Presented to University of Cal.
by Wm J. Lewis.

A TREATISE

ON

PLANE AND SPHERICAL TRIGONOMETRY;

INCLUDING THE

CONSTRUCTION OF THE AUXILIARY TABLES;

A CONCISE

TRACT ON THE CONIC SECTIONS,

AND THE

PRINCIPLES OF SPHERICAL PROJECTION.

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J. Fagan, Stereotyper.

PREFACE.

It will probably appear, to some, a work of supererogation to add another tract on Plane and Spherical Trigonometry to the great number already before the public; especially as some of those treatises are the productions of men whose talents and attainments were unquestionably of the highest order. Still, it has appeared to me that, however valuable many of those works must be considered, there are none of them exactly suited to the use of schools in which this branch of mathematics is traced to its principles. It is, indeed, no unusual thing to find young men who have studied this science in the way it is commonly taught, who are very imperfectly acquainted with the nature, and almost entirely ignorant of the construction, of the tables which they are continually using. And it must be admitted, that, when the nature and construction of logarithms, and of sines and tangents, are explained by Algebra and common Geometry, the processes are generally either so obscure, or so prolix, as to discourage the majority of students. The Differential Calculus is well known to furnish the most direct, if not the only direct, and simple method of investigating the formulæ by which those tables are most expeditiously computed. But that calculus itself, as commonly exhibited, presents so many refined speculations, that very few, except those who have a taste for mathematical studies, can avail themselves of its advantages.

As it is evidently unscientific to erect a system, either in theory or practice, upon unknown principles, it has been my object, in the following work, to trace every process which is required to be adopted, to principles which are supposed to be previously understood. The student is supposed to be already acquainted with Algebra and Geometry. If the student is master of the first six, and the eleventh books of Euclid; or, which is nearly the same thing, of the first six, and the second supplementary book of Playfair's Geometry; and of as much algebra as is contained in the first ninety pages of my treatise; he may proceed with confidence to the study of the following tract.

This work was intended to include as much only of the Differential Calculus, as the elucidation of the science of Trigonometry required. I have therefore confined myself to differentials of the first order; and, by the use of proper expedients, have deduced the requisite formulæ from those differentials. Some of the methods used in this work are supposed to be new; and, if so, they may be considered as improvements upon the labours of my predecessors. Of this character are the investigations of Gregory's theorems for computing an arc in terms of its tangent, and for computing logarithms.

The treatises on Spherical Trigonometry with which our schools are supplied, are nearly all of them destitute of anything on the subject of Spherical Projections. This appears to me an important defect. A small tract on that subject, which I added, more than thirty years ago, to a Philadelphia edition of Thomas Simpson's Plane and Spherical Trigonometry, is the only one, so far as I know, which is to be found in our schools; unless we consider Davies's Descriptive Geometry as one. Simpson's work being now out of print, and the work of Davies, notwithstanding its merits, not appearing calculated to supply the place of the appendix, I have revised, or rather written anew, that part of my early labours, and subjoined it to this work.

The present treatise being designed as an introduction or preliminary to Astronomy, a concise tract on the Conic Sections, including all the properties of the ellipse and parabola which are usually cited by writers on that science, has been introduced. This appeared requisite, because some of those properties were unavoidably referred to in the tract on Spherical Projections; and, among the treatises on that subject already in print, it was not easy to fix upon any one to which I could refer, that would be generally known to my readers. Besides, it appeared no difficult matter to include in this work all the information which the astronomic inquirer would need in the prosecution of his studies. The Conic Sections being, as the name implies, derived from the cutting of a cone, I thought it more direct to deduce the primary propositions from the section of the cone, than to lay down first a plane figure, derived in a different way; and, after demonstrating most of its properties, to prove at last its identity with a conic section.

In the practical examples, some astronomical terms are used which are not defined, because it was supposed that few young persons would study this work without some previous acquaintance with terms so generally understood.

The references to the properties of geometrical figures are made to Playfair's Euclid; but they will generally apply to Simson's translation of Euclid's Elements. This oldest work on Geometry, with the few improvements introduced by the Scotch geometer, is, in my opinion, better calculated to lead the attentive student into an accurate acquaintance with this noble science, than any modern treatise which has fallen into my way. Among those improvements, however, the substitution of the language of Algebra, in the fifth book, in place of that which Euclid made use of, can hardly be reckoned as one. Playfair's exposition of the fifth definition is certainly a good one; but, in other respects, I consider Simson's translation of that book greatly preferable to the form in which Playfair has left it.

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PLANE TRIGONOMETRY.

INTRODUCTION.

In the practice of Trigonometry there are several tables generally used, the construction and uses of which constitute an essential part of the science. But when the construction of these tables is deduced from Geometry and common Algebra, the subject is certainly presented to the student in a very discouraging shape. The rules by which these tables are most readily computed, are easily derived from the Differential Calculus; but that branch of science, when pursued to any considerable extent, involves many refined and difficult speculations. It therefore generally happens that the students of Trigonometry acquire a practical acquaintance with the auxiliary tables, but understand neither their construction nor nature.

As those parts of the Differential Calculus which the construction of all the tables commonly used in trigonometrical calculations absolutely demands, lie within a narrow compass, and involve no very difficult inquiries, I shall employ a few pages in explaining the elements of this science, so as to enable the student to understand the nature and origin of the formulæ which are commonly used in the computation of the trigonometrical tables

ARTICLE 1. The Differential Calculus is founded essentially upon the relation which variable and dependent quantities, considered as decreasing till they vanish, bear to each other in their evanescent or vanishing state. This ratio is called their *ultimate ratio.* Newton observes that the ultimate ratio of evanescent quantities is not the ratio which they have before they vanish, nor afterwards; but the ratio with which they vanish.* As the ultimate ratio of vanishing quantities is not the ratio which they have *before* they vanish, but at the instant *when* they vanish, it is most convenient to determine that ratio by supposing the evanescent quantities to have actually vanished. This may often be done in a way which leaves no room for error or doubt. One or two examples in common Algebra will render this obvious:

$$\frac{a^2-x^2}{a-x}=a+x;$$

$$\frac{a^3-x^3}{a-x}=a^2+ax+x^2.$$

Now, these results being strictly correct, whatever value may be assigned to a or x, let x be supposed at first less than a, and to increase till it becomes equal to a; a^2-x^2 and $a-x$, are thus rendered decreasing and evanescent quantities, whose ultimate ratio is the ratio which they have, not before x becomes equal to a, but at the instant when that equality is attained. But when $x=a$, $a+x=2a=\frac{2a}{1}$; and $a^2+ax+x^2=3a^2=\frac{3a^2}{1}$. Hence the ultimate ratio of $a-x$ to a^2-x^2, when x approximates and ultimately equals a, is the ratio of $1:2a$; and the ultimate ratio of $a-x$ to a^3-x^3, under like circumstances, is the ratio of $1:3a^2$.

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* Principia, Scholium, Book I., Art. 1.

Art. 2. In the investigations connected with the Differential Calculus, quantities are considered under two very different aspects: constant and variable.

Constant quantities are such as remain unchanged, or retain the same value throughout the investigation. Variable quantities are those which change their value in the course of the solution or demonstration. Constant quantities are usually denoted by the initial letters, a, b, c, &c.; variable ones by the final letters, x, y, z, &c.

Art. 3. When two variable quantities enter into an equation, so that the value of one depends upon the value of the other, and one of them is increased by any quantity or increment; the quantity which must be added to the other to preserve the equation, is called the corresponding increment. Thus, if $y=ax^2$, and we increase x by the increment h, so that x becomes $x+h$, the new value of y will be

$$a(x+h)^2=ax^2+2axh+ah^2;$$

hence the corresponding increments of x and y are h and $2axh+ah^2$.

The variable quantity from which a differential is deduced is called an integral.

Differentials of dependent variables may be considered as the corresponding increments of those variables, in the ultimate state of the increments, when they are diminished till they vanish. Or differentials may be defined to be quantities having to each other the ratio which is the ultimate ratio of the corresponding increments, those increments being supposed to decrease till they vanish.

Hence it follows that one differential cannot enter into an equation without another. We never inquire into the absolute, but merely into the relative values of differentials.

Art. 4. The differential of a variable quantity is denoted by prefixing the letter d; thus dx signifies the differential of x. When the integral is a compound quantity, it is enclosed in brackets, or a point is introduced between it and the letter d: thus the differential of x^n is expressed by $d(x^n)$, or $d.x^n$;

whereas dx^n signifies the n power of dx, the same as $(dx)^n$. The differential of $x+y$ is also expressed by $d(x+y)$. In these cases the letter d is not an algebraic quantity, but a sign, or an abridgement of the words *differential of.*

ART. 5. Let $y = x + z$, and let x and z be increased by the increments h and k respectively. Denote the new value of y by y'; so that

$$y' = x + h + z + k;$$

wherefore $$y' - y = h + k;$$

or the increment of $y =$ the sum of the increments of x and z. Now, this being true, whether the increments are taken in their vanishing state or any other, we evidently have

$$dy = dx + dz.$$

Again: Let $$y = ax,$$

and $$y' = a(x+h) = ax + ah;$$

then $$y' - y = ah;$$

which is also true, whether the increment h is taken in its vanishing or finite state: consequently,

$$dy = adx.$$

ART. 6. Let $$y = x^2,$$

and $$y' = (x + h)^2 = x^2 + 2xh + h^2;$$

whence $$y' - y = 2xh + h^2,$$

and $$\frac{y' - y}{h} = 2x + h = \frac{2x + h}{1}:$$

consequently, $$h : y' - y : : 1 : 2x + h.$$

But the ultimate ratio of $1 : 2x + h$ is the ratio of $1 : 2x$;

wherefore, $$dx : dy : : 1 : 2x;$$

or . . . . $$dy = 2xdx;$$

that is, $d.x^2 = 2xdx$. If the equation $dy = 2xdx$ be put into this form,

$$\frac{dy}{dx} = 2x,$$

the last member $2x$ is called the differential coefficient.

ART. 7. To find the differential of xz.

Put $\qquad y = x + z$;

then $\qquad y^2 = (x + z)^2 = x^2 + 2xz + z^2$;

and (Art. 5,) $\qquad dy = dx + dz$;

also $\qquad d.y^2 = d.(x + z)^2 =$

(Art. 6) $\quad 2(x + z)(dx + dz) = 2xdx + 2xdz + 2zdx + 2zdz.$

But $\qquad d.y^2 = d(x^2 + 2xz + z^2) =$

(Arts. 5 & 6) $\qquad 2xdx + 2d(xz) + 2zdz.$

Comparing these values of $d.y^2$, we have

$$2d.(xz) = 2xdz + 2zdx;$$

or . . . $\qquad d.xz = xdz + zdx.$

Hence, $\qquad d.xyz = xd.yz + yzdx = xydz + xzdy + yzdx.$

Again:

$$\frac{d.xyz}{xyz} = \frac{dz}{z} + \frac{dy}{y} + \frac{dx}{x}:$$

and the same thing may be proved, whatever be the number of variables.

ART. 8. To find the differential of x^n, n being a whole positive number.

It is obvious that $x^n = x.x.x.x$ to n terms. Hence, by Art. 7,

$$\frac{d.x^n}{x^n} = \frac{dx}{x} + \frac{dx}{x} + \text{\&c. to } n \text{ terms } = \frac{ndx}{x}:$$

wherefore, by clearing of fractions,

$$d.x^n = nx^{n-1}dx.$$

ART. 9. To find the differential of $\frac{x}{y}$, both numerator and denominator being variable.

Put $$z = \frac{x}{y};$$

whence, $$yz = x,$$

and, (Art. 7), $$ydz + zdy = dx:$$

consequently,

$$dz = \frac{dx - zdy}{y} = \frac{dx - \frac{xdy}{y}}{y} = \frac{ydx - xdy}{y^2};$$

that is, $$d.\frac{x}{y} = \frac{ydx - xdy}{y^2}.$$

ART. 10. To find $d.x^{\frac{m}{n}}$, m and n being positive integers.

Put $$y = x^{\frac{m}{n}};$$

then, . . . $$y^n = x^m;$$

and, (Art. 8) $$ny^{n-1}dy = mx^{m-1}dx;$$

whence,

$$dy = \frac{mx^{m-1}dx}{ny^{n-1}} = \frac{m}{n}.\frac{x^{m-1}}{x^{m-\frac{m}{n}}}dx = \frac{m}{n}x^{\frac{m}{n}-1}dx.$$

ART. 11. To find the differential of $x^{-n} = \frac{1}{x^n}$.

Put $$y = \frac{1}{x^n};$$

then $$yx^n = 1.$$

But 1 being invariable, its differential is 0; therefore,

$$d.yx^n = 0.$$

Or, $$yd.x^n + x^n dy = 0;$$

that is, $$nyx^{n-1}dx + x^n dy = 0:$$

whence,

$$dy = \frac{-nyx^{n-1}dx}{x^n} = -n\frac{x^{n-1}dx}{x^{2n}} = \frac{-ndx}{x^{n+1}} = -nx^{-n-1}dx;$$

or, $$d.x^{-n} = -nx^{-n-1}dx.$$

From this article, and Arts. 8 and 10, it is evident that

$$d.x^n = nx^{n-1}dx,$$

whether n is integral or fractional, positive or negative.

ART. 12. Let the two ascending series, $Ax^a + Bx^b + Cx^c + Dx^d +$ &c., and $Mx^m + Nx^n + Px^p + Qx^q +$ &c., be always equal; so that whatever value may be assigned to x, we shall still have,

$$Ax^a + Bx^b + Cx^c + Dx^d + \&c. = Mx^m + Nx^n + Px^p + Qx^q + \&c.;$$

then, $\qquad a = m,\ b = n,\ c = p,$ &c.;

and $\qquad A = M, B = N, C = P,$ &c.

For, if possible, let a be less than m, and divide the equation by x^a; then

$$A + Bx^{b-a} + Cx^{c-a} + Dx^{d-a} + \&c. =$$
$$Mx^{m-a} + Nx^{n-a} + Px^{p-a} + Qx^{q-a} + \&c.$$

But as the series are both ascending ones, $b, c, d, m, n, p, q,$ &c. are all greater than a. Hence, if $x = 0$, all the terms of this equation, except the first, will vanish; hence in that case $A = 0$, which is evidently absurd. Therefore, a is not less than m; and in a similar way it may be proved that m is not less than a; therefore $a = m$, and the above equation becomes

$$A + Bx^{b-a} + Cx^{c-a} + Dx^{d-a} + \&c. =$$
$$M + Nx^{n-m} + Px^{p-m} + Qx^{q-m} + \&c.$$

And making $x = 0$, $A = M$. Consequently, $Ax^a = Mx^m$; wherefore,

$$Bx^b + Cx^c + Dx^d + \&c. = Nx^n + Px^p + Qx^q + \&c.$$

Then, by the same process of reasoning, we find $b = n$; and $B = N$. Hence the proposition is manifest.

ART. 13. An important application of the property just announced may be exhibited in the demonstration of Newton's binomial theorem.

It is evident that in the general development of $(1 + x)^n$, the first term must be 1; for when $x = 0$, $(1 + x)^n = 1^n = 1$. We may therefore assume

$$(1 + x)^n = 1 + Ax^p + Bx^q + Cx^r + Dx^s + \&c.$$

in which A, B, &c., are unknown, but determinate coefficients; and p, q, r, &c., unknown exponents, integral or fractional, positive or negative. Suppose x a variable quantity. Then, differentiating both sides of this equation, and dividing by dx, we have,

$$n.(1+x)^{n-1} = p\mathrm{A}x^{p-1} + q\mathrm{B}x^{q-1} + r\mathrm{C}x^{r-1} + s\mathrm{D}x^{s-1} + \&c.$$

Multiplying by $1 + x$,

$$n.(1+x)^{n} = p\mathrm{A}x^{p-1} + q\mathrm{B}x^{q-1} + r\mathrm{C}x^{r-1} + s\mathrm{D}x^{s-1} + \&c.$$
$$p\mathrm{A}x^{p} + q\mathrm{B}x^{q} + r\mathrm{C}x^{r} + s\mathrm{D}x^{s} + \&c.$$

Then, from first equation, multiplying by n,

$$n(1+x)^{n} = n + n\mathrm{A}x^{p} + n\mathrm{B}x^{q} + n\mathrm{C}x^{r} + \&c.$$

These equations being identical, we have by transposition,

$$p\mathrm{A}x^{p-1} + q\mathrm{B}x^{q-1} + r\mathrm{C}x^{r-1} + s\mathrm{D}x^{s-1} + \&c. = n +$$
$$(n-p)\,\mathrm{A}x^{p} + (n-q)\,\mathrm{B}x^{q} + (n-r)\,\mathrm{C}x^{r} + (n-s)\,\mathrm{D}x^{s} + \&c.;$$

and by comparing, first the exponents, and then the coefficients, (Art. 12,) we have

$$p-1=0;\ q-1=p;\ r-1=q;\ \&c.;$$

or, $$p=1;\ q=2;\ r=3;\ s=4;\ \&c.;$$

and

$$\mathrm{A}=n;\ \mathrm{B}=\frac{n-1}{2}\mathrm{A}=n.\frac{n-1}{2};\ \mathrm{C}=\frac{n-2}{3}\mathrm{B}=n.\frac{n-1}{2}.\frac{n-2}{3};\ \&c.$$

Consequently,

$$(1+x)^{n} = 1 + nx + n.\frac{n-1}{2}x^{2} + n.\frac{n-1}{2}.\frac{n-2}{3}x^{3} + \&c.$$

From this we readily obtain the development of $(a+b)^{n}$. For,

$$(a+b)^{n} = a^{n}\left(1+\frac{b}{a}\right)^{n};$$

in which we have $\frac{b}{a}$ instead of x in the preceding. Consequently,

$$(a+b)^{n} = a^{n}\left\{1 + n\frac{b}{a} + n.\frac{n-1}{2}\frac{b^{2}}{a^{2}} + n.\frac{n-1}{2}.\frac{n-2}{3}\frac{b^{3}}{a^{3}}\&c.\right\}$$

$$= a^n + na^{n-1}b + n.\frac{n-1}{2}a^{n-2}b^2 + n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}b^3 \text{ \&c.}$$

As the equation $d.x^n = nx^{n-1}dx$, on which this demonstration is founded, is equally correct whether n is integral or fractional, positive or negative, it is evident that the preceding development of $(a+b)^n$ is also correct, whatever may be the value of n.

Of Logarithms.

The calculations which are connected with Trigonometry are much facilitated by the use of logarithms; it will therefore be proper, in a treatise on that science, to explain their nature and use.

Art. 14. If we take a series of numbers in geometrical proportion, beginning with a unit, as $1, a, a^2, a^3, a^4, a^5, a^6, a^7$, &c., it is manifest that the product of any two of these terms is a term whose exponent is the sum of the exponents of the factors. Thus,

$$a^4.a^3 = a^7; \quad a^m.a^n = a^{m+n}.$$

If then $A = a^m$, and $B = a^n$,

$$AB = a^{m+n}:$$

$$A^2 = A.A = a^{m+m} = a^{2m}:$$

$$A^3 = a^{m+m+m} = a^{3m}, \text{ \&c.}$$

On the other hand,

$$\frac{a^7}{a^3} = a^4; \frac{A}{B} = \frac{a^m}{a^n} = a^{m-n};$$

$$A^{\frac{1}{2}} = a^{\frac{m}{2}}; \qquad A^{\frac{1}{3}} = a^{\frac{m}{3}}.$$

Hence it appears that a, being assumed equal to any number at pleasure, if we can find such values of m, n, &c., that $a^m = A$, $a^n = B$, &c., A, B, being given numbers, then calling m the logarithm of A, n the logarithm of B, &c.; the logarithm of AB will be the sum of the logarithms of A and B.

The logarithm of $\frac{A}{B}$ will be the logarithm of A diminished by the logarithm of B. In other words, the business of multiplying and dividing by given numbers may be effected by the addition and subtraction of their logarithms.

As $a^m = A$, a given number; we readily perceive that, by assuming different values of a, we shall change the value of m; that is, we shall have different numbers to denote the logarithm of a given number A, by varying the value of a. Thus it appears there may be an indefinite variety of systems, according to the various values which may be taken for a. This quantity a is called the radix or base of the system.

Art. 15. To investigate a formula by which the logarithm of any given number may be computed, we may assume $a^x = y$; y being any given number whatever; then $x =$ logarithm of y: and the object in view is to find a general expression for x in terms of y. If we suppose x to be variable, it is manifest that $y = a^x$ must also be variable.

In the first place, if $x = 0$, then $y = a^0 = 1$, whatever value may be assigned to a; it is therefore evident that the logarithm of 1 is 0 in every system.

Now, let $$y' = a^{x+h} = a^x . a^h = y a^h:$$
and, to reduce this second member to a more manageable form, put $$1 + b = a;$$
then, (Art. 13,)

$$y' = y(1+b)^h = y + y \left\{ hb + h.\frac{h-1}{2}b^2 + h.\frac{h-1}{2}.\frac{h-2}{3}b^3 + \&c. \right\}$$

Therefore,

$$y' - y = y \left\{ hb + h.\frac{h-1}{2}b^2 + h.\frac{h-1}{2}.\frac{h-2}{3}b^3 + \&c. \right\};$$

and, consequently,

$$\frac{y'-y}{h} = y \left\{ b + \frac{h-1}{2}b^2 + \frac{h-1}{2}.\frac{h-2}{3}b^3 + \&c. \right\}$$

Now, when $h = 0$, the series in the second member of this equation becomes

$$b - \tfrac{1}{2}b^2 + \tfrac{1}{3}b^3 - \tfrac{1}{4}b^4 + \tfrac{1}{5}b^5 - \&c.;$$

and this is the value to which this series approximates as h is diminished, and to which it arrives only at the instant when $h = 0$. Put, then,

$$b - \tfrac{1}{2}b^2 + \tfrac{1}{3}b^3 - \tfrac{1}{4}b^4 + \tfrac{1}{5}b^5 - \&c. = \frac{1}{m};$$

and it will be $$\frac{y' - y}{h} = \frac{y}{m}.$$

Hence the ultimate ratio of h to $y' - y$ is the ratio of m to y;

consequently, $$\frac{dy}{dx} = \frac{y}{m};$$

or, ... $$\frac{mdy}{y} = dx. \qquad \text{(A)}$$

In this equation, the value of m depends upon the value of a; and as a may be assumed at pleasure, we may assign any value we please to m. This is more convenient than to assume a value of a, and from that assumption to find the value of m. It is usual to call m the modulus of the system. When m is taken $= 1$, the logarithms thence deduced are called hyperbolic logarithms, because they correspond with certain areas contained between the curve and asymptotes of an equilateral hyperbola. In Briggs', or the common logarithms, the radix a is assumed $= 10$; but m is computed by a method hereafter explained.

Art. 16. As no general method has been discovered by which to express the logarithm of a number in finite terms of the number itself, we are obliged to have recourse to infinite series. When numbers are to be computed by means of such series, it is of importance to have the series constructed in such manner that the successive terms shall become smaller and smaller; so that, a limited number of terms being introduced into the computation, the rest of the series may be rejected without sensible error.

To find the logarithm of $a + z$, a being constant, and z variable.

By Art. 15, if $x =$ log. of y,

$$dx = \frac{mdy}{y}.$$

Assume, then, log. of $a+z =$ log. $a + \mathrm{A}z^n + \mathrm{B}z^p + \mathrm{C}z^q + \mathrm{D}z^r +$ &c.; in which the exponents n, p, q, &c., as well as the coefficients A, B, C, &c., are indeterminate. In this equation, if $z = 0$, we have log. $a =$ log. a, as it evidently ought to be; and the quantities n, p, q, A, B, C, &c., being susceptible of any value, positive or negative, integral or fractional, the above equation must express the log. of $a + z$, if it can be expressed at all in terms of z.

Differentiating this equation, and dividing by dz, we have

$$\frac{m}{a+z} = n\mathrm{A}z^{n-1} + p\mathrm{B}z^{p-1} + q\mathrm{C}z^{q-1} + r\mathrm{D}z^{r-1} + \text{\&c.}$$

By multiplication and transposition,

$$\left.\begin{array}{l} na\mathrm{A}z^{n-1} + pa\mathrm{B}z^{p-1} + qa\mathrm{C}z^{q-1} + ra\mathrm{D}z^{r-1} + \text{\&c.} \\ -m + n\mathrm{A}z^n + p\mathrm{B}z^p + q\mathrm{C}z^q + \text{\&c.} \end{array}\right\} = 0.$$

Equating the exponents and the coefficients respectively of the corresponding terms,

$$n - 1 = 0, p - 1 = n, q - 1 = p, r - 1 = q, \text{\&c.}$$

$$na\mathrm{A} - m = 0, pa\mathrm{B} + n\mathrm{A} = 0, qa\mathrm{C} + p\mathrm{B} = 0, ra\mathrm{D} + q\mathrm{C} = 0, \text{\&c.}$$

Hence, $\quad n = 1, p = 2, q = 3$, &c.;

and $\quad \mathrm{A} = \frac{m}{a}, \mathrm{B} = -\frac{m}{2a^2}, \mathrm{C} = \frac{m}{3a^3}, \mathrm{D} = -\frac{m}{4a^4}$, &c.

Consequently,

$$\log. (a + z) = \log. a + \frac{mz}{a} - \frac{mz^2}{2a^2} + \frac{mz^3}{3a^3} - \frac{mz^4}{4a^4} + \text{\&c.}$$

Putting $-z$ for $+z$,

$$\log.\ (a - z) = \log.\ a - \frac{mz}{a} - \frac{mz^2}{2a^2} - \frac{mz^3}{3a^3} - \frac{mz^4}{4a^4} - \&c.$$

Therefore, $\log.\ \dfrac{a+z}{a-z} = 2m \left\{ \dfrac{z}{a} + \dfrac{z^3}{3a^3} + \dfrac{z^5}{5a^5} + \&c. \right\}$

When $a = 1$,

$$\log.\ \frac{1+z}{1-z} = 2m \left\{ z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \&c. \right\}$$

To find a number in terms of its hyperbolic log.

In this case, if $x = \log.\ y$,

$$\frac{dy}{dx} = y.$$

Put $y = 1 + Ax^n + Bx^p + Cx^q + Dx^r + \&c.$; where $y = 1$, when $x = 0$, as it ought to be.

Differentiate and divide by dx; and

$$\frac{dy}{dx} (= y) = nAx^{n-1} + pBx^{p-1} + qCx^{q-1} + rDx^{r-1} + \&c.$$

Comparing these values of y, and equating the coefficients and exponents respectively, we have

$$y = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \&c.$$

Art. 17. From the formulæ contained in the last article, the logarithms of small numbers are readily computed; and those of large ones are easily deduced from the smaller. A few examples are subjoined.

Required the logarithm of 2.

Here $z = \dfrac{2-1}{2+1} = \frac{1}{3}$.

$$\begin{aligned} z &= \tfrac{1}{3} &&= .3333333333 \\ z^3 &= \tfrac{1}{9} z &&= .0370370370 \\ z^5 &= \tfrac{1}{9} z^3 &&= .0041152263 \\ z^7 &= \tfrac{1}{9} z^5 &&= .0004572474 \\ z^9 &= &&\ .0000508053 \end{aligned}$$

$$
\begin{aligned}
z^{11} &= .0000056450\\
z^{13} &= 6272\\
z^{15} &= 697\\
z^{17} &= 77
\end{aligned}
$$

And

$$
\begin{aligned}
z &= 3333333333\\
\tfrac{1}{3}z^3 &= 123456790\\
\tfrac{1}{5}z^5 &= 8230453\\
\tfrac{1}{7}z^7 &= 653211\\
\tfrac{1}{9}z^9 &= 56450\\
\tfrac{1}{11}z^{11} &= 5132\\
\tfrac{1}{13}z^{13} &= 483\\
\tfrac{1}{15}z^{15} &= 46\\
\tfrac{1}{17}z^{17} &= 5\\
\hline
& .3465735903
\end{aligned}
$$

Therefore, the log. of $2 = .6931471806\,m$; and log. of $8 = 2.0794415418\,m$.

Next, let the log. of $\frac{5}{4}$ be required. Here,

$$z = \frac{\frac{5}{4} - 1}{\frac{5}{4} + 1} = \tfrac{1}{9}.$$

This number being substituted for z in the foregoing equation, produces

$$\text{log. of } \tfrac{5}{4} = .2231435514\,m.$$

Then the sum of these logarithms is the

$$\text{log. of } 8 \times \tfrac{5}{4} = 10.$$

Therefore, the log. of $10 = 2.3025850932\,m$.

If now we desire to compute the common logarithm of any number, the radix of that system being 10, the log. of 10 is $= 1$: putting, therefore, the log. of 10 just found equal 1, we find,

$$m = \frac{1}{2.3025850932} = .4342944819.$$

The coefficients of m, in the log. of 2 first found, being multiplied by this number, the product is the common log. of 2.

Hence, the common log. of 2 = .3010299956.

From this log. we may find the logarithm of any power of 2 by multiplication alone.

ART. 18. The common logarithm of any prime number may be readily computed when that of the next inferior integral number is known.

Let $p =$ the number whose log. is required; $q =$ the preceding whole number; $R = 2m = .8685889638$; and putting

$$\frac{p}{q} = \frac{1+z}{1-z};$$

we find, $$z = \frac{p-q}{p+q} = \frac{1}{p+q}.$$

As an example, let the log. of 9 be required; that of 8, the third power of 2, being known.

In this case, $p = 9$, $q = 8$, and $z = \frac{1}{17}$. Hence, the

$$\text{log. of } \tfrac{9}{8} = \frac{R}{17} + \tfrac{1}{3}\frac{A}{(17)^2} + \tfrac{1}{5}.\frac{B}{(17)^2} + \tfrac{1}{7}.\frac{C}{(17)^2} = .0511525224;$$

in which A is the preceding term; B, C the preceding terms without the divisors, 3, 5.

To the log. of $\frac{9}{8}$ add the log. of 8, or three times the log. of 2; the sum .9542425093 = log. of 9;

and its half, or .4771212546 = log. of 3.

From these logarithms, the logarithms of all the powers of 3, and of all the products of 2 and 3, and of all the products of their powers, may be obtained by multiplication and addition.

As a second example, let the log. of 49 or 7^2 be required, the log. of 48 being known from those of 2 and 3. Here,

$p = 49$; $q = 48$; $\dfrac{1+z}{1-z} = \frac{49}{48}$; and $z = \frac{1}{97}$. Hence,

$$\text{log. of } \tfrac{49}{48} = \frac{R}{97} + \tfrac{1}{3}.\frac{A}{(97)^2} = .0089548426;$$

to this add the log. of 48; and the sum $= 1.6901960797$ is the log. of 49, and its half $= .8450980398$ is the log. of 7.

Art. 19. Although the methods already explained are sufficient to enable the student to compute the logarithm of any given number, yet there are other expedients for abridging the labour of such computations; one of which is the following:

Let a, b, c, be three equi-different numbers, whose common difference is 1; so that $a = b - 1$, and $c = b + 1$; then $ac = b^2 - 1$, and $ac + 1 = b^2$; consequently,

$$\frac{ac+1}{ac} = \frac{b^2}{ac}.$$

If now we put the first member of this equation in place of y or $\frac{1+z}{1-z}$ in the general equation, (Art. 16,) we shall have

$$z = \frac{1}{2ac+1};$$

a quantity which will converge more rapidly, the greater a and c are. Finding, then, the log. of

$$\frac{ac+1}{ac},$$

we have the log. of

$$\frac{b^2}{ac}.$$

If, now, the logarithms of any two of these numbers a, b, c, are known, the log. of the third is immediately determined. For, put

A=log. of a; B=log. of b; C=log. of c; and S=log. of $\frac{ac+1}{ac}$;

then, since

$$\frac{ac+1}{ac} = \frac{b^2}{ac};$$

it follows that $S = 2B - A - C$;
whence either A, B, or C being required, is immediately determined by means of the others.

As the series for computing the log. of

$$\frac{ac+1}{ac},$$

converges more rapidly, when a, &c. are large numbers, than when they are small ones, the labour is frequently abridged by computing the log. of a power or multiple of the number whose log. is required.

Let the log. of 11 be required, those above computed being considered as known.

Here we may take $a = 98$, $b = 99$, and $c = 100$; whence

$$z = \frac{1}{2ac+1} = \frac{1}{19601};$$

and $$S = \text{log. of } \frac{ac+1}{ac} = \frac{R}{19601} = .0000443135,$$

the other terms being rejected, because they do not affect the result short of the fourteenth decimal. Now, the log. of 98 is known from those of 49 and 2, and the log. of 100 = 2, the log. of $(10)^2$. Consequently, in the equation

$$S = 2B - A - C;$$

the terms are all known except B. Therefore,

$$B = \tfrac{1}{2}(A + C + S).$$

$$\begin{array}{lrl}
A = \begin{cases} \\ \\ \end{cases} & \begin{array}{r} .3010299956 \\ 1.6901960797 \end{array} & \begin{array}{l} \ldots \text{ log. of } 2. \\ \ldots \text{ log. of } 49. \end{array} \\
C = & 2. & \ldots \text{ log. of } 100. \\
S = & .0000443135 & \\
\hline
 & 2\,)\,3.9912703888 & \\
\hline
 & 1.9956351944 & \ldots \text{ log. of } 99. \\
\text{But} & .9542425093 & \ldots \text{ log. of } 9. \\
\hline
\text{wherefore,} & 1.0413926851 & \ldots \text{ log. of } 11.
\end{array}$$

ART. 20. As the radix of the common system is 10, the log. of 10 = 1, the log. of 100 = 2, the log. of 1000 = 3, &c.; hence it follows that the log. of any number less than 10 consists wholly of decimals; the log. of a number which is more than 10, but less than 100, is more than 1, but less than 2; the log. of a number which is more than 100, but less than 1000, is more than 2, but less than 3, &c.; that is, if the number is between 1 and 10, the integral part of the log. is 0; if the number is between 10 and 100, the integral part of the log. is 1; if the number is 100 or more, but less than 1000, the integral part of the log. is 2. This integral part is usually termed the index of the logarithm.

As a number, when multiplied or divided by any power of 10, is still indicated by the same significant figures, the position of the decimal point only being changed by the process; so the logarithm of a number being increased or diminished by adding or subtracting the log. of any power of 10, suffers no change except in the index or integral part. Hence we readily perceive that the index of the log. will be 0, 1, 2, or 3, according as the first left-hand significant figure of the corresponding natural number denotes units, tens, hundreds, or thousands.

The log. of 1 being 0, the log. of a proper fraction must be negative; yet, as a decimal number is equivalent to an integral one divided by some power of 10, the log. of a decimal number differs in nothing but the index from the log. of a whole number which is indicated by the same significant figures. The relation between the logarithmic index and the power of 10 denoted by the left-hand digit of the corresponding natural number, may be illustrated by arranging the integral logarithms and their corresponding natural numbers in adjacent lines, as follows:

3	2	1	0	—1	—2	—3	—4	Log.
1000	100	10	1	$\frac{1}{10}=.1$	$\frac{1}{100}=.01$	$\frac{1}{1000}=.0001$	$\frac{1}{10000}=.0001$	natural numb.

Here it is evident that if a natural number falls between 1 and 10, its log. will fall between 0. and 1., or it will consist wholly of decimals. If the number is between .1 and 1., the log. will be between —1 and 0; that is, the index of the log will be —1., while the decimal part of it will be positive. In like manner, when the natural number lies between .01 and .1, the index of the log. must be —2, and the decimal part of it a positive quantity. Hence we observe that when the natural number consists wholly of decimals, the logarithmic index will be —1., —2., —3., &c., according as the left-hand significant figure of the natural number denotes tenths, hundredths, thousandths, &c.

In printing tables of logarithms, it is usual to omit the index, leaving it to be supplied in practice upon the principles above explained.

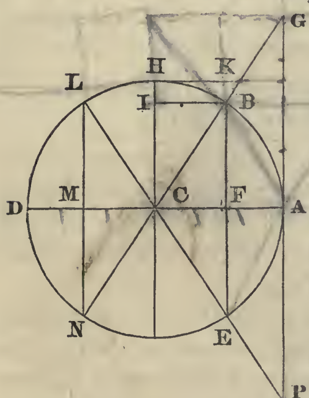
SECTION I.

PLANE TRIGONOMETRY.

The object of Plane Trigonometry is, when of the sides and angles of a plane triangle we have enough given to limit it, to determine the parts which are not given.

As every oblique angled triangle may be divided into two right angled ones, it is found most expedient to commence the subject by examining the relations and properties of triangles of the latter kind. The terms of the science are therefore adapted chiefly to right angled triangles.

Definitions.



Article 21. *Definition* 1. An *arc* of a circle is any part of the circumference, usually taken less than the whole. As AB, or BHD.

2. The *chord* of an arc is a right line drawn from one end of the arc to the other. Thus, BE is the chord of the arc BAE, or BDE.

3. The *sine* of an arc is a straight line drawn from one extremity of the arc, at right angles to the diameter, which passes through the other extremity. Thus, AD being a diameter to the circle, the line BF, at right angles to it, is the sine, or right sine, of the arc AB or DHB.

4. The *tangent* of an arc is the right line which touches

the circle at one extremity of the arc, and extends till it meets another right line, which is drawn from the centre through the other extremity. Thus AG, which touches the circle at A, is the tangent of AB.

5. The *secant* of an arc is the right line intercepted between the centre of the circle and the extremity of the tangent. Thus CG is the secant of the arc AB.

6. The *versed sine* of an arc is the part of the diameter intercepted between one end of the arc, and the sine which passes through the other end. Thus AF is the versed sine of AB, and DF is the versed sine of DHB.

7. The part by which an arc differs, in excess or defect, from a quadrant, or fourth part of the circumference, is called its *complement*. Thus, the arc ABH being a quadrant, HB is the complement of AB or of DHB.

8. The *cosine*, *cotangent* or *cosecant* of an arc, is the sine, tangent or secant of the complement of that arc. Thus BI, HK and CK, the sine, tangent and secant of HB, are termed the cosine, cotangent and cosecant of AB.

9. What an arc lacks of a semicircle, is called its *supplement*. Thus BHD is the supplement of AB.

10. The circumference of every circle is supposed to be divided into 360 equal parts, called *degrees*; each degree into 60 equal parts, called *minutes*; each minute into 60 equal parts, called *seconds*, &c. Degrees, minutes and seconds are designated thus, °, ′, ″.

11. As angles at the centre of a circle have to each other the same ratio as the arcs on which they stand (33.6); the latter are usually termed the *measures* of the former. Hence an angle at the centre of a circle is said to contain as many degrees, minutes and seconds, as the arc which subtends it. The sine, tangent, &c., of an arc, is also called the sine, tangent, &c. of the angle which is measured by the arc. Thus BF, the sine of AB, is called the sine of the angle ACB.

General Properties and Relations of Arcs, Sines, Tangents, &c.

ART. 22. If the arcs AH and DH are quadrants, and therefore equal, it follows (33.6) that the angles ACH and DCH are equal, and therefore are right angles. Hence the angle at the centre of a circle, subtended by a quadrant, is always a right angle.

ART. 23. The lines AG and HK, which touch the circle at A and H, are respectively at right angles to CA and CH (18.3); hence CAG and KHC are right angled triangles. Now the lines AG and CH, being at right angles to AC, are parallel to each other (28.1); consequently, the alternate angles HCK and CGA are equal; wherefore the triangles CHK and GAC are similar. The triangle CFB is also evidently similar to CAG; and CIB to CHK; therefore those four triangles are similar to each other. Also, the figure CFBI being a parallelogram, CI = FB, and CF = BI. From these triangles we have of course the following analogies:

1. As CF : FB : : CA : AG, or as cosine : sine : : radius : tang.
2. As CF : CB : : CA : CG, or as cosine : rad. : : radius : sec't.
3. As CI : CB : : CH : CK, or as sine : radius : : radius : cosec.
4. As AG : CA : : CH : HK, or as tang. : radius : : radius : cotan.
5. As CG : AG : : CB : BF, or as secant : tang. : : radius : sine.

In the algebraic formulæ used to express the relations of sines, tangents, &c., it is most convenient to assume the radius = 1. Making, therefore, this assumption, we may convert the foregoing analogies into the following equations.

1. $\dfrac{\text{sine}}{\text{cosine}} = \text{tangent}$; sine = cosine.tangent; $\text{cosine} = \dfrac{\text{sine}}{\text{tan}}$.

2. $\dfrac{1}{\text{cosine}} = \text{secant}$; cosine.secant = 1; $\cos = \dfrac{1}{\text{sec't}} = \dfrac{1}{\sqrt{1+\tan^2}}$

3. $\frac{1}{\text{sine}}$ = cosecant; sine.cosecant = 1.

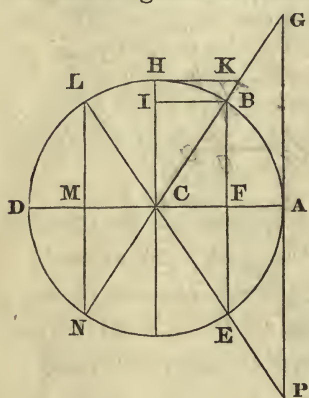
4. $\frac{1}{\text{tan}}$ = cotan; tan.cotan = 1; cotan $= \frac{1}{\text{tan}} = \frac{\text{cosine}}{\text{sine}}$.

Hence, taking P and Q any arcs, tan P. cotan P = tan Q. cotan Q; consequently (16.6),

tan P : tan Q : : cotan Q : cotan P.

5. sine $= \frac{\text{tangent}}{\text{secant}} = \frac{\text{tangent}}{\sqrt{1+\text{tan}^2}}$.

Art. 24. It is sometimes necessary to attend to the algebraic signs of these quantities, particularly when they are reduced to general formulæ.



An arc estimated in one direction is considered as positive, and in the opposite direction as negative. The same may be said of the sines, tangents, &c. Thus the arc AB, its sine FB, cosine CF, tangent AG, and secant CG, are considered as positive; but, when estimated in the opposite direction, they are considered as negative. Now, we readily perceive that when the arc is less than a quadrant, as AB is, the sine, tangent, &c., are all positive. But if we take the arc more than a quadrant, but less than a semicircle, as AL, the sine LM is still positive, but the cosine CM is negative, being measured from C in a direction opposite to CF. The tangent AP and secant CP are also negative; the former being drawn in a direction opposite to AG, and the latter not produced from C through L, the extremity of the arc, but in the opposite direction. If we take the arc more than a semicircle, but less than three quadrants, as AHDN; the sine MN becomes negative, the cosine CM also negative, the tangent AG positive; but the secant CG, not being produced through N, but in the opposite direction, is negative. If we take the arc more than three quadrants, but less than four, as AHDE; the sine EF is still negative, but the cosine CF and the secant CP are positive; the secant being produced from the centre, through the extremity of the arc, till it meets the tangent; but the tangent AP is negative.

These signs, when prefixed to the several quantities in the preceding equations, are found to be conformable to the algebraic rules for the adaptation of signs. In the first quadrant,

$$\frac{+\text{sine}}{+\text{cosin}}=+\text{tan};\ \frac{1}{+\text{cos}}=+\text{sec};\ \frac{1}{+\text{sine}}=+\text{cosec};\ \frac{1}{+\text{tan}}=+\text{cot}.$$

In the second quadrant,

$$\frac{+\text{sine}}{-\text{cosin}}=-\text{tan};\ \frac{1}{-\text{cos}}=-\text{sec};\ \frac{1}{+\text{sine}}=+\text{cosec};\ \frac{1}{-\text{tan}}=-\text{cot}.$$

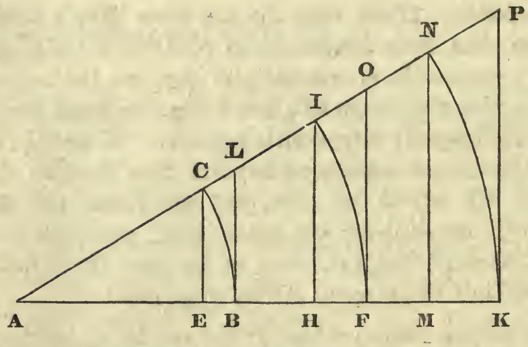
In the third quadrant,

$$\frac{-\text{sine}}{-\text{cosin}}=+\text{tan};\ \frac{1}{-\text{cos}}=-\text{sec};\ \frac{1}{-\text{sine}}=-\text{cosec};\ \frac{1}{+\text{tan}}=+\text{cot}.$$

In the fourth quadrant,

$$\frac{-\text{sine}}{+\text{cosin}}=-\text{tan};\ \frac{1}{+\text{cos}}=+\text{sec};\ \frac{1}{-\text{sine}}=-\text{cosec};\ \frac{1}{-\text{tan}}=-\text{cot}.$$

Art. 25. It is easily perceived that the sine, tangent, &c., of a given arc are limited, being dependent upon the length of the arc; but the sine, tangent, &c., of an angle, being the sine, tangent, &c., of the measuring arc, whatever may be the radius with which that arc is described, evidently admit various values. Thus EC, HI, MN, which are the sines of

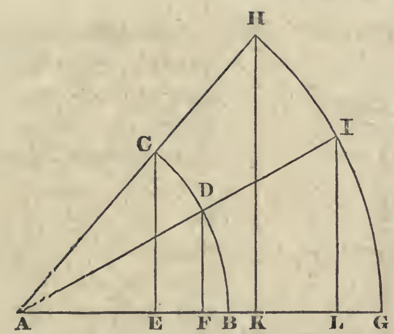


BC, FI, KN, respectively, are also the sines of the angle at A. The lines BL, FO, KP, which are the tangents of the same arcs, are likewise the tangents of the angle at A.

Art. 26. It appears from cor. to 15.4, that the side of a regular hexagon, inscribed in a circle, is equal to the radius of the circle. But the side of a regular hexagon, inscribed in a circle, subtends an arc of 60°; hence the chord of 60° is equal to the radius of the circle. Again, since a quadrant subtends a right angle at the centre of the circle (Art. 22), it is evident that the sine of a quadrant, or 90°, is the radius of the circle (see Fig. p. 34); thus HC the sine of AH, or ACH is the radius of the circle. Further, if we suppose CG to bisect the right angle ACH, we shall have CGA (which = HCG, by 29.1) = ACG; whence AG = CA; that is, the tangent of 45°, or half a right angle, = the radius. Thus it appears that the chord of 60°, the sine of 90°, and the tangent of 45°, are respectively equal to the radius of the circle.

Trigonometrical Propositions.

Art. 27. The sines of two angles adapted to any radius have to each other the same ratio as the sines of the same angles adapted to any other radius.



Let BAC and BAD be two angles, whose sines adapted to the radius AC or AB, are EC and FD; while the sines of the same angles adapted to the radius AG or AH, are KH and LI.

Since the angles at E, F, K and L, are right ones, it is evident that the triangles AEC and AKH are similar; as are also AFD and ALI. Consequently,

As AC : CE :: AH : HK;

and alternately,

AC : AH :: CE : HK.

