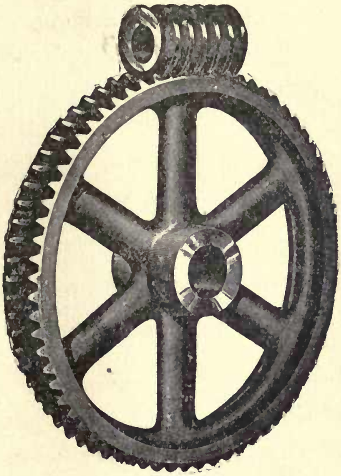
A TREATISE ON

GEAR WHEELS,

BY GEORGE B. GRANT.



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COPIED from the American Machinist, and published in full by the "English Mechanic and World of Science," and by the "Mechanical World."

ADOPTED and used as a text or reference book by Michigan and Cornell Universities, Rose Polytechnic Institute, and by many schools of mechanics and drafting.

A Working Course of Study.

It is not necessary that the student, especially if he is a workman, should learn all that is taught in this book, for it contains much that is not only difficult but also of minor practical importance.

The beginner is therefore advised to master only the following sections:

1, 2, 7 to 15, 22, 25, 31, 32 of the general theory;

35 to 47 of the spur gear;

53 to 64 of the involute tooth;

76, 77, 80 to 83, 89 of the cycloidal tooth;

91, 95, 97 of the pin tooth;

98, 99, 111, 113 to 119 of spiral and worm gears;

154, 157, 158, 161 to 169 of the bevel gear.

These include not half of the whole matter, but, knowing this much well, the student has a good outline knowledge of the whole, and he can then take the balance at leisure.

A TREATISE ON

GEAR WHEELS.

1. THEORY OF TOOTH ACTION.

1.—INTRODUCTORY.

The present object is practical, to reach and interest the man that makes the thing written of; the machinist or the millwright that makes the gear wheel, or the draftsman or foreman that directs the work, and to teach him not only how to make it, but what it is that he makes.

To most mechanics a gear is a gear.

> "A yellow primrose by the shore,
> A yellow primrose was, to him,
> And it was nothing more;"

and, in fact, the gear is often a gear and nothing more, sometimes barely that.

But, if the mechanic will look beyond the tips of his fingers, he will find that it can be something more; that it is one of the most interesting objects in the field of scientific research, and not the simplest one; that it has received the attention of many celebrated mathematicians and engineers; and that the study of its features will not only add to his practical knowledge, but also to his entertainment. There is an element in mathematics, and in its near relative, theoretical mechanics, that possesses an educating and disciplining value beyond any capacity for earning present money. The thinking, inquisitive student of the day is the successful engineer or manufacturer of the future.

2.—METHOD.

The method will be fitted to the object, and will be as simple and direct as possible. It is not possible to treat all the items in simple every-day fashion, by plain graphical or arithmetical methods, but where there is a choice the path that is the plainest to the average intelligent and educated mechanic will be chosen.

A thousand pages could be filled with the subject and not exhaust anything but the reader thereof, but what is written should receive and deserve attention, and must be condensed within such reasonable limits, that it shall not call for more time and labor than its limited application will warrant. Demonstrations and controversies will be avoided, and the matter will be confined as far as is possible to plain statements of facts, with illustrations. The simplest diagram is often a better teacher than a page of description.

First, we shall study the odontoid or pure tooth curve as applied to spur gears, then we shall consider the involute, cycloid, and pin tooth, special forms in which it is found in practice; then the modifications of the spur gear, known as the spiral gear, and the elliptic gear; then the bevel gear, and lastly the skew bevel gear.

3.—PARTICULARLY IMPORTANT.

Begin at the beginning.

The natural tendency is too often to skip first principles, and begin with more advanced and interesting matter, and the result is a trashy knowledge that stands on no foundation and is soon lost. When a fact is learned by rote it may be remembered, but when it follows naturally upon some simple principle it cannot be forgotten.

Therefore the student is urged to begin with and pay close attention to the odontoid or pure tooth curve, before going on to its special applications, for the apparently dry and trivial matter relating to it is really the foundation of the whole subject.

The usual course is to begin at once with the cycloidal tooth, to hurry over the involute tooth, and then, if there is room, it is stated that such curves are particular forms of some confused and indefinite general curve. Our course will be to study the undefined tooth curve first, and then take up its particular cases.

4.—LITERATURE.

It is impracticable to acknowledge all the sources from which information has been drawn, but it is in order to briefly mention the principal works devoted to the subject.

Professor Herrmann's section of Professor Weisbach's "Mechanics of Engineering and Machinery" is the most important work that can be named in this connection. It treats of much besides the teeth of gears, but its treatment of that branch is particularly valuable. It is not easy reading. Wiley, $5.00.

Professor Willis' "Principles of Mechanism" is a celebrated book, now many years behind the age, but it is, nevertheless, of the greatest value and interest in this matter. To Willis we are indebted for many of the most important additions to our knowledge of theoretical and practical mechanism. Longmans, $7.50. Out of print.

Professor Rankine's "Machinery and Millwork" should not be neglected by the student, for, although it is the dryest of books, its value is as great as its reputation. Griffin, $5.00.

Professor MacCord's "Kinematics" is a work that abounds in novelties, and is written in an attractive style. It contains many errors, and some hobbies, and needs a thorough revision, but the student cannot afford to avoid it, or even to slight it. Wiley, $5.00.

Mr. Beale's "Practical Treatise on Gearing" is really practical. Many of the so-called "practical" books are neither practical or theoretical, but we have in this small book a collection of workable information that should be within the reach of every man who pretends to be a machinist. We have drawn from it, by permission, particularly with regard to spiral and worm gears. Mr. Beale's experimental work, in connection with the spiral gear, has been of great service. The Brown & Sharpe Mfg. Co., cloth $1.00, paper 75c.

Professor Reuleaux's "Konstrukteur" is a justly celebrated work in the German language. A translation of it is now being published in an American periodical—*Mechanics.*

Professor Klein, the translator of Herrmann's work, has lately published the "Elements of Machine Design," a collection of practical examples, with illustrations. J. F. Klein, Bethlehem, Pa., $6.00.

"Mill Gearing," by Thomas Box, is a practical work by an engineer, and from it we have drawn much of our matter on the cloudy subject of the strength and horsepower of gearing. Spon, $3.00.

"Elementary Mechanism," by Professors Stahl and Woods, is a recent work of general merit. It is well designed as a text book, and treats the subject in a simple and interesting manner. Van Nostrand, $2.00.

In addition to the above works, reference may be made to numerous articles to be found in periodicals, notably in the "American Machinist," the "Scientific American Supplement," the "Journal of the Franklin Institute," "Mechanics," and the "Transactions of the American Society of Mechanical Engineers."

5.—KINEMATICS.

This, the science of pure mechanism, relates exclusively to the constrained and geometric motions of mechanism, and it has nothing to do with questions of force, weight, velocity, temperature, elasticity, etc. The path of a cannon ball is not within the field of kinematics, because it depends upon time and force. A belt and pulley are kinematic agents, because the contact between them can be assumed to be definite, and the action is therefore geometric, but the slipping and stretching of the belt is not kinematic. The action of gear teeth upon each other is purely kinematic, but we cannot consider whether the material is wood, or steel, or wax, whether the gears are lifting one pound or a ton, or whether they are running at one revolution per second or one per day.

6.—ODONTICS.

The name "odontics" may be selected for that limited but important branch of kinematics that is concerned with the transmission of continuous motion from one body to another by means of projecting teeth.

Even this restricted corner of the whole subject is too large for the present purpose, for it covers much that cannot be considered within our set limits, and gear wheels must, therefore, be defined as devices for transmitting continuous motion from one fixed axis to another by means of engaging teeth.

Thus confined, gear wheels may be conveniently divided into three general classes.

Skew bevel gears, transmitting motion between axes not in the same plane.

Bevel gears, transmitting motion between intersecting axes.

Spur gears, transmitting motion between parallel axes.

The last two classes are particular cases of the first; for, if the shafts may be askew at any distance, that distance may be zero, and if they intersect at any point, that point may be at infinity.

It would be scientifically more correct to first develop the skew bevel gear, and from that proceed to the bevel and spur gear, but practical clearness and convenience is often more to be admired than strict accuracy, and, as the true path is difficult to follow, we shall enter in the rear, and consider the spur gear first.

Odontics does not properly include the consideration of questions of strength, power and friction, but we must admit certain important items in that direction.

7.—PITCH SURFACES.

The fixed axes are connected with each other by imaginary surfaces called "axoids," or pitch surfaces, touching each other along a single straight line. We must imagine that the pitch surfaces roll on each other without slipping, as if adhering by friction.

The whole object of odontics is to provide these imaginary surfaces with teeth, by which they can take advantage of the strength of their material and transmit power that is as definite as the geometric motion.

The pitch surface of the skew bevel gear is the hyperboloid of revolution, which becomes a cone when the axes intersect, and a cylinder when the axes are parallel.

8.—NORMAL SURFACES.

An important adjunct of the pitch surface is the normal surface, or surface that is everywhere at right angles to both pitch surfaces of a pair of axes, and upon which the action of the teeth on each other may best be studied.

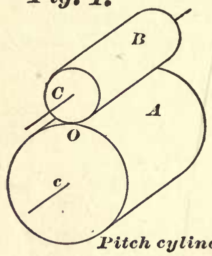
For the skew bevel gear there does not appear to be any normal surface. For the bevel gear the normal surface is a sphere, and for the spur gear the sphere becomes a plane.

9.—UNCERTAINTIES.

The theory of tooth action is not yet full and definite in all its parts, for there are some disputed points, and some confusion and clashing of rules and systems. This is particularly the case with the theory of spiral and skew bevel teeth, for much of the work that has been done is clearly wrong, and there is little that has been definitely decided.

10.—PITCH CYLINDERS.

Fig. 1.



Pitch cylinders

Two cylinders, *A* and *B*, Fig. 1, that will roll on each other, will transmit rotary motion from one of the fixed parallel axes *c* and *C* to the other, if their surfaces are provided with engaging projections.

When these projections are so small that they are imperceptible, the motion is said to be transmitted by friction, and it is practically uniform. But when they are of large size, and readily observed, the motion, although it is unchanged in nature, is said to be transmitted by direct pressure, and it is irregular unless the acting surfaces of the projections are carefully shaped to produce an even motion.

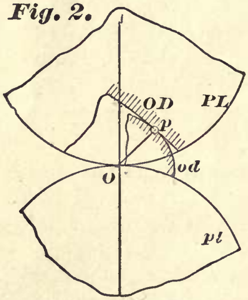
The whole object of odontics is to so shape these large projections or teeth that they shall transmit the same uniform motion between the rotating cylinders, as would be apparently transmitted by friction.

These cylinders are imaginary in actual practice, although they are one of the principal elements of the theory, and they are called the axoids, or pitch cylinders of the gears.

The normal surface (8) of the spur gear is a plane, and, as all sections by normal surfaces are alike, we can study the action on a plane figure easier than in the solid body of the gear.

11.—THE LAW OF TOOTH CONTACT.

Fig. 2.



Tooth action

With the above conditions given we can deduce the following law:

The common normal to the tooth curves must pass through the pitch point.

That is, in Fig. 2, if the tooth curves *OD* and *o d* are to transmit the same motion between the pitch lines *pl* and *PL* as would be transmitted by frictional contact at the pitch point *O*, they must be so shaped that their common normal *Op* at their common point *p* shall pass through that pitch point.

Conversely, if the tooth curves are so shaped that their common normal always passes through the pitch point, they will transmit the required uniform motion.

12.—THE ODONTOID.

This universal law enables us to define the "odontoid," or pure tooth curve, for the contact of the pitch lines at the pitch point is continuous and progressive, and, if the tooth curves are to transmit the same motion, their normals must be arranged in a continuous and progressive manner. The normals *nl*, as in Fig. 3, must be arranged without a break or a crossing, not only springing from the odontoid at consecutive points, but intersecting the pitch line at consecutive points. This arrangement may be called

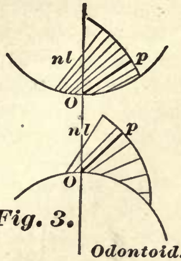


Fig. 3.

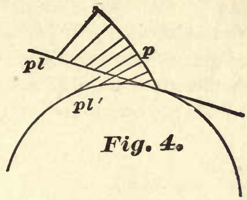


Fig. 4.

"consecutive," and the definition is not a law by itself, but an expression of the given universal law.

It is seen that the odontoid is inseparably connected with its pitch line, and that the same curve may be an odontoid with respect to one pitch line, and not with respect to some other. The curve Fig. 4 is an odontoid with respect to the pitch line *pl*, but not with respect to the pitch line *pl'* beyond the point *p* at which the normal is tangent to that pitch line.

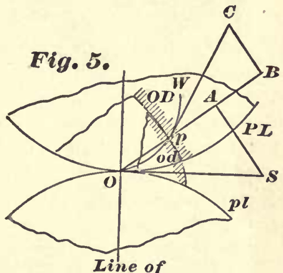
The odontoid, so far as defined, is not a definite thing, and, for practical purposes, it must be given some particular shape. It may be involute or cycloidal, or of other form, but must always have normals arranged in consecutive order.

13.—THE LINE OF ACTION.

As the tooth curves *od* and *OD*, Fig. 5, work together, the point of contact will travel along a line *O p W* called the "line of action."

There is a definite relation between the odontoid and the line of action, so that, if either one is given, the other is fixed. If the odontoid *OD* is given, with its pitch line *PL*, the line of action is determined without reference to the pitch line *pl* or its odontoid; and, conversely, if the pitch line and line of action are given, the odontoid to correspond is determined.

Fig. 5.



14.—INTERCHANGEABLE ODONTOIDS.

This feature leads at once to the broad and useful fact that all odontoids, on pitch lines of all sizes, that are formed from the same line of action, will work together interchangeably, any one working with any other.

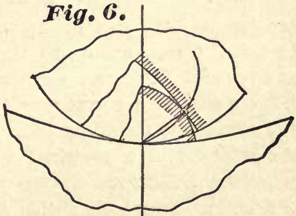
Therefore, to produce an interchangeable set of odontoids we can choose any one line of action, and form any desired number of them from it.

15.—INTERNAL CONTACT.

The pitch lines of Fig. 5 curve in opposite directions, and the contact is said to be "external." But the principles involved are independent of the direction of the pitch lines, and they may curve in the same direction, as in Fig. 6, in "internal" contact.

Tooth contact is between lines only, there being no theoretical need of a solid material on either side of the line, so that either side

Fig. 6.



of the tooth may be chosen as the practical working side.

Therefore the internal gear is precisely like the external gear of the same pitch diameter, working on the same lines of action, so far as the odontoids are concerned, as illustrated by Fig. 7.

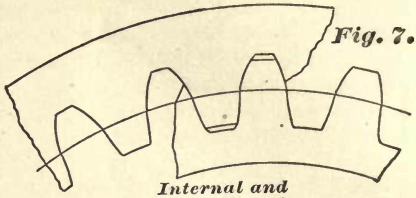


Fig. 7.

Internal and external teeth

16.—THE CUSP AND INTERFERENCE.

When, as in Fig. 8, the pitch circle *p l* is so small with respect to the line of action *O C′ C″ W*, that two tangent circles *C′ c′* and *C″ c″* can be drawn to the line of action from the center *C* of the pitch line, we shall have a troublesome convolution in the resulting flank curve *o d*. This convolution will be formed of two cusps, a first cusp *c′* on the inner tangent arc, the "base circle" *C′ c′*, and a second cusp *c″* on the outer tangent arc *C″ c″*.

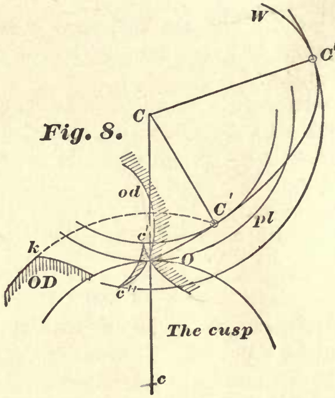


Fig. 8.

The cusp

This happens with any form of odontoid, although sometimes in disguised form, and creates a practical difficulty that can be avoided only by stopping the tooth curve at the first cusp *c′*.

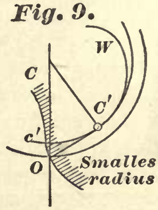
Furthermore, any odontoid *OD* that is to work with the odontoid *o d*, must be cut off at the point *k* on the "limit line" *C′ k* through the point *C′* from the center *c*.

If the odontoids, when the pitch line is so small that the cusps occur, are not cut off as required, the action will still be mathematically perfect, but, as the contact changes at a cusp, from one side of the curve to the other, the action is no longer practicable with solid teeth. The curves will cross each other, and there will be an interference.

17.—THE SMALLEST PITCH CIRCLE.

To determine the smallest pitch circle that can be used, and avoid the cusps altogether, find by trial the point *C*, Fig. 9, from which but one tangent arc *C′ c′* can be drawn to the line of action *O C′ W*. This point will be the center of the smallest pitch circle, and all points outside of it will avoid interference, while all inside of it will be subject to it.

Fig. 9.



Smallest radius

18.—THE TERMINAL POINT.

When a tangent arc can be drawn, from the pitch point *O* as a center, to the line of action at any point *T*, except the vertex *W*, Fig. 10, there will be a corresponding crossing of the normals to the odontoid commencing at the point *t*, and a termination of the action when the point *t* reaches the point *T*.

As the action approaches the terminal point *T* there will be two points of action,

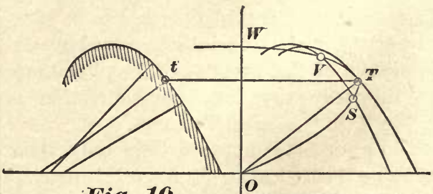


Fig. 10.

Terminal point

since the odontoid crosses the line of action at two points—one point of direct and ordinary action at S, and another point of retrograde and unusual action at V. These two points of action will come together at T, the odontoid will leave the line of action, and all tooth action will then cease. The retrograde action is theoretically and actually correct, but it is so oblique that it is of no practical value, and therefore the odontoid may as well be cut off at its terminal point t.

19.—SPEED OF THE POINT OF ACTION.

Lay off $O\ S$, Fig. 5, to represent the speed of the pitch lines, and draw $S\ A$ at right angles with the common normal $O\ p$. Draw $p\ C$ tangent to the line of action at the point of action p.

Lay off $p\ B$ equal to $O\ A$, and draw $B\ C$ at right angles to $O\ B$. Then $p\ C$ will be the speed of the point of action along the line of action.

When the line of action is a circle the angle $S\ O\ A$ is always equal to the angle $B\ p\ C$, and therefore the speed of the point of action is uniform, and equal to that of the pitch lines.

If the line of action is a straight line the angle $B\ p\ C$ will be constant—always zero—and therefore the speed of the point of action will be uniform and always equal to $O\ A$.

20.—NATURE OF THE TOOTH ACTION.

The nature of the action may be determined by a study of the normal intersections; the intersections of the normals with the odontoid being at uniform distances apart, their intersections with the pitch lines will indicate the action of the teeth. If the normal intersections, as in Fig. 3, are quite regular, the action of the teeth will be smooth and regular, while if they are crowded within a narrow space the action of the tooth will be crowded and jerky.

21.—THE SECONDARY LINE OF ACTION.

From the universal law of tooth contact stated in (11) we can reason that any point on the tooth curve is in position for contact whenever its normal passes through the pitch point O, and therefore that the point will then be upon a line of action.

In Fig. 11 the normal to the point p must cross the pitch line twice—at a primary intersection a, and at a secondary intersection b, and therefore there will be a point of action on a primary line of action $O\ M'$ at q, when the curve has moved so that the primary point of intersection a is at the pitch point O, and a point of action w on a secondary line of action, when the secondary point of intersection b has reached the pitch point.

Therefore there will generally be not only the primary line of action $O\ q\ M$ or $O\ q'\ M'$, but also a secondary line $O\ w\ Y$ or $O\ w'\ Y'$.

The secondary line of action must have the same property as the first, as a locus of contact, and therefore if we can so arrange two pitch lines with their odontoids that their secondary lines of action coincide, there will be secondary contact between the odontoids.

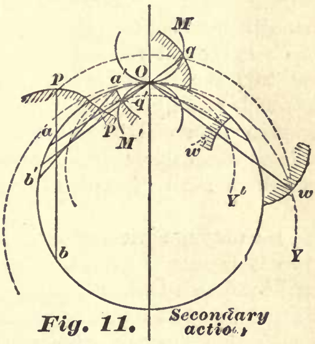


Fig. 11.

When it so happens that both primary and secondary lines coincide, we shall have double contact. Two points of contact will exist at the same time, one on the primary and the other on the secondary line of action.

The secondary lines of action cannot be made to coincide when the contact is external, but when it is internal they sometimes can be, so that the matter has an application to internal gears.

It is to be noticed that the primary line is independent of the pitch line, while the secondary is dependent upon it.

Secondary contact is an interesting feature of tooth action, but it is of small importance, and has been studied but little.

22.—THE INTERCHANGEABLE TOOTH.

The simple odontoid so far studied is the perfect solution of the problem from a mathematical point of view, for it will transmit the required uniform motion as long as it remains in working contact. But from a mechanical point of view it is still incomplete, as it works in but one direction, through but a limited distance, and, although the odontoids are interchangeable, the gears are not.

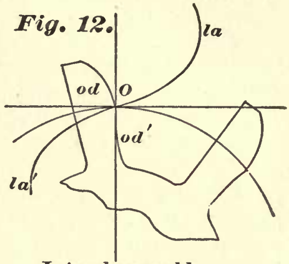
In order that the gears shall be fully interchangeable, it is necessary that the teeth shall have both faces and flanks, and that the line of action for the face shall be equal to that for the flank; that is, the tooth must have an odontoid on each side of the pitch line, the face *o d*, Fig. 12, outside, and the flank *o d'* inside of it, and the line of action *l a* for the faces must be like the line of action *l a'* for the flanks. If so made, any gear will work with any other, without regard to the diameters of the pitch lines.

But such a gear will run in but one direction, and to make it double-acting it must have odontoids facing both ways, as in Fig. 13. Gears so made will be both double-acting and interchangeable, and it is not necessary that both sides of the tooth shall be alike.

Again, the unsymmetrical gear of Fig. 13 fails when it is turned over, upside down, for then the unlike odontoids come together, and, to avoid this last difficulty, all four of the lines of action must be alike, producing the complete and practically perfect tooth of Fig. 14.

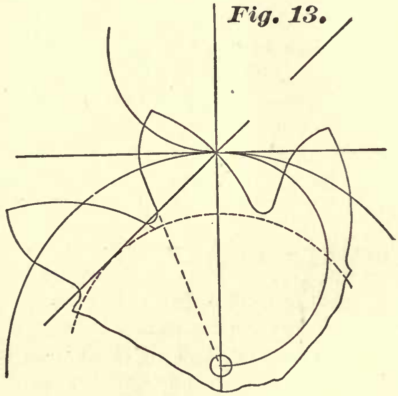
We can therefore define the completely interchangeable tooth, as the tooth that is formed from four like lines of action.

Fig. 12.



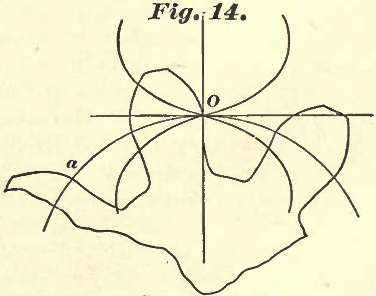
Interchangeable teeth

Fig. 13.



Unsymmetrical teeth

Fig. 14.



Complete teeth

23.—INTERCHANGEABLE RACK TOOTH.

When the pitch line is a circle the flanks of the tooth are not like the faces, but when it is a straight line there is no distinction between face and flank. We then have the important practical fact that the four odontoids of the interchangeable rack tooth are alike.

24.—CONSTRUCTION BY POINTS.

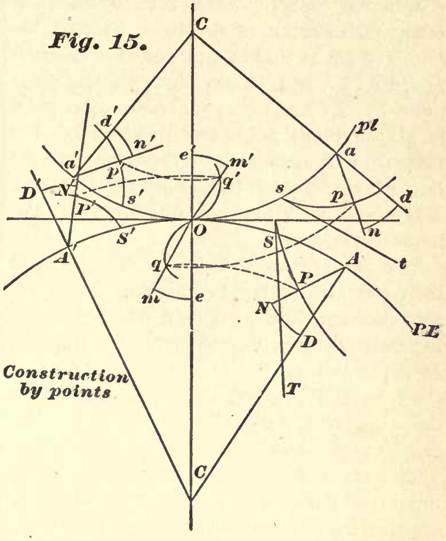
When we have an odontoid and its pitch line given, it is a very simple matter to construct either the line of action or the conjugate odontoid for any other pitch line.

We know, for example, the odontoid *s p*, Fig. 15, on the pitch line *p l*, and it is required to construct an odontoid on the pitch line *P L* that is conjugate to it.

As the odontoid is given we know or can construct its normals. Construct the normal *p a* from any chosen point *p*, draw the radial line *d a C*, lay off *A O* equal to *a O*, draw the radial line *A C*, lay off the angle *N A D* equal to the angle *n a d*, lay off *P A* equal to *p a*, and *P* will be a point in the required conjugate odontoid *S P*. *P A* will be a normal to the curve. Construct a number of points by this process, and draw the required curve through them. The tangents *s t* and *S T* make equal angles with the pitch lines, so that the required curve can often be fully determined by drawing its tangent and one or two points.

To construct the line of action, make the angle *m O e* equal to the angle *n a d*, and lay off *O q* equal to *p a*. The point *q* is on a circle from either *p* or *P* drawn from the centers *C*, and is the point at which *p* and *P* will coincide when the two curves are in working contact, the normals *p a* and *P A* then coinciding with the radiant *O q*.

Fig. 15.



When the line of action alone is given, the odontoids for given pitch lines are fully determined, but there seems to be no simple graphical method for constructing them except for special cases. They can be obtained by the use of the calculus (33), or drawn by the integrating instrument of (34).

The two tooth curves thus constructed are paired, and are said to be "conjugate" to each other.

25.—THE ARC OF ACTION.

The action between two teeth commences and ends at the intersections *m* and *N* of the line of action with the addendum lines of the two gears, *a l* and *A L*, Fig. 16. The arc of action is the distance *a b* on the pitch line that is passed over by the tooth while it is in action.

The arc *a O* passed over while the point of contact is approaching the pitch point, is called the arc of approach, and *O b*, that passed over while the action is receding from that point, is the arc of recess.

With a given line of action the arcs of approach and recess can be controlled by the addenda. If it is desirable to have a great recess and a small approach, the addendum of the gear that acts as a driver is to be increased. When there is a limit line (16), it limits the addendum and the arc of action.

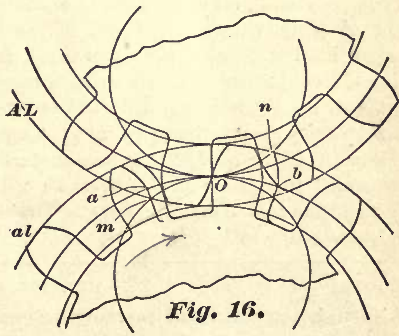


Fig. 16.

26.—OBLIQUITY OF THE ACTION.

When a pair of teeth bear upon each other, the direction of the force exerted between them is that of the common normal $O\,p$, Fig. 17, and passes through the pitch point O. Except when the point of contact is at the pitch point the direction of the pressure will deviate from the normal to the line of centers by the angle of obliquity $Z\,O\,p$, and with many forms of teeth the angle is never zero.

The force exerted between two teeth at their point of contact is found by laying off the tangential force $O\,H$ with which the driving gear D is turning, and drawing the line $H\,V$ parallel to the line of centers, to find the force $O\,V = P\,K$. It is proportional to the secant of the angle of obliquity, and increases rapidly with that angle.

The chief influence of the obliquity is upon the friction between the teeth, and consequent inefficiency of the gear, and upon the destruction by wearing. It is particularly important upon the approaching action, and a gear that is otherwise perfect may be inoperative on account of excessive obliquity.

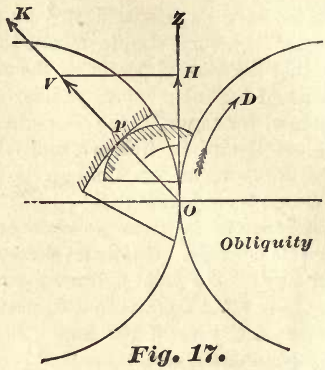


Fig. 17.

Although the direct pressure of the teeth upon each other at their point of contact will vary with the obliquity, the tangential force exerted to turn the gear is always uniform. Leaving friction out of the calculation, the two gears of a pair always turn with the same force at their pitch lines.

The obliquity of the action has an effect upon the direction and amount of the pressure of the gear upon its shaft bearing, but the usual variation is of little consequence.

It is desirable that the pressure between the teeth should be as uniform as possible, not only in amount, but in direction, and excessive obliquity is to be carefully avoided.

27.—CONSTRUCTION BY MOLDING.

The mode of action of the conjugate teeth upon each other, suggests a process by which a given tooth can be made to form its conjugate by the process of molding.

The given tooth, all of its normal sections being of some odontoidal form, is made of some hard substance, while the blank in which the conjugate teeth are to be formed is made of some plastic material. The shafts of the two wheels are given, by any means, the same motions as if their pitch surfaces were rolled together. The hard tooth will then mold the soft tooth into the true conjugate shape.

It matters not what shape is given the molding tooth, if its sections are all odontoidal, and a twisted or irregular shape will be as serviceable as the common straight tooth.

This process is continually in operation between a pair of newly cut teeth, or between rough cast teeth, until the badly matched surfaces have been worn to a better fit, but it is too slow for ordinary purposes, and is of little practical value.

Gears can be formed by this process, by rolling a steel forming gear against a white hot blank, but the process can hardly be called practical.

28.—MOLDING PLANING PROCESS.

Although the described molding process is of limited practical value, having but one direct application, it leads to a process of great value when the tooth is straight or of such a shape that it can be followed by a planing tool, its normal sections being alike.

The originating tooth is fixed in the shape of a steel cutting tool C, Fig. 18, which is rapidly reciprocated in guides G, in the direction of the length of the tooth, as the two pitch wheels A and B are rolled together. Although the tool has but a single cutting edge, its motion makes it the equivalent of the molding tooth, and it will plane out the conjugate tooth D by a process that is the equivalent of the more general molding process.

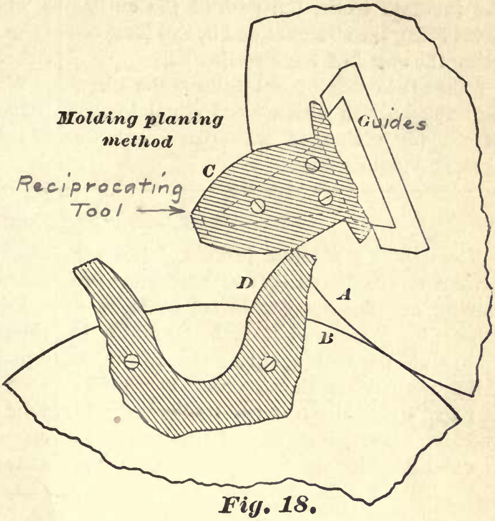


Fig. 18.

A simple graphical method is founded upon this molding process, the shaping tool taking the form of a thin template C, Fig. 19, that is repeatedly scribed about as the pitch wheels are rolled together, the marks combining to form the conjugate tooth curves D.

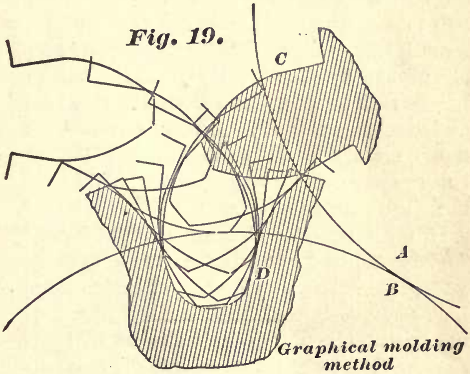
This mechanical process has the decided advantage over the process of construction by points (24), that the tooth is formed with a correct fillet (44), and is much stronger. The dotted lines show the tooth that would be constructed by points.

The only practicable method for forming the line of action when this method is used is by observing and marking a number of points of contact between the teeth. This method is applicable to all possible forms of spur teeth, either straight, twisted or spiral. It can be practically applied only to the octoid form of bevel tooth.

On account of the fillet (44) that is formed by this process, the tooth space cannot be used with a mating gear having more teeth than that of the forming gear, although it belongs to the same interchangeable set. The tooth space of the figure will not run with a tooth on a pitch line larger than the pitch line A.

Therefore the rack tooth must be used as the forming tooth, to allow of the use of all gears of the set up to the rack. Gears of the set thus formed will not work with internal gears.

Fig. 19.



29.—LINEAR PLANING PROCESS.

A second planing process, quite distinct from the molding process of (27), is founded upon the fact that the tooth curves are in contact at a single point which has a progressive motion along the line of action.

Therefore if a single cutting point p, Fig. 20, is caused to travel along the line of action with the proper speed relatively to the speed of the pitch line, it will trim the tooth outline to the proper odontoidal shape.

The figure shows the application to

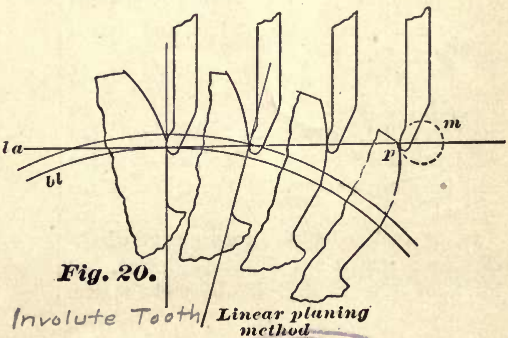


Fig. 20.

the involute tooth, the path of the cutting point being the straight line *l a*, and its speed being the speed of the base line *b l*.

When the cutting point follows the circular line of action with a speed equal to that of the pitch line, it will plane out the cycloidal tooth curve.

This process is applicable to all possible forms of gear teeth, either spur or bevel, in either external or internal contact.

When the curvature of the odontoid will permit, the milling cutter may take the place of the planing tool, and is the equivalent of it.

30.—THE RACK ORIGINATOR.

The molding planing process of (28) supplies a means for easily and accurately producing an interchangeable set of gears or cutters for gears, and it is best applied by means of the rack tooth as the originator. All four curves of the rack tooth being alike, the tooth is easily formed, particularly for the involute or the segmental systems, and it is a matter of less consequence that the curves shall be of some particular form, if care is taken that it is odontoidal.

It has been taught, and it is therefore sometimes considered, that any "four similar and equal lines in alternate reversion" will answer the purpose, but it is necessary that the four similar curves shall be odontoids. Four circular arcs, with centers on the pitch line, will answer the definition, but are not odontoids.

31.—PARTICULAR FORMS OF THE ODONTOID.

The odontoid, as so far examined, is undefined except as to one feature of the arrangement of its normals, and to bring it into practical use it is neccssary to give it some definite shape. This is most easily accomplished by choosing some simple curve for the rack odontoid, and from that making an interchangeable set. A more correct but much more difficult method would be to choose some definite line of action, and from that derive the odontoids.

If the rack odontoids are straight lines, Fig. 21, the common involute tooth system will be produced, and the line of action will be a straight line at right angles with the rack odontoid. For bevel teeth, as will be shown, the straight line odontoid produces the octoid tooth system, while to produce the involute system it is necessary to define the line of action as a straight line, and derive the system from that.

If the rack odontoids are cycloids, as in Fig. 22, the resulting tooth system will be the cycloidal, commonly misnamed the "epicycloidal" system. The line of action will be a circle equal to the roller of the cycloid.

If the rack odontoids are segments of circles from centers not on the pitch line, but inside of it, as in Fig. 23, the tooth system

Fig. 21.

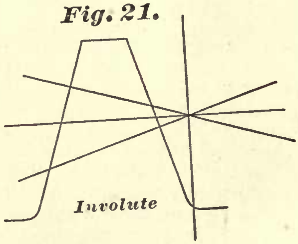
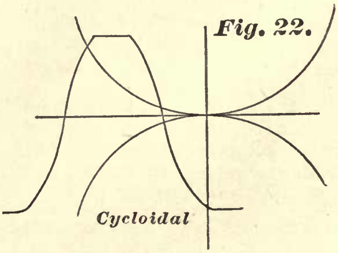


Fig. 22.



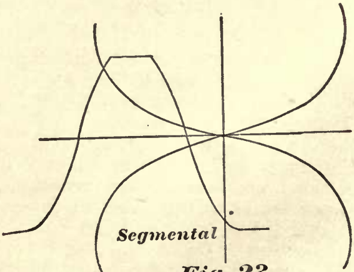


Fig. 23.

will be the segmental, and its line of action will be the loop of the "Conchoid of Nicomedes."

If we choose a parabola for the rack tooth, as in Fig. 24, the parabolic system will be formed with its peculiar "hour glass" line of action.

