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A TREATISE

ON THE

NATURE AND PROPERTIES

OF

ALGEBRAIC EQUATIONS,

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A TREATISE

ON THE

THEORY OF EQUATIONS.

CHAP. I.

ON THE STRUCTURE OF EQUATIONS.

1. By the term Equation is meant, in general, the algebraic expression of the equality existing between two quantities, without reference to the form in which that equality is presented, or to the distinction between the known and unknown quantities which are involved in it. But in the following pages the general term will be restricted to that class of equations only, which contains but one unknown quantity. This is to be understood in all cases, unless the contrary be stated. The indices of the unknown quantity will in all cases be supposed positive integers: and the coefficients will, in general, be supposed real; that is, either numbers or symbols which do not involve the imaginary sign. If the coefficients are imaginary in particular cases, they will be expressed in that form, or it will be stated that such a form is to be understood.

2. The general type of such equations, according to the usual arrangement of their terms, will be

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots\dots\dots + p_n = 0.$$

The degree, or dimensions, of the equation will be determined by the index of the highest power of the unknown quantity. Thus the preceding equation is of the n^{th} degree.

3. As it will frequently be necessary to speak of the quantity, or expression,

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots\dots\dots\dots + p_n,$$

without reference to its equality to zero, but merely considering it as a function of the symbol x, it will be convenient to designate it as the polynomial composing the equation. For the sake of brevity in our notation, as well as for the convenience of exhibiting the particular values of this polynomial, it will be denoted by $f(x)$. And similar advantages will attend the adoption of Lagrange's notation, for its differential coefficients taken with respect to x successively; which will accordingly be represented by

$$f'(x),\ f''(x),\ f'''(x) \dots\dots\dots\dots\dots f^n(x).$$

So that we can immediately express the result of substituting for x any known value a in this series of polynomials, by writing the series

$$f(a),\ f'(a),\ f''(a) \dots\dots\dots\dots\dots f^n(a).$$

4. Now it is evident, that by giving different values to x, the value of $f(x)$ will vary; and it may so happen that when x receives the particular value a, the particular value of $f(x)$ shall be zero; or that we have identically $f(a) = 0$.

In this case, the particular value a is said to satisfy, or to be a root of the equation

$$f(x) = 0.$$

And it is to the properties of these roots, as connected with the formation of the polynomial $f(x)$, that our attention is now to be directed.

5. The first object of inquiry which here presents itself is, whether under all forms of the polynomial $f(x)$ we shall have a right to conclude that there exists any such particular value, or root, expressible in a known form, possible or impossible? For we are immediately obliged to relinquish the hope of finding in all cases a real root of the equation, from our experience in the case of quadratic equations; as for instance,

$$x^2 + 1 = 0,$$

$$x = \pm\sqrt{-1}.$$

The question then is, does there in every case exist a root which is expressible by algebraic symbols? The following demonstration by M. Cauchy of the existence of such a root appears to be free from objection.

6. There will always exist some real values of ρ and θ, which shall render

$$\rho(\cos\theta + \sqrt{-1}\sin\theta)$$

a root of the equation $f(x) = 0$.

For the sake of brevity, denote by a the quantity

$$\rho(\cos\theta + \sqrt{-1}\sin\theta);$$

and let a receive an increment h of the same nature, namely

$$h = \sigma(\cos\phi + \sqrt{-1}\sin\phi).$$

Then, expanding by Taylor's theorem, after writing $a + h$ for x, we shall find

$$f(a+h) = f(a) + hf'(a) + \ldots\ldots\ldots\ldots$$

observing that the theorem cannot fail for a polynomial of this description. Here $f(a)$ is supposed not zero, or a would be a root; and $f'(a)$ is also supposed not zero, for if it be zero, then we may take the first term of the series which is not zero, and the reasoning will still be the same.

Now, for the convenience of the calculation, let the quantities involved in the above series be put under the same form as a and h, which can always be effected; so that we may assume

$$\begin{aligned} f(a) &= A(\cos \alpha + \sqrt{-1}\sin \alpha), \\ f'(a) &= A'(\cos \alpha' + \sqrt{-1}\sin \alpha'), \\ &\ldots\ldots\ldots\ldots \\ f(a+h) &= R(\cos \omega + \sqrt{-1}\sin \omega). \end{aligned}$$

Then, after the substitution, and reduction by the aid of Demoivre's theorem, we shall have

$$\left.\begin{aligned} R(\cos \omega + \sqrt{-1}\sin \omega) = {} & A(\cos \alpha + \sqrt{-1}\sin \alpha) \\ & + A'\sigma\{\cos(\alpha' + \phi) + \sqrt{-1}\sin(\alpha' + \phi)\} \\ & + \ldots\ldots\ldots\ldots\ldots \end{aligned}\right\},$$

from which, by the separate equality of the possible and impossible parts, we obtain the two equations

$$\begin{aligned} R\cos \omega &= A\cos \alpha + A'\sigma\cos(\alpha' + \phi) + \ldots\ldots \\ R\sin \omega &= A\sin \alpha + A'\sigma\sin(\alpha' + \phi) + \ldots\ldots \end{aligned}$$

and, by adding the squares of these equations,

$$R^2 = A^2 + 2AA'\sigma\cos(\alpha' + \phi - \alpha) + A'^2\sigma^2 + \ldots$$

Now the sign of $R^2 - A^2$ can be made to depend on its first term by diminishing σ. Hence we can in every case render $R^2 < A^2$, because ϕ is an arbitrary angle.

But on the other hand, since there must be some minimum amongst the quantities A^2, R^2, and all other similar quantities, we may suppose that ρ, θ, have been so chosen as to give A^2 this minimum value. Hence, after this choice of ρ and θ, we cannot make $R^2 < A^2$.

It follows then, that this minimum value must be zero; and that ρ, θ, are then so chosen that

$$\rho(\cos\theta + \sqrt{-1}\sin\theta)$$

is a root of the equation. For if A be not zero, we can in all cases make $R^2 < A^2$.

We may remark that the above proof applies, when the coefficients are impossible quantities.

Hence, in every case, we have a right to conclude that there exists a value of x of the form

$$\rho(\cos\theta + \sqrt{-1}\sin\theta),$$

which shall render $f(x) = 0$; or, in other words, that every equation has a root expressible by means of algebraical symbols.

7. The investigation of the roots of the equation $f(x) = 0$ is identical with the decomposition of the polynomial $f(x)$ into its factors. For, whenever any root a of the equation is found, the corresponding factor $x - a$ of the polynomial is determined: and vice versâ. The proof of this is at once evident, if we observe that Taylor's series can never fail for such a polynomial.

Now we have

$$f(x) = f(a + \overline{x - a}),$$
$$= f(a) + \overline{x - a}\, f'(a) + \ldots + (x - a)^n.$$

Hence, if $f(a) = 0$, that is, if a is a root, $x - a$ is a factor. And conversely, if $x - a$ is a factor, we must have $f(a) = 0$, and a will be a root.

8. Impossible roots enter equations by pairs, and each pair corresponds to a real quadratic factor of the polynomial.

For if one root of the equation be

$$\rho\,(\cos\theta + \sqrt{-1}\,\sin\theta),$$

then, by substitution and reduction,

$$\rho^n(\cos n\theta + \sqrt{-1}\sin n\theta) + p_1\rho^{n-1}(\cos \overline{n-1}\theta + \sqrt{-1}\sin\overline{n-1}\theta)$$
$$+\, p_2\rho^{n-2}(\cos\overline{n-2}\theta + \sqrt{-1}\sin\overline{n-2}\theta) + \ldots = 0;$$

and since the equality can be separated, we must also have

$$\rho^n(\cos n\theta - \sqrt{-1}\sin n\theta) + p_1\rho^{n-1}(\cos \overline{n-1}\theta - \sqrt{-1}\sin\overline{n-1}\theta)$$
$$+\, p_2\rho^{n-2}(\cos\overline{n-2}\theta - \sqrt{-1}\sin\overline{n-2}\theta) + \ldots = 0;$$

from which we conclude that

$$\rho\,(\cos\theta - \sqrt{-1}\,\sin\theta)$$

is also a root of the equation.

Now the two corresponding factors of the polynomial will be

$$x - \rho\cos\theta - \sqrt{-1}\,\rho\sin\theta,$$
$$x - \rho\cos\theta + \sqrt{-1}\,\rho\sin\theta;$$

and their product gives the quadratic factor

$$(x - \rho\cos\theta)^2 + \rho^2\sin^2\theta,$$
$$\text{or } x^2 - 2x\rho\cos\theta + \rho^2.$$

9. Every equation has as many roots as it has dimensions, and no more.

For it has been shown that $f(x) = 0$, has always one root, and therefore $f(x)$ has always some factor of the form $x - a_1$, which exactly divides it. Now the quotient $\frac{f(x)}{x - a_1}$ is a quantity perfectly similar to $f(x)$ in form, but of one dimension lower. It must then have a factor $x - a_2$, and the quotient $\frac{f(x)}{(x - a_1)(x - a_2)}$ will be a polynomial of $n - 2$ dimensions. Proceeding in this manner, we shall at length find a quotient of no dimensions, which will be unity; thus

$$\frac{f(x)}{(x - a_1)(x - a_2) \ldots\ldots (x - a_n)} = 1:$$

that is,

$$f(x) = (x - a_1)(x - a_2) \ldots\ldots (x - a_n).$$

It is evident that the quantities $a_1, a_2, \ldots\ldots a_n$, are roots of the equation $f(x) = 0$; and that no other quantity can be a root of it.

For any other quantity Q, when substituted for x, will give the result

$$f(\mathrm{Q}) = (\mathrm{Q} - a_1)(\mathrm{Q} - a_2) \ldots\ldots (\mathrm{Q} - a_n),$$

which is not zero; and consequently Q cannot be a root.

10. Connexion of the coefficients of an equation with its roots.

Let the roots be denoted by $a, b, c, \ldots\ldots l$; and let the equation be of n dimensions with the usual coefficients. Then, by the decomposition of the polynomial, which has been effected,

$$
\begin{aligned}
&x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots\ldots + p_n \\
&= (x-a)(x-b)(x-c)\ldots\ldots\ldots(x-l) \\
&= \left.\begin{aligned}
& x^n - x^{n-1}(a+b+c+d+\ldots\ldots\ldots) \\
&\quad + x^{n-2}(ab+ac+bc+\ldots\ldots\ldots) \\
&\quad - x^{n-3}(abc+acd+\ldots\ldots\ldots) \\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad + (-1)^n . abcd \ldots . l
\end{aligned}\right\} \\
&= x^n - x^{n-1}\Sigma(a) + x^{n-2}\Sigma(ab) - \ldots + (-1)^n abc \ldots l,
\end{aligned}
$$

denoting by Σ the sum of all quantities similar to the one to which it is prefixed.

Hence, generally, we shall have

$$(-1)^r\, p_r = \text{sum of products of every } r \text{ roots.}$$

11. By means of the relations just established between the coefficients and the roots of an equation, we can determine the values of any symmetrical functions of the roots, without knowing the roots themselves; as in the following examples:

Ex. 1. To find the sum of the squares of the roots of an equation.

It is manifest, that in multiplying $\Sigma(a)$ by itself, terms of two kinds will be produced; namely, a^2, ab; and while the term a^2 can be formed in one way only, the term ab can be formed by the a of either factor, and the b of the other, and will consequently appear twice. Wherefore

$$\Sigma(a) . \Sigma(a) = \Sigma(a^2) + 2\Sigma(ab),$$

$$\text{or } p_1^2 = \Sigma(a^2) + 2p_2,$$

$$\Sigma(a^2) = p_1^2 - 2p_2.$$

Ex. 2. To estimate $\Sigma(a^2bc)$.

Commence with forming the product

$$\Sigma(a)\,.\,\Sigma(abc);$$

and observe that its terms are of two kinds, a^2bc, $abcd$; also that a^2bc can enter but once, whilst $abcd$ can be produced in four different ways.

Hence

$$\Sigma(a)\,.\,\Sigma(abc) = \Sigma(a^2bc) + 4\,.\,\Sigma(abcd),$$

$$\text{or } p_1p_3 = \Sigma(a^2bc) + 4p_4,$$

$$\Sigma(a^2bc) = p_1p_3 - 4p_4.$$

Ex. 3. To find the value of $\Sigma(a^2b^2)$.

Adopting the same method as above,

$$\Sigma(ab)\,\Sigma(ab) = \Sigma(a^2b^2) + 2\Sigma(a^2bc) + 6\Sigma(abcd),$$

$$\text{or } p_2^2 = \Sigma(a^2b^2) + 2(p_1p_3 - 4p_4) + 6p_4,$$

$$\Sigma(a^2b^2) = p_2^2 - 2p_1p_3 + 2p_4.$$

Ex. 4. To find the value of $\Sigma\left(\frac{a}{b}\right)$.

$$\text{Here } \Sigma(a)\,\Sigma\left(\frac{1}{a}\right) = n + \Sigma\left(\frac{a}{b}\right).$$

$$\text{But } \Sigma\left(\frac{1}{a}\right) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \ldots + \frac{1}{l}$$

$$= \frac{\text{sum of prod. of every } \overline{n-1} \text{ roots}}{abc\ldots l}$$

$$= -\frac{p_{n-1}}{p_n}.$$

$$\text{Hence } \Sigma\left(\frac{a}{b}\right) = \frac{p_1p_{n-1}}{p_n} - n.$$

12. The same process would of course conduct us to the expressions for the sums of the powers of the roots. But these can be found in a more direct manner by the following methods:

Since we have the identity

$$f(x) = (x-a)(x-b)(x-c) \ldots\ldots (x-l);$$

$$\therefore \frac{f(x)}{x^n} = \left(1-\frac{a}{x}\right)\left(1-\frac{b}{x}\right)\left(1-\frac{c}{x}\right) \ldots\ldots \left(1-\frac{l}{x}\right),$$

and by taking the Naperian logarithms

$$\log \left\{\frac{f(x)}{x^n}\right\} = -\frac{\Sigma(a)}{x} - \frac{1}{2}\frac{\Sigma(a^2)}{x^2} - \frac{1}{3}\frac{\Sigma(a^3)}{x^3} - \ldots\ldots$$

But since we have

$$\frac{f(x)}{x^n} = 1 + \left(\frac{p_1}{x} + \frac{p_2}{x^2} + \ldots\ldots + \frac{p_n}{x^n}\right).$$

The first side of the above equation is capable of expansion in negative powers of x. After expanding, the equation of coefficients will give the sums of the powers of the roots.

A similar process will apply for finding the sums of the negative powers of the roots. For, as above,

$$f(x) = (x-a)(x-b)(x-c) \ldots\ldots (x-l);$$

and therefore, observing that $p_n = (-1)^n . abc \ldots\ldots l$,

$$\frac{f(x)}{p_n} = \left(1-\frac{x}{a}\right)\left(1-\frac{x}{b}\right)\left(1-\frac{x}{c}\right) \ldots\ldots \left(1-\frac{x}{l}\right),$$

$$\log . \left\{\frac{f(x)}{p_n}\right\} = -x\,\Sigma\left(\frac{1}{a}\right) - \frac{1}{2}x^2\,\Sigma\left(\frac{1}{a^2}\right) - \ldots\ldots\ldots$$

But since we have

$$\frac{f(x)}{p_n} = 1 + \frac{p_{n-1}x + p_{n-2}x^2 + \ldots\ldots + x^n}{p_n},$$

by expansion and equation of coefficients, we have the sums of the negative powers of the roots.

13. The above method is, however, only applicable with ease to the cases where n is not large, or at least where the number of terms of the equation is not large. In other cases it is best to calculate the sums of powers from each other in succession.

Since $f(x) = (x-a)(x-b)(x-c)\ldots.(x-l)$, by taking Naperian logarithms, and differentiating the equation, we shall have

$$\frac{f'(x)}{f(x)} = \frac{1}{x-a} + \frac{1}{x-b} + \ldots. + \frac{1}{x-l}$$

$$\left.\begin{aligned} &= \frac{1}{x} + \frac{a}{x^2} + \frac{a^2}{x^3} + \ldots\ldots \\ &+ \frac{1}{x} + \frac{b}{x^2} + \frac{b^2}{x^3} + \ldots\ldots \\ &+ \ldots\ldots\ldots\ldots\ldots \\ &+ \frac{1}{x} + \frac{l}{x^2} + \frac{l^2}{x^3} + \ldots\ldots \end{aligned}\right\},$$

$$\therefore \frac{f'(x)}{x^{n-1}} = \frac{f(x)}{x^n}\left\{n + \frac{\Sigma(a)}{x} + \frac{\Sigma(a^2)}{x^2} + \ldots\ldots\right\}.$$

But $f(x) = x^n + p_1 x^{n-1} + \ldots. + p_n$,

$$f'(x) = nx^{n-1} + \overline{n-1}\, p_1 x^{n-2} + \ldots. + p_{n-1};$$

and after the substitution of these values, the above equation will contain only negative powers, and we can compare the coefficients of like powers of x.

The comparison of the coefficient of $\frac{1}{x^r}$ will give the equation

$$(n-r)p_r = \Sigma(a^r) + p_1\Sigma(a^{r-1}) + \ldots + p_{r-1}\Sigma(a) + np_r,$$

$$\text{or,}\quad 0 = \Sigma(a^r) + p_1\Sigma(a^{r-1}) + \ldots\ldots + p_{r-1}\Sigma(a) + rp_r.$$

By making $r = 1, 2, 3, \ldots\ldots$ successively, we can find $\Sigma(a)$, $\Sigma(a^2)$, $\Sigma(a^3), \ldots\ldots$ in order.

We may remark that the formula still holds when r is greater than n, for in that case all the coefficients after p_n are to be considered as zeros.

A very slight modification of the preceding equation will give the values of the sums of the negative powers of the roots.

We obtained, by logarithmic differentiation, the equation

$$\frac{f'(x)}{f(x)} = \frac{1}{x-a} + \frac{1}{x-b} + \ldots + \frac{1}{x-l};$$

and if the second side be expanded in ascending powers of x, instead of descending powers, we shall have

$$\frac{f'(x)}{f(x)} = -\Sigma\left(\frac{1}{a}\right) - x\,\Sigma\left(\frac{1}{a^2}\right) - x^2\,\Sigma\left(\frac{1}{a^3}\right) - \ldots\ldots\ldots$$

from which we obtain

$$-xf'(x) = f(x)\left\{x\,\Sigma\left(\frac{1}{a}\right) + x^2\,\Sigma\left(\frac{1}{a^2}\right) + \ldots\ldots\right\},$$

and after supplying the values of $f(x)$ and $f'(x)$, the comparison of coefficients of x^r will give us

$$-rp_{n-r} = p_n\,\Sigma\left(\frac{1}{a^r}\right) + p_{n-1}\,\Sigma\left(\frac{1}{a^{r-1}}\right) + \ldots + p_{n-r+1}\,\Sigma\left(\frac{1}{a}\right),$$

or,

$$0 = p_n\,\Sigma\left(\frac{1}{a^r}\right) + p_{n-1}\,\Sigma\left(\frac{1}{a^{r-1}}\right) + \ldots + p_{n-r+1}\,\Sigma\left(\frac{1}{a}\right) + rp_{n-r}.$$

14. We have hitherto considered the roots $a, b, c, \ldots. l$, of the equation, without reference to any particular relation that may exist amongst them. There is, however, one very important case, that of equal roots, in which the equation may be reduced to a lower degree.

Let us suppose, then, that r roots of the equation become alike, and consequently that $(x-a)^r$ becomes a factor of $f(x)$. Then, since

$$f(x) = f(a) + (x-a)f'(a) + \ldots\ldots$$

it follows that we must have, besides the equation $f(a) = 0$, the $r-1$ additional equations

$$\left.\begin{array}{r} f'(a) = 0 \\ f''(a) = 0 \\ \ldots\ldots \\ f^{r-1}(a) = 0 \end{array}\right\}.$$

In fact, since we have

$$f(x) = (x-a)^r \cdot \phi(x),$$

it is manifest that $(x-a)^{r-1}$ will be a factor of $f'(x)$, or

$$f'(x) = (x-a)^{r-1}\,\psi(x);$$

$$\text{similarly } f''(x) = (x-a)^{r-2}\,\chi(x),$$

$$\ldots\ldots\ldots$$

and the first $r-1$ differential coefficients will vanish when $x = a$. And the same conditions must hold for any other repeated factor as $(x-b)^s$; so that if we have

$$f(x) = (x-a)^r \cdot (x-b)^s\,\phi(x),$$

we must also have

$$f'(x) = (x-a)^{r-1}\,(x-b)^{s-1}\,\psi(x),$$

and so on.

15. By means of the conditions which have been determined for the case of equal roots, we can reduce the degree of the equation containing them.

Thus, let $f(x)$ have some factors entering once, others entering twice, and so on: and let X_m denote the product of all the factors which enter m times into $f(x)$; so that X_m enters m times into $f(x)$. Then we shall have

$$f(x) = X_1 X_2^2 X_3^3 \dots\dots\dots\dots$$

And from what has preceded, it is manifest that the greatest common measure of $f(x)$ and $f'(x)$ will be

$$X_2 X_3^2 X_4^3 \dots\dots\dots\dots\dots\dots$$

which we shall call $f_1(x)$. Then treating the polynomial

$$f_1(x) = X_2 X_3^2 X_4^3 \dots\dots\dots\dots$$

in the same manner as $f(x)$ was treated, we shall find the greatest common measure of $f_1(x)$ and $f_1'(x)$ to be

$$f_2(x) = X_3 X_4^2 X_5^3 \dots\dots\dots$$

and so on, as long as the operation can be executed.

Now if we form from these successive polynomials $f(x)$, $f_1(x)$, $f_2(x)$, ... the successive quotients $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ... such that

$$\phi_1(x) = \frac{f(x)}{f_1(x)} = X_1 X_2 X_3 \dots\dots\dots$$

$$\phi_2(x) = \frac{f_1(x)}{f_2(x)} = X_2 X_3 X_4 \dots\dots\dots$$

$$\phi_3(x) = \frac{f_2(x)}{f_3(x)} = X_3 X_4 X_5 \dots\dots\dots$$

and so on, as far as possible; it will be evident that X_1, X_2, X_3, ... can be determined from these quotients by a second operation of division, in the same order; for we have

$$X_1 = \frac{\phi_1(x)}{\phi_2(x)},$$

$$X_2 = \frac{\phi_2(x)}{\phi_3(x)},$$

and so on, as far as the degree of repetition of factors admits.

The solution of the original equation

$$f(x) = 0$$

is thus reduced to the equations

$$\left.\begin{array}{l} X_1 = 0, \\ X_2 = 0, \\ X_3 = 0, \\ \dots\dots \end{array}\right\}$$

all of which are of less dimensions.

This method has also the advantage of pointing out which of the roots enter once, twice, or oftener.

The following is an example of the above process:

Ex. Required to solve the equation

$$x^5 - x^4 + 4x^3 - 4x^2 + 4x - 4 = 0,$$

which has equal roots.

In the first place,

$$f(x) = x^5 - x^4 + 4x^3 - 4x^2 + 4x - 4,$$
$$f'(x) = 5x^4 - 4x^3 + 12x^2 - 8x + 4;$$

and by the rule for finding the greatest common divisor of two algebraic quantities,

$$f_1(x) = x^2 + 2,$$
$$f_1'(x) = 2x;$$

and there is no common divisor, but

$$f_2(x) = 1,$$
$$f_3(x) = 1.$$

Secondly, to form the primary quotients,

$$\phi_1(x) = \frac{f(x)}{f_1(x)} = x^3 - x^2 + 2x - 2,$$

$$\phi_2(x) = \frac{f_1(x)}{f_2(x)} = x^2 + 2,$$

$$\phi_3(x) = \frac{f_2(x)}{f_3(x)} = 1,$$

$$\phi_4(x) = 1.$$

Lastly, the final quotients are

$$X_1 = \frac{\phi_1(x)}{\phi_2(x)} = x - 1,$$

$$X_2 = \frac{\phi_2(x)}{\phi_3(x)} = x^2 + 2,$$

$$X_3 = 1.$$

So that we have

$$f(x) = (x-1)(x^2+2)^2.$$

And the five roots are

$$\left.\begin{array}{lcl} \text{one root} & = & 1 \\ \text{two roots} & = & +\sqrt{-2} \\ \text{two roots} & = & -\sqrt{-2} \end{array}\right\}.$$

In general, the operations are to be carried on for each set of polynomials, until we arrive at one of no dimensions, or unity; after which every other will be unity.

16. When it is required to resolve any polynomial into its factors, the only difficulty is that of discovering all the roots of the equation formed by equating that polynomial to zero. We shall give some examples of this problem.

Ex. 1. To resolve $x^n - 1$ into its real component factors, n being odd.

