m\left\{1 + \phi(xy)\frac{dy}{dx}\right\} = \phi(xy) - \frac{dy}{dx},$$

which is of the first order and degree.

Cor. If the required curve cut the given curves at right angles,

$$\text{then } m = \frac{1}{0};\quad \therefore\ 1 + \phi(xy)\frac{dy}{dx} = 0;\quad \therefore\ \frac{dy}{dx} = -\frac{1}{\phi(xy)},$$

which is the equation to the *Orthogonal Trajectory*.

Ex. 1. Find the curve which will cut all the parabolas that have a common vertex and axis at right angles.

Let $y^2 = 2mx$ be equation to one of the parabolas;

$$\therefore\ \phi(xy) = \frac{m}{y} = \frac{y}{2x};$$

$$\therefore\ \frac{dy}{dx} = -\frac{2x}{y};\quad \therefore\ \frac{y^2}{2} = (c^2 - x^2),$$

the equation to an ellipse of which the centre is the common vertex of the parabolas, and the major axis is perpendicular to the common axis, the ratio of the axes being $\sqrt{2} : 1$; c being indeterminate shews that any ellipse of which the axes are in the given ratio will cut the parabolas at right angles.

(2) Find the curve which will cut at right angles all the ellipses that have a common centre, coincident major axes and the ratio of their axes constant.

Let $y^2 = n(a^2 - x^2)$ be the equation to one of the ellipses in which $\sqrt{n} = \dfrac{b}{a}$;

$$\therefore \frac{dy}{dx} = -\frac{\sqrt{n}\,x}{\sqrt{a^2 - x^2}} = -n\frac{x}{y} = -\frac{dx}{dy};$$

$$\therefore n\frac{dy}{y} = \frac{dx}{x};$$

$$\therefore n\log\left(\frac{y}{b}\right) = \log\left(\frac{x}{c}\right);$$

$$\therefore y^n = \frac{b^n}{c}x,$$

the equation to a parabola, of which the vertex is in the common centre of the ellipses.

If $n = 2$, $y^2 = \dfrac{b^2}{c}x$, the common parabola, this case is obviously the converse of the preceding problem.

(3) Find the curve which intersects at an angle of 45^0, all the straight lines drawn from the origin to meet it.

Let $y = ax$ be one of the lines;

$$\therefore \phi(xy) = a = \frac{y}{x}, \text{ and } m = 1;$$

$$\therefore 1 + \frac{y}{x}\frac{dy}{dx} = \frac{y}{x} - \frac{dy}{dx},$$

a homogeneous equation,

$$\text{whence } \log\left(\frac{\sqrt{x^2 + y^2}}{c}\right) = -\tan^{-1}\left(\frac{y}{x}\right).$$

Let $y = r\sin\theta$, $x = r\cos(\pi - \theta) = -r\cos\theta$;

$$\therefore\ r = \sqrt{x^2 + y^2}, \text{ and } \frac{y}{x} = -\tan\theta;$$

$$\therefore\ \log\left(\frac{r}{c}\right) = \theta; \quad \therefore\ r = ce^{\theta},$$

the equation to the logarithmic spiral.

91. To integrate, *Riccati's* equation, so called from its proposer,

$$\frac{dy}{dx} + by^2 = ax^m.$$

(1) If $m = 0$. Then $\frac{dy}{dx} = a - by^2$ which is easily integrable.

(2) If m be not $= 0$. We must proceed as follows.

CASE 1. Let $y = \frac{1}{bx} + \frac{z}{x^2}$;

$$\therefore\ dy = -\frac{dx}{bx^2} + \frac{dz}{x^2} - \frac{2z\,dx}{x^3},$$

$$by^2\,dx = \frac{dx}{x^2} + \frac{bz^2}{x^4}\,dx + \frac{2z\,dx}{x^3};$$

$$\therefore\ dy + by^2\,dx = \frac{dz}{x^2} + \frac{bz^2}{x^4}\,dx = ax^m\,dx;$$

$$\therefore\ \frac{dz}{dx} + \frac{bz^2}{x^2} = ax^{m+2},$$

which is homogeneous if $m + 2 = 0$, or $m = -2$, and the variables may be separated if $m = -4$; for then we have,

$$\frac{dz}{dx} + \frac{bz^2}{x^2} = \frac{a}{x^2}; \quad \therefore\ \frac{dz}{bz^2 - a} + \frac{dx}{x^2} = 0.$$

If m have any other value, make

$$z = \frac{1}{y_1}; \quad x^{m+3} = x_1; \quad \therefore\ dz = -\frac{dy_1}{{y_1}^2}; \quad x^{m+2}\,dx = \frac{dx_1}{m+3}.$$

$$\text{Also } \frac{1}{x} = x_1^{-\frac{1}{m+3}}; \quad \therefore \frac{dx}{x^2} = \frac{dx_1}{m+3} x_1^{-\frac{m+4}{m+3}},$$

$$\text{whence } -\frac{dy_1}{y_1^2} + \frac{b}{(m+3)y_1^2} x_1^{-\frac{m+4}{m+3}} dx_1 = \frac{a}{m+3} dx_1.$$

$$\text{Let } \frac{b}{m+3} = a_1; \quad \frac{a}{m+3} = b_1; \quad -\frac{m+4}{m+3} = m_1.$$

$$\text{Then } dy_1 + b_1 y_1^2 dx_1 = a_1 x_1^{m_1} dx_1,$$

which is of the same form as the original equation, and may be made homogeneous if $m_1 = -2$, and the variables may be separated by the preceding process if $m_1 = -4$.

By continuing the same methods it is evident that we shall have a similar equation,

$$dy_2 + b_2 y_2^2 dx_2 = a_2 x_2^{m_2} dx_2,$$

where $m_2 = -\dfrac{m_1+4}{m_1+3}$; and b_2, a_2 are derived from b_1 and a_1 as these were from b and a; which equation will be integrable if $m_2 = -4$.

And hence if among the series of indices

$$-m, \quad -\frac{m+4}{m+3}, \quad -\frac{m_1+4}{m_1+3}, \quad -\frac{m_2+4}{m_2+3}, \text{ \&c.}$$

any one becomes $= -4$, the equation is integrable. And by successively putting these indices $= -4$, we find that the values of m are, -4, $-\frac{8}{3}$, $-\frac{12}{5}$, $-\frac{16}{7}$, $-\frac{20}{9}$, &c., which are included under the general form $-\dfrac{4n}{2n-1}$, n being any whole number.