Only three of these tooth systems are in actual use, the involute and the cycloidal for spur gears, and the octoid for bevel gears only, and we will therefore confine the application of the theory to them.

Only one of the systems in common use for spur gears, the involute, should be in use at all, and we will pay principal attention to that.

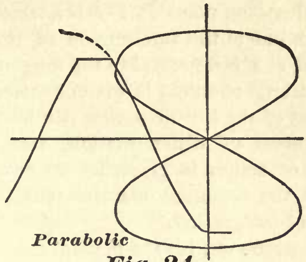


Fig. 24.

The segmental system would be superior to the cycloidal, and in many cases to the involute; but as there is already one system too many, we will not attempt to add another.

32.—THE ROLLED CURVE THEORY.

If any curve R, Fig. 25, is rolled on any pitch curve $p\ l$, a point p in the former will trace out on the plane of the latter a curve $s\ p\ z$, called a rolled curve.

The line $p\ q$, from the tracing point p to the point of contact q, is a normal to the curve $s\ p\ z$, and, as all the normals are arranged in "consecutive" order, that curve must be an odontoid. The converse of this statement is also true, that all odontoids are rolled curves; but the fact is generally very far fetched and of no practical importance.

It is also a property of all such curves that are rolled on different pitch lines, that they are interchangeable.

This accidental and occasionally useful feature of the rolled curve has generally been made to serve as a basis for the general theory of the gear tooth curve, and it is responsible for the usually clumsy and limited treatment of that theory. The general law is simple enough to define, but it is so difficult to apply, that but one tooth curve, the cycloidal, which happens to have the circle for a roller, can be intelligently handled by it, and the natural result is, that that curve has received the bulk of the attention.

For example, the simplest and best of

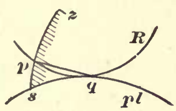


Fig. 25.

all the odontoids, the involute, is entirely beyond its reach, because its roller is the logarithmic spiral, a transcendental curve that can be reached only by the higher mathematics.

No tooth curve, which, like the involute, crosses the pitch line at any angle but a right angle, can be traced by a point in a simple curve. The tracing point must be the pole of a spiral, and therefore the tracing of such a curve is a mechanical impossibility. A practicable rolled odontoid must cross the pitch line at right angles.

To use the rolled curve theory as a base of operations will confine the discussion to the cycloidal tooth, for the involute can only be reached by abandoning its true logarithmic roller, and taking advantage of one of its peculiar properties, and the segmental, sinusoidal, parabolic, and pin tooth, as well as most other available odontoids, cannot be discussed at all.

33.—MATHEMATICAL RELATION OF ODONTOID AND LINE OF ACTION.

In Fig. 26 the odontoid on the pitch line $p\ l$ is connected with the line of action $l\ a$, by the relations $P\ T = p\ t = y$, and $T\ S = t\ O = x$, where $P\ S$ is the normal to the odontoid at the point P, $T\ S$ is a tangent to the pitch line at the intersection of the normal, and $P\ T$ is a normal to the tangent.

When any odontoid is given by its equation, that of the line of action can be found by a process of differentiation, and when the line of action is given by its equation, that of the odontoid can be found by a process of integration.

These processes, for the general case where the pitch line is curved, are quite intricate, but when the pitch line is a straight line, they are simple, and may be worked as follows.

To get the equation of the line of action from that of the given rack odontoid, arrange the equation of the odontoid in the form $x = f(y)$, and put its differential coefficient $\frac{dx}{dy}$ equal to $\frac{y}{x}$. Thus, the equation of the straight rack odontoid of the involute system is $y = x$ tan. A, from which $\frac{dx}{dy} = \frac{1}{\tan. A} = \frac{y}{x}$, and $y = \frac{x}{\tan. A}$ is the equation of the straight line of action at right angles to the odontoid. Again, the equation of the cycloid being $x =$ ver. sin.$^{-1}$ $y - \sqrt{2ry - y^2}$, $x =$ ver. sin. $^{-1}$

$$\frac{dx}{dy} = \frac{y}{\sqrt{2ry - y^2}} = \frac{y}{x},$$

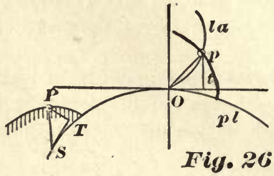


Fig. 26.

and $x^2 + y^2 = 2ry$ is the equation of the circular line of action.

To get the equation of the odontoid when that of the line of action is given, arrange the equation of the line of action in the form $\frac{y}{x} = f(y)$, put it equal to $\frac{dx}{dy}$, and integrate. Thus, the equation of the straight line of action being

$$y = \frac{x}{\tan. A},$$

we have

$$\frac{y}{x} = \frac{1}{\tan. A} = \frac{dx}{dy},$$

and $y = x$ tan. A is the equation of the straight odontoid at right angles to the line of action. Again, the equation of the circular line of action being $x^2 + y^2 = 2ry$, we have

$$\frac{y}{x} = \frac{y}{\sqrt{2ry - y^2}} = \frac{dx}{dy},$$

and $x =$ ~~ver. sin.$^{-1}$ $y - \sqrt{2ry - y^2}$~~ is the cycloidal odontoid. r ver $^{-1}\frac{y}{r} - \sqrt{2ry - y^2}$

34.—THE ODONTOIDAL INTEGRATER.

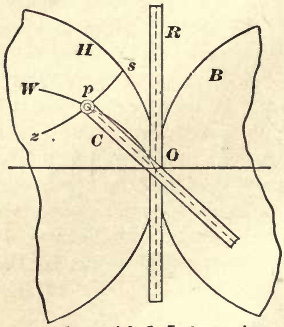
The form of the odontoid to correspond to a given line of action and a given pitch line can be determined only by the integral calculus (33), it evidently being impossible to contrive a general graphical or algebraic method.

But it can be directly drawn by an instrument, the principle of which is analogous to that of the well-known polar planimeter for integrating surfaces.

The bar R, Fig. 27, moves at right angles to the line of centers, and it moves the pitch wheel A, with the same speed at the pitch line. The bar C has a point p, that is confined to move in the given line of action $O\ p\ W$, and it is so guided that it always passes through the pitch point O.

The two bars bear upon each other by friction, and we must suppose that there is no other friction to oppose the motion of the bar C.

Fig. 27.



Odontoidal Integrater

Then the point p will trace out the odontoid $s\ p\ z$ upon the pitch wheel A, or upon any other pitch wheel B rolling with the bar R on either side of it.

2. THE SPUR GEAR IN GENERAL.

35.—THE CIRCULAR PITCH.

The distance *a O*, Fig. 14, covered by each tooth upon the pitch circle, is commonly called the "circular pitch," and often the "circumferential pitch." The term "pitch arc" is the most appropriate but is not in common use.

This was formerly the measurement by which the size of the tooth was always stated, a tooth being said to be of a certain "pitch," and all of its other dimensions being expressed in terms of that unit, but it is fast being replaced, and should be entirely replaced, by the more convenient "diametral pitch" unit.

The circumference of a circle is measured in terms of its diameter by means of an incommensurable fractional number 3.14159, called π (pi), and, therefore, if the tooth is measured upon the arc of the circle by means of the circular pitch, one of two inconveniences must be tolerated. Either the pitch must be an inconvenient fraction, or else the pitch diameter must be as inconvenient, for the gear cannot have a fractional number of teeth. The fractional calculations are so clumsy that a table of pitch diameters corresponding to given numbers of teeth should be used, and errors in the laying out of the work are of constant occurrence.

Again, outside of the liability of error in making calculations, the circular pitch system is a constant source of error in the hands of lazy or incompetent draftsmen or workmen, for there is a constant temptation, often yielded to, to force the clumsy figures a little to produce some desired result. For example, a millwright has to make a gear of fourteen inches pitch diameter with fourteen teeth. He finds by the usual computation that the circular pitch is 3.14 inches, and, as his odontograph has a table for three-inch pitch, he uses that with the remark that it is "near enough," laying the blame on the odontograph or on the iron founder if the resulting gear roars. His next order is for a gear of one-inch pitch to match others in use, and to be fourteen and a half inches diameter. The circumference of the pitch line is 45.53 inches, and he has his choice between 45 and 46 teeth, both wrong. Perhaps the most frequent cause of error is that the workman is apt to apply a rule directly to the teeth of a gear he is about to repair or match, to get the circular pitch, and the result is more likely to be wrong than right.

The best plan when using this unit is to get convenient pitch diameters and let the pitch come as it will, provided that gears that work together are of the same pitch, and that is simply a roundabout way of using the diametral pitch unit.

When the circular pitch must be used the following table will greatly assist the work and save calculation. For example, the pitch diameter of a gear of three-quarter-inch pitch and 37 teeth is three-quarters the tabular number 11.78, or 8.84 inches.

PITCH DIAMETERS.

FOR ONE INCH CIRCULAR PITCH.

FOR ANY OTHER PITCH MULTIPLY BY THAT PITCH.

T.	P. D.	T.	P. D.	T.	P. D.	T.	P. D.
10	3.18	33	10.50	56	17.83	79	25.15
11	3.50	34	10.82	57	18.14	80	25.47
12	3.82	35	11.14	58	18.46	81	25.79
13	4.14	36	11.46	59	18.78	82	26.10
14	4.46	37	11.78	60	19.10	83	26.42
15	4.78	38	12.10	61	19.42	84	26.74
16	5.09	39	12.42	62	19.74	85	27.06
17	5.41	40	12.73	63	20.06	86	27.38
18	5.73	41	13.05	64	20.37	87	27.70
19	6.05	42	13.37	65	20.69	88	28.01
20	6.37	43	13.69	66	21.01	89	28.33
21	6.69	44	14.00	67	21.33	90	28.65
22	7.00	45	14.33	68	21.65	91	28.97
23	7.32	46	14.64	69	21.97	92	29.29
24	7.64	47	14.96	70	22.28	93	29.60
25	7.96	48	15.28	71	22.60	94	29.92
26	8.28	49	15.60	72	22.92	95	30.24
27	8.60	50	15.92	73	23.24	96	30.56
28	8.91	51	16.24	74	23.56	97	30.88
29	9.23	52	16.55	75	23.88	98	31.20
30	9.55	53	16.87	76	24.19	99	31.52
31	9.87	54	17.19	77	24.51	100	31.83
32	10.19	55	17.51	78	24.83		

36.—THE DIAMETRAL PITCH.

This is not a measurement, but a ratio or proportion. It is the number of teeth in the gear divided by the pitch diameter of the gear. Thus, a gear of 48 teeth and 12 inches pitch diameter is of 4 pitch. The advantages of the diametral pitch unit are so apparent that it is fast displacing the circular pitch unit, and has almost entirely displaced it for cut gearing. It is so simple that a table of pitch diameters is entirely useless, although such useless tables have been published.

The diametral pitch is sometimes defined as the number of teeth in a gear of one inch diameter. It is a common, but bad practice, to designate diametral pitches by numbers, as No. 4, No. 16, etc.

37.—RELATION OF PITCH UNITS.

The product of the circular pitch by the diametral pitch is the constant number 3.1416, so that if one is given the other is easily calculated.

The following tables of equivalent pitches will be convenient in this connection.

38.—ACTUAL SIZES.

Figs. 28 and 29 show the actual sizes of standard teeth of the usual diametral pitches, and give a better idea of the actual teeth than can be given by any possible description. They are printed from cut teeth, and may be depended upon as accurate.

Diametral Pitch.	Circular Pitch.	Circular Pitch.	Diametral Pitch.
2	1.571 inch	2	1·571
2¼	1.396 "	1⅞	1 676
2½	1.257 "	1¾	1.795
2¾	1.142 "	1⅝	1.933
3	1.047 "	1½	2 094
3½	.898 "	1 7/16	2 185
4	.785 "	1⅜	2.285
5	.628 "	1 5/16	2.394
6	.524 "	1¼	2.513
7	.449 "	1 3/16	2.646
8	.393 "	1⅛	2.793
9	.349 "	1 1/16	2.957
10	.314 "	1	3.142
11	.286 "	15/16	3.351
12	.262 "	⅞	3.590
14	.224 "	13/16	3.867
16	.196 "	¾	4.189
18	.175 "	11/16	4.570
20	.157 "	⅝	5.027
22	.143 "	9/16	5.585
24	.131 "	½	6.283
26	.121 "	7/16	7 181
28	.112 "	⅜	8 378
30	.105 "	5/16	10 053
32	.098 "	¼	12.566
36	.087 "	3/16	16.755
40	.079 "	⅛	25.133
48	.065 "	1/16	50.266

39.—ADDENDUM AND DEDENDUM.

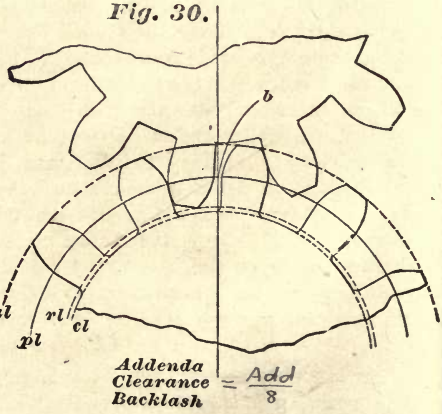
The tooth is limited in length by the circle *a l*, Fig. 30, called the addendum line, and drawn outside the pitch line at a given distance, called the addendum. Its depth is also limited by a line *r l*, called the dedendum or root line, drawn at a given distance inside of the pitch line.

The addendum and the dedendum are both arbitrary distances, but, for convenience in computation, they are fixed at simple fractions of the unit of pitch that is in use. When the circular pitch is used the addendum is one-third of the circular pitch.

When the diametral pitch unit is used the addendum is one divided by the pitch.

It is customary to make the addendum and the dedendum the same, except in certain cases where some special requirement is to be satisfied.

Fig. 30.



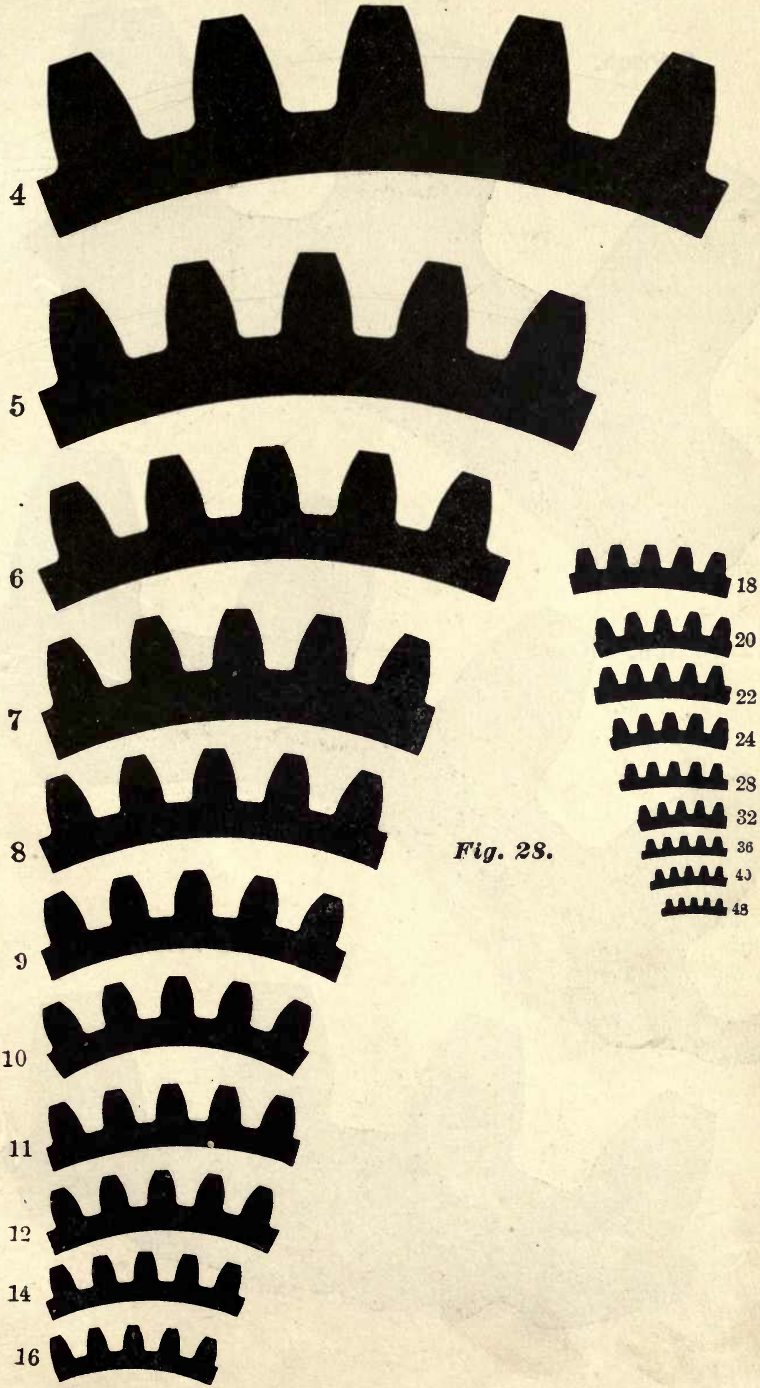


Fig. 28.

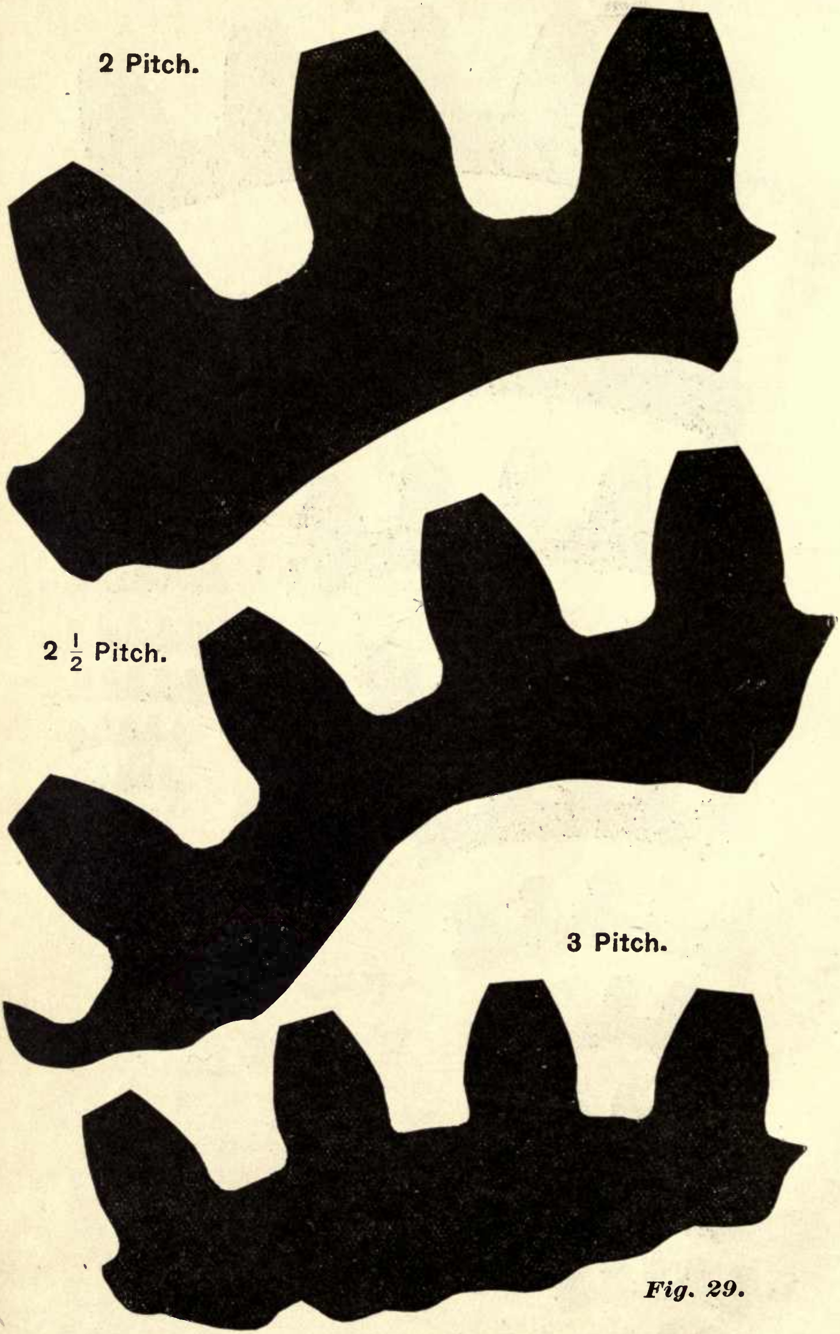


Fig. 29.

40.—THE CLEARANCE.

To allow for the inevitable inaccuracies of workmanship, especially on cast gearing, it is customary to carry the tooth space slightly below the root line to the clearance line *c l*, Fig. 30.

The clearance, or distance of the clearance line inside of the root line, is arbitrary, but it is convenient and customary to make it one-eighth of the addendum.

41.—THE BACK-LASH.

When rough wooden cogs or cast teeth are used, the irregularities of the surface, and inaccuracies of the shape and spacing of the teeth, require that they should not pretend to fit closely, but that they should clear each other by an amount *b*, Fig. 30, called the back-lash.

The amount of the back-lash is arbitrary, but it is a good plan to make it about equal to the clearance, one-eighth of the addendum.

Skillfully made teeth will require less back-lash than roughly shaped teeth, and properly cut teeth should require no back-lash at all. Involute teeth require less back-lash than cycloidal teeth.

42.—THE STANDARD TOOTH.

The tooth must be composed of odontoids, preferably of odontoids of which the properties are well known, and an advantage is gained if it is still further confined to a particular value of that odontoid. If the teeth are to be drawn by an odontograph some standard must be fixed upon, since the method will cover but one proportion of tooth.

For example, the standard involute tooth is that having its line of action inclined at an angle of obliquity of fifteen degrees. For the cycloidal system the standard agreed upon is the tooth having radial flanks on a gear of twelve teeth.

43.—ODONTOGRAPHS.

The construction of the tooth is generally not simply accomplished by graphical means, as it is generally required to find points in the curve and then find centers for circular arcs that will approximate to the curve thus laid out.

It is sometimes attempted to construct the curve by some handy method or empirical rule, but such methods are generally worthless.

An odontograph is a method or an instrument for simplifying the construction of the curve, generally by finding centers for approximating circular arcs without first finding points on the curve, and those in use will be described.

44.—THE FILLET.

When the teeth are laid out by theory there will be a portion of the tooth space at the bottom that is never occupied by the mating tooth. Fig. 31 shows a ten-toothed pinion tooth and space with a rack tooth in three of its positions in it, showing the unused portion by the heavy dotted line.

If this unused space is filled in by a "fillet" *f* the tooth will be strengthened just where it needs it the most, at the root.

The fillet is dependent on the mating tooth, and is therefore not a fixed feature of the tooth. If a gear is to work in an interchangeable set, it may at some time work with a rack, and therefore its fillet should be fitted to the rack; but if it is to work only with some one gear it may be fitted to that. The light dotted line shows the fillet that would be adapted to a ten-toothed mate. The fillet to match an internal gear tooth would be even smaller than that made by the rack.

Fig. 31.

The fillet

When the tooth is formed by the molding process of (27), or by the equivalent planing process of (28), the fillet will be correctly formed by the shaping tool, but not so when the linear process of (29) is used. When the tooth is drawn by theory or by an odontograph the fillet must be drawn in, and can be most easily determined by making a mating tooth of paper, and trying it in several positions in the tooth space, as in the figure.

Except on gears of very few teeth the strength gained will not warrant the trouble of constructing the fillet.

45.—THE EQUIDISTANT SERIES.

When arranging an odontograph for drafting teeth, or a set of cutters for cutting them, we must make one sizing value do duty for an interval of several teeth, for it is impracticable to use different values for two or three hundred different numbers of teeth. The object of the equidistant series is to so place these intervals that the necessary errors are evenly distributed, each sizing value being made to do duty for several numbers each way from the number to which it is fitted, and being no more inaccurate than any other for the extreme numbers that it is forced to cover.

This series is readily computed for any case that may arise, and with a degree of accuracy that is well within the requirements of practice; by the formula

$$t = \frac{a\,n}{n - s + \frac{a\,s}{z}}$$

in which a is the first and z is the last tooth of the interchangeable series to be covered; n is the number of intervals in the series, and s is the number in the series of any interval of which the last tooth t is required.

For example, it is required to compute the series here used for the cycloidal odontograph, having twelve tabular numbers to cover from twelve teeth to a rack.

Putting $a = 12$, $z =$ infinity, and $n = 12$, the formula becomes

$$t = \frac{12 \times 12}{12 - s + \frac{12\,s}{\infty}} = \frac{12 \times 12}{12 - s + 0} = \frac{144}{12 - s}$$

and then, by putting s successively equal to 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12, we get the series of last teeth, $13\frac{1}{11}$, $14\frac{2}{5}$, 16, 18, $20\frac{4}{7}$, 24, $28\frac{4}{5}$, 36, 48, 72, 144, and infinity. These give the required equidistant series of intervals.

12
13 to 14,
15 to 16,
17 to 18,
19 to 21,
22 to 24,
25 to 28
30 to 36,
37 to 48,
49 to 72,
73 to 144,
145 to a rack ;

and the method is as easily applied to any other practical example.

This formula and method is independent of the form and of the length of the tooth, and therefore is applicable to all systems under all circumstances. This is proper and convenient, for these elements can be eliminated without vitiating the results or destroying the "equidistant" characteristic of the series. The formula is an approximation based upon an assumption, but nothing more convenient or more accurate has so far been devised by laboriously considering all the petty elements involved.

The sizing value, or number for which the tabular number is computed, or the cutter is accurately shaped, can best be placed, not at the center of the interval, but by considering the interval as a small series of two intervals, and adopting the intermediate value. The sizing value for the interval from c to d is given by the formula

$$v = \frac{2\,c\,d}{c + d}.$$

Thus, the sizing value for the interval from 37 to 48 teeth should be 41.8, and that for the interval from 145 to a rack should be 290.

It is sometimes the practice to size the cutter for the lowest number in its interval, on the ground that a tooth that is considerably too much curved is better than one that is even a little too flat. This makes the last tooth of the interval much more inaccurate than if the medium number was used.

46.—THE HUNTING COG.

It is customary to make a pair of cast gears with incommensurable numbers of teeth so that each tooth of each gear will work with all the teeth of the other gear. If a pair of equal gears have twenty teeth each, each tooth will work with the same mating tooth all the time; but if one gear has twenty and the other twenty-one teeth, or any two numbers not having a common divisor, each tooth will work with all the mating teeth one after the other.

The object is to secure an even wearing action; each tooth will have to work with many other teeth, and the supposition is that all the teeth will eventually and mysteriously be worn to some indefinite but true shape.

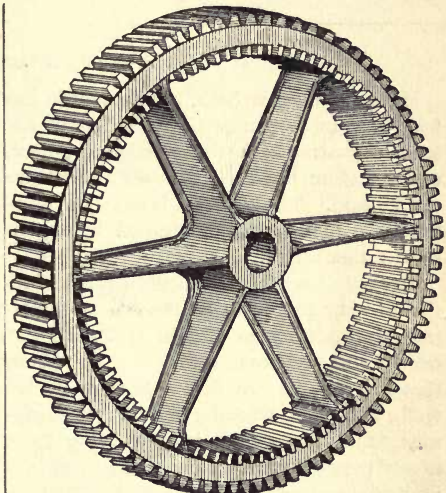
It would seem to be the better practice to have each tooth work with as few teeth as possible, for if it is out of shape it will damage all teeth that it works with, and the damage should be confined within as narrow limits as possible. If a bad tooth works with a good one it will ruin it, and if it works with a dozen it will ruin all of them. It is the better plan to have all the teeth as near perfect as possible, and to correct all evident imperfections as soon as discovered.

47.—THE MORTISE WHEEL.

Another venerable relic of the last century is the "mortise" gear, Fig. 32, having wooden teeth set in a cored rim, in which they are driven and keyed.

Where a gear is subjected to sudden strains and great shocks, the mortise wheel is better, and works with less noise than a poor cast gear, and will carry as much as or more power at a high speed with a greater durability. But in no case is it the equal of a properly cut gear, while its cost is about as great.

In times when large gears could not be cut, and when the cast tooth was not even approximately of the proper shape, the mortise wheel had its place, but now that the large cut gear can be obtained the mortise gear should be dropped and forgotten.



Mortise wheel
Fig. 32.

48.—THE FRICTION OF APPROACH.

When the point of action between two teeth is approaching the pitch point, that is, when the action is approaching, the friction between the two tooth surfaces is greater than when the action is receding. This extra friction is always present, but is most troublesome when the surfaces are very rough, as on cast teeth, giving little trouble when the teeth are properly shaped and well cut. When the roller pin gear (93) is used, the friction between the teeth is rolling friction, and is no greater on the approach than on the recess.

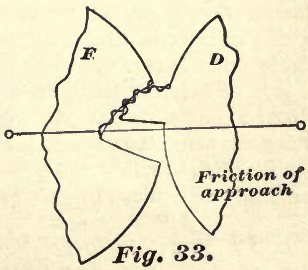


Fig. 33.

The difference in the friction is probably due to the difference in the direction of the pressure between the small inequalities to which all friction is due. When the gear D, Fig. 33, is the driver, the action between the teeth is receding, and the inequalities lift over each other easily, while if F is the driver, the action is approaching, and the inequalities tend to jam together.

In the exaggerated case illustrated, it is plain that the teeth are so locked together that approaching action is impossible, while it is equally plain that motion in the other direction is easy. The same action takes place in a lesser degree with the small inequalities of ordinary rough surfaces.

The action of the common friction pawl, which works freely in one direction and jams hard in the other, is upon the same principle. A weight may be easily dragged over a rough surface that it could not be pushed over by a force that is not parallel to the surface.

The extra friction of approaching action can be avoided by giving the driver the longest face. When the driver has faces only, and the follower has only flanks, the action is particularly smooth.

Teeth that are subject to excessive maximum obliquity, such as cycloidal teeth, should not be selected for rough cast gearing, for it is the maximum rather than the average obliquity that has the greatest influence.

49.—EFFICIENCY OF GEAR TEETH.

Much has been written, but very little has been done to determine the efficiency of the teeth of gearing in the transmission of power, and therefore but little of a definite nature can be said. The question is mostly a practical one, and should be settled by experiment rather than by analysis.

The only known experiments upon the friction of spur gear teeth are the Sellers experiments, more fully detailed in (112), and but one of these relates to the spur gear. From that one it is known that a gear of twelve teeth, two pitch, working in a gear of thirty-nine teeth, has an efficiency varying from ninety per centum at a slow speed to ninety-nine per centum at a high speed. That is, an average of five per centum of the power received is wasted by friction at the teeth and shaft bearings. This result is probably a close approximation to that for any ordinary practical case.

Although theory can do nothing to decide such a question as this, it can do much to indicate probable results.

If a pair of involute teeth, for example, move over a certain distance, w, either way from the pitch point, the distance being measured on the pitch line, they will do work that is theoretically determined by the formula :

$$\text{work done} = \frac{f\,P}{2} \cdot \frac{k \pm h}{k\,h}\, w^2$$

in which f is the coefficient of friction, P is the pressure, and k and h are the pitch radii of the gears. The positive sign is to be used for gears in external, and the negative sign for those in internal contact.

The loss by friction, as shown by the formula, decreases directly as the diameters increase, the proportion of the diameters being constant.

The loss increases rapidly with the distance of the point of action from the pitch point. When the contact is at the pitch point the teeth do not slide on each other, and there is no loss, but away from that point the loss is as the square of the distance in this case, and in a still greater proportion in the case of the cycloidal tooth. Therefore a short arc of action tends to improve the efficiency.

It has been satisfactorily determined that the loss is greater during the approaching than during the receding action. This is not shown by the formula, but it may be laid to a variation in the coefficient f.

The formula shows that the loss is independent of the width or face of the gear, and therefore strength can be increased by widening the face, without increasing the friction.

If the work of internal gearing is compared with that of external gearing of the same sizes, the losses are in the proportion,

$$\frac{k - h}{k + h}.$$

so that the internal gear is much the more economical, particularly when the gear and pinion are nearly of the same size. If the gear is twice the size of the pinion the loss is but one-third of the loss when both gears are external.

Small improvement can be effected, by putting a small pinion inside rather than outside of a large gear. A six-inch pinion working with a six-foot gear has but 1.18 times the loss by the same gears, when the gear is internal.

Theoretical efficiency is discussed at great length in the Journal of the Franklin Institute, for May, 1887: Also by Reuleaux, and again by Lanza, in the Transactions of the American Society of Mechanical Engineers for 1887, and the discussion has been carried far enough.

A series of experiments with gear teeth of various sizes and forms, of various metals, would add greatly to our knowledge of this important matter.

A true determination of the efficiency of the rough cast gear, as compared with that of the cut gear, would tend to discourage the use of the former for the transmission of power, for experiment would undoubtedly show that the power wasted by the cast gear would soon pay the difference in cost of the better article.

50.—STRENGTH OF A TOOTH.

The strength of a tooth is the still load it will carry, suspended from its point, and is to be carefully distinguished from the horse-power, or the load the gear will carry in motion.

The strength of a substance is not a fixed element, but will vary with different samples, and with the same sample under different circumstances; allowance must be made for the amount of service the sample has seen, concealed defects must be provided against, and therefore nothing but an actual test will surely determine its character.

Although no possible rule can be depended upon, the ultimate or breaking strength of a standard cast-iron tooth, having an addendum about equal to a third of the circular pitch, will average about three thousand five hundred pounds multiplied by the face of the gear and again by the circular pitch, both in inches.

But a tooth should never be forced up to its ultimate strength, and the best practice is to give it only about one-tenth of the load it might possibly bear, so that the following rule should be used: Multiply three hundred and fifty pounds by the face of the gear, and again by the circular pitch, both in inches, and the product will be the safe working load of one tooth.

Example: A cast-iron gear of one inch pitch, and two inches face, will safely lift $350 \times 2 \times 1 = 700$ pounds, although it would probably lift 7,000 pounds.

When there are two teeth always in working contact, it is safe to allow double the load, but care must be taken that both teeth are always in full contact.

A hard wood mortised cog has about one-third of the strength of a cast-iron tooth; steel has double the strength; wrought-iron is not quite as strong.

A small pinion generally has teeth that are weak at the roots, and then it will increase the strength to shroud the gear up to its pitch line, but shrouding will not strengthen a tooth that spreads towards its base, like an involute tooth, and when the face of the gear is wide compared with the length of the tooth the shroud is of little assistance.

It does not increase the strength of a tooth to double its pitch, for when the pitch is increased the length is also increased, and the strength is still in direct proportion to the circular pitch, while the increase has reduced the number of teeth in contact at a time.

Cut gears and cast gears are about equal as to actual strength, with the advantages in favor of the cut gear, that hidden defects are likely to be discovered, and that it is not as liable to undue strains on account of defective shape.

The rules for strength must not be used for gears running at any considerable speed, for they are intended only for slow service, as in cranes, heavy elevators, power punches, etc.

51.—HORSE-POWER OF CAST GEARS.

The horse-power of a gear is the amount of power it may be depended upon to carry in continual service.

It is very well settled that continual strains and impact will change the nature of the metal, rendering it more brittle, so that a tooth that is perfectly reliable when new may be worthless when it has seen some years of service. This cause of deterioration is particularly potent in the case of rough cast teeth, for they can only approximate to the true shape required to transmit a uniform speed, and the continual impact from shocks and rapid variations in the power carried must and does destroy the strength of the metal.

There are about as many rules for computing the power of a gear as there are manufacturers of gears, each foundryman having a rule, the only good one, which he has found in some book, and with which he will figure the power down to so many horses and hundredths of a horse as confidently as he will count the teeth or weigh the casting.

Even among the standard writers on engineering subjects the agreement is no better, as shown by Cooper's collection of twenty-four rules from many different writers, applied to the single case of a five-foot gear. See the "Journal of the Franklin Institute" for July, 1879. For the single case over twenty different results were obtained, ranging from forty-six to three-hundred horse-power, and proving conclusively that the exact object sought is not to be obtained by calculation.

This variety is very convenient, for it is always possible to fit a desired power to a given gear, and if a badly designed gear should break, it is a simple matter to find a rule to prove that it was just right, and must have met with some accident.

Although no rule can be called reliable, the one that appears to be the best is that given by Box, in his Treatise on Mill Gearing. Box's rule, which is based on many actual cases, and which gives among the lowest, and therefore the safest results, is by the formula:

$$\text{Horse-power of a cast gear} = \frac{12\, c^2 f \sqrt{d\, n}}{1{,}000}$$

in which c is the circular pitch, f is the face, d is the diameter, all in inches, and n is the number of revolutions per minute.

Example: A gear of two feet diameter, four inches face, two inches pitch, running at one hundred revolutions per minute, will transmit

$$\frac{12 \times 2 \times 2 \times 4 \times \sqrt{24 \times 100}}{1{,}000} = 9.4 \text{ h. p.}$$

For bevel gears, take the diameter and pitch at the middle of the face.