Put $x^n - 1 = 0$,

$$x^n = 1 = \cos 2\lambda\pi + \sqrt{-1}\sin 2\lambda\pi,$$

where λ is any integer whatever; and we obtain the values

$$x = \cos\frac{2\lambda\pi}{n} + \sqrt{-1}\sin\frac{2\lambda\pi}{n};$$

and by giving λ the values 1, 2, 3, ... n, we obtain n different values for x; but these values recur when λ is made greater than n, or less than 1.

Now, observing that we have

$$1 = \cos\frac{2n\pi}{n} + \sqrt{-1}\sin\frac{2n\pi}{n},$$

$$\cos\frac{2\pi}{n} - \sqrt{-1}\sin\frac{2\pi}{n} = \cos\frac{2(n-1)\pi}{n} + \sqrt{-1}\sin\frac{2(n-1)\pi}{n},$$

$$\cos\frac{4\pi}{n} - \sqrt{-1}\sin\frac{4\pi}{n} = \cos\frac{2(n-2)\pi}{n} + \sqrt{-1}\sin\frac{2(n-2)\pi}{n},$$

$$\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$

and that $n = 1 +$ an even number, we may write the roots thus;

First, the single real root $= 1$.

Secondly, the $\frac{n-1}{2}$ pairs of impossible roots, which are

$$\left.\begin{array}{l} \cos\frac{2\pi}{n} \pm \sqrt{-1}\sin\frac{2\pi}{n}, \\ \cos\frac{4\pi}{n} \pm \sqrt{-1}\sin\frac{4\pi}{n}, \\ \ldots\ldots\ldots\ldots \\ \cos\frac{(n-1)\pi}{n} \pm \sqrt{-1}\sin\frac{(n-1)\pi}{n} \end{array}\right\}.$$

And the corresponding real factors of $x^n - 1$ will be

First, the single factor $x - 1$.

Secondly, the $\frac{n-1}{2}$ quadratic factors,

$$\left.\begin{array}{l} x^2 - 2x\cos\frac{2\pi}{n} + 1, \\ x^2 - 2x\cos\frac{4\pi}{n} + 1, \\ \dots\dots\dots\dots \\ x^2 - 2x\cos\frac{(n-1)\pi}{n} + 1 \end{array}\right\}.$$

Ex. 2. To resolve $x^n - 1$, n being even.

The general form of the roots is the same as before; and taking away the last root, which $= 1$, we must unite the rest in pairs as before; but after taking away that last root, there will be an odd number left, which can be paired as before, only that the middle one corresponding to $\lambda = \frac{n}{2}$ is left single. This single root is

$$\cos\pi + \sqrt{-1}\sin\pi = -1.$$

Hence, for the corresponding real factors of $x^n - 1$, we have:

First, the single factors $(x - 1)$, $(x + 1)$.

Secondly, the $\frac{n}{2} - 1$ quadratic factors,

$$\left.\begin{array}{l} x^2 - 2x\cos\frac{2\pi}{n} + 1, \\ x^2 - 2x\cos\frac{4\pi}{n} + 1, \\ \dots\dots\dots\dots \\ x^2 - 2x\cos\frac{(n-2)\pi}{n} + 1 \end{array}\right\}.$$

Ex. 3. To resolve $x^n + 1$, when n is odd.

By changing x into $-x$, in Ex. 1, and changing the signs of the result, the factors of $x^n + 1$, will be,

First, the single factor $x + 1$.

Secondly, the $\frac{n-1}{2}$ quadratic factors,

$$\left.\begin{array}{r} x^2 + 2x\cos\frac{2\pi}{n} + 1, \\ x^2 + 2x\cos\frac{4\pi}{n} + 1, \\ \cdots\cdots\cdots \\ x^2 + 2x\cos\frac{(n-1)\pi}{n} + 1. \end{array}\right\}.$$

Ex. 4. To resolve $x^n + 1$, n being even.

Since $x^n + 1 = 0$, gives

$$\begin{aligned} x^n &= -1, \\ &= \cos(2\lambda+1)\pi + \sqrt{-1}\sin(2\lambda+1)\pi, \end{aligned}$$

$$x = \cos\frac{(2\lambda+1)\pi}{n} + \sqrt{-1}\sin\frac{(2\lambda+1)\pi}{n},$$

λ being any integer whatever; and by giving λ the values $0, 1, 2, \ldots\ldots \overline{n-1}$, we obtain n different values of x, all of which are impossible, and all different; but if λ be made greater than $\overline{n-1}$, or less than 0, the values recur.

Now observing that we have

$$\cos\frac{(2n-1)\pi}{n} + \sqrt{-1}\sin\frac{(2n-1)\pi}{n}$$

$$= \cos\frac{\pi}{n} - \sqrt{-1}\sin\frac{\pi}{n},$$

$$\cos\frac{(2n-3)\pi}{n} + \sqrt{-1}\sin\frac{(2n-3)\pi}{n}$$

$$= \cos\frac{3\pi}{n} - \sqrt{-1}\sin\frac{3\pi}{n},$$

. . . .

we shall be able to class the n impossible values of x into $\frac{n}{2}$ pairs,

$$\left.\begin{array}{c} \cos\frac{\pi}{n} \pm \sqrt{-1}\sin\frac{\pi}{n}, \\ \cos\frac{3\pi}{n} \pm \sqrt{-1}\sin\frac{3\pi}{n}, \\ \dots\dots\dots\dots \\ \cos\frac{(n-1)\pi}{n} \pm \sqrt{-1}\sin\frac{(n-1)\pi}{n}. \end{array}\right\}.$$

And the corresponding $\frac{n}{2}$ real quadratic factors will be

$$\left.\begin{array}{c} x^2 - 2x\cos\frac{\pi}{n} + 1, \\ x^2 - 2x\cos\frac{3\pi}{n} + 1, \\ \dots\dots\dots\dots \\ x - 2x\cos\frac{(n-1)\pi}{n} + 1. \end{array}\right\}.$$

In the preceding examples there has been no difficulty in the decomposition into factors, because we were at once certain of having obtained all the roots of the equation. This is also the case in the more general problem.

Ex. 5. To find the real factors of

$$x^{2n} - 2x^n\cos\theta + 1.$$

For the polynomial above given is evidently the product of

$$x^n - \cos\theta + \sqrt{-1}\sin\theta,$$

$$x^n - \cos\theta - \sqrt{-1}\sin\theta;$$

and by equating these separately to zero, we find the roots of

$$x^{2n} - 2x^n\cos\theta + 1 = 0.$$

These roots are evidently given by the single formula

$$x = \cos\frac{2\lambda\pi + \theta}{n} \pm \sqrt{-1}\cdot\sin\frac{2\lambda\pi + \theta}{n}.$$

And the required factors are

$$x^2 - 2x\cos\frac{2\lambda\pi + \theta}{n} + 1.$$

Here λ is to receive the n values, 1, 2, 3, ... n.

17. We shall now treat of the inverse problem, namely, that of finding the polynomial, when the roots of the equation, formed by equating that polynomial to zero, are known.

Thus, to find the polynomial of the n^{th} degree, which shall become zero for the n values of x,

$$a_1, a_2, a_3 \ldots\ldots a_n,$$

we have the factors of the polynomial

$$x - a_1, x - a_2, x - a_3, \ldots\ldots x - a_n;$$

and observing that a constant factor will not affect the roots of any equation, we conclude that the required polynomial is

$$f(x) = C(x - a_1)(x - a_2) \ldots (x - a_n),$$

denoting by C any constant quantity, which is indeterminate, until some other condition is proposed for $f(x)$ to satisfy, and for particularizing C.

Thus, if any value of $f(x)$ be known for a value of x not causing $f(x)$ to vanish, we can determine C in terms of the known value.

Let $f(a_0)$ be supposed known,

$$f(a_0) = \mathrm{C}(a_0 - a_1)(a_0 - a_2) \ldots\ldots (a_0 - a_n);$$

and by dividing the former equation by this, we obtain

$$\frac{f(x)}{f(a_0)} = \frac{x - a_1}{a_0 - a_1} \cdot \frac{x - a_2}{a_0 - a_2} \ldots\ldots \frac{x - a_n}{a_0 - a_n}.$$

In the particular case in which it is required to find a polynomial $f(x)$, which shall be zero for the values of x,

$$a_1, a_2, a_3, \ldots\ldots\ldots\ldots a_n,$$

and shall become $= 1$, when $x = a_0$, or $f(a_0) = 1$, the required polynomial is

$$f(x) = \frac{x - a_1}{a_0 - a_1} \cdot \frac{x - a_2}{a_0 - a_2} \ldots\ldots\ldots \frac{x - a_n}{a_0 - a_n}.$$

18. There is another problem of more generality, which can be reduced to the preceding. It is to find a polynomial of n dimensions, when the $n + 1$ values of the polynomial due to $n + 1$ values of the variable are known. That is, to find $f(x)$, having given the $n + 1$ values

$$f(a_0),\ f(a_1),\ f(a_2), \ldots\ldots f(a_n),$$

which are due to the values of x,

$$a_0, a_1, a_2, \ldots\ldots\ldots a_n.$$

Here we must assume

$$f(x) = \phi_0(x) f(a_0) + \phi_1(x) f(a_1) + \ldots\ldots + \phi_n(x) f(a_n),$$

$$\text{where } \phi_0(x), \phi_1(x), \ldots\ldots \phi_n(x),$$

are all polynomials of the n^{th} degree; and, consequently, $f(x)$ is of the same degree.

Now it is manifest that we shall have $\phi_0(x) =$ zero for the values

$$x = a_1, x = a_2, \ldots\ldots x = a_n,$$

but that $\phi_0(a_0)$ must $= 1$.

Hence $\phi_0(x) = \frac{x-a_1}{a_0-a_1}\cdot\frac{x-a_2}{a_0-a_2}\cdot\ldots\ldots\ldots\frac{x-a_n}{a_0-a_n}$.

And similar reasoning will hold for the determination of

$$\phi_1(x),\ \phi_2(x),\ \ldots\ldots\ldots\ \phi_n(x).$$

Hence we obtain, finally,

$$\left.\begin{aligned} f(x) = f(a_0)&\cdot\frac{x-a_1}{a_0-a_1}\cdot\frac{x-a_2}{a_0-a_2}\cdot\ldots\ldots\frac{x-a_n}{a_0-a_n} \\ +f(a_1)&\cdot\frac{x-a_0}{a_1-a_0}\cdot\frac{x-a_2}{a_1-a_2}\cdot\ldots\ldots\frac{x-a_n}{a_1-a_n} \\ +&\ \ldots\ldots\ldots\ldots\ldots \\ +f(a_n)&\cdot\frac{x-a_0}{a_n-a_0}\cdot\frac{x-a_1}{a_n-a_1}\cdot\ldots\ldots\frac{x-a_{n-1}}{a_n-a_{n-1}} \end{aligned}\right\}.$$

We may here remark, that if a be any other constant, then the polynomial $f(x) - f(a)$ will take the $n+1$ values

$$f(a_0) - f(a),\ f(a_1) - f(a),\ \ldots\ldots f(a_n) - f(a).$$

Hence we shall have the equation

$$\left.\begin{aligned} f(x)-f(a) = \{f(a_0)-f(a)\}&\cdot\frac{x-a_1}{a_0-a_1}\cdot\frac{x-a_2}{a_0-a_2}\cdot\ldots\ldots\frac{x-a_n}{a_0-a_n} \\ +\ \{f(a_1)-f(a)\}&\cdot\frac{x-a_0}{a_1-a_0}\cdot\frac{x-a_2}{a_1-a_2}\cdot\ldots\ldots\frac{x-a_n}{a_1-a_n} \\ +&\ \ldots\ldots\ldots\ldots\ldots\ldots \\ +\ \{f(a_n)-f(a)\}&\cdot\frac{x-a_0}{a_n-a_0}\cdot\frac{x-a_1}{a_n-a_1}\cdot\ldots\ldots\frac{x-a_{n-1}}{a_n-a_{n-1}} \end{aligned}\right\}$$

And as this holds for all values of a, we can profit by the introduction of this arbitrary constant to expel one of the terms on the second side; which will be effected by taking $a =$ one of the $n+1$ values of x,

$$a_0,\ a_1,\ a_2,\ \ldots\ldots,\ a_n.$$

As an example, take $n = 1$, and put $a = a_0$; we shall then have

$$f(x) - f(a_0) = \{f(a_1) - f(a_0)\} \frac{x - a_0}{a_1 - a_0};$$

which is the equation for a straight line passing through two given points, supposing x to denote the abscissa, and $f(x)$ the corresponding ordinate.

The general problem is one of considerable utility in interpolating the series

$$f(a_0),\ f(a_1),\ f(a_2), \ldots \ldots f(a_n),$$

corresponding to the values

$$a_0,\ a_1,\ a_2, \ldots \ldots \ldots a_n,$$

so as to obtain the term $f(x)$ corresponding to any intermediate value x.

CHAP. II.

ON THE TRANSFORMATION OF EQUATIONS; OR THE DETERMINATION OF EQUATIONS BY MEANS OF THE RELATIONS EXISTING BETWEEN THEIR ROOTS AND THE ROOTS OF GIVEN EQUATIONS.

19. The general problem is, having given the equation $f(x) = 0$, whose roots are $a_1, a_2, a_3, \ldots\ldots a_n$, to find the equation $F(y) = 0$, whose roots are to be all the combinations of the roots $a_1, a_2, a_3 \ldots a_n$, of which the general type is $\phi(a_1, a_2, \ldots\ldots a_r)$.

Here one value of y will be

$$y = \phi(a_1, a_2, \ldots\ldots a_r),$$

and since $a_1, a_2, \ldots\ldots a_r$, belong to the n roots of the equation $f(x) = 0$, we shall have the r equations of condition

$$\left.\begin{array}{c} f(a_1) = 0 \\ f(a_2) = 0 \\ \cdot\ \cdot\ \cdot\ \cdot \\ f(a_r) = 0 \end{array}\right\},$$

from which, and the equation

$$y = \phi(a_1, a_2, \ldots\ldots a_r),$$

we shall be enabled to eliminate the quantities $a_1, a_2, \ldots\ldots a_r$;

and the resulting equation $\Phi(y) = 0$ will have for its roots all the combinations similar to $\phi(a_1, a_2, \ldots\ldots a_r)$. This is evident from the consideration that the same resulting equation $\Phi(y) = 0$ would have been obtained by the assumption of any other combination,

$$\text{as } y = \phi(a_2, a_3, \ldots\ldots a_{r+1}).$$

But it must be remarked, that the final equation so found will also contain the roots

$$\phi(a_1, a_1, \ldots\ldots a_1),$$
$$\phi(a_2, a_2, \ldots\ldots a_2),$$
$$\ldots\ldots\ldots$$
$$\text{and } \phi(a_1, a_2, a_2, \ldots\ldots a_2),$$
$$\phi(a_2, a_1, a_1, \ldots\ldots a_1),$$
$$\ldots\ldots\ldots$$

with several other classes, all of which are equally foreign to the problem, if combinations without repetitions be required.

If, however, $r = 1$, or the values of y are dependent upon the values of x singly, there will not be any such difficulty. We then have

$$f(x) = 0,$$
$$y = \phi(x);$$

whence, by elimination, we have the correct final equation

$$F(y) = 0.$$

It will be seen in the sequel, that, in particular cases of r greater than 1, there will always be found a method of excluding the obnoxious roots from $\Phi(y) = 0$, and reducing it to the correct equation $F(y) = 0$; and this will be effected by some artifice peculiarly adapted to each case.

20. There is also another method of finding the equation $F(y) = 0$; for we can find its degree by means of the number of the combinations of the form

$$\phi(a_1, a_2, \ldots\ldots a_r);$$

and assuming indeterminate coefficients, we can write the equation $F(y) = 0$.

These coefficients will now be symmetrical functions of all the roots, and can therefore be estimated in terms of the coefficients of $f(x)$.

21. The following are examples of the case, where y depends on one root only of the equation $f(x) = 0$.

Ex. 1. To form the equation whose roots shall differ from those of $f(x) = 0$ in sign only.

Assume for $F(y) = 0$ the equation

$$y^n + q_1 y^{n-1} + q_2 y^{n-2} + \ldots\ldots + q_n = 0,$$

since $F(y)$ is evidently of the same degree as $f(x)$, that is, of the n^{th} degree.

Then, if $a, b, c, \ldots\ldots l$, are the roots of $f(x) = 0$, we have

$$\begin{aligned} q_1 &= -\Sigma(-a) = \Sigma(a) = -p_1, \\ q_2 &= \quad \Sigma(ab) \quad = p_2, \\ q_3 &= \quad \Sigma(abc) \quad = -p_3, \\ &\ldots\ldots\ldots\ldots \\ q_n &= (-1)^n p_n. \end{aligned}$$

And the required equation is

$$y^n - p_1 y^{n-1} + p_2 y^{n-2} - \ldots\ldots + (-1)^n p_n = 0.$$

The rule in this case is to change the signs of the alternate terms of the equation, beginning with the second.

Thus, the roots of the equation,

$$x^4 - 27x^2 + 14x + 120 = 0,$$

$$\text{or } x^4 + 0 \,.\, x^3 - 27x^2 + 14x + 120 = 0,$$

are 4, 3, -2, and -5.

But when the signs have been changed according to the above rule, as

$$x^4 - 0 \,.\, x^3 - 27x^2 - 14x + 120 = 0,$$

$$\text{or } x^4 - 27x^2 - 14x + 120 = 0,$$

the roots will have been changed to

$$-4, -3, 2, \text{ and } 5.$$

This transformation can also be effected by eliminating x, thus,

$$f(x) = 0,$$

$$y = -x,$$

and the resulting equation is

$$f(-y) = 0.$$

Ex. 2. To increase or diminish the roots of an equation by a given quantity.

$$\text{Put } y = x - \delta,$$

$$x = \delta + y:$$

then the roots will be diminished by δ, if δ be positive; or will be increased by $-\delta$, if δ be negative.

Hence, by substitution in $f(x) = 0$, we obtain the resulting equation in y,

$$f(\delta) + yf'(\delta) + \frac{y^2}{1 \,.\, 2}f''(\delta) + \cdot \;.\;.\;.\; + y^n = 0.$$

Thus, to diminish by 3 the roots

$$5,\ 2,\ -3,\ -4$$

of the equation before mentioned,

$$x^4 - 27x^2 - 14x + 120 = 0.$$

$$\begin{aligned} \text{Here } f(\delta) &= \delta^4 - 27\delta^2 - 14\delta + 120 &&= -84, \\ f'(\delta) &= 4\delta^3 - 54\delta - 14 &&= -68, \\ f''(\delta) &= 12\delta^2 - 54 &&= \quad 54, \\ f'''(\delta) &= 24\delta &&= \quad 72, \end{aligned}$$

and substituting these values, we obtain the final equation in y,

$$y^4 + 12y^3 + 27y^2 - 68y - 84 = 0;$$

whose four roots are

$$2,\ -1,\ -6, \text{ and } -7.$$

By means of this transformation we can take away any term of an equation. Thus, if we wish that the new equation shall want its second term, we should assume that the coefficient of y^{n-1} = zero, or

$$p_1 + n\delta = 0,$$

$$\delta = \frac{-p_1}{n};$$

and we must increase the roots of the equation, each by $\frac{p_1}{n}$.

Thus, for the equation

$$x^3 - 6x^2 + 5 = 0,$$

$$\text{assume } x = y + \delta = y + 2,$$

and the transformed equation is

$$y^3 - 12y - 11 = 0,$$

which wants its second term.

Ex. 3. To increase or diminish all the roots in a given ratio.

Assume the equation, whose roots are

$$ma, mb, \ldots \ldots ml,$$

and which is of the n^{th} degree, to be

$$y^n + q_1 y^{n-1} + q_2 y^{n-2} + \ldots \ldots + q_n = 0.$$

$$\text{Then } q_1 = -\Sigma(ma) = -m\,\Sigma(a) = mp_1,$$

$$q_2 = \Sigma(m^2ab) = m^2\Sigma(ab) = m^2p_2,$$

$$\ldots \ldots \ldots \ldots \ldots$$

$$q_n = m^np_n.$$

And the required equation is

$$y^n + mp_1 y^{n-1} + m^2p_2 y^{n-2} + \ldots \ldots + m^np_n = 0.$$

One chief advantage of the preceding transformation is, that it enables us to render the coefficients integers, when they are vulgar fractions, or can be reduced to that form; that is, when they are commensurable.

Thus, to render the coefficients of the equation

$$x^3 + 2x^2 + \frac{1}{4}x + \frac{1}{9} = 0,$$

integers, we must assume

$$m = 2 \cdot 3 = 6,$$

and the resulting equation will be

$$y^3 + 12y^2 + 9y + 24 = 0.$$

This transformation might also be effected by elimination, as follows:

$$f(x) = 0,$$

$$y = mx.$$

$$\text{Hence } x = \frac{y}{m},$$

$$\text{and } f\left(\frac{y}{m}\right) = 0.$$

Ex. 4. To find the equation whose roots are the squares of the roots of the given equation.

Here we have to eliminate x from the equations

$$f(x) = 0,$$
$$y = x^2;$$

and the result is to be an equation in y, not containing fractional powers of y; which it must do if the elimination were performed by substituting for x its values $\pm \sqrt{y}$.

Instead then of this elimination, we shall form a polynomial of the $2n^{th}$ degree, and containing only even powers of x; the form of the polynomial being the product of the factors

$$(x^2 - a^2)(x^2 - b^2)(x^2 - c^2) \ldots . (x^2 - l^2);$$

and then write y for x^2, and equate to zero.

Now, observing that we have

$$f(x) = (x - a)(x - b) \ldots .. (x - l),$$
$$f(-x) = (-1)^n . (x + a)(x + b) \ldots .. (x + l);$$

and we shall obtain the required polynomial

$$\mathrm{F}(x^2) = (x^2 - a^2)(x^2 - b^2) \ldots . (x^2 - l^2),$$

by forming the product

$$(-1)^n . f(x) . f(-x).$$

And if we put y in the place of x^2 in $\mathrm{F}(x^2)$, and equate to zero, we shall have the required equation,

$$\mathrm{F}(y) = 0;$$

which is evidently the same equation with

$$f(\sqrt{y}) f(-\sqrt{y}) = 0,$$

omitting the factor $(-1)^n = \pm 1$.

Thus, in the example

$$x^3 - 6x^2 + 8x - 10 = 0,$$

we shall form the product of

$$y^{\frac{3}{2}} - 6y + 8y^{\frac{1}{2}} - 10,$$

$$y^{\frac{3}{2}} + 6y + 8y^{\frac{1}{2}} + 10,$$

and equate to zero the product; which process gives the equation

$$y^3 - 20y^2 - 56y - 100 = 0.$$

The proof of the advantage of adopting the method here given is, that an analogous method will apply for any other powers of the roots, provided the index be an integer.

Ex. 5. To find the equation whose roots are the r^{th} powers of the roots of the given equation.

We have to form and equate to zero the polynomial of the n^{th} degree,

$$(y - a^r)(y - b^r) \ldots\ldots (y - l^r);$$

or, we may proceed to form the function

$$\mathrm{F}(x^r) = (x^r - a^r)(x^r - b^r) \ldots\ldots (x^r - l^r),$$

and then write y for x^r, and equate to zero; the only datum for the formation of this function being

$$f(x) = (x - a)(x - b) \ldots\ldots (x - l).$$

Now if we designate the roots of

$$z^r - 1 = 0,$$

by $\lambda_1, \lambda_2, \ldots\ldots \lambda_r$, we shall have

$$(z^r - 1) = (z - \lambda_1)(z - \lambda_2) \ldots\ldots (z - \lambda_r);$$

whence $\left(\frac{a^r}{x^r} - 1\right) = \left(\frac{a}{x} - \lambda_1\right)\left(\frac{a}{x} - \lambda_2\right) \ldots\ldots \left(\frac{a}{x} - \lambda_r\right)$,

$$\text{or } a^r - x^r = (a - \lambda_1 x)(a - \lambda_2 x) \ldots\ldots (a - \lambda_r x);$$

$$\text{similarly, } b^r - x^r = (b - \lambda_1 x)(b - \lambda_2 x) \ldots\ldots (b - \lambda_r x),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$l^r - x^r = (l - \lambda_1 x)(l - \lambda_2 x) \ldots\ldots (l - \lambda_r x).$$

And forming the product, we have,

$$\text{calling } \mathrm{F}(y) = (y - a^r)(y - b^r) \ldots\ldots (y - l^r),$$

$$(-1)^r . \mathrm{F}(x^r) = (-1)^{nr} f(\lambda_1 x) f(\lambda_2 x) \ldots\ldots f(\lambda_r x),$$

and the equation $\mathrm{F}(y) = 0$ is the same as the equation

$$f(\lambda_1 y^{\frac{1}{r}}) . f(\lambda_2 y^{\frac{1}{r}}) \ldots\ldots f(\lambda_r y^{\frac{1}{r}}) = 0,$$

in which only integer powers of y will appear.