CASE 2. Make in the original equation $y = \dfrac{1}{y_1}$;

$$\therefore -\frac{dy_1}{y_1^2} + \frac{b}{y_1^2} dx = a x^m dx;$$

$$\therefore dy_1 + a y_1^2 x^m dx = b dx.$$

Let $x^{m+1} = x_1$; $\therefore x = x_1^{\frac{1}{m+1}}$; $\therefore dx = \dfrac{1}{m+1} x_1^{-\frac{m}{m+1}} dx.$

$$\text{And } dy_1 + \frac{a}{m+1} y_1^2 \, dx_1 = \frac{b}{m+1} x_1^{-\frac{m}{m+1}} dx_1,$$

or putting $\dfrac{a}{m+1} = b_1$, $\dfrac{b}{m+1} = a_1$, and $-\dfrac{m}{m+1} = m_1$,

$$dy_1 + b_1 y_1^2 dx_1 = a_1 x_1^{m_1} dx_1,$$

which may be integrated by the former method if $m_1 = \dfrac{-4n}{2n-1}$; i.e. if $\dfrac{m}{m+1} = \dfrac{4n}{2n-1}$, whence $m = \dfrac{-4n}{2n+1}$. Hence Riccati's equation is integrable when m is of the form $\dfrac{-4n}{2n \mp 1}$. The first case belonging to the upper and the second to the lower sign.

Ex. 1. Integrate $dy + y^2 dx = \dfrac{a^2 dx}{x^{\frac{4}{3}}}$ (*Peacock's Examples*).

Here $-\frac{4}{3}$ is of the form $-\dfrac{4n}{2n+1}$;

$\therefore$ let $y = \dfrac{1}{y_1}$, and let $x^{m+1} = x^{-\frac{4}{3}+1} = x^{-\frac{1}{3}} = x_1$;

$$\therefore x = x_1^{-3}; \quad dx = -3x_1^{-4} dx_1; \quad x^{\frac{4}{3}} = x_1^{-4};$$

$$\therefore -\frac{dy_1}{y_1^2} - \frac{3}{y_1^2} x^{-4} dx_1 = -3a^2 dx_1,$$

$$dy_1 - 3a^2 y_1^2 dx_1 = -3x_1^{-4} dx_1.$$

Let $-3a^2 = b_1$, $-3 = a_1$;

$$\therefore dy_1 + b_1 y_1^2 dx_1 = a_1 x_1^{-4} dx_1.$$

Now let $y_1 = \dfrac{1}{b_1 x_1} + \dfrac{z_1}{x_1^2}$.

$$\text{Then } dz_1 + b_1 z_1^2 \frac{dx_1}{x_1^2} = a_1 x_1^{-2} dx_1,$$

$$\text{or } x_1^2 dz_1 = (a_1 - b_1 z^2) dx_1;$$

$$\therefore \frac{dx_1}{x_1^2} = \frac{dz_1}{a_1 - b_1 z_1^2} = \frac{dz_1}{3(a^2 z_1^2 - 1)},$$

$$\frac{3a}{x_1} = \int \sqrt{\frac{az_1 + 1}{az_1 - 1}} \times \frac{1}{c} = \int \frac{1}{c} \sqrt{\frac{3a^2 x_1^2 y_1 + x_1 + 3a}{3a^2 x_1^2 y_1 + x_1 - 3a}};$$

$$\therefore \text{ since } \frac{1}{x_1} = x^{\frac{1}{3}} \quad \text{and } y_1 = \frac{1}{y};$$

$$\therefore e^{6ax^{\frac{1}{3}}} = \frac{1}{c^2} \left\{ \frac{3a^2 x^{-\frac{2}{3}} + y(3a + x^{-\frac{1}{3}})}{3a^2 x^{-\frac{2}{3}} + y(x^{-\frac{1}{3}} - 3a)} \right\}$$

$$= \frac{1}{c^2} \left\{ \frac{3a^2 x^{-\frac{1}{3}} + y(1 + 3ax^{\frac{1}{3}})}{3a^2 x^{-\frac{1}{3}} + y(1 - 3ax^{\frac{1}{3}})} \right\}$$

$$\therefore c^2 = C = e^{-6ax^{\frac{1}{3}}} \left\{ \frac{3a^2 x^{-\frac{1}{3}} + y(1 + 3ax^{\frac{1}{3}})}{3a^2 x^{-\frac{1}{3}} + y(1 - 3ax^{\frac{1}{3}})} \right\}.$$

Ex. 2. Let $dy + y^2 dx = \dfrac{a^2 dx}{x^{\frac{8}{3}}}$.

Here $-\dfrac{8}{3}$ is of the form $\dfrac{-4n}{2n-1}$;

$$\therefore \text{ let } y = \frac{1}{x} + \frac{z}{x^2}; \quad \text{and } \because b = 1, \quad m = -\frac{8}{3};$$

$$\therefore dz + \frac{bz^2 dx}{x^2} = ax^{m+2} dx \text{ becomes}$$

$$dz + z^2 \frac{dx}{x^2} = \frac{a^2 x^2 dx}{x^{\frac{8}{3}}} = a^2 x^{-\frac{2}{3}} dx.$$

$$\text{Let } z = \frac{1}{y_1}, \quad \text{and } \because m + 2 = -\frac{2}{3}; \quad \therefore m + 1 = \frac{1}{3}.$$

$$\text{Let } x^{\frac{1}{3}} = x_1 ; \quad \therefore \; x^{-\frac{2}{3}} dx = 3\, dx_1 ;$$

$$\text{and } \frac{1}{x} = \frac{1}{x_1^{\,3}} ; \quad \therefore \; \frac{dx}{x^2} = \frac{3\, dx_1}{x_1^{\,4}} ;$$

$$\therefore \; \frac{-dy_1}{y_1^{\,2}} + \frac{1}{y_1^{\,2}} \frac{3\, dx_1}{x_1^{\,4}} = 3a^2 dx_1 ;$$

$$\therefore \; dy_1 + 3a^2 y_1^{\,2} dx_1 = 3 x_1^{-4} dx_1 ,$$

which, as has been shewn, is integrable.