It is perfectly allowable, although it is not good practice, to depend upon the gear for from three to six times the calculated power, if it is new, well made, and runs without being subjected to sudden shocks and variations of load.

The influence of impact and continued service will be appreciated when it is considered that the gear in the example, which will carry 9.4 horse-power, will carry seventy horse-power if impact is ignored, and the ultimate strength of the metal is the only dependence.

A mortise gear, with wooden cogs, will carry as much as, or more than a rough cast-iron gear will carry, although its strength is much inferior. The elasticity of the wood allows it to spring and stand a shock that would break a more brittle tooth of much greater strength. And, for the same reason, a gear will last longer in a yielding wooden frame than it will in a rigid iron frame.

52.—HORSE-POWER OF CUT GEARS.

We know a little, and have to guess the rest, as to the power of a cast gear, but with respect to that of a cut gear we are not as well posted, for there are no experimental data upon which a reliable rule can be founded.

Admitting, as we must, that impact is the chief cause of the deterioration of the cast gear, we are at liberty to assume that a properly cut and smoothly running cut gear is much more reliable.

No definite rule is possible, but we can safely assume that a cut gear will carry at least three times as much power as can be trusted to a cast gear of the same size.

The great reliance of those who claim that a cast gear is superior to a cut gear is upon the hard scale with which the cast tooth is covered. This scale is not over one-hundredth of an inch thick, is rapidly worn away, and is of no account whatever. From that point of view it is difficult to explain why a wooden tooth will outwear an iron one, although it is softer than the softest cut iron.

Assuming that a cut gear is about three times as reliable as a cast gear, we can compute its power by the formula :

$$\text{Horse-power of a cut gear} = \frac{c^2 f \sqrt{d\,n}}{30}$$

in which c is the circular pitch, f is the face, and d is the pitch diameter, all in inches, and n is the number of revolutions per minute.

3. THE INVOLUTE SYSTEM.

53.—THE INVOLUTE TOOTH.

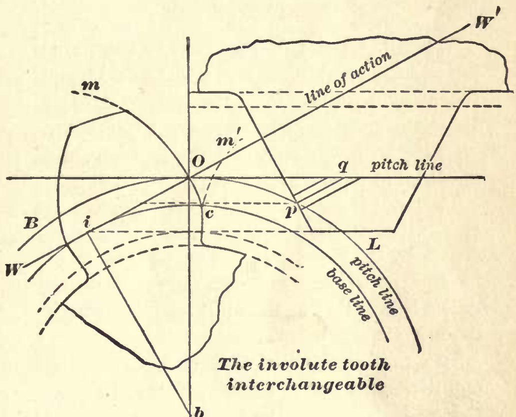
The simplest and best tooth curve, theoretically, as well as the one in greatest practical use for cut gearing, is the involute.

The involute tooth system is based on the straight rack odontoid, (31) and Fig. 21, and it is illustrated by Fig. 34. If the four odontoids of the rack outline are equally inclined to the pitch line, the resulting tooth system will be completely interchangeable; but if, as in Fig. 35, the face and flank are inclined at different angles of obliquity, $T\ S\ K$ and $T'\ S\ K'$, the system is not interchangeable, although otherwise perfect.

The rack odontoid cannot have a corner or change of direction anywhere except at the pitch line, without causing a break in the line of action.

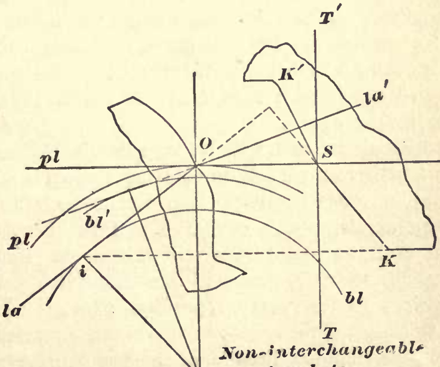
As the normals $p\ q$ are parallel, the line of action is a straight line $W'\ O\ W$ at right angles to the rack odontoid. The interchangeable line of action is continued in a straight line on both sides of the pitch line, but the non-interchangeable line changes direction at that line.

In accordance with the universal custom we will consider that the involute tooth is always interchangeable, having a single angle of obliquity.



The involute tooth interchangeable

Fig. 34.



Non-interchangeable involute

Fig. 35.

54.—THE CUSP.

As a circle $i\ c$, Fig. 34, can always be drawn tangent to the line of action at an interference point i, from the center b of any pitch line B, there will always be a cusp in the curve at the point c (16), and at that point the working part of the curve must stop. The working part of the rack tooth must end at the limit line $i\ L$ through the interference point i.

The working curves of any two teeth that work with each other must each end at the line drawn through the interference point of the other, Fig. 43, being limited by limit lines $l\ l$ and $L\ L$.

The second branch $c\ m'$ of the curve is equal to the first branch $c\ m$, but is reversed in direction. The second cusp is at infinity, and therefore has no practical existence.

The tangent circle $i\ c$, through the interference point and the cusp, is called the "base line."

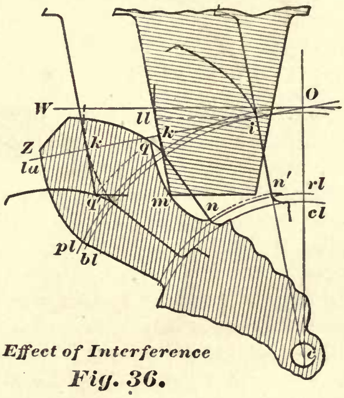
It is customary to continue the flank of the tooth inside the base line by a straight radial line, as far as may be necessary to allow the mating gear to pass.

55.—INTERFERENCE.

When the point of the tooth is continued beyond the limit line it will interfere with and cut away a portion of the working curve of the mating tooth. Fig. 36 shows a rack tooth working with the tooth of a small pinion, and cutting out its working curve.

This cut is not confined to the flank, but extends across the pitch line into the face, as shown by the line $q\ m\ n$. The rack tooth of the figure will not work with the pinion tooth unless it is cut off at the limit line $l\ l$ through the interference point i.

The mathematical action still continues, and the figure shows the rack tooth in action at k with the second branch of the curve.



Effect of Interference
Fig. 36.

56.—ADJUSTABILITY.

An interesting and in many cases a valuable feature of the involute curve, and one that is confined to it, is the fact that its position as a whole with regard to the mating curve is adjustable.

Two involutes, each with its base line, will work together in perfect tooth contact when they are moved with respect to each other, as long as they touch at all. The lines of action and the pitch lines will shift as the curves are moved, and will accommodate themselves to the varying position of the base lines.

But this valuable feature of the involute curve is not always available, and involute gears are not, as commonly supposed, necessarily adjustable, for the conditions are often such that the teeth will fail to act when the centers are moved, except within very narrow limits. Care must be taken that the arc of action is not so reduced by separating the centers of the gears that it is less than the circular pitch, for the former arc is variable and the latter is fixed. Care must also be taken that the working curve is not pushed over the limit line when the centers are drawn together.

In any limiting case, such as in Fig. 43, the centers are not adjustable. The gears of the standard set are either not adjustable at all or are so within very narrow limits, on account of the correction for interference.

57.—CONSTRUCTING THE INVOLUTE BY POINTS.

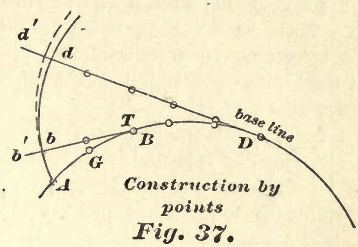
The simple involute curve can be constructed by points by the general method of (24), but it is much better to take advantage of the property that it is an involute of its base circle, and construct it by the rectification of that circle.

As in Fig. 37 any convenient small distance *A G* is taken on the dividers, and the points on the curve located by stepping along the circle and its tangent from any given point to any desired point.

This method is so accurate, if care is taken to step accurately on the line, that the curve seldom needs correction; but, when great accuracy is required, correction can be applied at the rate of one-thousandth of an inch to the step, if the length of the step is regulated by the diameter of the circle according to the following table:

Diameter of Circle:											
1	2	3	4	5	6	7	8	9	10	11	12
Length of Step:											
.17	.26	.37	.46	.53	.60	.67	.73	.76	.79	.82	.84

For example: If the circle of Fig. 37 is



Construction by points

Fig. 37.

four inches in diameter, and the dividers are set to .46 inch, the true curve, *A b′ d′*, will be outside of the constructed curve *A b d* by .002 inch at *b* and .005 inch at *d*.

From the table we can form the handy and sufficiently accurate rule that the length of the step should be about one-tenth of the diameter of the circle, for a correction of about one-thousandth of an inch per step.

Having thus found several points of the involute, we can draw it in by hand, or by constructing a template, or by finding centers from which approximately accurate circular arcs can be drawn.

58.—THE STANDARD INVOLUTE TOOTH.

The tooth that is selected for general use, and the one that is the best for all except a few special cases and limiting cases, is the interchangeable tooth having an angle of obliquity of fifteen degrees, an addendum of one-third the circular pitch, or one divided by the diametral pitch, and a clearance of one-eighth of the addendum.

The standard to which involute cutters are made is slightly different, having an angle of 14° 28′ 40″, the sine of which is one-quarter, and a clearance of one-twentieth of the circular pitch.

If the obliquity is 15° the smallest possible pair of equal gears have 11.72 teeth, and therefore 12 is the smallest gear of the interchangeable set.

The base distance, the distance of the base line inside of the pitch line, is about one-fifty-ninth of the pitch diameter, and one-sixtieth is a convenient fraction for practical use.

The limit points of the whole set must be determined by that of the twelve-toothed gear, for any gear of the set may be required to work with that one, and the working curve of each tooth must end at the point thus determined. As the limit point is always inside of the addendum line there must always be a false extension on the tooth, the point being rounded over outside of the limit point.

59.—THE INVOLUTE ODONTOGRAPH.

As the base line must always be drawn, it is advisable, to save work, to locate the centers of the approximate circular arcs upon that line. It is also necessary that the points of the teeth shall be rounded over, to avoid interference. These requirements made it impracticable to compute the positions of the centers, and an empirical rule had to be adopted instead.

Teeth were carefully drawn by the stepping method of (57) on a very large scale, one-quarter pitch, giving a tooth eight inches in length. These teeth were corrected for interference by giving them epicycloidal points that would clear the radial flanks of the twelve-toothed pinion.

Then the proper centers on the base line were determined by repeated trials, and tooth curves obtained that would agree with the true involute up to the limit point, and still clear the corrected point. The odontograph table is a record of these radii, which are believed to be as nearly correct as the given conditions will permit.

It was found that separate curves were required for face and flank up to thirty-six teeth, but that one curve would answer for teeth beyond.

It was found necessary to devise a separate method for drafting the rack tooth.

60.—TEN AND ELEVEN TEETH.

Theoretically the twelve-toothed pinion is the smallest standard gear that will have an arc of action as great as the circular pitch, but ten and eleven teeth may be used with an error that is not practically noticeable. Fig. 38 shows a pair of ten-toothed gears in action. They can be in correct action only when the point of contact is between the two interference points i and I, but they will be in practical contact for a greater and sufficient distance

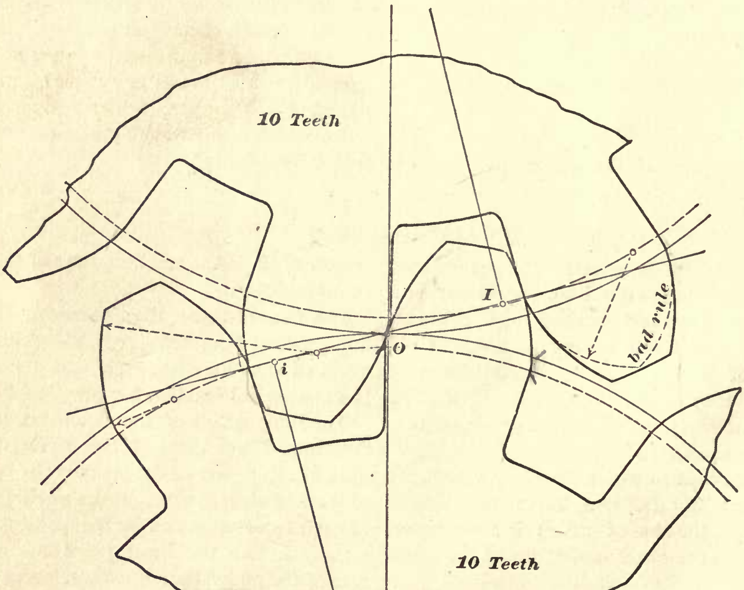


Fig. 38.

61.—A BAD RULE.

There is a simple and worthless rule for involute teeth that deserves notice only because it is considerably in use.

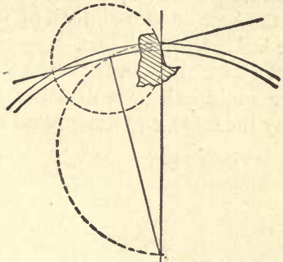
It constructs the whole tooth curve, face and flank, for all numbers of teeth, as a single arc from a center on the base line, and with a radius equal to one-quarter of the pitch radius, Fig. 39.

This is wonderfully convenient, but the convenience is purchased at the expense of ordinary accuracy, for the rule is not even approximately correct. It is handy, and nothing else.

Figs. 38 and 40 show the kind of teeth that are constructed by this rule on gears of ten and twelve teeth, where its error is the greatest, and it is reasonable that the involute tooth should not be in great favor with those who have been taught to draw it thus.

The error gradually decreases, until, for more than thirty teeth, it is tolerably correct, but it gives the rack with the straight, uncorrected working face that would interfere, as shown at q, Fig. 40.

As it is tolerable only for thirty or more teeth, and not good then, it may well be dropped altogether.



A bad rule

Fig. 39.

62.—USING THE INVOLUTE ODONTOGRAPH.

INVOLUTE ODONTOGRAPH.

STANDARD INTERCHANGEABLE TOOTH, CENTERS ON BASE ~~LINE.~~ Circle

(*For Table of Pitch Diameters see* 35.) p 15

Teeth.	Divide by the Diametral Pitch. Face Radius.	Divide by the Diametral Pitch. Flank Radius.	Multiply by the Circular Pitch. Face Radius.	Multiply by the Circular Pitch. Flank Radius.
10	2.28	.69	.73	.22
11	2.40	.83	.76	.27
12	2.51	.96	.80	.31
13	2.62	1.09	.83	.34
14	2.72	1.22	.87	.39
15	2.82	1.34	.90	.43
16	2.92	1.46	.93	.47
17	3.02	1.58	.96	.50
18	3.12	1.69	.99	.54
19	3.22	1.79	1.03	.57
20	3.32	1.89	1.06	.60
21	3.41	1.98	1.09	.63
22	3.49	2.06	1.11	.66
23	3.57	2.15	1.13	.69
24	3.64	2.24	1.16	.71
25	3.71	2.33	1.18	.74
26	3.78	2.42	1.20	.77
27	3.85	2.50	1.23	.80
28	3.92	2.59	1.25	.82
29	3.99	2.67	1.27	.85
30	4.06	2.76	1.29	.88
31	4.13	2.85	1.31	.91
32	4.20	2.93	1.34	.93
33	4.27	3.01	1.36	.96
34	4.33	3.09	1.38	.99
35	4.39	3.16	1.39	1.01
36	4.45	3.23	1.41	1.03
37—40	4.20		1.34	
41—45	4.63		1.48	
46—51	5.06		1.61	
52—60	5.74		1.83	
61—70	6.52		2.07	
71—90	7.72		2.46	
91—120	9.78		3.11	
121—180	13.38		4.26	
181—360	21.62		6.88	
Rack	½ Face Rad 2.10		½ Face Rad .67	

1.615

~~Draw the rack tooth by the special method.~~

To draft the tooth lay off the pitch, addendum, root, and clearance lines, and space the pitch line for the teeth, as in Fig. 40.

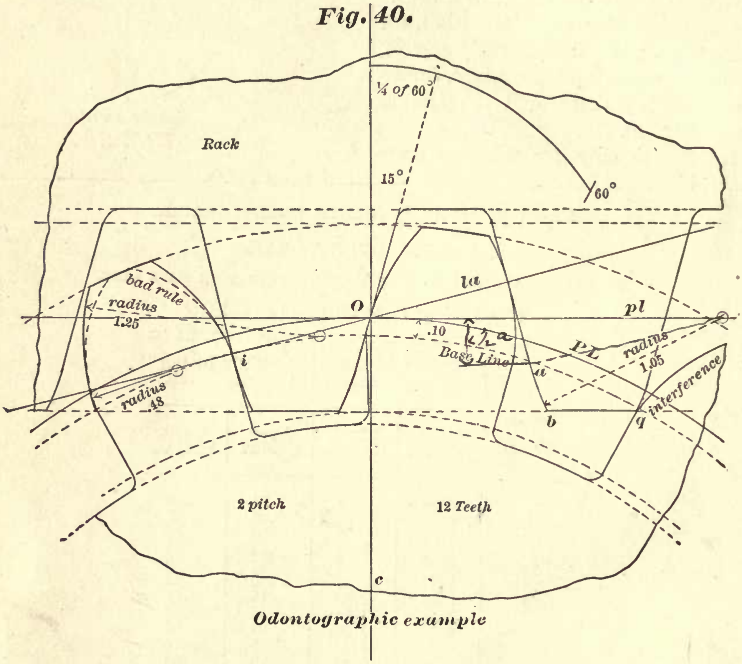
Draw the base line one-sixtieth of the pitch diameter inside the pitch line.

Take the tabular face radius on the dividers, after multiplying or dividing it as required by the table, and draw in all the faces from the pitch line to the addendum line from centers on the base line.

Set the dividers to the tabular flank radius, and draw in all the flanks from the pitch line to the base line.

Draw straight radial flanks from the base line to the root line, and round them into the clearance line.

Fig. 40.



Odontographic example

63.—SPECIAL RULE FOR THE RACK.

Draw the sides of the rack tooth, Fig. 40, as straight lines inclined to the line of centers *c O c* at an angle of fifteen degrees, best found by quartering the angle of sixty degrees,

Draw the outer half *a b* of the face, one-quarter of the whole length of the tooth, from a center on the pitch line, and with a radius of

2.10 inches divided by the diametral pitch.

.67 inches multiplied by the circular pitch.

64.—DRAFTING INTERNAL GEARS.

When the internal gear is to be drawn, the odontograph should be used as if the gear was an ordinary external gear. See Fig. 41.

But care must be taken that the tooth of the gear is cut off at the limit line drawn through the interference point *i* of the pinion. The point of the tooth may be left off altogether or rounded over to get the appearance of a long tooth.

The pinion tooth need not be carried in to the usual root line, but, as in the figure, may just clear the truncated tooth of the gear.

The curves of the internal tooth and of its pinion may best be drawn in by points (57), for the odontographic corrected tooth is not as well adapted to the place as the true tooth, and no correction for interference is needed on the points of the pinion teeth or on the flanks of those of the gear.

Care must be taken that the internal teeth do not interfere by the point a striking the point b, as they will if the pitch diameters are too nearly of the same size.

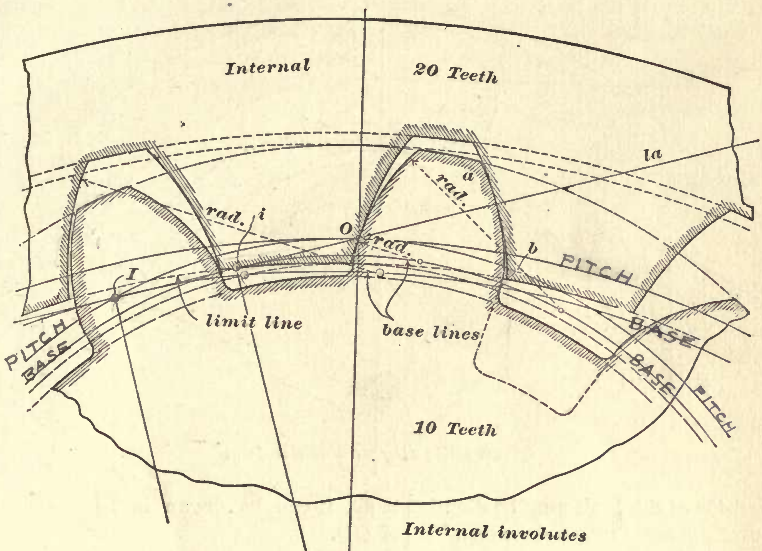


Fig. 41.

65.—INVOLUTE GEARS FOR GIVEN OBLIQUITY AND ADDENDA.

When the obliquity and addenda, as well as the pitch diameter and number of teeth in a gear are given, as is generally the case, we can proceed to draft the complete gear as follows:

Draw the pitch line $p\ l$, Fig. 42, the addendum line $a\ l$, the root line $r\ l$, and the clearance line $c\ l$, as given. Draw the line of action $l\ a$ at the given obliquity $W\ O\ Z = K$. Draw the base line $b\ l$ tangent to the line of action. Find the interference point i by bisecting the chord $O\ v$.

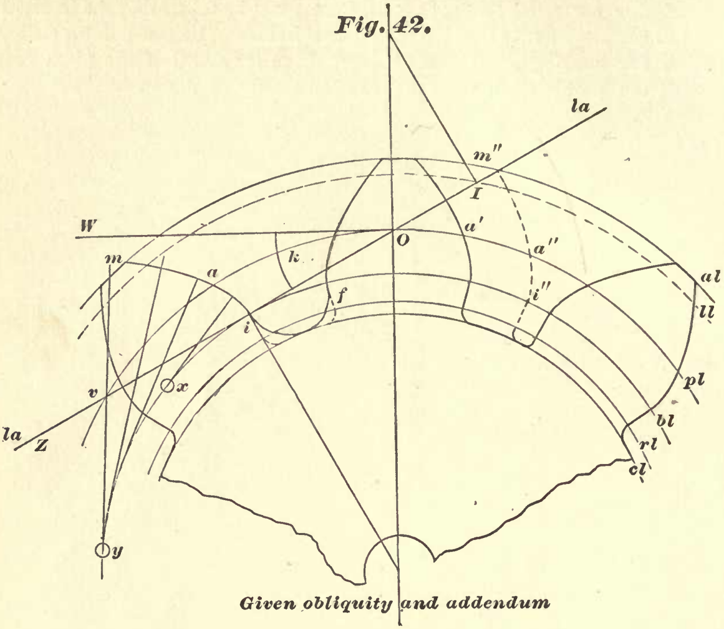
Draw the involutes $i\ a\ m$ and $i''\ a''\ m''$, and $a\ a''$ will be the maximum arc of action.

If the given arc of action $a\ a'$ is not greater than the maximum arc, the pitch line is to be spaced and the tooth curves drawn in from the base line to the addendum line. These tooth curves, when small, are best drawn as circular arcs from centers on or near the base line, one center x for the flank from the base line to the pitch line, and another center y for the face from the pitch line to the addendum line. One involute $i\ a\ m$ should be carefully constructed by points, and then the required centers can be found by trial. One center and arc will often answer for the whole curve, and it is only when great accuracy is required that more than two centers will be necessary.

Continue the flanks of the teeth toward center by straight radial lines, and round these lines into the clearance line.

If the interference point for the gear that the gear being drawn is to work with is at I, within the addendum line, the limit line $l\ l$ must be drawn through it, and the points of

Fig. 42.



Given obliquity and addendum

the teeth outside of this limit must be slightly rounded over, to avoid interference (55).

If a fillet f is desirable, to strengthen the tooth, it can be drawn in by the method of (44).

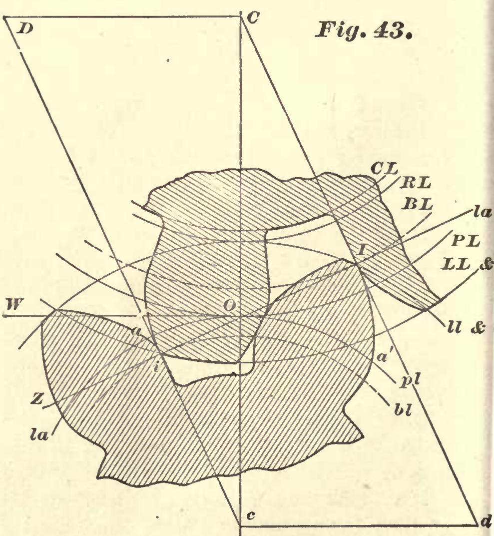
66.—INVOLUTE GEARS FOR GIVEN NUMBERS OF TEETH.

When the numbers of teeth and the pitch lines are the only given details, the shape and action of the tooth depends upon the obliquity, and the action will fail if the angle is too small. The principal object is to determine the least possible angle that is permitted by the given pitch diameters and numbers of teeth.

Draw the pitch lines $P\ L$ and $p\ l$, Fig. 43, lay off the given pitch arc, as a straight line $c\ d$ or $C\ D$, at right angles to the line of centers, and draw the line $C\ d$ or $c\ D$. Then the required line of action will be $l\ a$ passing through O at right angles to $c\ D$ or $C\ d$. The complete teeth can then be drawn in as previously directed.

In this case, the obliquity $W\ O\ Z$ being the least possible, the limit lines and the addendum lines must coincide, but the addenda may be reduced by increasing the angle.

Fig. 43.



Given numbers of teeth

67.—INVOLUTE GEARS FOR GIVEN OBLIQUITY.

When the pitch diameters and the obliquity are the only given details, the lines $C\ I$ and $c\ i$, Fig. 43, drawn from the centers at right angles to the line of action, will determine the limit lines. The maximum arc of action $a\ a'$ may be found either by drawing the involutes $i\ a$ and $I\ a'$, or by continuing the line $C\ I$ to the line $c\ d$, and measuring the required distance $c\ d$. Any arc of action less than $a\ a'$ may be used.

The drawings should always be made to a scale of one tooth to the inch radius, so that the pitch arc will be 2π. If the scale is one tooth to the inch of diameter, the pitch arc will be π.

68.—INVOLUTE GEARS WITH LESS THAN FIVE EQUAL TEETH.

The method of Fig. 43 and (66) will be found to apply to any given numbers of teeth not less than five, and to fail, if either gear has but three or but four teeth. Any external gear of five or more teeth will work with any external gear of five or more teeth, and with an internal gear of any number of teeth unless stopped by internal interference (64).

For example, if a pair having four and five teeth, Fig. 44, is tried, the four-toothed pinion will fail, because its tooth will come to a point upon the line of action before it has passed over the required pitch arc. The difficulty cannot be remedied by increasing the obliquity, for an angle that would allow the four-toothed pinion to act would also cause the five-toothed pinion to fail.

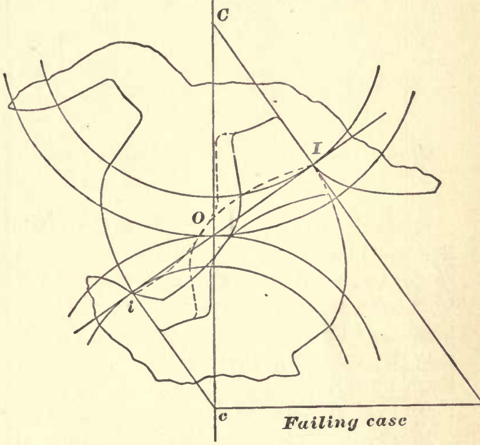
The practical limit is five teeth, but the mathematical limit is the pair having the fractional number 4.62 teeth, Fig. 45.

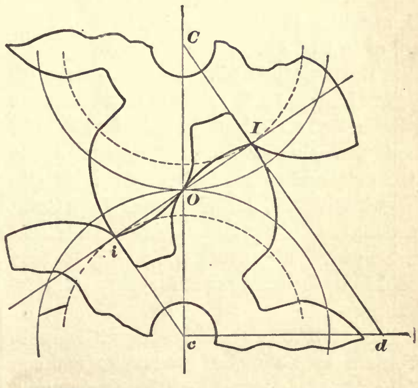
The four-toothed pinion will not work with any external gear, not even with a rack, but it will work with an internal gear that has about ten thousand teeth, and is practically a rack. It will work with any internal gear having less than ten thousand teeth, and Fig. 46 shows it working with an internal gear of six teeth. Internal interference will prevent its working with an internal gear of five teeth.

The three-toothed pinion has no practical action. It has a mathematical action with internal gears of 3.56 or less teeth, as shown by Fig. 47, but as its limit is less than four, it cannot work with any whole number. The figure shows the interference at a.

The extreme mathematical limit may be said to be the gear of 2.70 teeth, which has a theoretical action with an internal gear of the same size, coinciding with it.

Fig. 44.





4.62 X 4.62 limit for equal teeth

Fig. 45.

Fig. 46.

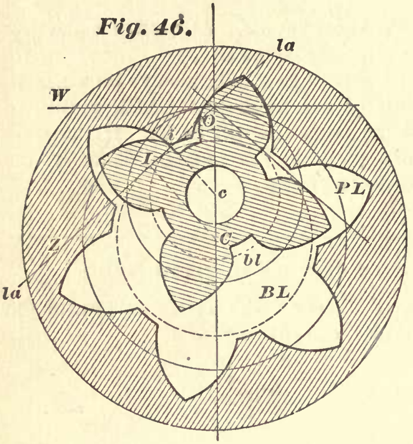
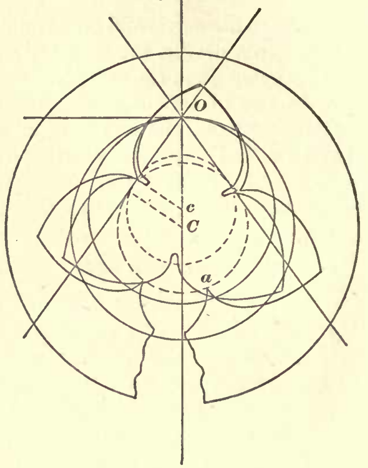


Fig. 47.



69.—INVOLUTE GEARS WITH LESS THAN FIVE UNEQUAL TEETH.

If we drop the condition that the pitch line must be equally divided into tooth and space arcs, we can make gears of three and of four teeth work with external gears by the method of (65). The failing case of Fig. 44 may be corrected by widening the failing tooth until it acts, and narrowing the other tooth to correspond, as shown in broken lines.

In this way a four-toothed pinion will work with any number of teeth not less than 5.57, at which limit both gears have pointed teeth, as in Fig. 48.

The three-toothed pinion will work with any gear having 10.17 or more teeth. Fig. 49 shows the 3x10.17 limiting pair, and Fig. 50 shows the three-toothed pinion working with an internal gear of five teeth. It will not work with an internal gear of four teeth, on account of internal interference, and therefore the combination shown by Fig. 50 may be said to be the least possible symmetrical involute pair.

A gear of 2.70 teeth will work with a rack, but there seems to be no way to make a pinion of two teeth work under any circumstanccs.

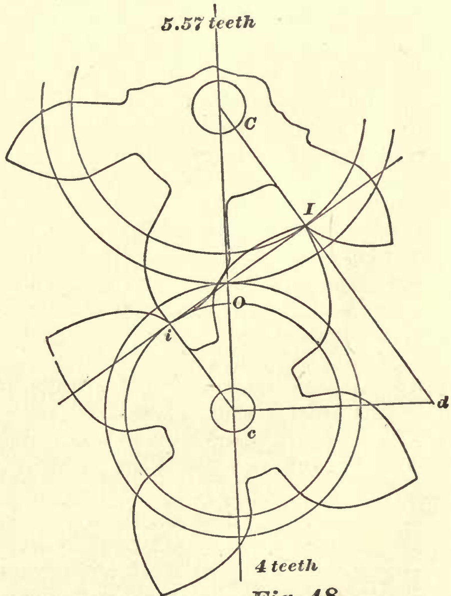
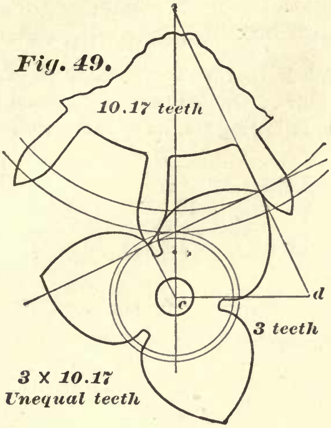
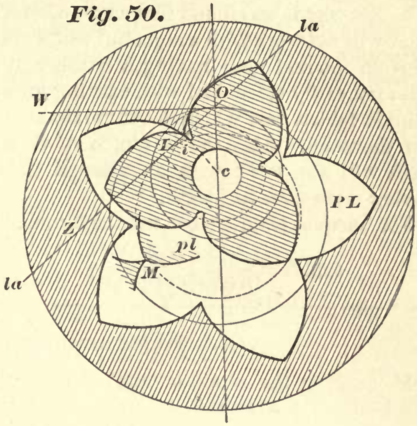


Fig. 48.





70.—THE MATHEMATICAL LIMITS.

The above results for low numbered pinions can be obtained by graphical means, but that method is not accurate enough to determine the limits with great precision, and in any case is tedious and laborious.

The mathematical process is not particularly difficult, and consists in repeated trials with given formulæ.

To determine the obliquity at which a limiting pinion will be pointed on the line of action, for tooth equal to space, we use the formulæ :

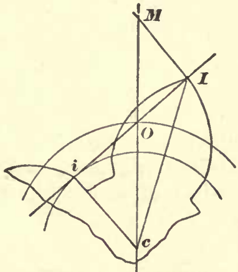
$$tan.\ h = \frac{2\,\pi\,M}{n\,(M+n)+4\,\pi^2}$$

$$\frac{4\,M}{M+n} - \frac{h\,n}{90} = 1,$$

in which n is the given number of teeth in the pointed gear, Fig. 51, M is the number in the gear having the radius $O\,M$, and h is the angle $O\,c\,I$. Knowing n, we assume a value for M, and from that find a value for h by means of the first formula. This value of h, tried in the second formula, will give an error. A second assumption for M will give a second error, and if the two errors are not too great a comparison will nearly locate the true value of M.

Knowing n and M, we find the obliquity from

$$tan.\ K = \frac{2\,\pi}{M+n}.$$



Pointed pinion
Fig. 51.

In this way the following values were determined :

n	M	K
2.695	1.26	57° 49′
3.	1.51	54° 20′
4.	2.86	42° 29′
4.62	4.62	34° 11′
5.	6.75	28° 8′
5.58	∞	0

Having determined the obliquity for the pointed pinion, we can determine the least number of teeth it will work with by means of the following formulæ :

$$Angle\ B = \frac{180}{\pi}\,\frac{n}{N}\,tan.\ K - \frac{90}{N} + K$$

$$tan.\ B = \frac{n}{N}\,tan.\ K + tan.\ K$$

in which N is the required least number.

In this way it was found that a gear of four teeth will not work with a rack, but will work with an internal gear having a number of teeth not easily calculated with existing logarithmic tables, but which is approximately ten thousand. Also that a pinion of three teeth will not work with an internal gear having more than 3.56 teeth.

For unequal teeth we can use the formulæ,

$$tan.\ h = \frac{2\,\pi\,N}{n\,(N + n) + 4\,\pi^2}$$

$$tan.\ H = \frac{2\,\pi\,n}{N\,(N + n) + 4\,\pi^2}$$

in which N and n are the numbers of teeth in the pair of pointed gears. By these formulæ the following results were determined,

n	N	K
2.695	∞	0
3.	10.17	25° 27′
4.	5.57	33° 17
4.62	4.62	34° 11

71.—MINIMUM NUMBERS FOR UNSYMMETRICAL TEETH.

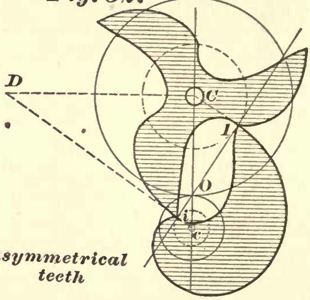
If we drop the condition that the fronts and backs of the teeth shall be alike we have an unimportant case that is similar to that already studied, but much more intricate.

If we carry this case to its extreme, and adopt single acting teeth, we have no minimum numbers at all, for any two numbers of teeth will then work together. Fig. 52 shows one tooth working with three teeth, and any other combination can be obtained. The minimum obliquity for a given pair is obtained, as in (66), by laying off the known pitch arc, $C\,D$, at right angles to $C\,c$, and drawing the line of action at right angles to the line $D\,c$. The obliquity is also given by the formula:

$$tan.\ K = \frac{2\,\pi}{N + n},$$

in which n and N are the numbers of teeth. When the obliquity is as great as is often

Fig. 52.



Unsymmetrical teeth

the case for very low numbers of teeth the action may be impracticable on account of the great friction of approach (48). The gears of Fig. 52 will not drive each other on the approach, unless the tooth surfaces are very smooth, and the power transmitted is almost nothing.

72.—MINIMUM NUMBERS FOR GIVEN ARC OF RECESS.

It has generally been assumed, although no good reason for the assumption has ever been given, that the minimum numbers of teeth occur when the tooth of one of the gears, Fig. 53, is pointed at the interference point I, and at the same time has passed over an arc of recess $O\ a$ that is a given part of the whole pitch arc $a'\ a$.