Ex. 6. To find the equation whose roots are the reciprocals of the roots of the given equation.

$$\text{Here} \qquad f(x) = 0,$$

$$y = \frac{1}{x};$$

from which equations we find

$$x = \frac{1}{y},$$

$$f\left(\frac{1}{y}\right) = 0;$$

$$\text{or } \frac{1}{y^n} + \frac{p_1}{y^{n-1}} + \frac{p_2}{y^{n-2}} + \ldots\ldots + p_n = 0,$$

$$\text{or } y^n + \frac{p_{n-1}}{p_n} y^{n-1} + \ldots\ldots + \frac{p_1}{p_n} y + \frac{1}{p_n} = 0;$$

and the rule is to reverse the order of the coefficients of the equation; and the equation may then be reduced to the usual form, by dividing by the coefficient of the highest power of y, if necessary.

We may remark here, that if the transformed equation be the same as the original one; that is, if the coefficients are the same when inverted, or differ only in the sign of the whole series; the roots of the transformed equation are the same as those of the given equation. Hence the roots of the given equation must consist of either $+1$, or -1, repeated any number of times, (these being the only quantities which are identical with their reciprocals,) and of pairs of roots of the form $a, \frac{1}{a}, b, \frac{1}{b}, \ldots\ldots\ldots$ every pair of which will give a similar pair in the transformed equation. Such equations are called Reciprocal Equations, and their solution is very much simplified by this property of their roots.

22. We shall now proceed to give examples of the case where the roots of the transformed equation are connected with the roots of the given equation, in such a manner that each one of the former depends upon more than one of the latter.

Ex. 1. To find the equation to the differences of the roots.

$$\text{Here } y = a - b,$$
$$\text{and } f(a) = 0,$$
$$f(b) = 0,$$

are the equations which, by the elimination of a, b, will give the final equation $F(y) = 0$.

Eliminating a, we have

$$f(b + y) = 0,$$
$$f(b) = 0,$$

from which b is to be eliminated.

If we had eliminated b first, we should have obtained the equations

$$f(a) = 0,$$
$$f(a - y) = 0,$$

from which a is to be eliminated.

Hence we see that the final equation

$$\mathrm{F}(y) = 0,$$

is to be obtained from the elimination of x between the equations

$$\left.\begin{aligned} f(x) &= 0 \\ f(x + y) &= 0 \end{aligned}\right\},$$

or, between the equations

$$\left.\begin{aligned} f(x) &= 0 \\ f(x - y) &= 0 \end{aligned}\right\}.$$

This shows that $\mathrm{F}(y)$ will not alter its form when y is changed into $-y$, and therefore $\mathrm{F}(y)$ contains only even powers of y; a fact which we may point out a priori. For if $\alpha, \beta, \gamma, \ldots\ldots$ be roots of the equation $\mathrm{F}(y) = 0$, and we suppose that

$$\alpha = a - b,$$
$$\beta = b - c,$$
$$\gamma = c - d.$$
$$\ldots\ldots\ldots$$

$$\text{then } -\alpha = b - a,$$
$$-\beta = c - b,$$
$$-\gamma = d - c,$$
$$\ldots\ldots\ldots$$

which are still roots of the equation;

consequently, we shall have

$$\begin{aligned} F(y) &= (y-a)(y+a)(y-\beta)(y+\beta), \dots \\ &= (y^2-a^2)(y^2-\beta^2) \dots \\ &= \Phi(y^2), \text{ suppose.} \end{aligned}$$

If we now put $y^2 = z$, we have

$$\Phi(z) = (z-a^2)(z-\beta^2) \dots$$

which is the equation to the squares of the differences of the roots; and therefore is of the $n \cdot \frac{n-1}{2}^{th}$ degree.

We have yet to perform the elimination of x between

$$\left.\begin{array}{r} f(x) = 0 \\ f(x+y) = 0 \end{array}\right\}.$$

If the elimination be performed directly, the resulting equation $F(y) = 0$ will contain n roots $=$ zero, and we shall have to suppress the factor y^n. These roots correspond to the differences

$$\begin{array}{c} a - a, \\ b - b, \\ c - c, \\ \dots \end{array}$$

or the combinations with repetitions.

These roots may, however, be suppressed in the commencement of the calculation, since we may take, instead of the two equations above, another pair of equations formed from them;

$$\left.\begin{array}{r} f(x) = 0 \\ f(x+y) - f(x) = 0 \end{array}\right\},$$

Now the second equation is satisfied by $y = 0$, and, therefore, is of the form

$$y \,.\, \psi(x, y) = 0\,;$$

and, by suppressing this factor y, we have the correct result, by obtaining the final equation in y, from

$$\left.\begin{aligned} f(x) &= 0 \\ \psi(x, y) &= 0 \end{aligned}\right\}.$$

In fact we have, by Taylor's series, after expanding the second equation, and suppressing the factor y,

$$\psi(x, y) = f'(x) + \frac{y}{1\,.\,2} f''(x) + \;.\;.\;.\;.\;.\;. + y^{n-1},$$

from which equation, and the equation

$$f(x) = 0,$$

we obtain the required equation

$$\mathrm{F}(y) = 0.$$

As this transformation, or rather the second transformation, by putting $y^2 = z$, and forming the equation whose roots are the squares of the differences of the roots of the original equation, is one of considerable importance; and because the process of elimination, though more beautiful in theory, is less commodious in practice; we shall proceed to form the equation in z, whose degree is known, by calculating its coefficients. This is the method adopted by Lagrange.

Assume the equation to be

$$z^m + q_1 z^{m-1} + \;.\;.\;.\;. + q_m = 0,$$

$$\text{where } m = \frac{n(n-1)}{2}.$$

Let $\alpha, \beta, \gamma, \;.\;.\;.\;.\;.\; \lambda$, be its roots; which are the squares of the differences of the roots $a, b, c, \;.\;.\;.\;.\; l$.

Now we have, for any integer k,

$$(x - a)^{2k} = x^{2k} - 2kx^{2k-1}a + \ldots . + a^{2k},$$
$$(x - b)^{2k} = x^{2k} - 2kx^{2k-1}b + \ldots . + b^{2k},$$
$$\ldots \ldots \ldots \ldots \ldots ,$$

whence, by addition, we have

$$\Sigma \{(x - a)^{2k}\} = nx^{2k} - 2k\Sigma(a)x^{2k-1} \ldots . + n\Sigma(a^{2k}).$$

And from this equation, writing for x successively $a, b, c, \ldots \ldots l$, we have, after adding all such equations, and observing that every member of the first side, as $(b-a)^{2k}$, will be repeated under the form $(a - b)^{2k}$,

$$2\Sigma(a^k) = n\Sigma(a^{2k}) - 2k\Sigma(a)\Sigma(a^{2k-1}) \ldots . + n\Sigma(a^{2k});$$

and collecting the terms of the second side in pairs, we obtain

$$\Sigma(a^k) = n\Sigma(a^{2k}) - 2k\Sigma(a)\Sigma(a^{2k-1}) \ldots . \left. - (-2)^{k-1}\frac{1.3.5 \ldots . (2k-1)}{1.2.3 \ldots . \quad k}\left\{\Sigma(a^k)\right\}^2 \right\},$$

observing that the middle term is left single.

Now we can find the sums of the powers of the roots,

$$\Sigma(a)\ \Sigma(a^2) \ldots \ldots \Sigma(a^{2k})$$

in terms of the coefficients

$$p_1, p_2, p_3, \ldots \ldots p_n;$$

hence $\Sigma(a^k)$ can be found in terms of the same quantities. But $\Sigma(a^k)$ can also be determined in terms of the new coefficients

$$q_1, q_2, q_3, \ldots \ldots \ldots q_m;$$

hence, by equating the two values, we have an equation connecting the new coefficients with the old ones; and, by putting $k = 1, 2, 3, \ldots m$, successively, we shall successively determine the coefficients of the required equation.

Even this process, however, becomes too complicated for practice when n is > 4.

Ex. 2. To find the equation whose roots shall be the sums of every two roots of the given equation.

$$\text{Here } y = a + b,$$
$$f(a) = 0,$$
$$f(b) = 0,$$

are the equations, from which we are to eliminate a and b, in order to obtain the required equation

$$\mathrm{F}(y) = 0.$$

First, let b be eliminated, and we have

$$\left.\begin{array}{r} f(a) = 0, \\ f(y-a) = 0 \end{array}\right\}.$$

Secondly, we have to eliminate a from these two equations; but if that operation be performed immediately, the result would contain the roots $2a, 2b, \ldots . 2l$, which are foreign to the question.

To avoid this, we must form the equations

$$\left.\begin{array}{r} f(a) = 0, \\ f(y-a) - f(a) = 0 \end{array}\right\},$$

the second of which is satisfied by $y = 2a$, and therefore contains the factor $y - 2a$, which is foreign to the question, and must be expunged.

In fact, expanding the second equation by the series of Taylor, we have

$$f(a + \overline{y - 2a}) - f(a) = 0,$$

$$\text{or } f'(a) + \frac{y-2a}{1.2} f''(a) + \ldots\ldots + (y-2a)^{n-1} = 0,$$

observing that we must suppress the factor $y - 2a$; and we shall now obtain the correct equation by eliminating a from this equation, and $f(a) = 0$. We shall then have to take the square root of $F(y) = 0$, since each root enters twice.

Ex. 3. To find the equation of which the roots shall express the ratios of every two roots of the given equation.

$$\left.\begin{aligned} \text{Here } y &= \frac{a}{b}, \\ f(a) &= 0, \\ f(b) &= 0. \end{aligned}\right\}$$

Proceeding as usual, we have

$$\left.\begin{aligned} f(a) &= 0 \\ f\left(\frac{a}{y}\right) &= 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} \text{or } f(b) &= 0 \\ f(by) &= 0 \end{aligned}\right\};$$

so that the final equation in y results equally from eliminating x between

$$\left.\begin{aligned} f(x) &= 0 \\ f\left(\frac{x}{y}\right) &= 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} \text{or between } f(x) &= 0 \\ f(xy) &= 0 \end{aligned}\right\}.$$

Hence we may remark, that the required equation

$$F(y) = 0$$

will not change, when we write $\frac{1}{y}$ for y; in other words, it is a reciprocal equation. This is also evident, from the consideration that every root $\frac{a}{b}$ has for a companion the inverse $\frac{b}{a}$.

The final equation resulting from the elimination will contain n roots equal to unity, being the ratios

$$\frac{a}{a}, \frac{b}{b}, \ldots \ldots \frac{l}{l};$$

to avoid this, take the equations

$$\left.\begin{array}{r} f(x) = 0 \\ f(xy) - f(x) = 0 \end{array}\right\},$$

and for the second equation, write

$$f\{x + x(y-1)\} - f(x) = 0,$$

or expanding and suppressing the factor $(y-1)\,x$,

$$f'(x) + \frac{x(y-1)}{2} f''(x) \ldots \ldots + x^{n-1}(y-1)^{n-1} = 0.$$

We give the following examples for practice:

Ex. 4. To find the equation whose roots shall be the products of every two of the roots of the given equation.

Ex. 5. Form the equation whose roots shall be all combinations of the form $p(a+b) + q \,.\, ab$, a and b being two roots of $f(x) = 0$, and p, q, constant quantities.

23. In particular cases of the general equation, particular artifices are preferable; thus, for the cubic

$$x^3 + px^2 + qx + r = 0,$$

whose roots are a, b, c, we can find the equation whose roots are

$$\frac{b+c}{a}, \frac{a+c}{b}, \frac{a+b}{c},$$

$$\text{or } \frac{a+b+c}{a} - 1, \frac{a+b+c}{b} - 1, \frac{a+b+c}{c} - 1,$$

$$\text{by putting } y = -\frac{p}{x} - 1,$$

$$\text{or } x = \frac{-p}{1+y},$$

and substituting in the given equation.

24. It will not be improper, in this place, to treat of a class of problems which can be most easily solved by the method of transformation; they are those in which there is a given relation between some of the roots of the given equation, and those roots are required. The general problem is, having given that one of the roots g depends on a certain number of the others, as for instance upon three by the relation

$$g = \phi(a, b, c),$$

required to find this root g of the equation.

Form the equation whose roots are all combinations similar to $\phi(a, b, c)$.

Let this be $F(x) = 0$; then one root of $F(x) = 0$ is g, which is also a root of $f(x) = 0$; and, consequently, $F(x)$, $f(x)$, have a common factor $x - g$, which can be found by the usual method.

If any other root h is in similar circumstances, they will have a common factor $(x - g)(x - h)$.

Ex. 1. Suppose that we knew that there were two roots of $f(x) = 0$, of the form $\pm a$, to find these roots.

Since $\pm a$ are two roots of

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_n = 0;$$

it follows that, on changing the signs of the roots of the equation, we shall have $\mp a$ still roots of

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots\ldots + p_n = 0.$$

We suppose n to be an even number; and it will be seen that a similar proof will apply when n is odd.

Now since $\pm a$ satisfy the two preceding equations, they will also satisfy their sum and difference; or, omitting the factor x of the difference, they will still be roots of the equations

$$\left.\begin{aligned} x^n + p_2x^{n-2} + \ldots\ldots + p_n &= 0 \\ p_1x^{n-2} + p_3x^{n-4} + \ldots\ldots + p_{n-1} &= 0 \end{aligned}\right\}.$$

Hence the two polynomials forming these equations must have a common factor $x^2 - a^2$ at least. And if there be any other pair of roots $\pm b$ in similar circumstances, $x^2 - b^2$ is also a factor.

Thus, for the equation

$$x^3 - 2x^2 - x + 2 = 0,$$

which has two such roots, we form the equation

$$x^3 + 2x^2 - x - 2 = 0;$$

and forming the sum and difference, after suppressing the factor x,

$$\left.\begin{aligned} x^2 - 1 &= 0 \\ x^2 - 1 &= 0 \end{aligned}\right\},$$

and the common factor is evidently

$$x^2 - 1;$$

so that these two roots are ± 1.

Also, since $\dfrac{x^3 - 2x^2 - x + 2}{x^2 - 1} = x - 2$, the remaining root is 2.

Ex. 2. The difference of two roots of the equation $f(x) = 0$ is δ, to find these roots.

Let the less of the two roots be a, and therefore the other be $a + \delta$. Increase the roots of the equation by δ, these two roots become $a + \delta$, $a + 2\delta$; so that $a + \delta$ is still a root of the transformed equation. In other words, $x - a - \delta$ is a factor of the two polynomials

$$f(x),$$

$$\text{and } f(x - \delta);$$

observing that the roots of the equation

$$f(x - \delta) = 0,$$

are those of $f(x) = 0$ increased by δ.

Thus, for example, knowing that the difference of two roots is 2 in the equation

$$x^3 - 2x^2 - x + 2 = 0,$$

and searching the common factor as above, we find

$$a + \delta = 1, \text{ and } a = 1 - \delta = -1.$$

Ex. 3. Having given the sum of two roots of $f(x) = 0$, to find the roots.

Ex. 4. Given the product of two roots, or their quotient, to find these roots.

Ex. 5. The sides and hypothenuse of a right-angled triangle are roots of $f(x) = 0$; find the hypothenuse.

These examples are given for the sake of rendering the subject familiar, by exciting individual practice in problems of this description. We shall add two other examples.

Ex. 6. The roots of $f(x) = 0$ are all in arithmetical progression; find them.

Ex. 7. The roots of $f(x) = 0$ are all in geometrical progression; find them.

These two examples are best solved by means of the coefficients.

25. There is, however, one class of equations whose roots are connected by certain relations, to which the ordinary process of transformation will not be applicable. For the method becomes illusory, when the transformation reproduces the original equation.

The most important case is that of reciprocal, or recurring equations. It was stated, that the roots of these equations consisted of single roots, either $+1$, or -1; and of pairs of the form $a, \frac{1}{a}, b, \frac{1}{b} \ldots\ldots$ We shall suppose that all the roots ± 1 have been expelled, which can always be done by trial, so that the equation is no longer satisfied by ± 1. In this state, which is the simplest of all recurring equations, the polynomial is of even dimensions, and its last term will be

$$= (-a)\left(-\frac{1}{a}\right)(-b)\left(-\frac{1}{b}\right) \ldots\ldots = 1.$$

Hence we shall write the equation

$$x^{2m} + p_1 x^{2m-1} + \ldots\ldots\ldots + p_1 x + 1 = 0.$$

And the polynomial

$$x^{2m} + p_1 x^{2m-1} + \ldots\ldots\ldots + p_1 x + 1$$

$$= (x-a)\left(x-\frac{1}{a}\right)(x-b)\left(x-\frac{1}{b}\right) \ldots\ldots$$

$$= \left\{x^2 - \left(a+\frac{1}{a}\right)x + 1\right\}\left\{x^2 - \left(b+\frac{1}{b}\right)x + 1\right\} \ldots\ldots$$

and, consequently, dividing by x^m, we have, after collecting the terms of the polynomial in pairs equidistant from the ends,

$$\left(x^m + \frac{1}{x^m}\right) + p_1\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + \ldots\ldots + p_m$$

$$= \left\{\left(x + \frac{1}{x}\right) - \left(a + \frac{1}{a}\right)\right\}\left\{\left(x + \frac{1}{x}\right) - \left(b + \frac{1}{b}\right)\right\}\ldots\ldots$$

$$= (\xi - \alpha)(\xi - \beta)\ldots\ldots$$

$$\left.\begin{array}{r} \text{putting } x + \frac{1}{x} = \xi \\ a + \frac{1}{a} = \alpha \\ b + \frac{1}{b} = \beta \\ \ldots\ldots\ldots \end{array}\right\}.$$

It follows, therefore, that the polynomial

$$\left(x^m + \frac{1}{x^m}\right) + p_1\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + \ldots\ldots + p_m$$

can be put under the form

$$\xi^m + q_1\xi^{m-1} + \ldots\ldots + q_m;$$

and that on equating it to zero in this form, its roots, or the values of ξ, will be $\alpha, \beta, \gamma \ldots\ldots$; so that the original equation will then be reduced to the m quadratics

$$\left.\begin{array}{r} x^2 - \alpha x + 1 = 0 \\ x^2 - \beta x + 1 = 0 \\ x^2 - \gamma x + 1 = 0 \\ \ldots\ldots\ldots \end{array}\right\},$$

where $\alpha, \beta, \gamma \ldots\ldots$ depend on an equation of the m^{th} degree only, since they are the roots of

$$\xi^m + q_1\xi^{m-1} + \ldots\ldots + q_m = 0.$$

We have yet to perform the transformation of the polynomial

$$\left(x^m+\frac{1}{x^m}\right)+p_1\left(x^{m-1}+\frac{1}{x^{m-1}}\right)+\ .\ .\ .\ .+p_m$$

$$\text{into } \xi^m+q_1\xi^{m-1}+\ .\ .\ .\ .\ .+q_m,$$

$$\text{where } \xi=x+\frac{1}{x}.$$

Now we have identically

$$(1-xt)\left(1-\frac{t}{x}\right)=1-\xi t+t^2;$$

and taking the Naperian logarithms, expanding, and equating the coefficients of t^n in the equivalent series, we have

$$-\frac{1}{n}\left(x^n+\frac{1}{x^n}\right)=\text{coefficient of } t^n \text{ in the series}$$

$$-\frac{1}{n}(\xi t-t^2)^n-\frac{1}{n-1}(\xi t-t^2)^{n-1}-\ .\ .\ .\ .\ .$$

$$\text{or } x^n+\frac{1}{x^n}=\text{coefficient of } t^n \text{ in the series}$$

$$t^n.(\xi-t)^n+\frac{n}{n-1}t^{n-1}.(\xi-t)^{n-1}+\ .\ .\ .\ .$$

$$=\xi^n-n\,\xi^{n-2}+n\frac{n-3}{2}\xi^{n-4}-\ .\ .\ .\ .$$

$$+(-1)^s\frac{n.(n-s-1)(n-s-2)\ldots(n-2s+1)}{1.2.3\ .\ .\ .\ .\ .\ s}\xi^{n-2s}\ .\ .\ .\ .\ .$$

Hence, by the application of this theorem to the cases $n=m$, $n=m-1$, ... successively, every term of

$$\left(x^m+\frac{1}{x^m}\right)+p_1\left(x^{m-1}+\frac{1}{x^{m-1}}\right)+\ .\ .\ .\ .+p_m$$

will be expanded in powers of ξ; and the whole will thus be reduced to

$$\xi^m+q_1\xi^{m-1}+\ .\ .\ .\ .\ .+q_m.$$

By way of exemplifying the process, we shall apply it to the equation

$$x^{10} - 3x^8 + 5x^6 - 5x^4 + 3x^2 - 1 = 0.$$

Here the first remark to be made is, that there are no odd powers, and therefore the roots are of the form $\pm \sqrt{y}$, where the values of y are given by the equation

$$y^5 - 3y^4 + 5y^3 - 5y^2 + 3y - 1 = 0.$$

Secondly, this equation is satisfied by $y = 1$, and hence the factor $y - 1$ is to be expunged before treating it as a reciprocal equation. This gives the equation

$$y^4 - 2y^3 + 3y^2 - 2y + 1 = 0,$$

which is no longer satisfied by ± 1.

Assume $\eta = y + \frac{1}{y}$; and since

$$\left(y^2 + \frac{1}{y^2}\right) - 2\left(y + \frac{1}{y}\right) + 3 = 0,$$

we shall have, by the expansion,

$$(\eta^2 - 2) - 2\eta + 3 = 0,$$

$$\eta^2 - 2\eta + 1 = 0,$$

and there are two roots $= 1$.

Hence the equation in y is reduced to two equal quadratics,

$$y^2 - y + 1 = 0;$$

whence we have, for the four values of y,

$$\text{two roots} = \frac{1 + \sqrt{-3}}{2} = \cos 60° + \sqrt{-1} \sin 60°,$$

$$\text{and two roots} = \frac{1 - \sqrt{-3}}{2} = \cos 60° - \sqrt{-1} \sin 60°.$$

Hence the ten values of x will be

$$\text{two roots, } \pm 1,$$

$$\text{two equal pairs, } \pm(\cos 30^\circ + \sqrt{-1}\,\sin 30^\circ),$$

$$\text{two equal pairs, } \pm(\cos 30^\circ - \sqrt{-1}\,\sin 30^\circ).$$

We may remark, that the general equation

$$x^n \pm 1 = 0$$

is of the same kind, and when n is not too large, may be solved in a similar manner.

But, in every case, we must recollect that it will be extremely advantageous to get rid of the roots ± 1, at the very commencement of the operation; for it is of the greatest importance that the equation in ξ should be of as low a degree as possible, on account of the troublesome expansion necessary in forming it.

CHAP. III.

ON THE THEORY OF THE LIMITS OF THE ROOTS, AS FAR AS IT WAS KNOWN PREVIOUS TO FOURIER.

26. It very early became an object of inquiry to assign the intervals in the series of all possible magnitudes from $-\infty$ up to $+\infty$, within which each of the roots of the equation ought to be sought. And although this question was never fully answered in a form adapted to practice, until the recent publication of Fourier's Treatise, yet various interesting as well as important theorems were the result of the researches of former analysts in this direction. The important determination of the number of impossible roots of the equation was also practically unaccomplished previous to Fourier's theory of the separation of the roots. But the investigation of a limit to the number of the positive and negative roots had been previously effected by the theorem, known under the name of the Rule of Signs, which was first given by Descartes. The properties of the limiting equation were also discovered, and the method of finding a superior and inferior limit. On this account, as well as for the sake of giving the method of Fourier in a connected form, we shall first demonstrate the principal theorems which had been brought to a complete state by previous analysts.

27. We shall commence with the rule of signs of Descartes, which is to be enunciated in the following manner:

There can be no more positive roots in any equation than there are changes of sign in the series of signs of the terms composing the equation; and no more negative roots than there are continuations of sign.

Let the signs of the equation $f(x) = 0$, that is, of the polynomial $f(x)$, be

$$+ - + + - - - + + - +,$$

and let a new real root be introduced into the equation.

First, let this be a positive root m; then $f(x)$ must be multiplied by the factor $x - m$; and setting down only the signs of the multiplication, we have

$$\begin{array}{cccccccccccc}
+ & - & + & + & - & - & - & + & + & - & + & \\
 & - & + & - & - & + & + & + & - & - & + & - \\
\hline
+ & - & + & \pm & - & \pm & \pm & + & \pm & - & + & - \\
\hline
\end{array}$$

Secondly, let the root be negative, as $-m$; then multiplying $f(x)$ by $x + m$, and retaining the signs only of the operation, we have

$$\begin{array}{cccccccccccc}
+ & - & + & + & - & - & - & + & + & - & + & \\
 & + & - & + & + & - & - & - & + & + & - & + \\
\hline
+ & \pm & \pm & + & \pm & - & - & \pm & + & \pm & \pm & + \\
\hline
\end{array}$$

Now it will easily be seen, that we have added at least one change of sign in the first case, and at least one continuation in the second, whatever be the signs of those terms where it is doubtful whether the sign is to be $+$ or $-$. So that every additional positive root introduces at least one additional change, and every additional negative root at least one additional continuation; and, consequently, there cannot be more positive roots than there are changes, or more negative roots than there are continuations of sign.