92. To integrate the differential equation of the first order and of any degree.

Let $\left(\dfrac{dy}{dx}\right)^n + P\left(\dfrac{dy}{dx}\right)^{n-1} + Q\left(\dfrac{dy}{dx}\right)^{n-2} + \&c. + U = 0$ be the equation; let it be solved with regard to $\dfrac{dy}{dx}$; and let X_1, X_2, X_3, &c. be the values of $\dfrac{dy}{dx}$, thus found; then each of the equations $p = X_1$, $p = X_2$, $p = X_3$, &c. when integrated will satisfy the proposed equation, as also will the equation formed of the product of all these integrals.

But as the original equation arises from the elimination of a single constant, raised to the n^{th} power, and since each simple integral introduces a constant, the solution will appear to contain n arbitrary constants, and therefore to be more general than that from which it has been derived. But if we consider that the constants are quite arbitrary, we may, by giving all values to them, make each equal to that particular constant which belonged to the primitive equation, and thus the result will be of the required form.

Ex. 1. Let $\dfrac{dy^2}{dx^2} = a^2$; $\therefore \; \dfrac{dy}{dx} = a$, and $\dfrac{dy}{dx} = -a$;

$$\therefore \; y = ax + c, \quad \text{and } y = -ax + c',$$

both of which satisfy the equation. Their product

$$(y - ax - c)(y + ax - c') = 0$$

will also satisfy it.

For differentiating we obtain

$$\left(\frac{dy}{dx} - a\right)(y + ax - c') + \left(\frac{dy}{dx} + a\right)(y - ax - c) = 0,$$

and making successively $y = ax + c$, and $y = -ax + c'$, we get the results $\frac{dy}{dx} = a$; $\frac{dy}{dx} = -a$; as we ought.

The integral $(y - ax - c)(y + ax - c') = 0$ contains two arbitrary constants, and appears to be more general than those of the other equations which involve but one constant; but we must remember that each factor ought to be separately considered, and that we obtain no other lines but those which would result from an integral including one constant only, of which constant this equation is also susceptible.

This equation may be obtained by observing that if we refer to the original equation we have $\frac{dy}{dx} = \pm a$; and integrating $y - c = \pm ax$, and squaring both sides, $(y-c)^2 = a^2 x^2$.

This equation gives two lines, inclined at different directions to the axis of x, but both cutting the axis of y in the same point; and by giving to (c) different values, we may have groups of such lines in pairs.

And the integral of $(y - ax + c)(y + ax - c')$ gives the same result, except that each factor only represents lines inclined in the same direction; but by giving to c and c' all possible values, and taking care to collect together those straight lines in which c and c' are equal, we shall find the solutions comprised in the equation $(y - c)^2 = a^2 x^2$, which is limited to the single constant c.

Ex. 2. Let $\frac{dy^2}{dx^2} = ax$, or $p = \pm\sqrt{ax}$;

$$\therefore \frac{dy}{dx} = \sqrt{ax}, \quad \text{and} \quad \frac{dy}{dx} = -\sqrt{ax};$$

$$\therefore\ y = \frac{2}{3}\sqrt{a}\,x^{\frac{3}{2}} + c, \qquad \text{and } y = -\frac{2}{3}\sqrt{a}\,x^{\frac{3}{2}} + c',$$

each of which is comprised in $(y - c)^2 = \frac{4}{9}ax^3$.

Ex. 3. Find the curve when $s = ax + by$.

$$\text{Here } \frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}} = a + b\frac{dy}{dx}.$$

$$\text{And } \because \frac{dy}{dx} \text{ is obviously constant, let } \frac{dy}{dx} = m;$$

$\therefore\ y = mx + c$, the equation to a straight line;

$$\therefore\ \frac{y-c}{x} = m, \quad \text{and } \sqrt{1 + \left(\frac{y-c}{x}\right)^2} = a + b\left(\frac{y-c}{x}\right).$$

93. When the equation only involves x and p, and the equation is easily solved with regard to x, we can integrate thus:

$$\text{Since } x = f(p) = P, \quad \text{and } \frac{dy}{dx} = p;$$

$$\therefore\ y = px - \textstyle\int_p x = pP - \int_p P;$$

whence y is a function of p, and therefore of x.

Ex. 1. Let $x + ap = b\sqrt{1 + p^2}$;

$$\therefore\ y = -ap^2 + bp\sqrt{1+p^2} - \textstyle\int_p(-ap + b\sqrt{1+p^2})$$

$$= -\frac{ap^2}{2} + \frac{bp}{2}\sqrt{1+p^2} - \frac{b}{2}\textstyle\int(p + \sqrt{1+p^2}) + c.$$

The elimination of p will give y in terms of x.

Ex. 2. Let $(1+p^2)x = 1$; $\therefore\ x = \dfrac{1}{1+p^2}$, and $p = \sqrt{\dfrac{1-x}{x}}$;

$$\therefore\ y = px - \int\frac{1}{1+p^2} = px - \tan^{-1}p + C$$

$$= \sqrt{x - x^2} - \tan^{-1}\sqrt{\frac{1-x}{x}} + C.$$

Ex. 3. Let $x\dfrac{dy}{dx} = \sqrt{1 + \dfrac{dy^2}{dx^2}}$;

$$\therefore\ px = \sqrt{1 + p^2};\ \therefore\ x = \frac{\sqrt{1 + p^2}}{p};$$

$$\therefore\ y = px - \int_p x = px - \int_p \frac{\sqrt{1 + p^2}}{p}$$

$$= px - \int_p \frac{1}{p\sqrt{1 + p^2}} - \int_p \frac{p}{\sqrt{1 + p^2}}$$

$$= px - \log\left(\frac{cp}{1 + \sqrt{1 + p^2}}\right) - \sqrt{1 + p^2};$$

$$\therefore\ y = \log\left(\frac{1 + \sqrt{1 + p^2}}{cp}\right) = \log\left(\frac{x + \sqrt{x^2 - 1}}{c}\right).$$

94. When the differential equation contains, y, x and p, and is homogeneous with respect to y, the variables can be separated by making $y = zx$; for then x will disappear, and we shall have $z = f(p)$.