The solution is simple enough, graphically by repeated trials, or by a formula that can be applied directly without the usual process by trial and error.

But, as involute teeth have a uniform obliquity, there is no necessity for assuming

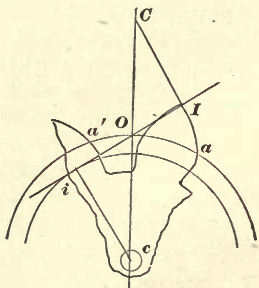


Fig. 53.

a definite arc of recess, and the condition on which the problem is based is unwarranted. No real limit is reached, and the matter is not worth examination at any length. The problem is investigated, for both bevel and spur gears, in either external or internal contact, in the Journal of the Franklin Institute for Feb., 1888, and it has received more attention than its slight importance entitles it to.

73.—EFFICIENCY OF INVOLUTE TEETH.

But little can be said in addition to the matter in (49), for both forms of teeth in common use are substantially equal with respect to the transmission of power.

From the formula of (49), which is the formula for the involute tooth, it is seen that the loss from friction is entirely independent of the obliquity, and, therefore, all systems of involute teeth are independent of the obliquity in this respect. This is contrary to the accepted idea that a great efficiency requires a small obliquity.

It has been stated on high authority that the involute tooth is inferior to the cycloidal tooth in efficiency, but the statement is not true. The difference in efficiency is minute, a small fraction of one per centum, but what little difference there is is always in favor of the involute tooth.

74.—OBLIQUITY AND PRESSURE.

The involute tooth action is in the direction of the line of action, and the obliquity is a constant angle. It is variable only when the shaft center distance is varied.

As the pressure is always equal to the product of the tangential force at the pitch line multiplied by the secant of the obliquity, (26), it is constant for the involute tooth.

Involute teeth, therefore, have a steady action that is not possessed by other forms; particularly by forms which, like the cycloidal, have a pressure and an obliquity that varies between great extremes.

75.—THE ROLLER OF THE INVOLUTE.

The involute odontoid, like all possible odontoids, can be formed by a tracing point in a curve that is rolled on the pitch line, and this roller is the logarithmic spiral with the tracing point at its pole, (32).

This feature is, however, more curious than useful, and it is not of the slightest importance in the study of the curve. Neither is the operation of rolling the involute mechanically possible, for the logarithmic roller has an infinite number of convolutions about its pole, and the tracing point would never reach the pitch line.

The involute is often considered to be a rolled curve, because it can be formed by a tracing point in a straight line that rolls on its base line; but, although that is the fact, it is a special feature and has nothing to do with the rolled curve theory. The rolled curve theory requires that the odontoid shall be formed by a roller that rolls on the pitch line only.

4. THE CYCLOIDAL SYSTEM.

76.—THE CYCLOIDAL SYSTEM.

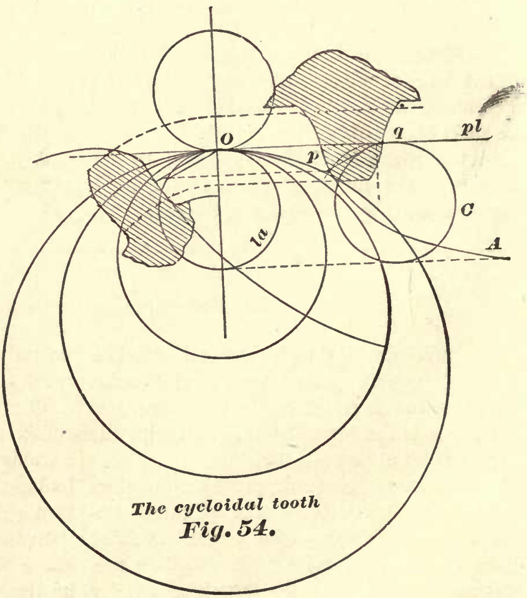
If the curve known as the cycloid is chosen as the determining rack odontoid, (31), the resulting tooth system will be cycloidal.

It is commonly called the "epicycloidal" system, because the faces of its teeth are epicycloids, but, as the flanks are hypocycloids, it seems as if the name "epihypocycloidal" would be still more clumsy and accurate.

There is no more need of two different kinds of tooth curves for gears of the same pitch than there is need of two different kinds of threads for standard screws, or of two different kinds of coins of the same value, and the cycloidal tooth would never be missed if it was dropped altogether. But it was first in the field, is simple in theory, is easily drawn, has the recommendation of many well-meaning teachers, and holds its position by means of "human inertia," or the natural reluctance of the average human mind to adopt a change, particularly a change for the better.

77.—THE CYCLOIDAL TOOTH.

The cycloid is the curve A that is traced by the point p in the circle C that is rolled on the straight pitch line $p\ l$, Fig. 54. The normal at the point p is the line $p\ q$ to the point of tangency of the rolling circle and the pitch line.



The cycloidal tooth
Fig. 54.

The line of action is the circle $l\ a$, of the same size as the roller C.

As no tangent arc can be drawn to the line of action from the pitch point O as a center, no terminal point (18) exists. As there is no point upon the line of centers from which a circle can be drawn tangent to the line of action, there will be no cusps, (16) except on the pitch line.

The cycloidal tooth can be drawn by the general method of (24), but there are several easier methods which will be described. There are numerous empirical rules and short cuts to save labor and spoil the tooth, which will not be described.

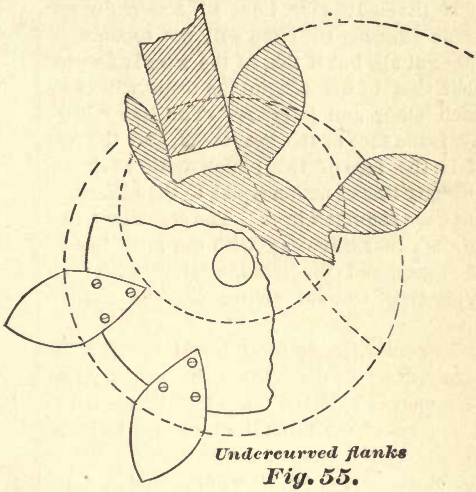
When the pitch line is of twice the diameter of the line of action, the flank of the tooth is a straight line. If the pitch line is less than twice as large as the line of action, the flank of the tooth will be under-curved, as shown by Fig. 55, and it is customary to avoid the resulting weak tooth by limiting the line of action to a diameter not greater than half that of the smallest gear to be used.

78.—SECONDARY ACTION.

The secondary line of action (21) is a circle, Fig. 56, differing from the pitch circle by the diameter of the primary line of action, either inside or outside of it.

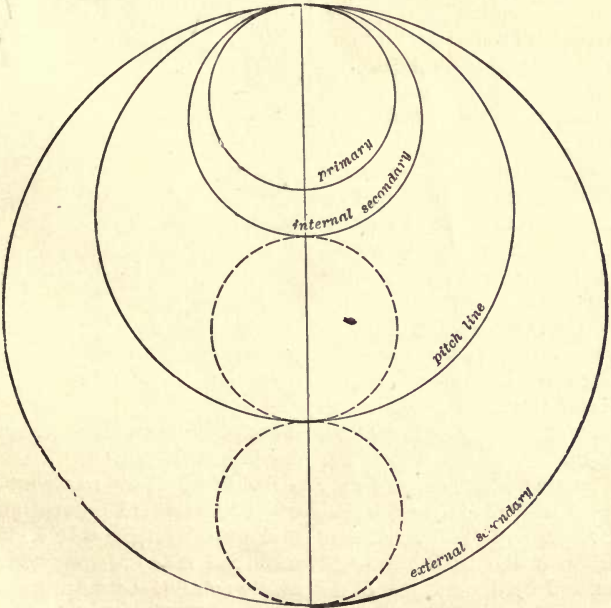
When the internal secondary line of action of an internal pitch line coincides with the external secondary line of action of its pinion, there will be secondary contact between the gears, the face of the gear working with the face of the pinion at a point of contact upon the combined secondaries. Fig. 57 shows this for the cycloidal tooth, the two faces working together at the point *a*. As both secondaries are circles they must coincide, and the secondary action will be continuous.

When the teeth are also in contact at *b* on the primary line of action, there will be double contact.



Undercurved flanks

Fig. 55.



Secondary lines of action

Fig. 56.

79.—INTERNAL INTERFERENCE.

If the secondary lines of action do not come together the teeth will not touch each other at all, but if that of the gear is smaller than that of the pinion the teeth will cross each other and interfere. The line *c*, Fig. 57, is the face of the gear tooth, and the line *d* is the face of the pinion tooth having a primary line of action equal to the difference between the pitch lines. The secondary line of each gear coincides with the pitch line of the other, and the faces interfere with each other the amount shown by the shaded space.

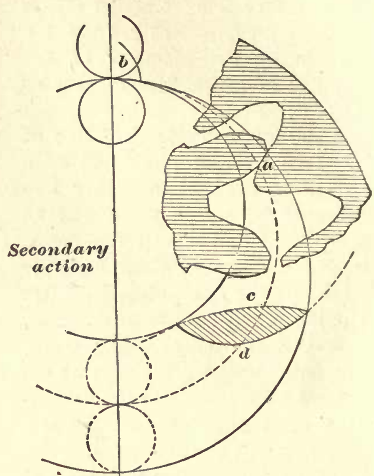


Fig. 57.

The only remedy for internal interference is to reduce the diameter of the primary line of action to half the difference between the diameters of the pitch lines, or else to leave off one of the faces of the teeth.

The discovery of the law of internal cycloidal interference is due to A. K. Mansfield, who published it in the "Journal of the Franklin Institute" for January, 1877. It was afterwards re-discovered by Professor MacCord, and most thoroughly applied and illustrated in his "Kinematics."

When interference is avoided by omitting one of the faces of the teeth the primary line of action may be enlarged, but it must not then be larger than the difference between the pitch diameters.

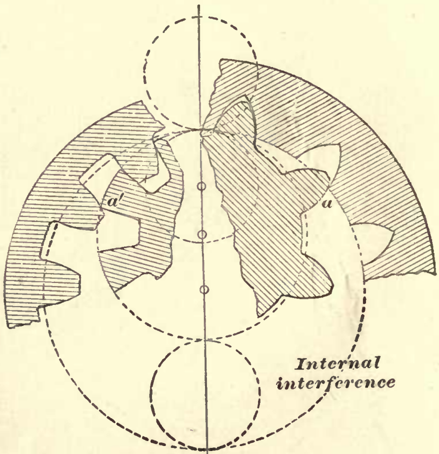


Fig. 58.

Fig. 58 shows on the right the action when the face of the gear is omitted, and on the left the action when the face of the pinion is left off. The teeth will just clear each other, each one touching the other at a single point *a* in its pitch line.

As the contact at *a* is not a point of practical action, care must be taken that the arc of action at the primary line of action is as great as the circular pitch, for otherwise, as in the figure, the gears will not be in continuous primary action.

The rule for internal interference, simply stated, is that the diameters of the pitch lines must differ by the sum of the diameters of the lines of action if the teeth have both faces and flanks, and by the diameter of the acting line of action if the face of either gear is omitted. For the standard interchangeable system the gears must differ by twelve teeth if both teeth have faces, and by six teeth if one face is omitted.

Fig. 62 shows the secondary contact in the case of a standard internal gear of twenty-four teeth working with a pinion of twelve teeth, and it is to be noticed that the teeth nearly coincide between the two points of contact. Where there is secondary contact the teeth practically bear on a considerable line instead of at a point.

80.—THE STANDARD TOOTH.

The standard tooth (42), selected for the cycloidal system, is by common consent the one having a line of action of half the diameter of a gear of twelve teeth, so that that gear has radial flanks.

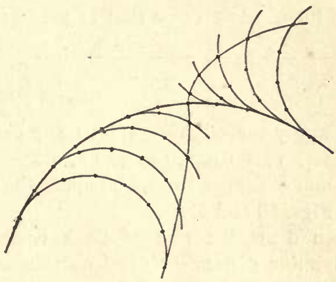
The standard adopted by manufacturers of cycloidal gear cutters is that having radial flanks on the gear of fifteen teeth, but it is not and should not be in use for other purposes. If any change is made, it should be made in the other direction, to make the set take in gears of ten teeth.

It must be borne in mind that the standard adopted does not limit the set to the stated minimum number of teeth, but that it simply requires that smaller gears shall have weak under-curved teeth.

81.—THE ROLLED CURVE METHOD.

It happens in this case, and in this case only, that the rolled curve method, which theoretically applies to all odontoids, can be actually put into practical use, for the generating roller is here the circle, the simplest possible curve.

As in Fig. 59, roll a circle of the diameter of the circle of action upon the outside of the pitch line for the faces, and upon the inside for the flanks, and a fixed point in it will trace the curve.



Construction by rolling
Fig. 59.

The method can be used by actually constructing pitch and rolling circles, but the same result can be reached more easily and quite as accurately by drawing several circles, and then stepping from the pitch point along the pitch line, and back on the circles to the desired point. If the length of the step is not more than one-tenth of the diameter of the circle, the error will not be over one-thousandth of an inch for each step.

This method is the best one to adopt, except for the standard tooth.

82.—THE THREE POINT ODONTOGRAPH.

It is a simple matter to draw the tooth curve by means of rolling circles, but such a method requires skill on the part of the draftsman. It is, moreover, nothing but a method for finding points in the curve for which approximate circular arcs are then determined.

The "three point" odontograph is simply a record of the positions of the centers of the circles which approximate the most closely to the whole curve of the standard tooth. The positions of two points, *a* at the center of the face or of the flank, Fig. 60, and *b* at the addendum point or root point of the curve, were carefully computed, and then the position of the center *C* of the circle which passes through these two points and the pitch point *O*, was calculated. The circle that passes through these three points is assumed to be as accurately approximate to the true curve as any possible circular arc can be.

The odontograph gives the radius "rad." of the circular arc, and the distance "dis." of the circle of centers from the pitch line, for the tooth of a given pitch, and their values for other pitches are easily found by simple multiplication or division.

The advantages of this method lie in the facts that the desired radius and distance are given directly, without the labor of finding them, and that as they are computed they are free from errors of manipulation. In point of time required, the advantage is with the odontograph in the ratio of ten to one.

The greatest error of the odontographic arc, shown greatly exaggerated by the dotted lines, is at the point *c* on the face, and it is greater on a twelve-toothed pinion than on any larger gear. For a twelve-toothed pinion of three-inch circular pitch, a large tooth, the actual amount of the maximum error is less than one one-hundredth of an inch, and its average for eight equidistant points on the face is about four-thousandths of an inch. Any error that is greater than that stated will be due to manipulation, and not to the method.

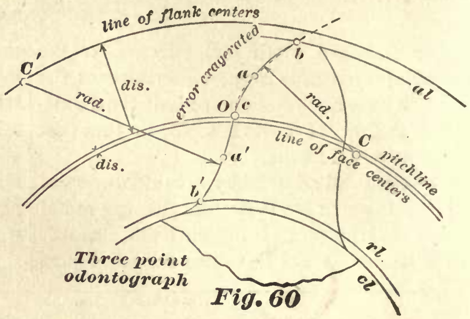


Fig. 60

83.—USING THE ODONTOGRAPH.

To apply the odontograph to any particular case, first draw the pitch, addendum, root, and clearance lines, and space the pitch line, Figs. 60 and 61.

Then draw the line of flank centers at the tabular distance "dis." outside of the pitch line, and the line of face centers at the distance "dis." inside of it. Take the face radius "rad." on the dividers, and draw in all the face curves from centers on the line of face centers; then take the flank radius "rad." and draw all the flank curves from centers on the line of flank centers.

THREE POINT ODONTOGRAPH.

STANDARD CYCLOIDAL TEETH.

INTERCHANGEABLE SERIES.

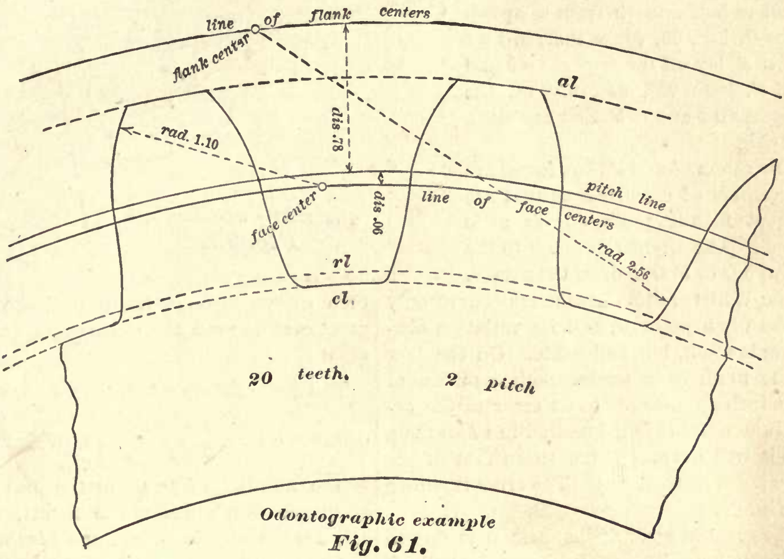
From a Pinion of Ten Teeth to a Rack.

NUMBER OF TEETH IN THE GEAR.		For One DIAMETRAL PITCH. For any other pitch divide by that pitch.				For One Inch CIRCULAR PITCH. For any other pitch multiply by that pitch.			
		Faces.		Flanks.		Faces.		Flanks.	
Exact.	Intervals.	Rad.	Dis.	Rad.	Dis.	Rad.	Dis.	Rad.	Dis.
10	10	1.99	.02	— 8.00	4.00	.62	.01	—2.55	1.27
11	11	2.00	.04	—11.05	6.50	.63	.01	—3.34	2.07
12	12	2.01	.06	∞	∞	.64	.02	∞	∞
13½	13—14	2.04	.07	15.10	9.43	.65	.02	4.80	3.00
15½	15—16	2.10	.09	7.86	3.46	.67	.03	2.50	1.10
17½	17—18	2.14	.11	6.13	2.20	.68	.04	1.95	.70
20	19—21	2.20	.13	5.12	1.57	.70	.04	1.63	.50
23	22—24	2.26	.15	4.50	1.13	.72	.05	1.43	.36
27	25—29	2.33	.16	4.10	.96	.74	.05	1.30	.29
33	30-36	2.40	.19	3.80	.72	.76	.06	1.20	.23
42	37—48	2.48	.22	3.52	.63	.79	.07	1.12	.20
58	49—72	2.60	.25	3.33	.54	.83	.08	1.06	.17
97	73—144	2.83	.28	3.14	.44	.90	.09	1.00	.14
290	145—300	2.92	.31	3.00	.38	.93	.10	.95	.12
∞	Rack	2.96	.34	2.96	.34	.94	.11	.94	.11

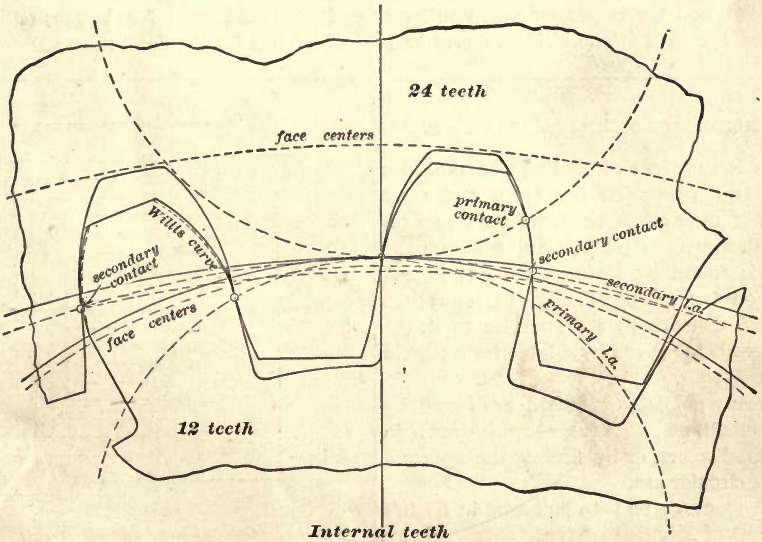
The table gives the distances and radii if the pitch is either exactly one diametral or one inch circular, and for any other pitch multiply or divide as directed in the table.

Fig. 61 shows the process applied to a practical case, with the distances given in figures.

Fig. 62 shows the same process applied to an internal gear of twenty-four teeth working with a pinion of twelve teeth. It illustrates secondary action and double contact. It also shows the actual divergence of the Willis odontographic arc from the true curve.



Odontographic example
Fig. 61.

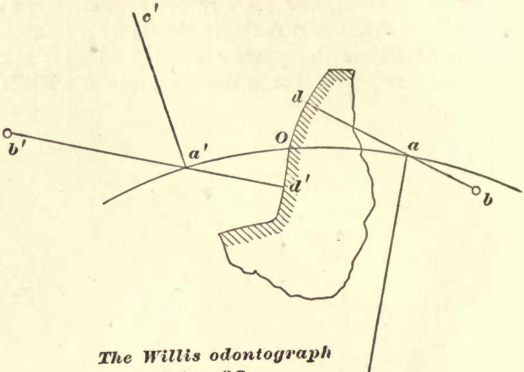


Internal teeth
Fig. 62.

84.—THE WILLIS ODONTOGRAPH.

This is the oldest and best known of all the odontographs, but it is inferior to several others since proposed, not only in ease of operation, but in accuracy of result.

To apply it, find the pitch points a and a' half a tooth from the pitch point O, Fig. 63, draw the radii $a\ c$ and $a'\ c'$, lay off the angles $c\ a\ b$ and $c'\ a'\ b'$, both 75°, and lay off the distances $a\ b$ and $a'\ b'$ that are given by table.



The Willis odontograph
Fig. 63.

The centers b and b' thus found are the centers of circular arcs that are tangent to the tooth curves at d and d'. The dividers are set to the radius $b\ O$ or $b'\ O$ to draw the curves.

The Willis arc touches the true curve only at the pitch point O, and its variation elsewhere is small, but noticeable. On the face of the tooth of a twelve-toothed pinion of three inch circular pitch, its error at the addendum point is four-hundredths of an inch, and it will average three times that of the three point method (82). The error is shown by Fig. 62.

The greatest error of the method is due to manipulation. The angle is usually laid off by a card, and the center measured in by a scale on the card. The circle of centers is then drawn through the center, and unless great care is used the chances of error are great.

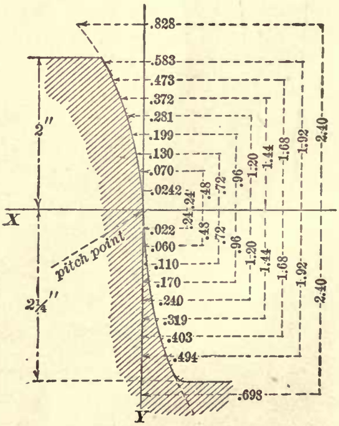
The angle $90° - c\ a\ b = W = \frac{180}{s}$, and the distance $a\ b = \frac{s\ c}{2\ \pi} \cdot \frac{t}{t \pm s}\ sin.\ W$, in which s is the number of teeth in the gear of the same set which has radial flanks, usually 12; c is the circular pitch, and t is the number of teeth in the gear being drawn. The positive sign is used for the face radius, and the negative for the flank radius.

85. — KLEIN'S CO-ORDINATE ODONTOGRAPH.

This is a method of finding the positions of several points on the tooth curve by means of their co-ordinates referred to axes through the pitch point. Any point on the curve is found by laying off a certain distance on the radius Y, Fig. 64, and then a certain distance at right angles to it, the distances being given by a table for a certain standard tooth.

As many points as required are found by this method, and then the curve is drawn in by curved rulers, or by finding the approximating circular arc.

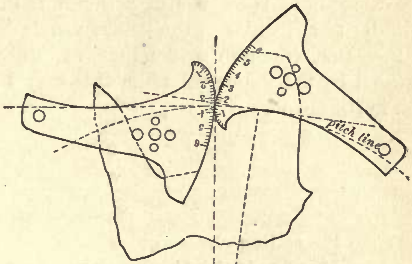
This odontograph is to be found in Klein's Elements of Machine Design.



Coordinate odontograph
Fig. 64.

86.—THE TEMPLET ODONTOGRAPH.

Prof. Robinson's templet odontograph is an instrument, not a method. It is a piece of sheet metal, Fig. 65, having two edges shaped to logarithmic spirals. It is laid upon the drawing, according to directions given in an accompanying pamphlet, and used as a ruler to guide the pen. It can be fastened to a radius bar, and swung on the center of the gear, to draw all the teeth. See Van Nostrand's Science Series, No. 24, for the theory of the instrument in detail.



The templet odontograph
Fig. 65.

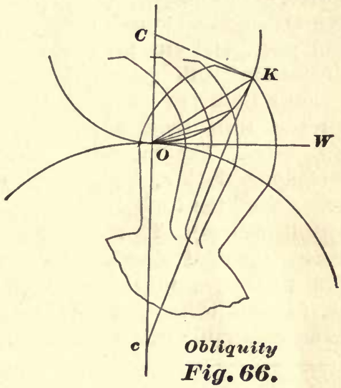
87.—OBLIQUITY OF THE ACTION.

When the point of contact between two teeth is at the pitch point *O*, Fig. 66, the pressure between the teeth is at right angles to the line of centers, but, as the point of contact recedes from the line, the direction of the pressure varies by an angle of obliquity which increases from zero until the point *K*, at the intersection of the addendum circle with the line of action, is reached.

The angle $K = K\ O\ W$, of the maximum obliquity, can be found by solving the triangle $C\ c\ K$, and for the standard set we have,

$$cos.\ 2\,K = \frac{2\,n + 17}{3\,n + 18},$$

in which n is the number of teeth in the gear.



Obliquity
Fig. 66.

For the smallest gear of the set, the one having twelve teeth, K is 20° 15′, and for the rack it is 24° 5′, so that it will always be between those two limits for external gears, and greater for internal gears.

The friction between two gear teeth increases with the angle of obliquity, but not in direct proportion. With the involute tooth the work done while going over a certain arc from the line of centers is proportional to the square of the arc, and for cycloidal teeth the increase with the arc is still more rapid. Therefore it is the maximum obliquity of the action that principally determines the injurious effects of friction.

88.—THE CUTTER LIMIT.

When the number of teeth in the gear is less than that in the gear having teeth with radial flanks, the flanks will be under-curved, and when too much so they cannot be cut with a rotary cutter. The teeth of Fig. 55 could not be cut with a rotary cutter beyond the points where the tangents to the two sides are parallel.

The limit is reached when the last point that is cut by the rotary cutter is also the last point that is touched by the tooth of the rack in action with it, not allowing for internal gears.

The diameter of the gear when this limit is reached is found by the formula,

$$D = 2d - \frac{c}{2\,sin.^{-1}\sqrt{\frac{a}{d}}}$$

in which D is the diameter of the gear, d is the diameter of the circle of action, c is the circular pitch, and a is the addendum

For the common addendum of unity divided by the diametral pitch this may be put in the shape,

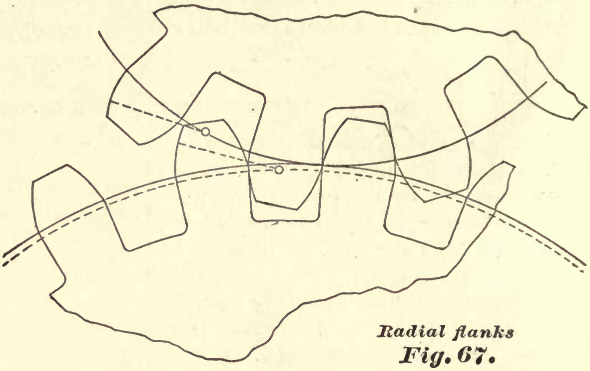
$$n = s - \frac{\pi}{2 \sin^{-1} \sqrt{\frac{2}{s}}}$$

in which s is the number of teeth in the radial flanked gear, and n is the number in the required cutter limit.

For the common series, where $s = 12$, we have $n = 8.26$; and for the cutter standard of $s = 15$, we have $n = 10.80$, so that cutters could easily be made to cut gears with less than s teeth.

89.—RADIAL FLANKED TEETH.

When the rolling circle for the faces is of half the diameter of the pitch line of the mating gear, the flanks of both gears will be straight radial lines, as in Fig. 67.



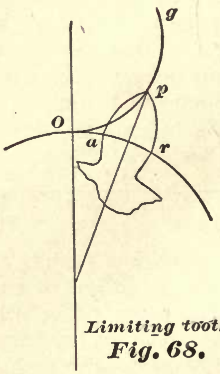
Radial flanks
Fig. 67.

Such gears are fitted to each other in pairs, and are not interchangeable with other sizes. Their teeth are more easily made than those of standard gears. The maximum obliquity is less, but the strength of the teeth is also less than usual. There is no reason for making such teeth in preference to the standard, although, for that reason probably, they are used to a considerable extent. It would be difficult to devise a form of tooth so whimsical that it would find no one to adopt and use it. see p. 19

90.—THE LIMITING NUMBERS OF TEETH.

When the number of teeth in a driving gear is small, the point p, Fig. 68, of its pointed tooth may go out of action by leaving the line of action $O\,g$ before a certain definite arc of recess $O\,r$ has been passed over, and the problem is to find the smallest number of teeth in the following gear that will just allow the given recess.

This question, which is not a particularly important one, is discussed at length, and applied to both bevel and spur gears, in either external or internal contact, in an article in the "Journal of the Franklin Institute" for Feb., 1888, and we will here consider only the case of the common spur gear.



Limiting tooth
Fig. 68.

The recess $O\,r$ is given as a times the circular pitch, and the thickness $a\,r$ of the tooth is given as b times the same. The diameter of the circle of action is q times that of the pitch line of the following gear. The number of teeth in the driving gear is d, and the number in the following gear is f.

M is an auxiliary angle equal to $\frac{360\,a}{q}$, and

W is an angle $\frac{360}{d}\left(a - \frac{b}{2}\right)$.

Then the required number f can be found by a process of trial and error with the formula,

$$\frac{\sin.\,(M + W)}{\sin.\,W} - \frac{d}{q\,f} - 1 = 0.$$

For an example, let the recess be $\frac{3}{4}$ of the pitch, the tooth equal to the space, and the flanks of the follower to be radial. Let the problem be to find a follower for a driver of seven teeth. This gives $a = \frac{3}{4}$, $b = \frac{1}{2}$, $q = \frac{1}{2}$, $d = 7$, and the formula becomes

$$\frac{\sin.\left(\frac{540°}{f} + 25°\ 43'\right)}{\sin.\ 25°\ 43'} - \frac{14}{f} - 1 = 0.$$

If we put f at random, at 20, we shall get, $+ .134 = 0$. Next, trying $f = 10$, we get, $- .132 = 0$, and the opposite signs show that f is between 20 and 10. Trying 12 the result is positive, and for 11 it is negative, showing that 12 is the required value of f. That is, 7 teeth will not drive less than 12 teeth with radial flanks, unless it is allowed an arc of recess greater than $\frac{3}{4}$ of the pitch.

For another example, test MacCord's value of 382 as the least driver for a follower of 10 teeth, when recess equals the pitch and the follower has radial flanks. Trying $d = 382$, the error is negative; for 383 it is also negative, but for 384 it is positive, and therefore the latter is the true number.

Extensive and sufficiently accurate tables of limiting values are given by MacCord in his "Kinematics."

5. THE PIN TOOTH SYSTEM.

91.—THE PIN GEAR TOOTH.

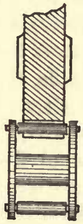
The theory of the pin gear tooth is entirely beyond the reach of the "rolled curve" method of treatment, and, therefore, writers who have adopted that method have had to depend more on special methods adapted to it alone than on general principles. The result is that its properties are often given incorrectly, or with an obscurity and complication that is bewildering to the student. Although the tooth is one of the oldest in use, its theory is so difficult that its defect was not discovered until within a very few years, by MacCord, about 1880, and it was not until it was examined by means of its normals that a remedy for that defect was discovered.

By treating the curve on the general principles here adopted, as a special form of the segmental tooth, it can be studied with ease, and its peculiarities developed in a complete and satisfactory manner. The method, in general terms, is to find the conjugate tooth curve of the gear, for the given circular tooth curve of the pinion, and it presents no new features or difficulties.

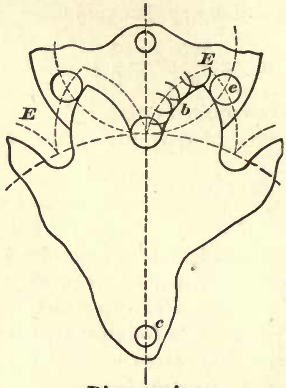
92.—APPROXIMATE FORM OF PIN TOOTH CURVE.

Considered roughly, but accurate enough for teeth of small size, the form of the gear tooth b, Fig. 69, is a simple parallel to the epicycloid E, formed by the center e of the pin, and is to be drawn tangent to any convenient number of circles having centers on the epicycloid.

The action is practically all on one side of the line of centers, the face of the gear tooth working with the part of the pin that is inside of its pitch line. It is, therefore, all approaching action when the pin drives and all receding action when the gear drives, and it is best to avoid the increased friction of the approaching action by always putting the pins on the follower.



Lantern wheel
Fig. 70.



Pin gearing
Fig. 69.

93.—ROLLER TEETH.

The pin gear is particularly valuable when the pins can be made in the form of rollers, Fig. 70, for then the minimum of friction is reached. The roller runs freely on a fixed stud, or on bearings at each end, and can be easily lubricated.

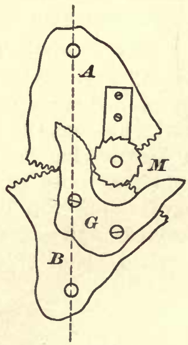
The friction between the tooth and pin, otherwise a sliding friction at a line bearing, is, with the roller pin, a slight rolling friction, and the sliding friction is confined to the surface between the roller and its bearings.

When the roller pin is used there can be no increased friction of approach, and the pin wheel can drive as well as follow.

For very light machinery, such as clock work, there is no form of tooth that is superior to the roller pin tooth, and, with the improvement to be explained, there is no better form for any purpose.

94.—CUTTING THE PIN TOOTH.

The pin gear tooth can be very easily and accurately shaped by mounting a revolving milling cutter *M*, Fig. 71, of the size of the pin, upon a wheel *A*, and causing it to roll with a wheel *B*, carrying the gear blank *G*. The mill will shape the teeth to the correct form.

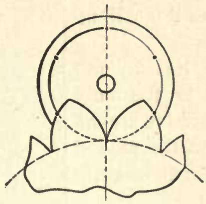


Pin gear cutter
Fig. 71.

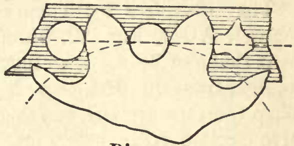
95.—PARTICULAR FORMS OF PIN GEARS.

When the pins are supported between two plates, as in Fig. 70, the wheel is called a "lantern" wheel, and is the most common form of clock pinion. The pins are sometimes called "staves," and are sometimes known as "leaves."

When the diameter of the pin is zero, Fig. 72, it being merely a point, the correct tooth curve will be a simple epicycloid.



Point gears
Fig. 72.

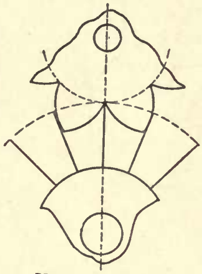


Pin rack
Fig. 73.

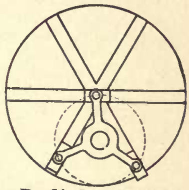
When the pin gear is a rack, Fig. 73, the tooth bears on the pin only at a single point on the pitch line, and the action is therefore very defective unless the roller form of pin is used. This form is more properly a particular case of the involute tooth, for the shape of the pin is immaterial if it does not interfere with the gear tooth. The circle with center on a straight line is not an odontoid at all, for, although it coincides as a whole and for a single instant with a circular space in the gear, it has no proper and continuous tooth action.

The gears of Fig. 74, sometimes classed with pin gearing, are not pin gears at all. An epicycloidal face working with a radial flank is a very common combination.

When the diameter of the pin wheel is half that of the internal gear with which it works, we have the combination of Fig. 75. The pins may run in blocks fitted to the straight slots.



Not pin gears
Fig. 74.



Radial pin teeth
Fig. 75.

96.—CORRECT FORM AND DEFECT OF PIN TEETH.

Although the pin tooth is apparently of a very simple form, a close examination will show that it is really quite complicated, and that its practical action is incomplete and defective. There is a cusp (16), and consequent failure in the action, that is of small importance when the teeth are small, but which is troublesome when they are large. This defect need not be considered when pinions for clock work are in view, but if pin wheels are to be used for large machinery and heavy power it is important.

If the pin *a*, Fig. 76, is examined as an odontoid, it will be seen that it is a true odontoid only within the line *TeT* that is tangent to the pitch line at the center of the pin, for all normals, as *pe*, from points outside of that line, intersect the pitch line at the center.