If all the roots are real, their number is equal to the number of changes together with the number of continuations. Hence it follows, that the number of positive roots is equal to the number of changes, and the number of negative roots equal to the number of continuations.

28. An even number of roots, or none, will lie between α and β, when the quantities $f(\alpha)$ and $f(\beta)$ have the same sign; and an odd number, when they have a different sign.

For if the roots of $f(x) = 0$ are

$$a,\ b,\ c, \ldots .\ \rho\,(\cos\theta \pm \sqrt{-1}\,\sin\theta) \ldots .$$

then we have the identity

$$f(x) = (x-a)\,(x-b) \ldots .\ \{(x-\rho\cos\theta)^2 + \rho^2\sin^2\theta\} \ldots .$$

and, consequently,

$$\frac{f(\alpha)}{f(\beta)} = \frac{\alpha-a}{\beta-a}\cdot\frac{\alpha-b}{\beta-b} \ldots . \frac{(\alpha-\rho\cos\theta)^2+\rho^2\sin^2\theta}{(\beta-\rho\cos\theta)^2+\rho^2\sin^2\theta} \ldots .$$

Now if no root lie between α and β, every factor of $\dfrac{f(\alpha)}{f(\beta)}$ will be positive; but every root which does lie between those limits will give one corresponding negative factor; since $\dfrac{\alpha-a}{\beta-a}$ is negative only when a lies between α and β.

Hence, if m roots lie between α and β, the sign of $\dfrac{f(\alpha)}{f(\beta)}$ depends on $(-1)^m$; that is, $f(\alpha)$ and $f(\beta)$ will have the same sign, if $m = 0$, or if m be even; but they will have different signs, if m be odd.

29. To find a limit greater than the greatest root.

Let a be this greatest root, and suppose α to be greater than a; then, taking the same equation, we have

$$f(\alpha) = (\alpha - a)(\alpha - b) \ldots . \{(\alpha - \rho\cos\theta)^2 + \rho^2\sin^2\theta\} \ldots .$$

which is positive, since all its factors are so. And the result is still positive for quantities greater than α.

Hence, if we can find a quantity α, such that $x = \alpha$, and every greater value of x, shall render $f(x)$ positive, then α shall be a limit superior to any root.

But the following method of finding a superior limit is preferable to the preceding, and is better adapted to practice:

Decrease the roots of the equation

$$f(x) = 0,$$

each by the quantity λ; which is done by putting

$$y = x - \lambda,$$
$$\text{or } x = y + \lambda;$$

and the final equation in y is

$$f(\lambda + y) = 0,$$

$$\text{or } f(\lambda) + y\, f'(\lambda) + \frac{y^2}{1.2} f''(\lambda) + \ldots\ldots + y^n = 0.$$

Now, by taking λ large enough, the signs of $f(\lambda)$, $f'(\lambda)$, $f''(\lambda)$, ... can be rendered all positive; and when this is the case there is no positive value of y, as there is no change of sign.

Consequently, there can be no positive value of $x - \lambda$; or, in other words, λ is a superior limit to the roots of the equation $f(x) = 0$.

Hence, the rule is to find a value λ for x, such as shall render positive the series of polynomials

$$f(x), f'(x), f''(x), \ldots\ldots f^{n-1}(x);$$

and that value is a superior limit for the roots of the equation

$$f(x) = 0.$$

We may remark, that the same quantity λ is also a superior limit to the roots of any of the equations

$$\left.\begin{array}{r} f'(x) = 0, \\ f''(x) = 0, \\ \cdots\cdots \\ f^{n-1}(x) = 0 \end{array}\right\}.$$

30. To find an inferior limit to the roots of the same equation, we must change the signs of the roots, and find a superior limit to the new roots: this limit, with its sign changed, will be an inferior limit to the original roots.

We can also find a superior limit to the negative roots, and an inferior limit to the positive roots. For if we change the roots to their reciprocals, and find the superior and inferior limits of the new roots, then the reciprocals of these limits will be respectively the inferior limit to the positive roots, and the superior limit to the negative roots of the original equation.

31. It was implied in the statement of Art. 26, that the theoretical separation of the real roots, or the assignment of the interval containing each root, had been accomplished previous to the method of Fourier. This was done by Lagrange, in the following manner:

First, let the equation be cleared of equal roots. Secondly, form the equation of the squares of the differences of the roots, and find its inferior limit δ^2; the quantity δ will be inferior to any difference of the roots, taking always the less from the greater. Lastly, find the superior and inferior limits of the roots, l and l', and it is evident that any interval of the series

$$l', \; l' + \delta, \; l' + 2\delta \ldots\ldots, \text{ up to } l,$$

will contain no root, or one only; and thus, by trial, we can find the intervals of the real roots, and consequently their number.

The method of trying the intervals of the series

$$l', l' + \delta, l' + 2\delta, \ldots\ldots, \text{ up to } l,$$

is to substitute the terms of that series successively in $f(x)$, and to write the signs of the results in order. Every change of sign in that row of signs, points out the interval containing one of the roots; and the number of such changes of sign gives the number of real roots of the equation $f(x) = 0$.

Hence also, by subtracting this number from n, we obtain the number of impossible roots.

But although perfect in theory, yet in practice this method is useless, except for equations below the fifth degree; since the equation of the squares of the differences cannot be formed. Besides, the process has the disadvantage of trying a large number of intervals, many of which the method of Fourier will at once exclude.

32. The real roots of the equation $f'(x) = 0$ lie between those of $f(x) = 0$; so that an odd number of the roots of $f'(x) = 0$ will be found between every two of the roots of $f(x) = 0$, when the real roots of both equations are written in one series in the order of magnitude.

For let $a, b, c, \ldots\ldots$ be the real roots of the equation $f(x) = 0$; then we have

$$f(x) = (x-a)(x-b)(x-c) \ldots \{(x-\rho\cos\theta)^2 + \rho^2\sin^2\theta\} \ldots$$

and if we take the logarithms of these expressions, and differentiate,

$$\frac{f'(x)}{f(x)} = \frac{1}{x-a} + \frac{1}{x-b} \cdots\cdots + \frac{2(x-\rho\cos\theta)}{(x-\rho\cos\theta)^2+\rho^2\sin^2\theta} \cdots\cdots$$

so that, on multiplying by $f(x)$, every fraction on the second side will become a polynomial; and we may remark that the factor $x - a$ will enter into every polynomial except $\frac{f(x)}{x-a}$, and the factor $x - b$ into every one but $\frac{f(x)}{x-b}$, and so on for all the real factors.

Hence, upon substituting for x, the value a, these polynomials will all vanish, except $\frac{f(x)}{x-a}$; and on putting $x = b$, the only one which does not vanish will be $\frac{f(x)}{x-b}$; and so on for all the real roots of $f(x) = 0$. And, consequently,

$$f'(a) = \text{value of } \frac{f(x)}{x-a} \text{ due to } x = a$$

$$= (a-b)(a-c) \cdots\cdots \{(a-\rho\cos\theta)^2 + \rho^2\sin^2\theta\} \cdots\cdots$$

and, by a similar process,

$$f'(b) = (b-a)(b-c) \cdots\cdots \{(b-\rho\cos\theta)^2 + \rho^2\sin^2\theta\} \cdots\cdots$$

$$f'(c) = (c-a)(c-b) \cdots\cdots \{(c-\rho\cos\theta)^2 + \rho^2\sin^2\theta\} \cdots\cdots$$

. . . .

And, by forming the successive quotients

$$\frac{f'(a)}{f'(b)} = (-1)\frac{a-c}{b-c}\cdot\frac{a-d}{b-d}\cdots\frac{(a-\rho\cos\theta)^2+\rho^2\sin^2\theta}{(b-\rho\cos\theta)^2+\rho^2\sin^2\theta}\cdots$$

$$\frac{f'(b)}{f'(c)} = (-1)\frac{b-a}{c-a}\cdot\frac{b-d}{c-d}\cdots\frac{(b-\rho\cos\theta)^2+\rho^2\sin^2\theta}{(c-\rho\cos\theta)^2+\rho^2\sin^2\theta}\cdots$$

. . . .

from which we conclude that all these quotients are negative, since no factor of any one of them is negative, except the factor -1; for we cannot have $\frac{a-c}{b-c}$ negative, unless c lies between a and b, which is not the case. Consequently the series of quantities

$$f'(a), f'(b), f'(c) \ldots\ldots.$$

are alternately positive and negative; and an odd number of roots of $f'(x) = 0$ lie between every two of the roots $a, b, c, \ldots$ of the equation $f(x) = 0$.

It is manifest that if r be the number of real roots of $f(x) = 0$, then the least number of real roots that $f'(x) = 0$ can have is $r - 1$, so that one root may lie between every two of the r roots of $f(x) = 0$.

The case of equal roots might be considered in a similar manner. But it is better to clear the equation of equal roots by the method previously given, and apply the present rule to the reduced equations given by that method.

We may remark, that when the limiting equation $f'(x) = 0$ can be solved, and we have its real roots $\alpha, \beta, \gamma, \delta, \ldots\ldots$ in the order of magnitude, then the roots $a, b, c, d, \ldots\ldots$ of the primitive equation $f(x) = 0$, lie singly between the terms of the series

$$\infty, \alpha, \beta, \gamma, \delta, \ldots\ldots, -\infty.$$

Hence, if we form the series of results

$$f(\infty), f(\alpha), f(\beta), f(\gamma), \ldots\ldots f(-\infty),$$

the number of changes of sign in the series of signs of these results will be the number of the real roots $a, b, c, d, \ldots\ldots$

CHAP. IV.

ON THE SEPARATION OF THE ROOTS BY THE METHOD OF FOURIER.

33. In order to investigate the intervals between which each root of the equation

$$f(x) = 0$$

is to be found, it will be necessary for us to consider, at one view, the series of polynomials

$$f(x), f'(x), f''(x), \ldots\ldots f^n(x),$$

which it will be more convenient to write in the inverse order,

$$f^n(x); f^{n-1}(x), \ldots . f''(x), f'(x), f(x).$$

Let us now give to x a determinate value a, positive or negative. And let the signs of the above series of quantities be set down, instead of the quantities themselves. A series of signs will thus be formed which will always commence with $+$; because $f^n(x) = 1 . 2 . 3 \ldots\ldots n$ is always positive. But the signs after the first will vary, as a varies in magnitude. If now we begin with $a = -\infty$, the series will consist of $+$ and $-$, alternately; since the dimensions of the polynomials are, respectively,

$$0, 1, 2, 3, \ldots\ldots, n,$$

and on substituting $-\infty$ for x, the sines will depend upon their first terms, that is, upon

$$(-1)^0, (-1)^1, (-1)^2, \ldots . (-1)^n.$$

And when a has gone through all stages of magnitude, up to $+\infty$, the series of signs will evidently contain no $-$. Hence the n changes of sign due to $-\infty$ have become continuations, when a has become $+\infty$.

For the sake of brevity we shall denote the series

$$f^n(a), f^{n-1}(a) \ldots . . f'(a), f(a),$$

by the symbol, result (a).

And the series of signs of the terms composing that series, will be denoted by the symbol, sign (a).

Also, the number of changes of sign in the series of signs, will be denoted by the symbol, change (a).

34. We shall now proceed to show, that as a continually increases from $-\infty$ to $+\infty$, change (a) will from time to time diminish by one or more units, but that it will never increase. And that change (a) will lose one unit every time that a becomes equal to a root of the equation $f(x) = 0$.

It is evident that sign (a) can receive no change of any of its terms, so long as the gradual increase of a does not at some instant render zero one or more of the terms of result (a). It becomes necessary, therefore, to examine what takes place for values of a, just before and just after that value, which has produced such zero or zeros.

In the first place, then, we shall suppose that there is only one such zero, and that this is the last term of result (a); in which case, $f(a) = 0$, and a is a root. Write for a successively $a \mp da$; then, since no difference can exist between sign $(a \mp da)$ and sign(a) if we omit their last terms, we have only to compare the last two terms of

$$\text{result } (a - da) \text{ and result } (a + da),$$

to estimate the difference between

$$\text{sign } (a - da) \text{ and sign } (a + da).$$

These two terms will be

$$\left.\begin{matrix} f'(a - da), f(a - da) \\ f'(a + da), f(a + da) \end{matrix}\right\}.$$

And since we are concerned with the signs of these quantities only, and not with their magnitudes, we may write these terms

$$\left.\begin{matrix} f'(a), \ -da f'(a) \\ f'(a), \ +da f'(a) \end{matrix}\right\}.$$

And so far as concerns the change or continuation of sign between these two terms, we may omit the factor $f'(a)$, so that we have

$$\left.\begin{matrix} 1, \ -da \\ 1, \ +da \end{matrix}\right\}.$$

And it is manifest that we shall have change $(a + da)$ less by unity than change $(a - da)$. This loss of one change corresponds to the root a.

Secondly, let there still be but one zero in result (a), but not in its last term; as for instance, $f^r(a) = 0$. Here it is manifest that sign $(a \mp da)$ and sign (a) will be perfectly alike, except so far as regards the term corresponding to $f^r(a)$. And to judge of the effect of this term upon

$$\text{sign } (a - da) \text{ and sign } (a + da),$$

we need only consider that term and its neighbours on either side, in the two series of numbers

$$\text{result } (a - da) \text{ and result } (a + da).$$

These three terms will be

$$\left.\begin{array}{l} f^{r+1}(a-da),\ f^{r}(a-da),\ f^{r-1}(a-da) \\ f^{r+1}(a+da), f^{r}(a+da),\ f^{r-1}(a+da) \end{array}\right\},$$

or so far as concerns the changes or continuations of sign of the corresponding terms of

$$\text{sign}\ (a-da) \text{ and sign } (a+da),$$

we may write these terms,

$$\left.\begin{array}{l} f^{r+1}(a),\ -da\,f^{r+1}(a),\ f^{r-1}(a) \\ f^{r+1}(a),\ +da\,f^{r+1}(a),\ f^{r-1}(a) \end{array}\right\};$$

or, dividing the terms by $f^{r+1}(a)$, and setting down q for the quotient of $f^{r-1}(a)$ divided by $f^{r+1}(a)$, we have

$$\left.\begin{array}{l} 1,\ -da,\ q \\ 1,\ +da,\ q \end{array}\right\}.$$

And it is manifest that change $(a-da)$ — change $(a+da)$ will be equal to 0 or 2, according as q is negative or positive; that is, according as $f^{r+1}(a)$ and $f^{r-1}(a)$ are of opposite or of similar signs.

Thirdly, let us suppose that there are r successive zeros, and that these are the r last terms of result (a); so that there are no other zeros than

$$f^{r-1}(a)=0, f^{r-2}(a)=0\ .\ .\ .\ .\ f'(a)=0, f(a)=0.$$

It will be necessary here to consider the effect of the substitution of $(a\mp\delta)$, where δ is very small, but not infinitely small; otherwise we should still have $r-1$ zeros. As in the previous cases, we shall have to consider only the terms

$$f^{r}(a\mp\delta), f^{r-1}(a\mp\delta),\ .\ .\ .\ .\ .\ f'(a\mp\delta), f(a\mp\delta);$$

or, expanding by Taylor's series, and observing the r zeros given above, as well as omitting positive numerical coefficients, inasmuch as the signs only are the object of investigation, we may write

$$\left.\begin{array}{l} f^r(a),\ -\delta f^r(a),\ +\delta^2 f^r(a), \dots (-\delta)^r f^r(a) \\ f^r(a),\ +\delta f^r(a),\ +\delta^2 f^r(a), \dots +\delta^r f^r(a) \end{array}\right\},$$

or, suppressing the common factor $f^r(a)$,

$$\left.\begin{array}{l} 1,\ -\delta,\ +\delta^2,\ -\delta^3,\ \dots,\ (-\delta)^r \\ 1,\ +\delta,\ +\delta^2,\ +\delta^3,\ \dots,\ +\delta^r \end{array}\right\};$$

and, consequently, in passing from sign $(a-\delta)$ to sign $(a+\delta)$, there are lost exactly r changes of sign; which correspond to the r equal roots (a) of the equation, pointed out by the circumstance of having at once

$$f(a) = 0, f'(a) = 0, \dots f^{r-1}(a) = 0.$$

Fourthly, let there be r successive zeros, not the last r terms of result (a); as, for instance, if we have

$$f^{m-1}(a) = 0, f^{m-2}(a) = 0, \dots f^{m-r}(a) = 0;$$

and let there be no other zeros.

Here, by proceeding on the same plan, we shall have to consider the terms

$$f^m(a \mp \delta), \dots \dots f^{m-r-1}(a \mp \delta);$$

and by expanding and simplifying, as in the preceding cases, and setting down q for the quotient of $f^{m-r-1}(a)$ divided by $f^m(a)$, we shall at last reduce these terms to

$$\left.\begin{array}{l} 1,\ -\delta,\ +\delta^2,\ -\delta^3,\ \dots,\ (-\delta)^r,\ q \\ 1,\ +\delta,\ +\delta^2,\ +\delta^3,\ \dots,\ +\delta^r,\ q \end{array}\right\},$$

so that there will be a loss of exactly r changes in passing from sign $(a-\delta)$ to sign $(a+\delta)$, when r is even; and when r is odd, we shall have

$$\text{change } (a-\delta) - \text{change } (a+\delta) = r \mp 1,$$

according as q is negative or positive, that is, according as $f^m(a)$ and $f^{m-r-1}(a)$ have opposite or similar signs.

But in none of these cases can any change of sign be gained in passing from sign $(a - \delta)$ to sign $(a + \delta)$.

Finally, if there be several such zeros, or sets of successive zeros, in different parts of result (a), the above reasoning will apply to every such zero, or set of successive zeros. So that we are now able to draw the following conclusions:

1st. There is a continual diminution of change (a), by one or more units at a time, during the increase of a from $-\infty$ to $+\infty$; and change (a) never increases during that increase of a.

The limits of change (a) are

$$\left.\begin{aligned} \text{change}\ (-\infty) &= n \\ \text{change}\ (+\infty) &= 0 \end{aligned}\right\}.$$

2ndly. Every time that a becomes $=$ a root of the equation, or $=$ each of a set of equal roots, change (a) will lose, in the passage of a through that value, one unit for every such root. And change (a) must thus lose, during the increase of a from $-\infty$ to $+\infty$, as many units as there are real roots.

3rdly. If a has such a value as shall render zeros any number of terms of result (a), though a is not a root of the equation, then change (a) may lose, in the passage of a through that value, an even number of units, or may not lose any. And from the preceding conclusion, with respect to the real roots, it is evident that every loss of a pair of changes in this manner, must correspond to a pair of impossible roots of the given equation.

35. The preceding conclusions at once give us a rule for avoiding those intervals, in our investigation of the roots, which cannot contain any roots of the equation under con-

sideration. For change (a) must lose one of its units for every root that a passes in its increase; and it may lose besides these units, an even number more corresponding to imaginary roots. Hence there cannot be more real roots between a and b, than there are units in the number

$$\text{change } (a) - \text{change } (b);$$

supposing b to be greater than a.

If, therefore, we find that

$$\text{change } (a) = \text{change } (b),$$

we may rest assured that no real root lies between a and b.

If we have

$$\text{change } (a) = \text{change } (b) + 1,$$

there will evidently be one real root of the equation between a and b, and only one.

And, generally, if we have

$$\text{change } (a) = \text{change } (b) + \text{an odd number } \lambda,$$

there is at least one root between a and b, and there may be 3, or 5, ... or λ, roots between those limits.

But if, on the contrary, we have

$$\text{change } (a) = \text{change } (b) + \text{an even number } \mu,$$

we cannot affirm that there is any root of the equation between a and b, though there may be 2, or 4, or 6, ... or μ, roots between those limits.

36. We may remark, that by taking the limits $-\infty$ and 0, or 0 and $+\infty$, we find the limits which the numbers of negative or positive roots cannot exceed; and, in fact, we obtain the rule of signs of Descartes.

For we shall have the limit of the number of positive roots of the equation, or of the roots between 0 and ∞, equal to

$$\begin{aligned} &\text{change } (0) - \text{change } (\infty) \\ &= \text{change } (0). \end{aligned}$$

Now if we consider the series, sign (0), we shall find that its signs are the same as those of the coefficients of the original equation, and, consequently,

$$\text{change } (0) = \text{number of changes in } f(x);$$

so that there are not more positive roots than changes of sign in $f(x)$.

And the limit of the number of negative roots will evidently be equal to

$$\begin{aligned} &\text{change } (-\infty) - \text{change } (0) \\ &= n - \text{number of changes in } f(x) \\ &= \text{number of continuations in } f(x), \end{aligned}$$

which accords with the rule of signs.

37. There is one important remark to be made before proceeding to the application of the theorem with respect to the number, change (a).

If it happens that there are any zeros in the series, sign (a), then it is not possible to estimate properly the number, change (a). In this case we must have recourse to the two series, sign $(a \mp \delta)$, and estimate the two numbers, change $(a \mp \delta)$. And when it is required to compare the numbers, change $(a - b)$, change (a), change $(a + b)$, we must compare change $(a - b)$ with change $(a - \delta)$, and change $(a + b)$ with change $(a + \delta)$; b being any positive quantity, and δ a very small positive quantity.

The rule for deducing sign $(a \pm \delta)$ from sign (a), which contains zeros, may be collected from the preceding theory.

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It may be enunciated thus:

To form the series sign $(a - \delta)$ from sign (a), we must commence copying the signs of sign (a) from left to right; but when we arrive at a zero in sign (a), we are to set down the sign opposite to the one just written, for the corresponding term of sign $(a - \delta)$.

To form the series for sign $(a + \delta)$, we must proceed as before; only that for the zero we are to set down the same sign as the one just written.

The following example of the rule of the double sign will best explain its meaning:

$$\left.\begin{array}{lllllllllllll} \text{sign } (a-\delta) & = & + & + & - & + & - & + & + & - & + & - & - \\ \text{sign } (a) & = & + & + & - & 0 & 0 & 0 & + & - & + & 0 & - \\ \text{sign } (a+\delta) & = & + & + & - & - & - & - & + & - & + & + & - \end{array}\right\}.$$

38. The following are examples of the application of the preceding method to particular equations:

Ex. 1. $x^5 - 3x^4 - 24x^3 + 95x^2 - 46x - 101 = 0.$

Here we have the polynomials

$$\begin{aligned} f(x) &= x^5 - 3x^4 - 24x^3 + 95x^2 - 46x - 101, \\ f'(x) &= 5x^4 - 12x^3 - 72x^2 + 190x - 46, \\ f''(x) &= 20x^3 - 36x^2 - 144x + 190, \\ f'''(x) &= 60x^2 - 72x - 144, \\ f^{iv}(x) &= 120x - 72, \\ f^{v}(x) &= 120. \end{aligned}$$

And if we substitute for x, successively,

$$\ldots\ldots, -10, -1, 0, 1, 10, \ldots\ldots$$

we shall find

$$\left.\begin{array}{ll} \text{sign}\,(-10) & = + - + - + - \\ \text{sign}\,(-1) & = + - + - + + \\ \text{sign}\,(0) & = + - - + - - \\ \text{sign}\,(1) & = + + - + + - \\ \text{sign}\,(10) & = + + + + + + \end{array}\right\},$$

from which we obtain the numbers

$$\left.\begin{array}{ll} \text{change}\,(-10) & = 5 \\ \text{change}\,(-1) & = 4 \\ \text{change}\,(0) & = 3 \\ \text{change}\,(1) & = 3 \\ \text{change}\,(10) & = 0 \end{array}\right\}.$$

Hence we conclude, that one root of the equation lies between -10 and -1; a second lies between -1 and 0; a third between 1 and 10; and the remaining pair, if they are real, are also comprised between the limits 1 and 10. At present we cannot determine the question concerning these roots, whether they are real or not; but we shall hereafter give a rule applicable to such cases.

Ex. 2. Let the proposed equation be

$$x^4 - 4x^3 - 3x + 23 = 0.$$

The series of polynomials will be

$$\begin{aligned} f(x) &= x^4 - 4x^3 - 3x + 23, \\ f'(x) &= 4x^3 - 12x^2 - 3, \\ f''(x) &= 12x^2 - 24x, \\ f'''(x) &= 24x - 24, \\ f^{iv}(x) &= 24. \end{aligned}$$

And if we substitute for x, successively,

$$0,\ 1,\ 10,$$

we shall find

$$\left.\begin{array}{ll}\text{sign } (0) & = + - 0 - + \\ \text{sign } (1) & = + \ 0 - - + \\ \text{sign } (10) & = + + + + +\end{array}\right\};$$

and applying the rule of the double sign, we shall write these series

$$\left.\begin{array}{ll}\text{sign } (\mp \delta) & = + - \pm - + \\ \text{sign } (1 \mp \delta) & = + \mp - - + \\ \text{sign } (10) & = + + + + +\end{array}\right\},$$

δ representing a very small fraction.