In like manner,

As AD : AI :: DF : IL;

wherefore, CE : HK :: DF : IL;

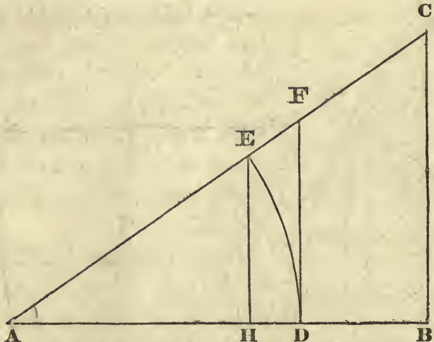
and again alternately,

CE : DF :: HK : IL.

Q. E. D.

Cor. If we substitute the word tangent or secant in place of sine, the proposition will still be true; and the demonstration will be made out in the same manner by drawing tangents to the circles at B and G, and using those tangents, or their secants, instead of the sines.

Art. 28. In any right angled plane triangle, as the hypothenuse is to the perpendicular, so is radius to the sine of the angle at the base; as the hypothenuse is to the base, so is radius to the cosine of the angle at the base; and as the base is to the perpendicular, so is radius to the tangent of the angle at the base.



Let ABC be a triangle, right angled at B; from A, with the radius AD, describe the arc DE, measuring the angle A; through E and D, draw the sine EH and tangent DF. Then the triangles ABC, ADF and AHE, being similar,

As AC : BC :: AE : HE;

As AC : AB :: AE : AH;

and AB : BC :: AD : DF;

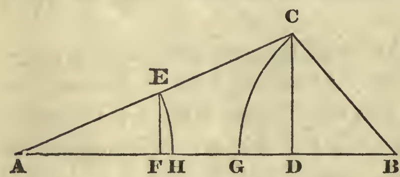
that is, As AC : BC :: radius : the sine of A;

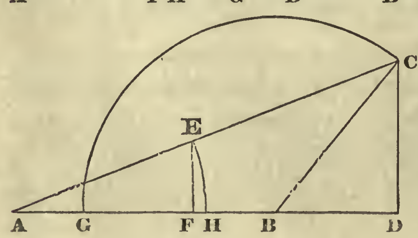
As AC : AB :: radius : the cosine of A;

and As AB : BC :: radius : the tangent of A.

Q. E. D.

Art. 29. In any right lined triangle, the sides have to each other the same ratio as the sines of the opposite angles.





Let ABC be a triangle; make AE = BC; from the centres B and A, with the radii BC and AE, describe the arcs CG and EH; from C and E, let fall on AB (produced if necessary) the perpendiculars CD and EF; these perpendiculars are the sines of CG and EH, or of the angles B and A to the radius BC or AE. Now, from the similar triangles ACD, AEF;

As AC : CD :: AE : EF (4.6);

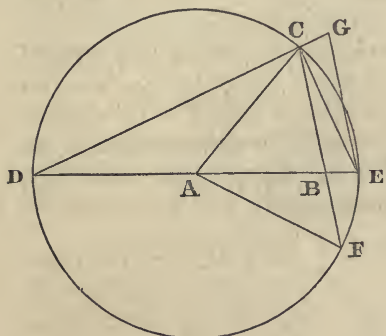
and alternately,

AC : AE :: CD : EF, :: sine of B : sine of A;

these sines being suited to any radius whatever (Art. 27).

Q. E. D.

Art. 30. In any right lined triangle, the sum of any two sides is, to their difference, as the tangent of half the sum of the angles, opposite to those sides, to the tangent of half their difference.



Let ABC be the triangle; AC, AB, the sides. From the centre A, with the distance AC, describe the circle DCEF; meeting AB, produced in D and E; and CB, produced in F; join AF, DC; and through E draw EG parallel to BC, meeting DC produced in G. Then it is evident that DB is the sum, and BE the difference, of AC and AB. The outward angle CAD of the triangle ABC, is equal to the two inward and opposite angles, ABC and ACB (32.1). But AEC, at the circumference, is equal to half the angle CAD at the centre (20.3); that is, AEC = half the sum of ABC and ACB. Again, AC = AF; therefore, AFB = ACB (5.1). But,

$$ABC = AFB + BAF \text{ (32.1)} = ACB + BAF;$$

consequently, BAF = the difference between ABC and ACB; and therefore ECF = half that difference (20.3). But EG being parallel to BC, the angle CEG = ECF. Furthermore, the angle DCE in a semicircle being a right one (31.3), ECG is also a right angle. Now, because BC is parallel to EG;

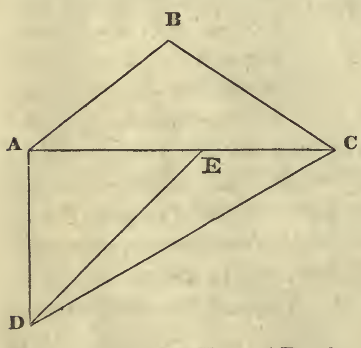
DB : BE :: DC : CG (2.6).

But CD is the tangent of CED, and CG the tangent of CEG, suited to the radius EC; and these tangents have to each other the same ratio as the tangents of the same angles adapted to any other radius (Art. 27). Hence,

$$AC + AB : AC - AB :: \text{tang of } \tfrac{1}{2}(ABC + ACB) : \text{tang of } \tfrac{1}{2}(ABC - ACB).$$

Q. E. D.

Art. 31. In any right lined triangle, having two unequal sides; as the less of those sides is to the greater, so is radius to the tangent of an angle; and as radius is to the tangent of the excess of that angle above half a right angle, so is the tangent of half the sum of the angles opposite to those sides, to the tangent of half their difference.



Let ABC be the triangle; AB the less, and AC the greater side. Draw AD at right angles to AC, and equal to AB; cut off AE, also = AB; and join DE and DC. Then, DAC being a right angle,

DA : AC :: rad : tangent of ADC (Art. 28).

Now, because AE = AD, the angle ADE = AED; hence each of those angles is half a right angle. Since the triangles

ADE and ADC have the angle at A common, the angles

$$ADC + ACD = ADE + AED = 2ADE.$$

Again, since

$$ADC = ADE + EDC = AED + EDC;$$

and $$AED = ACD + EDC \ (32.1):$$

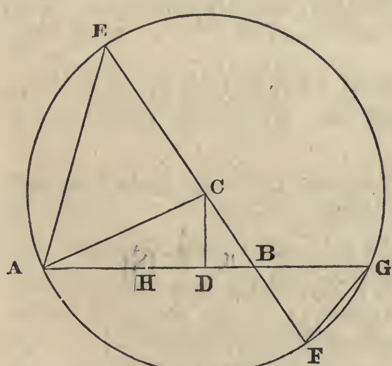
it follows that $$ADC - ACD = 2EDC;$$

that is, ADE is half the sum, and EDC half the difference, of ADC and ACD. Hence (Art. 30),

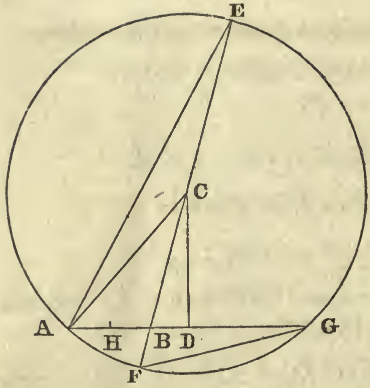
As tan of ADE : tan of EDC :: AC + AD : AC — AD :: tan $\frac{1}{2}$ (ABC + ACB) : tan $\frac{1}{2}$ (ABC — ACB).

But the tangent of ADE = radius (Art. 26); hence the above analogies are the same as those announced at the beginning of this article. *Q. E. D.*

Art. 32. In any plane triangle, as the base is to the sum of the sides, so is the difference of the sides to twice the distance between the middle of the base and the perpendicular falling upon it from the vertex of the triangle.



Let ABC be a triangle, whose base is AB. From the vertex C, with the greater side AC, describe the circle AEGF, cutting BC produced in E and F, and AB produced in G; join AE, FG; bisect AB in H, and draw CD at right angles to AB. Then, since CD, which passes through the centre of the circle, cuts AG at right angles,



$AD = DG$ (3.3);

or $AG = 2AD$;

and $AB = 2AH$;

therefore,

$$BG = 2HD.$$

Now, the angle $BAE = BFG$; and $AEB = BGF$ (21.3); consequently, the triangles ABE, FBG, are similar; wherefore,

$$AB : BE :: BF : BG;$$

that is, $AB : AC + BC :: AC - BC : 2HD$.

Q. E. D.

Art. 33. If the half difference of two unequal magnitudes be added to the half sum, the result is the greater magnitude; but if the half difference be subtracted from the half sum, the result is the less magnitude.

Let AB and BC denote any two unequal magnitudes, whose sum is AC and half the sum AE or EC; make $AD = BC$; then $DE = EB$, the half difference. Now, $AB = AE + EB$; and $BC = EC - EB$. *Q. E. D.*

Art. 34. When the sides of a triangle are given, we have the three following proportions for finding either of the angles.

Find half the sum of the three sides, and from that half sum subtract the sides severally. Then,

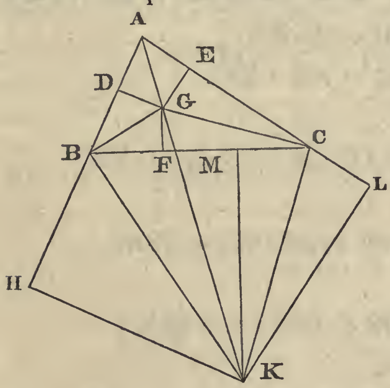
1. As the rectangle of the half sum, and the excess thereof above the side opposite the proposed angle, is to the rectangle of the other two remainders; so is the square of radius to the square of the tangent of half the angle.

2. As the rectangle of the sides containing the required angle is to the rectangle of the excesses of the half sum

above those sides respectively; so is the square of radius to the square of the sine of half the angle.

3. As the rectangle of the sides containing the required angle is to the rectangle of the half sum, and the excess thereof above the side opposite to the proposed angle; so is the square of radius to the square of the cosine of half the angle.

Let ABC be the triangle; produce AB, AC, to H and L. Bisect the angles BAC, ABC and HBC, by the lines AG, BG and BK, respectively; and let AG, BG, meet in G. Now, since the angle CBH is greater than BAC (16.1), it is obvious that HBK is greater than BAG; and, therefore, (13.1,) BAG + ABK is less than two right angles; consequently (cor. 29.1), BK and AG produced will meet. Let them meet in K; and draw KH, KM, KL, and GD, GF, GE, respectively, perpendicular to AB, BC and AC. Then it is obvious (4.4) that DG, FG and EG are equal; as well as KH, KM and KL.



Now, in the triangles ADG, AEG, the side AG being common, and the perpendiculars DG, EG, equal, we have (47.1) AD = AE. For a like reason, BD = BF, CE = CF, BH = BM, CL = CM, and AH = AL. As BH = BM, and CM = CL, it follows that

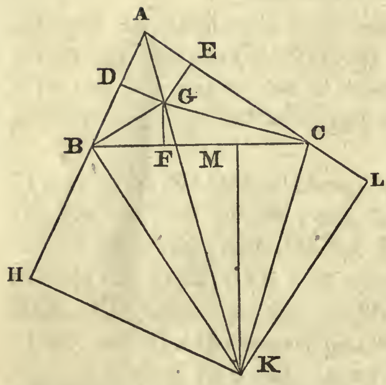
$$AH + AL = BC + AB + AC.$$

Hence AH or AL is equal to half the sum of the sides. But that half sum $= AD+BD+CF=AD+BC=BD+AC$.

Hence, $$AH - BC = AD;$$

$$AH - AC = BD;$$

$$AH - AB = BH.$$



Now, the angles ABC and CBH, being together equal to two right angles (13.1), DBG + HBK = one right angle = DBG + BGD. Consequently, the triangles DBG, HBK, are equiangular. Also, the triangles ADG and AHK are equiangular.

Hence (4.6)

$$BD : DG :: HK : HB;$$

wherefore (16.6) $BD.HB = HK.DG$.

Also, $AD : DG :: AH : HK$;

consequently (23.6),

$$AD^2 : DG^2 :: AH.AD : HK.DG, \text{ or } BD.HB.$$

But (Art. 28 and 22.6),

$$AD^2 : DG^2 :: \text{rad}^2 : \tan^2 DAG \text{ or } \tfrac{1}{2}BAC.$$

Therefore,

$$AH.AD : BD.HB :: \text{rad}^2 : \tan^2 \tfrac{1}{2}BAC.$$

This is the first proportion.

Again, it has been proved that CF = CE, and GF = GE, CG being common; hence (8.1) the angle GCF = GCE; wherefore,

$$GCA + GAC \ (=CGK) = \tfrac{1}{2}BCA + \tfrac{1}{2}BAC = \tfrac{1}{2}HBC \ (32.1) = HBK.$$

Consequently (13.1), AGC = ABK; these angles being the supplements of CGK, HBK. Also the angle GAC = BAK. Therefore, the triangles AGC and ABK are equiangular; whence (4.6),

$$AB : AK :: AG : AC \therefore AK.AG = AB.AC.$$

Also (4.6), $AG : GD :: AK : HK$;

wherefore (23.6),

$$AG^2 : GD^2 :: AK.AG : HK.DG :: AB.AC : BD.HB.$$

But (Art. 28 and 22.6)

$$AG^2 : GD^2 :: \text{rad}^2 : \sin^2 DAG \text{ or } \sin^2 \tfrac{1}{2}BAC.$$

Hence, $AB.AC : BD.HB : \text{rad}^2 : \sin^2 \frac{1}{2}BAC$;

which is proportion 2d.

Further:

$$AG : AD :: AK : AH \therefore (23.6)\ AG^2 : AD^2 :: AK.AG : AH.AD :: AB.AC : AH.AD.$$

But (Art. 28 and 22.6),

$$AG^2 : AD^2 :: \text{rad}^2 : \cos^2 DAG \text{ or } \cos^2 \tfrac{1}{2}BAC.$$

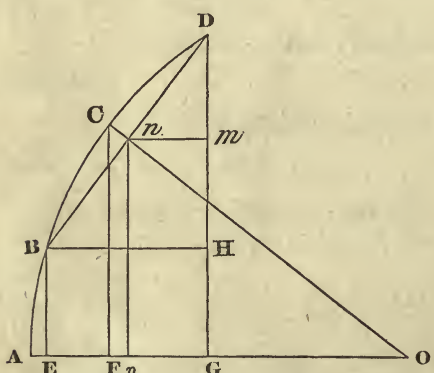
Consequently,

$$AB.AC : AH.AD :: \text{rad}^2 : \cos^2 \tfrac{1}{2}BAC;$$

which is the third proportion.

SECTION II.

The properties of plane triangles, which are explained in the preceding section, are sufficient, with the aid of the usual auxiliary tables, to enable the student to solve all the common cases in Plane Trigonometry. But for the solution of more complex problems, and particularly for the purpose of understanding the manner in which the trigonometrical tables are computed, it is necessary to investigate other theorems. This is most readily effected by the analytical method. In what follows, the radius to which the sines, tangents, &c., are adjusted, is always taken $= 1$. But it may be observed that whenever it is required to apply the results, here obtained, to the case where the radius is denoted by any other number, nothing more is necessary than to change all the trigonometrical lines in the same ratio in which the radius is changed.



Article 35. Let AB, AC, AD, be three equidifferent arcs, whose common difference is BC or CD. From the centre O, draw OA, OC; from B, C, D, draw BE, CF, DG, at right angles to OA; join BD, meeting OC in n; through B, n, draw BH, nm, parallel to AO; and np parallel to CF. Then, since the arc BC = CD, if we suppose BO and DO joined, those arcs will subtend equal angles at O (27.3).

Hence (4.1) Bn = Dn; and BnO = DnO; consequently, DnO is a right angle. Hence, BE = sin AB; CF = sin AC; DG = sin AD; Dn = sin CD or BC; OE = cos AB; OF = cos AC; OG = cos AD; On = cos CD or BC. Since np is parallel to CF, and nm to BH, it is obvious that the triangle Onp is similar to OCF; and Dnm to DBH; and as DB = 2Dn, it follows that BH = 2nm; and DH = 2Dm. Since DnO = mnp, both being right angles, Dnm = Onp; and the angles at m and p are right ones; therefore Dnm, Onp, are similar triangles. Of course, the three Dnm, Onp, OCF, are similar. Hence, we have the following proportions:

$$\text{As OC : CF :: O}n : np.$$

$$\text{As OC : OF :: O}n : \text{O}p.$$

$$\text{As OC : OF :: D}n : \text{D}m.$$

$$\text{As OC : CF :: D}n : nm.$$

Taking now OC = 1, and substituting for CF, OF, &c., sin AC, cos AC, &c., these proportions furnish the following equations:

$$np = \sin \text{AC}. \cos \text{CD}. \quad (A)$$

$$\text{O}p = \cos \text{AC}. \cos \text{CD}. \quad (B)$$

$$\text{D}m = \cos \text{AC}. \sin \text{CD}. \quad (C)$$

$$nm = \sin \text{AC}. \sin \text{CD}. \quad (D)$$

From equations A and C,

$$\sin \text{AD}(=np+\text{D}m)=\sin \text{AC}. \cos \text{CD}+\cos \text{AC}. \sin \text{CD}. \quad (1)$$

and

$$\sin \text{AB}(=np-\text{D}m)=\sin \text{AC}. \cos \text{CD}-\cos \text{AC}. \sin \text{CD}. \quad (2)$$

From equations B and D,

$$\cos \text{AD}(=\text{O}p-nm)=\cos \text{AC}. \cos \text{CD}-\sin \text{AC}. \sin \text{CD}. \quad (3)$$

and

$$\cos \text{AB}(=\text{O}p+nm)=\cos \text{AC}. \cos \text{CD}+\sin \text{AC}. \sin \text{CD}. \quad (4)$$

By adding equations 1, 2;

$$\sin AD + \sin AB = 2 \sin AC. \cos CD. \quad (5)$$

By subtracting,

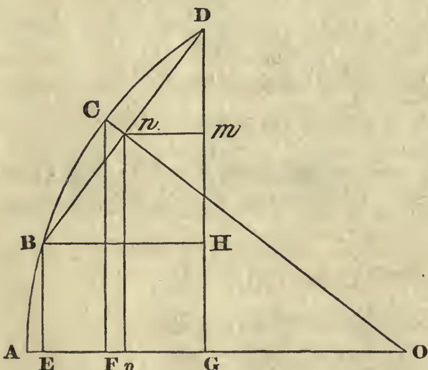
$$\sin AD - \sin AB = 2 \cos AC. \sin CD. \quad (6)$$

By adding equations 3, 4;

$$\cos AB + \cos AD = 2 \cos AC. \cos CD. \quad (7)$$

By subtracting,

$$\cos AB - \cos AD = 2 \sin AC. \sin CD. \quad (8)$$



Art. 36. From these equations, a number of others may be deduced.

Suppose AC = CD; and put AC = a; then AD = $2a$, and AB = 0; and equation 1 becomes $\sin 2a = 2 \sin a. \cos a.$ (1)

equation 3, $\cos 2a = \cos^2 a - \sin^2 a.$ (2)

equation 4, $\cos 0 = 1 = \cos^2 a + \sin^2 a,$

which corresponds with 47.1.

From the equations for $\sin 2a$ and $\cos 2a$, it is manifest that

$$\sin a = 2 \sin \tfrac{1}{2}a. \cos \tfrac{1}{2}a. \quad (3)$$

and

$$\cos a = \cos^2 \tfrac{1}{2}a - \sin^2 \tfrac{1}{2}a. \quad (4)$$

But $\cos^2 \frac{1}{2}a = 1 - \sin^2 \frac{1}{2}a. \therefore \cos a = 1 - 2 \sin^2 \frac{1}{2}a.$ (5)

From the last, $2 \sin^2 \frac{1}{2}a = 1 - \cos a.$ (6)

In equation 4, substitute for $\sin^2 \frac{1}{2}a$ its equal $1 - \cos^2 \frac{1}{2}a$, and the equation becomes

$$\cos a = 2 \cos^2 \tfrac{1}{2}a - 1. \quad (7)$$

From this equation,

$$2 \cos^2 \tfrac{1}{2}a = 1 + \cos a. \quad (8)$$

By Art. 23.1,

$$\tan \tfrac{1}{2}a = \frac{\sin \frac{1}{2}a}{\cos \frac{1}{2}a} = \frac{2 \sin \frac{1}{2}a . \cos \frac{1}{2}a}{2 \cos^2 \frac{1}{2}a} = \text{(eq. 3, 8)}$$

$$\frac{\sin a}{1 + \cos a}. \quad (9)$$

By Art. 23.4,

$$\cot \tfrac{1}{2}a = \frac{\cos \frac{1}{2}a}{\sin \frac{1}{2}a} = \frac{2\cos \frac{1}{2}a . \sin \frac{1}{2}a}{2\sin^2 \frac{1}{2}a} = \text{(eq. 3, 6)} \frac{\sin a}{1 - \cos a}. \quad (10)$$

$$\tan^2 \tfrac{1}{2}a = \frac{\sin^2 \frac{1}{2}a}{\cos^2 \frac{1}{2}a} = \frac{2\sin^2 \frac{1}{2}a}{2\cos^2 \frac{1}{2}a} = \text{(eq. 6, 8)} \frac{1 - \cos a}{1 + \cos a}. \quad (11)$$

$$\cot^2 \tfrac{1}{2}a = \frac{\cos^2 \frac{1}{2}a}{\sin^2 \frac{1}{2}a} = \frac{2 \cos^2 \frac{1}{2}a}{2 \sin^2 \frac{1}{2}a} = \text{(eq. 6, 8)} \frac{1 + \cos a}{1 - \cos a}. \quad (12)$$

ART. 37. Take now $AC = a$, $CD = b$; whence

$$AD = a + b;$$

and

$$AB = a - b;$$

and equations 1 and 2, Art. 35, become

$$\sin (a \pm b) = \sin a . \cos b \pm \cos a . \sin b. \quad (1)$$

3 and 4 become,

$$\cos (a \pm b) = \cos a . \cos b \mp \sin a . \sin b. \quad (2)$$

Now,

$$\tan (a \pm b) = \frac{\sin(a \pm b)}{\cos(a \pm b)} = \frac{\sin a . \cos b \pm \cos a . \sin b}{\cos a . \cos b \mp \sin a . \sin b} = \text{(di-}$$

viding numerator and denominator by cos a. cos b, and using tan for $\frac{\sin}{\cos}$) $\frac{\tan a \pm \tan b}{1 \mp \tan a.\ \tan b}$. (3)

Equations 5, 6, 7, 8, also become

$$\sin(a+b)+\sin(a-b)=2\sin a.\cos b. \quad (4)$$

$$\sin(a+b)-\sin(a-b)=2\cos a.\ \sin b. \quad (5)$$

$$\cos(a-b)+\cos(a+b)=2\cos a.\cos b. \quad (6)$$

$$\cos(a-b)-\cos(a+b)=2\sin a.\ \sin b. \quad (7)$$

By Art. 23.1,

$$\tan a \pm \tan b = \frac{\sin a}{\cos a} \pm \frac{\sin b}{\cos b} = \frac{\sin a.\cos b \pm \cos a.\sin b}{\cos a.\cos b.} =$$

(eq. 1) $\frac{\sin(a \pm b)}{\cos a.\cos b}$. (8)

By Art. 23.4,

$$\cot b \pm \cot a = \frac{\cos b}{\sin b} \pm \frac{\cos a}{\sin a} = \frac{\sin a.\cos b \pm \cos a.\sin b}{\sin a.\ \sin b} =$$

$\frac{\sin(a \pm b)}{\sin a.\ \sin b}$. (9)

By changing our notation, other equations may be deduced.

Let AD $= a$; AB $= b$; then AC $= \frac{1}{2}(a+b)$, and DC or BC $= \frac{1}{2}(a-b)$. With this notation, equations 5, 6, 7, 8, Art. 35, become

$$\sin a + \sin b = 2\sin\tfrac{1}{2}(a+b).\cos\tfrac{1}{2}(a-b). \quad (10)$$

$$\sin a - \sin b = 2\cos\tfrac{1}{2}(a+b).\sin\tfrac{1}{2}(a-b). \quad (11)$$

$$\cos b + \cos a = 2\cos\tfrac{1}{2}(a+b).\cos\tfrac{1}{2}(a-b). \quad (12)$$

$$\cos b - \cos a = 2\sin\tfrac{1}{2}(a+b).\sin\tfrac{1}{2}(a-b). \quad (13)$$

Also,

$$\tan \tfrac{1}{2}(a+b) = \frac{\sin \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a+b)} = \frac{2 \sin \frac{1}{2}(a+b) . \cos \frac{1}{2}(a-b)}{2 \cos \frac{1}{2}(a+b) . \cos \frac{1}{2}(a-b)} =$$

$$\text{(eq. 10, 12) } \frac{\sin a + \sin b}{\cos a + \cos b}. \qquad (14)$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a-b)} = \frac{2 \sin \frac{1}{2}(a-b) . \cos \frac{1}{2}(a+b)}{2 \cos \frac{1}{2}(a-\mathrm{b}) . \cos \frac{1}{2}(a+b)} =$$

$$\text{(eq. 11, 12) } \frac{\sin a - \sin b}{\cos a + \cos b}. \qquad (15)$$

$$\operatorname{cotan} \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a+b)} = \frac{2 \cos \frac{1}{2}(a+b) . \sin \frac{1}{2}(a-b)}{2 \sin \frac{1}{2}(a+b) . \sin \frac{1}{2}(a-b)}$$

$$= \text{(eq. 11, 13) } \frac{\sin a - \sin b}{\cos b - \cos a}. \qquad (16)$$

$$\operatorname{cotan} \tfrac{1}{2}(a-b) = \frac{\cos \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a-b)} = \frac{2 \cos \frac{1}{2}(a-b) . \sin \frac{1}{2}(a+b)}{2 \sin \frac{1}{2}(a-b) . \sin \frac{1}{2}(a+b)}$$

$$= \text{(eq. 10, 13) } \frac{\sin a + \sin b}{\cos b - \cos a}. \qquad (17)$$

$$\frac{\tan \frac{1}{2}(a-b)}{\tan \frac{1}{2}(a+b)} = \text{(eq. 14, 15)} \frac{\sin a - \sin b}{\sin a + \sin b}. \qquad (18)$$

$$\frac{\tan \frac{1}{2}(a-b)}{\cot \frac{1}{2}(a+b)} = \text{(eq. 15, 16)} \frac{\cos b - \cos a}{\cos a + \cos b}. \qquad (19)$$

From equations 15 and 16,

$$\frac{\cot \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-\mathrm{b})} = \frac{\cos a + \cos b}{\cos b - \cos a}. \qquad (20)$$

The following is an analytical investigation of the rules, already given in Art. 34, for finding an angle of a plane triangle, when the sides are known.

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Let ABC be the triangle; and put the angle ABC = B; the side AB = c; AC = b; BC = a; BD = d; the line CD being at right angles to AB. Then (12, 13.2),

$$c^2 + a^2 = b^2 \pm 2cd;$$

the sign + being used when B is acute, as in Fig. 1; and the sign — when B is obtuse, as in Fig. 2.

Fig. 1.

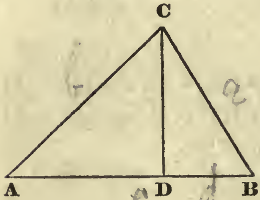
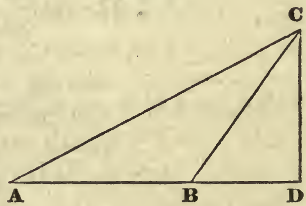


Fig. 2.



Hence, $$\pm d = \frac{c^2 + a^2 - b^2}{2c}.$$

But (Art. 28) $$\frac{d}{a} = \cos B;$$

the cos B being positive or negative, according as the angle is acute or obtuse (Art. 24). Consequently,

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac};$$

and

$$1 + \cos B = 1 + \frac{c^2 + a^2 - b^2}{2ac} = \frac{c^2 + 2ac + a^2 - b^2}{2ac} = \frac{(c+a)^2 - b^2}{2ac}$$

$$= (\text{cor. to } 5.2) \frac{(c + a + b)\,(c + a - b)}{2ac}.$$

Now, Art. 36, Form. 8,

$$1 + \cos B = 2 \cos^2 \tfrac{1}{2}B;$$

wherefore,

$$\cos^2 \tfrac{1}{2}B = \frac{(c+a+b)\,(c+a-b)}{4ac} = \frac{\frac{1}{2}(c+a+b).\ \frac{1}{2}(c+a-b)}{ac}$$

$$= \left(\text{putting } s = \frac{c+a+b}{2}\right) \frac{s.(s-b)}{ac}; \qquad \text{(A)}$$

which is Rule 3, Art. 34.

Again, from the equation $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$, we have

$$1 - \cos B = 1 - \frac{c^2+a^2-b^2}{2ac} = \frac{2ac+b^2-c^2-a^2}{2ac} =$$

$$\frac{b^2-(c^2-2ac+a^2)}{2ac} = \frac{b^2-(c-a)^2}{2ac}.$$

But, Art. 36, Form. 6,

$$1 - \cos B = 2 \sin^2 \tfrac{1}{2}B;$$

consequently,

$$\sin^2 \tfrac{1}{2}B = \frac{b^2-(c-a)^2}{4ac} = \frac{\frac{1}{2}(b+c-a).\ \frac{1}{2}(b+a-c)}{ac} =$$

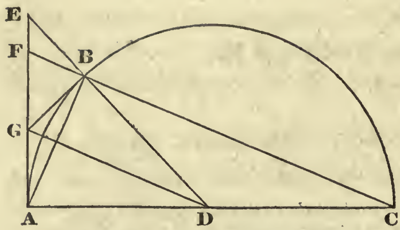
$$\frac{(s-a).(s-c)}{ac}; \qquad \text{(B)}$$

which is Rule 2, Art. 34.

Since $\tan = \dfrac{\text{sine}}{\text{cosine}}$ (Art. 23); $\tan^2 B = \dfrac{\sin^2 B}{\cos^2 B} =$ (by eq.

A and B) $\dfrac{(s-a).(s-c)}{s.(s-b)}$;

which is Rule 1, Art. 34.



Art. 38. Let ABC be a semicircle; ADC its diameter; D the centre; AB the chord of an arc; AE, BG, lines touching the circle in A and B, and meeting in G. Join DG, DB, CB; and produce DB, CB, to meet AE in E and F.

Now, the angle ABC in a semicircle being a right one (31.3), the adjacent angle ABF is also a right one (13.1). Again, since AG and BG touch the circle, each of the angles GAB, GBA = the angle ACB in the alternate segment (32.3); hence GAB = GBA, and GB = GA (6.1). Furthermore, since AFB + FAB = ABC (32.1) = ABF (31.3 and 13.1) = GBF + GBA; it follows that GFB = GBF; whence GB = GF; and AG + GB = AF. But the triangles ADG, BDG, being evidently equal, the line AG is the tangent of half the arc intercepted between A and B; hence AF = twice the tangent of that half arc. The chord AB is also twice the sine of the same half arc. Now, the triangles ACB, FAB, being right angled at B, and having the angle ACB = FAB, are similar; whence AC : CB :: AF : AB :: AG : $\frac{1}{2}$AB :: tan of $\frac{1}{2}$ arc : sine of the same half arc.

Let, now, the point B move along the arc towards A; the lines which pass through B moving with it: then, as the angle at C diminishes, the line CB must approximate to AC, and ultimately become equal to it. Consequently, the ratio of AF to AB, or of the tangent to the sine of half the arc between A and B, is ultimately a ratio of equality.

Again. The angle ADG = $\frac{1}{2}$ADB = ACB (21.3); consequently, DG is parallel to CF (28.1); and therefore,

$$EB : BD :: EF : FG \ (2.6);$$

or, doubling the consequents,

$$EB : AC :: EF : FA;$$

whence, by composition (18.5),

$$EB + AC : AC :: AE : AF.$$

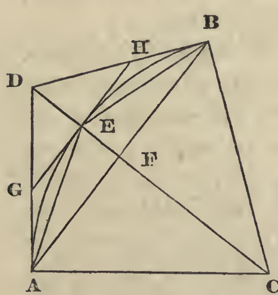
But $$AC : CB :: AF : AB \ (4.6);$$

therefore, *ex equali*,

$$AC + BE : CB :: AE : AB \ (22.5).$$

But as the point B approaches A, the line BE decreases and ultimately vanishes. The angle at C ultimately vanishing, the line CB becomes finally equal to AC. Hence the ratio of AC + BE to CB, and consequently of the tangent AE to the chord AB, becomes ultimately a ratio of equality.

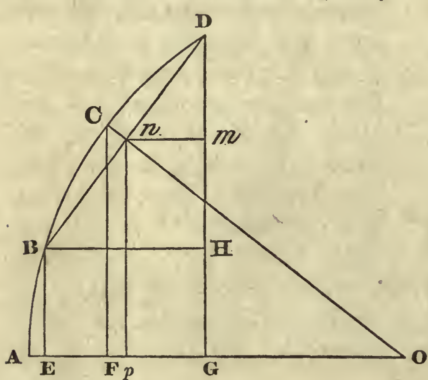
Hence it is manifest that the ultimate ratio of the tangent of an evanescent arc to its sine, or to its chord, is a ratio of equality.



Let AEB be a circular arc, whose centre is C; AD, BD, two right lines touching the circle in A and B. Join CD, AB; and let CD cut the arc in E, and the chord AB in F. Through E, draw GH parallel to AB; and join AE, BE. Then, from what is above proved, AD = BD; the angle ACF = BCF; and consequently (4.1), AFC = BFC; whence GH, being parallel to AB, and therefore at right angles to CE, touches the circle in E. Also, AD is the tangent and AF the sine of the arc AE.

Since AFE is a right angle, the angles AEF and ADF are each less than a right angle (17.1). But AED = AFE + EAF (32.1) is greater than a right angle. Hence, AD is greater than AE, and AE than AF (19.1). That is, the tangent is greater than the chord, and the chord than the sine. Again, since GD + DH are greater than GH (20.1), it is obvious that AD + DB must be greater than AG + GH + HB. But AG + GE being greater than AE (20.1), and EH + HB than EB; AG + GH + HB must be greater than AE + EB. Also AE + EB are greater than AB.

If we were to join CG, CH, and draw the tangents and chords to the intercepted arcs, we might demonstrate, in the same manner, that the sum of the tangents thus drawn would be less than AG + GH + HB, and the sum of the chords greater than AE + EB. By continued bisections, we thus find the sum of the tangents continually decreasing, and the sum of the chords always increasing. But the tangents and chords are, by this process, brought to approximate still more and more nearly to the circular arc which lies between them. Hence we infer that when, by the evanescence of the arc, the ratio of the tangent to the chord or sine becomes a ratio of equality, the ratio of the arc itself to the tangent, chord or sine, is a ratio of equality.



Art. 39. As nearly all trigonometrical calculations are usually performed by means of auxiliary tables, it becomes necessary to explain the nature and origin of those tables. This is most expeditiously effected by the Differential Calculus.

Let AB $= z$; BE $= y$; OE $= x$; BCD $= h$; the radius OC being $= 1$. Then, as proved in Art. 35, equation C,

$$\mathrm{D}m = \cos(z + \tfrac{1}{2}h).\sin\tfrac{1}{2}h;$$

whence, $\sin(z + h) = \sin z + 2\cos(z + \tfrac{1}{2}h).\sin\tfrac{1}{2}h$;

consequently,

$$\frac{\sin(z+h)-\sin z}{h}=\frac{2\cos(z+\frac{1}{2}h).\sin\frac{1}{2}h}{h}=\frac{\cos(z+\frac{1}{2}h).\sin\frac{1}{2}h}{\frac{1}{2}h}$$

But (Art. 38) when h becomes evanescent, the ultimate ratio of $\frac{1}{2}h$ to the sin $\frac{1}{2}h$ is a ratio of equality. Also the ultimate ratio of cos $(z+\frac{1}{2}h)$ to cos z is a ratio of equality. Hence,

$$\frac{d.\sin z}{dz}=\left(\frac{dy}{dz}\right)=\cos z=x;$$

consequently, $$dy=xdz.$$

Again, $x^2+y^2=1$; whence, $2xdx+2ydy=0$; or

$$dx=\frac{-ydy}{x}=\frac{-yxdz}{x}=-ydz.$$

Now, let $t=\tan z=\frac{y}{x}$; then

$$dt=d.\frac{y}{x}=\frac{xdy-ydx}{x^2}=\frac{x^2dz+y^2dz}{x^2}=\text{(since } x^2+y^2=1)$$

$$\frac{dz}{x^2}=(1+t^2)\,dz,\text{ because}\frac{1}{x^2}=1+t^2 \text{ (Art. 23, 2.)}$$

Art. 40. To find the length of an arc z in terms of its tangent, t. No formula has been discovered for expressing an arc in finite terms of its tangent; recourse must therefore be had to infinite series.

Let, then, $z=\mathrm{A}t^n+\mathrm{B}t^p+\mathrm{C}t^q+\mathrm{D}t^r$, &c., in which the exponents and coefficients are unknown. As the tangent of an arc becomes nothing at the same time the arc itself vanishes, the series here assumed must express the arc z, if it can be expressed at all in terms of t.

From this equation,

$$\frac{dz}{dt}=n\mathrm{A}t^{n-1}+p\mathrm{B}t^{p-1}+q\mathrm{C}t^{q-1}+r\mathrm{D}t^{r-1}+\text{\&c.}$$

But (Art. 39),

$$\frac{dz}{dt} = \frac{1}{1+t^2} = (1+t^2)^{-1} = 1 - t^2 + t^4 - t^6 + \&c.$$

Comparing these values of $\frac{dz}{dt}$; $n - 1 = 0$, $p - 1 = 2$, $q - 1 = 4$, $r - 1 = 6$, &c.; and $nA = 1$; $pB = -1$; $qC = 1$; $rD = -1$, &c.; from which $n = 1$; $p = 3$; $q = 5$, &c.; $A = 1$; $B = -\frac{1}{3}$; $C = \frac{1}{5}$; $D = -\frac{1}{7}$, &c.; which values, substituted in the primitive equation, give

$$z = t - \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 - \tfrac{1}{7}t^7 + \tfrac{1}{9}t^9 + \&c.$$

The length of the arc being thus found in terms of the tangent, the next object is to find a value of t which will converge rapidly in this series, and at the same time correspond to a known part of the circumference of a circle. One of the most convenient methods yet discovered of effecting this object, is the following:

Let a and b denote two circular arcs; such that $\tan a = \frac{1}{2}$, and $\tan b = \frac{1}{3}$; then (Art. 37, Form. 3),

$$\tan(a+b) = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}} = 1.$$

Now (Art. 26) the tangent of $45° = 1$. Hence, $a + b = 45°$.

$$\text{Now } a = \frac{1}{2} - \frac{1}{3}\cdot\frac{1}{2^3} + \frac{1}{5}\cdot\frac{1}{2^5} - \frac{1}{7}\cdot\frac{1}{2^7}\ \&c. = .463647609001$$

$$\text{and } b = \frac{1}{3} - \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} - \frac{1}{7}\cdot\frac{1}{3^7}\ \&c. = .321750554397$$

$$a + b = (45°) = \quad . \quad . \quad . \quad . \quad .785398163398$$

From which we find the semicircumference=3.141592653592; and this is the whole circumference very nearly true to twelve decimals, when the diameter is = 1.

Art. 41. The length of an arc of 45° being thus determined, the length of an arc of 1°, or of any fraction of it, is readily found. But, from this datum, to find a general formula for the sine and cosine, requires the aid of the Differential Calculus. And no method has been discovered to denote the sine or cosine in terms of the arc, without recourse to infinite series.

Let z = an arc; $y = \sin z$; $x = \cos z$; and assume

$$x = 1 + Az^n + Bz^p + Cz^q + Dz^r + \&c.;$$

in which the first term of the series is taken $= 1$; because, when $z = 0$, $x = 1$. From this equation,

$$\frac{dx}{dz} = nAz^{n-1} + pBz^{p-1} + qCz^{q-1} + rDz^{r-1} + \&c.$$

But (Art. 39),

$$\frac{dx}{dz} = -y.$$

Therefore,

$$y = -nAz^{n-1} - pBz^{p-1} - qCz^{q-1} - rDz^{r-1} - \&c.;$$

and

$$\frac{dy}{dz} = -(n-1).nAz^{n-2} - (p-1).pBz^{p-2} - (q-1).qCz^{q-2} - (r-1).rDz^{r-2} - \&c.$$

But (Art. 39),

$$\frac{dy}{dz} = x = 1 + Az^n + Bz^p + Cz^q + \&c.$$

Comparing the corresponding terms of these series,

$$n-2=0;\ p-2=n;\ q-2=p;\ r-2=q, \&c.$$
$$(n-1).nA = -1;\ (p-1).pB = -A;\ \&c.$$

Whence $n=2$; $p=4$; $q=6$; $r=8$; &c.

$$A = -\frac{1}{2};\ B = \frac{1}{2.3.4};\ C = -\frac{1}{2.3.4.5.6};\ D = \frac{1}{2.3.4.5.6.7.8}\ \&c.$$

Consequently,

$$x = 1 - \frac{z^2}{2} + \frac{z^4}{2.3.4} - \frac{z^6}{2.3.4.5.6} + \&c. \qquad (A)$$

$$y = z - \frac{z^3}{2.3} + \frac{z^5}{2.3.4.5} - \frac{z^7}{2.3.4.5.6.7} + \&c. \qquad (B)$$

To exemplify these series, let the sine and cosine of 1° be required. In that case,

$$z = .017453292520; \frac{z^2}{2} = .000152308710; \frac{z^3}{2.3} = .000000886080;$$

$$\frac{z^4}{2.3.4} = .000000003866; \frac{z^5}{2.3.4.5} = .000000000013.$$

Substitute these values in series A and B. Whence $x = .999847695156$; and $y = .017452406453$.

If, instead of these values of $z, \frac{z^2}{2}, \frac{z^3}{2.3}$, &c., we substitute in the series A and B, $\frac{1}{2}$ of the first, $\frac{1}{4}$ of the second, $\frac{1}{8}$ of the third, &c., we shall obtain the cosine and sine of 30′; and from these results we may, by a similar process, find the cosine and sine of 15′.

Thus, cos of

$30' = 1 - .000038077177 + .000000000242 = .999961923095$;

and sin of

$30' = .008726646260 - .000000110760 = .008726535500$;

Also, cos of

$15' = 1 - .000009519299 + .000000000015 = .999990480728$;

and sin of

$15' = .004363323130 - .000000013845 = .004363309285$.

If, instead of the values of $z, \frac{z^2}{2}$, &c., first found, we take $\frac{1}{60}$ of the first, $\frac{1}{3600}$ of the second, &c., and substitute them in equations A and B, we shall have the cosine and sine of 1′.

Thus, cos of $1' = 1. - .000000042308 = .999999957692$; and sin of

$1' = .000290888209 - .000000000004 = .000290888205$.

When the sines and cosines of two arcs are known, the sine and cosine of their sum, or difference, are readily found from equations 1 and 2, Art. 37. Or the sine and cosine of any arc, v, being found, the sine and cosine of $2v, 3v, 4v$, may be determined in the following manner:

From Art. 37, Form. 4 and 7, we have, by transposition,

$$\sin(a+b) = 2\sin a . \cos b - \sin(a-b) \qquad \text{(C)}$$

$$\cos(a+b) = \cos(a-b) - 2\sin a . \sin b \qquad \text{(D)}$$

Taking, then, b = any arc v; and a successively $= v, 2v, 3v$, &c.; these equations become

$$\sin 2v = 2\sin v . \cos v;$$

2 S v Cos v - ...

$$\cos 2v = 1 - 2\sin^2 v;$$

$$\sin 3v = 2\sin 2v . \cos v - \sin v;$$

$$\cos 3v = \cos v - 2\sin 2v . \sin v;$$

$$\sin 4v = 2\sin 3v . \cos v - \sin 2v;$$

$$\cos 4v = \cos 2v - 2\sin 3v . \sin v.$$

Hence, the sine and cosine of v being known, the sine and cosine of any multiple of v may be found.

Art. 42. The sine of an arc, being half the chord of twice the arc, and the chord of $60° = 1$, (Art. 26,) the sine of $30° = \frac{1}{2}$; consequently, if we take $a = 30°$, in the equations C and D of the last article, we shall have

$$\sin(30° + b) = \cos b - \sin(30° - b);$$

$$\cos(30° + b) = \cos(30° - b) - \sin b.$$

If, then, the sine and cosine of every degree and minute, as far as 30°, were computed by the preceding methods,

4 sin v cos v - sin ...

these equations furnisn a mode of computing the sines and cosines of the remaining arcs by subtraction only.

To find the sine and cosine of 31°,

$$\begin{array}{rr}
\cos 29^\circ = \cos 30^\circ . \cos 1^\circ + \sin 30^\circ . \sin 1^\circ = & .874619707108 \\
\sin 1^\circ = & .017452406453 \\
\hline
\cos 31^\circ = & .857167300655 \\
\hline
\cos 1^\circ = & .999847695156 \\
\sin 29^\circ = \sin 30^\circ . \cos 1^\circ - \cos 30^\circ . \sin 1^\circ = & .484809620238 \\
\hline
\cos . 31^\circ = & .515038074918
\end{array}$$

The sines, computed as above explained, and arranged in a table, constitute a table of natural sines. Those sines, as put down in the tables, are seldom extended to more than seven decimals, and frequently not even so far; but, in computing such tables, it is necessary to extend the sines of the primary arcs considerably further than the number of decimals intended to be retained, in order to render the numerous deductions from them sufficiently correct.

The tangents may be found from the sines and cosines, by simple division; for $\tan = \frac{\text{sine}}{\text{cosine}}$ (Art. 23). The secants are also deduced from the cosines; for $\text{secant} = \frac{1}{\text{cosine}}$.

Art. 43. The tables of sines, tangents, &c., which are commonly used in trigonometrical calculations, are logarithmic, and are easily deduced from a table of logarithms and of natural sines. But it may be observed that the sines computed to a radius 1, are all decimals except the sine of 90°. Hence, the logarithms of those sines, if the common logarithms are used, must all have negative indices, excep the sine 90°, whose logarithm is 0. To avoid this inconve nience the decimal point in each of the sines is removed ten places towards the right, which is equivalent to finding the sines to a radius of 10000000000. The logarithm of this number is 10; and the sine of 1″ computed to this radius is 48481.37, whose log. = 4.6855749. From which it appears that all arcs or angles which can occur in practice, have their logarithmic sines positive.