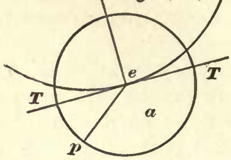
Drawing the normals, which are radii of the pin, we can easily construct the line of action and the conjugate tooth curve. The line of action, commencing at the pitch point *O*, Fig. 77, is there tangent to the line *eOm*, which passes through the center *e* of the pin, curves toward *Oh*, the tangent to the pitch line at the pitch point, and touches it at the point *h*, at the distance *Oh*, equal to the radius of the pin. From the point *h* it follows the circle *hJh'* to the point *h'*, thence returning to the pitch point and forming the loop *OKL*.

From the center *c* of the gear, Fig. 78, we can always draw a tangent arc *FN* to the line of action at the point *F*, and therefore there will always be a cusp at *N* on the tooth curve. The tooth curve must end at the cusp, and, to avoid interference, the pin must be cut off at the arc *W*, drawn through the point *F*, from the center *C*.

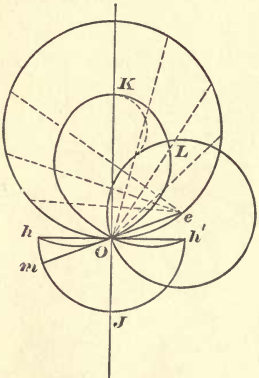
The whole pin is generally used, and when it is a roller it must be whole, and then interference can be avoided only by cutting away the tooth curve until it will allow it to pass.

The complete tooth curve has a first branch *NOM*, Fig. 78, which is the only part that can be used, an inoperative second branch from the first cusp *N* to the second cusp *Q* on the arc *EQ*, and thence an inoperative circle *ORQ'*.

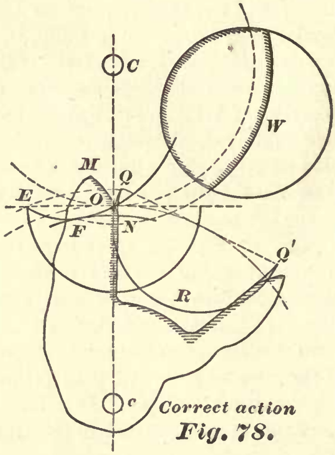
Fig. 76.



Pin odontoid



Line of action
Fig. 77.



Correct action
Fig. 78.

97.—AN IMPROVED PIN TOOTH.

The cause of the broken action of the pin tooth is the cusp, which is always present when the center of the pin is on the pitch line, and it can be avoided by placing the center back, as in Fig. 79, to such a distance inside the pitch line that the cusp does not occur.

When the center of the pin is inside the pitch line, the whole circle of the pin is a true odontoid, and the distance *en* of the center from the pitch line can be so chosen that the cusp is not formed.

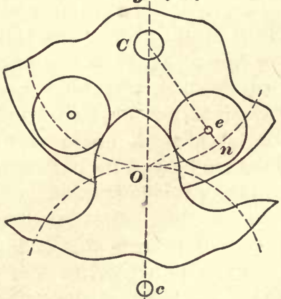
This distance does not appear to be subject to any simply stated rule, but in the single case of the pin rack it is determined by the formula:

$$x = \frac{2}{27} \frac{d^2}{D},$$

in which x is the required distance *en*, D is the diameter of the gear, and d is the diameter of the pin.

If the angle *CeO*, Fig. 79, is not less than a right angle, there will be no cusp on the gear tooth if the diameter of the gear is greater than that of the pin.

Fig. 79.



Corrected pin gear

6. TWISTED, SPIRAL, AND WORM GEARS.

98.—STEPPED GEARS.

When two or more gears, Fig. 80, of the same pitch diameter, are placed in contact on the same shaft, they will evidently act as independently of each other as if they were some distance apart, while they appear to act together as a single gear with irregular teeth. They are known as "Hooke's Gears."

It matters not how many different kinds or numbers of teeth the several gears may have, or in what order they are arranged, if those that work together on opposite shafts are matched. They may be given an irregular arrangement, as in Fig. 80 ; a spiral arrangement, as in Fig. 81 ; a double spiral, or "herring-bone" arrangement, as in Fig. 82 ; a circular arrangement, as in Fig. 83, or otherwise at will.

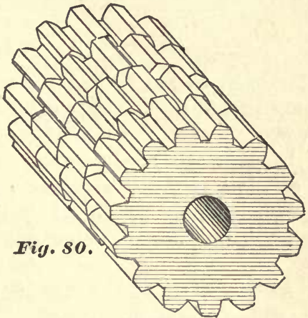
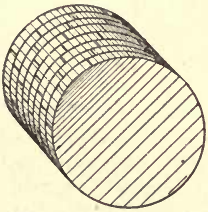
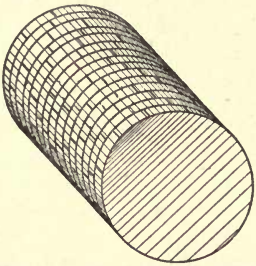


Fig. 80.

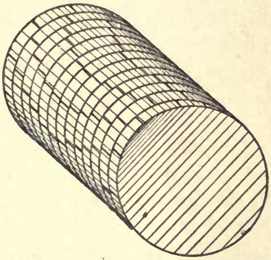
Stepped gear



Spiral arrangement
Fig. 81.



Double spiral arrangement
Fig. 82.

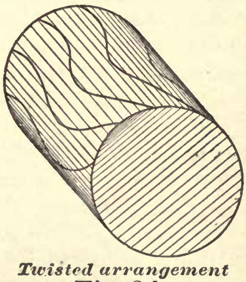


Circular arrangement
Fig. 83.

99.—TWISTED TEETH.

The thickness of the component gears has nothing to do with the theoretical action of the stepped gear as a whole, and therefore we can have them as thin as required. If the thickness is infinitesimal the component character of the gear is not apparent, and it is known as a twisted gear, Fig. 84.

When the teeth are twisted there may always be one or more points of contact at the line of centers, where the theoretical friction is nothing, and therefore they are particularly well suited for rough cast teeth. Furthermore, if the teeth are badly shaped



Twisted arrangement
Fig. 84.

the twisted arrangement tends to distribute the errors so that they are not as noticeable.

The oblique action of twisted teeth tends to produce a longitudinal motion of the gears upon their shafts, which must be guarded against. This end thrust may be avoided by so forming the twist that there are always two oblique bearings between the teeth, acting in opposite directions, as in the herring-bone arrangement.

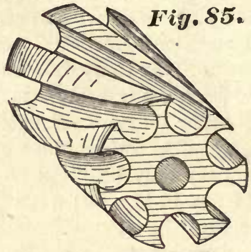
The twisted form of tooth is seldom found in practice, except in the form of spiral and double spiral teeth, for the difficulty of forming other twists is great.

100.—EDGE TEETH.

If the twist of the twisted tooth is such that some part of the twist at the pitch cylinder is always upon the line of centers, the gears will always be in action whether there are full teeth or not, and they will work with theoretical accuracy if they are reduced to edges in the pitch cylinder, as in Fig. 85.

The friction of the edge tooth is theoretically nothing, as there is no sliding of the teeth on each other. There is but one point of contact, and that is always upon the line of centers; but if any power is carried the pressure will soon destroy the single point of contact.

Fig. 85.



Edge teeth

If the edges are thick the action will be stronger, but there will still be but one point of contact.

101.—INVOLUTE TWISTED TEETH.

When the form of the tooth is the involute, and the twist is such that some part of it on the pitch cylinder always crosses the line of centers, the teeth will remain in contact, when the parallel axes are separated, until their points are separated, although the contact may sometimes be very short or even point contact. The straight involute tooth will fail as soon as the arc of contact is less than the tooth arc.

Twisted involute teeth are therefore particularly valuable for gears for driving rolls, or for other purposes where the shaft distance is variable.

102.—FORMATION OF THE TWISTED TOOTH.

When the twist is a uniform spiral there are convenient methods for shaping the tooth, but the twisted tooth in general can be formed only by the processes of (27), (28) and (29), and then only when the twist is not very irregular.

The principle of the linear planing operation of (29) is the same as for the straight tooth, but the blank must be rotated according to the form of the twist adopted, while the tool is cutting. The twisting motions are independent of the feeding motion, and are repeated at every stroke.

103.—SPIRAL GEARS.

The spiral gear is that particular form of the twisted gear which has uniformly twisted teeth, and it is, therefore, a particular form of the common spur gear. It has such peculiar properties that it is often classed by itself as a separate form of tooth.

The normal spiral section is that section of the teeth of the spiral gear that is made by a spiral surface, called a helix, that is at right angles with the teeth. It is the equivalent, for spiral teeth, of the normal section of the spur gear that is made by a plane, or of the normal section of the bevel gear that is made by a sphere. As with spur and bevel gears, the action of the teeth on each other should be studied upon this normal surface. As the helix cannot be represented upon a plane figure it must be imagined, and as it is obscure it requires close attention.

Any two spiral teeth will work together, provided their normal spiral sections are conjugate (24), and, as the shape of the normal spiral section is independent of the angle of the spiral, two spiral gears will work together, approximately, on shafts that are askew. This will be seen more clearly if the spiral section is imagined to be a flexible sheet-metal toothed helix, which can be coiled about the shaft of the gear, for it can evidently be coiled close or loose without affecting the shape of its teeth. If coiled close, with a short lead, it runs nearly at right angles to the shaft, and the gear approximates to the spur gear, while if the lead is long the gear approximates to the screw.

As the diameter of the spiral gear increases, the teeth straighten, and when the diameter is infinite and it is a rack, they are straight and in no way different from those of a common rack.

104.—THEORY OF SPIRAL TOOTH ACTION.

The Willis theory of the action of spiral teeth is the one generally accepted, but it is not correct. It assumes that the action between the gears is upon a section by a plane through the axis of the gear and the common normal to the two axes, and that the section of the two gears made by the plane act together like a rack and gear.

When the axes are at right angles, and the spiral angle is great, this theory is apparently correct, the error being practically imperceptible, but, as the axes become more nearly parallel, the error is more apparent, until, when they are parallel, the error is plain enough. Willis applied his theory to worms and worm gears, on axes at right angles, and evidently did not consider the spiral gear in general.

The action between spiral teeth is not upon the axial section, and it is not that of a rack and gear, but when there is any action at all it is upon the normal spiral section. See the AMERICAN MACHINIST for May 19th, 1888.

When the axes are parallel the normal spiral sections, as well as the sections made by a plane normal to the axes, are conjugate, and therefore the action is correct and along a line of action. The action is also continuous when the axes intersect and the gears are bevel gears.

When, however, the axes are askew, the normal spiral sections are not necessarily conjugate, for they coincide only on one line, the common normal to the two axes. Therefore, there is no continuous tooth contact, except in one particular case, the teeth being in contact only for an instant as they pass the normal.

The special case for which spiral teeth on askew axes have continuous tooth contact, is that case of the involute tooth when the base cylinders are tangent and the gears become spiraloidal skew bevel gears. See (175) and (176). In that particular case the teeth have a sliding conjugate action on each other. As the spiraloidal gear is fully described in its place, it will not be further considered here.

This theory is corroborated by experimental gears made for the Brown & Sharpe Manufacturing Company, for whom Mr. O. J. Beale, to whom the theory of the spiral gear is much indebted, made a pair of theoretically perfect spiral gears, exactly alike, with a spiral angle of 45°, working on shafts at right angles, and of such a large size that the action of the teeth could be plainly observed. See Figs. 88 and 90.

Beale's gears cannot be made to run together properly at any shaft distance, but if their ends are brought to the common normal, and their base cylinders are in contact, they are skew bevel gears and show the action required by Olivier's theory.

But, although the action of spiral gear teeth is intermittent, and their contact is theoretically perfect at one instant only, when they are passing the common normal, they are very nearly in contact all the time and the action is practically perfect. Spiral teeth of ordinary sizes work together with a remarkably smooth action.

105.—FORMATION OF THE SPIRAL TOOTH.

As the spiral rack has an ordinary straight tooth, we can conveniently derive the spiral tooth in general from it by a method that is a form of the molding method of (27) for spur gears.

If a plane is moved in any direction upon a cylinder it will move it, as if by friction, with a speed that depends upon the direction of the motion. If we imagine the same resulting motion between the plane and the pitch cylinder, and assume that the plane is provided with hard and straight teeth running in any direction, it will mold the plastic substance of the cylinder and form spiral teeth upon it. All spiral teeth formed by the same rack will have normal spiral sections that are approximately conjugate to each other, and they will work together interchangeably.

This process may be put into practical shape by a modification of the process of (28) for spur gears, by substituting a planing tooth for the molding rack tooth. The tooth has the shape of the normal section of the rack, and, as it is reciprocated at an angle with the axis of the gear blank being shaped, both the tool and the gear blank receive the motion of the plane and pitch cylinder. The cutting face of the tool is normal to the direction of its motion, which motion is tangent to the direction of the tooth spiral.

The linear process of (29) may be used, the plane of Fig. 20 representing, approximately, the normal spiral section of the gear. Thus, if the planing tool or the equivalent milling cutter receives a motion as if in a plane rolling upon the base cylinder, the involute tooth will be produced.

The spiral tooth may be formed by the linear planing process of (29), directly applied on the principle that the spiral tooth is a twisted spur tooth. The planing tool receives a planing motion in the direction of the axis of the gear blank, and both tool and blank receive the feeding rolling motion that would produce the spur tooth of the section that is normal to the axis. In addition, the blank receives a motion of rotation while the tool moves, that is repeated for every troke of the tool. The cutting edge of the tool is set normal to the axis of the gear.

The spiral tooth may also be formed by a tool that is formed to the true shape of some section of the tooth, preferably its normal section, and which is guided in the tooth spiral. This is the process used to shape a worm, the tool being guided by a screw-cutting lathe.

The process generally used to mill the teeth of the spiral gear is the equivalent of the operation last described. The milling cutter is shaped to the normal section of the tooth space, and is guided in the tooth spiral by a special feeding device that rotates the blank while the cutter is working in it.

Of these processes the planing process of (28) is the best, as it produces the tooth with theoretical perfection, and because all gears formed with the same tool are conjugate and interchangeable. But the screw-cutting and milling processes are most in use, for the reason that they are more expeditious and better adapted to the common machine tools, and it is therefore necessary to study the shape of the normal section of the tooth with some care.

106.—THE NORMAL PITCH.

The real pitch of the spiral gear is measured on a section that is normal to its axis, and, as in the case of the spur gear, it is found by dividing the number of teeth by the pitch diameter, but the shape of the tooth must be regulated by the normal pitch, or pitch of its normal section.

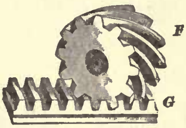
The normal pitch is found by dividing the real pitch by the cosine of the angle made by the tooth spiral with the axis of the gear. Thus, if the pitch is 8, and the angle is 45°, the normal pitch is 8, divided by .707, or 11.3.

The normal circular pitch is found by multiplying the real circular pitch by the cosine of the spiral angle.

107.—THE ADDENDUM.

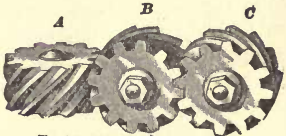
The addendum of the spiral gear should not be determined by its real pitch, but by its normal pitch, for it is then usually possible to mill the tooth with a milling cutter that is made for a standard spur gear. A gear of 8 pitch and 45° angle should have an addendum of $\frac{1}{11.3} = .089''$.

If the addendum is determined by the true pitch when the angle is considerable, the tooth will be long and thin. Fig. 86 shows the normal pitch section of a rack to run with a pinion of 45° angle, while Fig. 87 shows the true pitch of the same rack. Fig. 88 also shows the true pitch of the pinion, and, although the tooth appears to be stunted, it is really of the standard shape.



Normal Rack.
Fig. 86.

Spiral Rack.
Fig. 87.



Beale's Experimental Gears.
Fig. 88.

108.—THE AXIAL PITCH.

The section of the spiral gear by a plane through the axis is that of a rack, and the axial pitch, or pitch of the rack, is found by dividing the true pitch by the tangent of the spiral angle. Thus, if the angle is 45°, the axial pitch is the same as the true pitch, but the axial pitch of a 70° spiral tooth is but .364 of the true pitch.

109.—SHAPING THE TOOL.

When the spiral gear is cut in a milling machine, or turned in a lathe, it is necessary to give the tool the shape of the normal section of the tooth to be cut, and this is most readily accomplished by shaping it for the spur gear that most nearly coincides with that normal section.

The number of teeth in the gear that is osculatory to the normal spiral, and therefore most nearly coincides with it, is found by dividing the actual number of teeth in the gear by the third power of the cosine of the spiral angle.

For example, if we are to cut a gear of 4″

diameter, 6 pitch, and 24 teeth, at a spiral angle of 45°, the cutter should be shaped to cut a spur gear of $\frac{24}{.707^3} = \frac{24}{.35} = 69$ teeth of $\frac{6}{.707} = 8.5$ pitch. If the gear has 28 teeth of 4 pitch, and an angle of 10°, the equivalent spur gear has 29 teeth of 4.08 pitch, as the gear varies but little from a spur gear. If the gear is of 5 pitch, and 15 teeth, with an angle of 80°, the equivalent spur gear has 2,830 teeth of 28.7 pitch, and in general, when the gear has a great angle it is a worm, the section is practically that of a rack. Care must be taken, when the gear is a screw, and is turned in the lathe, that the tool should be set with its cutting edge normal to the thread of the screw, if it is shaped by the above rule. If the tool is set in the axial section of the screw, and it generally is, it should be shaped to the axial section of the worm, and have the axial pitch and addendum. But when the lead of the thread of the screw is small compared with its diameter the difference between the normal and axial sections is not noticeable.

110.—VELOCITY RATIO OF SPIRAL GEARS.

The spiral gear does not follow the well-known rule of spur gears, that the velocities in revolutions in a given time are inversely proportional to the pitch diameters, but requires that ratio to be multiplied by the ratio of the cosines of the spiral angles.

In the formula

$$\frac{v}{V} = \frac{D}{d} \; \frac{\cos. A}{\cos. a}$$

D and d are the diameters of the gears, A and a are their spiral angles, and V and v are their velocities in revolutions.

If the angles are equal, the velocity ratio is the same as for spur gears of the same diameters. Fig. 88 shows a pair of gears B and C that are of the same size and have the same angle in opposite directions, requiring the shafts to be parallel. See also Fig. 89. The pair of gears A and B are exactly alike, with equal angles in the same direction, requiring the shafts to be at an angle equal to twice the spiral angle. See also Fig. 90. The statement that like spiral gears will not run together is founded on the Willis theory of spiral gear contact, and is wrong.

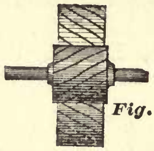
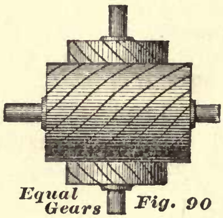


Fig. 89

Spiral Spur Gears



Equal Gears *Fig. 90*

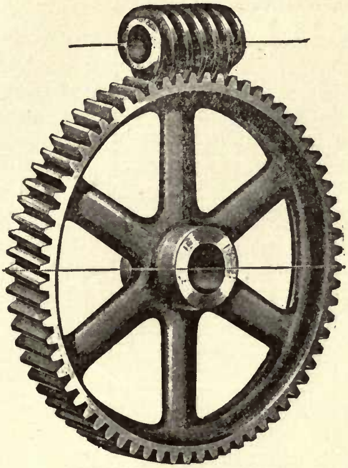
111.—SPIRAL WORM AND GEAR.

When the shafts are at right angles, and the angle on one is so great that it is a screw, the combination is known as a worm gear and worm, Figs. 91 and 92, and is much used for obtaining slow and powerful motions. It is also too much used for wasting power and wearing itself out, for its friction is very great and consumes from one-quarter to two-thirds of the power received.

When the screw has a single thread, the velocity ratio is simply the number of teeth in the gear, and if th re are two or three threads it must be modified accordingly.

The spiral worm is adjustable in its gear both laterally and longitudinally, so that it will change its position as required by wear in the shaft bearings.

It is an excellent substitute for the hobbed worm and gear, and in most cases will serve practical purposes quite as well.



Spiral Worm Gear and Worm.
Fig. 91.

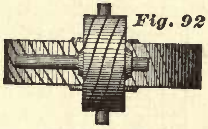


Fig. 92

Worm Gears

112.—EFFICIENCY OF SPIRAL AND WORM GEARING.

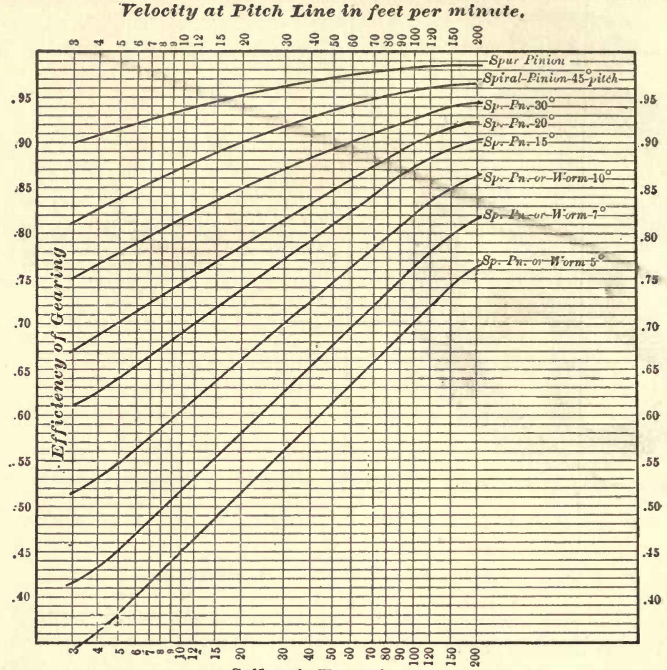
Unless the shafts are parallel the teeth of a pair of spiral gears are moving in different directions, and therefore they cannot pass each other without sliding on each other an amount that increases rapidly with the angle of divergence of the directions of motion, that is, the shaft angle.

This sliding action creates friction and tends to wear the teeth, and to a very much greater extent than is generally supposed. The friction is so great, in fact, that such gears, particularly worm gears, should be used only for conveying light powers. They are extensively used, or rather misused, for driving elevators, and are even found in milling machines, gear cutters, planers, and similar places, in evident ignorance that they waste from a quarter to two-thirds of the power received.

The most extensive experiments on the efficiency of spiral and worm gears ever made were made by Wm. Sellers & Co., and they may be found described in great detail in a paper by Wilfred Lewis in the Transactions of the American Society of Mechanical Engineers, vol. vii. Space will not permit extensive quotations from this valuable paper, but the general result of the experiments is shown by the diagram, Fig. 93. The diagram shows that a common cast-iron spur gear and pinion on parallel shafts have an efficiency of from ninety to ninety-nine per cent., according to the speed at which they are working; that a spiral pinion of 45°, angle working in a spur gear, with shafts at 45°, has an efficiency of from 81 to 97 per cent.; that the efficiency decreases as the angle of the shafts increases, until, for a worm of a spiral angle of 5°, at a shaft angle of 85°, it goes as low as 34, and does not rise higher than 77 per cent. This includes the waste of power at the shaft bearings as well as that at the teeth of the gears. The efficiency is lowest for slow speeds, and rises with the speed. The diagram may be relied upon to give its true value, under ordinary conditions, within five per cent.

The same experiments developed the fact that the velocity of the sliding motion of the cast-iron teeth on each other should not be over two hundred feet per minute in continuous service, to avoid cutting of the surfaces. It may be assumed that the efficiency will be higher when the worm is of steel, particularly when the gear is of bronze.

Diagram, Fig. 94, shows the result of simi-



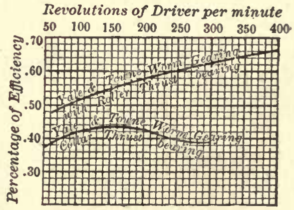
Sellers' Experiments

Fig. 93.

lar experiments by Prof. Thurston, with a worm of 6″ diameter and one inch circular pitch running in a gear of 16″ diameter, both cast-iron.

It is to be observed that it is the shaft angle, and not the angle of the spiral, that determines the efficiency. A pair of spiral gears on parallel shafts are practically as efficient as gears with straight teeth.

The great friction of worm gearing is of advantage for one purpose, and for one only, to secure safety and prevent undesired motion of the gears. The worm of Fig. 97 will easily move the gear, but the gear must be moved with great force to start the worm. When the angle of the worm is as small as the "angle of repose" for the metals in contact, it is impossible for the gear to drive the worm. This may be an excuse for the use of the worm gear in elevators, but it would seem that the safety of the cage should depend on devices attached to the cage itself, rather than to the hoisting machinery or other distant part.



Yale & Towne Experiments

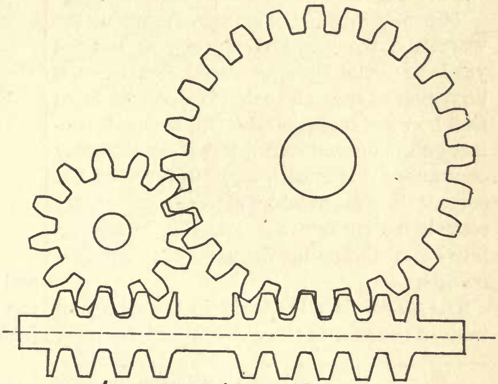
Fig. 94.

Unless the friction of the gears must be depended upon for safety, the worm gear should be used only for purposes of adjustment, or when speed must be greatly reduced or power increased within a small compass, and not for conveying power.

113.—THRUST OF SPIRAL TEETH.

The oblique action of the teeth of spiral gears on each other tends to throw the gears bodily in the direction of their axes, and this tendency creates a thrust that must be opposed by thrust bearings. The end pressure on the shaft of a worm is greater than that exerted on the teeth of the worm gear it is driving.

When the shafts are parallel the thrust may be completely avoided by the use of double spiral or "herring-bone" teeth, Fig. 82 or 83, which act in opposite directions, and neutralize each other.



Arrangement to avoid thrust
Fig. 95.

When the shafts are at right angles the thrust may be neutralized by opposing a second gear in the manner shown in section by Fig. 95. The two worms with opposite spirals run in two spiral worm gears that also work with each other, and, as the pressure on one gear is opposite that on the other, there is no thrust on the shaft. When this combination is made with worm gears having concave teeth, the teeth can bear only at their ends.

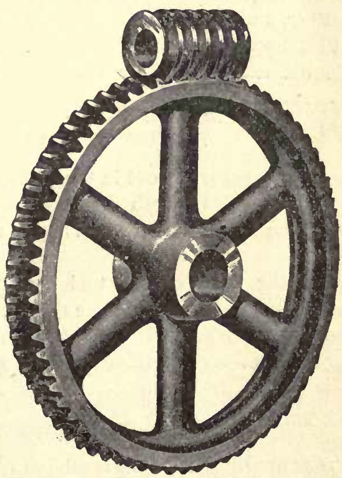
When the thrust cannot be avoided it should be taken by a roller bearing, rather than by the common collar bearing. The diagram, Fig. 94, shows the superior efficiency of the roller bearing as compared with the collar bearing, the gain being from ten to twenty per cent.

114.—THE HOBBED OR CONCAVE WORM GEAR.

If a spiral gear is made of steel, provided with cutting edges by making slots across its teeth, and hardened, it will be a practical cutting tool called a spiral milling cutter or hob. Fig. 96 shows a spiral milling cutter, having a great spiral angle, and therefore called a worm.

If this cutting spiral gear is mounted in connection with an uncut blank so that both are rotated with the proper speeds, and the shafts of the two gears are gradually brought together while they are revolving, the edge of the blank will be formed with concave teeth that curve upwards about the sides of the cutting gear. If the hob is then replaced with a spiral gear that is a duplicate of it, except that it has no cutting teeth, we shall have the familiar worm and worm gear of Fig. 97.

The principle of the concave gear applies to any pair of spiral gears, on shafts at any



Concave Worm Gear and Worm.
Fig. 97.

angle, but in practice it is confined to the worm and gear on shafts at right angles.

The nature of the contact between the worm and the concaved worm gear has not yet been definitely determined, but there is no reason to suppose that it is different from that between plain spiral teeth, a point contact on the normal spiral, but it is probably continuous. It is certain, however, that the contact is considerably closer, more nearly resembling surface contact, and being surface contact when the diameter of the gear is infinite.

The worm is adjustable in the concaved teeth of the gear in the direction of its axis, and will change its position as required by the wear of the thrust bearing. It is not adjustable laterally.

A Hob. Fig. 96.

115.—HOBBING THE WORM GEAR.

When the hob is provided it is a simple matter to cut the gear. The gear is generally provided with the desired number of notches in its edge, that are deep enough to receive the points of the teeth of the hob, and the hob will then pull it around as it revolves.

It is a too common practice to make the hob do its own nicking, for, if it is forced into the face of the gear as it revolves, it will pull it around by catching its last teeth in the nicks made by the first.

If luck is good these nicks will run into each other, and the gear will be cut with teeth that appear to be correct, but, as the outside diameter is greater than the pitch diameter, there will be one, two, or three teeth too many. The teeth of the finished gear are therefore smaller than those of the worm by an amount that is ruinous if the gear is small, although it is not noticeable when the diameter is large. If there are 12 teeth where there should be but 10, each tooth will be too small by two-twelfths of itself; but if there are 102 teeth where there should be but 100, each tooth is too small by but two-hundredths of itself. This handy makeshift process will do very well on large gears, but not on small ones, unless the worm to run in the gear is made to fit the tooth, with a tooth that is smaller, and lead that is shorter than that of the hob.

116.—INVOLUTE WORM TEETH.

Worms are generally cut in the lathe, and as a straight-sided tooth is most easily formed, the involute tooth is generally adopted.

Strictly, the form of the tool should be that of the normal section of the thread, and it should always be set in the lathe with its cutting face at right angles to the thread. But custom and convenience allow the tooth to have straight sides, and to be set with its face parallel with the axis of the worm, and the real difference is not generally noticeable.

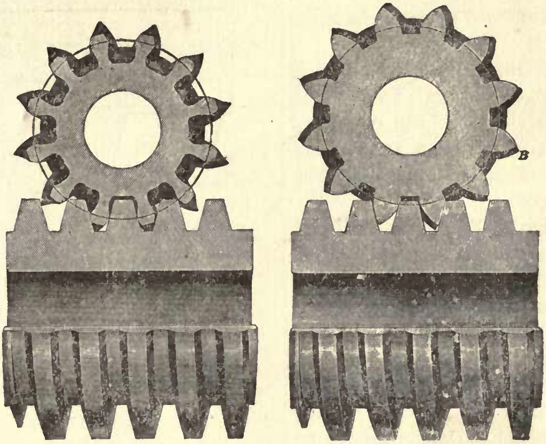
The standard tool has its sides inclined at an angle of 30°, and has a length and a width dependent upon the pitch.

117.—INTERFERENCE OF INVOLUTE WORM TEETH.

There is one difficulty that is seldom recognized, but which must be carefully guarded against if properly running gears are expected, and that is interference. The teeth of worm gears will interfere with each other when the conditions are right for interference, just as spur involute teeth will interfere, as shown by Fig. 36. Fig. 98 shows the gear that would be formed by the usual process.

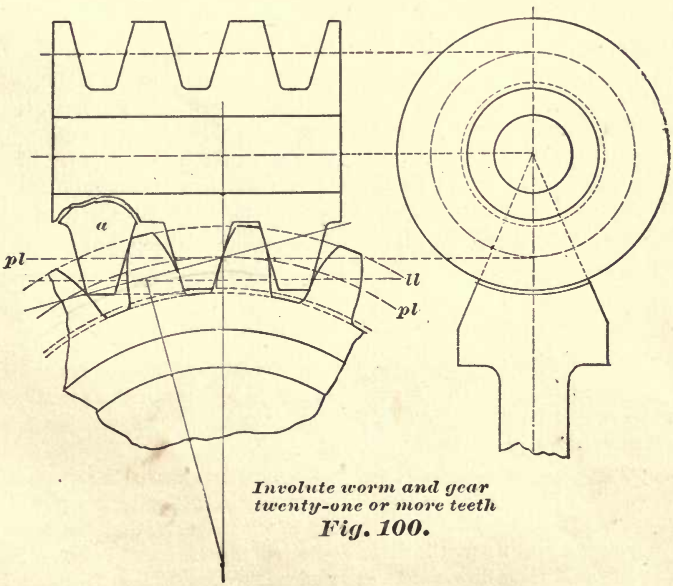
The difficulty can be remedied by rounding over the tops of the teeth of the hob and worm, as described in (55).

It is also a simple matter to avoid the interference by enlarging the outside diameter of



Interfering Worm.
Fig. 98.

Interference Avoided.
Fig. 99.



Involute worm and gear
twenty-one or more teeth
Fig. 100.

the worm gear. If, as shown by Fig. 99, the tooth has but a short flank, or none at all, and the addendum of the gear is about twice that by the usual rule, the action will be confined to the face of the gear and the flank of the worm, and there can be no interference. By adopting an obliquity greater than 15°, interference can be avoided without changing the addendum.

This method has the advantage that the same straight-sided worm and hob can be used for small gears as for large ones, and the disadvantage that the action is confined to the approach and subject to greater friction (48).

When the standard 30° tool is used, all gears of 26 teeth, or smaller, should be made in this way, but the correction is not strictly necessary for gears of more than 20 teeth, unless particularly nice work is required.

Fig. 100 shows the proper construction of a gear of 21 or more teeth, and Fig. 101 shows that of a gear of less than 21 teeth. In the former case, the teeth of the worm should be limited by the limit line *ll*, but the interference for 21 to 25 teeth is not noticeable.

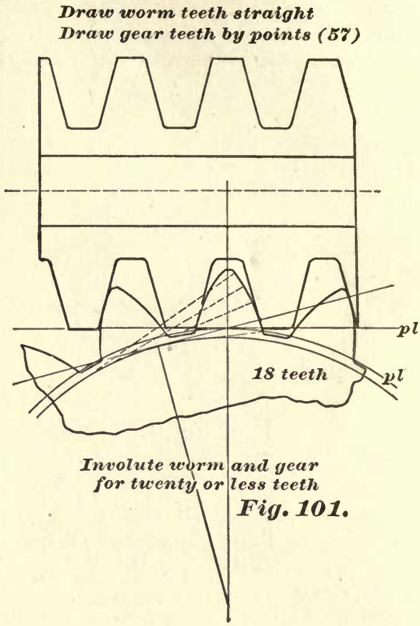


Fig. 101.

118.—CLEARANCE OF WORM TEETH.

There is another practical point that is seldom recognized, and that is that worm teeth should have clearance (40), for there is no reason for clearing spur teeth that will not apply quite as well to any other kind.

The clearance is easily obtained by making the tooth of the hob a little longer than that of the worm, as shown by the tooth *a* of Fig. 100. For the same reason the hob should have no clearance at the bottom of its thread, so that the tops of the gear teeth will be formed of the proper length. The custom of making the hob and worm of exactly the same diameters will apply only when the worm "bottoms" in the gear and the gear bottoms in the worm.

119.—CIRCULAR PITCH WORM TEETH.

The old and clumsy circular pitch system is in universal use for worm teeth, for the reason that worms are generally made in the lathe, and lathes are never provided with the proper change gears for cutting diametral pitches. The error is so firmly rooted that it is useless to attempt to dislodge it.

It is therefore necessary to figure the diameters of worm gears as if their throat sections were the same as those of common spur gears and racks on the circular pitch system. The table of diameters (35) will be of great assistance.

One great objection to the use of the circular pitch system for spur gears does not apply to worm gears, that the center distance between the shafts will always be an inconvenient fraction, unless the pitch is as inconvenient. The worm can be made of any diameter, and can therefore be made to suit the pitch diameter of the gear and the center distance at the same time.

The sides of the tool for circular pitches should come together at an angle of thirty degrees, and the width of the point, as well as the depth to be cut in the worm or in the hob, should be taken from the following table. The diameter of the hob should be greater than that of the worm by the "increase" given.

Make the tool with the proper width at the point to thread the worm, and then, after making the worm, grind off half the "increase" from the length of the tool, and use it to thread the hob.

TABLE FOR CIRCULAR PITCH WORM TOOLS.

Circular pitch	2	1¾	1½	1¼	1⅛
Point of hob tool	.644	.564	.483	.402	.362
Point of worm tool	.620	.542	.466	.388	.349
Depth of cut in worm or hob	1.416	1.240	1.062	.886	.797
Increase	.166	.146	1.249	.104	.094
Circular pitch	1	⅞	¾	⅝	½
Point of hob tool	.322	.282	.241	.201	.161
Point of worm tool	.310	.271	.233	.194	.155
Depth of cut in worm or hob	.708	.620	.531	.443	.354
Increase	.083	.073	.062	.052	.042
Circular pitch	7/16	⅜	5/16	¼	3/16
Point of hob tool	.141	.121	100	.080	.060
Point of worm tool	.135	.116	.097	.078	.058
Depth of cut in worm or hob	.310	.265	.222	.177	.133
Increase	.036	.031	.026	.021	.016

120.—DIAMETRAL PITCH WORM TEETH.