Hence we obtain the numbers

$$\left.\begin{array}{ll}\text{change } (-\delta) & = 4 \\ \text{change } (+\delta) & = 2 \\ \text{change } (1-\delta) & = 2 \\ \text{change } (1+\delta) & = 2 \\ \text{change } (10) & = 0\end{array}\right\},$$

from which we conclude, that there are two impossible roots, because two changes are lost at once between the substitution of $-\delta$, and of $+\delta$, although 0 is not a root; and that the remaining two roots, if they are real, must lie in the interval from 1 to 10.

Ex. 3. The cubic equation

$$x^3 + 2x^2 - 3x + 2 = 0$$

gives the following table:

$$\begin{aligned} f(x) &= x^3 + 2x^2 - 3x + 2, \\ f'(x) &= 3x^2 + 4x - 3, \\ f''(x) &= 6x + 4, \\ f'''(x) &= 6, \end{aligned}$$

$$\left.\begin{array}{ll} \text{sign}\,(-10) & = + - + - \\ \text{sign}\,(-1) & = + - - + \\ \text{sign}\,(0) & = + + - + \\ \text{sign}\,(1) & = + + + + \end{array}\right\},$$

$$\left.\begin{array}{ll} \text{change}\,(-10) & = 3 \\ \text{change}\,(-1) & = 2 \\ \text{change}\,(0) & = 2 \\ \text{change}\,(1) & = 0 \end{array}\right\};$$

one root lies between -10 and -1; and the other two lie, if they are real, between 0 and 1.

Ex. 4. When the rule is applied to binomial equations, of the form

$$x^n \pm 1 = 0,$$

we shall immediately find the number of impossible roots.

Thus, if we have the equation

$$x^6 - 1 = 0,$$

we shall form the table

$$\begin{aligned} f(x) &= x^6 - 1, \\ f'(x) &= 6x^5, \\ f''(x) &= 6\,.\,5\,.\,x^4, \\ f'''(x) &= 6\,.\,5\,.\,4\,.\,x^3, \\ f^{IV}(x) &= 6\,.\,5\,.\,4\,.\,3\,.\,x^2, \\ f^{V}(x) &= 6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,x, \\ f^{VI}(x) &= 6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1\,; \end{aligned}$$

and the substitution of $-1, 0, +1$, successively, will give

$$\left.\begin{array}{ll} \text{sign}(-1) & = + - + - + - 0 \\ \text{sign}(0) & = + \; 0 \; 0 \; 0 \; 0 \; 0 \; - \\ \text{sign}(1) & = + + + + + + 0 \end{array}\right\};$$

and upon applying the rule of the double sign, we obtain

$$\left.\begin{array}{ll} \text{sign}(-1 \mp \delta) & = + - + - + - \pm \\ \text{sign}(\mp \delta) & = + \mp + \mp + \mp - \\ \text{sign}(1 \mp \delta) & = + + + + + + \mp \end{array}\right\}.$$

Here we conclude, from the corresponding numbers,

$$\left.\begin{array}{ll} \text{change}(-1-\delta) & = 6 \\ \text{change}(-1+\delta) & = 5 \\ \text{change}(-\delta) & = 5 \\ \text{change}(+\delta) & = 1 \\ \text{change}(1-\delta) & = 1 \\ \text{change}(1+\delta) & = 0 \end{array}\right\},$$

that there are only two real roots, -1 and $+1$; and that the remaining four impossible roots correspond to the simultaneous loss of four changes in passing through zero.

Ex. 5. Let the equation be

$$x^5 + 3x^4 + 2x^3 - 3x^2 - 2x - 2 = 0.$$

$$f(x) = x^5 + 3x^4 + 2x^3 - 3x^2 - 2x - 2,$$

$$f'(x) = 5x^4 + 12x^3 + 6x^2 - 6x - 2,$$

$$f''(x) = 20x^3 + 36x^2 + 12x - 6,$$

$$f'''(x) = 60x^2 + 72x + 12,$$

$$f^{IV}(x) = 120x + 72,$$

$$f^{V}(x) = 120.$$

We shall therefore have the following table, writing the double sign for a zero according to the rule:

	V	IV	III	II	I	0
sign (– 1)	+	–	±	–	+	–
sign (0)	+	+	+	–	–	–
sign (1)	+	+	+	+	+	–
sign (10)	+	+	+	+	+	+.

Here two roots are impossible; two may lie between – 1 and 0; and the remaining root lies between 1 and 10.

The above is the abbreviated form of the table best adapted for practice.

Ex. 6. Let the equation be

$$x^5 - 10x^3 + 6x + 1 = 0.$$

$$f(x) = x^5 - 10x^3 + 6x + 1,$$
$$f'(x) = 5x^4 - 30x^2 + 6,$$
$$f''(x) = 20x^3 - 60x,$$
$$f'''(x) = 60x^2 - 60,$$
$$f^{IV}(x) = 120x,$$
$$f^{V}(x) = 120.$$

	V	IV	III	II	I	0
sign (– 10)	+	–	+	–	+	–
sign (– 1)	+	–	±	+	–	+
sign (0)	+	∓	–	±	+	+
sign (1)	+	+	∓	–	–	–
sign (10)	+	+	+	+	+	+.

Here one root lies between – 10 and – 1; two roots, if they are real, lie between – 1 and 0; one root lies between 0 and 1; and the remaining two, if real, between 1 and 10.

39. We have now completely determined the intervals in which the roots of the equation must be sought; and have excluded from the investigation all those intervals which by the rule of signs cannot contain any root. These latter intervals are of much greater extent than the former; and it is in the exclusion of these that the method of Fourier possesses one of its chief advantages over that of Lagrange.

But there is still another, and a very important question, which here presents itself, in those cases where any interval contains more than one root: are those roots all real?

The first example offers an instance of this inquiry. It was found that between 1 and 10, three roots of the equation were to be sought. One of these must be real; but the remaining pair, so far as we know at present, may be imaginary. We might subdivide the interval into several smaller intervals, and by continuing such a process we should at last separate the three roots, if they are real, and we should have each of them comprised within a separate interval. But if two of the roots are imaginary, the process of subdivision would not come to a conclusion; and we should still be ignorant, whether the separation was impossible because the roots were imaginary, or was only very much delayed because their difference was extremely small.

This difficulty is theoretically surmounted by the method of Lagrange; for if we can find the inferior limit of the differences of the roots, we shall know when to stop in our subdivision of intervals. But the practical determination of that limit is impossible for equations whose degree is greater than 3 or 4; as the calculation of it becomes complicated more and more for each additional unit in the degree of the equation, and the difficulty increases at a rapid rate. Hence we must seek some other method, as a criterion of the reality of the included roots. Such a criterion, in a form readily applicable to practice, has been given by Fourier.

40. It will be convenient, before proceeding to the demonstration of the criterion of Fourier, to adopt an abbreviated and expressive notation for the number change (a) — change (b), which indicates the number of roots of the equation $f(x) = 0$ to be sought in the interval from a to b. This will be effected by writing for the number change (a) — change (b), the expression, root-index (a, b). The method of finding this root-index for any equation, and for any interval, is obvious from what has preceded. We need only refer to Example (1) of Art. (37). For that equation we find the following root-indices:

$$\left.\begin{array}{ll}\text{root-index }(-10, -1) & = 1\\ \text{root-index }(-1, 0) & = 1\\ \text{root-index }(1, 10) & = 3\end{array}\right\}.$$

But further, in considering the interval from a to b, it will not be sufficient to determine for the polynomial $f(x)$, the quantities change (a), change (b), and their difference root-index (a, b). We must apply the same process to every one of the derivatives

$$f'(x), f''(x), \ldots\ldots f^n(x).$$

The three series of numbers so obtained are to be written in reverse order, that is, in the order of

$$f^n(x), f^{n-1}(x), \ldots\ldots f'(x), f(x),$$

and will form what we shall term change-series (a), change-series (b), and index-series (a, b).

Referring again to Art. (37), and to Example (1), for illustration, we shall have the following table for the interval (1, 10).

	V	IV	III	II	I	0
sign (1)	+	+	−	+	+	−
sign (10)	+	+	+	+	+	+
change-series (1)	0	0	1	2	2	3
change-series (10)	0	0	0	0	0	0
index-series (1, 10)	0	0	1	2	2	3,

the last series is formed by subtracting the terms of the two preceding series vertically.

In practice it is usual to write only the index-series, and its terms are set down between the rows of signs, thus:

	V	IV	III	II	I	0
sign (1)	+	+	−	+	+	−
	0	0	1	2	2	3
sign (10)	+	+	+	+	+	+.

As a complete acquaintance with the preceding notation will be necessary, in order to follow the train of reasoning, we add the following examples, in which the process of forming the index-series is shown:

Ex. 1. $x^7 - 2x^5 - 3x^3 + 4x^2 - 5x + 6 = 0.$

$$f(x) = x^7 - 2x^5 - 3x^3 + 4x^2 - 5x + 6,$$
$$f'(x) = 7x^6 - 10x^4 - 9x^2 + 8x - 5,$$
$$f''(x) = 42x^5 - 40x^3 - 18x + 8,$$
$$f'''(x) = 210x^4 - 120x^2 - 18,$$
$$f^{IV}(x) = 840x^3 - 240x,$$
$$f^{V}(x) = 2520x^2 - 240,$$
$$f^{VI}(x) = 5040x,$$
$$f^{VII}(x) = 5040,$$

from which we obtain the following table of signs, and can thence form the index-series for each interval, as follows:

	VII	VI	V	IV	III	II	I	0
sign (− 10)	+	−	+	−	+	−	+	−
		0	0	0	0	1	1	1
sign (− 1)	+	−	+	−	+	+	−	+
		0	1	1	1	0	0	0
sign (0)	+	∓	−	±	−	+	−	+
		0	1	1	1	1	2	2
sign (1)	+	+	+	+	+	−	−	+
		0	0	0	0	1	1	2
sign (10)	+	+	+	+	+	+	+	+.

We have here placed the indices between the rows of signs corresponding to the interval; the method of finding any index, as, for instance, root-index $(-1, 0)$, corresponding to $f^{IV}(x)$, is to consider the signs as far as that term only for the required interval,

$$\begin{array}{lcccc} \text{sign } (-1) & + & - & + & - \\ \text{sign } (0) & + & - & - & +. \end{array}$$

of which the first line counts three changes, and the second two; their difference 1 is the required root-index $(-1, 0)$ for $f^{IV}(x)$.

Ex. 2. $\quad x^3 + 2x^2 - 3x + 2 = 0.$

$$f(x) = x^3 + 2x^2 - 3x + 2,$$
$$f'(x) = 3x^2 + 4x - 3,$$
$$f''(x) = 6x + 4,$$
$$f'''(x) = 6,$$

from which we have the table

	III	II	I	0
sign (− 10)	+	−	+	−
		0	1	1
sign (− 1)	+	−	−	+
		1	0	0
sign (0)	+	+	−	+
		0	1	2
sign (1)	+	+	+	+.

We may remark, in general, that the difference of two successive terms of any index-series is always either 0, or ± 1; this is a consequence of the formation of these indices.

41. After having thus defined at length the quantities which we are about to consider, and expressed them by a notation convenient at once by its brevity and perspicuity, we shall be the better able to bring under one view the rule given by Fourier, for distinguishing the nature of the roots indicated in any interval, supposing more than one to be so indicated.

When the index-series is formed, we are to choose that term of it in which the index 1 appears for the last time in the series. The following index must be 2; for it is necessarily one of the three numbers 0, 1, 2. If it was 0, then there must be some index 1 later in the series, as the last index is not 0. If it was 1, then the index 1 first chosen was not the latest index 1, as it ought to have been. Both these cases are therefore excluded, and the index 1 first chosen is followed by an index 2. We shall suppose that the preceding index is p; and that these three indices correspond to the three polynomials

$$\begin{array}{ccc} f^{r+1}(x), & f^{r}(x), & f^{r-1}(x) \\ p, & 1, & 2. \end{array}$$

The index p is either 0, 1, or 2. If it be not 0, we can reduce it to 0 in the following manner. Since $f^r(x) = 0$ has only one root lying in the interval (a, b), $f^{r+1}(x) = 0$ cannot have a root equal to that root; for then $f^r(x) = 0$ must have equal roots lying in the interval (a, b), which is not the case. Hence there can always be found a new interval (a', b') included within the larger one under consideration, for which new interval the index p is zero, and the succeeding index is still unity. The larger interval is thus divided into the three intervals

$$(a, a'), (a', b'), (b', b),$$

which give the corresponding series of indices as far as these three terms are concerned,

	$f^{r+1}(x)$,	$f^r(x)$,	$f^{r-1}(x)$
(a, a')	p	0	q
(a', b')	0	1	t
(b', b)	p	0	q,

where the indices p, q may be either 0 or 1; and t may be either 0, 1, or 2.

We may now discard the extreme intervals (a, a'), (b', b), since for these there will be some index 1 at a later part of the series; and for those intervals, therefore, the separation of roots has been carried towards the end of the series, past the term $f^r(x)$. The object of the whole process is evidently to reduce the last index to 0 or 1; and this is effected by choosing, as we have done, the latest index 1, and moving it, if possible, nearer to the end of the series. The remaining interval (a', b') is the only one for which this removal of the latest index 1 may not have been effected; and we shall now proceed to investigate the possibility of such a removal in this case, that is, when $t = 2$.

42. The only case now left for our consideration, is that presented by the table

$$\begin{array}{cccc} & f^{r+1}(x), & f^{r}(x), & f^{r-1}(x) \\ (a, b) & 0 & 1 & 2, \end{array}$$

and the object is, if possible, to separate the two roots of the equation

$$f^{r-1}(x) = 0,$$

indicated by the above table.

For this purpose, we must seek a value c intermediate to a and b, such that $f^{r-1}(c)$ shall have a different sign from that of $f^{r-1}(a)$ and $f^{r-1}(b)$; or such that

$$\left.\begin{aligned} \frac{f^{r-1}(c)}{f^{r-1}(a)} &= \text{neg.} \\ \frac{f^{r-1}(c)}{f^{r-1}(b)} &= \text{neg.} \end{aligned}\right\}.$$

Let us now put

$$\left.\begin{aligned} c &= a + h \\ c &= b - k \end{aligned}\right\},$$

$$\text{whence } b - a = h + k,$$

and expand the preceding conditions by Taylor's series, so that the term of the second order shall include the remainder of the series; that is, let us make the substitutions

$$f^{r-1}(c) = f^{r-1}(a) + h f^{r}(a) + \frac{h^2}{2} f^{r+1}(\lambda),$$

$$f^{r-1}(c) = f^{r-1}(b) - k f^{r}(b) + \frac{k^2}{2} f^{r+1}(\mu),$$

where λ is some value intermediate to a and c; and μ lies between c and b.

The preceding conditions will then take the following form:

$$\left.\begin{aligned} 1 + h\frac{f^r(a)}{f^{r-1}(a)} + \frac{h^2}{2}\frac{f^{r+1}(\lambda)}{f^{r-1}(a)} &= \text{neg.} \\ 1 - k\frac{f^r(b)}{f^{r-1}(b)} + \frac{k^2}{2}\frac{f^{r+1}(\mu)}{f^{r-1}(b)} &= \text{neg.} \end{aligned}\right\}.$$

Now, observing the table

$$\begin{array}{cccc} & f^{r+1}(x), & f^r(x), & f^{r-1}(x) \\ (a,\, b) & 0 & 1 & 2, \end{array}$$

it will be evident that we shall always have a positive quotient for the two fractions

$$\frac{f^{r+1}(\lambda)}{f^{r-1}(a)} \text{ and } \frac{f^{r+1}(\mu)}{f^{r-1}(b)}.$$

Hence, transposing these terms, the second sides of the equations may still be written as before; thus we shall have

$$\left.\begin{aligned} 1 + h\frac{f^r(a)}{f^{r-1}(a)} &= \text{neg.} \\ 1 - k\frac{f^r(b)}{f^{r-1}(b)} &= \text{neg.} \end{aligned}\right\}.$$

Again, by inspection of the above table, we shall find

$$\left.\begin{aligned} \frac{f^{r-1}(a)}{f^r(a)} &= \text{neg.} \\ \frac{f^{r-1}(b)}{f^r(b)} &= \text{pos.} \end{aligned}\right\},$$

and multiplying the preceding equations respectively by these latter, we find the conditions reduced to

$$\left.\begin{aligned} \frac{f^{r-1}(a)}{f^r(a)} + h &= \text{pos.} \\ \frac{f^{r-1}(b)}{f^r(b)} - k &= \text{neg.} \end{aligned}\right\},$$

which give, by subtraction,

$$\frac{f^{r-1}(a)}{f^r(a)} - \frac{f^{r-1}(b)}{f^r(b)} + h + k = \text{pos.}$$

If we call Q the sum of the quotients for the fractions

$$\frac{f^{r-1}(a)}{f^r(a)} \text{ and } \frac{f^{r-1}(b)}{f^r(b)},$$

neglecting the signs of those quotients,

$$\text{or, } Q = \frac{f^{r-1}(b)}{f^r(b)} - \frac{f^{r-1}(a)}{f^r(a)};$$

and also call D the difference of the interval (a, b),

$$\text{or, } D = b - a = h + k;$$

then the preceding condition becomes

$$D - Q = \text{pos.}$$

$$\text{or, } D > Q.$$

It follows at once, from this condition, that when we find

$$Q = \text{or} > D,$$

no such value of c can exist.

If, on the contrary, we find that

$$D > Q,$$

then such a value of c may exist; but it does not follow that such a value must necessarily exist.

43. We shall now proceed to the discussion of the first case, when

$$Q = \text{or} > D.$$

Since no value c can lie between the two roots of the equation

$$f^{r-1}(x) = 0,$$

which may lie in the interval (a, b), it follows, that if these roots exist they are equal. Suppose then that they do exist, and that each $= \gamma$. Then $f^{r-1}(x)$ and $f^{r}(x)$ must have a common divisor $\phi(x)$, of which there is one factor $x - \gamma$, and only one; and the other factors, if there are any, are also factors entering into $f^{r}(x)$; consequently, $\phi(x)$ can only have the root γ lying in the interval (a, b). And we shall then have $\frac{\phi(a)}{\phi(b)}$ negative. If then we find that such a divisor $\phi(x)$ exists, and that $\frac{\phi(a)}{\phi(b)}$ is negative, we shall know that the two roots of

$$f^{r-1}(x) = 0$$

are equal; and if γ be not a root of $f(x) = 0$, entering $r + 1$ times, the series of signs denoted by sign (γ) will give at least two consecutive zeros, included by terms which are not zero; and the rule of the double sign points out immediately two impossible roots of the equation

$$f(x) = 0.$$

If therefore we apply the method of equal roots to $f(x)$, we shall discover whether any such root γ enters $r + 1$ times, or we are to conclude that there are two impossible roots at least of the equation $f(x) = 0$. If the latter case happens, we can diminish by 2 all the succeeding root-indices. For in the succeeding root-indices, a part of every term is formed by the two changes which are lost by the series sign (γ) in passing through γ.

Again, if the two roots of the equation

$$f^{r-1}(x) = 0$$

are impossible, then there will be a loss of two signs in sign (γ) when γ becomes equal to the single root of

$$f^{r}(x) = 0,$$

lying in the interval (a, b). For there will be one zero between like signs. Hence we may, as before, subtract 2 from the succeeding root-indices.

If therefore we find Q $=$ or $>$ D, and there are not $r+1$ equal roots of

$$f(x)=0$$

in the interval (a, b), we are in all cases to subtract 2 from the succeeding root-indices.

44. But there remains the second case for consideration, when

$$D > Q.$$

We must choose at pleasure in the interval (a, b) a new value c. If we find that $f^{r-1}(c)$ differs in sign from $f^{r-1}(a)$ and $f^{r-1}(b)$, the two roots are separated. But if the contrary take place, we are to choose of the two intervals (a, c), (c, b), that one which still gives the table

$$\begin{array}{ccc} f^{r+1}(x), & f^{r}(x), & f^{r-1}(x) \\ 0, & 1, & 2, \end{array}$$

and form the criterion anew, with a less value of D. We must however first enquire, whether the two roots are equal; for if they are so, we need proceed no further.

By this process we shall at last be able to separate all the real roots of the equation

$$f(x)=0$$

into their respective intervals; and, of course, know their number.

45. We shall now give some examples of the application of the criterion, in order that the process may become more familiarized than it can possibly be under the shape of a general theorem.

Ex. 1. $$x^3 + 2x^2 - 3x + 2 = 0.$$

$$f(x) = x^3 + 2x^2 - 3x + 2,$$
$$f'(x) = 3x^2 + 4x - 3,$$
$$f''(x) = 6x + 4,$$
$$f'''(x) = 6.$$

The interval (0, 1) gives the table

	III	II	I	0
(0)	+	+	−	+
	0	0	1	2
(1)	+	+	+	+.

Hence, observing the series of root-indices, we shall have to consider whether the interval 1 is greater or less than Q, the sum of the quotients $\frac{f(0)}{f'(0)}$, and $\frac{f(1)}{f'(1)}$, neglecting their signs. Now we have

$$f(0) = \quad 2,$$
$$f'(0) = -3,$$
$$f(1) = \quad 2,$$
$$f'(1) = \quad 4,$$

and we find that

$$Q = \frac{2}{3} + \frac{2}{4} = \frac{7}{6},$$

consequently Q is greater than the interval unity; and we conclude that the two roots indicated by the figure 2 for the interval (0, 1) are impossible, as there are not equal roots.

The abbreviated form of the table for the whole process, is

	III	II	I	0
(0)	+	+	−	+
			3	2
	0	0	1	2
(1)	+	+	+	+.
			4	2

Here $\frac{2}{3} + \frac{2}{4} > 1$; and the two roots are impossible, as there are not equal roots.

Ex. 2. $x^5 - 3x^4 - 24x^3 + 95x^2 - 46x - 101 = 0.$

For the interval (1, 10), we have

	V	IV	III	II	I	0
(1)	+	+	−	+	+	−
			156	30		
	0	0	1	2	2	3
(10)	+	+	+	+	+	+.
			5136	15150		

$$\text{Here } \frac{30}{156} + \frac{15150}{5136} < 9;$$

and we must divide the interval into two parts by forming the series of signs for some intermediate number, as 3 for instance. But previously we must find whether there is a divisor common to $f''(x)$, $f'''(x)$. There being no such common divisor, we proceed to form the new table as follows:

	V	IV	III	II	I	0
(1)	+	+	−	+	+	−
	0	0	1	1	1	2
(3)	+	+	+	−	−	−
	0	0	0	1	1	1
(10)	+	+	+	+	+	+.

Hence, we conclude that there is one root of the equation only in the interval (3, 10), and that there may be two in the interval (1, 3). This last interval then is to be examined; and the first remark is, that the latest index 1 is not preceded by 0, so that the interval must yet be subdivided, as follows:

	V	IV	III	II	I	0
(1)	+	+	−	+	+	−
	0	0	0	1	0	0
(2)	+	+	−	−	+	−
					30	21
	0	0	1	0	1	2
(3)	+	+	+	−	−	−.
					43	32

It is, therefore, to the last of these intervals only that we are to attend; and the criterion gives

$$\frac{21}{30}+\frac{32}{43}>1,$$

whence we conclude that the roots are impossible.

Ex. 3. $\quad x^5+x^4+x^3-2x^2+2x-1.$

The interval (0, 1) gives the table

	V	IV	III	II	I	0
(0)	+	+	+	− 4	+ 2	−
	0	0	0	1	2	3
(1)	+	+	+	+ 36	+ 10	+.

$$\text{Here } \frac{2}{4}+\frac{10}{36}<1,$$

and $f'(x)$ and $f''(x)$ have no common divisor.

The interval is to be separated.

	V	IV	III	II	I	0
(0)	+	+	+	− 4	+ 2	−
	0	0	0	1	2	2
($\frac{1}{2}$)	+	+	+	+	+	−
	0	0	0	0	0	1
(1)	+	+	+	+	+	+.

Here the single quotient

$$\frac{2}{4}=\frac{1}{2},$$

and, consequently, the roots due to the interval $\left(0, \frac{1}{2}\right)$ are impossible.

46. Instead of giving any more examples of the application of the criterion, we shall give several examples of the complete separation of the roots, combining both processes, the rule of signs, and the criterion. These will be best exhibited in the tabular form adapted for practice. The operations of subdividing the intervals, when necessary, are here expressed at once. To become perfectly acquainted with the method, so as to perceive at one view its several parts and their relations, it cannot be too much recommended that every one should go through the working of these examples for himself.