The sines computed according to the preceding articles have the decimal point, in each case, removed ten places to the right; the logarithms of the results are then taken from a table of logarithms, and arranged in a table. This composes a table of logarithmic or artificial sines. Then, since

cos : sine :: rad : tang (Art. 23);

the index of the logarithmic sine being increased by 10, and the logarithmic cosine subtracted, the remainder will be the logarithmic tangent. And, since

cos : rad :: rad : secant (Art. 23);

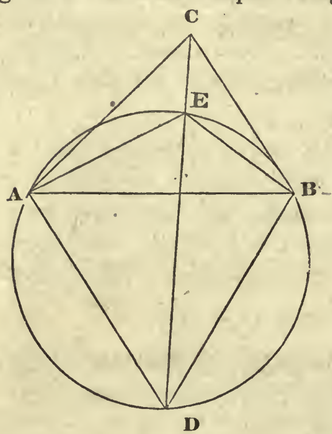
if we subtract the logarithmic cosine from 20 (twice the log. of radius), the remainder will be the secant. Again,

tang : rad :: rad : cotan;

consequently, the logarithmic tangent of an arc, being subtracted from 20, will leave the logarithmic cotangent.

In trigonometrical calculations where logarithms are used, it is most convenient to take the arithmetical complements of the logarithms which are to be subtracted, (that is, what those logarithms want of 10 or 20,) and add them with the other additive logarithms, rejecting as many tens or twenties from the result as there are complements used. When the subtractive numbers are logarithmic sines, tangents or secants, the arithmetical complements can be taken immediately from the table; for the cosecant is the arithmetical complement of the sine; the cotangent of the tangent; and the cosine of the secant. All this is manifest from the nature of logarithms, and the analogies in Art. 23.

ART. 44. A few trigonometrical problems will now be given to exercise the preceding rules.



1. Given, AB 35, AC 30, BC 25, the three sides of a triangle; to find the distances from the several angles to a point E within the triangle, such that the angles AEB, AEC and BEC shall be equal to each other.

Construction. On AB, one of the sides, describe the equilateral triangle ABD; and about that triangle describe a circle; join DC, cutting the circle in E; then is E the point required.

Join AE, BE; then, since the angles of the triangle ABD are all equal (cor. to 5.1), each of them contains 60°, or $\frac{1}{3}$ of two right angles (32.1). But AED=ABD; and BED=BAD (21.3): therefore AED = 60°, and AEC = 120°. Also BED = 60°, and BEC = 120°.

Calculation. With the three sides, the angle BAC may be found by Art. 34, Rule 2.

AB	35	AC	8.4559320
AC	30	AC	8.5228787
BC	25		
sum	90		
$\frac{1}{2}$ sum	45		
$\frac{1}{2}$ sum — AB	10	log.	1.
$\frac{1}{2}$ sum — AC	15	"	1.1760913
		2)	19.1549020
sin $\frac{1}{2}$BAC	22° 12$\frac{1}{2}$′		9.5774510

BAC	44° 25′
BAD	60°
DAC	104° 25
$\frac{1}{2}$DAC	52° $12\frac{1}{2}'$
$\frac{ACD + ADC}{2}$	37° $47\frac{1}{2}'$

Then, by Art. 30,

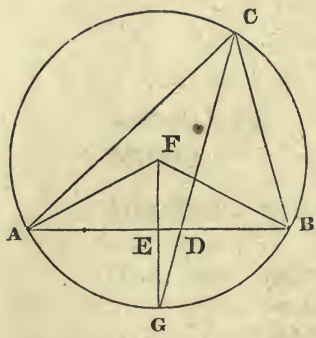
As	AD + AC	65	AC	8.1870866
is to	AD — AC	5		.6989700
So is	$\tan\frac{ACD + ADC}{2}$	37° $47\frac{1}{2}'$		9.8895519
to	$\tan\frac{ACD - ADC}{2}$	3° $24\frac{1}{2}'$		8.7756085
	ACD	41° 12′		
	ADC, ABE	34° 23′		
	CAE	18° 48′ = AED — ACE.		
	EAB	25° 37′ = CAB — CAE.		

From Art. 29,

As	sin AEB	120°	AC	.0624694
is to	sin ABE	34° 23′		9.7518385
So is	AB	35		1.5440680
to	AE	22.823		1.3583759

As	sin AEB	120°	AC	.0624694
is to	sin BAE	25° 37′		9.6358335
So is	AB	35		1.5440680
to	BE	17.473		1.2423709

As	sin AEC	120°	AC	.0624694
is to	sin CAE	18° 48′		9.5082141
So is	AC	30		1.4771213
to	CE	11.164		1.0478048



2. Given, the vertical angle ACB 70°, the segments into which the base is divided by the line CD bisecting the vertical angle, viz. AD 30, and DB 20, to determine the angles and sides of the triangle, and the line which bisects the vertical angle.

Construction. Bisect the base AB in E; through E draw FEG at right angles to AB; make the angle EAF = complement of the vertical angle, above AB when that angle is acute, but below when it is obtuse. From the centre F, where the line AF meets the perpendicular, describe a circle passing through A, and cutting FEG in G; join GD, and produce it to cut the circle in C; join CA, CB; and ACB will be the triangle required.

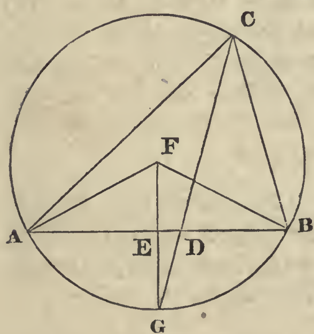
Since AE = BE, and the angles at E are right ones, the line AF = BF (4.1); consequently, the circle must pass through B. The angle AFE being equal to BFE, the arc AG = BG (26.3); consequently, ACG = BCG (27.3). Also, the angle AFE = twice ACG (20.3) = ACB; therefore, ACB is the complement of EAF.

Calculation. In the right angled triangle EAF we have, besides the right angle, the side AE = 25, and the angle EAF = 20°; from which we find, by Art. 28, AF = 26.604, and EF = 9.099; whence EG = 17.505; then, in the right angled triangle GED, we have EG; and ED = 5; from which we find the angle EDG or CDB = 74° 3′; then

$$CAD = CDB - ACD\ (32.1) = 39° 3'.$$

In the triangle ADC we then have AD = 30, and all the angles, from which, by Art. 29, we find AC = 50.29, and DC = 32.951. Lastly, from 3.6, we have

$$AD : DB :: AC : BC = 33.527.$$



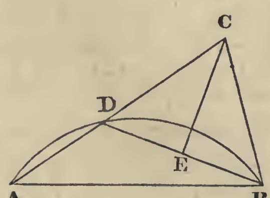
3. Given, the base AB 70; the vertical angle ACB 75°; and the ratio of the sides, viz., AC : BC :: 4 : 3, to determine the rest.

Divide AB in D, so that AD : DB :: 4 : 3 (10.6); then the construction will be the same as in the last example.

The calculation will likewise be similar to the last.

The results are, AF = 36.235; EF = 9.378; BDC = 79° 27′; CAB = 41° 57′; ABC = 63° 3′; AC = 64.596; BC = 48.447; DC = 43.927.

4. Given, the base AB 500; the difference of the sides 100; and the vertical angle ACB 72°, to determine the rest.



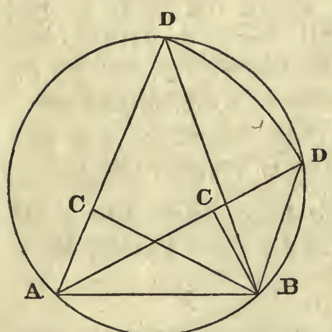
Construction. On the base AB describe the segment of a circle containing an angle equal 90° + ½ the vertical angle (33.3); place in this circle the right line AD = the difference of the sides; produce AD; join DB; and make the angle DBC = BDC; then is ABC the triangle proposed.

Draw CE at right angles to DB; then, CD being = CB (because the angle CBD = CDE), the angle DCB is evidently bisected; and the angle

$$ADB = DEC + DCE = 90° + \tfrac{1}{2}ACB;$$

also, AD = AC — BC: hence the construction is manifest.

Calculation. In the triangle ABD, AB, AD, and the angle ADB, are known; whence the angle BAD may be found; from which and the given angle ACB, the angle ABC becomes known. Then AB and all the angles of the triangle being known, AC and BC are determined. The results are BAC = 44° 41′, ABC = 63° 19′, AC = 469.74; and BC = 369.74.



5. Given, the base AB 465, the vertical angle ACB 75°; and the sum of the sides 760, to determine the rest.

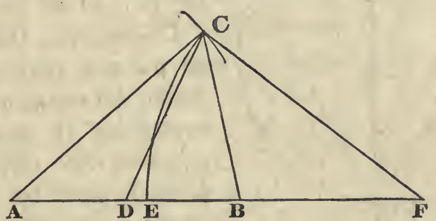
Construction. On AB describe a segment of a circle containing half the vertical angle; from A, place the line AD = the sum of the sides, in this segment; join DB; and make the angle DBC = BDC. Then will ACB be the triangle proposed.

Because the angle DBC = BDC; DC = BC; and the exte rior angle ACB = twice ADB; also AC + BC = AD.

Calculation. With the sides AB, AD, and angle ADB; the angles ABD and BAD may be found; whence ABC becomes known. Then AC and BC are determined. Results: ABC

$= 46° 45'$, or $58° 15'$; AC = 350.64, or 409.36; BC = 409.36 or 350.64

6. Given, the base AB 50; the line DC, drawn from the middle of the base to the vertex 40; and the ratio of the sides, AC : BC :: 3 : 2, to determine the sides



Construction. Divide AB in E, so that AE : EB in the proposed ratio of AC : BC; produce AB to F, so that BF shall be a third proportional to AE — EB and EB; from the centre F, through E, describe the arc EC; from the centre D, with the given distance DC, describe an arc, cutting the former in C; join AC, DC and BC; then ABC is the triangle proposed.

From the proportion AE — EB : EB :: EB : BF, we have (18.5), AE : EB :: EF : BF;

therefore (12.5) AE : EB :: AF : EF;

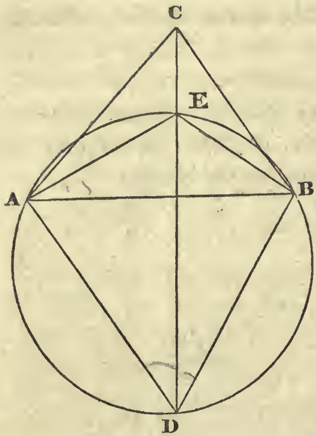
consequently (19.5),

AF : EF :: EF : BF;

whence (F. 6), AC : CB :: AE : EB.

Calculation. In the triangle CDF, we have all the sides to find $\frac{1}{2}$ the angle FDC, which is the $\frac{1}{2}$ sum of DAC and DCA; then, with that $\frac{1}{2}$ sum and the sides AD, DC, the angle DAC and side AC may be found.

Result: AC = 55.50; BC = 37.00.



7. Given, the sides of the triangle ABC, viz., AB 90, AC 80, BC 70, to determine the distances AD, CD and BD, to a point D, which is so situated that the angles ADB and ADC are 70° and 40° respectively.

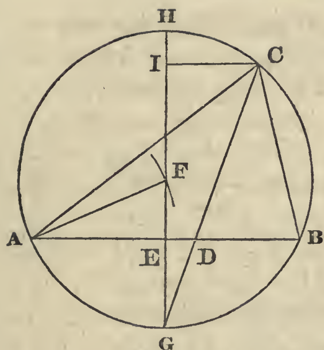
Construction. On AB, and on the side opposite to C, describe the segment of a circle containing an angle of 70°; complete the circle; at the point B make the angle ABE = 40°; from C, through E, where BE cuts the circle, draw CE, and produce it till it cuts the circle again in D, the point required; join DA, DB, and the work is done.

The angle ADE = ABE (21.3) = 40°; whence the construction is manifest.

Calculation. Join AE; then the angles ABE, BAE, and the side AB, are known; whence AE may be found. The angle CAB may also be determined from the three sides; hence CA, AE, and the contained angle, become known; from which ACE and AEC, and consequently AED, may be found. But AED = ABD (21.3); therefore the angles of the triangle ABD become known, as well as those of ADC; from which and the given sides, AD, BD and CD may be found.

Result: AD = 82.915; DB = 73.406; DC = 123.178; and DAB = 50° 2′ 6″.

8. Given, the base AB 50; radius of the circumscribing circle 30; and ratio of the sides, AC : BC :: 3 : 2, to find the sides.



Construction. Divide AB in D so that AD : DB :: 3 : 2; bisect AB in E; draw EF at right angles to AB; from the centre A, with the radius of the circumscribing circle, cut EF in F; from the centre F, with the radius FA, describe a circle cutting FE produced in G; join GD, and produce it to meet the circle in C; join AC, BC; then ABC will be the triangle required. The circle will pass through B (1.3); and the arc AGB is bisected in G; consequently, the angle ACB is bisected by the line CD (27.3); wherefore (3.6),

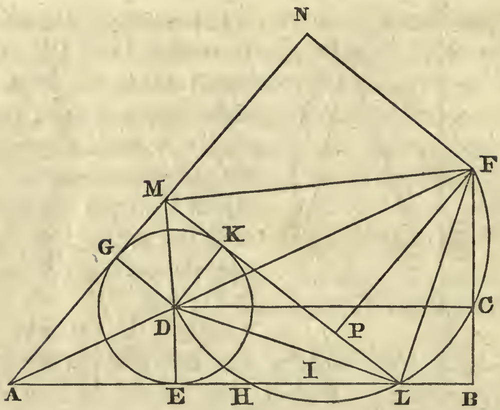
$$AC : BC :: AD : DB.$$

Calculation. In the right angled triangle AEF, AE and AF are given, from which EF and the angle AFE are determined; but AFE = 2ACD = ACB. In the triangle EDG; EG and ED being known, the angle EDG or BDC is determined; whence the angles of the triangle ABC are known, and the sides AC, BC determined.

Result: AC = 59.447; BC = 39.632.

9. Given, one angle of a triangle 50°; the sum of the three sides 120; and the radius of the inscribed circle 10, to determine the sides of the triangle.

Construction. Make AB = $\frac{1}{2}$ the sum of the sides, 60; the angle BAN = the given angle 50°; bisect that angle by the line AF, meeting BF, which is drawn at right angles to AB; make BC = the given radius of the inscribed circle, 10; through C draw CD parallel to BA, meeting AF in D; draw DE, DG at right angles to AB, AN respectively; from the centre D, with the radius DE, describe the circle: that circle



will touch the lines AB, AN, in E and G (4.4). On the diameter DF, describe the semicircle DHLF, cutting AB in H and L; from either of these points L draw LK, touching the circle GEK in K, and cutting AN in M; then ALM is the triangle proposed.

Join DL, DK, DM; LF, MF; and draw FN, FP at right angles to AN, LK respectively. Now, since DE = DK, the angles at E and K are right ones, and DL is common to the triangles DEL and DKL, it follows that the angle DLE = DLK. But DLF is a right angle (31.3); hence, DLE+FLB =DLF (13.1); from these equals, take the equals DLE and DLK; and we have BLF = KLF. Thence, the angles at B and P being right ones, it is obvious that LB = LP, and FB = FP. Again, in the triangles FAB and FAN, we have AF common, and the angles of the one respectively equal to those of the other; hence AN = AB, and FN = FB = FP; consequently (47.1), MN = MP.

Now, it has been proved that LP = LB; consequently,

$$ML = MN + LB;$$

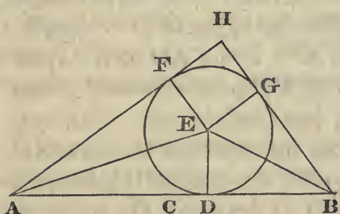
and, therefore,

$$AM + AL + ML = AB + AN = 2AB$$

Calculation. In the right angled triangles AED, ABF, we have the angle at A, and the lines ED and AB given; from which AE, EB and BF are determined. Then the angle DCF being a right one, the semicircle on DF must pass through C (converse 31.3); consequently, CB.BF = LB.BH (cor. 36.3). If, now, we suppose a line drawn from the centre of the semicircle, cutting EB at right angles in I, it is manifest (2.6 and 3.3) that EB and HL are both bisected in I; whence EL = HB; and EL.LB = CB.BF, a known rectangle. Hence IL^2 $\left(= IB^2 - EL.LB \text{ (5.2)}\right)$ becomes also known.

Result: AL=50.306; AM=31.139; LM=38.555.

10. Given, the base 50; difference of the other sides 10; and radius of the inscribed circle 12; to determine the sides.



Construction. Make AB= the base 50, and bisect it in C; lay down CD = $\frac{1}{2}$ the difference of the sides; at D erect a perpendicular DE= radius of the inscribed circle 12; from E, with the distance ED, describe a circle; through A and B draw the lines AH, BH, touching the circle in F and G; ABH is the triangle required.

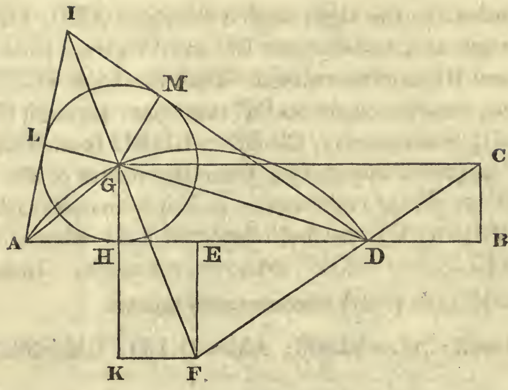
Join EF, EG; then (47.1) AD = AF; BD = BG; FH = GH; consequently,

$$AH - BH = AD - BD = 2CD.$$

Calculation. With AD, DE, and BD, DE, find the angles BAE, ABE, from which BAH and ABH are known; and thence the sides AH and BH.

Result: AH = 45.79; BH = 35.79.

11. Given, the perimeter of a triangle 120; radius of the inscribed circle 10; and vertical angle 70°, to determine the sides.



Construction. Make AB=$\frac{1}{2}$ the perimeter 60; at B erect the perpendicular BC=radius of the inscribed circle 10; through C, draw CG parallel to AB; make the angle BCD = complement of half the vertical angle, 55°; bisect AD in E; draw EF at right angles to AD, meeting CD produced in F; from F, as a centre, through A or D, describe the arc AGB, cutting the line GC; from one of the intersections G, draw GH at right angles to AD; from G, with the radius GH, describe the circle HLM; through A and D draw AI, DI, touching the circle in L and M; then ADI is the triangle required.

Join AG, DG and IG; then it is obvious that those lines bisect the angles DAI, ADI and AID (26.1); consequently,

$$DAG + ADG + AIG = \text{a right angle } (32.1);$$

that is, AIG is the complement of DAG + ADG. Now, since EFD = the angle at the circumference, which stands on AGD; it follows that

$$EFD + AGD = 2 \text{ right angles } (22.3) = GAD + GDA + AGD$$

(32.1); wherefore.

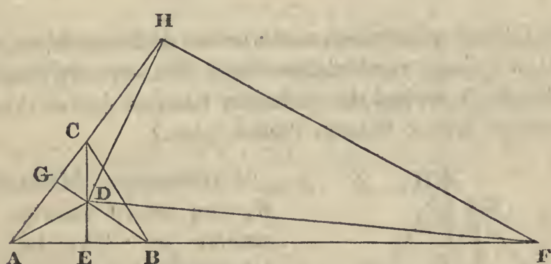
$$EFD = GAD + GDA;$$

their complements are therefore equal; that is, EDF or CDB = LIG. Consequently, the whole angle AID is twice the complement of DCB. Also, the angles LIG and ILG, being respectively equal to BDC and DBC, and LG = BC, the side IL = DB (26.1); hence the semiperimeter of the triangle ADI = AB.

Calculation. Draw FK parallel to AB, meeting GH produced in K, and join FG; then, in the triangle DBC, having BC and all the angles, BD is found; whence ED becomes known; from which, and the angles, DF and EF are found; then, in the right angled triangle FGK, FG and GK are known; whence FK or EH becomes known; whence AH and BH are known.*

Result: AD=45.719; AI=27.02; DI=47.261.

12. Given, the sides of the triangle ABC, viz.: AB 464, AC 418, and BC 385; it is required to find a point D within the triangle, such that AD, BD and CD shall be to each other in the ratio of 7, 6 and 5 respectively.



Construction. Divide AB in the point E, and AC in G, so that AE : EB : : 7 : 6; and AG : GC : : 7 : 5; produce AB and AC to F and H, so that BF and CH shall be third proportionals to AE — EB and EB, and to AG — GC and GC respectively; from the centres F and H, with the distances FE and HG, describe arcs cutting each other in D; join AD

* Examples 9 and 11 are essentially of the same nature, and might have been solved by the same method; the two solutions furnish a little variety.

BD and CD; and the figure is constructed. For, as was proved in the 6th example,

AD : BD :: AE : EB :: 7 : 6;

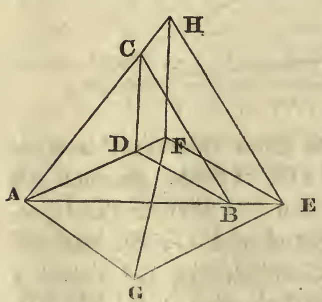
and AD : CD :: AG : GC :: 7 : 5;

whence, BD : CD :: 6 : 5.

Calculation. Join DF, DH and FH; then, in the triangle ABC, we have all the sides to find the angle BAC; then, in the triangle AFH, we have the sides AF, AH, and the included angle, to find the angle AFH and side FH; in the triangle FDH, the three sides are then known to find the angle DFH; whence the angle AFD becomes known: then, in the triangle AFD, we have the sides AF, FD, and the contained angle, to find the angle FAD and the side AD; from which BD and CD are found from the given ratios.

Results: BAD = 25° 59′ 8″; CAD = 25° 27′ 15″; AD = 283.688; BD = 243.161; CD = 202.635.

[The following ingenious construction of this problem, which admits of a simpler calculation than that already given, has been kindly furnished the author by Samuel Alsop, Principal of Friends' Select School, Philadelphia.]



Construction. Make FE = 6; and on it describe FEG similar to CBA, the given triangle, making

FE : EG : FG :: BC : BA : AC.

On FG describe the triangle FGA, making FA = 7, and AG a fourth proportional to BC, AB and 5. Upon AE, on the same or any other scale, lay down AB = 464, and complete the triangle ABC. Draw BD parallel to EF, cutting AF in D, which will be the point required. For, join CD;

draw EH parallel to BC; and join FH. Then, since AEH is similar to FEG, being both similar to BAC;

$$AE : EG :: EH : EF;$$

therefore (6.6), AEG and HEF are similar; and

$$AB : BC :: AE : EH :: AG : FH.$$

But $$AB : BC :: AG : 5;$$

therefore, FH = 5. Consequently,

$$AD : BD : CD :: AF : FE : FH :: 7 : 6 : 5.$$

Calculation. In the triangle ABC, with the given sides, find the angle BAC = FGE; also find AG, GF and GE. From the three sides of the triangle AGF, find the angle AGF; whence AGE becomes known. In the triangle AGE, find AE; then

$$AE : EF :: AB : BD;$$

from which AD and CD are found from the given ratios.

13. In a right-angled isosceles triangle, the hypothenuse is 30 yards longer than one of the sides; what are the sides?
Ans.: hypoth. 102.4264 : sides 72.4264.

14. The hypothenuse of a right-angled triangle is 75, and the sum of the sides is 105; what are the sides?
Ans.: 60 and 45.

15. The sides of a triangle are in the ratio of 4, 6, and 7; and the line bisecting the greatest angle is 20; required the sides. Result: 22.87, 34.31, 40.02.

16. Given the perimeter of a right-angled triangle 120, and the radius of the inscribed circle 10; required the sides of the triangle? Ans.: 50, 40, and 30.

17. From a position in a horizontal plane, I observe the angle of elevation of a tower, which is 100 feet high, to be 60°; how far must I measure back, to obtain a position from which the elevation shall be 30°? Ans. 115.47 feet.

18. A person on the top of a tower which is 50 feet in height, observes the angles of depression of two objects on the horizontal plane, which are in the same straight line with the bottom of the tower, to be 30° and 45°. Determine their distance from each other and from the observer.

Ans. Distance from each other 36.60 feet.
From the observer 70.71, and 100 feet.

19. From the top of a tower, whose height is 108 feet, the angles of depression of the top and bottom of a vertical column, standing in the horizontal plane, are found to be 30° and 60° respectively. Required, the height of the column.

Ans. 72 feet.

20. Suppose the angle of elevation of the top of a steeple to be 40°, when the observer's eye is level with the bottom, and that from a window 18 feet directly above the first station the angle of elevation is found to be 37° 30′. Required, the height and distance of the steeple.

Ans. Height, 210.44 feet.
Distance, 250.79 feet.

21. Two columns, 80 and 100 feet in height, standing on a horizontal plane, are distant from each other 220 feet; it is required to find a point in the line joining their bases, from which the angles of elevation of the two columns shall be equal. Ans. The point is $122\frac{2}{9}$ feet from the higher column.

22. The altitude of a cloud was observed to be 34° 20′, and that of the sun in the same direction 50°; also the distance of the shadow of the cloud from the station of the observer measured 375 yards. Determine the height of the cloud.

Ans. 600 yards.

23. In a plane triangle there are given, the base 60, an adjacent angle 55° 30′, and the ratio of the side opposite the given angle to the other unknown side 6 to 5; to determine these sides. Ans. 50.047, and 41.706.

24. From a station in a horizontal plane, I observed the angle of altitude of the summit of a cliff which bore exactly north to be 47° 30′. I then measured N. 87 W. 283 feet, and again taking the angle of altitude, found it to be 40° 12′. What was the height of the cliff? Ans. 354.53 feet.

Remark. In the solution of the preceding problem, it will assist the pupil if he will observe, that when two right angled triangles have the same perpendicular, their bases are to each other as the cotangents of the angles at the base.

25. Three ships sailed from the same place to different ports in the same parallel of latitude; the first sailed directly south 55 leagues, when she arrived at the desired port; the other two sailed upon different courses, between the south and west, till they arrived at their destined ports, which were 57 leagues asunder, and the angle included by their courses at the port sailed from was 38°. Required, the course and distance run by each of the two latter vessels.

Ans. S. 52° 12′, W. 89.75 leagues;
and S. 14° 12′, W. 56.73 leagues.

26. Walking on shore, I was surprised by the flash of a gun, at sea, bearing S. 56° 15′ E.; seven seconds after the flash I heard the report, and four seconds after that I heard the echo from a castle bearing from me S. 56° 15′ W. Required, the distance of the gun and castle; sound being estimated to pass over 1142 feet in one second of time.

Ans. Distance of gun, 7994 feet; of castle, 3005.51 feet.

27. In a right angled triangle there are given, one of the legs 94, and the segment of the hypothenuse adjacent to the other leg, made by a perpendicular from the right angle, 66, to determine the triangle.

Ans. The other leg is 93.56, and the hypothenuse, 132.62.

28. Having given two sides of a triangle, 40 and 50, and the line drawn from the included angle to the middle of the third side, 34; to determine the third side. Ans. 59.80.

Construction.— Form the triangle BAE, making AB 40, AE 50, and BE 68; complete the parallelogram ABCE, draw the diagonal AC, and ABC will be the required triangle.

29. At three points in the same horizontal straight line, the angles of elevation of an object were found to be 36° 50′, 21° 24′, and 14°, the middle station being 84 feet from each of the others. Required, the height of the object.

Ans. 53.96 feet.

30. In a level garden there are two lofty firs, having their tops ornamented with gilt balls: one is 100 feet high, the other 80, and they are 120 feet distant at the bottom. Now, the owner wants to place a fountain in a right line between the trees, to be equally distant from the top of each, and to make a walk or path from the fountain, in every point of which he shall be equally distant from each of the balls; also, at the end of the walk he would fix a pleasure-house, which should be at the same distance from each ball, as the two balls are from each other. How must this be done?

Ans. From bottom of taller tree to fountain, 45 feet.
From ball to ball, 121.655 "
Length of the walk, 52.678 "
From bottom of taller tree to house, 69.282 "

31. Three objects, A, B, and C, are situated in the same straight line, and are distant from D, 312, 150, and 123 yards; also, the distance of A from B is to the distance of C from B as 22 to 13. How far is B from A and C?

Ans. From A 198 yards, from C 117 yards

Construction.—Make AD = 312, and divide it in E so that AE : ED : : 22 : 13. On DE form the triangle DEB, making DB 150, and EB = $\frac{22}{35}$ DC. Join AB, produce it till it meets DC drawn parallel to BE in C, and the figure is constructed.

32. Given, the angles of elevation of an object taken at three positions, A, B, and C, in the same horizontal straight line, 17° 46′, 33° 41′, and 39° 6′, respectively; also, from A to B is 264 feet, and from B to C 156 feet. Required, the height of the object. Ans. 133.33 feet.

33. There are three towns, A, B, and C, whose distances apart are as follows: from A to B 6 miles; from A to C 22 miles; and from B to C 20 miles. A messenger is despatched from B to A, and has to call at a town D in a direct line between A and C. Now, in travelling from B to D, he walks uniformly at the rate of 4 miles an hour, and from D to A at the rate of 3 miles an hour. Supposing him to perform his journey in three hours, it is required to determine the position of the town D.

Ans. The distance of D from A is 4.72 miles.

In the above example, we have $\frac{1}{3}$ AD + $\frac{1}{4}$ BD = 3, or AD + $\frac{3}{4}$ BD = 9. On AC lay off AE = 9, and join BE; then in the triangle BDE the side BE and the angle BED become known, and ED : DB : : 3 : 4. Hence the point D is readily determined.

34. The lengths of three lines drawn from a given point to three angles of a square are, 35, 46, and 50 yards; to determine a side of the square. Ans. 59.95 yards.

35. Wishing to ascertain the length of a tree which leaned in the plane of the meridian, I measured from the foot of the tree north 85 feet, when I found the angle of elevation of the top to be 35°. I then took a second station 50 feet east of the former, at which the elevation was 30°. Required, the length of the tree. Ans. 52.44 feet.

SECTION III.

SPHERICAL TRIGONOMETRY.

Article 45. The business of Spherical Trigonometry is, to investigate the properties of triangles formed on the surface of a sphere, by the arcs of circles whose planes pass through the centre.

As the diameter of a circle is the greatest straight line in it (15.3), so the diameter of a sphere is necessarily the greatest straight line in it. Hence, when a plane passes through the centre of the sphere, the diameter of the circle which is formed by the section of this plane and the sperical surface, is greater than any other line in the sphere which is not a diameter.

A plane cutting the sphere, but not passing through its centre, forms, by its section with the spherical surface, a circle whose diameter is less than the diameter of the sphere. That the section is a circle, is readily inferred from 14.3; and that the diameter of that circle is less than the diameter of the sphere, is plain from 15.3.

Definition 1. Those circles whose planes pass through the centre of the sphere, are called great circles; but circles whose planes do not pass through the centre of the sphere, are called less circles.

Corollary 1. The diameter of every great circle is also a diameter of the sphere.

Cor. 2. The common section of the planes of two great circles, is a diameter to each of those circles.

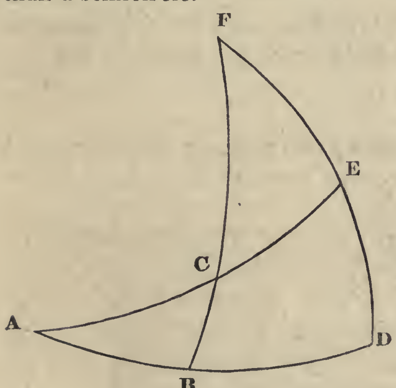
Cor. 3. Every great circle in the sphere divides every other great circle into two equal parts.

Def. 2. The axis of a circle is the right line which passes through its centre, and is at right angles to the plane of the circle; and the poles of a circle are the points where its axis meets the surface of the sphere.

Def. 3. A spherical angle, or the angle formed by two great circles, is the inclination of their planes.

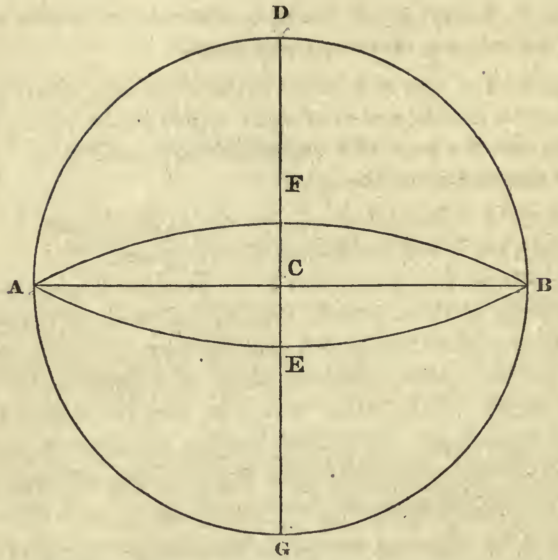
Cor. When two great circles are at right angles to each other, each of them passes through the poles of the other; and if they pass through the poles of each other, they are at right angles. Also, when the plane of a great circle is at right angles to the plane of a less one, the former circle passes through the poles of the latter. For the axis of every circle passes through the centre of the sphere, and is at right angles to the plane of its own circle.

Def. 4. A spherical triangle is formed by the arcs of three great circles, each of which cuts the other two, but in such manner that each of the arcs composing the triangle is less than a semicircle.



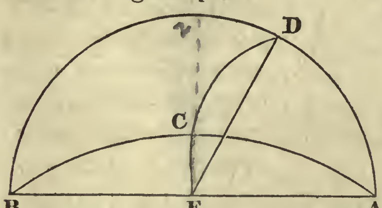
Def. 5. If AD and DF, two quadrants of great circles, are placed at right angles to each other; and through the points A, F, two other great circles, AE, FB, are described, cutting each other in C; the triangles ABC, FEC are called complemental triangles.

Art. 46. The arc of a great circle, intercepted between another great circle and its pole, is a quadrant.



Let AEBF be a great circle, whose centre is C, and axis DCG, its poles being D and G; DAGB another great circle passing through the axis DCG; these great circles are at right angles to each other (17.2 sup.), and CA their common section at right angles to CD; hence the arcs AD, BD, AG and BG, are quadrants.

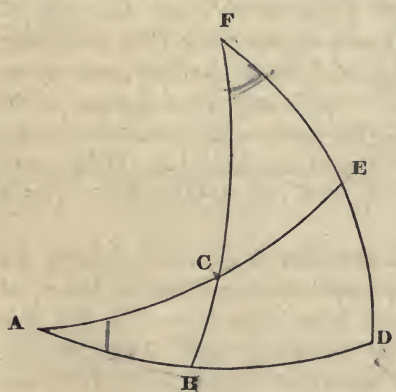
Art. 47. The angle made by two great circles is measured by the arc intercepted between them, at the distance of 90° from the angular point.



Let ACB, ADB, be two semicircles, whose common section passes through E, the centre of the sphere; from E, draw EC, ED, at right angles to AB, one in the the plane ACB, the other in the plane ADB; and let the plane passing through EC, ED, cut the surface of the sphere in the arc CD; CD is part of a great circle (Def. 1), and AE is at right angles to its plane (4.2 sup.); consequently, AC and AD are quadrants; and A, B are the poles of CD. Also, the inclination of the planes ACB, ADB is the angle CED (Def. 4.2 sup.); and that angle is measured by the arc CD.

Cor. Since the plane of CED is at right angles to AB, and consequently to each of the planes ADB, ACB (17.2 sup.), it must pass through the axes of those planes; and therefore the circle DC, continued, must pass through the poles of ADB and ACB. Those poles being 90 degrees from their respective circles, the arc intercepted between them is manifestly equal to CD, the measure of the spherical angle CAD.

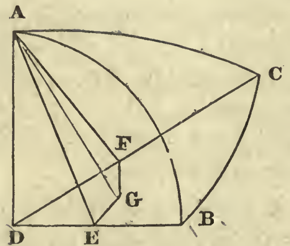
Art. 48. In the complemental triangles ABC, FCE; AC is the complement of CE; BC is the complement of FC; AB of the angle at F; and the angle at A of the side FE.



For, since FD and AD are quadrants at right angles to each other, F is the pole of AD, and A is the pole of FD (Art. 46); hence FB, AE are also quadrants; consequently, BD is the measure of the angle at F, and DE of the angle at A; whence the proposition is obvious.

Art. 49. In isosceles spherical triangles, the angles opposite the equal sides are equal.

Let ABC be a spherical triangle, whose sides AB and AC are equal; it is to be proved that the angles ABC and ACE are also equal.



Take D the centre of the sphere, and join DA, DB and DC; and in the plane of ADB, draw AE at right angles to DB. In like manner, in the plane ADC, draw AF at right angles to DC. Then, since the arc AB is equal to AC, the angle ADB at the centre of the sphere is equal to the angle ADC; therefore the triangles ADE, ADF, right angled at E and F, having two angles of the one respectively equal to two angles of the other, and the side AD, opposite the right angle in each, common to both; have the sides AE, AF, adjacent to the right angles, also equal (26.1).

Again, in the plane BDC, draw EG and FG at right angles to DB and DC respectively, and let them meet in G. Then, because AE and EG are both at right angles to DB, the line DB is at right angles to the plane which passes through AE and EG (4.2 sup.); and therefore the plane DBC is at right angles to the plane AEG (17.2 sup.). In like manner, the plane DBC is proved to be at right angles to the plane AFG; consequently, the line AG, the common section of the planes AEG, AFG, is at right angles to the plane DBC (18.2 sup.); wherefore the angles AGE, AGF are right angles (1 Def. 2 sup.)

Now, the right angled triangles AGE, AGF, having the perpendicular AG common, and the hypothenuse AE equal the hypothenuse AF, must have their bases EG, FG, also equal (47.1); and therefore the angles AEG, AFG, likewise equal (8.1); that is, the spherical angles ABC, ACB, are equal. *Q. E. D.*

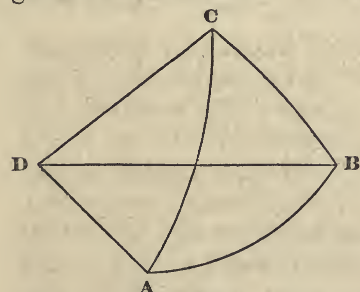
Art. 50. If two angles of a spherical triangle are equal, the sides opposite to them are also equal.

Let the spherical angles ABC, ACB be equal; then the sides AB, AC shall be also equal.

Making the same construction as in the last article, w have, as before, the angles AGE, AGF, both right angles; also the angles AEG and AFG, which are the same as the spherical angles ABC and ACB, likewise equal, and the side AG common to the triangles AGE, AGF; therefore AE is equal to AF (26.1). Then, in the right angled triangles ADE, ADF, we have the perpendiculars AE, AF equal, and the hypothenuse AD common; wherefore DE is equal to DF (47.1), and consequently the angle ADE equal to ADF (8.1): whence AB is equal to AC (26.3).

Q. E. D.

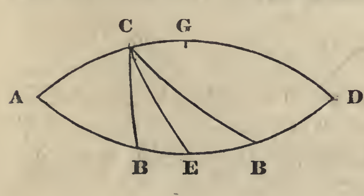
Art. 51. Any two sides of a spherical triangle are together greater than the third.



Let ABC be a spherical triangle; any two of its sides taken together are greater than the third.

Take D the centre of the sphere, and join DA, DB and DC. Then the solid angle at D is contained by the three plane angles ADB, ADC and BDC, of which any two taken together are greater than the third (20.2 sup.); therefore any two of the arcs which measure those angles are likewise together greater than the third. *Q. E. D.*

Art. 52. The three sides of a spherical triangle are together less than the circumference of a circle.



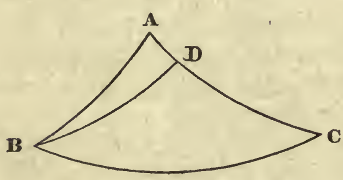
Let ABC be a spherical triangle; the sides AB + AC + BC are less than 360°. Continue two of those sides AC, AB, till they meet in D; then ACD and ABD are semicircles (Art. 45, Cor. 3, Def. 1). But BD + CD are greater than BC. If to these unequal quantities we add AB + AC, we have ABD + ACD greater than BC + AB + AC; that is, AB + AC + BC are less than two semicircles, or 360°.

Q. E. D.

Art. 53. In any spherical triangle having unequal angles, the greater angle has the greater side opposite to it.

In the spherical triangle ABC, let the angle ABC be greater than ACB; and take CBD = BCA; then (Art. 50) BD=CD; consequently, AC = BD + AD; but (Art. 50) BD + AD are greater than AB; that is, AC is greater than AB.

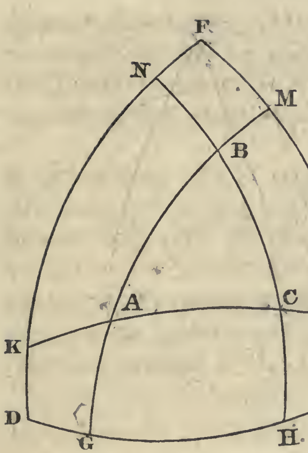
Q. E. D.



Conversely: If the side AC is greater than AB, the angle ABC is greater than ACB. For if it is not greater, it is equal or less. If the angles were equal, the opposite sides would also be equal (Art. 49); and if ACB was greater than ABC, the side AB would be greater than AC.

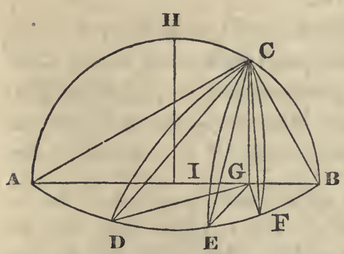
Art. 54. If the angular points of a spherical triangle are made the poles of three great circles, these three circles, by their intersections, will form a triangle, which is said to be supplemental to the former; and the two triangles are such, that the sides of the one are the supplements of the arcs which measure the angles of the other.

Let A, B, C, the angular points of the triangle ABC, be the poles of the great circles FE, DE, DF, which form the triangle FED; and let the sides of the former triangle be produced till they meet those of the latter. Now, since A and C are the poles of EF and DF respectively, the distances



from F to A and from F to C are quadrants (Art. 46); hence F is the pole of KACL. In the same manner it is proved that D and E are the poles of NBCH and GABM; consequently, EM + LF = 180°; that is, EF and LM are supplements to each other. In like manner it may be proved that DE and GH; DF and KN; AC and KL; AB and GM; BC and NH, are respectively supplements to each other. But ML, GH and KN are the measures of the angles A, B and C; also KL, NH and GM are the measures of the angles F, D and E (Art. 47). Hence the proposition is manifest. *Q. E. D.*

Cor. Since the sides FE, FD and DE, together with the measures of the angles A, B and C, are equal to three semicircles, or 540°; and the three sides of any spherical triangle are together less than two semicircles, or 360° (Art. 52), it follows that the three angles of the triangle ABC are more than 180°, but less than 540°.



Art. 55. Let AEB and AHB be semicircles, whose planes are at right angles to each other; and AB the common section of those planes, a diameter to the sphere; AH, HB, quadrants; and C any other point than H in the semicircle AHB; then CD, CE, CF, being arcs of great circles, intercepted between the point C

and the semicircle AEB; the arc CA, which passes through H, is greater, and CB, the remaining part of the semicircle, is less, than any other arc contained between C and AEB; also those nearer to CHA are greater than those which are more remote.

Draw CG and HI at right angles to AB; then is CG at right angles to the plane AEB (Def. 2, 2 sup.); hence, GD, GE, GF, being drawn, the angles CGD, CGE, CGF, are all right ones. And AH, BH, being quadrants, H is evidently the pole, and I the centre, of AEB; consequently, GA is the greatest, and GB the least, of all the straight lines drawn from G to the circumference; and GD is greater than GE; and GE than GF (7.3). Now,

$$AC^2 = CG^2 + GA^2;$$

and

$$DC^2 = CG^2 + GD^2;$$

of which GA^2 is greater than GD^2; wherefore AC^2 is greater than DC^2, and AC greater than DC. But the arcs AC and DC are each less than a semicircle; and, therefore, the greater chord subtends the greater arc; that is, the arc AC is greater than DC. In the same manner it may be proved that the arc DC is greater than EC, that EC is greater than FC, and FC greater than BC. *Q. E. D.*

Art. 56. In a right angled spherical triangle, the sides containing the right angle are of the same affection as the angles opposite to them.*

Let ACB be a spherical triangle, right angled at A; and let AC, AB be continued till they meet in D; and bisect ABD in E; then ACD, ABD are semicircles (Art. 45, Cor. 3 to Def. 1); and AE is a quadrant. But the angle at A being a right one, AEB passes through the pole of AC (Art. 45, Cor.

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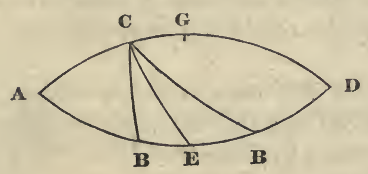
* Sides are said to be of the same *affection* when they are both less or both greater than quadrants; the same is said of angles when they are both less or both greater than right angles. A side and an angle are also of the same affection when the former is less or greater than a quadrant, and the latter less or greater than a right angle.

to Def. 3). Consequently E is the pole of AC (Art. 46); CE is a quadrant, and ACE a right angle.



Now, AC being taken less than a quadrant, the angle ACB will be less or greater than ACE, according as B lies between A and E, or between E and D; that is, when AB is less than a quadrant, the angle ACB is less than a right angle; and when AB is more than a quadrant, ACB is more than a right angle. And if we suppose ACB to be less than a right angle, it is manifest that AB is less than a quadrant; and, if greater, greater.

Again, in the right angled triangle DCB, in which DC is greater than a quadrant, it is manifest that the angle DCB is greater or less than a right angle, according as DB is greater or less than a quadrant; and vice versa. *Q. E. D.*

Art. 57. When the sides of a right angled spherical triangle, about the right angle, are of the same affection, the hypothenuse is less than a quadrant; but when those sides are of different affections, the hypothenuse is more than a quadrant.

Retaining the construction used in the last article, and bisecting ACD in G, we have G the pole of ABD, and CE a quadrant as before. But CB is either greater or less than CE, according as it is nearer to or farther from CGD than CE is (Art. 55); that is, the hypothenuse is less than a quadrant when the sides are both less or both greater than a quadrant; but the hypothenuse is greater than a quadrant when one side is less, and the other greater, than a quadrant.

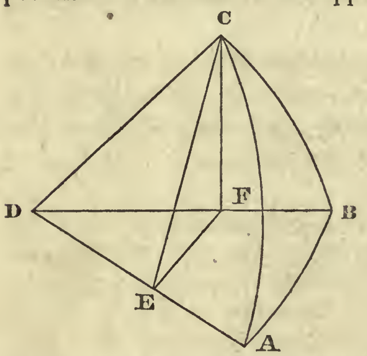
Q. E. D.

Cor. 1. Conversely, when the hypothenuse of a right angled spherical triangle is less than a quadrant, the sides are of the same affection; but, when the hypothenuse is greater than a quadrant, the sides are of different affections.

Cor. 2. Since the oblique angles of a right angled spherical triangle are of the same affection as the opposite sides (Art. 56); therefore, according as the hypothenuse is greater or less than a quadrant, the oblique angles will be of different, or the same affection.