But $\because\ y = xz;\ \therefore\ p - z = x\dfrac{dz}{dx}$;

$$\therefore\ \frac{1}{x}\frac{dx}{dz} = \frac{1}{p - z};$$

$$\therefore\ \frac{1}{x}\frac{dx}{dp} = \frac{\frac{dz}{dp}}{p - f(p)} = \frac{f'(p)}{p - f(p)}.$$

And x being found a function of p; $y = px - \int x\dfrac{dp}{dx}$, y may be determined in terms of p and therefore in terms of x.

Ex. Let $y - px = x\sqrt{1 + p^2}$.

Make $y = xz$; $\therefore\ z - p = \sqrt{1 + p^2}$;

$$\therefore\ z^2 - 2zp = 1,\ \text{or}\ p = \frac{z}{2} - \frac{1}{2z} = z + x\frac{dz}{dx};$$

$$\therefore\ x\frac{dz}{dx} = -\frac{z}{2} - \frac{1}{2z} = -\frac{1}{2}\cdot\frac{z^2+1}{z};$$

$$\therefore\ \frac{dx}{x\,dz} = -\frac{2z}{1+z^2};$$

$$\therefore\ \log\left(\frac{x}{2c}\right) = \log\frac{1}{1+z^2} = \log\frac{x^2}{x^2+y^2};$$

$$\therefore\ x^2 + y^2 - 2cx = 0,$$

the equation to a circle the origin being in the circumference. This is the solution of the problem; find the curve in which the perpendicular from the origin upon the tangent is equal to the abscissa.

95. *Integration of the equation, called Clairaut's Formula.*

$$y = px + f(p) = px + P.$$

Differentiate, when we have

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{dP}{dx};$$

$$\therefore\ \text{since } \frac{dy}{dx} = p, \text{ and } \frac{dP}{dx} = P'\frac{dp}{dx}, \text{ we have}$$

$$0 = (x + P')\frac{dp}{dx};\ \therefore\ \frac{dp}{dx} = 0, \text{ or } x + P' = 0.$$

$$\text{If we make } \frac{dp}{dx} = 0;\ p = c;\ \therefore\ y = cx + c'.$$

This equation appears to have two arbitrary constants; but if we put c for p in the original equation, and C for P, C being what P becomes when c is substituted for p, we shall have $y = cx + C$; $\therefore\ C = c'$, and the equation has but one arbitrary constant. This is the general solution of the differential equation.

Again from $x + P' = 0$, a value of p will be obtained which is a function of x or y, and does not introduce into the original equation the constant by the elimination of which the differential equation was formed, such a solution of the equation is called a singular or particular solution.

The particular value may, however, be derived from the general solution, by making c to vary; and as $y = cx + C$ is the equation to a straight line, we evidently see that the particular solution gives the equation to the curve which is the locus of the intersections of the straight lines denoted by the general solution.

Ex. 1. $y - px = a\sqrt{1 + p^2}$;

$$\therefore \frac{dy}{dx} - p - x\frac{dp}{dx} = \frac{ap}{\sqrt{1 + p^2}}\frac{dp}{dx};$$

$$\therefore \left\{x + \frac{ap}{\sqrt{1 + p^2}}\right\} = 0, \text{ and } \frac{dp}{dx} = 0;$$

$$\therefore p = c, \text{ and } y = cx + a\sqrt{1 + c^2},$$

which is the general solution.

$$\text{But } x = \frac{-ap}{\sqrt{1 + p^2}}; \quad \therefore \frac{a^2}{x^2} = \frac{1 + p^2}{p^2}; \quad \therefore \frac{\sqrt{a^2 - x^2}}{x} = \frac{1}{p};$$

$$\therefore p = \frac{x}{\sqrt{a^2 - x^2}}; \quad \sqrt{1 + p^2} = \frac{-ap}{x} = \frac{-a}{\sqrt{a^2 - x^2}};$$

$$\therefore y = \frac{x^2}{\sqrt{a^2 - x^2}} - \frac{a^2}{\sqrt{a^2 - x^2}} = -\frac{a^2 - x^2}{\sqrt{a^2 - x^2}}$$

$$= -\sqrt{a^2 - x^2}; \quad \therefore y^2 + x^2 = a^2.$$

which is the solution to the following problem. "Find the curve, in which each of the perpendiculars drawn from a given point upon the tangent, is equal to a given line:" and we find (see Art. 178, Diff. Calculus); that it is the curve which is formed by the intersections of the line defined by

$$y = cx + a\sqrt{1 + c^2}.$$

Ex. 2. Let $y = px + \frac{a}{p}(1 + p^2)$; $\therefore\ y^2 = 4a(a + x)$.

Ex. 3. Let $(y - px)^m = a^{m-1}\left(\frac{y}{p} - x\right)$;

$$\therefore\ \left(\frac{y}{m}\right)^m = \left(\frac{a}{m-1}\right)^{m-1} x;$$

this is the curve in which $AD^m = a^{m-1}AT$.

Ex. 4. Find the equation to the curve, when the rectangle contained by the perpendiculars on the tangent, one from the origin and the other from a point in the axis of x, at a distance $2c$ from the origin shall equal a given quantity b^2.

$$\text{Here } \frac{y - px}{\sqrt{1 + p^2}} \times \frac{y + (2c - x)p}{\sqrt{1 + p^2}} = b^2,$$

whence if $b^2 + c^2 = a^2$, $y^2 = \frac{b^2}{a^2}\left\{a^2 - (c - x)^2\right\}$.

Integration of differential equations of the second and higher orders.

96. The integration of differential equations of the higher order is effected only in a few instances. We shall begin with the most simple.

97. To integrate $\frac{d^n y}{dx^n} = X$, X being a function of x.

(1) Let $\frac{d^2 y}{dx^2} = X$; $\therefore\ \frac{d}{dx}\left(\frac{dy}{dx}\right) = X$;

$$\therefore\ \frac{dy}{dx} = \textstyle\int_x X; \quad \therefore\ y = \int_x \int_x X.$$

(2) Let $\frac{d^3 y}{dx^3} = X$;

$$\therefore\ \frac{d^2 y}{dx^2} = \textstyle\int_x X; \quad \frac{dy}{dx} = \int_x \int_x X; \quad \therefore\ y = \int_x \int_x \int_x X,$$

and so on; the constants have been omitted.