Ex. 1. $x^5 - 3x^4 - 24x^3 + 95x^2 - 46x - 101 = 0.$

$$f(x) = x^5 - 3x^4 - 24x^3 + 95x^2 - 46x - 101,$$
$$f'(x) = 5x^4 - 12x^3 - 72x^2 + 190x - 46,$$
$$f''(x) = 20x^3 - 36x^2 - 144x + 190,$$
$$f'''(x) = 60x^2 - 72x - 144,$$
$$f^{IV}(x) = 120x - 72,$$
$$f^{V}(x) = 120.$$

	V	IV	III	II	I	0
(− 10)	+	−	+	−	+	−
						1
(− 1)	+	−	+	−	+	+
						1
(0)	+	−	−	+	−	−
						0
(1)	+	+	−	+	+	−
						0
(2)	+	+	−	−	+	−
					30	21
				0	1	2
(3)	+	+	+	−	−	−
					43	32
						1
(10)	+	+	+	+	+	+.

Here one root lies between -10 and -1; a second between -1 and 0; a third between 3 and 10: but the two remaining roots due to the interval from 2 to 3 are impossible, because we have $\frac{21}{30} + \frac{32}{43} > 1$, and there are not equal roots.

Ex. 2. $$x^4 - 4x^3 - 3x + 23 = 0.$$

$$f(x) = x^4 - 4x^3 - 3x + 23,$$
$$f'(x) = 4x^3 - 12x^2 - 3,$$
$$f''(x) = 12x^2 - 24x,$$
$$f'''(x) = 24x - 24,$$
$$f^{\text{iv}}(x) = 24.$$

	IV	III	II	I	0
(0)	+	−	±	−	+
					0
(1)	+	∓	−	−	+
					0
(2)	+	+	∓	−	+
					1
(3)	+	+	+	−	−
					1
(10)	+	+	+	+	+.

Hence two roots are impossible, by the rule of the double sign; and one root lies between 2 and 3, and the remaining one between 3 and 10.

Ex. 3. $$x^3 + 2x^2 - 3x + 2 = 0.$$

$$f(x) = x^3 + 2x^2 - 3x + 2,$$
$$f'(x) = 3x^2 + 4x - 3,$$
$$f''(x) = 6x + 4,$$
$$f'''(x) = 6.$$

	III	II	I	0
(− 10)	+	−	+	−
				1
(− 1)	+	−	−	+
				0
(0)	+	+	−	+
			3	2
		0	1	2
(1)	+	+	+	+.
			4	2

Here one root lies between − 10 and − 1 ; and the two roots due to the interval from 0 to 1 are imaginary, because $\frac{2}{3} + \frac{2}{4} > 1$, and there are not equal roots.

Ex. 4. $$x^4 - x^3 + 4x^2 + x - 4 = 0.$$

$$f(x) = x^4 - x^3 + 4x^2 + x - 4,$$
$$f'(x) = 4x^3 - 3x^2 + 8x + 1,$$
$$f''(x) = 12x^2 - 6x + 8,$$
$$f'''(x) = 24x - 6,$$
$$f^{IV}(x) = 24.$$

	IV	III	II	I	0
(− 10)	+	−	+	−	+
					1
(− 1)	+	−	+	−	−
					0
(0)	+	−	+	+	−
				6	8
	0	1	2	2	3
(1)	+	+	+	+	+.

Here the only real root lies between − 10 and − 1 ; and the two due to the interval 0 and 1 are impossible, since we have $\frac{8}{6} > 1$, and there are not equal roots.

Ex. 5. $x^6 - 12x^5 + 60x^4 + 123x^2 + 4567x - 89012 = 0.$

$$f(x) = x^6 - 12x^5 + 60x^4 + 123x^2 + 4567x - 89012,$$

$$f'(x) = 6x^5 - 60x^4 + 240x^3 + 246x + 4567,$$

$$f''(x) = 30x^4 - 240x^3 + 720x^2 + 246,$$

$$f'''(x) = 120x^3 - 720x^2 + 1440x,$$

$$f^{iv}(x) = 360x^2 - 1440x + 1440,$$

$$f^{v}(x) = 720x - 1440,$$

$$f^{vi}(x) = 720.$$

	VI	V	IV	III	II	I	0
(-10)	+	−	+	−	+	−	+
							1
(-1)	+	−	+	−	+	−	−
							0
(0)	+	−	+	$\mp$	+	+	−
							0
(1)	+	−	+	+	+	+	−
	0	1	2	2	2	2	3
(10)	+	+	+	+	+	+	+.

Here $f^{iv}(x) = 0$ has equal roots in the last of the intervals, and these are not roots of $f(x) = 0$; hence there is only one root between 1 and 10; there is also one root between -10 and -1: the other four are imaginary, by the rule of the double sign, and the criterion.

Ex. 6. $x^5 + x^4 + x^2 - 25x - 36 = 0.$

$$f(x) = x^5 + x^4 + x^2 - 25x - 36,$$
$$f'(x) = 5x^4 + 4x^3 + 2x - 25,$$
$$f''(x) = 20x^3 + 12x^2 + 2,$$
$$f'''(x) = 60x^2 + 24x,$$
$$f^{IV}(x) = 120x + 24,$$
$$f^{V}(x) = 120.$$

	V	IV	III	II	I	0
(− 10)	+	−	+	−	+	−
						1
(− 2)	+	−	+	−	+	+
						1
(− 1)	+	−	+	−	−	−
						0
(0)	+	+	∓	+	−	−
						0
(1)	+	+	+	+	−	−
						1
(10)	+	+	+	+	+	+.

The three real roots lie in the intervals from − 10 to − 2, from − 2 to − 1, and from 1 to 10; the remaining two are imaginary, by the rule of the double sign.

Ex. 7. $x^5 + x^4 + x^3 - 2x^2 + 2x - 1 = 0.$

$$f(x) = x^5 + x^4 + x^3 - 2x^2 + 2x - 1,$$
$$f'(x) = 5x^4 + 4x^3 + 3x^2 - 4x + 2,$$
$$f''(x) = 20x^3 + 12x^2 + 6x - 4,$$
$$f'''(x) = 60x^2 + 24x + 6,$$
$$f^{IV}(x) = 120x^2 + 24,$$
$$f^{V}(x) = 120.$$

	V	IV	III	II	I	0
(-1)	+	−	+	−	+	−
						0
$(-\frac{1}{2})$	+	−	+	−	+	−
		36	9			
	0	1	2	2	2	2
(0)	+	+	+	−	+	−
		24	6	4	2	
	0	0	0	1	2	2
$(\frac{1}{2})$	+	+	+	+	+	−
						1
(1)	+	+	+	+	+	+.

Here there is but one real root, which lies between $\frac{1}{2}$ and 1; the four due to the intervals $\left(-\frac{1}{2}, 0\right)$, and $\left(0, \frac{1}{2}\right)$ being impossible, since we have

$$\frac{9}{36} + \frac{6}{24} = \frac{1}{2}, \text{ and } \frac{2}{4} = \frac{1}{2}.$$

47. It appears from what has preceded, that there can be no imaginary roots of the equation, unless for some value of a we find that sign (a) presents a single zero included between like signs, or a set of consecutive included zeros. We shall proceed to apply this rule to the equations

$$\sin x = 0,$$

$$\cos x = 0.$$

It is at once seen, that for these equations no such single zero or set of zeros can exist. Hence they can have no impossible roots. And neither of them contains equal roots, since we can never satisfy both equations together by any real value of x.

We can now proceed accurately in the resolution of $\sin x$ and $\cos x$ into their factors, all of which are of the first degree, and enter only once.

First, to find the factors of $\sin x$.

The roots of the equation

$$\sin x = 0$$

are $0, \ \pm \pi, \ \pm 2\pi, \ \pm 3\pi, \ \ldots\ldots$ to ∞.

We may therefore assume

$$\begin{aligned}\sin x &= k \,.\, x\,(x+\pi)(x-\pi) \ldots\ldots \\ &= k \,.\, x\,(x^2-\pi^2)(x^2-2^2\pi^2) \ldots .\end{aligned}$$

where k denotes some constant.

The preceding equation may be put into the form

$$\sin x = k'x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right) \ldots\ldots$$

where k' is still some constant.

For the determination of k', we may observe that, when $x = 0$, then $\sin x = x$; or, in other words, that the limit of $\frac{\sin x}{x}$, when x becomes zero, will be unity. But this will not be the case in the above equation, unless $k' = 1$.

Hence, we have finally,

$$\sin x = x \,.\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right) \ldots\ldots$$

the series of factors being continued to infinity.

By a process of exactly the same nature, we find that

$$\cos x = \left(1-\frac{2^2x^2}{\pi^2}\right)\left(1-\frac{2^2x^2}{3^2\pi^2}\right) \ldots\ldots.$$

the factors, as before, being continued to infinity.

48. We shall conclude the present chapter by a recapitulation of the rules to be observed in the process of the separation of the roots. This process consist of two parts: the rule of signs for any interval, and the criterion of the existence of the roots indicated for an interval which may contain more than one root.

When the equation $f(x) = 0$ is proposed for investigation, the first step is to form, by successive differentiation, the polynomials

$$f'(x), f''(x), \ldots\ldots f^n(x).$$

These polynomials, with the given polynomial $f(x)$, are to be written in the inverse order

$$f^n(x), f^{n-1}(x), \ldots\ldots\ldots f'(x), f(x).$$

We are then to substitute for x, in the series so formed, the successive terms of the series

$$\ldots\ldots -100, -10, -1, 0, 1, 10, 100, \ldots\ldots\ldots$$

and write down the series of signs corresponding to these substitutions; the number of such substitutions is to be limited by those two, of which one gives only changes, and the other only continuations of sign. In case of a zero occurring in the series of signs, it is to be replaced according to the rule of the double sign; or, in other words, the series of signs is to be taken on each side of the transition through that zero, instead of the series at the point of transition.

All intervals in which no change of sign is lost, are immediately to be rejected, as containing no root of the equation.

All intervals in which only one change of sign is lost, are to be set down as containing a single real root of the equation.

Any interval in which more than one change of sign is lost, may contain as many roots as there are lost changes of sign. But to discover whether this is the case or not, we must apply the criterion. The first step is to form the index-series for that interval, according to the rules already laid down.

The index corresponding to any term $f^r(x)$ of the series of polynomials, denotes the number of changes of sign lost in that interval by the partial series of polynomials

$$f^n(x), f^{n-1}(x) \ldots \ldots f^r(x);$$

or, in other words, that index expresses the number of roots that $f^r(x) = 0$ may have in the interval under consideration. By this means a series of indices will be formed beginning with 0, and whose last term is greater than 1. The difference between two successive terms of the index-series will in all cases be 0, or $\pm$ 1.

Considering now any one such series, we must fix upon that term in which the index 1 occurs for the last time in the series. This last index 1 is necessarily followed by 2. If it be not preceded by 0, then we are to subdivide the interval; and by this subdivision, the last index 1 will in all cases occur later in the series, unless it happen that we find for any interval the three successive indices

$$0, 1, 2.$$

When this condition presents itself, we cannot be certain of moving the index 1 towards the end of the series, by the process of subdividing the interval. So that we are now warned to apply the criterion.

Denoting by

$$\begin{array}{ccc} f^{r+1}(x), & f^r(x), & f^{r-1}(x) \\ 0, & 1, & 2 \end{array}$$

the part of the table to which the criterion is to be applied, we are to form the sum of the two quotients

$$\frac{f^{r-1}(a)}{f^r(a)}, \frac{f^{r-1}(b)}{f^r(b)},$$

neglecting the algebraic signs; and we must examine whether that sum be equal to the difference $b - a$ of the limits; if it is equal to that difference, or greater than it, then the two roots indicated for $f^{r-1}(x) = 0$ are either equal or imaginary.

If they are equal, there ought to be a common factor $\phi(x)$ of the two polynomials $f^{r-1}(x)$ and $f^{r}(x)$, such that $\frac{\phi(a)}{\phi(b)}$ is a negative quantity.

Whenever this is the case, we must examine whether these two roots are also roots of the equations formed by equating to zero each of the succeeding polynomials

$$f^{r-2}(x), f^{r-3}(x), \ldots . f(x);$$

that is, we must seek, by the method of equal roots, whether the equation $f(x) = 0$ has a set of $r + 1$ equal roots, which lie in the interval (a, b). We may remark that this operation will be unnecessary, when the last index of the series is less than $r + 1$. If there is not such a set of equal roots of the equation $f(x) = 0$, we conclude that this equation has two impossible roots.

If the two roots of the equation $f^{r-1}(x) = 0$ are impossible, then we also conclude that the equation $f(x) = 0$ contains two impossible roots.

In both cases, therefore, the index-series may have all its terms diminished by 2, after the term corresponding to $f^{r}(x)$. And the last index 1 will have been removed towards the end of the series.

Lastly, when the difference $b - a$ is greater than the sum of the quotients above mentioned, we must reduce this difference by subdividing the interval, in order either to separate the two roots of $f^{r-1}(x) = 0$, or to obtain an interval proving the impossibility of that separation.

The whole operation of the separation of the roots will come to a conclusion when the index-series all end with 0 or 1. The number of such final indices 1 will give the number of real roots of the equation $f(x) = 0$; and their positions will point out where each of those real roots is to be sought.

We may remark that, whenever a single zero occurs between like signs, or whenever several successive included zeros occur, in the series of signs corresponding to any definite value of the variable, there is an indication of impossible roots. These values then ought to be as much the object of our search as the roots of the equation, at least as far as concerns their existence. Now it is the precise object of the criterion to point out the existence of these values indicative of impossible roots, or to demonstrate the possibility of the roots, by separation.

CHAP. V.

ON THE METHOD OF DIVISORS.

49. The method of Divisors was proposed by Newton for the discovery of the integral roots of any equation, if such roots existed. It will be shown that this process can be applied to the discovery of all the roots of the equation, which are commensurable, when the coefficients of the equation are commensurable quantities; that is, all the roots which are either integers, or can be expressed in the form of vulgar fractions. Although the method of Fourier will determine such roots exactly, yet, as the method of divisors forms a complete theory for roots of this kind, and is also practically applicable, it deserves still to hold a place in the Theory of Equations.

50. We shall first show, that by transformation all the commensurable roots of any equation, whose coefficients are commensurable, can be rendered integers. For we can, by transformation, render the coefficients of the equation integers. Now in this state of the equation there cannot exist any root expressible in the form of a vulgar fraction. For if we should suppose that one root of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_n = 0,$$

whose coefficients are all integers, could be expressed by the fraction $\frac{a}{b}$ in its lowest terms, that is, when the integers a and b are prime to each other, we should have

$$\left(\frac{a}{b}\right)^n + p_1\left(\frac{a}{b}\right)^{n-1} + \ldots\ldots + p_n = 0,$$

$$\text{or } \frac{a^n}{b} + p_1 a^{n-1} + \ldots\ldots + p_n b^{n-1} = 0;$$

that is, we should arrive at the absurd equation,

$$\text{fraction} + \text{integer} = 0.$$

Hence the commensurable roots, if any exist, must now be integers.

51. In order to determine the integer roots of any equation $f(x) = 0$, let us suppose that one of its roots is the integer a; then we have

$$f(a) = 0,$$

$$\text{or } a^n + p_1 a^{n-1} + p_2 a^{n-2} + \ldots + p_n = 0,$$

in which the coefficients of a are all integers. We shall write this equation in the inverse order, thus

$$p_n + p_{n-1}\,a + p_{n-2}\,a^2 + \ldots + p_1 a^{n-1} + a^n = 0.$$

Dividing this equation by a, we have

$$\frac{p_n}{a} + p_{n-1} + p_{n-2}\,a + \ldots + p_1 a^{n-2} + a^{n-1} = 0;$$

so that the quantity $\frac{p_n}{a}$ cannot but be an integer; let this integer be denoted by q_1, and we obtain

$$q_1 + p_{n-1} + p_{n-2}\,a + \ldots\ldots + p_1 a^{n-2} + a^{n-1} = 0.$$

Again, dividing by a, we obtain

$$\frac{q_1 + p_{n-1}}{a} + p_{n-2} + p_{n-3}\,a + \ldots + p_1 a^{n-3} + a^{n-2} = 0;$$

so that the quantity $\frac{q_1 + p_{n-1}}{a}$ cannot but be an integer; and we may denote this integer by q_2.

Proceeding to form the new equations

$$q_2 + p_{n-2} + p_{n-3}\, a + \ldots\ldots + p_1 a^{n-3} + a^{n-2} = 0,$$

$$\frac{q_2 + p_{n-2}}{a} + p_{n-3} + \ldots\ldots + p_1 a^{n-4} + a^{n-3} = 0;$$

we find that we must have

$$\frac{q_2 + p_{n-2}}{a} = \text{an integer } q_3.$$

And in this manner we shall continue to find a series of conditions which a must satisfy, the last of which will be obtained by the equation

$$q_{n-1} + p_1 + a = 0,$$

$$\text{or } \frac{q_{n-1} + p_1}{a} + 1 = 0;$$

that is, we must have the quantity

$$q_n, \text{ or } \frac{q_{n-1} + p_1}{a} = -1.$$

Hence the conditions, in order that a may be an integer root of the equation $f(x) = 0$, are

$$\frac{p_n}{a} = q_1 = \text{integer},$$

$$\frac{q_1 + p_{n-1}}{a} = q_2 = \text{integer},$$

$$\frac{q_2 + p_{n-2}}{a} = q_3 = \text{integer},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{q_{n-2} + p_2}{a} = q_{n-1} = \text{integer},$$

$$\frac{q_{n-1} + p_1}{a} = q_n = -1.$$

We shall now show the practical method of applying the above conditions to the determination of the integer roots of the equation. But it will be better in the first instance to find by immediate trial, whether there are any roots $= \pm 1$, and if there are such roots to expel them from the equation; so that the reduced equation is no longer satisfied by ± 1.

We shall now form a table of the divisors of p_n; to each of which the double sign $\pm$ is to be affixed. For the first condition shows that in this table all the values of a are to be sought, since we must have a dividing p_n exactly.

We shall next form a corresponding table of the values of q_1. And we can now commence forming the table of values of q_2; setting down only those values of q_2 which prove to be integers.

We may now erase from the first table of divisors, or values of a, all those which give no term in the table of values of q_2; for such values of a cannot be roots, by the second condition.

Proceeding now to form the table for q_3 on the same plan, that is, setting down only integer values of q_3, and whenever q_3 is not an integer, erasing the corresponding value of a from the first table, we shall have preserved only those values of a which satisfy the condition $q_3 =$ integer.

In this manner the table of values of a will at each step contain only such values as may still be roots of the equation. And when we arrive at the table for q_n, we are to erase from the first table all values which do not give $q_n = -1$. The table then remaining for the values of a will contain the integer roots of the equation.

We may remark, that at the outset of the operation, the table for a need not contain any values which do not lie between the inferior and superior limits of the roots.

We may also remark, that by increasing or diminishing all the roots by any integer, the integer roots will still remain integers. Nor if p_n is such a quantity as to give a large table of divisors, the equation may be transformed in the above manner, so that the last term shall give a smaller number of divisors. But if we put

$$y = x - m,$$
$$\text{or } x = y + m,$$

the last term of the new equation is $f(m)$; it will therefore be our object to find some positive or negative integer value for m, such that $f(m)$ shall have a small number of divisors, and then decrease the roots by that integer value.

But instead of effecting such a transformation, it will be sufficient to remark, that if a be an integer root of the original equation, then $a - m$ is a root of the transformed equation, and, consequently, we shall have

$$\frac{f(m)}{a-m} = \text{integer}.$$

Now, by giving m several integer values, we can form several new criteria, to be satisfied by the table of values of a, previous to the trial of the conditions above given. So that we can, at the commencement of the operation, reduce the table of values of a to a considerable extent. The best criteria of this kind are those most easily calculated, namely,

$$\frac{f(\pm 1)}{a \mp 1} = \text{integer},$$
$$\frac{f(\pm 10)}{a \mp 10} = \text{integer},$$

and so on, to any extent that may be required in reducing the table for a.

52. The following is an example of the method of divisors:

$$x^4 - x^3 - 13x^2 + 16x - 48 = 0.$$

Here the superior limit is 14, and the inferior limit is -8; hence the table of divisors is, at first,

(a) 12, 8, 6, 4, 3, 2, -2, -3, -4, -6.

We have omitted the divisors ± 1, as they are found by trial not to be roots.

Now, applying the criteria,

$$\frac{f(\pm 1)}{a \mp 1} = \text{integer},$$

the corrected table of divisors is

(a) 4, 2, -2, -4.

We may now proceed to form the table for q_1, or $\dfrac{-48}{a}$,

(q_1) -12, -24, 24, 12.

Again, forming the table for q_2, or $\dfrac{q_1 + 16}{a}$, we obtain

(q_2) 1, -4, -20, -7.

By a similar process for $q_3 = \dfrac{q_2 - 13}{a}$, we obtain the table

(q_3) -3, f, f, -5,

denoting by f any fractional quantity.

Hence we now obtain the corrected table of divisors

(a) 4, -4,

and the corresponding table for q_3 is

(q_3) -3, -5.

Lastly, forming the table for q_4 or $\frac{q_3 - 1}{a}$, we obtain

$$(q_4) \qquad -1, -1.$$

Hence $\pm$ 4 are the only integer roots of the proposed equation.

The best form of the table for practice is the following:

(a)	4,	2,	-2,	-4
$-48\ (q_1)$	-12,	-24,	24,	12
$+16\ (q_2)$	1,	-4,	-20,	-7
$-13\ (q_3)$	-3,			-5
$-1\ (q_4)$	-1,			-1.

In the above table the coefficients of the equation are placed in order, as they are wanted in the operation, by the side of the quantities q_1, q_2, q_3, and q_4. And whenever a fractional value occurs in the table, or any integer but -1 in the last line, it is not marked. So that we may neglect all the divisors standing at the head of those columns which do not reach to the last line.

53. The method of divisors is, however, still defective in one point, to which we must now give our attention. Although we have found the values of all the integer roots of the proposed equation, yet some of these roots may enter into that equation more than once. We must, therefore, either apply directly the method of equal roots to the proposed equation; or we must again apply the method of divisors to the limiting equation, commencing with the table remaining for the integer values of a, from the operations of the first application of the method; so that we may find whether any of the integer roots so determined, are also roots

of the limiting equation. By continuing this process as far as is necessary, with regard to the successive derived equations, the solution will be rendered complete.

The following are examples to which it will be necessary to apply the last process, in order to complete the solution given by the method of divisors:

Ex. 1. To find the integer roots of the equation

$$x^4 - 8x^3 + 8x^2 - 16 = 0.$$

Ex. 2. To find the commensurable roots of the equation

$$x^4 - 9x^3 + \frac{45}{4}x^2 + \frac{27}{2}x - \frac{81}{4} = 0.$$

Ex. 3. Solve, by the method of divisors, the equation

$$x^3 - 2x^2 - 4x + 8 = 0.$$

CHAP. VI.

ON THE METHOD OF NEWTON FOR OBTAINING APPROXIMATELY THE REAL ROOTS OF ANY EQUATION, SO FAR AS IT HAD BEEN DEVELOPED PREVIOUS TO ITS COMPLETION BY FOURIER.

54. The method of approximation given by Newton supposed that two limits a and b had been found, between which a root of the equation must lie. It was then easy to reduce by trial the difference $b - a$ of these limits, until it became a fraction, whose square might be neglected in the process of approximation.

Let us suppose then, that an approximate value of the root in question is a; so that the difference between the correct value α of that root, and the approximate value a, must be a small fraction δ. Then we have

$$f(\alpha) = 0,$$

since α is supposed to be the exact value of the root: and by substituting for α its value $a + \delta$, we obtain

$$f(a + \delta) = 0;$$

from which equation, by expanding in a series of powers of

δ, and neglecting all its powers beyond the first, we obtain the approximate equation

$$f(a) + \delta . f'(a) = 0.$$

From this latter equation we obtain an approximate value of δ, which we may denote by δ_1, namely,

$$\delta_1 = -\frac{f(a)}{f'(a)}.$$

And we have thus arrived at a new approximate value for the root in question, which we may denote by a_1, so that

$$a_1 = a + \delta_1;$$

and in general we may suppose that a_1 will be a nearer approximation to the root α, than the original approximate value a.

We may now repeat the process, commencing with the approximate value a_1, in order to arrive at a nearer approximation a_2. For this purpose we must put

$$\alpha = a_1 + \delta,$$

and we obtain the equation

$$f(a_1 + \delta) = 0,$$

which gives the approximate equation

$$f(a_1) + \delta . f'(a_1) = 0.$$

If we now call δ_2 the approximate value of δ derived from this last equation, we obtain

$$\delta_2 = -\frac{f(a_1)}{f'(a_1)},$$

and the new approximate value will be known, since we shall have

$$a_2 = a_1 + \delta_2.$$

In this manner the approximation can be carried on to any required degree of accuracy.