Cor. 3. Because the sides are of the same affection as their opposite angles; therefore, when an angle and the side adjacent are of the same affection, the hypothenuse is less than a quadrant, and vice versa.

Art. 58. In any right angled spherical triangle, as radius is to the sine of an oblique angle, so is the sine of the hypothenuse to the sine of the opposite side.



Let ABC be the triangle, right angled at B; take D the centre of the sphere; join DA, DB, and DC; in the plane ADC, draw CE at right angles to DA; from E, draw in the plane ADB, the line EF at right angles to DA, meeting DB in F; and join CF.

Then, the lines EC and EF being both at right angles to DA, the plane CEF is at right angles to DA (4.2 sup.); consequently, the planes ADB and CEF are at right angles to each other (17.2 sup.). But the plane DBC is, by hypothesis, at right angles to DAB; hence the planes CEF and DBC, being both at right angles to DAB, their common section FC is also at right angles to the same plane (18.2 sup.); wherefore DFC and EFC are right angles. Hence CF is the sine of CB; also CE is the sine of CA; and the angle CEF is the inclination of the planes CDA and BDA; that is, CEF = the spherical angle CAB. Now (Art. 28),

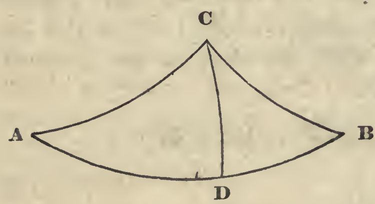
As radius : sine of CEF :: CE : CF;

that is,

As radius : sine of CAB :: sine of AC : sine of BC.

Q. E. D.

Art. 59. In any oblique angled spherical triangle, the sines of the angles are to each other as the sines of the opposite sides.



Let ABC be the triangle; and through C describe the arc CD of a great circle, at right angles to AB; then, by last article,

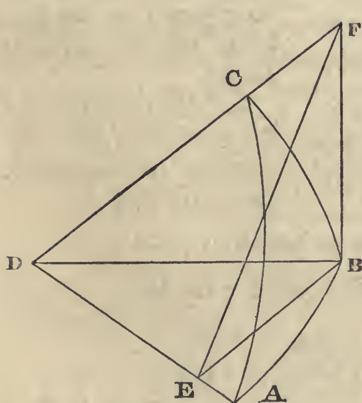
As sin of A : radius :: sin DC : sin AC;

radius : sin B :: sin BC : sin DC;

therefore (23.5),

sin A : sin B :: sin BC : sin AC.

Q. E. D.



Art. 60. In any right angled spherical triangle, as radius is to the sine of one of the sides, so is the tangent of the adjacent angle to the tangent of the opposite side.

Let ABC be the triangle, right angled at B; take D the centre of the sphere, and join DA, DB, DC; in the plane ADB, draw BE at right angles to DA; from E, draw EF, in the plane ADC, at right angles to AD, and meeting DC produced in F; and join FB. Then EB and EF being at right angles to DE, the plane FEB is at right angles to DE (4.2 sup.); consequently, the plane ADB, which passes through DE, is at right angles to the plane FEB (17.2 sup.); therefore the common section BF of the planes EBF and DBC, is at right angles to the plane DAB (18.2 sup.); whence EBF and DBF are right angles; and, consequently, BF is the tangent of BC; BE is also the sine of AB; and the angle BEF the same as the spherical angle BAC. Now (Art. 28),

As radius : tan BEF :: BE : BF;

that is,

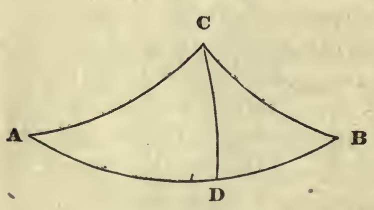
As radius : tan BAC :: sin AB : tan BC;

and alternately (16.5),

As radius : sin AB :: tan BAC : tan BC.

Q. E. D.

Art. 61. If two right angled spherical triangles have the same perpendicular, the sines of the bases are to each other reciprocally as the tangents of the adjacent angles.



Let ADC and BDC be the triangles; DC the common perpendicular; then (Art. 60),

As sin AD : radius :: tan DC : tan A;

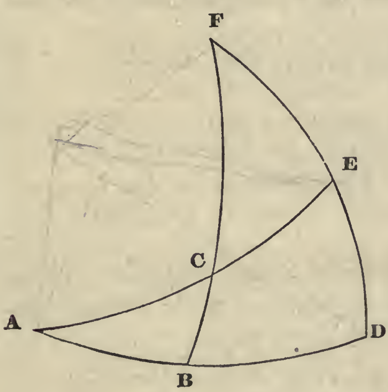
and As radius : sin BD :: tan B : tan DC;

whence (23.5),

sin AD : sin BD :: tan B : tan A.

Q. E. D.

Art. 62. In any right angled spherical triangle; as radius is to the cosine of the angle at the base, so is the tangent of the hypothenuse to the tangent of the base.



Let ABC be the triangle; B the right angle; and FCE the complemental triangle.

Then (Art. 60),

As radius : sin FE :: tan F : tan CE;

that is (Art. 48),

As radius : cos A :: cotan AB : cotan AC.

But (Art. 23.4),

As tan P : tan Q :: cotan Q : cotan P;

consequently,

As radius : cos A :: tan AC : tan AB.

Q. E. D.

Art. 63. If two right angled spherical triangles have the same perpendicular, the cosines of the vertical angles are to each other reciprocally as the tangents of the hypothenuses.

Let ADC, BDC (see fig. on opposite page), be the triangles right angled at D; then (Art. 62),

As cos ACD : radius :: tan DC : tan AC;

and As radius : cos BCD :: tan BC : tan DC;

therefore (23.5),

As cos ACD : cos BCD :: tan BC : tan AC.

Q. E. D

Art. 64. In any right angled spherical triangle; as radius is to the cosine of the hypothenuse, so is the tangent of either angle to the cotangent of the remaining angle.

In the triangle FCE (see fig. on opposite page), we have (Art. 60),

As radius : sin CE :: tan FCE : tan FE;

that is (Art. 48),

As radius : cos AC :: tan ACB : cot BAC :: (Art. 23.4) tan BAC : cot ACB.

Q. E. D.

Art. 65. In any right angled spherical triangle; as radius is to the cosine of one of the sides, so is the cosine of the other side to the cosine of the hypothenuse.

In the right angled triangle FCE, we have (Art. 57),

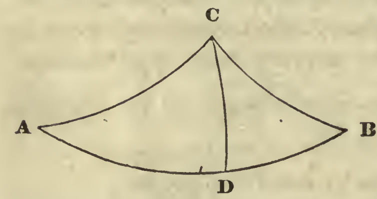
As radius : sin F :: sin FC : sin CE;

that is (Art. 48),

As radius : cos AB :: cos BC : cos AC.

Q. E. D.

Art. 66. If two right angled spherical triangles have the same perpendicular, the cosines of the hypothenuses are to each other as the cosines of the bases.



Taking ADC and BDC as the triangles right angled at D, we have (Art. 65),

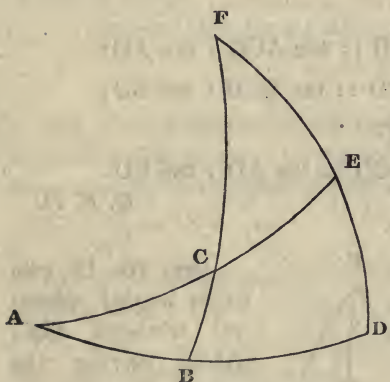
As radius : cos DC :: cos AD : cos AC;

and As radius : cos DC :: cos BD : cos BC;

whence (16.5),

cos AD : cos BD :: cos AC : cos BC.

Q. E. D.



ART. 67. In any right angled spherical triangle; as radius is to the sine of either oblique angle, so is the cosine of the adjacent side to the cosine of the opposite angle.

In the right angled triangle FCE, we have (Art. 58),

As radius : sin FCE :: sin CF : sin FE;

that is (Art. 48),

As radius : sin ACB :: cos BC : cos BAC.

Q. E. D.

ART. 68. In two right angled spherical triangles, ACD, BCD (fig. p. 92), having the same perpendicular CD, the cosines of the angles at the base are to each other as the sines of the vertical angles.

By Art. 67 and A. 5,

As cos DAC : cos DC :: sin ACD : radius;

and, by same article,

As cos DC : cos DBC :: radius : sin BCD;

consequently (22.5),

As cos DAC : cos DBC :: sin ACD : sin BCD.

Q. E. D.

ART. 69. The same things being supposed as in the last article, the tangents of the bases are to each other as the tangents of the vertical angles.

By Art. 60,

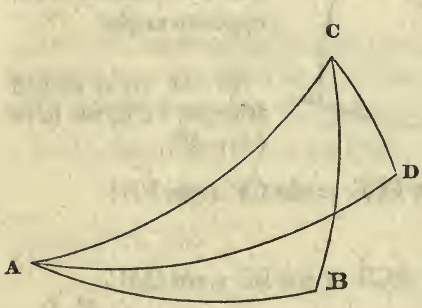
As radius : sin CD :: tan ACD : tan AD;

and As radius : sin CD :: tan BCD : tan BD;

consequently (11 and 16.5),

tan ACD : tan BCD :: tan AD : tan BD.

Q. E. D.



ART. 70. In two right angled spherical triangles ABC, ADC, having the same hypothenuse AC, the cosines of the bases are to each other reciprocally as the cosines of the perpendiculars.

For (Art. 65),

As radius : cos AB :: cos BC : cos AC;

and from the same article inverted (A 5),

As cos AD : radius :: cos AC : cos DC;

hence (23.5),

As cos AD : cos AB :: cos BC : cos DC.

Q. E. D.

ART. 71. The same things being supposed as in the last article, the tangents of the bases are to each other as the cosines of the adjacent angles. For (Art. 62),

As radius : cos CAB :: tan AC : tan AB;

and, by inversion,

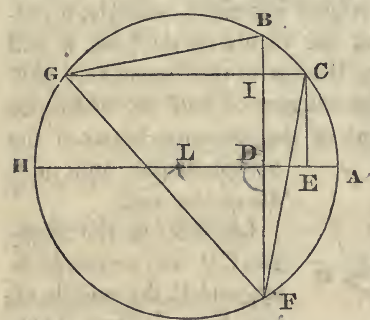
As cos CAD : radius :: tan AD : tan AC;

therefore (22.5),

As cos CAD : cos CAB :: tan AD : tan AB.

Q. E. D.

Art. 72. As the sum of the sines of any two unequal arcs is to their difference, so is the tangent of half the sum of those arcs, to the tangent of half their difference.



Let AB, AC be the arcs; L the centre of the circle; and AH the diameter passing through A; make AF=AB; join BF, and let BF cut AH in D; draw CE parallel to BF, and CG to AH; let CG meet BF in I; join GB, GF and CF. Then, since AF = AB, FD = BD; and HDF = HDB; hence BD is the sine of AB; and CE, which is parallel to BD, is the sine of AC; therefore, FI is the sum, and BI the difference, of the sines of AB and AC.

Again, the arc CF is the sum, and CB the difference, of AB and AC; therefore the angle FGC is measured by half the sum, and BGC by half the difference, of AB and AC (20.3). But the angle GIF = HDF (29.1), and is therefore a right angle; consequently, IF is the tangent of CGF, and IB the tangent of CGB, to the radius GI; therefore, for any other radius (Art. 27, Cor.),

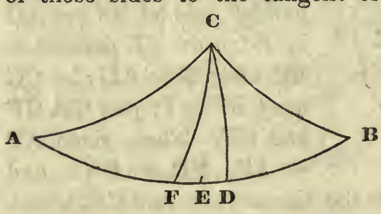
As IF : IB :: tan CGF : tan CGB :: tan $\frac{1}{2}$(AB+AC) : tan $\frac{1}{2}$(AB — AC). *Q. E. D.*

Art. 73. The sum of the cosines of two unequal arcs is, to their difference, as the cotangent of half their sum is to the tangent of half their difference.

Retaining the construction of the last article, it is easily perceived that GI is the sum, and IC the difference, of the cosines of AB and AC; also the angle GFI is the complement of CGF; and IFC = BGC; hence,

$\cos AB + \cos AC : \cos AC - \cos AB :: \text{cotan} \frac{1}{2}(AB+AC) : \tan \frac{1}{2}(AB - AC)$. *Q. E. D.*

ART. 74. In any oblique angled spherical triangle, a perpendicular being let fall from the vertex on the base, it will be, as the tangent of half the base is to the tangent of half the sum of the sides, so is the tangent of half the difference of those sides to the tangent of the distance between the perpendicular and middle of the base.



Let ABC be the triangle, CD the perpendicular, and E the middle of the base. Then (Art. 66),

$$\text{As } \cos AC : \cos BC :: \cos AD : \cos BD;$$

whence (E 5),

$$\text{As } \cos AC + \cos BC : \cos BC - \cos AC :: \cos AD + \cos BD ; \cos BD - \cos AD;$$

consequently (Art. 73),

$$\text{As cotan} \tfrac{1}{2}(AC + BC) : \tan \tfrac{1}{2}(AC - BC) :: \text{cotan} \tfrac{1}{2}(AD + DB) : \tan \tfrac{1}{2}(AD - DB) :: \text{cotan } AE : \tan ED;$$

and, alternately,

$$\text{As cotan} \tfrac{1}{2}(AC+BC) : \text{cotan } AE :: \tan \tfrac{1}{2}(AC-CB) : \tan ED.$$

But (Art. 23.4),

$$\text{cotan} \tfrac{1}{2}(AC+BC) : \text{cotan } AE :: \tan AE : \tan \tfrac{1}{2}(AC+BC);$$

therefore,

As $\tan AE : \tan \frac{1}{2}(AC+BC) :: \tan \frac{1}{2}(AC-BC) : \tan ED$.

Q. E. D.

Art. 75. In any oblique angled spherical triangle, a perpendicular being let fall from the vertex on the base, and an arc described bisecting the vertical angle; it will be, as the cotangent of half the sum of the angles at the base is to the tangent of half their difference, so is the tangent of half the vertical angle to the tangent of the angle formed by the perpendicular and the arc bisecting the vertical angle.

Let ABC be the triangle, CD the perpendicular, and CF the arc bisecting the vertical angle; then (Art. 68),

$$\text{As } \cos A : \cos B :: \sin ACD : \sin BCD;$$

hence (E 5)

$$\cos A+\cos B \,.\, \cos A-\cos B :: \sin ACD + \sin BCD : \sin ACD - \sin BCD;$$

therefore (Arts. 72, 73)

$$\text{As } \cot \tfrac{1}{2}(A+B) : \tan \tfrac{1}{2}(B-A) :: \tan \tfrac{1}{2}(ACD+BCD) : \tan \tfrac{1}{2}(ACD-BCD) :: \tan ACF : \tan DCF.$$

Q. E. D.

Scholium. From the analogies demonstrated in Articles 60, 62, 63, 65, 66 and 67, we may frequently determine the affections of the sides and angles of the triangles, by adverting to the signs of the terms, as explained in Art. 24. Thus, in Art. 60, the base AB being always less than a semicircle, the sin AB is positive; hence the tan BAC and tan BC are both positive or both negative; consequently, BAC and BC are both less or both more than 90°. In Art. 62, when the angle at the base is acute, its cosine is positive; consequently, the tangents of the hypothenuse and base will be both positive or both negative; therefore the arcs themselves will be both more or both less than 90°; that is, they will be of the same affection. But when the angle at the base is obtuse, its cosine will be negative; and therefore the tangent of the hypothenuse and that of the base will be one positive, and the other negative; consequently, the arcs themselves will be of different affections. In Art. 63, when the vertical angles are of the same affection, their cosines have the same sign; consequently, the tangents of the adjacent sides will have the same sign, and will therefore be of the same affection. The same principles are applicable to the other cases. The conclusions thus obtained are consonant to those obtained in a different manner in Arts. 56, 57.

The analogies above demonstrated are sufficient to enable the student to calculate the sides and angles of spherical triangles from the usual data; yet there are various useful forms, hereafter demonstrated, which are applicable to particular cases. We have also two concise rules, discovered by Baron Napier, the celebrated inventor of logarithms, by which the cases in right angled spherical triangles are conveniently solved; and, being easily remembered, they are frequently used in practice.

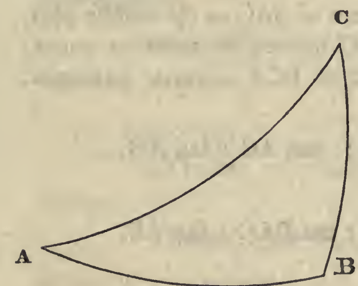
Art. 76. In right angled spherical triangles, there are five parts which may have different values assigned to them without changing the right angle, viz.: the hypothenuse, the two sides, and the two oblique angles. Now, the sides, and the complements of the hypothenuse and oblique angles, are called the five *circular parts;* one of which being assumed as the *middle part*, the two which lie contiguous to this middle part are called the *adjacent extremes;* and the other two are termed the *opposite extremes*. Then Napier's rules are:

1. The rectangle of radius and the sine of the middle part is equal to the rectangle of the tangents of the adjacent extremes.

2. The rectangle of radius and the sine of the middle

part is equal to the rectangle of the cosines of the opposite extremes.

These rules may be explained and demonstrated in the following manner:



Let ABC be the triangle, right angled at B. Then, assuming AB as the middle part, the side BC and complement of BAC are the adjacent extremes; and the complements of AC and ACB are the opposite extremes. Now (Art. 60),

As radius : sin AB :: tan BAC : tan BC;

and, alternately,

As radius : tan BAC :: sin AB : tan BC.

But (Art. 23),

As radius : tan BAC :: cotan BAC : radius.

Hence,

cotan BAC : radius :: sin AB : tan BC;

therefore (16.6),

radius . sin AB = cotan BAC . tan BC;

which is Napier's first rule.

Again (Art. 58),

As radius : sin ACB :: sin AC : sin AB;

whence (16.6),

radius . sin AB = sin ACB . sin AC;

which is Napier's second rule.

If we assume BC the middle part, AB and the complement of ACB become the adjacent extremes; and the complements of BAC and AC, the opposite extremes. Napier's rules may then be demonstrated in that case exactly as before.

Assuming next the complement of BAC as the middle part, AB and the complement of AC become adjacent extremes; and BC and the complement of BCA opposite extremes. Then (Art. 62),

As radius : cos BAC :: tan AC : tan AB;

alternately,

As radius : tan AC :: cos BAC : tan AB.

Hence (Art. 23),

As cotan AC : radius :: cos BAC : tan AB;

consequently (16.6),

radius . cos BAC = cotan AC . tan AB;

which is rule first.

Again (Art. 67),

As radius : sin BCA : cos BC : cos BAC;

whence,

radius . cos BAC = sin BCA . cos BC;

which is rule second.

In the same manner, the rule is demonstrated, when the complement of ACB is taken as the middle part.

Lastly, assuming the complement of AC as the middle part, the complements of BAC and BCA are the adjacent extremes; and AB, BC, the opposite extremes. Then (Art. 64),

As radius : cos AC :: tan ACB : cotan BAC;

alternately,

As radius : tan ACB : : cos AC : cotan BAC;

wherefore (Art. 23),

As cotan ACB : radius : : cos AC : cotan BAC;

consequently,

radius . cos AC = cotan ACB . cotan BAC;

which is rule first.

And (Art. 65),

As radius : cos AB : : cos BC : cos AC;

whence,

radius . cos AC = cos AB . cos BC;

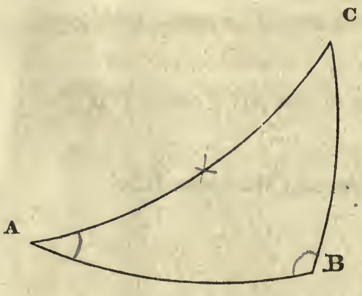
which is rule second.

The following table exhibits the different cases, and the equations arising from Napier's rules:

Cases	Middle part.	Adjacent Extremes.	Opposite Extremes.	Equations.
1	AB	Comp BAC BC	Comp ACB Comp AC	rad.sinAB = { cot BAC.tan BC sin ACB.sin AC
2	BC	Comp BCA AB	Comp BAC Comp AC	rad.sinBC = { cot BCA.tan AB sin BAC.sin AC
3	Comp BAC	Comp AC AB	Comp ACB BC	rad.cosBAC = { cot AC.tan AB sin ACB.cosBC
4	Comp BCA	Comp AC BC	Comp BAC AB	rad.cosBCA = { cot AC.tan BC sin BAC.cosAB
5	Comp AC	Comp BAC Comp BCA	AB BC	rad.cosAC = { cot BAC.cotBCA cos AB.cos BC

When any two of these circular parts are given to find a third, we must assume such one to be the middle part as will make the other two either both adjacent or both opposite extremes.

The following practical examples will serve to exercise the preceding theory:



Example 1. In the spherical triangle ABC, right angled at B, given the side AC 52° 15′, and the angle A 23° 28′, to find the other sides and the remaining angle.

The perpendicular BC may be found by Art. 58; the side AB, by Art. 62; and the angle C, by Art. 64. Or, using Napier's circular parts, we find BC by the second equation, case 2, in the foregoing table; AB by the first equation, case 3; and the angle C by the first equation, case 5. The results are, BC 18° 21′ 9″; AB 49° 49′ 57″; C 75° 6′ 58″.

Ex. 2. Given, the base AB 61° 25′, and the adjacent angle A 32° 45′, to determine the rest.

The hypothenuse AC may be found by Art. 62; the perpendicular BC by Art. 60; and the angle C by Art. 67; or by cases 3, 1, 4 of the circular parts.

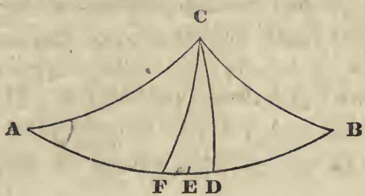
The results are, AC 65° 22′ 52″; BC 29° 27′ 32″; and C 75°.

Ex. 3. Given, the base AB 75° 28′, and the perpendicular BC 41° 15′, to find the rest.

The results are, AC 79° 7′ 30″; A 42° 10′ 32″; C 80° 18′ 1″.

Ex. 4. Given, the angle A 23° 28′ 30″, and angle C 75° 22′, to find the rest.

The results are, AC 53° 2′ 36″; AB 50° 38′ 22″; BC 18° 33′ 40″.



Ex. 5. In the oblique angled triangle ABC, given the sides AB 70°, AC 58°, and the angle CAB 52° 30′, to find the rest.

Suppose the arc CD of a great circle at right angles to AB to pass through C; then the given triangle will be divided into two right angled ones, ADC and BDC.

In the triangle ADC, the side AD and angle ACD may be computed by Arts. 62 and 64. Hence BD is known. Then the angle B, the side BC, and the angle BCD, may be found by Arts. 61, 66 and 69.

Results: B 64° 28′; BC 48° 12′ 46″; ACB 91° 0′ 21″.

Ex. 6. In the spherical triangle ABC, given the angle BAC 50° 15′, ACB 92°, and side AC 57° 30′, to find the rest.

The arc CD being made perpendicular to AB, the side AD and angle ACD may be found as in the last example; whence BD, BC, and the angle at B, may be computed by Arts. 69, 63 and 68.

Results: BA 69° 25′ 2″; BC 46° 4′ 16″; ABC 64° 12′ 16″.

Ex. 7. In the triangle ABC, given AB 71° 30′; AC 59° 20′; BC 50° 10′; to find the angles.

Drawing CD at right angles to AB, and taking AE $= \frac{1}{2}$AB, the arc ED is found by Art. 74; from which AD and BD become known; and thence the angles may be found by Arts. 62 and 59.

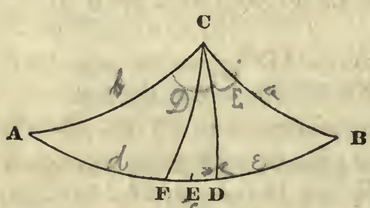
Results: BAC 54° 3′ 51″; ABC 65° 5′ 4″; ACB 90° 48′ 47″.

Ex. 8. In the triangle ABC, given the angle BAC 51°; ABC 58°; and ACB 110°; to find the sides.

Using the construction of the last example, and describing CF so as to divide the vertical angle into two equal angles, the angle FCD may be found by Art. 75; whence the angles ACD and BCD become known; and thence the sides AC, BC may be determined by Art. 64; and AD, BD, by Art. 67.

Results: AC 64° 28′ 31″; BC 55° 47′ 13″; AB 90° 44′ 26″.

Art. 77. It has been already mentioned that there are various useful forms which are applicable to particular cases. By means of these, the necessity of dividing an oblique angled triangle into two right angled ones, is always obviated. Of these forms, the following are the most important. They are investigated most conveniently by algebra.



Let ABC be a spherical triangle; CD a perpendicular upon AB; and, to accommodate the expressions to the language of algebra, let the capital letters A, B, C denote the angles, and the small letters a, b, c, the opposite sides; the segments AD, BD, being represented by d, e, and the opposite angles ACD and BCD by D and E respectively. In these investigations, the radius is taken $= 1$.

Now (Art. 62),

$$1 : \cos A :: \tan b : \tan d.$$

But (Art. 23),

$$\frac{\sin}{\text{cosin}} = \tan \therefore \frac{\sin d}{\cos d} = \cos A . \frac{\sin b}{\cos b}.$$

Again (Art. 66).

$$\cos b : \cos a :: \cos d : \cos (c - d);$$

wherefore,

$$\frac{\cos a}{\cos b} = \frac{\cos (c-d)}{\cos d} = \text{(Art. 37, Form. 2)} \frac{\cos c.\cos d + \sin c.\sin d}{\cos d}$$

$= \cos c + \sin c.\dfrac{\sin d}{\cos d} = \cos c + \cos A.\sin c.\dfrac{\sin b}{\cos b}$; by putting $\cos A.\dfrac{\sin b}{\cos b}$ instead of $\dfrac{\sin d}{\cos d}$.

Clearing this equation of fractions,

$$\cos a = \cos c.\cos b + \cos A.\sin c.\sin b. \qquad (1)$$

By Art. 64,

As $1 : \cos a :: \tan B : \cot E ::$ (Art. 23) $\dfrac{\sin B}{\cos B} : \dfrac{\cos E}{\sin E}$ $\therefore$

$$\frac{\cos E}{\sin E} = \cos a.\frac{\sin B}{\cos B}.$$

By Art. 68,

As $\cos B : \cos A :: \sin E : \sin (C - E)$ $\therefore$

$$\frac{\cos A}{\cos B} = \frac{\sin (C-E)}{\sin E} = \text{(Art. 37, F. 1)} \frac{\sin C.\cos E - \cos C.\sin E}{\sin E}$$

$= \sin C.\dfrac{\cos E}{\sin E} - \cos C. = \cos a.\sin C.\dfrac{\sin B}{\cos B} - \cos C$; substituting for $\dfrac{\cos E}{\sin E}$. Then, clearing of fractions,

$$\cos A = \cos a.\sin C.\sin B - \cos C.\cos B. \qquad (2)$$

By Art. 62,

As $1 : \cos B :: \tan a : \tan e ::$ (Art. 23, 4) $\text{cotan } e : \text{cotan } a$

$$\therefore \text{cotan } a = \cos B.\text{cotan } e = \cos B.\frac{\cos e}{\sin e};$$

wherefore, $$\frac{\text{cotan } a}{\cos B} = \frac{\cos e}{\sin e}.$$

But (Art. 61),

$$\text{As } \tan A : \tan B :: \sin e : \sin (c-e);$$

therefore,

$$\frac{\tan B}{\tan A} = \frac{\sin (c-e)}{\sin e} = (\text{Art. 37, F. 1}) \ \frac{\sin c.\cos e - \cos c.\sin e}{\sin e}$$

$$= \sin c.\frac{\cos e}{\sin e} - \cos c = \sin c.\frac{\cot a}{\cos B} - \cos c.$$

Consequently, by clearing of fractions,

$$\cos B.\tan B = \cot a.\sin c.\tan A - \cos c.\cos B.\tan A.$$

Now,

$$\cos B.\tan B = \sin B \ (\text{Art. 23}) \text{ and } \frac{1}{\tan A} = \cot A.$$

Hence $$\cot a = \frac{\sin B.\cot A + \cos c.\cos B}{\sin c}. \qquad (3)$$

By Art. 64,

$$\text{As } 1 : \cos b :: \tan D : \cot A = \cos b.\tan D = \cos b.\frac{\sin D}{\cos D}$$

By Art. 63,

$$\text{As } \tan a : \tan b :: \cos D : \cos (C-D);$$

$$\therefore \frac{\tan b}{\tan a} = \frac{\cos (C-D)}{\cos D} = (\text{Art. 37, Form. 2}) \cos C$$

$$+ \sin C.\frac{\sin D}{\cos D} \therefore \frac{\sin D}{\cos D} = \frac{\tan b.\cot a - \cos C}{\sin C}$$

Consequently,

$$\cot A = \frac{\sin b.\cot a - \cos C.\cos b}{\sin C}. \qquad (4)$$

These forms are not suited to logarithmic computations; but they are useful in the investigation of other equations, to which logarithms are conveniently applied.

From Form. 1, above given, we find, by transposition and division,

$$\cos A = \frac{\cos a - \cos c.\cos b}{\sin c.\sin b};$$

hence,

$$1 - \cos A = 1 - \frac{\cos a - \cos c.\cos b}{\sin c.\sin b} =$$

$$\frac{\cos c.\cos b + \sin c.\sin b - \cos a}{\sin c.\sin b} = \text{(Art. 37, Form. 2)}$$

$$\frac{\cos (c - b) - \cos a}{\sin c.\sin b}.$$

But (Art. 36, Form. 6),

$$1 - \cos A = 2 \sin^2 \tfrac{1}{2}A;$$

and (Art. 37, Form. 13),

$$\cos (c - b) - \cos a = 2 \sin \tfrac{1}{2}(a + c - b).\sin \tfrac{1}{2}(a + b - c).$$

$$\therefore \sin^2 \tfrac{1}{2}A = \frac{\sin \tfrac{1}{2}(a+c-b).\sin \tfrac{1}{2}(a+b-c)}{\sin c.\sin b}. \qquad (5)$$

Again, from the equation $\cos A = \dfrac{\cos a - \cos c.\cos b}{\sin c.\sin b}$, we have

$$1 + \cos A = 1 + \frac{\cos a - \cos c.\cos b}{\sin c.\sin b} =$$

$$\frac{\cos a - \cos c.\cos b + \sin c.\sin b}{\sin c.\sin b} = \text{(Art. 37, Form. 2)}$$

$$\frac{\cos a - \cos (c + b)}{\sin c.\sin b}.$$

But (Art. 36, Form. 8)

$$1 + \cos A = 2 \cos^2 \tfrac{1}{2}A;$$

$$\therefore \cos^2 \tfrac{1}{2}A = \frac{\sin \frac{1}{2}(c+b+a) \sin \frac{1}{2}(c+b-a)}{\sin c . \sin b}. \qquad (6)$$

Again:

$$\tan^2 \tfrac{1}{2}A = \frac{\sin^2 \frac{1}{2}A}{\cos^2 \frac{1}{2}A} = \frac{\sin \frac{1}{2}(a+c-b) \sin \frac{1}{2}(a+b-c)}{\sin \frac{1}{2}(c+b+a) \sin \frac{1}{2}(c+b-a)}. \qquad (7)$$

Equations 5, 6 and 7 furnish convenient expressions for finding an angle, when the three sides are given.

From Form. 2,

$$\cos a = \frac{\cos A + \cos C . \cos B}{\sin C . \sin B};$$

wherefore,

$$1 - \cos a = 1 - \frac{\cos A + \cos C . \cos B}{\sin C . \sin B} =$$

$$\frac{\sin C . \sin B - \cos C . \cos B - \cos A}{\sin C . \sin B} =$$

$$\frac{-(\cos C . \cos B - \sin C . \sin B) - \cos A}{\sin C . \sin B} = \frac{-\cos (B+C) - \cos A}{\sin C . \sin B}$$

$$= -\frac{\cos (B+C) + \cos A}{\sin C . \sin B} = \text{(Art. 37, Form. 12)}$$

$$\frac{-2 \cos \frac{1}{2}(B+C+A) . \cos \frac{1}{2}(B+C-A)}{\sin B . \sin C}.$$

But (Art. 36, Form. 6),

$$1 - \cos a = 2 \sin^2 \tfrac{1}{2}a;$$

Wherefore,

$$\sin^2 \tfrac{1}{2}a = \frac{-\cos \frac{1}{2}(B+C+A) . \cos \frac{1}{2}(B+C-A)}{\sin B . \sin C}.* \qquad (8)$$

* Since the three angles of a spherical triangle are together greater than 180°, but less than 540° (Art. 54, Cor.), $\cos \frac{1}{2}(B+C+A)$ will always be a negative quantity, and consequently —cosine a positive quantity.

Again:

$$1 + \cos a = 1 + \frac{\cos A + \cos C.\cos B}{\sin C.\sin B} =$$

$$\frac{\cos C.\cos B + \sin C.\sin B + \cos A}{\sin C.\sin B} = \frac{\cos (C - B) + \cos A}{\sin C.\sin B}$$

(Art. 37, Form. 2).

But (Art. 36, Form. 8),

$$1 + \cos a = 2 \cos^2 \tfrac{1}{2}a;$$

and (Art. 37, Form. 12)

$$\cos (C - B) + \cos A = 2 \cos \tfrac{1}{2}(A + C - B).\cos \tfrac{1}{2}(A + B - C);$$

whence,

$$\cos^2 \tfrac{1}{2}a = \frac{\cos \frac{1}{2}(A + C - B).\cos \frac{1}{2}(A + B - C)}{\sin C.\sin B}. \qquad (9)$$

Further:

$$\tan^2 \tfrac{1}{2}a = \frac{\sin^2 \frac{1}{2}a}{\cos^2 \frac{1}{2}a} = \frac{-\cos \frac{1}{2}(B + C + A).\cos \frac{1}{2}(B + C - A)}{\cos \frac{1}{2}(A + C - B).\cos \frac{1}{2}(A + B - C)}. \qquad (10)$$

Equations 8, 9 and 10 may be conveniently used for determining the sides, when all the angles are given.

By Art. 59,

$$\text{As } \sin b : \sin a :: \sin B : \sin A;$$

wherefore (E. 5),

$$\sin b + \sin a : \sin b - \sin a :: \sin B + \sin A : \sin B - \sin A;$$

consequently, by Art. 72,

$$\text{As } \tan \tfrac{1}{2}(b + a) : \tan \tfrac{1}{2}(b - a) :: \tan \tfrac{1}{2}(B + A) : \tan \tfrac{1}{2}(B - A). \qquad (N)$$

wherefore,

$$\tan \tfrac{1}{2}(B - A) = \tan \tfrac{1}{2}(B + A).\frac{\tan \frac{1}{2}(b - a)}{\tan \frac{1}{2}(b + a)} \qquad (P)$$

Again (Art. 68),

$$\text{As } \cos B : \cos A :: \sin E : \sin D;$$

hence (E. 5),

$$\text{As } \cos B+\cos A : \cos A-\cos B :: \sin E+\sin D : \sin D-\sin E;$$

consequently (Arts. 73.72),

$$\text{As } \operatorname{cotan} \tfrac{1}{2}(B+A) : \tan \tfrac{1}{2}(B-A) :: \tan \tfrac{1}{2}C : \tan \tfrac{1}{2}(D-E);$$

whence

$$\tan \tfrac{1}{2}(D-E) = \tan \tfrac{1}{2}C.\frac{\tan \frac{1}{2}(B-A)}{\operatorname{cotan} \frac{1}{2}(B+A)} = \text{(Art. 23.4)}$$

$$\tan \tfrac{1}{2}C.\tan \tfrac{1}{2}(B-A).\tan \tfrac{1}{2}(B+A).$$

By Art. 63,

$$\text{As } \tan b : \tan a :: \cos E : \cos D;$$

wherefore (E. 5),

$$\text{As } \tan b+\tan a : \tan b-\tan a :: \cos E+\cos D : \cos E-\cos D.$$

But (Art. 37, equation 8),

$$\tan b \pm \tan a = \frac{\sin (b \pm a)}{\cos a.\cos b};$$

consequently,

$$\sin (a+b) : \sin (b-a) :: \operatorname{cotan} \tfrac{1}{2}C : \tan \tfrac{1}{2}(D-E);$$

wherefore,

$$\tan \tfrac{1}{2}(D-E) = \operatorname{cotan} \tfrac{1}{2}C.\frac{\sin (b-a)}{\sin (b+a)} = \text{(Art. 36, eq. 3)}$$

$$\operatorname{cotan} \tfrac{1}{2}C.\frac{2 \cos \frac{1}{2}(b-a).\sin \frac{1}{2}(b-a)}{2 \cos \frac{1}{2}(a+b).\sin \frac{1}{2}(a+b)}.$$

Equating these values of $\tan \frac{1}{2}(D-E)$,

$$\tan \tfrac{1}{2}C.\tan \tfrac{1}{2}(B-A).\tan \tfrac{1}{2}(B+A) =$$

$$\operatorname{cotan} \tfrac{1}{2}C.\frac{\cos \frac{1}{2}(b-a).\sin \frac{1}{2}(b-a)}{\cos \frac{1}{2}(a+b).\sin \frac{1}{2}(a+b)}.$$

In this equation, substitute for tan $\frac{1}{2}$(B — A), its value given in equation P; then,

$$\tan \tfrac{1}{2}C.\tan^2 \tfrac{1}{2}(B + A).\frac{\tan \frac{1}{2}(b - a)}{\tan \frac{1}{2}(b + a)} =$$

$$\text{cotan } \tfrac{1}{2}C.\frac{\cos \frac{1}{2}(b - a).\sin \frac{1}{2}(b - a)}{\cos \frac{1}{2}(a + b).\sin \frac{1}{2}(a + b)}.$$

Hence,

$$\tan^2 \tfrac{1}{2}(B+A) = \frac{\text{cotan } \frac{1}{2}C.\cos \frac{1}{2}(b-a).\sin \frac{1}{2}(b-a).\tan \frac{1}{2}(b+a)}{\tan \frac{1}{2}C.\cos \frac{1}{2}(a+b).\sin \frac{1}{2}(a+b).\tan \frac{1}{2}(b-a)}.$$

Now (Art. 23.1),

$$\cos = \frac{\sin}{\tan} \therefore \frac{1}{\cos} = \frac{\tan}{\sin};$$

and (Art. 23.4),

$$\text{cotan} = \frac{1}{\tan} \therefore \frac{\text{cotan}}{\tan} = \text{cotan}^2.$$

Our equation is therefore reducible to this:

$$\tan^2 \tfrac{1}{2}(B+A) = \text{cotan}^2 \tfrac{1}{2}C.\frac{\cos^2 \frac{1}{2}(b - a)}{\cos^2 \frac{1}{2}(a+b)};$$

consequently,

$$\tan \tfrac{1}{2}(B+A) = \text{cotan } \tfrac{1}{2}C.\frac{\cos \frac{1}{2}(b - a)}{\cos \frac{1}{2}(a + b)}. \qquad (11)$$

From equation P,

$$\tan \tfrac{1}{2}(B - A) = \text{cotan } \tfrac{1}{2}C.\frac{\cos \frac{1}{2}(b - a).\tan \frac{1}{2}(b - a)}{\cos \frac{1}{2}(b + a).\tan \frac{1}{2}(b + a)} =$$

$$\text{cotan } \tfrac{1}{2}C.\frac{\sin \frac{1}{2}(b - a)}{\sin \frac{1}{2}(b + a)}. \qquad (12)$$

From proportion N,

As $\tan \frac{1}{2}(B+A) : \tan \frac{1}{2}(B-A) :: \tan \frac{1}{2}(b+a) : \tan \frac{1}{2}(b-a)$

$$\therefore \tan \tfrac{1}{2}(b - a) = \tan \tfrac{1}{2}(b + a) . \frac{\tan \frac{1}{2}(B - A)}{\tan \frac{1}{2}(B + A)}. \qquad (Q)$$

By Art. 66,

As $\cos a : \cos b :: \cos e : \cos d$;

therefore (E. 5 and Art. 73),

As $\cot \tfrac{1}{2}(b+a) : \tan \tfrac{1}{2}(b - a) :: \cot \tfrac{1}{2}c : \tan \tfrac{1}{2}(d - e)$;

whence,

$$\tan \tfrac{1}{2}(d - e) = \cot \tfrac{1}{2}c . \frac{\tan \frac{1}{2}(b - a)}{\operatorname{cotan} \frac{1}{2}(b + a)} = \text{(Art. 23, eq. 4)}$$

$$\cot \tfrac{1}{2}c . \tan \tfrac{1}{2}(b + a) . \tan \tfrac{1}{2}(b - a).$$

Again (Art. 61),

As $\tan B : \tan A :: \sin d : \sin e$;

therefore (E. 5 and Art. 72),

As $\tan B + \tan A : \tan B - \tan A :: \tan \tfrac{1}{2}c : \tan \tfrac{1}{2}(d - e)$;

$$\therefore \tan \tfrac{1}{2}(d - e) = \tan \tfrac{1}{2}c . \frac{\tan B - \tan A}{\tan B + \tan A} = \text{(Art. 37, eq. 8)}$$

$$\tan \tfrac{1}{2}c . \frac{\sin (B - A)}{\sin (B + A)} = \text{(Art. 36, eq. 3)}$$

$$\tan \tfrac{1}{2}c . \frac{\sin \frac{1}{2}(B - A) . \cos \frac{1}{2}(B - A)}{\sin \frac{1}{2}(B + A) . \cos \frac{1}{2}(B + A)}.$$

Equating these values of $\tan \tfrac{1}{2}(d - e)$, and substituting for $\tan \tfrac{1}{2}(b - a)$ its value in equation Q;

$$\cot \tfrac{1}{2}c . \tan^2 \tfrac{1}{2}(b + a) \frac{\tan \frac{1}{2}(B - A)}{\tan \frac{1}{2}(B + A)} =$$

$$\tan \tfrac{1}{2}c \frac{\sin \frac{1}{2}(B - A)}{\sin \frac{1}{2}(B + A)} . \frac{\cos \frac{1}{2}(B - A)}{\cos \frac{1}{2}(B + A)} \therefore \tan^2 \tfrac{1}{2}(b + a) =$$

$$\frac{\tan \frac{1}{2}c.\sin \frac{1}{2}(B-A).\cos \frac{1}{2}(B-A).\tan \frac{1}{2}(B+A)}{\cot \frac{1}{2}c.\sin \frac{1}{2}(B+A).\cos \frac{1}{2}(B+A).\tan \frac{1}{2}(B-A)} =$$

$$\tan^2 \tfrac{1}{2}c.\frac{\cos^2 \frac{1}{2}(B-A)}{\cos^2 \frac{1}{2}(B+A)};$$

consequently,

$$\tan \tfrac{1}{2}(b+a)=\tan \tfrac{1}{2}c.\frac{\cos \frac{1}{2}(B-A)}{\cos \frac{1}{2}(B+A)}. \qquad (13)$$

From equation Q,

$$\tan \tfrac{1}{2}(b-a)=\tan \tfrac{1}{2}c.\frac{\sin \frac{1}{2}(B-A)}{\sin \frac{1}{2}(B+A)}. \qquad (14)$$

Equations 11 and 12 may be used when two sides and the included angle are given to find the other angles; and equations 13 and 14 when two angles and the side between them are given to find the other sides.*

From these four last equations, the following are derived by a very simple process:

$$\operatorname{cotan} \tfrac{1}{2}C=\tan \tfrac{1}{2}(B+A).\frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(b-a)}. \qquad (15)$$

$$\operatorname{cotan} \tfrac{1}{2}C=\tan \tfrac{1}{2}(B-A).\frac{\sin \frac{1}{2}(b+a)}{\sin \frac{1}{2}(b-a)}. \qquad (16)$$

$$\tan \tfrac{1}{2}c=\tan \tfrac{1}{2}(b+a).\frac{\cos \frac{1}{2}(B+A)}{\cos \frac{1}{2}(B-A)}. \qquad (17)$$

$$\tan \tfrac{1}{2}c=\tan \tfrac{1}{2}(b-a).\frac{\sin \frac{1}{2}(B+A)}{\sin \frac{1}{2}(B-A)}. \qquad (18)$$

A few examples are given to exercise these equations.

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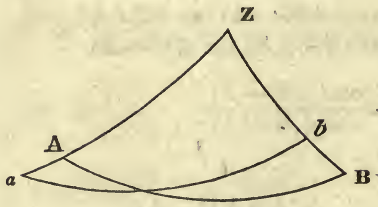
* The discovery of these four equations is attributed to Baron Napier.



Ex. 1. In the spherical triangle ABZ, given AZ=54° 10′, BZ = 39° 25′, AB = 72° 36′, in ZA produced Aa = 2′, and in ZB, Bb = 35′, to find ab.

First, with the three sides, find the angle Z, by equation 5.

ZA	54° 10′	cosec	.0911273
ZB	39° 25′	cosec	.1972569
AB	72° 36′		
	2) 166° 11′		
S	83° 5′ 30″		
S—ZA	28° 55′ 30″	sine	9.6845440
S—ZB	43° 40′ 30″	sine	9.8392057
			2) 19.8121339
½Z	53° 39′ 32″	sine	9.9060669
BZA	107° 19′ 4″		

Then, by equations 11 and 12,

½(Za+Zb)	46° 31′	sec	.1623210	cosec	.1393179
½(Za—Zb)	7° 41′	cos	9.9960834	sin	9.1261246
½BZA	53° 39′ 32″	cot	9.8666880	cot	9.8666880
½(b+a)	46° 39′ 15″	tan	10.0250924		
½(b—a)	7° 43′ 12″				9.1321305
Zba	54° 22′ 27″				

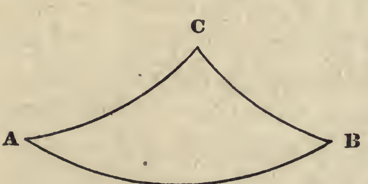
By Art. 58,

Zba	54° 22′ 27″	cosec	.0899959
bZa	107° 19′ 4″	sin	9.9798526
Za	54° 12′	sin	9.9090550
ab	72° 17′ 8″	sin	9.9789035

Or the side ab may be found by equation 18.

$\frac{1}{2}(b-a)$	7° 43′ 12″	cosec	.8718203
$\frac{1}{2}(b+a)$	46° 39′ 15″	sine	9.8616681
$\frac{1}{2}(Za-Zb)$	7° 41′	tan	9.1300413
$\frac{1}{2}ab$	36° 8′ 33″	tan	9.8635297
ab	72° 17′ 6″		

This is the direct solution of the celebrated problem of clearing the observed distance between the moon and the sun, or a star, from the effect of parallax and refraction.



Ex. 2. In the spherical triangle ABC, given AC 46° 18′, AB 100° 26′, and the angle A 39° 50′, to find the rest. Result: ACB 136° 0′ 54″; ABC 30° 41′ 54″. BC 65° 6′ 34″.