98. Next to integrate $\dfrac{d^2y}{dx^2} = Y$.

$$\text{Let } \frac{dy}{dx} = p; \quad \therefore \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\,\frac{dy}{dx} = p\,\frac{dp}{dy};$$

$$\therefore p\,\frac{dp}{dy} = Y, \text{ and } \frac{p^2}{2} = C + \textstyle\int_y Y.$$

Ex. 1. Let $\dfrac{d^3y}{dx^3} = x^3$; make $\dfrac{dy}{dx} = p$, $\dfrac{d^2y}{dx^2} = q$;

$$\therefore \frac{dq}{dx} = x^3; \quad \therefore q = \frac{x^4}{4} + C = \frac{dp}{dx};$$

$$\therefore p = \frac{x^5}{4\,.\,5} + cx + c'; \text{ and } y = \frac{x^6}{4\,.\,5\,.\,6} + \frac{cx^2}{2} + c'x + c''.$$

Ex. 2. Integrate $\dfrac{d^2y}{dx^2} = \dfrac{1}{\sqrt{ay}}$;

$$\therefore \frac{dp}{dx} = p\,\frac{dp}{dy} = \frac{1}{\sqrt{ay}};$$

$$\therefore \frac{p^2}{2} = 2\sqrt{\frac{y}{a}} + c = 2\,\frac{\sqrt{y} + \sqrt{b}}{\sqrt{a}} \text{ by substitution};$$

$$\therefore \frac{dx}{dy} = \frac{\sqrt[4]{a}}{2\sqrt{\sqrt{y} + \sqrt{b}}};$$

which may be integrated by making $\sqrt{y} + \sqrt{b} = z$.

99. To integrate equations involving p and q, we must put $\dfrac{dp}{dx}$ instead of q, and the equations will be transformed to those of the first order.

Ex. 1. Find the curve in which the radius of curvature is inversely as the abscissa.

$$\text{Here } \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}} = \frac{a^2}{2x};$$

$$\text{or putting } \frac{dy}{dx} = p; \quad \therefore \frac{d^2y}{dx^2} = \frac{dp}{dx};$$

$$\frac{\frac{dp}{dx}}{(1+p^2)^{\frac{3}{2}}} = -\frac{2x}{a^2};$$

$$\therefore \frac{p}{\sqrt{1+p^2}} = C - \frac{x^2}{a^2} = \frac{b^2 - x^2}{a^2};$$

$$\therefore \frac{1}{p^2} = -1 + \frac{a^4}{(b^2-x^2)^2} = \frac{a^4 - (b^2-x^2)^2}{(b^2-x^2)^2};$$

$$\therefore p = \frac{b^2 - x^2}{\sqrt{a^4 - (b^2-x^2)^2}}.$$

Ex. 2. Find the same when radius of curvature $= \dfrac{x^2}{a}$.

$$\therefore \frac{\frac{dp}{dx}}{(1+p^2)^{\frac{3}{2}}} = -\frac{a}{x^2}.$$

$$\frac{p}{\sqrt{1+p^2}} = \frac{a}{x} + c = \frac{a+cx}{x};$$

$$\therefore \frac{1}{p^2} = \frac{x^2 - (a+cx)^2}{(a+cx)^2};$$

$$\therefore p = \frac{a+cx}{\sqrt{x^2 - (a+cx)^2}}.$$

Ex. 3. Integrate $\dfrac{d^2y}{dx^2} = g + m\dfrac{dy^2}{dx^2}$.

Let $\frac{dy}{dx} = p$; $\therefore \frac{dp}{dx} = g + mp^2$;

$$\therefore \frac{dx}{dp} = \frac{1}{g + mp^2};$$

$$\text{and } \frac{dy}{dp} = \frac{p}{g + mp^2};$$

both of which are integrable, and a relation between y and x may be found.

Ex. 4. Integrate $\frac{d^2y}{dx^2} = Y + m\frac{dy^2}{dx^2}$.

$$\text{Make } \frac{dy}{dx} = p; \quad \therefore \frac{dp}{dx} = \frac{d^2y}{dx^2};$$

$$\therefore \frac{dp}{dx} = Y + mp^2.$$

$$\text{But } \frac{dp}{dx} = p\frac{dp}{dy} = Y + mp^2.$$

$$\text{Let } p^2 = 2z; \quad \therefore p\frac{dp}{dy} = \frac{dz}{dy};$$

$$\therefore \frac{dz}{dy} - 2mz = Y;$$

a linear equation of the first order and degree.

This equation is used, to find the velocity of a body moving down a circular arc, in a resisting medium.

100. Next to solve the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0.$$

Make $y = e^{\int_x u}$; $\therefore \frac{dy}{dx} = ue^{\int_x u}$; $\frac{d^2y}{dx^2} = \left(\frac{du}{dx} + u^2\right) e^{\int_x u}$;

$$\therefore \; e^{\int_x u} \left\{ u^2 + Pu + Q + \frac{du}{dx} \right\} = 0;$$

$$\text{whence } u^2 + Pu + Q + \frac{du}{dx} = 0,$$

an equation of the first degree and order; but which is seldom integrable when P and Q are functions of x.

101. Let P and Q be constant, and let $P = A$; $Q = B$.

$$\therefore \; \frac{du}{dx} + u^2 + Au + B = 0;$$

$$\text{or } \frac{du}{dx} + (u - a)(u - b) = 0;$$

an equation which is satisfied by making $u = a$ and $u = b$;

$$\therefore \; y = c^{\int_x a} = e^{ax + c'} = c_1 e^{ax},$$

$$\text{and } y = e^{\int_x b} = e^{ba + c''} = c_2 e^{bx};$$

either of these values when substituted for y will satisfy the conditions of the differential equation, but the complete solution, which must comprise two constants is

$$y = c_1 e^{ax} + c_2 e^{bx},$$

for by substitution we find that their value also satisfies the condition required.