55. The following is the example to which Newton applied the process just described, for the sake of showing the practical advantages of the method of approximation he proposed.

Ex. $x^3 - 2x - 5 = 0$.

Here one root lies between 2 and 3, so that we may assume

$$a = 2 + \delta.$$

And we find that the first approximation will give us

$$a_1 = 2 + \delta_1,$$

$$\text{where } \delta_1 = -\frac{f(2)}{f'(2)}.$$

Now we have

$$\begin{aligned} f(x) &= x^3 - 2x - 5 \\ f'(x) &= 3x^2 - 2, \\ \text{whence } f(2) &= -1 \\ f'(2) &= 10; \end{aligned}$$

so that we obtain

$$\delta_1 = \frac{1}{10} = 0,1,$$

$$\text{and } a_1 = 2,1.$$

Again, we shall find, after a second application of the process,

$$\begin{aligned} \delta_2 &= -\frac{f(2,1)}{f'(2,1)} \\ &= -\frac{0,061}{11,23} \\ &= -0,0054 \ldots \text{ nearly}, \end{aligned}$$

so that we obtain

$$\begin{aligned} a_2 &= a_1 + \delta_2 \\ &= 2,1 - 0,0054 \\ &= 2,0946. \end{aligned}$$

Continuing the approximation, we find that

$$\delta_3 = -\frac{f(2,0946)}{f'(2,0946)}$$

$$= -\frac{0,000541708}{11,16196}$$

$$= -0,00004853 \text{ nearly,}$$

so that we obtain

$$a_3 = 2,0946 - 0,00004853$$

$$= 2,09455147.$$

And the approximation might be continued if required.

56. It was stated, that in general the approximate values $a_1, a_2, a_3, \ldots$ would converge towards the true value of the root a. This however depends on the possibility of rejecting the succeeding terms of the equation in comparison with the second term, in the approximation to the quantity δ, which is the error of the preceding approximate value of the root. Now if we denote the root by $a + \delta$, the two first terms of the equation will be $f(a) + \delta f'(a)$. But the succeeding terms will contain higher powers of δ, so that in general they will be much smaller than the second term. If, however, we happen to have $f'(a)$ very small, the second term may be of the same order as the third, or even of an inferior order: so that our mode of approximation is in this case totally incorrect. Now this will take place whenever the original equation $f(x) = 0$ has another root nearly equal to a; for then $f'(x) = 0$ has a root nearly equal to a, and therefore not differing much from a: so that $f'(a)$ will be a very small quantity. For if we call that root $a + \mu$, where μ is very small, we obtain $f'(a + \mu) = 0$, or

$$f'(a) + \text{terms involving } \mu = 0,$$

so that $f'(\alpha)$ is of the order of μ. There are also other cases in which $f'(a)$ may be very small. It is therefore necessary to find some criterion with respect to the applicability of Newton's method. The following is the reasoning of Lagrange on this subject; but no certain criterion was given, until the aspect of this branch of analysis was so completely changed by the discoveries of Fourier.

57. In order that the method of approximation may be applied with safety, it is necessary that the succeeding approximation shall give a result $a + \delta$, which is nearer the truth than the preceding result a: so that the condition of applying the method with certainty is, that we shall have $a + \delta$ differing less from the root α than a does, the algebraical sign of that difference being immaterial. In other words, we must have

$$(a + \delta - \alpha)^2 < (a - \alpha)^2,$$

$$\text{or } 2\delta . (a - \alpha) + \delta^2 = \text{neg.}$$

$$\frac{1}{2} + \left(\frac{a - \alpha}{\delta}\right) = \text{neg.}$$

$$\frac{1}{2} - (a - \alpha)\frac{f'(a)}{f(a)} = \text{neg.}$$

Now, if α, α', α'' . . . . denote the roots of the equation

$$f(x) = 0,$$

we have the identity

$$f(x) = (x - \alpha)(x - \alpha') \ldots .$$

and by differentiating this equation, after taking the logarithms of both sides, we find

$$\frac{f'(x)}{f(x)} = \frac{1}{x - \alpha} + \frac{1}{x - \alpha'} + \ldots \ldots$$

$$\text{and } \frac{f'(a)}{f(a)} = \frac{1}{a-\alpha} + \frac{1}{a-\alpha'} + \ldots\ldots$$

$$= \frac{1}{a-\alpha} + \text{R suppose.}$$

And our condition will now take the form

$$\frac{1}{2} - 1 - \text{R}\,(a-\alpha) = \text{neg.}$$

$$\text{or } \frac{1}{2} + \text{R}\,(a-\alpha) = \text{pos.}$$

Hence we must either have the two quantities

$$\text{R, and } a-\alpha,$$

of the same sign; or if they have different signs, their product must be less than $\frac{1}{2}$ neglecting its sign.

It is easy to form à posteriori equations, where that condition will not hold; we have only to suppose that the difference between α and α' is very small, and that a lies between them, but very near to α'; the rest of the roots being such as to give small terms in the value of R. Now in this case the signs of R, and $a-\alpha$, will be different; for the sign of R will depend on that of its first term $\frac{1}{a-\alpha'}$, which differs in sign from $a-\alpha$: also we may suppose R so large, that the product R $(a-\alpha)$ shall be numerically greater than $\frac{1}{2}$.

Again, if there are two impossible roots of the form $\rho\,(\cos\theta \pm \sqrt{-1}\sin\theta)$, we shall have two corresponding terms of R,

$$\frac{1}{a-\rho\cos\theta-\sqrt{-1}\,\rho\sin\theta} + \frac{1}{a-\rho\cos\theta+\sqrt{-1}\,\rho\sin\theta}$$

$$= \frac{2\,.\,(a-\rho\cos\theta)}{(a-\rho\cos\theta)^2+\rho^2\sin^2\theta};$$

and if we have sin θ very small, the two terms will be nearly

$$=\frac{2}{(a-\rho\cos\theta)};$$

so that if a lies between α and ρ cos θ, and is very nearly equal to the latter, the value of R is of the same kind as before, and the condition will not be satisfied.

It seems impossible, then, to establish a criterion, without a previous knowledge of the roots of the equation. But there is one case in which it may be shown that the method will necessarily be applicable. If a be greater than all the real roots $\alpha, \alpha', \alpha''$ . . . . and also greater than the possible parts of the imaginary roots, then every term of R corresponding to a real root, and the sum of every pair corresponding to a pair of impossible roots, will be positive; that is, R will be of the same sign as $a-\alpha$. And, on the contrary, if a is less than all the real roots, and the possible parts of the imaginary roots, every term of R will be negative; but $a-\alpha$ is also negative, so that the condition is still satisfied.

CHAP. VII.

ON THE COMPLETION OF NEWTON'S METHOD OF APPROXIMATION BY FOURIER.

58. Before entering upon the method of Fourier, it will be expedient to remark, that the separation of the roots has now been completely effected; so that by following his plan we can now be assured that only one root of the equation $f(x) = 0$ lies within any interval (a, b) proposed for examination. The object of the approximation is then to determine values nearer and nearer to the root lying between those limits, by a method at once regular in its plan and rapid in its effects. In this manner all the digits of the root will at last be found, if the number of those digits be limited; or the approximation may be pushed as far as may be thought fit, in case the number of digits of the root be infinite. The process of approximation is that given by Newton, so far as regards the nature of the operations. But it is only under certain limitations that this method can be employed with confidence of success. The question which must first be solved as a preliminary to the application of Newton's method, involves one of these limitations. It is thus stated by Fourier.

59. Though the limits (a, b) do not comprise more than one root of the given equation, yet a criterion is still wanted to point out whether the interval is sufficiently small to permit the commencement of the method of approximation at either of the limits (a, b). What then is the nature of this criterion?

60. We shall first remark, that it will always be possible to reduce the last three terms of the index-series for the interval (a, b) to the three indices

$$0, \quad 0, \quad 1.$$

For since $f(x) = 0$ has only one root between the limits (a, b), it follows that $f'(x) = 0$ cannot have a root equal to that root of the proposed equation $f(x) = 0$. Hence we can always obtain an interval, which shall include the root of $f(x) = 0$, and not include any root of $f'(x) = 0$. In other words, the last two terms of the index-series can always be rendered 0 and 1.

But we are not able to state the same proposition with respect to the equation $f''(x) = 0$; since in some particular cases it may happen that the root of $f(x) = 0$, comprised within the interval (a, b), is also a root of $f''(x) = 0$. We must inquire therefore, whether there be any common divisor $\phi(x)$ of the polynomials $f(x)$ and $f''(x)$; and if so, whether the equation $\phi(x) = 0$ has any root within the interval (a, b). Now if $\phi(x)$ does not exist, or if it exists, but $\phi(x) = 0$ has no root within the interval (a, b), then $f''(x) = 0$ can have no root equal to the root of $f(x) = 0$ which we are investigating: and it will be possible to include this root of $f(x) = 0$ within some interval which excludes all the roots of $f''(x) = 0$. In this case then we can reduce the three last terms of the index-series to the indices

$$0, \quad 0, \quad 1.$$

But if the common divisor $\phi(x)$ is found to exist, and also $\phi(x) = 0$ to contain a root between the limits (a, b), then there can be no other such root than the very root of $f(x) = 0$, of which we are in search.

For all the roots of $\phi(x) = 0$ are roots of $f(x) = 0$, which has only one root within the interval (a, b). We may, therefore, discover this root by commencing anew with the equation $\phi(x) = 0$, which is of lower dimensions than $f(x) = 0$. By this means we can discard such a particular case, inasmuch as it can be reduced to the general case of an inferior equation.

In all cases therefore we shall consider it possible to reduce the three last indices to

$$0, \quad 0, \quad 1.$$

61. We shall now proceed to show, that when this reduction of the index-series has been made, we may apply with confidence the method of approximation given by Newton, commencing the operation from one of the limits (a, b).

Suppose then that for the interval (a, b) we find that the last three terms of the index-series are

$$\begin{array}{ccc} f''(x), & f'(x), & f(x), \\ 0, & 0, & 1. \end{array}$$

And let the root comprised in this interval be γ; of which c is an approximate value, not lying beyond that interval, but whose extreme values are the limits a and b. We shall seek the condition that c must satisfy, in order that the approximation may be carried on with safety, commencing with c.

Now if we assume

$$\gamma = c + h,$$

$$\text{then } 0 = f(\gamma) = f(c + h);$$

and by the Newtonian approximation we find the new quantity

$$c_1 = c + h_1,$$

by using the approximate equation

$$0 = f(c) + h_1 f'(c)$$

to obtain the value of h_1, which is supposed to be nearly the correct value of h.

But the correct equation would have been, according to the rule of the remainder of Taylor's series,

$$0 = f(c) + hf'(\lambda),$$

where λ lies between c and γ.

Now, in order that the approximation may proceed with success, we must have the error of c_1 less than that of c, neglecting signs. Or, in other words, we must have

$$(\gamma - c)^2 > (\gamma - c_1)^2,$$

$$\text{or } h^2 > (h - h_1)^2,$$

$$\text{or } 2hh_1 - h_1^2 > 0,$$

$$\text{or } \frac{h}{h_1} - \frac{1}{2} = \text{pos.};$$

that is, we must have

$$\frac{f'(c)}{f'(\lambda)} - \frac{1}{2} = \text{pos.}$$

The first remark to be made here is, that it will not be safe to employ an interval (a, b) in which $f'(x)$ changes its sign. For as λ is unknown, we could not say à priori that $\frac{f'(c)}{f'(\lambda)}$ was itself a positive quantity, unless we knew that $f'(x)$ did not change its sign during that interval. Hence it becomes necessary to have the last index but one reduced to 0.

62. But this condition is not sufficient, for we must also have

$$\frac{f'(c)}{f'(\lambda)} - \frac{1}{2} = \text{pos.}$$

Now the true equation

$$0 = f(c+h)$$

may be written either in the form

$$0 = f(c) + h f'(\lambda),$$

or, expanding to one more term, in the form

$$0 = f(c) + h f'(c) + \frac{h^2}{2} f''(\mu),$$

where μ is also a quantity lying between c and γ. By the elimination of h from these two equations, we obtain

$$0 = 1 - \frac{f'(c)}{f'(\lambda)} + \frac{1}{2}\frac{f(c)\,f''(\mu)}{\{f'(\lambda)\}^2},$$

so that we have

$$\frac{f'(c)}{f'(\lambda)} - \frac{1}{2} = \frac{1}{2} + \frac{1}{2}\frac{f(c)\,f''(\mu)}{\{f'(\lambda)\}^2},$$

and our condition, that

$$\frac{f'(c)}{f'(\lambda)} - \frac{1}{2} = \text{pos.}$$

will manifestly be satisfied, if we have

$$\{f'(\lambda)\}^2 + f(c)\,f''(\mu) = \text{pos.}$$

The second remark to be made here is, that we cannot with safety use any interval in which $f''(x)$ changes its sign. For as μ is unknown, we cannot say à priori whether $f''(\mu)$ will satisfy any condition relating to signs, or not; unless we knew the sign of $f''(\mu)$, by knowing that $f''(x)$ keeps its sign for the whole of the interval. Hence the last index but two must be reduced to zero.

63. But we have still to satisfy the condition

$$\{f'(\lambda)\}^2 + f(c)f''(\mu) = \text{pos.}$$

Now as the values of λ, μ, are unknown, the only mode of satisfying with certainty the preceding condition, is to make

$$f(c)\ f''(\mu) = \text{pos.},$$

$$\text{or } f(c)\ f''(c) = \text{pos.};$$

observing that $f''(x)$ preserves its sign during the interval (a, b).

But we have, by the table of signs,

$$\left.\begin{array}{l} \dfrac{f''(a)}{f''(b)} = \text{pos.} \\[2ex] \dfrac{f(a)}{f(b)} = \text{neg.} \end{array}\right\};$$

$$\text{whence } \frac{f(a)\ f''(a)}{f(b)\ f''(b)} = \text{neg.},$$

and either the numerator or denominator of this fraction must be positive, whilst the other is negative.

The third remark now to be made is, that the method of Newton cannot be safely applied to both limits alike; but that only one of the limits can be employed with certainty of success. And the criterion for choosing that one of the two limits, suppose c, is that in the series denoted by sign (c) the last term and the last but two are alike.

64. We have now shown that there can always be found a limit c, such that the Newtonian approximation can be commenced with certainty of success. In other words, we shall always be able to deduce from the approximate value c a second approximation c_1, whose error shall be less than the error of c. But there remains still the question, whether we

can proceed with certainty to a third approximation c_2, commencing at the second approximation c_1. For this purpose it will be necessary to inquire into the nature of the approximation c_1, which has been obtained by the application of Newton's method under the limitations given by Fourier. Recurring then to the expanded forms of the true equation

$$0 = f(c+h),$$

we find, on the elimination of h,

$$0 = 1 - \frac{f'(c)}{f'(\lambda)} + \frac{1}{2}\frac{f(c)\, f''(\mu)}{\{f'(\lambda)\}^2};$$

or, if we take into consideration the choice of the limit c, we can write this equation in the form

$$\frac{f'(c)}{f'(\lambda)} - 1 = \text{pos.};$$

$$\text{that is, } \frac{h}{h_1} - 1 = \text{pos.},$$

$$\text{or } \frac{h - h_1}{h_1} = \text{pos.};$$

and expressing this condition in terms of c, c_1, and γ, we find that

$$\frac{\gamma - c_1}{c_1 - c} = \text{pos.};$$

hence c_1 lies between c and γ.

The fourth remark, then, to be made on the Newtonian process is, that the approximate value given by that method, under the limitations of Fourier, is always on the same side of the root as the limit from which the approximation commenced. Hence we have only to write the new limit c_1 for the old limit c, and the whole of the preceding limitations will of course be satisfied by the new interval; so that the process may be renewed with confidence, commencing at the last approximation c_1. And in this manner the process can be continued to any extent with certainty of success.

65. There is yet one question which must be solved before the Newtonian approximation can be considered complete. This relates to the measure of the degree of approximation at any stage of the operation. We must therefore seek some convenient expression for the limit, which the error cannot exceed. This limit will, in the first instance, be the difference $b - a$ of the interval, so that we have $h < b - a$; or rather $h^2 < (b-a)^2$, since we are speaking of the magnitude of the error, and not of its sign. The error of the next approximation will be $h - h_1$. Now we have

$$\left.\begin{aligned} 0 &= f(c) + h_1 f'(c) \\ 0 &= f(c) + hf'(c) + \frac{h^2}{2} f''(\mu) \end{aligned}\right\},$$

where μ lies between c and $c + h$, and therefore of course between a and b. But by subtracting the above equations we obtain

$$0 = (h - h_1) f'(c) + \frac{h^2}{2} f''(\mu),$$

$$\text{or } (h - h_1)^2 = \frac{h^4}{4} \left\{\frac{f''(\mu)}{f'(c)}\right\}^2;$$

that is, the error $h - h_1$ is of the order h^2, since we shall not, except in particular cases, find $\dfrac{f''(\mu)}{f'(c)}$ to be very large.

The best method for calculating a limit of the error $h - h_1$, is to take a coefficient K not far from the true value of $\dfrac{f''(\mu)}{f'(c)}$, yet always greater than that true value numerically, and then to consider $K^2 \dfrac{h^4}{4}$ as the limit of the square of the error $h - h_1$.

The necessity of taking notice of the limit of the error is, that we may not perform any useless arithmetical operation, by finding the successive approximations to too many places of decimals. For in finding the value of h_1, which is $\dfrac{-f(c)}{f'(c)}$, we ought to carry on the division only to so many decimal places as must of necessity be correct; and we must consequently stop before the digit which is of the same order as the probable error. Now, inasmuch as every error is of the same order as the square of the preceding one, it follows that at every new approximation the number of correct decimals given by the process will become doubled. Thus the process is not only perfectly certain to succeed, but the rapidity of approximation is accelerated continually; and it is on this account that it is preferable to any other method of approximation whatever.

CHAP. VIII.

ON THE METHOD OF APPROXIMATION GIVEN BY LAGRANGE, AS SIMPLIFIED BY THE THEOREMS OF FOURIER.

66. We shall suppose in general that the interval, whose limits are the successive integers c and $c+1$, contains at least one root γ of the equation under consideration; in some cases it may contain other roots besides γ, and the method of Fourier will always discover how many. The object then of this approximation is to discover immediately values nearer and nearer to each of the roots γ, δ, ... so contained in the interval $(c, c+1)$, commencing with the value c.

For this purpose, then, assume

$$x = c + \frac{1}{x'};$$

and substitute in the proposed equation

$$f(x) = 0;$$

the transformed equation will be

$$0 = f\left(c + \frac{1}{x'}\right)$$

$$= f(c) + \frac{1}{x'}f'(c) + \ldots + \frac{1}{x'^n};$$

or, when reduced to the ordinary form,

$$x'^n + x'^{n-1}\frac{f'(c)}{f(c)} + \ldots\ldots + \frac{1}{f(c)} = 0.$$

And the roots of this new equation correspond to the roots of the given equation, being connected with them respectively by the equation

$$x = c + \frac{1}{x'}.$$

Now if there are k values of x lying between c and $c + 1$, namely γ, δ, ..., it follows that there will be exactly k values of x', which are greater than unity, and which we shall denote by γ', δ', ... as corresponding to γ, δ, ... These values are roots of the equation in x', and the integers c', d', ... next below them may be found by the method of Fourier. If these integers are all different, the process now becomes separated for each of the roots; if all or any part of them are alike, the process is not yet separated for the corresponding roots. In all cases we are to proceed separately with each one of the different integers amongst the series c', d', . . . The following is the process for c'.

Assume

$$x' = c' + \frac{1}{x''},$$

and transform the equation as before. Then as many values as there are of x' lying between c' and $c' + 1$, so many values will there be of x'' greater than unity. Hence, if the method of Fourier points out k' roots of the equation in x', we shall have to seek for k' integers c'', d'', ... next below the roots of the equation in x''.

If there are not k' different integers, we cannot completely separate the process at present, but we are to proceed separately with each one of the different integers of the series

c'', d'', ... It is manifest that this process can be continued to any extent.

We shall then have, for the root γ,

$$\gamma = c + \cfrac{1}{c' + \cfrac{1}{c'' + \ldots}}.$$

Now if any of the integers c, c', c'', c''', ... be an exact root of the corresponding equation, instead of being merely a limit of the root, the continued fraction will close with that integer, and give the exact value of the root. In all other cases we must conclude that the root in question is incommensurable; and we may carry the approximation as far as we choose.

67. The successive convergents will be

(1.) $\quad c,$

(2.) $\quad c + \dfrac{1}{c'},$

(3.) $\quad c + \cfrac{1}{c'' + \cfrac{1}{c'''}},$

. . . . . .

or, if we assume

$$\begin{array}{ll} p_1 = c, & q_1 = 1, \\ p_2 = c'p_1 + 1, & q_2 = c'q_1, \\ p_3 = c''p_2 + p_1, & q_3 = c''q_2 + q_1, \\ p_4 = c'''p_3 + p_2, & q_4 = c'''q_3 + q_2, \\ . = . \; . \; . \; . & . = . \; . \; . \; . \end{array}$$

the successive convergents will be denoted by the vulgar fractions

$$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \frac{p_4}{q_4}, \ldots \ldots$$

Now, by eliminating c''' from the equations

$$\left.\begin{array}{l} p_4 = c'''p_3 + p_2 \\ q_4 = c'''q_3 + q_2 \end{array}\right\},$$

we have

$$\frac{p_4}{p_3} - \frac{q_4}{q_3} = \frac{p_2}{p_3} - \frac{q_2}{q_3},$$

$$\text{or } q_3p_4 - p_3q_4 = p_2q_3 - q_2p_3.$$

But, by a similar elimination of c'',

$$q_2p_3 - p_2q_3 = p_1q_2 - q_1p_2.$$

And lastly, by the elimination of c', we find that

$$q_1p_2 - p_1q_2 = q_1 = 1.$$

Hence we have

$$\frac{p_2}{q_2} - \frac{p_1}{q_1} = \frac{1}{q_1q_2},$$

$$\frac{p_3}{q_3} - \frac{p_2}{q_2} = \frac{-1}{q_2q_3},$$

$$\frac{p_4}{q_4} - \frac{p_3}{q_3} = \frac{1}{q_3q_4},$$

$$\cdot \quad \cdot \quad \cdot = \cdot \quad \cdot$$

so that we may infer generally

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(-1)^{n+1}}{q_n \cdot q_{n+1}}.$$

Hence the difference of the two convergents $\frac{p_{n+1}}{q_{n+1}}, \frac{p_n}{q_n}$, neglecting its sign, will be $\frac{1}{q_nq_{n+1}}$; and of course less than $\frac{1}{q_n{}^2}$.

But by observing that

$$\gamma = c + \cfrac{1}{c' + \cfrac{1}{c'' + \dots}},$$

$$\text{and } \frac{p}{q} = c,$$

$$\frac{p_1}{q_1} = c + \frac{1}{c'},$$

$$\frac{p_2}{q_2} = c + \cfrac{1}{c' + \cfrac{1}{c''}},$$

$$. = . \quad . \quad . \quad . \quad .$$

we perceive that the convergents are alternately less and greater than the true value γ. Hence γ will always lie between $\frac{p_{n+1}}{q_{n+1}}$ and $\frac{p_n}{q_n}$; and consequently the error of $\frac{p_n}{q_n}$ will be less than $\frac{1}{q_n{}^2}$.

68. The example to which Lagrange applies his method is the one Newton had chosen for the illustration of his own process of approximation.

The equation is

$$x^3 - 2x - 5 = 0.$$

An approximate value, namely the integer next below one of the roots, is 2. And the method of Fourier shows that the other two roots are imaginary. Putting then

$$x = 2 + \frac{1}{x'},$$

the transformed equation in x' will be

$$x'^3 - 10x'^2 - 6x' - 1 = 0.$$

Here the integer next below the required value of x' is 10.

Putting therefore

$$x' = 10 + \frac{1}{x''},$$

we find

$$61x''^2 - 94x''^2 - 20x'' - 1 = 0.$$

And the integer to be taken for the next below x'' is 1.

Assume therefore

$$x'' = 1 + \frac{1}{x'''},$$

and proceed as before.

We find for the value of the required root of the proposed equation

$$2 + \cfrac{1}{10 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \dots}}}}$$

So that the convergents are

$$\frac{2}{1}, \frac{21}{10}, \frac{23}{11}, \frac{44}{21}, \dots\dots$$

The tenth convergent is $\frac{16415}{7837}$; and of course the error is less than $\left(\frac{1}{7837}\right)^2$, that is, less than 0.00000001.

Hence the value of that convergent is correct to seven places of decimals : this value is

$$2.0945514,$$

which agrees with Newton's method.

It is however easily seen that this method is far inferior in brevity, in facility, and in regularity, to the process of Newton.

CHAP. IX.