Ex. 3. In the triangle ABC, given AB 112° 56′, BAC 40° 16′, ABC 54° 20′, to find the rest.

Result: AC 79° 44′ 58″; BC 51° 31′ 30″; ACB 130° 30′ 20″.

Ex. 4. Given, the side AB 96° 12′, AC 57° 16′, BC 49° 8′, to find the angles.

Result: BAC 31° 32′ 42″; ABC 35° 35′ 15″; ACB 136° 32′ 48″.

Ex. 5. Given, the angle BAC 50°, ABC 60°, ACB 85°, to find the sides.

Result: AB 51° 59′ 16″; AC 43° 13′ 48″; BC 37° 17 26″.

SECTION IV.

CONIC SECTIONS.

Article 78. *Definition* 1. If, from a point in the circumference of a circle, a right line be drawn to pass through a fixed point which is not in the plane of that circle, and then caused to revolve round that fixed point so as to describe the whole circumference of the circle; the curve surface, described by this revolving line, is called a *conical surface;* and the solid included between this curve surface and the generating circle, is called a *cone.*

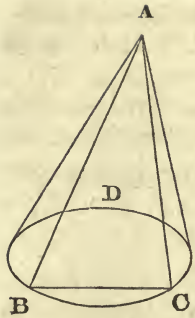
Def. 2. The circle described by the revolving line is called the *base,* and the fixed point the *vertex,* of the cone.

Def. 3. The straight line drawn from the vertex to the centre of the base, is called the *axis* of the cone.

Def. 4. When the axis is at right angles to the plane of the base, the cone is called a *right cone;* but when the axis is oblique to that plane, the solid is termed a *scalene cone.*

As the line which, by its revolution, describes the conical surface, may be indefinitely extended, two cones having a common vertex, and equal solid angles at the vertex, may be generated by the same revolution.

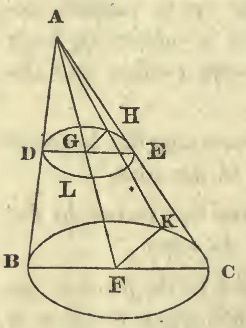
Art. 79. Let the cone ABCD be cut by a plane which passes through its vertex A, and cuts the base in the right line BC; the common section of this plane with the surface



of the cone, will be a triangle. The common section of the base and cutting plane is a right line (3.2 sup.); and the right lines drawn from B and C to the vertex, are in the cutting plane (2.2 sup.); and those lines correspond to the position of the revolving line when it passes through B and C; they are therefore in the conical surface.

Q. E. D.

Art. 80. Let the cone ABC be cut by a plane which is parallel to the plane of the base; then the section of this cutting plane with the conical surface, is a circle whose centre is in the axis of the cone.



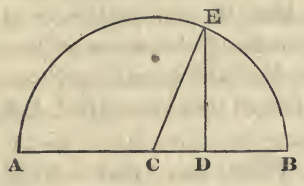
Let AF be the axis of the cone; DLE the cutting plane. In the circumference of the base take any point K; join FK; and through AF, FK, suppose a plane to pass, cutting the conical surface in AK, and the cutting plane in GH; then (Art. 79, and 3.2 sup.) AK and GH are right lines. Let also another plane ABC pass through the axis; its section with the base will be a diameter, because F is the centre of the circle; and the section of this plane with the conical surface is a triangle (Art. 79). Take BC and DE, the sections of this plane with the parallel planes BCK and DLE; then (14.2 sup.) DE and GH are respectively parallel to BC, FK; consequently,

As AF : AG :: BF : DG :: FC : GE :: FK : GH.

But BF, FC and FK are all equal; therefore, DG, GE and GH are also equal; consequently (9.3), DLEH is a circle whose centre is G.

Q. E. D.

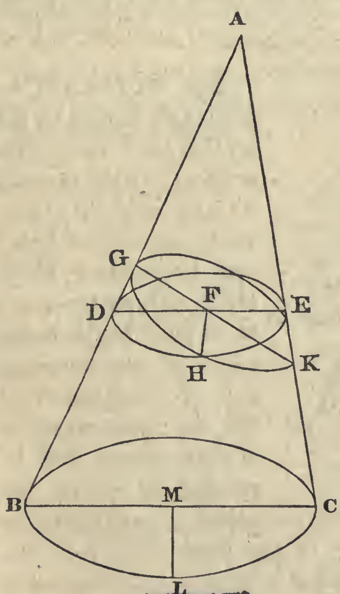
Art. 81. Let AB, DE, two lines at right angles to each other, such that AD.DB = DE²; then a semicircle described on the diameter AB will pass through the point E.



Bisect AB in C; then, since AD.DB = DE²,

$$AD.DB + CD^2 = DE^2 + CD^2;$$

that is (5.2 and 47.1), $CB^2 = CE^2 \therefore CB = CE$. Consequently, a circle described from the centre C, with the radius CB, will pass through E. *Q. E. D.*



Art. 82. Let ABLC be a scalene cone; ABC the triangle formed by the section of the conical surface with a plane which passes through the axis, and stands at right angles to the plane of the base; and let another plane GHK, at right angles to the plane ABC, cut that plane in GK, making the angle AGK = ACB, and AKG = ABC;* then the plane GHK cuts the conical surface in a circle.

In the section of the cutting plane and conical surface, take any point H; through H let a plane DHE pass, parallel to the base of the cone, cutting the planes GHK and ABC in the lines HF and DE respectively. Then, since the plane

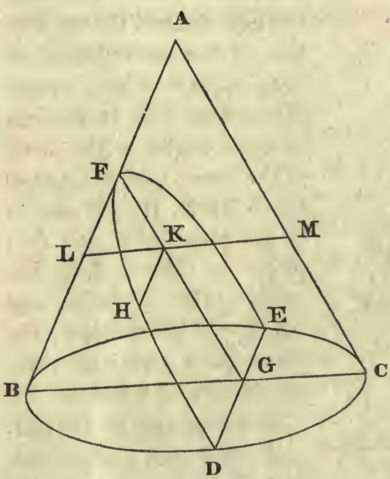
* This is called a sub-contrary section.

DHE is parallel to the plane of the base, it is at right angles to the plane ABC (15.2 sup.). But the plane GHK is at right angles to the same plane; therefore the common section HF is at right angles to the plane ABC (18.2 sup.), and consequently to the lines DE and GK in that plane (Def. 1, 2 sup.)

Now, the angle GDF being = FKE; and DFG = KFE (15.1); the triangles GDF, EKF, are similar; therefore,

As DF : FG :: FK : FE (4.6)

consequently, DF.FE=GF.FK (16.6). But DHE is a circle (Art. 80); therefore, DF.FE = HF^2 (35.3). Hence, GF.FK = HF^2; and consequently GHK is a circle (Art. 81), whose diameter is GH. *Q. E. D.*



Art. 83. Let ABC be a triangle formed by the section of a cone with a plane passing through its axis at right angles to the plane of its base; and let another plane DFE, cutting the cone, be at right angles to the plane of the triangle, and so situated that FG, the common section of these planes, shall be parallel to AC, the opposite side of the triangle; then the common section of the plane DFE with the conical surface, is a curve called a *parabola*; the general property of which this article is intended to exhibit.

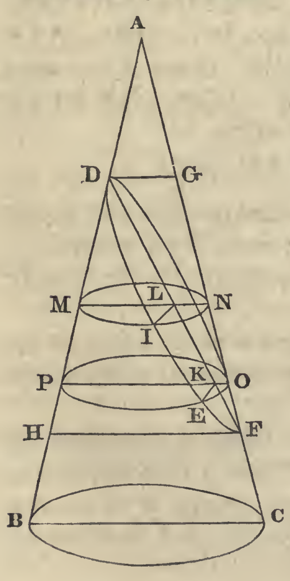
In this curve take any point H, and through H let a plane, parallel to the base of the cone, be passed; and let this plane cut the plane of ABC in the line LM, and the plane DFE in

HK. Then, because the planes LHM and DFE are at right angles to ABC, their common section HK is at right angles to the same plane (18.2 sup.); it is therefore at right angles to FG and LM. Now, the section of the plane LHM with the conical surface, is a circle (Art. 80); wherefore, LK.KM $= HK^2$ (35.3). In like manner, BG.GC $= DG^2$. The plane LHM being parallel to the base, LK is parallel to BG (14.2 sup.); therefore,

As FG : FK :: BG : LK :: (1.6) BG.GC : LK.KM :: $DG^2 : HK^2$.

Def. 5. The line FG is called the *axis*, and F the *vertex*, of the parabola; any segment of the axis FK, reckoned from the vertex, is called an *abscissa;* and a perpendicular KH, on the axis, is called an *ordinate.*

The demonstration in this article, therefore, shows that any two abscissas are to each other as the squares of the corresponding ordinates.



Art. 84. Let ABC be a triangle, formed by the common section of a cone and a plane through its axis, at right angles to the plane of the base; DIEF the common section of the conical surface, and a plane which is at right angles to the plane of the triangle ABC, passing through its opposite sides, but neither parallel to the base, nor sub-contrarily situated; the curve DIEF is called an *ellipse;* the general property of which this article is designed to explain.

Bisect DF, the common section of DIEF and ABC, in L; take any other point K in DF; and through L and K let the planes MIN and PEO pass parallel to the base of the cone cutting ABC in MN, and PO, and DIEF in LI and KE. Through D and F draw the lines DG and HF parallel to BC.

Now, since (15.2 sup.) the planes MIN and PEO are at right angles to the plane ABC; LI and KE, the common intersections of these planes and the plane DIEF, are at right angles to the plane ABC (18.2 sup.); and consequently to the lines MN, PO and DF in that plane (Def. 1.2 sup). Also the common sections of the planes MIN and PEO, and the conical surface, are circles, of which MN and PO are diameters (Art. 80). Therefore, $ML.LN = LI^2$, and $PK.KO = KE^2$. Now, the lines MN and PO are parallel to BC (14.2 sup). Hence, by similar triangles,

$$\text{As } DL : DK :: ML : PK;$$

and

$$\text{As } LF : KF :: LN : KO;$$

therefore (23.6),

$$\text{As } DL.LF : DK.KF :: ML.LN : PK.KO :: LI^2 : KE^2.$$

As the line $DL = LF$, it is obvious that $ML = \frac{1}{2}HF$, and $LN = \frac{1}{2}DG$; therefore $ML.LN = \frac{1}{4}DG.HF$. Hence IL is a mean proportional between $\frac{1}{2}DG$ and $\frac{1}{2}HF$. As $DL = LF$, $DL.LF = LF^2$. Consequently, the above analogy is,

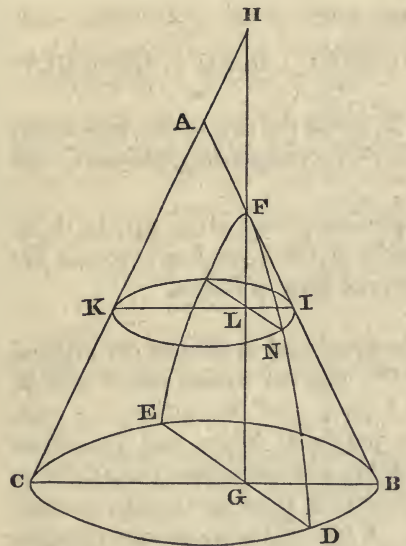
$$\text{As } LF^2 : LI^2 :: DK.KF : KE^2.$$

Def. 6. The lines LF, LI, are called the first and second semi-axes; DK, KF, the abscissas; and KE an ordinate.

The property of the ellipse, demonstrated in this article, therefore, is this:

As the square of the first semi-axis is to the square of the second, so is the rectangle of the two abscissas to the square of the ordinate.

It is observable that if the plane DIEF is parallel to the base, or sub-contrarily situated, all that is demonstrated in this article continues to be true; but in either of those cases the curve becomes a circle (Arts. 80, 82): and therefore $LI = LF$, and $DK.KF = KE^2$.



ART. 85. Let ABC be a triangle, formed by the section of a cone and a plane passing through its axis, at right angles to the plane of its base; and let DFE be a plane at right angles to the plane of the triangle, so situated that GF, the common section of these planes, being produced, will meet CA, the opposite side of the triangle also produced, beyond the vertex A; then the curve which is the common section of the conical surface and the plane DFE, is called an *hyperbola;* the general property of which is to be shown.

In this section, take any point N; through which let a plane pass parallel to the plane of the base; and let ILK be the common section of this plane with the plane of the triangle, and NL its section with the plane DFE.

Now (18.2 sup.), DG, the section of DFE and the base of the cone, and NL, are both at right angles to the plane of the triangle. Also (Art. 80), the common section of the conical surface, and the plane which passes through NL, is a circle; consequently, $IL.LK = LN^2$, and $BG.GC = GD^2$ (35.3). Since IK is parallel to BC (14.2 sup.), by similar triangles,

$$\text{As } FG : FL :: BG : IL;$$

and

$$\text{As } HG : HL :: GC : LK;$$

consequently (23.6),

As $FG.HG : FL.HL :: BG.GC : IL.LK :: GD^2 : NL^2$.

Def. 7. The line HG is called the *axis* of the hyperbola; HL, FL, as likewise HG, FG, corresponding *abscissas;* and DG, NL, the *ordinates.*

The property of the hyperbola, proved in this article, therefore, is this: The rectangles of corresponding abscissas are to each other as the squares of their ordinates.

From what has been demonstrated in the last six articles, it appears that there are five different figures which may be formed by the section of a plane and the surface of a cone, viz., the triangle, circle, parabola, ellipse and hyperbola. The properties of the triangle and circle being explained in common Geometry, the remaining three are usually denominated the *Conic Sections.* A few of the most useful properties of these figures, deduced from the general relations above demonstrated, are subjoined.

Of the Parabola.

Art. 86. Let BAC be a parabola; AD, part of the axis, an abscissa; DC an ordinate; $AE = AF = \frac{DC^2}{4AD}$; EG perpendicular, and CG parallel to ED; then, FC being joined, FC shall be equal to CG.

Since EF is bisected in A,

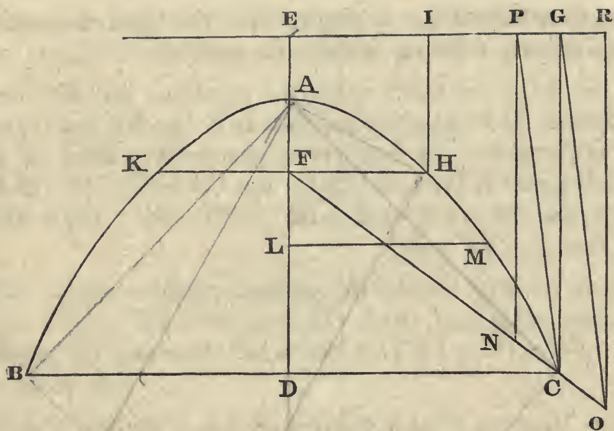
$$4AF.AD + DF^2 = ED^2 \text{ (8.2)} = CG^2.$$

But (47.1),

$$FC^2 = DC^2 + DF^2 = \text{(by construction) } 4AF.AD + DF^2.$$

Therefore, $CG^2 = CF^2$; and $CG = CF$. *Q. E. D.*

Def. 8. The line EG is called the *directrix;* 4AE, the *latus rectum;* and the point F, the *focus.*



The proposition demonstrated in this article, therefore, may be enunciated: If from any point in the parabola, two straight lines be drawn, the one to the focus and the other at right angles to the directrix, they will be equal to each other.

Art. 87. A right line KFH, drawn through the focus parallel to the directrix, bounded at both ends by the parabola, is equal to the latus rectum; and the rectangle of the latus rectum and abscissa is equal to the square of the corresponding ordinate.

It is evident from Art. 83, that $KF = FH$; and (Art. 86), $FH = HI = 2AF$; therefore, $KH = 4AE$.

Draw any other ordinate LM; then (Art. 83),

As $DC^2 : LM^2 :: AD : AL :: (1.6)\ AD.KH : AL.KH$.

But (by construction of Art. 86),

$$4AD.AF = AD.KH = DC^2 \therefore AL.KH = LM^2.$$

Q. E. D.

Art. 88. If a point be taken either within or without a parabola, and from it a straight line be drawn to the focus, and another at right angles to the directrix; the former of these lines will be less or greater than the latter, according as the point is within or without the parabola.

First, let N be taken *within* the parabola; join FN, and produce it till it meets the parabola in C; let NP and CG be at right angles to the directrix; and join CP. Then CP is greater than CG (17.1 and 19.1); but $CN+NP>CP$. (20.1) $>CG$; and $CG=CF=CN+NF$ $\therefore$ $CN+NP>CN+NF$; and therefore $NP>NF$.

Next, let O be without the parabola; then, a similar construction being used, $OG>OR$; but $OC+CG>OG$. Also $OC+CG=OC+CF$ (Art. 86) $=OF$; therefore, $OF>OR$.

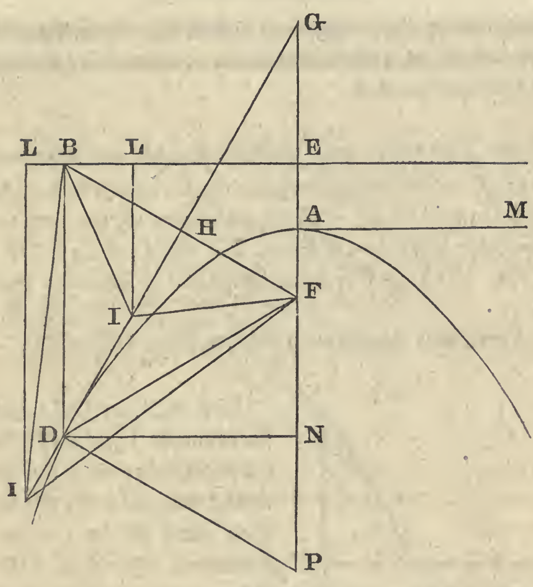
Q. E. D.

Cor. Hence, a point is either in, within or without a parabola, according as the line drawn from it to the focus is equal to, less or greater than the perpendicular falling from it upon the directrix.

Art. 89. Let D be a point in the parabola; DF the line to the focus; DB the perpendicular to the directrix; and DG a line bisecting the angle FDB; then DG touches the parabola.

In DG, take any other point I; and join IF, IB; then the angle FDB being bisected, we have (Art. 86) the sides BD, DI, and the contained angle BDI, severally equal to FD,DI, and the contained angle FDI; consequently, $BI=FI$ (4.1). But BI is evidently greater than IL, the perpendicular from I to the directrix (19.1); hence the point I is without the parabola (Art. 88, Cor.); and therefore DG touches the parabola. *Q. E. D.*

Cor. 1. A right line through the vertex, at right angles to the axis, is a tangent to the parabola. For AM being drawn through the vertex A, at right angles to EF, it is evident that every point in AM, except the point A, is farther from F than from the directrix.



Cor. 2. If FB be drawn from the focus to the point where the line through D parallel to the axis meets the directrix, it is manifest that FB is bisected and cut at right angles by the tangent DG. For the triangles BDH and FDH are in every respect equal (4.1).

Art. 90. Let DG touch the parabola in D, and meet the axis in G; then DF being drawn to the focus, and DN at right angles to the axis, FG = DF, and AG = AN.

Because DB is parallel to FG, the angle FGD = BDG (29.1) = GDF (Art. 89). Hence FG = FD (6.1).

Again, since FD = DB (Art. 86) = EN; and EA = AF, AG = AN. *Q. E. D.*

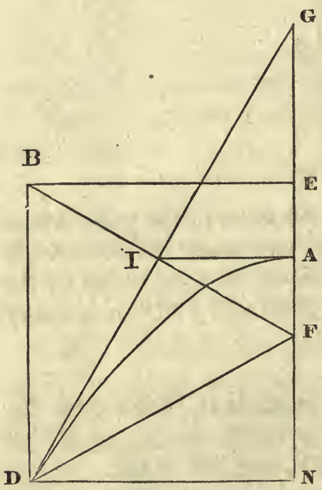
Def. 9. The line NG is called the *subtangent;* the second part of this article, therefore, shows that the subtangent is double the abscissa; or GN = 2AN.

Def. 10. The line DP, drawn from the curve to the axis, perpendicular to the tangent, is called the *normal;* and NP, the segment of the axis between the ordinate and the normal, is called the *subnormal.*

Art. 91. The subnormal is equal to half the latus rectum.

The angle GDP being a right angle, is equal to DGF + DPF (32.1); and GDF = DGF, as shown in the last article; the remainder FDP = DPF; ∴ FP = FD (6.1) = DB (Art. 86) = NE. Taking FN from each, NP = EF = $\frac{1}{2}$ the latus rectum. *Q. E. D.*

Cor. From this demonstration, we have DF = FP.



Art. 92. Let DG touch the parabola in D, and FI be the perpendicular from the focus F to the tangent; A, the vertex; then shall FI be a mean proportional between DF and FA.

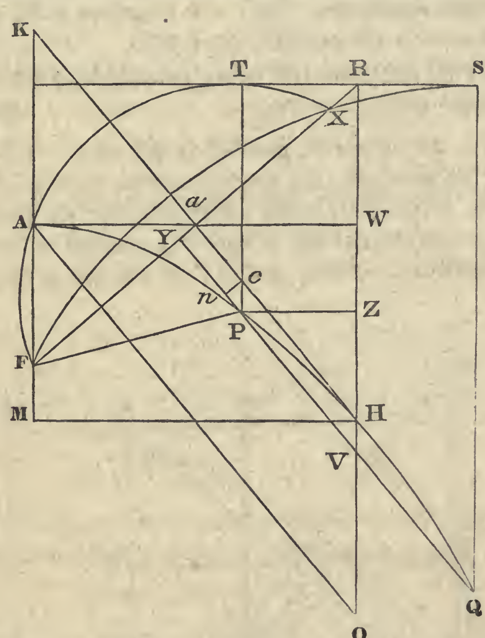
Join AI, and draw through D the ordinate DN. Then, since the angle DIF = GIF; and DF = GF (Art. 90); DI = IG. Also, AN = AG; consequently, AI is parallel to DN (2.6); and therefore the angles at A are right angles; wherefore the triangle IAF is similar to GIF or DIF (8.6). Hence,

As DF : FI :: FI : FA.

Q. E. D.

Cor. As DF : FA :: DF^2 : FI^2 (cor. 2.20.6).

Art. 93. Let HK touch the parabola in H, and HR be parallel to the axis AM; from any point Q in the parabola, let QV be drawn parallel to HK; then QV, produced, will meet the parabola in another point; and the line between its points of section with the curve will be bisected by RH.

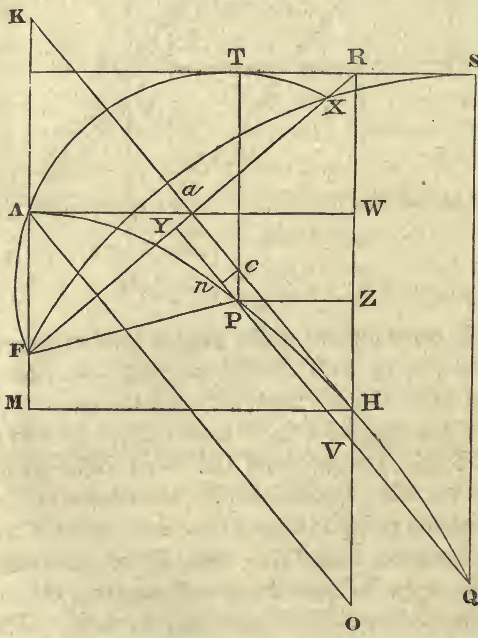


From Q, draw QS at right angles to the directrix; and from the centre Q, with the distance QS, describe a circle; this circle will evidently pass through the focus F (Art. 86), and touch the directrix in S (cor. 16.3). Join FR, and let QV cut FR in Y; then, since HK is at right angles to FR, and also bisects it (Art. 89, cor. 2); the angles at Y are right angles; and the point X, where the circle cuts FY the second time, lies between F and R. Also, FY = YX (3.3). Make RT = RS; draw TP parallel to AM, meeting QV in P; and through the points F, T, X, describe a circle. Then, since FR cuts the circle FXS, and RS touches it, $FR.RX = RS^2$ (36.3) $= RT^2$; consequently, RT touches the circle FTX in

T (37.3). Hence the centre of that circle is in TP, which is at right angles to RT (19.3); it is also in VY, which bisects FX at right angles (cor. 1.3); it is therefore in P; the point P is of course in the parabola (Art. 86).

Now, PT, HR and QS being parallel, and TR = RS, it follows (2.6) that PV = VQ. *Q. E. D.*

Def. 11. Any line OR, parallel to the axis, is called a *diameter* of the parabola; the point H, where the diameter meets the curve, is called its *vertex;* 4HR is called the *latus rectum* of that diameter; the line PV or VQ, parallel to the tangent HK, is called an *ordinate* and HV an *abscissa to the diameter* OH.



Art. 94. Let PZ be drawn at right angles to the diameter OR, and PV parallel to the tangent HK; then $PZ^2 = 4AF.VH$.

Retaining the construction of the last article, let the tangent HK meet FR in a, and the diameter TP in c; draw cn parallel to FR, and join Aa. Then, since FR is bisected in a (Art. 89, cor. 2); and FX in Y (3.3); Ya or $nc = \frac{1}{2}$XR: also, Pc = VH (34.1). But (as was proved in Art. 92), the triangle FAa is similar to FaK or Pcn. Consequently,

$$\text{As } \mathrm{F}a : \mathrm{FA} :: \mathrm{P}c : cn;$$

whence FA.Pc (or FA.VH) = Fa.nc (16.6). Hence,

$$4\mathrm{FA.VH} = \mathrm{FR.RX} = \mathrm{TR}^2 = \mathrm{PZ}^2$$

Q. E. D.

Art. 95. If, from two points P, A, ordinates, PV, AO, be drawn to any diameter OR, the squares of those ordinates shall be to each other in the same ratio as their abscissas; that is,

$$\text{As } \mathrm{PV}^2 : \mathrm{AO}^2 :: \mathrm{VH} : \mathrm{OH}.$$

Draw PZ, AW, at right angles to OR; then (Art. 94) $\mathrm{PZ}^2 = 4\mathrm{FA.VH}$; and $\mathrm{AW}^2 = 4\mathrm{FA.OH}$; consequently (1.6),

$$\mathrm{PZ}^2 : \mathrm{AW}^2 :: \mathrm{VH} : \mathrm{OH}.$$

But the triangles PVZ, AOW, being similar,

$$\mathrm{PV}^2 : \mathrm{AO}^2 :: \mathrm{PZ}^2 : \mathrm{AW}^2 :: \mathrm{VH} : \mathrm{OH}.$$

Q. E. D.

Art. 96. The square of any ordinate is equal to the rectangle of its abscissa, and the latus rectum of the diameter.

Let PV be an ordinate to the diameter OH; from the vertex of the axis let AO be drawn to the same diameter, parallel to PV; from H, draw HM at right angles to the axis. Then, since AOHK is a parallelogram,

$$\mathrm{OH} = \mathrm{AK} = (\text{Art. } 90)\ \mathrm{AM} = \mathrm{HW}\ \therefore\ \mathrm{OW} = 2\mathrm{OH}.$$

Now,

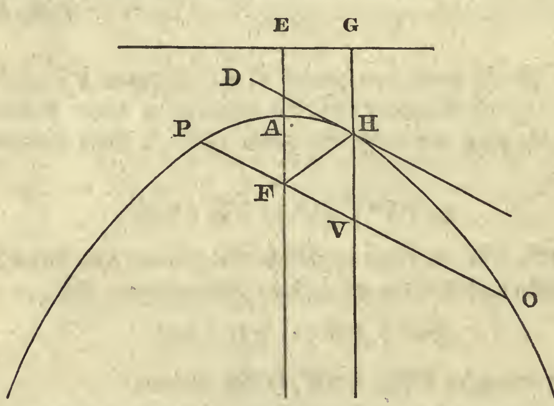
$AO^2 = AW^2 + OW^2$ (47.1) $= 4AF.OH + 4OH^2$ (Art. 94, and cor. 2.8.2) $= 4RW.OH + 4WH.OH = 4RH.OH$.

But (Art. 95),

As $AO^2 : PV^2 :: OH : VH ::$ (1.6) $4RH.OH : 4RH.VH$.

Hence, $PV^2 = 4RH.VH$. *Q. E. D.*

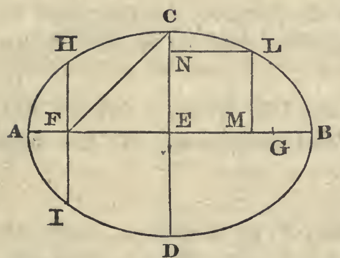
Art. 97. A double ordinate passing through the focus of a parabola, is equal to the latus rectum of the diameter to which that ordinate is applied.



Let F be the focus; VH a diameter; EG the directrix; HD a tangent to the parabola; PFO, the line through the focus parallel to DH. Join FH; then, PV being parallel to DH, the angle DHG = FVH; and DHF = HFV (29.1). But DHG = DHF (Art. 89); therefore, FVH = HFV; and HV = FH (6.1) = HG (Art. 86). Now, PV = VO (Art. 93), and $PV^2 = 4GH.HV$ (Art. 96) $= 4HG^2$. Hence PV = 2HG (cor. 2.8.2). Therefore, PO = 4HG.

The case of the double ordinate applied to the axis, is proved in Art. 87. *Q. E. D.*

Of the Ellipse.



Def. 12. Let ACBD be an ellipse; AE, CE, the semi-axes; and from C, the extremity of the less, with a distance equal to the greater, let an arc be described cutting AB in F and G; each of those points, F, G, is called the *focus* of the ellipse; and the line HI, passing through either focus at right angles to AE, meeting the ellipse in H and I, is called the *latus rectum* of the ellipse.

Art. 98. The rectangle of the abscissas AF.FB, into which the axis AB is divided by the focus, is equal to the square of the semi-axis CE.

Since AB is bisected in E, and divided unequally in F;

$AF.FB + FE^2 = AE^2$ (5.2) $= FC^2 = FE^2 + CE^2$ (47.1);

therefore, $AF.FB = EC^2$. *Q. E. D.*

Cor. Hence, HF is a third proportional to AE, EC. For (Art. 84),

$$\text{As } AE^2 : EC^2 :: AF.FB\ (EC^2) : FH^2;$$

whence $$AE : EC :: EC : FH.$$

Art. 99. If from L, any point in the ellipse, a line LN be drawn at right angles to the second or minor axis CD; then,

$$\text{As } EC^2 : EB^2 :: CN.ND : NL^2.$$

Draw LM at right angles to AB; then (Art. 84),

$$\text{As } EB^2 : EC^2 :: AM.MB\ (EB^2 - EM^2) : ML^2 \text{ or } EN^2.$$

Therefore (19.5),

$$\text{As } EB^2 : EC^2 :: EM^2 : EC^2 - EN^2 \text{ or } CN.ND \text{ (5.2)};$$

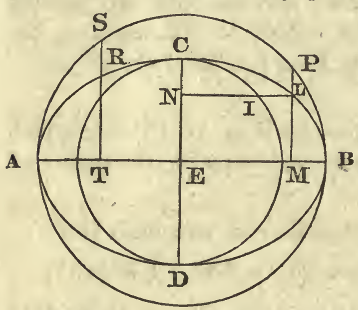
hence, by inversion,

$$\text{As } EC^2 : EB^2 :: CN.ND : EM^2 \text{ or } NL^2.$$

It is therefore manifest that the property demonstrated in Art. 84 is equally true, whichever axis is divided by an ordinate.

Cor. Since CN.ND is less than CE^2 (27.6), it follows that NL^2 is always less than EB^2, and consequently NL less than EB.

Art. 100. If, on either axis of an ellipse, a circle be described; and from any point in the ellipse a perpendicular be drawn to that axis, meeting the circle; it will be, as the axis on which the circle is described is to the other axis, so is the ordinate to the circle, to the ordinate to the ellipse.



Let ACBD be the ellipse, and first let the circle be described on the greater axis AB, and let LM be the ordinate; then (Art. 84),

$$\text{As } EB^2 : EC^2 :: AM.MB : ML^2.$$

But $AM.MB = MP^2$ (35.3). Hence (22.6),

$$\text{As } EB : EC :: MP : ML.$$

Again, let the circle be described on CD, and the ordinate LN meet the circle in I; then (Art. 99),

$$\text{As } CE^2 : EB^2 :: CN.ND : NL^2.$$

But $CN.ND = NI^2$ (35.3); therefore (22.6),

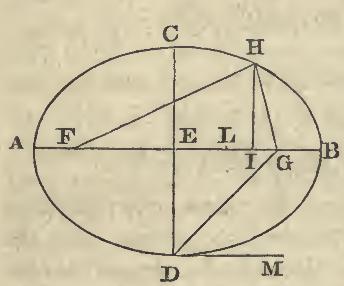
$$CE : EB :: NI : NL.$$

Now, $\quad CE : EB :: CD : AB.$

Hence the proposition is manifest.

Cor. Two ordinates, ML, TR, being drawn to the same axis, $ML : TR :: MP : TS$.

Art. 101. The sum of the lines FH, GH, drawn from any point in the ellipse to the two foci, is equal to the greater axis AB.



Take E the middle of AB,* and through it draw the perpendicular CD; this will be the less axis. From H draw the ordinate HI, and take EL a fourth proportional to EB, EG and EI. Then (22.6),

As $EB^2 : EG^2 :: EI^2 : EL^2$.

Consequently (19.5),

$EB^2 : EG^2 :: EB^2 - EI^2 : EG^2 - EL^2$ (that is, 5.2) $:: AI.IB : FL.LG$.

Hence (17.5 and Def. 12),

$$EB^2 : EC^2 :: AI.IB : AI.IB - FL.LG.$$

Therefore (Art. 84),

$$AI.IB - FL.LG = IH^2.$$

Again, $BE^2 + EL^2 = 2BE.EL + BL^2$ (7.2);

also, $GE^2 + EI^2 = 2GE.EI + GI^2$.

Taking the latter of these equations from the former, and remembering that $BE^2 - EI^2 = AI.IB$ (5.2); that $GE^2 - EL^2 = FL.LG$; and that $BE.EL = GE.EI$ (16.6); we have

$$AI.IB - FL.LG = BL^2 - IG^2;$$

that is, $IH^2 = BL^2 - IG^2$. Hence, $IH^2 + IG^2 = BL^2$. Consequently (47.1), $GH^2 = BL^2$; and $GH = BL$.

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* The middle of the axis is usually called the centre of the ellipse.

Taking $AE^2 + EL^2 + 2AE.EL = AL^2$ (4.2), and $FE^2 + EI^2 + 2FE.EI = FI^2$, and proceeding as before, we have

$$IH^2 = AL^2 - FI^2 \therefore FI^2 + IH^2 = AL^2;$$

whence (47.1), $FH^2 = AL^2$; therefore, $FH = AL$. Consequently,

$$FH + GH = AL + BL = AB.$$

Q. E. D.

Cor. 1. Since $FH = AL$, and $GH = BL$, it is manifest that

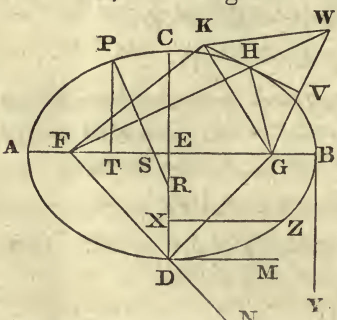
$$EL = FH - AE = AE - GH.$$

Hence, $FH - GH = 2EL$; or $EL = \frac{1}{2}(FH - GH)$.

Cor. 2. If, from a point without the ellipse, two right lines be drawn to the foci, their sum will be greater than the greater axis of the ellipse; but if a point be taken within the ellipse, the sum of the lines drawn from it to the foci will be less than the greater axis. This is evident, from what is above proved and 21.1.

Cor. 3. Conversely: A point is either in, without or within an ellipse, according as the sum of the lines drawn from it to the foci is equal to, greater, or less than the greater axis.

Art. 102. If, from any point P of an ellipse, a straight line $PR = AE$, half the greater axis, be applied to the less axis



CD, cutting the greater axis in S; then shall $SP = EC$, half the less axis.

From P draw PT at right angles to AB; then, because of the similar triangles TSP, ESR;

As $PR : PS :: ET : ST$;

hence (22.6 and 19.5),

As $PR^2 (= AE^2) : PS^2 :: AE^2 - ET^2 : PS^2 - ST^2 ::$ (5.2 and 47.1) $AT.TB : TP^2 ::$ (Art. 84) $AE^2 : EC^2$;

therefore, $PS = EC$. *Q. E. D.*

The ellipsograph, or instrument for describing an ellipse, is founded upon this property.

Art. 103. From a point H in an ellipse, two lines HF, HG, being drawn to the foci, and one of them FH produced, the line HV, which bisects the exterior angle, is a tangent to the ellipse.

Make HW, in FH produced, = HG; join GW, cutting the bisecting line in V; in VH, take any point K; and join KG, KW and KF. Then the triangles GHV, WHV, have GH= WH, and the angle GHV=WHV; hence (4.1), GV=WV, and the angle GVH = WVH. Consequently, GK =WK; and, therefore,

$$FK + GK = FK + WK.$$

But $FK + WK > FW$ (20.1), and $FW = FH + HG = AB$ (Art. 101); therefore $FK + GK > AB$; and (Cor. 3, Art. 101) the point K is without the ellipse. This being true of every point in HV, except H, the line HV is a tangent to the ellipse.

Q. E. D.

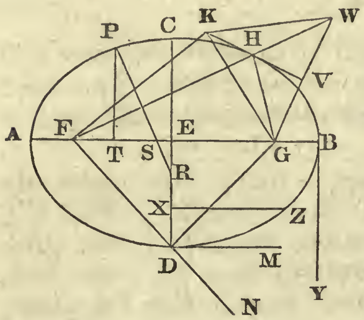
Cor. 1. From this demonstration it is obvious that the line HV, bisecting the exterior angle GHW, also bisects GW, and cuts it at right angles in V.

Cor. 2. Hence the angles FHK, GHV, which the lines from the foci to the point of contact make with the tangent, are equal. For WHV = KHF (15.1).

Art. 104. A right line through the vertex of either axis, parallel to the other axis, is a tangent to the ellipse.

The line DM parallel to AB, makes the angle GDM=DGF, and MDN = GFD (29.1), and DGF = GFD (5.1); consequently, DM bisects the angle GDN, and is therefore a tangent to the ellipse (Art. 103).

Next, take BY parallel to CD, and through any point X in CD draw XZ parallel to BE, meeting the ellipse in Z; then XZ is less than EB, (Cor., Art. 99); that is, Z lies between DC and BY; therefore, the line BY is without the ellipse.



Art. 105. Let the right lines FW and GW, of which FW=AB, the greater axis, be drawn from the foci to meet in W; and let VH, bisecting GW at right angles in V, cut FW in H; then HV touches the ellipse in H. Join HG; then it is evident (4.1) that GH =WH, and the angle GHV=WHV; hence the angle GHW is bisected by the line HV; and since HG = HW,

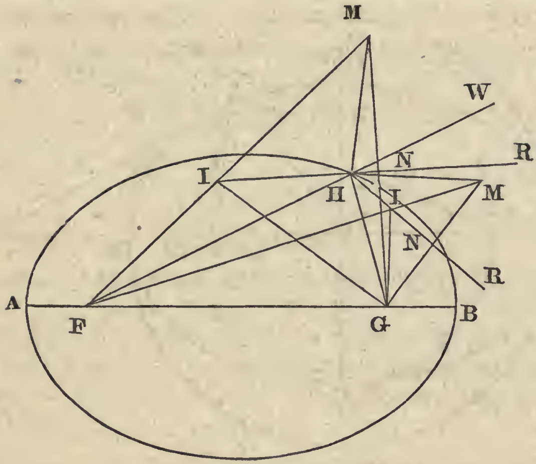
$$FH + HG = FW = AB;$$

therefore (Art. 101, Cor. 3), the point H is in the ellipse; and (Art. 103) HV is a tangent to the ellipse.

Cor. A line which cuts GW at right angles, but does not bisect it, is not a tangent to the ellipse. For this line is parallel to HV, and, if it cuts GW between V and W, it does not meet the ellipse; but if it falls between G and V, it must cut the ellipse.

Art. 106. A straight line which meets an ellipse, but does not bisect the exterior angle formed by the lines drawn to the foci, is not a tangent to the ellipse.

Let HR meet the ellipse in H, but not bisect the exterior angle GHW. On HR let fall the perpendicular GN; and produce it to M, making NM = GN; join FM, meeting HR in I; join also GI, GH and HM. Then (4.1), GI = IM, GH =HM, and the angle GHN=MHN. Hence the angle GHM



is bisected by the line HR, but by supposition the angle GHW is not; consequently, HM does not coincide with HW; and therefore FHM is a triangle, of which FM is less than FH + HM (20.1). But

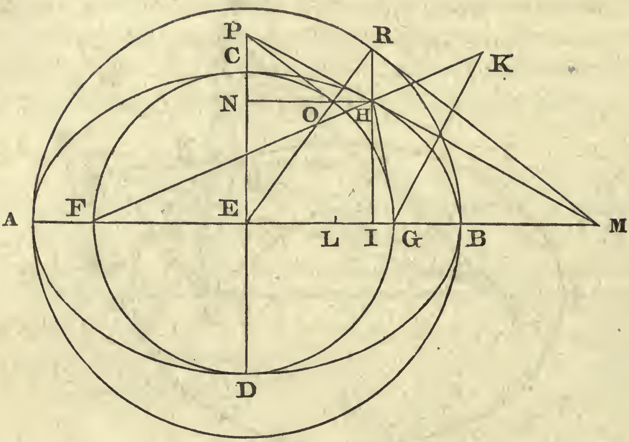
$$FH + HM = FH + HG = AB \text{ (Art. 101).}$$

Also FM = FI + IG. Therefore, FI + IG are less than AB, and the point I is within the ellipse (Art. 101, Cor. 3). Hence HR is not a tangent.

Cor. From this and Art. 103, it is evident that the tangent must bisect the exterior angle formed at the point of contact by right lines drawn to the foci; and that only one right line can touch the ellipse at a given point.

Art. 107. Let HM touch the ellipse in H, and meet the greater axis AB produced in M; and from H let HI be drawn at right angles to AB; then, E being the centre, EI, EB and EM shall be proportionals.

Take EL a fourth proportional to EB, EG and EI; this line is half the difference between FH and GH (Art. 101,



Cor. 1). Now, the tangent HM bisects the exterior angle of the triangle FHG (Cor. 2, Art. 103), and meets the base produced; therefore (A. 6),

As FH : HG :: FM : MG;

consequently (E. 5 and Art. 101),

As 2EB : 2EL :: FM+MG (2EM) : FG (2EG);

wherefore (15.5),

As EB : EL :: EM : EG;

consequently (1.6),

As EB^2 : EB.EL :: EM.EI : EG.EI.

But EB.EL = EG.EI (16.6); therefore, EB^2 = EM.EI; and (17.6), EI : EB :: EB : EM.

Q. E. D.

Cor. If the tangent MH meet the less axis EC produced in P, and HN be drawn at right angles to EC; then shall EN, EC and EP be proportionals.

By similar triangles,

As PE : PN :: EM : NH or EI :: (1.6) EM.EI : EI^2 :: (by this article) EB^2 : EI^2.

Hence (D. 5),

As PE : EN :: EB^2 : $EB^2 - EI^2$ :: (5.2, Art. 84 and 16.5) EC^2 : EN^2.

Again (1.6), As PE : EN :: PE.EN : EN^2.

Consequently, PE.EN = EC^2; or EN : EC :: EC : EP.

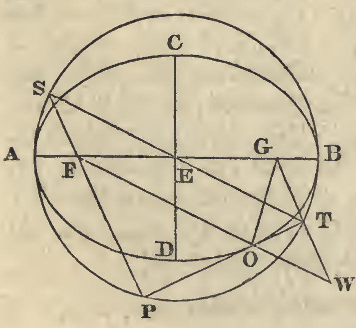
Art. 108. Let MHP touch the ellipse in H, and meet the axes in M and P; then if, on the axes, circles be described, cutting the ordinates HI and HN in R and O, the lines MR and PO shall touch the circles.

Join ER; then, since ER = EB, we have (Art. 107),

As EI : ER :: ER : EM.

Hence (6.6), the triangles EIR, ERM, are similar; consequently, ERM is a right angle; wherefore (cor. 16.3), RM touches the circle. In the same manner it may be proved that OP touches the circle.

Art. 109. Let PT be a right line touching the ellipse in O; FP, GT, perpendiculars falling upon it from the foci F and G; then shall FP.GT = EC^2, the square of the less semi-axis.



Join FO, and let FO, GT, produced, meet in W; and join OG, ET. Then, since the angle GOT = WOT; and GTO=WTO; the side OG = OW, and GT =WT (26.1). Hence FW = FO + OG = (Art. 101) AB. Now, in the triangles FGW, EGT, FG

$= 2EG$, $GW = 2GT$, and the angle at G common; therefore (6.6), $FW = 2ET$, or $EB = ET$. Consequently, the circle described on AB will pass through T. In the same manner it may be proved that it will pass through P.

Next, produce TE till it meets the circle in S, and join FS; then, because $ES = ET$, $EF = EG$, and the angle SEF $=$ TEG (15.1); the side FS must $= GT$, and the angle EFS $=$ EGT (4.1); consequently, FS is parallel to GT (27.1)· that is, FS is in the same straight line with PF. For GT and FP are at right angles to the same line, and are therefore parallel.

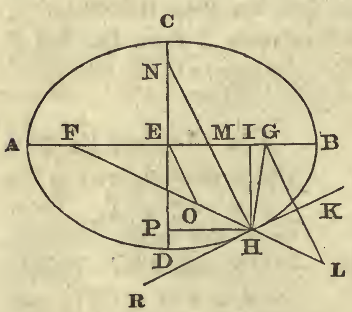
Now, $FP.FS = AF.FB$ (35.3) $= EC^2$ (Art. 98);

therefore, $FP.GT = EC^2$. *Q. E. D.*

Art. 110. Let RK be a tangent to the ellipse in H; the line HN a perpendicular to RK, meeting the greater axis in M, and the less in N; HI, HP, ordinates to the axes AB, CD; then it will be,

As $EB^2 : EC^2 :: EI : IM$;

and As $EC^2 : EB^2 :: EP : PN$.