Cor. 1. If the roots of the equation $u^2 + Au + B$ be impossible, then

$$a = \alpha + \beta\sqrt{-1}, \text{ and } b = \alpha - \beta\sqrt{-1};$$

$$\therefore \; y = e^{\alpha x}\left\{c_1 e^{\beta x\sqrt{-1}} + c_2 e^{-\beta x\sqrt{-1}}\right\}$$

$$= e^{\alpha x}\left\{(c_1 + c_2)\cos\beta x + (c_1 - c_2)\sqrt{-1}\sin\beta x\right\}.$$

Make $c_1 + c_2 = A\sin\delta$, $(c_1 - c_2)\sqrt{-1} = A\cos\delta$;

$$\therefore \; y = Ae^{\alpha x}\left\{\sin\delta\cos\beta x + \cos\delta\sin\beta x\right\} = Ae^{\alpha x}\sin(\beta x + \delta).$$

COR. 2. Let the roots be equal; or $a = b$.

Then $y = e^{ax}(c_1 + c_2) = c^1 e^{ax}$ which has but one constant. To find the second constant.

$$\text{Suppose } b = a + h; \quad \therefore \ y = c_1 e^{ax} + c_2 e^{ax + hx}$$

$$= e^{ax}\{c_1 + c_2 e^{hx}\} = e^{ax}\left\{c_1 + c_2 + c_2 h x + \frac{c_2 h^2 x^2}{1 \cdot 2} + \&c.\right\};$$

$$\text{make } c_1 + c_2 = c', \ c_2 h = c'', \text{ and } h = 0;$$

$$\therefore \ y = e^{ax}(c' + c'' x).$$

102. The equation

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = 0,$$

is seldom integrable when P and Q are functions of x; it can however be solved when

$$P = \frac{A}{a + bx}; \quad \text{and } Q = \frac{B}{(a + bx)^2}.$$

For make $a + bx = e^{bz}$;

$$\therefore \ \frac{dz}{dx} = \frac{1}{a + bx}; \quad \therefore \ \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{a + bx};$$

$$\therefore \ \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2}\frac{dz}{dx}\frac{1}{a + bx} - \frac{dy}{dz}\frac{b}{(a + bx)^2}$$

$$= \left(\frac{d^2 y}{dz^2} - b\frac{dy}{dz}\right)\frac{1}{(a + bx)^2};$$

whence by substitution, and multiplying by $(a + bx)^2$,

$$\frac{d^2 z}{dx^2} + (A - b)\frac{dy}{dz} + By = 0;$$

which may be integrated by the preceding methods.

103. To integrate the general equation

$$\frac{d^n y}{dx^n} + A\frac{d^{n-1}y}{dx^{n-1}} + B\frac{d^{n-2}y}{dx^{n-2}} + \&c. + Ly = 0;$$

where A, B, C, &c. L, are constant.

$$\text{Let } y = e^{mx}; \quad \therefore \frac{dy}{dx} = me^{mx}; \quad \frac{d^2y}{dx^2} = m^2 e^{mx}, \ \&c.$$

$$\therefore m^n + Am^{n-1} + Bm^{n-2} + Cm^{n-3} + \&c. + L = 0.$$

Let a, b, c, &c. be the roots of this equation; then

$$y = e^{ax}, \quad y = e^{bx}, \quad y = e^{cx}; \quad \&c.$$

will be particular integrals of the general equation, and the substitution of each in it will satisfy it. Hence the complete integral will be, by the introduction of n constants

$$y = c_1 e^{ax} + c_2 e^{bx} + c_3 e^{cx} + \&c.$$

Cor. 1. Should any of the roots be equal, as $a = b$; then for $c_1 e^{ax} + c_2 e^{bx}$, we must put $e^{ax}(c_1 + c_2 x)$;

$$\therefore y = e^{ax}(c_1 + c_2 x) + c_3 e^{cx} + \&c.$$

And if three roots be equal, and a be the equal root, we must put for

$$c_1 e^{ax} + c_2 e^{bx} + c_3 e^{cx},$$

the term

$$e^{ax}(c_1 + c_2 x + c_3 x^2),$$

and so on for any number of equal roots.

Cor. 2. If pairs of roots be impossible, we must substitute for the impossible exponential functions, the cosines and sines of the circular arcs, to which they are equivalent.

Ex. 1. $\dfrac{d^2u}{d\theta^2} + n^2 u = 0.$

$$\text{Let } u = e^{m\theta}; \quad \therefore \frac{du}{d\theta} = m e^{m\theta}; \quad \frac{d^2u}{d\theta^2} = m^2 e^{m\theta};$$

$$\therefore m^2 e^{m\theta} + n^2 e^{m\theta} = 0; \quad \therefore m^2 + n^2 = 0, \text{ and } m = \pm n\sqrt{-1};$$

$$\therefore u = c' e^{n\theta\sqrt{-1}} + c'' e^{-n\theta\sqrt{-1}}$$

$$= (c' + c'')\cos n\theta + (c' - c'')\sqrt{-1}\sin n\theta$$

$$= A\cos(n\theta + B).$$

If $c' + c'' = A\cos B$, and $(c' - c'')\sqrt{-1} = -A\sin B$.

Ex. 2. $\dfrac{d^2u}{d\theta^2} + n^2u + a^2 = 0.$

Make $a^2 = n^2\beta$, and $u + \beta = w$, whence we have

$$\frac{d^2w}{d\theta^2} + n^2w = 0,$$

and the solution is performed as in the preceding example.

Ex. 3. $\dfrac{d^2s}{dt^2} + 2k\dfrac{ds}{dt} + fs = 0.$

$$\text{Make } s = e^{mt}; \quad \therefore m^2 + 2km + f = 0;$$

$$\therefore m = -k \pm \sqrt{-1}\sqrt{f - k^2} = -k \pm a\sqrt{-1};$$

$$\therefore s = e^{-kt}\left(c' e^{at\sqrt{-1}} + c'' e^{-at\sqrt{-1}}\right) = Ae^{-kt}\cos(at + B).$$

Examples (1) and (2) are useful in *Physical Astronomy*, Ex. (3) gives the space a function of the time, when a body moves through the arc of a cycloid, the resistance varying as the velocity.