If the proper change gears are provided, it is as easy to cut diametral pitch worm teeth as any. The proper gears can always be easily calculated by the rule that the screw gear is to the stud gear as twenty-two times the pitch of the lead screw of the lathe is to seven times the diametral pitch of the worm to be cut.

For example, it is required to cut a worm of twelve diametral pitch, on a lathe having a leading screw cut six to the inch. We have

$$\frac{\text{screw gear}}{\text{stud gear}} = \frac{22 \times 6}{7 \times 12} = \frac{11}{7};$$

and any change gears in the proportion of 11 and 7 will answer the purpose with an error of $\frac{1}{10{,}000}$ of an inch to the thread of the worm.

If 22 and 7 give inconvenient numbers of teeth, the numbers 69 and 22 can be used with sufficient accuracy, and 47 and 15, or even 25 and 8 may do in some cases.

To save calculation and study, the table of change gears for diametral pitches is provided, and it will give the proportion of screw gear to stud gear to be used for all ordinary cases.

The pair on the left will give the proper pitch within less than a thousandth of an inch, and that on the right will serve with an error always less than a hundredth of an inch, and sometimes less than two or three thousandths of an inch.

Having the change gears, figure the pitch diameter of the gear as if the throat section is a spur gear on the diametral pitch system.

The sides of the tool should come together at an angle of thirty degrees, and the width of the point of the tool, as well as the depth to be cut in the worm or in the hob, should be taken from the following table. The diameter of the hob should be greater than that of the worm by the "increase" given.

TABLE FOR DIAMETRAL PITCH WORM TOOLS.

Diametral pitch	1	2	3	4
Point of hob tool	1.035	.517	.345	.258
Point of worm tool	.968	.484	.323	.242
Depth of cut in worm or hob	2.125	1.063	.708	.532
Increase	.250	.125	.083	.063
Diametral pitch	5	6	7	8
Point of hob tool	.207	.173	.148	.129
Point of worm tool	.194	.162	.138	.121
Depth of cut in worm or hob	.425	.354	.304	.266
Increase	.050	.042	.036	.032
Diametral pitch	10	12	14	16
Point of hob tool	.104	.086	.074	.065
Point of worm tool	.097	.081	.069	.060
Depth of cut in worm or hub	.213	.177	.152	.133
Increase	.025	.021	.018	.016

Make the tool with the proper width at the point to thread the worm; and then, after making the worm, grind off half the "increase" from the length of the tool, and use it to thread the hob.

Pitch of Leading Screw.

Diametral Pitch to be Cut.	2	3	4	5	6	7	8	10
3	44/21 · 23/11	22/7 ·	88/21 · 46/11	110/21 ·	44/7 · 69/11	21/3 ·	176/21 · 92/11	220/21 ·
4	11/7 ·	33/14 ·	22/7 ·	55/14 ·	33/7 ·	11/2 ·	44/7 ·	55/7 ·
5	44/35 · 5/4	66/35 · 15/8	88/35 · 5/2	22/7 · 25/8	132/35 · 15/4	22/5 · 35/8	176/35 · 5/1	44/7 · 25/4
6	22/21 ·	11/7 · 30/19	44/21 · 40/19	55/21 · 50/19	22/7 · 60/19	11/3 · 70/19	88/21 · 80/19	110/21 · 100/19
7	44/49 · 9/10	66/49 · 27/20	88/49 · 9/5	110/49 · 9/4	132/49 · 27/10	22/7 · 63/20	176/49 · 18/5	220/49 · 9/2
8	11/14 · 4/5	33/28 · 6/5	11/7 · 8/5	55/28 · 2/1	33/14 · 12/5	11/4 · 14/5	22/7 · 16/5	55/14 · 4/1
10	22/35 · 5/8	33/35 · 15/16	44/35 · 5/4	11/7 · 25/16	66/35 · 15/8	11/5 · 35/16	88/35 · 5/4	22/7 · 25/8
12	11/21 · 10/19	11/14 · 15/19	22/21 · 20/19	55/42 · 25/19	11/7 · 30/19	11/6 · 35/19	44/21 · 40/19	55/21 · 50/19
14	22/49 · 4/9	33/49 · 2/3	44/49 · 8/9	55/49 · 10/9	66/49 · 4/3	11/7 · 14/9	88/49 · 16/9	110/49 · 20/9
16	11/28 · 2/5	33/56 · 3/5	22/28 · 4/5	55/56 · 1/1	33/28 · 6/5	77/56 · 7/5	11/7 · 8/5	55/28 · 2/1

Exact numbers on the left. Approximate on the right.

TABLE OF CHANGE GEARS FOR DIAMETRAL PITCH WORMS.

121.—WIDTH OF WORM GEAR FACE.

The bearing between the tooth of the worm and that of the gear is near the center of the gear, and it is quite small (104). It is, therefore, useless to make the gear with a wide face. If the face is half the diameter of the worm it will have all the bearing that can be obtained, and any extra width will simply add to the weight and cost of the gear.

The length of the worm need be no more than three times the circular pitch, for there are seldom more than two teeth in contact at once. If, however, the worm is made long, it can be shifted when it becomes worn, so as to bring fresh teeth into working position. This provision is wise, for the reason that the worm is always worn more than the gear.

122.—THE HINDLEY WORM AND GEAR.

If the cutting hob and the worm is shaped by the tool *a*, and the process indicated by Fig. 102, the resulting pair of gears is known as the Hindley worm and gear. The worm is often called the "hour-glass" worm.

It is commonly but erroneously stated that this worm fits and fills its gear on the axial section, the section that is made by a plane through the axis of the worm and normal to the axis of the gear. It has even been stated that the contact is between surfaces, the worm filling the whole gear tooth.

The real contact is not yet certain, but it is certain that it is not a surface contact. It is also certain that it is on the normal and not on the axial section, and that the Hindley worm hob will not cut a tooth that will fill any section of it. The contact may be linear, along some line of no great length, but it is probably a point contact on the normal section. The order of the contact is certainly very close, resembling that of two surfaces.

The worm is limited in length, for the sides of the teeth cannot slant inward from the normal to the axis. The end tooth m in Fig. 102 cannot be used, for it will destroy the teeth of the gear as it is fed towards this axis in the operation of hobbing.

It has the one great defect that it is not adjustable in any direction, and, therefore, cannot change its position when the shaft bearings wear, unless it is itself worn the same amount. It is doubtful if this form of gearing has any advantage over the plain spiral gearing, except when new and in perfect adjustment.

The Hindley Worm Gear

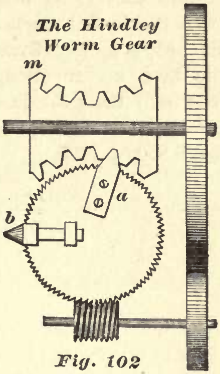


Fig. 102

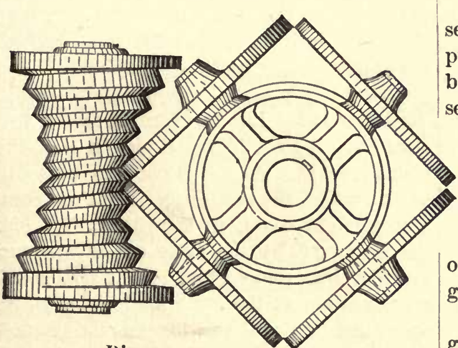
123.—THE PIN WORM AND GEAR.

If the hob and the worm are shaped by the pin-shaped revolving milling tool b of Fig. 102, the gearing produced will have linear bearing between the teeth.

The action will be the same as between a series of pin teeth like the milling tool, each pin being in the axial section of the worm, but having a linear bearing on the normal section of its teeth.

This form of gearing, which is a modification of the Hindley form, may take the shape of pin gearing, the teeth being round pins like the milling tool. If the pins are mounted on studs, so as to revolve, a roller pin worm gear will be produced.

Fig. 103 shows a form of roller pin gearing in which the pins have been enlarged.



Pin worm gear.
Fig. 103.

124.—THE WHITWORTH HOBBING MACHINE.

When the amount of work to be done will warrant the use of a special machine, the hobbing machine of Sir Joseph Whitworth may be used. It was invented in 1835, and has not been materially improved since then, although there are numerous patents relating to it. The worm gear to be hobbed is fixed upon the same spindle with a master worm-wheel. A driving worm runs in the master wheel, and it is connected by a train of gearing with a hob that is so mounted on a carriage that it can be fed towards the gear blank. The hob is slowly forced into the blank, while both are revolving with the proper speeds, and the gear is cut without the assistance of previously made nicks. See British patent 6,850 of 1835.

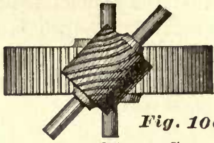


Fig. 106
Spiral and Spur Gear.

125.—THE CONJUGATOR.

This is a machine for cutting spur or spiral gears by means of a hob, and its principle is an extension of that of the Whitworth worm gear hobbing machine.

If, when the hob in the Whitworth machine has reached the full depth of the tooth, it receives a new motion in the direction of the tangent to its pitch spiral, it will continue the tooth to the edge of the gear, and form the plain spiral gear of Fig. 91.

Conjugator. Elevation

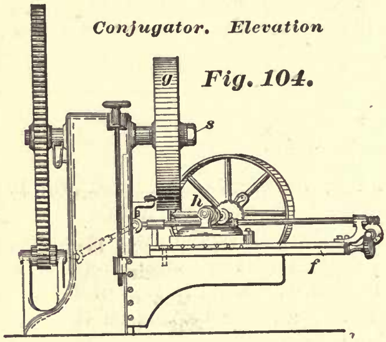
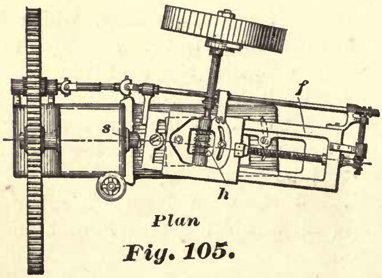


Fig. 104.



Plan
Fig. 105.

Fig. 104 is an elevation of the machine, and Fig. 105 is a plan. The hob *h* is mounted upon an arbor that is connected by a train of gearing with the spindle *s* that carries the blank gear *g* to be cut, so that the hob and blank revolve together with any definite proportionate speed. The hob is carried upon a carriage that is fed on a frame *f*. The hob swivels upon the carriage, so that the tangent to its pitch spiral can be set parallel with the direction of the feed, and the frame swivels so that the tooth can be cut at any angle with the gear spindle.

As the blank and the hob are revolving, the latter is fed into the former, and it will cut a perfect tooth in the direction that the frame is set at. As the frame can be set in any direction, the machine will cut the common straight tooth, as shown by Fig. 106. All gears cut by the same cutter will run together interchangeably, and if two spiral gears are cut at the same angle in opposite directions they will run together on parallel shafts. See U. S. patent number 405,030, June 11th, 1889.

7. IRREGULAR AND ELLIPTIC GEARS.

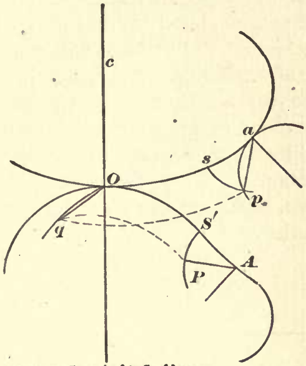
126.—NON-CIRCULAR PITCH LINES.

The consideration of pitch lines that are not circular, and of the teeth that are fitted for them, is an interesting but not particularly important branch of odontics. Such pitch lines are largely used for producing variations of speed and power, but have no other practical applications.

127.—THE IRREGULAR PITCH LINE.

The most general case is that of two indefinite irregular curves rolling together, Fig. 107, the only condition being that they shall be so shaped that they will roll together continuously.

As the practical importance of the free pitch line is very small, we shall not examine it in detail.



Irregular pitch lines
Fig. 107.

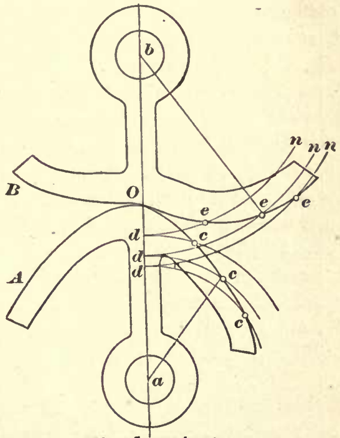
128.—PITCH LINES ON FIXED CENTERS.

When we attach the condition that the two pitch lines shall revolve in rolling contact on fixed centers, we have a definite problem of more interest and importance than that of the free pitch line.

If, as in Fig. 108, we have a pitch line *A* revolving upon a fixed center *a*, we can construct a pitch line *B* that will roll with it, and revolve on the given fixed center *b*, by the following process.

From any pitch point *O*, step off equal arcs *Oc, cc, cc;* draw circular arcs *cd* from the center *a;* draw circular arcs *dn* from the center *b;* step off the same equal arcs *Oe, ee, ee,* then *Oeee* will be the required mating pitch line.

These curves will always be in rolling contact at a point on the line of centers *ab*, the pitch point and the angle of the curves with the line of centers continually changing.



Fixed centers
Fig. 108.

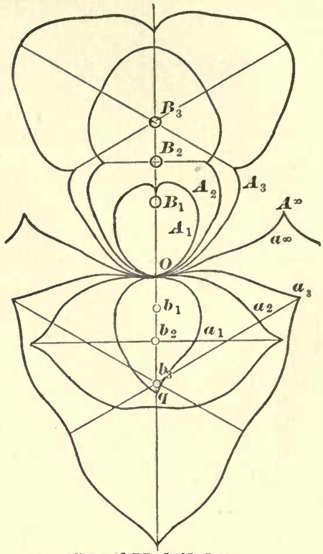
The velocity ratio of the curves will be variable, and always equal to the inverse proportion of any two mating radiants, *ac* and *be*.

129.—CLOSED PITCH LINES.

When one of the curves of Fig. 108 is a closed curve, the other will in general not be closed, but by trying different centers, a curve can be found that will be closed.

If the closed curve a_1, Fig. 109, is taken, the mating curve A_1 will be closed when the center is chosen at a certain point B_1, that can be found by repeated trials.

The mating closed curves thus constructed will seldom be alike, but will always have points of similarity. A salient point q on one will be paired with a reversed point or notch on the other, and lobes on one will be represented by depressions on the other. Half a revolution of one of the curves, from any position, will turn the other through half a revolution.



Set of Multilobes
Fig. 109.

130.—MULTILOBES.

If, after finding the center B_1, Fig. 109, for the closed mating curve, other centers are tried, second, third, and succeeding centers, B_2, B_3, B_4, will be found, about which the mating curves will also be closed.

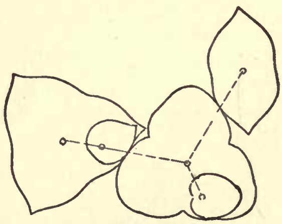
These closed curves, called multilobes, will be each divided into like lobes, the second curve, or bilobe, into two lobes; the third, or trilobe, into three lobes, and so on.

If the center is placed at infinity, the rack lobe A_∞ will be formed.

If the center be taken negatively, on the same side as the original center b_1, at b_2, b_3, b_4, etc., negative multilobes a_2, a_3, a_4, etc., will be formed about the original curve a_1.

All these multilobes, positive and negative, will roll together collectively about their fixed centers, in rolling contact at a common and shifting pitch point O.

Any two, of the same sign, will roll in internal contact, and any two of opposite signs will roll in external contact, so that they can be formed in train, Fig. 110.



Train of multilobes
Fig. 110.

When it so happens, as it does with the ellipse revolving on its focus, or the logarithmic spiral revolving on its pole, is taken, that the first derived pair of curves, or unilobes, are exactly alike, all the multilobes will be alike; the positive trilobe like the negative trilobe, and so on, so that any two curves of such a set will work together in either internal or external contact, Fig. 111.

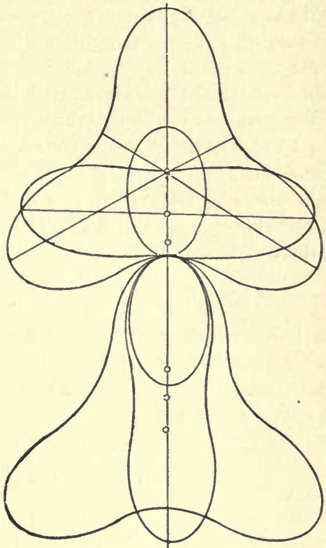
131.—CONIC SECTION PITCH LINES.

If two like conic sections are mounted upon their foci, they will roll together.

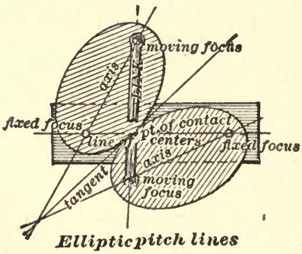
Their free foci will revolve at a fixed distance from each other, and may be connected by a link. The line of the free foci will intersect the line of the fixed foci at the point of contact of the pitch lines.

Fig. 112 shows a pair of ellipses, Fig. 113 a pair of parabolas, and Fig. 114 a pair of hyperbolas.

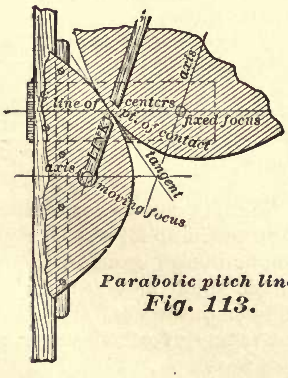
The elliptic pitch line is the only one known that will revolve with its equal, and make a practical and complete revolution.



Elliptic multilobes
Fig. 111.



Ellipticpitch lines
Fig. 112.



Parabolic pitch lines
Fig. 113.

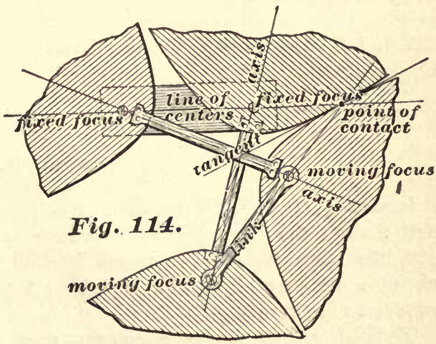


Fig. 114.

Hyperbolic pitch lines

132.—THE LOGARITHMIC SPIRAL.

If the radiants *a, b, c, d, e,* Fig. 115, make equal angles with each other, and each one is equal to the adjacent one multiplied by a constant number, their extremities will determine a logarithmic spiral.

If the first radiant *a* is given, with the constant multiplier *n*, the second radiant will be *na*, the third will be $n^2 a$, the fourth will be $n^3 a$, and so on.

If the first and last radiants, *a* and *e*, are given, and there are *p* equal angles between them, the constant is

$$n = \sqrt[p]{\frac{e}{a}}$$

so that it is a simple matter to construct a logarithmic spiral to connect any two given radiants at any given angle with each other.

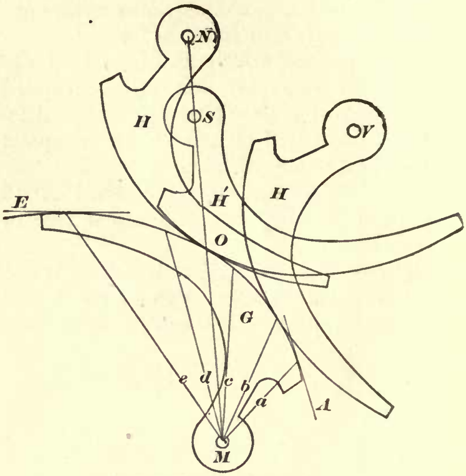
The curve possesses the singular property that all tangents, *A* or *E*, make the same angle with the radiants at their points of contact. The curves are always inclined to the line of centers at the constant angle.

The curve continually approaches the center *M*, or "pole," making an infinite number of turns about it, but never reaching it.

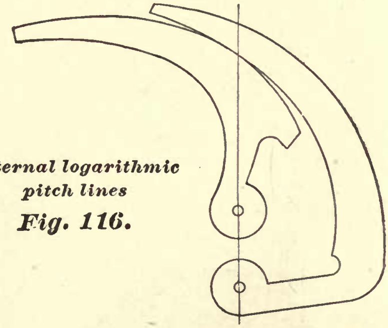
It also has the entirely useless property that the pole will trace an involute of the base circle if it is rolled upon the pitch circle (75).

It possesses the property, not possessed by any other curve, that it will roll with an equal mate on fixed centers that can be varied in position. The curve *H* will roll with the curve *C*, whether its pole is at *N*, or at *S*, or at *V*.

Fig. 116 shows a pair of logarithmic spirals in internal contact.



Logarithmic pitch lines
Fig. 115.



Internal logarithmic pitch lines
Fig. 116.

133.—COMPOSITE PITCH LINES.

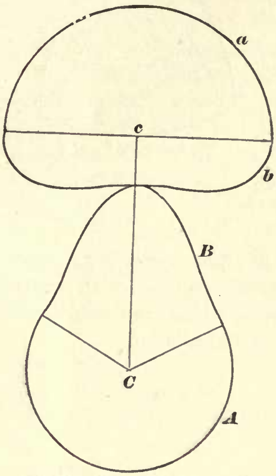
Instead of drawing a curve at random, and finding the mate to run with it, Fig. 108, the complete pitch line may be built up of a number of curves, of which the properties are known.

Thus, Fig. 117 shows composite gears, consisting of circular parts *A* and *a*, and an elliptic trilobe *B*, working with an elliptic bilobe *b*. Fig. 118 shows a combination of a pair of logarithmic spiral arcs *A* and *a*, a pair of elliptic bilobal arcs *B* and *b*, a pair of logarithmic spiral arcs *D* and *d*, and a pair of elliptic quadrilobal arcs *E* and *e*. An endless variety of combinations can be made in this way.

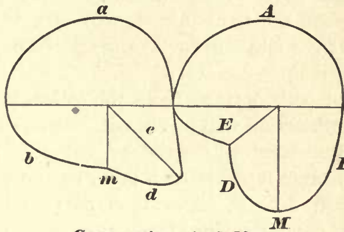
It is not necessary that the component

curves be tangent, if they succeed each other continuously. Fig. 119 shows a pair of equal logarithmic spirals with a break at *ab*, the action at *b* commencing just as it ends at *a*.

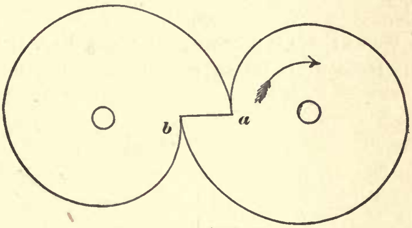
Care should be taken to avoid salient points, breaks, and interruptions of the continuity of the curve, for there must be defective tooth action at such points. The curves should run smoothly into each other with gradual changes of curvature.



Composite pitch lines
Fig. 117.



Composite pitch lines
Fig. 118.



Broken pitch lines
Fig. 119.

134.—TEETH OF NON-CIRCULAR PITCH LINES.

The action of the teeth of non-circular pitch lines does not, at first sight, appear to follow the laws pertaining to circular lines, but there is really very little difference.

If we consider the two pitch lines to be free, and to be so moved while they roll together that the pitch point O, Fig. 107, is fixed, and so that the fixed line $c\ C$ is always at right angles to both curves at their common point O, the laws of the tooth action will be almost precisely the same as laid down for the circular pitch line. Fig. 107 may be easily applied to (24) as illustrated by Fig. 15.

When the centers are fixed, the same tooth action takes place, but the line of action and the pitch point continually change their positions.

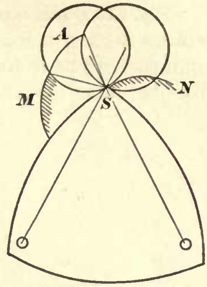
The teeth of non-circular pitch lines can therefore be formed either by conjugating a given odontoid, as in (24), or by the rolled curve theory of (32).

By all means the most practicable method, when the circumstances will permit, is to make up the curve by joining approximating circular arcs, and to provide each circular arc with teeth in the ordinary way. See this process as applied to the elliptic pitch line at Figs. 129 and 130.

135.—TEETH AT SALIENT POINTS AND BREAKS.

When there is a salient point, or other interruption of the continuity of the action, as at *q*, Fig. 109, or at *Mm*, Fig. 118, there must be an interruption in the arrangement of the normals of any tooth curve, and a consequent failure of the tooth action.

Fig. 120 shows a cycloidal tooth curve *M*, at a corner or salient point *S*, between two circular pitch arcs. There is a circular arc *A* on the odontoid made while the describing circle is turning about the point *S*, and that arc can have no continuous tooth action. Therefore the tooth action will fail, unless the next tooth curve *N* springs from the salient point.



The salient point
Fig. 120.

If a tooth springs from the salient point, the tooth action will be correct, but mechanically imperfect, as the arc of action of two teeth cannot lap over each other to allow for practical defects. And then, as two tooth curves cannot spring from the same pitch point in opposite directions, such gears can run in but one direction, and are not reversible.

When there is a break, as at *ab*, Fig. 119, the teeth must be so cut off that they will separate at *a* just as they engage at *b*, for there is a sudden change in the velocity ratio. Such combinations are practicable, but in every way undesirable.

136.—THE ELLIPTIC GEAR.

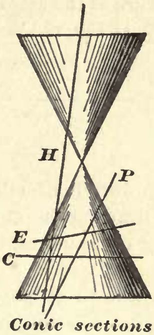
The principal, and almost the only use of the irregular gear, is to produce a variation of speed between certain given limits, without conditions as to the variations of speed and details of the motion between the limits. When that is the only object, the elliptic pitch line is the only one that is required, and it is chosen because it is the only known continuous closed curve that will work in rolling contact with an equal mate, and because it is, next to the circle, the simplest known curve. Of the elliptic multilobes, the unilobe, or simple ellipse, revolving on one of its foci as a center, is the only one used to any appreciable extent, and therefore is the only one that requires examination in detail.

The use of the elliptic gear is practically confined to producing a simple variation of speed between known limits, and to producing a "quick return motion" for planers, shapers, slotters, and similar cutting tools, as well as for pumps, shears, punches, shingle machines, and others where the work is done mostly during one-half of the stroke of a reciprocating piece. The work of a planer tool or of the plunger of a single acting pump, is all done during the motion of the tool or of the plunger in one direction, and the only object on the return is to get the piece ready for the next useful operation in the quickest possible time.

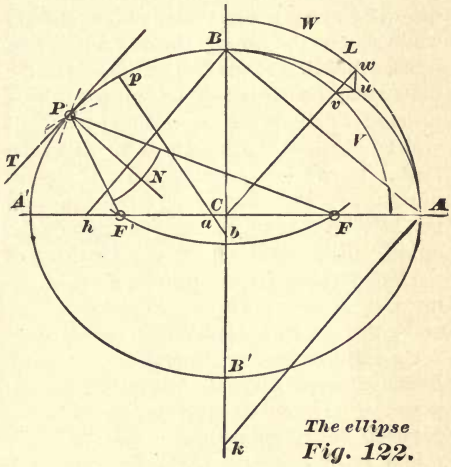
For an example, the bobbin of a spinning machine is to be wound in a conical form, the thread being fed to it through a moving guide, and the necessary variable motion of the guide, fast at the point of the cone, and slow at its base, is best given to it by a pair of elliptic gears. For another example, the motion of the platen of a printing press should be rapid when the press is open, and slow and powerful when the impression is being taken, and the object can be reached best by a pair of elliptic gears operating the platen.

The practical uses of the elliptic gear are endless, and it would be in greater use and favor, if it were not for the fact that its production, by the means ordinarily in use for that purpose, is as difficult and costly as the resulting gear is unsatisfactory.

137.—THE ELLIPSE.



Conic sections
Fig. 121.



The ellipse
Fig. 122.

To thoroughly understand the construction and operation of the ellipse, it is necessary to learn but a few of its many properties.

The mechanical definition of the ellipse is that it is one of the "conic sections." If the cone, Fig. 121, is cut by a plane C at right angles with its axis, the outline of the section will be a circle; if the plane E cuts the cone at an angle, the section will be an ellipse; if the plane P is parallel with the side of the cone, the section is a parabola, and if the plane H is at such an angle that it cuts both nappes of the cone, the section is a hyperbola. All these curves will roll together when fixed on centers at certain points called foci, but the ellipse, and its special case, the circle, are the only ones that are capable of continuous motion.

In the ellipse, Fig. 122, the point C is the center, the longest diameter, AA', is the major axis, the shortest diameter, BB', is the minor axis; A and A' are the major apices, and B and B' are the minor apices.

If an arc be drawn from the minor apex, with a radius equal to the major semi-axis, it will cut the major axis at points F and F, called the foci, and one focus must be chosen as the center, about which the curve is to revolve if used as the pitch line of a gear.

It is a property of the curve that the sum of the distances, PF and PF', from any point to the foci is equal to the major axis, AA', and this feature is used as a means of constructing the curve by points. Draw any arc at random from one focus with radius FP. Draw an arc from the other focus with a radius equal to AA'—FP, and it will cut the first arc at a point of the ellipse. When the point P is near either major apex, the arcs intersect at such a sharp angle that the method is nearly useless.

Another, and much the best known method for constructing the ellipse by points, is to draw any radial line L, and also circular arcs W and V, from the center through the apices. From the intersections, w and v, of the radial line and the circles, draw lines parallel to the axes, and they will intersect, always at right angles, at a point u on the curve. This method is very accurate, and has no failing position.

Another valuable property of the ellipse is that if the line pab be so drawn that the distance pa is equal to BC, and pb to AC, the point p will be upon the curve if the points a and b are upon the axes.

The curvature of the ellipse is an important feature in connection with its use as a gear pitch line. It is sharpest at the major axis A, and flattest at the minor apex B, elsewhere varying between the two limits.

The radius of curvature at either apex, that is, the radius of the circle that most nearly coincides with the curve, is found by drawing the lines Bh and Ak at right angles with the chord AB. The distance Ch is the radius of curvature at the major apex A, and the distance Ck is the radius at the minor apex B.

The normal PN to the curve at any point P bisects the angle FPF' between the focal lines, and the tangent PT is at right angles to the normal.

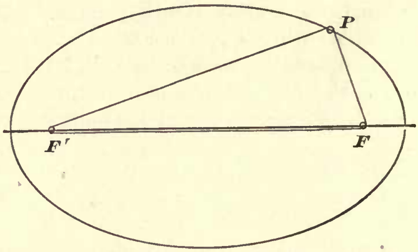
138.—ELLIPTOGRAPHS.

There are a multitude of elliptographs, or instruments for drawing the ellipse, but only two of them are of practical application in this connection.

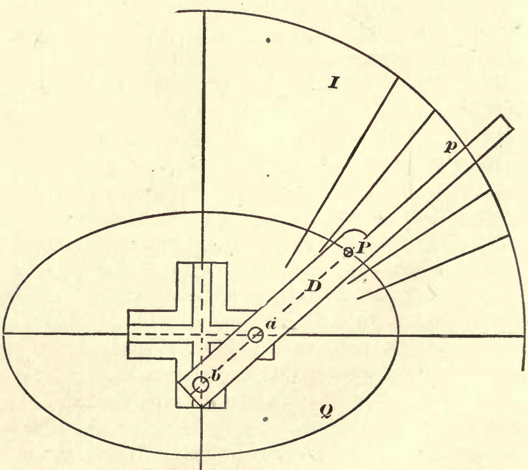
The simplest known elliptograph consists of a couple of pins, a thread, a pencil, and a stock of patience. The pins are inserted at the foci, as in Fig. 123, and the curve is drawn by moving the pencil with a uniform strain against the string. After a number of trials, depending in number on the skill of the draftsman, the curve may be induced to pass through the desired points. The best result will be obtained by the use of a well waxed thread running in a groove near the point of a hard pencil. The pencil should be long, and held by the end, so that the strain on the string will be uniform, for the elasticity of the string is the greatest source of error. This "gardener's ellipse" will generally be accurate enough for a tulip patch, but should not be relied upon for mechanical purposes, unless one or more points between the apices are tested and found to be correct. If the two pins and the pencil are circular, and of the same diameter, the accuracy of the ellipse is independent of their diameter.

The best elliptograph is the "trammel," Fig. 124, which takes a variety of shapes, but which in its simplest condition consists of a cross, with two grooves at right angles, and a bar D with two pins a and b, and a tracing point P placed in line. The distance Pb being set to the major semi-axis, and the distance Pa to the minor semi-axis, the point P will trace the ellipse if the pins are confined to move in the grooves. If carefully made, the instrument works with great precision, is easily handled and set, and, if the curve drawn is not very flat, it may be inked. The cheap wooden trammel should not be tolerated, for the string and two pins cost less and are more reliable.

Fig. 123.



Gardener's ellipse.

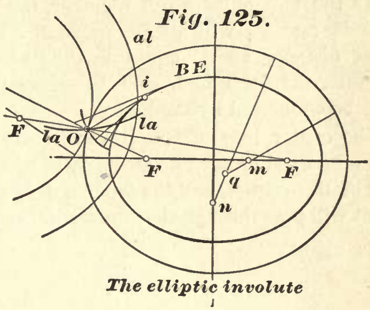


The trammel
Fig. 124.

139.—APPROXIMATE CIRCULAR ARCS.

If a well-made trammel is not at hand, the best plan is to draw the ellipse with a string, through several constructed points, and then to ink it by finding centers for approximate arcs, as shown by Fig. 125. An arc from a center m on the major axis, will coincide very well with the curve near the major apex, a similar arc n from a center on the minor axis will serve near the minor apex, and a third center q can be found for an arc to join the first two. More than three centers will seldom be required, and when the ellipse is not very flat the two centers on the axes will be sufficient.

Fig. 125.



The elliptic involute

140.—FOUR CENTER ELLIPSE.

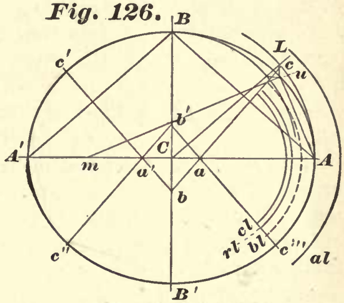
When the ratio of the axes is not less than eight to ten, as is generally the case, a practically perfect ellipse may be drawn from four centers by the following method.

Draw the line CL, Fig. 126, parallel to $A'B$, and construct the point u on the ellipse by the method of (137). Find a point a on the major axis, from which an arc from A will pass through u, and it will be the major center. It may be found by trial, or by drawing um at right angles to uA, and bisecting Am in a.

Through u draw ac at right angles to AB, and its intersection with the minor axis will be the minor center b. Lay off Ca' and Cb' equal to Ca and Cb, and draw bc', $b'c''$, and $b'c'''$.

From the centers a draw the arcs cAc''', and $c'A'c''$, and from the centers b draw the arcs cBc' and $c''B'c'''$.

Fig. 126.



Four center method

Lines that are parallel to the pitch line, such as the addendum, root, clearance, and base lines, are to be drawn from the same centers.

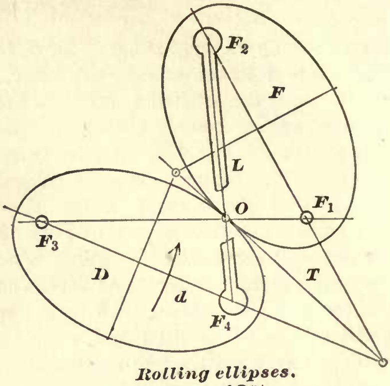
141.—ROLLING ELLIPSES.

When two equal ellipses, Fig. 127, are arranged to revolve on their foci as centers, with a center distance equal to the major axis, they will roll together perfectly, and be fitted to act as the pitch lines of gear wheels.

When the driver D turns in the direction of the arrow d, it will drive the follower F by direct contact of the pitch ellipses, but when turning in the other direction with respect to the follower, as it must during half of its revolution, it has no direct driving action, and the follower must be kept in contact by some other force.

As the two ellipses roll together, the free foci F_3 and F_4 will always move at a constant distance apart, equal to the distance between the fixed foci, and therefore they may be connected by the link L.

The center line of the link will always cross the fixed center line at the point of contact of the ellipses, and the tangent T at that point will pass through the intersection of the axes.



Rolling ellipses.
Fig. 127.

142.—SPACING THE ELLIPSE.

As the ellipses roll together it is essential that the axes come in line, and therefore, if the teeth of one gear are fixed at random, those of the other must be fixed to correspond. If this requirement is satisfied, it makes no difference where the teeth are placed.

It is, however, very desirable that the two gears shall be exactly alike, so that they can be cut at one operation while mounted together on an arbor through their focus holes, and to do this, it is necessary to start the teeth at different points, according to whether their number is odd or even.

If the number of teeth is even, one tooth must spring from the major axis, as shown by Fig. 128.

If the number of teeth is odd, the major axis must bisect a tooth and a space, as shown by Fig. 129. In this case, if one of the gears can be turned over, or if its other focus hole can be used as a center, it may have a tooth springing from the major axis.

The simplest method of spacing the ellipse is to step about it with the dividers. If the curve is flat, the dividers should be set to less than a whole tooth, for equal chords will not measure equal arcs of the curve.