Join FH, GH; and produce FH to L, making $HL = HG$; join GL, and draw EO parallel to HM. Then GL is at right angles to HK (Cor., Art. 103); and therefore parallel to HM or OE. Hence EO, which bisects FG, also bisects FL (2.6); wherefore $FO = EB$. Then, HL being $= HG$, and $OL = FO$; $HO =$ half the difference of FH and HG. Consequently (Art. 32),

As $EB : EG :: EI : OH$;

also (2.6),

$$\text{As } FO\ (EB) : FE\ (EG) :: OH : EM;$$

hence (23.6),

$$\text{As } EB^2 : EG^2 :: EI : EM;$$

therefore (D. 5),

$$\text{As } EB^2 : EB^2 - EG^2 :: EI : IM.$$

But $EB^2 - EG^2 = EC^2$ (see def. 12). Consequently,

$$\text{As } EB^2 : EC^2 :: EI : IM.$$

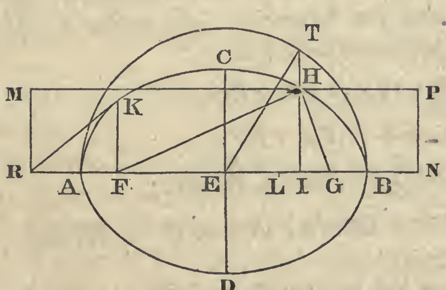
Again, from similar triangles,

$$\text{As } EI\ (PH) : IM :: NP : HI\ (PE);$$

and by inversion,

$$IM : EI :: PE : PN.$$

Hence, $EC^2 : EB^2 :: PE : PN.$ *Q. E. D.*



ART. 111. Let FK be the ordinate through the focus F; KR a tangent to the ellipse at K, cutting the greater axis AB produced in R; and RM a perpendicular to AB; then, if from any point H in the ellipse, HF be drawn to the focus, and HM at right angles to RM, it will be,

$$\text{As } EA : EF :: HM : HF :: AR : AF.$$

From H draw HI, at right angles to AB; and take EL a fourth proportional to AE, FE and EI. Then (Art. 101), $FH = AL$; also (Art. 107),

$$\text{As } ER : EA :: EA : EF :: \text{(by construction) } EI : EL;$$

therefore (12.5),

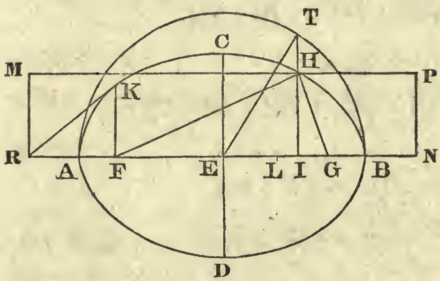
$$\text{As ER : EA :: ER + EI : EA + EL.}$$

But ER + EI = HM; and EA + EL = FH. From the analogy ER : EA :: EA : EF, we have (19.5),

$$\text{ER : EA :: AR : AF.}$$

Hence, $\quad$ EA : EF :: HM : HF :: AR : AF.

Q. E. D.



Def. 13. If we make EN = ER, and draw NP parallel to RM, each of the lines RM, NP, is called a *directrix* to the ellipse; and if HP, HG, are drawn, the latter to the focus and the former at right angles to the directrix, it may be proved as before that

$$\text{HP : HG :: BN : BG.}$$

Cor. Because ER : EA :: EA : EF;

(16.6) $\qquad EA^2 = ER.EF = EF^2 + EF.FR;$

therefore, $AE^2 - EF^2 = EF.FR$; that is, $EC^2 = EF.FR$.

Art. 112. Retaining the construction of the last article, it will be

$$FH = \frac{EC^2}{AE - EF.\cos BFH} = \frac{EC^2}{AE + EF.\cos AFH}$$

From Art. 28,

$$\text{As } 1 : \cos BFH :: FH : FI = FH.\cos BFH.$$

Now,

$$MH = RF + FI = RF + FH.\cos BFH;$$

therefore (Art. 111),

$$\text{As } AE : FE :: RF + FH.\cos BFH : FH;$$

consequently (16.6),

$$FH.AE = RF.FE + FH.FE.\cos BFH;$$

whence,

$$FH.AE - FH.FE.\cos BFH = RF.FE = (\text{Cor., Art. } 111)\ EC^2.$$

Consequently,

$$FH = \frac{EC^2}{AE - FE.\cos BFH} = \frac{EC^2}{AE + FE.\cos AFH}$$

For, $\cos AFH = -\cos BFH$. *Q. E. D*

Cor. If IH be produced to meet the semicircle on AB in T, and ET be joined, then shall

$$FH = AE + EF.\cos BET = AE - EF.\cos AET.$$

By Art. 28,

As 1 : cos BET :: ET (=AE) : EI : : EF : EL=EF.cos BET.

But

FH=AL (Art. 101)=AE+EF.cos BET=AE—EF.cos AET.

Art. 113. Every diameter* to an ellipse is bisected in the centre.

Let HI, passing through the centre E, meet the ellipse in H and I; then is EI = EH. For if it is not, take EP=EH; and from H, I, P, draw lines to the foci F and G. Then we shall have EH = EP; EF = EG; and the angle HEF = PEG (15.1); therefore (4.1), FH = PG. In like manner, GH = FP; therefore,

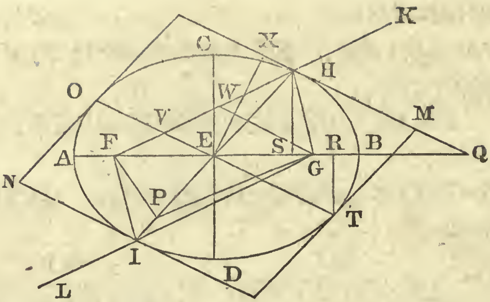
$$FH + HG = FP + PG.$$

But, $FH + HG = FI + IG$ (Art. 101);

consequently, $FP + PG = FI + IG$;

which is absurd (21.1).

* Any right line passing through the centre, limited at both extremities by the ellipse, is called a diameter.



Art. 114. The tangents to an ellipse passing through the extremities of any diameter, are parallel to each other.

Let HM and IN touch the ellipse in H and I, the extremities of the diameter HI; and produce FH, GI, the lines from the foci, to K and L; then the angles GHK, FIL are bisected by the tangents HM, IN (Cor. 1, Art. 106). And since GE, EH are respectively equal to FE, EI (Art. 112), and the angle GEH = FEI (15.1), the angle EHG = EIF, and the side HG = FI (4.1). In the same manner, FHE = GIE; consequently, the whole angle FHG = FIG; and, therefore, (13.1) GHK = FIL; whence GHM = FIN. But EHG = EIF; therefore, EHM = EIN; whence (27.1) HM is parallel to IN.

Cor. Hence, if tangents be drawn through the extremities of any two diameters, they will form a parallelogram.

Def. 14. If the diameter OT be drawn parallel to the tangents through the extremities of IH, then OT is said to be *conjugate* to IH.

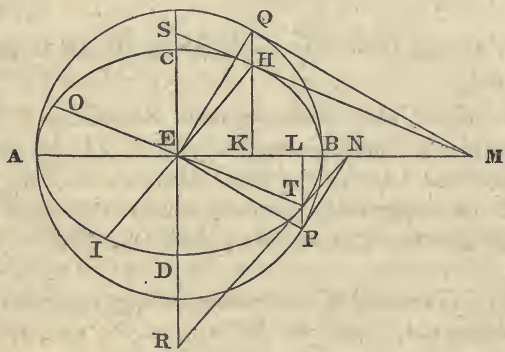
Art. 115. Let OT, which is conjugate to IH, cut the radius vector, or line from the focus to the curve, FH in V; then is HV = AE, half the greater axis.

Through the other focus G, draw GW parallel to OT, cutting FH in W. Then, because OT, and consequently WG, is parallel to the tangent HM, the angle HWG = KHM;

and HGW=GHM (29.1); therefore (Cor. 2, Art. 103), HWG =HGW; and consequently HG = HW (6.1). Since EV is parallel to GW, and FE = EG; therefore (2.6), FV = VW. Hence, VH=FV+HG; and therefore (Art. 101) VH=AE.

Q. E. D.

Art. 116. If the diameter OT is parallel to HM, the tangent at H, then the diameter HI shall be parallel to TN, the tangent at T.



Let HM, TN, meet the greater axis AB produced in M and N; through H and T draw the ordinates HK, TL, to that axis; and let them meet the circle described on AB, in Q and P; join QM, QE, PN and PE; then QM and PN are tangents to the circle (Art. 108). Now, ET being parallel to HM, the angle TEL = HMK (29.1), and ELT = MKH; therefore (4.6),

As EL : LT :: MK : KH;

and, alternately,

EL : MK :: LT : KH :: (Cor., Art. 100) LP : KQ.

Hence, As EL : LP :: MK : KQ;

and the angles at L and K are equal; therefore (6.6), the angle LEP = KMQ; consequently, EP is parallel to QM

(27.1). But EQM is a right angle (18.3); therefore, QEP is also a right angle (29.1); and EPN is a right angle (18.3); therefore, PN is parallel to EQ (28.1). Consequently, the angle LNP = KEQ (29.1); and as NLP = EKQ, we have,

As LN : EK :: LP : QK :: (Cor., Art. 100) LT : KH.

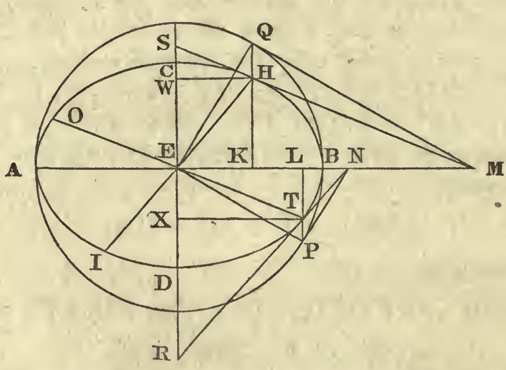
Hence, LN : LT :: EK : KH;

consequently, the triangles LNT, KEH are similar (6.6), and the angle LNT = KEH; therefore (27.1), TN is parallel to EH. *Q. E. D.*

Cor. 1. Hence, OT being conjugate to IH, IH is also conjugate to OT.

Cor. 2. Hence, also, if through the extremities H and T of two conjugate diameters, ordinates, HK, TL, be drawn to the greater axis, meeting the circle described on that axis in Q and P; the tangent QM to the circle is parallel to the radius EP, and the tangent PN to the radius EQ.

Art. 117. The sum of the squares of any two semi-conjugate diameters, is equal to the sum of the squares of the semi-axes.



Let OT, IH, be conjugate diameters; then, retaining the construction of the last article, the triangle EMH is similar to NET, because EH is parallel to TN, and HM to ET. The triangle HMK is likewise similar to TEL. Consequently,

As EM : MH :: EN : ET;

and As MH : MK :: ET : EL;

therefore (22.5),

As EM : MK :: EN : EL :: (Art. 107, and Cor. 2 to 20.6) $EB^2 : EL^2$.

Also, As EM : EK :: $EB^2 : EK^2$.

Consequently (24.5),

$$EM : MK + EK :: EB^2 : EL^2 + EK^2;$$

wherefore, $EB^2 = EL^2 + EK^2$.

If the tangents MH, NT, produced, meet the other axis CD produced, in S and R; and the ordinates HW and TX be drawn to that axis; we have, in like manner, the triangles ESH, SHW, respectively similar to RET and ETX; whence, as before,

As ES : SW :: $ED^2 : EX^2$;

and As ES : EW :: $EC^2 : EW^2$.

Consequently,

$$EC^2 = EX^2 + EW^2 = LT^2 + KH^2.$$

Wherefore,

$$EB^2 + EC^2 = EL^2 + LT^2 + EK^2 + KH^2 = (47.1)\ ET^2 + EH^2.$$

Q. E. D.

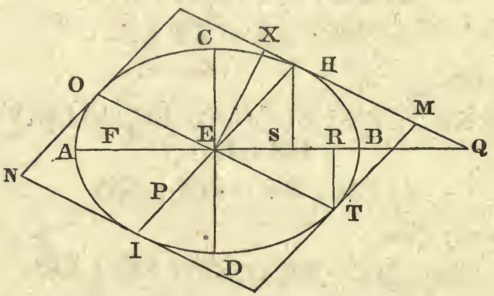
Cor. Since $EK^2 + EL^2 = EB^2 = EQ^2 = EP^2$, it follows (47.1) that EL = KQ, and EK = LP. Hence,

As EB : EC :: EL : KH :: EK : TL.

In like manner,

As EC : EB :: EX : WH :: EW : XT.

Art. 118. The parallelogram formed by the tangents through the extremities of any two conjugate diameters, is equal to the rectangle of the axes.



Let ET, EH be two conjugate semidiameters; TM, HM, tangents to the ellipse passing through their extremities; Q, the intersection of HM with the greater axis produced. Draw HS, TR at right angles to AB, the greater axis; and EX at right angles to HM. Then (Cor., Art. 117),

$$EC : EB :: TR : ES;$$

and (Art. 107), $EB : EQ :: ES : EB;$

therefore (22.5), $EC : EQ :: TR : EB.$

But in the similar triangles EQX, TER;

$$\text{As } EQ : EX :: ET : TR;$$

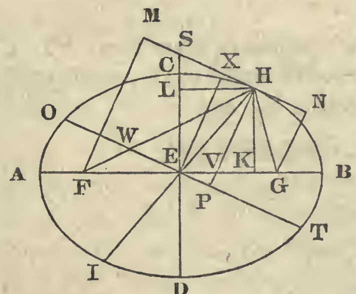
therefore (23.5), $EC : EX :: ET : EB;$

consequently (16.6),

$$EB.EC = ET.EX = \text{the parallelogram } ETMH.$$

Now, this parallelogram being one-fourth of that which circumscribes the ellipse, and EB.EC one-fourth of the rectangle of the axis, it is obvious that the circumscribing parallelogram is equal to the rectangle AB.CD.

ART. 119. Let OT, IH be conjugate diameters; MN, a tangent at H; FH, GH, lines from the foci to the point of contact; then shall the rectangle $FH.HG=ET^2$. Draw FM, GN, HP and EX at right angles to MN or OT; and let OT cut FH in W; then is WH=AE (Art. 115); and OT being parallel to MN, the angle HWE = WHM (29.1). But WHM =GHN (Cor. 2, Art. 103); consequently, the triangles FHM, GHN and HWP, being right angled at M, N and P, are similar; therefore (4.6),



As HP (or EX) : HW (or EB) :: FM : FH :: GN : GH;

consequently (22.6),

As $EX^2 : EB^2 :: FM.GN : FH.GH ::$ (Art. 109) $EC^2 : FH.GH$.

But (Art. 118 and 22.6),

$$\text{As } EX^2 : EB^2 :: EC^2 : ET^2;$$

therefore, $FH.GH = ET^2$.

COR. From the similar triangles FHM, GHN; we have,

FH : GH :: FM : GN :: (1.6) $FM^2 : FM.GN$ or EC^2.

ART. 120. Let OT, IH be conjugate diameters; HP, a perpendicular to OT, cutting the greater axis in V; then shall $HP.HV = EC^2$, the square of the less semi-axis.

Let the tangent at H meet the less axis produced in S; and draw HK, HL at right angles to the axes; and EX parallel to PH. Then the line ES being parallel to KH, and EX to VH, the angle SEX = VHK (29.1), and the angles at X and K are right ones; therefore,

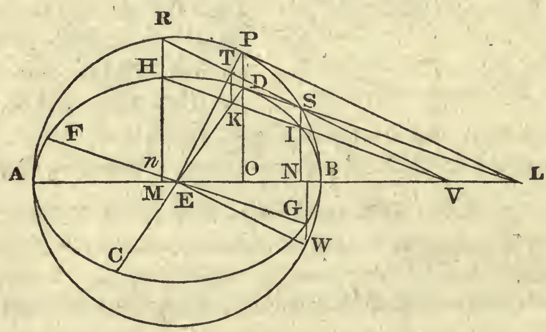
SE : EX (or PH) :: HV : HK or EL.

Therefore,

$$PH.HV = SE.EL = EC^2 \text{ (Cor., Art. 107).}$$

ART. 121. Let CD, FG be conjugate diameters; DL, a tangent to the ellipse; HI, a chord parallel to DL, cutting CD in K; then shall HI be bisected in K; and

$$\text{As } DE^2 : EG^2 :: DK.KC : HK^2.$$



Let DL, HI meet the greater axis AB produced in L and V; through H, D, I and G, draw HM, DO, IN and GW at right angles to AB, meeting the circle on AB, in R, P, S and W; join PL, RS, SV, EP and EW; and let RS cut EP in T; and join TK. Now, since IV is parallel to DL, the angle NVI = OLD (29.1), and VNI = LOD, both being right angles; therefore (4.6),

$$\text{As } VN : NI :: LO : OD.$$

But (Cor., Art. 100),

$$\text{As } NI : NS :: OD : OP;$$

therefore (22.5),

$$\text{As } VN : NS :: LO : OP;$$

consequently (6.6), the angle NVS = OLP. Hence (28.1), VS is parallel to LP.

Again, from similar triangles,

$$\text{As } VM : MH :: VN : NI;$$

and (Cor., Art. 100),

$$MH : MR :: NI : NS;$$

therefore (22.5),

$$\text{As } VM : MR :: VN : NS.$$

If, therefore, we suppose VR joined, the angle MVR = NVS (6.6); consequently, VS and SR are in the same straight line. Now, as DL touches the ellipse in D, PL touches the circle in P (Art. 108). Hence, EPL is a right angle (18.3); and RS, which is parallel to PL, is bisected in T (3.3). Since HV is parallel to DL, and TV to PL; we have (2.6),

$$\text{As } EK : KD :: EV : VL :: ET : TP;$$

hence TK is parallel to PD (2.6), and consequently to SI and RH; wherefore,

$$ST : TR :: IK : KH.$$

But ST = TR; therefore, IK = KH.

Again, since EW is parallel to PL (Cor. 2, Art. 116), and therefore to RV; and EG to DL or HV; the angle EVR = VEW, and EVH = VEG; hence, HVR = GEW. Also, HR being parallel to GW, the angle VHR = EGW; for each of them is equal to GnR (29.1). Hence,

$$EW : EG :: VR : VH :: RT : HK.$$

Now, from similar triangles,

$$\text{As } ED : EP :: EK : ET;$$

and, therefore (22.6),

$$\text{As } ED^2 : EP^2 :: EK^2 : ET^2 :: (19.5)$$

$$ED^2 - EK^2 : EP^2 - ET^2 :: (5.2 \text{ and } 47.1)\ DK.KC : RT^2.$$

But $\qquad EW^2 \text{ or } EP^2 : EG^2 :: RT^2 : HK^2;$

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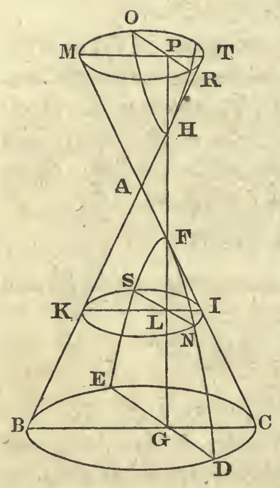
therefore (22.5),

$$\text{As } ED^2 : EG^2 :: DK.KC : HK^2.$$

Q. E. D.

Of the Hyperbola.



ART. 122. Let ABC be a triangle formed by the section of a cone and a plane passing through its axis, at right angles to the plane of its base; and let a plane at right angles to the plane of the triangle cut the cone in DFE and the opposite cone, made by the extension of the sides of the given cone on the other side of A, in OHR; then (Art. 85) the curves DFE and OHR form *opposite hyperbolas.*

The two branches of either hyperbola, as likewise the two opposite hyperbolas, are like figures, and equal to each other.

Let BDC be the base of the cone; DE, the intersection of BDC and the plane DFE; and GFHP, the intersection of the cutting plane and the plane of ABC.

In GF take any point L, make HP = FL, and through L and P let planes pass parallel to the base of the cone; then the sections KNI, MRT, of these planes and the conical surface, are circles, whose centres are in KI, MT (Art. 80), the intersections of these planes with the plane of ABC, which passes through the axis of the cone. But (18.2 sup.) SLN and OPR are at right angles to the plane of ABC, and therefore (Def. 1.2 sup.) to KI and MT in that plane; therefore (3.3), SL = LN, and OP = PR. Hence, the two branches of either hyperbola are equal to each other.

Again, since the planes MOT and KSI are parallel, MT is parallel to KI (14.2 sup). Hence the triangles FLI and MFP are similar, as likewise the triangles KHL and PHT; therefore,

As FL : FP :: LI : MP;

and As LH : HP :: KL : PT;

consequently,

As FL.LH : FP.HP :: LI.KL : MP.PT :: (35.3) $LN^2 : PR^2$.

But, by construction, FL = HP; therefore, LH = FP, and FL.LH = FP.HP; $\therefore LN^2 = PR^2$, or LN = PR. Consequently, equal ordinates corresponding to equal abscissas, the opposite hyperbolas must be like figures and equal to each other. *Q. E. D.*

Art. 123. Let MAN, PBQ (see fig. on page 156) be two hyperbolas, formed by the sections of a cone and a plane, as described in Art. 122; AB, the interval between the points where the cutting plane cuts the opposite cones, corresponding to HF in the figure on page 154.

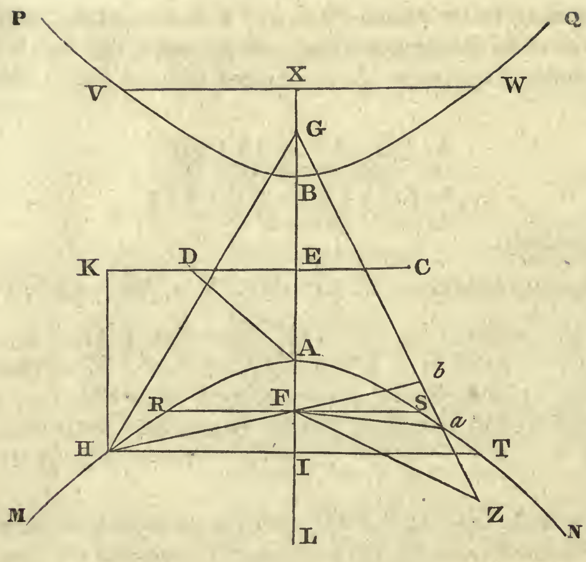
Bisect AB in E; draw DEC at right angles to AB; and, drawing the ordinate HI from any point H in the curve, to meet BA produced in I, make ED and EC such that

AI.IB : IH^2 :: EA^2 : ED^2 or EC^2;

then AB is called the *first*; and DC the *second, axis* of the hyperbola. The points A, B, are called the *vertices*, and E the *centre*, of the hyperbola.

From this construction and Art. 85, it is obvious that AE^2 is to ED^2 as the rectangle of any corresponding abscissas is to the square of their ordinate.

Art. 124. *Def.* 15. Join AD; and make EF, EG, each equal to AD; then F and G are called the foci of the hyper-



bolas; and the double ordinate RFS, passing through the focus, is called the latus rectum.

The latus rectum is a third proportional to the first and second axes. For BA is bisected in E; therefore,

$$BF.FA + AE^2 = EF^2 \ (6.2) = AD^2 = AE^2 + ED^2 \ (47.1);$$

consequently, $BF.FA = ED^2$; hence (Art. 123),

$$AE^2 : ED^2 :: ED^2 : RF^2 \text{ or } FS^2;$$

wherefore (22.6),

$$AE : ED :: ED : RF \text{ or } FS;$$

and $$AB : DC :: DC : RS.$$

Cor. From this demonstration, it is obvious that

$$ED^2 = AF.FB = BG.GA.$$

ART. 125. If from any point H in the hyperbola, a perpendicular HK be let fall on the second axis, it will be,

$$\text{As } ED^2 : AE^2 :: ED^2 + EK^2 : HK^2.$$

For (Art. 123),

$$\text{As } AE^2 : ED^2 :: AI.IB : IH^2 :: (12.5)$$
$$AE^2 + AI.IB : ED^2 + IH^2 \text{ or } ED^2 + EK^2.$$

But (6.2), $\quad AE^2 + AI.IB = EI^2.$

Hence, by inversion,

$$ED^2 : AE^2 :: ED^2 + EK^2 : EI^2 \text{ or } HK^2.$$

Q. E. D.

ART. 126. The difference of two right lines, HG, HF, drawn from any point H in the hyperbola to the foci, is equal to AB, the first axis.

Take EL a fourth proportional to EA, EF and EI; HI being an ordinate to the point H. Then (22.6),

$$\text{As } EA^2 : EF^2 :: EI^2 : EL^2;$$

therefore (17.5),

$$\text{As } EA^2 : EF^2 - EA^2 \; (ED^2) :: EI^2 : EL^2 - EI^2 :: (19.5)$$
$$EI^2 - EA^2 : EL^2 - EI^2 - ED^2.$$

But (6.2), $\quad EI^2 - EA^2 = AI.BI;$

and (Art. 123),

$$\text{As } EA^2 : ED^2 :: AI.IB : IH^2;$$

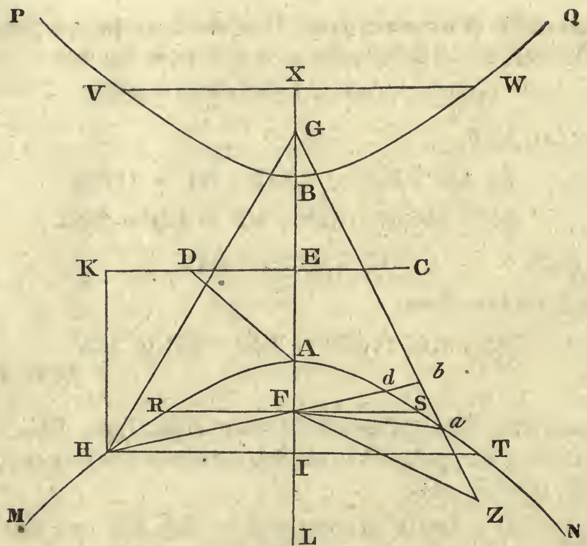
therefore, $\quad EL^2 - EI^2 - ED^2 = IH^2.$

Again, $\quad EA^2 + EL^2 = 2AE.EL + AL^2 \; (7.2);$

and $\quad EF^2 + EI^2 = 2EF.EI + FI^2.$

Taking the difference of these equations, and remembering that $EF^2 - EA^2 = ED^2$; and $AE.EL = EF.EI$, because $AE : EF :: EI : EL$; we have,

$$EL^2 - EI^2 - ED^2 = AL^2 - FI^2.$$



Hence, $AL^2 - FI^2 = IH^2$; or $AL^2 = IH^2 + FI^2 = (47.1)\ FH^2$; or $AL = FH$.

Further, $EL^2 + EB^2 + 2BE.EL = BL^2$ (4.2);

also, $EI^2 + EG^2 + 2IE.EG = IG^2$.

Taking the difference as before,

$$EL^2 - EI^2 - ED^2 = BL^2 - IG^2.$$

But $EL^2 - EI^2 - ED^2 = IH^2$;

therefore, $BL^2 - IG^2 = IH^2$;

and, therefore,

$$BL^2 = IG^2 + IH^2 = (47.1)\ GH^2;\ \text{or}\ BL = GH.$$

Consequently,

$$BA = (BL - AL =)\ GH - FH.$$

Q. E. D.

Art. 127. If, from a point Z within an hyperbola, two right lines, FZ, GZ, be drawn to the foci, the difference of these lines is greater than AB, the first axis; but if from a point b without the hyperbola, two right lines, bG, bF, be drawn to the foci, the difference of these lines will be less than the axis AB.

First, let ZG meet the hyperbola in a, and join Fa; then, since FZ is less than Fa + aZ, the difference between GZ and FZ is greater than between GZ and Fa + aZ; that is, than Ga — Fa. But (Art. 126) Ga — Fa = AB; therefore, GZ — FZ is greater than AB.

Next, let Fb meet the hyperbola in d, and suppose Gd joined; then, because Gd is greater than Gb — bd, Gd — dF; that is, AB is greater than Gb — bF, or Gb — bF is less than AB. *Q. E. D.*

Cor. Hence a point is either in, within, or without an hyperbola, according as the difference of the lines drawn from it to the foci is equal to, greater, or less than the first axis.

Art. 128. If, from any point H (see fig. on page 160) in the hyperbola, a right line HM be drawn bisecting the angle FHG, made by lines to the foci F, G, the line HM will be a tangent to the hyperbola.

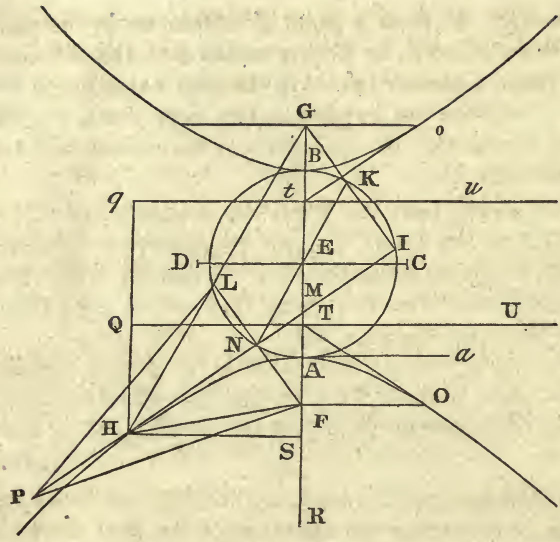
Take on HG, the line HL = HF; and take in MH any other point P, and join PL, PF; then (4.1), FP=LP. Now, since HF = HL, LG must be equal to AB (Art. 126); hence,

$$AB = PL + LG — FP.$$

But PL + LG are greater than the right line joining P and G; hence the excess of that line above PF is less than AB; consequently, the point P is without the hyperbola (Cor., Art. 127). And this being true of every point in PM except the point H, that line must be a tangent to the hyperbola.

Q. E. D.

Cor. 1. From this we may infer that the tangent must bisect the angle FHG; for if it was possible to draw a tan-



gent through H which did not bisect the angle, we might have two right lines touching the same curve in the same point, and yet not coinciding with each other.

Cor. 2. The line A*a* through the vertex of the hyperbola, at right angles to GF, is a tangent; for the angles GA*a*, FA*a*, are equal.

Art. 129. Let FN, GI be perpendiculars falling from the foci F, G, upon a tangent HI; then shall $FN.GI = ED^2$, the square of the second semi-axis.

Take, as in the last article, HL = HF; join LN, NE; and produce NE to meet GI in K. Then, since HF = HL; and the angle FHN = LHN (Cor. 1, Art. 128); the angle HNF must be equal to HNL (4.1): consequently, HNL = a right angle, and therefore FNL is a right line (14.1). Now, in the triangles NFE, LFG, we have the angle at F common, and the sides NF, FE, the halves of LF, FG, respectively; whence (6.6) the angle FNE = FLG, and NE = half LG;

consequently (28.1), NK is parallel to LG. But LN is parallel to GK; hence (34.1), LN = GK, and NK = LG = AB (Art. 126). Hence,

$$EN = EK = \tfrac{1}{2}AB = EA.$$

Consequently, a circle described from the centre E, at the distance EA, will pass through N and K; it will also pass through I, because KIN is a right angle (converse of 31.3). Therefore (cor. 36.3), AG.GB = IG.GK; that is, (Cor., Art. 124), $ED^2 = FN.GI$. *Q. E. D.*

Art. 130. Let HM touch the hyperbola in H, and meet the first axis AB in M, and HS be an ordinate to that axis; then it shall be,

$$\text{As } ES : EA :: EA : EM.$$

Take ER a fourth proportional to EA, EF and ES; then, as proved in Art. 126, AR = FH, and BR = GH. Then, the vertical angle of the triangle GHF being bisected by the line HM (Cor. 1, Art. 128),

$$\text{As } GH : HF :: GM : MF \text{ (3.6)};$$

hence (E. 5)

$$GH + HF : GH - HF :: GM + MF : GM - MF;$$

then, taking the halves of these quantities,

$$ER : EA :: EF : EM;$$

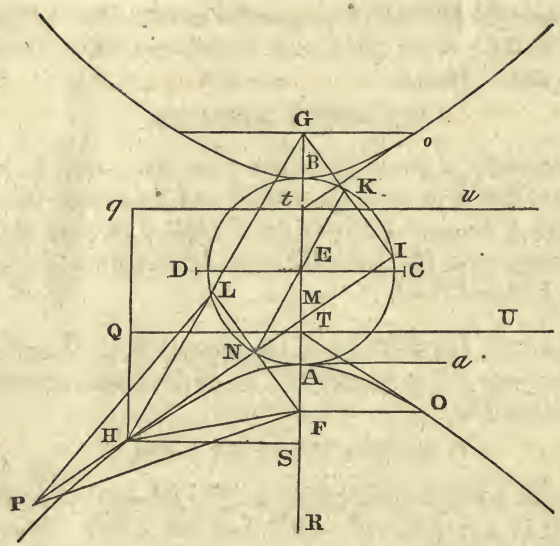
and, alternately (16.5),

$$ER : EF :: EA : EM.$$

But $$ER : EF :: ES : EA;$$

therefore (11.5), $ES : EA :: EA : EM.$ *Q. E. D.*

Art. 131. Let FO, G*o*, be the ordinates through the foci; OT, *ot*, tangents to the hyperbolas at O and *o*, cutting the first axis in T and *t*; QTU and *qtu*, perpendiculars to AB;



then, taking any point H in the hyperbola, and drawing HQq parallel to the axis AB, it will be,

As FH : HQ :: FA : AT :: GH : Hq :: GA : BT.

Draw HS at right angles to the axis; and take ER a fourth proportional to EA, EF, ES; then (Art. 126) AR = FH, and BR = GH. Now, by construction,

ER : ES :: EF : EA :: (Art. 130) EA : ET :: (19.5)
ER—EA : ES—ET :: EF—EA : EA—ET;

for the last four terms substituting their equals,

FH : HQ :: AF : AT.

Again (12.5),

ER : ES :: ER+EA : ES+ET :: EF+EA : EA+ET.

Hence, GH : Hq :: FB (GA): BT.

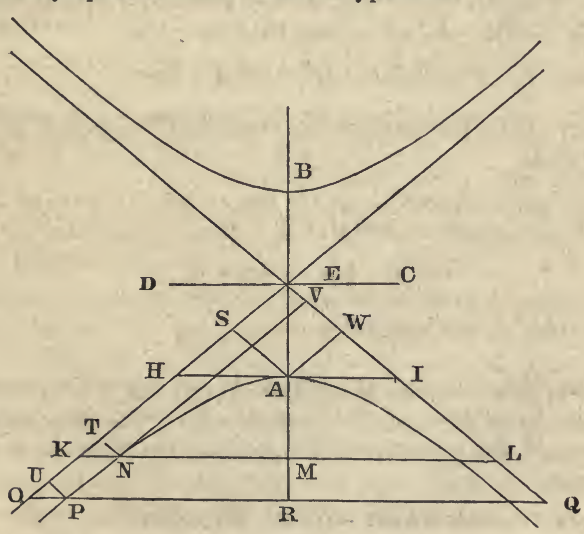
And these ratios in both cases are the same as EF : EA.

Each of the lines QU and *qu* is called the *directrix* of the hyperbola.

This article being compared with articles 86 and 111, it is manifest that from any point in either of the three conic sections, two straight lines being drawn, one of them to the focus and the other at right angles to the directrix, they will have to each other a constant ratio. In the parabola, the perpendicular upon the directrix is equal to, in the ellipsis it is greater, and in the hyperbola it is less, than the radius, vector, or line from the focus to the curve.

Art. 132. Let AB, DC be the axes of the hyperbolas; AH, AI, at right angles to AB, and each equal to half DC; then, right lines being drawn through E, the middle of AB, and the points H and I, and indefinitely extended, they are called the *asymptotes;* the property of which, to be demonstrated hereafter, is, that they continually approach the hyperbola, but do not meet it.

The asymptotes do not meet the hyperbola.



From any point N in the hyperbola, draw NM at right angles to the axis, and let it meet the asymptote in K; then (from similar triangles and 22.6),

$$\text{As } EA^2 : AH^2 :: EM^2 : MK^2.$$

But (Art. 123),

$$\text{As } EA^2 : ED^2\ (AH^2) :: AM.MB : MN^2;$$

and EM^2 is greater than AM.MB (6.2); therefore, MK^2 is greater than MN^2, and MK greater than MN.

Art. 133. Retaining the construction in the last article, produce KM to meet the asymptote EQ in L; then shall $KN.NL = ED^2$.

As in last article, we have,

$$EA^2 : ED^2 :: EM^2 : MK^2 :: AM.MB : MN^2 :: (19.5)$$

$$EM^2{-}AM.MB : MK^2{-}MN^2 :: (6.2 \text{ and } 5.2)\ AE^2 \,.\, KN.NL.$$

Hence, $$KN.NL = ED^2 = HA.AI.$$

Cor. Hence, OPQ being drawn parallel to KL, the rectangle $OP.PQ = KN.NL$; and, therefore (16.6),

$$\text{As } KN : OP :: PQ : NL.$$

Art. 134. The asymptotes continually approach to the hyperbola.

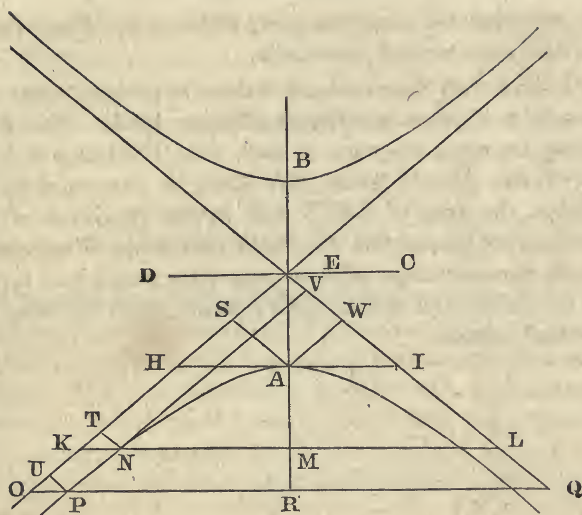
Taking KL and OQ as in the last article, it is evident that, ER being greater than EM, PQ is greater than NL; but,

$$KN : OP :: PQ : NL;$$

hence, OP is less than KN.

Art. 135. Through the vertex A, and any other point N of the hyperbola, let the lines AS, NT be drawn parallel to one of the asymptotes EQ, meeting the other in S and T; then.

$$\text{As } ES : ET :: TN : SA.$$



Draw AW, NV parallel to EO; then the triangles IAW, LNV, are similar; as are also HAS, KNT; hence the following analogies:

As AW : NV :: AI : NL :: (Art. 133) KN : HA :: TN : AS.

Therefore, As ES : ET :: TN : SA.

Cor. If PU be drawn parallel to EQ, then

EU : ET :: TN : UP.

Scholium. From the property demonstrated in Art. 135 is deduced a relation between logarithms and the areas contained between the hyperbolic curve and its asymptote. Let EA = ED, and consequently ES = SA, and SEW a right angle. The hyperbola is then called an equilateral or rectangular one. If in that case we assume ES = 1; and of course the square SW also = 1; then ET being estimated in units of ES, and the area ASTN in units of SW, it is proved by writers on differentials that ASTN is the logarithm of

ET, provided the modulus (Art. 15) $m = 1$. Hence, those logarithms are termed hyperbolic.

It is, however, observable, that these hyperbolic areas may be made to express logarithms of other kinds. For, if the relation between the axes is such that, ES being $= 1$, the area of the parallelogram SW shall be expressed by the modulus, the area of ASTN will be the logarithm of ET, according to the system to which that value of m belongs. But the demonstration of these properties would lead further into the differential and integral calculus, than the design of this work admits.

SECTION V.

SPHERICAL PROJECTIONS.

ARTICLE 136. The business of Spherical Projections is to represent by lines, drawn or described on a plane given in position, the circles which are described on the surface of a sphere. The lines thus drawn or described on the plane, are called the projections of the circles which they represent; and are so framed that, to an eye properly located, every circle on the sphere will appear coincident with its representative.

Def. 1. The plane on which the circles of the sphere are represented, is called the *plane of projection;* and the point where the eye is supposed to be located, the *projecting point.* A right line drawn from the projecting point to any point on the sphere, and extended to meet the plane of projection, is termed a *projecting line.*

Def. 2. Every circle on the sphere is called an *original circle;* and the figure which represents it on the plane of projection, a *projected circle.*

Def. 3. In the orthographic and stereographic projections, the plane of projection is supposed to pass through the centre of the sphere. Then the common section of this plane and the spherical surface is a circle, which is called the *primitive circle.* This circle is evidently a great one (Art. 45); and, being both on the sphere and plane of projection, may be considered as an original circle, projected into itself.

Def. 4. In the orthographic projection, the projecting point is in the axis of the primitive circle; but so remote, that all the projecting lines drawn to the different points of the sphere may be considered as parallel.

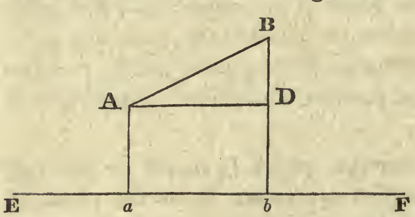
Def. 5. In the stereographic projection, the projecting point is at one of the poles of the primitive circle.

Def. 6. The *line of measures* of any circle which is to be projected, is the common section of the plane of projection, and another plane which passes through the axes, both of the primitive circle and of the circle to be projected.

Def. 7. The *semitangent* of an arc is the tangent of half the arc; not half the tangent of the arc.

Of the Orthographic Projection.

Art. 137. If a right line AB be projected orthographically upon a plane, the projection will be a right line; and the original line will be to its projection, as radius to the cosine of the inclination of the original to the plane of projection.



Let EF be the plane of projection, seen edgewise; Aa, Bb, the projecting lines, through the extremities of AB, meeting the plane of projection in a, b. Conceive a plane to pass through Aa, Bb; this plane will include AB, and be at right angles to the plane of projection (def. 4 and 17.2 sup). The common section of these planes will evidently be the projection of AB; but this section is a straight line (3.2 sup.) contained between Aa and Bb; that is, it is the line ab.

Through A draw AD parallel to ab; then is DAB the inclination of AB to the plane of projection. And (Art. 28)

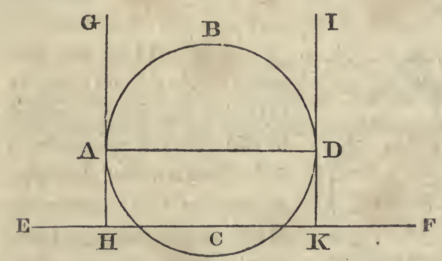
radius : cosine DAB : : AB : AD or ab.

Cor. 1. When a line is parallel to the plane of projection, its projection is equal to the original line.

Cor. 2. When two lines which make an angle with each other are parallel to the plane of projection, their projections make an equal angle with each other. This is obvious from 9.2 sup.

Cor. 3. Any figure which is delineated on a plane parallel to the plane of projection, is projected into a figure similar and equal to itself.

Art. 138. A circle whose plane is at right angles to the plane of projection, is projected into a right line equal to its diameter.



Let EF be the plane of projection, seen edgewise; ABDC, the circle; GH, IK, right lines perpendicular to the plane of projection, touching the circle in A and D; then the plane GHKI will evidently include the circle; and its intersection HK with the plane of projection, must be the projection of the circle: but HK = AD, the diameter of the circle.

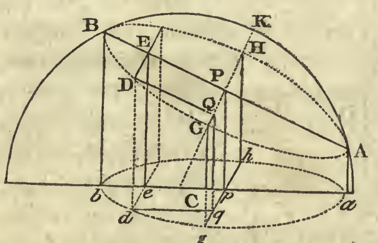
Cor. Hence any figure which is delineated on a plane at right angles to the plane of projection, is projected into a right line.

Art. 139. A circle of the sphere, whose plane is parallel to the plane of projection, is projected into a circle equal to itself, and concentric with the primitive circle.

For every circle of the sphere which is parallel to the plane of projection, has the same axis with the primitive circle; and that axis being at right angles to the plane of projection, is a projecting line (def. 4). Consequently, the centre of the original circle must be projected into the centre of the primitive. And every radius of that circle is projected into a line equal to itself (Cor. 1, Art. 137).

Cor. As the radius of any circle on the sphere is the sine of its distance from its own pole; the radius of a projected circle whose original is parallel to the primitive, is the sine of the distance of that original circle from its pole.

Art. 140. A circle whose plane is inclined to the plane of the primitive, is projected into an ellipse, whose major axis is equal to the diameter of the original circle, and whose minor axis is to the major, as the cosine of the inclination of the planes is to radius.



Let AGBH be the original circle; P, its centre; GH, its diameter parallel to the plane of projection; AB, the diameter at right angles to GH; AB*ba*, a plane at right angles both to the plane of projection and to the plane of AGBH. From any point D in the original circle, let DE be drawn perpendicular, and DQ parallel to BA; and let the given circle be projected into the figure *agbh*; the lines A*a*, G*g*, B*b*, H*h*, P*p*, Q*q*, D*d*, E*e*, being the projecting lines, which (def. 4) are necessarily at right angles to the plane of projection. Then, since GH and DE are parallel to the plane of projection, their projections are equal to the lines themselves (Cor. 1, Art. 137); that is, $gp = \mathrm{GP}$; $ph = \mathrm{PH}$; $de = (qp =)\ \mathrm{DE} = \mathrm{QP}$. Hence,

$$gq.qh = \mathrm{GQ.QH} = (35.3)\ \mathrm{DQ}^2.$$

But P*p*, E*e*, and B*b*, in the plane *a*AB*b*, being parallel,

$$\mathrm{BP} : \mathrm{EP} :: bp : ep\ (2.6);$$

consequently (22.6),

$$\mathrm{BP}^2 : \mathrm{EP}^2 \ (\text{or } \mathrm{DQ}^2) :: bp^2 : ep^2 \text{ or } dq^2;$$

that is, $$gp^2 : gq.qh :: bp^2 : dq^2;$$

wherefore *agbh* is an ellipse, whose major axis is $gh = \mathrm{GH}$, the diameter of the original circle (Art. 84). Also, gh $(= \mathrm{AB}) : ab ::$ radius : cosine of the inclination of AB to the plane of projection (Art. 137). Now, the plane AGBH and the plane of projection are both at right angles to the plane *a*AB*b*; hence, if those planes were extended so as to meet, their common section would be at right angles to the same plane (18.2 sup.), and therefore at right angles to AB. Hence, the inclination of AB to the plane of projection is the inclination of the original circle AGBH to the same plane.

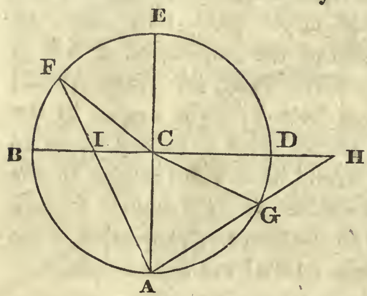
Cor. 1. The major axis of the ellipse into which a circle is projected, is twice the sine of the arc of a great circle intercepted between the original circle and its own pole.

Cor. 2. The minor axis of the same ellipse, or that axis produced, passes through the centre of the primitive circle. For the plane AB*ba*, being perpendicular to the circle AGBH, and passing through its centre, must pass through its axis; and that axis passes through the centre of the primitive circle.