Ex. 4. $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0.$

$$\text{Let } y = e^{mx}; \quad \therefore m^3 - 6m^2 + 11m - 6 = 0,$$

the roots of which are 1, 2, 3;

$$\therefore y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}.$$

Ex. 5. Let $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0.$

Here if $y = e^{mx}$, $m^3 - 3m^2 + 3m - 1 = 0$, or $(m-1)^3 = 0.$

$$\text{And } \therefore\ y = e^x(c_1 + c_2x + c_3x^2).$$

Ex. 6. $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0;\ \therefore\ y = e^{-4x}(c_1 + c_2x).$

Ex. 7. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 34y = 0;\ \therefore\ y = Ae^{3x}\cos(B + 5x).$

Ex. 8. $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + \frac{y}{x^2} = 0.$

$$\text{Make } x = e^z;\ \therefore\ \frac{dz}{dx} = \frac{1}{x};\ \frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz},$$

$$\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\frac{1}{x^2};$$

$$\therefore\ \frac{d^2y}{dz^2} - 2\frac{dy}{dz} + y = 0;$$

$$\therefore\ y = e^z(c_1 + c_2z) = x(c_1 + c_2\log x).$$

Ex. 9. $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 0,$

$$\text{making } x = e^z, \text{ we have } \frac{d^2y}{dz^2} - y = 0;$$

$$.\ y = c_1e^z + c_2e^{-z} = c_1x + \frac{c_2}{x}.$$

Ex. 10. Integrate $\frac{d^4y}{dx^4} - a^4y = 0.$

Here $m^4 - a^4 = 0$, the roots are $\pm a$, $\pm a\sqrt{-1}$;

$$\therefore\ y = c_1e^{ax} + c_2e^{-ax} + A\cos(B + ax).$$

Ex. 11. Integrate $\dfrac{d^4 y}{dx^4} + a^4 y = 0$.

Here $m^4 + a^4 = 0$; $\dfrac{m}{a} = \dfrac{1 \pm \sqrt{-1}}{\sqrt{2}}$, and $= \dfrac{-1 \pm \sqrt{-1}}{\sqrt{2}}$;

$$\therefore \; y = A e^{\frac{ax}{\sqrt{2}}} \cos\left(B + \frac{ax}{\sqrt{2}}\right) + A_1 e^{-\frac{ax}{\sqrt{2}}} \cos\left(B_1 + \frac{ax}{\sqrt{2}}\right).$$

Ex 12. Integrate $\dfrac{d^4 y}{dx^4} = \dfrac{1}{a^2}\dfrac{d^2 y}{dx^2}$.

Let $y = e^{mx}$; $\therefore \; m^4 = \dfrac{m^2}{a^2}$.

The roots of which are, $\dfrac{1}{a}$ $-\dfrac{1}{a}$, 0, 0;

$$\therefore \; y = c_1 e^{\frac{x}{a}} + c_2 e^{-\frac{x}{a}} + c_3 + c_4 x.$$

Ex. 13. Integrate $\dfrac{d^n y}{dx^n} - y = 0$.

Make $y = e^{mx}$; $\therefore \; m^n - 1 = 0$,

let $1, a_1, a_2, a_3, a_4$, &c. a_{n-1}, be the roots of this equation;

$$\therefore \; y = c_1 e^x + c_2 e^{a_1 x} + c_3 e^{a_2 x} + \text{\&c.} + c_n e^{a_{n-1} x}.$$

104. To solve the equation,

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = R \;\ldots\ldots\ldots\ldots (1)$$

We shall shew that the solution of this equation may be made to depend upon that of the equation,

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \;\ldots\ldots\ldots\ldots (2)$$

To effect this, we proceed to apply to this equation, a method called by Lagrange, "The Variation of the Parameters;" which consists in this, that if $y = c'y_1 + c''y_2$ be the solution of the equation (2), we may assume it to be that of equation (1), if c' and c'' be considered no longer constant but functions of x.

Let $\therefore\ y = c'y_1 + c''y_2$ be the solution of (1);

$$\therefore\ \frac{dy}{dx} = c'\frac{dy_1}{dx} + c''\frac{dy_2}{dx} + y_1\frac{dc'}{dx} + y_2\frac{dc''}{dx}.$$

But as we have made but one supposition to determine c' and c'', we may make another, let therefore

$$y_1\frac{dc'}{dx} + y_2\frac{dc''}{dx} = 0;\ \therefore\ \frac{dy}{dx} = c'\frac{dy_1}{dx} + c''\frac{dy_2}{dx};$$

$$\therefore\ \frac{d^2y}{dx^2} = c'\frac{d^2y_1}{dx^2} + c''\frac{d^2y_2}{dx^2} + \frac{dc'}{dx}\frac{dy_1}{dx} + \frac{dc''}{dx}\frac{dy_2}{dx},$$

whence by substitution in the original equation (1),

$$c'\left(\frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + Qy_1\right) + c''\left(\frac{d^2y_2}{dx} + P\frac{dy_2}{dx} + Qy_2\right)$$

$$+ \frac{dc'}{dx}\frac{dy_1}{dx} + \frac{dc''}{dx}\frac{dy_2}{dx} = R,$$

which by means of equation (2) is reduced to

$$\frac{dc'}{dx}\frac{dy_1}{dx} + \frac{dc''}{dx}\frac{dy_2}{dx} = R,$$

$$\text{or } \because\ \frac{dc''}{dx} = -\frac{y_1}{y_2}\frac{dc'}{dx},$$

$$\frac{dc'}{dx}\left(\frac{dy_1}{dx} - \frac{y_1}{y_2}\frac{dy_2}{dx}\right) = R;$$

whence $\frac{dc_1}{dx}$ is found to be a function of x, and $c' = X_1 + C_1$, also similarly $c'' = X_2 + C_2$;

$$\therefore\ y = C_1y_1 + C_2y_2 + y_1X_1 + y_2X_2.$$

A similar proof of this proposition applies to equations of a higher order.

Ex. 1. Integrate $\dfrac{d^2y}{dx^2} + a^2 y = \cos \beta x$.