ON THE INDIRECT RULES FOR THE SOLUTION OF EQUATIONS OF LOW DEGREES, WHICH HAVE BEEN ACCIDENTALLY DISCOVERED: WITH THE TRUE THEORY CONNECTING THESE METHODS, NAMELY THE APPLICATION OF THE METHOD OF SYMMETRICAL FUNCTIONS OF THE ROOTS TO THE SOLUTION OF THE EQUATION ITSELF: AND, LASTLY, THE REASON WHY THIS METHOD CANNOT BE EXTENDED BEYOND THE FOURTH DEGREE.

69. On the solution of quadratic equations by the method of completing the square.

Suppose that the given equation is

$$x^2 + px + q = 0.$$

The rule directs us to transpose q, and add $\frac{p^2}{4}$ to both sides, in order to render the first side a complete square: after this, the extraction of the square root of both members of the equation will reduce the quadratic to two simple equations, owing to the ambiguity of sign on either side, after extracting the square root.

The process is thus indicated:

$$x^2 + px + \frac{p^2}{4} = \frac{p^2}{4} - q,$$

$$\pm\left(x + \frac{p}{2}\right) = \pm\sqrt{\frac{p^2}{4} - q};$$

which last equation is equivalent to the two equations

$$x + \frac{p}{2} = +\sqrt{\frac{p^2}{4} - q},$$

$$x + \frac{p}{2} = -\sqrt{\frac{p^2}{4} - q};$$

and the two roots obtained are expressed by

$$-\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

Now this is obviously identical with the process of taking away the second term of the equation, by the method of transformation.

For let $x = y - \frac{p}{2}$,

then the transformed equation is

$$y^2 - \frac{p^2}{4} + q = 0,$$

$$y = \pm\sqrt{\frac{p^2}{4} - q},$$

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

70. On the solution of cubic equations by the method of Cardan.

Let the second term of the proposed cubic be taken away by transforming the equation, if necessary. The equation is then of the form

$$x^3 + qx + r = 0.$$

Suppose now that for the single symbol x we substitute the sum of two symbols, as $\alpha + \beta$. Then we shall have the equation

$$(\alpha + \beta)^3 + q(\alpha + \beta) + r = 0,$$

$$\text{or, } \left.\begin{array}{l} \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta) \\ \qquad + q(\alpha + \beta) \\ \qquad + r \end{array}\right\} = 0.$$

Now as x can be divided into two parts in an infinite number of ways, we may make a second assumption concerning these parts; and we shall suppose that they satisfy the condition

$$3\alpha\beta + q = 0.$$

The equation between α and β is thus divided into the two others

$$\left.\begin{array}{r} 3\alpha\beta + q = 0 \\ \alpha^3 + \beta^3 + r = 0 \end{array}\right\}.$$

Eliminating β, we have

$$\alpha^3 - \frac{q^3}{27\alpha^3} + r = 0$$

$$\alpha^6 + r\alpha^3 = \frac{q^3}{27}$$

$$\alpha^3 = \frac{-r}{2} \pm \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}.$$

Now since α and β enter into the two equations symmetrically, it follows that we should find the same values for β^3; so that the two roots may be considered as representing indifferently the two quantities a^3 and β^3. And we may write

$$a^3 = -\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}},$$

$$\beta^3 = -\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}};$$

and the value of x will be

$$\left(-\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)^{\frac{1}{3}}.$$

In finding the value of a, we must recollect that every cube root of any quantity will have three values, of which two are always imaginary; and the third will also be imaginary, if the quantity whose cube root is to be extracted, be itself imaginary. Again, β will have three values, which will correspond respectively to the three values of a. Hence x will have three values; so that the three roots of the cubic are found by one operation.

Suppose a', β', to be any one pair of values of a and β. Then, denoting by ρ the quantity

$$\cos\frac{2\pi}{3} + \sqrt{-1}\sin\frac{2\pi}{3},$$

we shall have

$$\rho^2 = \cos\frac{4\pi}{3} + \sqrt{-1}\sin\frac{4\pi}{3}.$$

And the cube roots of 1 are

$$1,\ \rho,\ \rho^2.$$

Hence the three values of x are

$$\left.\begin{array}{l} a' + \beta', \\ \rho a' + \rho^2\beta', \\ \rho^2 a' + \rho\beta'. \end{array}\right\}$$

71. We shall in all cases be able to find by the processes of arithmetic, this one pair of values α', β', except the quantity $\frac{r^2}{4} + \frac{q^3}{27}$ be negative.

Putting the sign of q in evidence, we are now to consider the solution of

$$x^3 - qx + r = 0,$$

in the case where we have

$$\frac{r^2}{4} - \frac{q^3}{27} = \text{neg.},$$

$$\text{or } \frac{q}{3} > \frac{r}{2},$$

neglecting the sign of r.

Now, in this case, we have

$$\alpha^3 = \frac{-r}{2} + \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}$$

$$= -\frac{r}{2} + \sqrt{-1}\sqrt{\frac{q^3}{27} - \frac{r^2}{4}}$$

$$= \frac{q^{\frac{3}{2}}}{3\sqrt{3}}\left\{-\frac{3\sqrt{3}r}{2q^{\frac{3}{2}}} + \sqrt{-1}\sqrt{1 - \frac{27r^2}{4q^3}}\right\}.$$

Assume now $\cos\theta = -\frac{3\sqrt{3}r}{2q^{\frac{3}{2}}}$; and we shall obtain

$$\alpha^3 = \frac{q^{\frac{3}{2}}}{3\sqrt{3}}\{\cos\theta + \sqrt{-1}\sin\theta\},$$

$$\alpha' = \sqrt{\frac{q}{3}}\left\{\cos\frac{\theta}{3} + \sqrt{-1}\sin\frac{\theta}{3}\right\};$$

obtaining one value by Demoivre's theorem.

And the corresponding value for β will be

$$\beta' = \frac{q}{3} \cdot \frac{1}{a}$$
$$= \sqrt{\frac{q}{3}} \left\{ \cos\frac{\theta}{3} - \sqrt{-1} \sin\frac{\theta}{3} \right\}.$$

Hence the three values of x are

$$\left. \begin{array}{l} 2\sqrt{\frac{q}{3}} \cdot \cos\frac{\theta}{3}, \\ 2\sqrt{\frac{q}{3}} \cdot \cos\frac{2\pi + \theta}{3}, \\ 2\sqrt{\frac{q}{3}} \cdot \cos\frac{4\pi + \theta}{3}. \end{array} \right\}$$

This case was termed by the old algebraists the irreducible case of Cardan's rule. We may remark that this happens when the three roots are all real.

72. The solution of a biquadratic equation by the method of Ferrari.

Let the second term of the given equation be taken away, if necessary: the equation is then of the form

$$x^4 + qx^2 + rx + s = 0.$$

Transpose all the terms but the first, and then add to both sides the quantity

$$2nx^2 + n^2.$$

The equation will then take the form

$$(x^2 + n)^2 = (2n - q)\, x^2 - rx + n^2 - s;$$

and since n is arbitrary, it will now be our object to determine n, so that the second side shall be an exact square; and then, by extracting the square root, the biquadratic is reduced to the two quadratic equations

$$x^2 + n = \pm \{(2n - q)^{\frac{1}{2}} x - (n^2 - s)^{\frac{1}{2}}\}.$$

Now the condition for the quantity

$$(2n - q)\,x^2 - rx + n^2 - s,$$

to be a perfect square, is that

$$(2n - q)\,(n^2 - s) = \frac{r^2}{4},$$

$$\text{or } \left(n - \frac{q}{2}\right)(n^2 - s) = \frac{r^2}{8};$$

that is, n must be a root of the cubic equation

$$n^3 - \frac{q}{2}\,n^2 - sn + \frac{qs}{2} - \frac{r^2}{8} = 0.$$

There will always be one real value of n, and this can be found by Cardan's rule. And, consequently, the biquadratic can always be reduced to two quadratics, and its four roots determined.

This method has sometimes been ascribed to Waring, instead of its real author, who was Cardan's pupil.

73. We shall now point out why the equation for the determination of n is a cubic.

If we examine the two quadratics to which the biquadratic has been reduced, we find for the last term of one of them, the quantity

$$n - \sqrt{n^2 - s},$$

and, consequently, this quantity is the product of the two roots of that quadratic; that is, of two of the four roots of the given biquadratic. Hence, if we denote these four roots by a, b, c, d, we shall have

$$n - \sqrt{n^2 - s} = ab.$$

Also, the last term of the other quadratic will be equal to the product of the remaining two roots; that is,

$$n + \sqrt{n^2 - s} = cd.$$

Hence, by addition, we find that

$$2n = ab + cd.$$

But since there was no reason for choosing the particular pair a, b of the four roots for one of the quadratics, and leaving the pair c, d for the other, it is evident that n must have the three values

$$\tfrac{1}{2}(ab + cd),\ \tfrac{1}{2}(ac + bd),\ \tfrac{1}{2}(ad + bc);$$

and, consequently, the equation for determining n is a cubic. These three values correspond to the three different pairs of quadratics to which the biquadratic can be reduced; and which are thus indicated by writing their roots

$$\begin{Bmatrix} ab \\ cd \end{Bmatrix} \quad \begin{Bmatrix} ac \\ bd \end{Bmatrix} \quad \begin{Bmatrix} ad \\ bc \end{Bmatrix}.$$

74. The solution of a biquadratic by the method of Des Cartes.

Let the proposed biquadratic, deprived of its second term, be

$$x^4 + bx^2 + cx + d = 0.$$

Now, since the first side of this equation may always be considered as the product of two quadratic factors with real coefficients, and since the coefficient of the second term will be the sums of the coefficients of the second terms of the quadratic factors; it follows that, because the second term has been taken away by transformation, we may assume

$$x^4 + bx^2 + cx + d = (x^2 + \sqrt{y}x + z)(x^2 - \sqrt{y}x + z'),$$

where y is some positive quantity, and z, z' are either positive or negative. Hence, performing the multiplication and equating the coefficients, we must have

$$\begin{aligned} z + z' - y &= b, \\ (z' - z)\sqrt{y} &= c, \\ zz' &= d: \end{aligned}$$

which three equations are sufficient for determining y, z, and z'.

For, by the first equation,

$$(z + z') \sqrt{y} = (y + b) \sqrt{y};$$

and, by the second,

$$(z' - z) \sqrt{y} = c;$$

$$\left.\begin{aligned} \text{whence, } 2z' \sqrt{y} &= (y + b) \sqrt{y} + c, \\ \text{and } 2z \sqrt{y} &= (y + b) \sqrt{y} - c \end{aligned}\right\};$$

and, by multiplication,

$$4zz'y = (y + b)^2 y - c^2,$$

or, observing that $zz' = d$,

$$y^3 + 2by^2 + (b^2 - 4d) y - c^2 = 0.$$

Now as the last term of this equation is essentially negative, there must be at least one real root which is positive; which may be found. Also, when y is found, z', z are known by the preceding equations: and the solution is reduced to that of the two quadratics

$$x^2 + \sqrt{y}.x + z = 0,$$
$$x^2 - \sqrt{y}.x + z' = 0;$$

which give the four roots required.

We observe that $+ \sqrt{y}$ is the sum of two of the roots of the proposed equation, and as there are $\frac{4.3}{1.2}$, or six ways of combining the roots two and two, and since every such combination leaves a supplementary combination of the other two roots, there are three ways of dividing the four roots into two pairs; that is, there are three ways of dividing the first side of the equation into two quadratic factors. It is on this account that the equation for finding y rises to the third degree.*

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* On this subject, see the excellent remarks in the note upon Art. 768 of Peacock's Treatise on Algebra.

75. The solution of a biquadratic by the method of Euler.

As before, let the equation, deprived of its second term, be

$$x^4 + bx^2 + cx + d = 0,$$

and suppose x to consist of three parts α, β, γ;

$$\therefore x = \alpha + \beta + \gamma,$$

$$\therefore x^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\beta\gamma + \alpha\gamma + \alpha\beta),$$

$$= \text{P} + 2(\beta\gamma + \alpha\gamma + \alpha\beta), \text{ for brevity.}$$

Transposing P, and squaring and writing Q for $\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2$, we have

$$x^4 - 2\text{P}x^2 + \text{P}^2 = 4\text{Q} + 8\alpha\beta\gamma(\alpha + \beta + \gamma)$$

$$= 4\text{Q} + 8\alpha\beta\gamma \cdot x;$$

$$\therefore x^4 - 2\text{P}x^2 - 8\alpha\beta\gamma \cdot x + \text{P}^2 - 4\text{Q} = 0,$$

which, compared with the proposed equation, gives

$$\alpha^2 + \beta^2 + \gamma^2 = \text{P} = -\tfrac{1}{2}b,$$

$$\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2 = \text{Q} = \tfrac{1}{4}(\text{P}^2 - d), = \tfrac{1}{16}(b^2 - 4d),$$

$$\text{and } \alpha\beta\gamma = -\frac{c}{8}.$$

From these equations it is evident that α^2, β^2, γ^2, are the roots of the auxiliary equation

$$y^3 + \frac{b}{2}y^2 + \frac{b^2 - 4d}{16}y - \frac{c^2}{64} = 0;$$

from which equation they may be found, and thence the values of x determined.

Since the last term of the cubic is negative, there is at least one real positive root, l suppose; and the other two are either both positive, both negative, or both impossible; and they may be denoted, in the three cases, by m, n, by $-m$, $-n$, and by $\rho(\cos\theta \pm \sqrt{-1}\sin\theta)$, where ρ is a positive number.

Hence we shall have $a = \pm \sqrt{l}$, and β, γ, respectively equal to

$$\pm \sqrt{m},\ \pm \sqrt{n} \text{ in the first case,}$$

$$\pm \sqrt{-m},\ \pm \sqrt{-n} \text{ in the second;}$$

and, in the third case,

$$\beta = \pm \sqrt{\rho}\left(\cos\frac{\theta}{2} + \sqrt{-1}\sin\frac{\theta}{2}\right),$$

$$\gamma = \pm \sqrt{\rho}\left(\cos\frac{\theta}{2} - \sqrt{-1}\sin\frac{\theta}{2}\right).$$

Now $x = a + \beta + \gamma$; and the only restriction upon the values of a, β, γ, is given by the equation $a\beta\gamma = -\frac{c}{8}$, which shows that the product $a\beta\gamma$ must have a different sign from that of c.

When r is negative, the product $a\beta\gamma$ must be positive; and applying this restriction to each of the three cases, we have, in the first case,

$$x = \sqrt{l} \pm (\sqrt{m} + \sqrt{n}), \text{ or } -\sqrt{l} \pm (\sqrt{m} - \sqrt{n});$$

in the second case,

$$x = \sqrt{l} \pm \sqrt{-1}\,(\sqrt{m} - \sqrt{n}), \text{ or } -\sqrt{l} \pm \sqrt{-1}\,(\sqrt{m} + \sqrt{n});$$

and in the third case,

$$x = \sqrt{l} \pm 2\sqrt{\rho}\cos\frac{\theta}{2}, \text{ or } -\sqrt{l} \pm 2\sqrt{-1}\,\sqrt{\rho}\sin\frac{\theta}{2}.$$

And, on the contrary, when r is positive, it will be found that the values of x are the same as the above, changing only the sign of $\sqrt{l}$ in each of them.

Hence we have only four roots in each case, although there were apparently eight roots if there had been no restriction to the values of a, β, γ.

76. We shall now proceed to show the general principle upon which these methods proceed. It is that of symmetrical functions, as before stated.

In general, we have one simple equation between the roots of the given equation, namely, that their sum is equal to the coefficient of the second term with its sign changed. Now the object of the general method of solution, by means of symmetrical functions, is to form $n-1$ other simple equations between the roots. This is done by assuming z equal to some linear function of the roots, and forming the equation in z, which must be one of $n-1$ dimensions, if possible, in order that we may have $n-1$ values of z.

By substituting these $n-1$ values of z, and permuting the roots of the given equation in the equation

$$z = \text{linear function of the roots},$$

we shall obtain $n-1$ simple equations between the roots, and thus be enabled, with the addition of the equation

$$-p_1 = \text{sum of the roots},$$

to find all the roots of the given equation. The process will be best exemplified in the equations of low degrees; and it will be shown that it fails to be of any practical utility for equations beyond the fourth degree.

The above method is due to Lagrange, who also demonstrated its failure for higher orders.

77. Solution of a cubic by the method of Lagrange.

Let the three roots of the cubic equation be a, b, c; the equation itself being

$$x^3 + qx + r = 0.$$

Then we have $a + b + c = 0$; and if we can form two other simple equations involving a, b, c, we shall be able to find the roots by the known method of elimination between simple equations.

Let one of these simple equations be

$$\tfrac{1}{3}(a + b\rho + c\rho^2) = z;$$

then, since there is nothing to distinguish one root from another, it follows that the elimination, which gives the final equation in z, must also gives z the values

$$\tfrac{1}{3}(c + a\rho + b\rho^2),$$
$$\tfrac{1}{3}(b + c\rho + a\rho^2),$$
$$\tfrac{1}{3}(a + c\rho + b\rho^2),$$
$$\tfrac{1}{3}(b + a\rho + c\rho^2),$$
$$\tfrac{1}{3}(c + b\rho + a\rho^2);$$

and the final equation will therefore be of six dimensions.

But if we assume

$$\rho = \cos\frac{2\pi}{3} + \sqrt{-1}\,\sin\frac{2\pi}{3}, \text{ (whence } \rho^3 = 1),$$
$$\tfrac{1}{3}(a + b\rho + c\rho^2) = z_1,$$
$$\tfrac{1}{3}(a + c\rho + b\rho^2) = z_2,$$

it will be evident, by inspection, that the six values of z are

$$z_1,\ z_1\rho,\ z_1\rho^2,$$
$$z_2,\ z_2\rho,\ z_2\rho^2;$$

and that u $(= z^3)$ has only the two values z_1^3, z_2^3. Also we shall have

$$\tfrac{1}{3}(a + b + c) = 0,$$
$$\tfrac{1}{3}(a + b\rho + c\rho^2) = z_1,$$
$$\tfrac{1}{3}(a + c\rho + b\rho^2) = z_2;$$

and if we observe that ρ is an impossible root of $\rho^3 - 1 = 0$, and therefore satisfies $\dfrac{\rho^3 - 1}{\rho - 1} = \rho^2 + \rho + 1 = 0$, we shall have

$$a = z_1 + z_2,$$
$$b = z_1\rho^2 + z_2\rho,$$
$$c = z_1\rho + z_2\rho^2.$$

Now let $u^2 + \mathrm{P}u + \mathrm{Q} = 0$ be the equation, in which u has the values z_1^3, z_2^3; then

$$-\mathrm{P} = z_1^3 + z_2^3,$$
$$= \tfrac{1}{27}\{2\Sigma(a^3) + 12abc + 3(\rho + \rho^2)\,\Sigma(a^2b)\},$$

and, since $abc = -r$, $\Sigma(a^3) = -3r$,

$$\rho + \rho^2 = -1,\ \Sigma(a^2b) = 3r,$$
$$\therefore \mathrm{P} = +r.$$

Again, $\mathrm{Q} = z_1^3 z_2^3$;

but we have $z_1 z_2 = \tfrac{1}{9}\{\Sigma(a^2) + (\rho + \rho^2)\,\Sigma(ab)\}$,

and, since $\Sigma(a^2) = -2q$, $\Sigma(ab) = q$,

$$\therefore \mathrm{Q} = \frac{-q^3}{27};$$

and the equation determining u is

$$u^2 + ru - \frac{q^3}{27} = 0;$$

$$\text{whence } u = -\frac{r}{2} \pm \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}.$$

$$\text{Hence } z_1 = \sqrt[3]{\left(-\frac{r}{2} \pm \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)},$$

$$\text{and } z_2 = \sqrt[3]{\left(-\frac{r}{2} \mp \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)}.$$

Also, the three roots are

$$z_1 + z_2,\ z_1\rho + z_2\rho^2,\ z_1\rho^2 + z_2\rho.$$

It is evident that the three roots cannot be possible, unless $z_1 z_2$ be both impossible. But in this case they will be so. For we may represent z_1, z_2, by $R(\cos\theta \pm \sqrt{-1}\sin\theta)$; and comparing $u^2 + ru - \frac{q^3}{27} = 0$, with $(u - z_1{}^3)(u - z_2{}^3)$, we have

$$2R^3 \cos 3\theta = -r,$$

$$R^6 = \frac{-q^3}{27},$$

whence R and θ can be found, since q is, in this case, negative. And the values of x will be $2R\cos\theta$, $2R\cos\left(\frac{2\pi}{3} + \theta\right)$ and $2R\cos\left(\frac{4\pi}{3} + \theta\right)$.

Let the given biquadratic equation be

$$x^4 + qx^2 + rx + s = 0;$$

and, as before, assume one of the simple equations to be

$$\tfrac{1}{4}(a + b\rho + c\rho^2 + d\rho^3) = z;$$

and there will in general be 24 values of z, by interchanging a, b, c, d.

But if we assume that $\rho = -1$, and that

$$z_1 = \tfrac{1}{4}\{(a + b) - (c + d)\},$$

$$z_2 = \tfrac{1}{4}\{(a + c) - (b + d)\},$$

$$z_3 = \tfrac{1}{4}\{(a + d) - (b + c)\},$$

then it is evident, from inspection, that z has only the six values

$$\pm z_1, \pm z_2, \pm z_3;$$

and, therefore, $u = z^2$ only has three values, z_1^2, z_2^2, z_3^2.

Let $u^3 + Pu^2 + Qu + R = 0$ be the equation determining u; then, since $a + b + c + d = 0$,

$$z_1 = \tfrac{1}{2}(a + b),$$

$$z_2 = \tfrac{1}{2}(a + c),$$
$$z_3 = \tfrac{1}{2}(a + d);$$

whence $z_1^2 + z_2^2 + z_3^2 = - \text{P},$

$$= \tfrac{1}{4}\{3a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad)\},$$

and, as it must involve the roots symmetrically, we may interchange a with $b, c, d,$ successively; whence, adding the four identical values of P, and taking the fourth part,

$$\text{P} = \frac{-1}{16}\{4\Sigma(a^2) + 4\Sigma(ab)\},$$
$$= \frac{+q}{2};$$

again, by a similar process, we have

$$\text{Q} = \frac{q^2 - 4s}{16};$$

also $z_1 z_2 z_3 = \tfrac{1}{8}(a + b)(a + c)(a + d),$

$$= \tfrac{1}{8}\{a^3 + a^2(b + c + d) + \Sigma(abc)\},$$
$$= -\frac{r}{8}, \text{ since } b + c + d = -a;$$

whence $\text{R} = \dfrac{-r^2}{64}.$

And the equation in u is

$$u^3 + \frac{q}{2}u^2 + \frac{q^2 - 4s}{16}u - \frac{r^2}{64} = 0.$$

The roots of this equation can be found, and we then have

$$0 = \tfrac{1}{4}(a + b + c + d),$$
$$z_1 = \tfrac{1}{2}(a + b - c - d),$$
$$z_2 = \tfrac{1}{2}(a + c - b - d),$$
$$z_3 = \tfrac{1}{2}(a + d - b - c);$$

whence a, b, c, d will be known.

The reducing cubic, in this case, is the same with that of Euler; and we must apply the same restriction to the double values of z_1, z_2, z_3, as before. For we have $z_1 z_2 z_3 = -\frac{r}{8}$, and therefore we must have such values of z_1, z_2, z_3, as shall make their product of a different sign from that of the coefficient r.

The theory of symmetrical combinations, which is found successful in resolving equations of the third and fourth degree, can be applied to those of higher orders. But, to use the words of Lagrange, "passé le quatrième degré, la méthode, quoiqu' applicable en général, ne conduit plus qu' à des equations résolvantes de degrés supérieurs à celui de la proposée." Thus, in the case of equations of the fifth degree, the theory leads to a reducing biquadratic; but to obtain its coefficients we must solve an equation of the sixth degree. So that the method is useless in practice.

There is, however, no doubt that the doctrine of permutations, and of symmetrical combinations of the roots, contains the principles from which we are to expect the resolution of equations of the higher orders, if that problem be possible.

In the 12th volume of the *Italian Society*, and in a work published at Modena in 1813, M. Paolo Ruffini has proved, that no function of five letters is susceptible of only three or four different values by the interchange of the letters. And M. Cauchy, in the 16th volume of the *Journal de l' Ecole Polytechnique,* has shown, that if a function of n letters has more than two values, it has at least k values, k being the prime number next below n.

On these grounds it has been inferred, that the resolution of equations of the fifth degree, and consequently of the higher orders, is in reality an impossible problem. (Lacroix, *Compt. des Elem. d' Algèbre*, p. 61). But it must be observed, that it is here assumed that the coefficients of the reducing equation are symmetrical functions of *all* the roots. It may, however, be possible that the resolution might be effected by means of equations, whose coefficients are only partial expressions susceptible of several values. On this supposition the problem may perhaps be considered as not altogether impossible.

THE END.

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