Cor. 3. The distances of the extremities of the minor axis from the centre of the primitive circle, are the sines of the greatest and least distance of the original circle from the pole of the primitive.

Of the Stereographic Projection.

Art. 141. To explain the nature of this projection, let ABED be a circle formed by the section of a spherical surface and a plane which passes through the centre of the sphere; this plane is here represented by the plane of the paper; take C the centre of this circle, then C is also the centre of the sphere; draw the diameter ACE; and suppose another plane, perpendicular to AE, to pass through C, the centre of the sphere; this plane will be perpendicular to the plane ABED (17.2 sup.), and its common section with the spherical surface will be a great circle; which circle, seen edgewise, may be represented by BD, a diameter to the circle ABED at right angles to AE. If, then, A be taken as the projecting point, the circle represented by BD will be the primitive circle, whose centre is C; and as the pole E, opposite to the projecting point, is projected in C, the pole of the primitive circle is projected in its centre.



Art. 142. Any point on the sphere is projected into a point distant from the centre of the primitive, the semitangent of the arc of a great circle intercepted between the given point and the pole of the primitive, opposite the projecting point.

Retaining the construction of the last article, let F and G be two points to be projected; join AF, AG; and let AF, AG, produced if necessary, cut the plane of the primitive in I and H; these points are evidently the projections of F and G. But IC is the tangent of CAI, and CH the tangent of

CAH; that is, CI is the semitangent of ECF (20.3) or of EF; and CH the semitangent of ECG, or of EG. *Q. E. D.*

Cor. Any point in the hemisphere BED, opposite to the projecting point, will be projected within the primitive circle; but a point on the hemisphere BAD, adjacent to the projecting point, will be projected without the primitive. For the distance of any point in the first hemisphere from the pole E, is less than a quadrant; but in the second it is greater; and the semitangent of 90° = tangent of 45° = radius.

Art. 143. Every circle of the sphere which passes through the projecting point, is projected into a right line at right angles to its line of measures.*

The original circle, being in the same plane as the projecting point, cannot be projected out of that plane; it will, therefore, be projected into the common section of that plane and the plane of projection: but that section is a right line (3.2 sup.); therefore, the projection of the circle is a right line. And as the plane of the circle projected, and the plane of projection, are at right angles to the plane which forms the line of measures (Art. 136, Def. 6), their common section is at right angles to that plane (18.2 sup.); and, therefore, the line of measures is at right angles to the projection of the given circle.

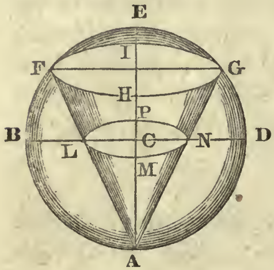
Cor. 1. A circle which passes through the poles of the primitive, is projected into a right line which passes through the centre of the primitive, at right angles to its line of measures.

Cor. 2. Every circle which passes through the poles of the primitive is projected into a line of semitangents. This is evident from Art. 142.

* The projection of a circle which passes through the poles of the primitive, is usually called a right ~~circle~~. line

Art. 144. Every circle of the sphere which does not pass through the projecting point, is projected into a circle.

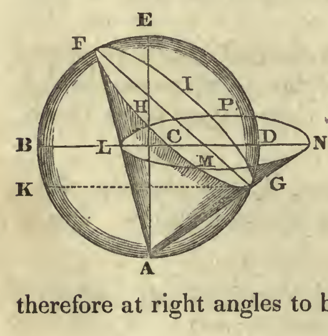
Case 1. When the plane of the original circle is parallel to the plane of projection.



Let A be the projecting point; E, the opposite pole; BCD, the plane of projection at right angles to AE; C, the centre of the primitive circle; FHGI, the circle to be projected; LMNP, its projection. If, now, while the point A remains fixed, we suppose the line AF carried round with a conical motion, so as to describe the circle FHGI; the common section of the conical surface and the plane of projection will be LMNP, the projection of the circle FHGI. But the plane of that section, being parallel to the plane of the base, is a circle (Art. 80).

Cor. The radius, CN or CL, of the projected circle is the semitangent of EG or EF, the distance of the original circle from the pole opposite to the projecting point.

Case 2. When the circle to be projected is not parallel to the plane of projection.



Let A be the projecting point; E, the opposite pole; FHGI, the circle to be projected; LMNP, its projection; ABED, the common section of the spherical surface and a plane which passes through the axes both of the primitive and of the circle FHGI, and therefore at right angles to both these planes. Then BN, the common section of this plane and the plane of projection, is the line of measures for the circle FHGI (Art. 136, Def. 6).

Supposing, as before, the line AF to be carried round the circle FHGI, it will describe a conical surface, whose common section with the plane of projection will be LMNP, the projection of FHGI.

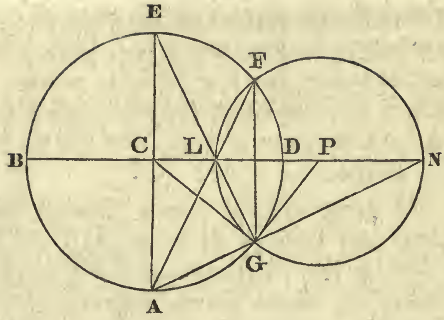
Because the plane ABED passes through the axis of FHGI, it must pass through its centre and the axis of the cone; therefore the line FG, the common section of this plane and the plane of the circle, is a diameter, which is projected into the line LN.

Draw GK parallel to NB. Then the angle LNA = KGA (29.1) = AFG (21.3), because AK=AG; hence the triangles AFG, ANL, which have the angle at A common, are equiangular to each other; and the section LMNP is a subcontrary section, and therefore (Art. 82) is a circle, whose diameter is LN.

Cor. The projected pole and centre of the projected circle are both in the line of measures.

Art. 145. The centre of a projected less circle, at right angles to the primitive, is in the line of measures, distant from the centre of the primitive the secant of the circle's distance from its own pole; and the radius of the projected circle is the tangent of the same distance.

Let A (see fig. on p. 176) be the projecting point; ABED, as before, the common section of the spherical surface and a plane which passes through the centre of the sphere, and is at right angles both to the plane of projection and the plane of the circle to be projected; BCDN, the plane of projection, seen edgewise; C, the centre of the sphere; FG, the common section of the circle to be projected and the plane ABED; FG will then represent that circle, seen edgewise. Join AF, AG, CG and EG; and let AF, AG meet BN in L and N; these points will then be the projections of F and G; and, consequently,

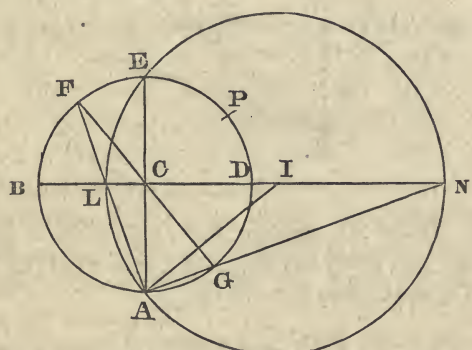


the line LN will be the projection of FG, the diameter of the circle to be projected. As the circle to be projected is perpendicular to the plane of projection, that plane must pass through its poles; hence the point D, where that plane cuts the circle ABED, is one of the poles; and, therefore, FD = DG; also, BN is the line of measures. Draw GP touching the circle ABED in G, and cutting BN in P; then, as proved in Art. 144, the angle ANL = AFG. And since GP touches the circle AGDE, and GA cuts it, the angle PGN = GFA (32.3) = GNP; consequently, PN = PG (6.1).

Again, since ACD = ECD, being both right angles; and CAF=CEG; it is plain (26.1) that EG cuts CD in L; then, since CG=CE, the angle CGL=CEL. Taking these equals from the right angles CGP, LCA; the angle LGP=CLE= PLG; hence, LP=PG. Consequently, PG = the radius of the circle described on the diameter LN; but GP is the tangent, and CP the secant of GD.

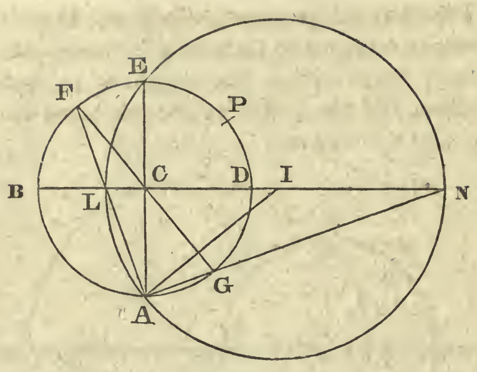
If now we suppose the figure to revolve on BN until the plane of ABED becomes perpendicular to CA, the circle ABED will be the primitive circle; and the points L, D, P, N, will remain unchanged: consequently, the circle described from the centre P with the radius PL = PG, will be the projected circle proposed.

ART. 146. Any oblique* great circle will be projected into a circle whose centre is in the line of measures, distant from the centre of the primitive, the tangent of its inclination to the primitive, and the radius of the projected circle is the secant of that inclination.



Let, as before, A be the projecting point; ABED, the great circle at right angles to the primitive, and to the circle to be projected; BN and FG, the common sections of the plane of this circle with the plane of projection, and with the plane of the circle to be projected, respectively. Then BN will represent the plane of projection, and FG the circle to be projected, both seen edgewise; the line BN will also be the line of measures. Join AF, AG, meeting BN in L and N; these will be the projections of F and G, and LN the diameter of the projected circle. Now, the plane of the primitive circle, and of the circle to be projected, being both at right angles to the plane of ABED, their common section, which passes through C, the centre of the sphere (Art. 45), is at right angles to that plane (18.2 sup.); hence CB and CF are at right angles to that common section; consequently, the angle FCB is the inclination of the circle FG to the primitive (def.

* A circle whose plane makes an oblique angle with the plane of projection, is called an oblique circle.



4.2 sup). Draw AI, making the angle CAI = FCB. To these equals add CAL = CFL, and LAI=FLB (32.1) = ILA (15.1). Consequently, LI = AI (6.1).

Again, the angle FAG in a semicircle being a right angle (31.3), is equal to BCF+CAF+CFA (32.1) = LAI + CFA. Hence, IAN = CFA. But ANI=AFC, by subcontrary section (Art. 144); wherefore, ANI=IAN, and AI=IN. Hence the radius of the circle described on LN, that is, the projection of FG, is equal to AI. But AI is the secant, and CI the tangent, of CAI or BCF, the inclination of the circle FG to the primitive.

If, then, as before, we suppose the figure to revolve on BN until the plane of ABED becomes perpendicular to AC, the circle ABED will be the primitive; and the circle described from the centre I, with the radius IA or IL, will be the projection proposed.

Cor. 1. Hence an oblique great circle being projected on the plane of the primitive; and, from the point where the projected circle cuts the primitive, two right lines being drawn to the centre of the primitive and of the projected circle; the inclination of those lines is the same as the inclination of the original circle to the primitive.

Cor. 2. Of all projected great circles, the primitive is the least; for the secant of any arc is greater than the radius.

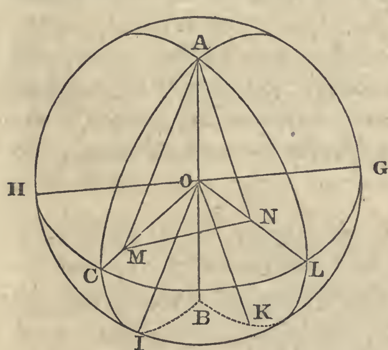
Def. 8. The angle made by two circles, whether on the same or different planes, is the angle made by their tangents passing through the point of intersection.

When the circles are both great circles, the tangents are at right angles to the diameter of the sphere, passing through the point of intersection, which is the common section of the planes of these circles; and, consequently, the angle made by the tangents measures the inclination of the planes.

The definition contained in Art. 45 is therefore but a particular application of the more general one now given.

Cor. The angle made by the radii (drawn to the point of intersection) of two circles on the same plane, is equal to the angle made by the circles.

Art. 147. The angle made by two great circles on the surface of the sphere, is equal to the angle made by their representatives on the plane of projection.



Let A be the place of the eye or projecting point; B, the opposite pole; O, the centre of the sphere; HCLG, the primitive circle; I and K, the poles of the proposed great circles; ACIB, a great circle passing through A, B and I; BKLA, another great circle passing through A, B and K.

Draw AOB, the axis of the primitive; and OI, OK, the axes of the proposed circles. Let the plane ACIB cut the plane of projection in the line OC; and BKLA cut it in OL; then (def. 6), OC and OL will be the lines of measures of the circles, whose poles are I and K respectively. Also (Cor., Art. 47), the angles IOB, KOB and IOK are respectively equal to the angles which the proposed circles make, on the surface of the sphere, with the primitive and with each other. In the plane ACIB, suppose the line AM to be drawn parallel to OI, meeting OC in M. Then, since the angle OAM=BOI (29.1), the angle which the great circle, whose pole is I, makes with the primitive; and M is in the line of measures of that circle; it follows (Art. 146) that M is the centre, and MA the radius, of the representative of that circle on the plane of projection. In like manner, suppose AN, in the plane BKLA, drawn parallel to the axis OK, and meeting the line OL in N; then N will be the centre, and NA the radius, of the circle on the plane of projection which represents the original circle, whose pole is K. Now, since AM is parallel to OI, and AN to OK, it follows (9.2 sup.), that the angle MAN = IOK, the inclination of the proposed circles.

Lastly, the points M, N, being in the plane of projection, let the triangle MAN revolve on MN, till the point A falls into the same plane, and its position will evidently be the point where the projected circles intersect each other; and as the radii of those circles make, at the point of intersection, the same angle as the original circles on the surface of the sphere, the projected circles themselves make the same angle (cor., def. 8). *Q. E. D.*

Art. 148. If a tangent to an original circle be projected, the projected tangent will be a tangent to the projected circle, provided the original circle does not pass through the projecting point.

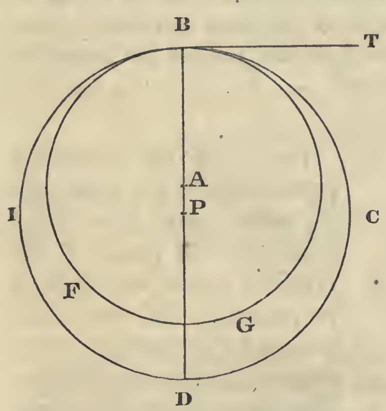
A projected circle is the intersection of the plane of the primitive and a cone, whose vertex is the projecting point, and base the original circle. If a plane be supposed to pass both through the projecting point and a tangent to the original circle, this plane will evidently touch the surface of the cone; and the intersection of this plane and the plane of projection will be a tangent to the intersection of the cone and the plane of projection. But the former of these intersections is the projected tangent, and the latter the projected circle.

Q. E. D.

Cor. If two original circles have a common tangent, the projections of these circles will have their radii drawn to the point of contact, in the same straight line.

For the radii of both the projected circles is in a line drawn through the point of contact at right angles to the projected tangent.

Art. 149. The angle made by any two circles on the surface of the sphere, is equal to the angle made by their representatives on the plane of projection.



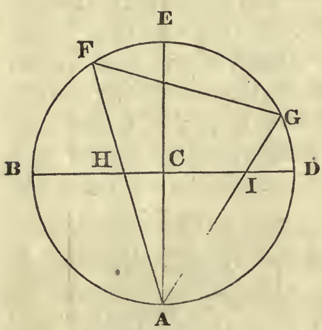
Let DIBC be a great circle of the sphere; BT, a straight line touching it at the point B; through BT let another plane pass, cutting the sphere in the circle BGF; this circle is, by Def. 1, Art. 45, a less circle. As BT is a tangent to the great circle, it is a tangent to the sphere, and therefore to the circle BGF. Consequently, if these circles are projected, their projections will have their radii, which are drawn to the point of contact, in the same straight line (Cor., Art. 148). If, then, through the point B, another great circle and a less one, having a common tangent, be supposed to pass, these circles, when delineated on the plane of projection, will have their radii, which are drawn to the point of contact, also in the same straight line.

But (Art. 147) the angle made by two great circles on the surface of the sphere, is equal to the angle made by their representatives on the plane of projection, or by the radii of those representatives drawn to the point of intersection. And (def. 8) the angle made by two circles is the angle made by their tangents passing through the point of intersection; hence it is obvious that the angle made by the two less circles, or by either of them, with a great circle touching the other at the point of intersection, is equal to the angle made by the two great circles.

Therefore the truth of the proposition is manifest.

Q. E. D.

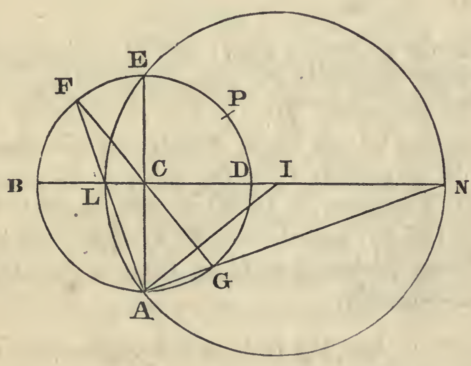
Art. 150. The extremities of the diameter of a projected circle are in the line of measures, distant from the centre of the primitive, the semitangents of the least and greatest distances of the original circle from the pole of the primitive opposite to the projecting point.



Let A be the projecting point; ABED, the great circle whose plane is at right angles to the plane of the primitive and of the circle to be projected; BD the primitive, and FG the circle to be projected, both seen edgewise. Then the right line BD, which is the common section of the plane ABED and the plane of the primitive, is the line of measures of the circle FG (Art. 136, Def. 6). As the extremities F and G are projected into H and I, the line IH, which is in the line of measures BD, is evidently the diameter of the projected circle. Also, E being the pole opposite to the projecting point, EF and EG are the least and greatest distances of the circle FG from that pole; and CH, CI are the semitangents of EF, EG (Art. 142).

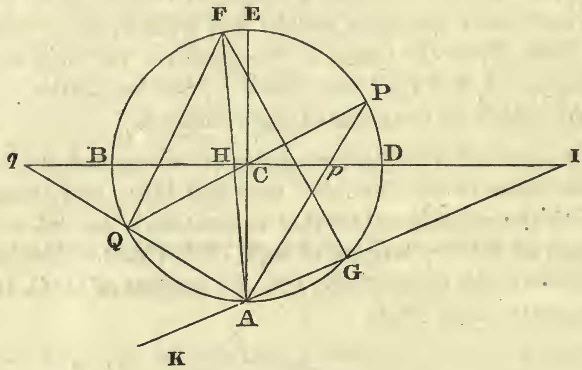
Cor. 1. The points where a projected oblique great circle cuts the line of measures, within and without the primitive, are distant from the centre of the primitive the tangent and cotangent of half the complement of the inclination of the original circle to the plane of the primitive.

The angle BCF is the inclination of the original circle FG to the plane of the primitive (see Art. 146); and therefore FCE is the complement of that inclination. But CL is the tangent of LAC = tangent of half FCE (20.3). Also, since FAG is a right angle (31.3), CN, the tangent of CAG, is the cotangent of half FCE.



Cor. 2. The centre of a projected circle is in the line of measures, distant from the centre of the primitive, half the difference of the semitangents of the greatest and least distance from the pole opposite to the projecting point, when the circle encompasses that pole; but half the sum of the semitangents, when the circle lies wholly on one side of the pole.

Art. 151. Any circle and its poles being projected on the plane of the primitive, the segments of the diameter intercepted between its extremities and one projected pole, have to each other the same ratio as the segments between the same extremities and the other pole.



Let A be the projecting point; ABED, as before, the great circle at right angles to the plane of projection and of the circle to be projected; BD, the line of measures; FG, the common section of the plane ABED and the plane of the circle to be projected; P, Q, the poles of the same circle. Then PQ is a diameter to ABED at right angles to FG (Art. 45, Cor. to Def. 3); and the arc PF is equal to PG. Hence F, P, G, being projected to H, p, I, the angle H$\text{A}p$ = I$\text{A}p$ (21.1); consequently,

$$\text{I}p : \text{H}p :: \text{IA} : \text{HA} \ (3.6).$$

Again, producing GA to K, and joining QF, the angle QAK = QFG (22.3 and 13.1) = QAF (21.3), because QG = QF. Hence (A. 6),

$$\text{I}q : q\text{H} :: \text{IA} : \text{AH};$$

consequently, $$\text{I}p : \text{H}p :: \text{I}q : q\text{H}.$$

Cor. Hence, of the two segments into which the diameter is divided by the projected pole, the greater is that which is more remote from the centre of the primitive circle;

and therefore the centre of the projected circle is further from the centre of the primitive, than the projected pole.

Art. 152. The projected poles of any circle are in the line of measures, within and without the primitive; and distant from its centre, the tangent and cotangent of half its inclination to the primitive.

Retaining the construction of the last article, the angle ECP = the inclination of the primitive to the circle, whose poles are P and Q (Art. 47, Cor.); and these poles being in the circle ABED, are projected to p and q in the line of measures BD. But

$$Cp = \tan CAp = \tan \tfrac{1}{2}ECP \text{ (20.3)};$$

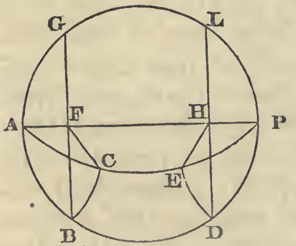
and pAq being a right angle (31.3),

$$Cq = \text{cotan } CAp = \text{cotan } \tfrac{1}{2}ECP.$$

Cor. The projected pole of the primitive is its centre; and the projected pole of a right circle lies in the primitive.

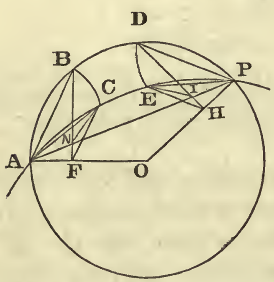
Art. 153. If two planes cut the sphere, and also intersect each other, and from the points where their common section meets the spherical surface, taken as poles, two circles be described at equal distances from those poles; the arcs of these circles, intercepted between the cutting planes, on the same side of the common section, are equal to each other

Case 1. When the cutting planes both pass through the centre of the sphere.



Let ABDPLG, ACEP, be the cutting planes; AP, their common section; BC, DE, the intercepted arcs; BF, CF, DH, EH, the common sections of their planes and the cutting planes. It is to be proved that BC=DE. Because the cutting planes pass through the centre of the sphere, their common section AP is a diameter to each of the circles formed by the spherical surface and the cutting planes. And since A, P, are the poles of the circles BC and DE, the line AP is perpendicular to the planes of those circles (Art. 45, Def. 2); therefore, AFB, AFC, PHD, PHE are right angles (def. 1.2 sup.); whence FB is parallel to HD, and FC to HE (28.1); consequently, the angle BFC = DHE (9.2 sup). The lines BF, CF, DH, EH, are also equal, because they are sines of equal arcs; wherefore, BC = DE (26.3).

Case 2. When one of the cutting planes passes through the centre of the sphere, and the other does not.



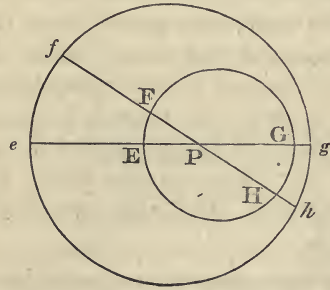
Let ABDP be the common section of the spherical surface and the cutting plane, which passes through the centre; ACEP, the common section of the same surface and the other plane; AP, the common section of the planes; BC and DE, as before, the intercepted arcs of the circles described from A and P; O, the centre of the sphere, and consequently of ABDP; BF, DH, the common sections of the planes of the circles BC, DE, and the plane of ABDP; N, I, the intersections of BF, DH, with the line AP. Join AO, PO, FC, NC, HE, IE, AC, PE; then, as in the first case, BF, CF, DH, EH, being sines of equal arcs, are equal to each other; and AFB, PHD, right angles: AF, PH, are also equal, being versed sines of equal arcs. Now, in the triangle AOP, the side AO = OP; wherefore, OAP = OPA; then, in the triangles AFN, PHI, we have AF=PH; the angle FAN=HPI, and AFN=PHI; whence (26.1), AN = PI, and FN=HI. Then, in the triangles ANC, PIE, we have AC = PE (29.3), AN = PI, and the angle NAC = IPE

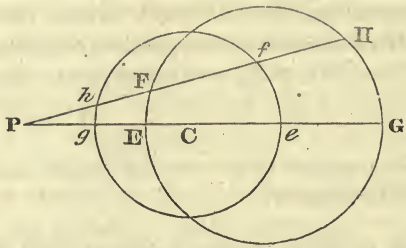
(21.3); consequently, NC = IE (4.1). Lastly, in the triangles CFN, EHI, the sides are respectively equal; wherefore the angle NFC = IHE (8.1), and consequently the arc BC = DE (26.3).

Case 3. When neither of the cutting planes passes through the centre of the sphere.

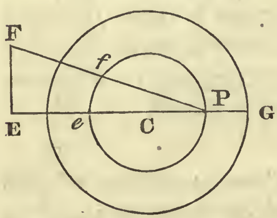
Through the common section of these planes and the centre of the sphere, let a third plane pass; then, by the last case, the arcs of one of those equidistant circles, intercepted between the third plane and each of the others, are respectively equal to the corresponding arcs of the other similarly intercepted: and, therefore, their sums or differences are also equal. But, when the third plane passes between the other two, the sum of the arcs contained between it and the other planes is the arc in question. When it passes on the same side of them, the difference is the arc proposed.

Art. 154. Let EFGH, *efgh*, be the projections of two equal circles, of which EFGH is as far from its pole P as *efgh* is from the projecting point; then any two right lines EP, FP, drawn through P, will intercept the representatives of equal arcs of those circles; on the same side, if P falls within the circles; but on the contrary side, if it falls without; that is, EF = *ef*, and GH = *gh*.





For (Art. 153) two planes passing through the projecting point and the pole of the original circle, which is represented by EFGH, will cut off equal arcs from those circles. And those planes will (Art. 143) be projected into right lines, which will evidently pass through the projected pole P.



Cor. 1. Let a circle be projected into a right line EF at right angles to the line of measures EG; and from C, the centre of the primitive, let a circle be described through P, the projected pole of EF; then any two lines PE, PF, will cut off from the circle an arc, *ef*, containing the same number of degrees as the arc which is represented by EF. And the arc, intercepted between PE and PF, of any other circle which passes through P, will contain the same number of degrees.

For any circle which is projected into a right line, must pass through the projecting point (Arts. 143, 144); and, therefore, the distance of that circle from its pole is the same as the distance of the pole from the projecting point. Consequently, the projected circle through P represents an original circle, as far from the projecting point as the circle which is projected into EF is from its own pole. Hence, EF and *ef* represent equal arcs. The latter part of the corollary is evident from 26.3.

Cor. 2. If two right lines be drawn through the projected pole of a great circle, the intercepted arc of that circle will contain the same number of degrees as the intercepted arc of the primitive.

For any great circle is distant 90° from its pole; and the primitive is 90° from the projecting point.

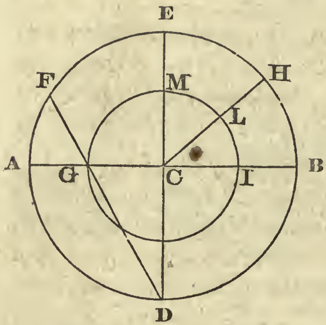
Cor. 3. If, from the point where two projected great circles cut each other, two right lines are drawn through the projected poles of those circles, the intercepted arc of the primitive circle will measure the spherical angle made by those circles at the point of their intersection.

For the arc of a great circle, contained between the poles of two other great circles, is the measure of the angle which the axes of those circles make with each other; and that angle is the same as the inclination of the planes of those circles (Cor., Art. 47).

Scholium. If the circles of the sphere were to be projected on a plane parallel to the plane of the primitive, the projections would be similar to those on the plane of the primitive itself; for the projecting line, when carried round on the circumference of a circle which does not pass through the projecting point, forms a conical surface; and that surface being cut by the plane of the primitive, and by any other plane parallel thereto, the sections are similar, but of greater or less dimensions. Thus projecting on the plane of a less circle parallel to the primitive, instead of projecting on the primitive itself, would be only changing the scale. In the subsequent parts, however, of this section, the plane of the primitive will be used.

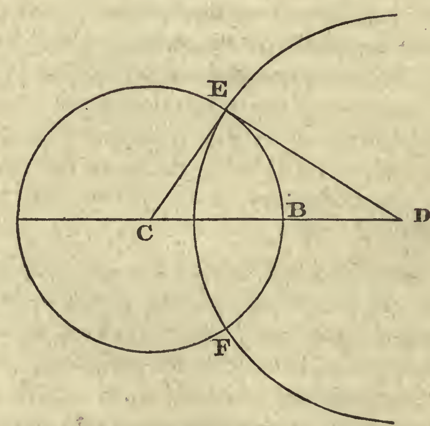
In the following problems, the primitive circle is always supposed to be described with the chord of 60°; and the secant, tangent, and semitangent referred to, are such as correspond to the scale used for the primitive. These different lines are frequently marked by the side of the scale of chords, on the small scales introduced into boxes of mathematical instruments; but more frequently on the foot or two feet scales, which contain Gunter's lines.

Art. 155. Problem. To describe a circle parallel to the primitive, at a given number of degrees from its pole.



From the centre of the primitive, with a radius equal to the semitangent of the given distance of the circle from its own pole, describe the circle required. Or draw the diameters AB, DE at right angles to each other; and from the extremity E of one of them, lay off the proposed number of degrees on the primitive as EF; join DF, cutting AB in G; from the centre C, at the distance CG, describe the circle required.

The radius CG is the tangent of CDG, or semitangent of EF, as it ought to be (Cor., Case 1, Art. 144).

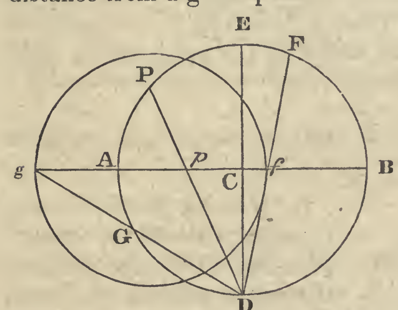


ART. 156. Prob. To describe a less circle at right angles to the primitive, and at a given distance from its own pole.

Let B be the pole of the circle proposed; through C, the centre of the primitive, and the pole B, draw the right line CD; from C to D, lay down the secant of the given distance; from the centre D, with a radius equal to the tangent of the same distance, describe the circle proposed. Or, from B, lay down BE on the primitive; join CE; draw ED touching the circle; from D, with the distance DE, describe the circle proposed.

It is obvious that DE is the tangent, and CD the secant, of BE, as they ought to be, Art. 145.

ART. 157. Prob. To describe an oblique circle at a given distance from a given pole.



Let p be the projected pole; through p draw the line of measures ApB; apply Cp, the distance of the given pole from the centre of the primitive, to the line of semitangents; and, having found the number of degrees, thus measured, in Cp, take the sum and difference of this number and the distance of the proposed circle from its own pole; lay down these results taken from the semitangents, on the line of measures, from C to g and f; on the diameter fg describe the circle required. Or, having drawn the line of measures, draw the diameter DCE at right angles to it; draw Dp to meet the primitive in P; from P, lay down on the primitive PF, PG, each equal to the given distance of the circle from its pole; draw DF, DG, cutting the line of measures in f and g; on fg describe a circle; it will be the circle required.

This construction follows from Art. 150.

Scholium. This method is applicable to great circles as well as less; but the former cases are conveniently managed by other methods hereafter given.

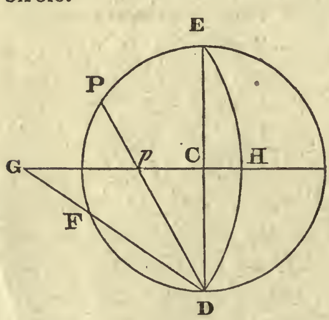
Art. 158. Prob. To describe a great circle, the projected pole of which is given in position.

Case 1. When the given pole is in the primitive circle.

Through the given pole draw the line of measures; and at right angles thereto draw a diameter to the primitive circle; this diameter will represent the circle proposed.

Because the pole is in the primitive, the original circle is at right angles to the primitive (Art. 45, Cor., Def. 3); and, being a great circle, it must pass through the poles of the primitive. Consequently (Art. 143, Cor. 1), it is represented by a right line through the centre of the primitive, at right angles to the line of measures.

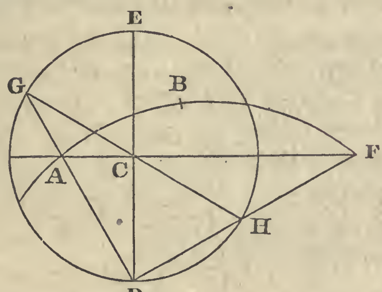
Case 2. When the given pole is within the primitive circle.



Let p be the projected pole; through p draw the line of measures CpG; apply Cp to the line of semitangents; take CG equal to the tangent of the number of degrees in Cp; from G, as a centre, with the secant of the same number of degrees, describe the circle DHE, which will be the one required.

Or, draw the diameter DCE at right angles to the line of measures; join Dp, and produce it to the circumference in P; make PF = PE; draw DF cutting the line of measures in G; from the centre G, with the radius GD, describe the circle DHE, the circle proposed. The line Cp is the projection of an arc of a great circle, intercepted between the pole of the primitive and the pole of the circle proposed; and that arc measures the inclination of those circles (Art. 47, Cor). The arc is also projected into a line of semitangents (Art. 143, Cor. 2); hence Cp, measured on the semitangents, or the arc PE, indicates the same inclination. But, by the construction, CG is the tangent and GD the secant of PE; consequently (Art. 146), DHE is the circle, whose pole is p.

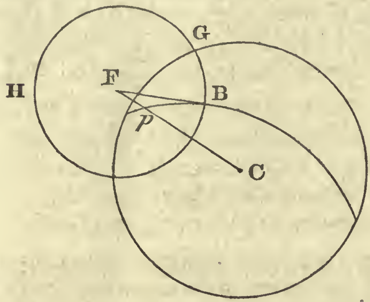
Art. 159. Prob. Through two given points, to describe a great circle.



Let A, B be the given points; through the centre of the primitive and one of the given points draw the right line ACF. If that line passes through B, the business is done; for ACF is the projection of a great circle at right angles to the primitive (Art. 143, Cor. 1). But if ACF does not pass through B, apply CA to the line of semitangents, and make CF the semitangent of the supplement of CA; through ABF describe a circle, and the thing is done. Or, draw the diameter DCE at right angles to ACF; draw DAG cutting the primitive in G; draw the diameter GCH; join DH; and let DH, produced if necessary, cut AF in F; and through ABF describe the circle ABF, as before.

From the construction, it is obvious that AF is the projection of a semicircle; consequently, any circle which passes through A and F must be a great one (converse to Cor. 3, Def. 1, Art. 45).

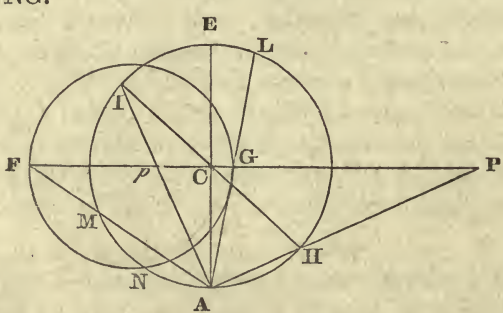
ART. 160. Prob. About a given pole, to describe a circle through a given point.



Let p be the given pole and B the given point; through p and B describe a great circle (Art. 159); and draw BF touching it in B (17.3); draw, from the centre of the primitive, the right line CpF, meeting the tangent in F; from the centre F, at the distance FB, describe the circle BGH; and the work is done.

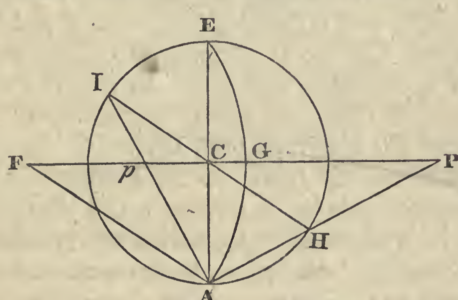
The centre of a circle whose pole is p is in the line of measures CpF (Cor. to Case 2, Art. 144). The circle pB, passing through the pole p of the proposed circle, is at right angles to it (Cor. to Def. 3, Art. 45); hence the radii of their projections, drawn to the point of their intersection, must also be at right angles to each other (Art. 149); consequently, the centre of the required circle is in the tangent BF (18.3); it is, therefore, at the intersection of Cp and BF.

ART. 161. Prob. To find the poles of a given projected circle FNG.

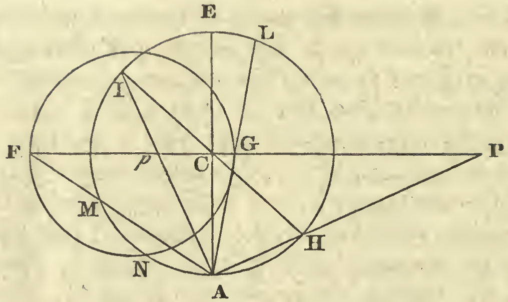


Through the centre of the given circle and centre C of the primitive, draw the right line FCGP, cutting the given circle in F and G. Measure CF and CG on the line of semitangents; take the half sum or half difference of these measures, according as F and G are on the same or opposite sides of C, and lay its semitangent from C to p; then is p one of the poles required: observing, however, that p must be on the same side of C as the centre of the given circle. Lay down CP from the semitangents equal to the supplement of Cp, and on the opposite side of C; then is P the other pole required. Or, draw the diameter ACE at right angles to FG; draw also AG, AF, cutting the primitive in L and M; bisect LM in I; join AI, cutting FG in p; draw the diameter IH to the primitive circle; draw AH, produced if necessary, to meet FG produced in P; then p and P are the poles required.

When the circle is a great one, as AGE, the poles may be found with more facility in a different manner.

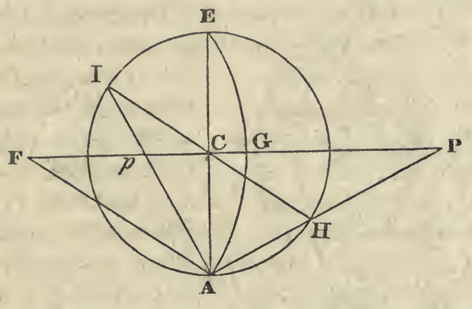


Draw FC from the centre of the given circle to the centre of the primitive, and produce it. Measure CF on the line of tangents: take half the number of degrees thus found, and lay down the tangent of this result from C towards F to p; and its cotangent in the opposite direction to P; then will p and P be the poles required. Or, draw the diameter ACE at right angles to FC; join AF; bisect the angle CAF by the line ApI, cutting FC in p and the primitive circle in I; draw the diameter IH and the line AHP as before; then p and P are the poles required.



In the case of the less circle, CG is the semitangent of EL, and CF the semitangent of EM; consequently (Art. 150) EL and EM are equal to the least and greatest distances of the original circle represented by FNG, from the pole of the primitive opposite the projecting point. Hence EI, half the difference of EM and EL, when F and G are on opposite sides of C (as in the figure), or half the sum when F and G are on the same side, is the distance of the pole of the original circle from the pole of the primitive. C*p*, the semitangent of EI, is therefore the distance of the projected pole of FNG from the centre of the primitive (Art. 142). And the projected pole lies in the line which joins the centre of the projected circle and the centre of the primitive (Art. 144, Cor. to Case 2). Also, *p* and P are on the same projected great circle, at the distance of 180°; hence, P is the pole opposite to *p*.

In the case of the great circle, CF is the tangent of CAF, the inclination of AGE to the primitive (Art. 146, Cor. 1). But, by the construction, ECI = CAF; consequently, the arc EI measures the distance of the pole of the original circle, represented by AGE, from the pole of the primitive (Cor. to Art. 47). Hence C*p*, the semitangent of EI, is equal to the distance of the projected pole of AGE from the centre of the primitive (Art. 142).



ART. 162. Prob. To describe a great circle making a given angle with the primitive at a given point A.

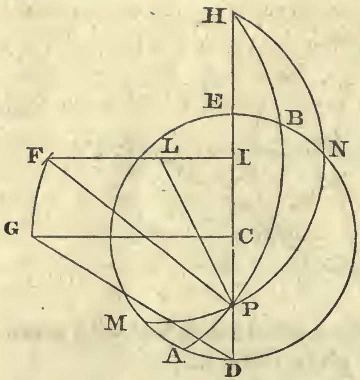
Draw through the given point the diameter ACE; and through C, the centre of the primitive circle, draw a line CF at right angles to AE; make the angle CAF = the angle proposed; and from F, the intersection of AF, and CF, describe the circle AGE; which will be the circle required. Or, from the centre of the primitive, with the tangent of the given angle, describe an arc; from the point A, with the secant of the same angle, describe an arc cutting the former · and from the point of intersection describe, through the point A, the circle AGE.

This construction is obvious from Art. 146.

When the given angle is a right one, the lines AF and CF are parallel; hence, in that case, the centre of the required circle is at an infinite distance; consequently, the circle becomes a right line passing through the centre of the primitive. See Cor. 1, Art. 143.

ART. 163. Prob. Through a given point P, to describe a great circle, making a given angle with the primitive.

From the centre of the primitive, with the tangent of the given angle, describe an arc; from the given point, with the secant of the same angle, describe an arc cutting the former in F; from the centre F, through P, describe the circle APB; this will be the circle required.



Or, through P draw the diameter DE; at the centre C erect a perpendicular CG; make the angle CDG = the given angle; from C, through G, describe an arc; and from P, with the distance DG, describe another, cutting the former in F; then F is the centre of the required circle.

This, like the last, depends upon Art. 146.

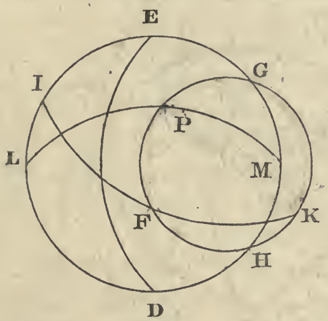
N. B. If the circles described from C and P do not meet, the problem is then impossible; and this case occurs when the required angle is less than that which would be measured by PD, taken on the scale of semitangents from 90° towards the beginning of the scale.

Art. 164. Prob. To describe a great circle, making at a given point P a given angle with a given great circle APB.

Through the given point P draw the diameter DE, meeting the circle again in H; find F the centre of the given circle; draw FI at right angles to DE; join PF; make the angle FPL = the given angle; then, from L, describe the circle MPN; this is the circle required.

The line FI, being at right angles to PH, bisects it (3.3); consequently, every circle whose centre is in FI, and which passes through P, will also pass through H. Now, as APB and DCE are the projections of great circles, PBH is the projection of a semicircle (Cor. to Def. 3, Art. 45); hence, any other circle passing through P and H must be a great circle. MPN is therefore a great circle; also (Art. 147) the angle BPN = FPL.

Art. 165. Prob. Through a given point P, to describe a great circle making a given angle with a given great circle DE.



Find the pole of the given circle (Art. 161); about that pole describe a circle GFH at a distance equal to the measure of the given angle (Arts. 155–6–7); about the point P as a pole, describe a great circle IFK, cutting GFH in F and K (Art. 158); about one of those points F as a pole, describe the great circle LPM; and the work is done.

Since P is the pole of IFK, every point in that circle is 90° from P; consequently, the great circle whose pole F is in IFK, must pass through P. And as the distance of the poles of two great circles is the measure of the angle which those circles make with each other, the construction is manifest.

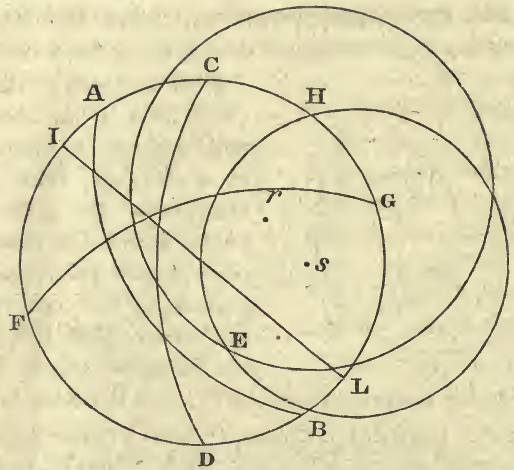
If the given angle is a right one, the circle must be described through P and the pole of DE, by Art. 159.

Art. 166. Prob. To describe a great circle making given angles with two given great circles AB and CD. See fig. on page 200.

Find the poles r, s of the given circles (Art. 161); describe about r and s less circles, at distances respectively equal to the measures of the given angles (Art. 157); from the intersection E of these circles, as a pole, describe the great circle FG (Art. 158); that circle is the one required.

This construction evidently depends upon the principle, that the distance between the poles of two great circles is the measure of their inclination (Cor. to Art. 47).

If the circle to be described is to be at right angles to each of those which are given, it must be described through their poles, by Art. 159.



Art. 167. Prob. To describe a right circle (that is, a great circle at right angles to the primitive) making a given angle with a given great circle CD.

Find s, the pole of CD; about s describe a circle at a distance equal to the measure of the given angle; from the point H, where this circle cuts the primitive, lay down HI, on the primitive, $= 90°$; through I draw the diameter IL, the right circle required.

Every great circle at right angles to the primitive is projected into a right line through its centre (Art. 143, Cor. 1).

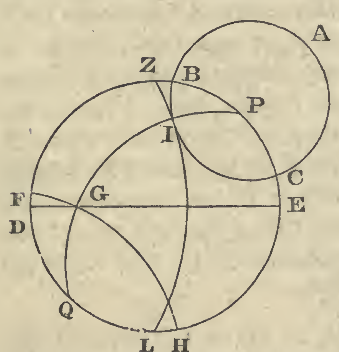
[*Note.* There is a limit in this and the last article. If, in Art. 166, the circles about r and s do not meet; or, in this article, if the circle about s does not meet the primitive; the problem is impossible.]

Scholium. When the proposed angle is a right one, lay down 90° from C on the primitive circle; and through the point thus found draw a diameter for the right circle required.

Art. 168. Prob. Through a given point Z, to describe a great circle which shall touch a given less circle ABC.

From Z, as a pole, describe the great circle DGE (Art

158); find P the internal pole of ABC (Art. 161), and Q the opposite pole; about Q describe a circle FGH, at a distance from Q equal to the complement of PB, the distance of ABC from its own pole (Arts. 155–6–7); from the point G, where these circles cut each other, taken as a pole, describe the great circle ZIL (Art. 158); this circle will touch the given circle ABC. Through PGQ describe a circle cutting ABC in I; this will be a great circle, because P and Q are opposite poles, and PI + QG are by construction = 90°; hence GI = 90°: and G being the pole of ZIL, that circle must pass through I, and make the angle ZIP a right angle. Hence (Art. 55) PI is less than any other arc of a great circle contained between P and ZIL; therefore, those circles touch each other at I.