The solution of the equation $\dfrac{d^2y}{dx^2} + a^2 y = 0$ is

$$y = c' \cos ax + c'' \sin ax;$$

$\therefore$ assume this to be the solution of the proposed equation;

$$\therefore \frac{dy}{dx} = -c'a \sin ax + c''a \cos ax + \frac{dc'}{dx}\cos ax + \frac{dc''}{dx}\sin ax$$

$$= -c'a \sin ax + c''a \cos ax.$$

Since we make $\dfrac{dc'}{dx}\cos ax + \dfrac{dc''}{dx}\sin ax = 0$;

$$\therefore \frac{d^2y}{dx^2} = -c'a^2 \cos ax - c''a^2 \sin ax - a\frac{dc'}{dx}\sin ax + a\frac{dc''}{dx}\cos ax$$

$$= -a^2 y - a\frac{dc'}{dx}\sin ax + a\frac{dc''}{dx}\cos ax;$$

$$\therefore \; -a\frac{dc'}{dx}\sin ax + a\frac{dc''}{dx}\cos ax = \cos \beta x.$$

$$\text{But } \frac{dc''}{dx} = -\frac{\cos ax}{\sin ax}\frac{dc'}{dx};$$

$$\therefore \; -a\frac{dc'}{dx}\left(\sin ax + \frac{\cos^2 ax}{\sin ax}\right) = \cos \beta x;$$

$$\therefore \frac{dc'}{dx} = -\frac{1}{a}\cos \beta x \sin ax = -\frac{1}{2a}\{\sin(a+\beta)x + \sin(a-\beta)x\},$$

$$\text{and } \frac{dc''}{dx} = \frac{1}{a}\cos \beta x \cos ax = \frac{1}{2a}\{\cos(a+\beta)x + \cos(a-\beta)x\};$$

$$\therefore\ c' = c_1 + \frac{1}{2a}\left(\frac{\cos(a+\beta)x}{a+\beta} + \frac{\cos(a-\beta)x}{a-\beta}\right),$$

$$c'' = c_2 + \frac{1}{2a}\left(\frac{\sin(a+\beta)x}{a+\beta} + \frac{\sin(a-\beta)x}{a-\beta}\right);$$

$$\therefore\ y = c_1 \cos ax + c_2 \sin ax + \frac{1}{2a}\left(\frac{\cos\beta x}{a+\beta} + \frac{\cos\beta x}{a-\beta}\right)$$

$$= c_1 \cos ax + c_2 \sin ax + \frac{\cos\beta x}{a^2-\beta^2}$$

$$= A\cos(B+ax) + \frac{\cos\beta x}{a^2-\beta^2},$$ making the proper substitutions.

Ex. 2. Integrate $\dfrac{d^2y}{dx^2} + a^2y = X$.

Let $y = c'\cos ax + c''\sin ax$, which is the solution of $\dfrac{d^2y}{dx^2} + a^2y = 0$, be that of the proposed equation. Proceeding as in Example 1, we have

$$\frac{dc'}{dx}\sin ax - \frac{dc''}{dx}\cos ax = -\frac{X}{a}.$$

$$\text{And}\ \frac{dc''}{dx} = -\frac{dc'}{dx}\,\frac{\cos ax}{\sin ax};$$

$$\therefore\ \frac{dc'}{dx} = -\frac{1}{a}X\sin ax, \quad \text{and}\ \frac{dc''}{dx} = \frac{1}{a}X\cos ax;$$

$$\therefore\ c' = c_1 - \frac{1}{a}\int_x X\sin ax,$$

$$c'' = c_2 + \frac{1}{a}\int_x X\cos ax;$$

$$\therefore\ y = c_1\cos ax + c_2\sin ax - \frac{\cos ax}{a}\int_x X\sin ax$$

$$+ \frac{\sin ax}{a}\int_x X\cos ax.$$

This case includes Example 1.

Ex. 3. Integrate $\dfrac{d^2y}{dx^2} + A\dfrac{dy}{dx} + By = X$, X being a function of x.

Let a and b be the roots of the equation $m^2 + Am + B = 0$; $\therefore\ y = c'e^{ax} + c''e^{bx}$ may be assumed the solution of the differential equation;

$$\therefore\ \frac{dy}{dx} = ac'e^{ax} + bc''e^{bx} + e^{ax}\frac{dc'}{dx} + e^{bx}\frac{dc''}{dx}.$$

$$\text{Make } e^{ax}\frac{dc'}{dx} + e^{bx}\frac{dc''}{dx} = 0;$$

$$\therefore\ \frac{dy}{dx} = ac'e^{ax} + bc''e^{bx}.$$

$$\frac{d^2y}{dx^2} = a^2c'e^{ax} + b^2c''e^{bx} + ae^{ax}\frac{dc'}{dx} + be^{bx}\frac{dc''}{dx};$$

$$\therefore\ c'e^{ax}(a^2 + Aa + B) + c''e^{bx}(b^2 + Ab + B)$$

$$+ ae^{ax}\frac{dc'}{dx} + be^{bx}\frac{dc''}{dx} = X.$$

And $a^2 + Aa + B = 0$; $b^2 + Ab + B = 0$;

$$\therefore\ ae^{ax}\frac{dc'}{dx} + be^{bx}\frac{dc''}{dx} = X.$$

But $be^{ax}\dfrac{dc'}{dx} + be^{bx}\dfrac{dc''}{dx} = 0$;

$$\therefore\ (a-b)e^{ax}\frac{dc'}{dx} = X, \quad \text{and} \quad -(a-b)e^{bx}\frac{dc''}{dx} = X;$$

$$\therefore\ c' = c_1 + \frac{1}{a-b}\int_x Xe^{-ax}, \quad c'' = c_2 - \frac{1}{a-b}\int_x Xe^{-bx};$$

$$\therefore\ y = c_1e^{ax} + c_2e^{bx} + \frac{e^{ax}}{a-b}\int_x Xe^{-ax} - \frac{e^{bx}}{a-b}\int_x Xe^{-bx}.$$

Ex. 4. Integrate $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = x$.

$$\text{Here } a = 3, \quad b = 2, \quad X = x;$$

$$\therefore\ y = c_1 e^{3x} + c_2 e^{2x} + \frac{1}{6}\left(x + \frac{10}{3}\right).$$

For further information on the subject of these equations, the reader is referred to the *Differential Calculus* of LACROIX, and to the *Collection of Examples* by PROFESSOR PEACOCK; to both of which works the author of the present treatise has been greatly indebted.

FINIS.