But this stepping method, although it is sufficient and convenient for drafting purposes, is wholly unfit for mechanical purposes, and therefore we must have a method that is not dependent on personal skill.

If the ellipse is drawn by means of the trammel, Fig. 124, it can be accurately spaced by means of a graduated index circle I, having a diameter equal to the sum of the diameters of the ellipse, for then the center line of the bar will pass over an arc on the ellipse that at the apices is exactly equal to half the arc passed over at the same time on the circle,

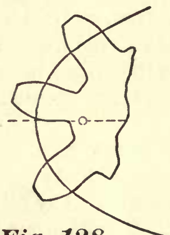
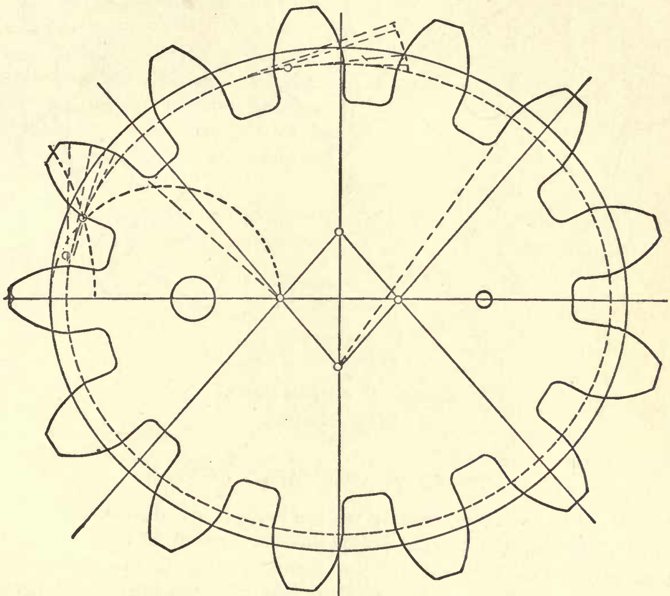


Fig. 128.

and that is elsewhere very nearly in the same proportion.

This method is not mathematically exact, but its accuracy is very far within the requirements of practice. The space on the quarter, at Q, will be greater than anywhere else, but the maximum error will in general be very minute.

For an example, take an extreme practical case, a gear with axes eight and ten inches long, and with seventy-two teeth The maximum error, the difference between the longest and shortest tooth arcs, will be not over one five-hundredth of an inch. In the more common practical case of a gear of nine and ten inches axes, and seventy-two teeth, the maximum error is about one two-thousandth of an inch. In both these cases, the difference between the tooth arc at the major apex and that at the minor apex is too small to be readily calculated, but will be about one twenty-thousandth of an inch. In all cases likely to be met with in practice, the inevitable mechanical errors are greater than the theoretical errors of the method, and it is serviceable on ellipses as flat as three to one.



Involute elliptic teeth.
Fig. 129.

143.—INVOLUTE ELLIPTIC TEETH.

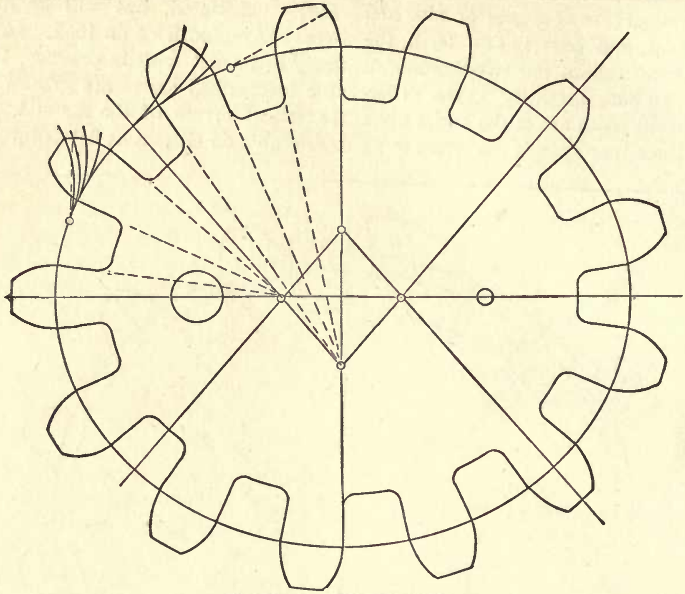
As in the case of the circular gear, the best form of tooth for the elliptic gear is the involute, and for the same reasons.

The base line of the involute tooth is any ellipse *B E*, Fig. 125, which is drawn from the same foci as the pitch ellipse ; the limit point *i* is the point of tangency of a tangent from the pitch point *O*, and the addendum line *a l* of the mating gear must not pass beyond that point. The method of laying out the tooth and drafting it is so exactly like the process for the circular gear that the description need not be repeated.

The centers of involute elliptic gears can be adjusted without affecting the perfection of the motion transmitted, but, as the focal distance remains fixed, the ratio of the axes will be altered.

The work of drawing the teeth can be much abbreviated by the process illustrated by Fig. 129. Find the centers for approximate circular arcs, preferably by the method of (140), and then consider the gear as made up of four circular toothed segments. It is then necessary to construct but two tooth curves, one for the major and one for the minor segment, and the flanks will be radii of the circular arcs.

The line of action, *la*, Fig. 125, is not a straight line, and it is not the same for all the teeth. It is not fixed when the pitch point *O* and the line of centers is fixed (134).



Cycloidal elliptic teeth.
Fig. 130.

144.—CYCLOIDAL ELLIPTIC TEETH.

The cycloidal tooth is drawn, exactly as upon a circular pitch line, by a tracing point in a circle that is rolled on both sides of the pitch line. The line of action is not a circle, and it is not the same curve for all the teeth.

That the flanks shall not be under-curved, the diameter of the rolling circle should not be greater than the radius of curvature at the tooth being drawn, and when, as usual, the same roller is used for all the teeth, its diameter should not be greater than the radius of curvature at the major apex, the distance *Ch* of Fig. 122.

Fig. 130 shows a cycloidal gear drawn as four circular segments, by the methods of (140) and (83).

145.—IRREGULAR TEETH.

It is most convenient to draw all the teeth alike, with the same rolling circle, or from the same base line, and also to uniformly space the pitch line, but such uniformity is not essential.

The only requirement is that each tooth curve shall be conjugate to the tooth curve that it works with, and if that condition is satisfied the teeth may be of all sorts and sizes.

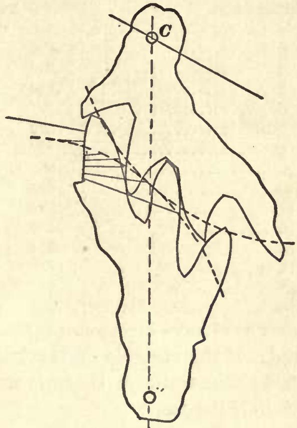
146.—FAILURE IN THE TOOTH ACTION.

When the major axes are in line the action of the teeth on each other is nearly direct, but when the minor axes are in line the action is more oblique, as shown by Fig. 127. The teeth tend to jam together when the driver is pushing the follower, and to pull apart when the follower is being pulled, and when the ellipse is very flat this tendency is so great that the teeth fail to act serviceably.

At first glance it might appear that this difficulty in the tooth action of very eccentric gears might be overcome by making the teeth radial to the focus, as shown by Fig. 131, but examination will show that but little can be gained in that way.

The teeth on the gear C were obtained by the method of (28) from the assumed tooth on the gear c, and the effect of the defective shape of one side of the assumed tooth was to cut away the conjugate curve of the derived tooth.

Such teeth would not work as well as the ordinary form, and their construction would be very difficult.



Radial teeth
Fig. 131.

147.—THE LINK.

When the teeth of the elliptic gear fail to properly engage, on account of the obliquity of the action, the difficulty can be entirely overcome by connecting the free foci by a link (141), as shown by Fig. 127.

This link works to the best advantage when the teeth are working at the worst, and when it fails to act, as it passes the centers, the teeth are working at their best. Therefore gears that are connected by a link need teeth only at the major apices.

When the tooth action is imperfect by reason of its obliquity, and the link is not available or desirable, the difficulty can be overcome by using three or more gears in a train, as shown by Fig. 137, for then the same result can be obtained by the use of gears that are much more nearly circular.

148.—VARIABLE SPEED AND POWER.

If the shaft c, Fig. 132, turns uniformly, the slowest speed of the shaft C will occur when the gears are in the position of the figure, and the proportion between the two speeds will be the proportion between the distances cO and CO. The greatest speed of the driven shaft will occur when the shafts have turned through a half revolution from the position of the figure, and the relative speed will be the same, reversed.

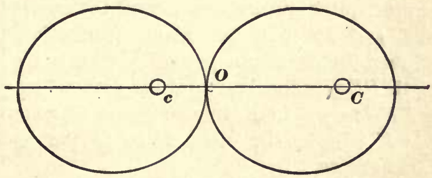


Fig. 132.

The ratio of speed, the ratio of the greatest speed to the slowest speed, is the square of the ratio between the speed of the driving shaft and the greatest or the least speed of the driven shaft, so that it requires but a slight variation of the axes to produce a decided variation of the speed.

The following table will give the proportion of minor to major axes that will give any desired ratio of speeds.

Ratio of Speeds.	Ratio of Axes.
2	.985
3	.962
4	.952
5	.924
6	.907
7	.892
8	.878
9	.868
10	.854
11	.844
12	.834
13	.824
14	.817
15	.807
16	.800

The power is always inversely proportional to the speed. If the variable shaft is running twice as fast as the uniform shaft, it will exert but one-half the force.

When the gears are arranged in a train, as in Fig. 137, the speed ratio for the second, third, and following gears will be in the proportion of the first, second, third and following powers of the first ratio.

Thus, the ratio for a pair of gears with axes in the proportion of .952 to 1 being 4 for the second gear, will be 16 for the third gear, 64 for the fourth gear, and so on.

The use of gears of troublesome eccentricity can be avoided by this means. A train of three gears of .952 axes, Fig. 137, is equivalent to a single pair of very flat gears with .800 axes, Fig. 138, and, in general, three gears that are nearly circular are equivalent to a single very flat pair.

149.—ELLIPTIC QUICK RETURN MOTION.

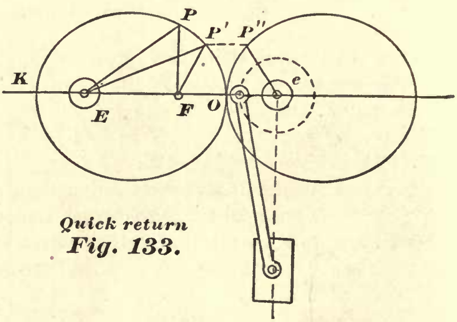
If the gears are arranged with respect to the piece to be reciprocated, in the manner shown by Fig. 133, the time of the cutting stroke will be to the time of the return stroke, as the angle *PEK* is to the angle *PEF*, where *E* and *F* are the foci of the ellipse.

The following table will show the ratio of axes that must be adopted to produce a required ratio of stroke to return.

Quick Return.	Ratio of Axes.
2 to 1	.964
3 to 1	.910
4 to 1	.861
5 to 1	.817
6 to 1	.778

To determine the ellipse that will give a required quick return, we lay off the angles *PEK* and *PEF* in the given proportion, and then find by trial a point *P* such that the length *PE* plus the length of the perpendicular *PF* is equal to the known center distance *Ee*. *F* will be the other focus of the required ellipse.

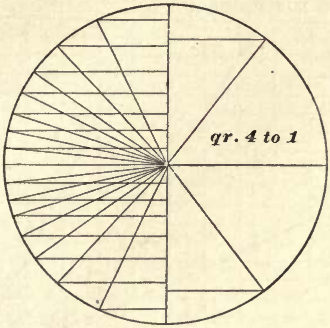
When the driving gear has turned through the angle *P'EF*, from the position of the figure at the middle of the return, the variable gear will have turned through the angle $P''eO = P'FO$, and we can study the action of the tool by drawing equi-distant radii about *E*, and finding the corresponding radii about *F*.



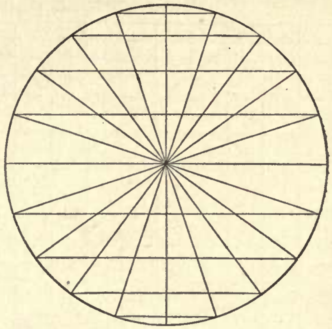
Quick return
Fig. 133.

Fig. 134 shows the arrangement of the radii ($P'F = P''e$ of Fig. 133) in the case of a four to one quick return, and it is seen, by the parallel lines, that the motion of the tool is very uniform, coming quickly to its maximum speed, and holding a quite uniform speed until near the end of the stroke. Fig. 135 shows that the same motion derived from a simple crank is not as uniform.

When the gears are arranged in a train, Fig. 137, the quick return ratios can be determined by the construction shown by Fig. 136. Draw *Fc* at right angles to *AA'*, and draw *cEd* through the other focus. The quick return ratio of the second gear will be the ratio of the angles a_2 and b_2. Draw *dFe*, and the ratio for the third gear will be



Quick return crank
Fig. 134.

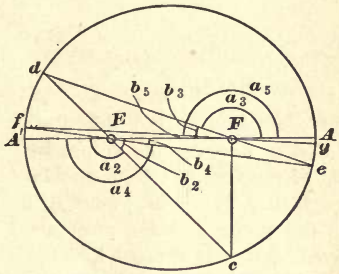


Ordinary crank
Fig. 135.

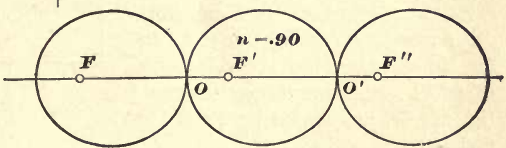
that of the angles a_3 and b_3. Draw eEf, and a_4 and b_4 will give the ratio for the fourth gear. And so on, in the same manner, as far as desired, the ratio being greatly increased by each gear that is added to the train.

If carefully performed, the graphical process is quite accurate. The case of axes in the proportion of .98 to 1 gave a quick return of 1.6 for the second gear, and 2.8 for the third gear, while their true computed values are 1.66 and 2.74.

The chart will solve quick return train questions involving gears not flatter than .80, as accurately as need be. For example, the ratio of axes of .95 will give a quick return of 2.25 for the second gear, 4.85 for the third gear, 9.80 for the fourth gear, and 19.70 for the fifth gear. Again, the proportion of axes to give a quick return of 5 for the third gear is .948.



Quick return train
Fig. 136.



Elliptic train ***Fig. 137.***

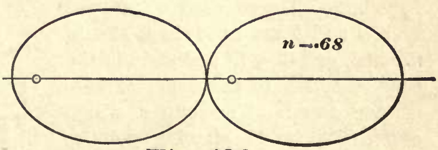
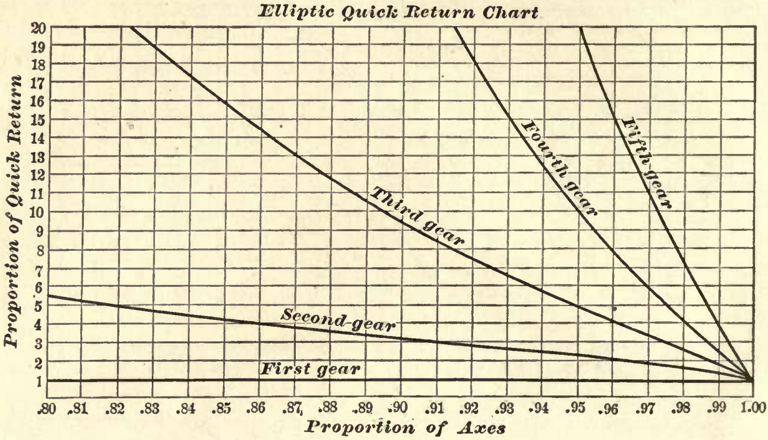


Fig. 138.

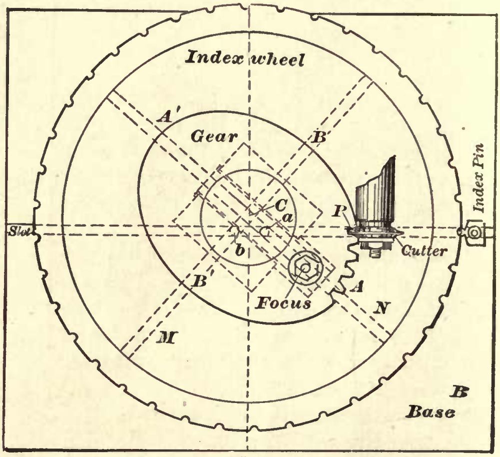


150.—THE ELLIPTIC GEAR CUTTING MACHINE.

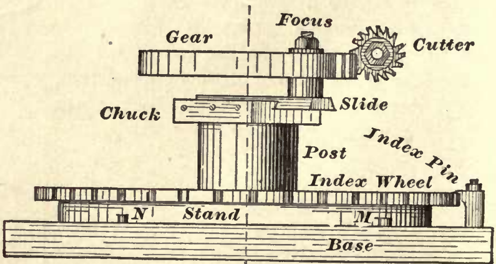
The conditions of the described operation of drawing the ellipse by means of the trammel (138) may be reversed, the bar being held still while the paper and the cross are revolved, and it is evident that the result will be the same ellipse on the paper as if the bar is revolved as described.

By thus reversing the process of describing the ellipse, and by adopting the improved spacing device of (142), we can construct a machine for accurately cutting the teeth in an elliptic gear, the main features of which, omitting various unessential details, are shown by Figs. 139 and 140.

The blank to be cut is fastened upon a trammel stand, which corresponds to the paper in the graphical process, and revolves upon the fixed base. The adjustable trammel pins *a* and *b* are fixed in a slot in the bed, and they fit and slide in the slots *M* and *N* in the under surface of the stand. The cutter which corresponds to the tracing point is fixed with the pitch center of its



Plan
Fig. 139.



Elevation
Fig. 140.

tooth curve directly over the point P in the line of the pins. The index plate has a diameter equal to the sum of the axes of the ellipse, and it is held by an index pin p, which slides in the slot, and is always in the line of the pins.

Thus arranged, the machine will always cut its tooth in the true ellipse, and the teeth will be accurately spaced.

The direction of the tooth will be substantially at right angles to the pitch line, and a simple arrangement can be applied to make it exactly so. An index plate of a fixed diameter may be used for all sizes of gears, if the index pin is carried by an arm which swings about the center of the gear, and has an adjustable pin that slides in the slot.

The tops of the teeth are trued by a cutter having a square edge, and the line of the tops will be substantially parallel to the pitch line.

The blank is held by an arbor through its focus hole, and the arbor is held by a slide, which slides in a chuck upon the stand, so that the focus can be accurately set in the major axis at the proper distance from the center.

151.—CHOICE OF CUTTERS.

Theoretically, the teeth are of different shapes, as they are in different positions upon the ellipse, and, therefore, each space should be cut with a cutter that is shaped for that particular space. But as this is impracticable, it is necessary to choose the cutter that will serve the best on the average.

Strictly, the cutter should be the one that is fitted to cut a spur gear having a pitch radius equal to the radius of curvature of the ellipse at the major apex, but as that cutter will be much too rounding for the minor apex, it is better to choose the one that is fitted for the medium radius of curvature.

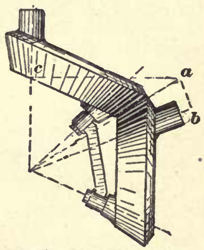
The two radii of curvature are the distances Ch and Ck, Fig. 122, and the cutter should be chosen for the radius half way between the two, approximately half the sum of the two.

152.—THE ELLIPTIC BEVEL GEAR.

An ellipse may be drawn on the surface of a sphere by means of a string and two pins, according to the method of (138), and a pair of such spherical ellipses will roll on each other while fixed on their foci, their free foci moving at a constant distance apart.

Therefore we can have elliptic bevel gears that are very similar to elliptic spur gears, as shown by Fig. 141. The two gears revolve on radial shafts through their foci, and the link connects radial shafts through the free foci. The velocity ratio is the ratio of the perpendiculars $a\ b$ and $a\ c$. The elliptic bevel gear is the invention of Professor MacCord.

The spherical ellipse cannot be drawn by the trammel method of (138), and therefore the method of spacing of (142), as well as



Elliptic bevel gears
Fig. 141.

the gear cutting machine of (150), does not apply.

153.—MATHEMATICAL TREATMENT.

If the major semi-axis is a, and the minor semi-axis is b, the equation of the curve from the origin at C is

$$a^2 y^2 + b^2 x^2 = a^2 b^2,$$

the major axis being the axis of X.

The distance CF from the center to the focus will be

$$c = \sqrt{a^2 - b^2} = a \sqrt{1 - n^2},$$

in which n is the ratio of axes $= \frac{b}{a}$.

The radius of curvature at the major apex is $\frac{b^2}{a}$, and that at the minor apex is $\frac{a^2}{b}$.

There is no practicable formula for the rectification of the curve, as the length is expressible only by a series.

The special spacing method of (142) is true only at the instant of passing either apex, for the tracing point describes half the arc described by the line of the bar on the index circle only when the bar is at right angles with the curve. The error will be at its maximum when the bar is at the maximum angle with the normal, which is at about an angle of forty-five degrees with the major axis. The difference between an ordinary tooth space at the major apex, and that at the minor apex, is very minute. A very careful calculation of the length of the chord of a gear of seventy-two teeth, and eight and ten inch axes, gave a chord of .41433″ at the major apex, a chord of .41495″ at 45° for the maximum, and a chord of .41441″ at the minor apex. The difference between the chords at the apices is .00008″, but as the curvature at the major apex is greater than at the minor apex, the difference between the arcs would be less, perhaps not over .00004″.

The ratio of speeds (148), is

$$\left(\frac{1 + \sqrt{1 - n^2}}{1 - \sqrt{1 - n^2}}\right)^2$$

The ratio of quick return being given as qr, the value of n is

$$n = \sqrt{2 \sqrt{d^2 + d^4} - 2d^2},$$

in which $d = tan. \left(\frac{180}{qr + 1}\right)^\circ$.

When the gears are in a train, there seems to be no simple method for computing the ratio of axes to produce a given quick return, but, when the ratio is given, the quick return for each gear can be computed best by trial and error with the formula

$$\frac{sin. (M - N)}{sin. M + sin. N} = \sqrt{1 - n^2}$$

in which M is any known angle b, Fig. 136, and N is the angle b for the next following gear in the train. Thus, assuming $n = .98$, and $M_1 = 90°$, we find $N_2 = 67° \; 28'$. Then putting $M_2 = 67° \; 28'$, we find $N_3 = 48° \; 5'$.

Knowing the angles, we compute the quick return ratio from

$$qr = \frac{180}{N} - 1,$$

which, for $n = .98$ gives qr for two gears equal to 1.66, and for three gears equal to 2.74. The graphical process of Fig. 136 should first be employed to fix the angles approximately.

8. THE BEVEL GEAR.

154.—THE BEVEL GEAR.

The theory of the bevel gear cannot be properly represented, and can be studied only with the greatest difficulty, upon a plane surface. It is essentially spherical in nature, and should be shown upon a spherical surface, as in Figs. 143 and 144.

This is best done upon a spherometer, which is simply a painted sphere fitted in a ring. The sphere rests upon a support, so that the ring coincides with a great circle upon it, and the ring is graduated to 360°. A very roughly made wooden sphere and plain ring will be found to answer the general purpose very well, and should be provided if the study of the bevel gear is seriously intended. If painted, ink marks can be scrubbed off, and pencil marks removed with a rubber.

The mathematical treatment is unapproachable without a knowledge of the common principles of spherical trigonometry.

A wide, interesting, and difficult field of study is offered, but space will permit but a brief examination of the more prominent and practical points. A careful examination would require ten times the available space.

155.—THE GENERAL THEORY.

When thus represented upon the spherical surface, the theory of the bevel gear is so similar to that of the spur gear, as represented upon a plane surface, that any detailed description would be mostly a repetition of what has already been stated.

All straight lines of the spur theory are represented by great circles, the crown gear being the rack among bevel gears, and all distances are measured in degrees.

Irregular pitch lines and multilobes are managed substantially as for spur gearing. The elliptic bevel gear has been described in connection with elliptic spur gears (152).

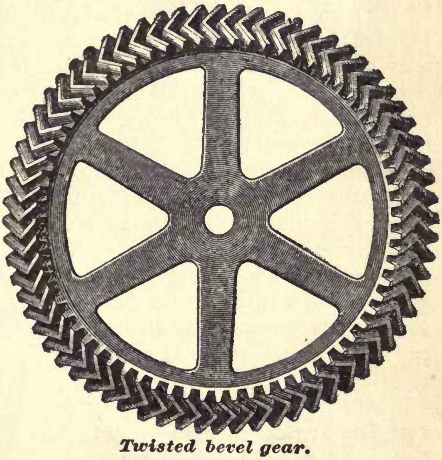
The tooth surfaces of the bevel gear are generally formed by drawing straight lines from the spherical outline to the center of the sphere, as in Figs. 143 and 144, the pitch lines and tooth outlines being the bases of cones with a common apex.

When limited in width, as is usually the case, it is by a sphere concentric with the outside sphere, so that a spherical shell is formed.

These concentric spherical shells can be moved on their axes to form twisted and spiral teeth, Fig. 142, precisely as described for spur gears (99).

The molding process of (27) will apply perfectly, but it has but one practical application.

Fig. 142.



Twisted bevel gear.

The planing process of (28) will fail, for practical purposes, except for one particular form of tooth, because the shape of the cutting tool cannot in the general case be changed in form as it approaches the apex, and therefore the tooth will not be conical.

The planing process of (29) will apply perfectly, the strokes of the tool being radial, and on this method we must depend for the accurate construction of all forms of bevel gear teeth except the octoid and the pin tooth.

As the diameter of the sphere is increased, the radii become more nearly parallel, until, when the diameter is infinite, they are parallel. Therefore the spur gear is a particular case of the bevel gear, and all formulæ and processes that apply to the bevel gear will apply to the spur gear if the diameter of the sphere is made infinite. The most scientific method of study would be to develop the theory of the bevel gear, and from that proceed to that of the spur gear, but such a method would be difficult to clearly carry out, and is best abandoned for the more confined process here adopted.

156.—PARTICULAR FORMS OF BEVEL TEETH.

As in the case of spur gearing, there can be an infinite number of tooth curves for bevel gearing (31), each form having its own line of action, but as there are only four forms that are available for practical use by means of simple processes of construction, our attention will be confined to them.

These four particular forms are, first, the involute tooth, having a great circle line of action; second, the cycloidal tooth, having a circular line of action; third, the octoid tooth, having a plane crown tooth, and a "figure eight" line of action; and, fourth, the pin tooth, for which one gear of a pair has teeth in the form of round pins.

157.—THE INVOLUTE BEVEL TOOTH.

The spherical involute must be studied as a whole if its form is to be clearly seen.

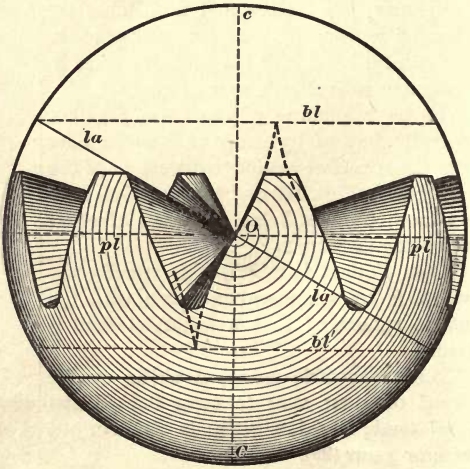
Its definition is that it is the tooth curve having a great circle for a line of action. In Fig. 143 the great circle line of action *la* extends around the sphere at an angle with the crown pitch line *pl*, and it is tangent to two base lines *bl* and *bl'*, that are parallel with the crown line.

The most convenient method of drawing the tooth curve is by rolling the line of action on the base line, while a point in it describes the curve on the surface of the sphere. The equivalent graphical process is to step along the base line and any two tangent great circles, from any point on the curve to any desired point.

It will take the form shown by the dotted lines; rising at right angles to the base line, it curves until the crown line is reached, there reversing its curvature and bending the other way until it meets the other base line. At the base line it has a cusp, and rises from it to repeat the same course indefinitely.

Fig. 143 shows a crown gear or rack. The pitch line is the great circle *pl*. The line of centers *cOC* is a great circle at right angles with the crown line *pl*. The line of action is the great circle *la* set at a given angle of obliquity with the crown line. The base

Fig. 143.
The involute Tooth



circles are the small circles bl and bl'. The spherical involutes have the same property of adjustability as have the spur involutes, the motion being confined to the sphere, and therefore the gears are adjustable as to their shaft angle, the apex remaining common to both.

158.—THE CYCLOIDAL BEVEL TOOTH.

The definition of the cycloidal tooth is that it is that form which has a circular line of action.

The rolled curve method of treatment (32) applies, and is the best means of studying the curve.

There is no gear with radial flanks, the flank formed by a roller of half the angular diameter of the gear being nearly but not exactly a plane.

The theory differs so little from that of the spur gear, that but little of interest can be found, and the curve will not be considered further.

159.—THE OCTOID BEVEL TOOTH.

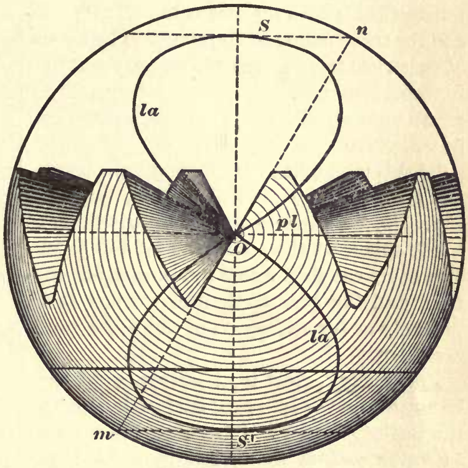
The definition of this tooth system is that it is the conjugate system derived from the crown gear having great circle odontoids.

In Fig. 144 the crown gear has plane teeth cutting the sphere in great circles, mOn, while a pinion would have convex tooth curves conjugate to the great circles of the crown tooth.

The line of action, from which the tooth derives its name, is the peculiar "figure eight" curve la, which is at right angles to the tooth curve at the crown line pl, and tangent to the polar circles S and S', to which the great circle crown odontoids are also tangent.

This tooth owes its existence to the fact that it is the only known tooth, and probably the only possible tooth, that can be practically formed by the molding planing process of (28).* The cutting edge of the tool being straight, no change is required while it is in motion, except in its position, and that is accomplished by giving it a motion in such a direction that its corner moves in the radial line of the corner of the bottom of the tooth space.

The octoid tooth, together with an ingenious machine for planing it, was invented by Hugo Bilgram, but it has always been mistaken for the very similar true involute tooth.



The Octoid Tooth
Fig. 144.

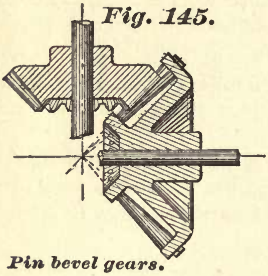
Bilgram's machine is described in the AMERICAN MACHINIST for May 9th, 1885, and in the *Journal of the Franklin Institute* for August, 1886.

* Since this statement was made, another bevel tooth, the "planoid" tooth, has been invented and practically constructed by the process of (28).

160.—THE PIN BEVEL TOOTH.

If the tooth of one gear of a pair is a conical pin, Fig. 145, with apex at the center of the sphere, that of the other will be conjugate to it, and the combination deserves notice because it is one of the few forms that are easily constructed. It may be said that its practical construction is simpler and easier than that of any other form of bevel gear tooth except the skew pin tooth of (180).

Fig. 145.



Pin bevel gears.

The tooth is preferably, but not necessarily, of the conical form, for other forms of circular pins would serve the theoretical purpose.

Its theory is, in the main, the same as that of the spur pin tooth. It has the same troublesome cusp, which can be avoided in the same way, by setting the center of the pin back from the pitch line.

It is the only known form of tooth that can be formed in a practical manner by the molding process of (27). If the cutting tool is a conical mill, it will form the conjugate tooth while the two pitch wheels are rolled together.

The pins may be mounted on bearings at their ends, forming roller teeth. They would be weak, but would run with the least possible friction, all the rubbing friction being confined to the bearings.

161.—TREDGOLD'S APPROXIMATION.

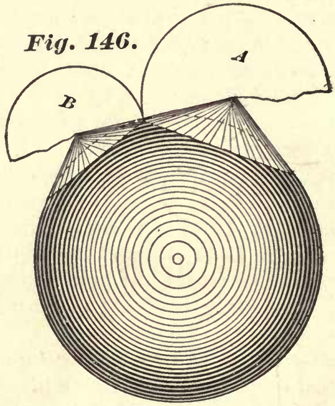
The construction of the true bevel gear tooth curve upon the true spherical surface is impracticable with the means in ordinary use, and the true method of computation by means of spherical trigonometry is equally unfitted for common use. But, by adopting Tredgold's approximate method the difficulties can be overcome.

By this method the tooth curves are drawn, not on the true spherical surface, but, as in Fig. 146, on cones A and B drawn tangent to the sphere at the pitch lines of the gears. The cones are then rolled out on a plane surface, and the gear teeth drawn upon them precisely as for spur gears of the same pitch diameter.

Practically correct tooth curves could thus be drawn on the spherical surface by cutting the teeth to shape, and bending them down to scribe around them, but in practice the back rims of the gears are shaped to the tangent cones so that the teeth lie directly upon the conical surface.

This method is called approximate, but its real error would be difficult to determine, and is certainly not as great as the inevitable errors of workmanship of any graphical process. The tooth outline drawn by it upon the spherical surface may be considerably different from that which would be drawn directly upon it, but it does not follow that it is therefore incorrect. The only requirement is that the engaging curves shall be

Fig. 146.



Tredgold's method.

conjugate odontoids, and it is a matter of very small consequence whether or not the curve on the sphere is the same kind of curve as that upon the cone. If the true plane involute curve is drawn upon the developed cone, the corresponding curve on the sphere will not be an exact spherical involute, but its divergence from some true odontoidal shape must be minute, even when the teeth are very large indeed. In ordinary cases it cannot be sufficient to affect materially the constancy of the velocity ratio. What is sometimes given as its error is mostly the "difference in shape" between the plane and the spherical teeth.

162.—DRAFTING THE BEVEL GEAR.

The practical application of Tredgold's method is illustrated by Fig. 147.

Draw the axes *CA* and *CB* at the given shaft angle *ACB*. Lay off the given pitch radii *a* and *b*, and draw the lines *c* and *d* intersecting at the pitch point *O*. Dra the center line *OC*, and lay off the face *Of*.

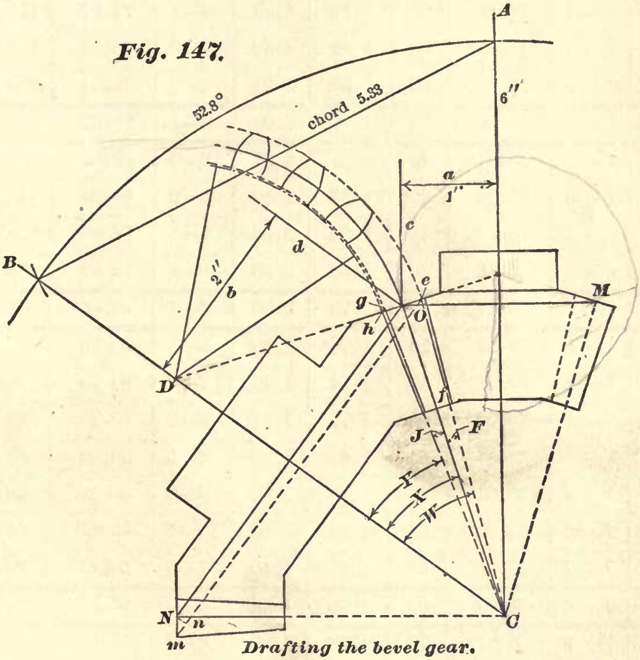
The pitch diameters are *ON* and *OM*, and *NCO* and *MCO* are the pitch cones.

Draw the back rim line *OD* at right angles with the center line, lay off the addenda *Oe* and *Og*, and the clearance *gh*. Draw the front rim line parallel to the back rim line.

The center angle is *X*, the face increment is *F*, and *W* is the face angle. The cutting decrement is *J*, and *Y* is the cutting angle. Twice the distance *mn* is the diameter increment, and *em* is the outside diameter.

The pitch radius of the Tredgold back cone is *OD*, and the figure shows the construction of the gear teeth on this cone developed. The teeth are represented as drawn upon the figure, but it is better to use a separate sheet. The odontograph should be used, calculating the number of teeth in the full circle of the developed cone.

Fig. 147.



Drafting the bevel gear.

163.—THE BEVEL GEAR CHART.

The drafting of the bevel gear blanks by means of the method of (162) is simple, but the method requires drafting instruments, not always at hand, as well as the ability to use them accurately. The drawing must be carefully made, to give correct results, particularly when the gears are small. After the drawing is made the various angles and diameters must be taken off for use at the lathe, and that is by no means a simple matter.