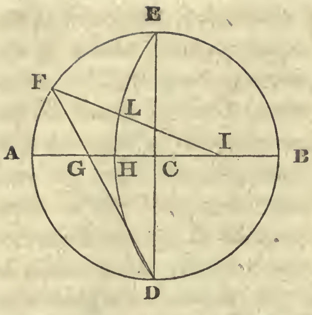


ART. 169. Prob. To lay down a given number of degrees on a given great circle, or to measure an arc of it.

Case 1. When the given circle is the primitive, lay down or measure the arc by the scale of chords.

For the primitive is an original circle, and is therefore measured as in common Geometry.

Case 2. When the given circle is a right one, that is, one passing through the centre of the primitive, lay down or measure the arc on the scale of semitangents (Art. 143, Cor. 2), observing that an arc beginning at the centre of the primitive must be measured from the beginning of the scale; but one beginning at the primitive must be measured from the 90th degree on the scale towards the beginning or end of the scale, according as the arc extends *towards* or *from* the centre of the primitive.



Or, let ACB be the right circle; draw the diameter DCE at right angles to AB; then, to lay down any proposed number of degrees from A or C, lay them on the primitive from A or E to F; join DF, cutting AB in G; or, to measure AG or CG, draw DG to cut the primitive in F; then AF is the measure of AG, and EF of CG.

Suppose the figure to revolve on AB till CD becomes perpendicular to the plane of projection; then is D the projecting point, and AEB the semicircle passing through the pole of the primitive; consequently, AG is the projection of AF, and CG of EF.

Case 3. When the given circle is an oblique one.

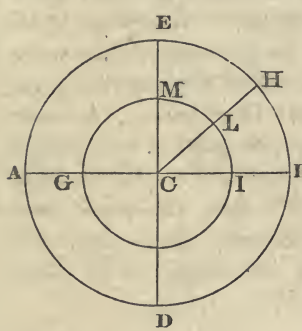
Let DHE be the circle; find its internal pole I (Art. 161); then, to lay down an arc HL or EL, lay the proposed number of degrees on the primitive from A or E to F, and join IF, cutting the given circle in L, the point required; or, to measure HL or EL, join IL, and produce it to F in the primitive; then AF is the measure of HL, and EF of EL.

The primitive circle is as far from the projecting point, as DHE is from its pole; therefore (Art. 154), the right lines IA, IF, cut off corresponding arcs AF, HL.

Art. 170. Prob. To lay down any proposed number of degrees on a less circle, or to measure a given arc of it.

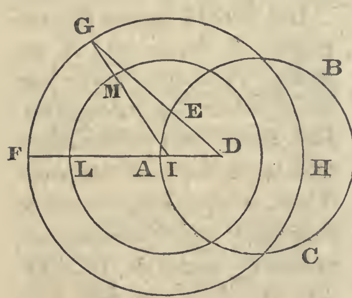
Case 1. When the given less circle is parallel to the primitive.

Lay the proposed number of degrees on the primitive circle; and through the extremities of the arc draw right lines to the centre; the intercepted arc of the less circle is that proposed; or, to measure the arc, draw right lines from



the centre, through its extremities, to the primitive, and measure the intercepted arc of the latter. Thus BH is the measure of IL, and EH of ML; for the projected less circle parallel to the primitive is formed by cutting the conical surface by a plane parallel to the base; consequently, the projected circle differs from its original in nothing but its dimensions.

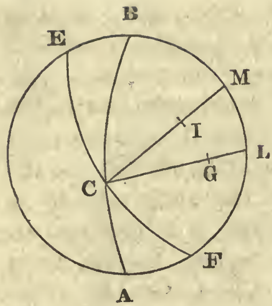
Case 2. When the circle is not parallel to the primitive.



Let ABC be the less circle; find its pole D (Art. 161); describe a circle FGH, as far from the projecting point as ABC is from its pole (Art. 155); then any arc of ABC may be laid down or measured by the aid of FGH as an arc of a great circle is, by means of the primitive in Art. 169, Case 3; observing that an arc of FGH is laid down or measured as directed in Case 1. Thus, I being the centre of the primitive, ML, a part of its circumference, is the measure of FG; and FG the measure of AE (Art. 154).

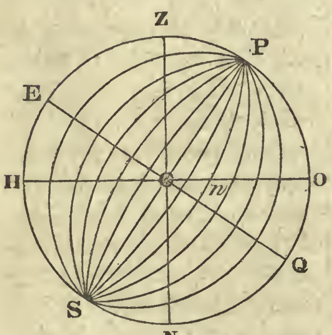
Art. 171. To measure the angle made by two great circles whose position is given.

Find the poles of the circles (Art. 161); from the angular point, through those poles, draw two right lines; and the intercepted arc of the primitive is the measure required.



Let ACB, ECF, be the circles; G, I, their poles; then the arc of a great circle, contained between I and G, would measure the angle ACF (Art. 154, Cor. 3). Or, draw lines from the point of intersection C to the centres of the circles ACB and ECF; then the angle contained between these lines will measure the angle ACF (Cor., Def. 8).

Art. 172. Prob. To form a general projection of the sphere on the plane of a meridian.*



Let ZONH denote the meridian; Z, the zenith; N, the nadir; P, S, the north and south poles; EQ, the equator; HO, the horizon; then ZE = OP, the latitude of the place. Then, circles being described through P and S, making successively angles of 15°, 30°, 45°, 60°, 75° and 90°, with the primitive; these will be the meridians, or hour circles, for the different hours.

A few examples will now be given to exercise the student in Spherical Projections and Calculations.

Of Rectangular Spherical Triangles.

1. Given, the hypothenuse 70° 15′, and the adjacent angle 30° 30′, to find the rest.

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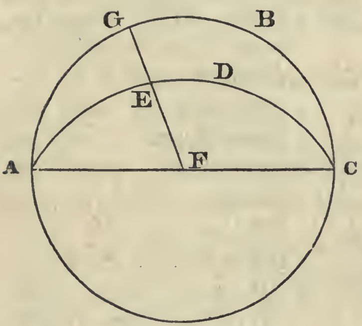
* See Definitions, page 217.



Construction. Describe the primitive circle ABC, and the oblique great circle ADC, making the given angle with the primitive at the point A (Art. 162); on AD lay AE equal to the given hypothenuse (Art. 169); and through F, the pole of the primitive, and the point E, describe the great circle FEG, cutting the primitive circle in G; then AGE is the triangle proposed, of which G is the right angle.

Calculation.

As rad : cos A :: tan AE : tan AG (Art. 62) $= 67° \; 23'$.

As rad : sin A :: sin AE : sin EG (Art. 58) $= 28° \; 32'$.

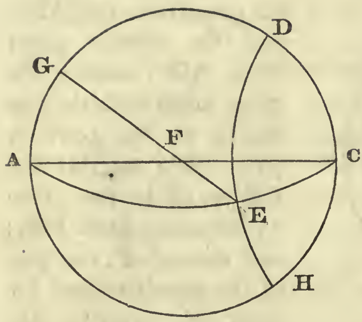
As rad : cos AE :: tan A : cot AEG (Art. 64) $= 78° \; 45'$.

It is obvious, from the construction, that there is no ambiguity in this problem; for the points E and F being given, the great circle passing through them can have but one position.

Again, the side EG is of the same affection as the angle A (Art. 56); also, AG is of the same affection with EG, or a different one, according as AE is less or greater than a quadrant (Art. 57); and the angle AEG is of the same affection as AG (Art. 56).

2. Given, the hypothenuse 125° 25′, and one leg 37° 40′, to find the rest.

Construction. Having described the primitive circle ABC, lay AG on it equal the given leg; through G and the

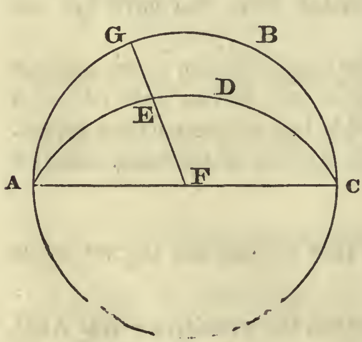


pole F, draw the right circle GFE; from the point A, as a pole, at a distance equal to the hypothenuse (or from the opposite pole C with its supplement), describe a less circle DEH (by Art. 156), cutting GFE in E; through A and E describe the great circle AEC (Art. 159); then AGE is the triangle proposed.

Calculation.

As tan AE : tan AG :: rad : cos EAG (Art. 62) = 123° 18′.
As sin AE : sin AG :: rad : sin AEG (Art. 58) = 48° 34′.
As cos AG : cos AE :: rad : cos EG (Art. 65) = 137° 4′.

In this problem there is no ambiguity; for the angle AEG is of the same affection with the side AG (Art. 56); EG is of the same affection with AG, when AE is less than a quadrant, and of a different one when AE is greater (Art. 57); and the angle EAG is of the same affection as EG (Art. 56).



3. Given, one leg 75° 26′, and the adjacent angle 40° 10′, to find the rest.

Construction. Describe the primitive circle ABC, and the oblique great circle AEC, making BAC equal to the given angle (by Art. 162); on the primitive lay AG equal the given leg; and through

G and the pole of the primitive, draw the right circle GEF, cutting AC in E; then is AGE the triangle in question.

Calculation.

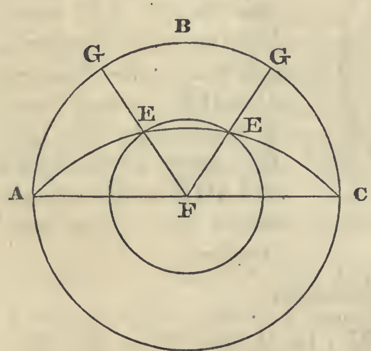
As rad : sin AG :: tan A : tan EG (Art. 60) = 39° 15′.

As cos A : rad :: tan AG : tan AE (Art. 62) = 78° 46′.

As rad : sin A :: cos AG : cos AEG (Art. 67) = 80° 40′.

This problem includes no ambiguous case; for the side EG and the angle at E are respectively of the same affection with the angle A and the side AG (Art. 56); and the hypothenuse is less or greater than a quadrant, according as AG and GE are of the same or different affections (Art. 57).

4. Given, one leg 36° 45′, and the opposite angle 42° 16′, to find the rest.



Construction. Describe the primitive circle ABC, and the oblique great circle AC, making at the point A an angle equal to the given one (Art. 162); about the pole F of the primitive, at a distance equal to the complement of the given leg, describe a less circle, cutting the oblique circle AC in E (Art. 155); through E and the pole F describe the great circle GEF (Art. 159), cutting the primitive in G; then AGE is the triangle proposed.

Calculation.

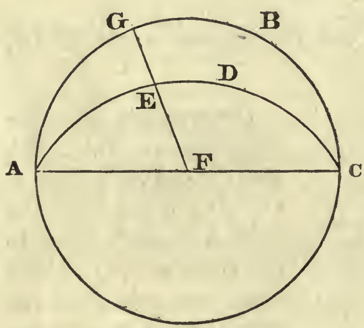
As sin A : sin EG :: rad : sin AE (Art. 58) = $\begin{cases} 62° \ 49′. \\ 117° \ 11′. \end{cases}$

$$\text{As } \tan A : \tan EG :: \text{rad} : \sin AG \text{ (Art. 60)} = \begin{cases} 55° \; 15'. \\ 124° \; 45'. \end{cases}$$

$$\text{As } \cos EG : \cos A :: \text{rad} : \sin AEG \text{ (Art. 67)} = \begin{cases} 67° \; 27'. \\ 112° \; 33'. \end{cases}$$

The cases contained in this problem are ambiguous, as is obvious from the construction; the unknown sides and angle being susceptible of two values, which are supplemental to each other.

5. Given, the two legs, 70° 29′ and 30° 16′, to find the rest.



Construction. On the primitive circle lay down AG equal to one of the legs; and through G and the pole of the primitive describe the right circle GF; on which lay down GE equal to the other leg (Art. 169); through A and E describe (by Art. 159) the great circle AEC; and AGE is the triangle which was to be constructed.

Calculation.

As sin AG : rad :: tan EG : tan A (Art. 60) = 31° 46′.

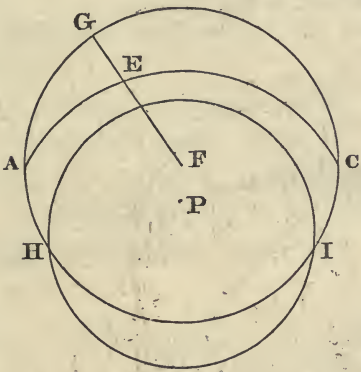
As sin EG : rad :: tan AG : tan E (Art. 60) = 79° 52′.

As rad : cos AG :: cos EG : cos AE (Art. 65) = 73° 14′.

This problem contains no ambiguity; for the angles at A and E are of the same affections as the opposite sides (Art. 56); and AE is less than a quadrant, when AG and GE are of the same affection (Art. 57).

6. Given, the two oblique angles, 28° 19′ and 75° 15′, to find the sides.

Construction. Describe the oblique great circle AC, making with the primitive at A an angle equal to one of those given (Art. 162); about the pole P of this circle, at a distance equal to the measure of the other given angle, or of its supplement if obtuse, describe a less circle cutting the primitive in H and I (Art. 157); from H nearest to A, when the second angle is acute, or from I the more remote point of intersection, when the angle is obtuse, lay down HG on the primitive equal a quadrant; through G describe a great circle at right angles to AG, cutting AC in E; then AGE is the triangle proposed. For, H is evidently the pole of GEF; and the distance HP is the measure of the angle AEG.



Calculation.

As tan A : cotan E :: rad : cos AE (Art. 64) = 60° 45′.

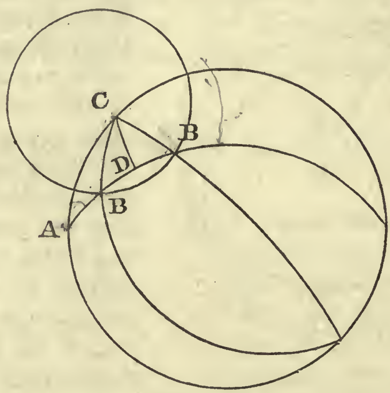
As sin A : rad :: cos E : cos AG (Art. 67) = 57° 32′.

As sin E : rad :: cos A : cos EG (Art. 67) = 24° 27′.

This problem contains no ambiguity; for the sides AG and EG are of the same affection as the opposite angles (Art. 56); and when those sides, or their opposite angles, are of the same affection, the hypothenuse is less than a quadrant (Art. 57).

Of Oblique Angled Spherical Triangles.

1. Given, two sides, AC 45° 30′, BC 30° 30′, and the angle A opposite one of them, 36° 45′, to find the rest.



Construction. On the primitive circle lay down AC, one of the given sides; about the pole C describe a less circle, at the distance BC of the other given side; describe a great circle, making, at the point A, with the primitive, an angle equal to the given one (Art. 162), intersecting the less circle in B; through C and B describe a great circle (Art. 159); and ABC is the triangle to be made.

When BC, the side opposite the given angle, is the less of the two, there may be two points of intersection, and, consequently, two positions of B. Hence, in that case, the problem is ambiguous.

Calculation. Describe the great circle CD through the pole of AB; then ADC, BDC are rectangular triangles. Hence,

As rad : cos A : : tan AC : tan AD (Art. 62) = 39° 12′.

As cos AC : cos BC : : cos AD : cos BD (Art. 66) = 17° 41′.

As rad : cos AC :: tan A : cotan ACD (Art. 64)$=62° \ 22'$.

As tan BC : tan AC :: cos ACD : cos BCD (Art. 63)$=36° \ 46'$.

As sin BC : sin AC :: sin A : sin ABC (Art. 59)$=57° \ 14'$ or $122° \ 46'$.

Hence, AB $= 56° \ 53'$, or $21° \ 31'$; and ACB $= 99° \ 8'$, or $25° \ 36'$.

Or, without a perpendicular.

Find ABC as above. Then,

As cos $\frac{1}{2}$(AC — BC) : cos $\frac{1}{2}$(AC+BC) :: tan $\frac{1}{2}$(ABC+BAC) : cotan $\frac{1}{2}$ACB (Art. 77, eq. 15)$=49° \ 34'$, or $12° \ 48'$.

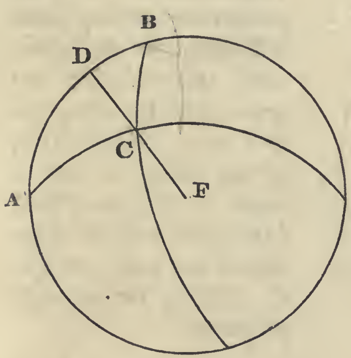
Hence, ACB $= 99° \ 8'$, or $25° \ 36'$.

And,

As cos $\frac{1}{2}$(ABC — BAC) : cos $\frac{1}{2}$(ABC+BAC) :: tan $\frac{1}{2}$(AC+BC) : tan $\frac{1}{2}$AB (Art. 77, eq. 17)$=28° \ 26\frac{1}{2}'$, or $10° \ 45\frac{1}{2}'$.

Whence, AB $56° \ 53'$, or $21° \ 31'$.

2. Given, two sides, $75° \ 20'$ and $60° \ 16'$, and the included angle $40° \ 18'$, to find the rest.



Construction. Describe a great circle, making at the given point A an angle with the primitive equal to the given one (Art. 162). Make AB, AC, respectively, equal to the given sides (Art. 169); and describe a great circle through B and C (Art. 159); then ABC is the triangle proposed.

In this problem, there is no ambiguity.

Calculation. Draw CD at right angles to AB. Then,

As rad : cos A :: tan AC : tan AD (Art. 62) = 53° 10′.

As sin BD : sin AD :: tan A : tan B (Art. 61) = 60° 56′.

As cos AD : cos BD :: cos AC : cos BC (Art. 66) = 39° 59′.

As rad : cos AC :: tan A : cotan ACD (Art. 64) = 67° 11′.

As tan AD : tan BD :: tan ACD : tan BCD (Art. 69) = 35° 57′.

Hence, ACB = 103° 8′.

Or, without a perpendicular,

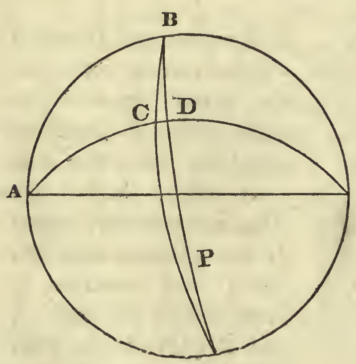
As cos $\frac{1}{2}$(AB + AC) : cos $\frac{1}{2}$(AB — AC) :: cot $\frac{1}{2}$BAC : tan $\frac{1}{2}$(ACB + ABC) (Art. 77, eq. 11) = 82° 2′.

As sin $\frac{1}{2}$(AB + AC) : sin $\frac{1}{2}$(AB — AC) :: cotan $\frac{1}{2}$BAC : tan $\frac{1}{2}$(ACB — ABC) (Art. 77, eq. 12) = 21° 6′.

Whence, ACB = 103° 8′, and ABC = 60° 56′.

As sin ABC : sin BAC :: sin AC : sin BC (Art. 59) = 39° 59′.

3. Given, one side 80° 44′, and the two adjacent angles, 40° 50′ and 70° 12′, to find the rest.



Construction. On the primitive circle, lay down AB equal to the given side; and describe two great circles, making angles with the primitive, at the points A and B, equal to the given ones (Art. 162), and let those circles cut each other in C; ABC is the triangle proposed.

Calculation. Through B, the extremity of a given side, and P the pole of AC, describe a great circle, cutting AC (produced, if necessary) in D. Then,

As rad : cos A :: tan AB : tan AD (Art. 62) = 77° 50′.

As rad : cos AB :: tan A : cotan ABD (Art. 64) = 82° 5′.

As cos CBD : cos ABD :: tan AB : tan CB (Art. 63) = 40° 49′.

As tan ABD : tan CBD :: tan AD : tan CD (Art. 69) = 7° 44′.

As sin ABD : sin CBD :: cos A : cos BCD (Art. 68) = 80° 57′.
Hence, AC = 70° 6′, and ACB = 99° 3′.

Or, without a perpendicular,

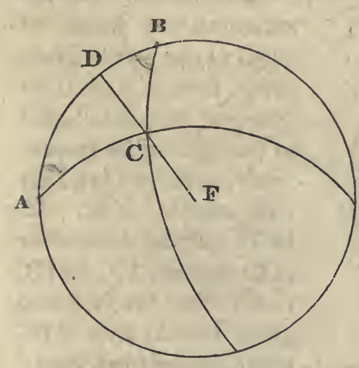
As cos $\frac{1}{2}$(B + A) : cos $\frac{1}{2}$(B — A) :: tan $\frac{1}{2}$AB : tan $\frac{1}{2}$(AC + BC) (Art. 77, eq. 13) = 55° 27′.

As sin $\frac{1}{2}$(B + A) : sin $\frac{1}{2}$(B — A) :: tan $\frac{1}{2}$AB : tan $\frac{1}{2}$(AC — BC) (eq. 14) = 14° 39′.

As cos $\frac{1}{2}$(AC — BC) : cos $\frac{1}{2}$(AC + BC) :: tan $\frac{1}{2}$(B + A) : cot $\frac{1}{2}$ACB (eq. 15) = 49° 31′.

Whence, AC = 70° 6′, BC = 40° 48′, and ACB = 99° 2′.

4. Given, two angles, 50° 16′ and 60° 36′, and a side 42° 34′, opposite one of them, to find the rest.



Construction. Describe (Art. 162) a great circle, making with the primitive, at the point A, an angle equal to the given one, to which the given side is adjacent. On that circle lay down AC equal to the given side (Art. 169); through C describe (by Art. 163) a great circle, making with the primitive an angle equal to the other given one; and let that circle cut the primitive in B: then ABC is the triangle.

Calculation. Draw CD at right angles to AB. Then,

As rad : cos A :: tan AC : tan AD (Art. 62) = $30° 25'$.

As tan B : tan A :: sin AD : sin BD (Art. 61) = $20° 4' 21''$.

As rad : cos AC :: tan A : cotan ACD (Art. 64) = $48° 27' 26''$.

As cos A : cos B :: sin ACD : sin BCD (Art. 68) = $35° 5' 9''$.

As sin B : sin A :: sin AC : sin BC (Art. 59) = $36° 39' 46''$.

Wherefore, AB = $50° 29' 21''$, and ACB = $83° 32' 35''$.

Or, without a perpendicular.

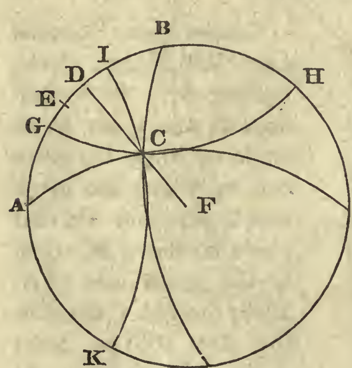
Find BC as above. Then,

As cos $\frac{1}{2}$(B — A) : cos $\frac{1}{2}$(B + A) :: tan $\frac{1}{2}$(AC + BC) : tan $\frac{1}{2}$AB (Art. 77, eq. 17) = $25° 14' 42''$.

As cos $\frac{1}{2}$(AC — BC) : cos $\frac{1}{2}$(AC + BC) :: tan $\frac{1}{2}$(B + A) : cotan $\frac{1}{2}$ACB (eq. 15) = $41° 46' 17''$.

Wherefore, AB = $50° 29' 24''$, and ACB = $83° 32' 34''$.

5. Given, the three sides, $80° 16'$, $60° 44'$, and $50° 20'$, to find the angles.



Construction. On the primitive, lay down AB equal to one of the given sides; from A and B, as poles, at distances equal to the other sides respectively, describe (Art. 156) two circles, ICK and GCH, cutting each other in C; through A, C, and B, C, describe (Art. 159) two great circles; then ABC is the triangle proposed.

Calculation. Through C, describe the great circle CD at right angles to AB, and bisect AB in E, then,

As tan AE : tan $\frac{1}{2}$(AC + BC) : : tan $\frac{1}{2}$(AC — BC) : tan ED (Art. 74) = 8° 56′ 14″.

Whence, AD = 49° 4′ 14″, and BD = 31° 11′ 46″.

As tan AC : tan AD : : rad : cos A (Art. 62) = 49° 44′ 18″.

As tan BC : tan BD : : rad : cos B = 59° 51′ 33″.

As sin AC : sin AD : : rad : sin ACD (Art. 58) = 60° 0′ 17″.

As sin BC : sin BD : : rad : sin BCD = 42° 17′ 25″

Wherefore, ACB = 102° 17′ 42″.

Or, without a perpendicular.

Take P = half the sum of the sides; then,

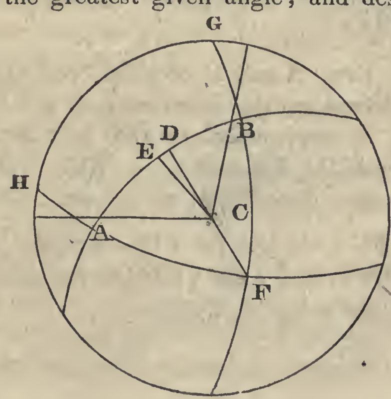
As sin P.sin (P — BC) : sin (P — AC).sin (P — AB) : : rad^2 : $\tan^2 \frac{1}{2}$A (Art. 77, eq. 7) = 24° 52′ 9″.

Wherefore, A = 49° 44′ 18″.

And (Art. 59), B = 59° 51′ 34″, and C = 102° 17′ 42″.

6. Given, three angles, 60° 36′, 66° 20′, and 99° 50′, to find the sides.

Construction. At the centre of the primitive, make an angle C equal to the greatest given angle; and describe a great



circle, making, with the right circles including that angle, two angles respectively equal to the other two given ones (Art. 166); and let A and B be those angles: then ABC is the triangle proposed.

Calculation. Through C let two great circles, CD and CE, pass; the former at right angles to AB, and the latter bisecting the angle ACB. Then,

As cotan $\frac{1}{2}(B + A)$: tan $\frac{1}{2}(B - A)$ :: tan ACE : tan ECD (Art. 75) $= 6° \ 47' \ 44''$.

Whence, ACD $= 56° \ 42' \ 44''$, and BCD $= 43° \ 7' \ 16''$.

As tan A : cotan ACD :: rad : cos AC $=$ $68° \ 17' \ 13''$.

As tan B : cotan BCD :: rad : cos BC (Art. 64) $= 62° \ 5' \ 42''$.

As sin A : rad :: cos ACD : cos AD (Art. 67) $= 50° \ 57' \ 5''$.

As sin B : rad :: cos BCD : cos BD $=$ $37° \ 9' \ 41''$.

AB $= 88° \ 6' \ 46''$.

Or, without a perpendicular,

As sin B.sin C : cos $\frac{1}{2}(A + C - B)$.cos $\frac{1}{2}(A + B - C)$ :: rad^2 : cos^2 $\frac{1}{2}$BC (Art. 77, eq. 9);

whence, BC $= 62° \ 5' \ 58''$. And (Art. 59), AB $= 88° \ 6' \ 46''$, AC $= 68° \ 17' \ 12''$.

Otherwise. From A and B, as poles, describe the great circles FG, FH, cutting the primitive in G and H; then the triangle FGH is supplemental to ABC (Art. 54). That is, GH $= 180° -$ ACB; FH $= 180° -$ ABC; FG $= 180° -$ BAC; whence the sides of FGH are known. Then,

As sin FH.sin FG : sin $\frac{1}{2}$(HG + FH — FG).sin $\frac{1}{2}$(HG + FG — FH) :: rad^2 : sin^2 $\frac{1}{2}$HFG or cos^2 $\frac{1}{2}$AB;

whence, AB $= 88° \ 6' \ 46''$, as before.

Promiscuous Examples.

The following astronomical terms being occasionally used in the succeeding examples, it is deemed advisable to insert their definitions.

The *axis* of the earth is the line through the centre on which it revolves; the points where the axis meets the surface of the earth are the *poles* of the earth; and the points where the axis produced meets the concave surface of the visible heavens, are the *celestial poles.*

The common section of the earth's surface and a plane passing through its centre at right angles to its axis, is termed the *terrestrial equator;* and the section of the same plane and the concave surface of the visible heavens, is called the *celestial equator*, or *equinoctial circle.*

Meridians are great circles passing through the poles, and, consequently, cutting the equator at right angles.

A right line drawn from the centre of the earth, through the place of an observer, and continued till it meets the celestial sphere, cuts it in a point which is termed the *zenith* of the place. The point where the same line, extended beyond the centre, meets the celestial sphere, is termed the *nadir.*

The great circle of which the zenith and nadir are the poles, is termed the *horizon.*

A meridian passing through the zenith of any place, is called the meridian of that place.

The angle formed by the meridian of a place, and a great circle passing through the zenith, and a celestial object, is called the *azimuth.*

The distance, reckoned in degrees, minutes, &c., on the meridian, between the equator and a place on the earth's surface, is termed the *latitude* of that place.

The angle contained between the meridian of a place, and some other assumed as a first meridian, is termed the *longitude* of the place. The longitude is usually reckoned eastward or westward as far as 180°.

The common section of the plane of the earth's orbit and the celestial sphere, is called the *ecliptic*. This circle is the sun's apparent annual path.

The points where the ecliptic cuts the celestial equator are termed the *equinoxes*. The point in which the sun appears when passing from the southern to the northern side of the equator, is termed the *vernal equinox*.

The arc of the celestial equator, reckoning eastward, between the vernal equinox and the point where a meridian through any celestial object cuts the equator, is called the *right ascension* of that object; and the arc of the meridian between the object and the equator, is termed the *declination*.

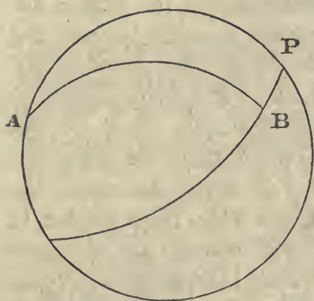
The arc of the ecliptic, reckoned eastward, between the vernal equinox and the point where a great circle, passing through the pole of the ecliptic and any celestial object, cuts the ecliptic. is termed the *longitude* of that object; and the arc of that great circle, between the object and the ecliptic, is termed its *latitude*.

The angle formed by the ecliptic and the celestial equator is termed the *obliquity* of the ecliptic.

The right ascension of the point in the celestial equator which is cut by the meridian of a place, is called the right ascension of the mid-heaven.

The point of the ecliptic which is cut by a great circle passing through its pole and the zenith of a place, is called the *nonagesima degree*.

Ex. 1. Required, the distance on a great circle of the earth between Point Venus in Otaheite, and Edinburgh; also, the direction of each from the other; the latitude of the former being 17° 29′ South, and longitude 149° 29′ West; and the atitude of the latter 55° 57′ North, and longitude 3° 11′ West.



Construction. Assume the primitive circle as the meridian of Point Venus; and take P as the north pole; from P lay down PA = the polar distance, 107° 29′, of the place; through P describe a great circle, making, with the primitive at P, an angle 146° 18′, equal to the difference of longitudes (Art. 162); on this circle lay down PB 34° 3′ = the polar distance of Edinburgh (Art. 169); through A and B describe a great circle. Then AB is the arc, and PAB, PBA the angles required.

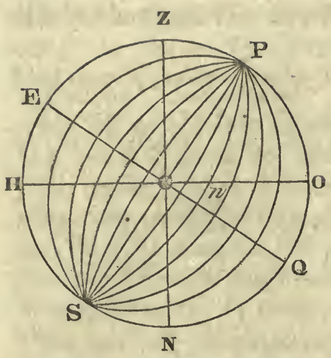
Computation. The sides AP, BP, and the angle APB, being given, the angles at A and B are found (Art. 77, eq. 11, 12), viz.: PAB = 25° 32′, PBA = 47° 15′. Then (eq. 18), AB = 133° 53′. Consequently, Edinburgh bears from Point Venus, N. 25° 32′ E.; and Point Venus bears from Edinburgh, N. 47° 15′ W.; distance 133° 53′.

Ex. 2. Required, the bearing and distance on a great circle of the Observatory at Greenwich from the Capitol at Washington: the latitude of the former being 51° 28′ 40″ N.; the latitude of the latter 38° 53′ N., and longitude 77° 2′ W. from the meridian of Greenwich.

Ans. N. 49° 20′ E.; dist. 53° 8′.

Ex. 3. When the sun's declination is 23° 28′ N., how long after midnight does it rise in latitude 39° 57′ N., and how far from the northern point of the horizon?

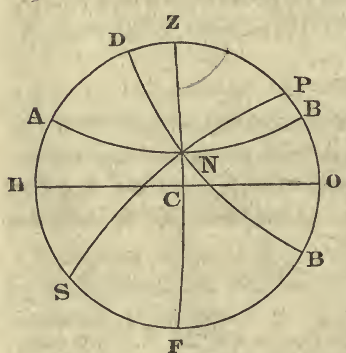
The construction of this problem is readily understood



from the figure; taking HO for the horizon; $OP=39° 57'$, the latitude of the place; describing a circle about the pole P, at the distance 66° 32′, the sun's polar distance; supposing this circle to cut HO in n; then, the great circle PnS being described, the triangle POn, right angled at O, will contain the elements required; the angle nPO, converted into time at the rate of one hour to 15°, or four minutes to 1°, will be the time required; and nO, the required distance from the northern part of the horizon.

Result: On 58° 42′, OPn 68° 41′.

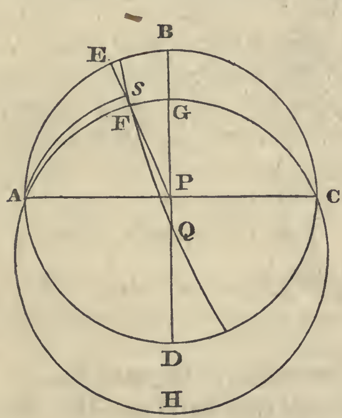
Ex. 4. Given, the latitude of the place, 39° 56′ N.; declination of the sun, 23° 28′ N.; and zenith distance, 60° 30′; to find the azimuth and polar angle.



To construct this on the plane of the meridian, take the primitive circle HZPO for the meridian; HO, the horizon; Z, the zenith; P, the north pole; and, therefore, PO = the latitude of the place. About P describe a circle at a distance = 66° 32′, the sun's polar distance (Art. 156); and about Z, at a distance = 60° 30′, the sun's zenith distance, describe another circle, cutting the former in N; then describe great circles through NZ and NP (Art. 159); and in the triangle NPZ, the angle PZN is the azimuth, and ZPN the polar angle required. Those angles may be computed by Art. 77, eq. 5, 6, or 7.

Result: NZP 82° 56′, ZPN 70° 20′.

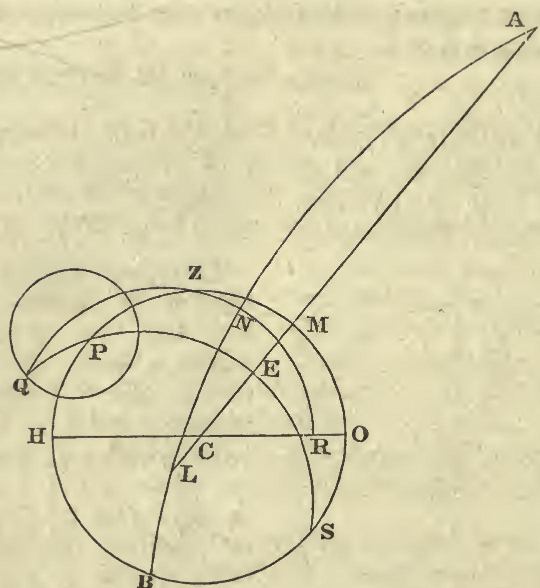
Ex. 5. In the beginning of 1842, the right ascension of Aldebaran (the bull's eye) was 66° 42′ 50″, and the declination 16° 8′ 36″ N. Required, the longitude and latitude at that time; the obliquity of the ecliptic being 23° 27′ 40″.



To project this on the plane of the celestial equator, take the primitive ABCD to denote that circle; A and C being the equinoxes, and P the north pole: through A, C, describe the circle AGCH, making an angle of 23° 27′ 40″ with the primitive; then AGCH will represent the ecliptic; take Q the pole of AGCH; make AE = the star's right ascension; through E, P, draw the line EP; this will represent the meridian passing through the star. On EP lay down ES (Art. 169) = the star's declination; then S is the place of the star. Through SQ describe (Art. 159) a great circle cutting the circle AGC in F; then AF is the longitude, and FS the latitude required. Through AS describe a great circle; then the angle EAS may be computed by Art. 60, and thence AF and FS by Arts. 71 and 70.

Result: long. 67° 34′ 23″; lat. 5° 31′ 18″ S.

Ex. 6. Given, the latitude of the place, 40° north; the obliquity of the ecliptic, 23° 28′; and the right ascension of the mid-heaven, 60°; to find the longitude and altitude of the nonagesima degree.



To project this example on the plane of the meridian, let the primitive circle HZMO denote the meridian; HO, the horizon; Z, the zenith, or pole of HO. On the primitive circle lay down HP, ZM, each equal to the latitude of the place, or 40°; then P will be the pole of the equator. About P, at the distance 23° 28′, the obliquity of the ecliptic, describe a less circle; through M and C (the centre of the primitive), draw the right circle CMA; this will represent the equator. Make MA = 60°, the right ascension of the mid-heaven; the point A will be the vernal equinox. Make CL = 30°; then L will be the autumnal equinox, for CM = 90°. About the pole L describe the great circle QPS, cutting the less circle in Q; then Q is the pole of the ecliptic. Through L describe a great circle having Q for its pole; this circle will denote the ecliptic, and pass through A. Lastly, through Q and Z describe a great circle QZNR, cutting the ecliptic in N, and the horizon in R; N is the nonagesima, whose longitude is AN, and altitude NR or QZ.

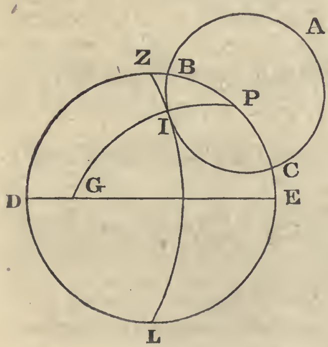
Calculation. Let QP cut the equator in E; then AE—AM =EM=ZPE. Hence, QPZ becomes known = 150°; then, in the triangle ZPQ, ZP = 50°, PQ = 23° 28′; with which and the contained angle, we may find the angle ZQP = 23° 53′ 44″, and side QZ = NR = 71° 0′ 18″. Consequently, the longitude of the point N = 66° 6′ 16″.

Ex. 7. Given, the latitude of the place 41° 36′ north, and the sun's declination 22° 10′ north, to find the time from noon, or the polar angle, when the sun is on the vertical circle which passes through the east and west points of the horizon, and the sun's altitude at the same time.

Result: Polar angle 62° 41′, or time from noon 4 hours 11 minutes; altitude 34° 38′.

Ex. 8. On the first day of the year 1836, when it was noon at Greenwich, the right ascension of Jupiter was 101° 55′, and declination 23° 4′ 2″ north; at the same time, the right ascension of Saturn was 212° 17′, and the declination 10° 30′ 50″ south. What was their distance on the arc of a great circle?

Ans. 112° 44′ 59″.



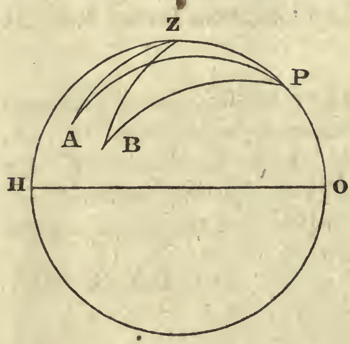
Ex. 9. In the beginning of 1842, the declination of the polar star was 88° 28′ north. What was its azimuth, when its elongation from the meridian was the greatest, the observer being in latitude 40° north?

Ans. 2° 0′ 6″.

Ex. 10. In latitude 40° north, required the duration of twilight at the several times when the sun's declination is 23° 28′ south, 5° 50′ south, and 23° 28′ north; the twilight being supposed to begin when the sun's centre is 49′ below the horizon,* and to end when it is 18° below.

Ans. 1 h. 35 m.; 1 h. 29 m.;† and 2 h. 4 m.

Ex. 11. Given, two zenith distances of the sun's centre, 65° 20′ and 60° 18′, taken at the same place, both being in the forenoon; the interval between the observations, measured by a good time-piece, 1 hour 32 minutes; the sun's declination 20° south, and the approximate latitude of the place 40° 15′ north; to find the time of the last observation, and the correct latitude of the place.



Let HO be the horizon; HZO, the meridian; Z, the zenith; P, the north pole; BZ and AZ, the given zenith distances. Then PZ is the colatitude; and PB, PA, the sun's polar distance.

Now (by Art. 77, eq. 1),

$$\cos AZ = \cos ZPA . \sin AP \sin ZP + \cos AP . \cos ZP;$$

and

$$\cos BZ = \cos ZPB . \sin BP . \sin ZP + \cos BP . \cos ZP.$$

Hence (AP being = BP),

$$\cos AZ - \cos BZ = (\cos ZPA - \cos ZPB) \sin AP . \sin ZP;$$

* The refraction of the sun's light, when in the horizon, is 33′, and apparent semi-diameter about 16′; hence his upper limb is just visible when the centre is 49′ below the horizon.

† This is the shortest twilight in latitude 40° N., and occurs twice in the year, viz., in spring and autumn. when the sun's declination is 5° 50′ S.

consequently (Art. 37, eq. 13),

$$\sin \tfrac{1}{2}(BZ + AZ).\sin \tfrac{1}{2}(BZ - AZ) = \sin \tfrac{1}{2}(BPZ + APZ).\ \sin \tfrac{1}{2}(BPZ - APZ) \sin AP.\sin ZP;$$

of which, BPZ—APZ is given from the elapsed time. Hence,

$$\sin \tfrac{1}{2}(BPZ + APZ) = \frac{\sin \tfrac{1}{2}(BZ + AZ).\sin \tfrac{1}{2}(BZ - AZ)}{\sin AP.\sin ZP.\sin \tfrac{1}{2}(BPZ - APZ)} = 15^\circ\ 51'\ 11''.$$

Whence APZ = 4° 21′ 11″, and (by Art. 59) AZP = 175° 17′ 24″; from which we find (by Art. 77, eq. 18), ZP = 49° 50′; and therefore PO, the corrected latitude, 40° 10′;* and time from noon, when the least zenith distance was taken, 17 minutes 24 seconds.

Ex. 12. Given, the approximate latitude, 39° 26′ N.; the sun's declination, 20° 41′ N.; sun's corrected zenith distance at 11 h. 30′ 15″, by watch, 21° 30′; and at 12 h. 26′ 28″, by watch, 18° 52′. Required, the corrected latitude, and error of the watch.

Ans. Lat. 39° 29′; watch too fast, 18 min. 57 sec.

Ex. 13. At the time when Sirius and Aldebaran were in the same vertical circle, the true zenith distance of the latter was found to be 30° 16′; the stars being on the east of the meridian. Required, the latitude of the place, and right ascension of the mid-heaven; the right ascension of Sirius being 99° 33′ 30″, and its declination 16° 30′ 18″ south; the right ascension of Aldebaran 66° 43′ 45″, and its declination 16° 11′ 12″ north.

Ans. Lat. 35° 8′ 42″ N.; right ascension of mid-heaven 39° 17′ 23″.

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* When the latitude thus found differs considerably from the approximate latitude, the computation ought to be repeated, with the result first obtained substituted for the approximate latitude.

Examples of a Mixed Character.

Ex. 1. From the top of a cliff near a river, two buoys at anchor being observed, whose distance from each other was known to be 300 yards, their angles of depression below the plane of the observer were found to be 30° and 40° respectively; and the angle at the eye, subtended by the line joining them, was 37°. Required, the distance of each buoy from the observer, and the altitude of the cliff above the level of the water.

The observed depressions, each increased by 90°, form two sides; and the angular distance, the base of a spherical triangle, with which the angle opposite the base is found $=44°$. This is the horizontal angle, subtended by the line joining the buoys.

Drawing then a vertical line through the position of the observer to meet the plane of the water; and, from the point where it meets that plane, drawing lines to the buoys; those lines will be to each other as the cotangents of the given angles of depression; and the angle which they make with each other will be 44°, as above found. The construction is this:

Take $AB=300$, the given distance; on AB describe a segment of a circle ACB, containing an angle of 44°; complete the circle, and bisect the arc AEB in E; make the angles ABF and $BAF = 30°$ and 40° respectively; draw FG at right angles to AB; join EG, and produce it to meet the circle in C; join CA, CB; then, since the angle ACB is bisected by the line CG,

As AC : CB :: AG : BG (3.6) :: cot BAF : cot ABF.

Consequently, C is the point in the plane of the water which is cut by a vertical line passing through the place of the observer.

The calculation is easily made. For (Art. 28 and Art. 37, eq. 8),

As sin (BAF+ABF) : sin (BAF — ABF) :: AB : BG—AG.

Hence AG is known. If we join AE, BE, the side AB, and all the angles of the triangle ABE, are given; whence AE becomes known. Then, in the triangle AGE, the sides AE, AG, and the contained angle, are known; from which the angle AEG = ABC is found. In the triangle ABC, we then have the base AB and all the angles, to find AC and BC. Then, from either of these and the angle of depression, the altitude of the cliff may be found. Lastly, with the distances AC, BC, and the angles of depression, the distances from A and B to the place of the observer are determined

Result: Altitude of cliff, 249; distances, 388 and 498

Ex. 2. Suppose an observer on a frozen lake takes the altitudes and angular distance of two cliffs on the shore, as follows: altitude of first, 50°; of second, 55° 30′; angular distance, 25° 20′. Then, advancing on the ice 500 yards, in the vertical plane which passes through the first cliff, the altitudes are 57° and 59° respectively. Required, the distance of the cliffs from each other, and their respective altitudes above the surface of the lake.

Ans. Distance of cliffs, 2347 or 2850 yards; altitude of first, 2636 yards; of second, 4069 or 552 yards.

Ex. 3. The crew of a vessel at sea discovering a light in the horizon, which they suppose to be a vessel on fire, sail directly towards it, over 1° of a great circle, when they perceive that it is a fire on a mountain, which is then 1° 30′ above the horizon. Required, the distance of the light when first seen, and the height of the mountain above the level of the sea; the earth being considered as a sphere whose radius is 3968 miles.

Answer. Distance, 138.5 miles; height, 2.4 miles.

Ex. 4. Given, AB, a horizontal line, 1785 yards in length, running exactly north; D, C, two elevated peaks, eastward from AB, such that the elevation of C above the plane of the horizon, seen from A, is 16° 30′, and the elevation of D 20° 40′. But, seen from B, the elevation of C is 14° 25′, and the elevation of D 13° 15′. Also, the angle BAC, taken in the oblique plane passing through AB and C, is 38° 16′; the angle CAD, taken in the plane which passes through A, C and D, 87° 20′. Required, the distance and bearing of DC when reduced to the horizontal plane on which AB lies.

Result: DC, N. 29° W.; 2,556 yards.

THE END