So great are the practical difficulties that any one who has a knowledge of simple arithmetic will find it not only easier, but more accurate to use the chart and method by means of the following rules.

THE BEVEL GEAR CHART.

Shafts at 90° Proportion.		Center Angle.	Angle Incr. / Div. by Teeth	Diam. Incr. / Div. by P.	Shafts at 90° Proportion.		Center Angle.	Angle Incr. / Div. by Teeth	Diam. Incr. / Div. by P.
.10	1—10	5.72	11	2.00	10.00	10—1	84.28	114	.20
.11	1— 9	6.33	13	2.00	9.00	9—1	83.67	114	.22
.13	1— 8	7.12	14	1.99	8.00	8—1	82.88	113	.25
.14	1— 7	8.13	16	1.98	7.00	7—1	81.87	113	.28
.17	1— 6	9.47	19	1.97	6.00	6—1	80.53	113	.33
.20	1— 5	11.32	23	1.96	5.00	5—1	78.68	112	.39
.22	2— 9	12.53	25	1.95	4.50	9—2	77.47	111	.43
.25	1— 4	14.03	28	1.94	4.00	4—1	75 97	111	.49
.29	2— 7	15.95	32	1 92	3.50	7—2	74.05	110	.55
.30	3—10	16.70	33	1.92	3.33	10—3	73.30	109	.58
.33	1— 3	18.44	36	1.90	3.00	3—1	71.57	109	.63
.38	3— 8	20.55	40	1.87	2 67	8—3	69.45	107	.70
.40	2— 5	21.80	43	1 86	2.50	5—2	68.20	106	.74
.43	3— 7	23.20	45	1.84	2.33	7—3	66.80	105	.79
.44	4— 9	23.97	46	1.83	2.25	9—4	66.03	104	.81
.50	1— 2	26.57	51	1.79	2.00	2—1	63.43	103	.89
.56	5— 9	29.05	56	1.74	1.80	9—5	60.95	101	.97
.57	4— 7	29.75	57	1.74	1.75	7—4	60.25	99	.99
.60	3— 5	30.97	59	1 72	1.67	5—3	59.03	98	1.03
.63	5— 8	32.00	61	1.69	1.60	8—5	58.00	97	1.06
.67	2— 3	33.68	64	1.66	1.50	3—2	56.32	95	1.11
.70	7—10	34.99	66	1.64	1.43	10—7	55.00	94	1.15
.71	5— 7	35.53	67	1.63	1.40	7—5	54.47	93	1.16
.75	3— 4	36.87	69	1.60	1.33	4—3	53.13	92	1.20
.78	7— 9	37.87	70	1.58	1.29	9—7	52.13	91	1 22
.80	4— 5	38.67	72	1.56	1.25	5—4	51.33	90	1.25
.83	5— 6	39.80	73	1.54	1.20	6—5	50.20	88	1.28
.86	6— 7	40.60	75	1.52	1.17	7—6	49.40	87	1.31
.88	7— 8	41.18	76	1.50	1.14	8—7	48.82	86	1.32
.89	8— 9	41.63	76	1.49	1.13	9—8	48.37	86	1.33
.90	9—10	41.98	77	1.49	1.11	10—9	48.02	85	1.34
1.00	1— 1	45.00	81	1.41	1.00	1—1	45.00	81	1.41

Fig. 148.

Sample Computation.

SHAFTS AT
A RIGHT ANGLE.

Pitch = 3	Prop. = 7 — 5	Shaft ang. 90
Teeth = 42) 93	(2.22 = face incr.	
84	.37 + $\frac{1}{6}$	
90	2.59 = cut decr.	
84		
60		
Center angles =	54.47	35.53
+ incr.........	2.22	2.22
Face angles....	56.69	37.75
Center angles =	54.47	35.53
— decr	2.59	2.59
Cut angles.....	51.88	32.94
Pitch =	3) 1.16	3) 1.63
Diam. incr. =	.39	.54
+ p. diams. =	14.	10.
o. diams. =	14.39	10.54

Fig. 149.

Sample Computation.

SHAFTS AT
ANY ANGLE.

Pitch = 5	Prop. = ×	Shaft ang. 52.8
Teeth = 20) 66	(3.30 = face incr.	
	.55 + $\frac{1}{6}$	
	3.85 = cut decr.	
Center angles =	35.80	17.00
+ incr.........	3.30	3.30
Face angles =	39.10	20.30
Center angles =	35.80	17.00
— decr.........	3.85	3.85
Cut angles =	31.95	13.15
Pitch =	5) 1.66	5) 1.91
Diam. incr.....	.33	.38
+ p. diams....	4.	2.
o. diams....	4.33	2.38

164.—SHAFTS AT RIGHT ANGLES.

1st.—Divide the pitch diameter by that of the other gear of the pair, or else the number of teeth by that of the other gear, to get the proportion. Enter the table by means of the proportion. All numbers for that pair will be found on the same horizontal line in the two columns.

2d.—The center angles are given directly by the table at the proper proportion.

3d.—Divide the tabular angle increment by the number of teeth in the gear, to get the angle increment. This need be done for but one gear of a pair, as the increment is the same for both.

4th.—Add the angle increment to the center angle, to get the face angle.

5th.—Increase the angle increment by one-sixth of itself, to get the cutting decrement, and subtract this decrement from the center angle, to get the cutting angle.

6th.—Divide the tabular diameter increment by the diametral pitch, to get the diameter increment, and add that to the pitch diameter, to get the outside diameter.

Fig. 148 is a sample computation for shafts at right angles.

165.—SHAFTS NOT AT RIGHT ANGLES.

The table cannot be entered by means of the proportion, and the numbers for the two gears of the pair will not be found on the same horizontal line, and it will be necessary to determine the center angles.

As in Fig. 147, draw the axes, at the given shaft angle, and find the center angles, by the method described in (162).

Then enter the table, for each gear by itself, by means of the center angles, and proceed as for shafts at right angles. The angle increment and decrement is the same for both gears of a pair.

Fig. 149 is a sample computation applied to the case of Fig. 147, the center angles being found by means of the table of chords. If preferred, the center angles can be found by means of the formula,

$$tan.\ C = \frac{sin.\ S}{\frac{1}{P} + cos.\ S}$$

in which C is the center angle of the gear, P is the proportion found by dividing the number of the teeth in the gear by the number in the other gear, and S is the shaft angle. Having found one center angle, subtract it from the shaft angle to get the other center angle.

166.—THE TABLE OF CHORDS AT SIX INCHES.

When the lathesman is provided with a graduated compound rest which feeds the tool at any angle, nothing but the computation is required; but when there is nothing but the common square feed, the faces must be scraped with a broad tool. A templet for guiding the work can easily be made by means of the table of chords at six inches.

To lay out a given angle, draw an arc with a radius of six inches, draw a chord of the length given by the table for the angle, and then draw the sides *oc* and *ob* of the angle *boc*, Fig. 150.

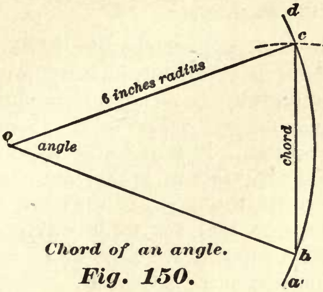
For tenths of a degree use the small tables. The chord of 37.5° is 3.81 + .05 = 3.86 inches.

Fig. 151 shows the manner of using the angle templet at the lathe.

This table of chords is very convenient for many purposes not connected with gearing, and it is more accurate than the common horn or paper protractor.

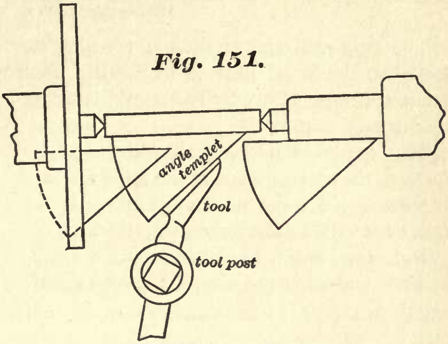
167.—BILGRAM'S CHART.

A graphical method for determining the angle and diameter increments, the invention of Hugo Bilgram, is described in the AMERICAN MACHINIST for November 10, 1883. It determines the required values by the intersections of lines and circles, and requires no computation.



Chord of an angle.
Fig. 150.

Fig. 151.



Using the templet

TABLE OF CHORDS OF ANGLES,

AT RADIUS OF SIX INCHES.

Degrees.	Chord.	Tenths.	Degrees.	Chord.	Tenths.	Degrees.	Chord.	Tenths.
1	.10		31	3.20		61	6.10	
2	.20		32	3.31		62	6.19	
3	.31		33	3.41		63	6.28	
4	.42		34	3.51		64	6.36	
5	.52		35	3.61		65	6.45	
6	.62		36	3.71		66	6.54	
7	.73		37	3.81		67	6.62	
8	.84		38	3.91		68	6.71	
9	.94		39	4.01		69	6.80	
10	1.04		40	4.10		70	6.89	
11	1.15		41	4.20		71	6.97	
12	1.26		42	4.30		72	7.06	
13	1.36	.1—.01	43	4.40	.1—.01	73	7.14	.1—.01
14	1.46	.2—.02	44	4.50	.2—.02	74	7.22	.2—.02
15	1.57	.3—.03	45	4.60	.3—.03	75	7.31	.3—.02
16	1.67	.4—.04	46	4.69	.4—.04	76	7.39	.4—.03
17	1.77	.5—.05	47	4.79	.5—.05	77	7.47	.5—.04
18	1.87	.6—.06	48	4.88	.6—.05	78	7.55	.6—.05
19	1.98	.7—.07	49	4.98	.7—.06	79	7.63	.7—.06
20	2.08	.8—.08	50	5.08	.8—.07	80	7.71	.8—.06
21	2.18	.9—.09	51	5.17	.9—.08	81	7.79	.9—.07
22	2.29		52	5.26		82	7.87	
23	2.39		53	5.35		83	7.95	
24	2.49		54	5.45		84	8.03	
25	2.59		55	5 54		85	8.11	
26	2.70		56	5.63		86	8.18	
27	2.80		57	5.72		87	8.26	
28	2.90		58	5.82		88	8.34	
29	3.00		59	5.91		89	8.41	
30	3.10		60	6.00		90	8.48	

168.—ROTARY CUT BEVEL TEETH.

The most common method of forming the teeth of the bevel gear is by cutting them from the solid blank by the use of the common rotary cutter.

The cutter should be shaped to cut the tooth of the correct shape at the large end, and the small end must be shaped either by another cut with a different cutter, or with a file.

It is impossible to cut the tooth correctly at both ends, for the simple reason that the shape of the tooth changes, while that of the cutter is invariable. Therefore the result must always be an approximation depending upon the personal skill and experience of the workman. It is a too common practice to make the teeth fit at the large ends, and to increase the depth of the tooth toward the point, so that the teeth will pass without filing, but such teeth can be in working contact only at the large ends.

169.—THE TEMPLET GEAR PLANER.

The most common method of planing the teeth of bevel gears is by means of devices adapted to guide the tool by a templet that has previously been shaped, as nearly as may be, to the true curve. The arm that carries the tool is hung by a universal joint at the apex of the gear, so that all of its strokes are radial, and a finger placed in the line of the stroke of the cutting point of the tool is held against the templet. There are many different arrangements for the purpose, but they are all founded on the same principles, and differ only as to details.

The invention of the templet gear planer is commonly credited to George H. Corliss, who patented it in 1849, and was the first to use it in this country. But it was patented in France, by Glavet, in 1829, and may be even older.

It is largely used for planing the teeth of heavy mill gearing, but has not been, and cannot be, profitably applied to common small gear work. Its product is, in any case, superior to the rough cast tooth, but its accuracy is dependent on that of the templet, and is therefore dependent on personal skill.

9. THE SKEW BEVEL GEAR.

170.—THE SKEW BEVEL GEAR.

When a pair of shafts are not parallel, and do not intersect, they are said to be askew with each other, and they may be connected by a pair of skew bevel gears, having straight teeth, which bear on each other along a straight line. Such gears are to be carefully distinguished from spiral gears, used for the same purpose but having spiral teeth bearing on each other at a single point only.

We will endeavor to describe the skew bevel gear so that its general nature can be understood, but it is impossible to do so in simple language. It is the most difficult object connected with the subject. The theory cannot even be considered as yet settled, for writers upon theoretical mechanism do not agree upon it, and there are points yet in controversy.

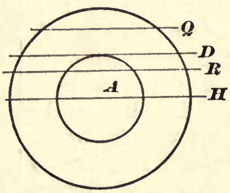
In the theory of the bevel gear the surface of reference is the spherical surface upon which the tooth outlines are drawn, and upon which the laws of their action may be studied, for spheres of reference of two separate gears may be made to coincide so that the lines upon one will come in contact with those upon the other. For the spur gear, the spheres become planes and the process is the same. But for the skew bevel gear there is no analogous process, for it is impossible to imagine a surface of such a nature that it can be made to coincide with a similar surface when both are attached to revolving askew shafts. There are spiral surfaces which will approximately coincide, and are analogous to the Tredgold tangent cones of bevel gears (161), but any tooth action developed upon such approximate surfaces must, of necessity, be not only approximate, but also very difficult to define and formulate.

Of all the skew tooth surfaces that have been proposed, there is but one, the Olivier involute spiraloid, that can be proved to be theoretically correct.

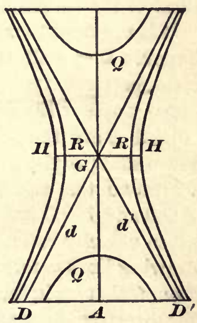
171.—THE HYPOID.

The pitch surface of the skew bevel gear is the surface known as the "hyperboloid of revolution," and it is so intimately connected with the subject that it must be thoroughly understood before going further. The clumsy name may be abbreviated to "hypoid."

If a line *D*, Figs. 152 and 153, called a generatrix, is attached to a revolving shaft *A*, so that it revolves with it, it will develop or "sweep up" the hypoid *H* in the space surrounding the shaft. A section of the surface by any plane normal to the axis is a circle. The common normal to the generatrix and the axis is the gorge radius *G*, and circular section through that line is the gorge circle. A section by a plane *D*, Fig. 152, parallel to the axis, at the gorge distance from the axis, will be the pair of straight lines *d* and *d'*, Fig. 153, either one of which is an element of the surface, and will form it if used as a generatrix. A section by any other plane parallel to the axis will be a hyperbola, to which the elements *d* and *d'*



Hypoidal sections.
Fig. 152.

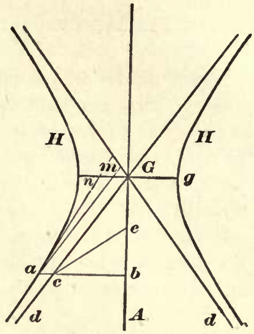


Hyperbolic sections.
Fig. 153.

are assymptotes, or lines which the curves continually approach, but reach only at infinity. Fig. 153 shows at Q the hyperbolas cut by the plane Q of Fig. 152, and at R those cut by the plane R. The principal hyperbola H is the only one with which we are concerned.

The hypoid is best studied as projected upon a plane parallel to the axis, as in Fig. 154, in which A is the projection of the axis, d is that of the generatrix, dGA is the skew angle, and H is the principal hyperbola.

When the skew angle and the gorge radius are given, the hyperbola is easily constructed by points. Any line ab is drawn normal to the axis and the gorge distance $be = Gg$ is laid off from b, the distance ab is made equal to ec, and a is then a point on the curve. The curve is to be drawn through several points thus constructed.



The hyperbola.
Fig. 154.

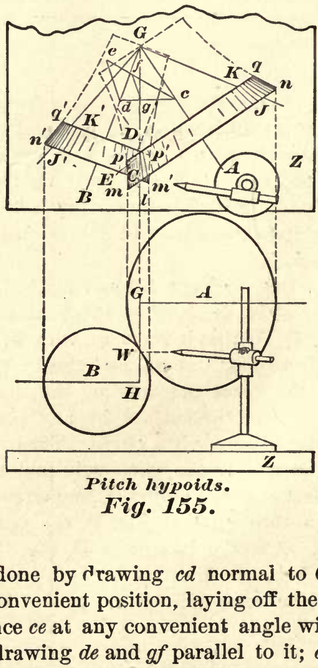
To draw a tangent to the curve at any point a, draw a line am parallel to the assymptote d, lay off mn equal to Gm, and draw the tangent an.

172.—THE PITCH HYPOIDS.

The utility of the hypoid as the pitch surface of the skew gear depends upon the peculiar property that any number of such surfaces will roll together, and drive each other by frictional contact with velocity ratios in the proportions of the sines of their skew angles, if their gorge radii are in the proportions of the tangents of their skew angles.

It is required to construct a pair of rolling hypoids that will transmit a given velocity ratio between two shafts that are set at a given angle with each other. In Fig. 155, A and B are the given axes, and AGB the given shaft angle. The directrix D is to be so drawn that the sines of the skew angles AGD and BGD are in the proportion of the given velocity ratio, and this is best done by drawing lines parallel to the axes, at distances from G that are in the given ratio, and drawing the directrix through their intersection D.

In the figure the axes are situated one over the other at a distance GH called the gorge distance, and the directrix D is situated between them so as to pass through the gorge line and divide the gorge distance into gorge radii, GW and HW, which are in proportion to the tangents of the skew angles. This is



Pitch hypoids.
Fig. 155.

best done by drawing cd normal to GD in any convenient position, laying off the gorge distance ce at any convenient angle with cd, and drawing de and gf parallel to it; cf will be the gorge radius GW for the axis GA, and *fe* will be the gorge radius *HW* for the axis *GB*.

Then, if the directrix, thus situated, is attached first to one shaft and then to the other, and used as a generatrix, it will sweep up a pair of pitch hypoids that will be in contact at the directrix, and which will roll on each other.

They will not only roll on each other in contact at the directrix, but they will also have a sliding motion on each other along that line, the two motions combining to form a resulting motion that must be seen to be understood. It is this sliding motion that makes all the difficulty in the construction of the teeth, for they must be so constructed as to allow it. It is also the cause of the great inefficiency of such teeth in action, for any possible form must have a lateral sliding motion, with the consequent friction and destruction.

If we draw two diameters *mn* and *m'n'* through the same point *C* on the directrix, they will be the diameters of circles that will touch each other while revolving, and may be called pitch circles. If they are thin, and provided with teeth in the given velocity ratio, they will drive each other with a contact that is approximately correct, and if there are several pairs of such thin gears set so far apart that they do not interfere with each other, they will serve light practical purposes fairly well.

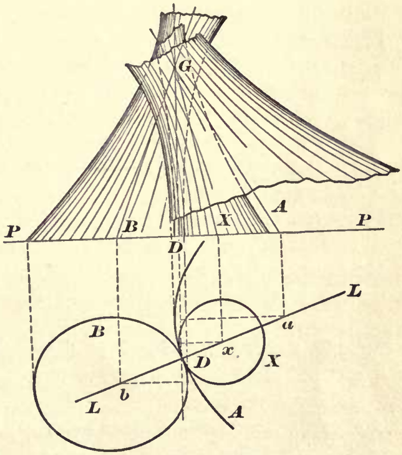
If a face distance *CE* is laid off on the directrix and another pair of pitch circles constructed, the frustra of the hypoids included between the circles may be called pitch frustra, and they will roll together in contact at the directrix.

It is to be noticed that the pitch diameters thus determined are not, as in spur and bevel gearing, in the inverse proportion of the velocity ratio of the axes, and therefore if one diameter of a pair of skew gears to have a given velocity ratio is given, the other must be constructed. When the skew angles are equal, the pitch diameters are equal, but otherwise the proportion cannot be expressed in simple terms, and must be determined by making the drawing.

173.—THE LOCUS OF AXES.

The rolling hypoids may be examined from another and most interesting point of view. In Fig. 156 the gorge line *G* is normal, and the directrix *D* is parallel to the plane of the figure. The plane *P* is normal to the directrix, and below is a front view of it. On the plane *P* draw any straight line *L* through the directrix. From any two points *a* and *b* on this line draw lines *A* and *B* normal to the gorge line *G*, and they will be axes of pitch hypoids that will roll on each other in contact at the directrix.

Axes drawn from all points of the line *L* will form a continuous surface called a "hyperbolic paraboloid," which will be the locus of all the axes of a set of hypoids that will roll together collectively in contact at the directrix.

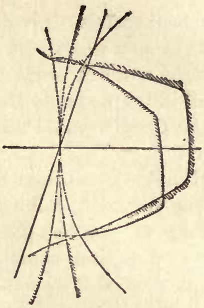


The locus of axes.
Fig. 156.

174.—CYCLOIDAL TEETH FOR SKEW GEARS.

As any number of hypoids, on axes in the same locus of axes, will roll together in either external or internal contact at the directrix, it might be supposed that a tooth similar to the cycloidal tooth for bevel and spur gears might be formed by an element in an auxiliary hypoid X, Fig. 156, which rolls inside of one and outside of the other pitch hypoid.

This is such a plausible supposition that it long passed for the truth, not only with its inventor, the celebrated Professor Willis, but with many other prominent writers, until shown by MacCord to be wrong. It serves to illustrate the confusion in which the whole subject has been and now is.



Cycloidal tooth Curves
Fig. 157.

The tooth surfaces which Willis supposed to be tangent at the generating element of the auxiliary hypoid really intersect at that line, and Fig. 157 shows a pair of such intersecting teeth. The curves of the figure were drawn by an instrument made for the purpose, and are, therefore, a better proof of the intersection of the surfaces than solid teeth would be.

The cycloidal tooth is examined at considerable length, and the instrumental proof of its failure is given in the AMERICAN MACHINIST for September 5th, 1889.

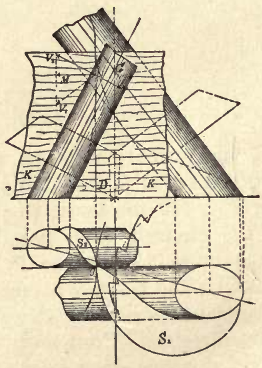
175.—INVOLUTE TEETH FOR SKEW BEVEL GEARS.

Herrmann's form of the Olivier spiraloidal tooth is constructed with the directrix of (172) as a generatrix, as follows:

Suppose that cylinders are constructed upon the gorge circles of a pair of pitch hypoids, Fig. 158, and suppose a plane K to be placed between them. This plane will be tangent to both cylinders, and will contain the directrix, and if moved will move the cylinders as if by friction. Then imagine the plane to move in a direction normal to the directrix, and it will carry that directrix with it as a generatrix always parallel to its first position. It will sweep up the spiraloid tooth surfaces S_1 and S_2 imperfectly shown by the figure, or by Fig. 159, and they will be correct tooth surfaces always in tangent contact.

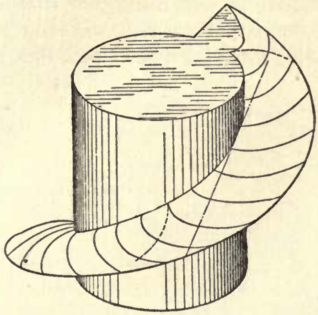
Fig. 159 shows a full involute tooth surface or "spiraloid," and Fig. 160 is a full Olivier skew bevel gear.

The particular involute skew tooth above described is not the only possible form, but it has the least possible sliding action, and is, therefore, the best.

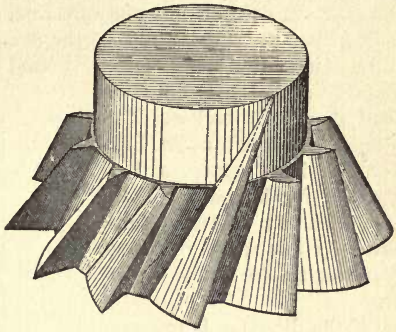


Involute tooth action
Fig. 158.

If the plane K has a generatrix line at any angles with the axes of the gears, and is moved in a direction at right angles with that line, correct tooth surfaces will be swept up. In fact, any two spiraloids on any two cylinders will work correctly with each other, and therefore any two spiraloidal gears of the same normal pitch will work correctly together.



A spiraloid
Fig. 159.



Olivier Involute skew Bevel Gear
Fig. 160.

176.—HERRMANN'S LAW.

Herrmann gives a law, and claims it to be universal, to the effect that the skew bevel tooth must be swept up by a straight line generatrix that is always parallel to the directrix. He mentions the Olivier tooth, and claims that it cannot be correct, evidently not understanding that Olivier's theory clearly includes the form he himself proposes. His form of tooth, claimed to be the only possible form, is really only the best form of the Olivier tooth.

We will not undertake to state wherein Herrmann's law is incorrect, but that it is wrong is clearly shown by the most convincing of all proofs, the reduction to practice. Beale, for the Brown & Sharpe Mfg. Co., has made working Olivier gears on a large scale, which are directly contrary to Herrmann's law, but which work perfectly, and demonstrate the truth of Olivier's theory in a way that admits of no question.

Indeed, the closest possible scrutiny of Olivier's theory, without the aid of Beale's experimental work, fails to detect a flaw in it. Herrmann's condemnation of it is not based on direct consideration, but simply on the fact that it does not agree with his own law.

177.—BEALE'S SKEW BEVEL GEARS.

Beale's gears are the same as Olivier's gears in general theory, but the improvement in practical form and application is so great that they may be considered a distinct invention.

Fig. 161 is a section through one axis, and at right angles to the other axis of a pair of Beale gears. Both surfaces of the teeth are true Olivier spiraloids of Fig. 159, and the gears will run in either direction. When corrected for interference they are reversible, like spur or bevel gears. The gorge cylinders are tangent to each other, and are so cut away inside as to allow the teeth of the mating gear to pass.

The Olivier theory requires the teeth to vanish at the gorge, as shown by the single full tooth of Fig. 160, in order to pass, while the Beale gear is cylindrical in form as a whole, and passes the full tooth at the gorge, with action over its whole surface. The difference is practically very great.

When in action a pair of uncorrected Beale gears must be placed as shown by Figs. 161 to 163, and Fig. 169, with one end of each at the gorge, and they will not run together if placed at random. If either gear extends beyond the gorge line there is an interference between the involute spiraloids which is the same in kind as that between the involute curves of common spur gear teeth.

Each gear can drive in but one direction, depending upon the position of the gear and the direction of the spiral, and if turned backwards the action is intermittent and practically useless. The gears must be placed as in Fig. 162 for right-hand spirals, and as in Fig. 163 for left-hand spirals, and the direction of the rotation is shown by the arrow *D*, when the gear bearing the arrow is the driver.

But, if the direction is to be reversed, the gears can be arranged as in Fig. 162*a*, or as in Fig. 163*a*. This resetting is the same in effect as turning the gears half around, except that opposite sides of the teeth are in contact in the two positions of the same gears.

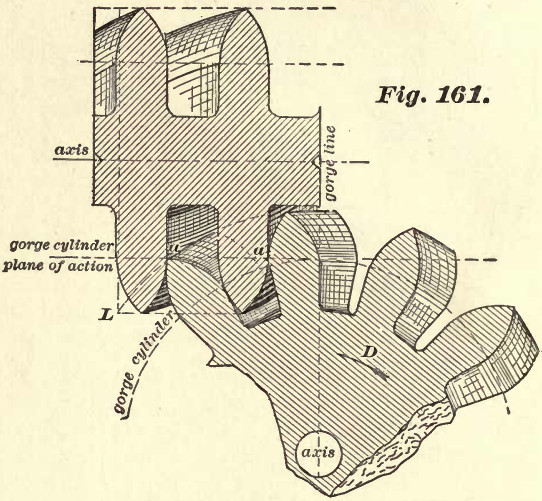
If, however, the interfering parts of the tooth surface are removed, the gears will run together perfectly and in either direction when put together at random as in Fig. 168.

In the cases shown by the figures, the spirals make the angles of forty-five degrees with the shafts, contrary to Herrmann's law, but the action will be smoother, and the sliding of the teeth on each other will be less, if Herrmann's angles are adopted. These angles are the same as those made by the conical face of common bevel gears of the same proportion with the axes, and the best angles for the two-to-one proportion of figures are those of the line *X* of Fig. 162, making the angles 26° 34′ and 63° 26′ with the axes.

The Olivier gear of Fig. 160 is perfect in theoretical action, but the teeth must be taken so far from the gorge that the obliquity of the action is excessive, and the arc of action is so limited that the teeth must be small. The sliding and wedging action is so great that the gears are practically useless.

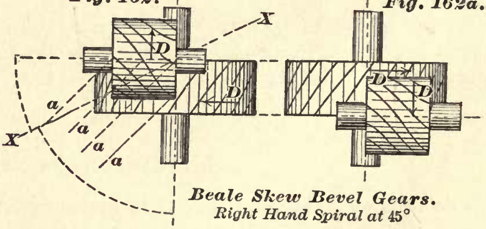
In comparison, the Beale gear is taken so near the gorge that it is practical and serviceable, having large teeth and small obliquity.

Fig. 161.



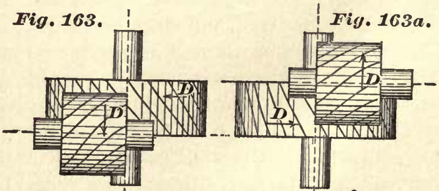
Section of Beale's Gears

Fig. 162. ***Fig. 162a.***



Beale Skew Bevel Gears.
Right Hand Spiral at 45°

Fig. 163. ***Fig. 163a.***



Left Hand Spiral at 45°

The working length of each gear is as determined by the line *L* of Fig. 161, and the whole surface of the tooth within that limit will be swept over by the line of contact. If the length of each gear is equal to the radius of the other gear it will always be long enough.

The action between two gears will be at the straight, equidistant, parallel lines *a a*, Figs. 161 and 162, in the plane of action tangent to both gorge cylinders.

The shafts of a pair of skew bevel gears should be as near together as possible, just far enough apart to allow the shafts to pass, so as to avoid the excessive sliding action. In that case both Beale and Olivier gears are practically useless, the former on account of the small size of the teeth, and the latter on account of the great obliquity of the action.

The common bevel gear becomes the spur gear when the shaft angle becomes zero, but the analogous transformation of the skew bevel gear into a bevel gear by reducing the gorge distance to zero is not possible.

The skew bevel gear becomes a spur gear if we imagine the axes to be brought parallel by removing the gorge to an infinite distance, for the spiraloids on the gorge cylinders then become involute surfaces on base cylinders. But, and it is a curious circumstance, when the shafts are brought parallel by imagining the shaft angles to become zero without changing the position of the gorge, the gorge cylinders become tangent and the gears do not become spur gears.

Involute skew bevel gears do not appear to have any possible adjustment corresponding to the adjustment of the shaft distance of involute spur gears, or of the shaft angle of involute bevel gears, (56) and (157).

Beale's gears are fully described in the AMERICAN MACHINIST of Aug. 28th, 1890.

178.—TWISTED SKEW TEETH.

As no two surfaces of reference attached to a pair of revolving askew shafts can be made to coincide with each other, like the planes of spur gears or the spheres of bevel gears, the twisted or spiral tooth is impossible, for such a tooth would not permit the required sliding action.

But, if a line is drawn upon one pitch hypoid of a pair, a corresponding line may be drawn upon the other, as if the given line could leave an impression. Therefore a tooth having edge contact (100) may be constructed, provided the twist is such that one pair of lines always crosses the directrix. These teeth are purely imaginary, but if the edges are thick they will have an action upon each other, at a single point of contact, that is closely approximate to the theoretical action, and they will serve the general purpose, if the power carried is inconsiderable.

179.—APPROXIMATE SKEW TEETH.

As the true involute skew tooth is difficult to construct, and in many cases is of small practical utility, and all other proposed forms are incorrect, it follows that we must depend for practical purposes mostly upon some approximation, provided it is not possible to avoid the skew gear altogether.

The blanks can be constructed by a definite process. Construct the frustra of the pitch hypoids by the method of (172) and Fig. 155. Consider the end sections *mn* and *pq* as ends of a frustrum of a pitch cone, and on this pitch cone construct the blank gear exactly as for a common bevel gear.

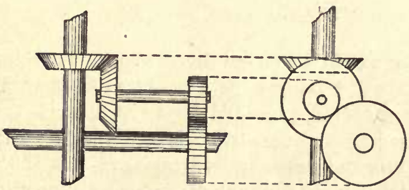
Having constructed the blanks, the general direction of the tooth is to be marked upon them. Mount each blank as in Fig. 155, with its axis parallel with a plane surface *Z*. Set a surface gauge with its point at the line of the directrix *W*, and with it mark the position of the directrix on the pitch line at each end of the blank.

The tooth must then be cut so that its direction follows the directrix, and it is to be noticed that it is not only askew with the axis, but that the tooth outline twists. The appearance of the tooth on either rim, as well as upon any section between the two rims, is the same as upon a common bevel gear, symmetrical, and not canted to one side, as is sometimes taught.

The approximate tooth is very similar to the twisted bevel tooth, see (155) and (99), with the twist following a straight line set askew with the axis, and as the line of the twist is not parallel with the conical face, that face should be as short as possible.

The process of cutting is not capable of description, for it depends upon personal skill and judgment. The workman must imagine that he sees the twisted cut in the body of the blank, and then must persuade the cutter to follow it. Gear cutting machines are seldom so made that the cutter can be turned while it feeds, and therefore it must be set to a medium path, and reset two or three times to get the desired form. The beginner will fail the first time, and there may be several failures. The best possible result can be bettered with a file, after running the cut gears together to find where they interfere.

In the hands of a skillful workman, a passable approximation can be reached, and if the axes are very near together compared with the diameters of the gears, the teeth are small, and the face is short, the result is satisfactory. In fact, when the conditions are favorable, this approximate tooth is more serviceable than the true tooth.



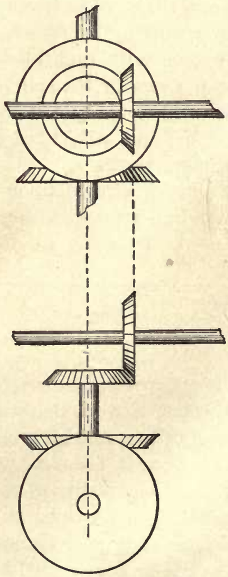
Substitute train.

Fig. 165.

180.—SUBSTITUTES FOR THE SKEW BEVEL GEAR.

When there is a chance to introduce an intermediate shaft, the skew bevel gear can be avoided, and it is not only better, but cheaper to avoid the objectionable gear at the cost of the extra mechanism.

Fig. 164 shows how to place an intermediate shaft and gears, when the shafts are so far apart that the shortest or gorge distance can be used. Fig. 165 shows how the skew shafts can be connected by one pair of bevel gears and one pair of spur gears, and that is the best device for general purposes.



Substitute train.

Fig. 164.

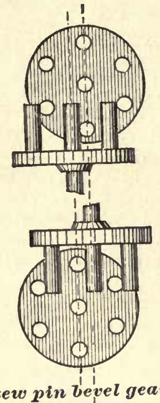
181.—SKEW PIN GEARING.

Fig. 166 shows a pair of skew pin gears commonly called face gears. They will run together with a uniform velocity ratio if they are exactly alike and at right angles with shafts at a distance apart equal to the diameter of the pins.

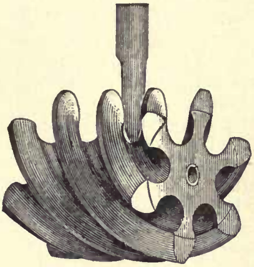
If the gears are not alike or not at right angles, the teeth on one may be straight pins, but those on the other must be shaped to correspond.

Such gears are objectionable because they have but a single point of contact for each pair of teeth, at which they slide on each other with great friction.

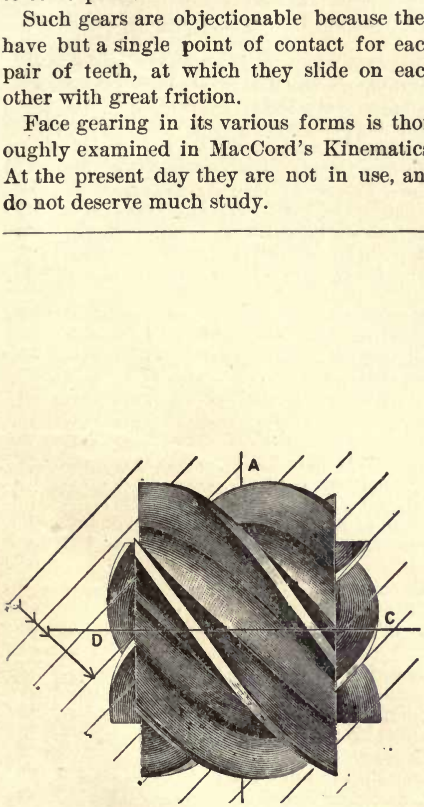
Face gearing in its various forms is thoroughly examined in MacCord's Kinematics. At the present day they are not in use, and do not deserve much study.



Skew pin bevel gears.
Fig. 166.



Formation of Beale gear.
Fig. 167.



Beale gears corrected for interference.
Fig. 168.

Uncorrected Beale gears.
Fig. 169.

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